
ON THE DIOPHANTINE EQUATION $9^x + 9^y + n^z = w^2$

Apirat Siraworakun and Suton Tadee*

Department of Mathematics, Faculty of Science and Technology,
Thepsatri Rajabhat University

*E-mail: suton.t@lawasri.tru.ac.th

Abstract

In this work, we give some conditions for the non-existence of solutions of the Diophantine equation $9^x + 9^y + n^z = w^2$, where n is a positive integer and x, y, z, w are non-negative integers. Moreover, we find all solutions of this equation for some positive integer n .

Keywords: Diophantine equation, Integer solution, Congruence

Introduction

The Diophantine equation is a well-known concept in number theory with numerous variations. One notable form is an exponential Diophantine equation $a^x + b^y = w^2$. Many authors study this equation by giving a and b as explicit positive integers, see (Asthana & Singh, 2020), (Sroysang, 2014). Moreover, a or b are given as variables under certain condition in various articles, see (Chotchaisthit, 2012), (Siraworakun & Tadee, 2023), (Tadee, 2023) and (Tangjai & Chubthaisong, 2021).

The Diophantine equation $a^x + b^y + c^z = w^2$, where a, b and c are positive integers, has been formulated and analyzed in subsequent studies. For example, Bacani and Rabago (2014) solved the Diophantine equation $3^x + 5^y + 7^z = w^2$, when x, y, z, w are non-negative integers. Analogously, a, b and c can be considered as variables subjected to certain conditions.

In addition, the non-negative integer solution of the equation $a^x + b^y + c^z = w^2$ is investigated, when all exponents are 1 or 2. For example, Burshtein (2020) showed all solutions of the equation $p^x + (p+1)^y + (p+2)^z = M^2$ for primes $p \geq 2$ when $1 \leq x, y, z \leq 2$. Pandichelvi and Sandhya (2022) investigated integer solutions to the Diophantine equation $p_1^x + p_2^y + p_3^z = M^2$, where p_1, p_2, p_3 are prime numbers and $x, y, z \in \{1, 2\}$.

In this research, we investigate some conditions regarding the non-existence of non-negative integer solutions of the Diophantine equation

$$9^x + 9^y + n^z = w^2, \tag{1}$$

where n is a positive integer and x, y, z, w are non-negative integers. Furthermore, we explore all solutions of this equation for some positive integer n . Without loss of generality, we may assume that $x \leq y$.

Main Results

In the beginning, we provide some conditions for the non-existence of non-negative integer solutions of the Diophantine equation (1).

Theorem 1. The Diophantine equation (1) has no non-negative integer solution if

1. $n \equiv 0, 1 \pmod{4}$ or
2. $n \equiv 2 \pmod{4}$ and $z \neq 1$ or
3. $n \equiv 3 \pmod{4}$ and z is even.

Proof. It is easy to see that $w^2 \equiv 0, 1 \pmod{4}$ and $9^x \equiv 9^y \equiv 1 \pmod{4}$.

1. Assume that $n \equiv 0, 1 \pmod{4}$. Suppose that (1) has a non-negative integer solution. We obtain that $w^2 \equiv 2, 3 \pmod{4}$. It is a contradiction. Thus, (1) has no non-negative integer solution.

2. Assume that $n \equiv 2 \pmod{4}$ and $z \neq 1$. Suppose that (1) has a non-negative integer solution. Since $z \neq 1$, we get $z = 0$ or $z \geq 2$. From (1), we have $w^2 \equiv 3 \pmod{4}$ or $w^2 \equiv 2 \pmod{4}$, respectively. It is a contradiction. Thus, (1) has no non-negative integer solution.

3. Assume that $n \equiv 3 \pmod{4}$ and z is even. Suppose that (1) has a non-negative integer solution. Then, $n^z \equiv 1 \pmod{4}$. From (1), we have $w^2 \equiv 3 \pmod{4}$. It is a contradiction. Thus, (1) has no non-negative integer solution. ■

Next, we give some results about the Diophantine equation (1), when $z = 1$.

Theorem 2. Let $x = y = 0$ and $z = 1$. The Diophantine equation (1) has no non-negative integer solution if $n \equiv 0 \pmod{3}$.

Proof. Assume that $n \equiv 0 \pmod{3}$. Suppose that (1) has a non-negative integer solution. Then $w^2 \equiv 2 \pmod{3}$. It is a contradiction since $w^2 \equiv 0, 1 \pmod{3}$. ■

Theorem 3. Let $x, y \geq 1$ and $z = 1$. The Diophantine equation (1) has no non-negative integer solution if

1. $n \equiv 2 \pmod{3}$ or
2. $n \equiv 3, 6 \pmod{9}$.

Proof. 1. Assume that $n \equiv 2 \pmod{3}$. Suppose that (1) has a non-negative integer solution. It implies that $w^2 \equiv 2 \pmod{3}$. It is a contradiction since $w^2 \equiv 0, 1 \pmod{3}$.

2. Assume that $n \equiv 3, 6 \pmod{9}$. Suppose that (1) has a non-negative integer solution. Since $n \equiv 3, 6 \pmod{9}$, we have $3|n$. From (1), we have $3|w$, $3|(w - 3^x)$ and $3|(w + 3^x)$. Then $9|(w - 3^x)(w + 3^x)$ and $9|(9^y + n)$. Thus $9|n$. It is a contradiction. ■

Theorem 4. Let $x = 0$ and $y \geq 0$ and $z = 1$. Then the equation (1) has a non-negative integer solution if and only if $1 + n = a(2 \cdot 3^m + a)$ for some positive integer a and non-negative integer m . Moreover, if the equation (1) has a non-negative integer solution, then $(n, x, y, z, w) = (a(2 \cdot 3^m + a) - 1, 0, m, 1, a + 3^m)$.

Proof. First, assume that (1) has a non-negative integer solution. Then $1 + n = w^2 - 3^{2y} = (w - 3^y)(w + 3^y)$. Choose $a = w - 3^y$ and $m = y$. Thus, $1 + n = a(2 \cdot 3^m + a)$. Conversely, assume that $1 + n = a(2 \cdot 3^m + a)$ for some positive integer a and non-negative integer m . Then

$$1 + 9^m + n = a(2 \cdot 3^m + a) + 9^m = a^2 + 2 \cdot 3^m a + 3^{2m} = (a + 3^m)^2.$$

Thus, (1) has a non-negative integer solution, which is $(n, x, y, z, w) = (a(2 \cdot 3^m + a) - 1, 0, m, 1, a + 3^m)$. ■

From Theorem 1, it remains to study 2 cases: 1) $n \equiv 2 \pmod{4}$ and $z = 1$, 2) $n \equiv 3 \pmod{4}$ and z is odd. Next, we investigate the non-negative integer solutions of the Diophantine equation (1) for $n \equiv 2 \pmod{4}$ and $z = 1$.

Theorem 5. Let $x, y \geq 1$ and $z = 1$. If $n \equiv 2 \pmod{12}$, then the Diophantine equation (1) has no non-negative integer solution.

Proof. Assume that $n \equiv 2 \pmod{12}$. Then $n \equiv 2 \pmod{3}$. By Theorem 3(1), we can conclude that (1) has no non-negative integer solution. ■

By Theorem 1(2), Theorem 4 and Theorem 5, we found all solutions of the Diophantine equation (1) for some positive integer n .

Example 6. The Diophantine equation $9^x + 9^y + 2^z = w^2$ has only one non-negative integer solution, i.e., $(x, y, z, w) = (0, 0, 1, 2)$.

Example 7. The Diophantine equation $9^x + 9^y + 14^z = w^2$ has only one non-negative integer solution, i.e., $(x, y, z, w) = (0, 0, 1, 4)$.

Example 8. The Diophantine equation $9^x + 9^y + 26^z = w^2$ has only one non-negative integer solution, i.e., $(x, y, z, w) = (0, 1, 1, 6)$.

Theorem 9. Let $x, y \geq 1$ and $z = 1$. If $n = 12k + 6$ for some non-negative integer k with $k \equiv 0, 2 \pmod{3}$, then the Diophantine equation (1) has no non-negative integer solution.

Proof. Assume that $n = 12k + 6$ for some non-negative integer k with $k \equiv 0, 2 \pmod{3}$. Therefore $n \equiv 3, 6 \pmod{9}$. By Theorem 3(2), we can conclude that (1) has no non-negative integer solution. ■

By Theorem 1(2), Theorem 4 and Theorem 9, we found all solutions of the Diophantine equation (1) for some positive integer n with above conditions.

Example 10. The Diophantine equation $9^x + 9^y + 6^z = w^2$ has only one non-negative integer solution, i.e., $(x, y, z, w) = (0, 1, 1, 4)$.

Example 11. The Diophantine equations $9^x + 9^y + 30^z = w^2$, $9^x + 9^y + 42^z = w^2$ and $9^x + 9^y + 66^z = w^2$ have no non-negative integer solution.

We can ensure that the Diophantine equation (1) has no non-negative solution in the following theorem.

Theorem 12. Let $z = 1$. If $n = 4 \cdot 3^k + 2$, where $k \geq 2$, then the Diophantine equation (1) has no non-negative integer solution.

Proof. Assume that $n = 4 \cdot 3^k + 2$, where $k \geq 2$. Suppose that (1) has a non-negative integer solution. Then $n \equiv 2 \pmod{12}$. By Theorem 5, we obtain that $x = 0$ and $y \geq 0$. Assume that $y \geq 1$. From (1), we get $3 \mid w$, $3 \mid (w - 3^y)$ and $3 \mid (w + 3^y)$. This implies that $9 \mid (4 \cdot 3^k + 3)$. It is a contradiction with the fact that $k \geq 2$. Therefore $y = 0$. From (1), we have $3^k = \left(\frac{w}{2} - 1\right)\left(\frac{w}{2} + 1\right)$. There exists a non-negative integer u such that

$\frac{w}{2} - 1 = 3^u$ and $\frac{w}{2} + 1 = 3^{k-u}$. Thus $2 = 3^u(3^{k-2u} - 1)$. Then $u = 0$ and $k = 1$. It is a contradiction. Therefore, (1) has no non-negative integer solution. ■

By Theorem 1(2) and Theorem 12, we have the following example.

Example 13. The Diophantine equations $9^x + 9^y + 38^z = w^2$, $9^x + 9^y + 110^z = w^2$ and $9^x + 9^y + 326^z = w^2$ have no non-negative integer solution.

For another cases of n , we cannot guarantee all solution. However, we show that the Diophantine equation (1) has the unique non-negative integer solution, when $x \leq 1$ and $n = 10$.

Theorem 14. If $x \leq 1$ and $n = 10$, then the Diophantine equation (1) has the unique non-negative integer solution, i.e., $(x, y, z, w) = (1, 2, 1, 10)$.

Proof. Since $n = 10$ and Theorem 1(2), we have $z = 1$. We consider the following cases.

Case 1. $x = 0$ and $y = 0$. From (1), we have $w^2 = 12$. This is impossible.

Case 2. $x = 0$ and $y \geq 1$. From (1), we have $11 = (w - 3^y)(w + 3^y)$. Then $w - 3^y = 1$, $w + 3^y = 11$ and $2 \cdot 3^y = 10$. Thus $3^y = 5$. It is a contradiction since y is an integer.

Case 3. $x = 1$ and $y \geq 1$. From (1), we get $w^2 = 9^y + 19$. It implies that $w^2 \equiv (-1)^y - 1 \pmod{10}$. Since $w^2 \equiv 0, 1, 4, 5, 6, 9 \pmod{10}$, we obtain that y is even. There exists a positive integer k such that $y = 2k$. Therefore $19 = (w - 9^k)(w + 9^k)$ and so $w - 9^k = 1$ and $w + 9^k = 19$. Then $2 \cdot 9^k = 18$. Thus $k = 1$ and $y = 2$. Hence $(x, y, z, w) = (1, 2, 1, 10)$. ■

Finally, we consider $n \equiv 3 \pmod{4}$. Then $n \equiv 3, 7 \pmod{8}$.

Theorem 15. If $n \equiv 3 \pmod{8}$, then the Diophantine equation (1) has no non-negative integer solution.

Proof. Assume that (1) has a non-negative integer solution. Since $n \equiv 3 \pmod{8}$, we get $n \equiv 3 \pmod{4}$. By Theorem 1(3), it implies that z is odd. Then $n^z \equiv 3 \pmod{8}$. From (1), $w^2 \equiv 5 \pmod{8}$. It is a contradiction since $w^2 \equiv 0, 1, 4 \pmod{8}$. ■

For $n \equiv 7 \pmod{8}$ and z is odd, the existence of the solutions cannot be confirmed. However, we show the solutions of the Diophantine equation (1), when $z = 1$.

Theorem 16. Let $x, y \geq 1$ and $z = 1$. If $n \equiv 23 \pmod{24}$, then the Diophantine equation (1) has no non-negative integer solution.

Proof. Assume that $n \equiv 23 \pmod{24}$. Then $n \equiv 2 \pmod{3}$. By Theorem 3(1), we can conclude that (1) has no non-negative integer solution. ■

By Theorem 4 and Theorem 16, we found all solutions of the Diophantine equation (1) for $z = 1$ and some positive integer n .

Example 17. The Diophantine equation $9^x + 9^y + 23 = w^2$ has only one non-negative integer solution, i.e., $(x, y, w) = (0, 0, 5)$.

Example 18. The Diophantine equation $9^x + 9^y + 47 = w^2$ has only one non-negative integer solution, i.e., $(x, y, w) = (0, 0, 7)$.

Example 19. The Diophantine equation $9^x + 9^y + 71 = w^2$ has only one non-negative integer solution, i.e., $(x, y, w) = (0, 1, 9)$.

Example 20. The Diophantine equation $9^x + 9^y + 95 = w^2$ has no non-negative integer solution.

Theorem 21. Let $x, y \geq 1$ and $z = 1$. If $n = 24k + 15$ for some non-negative integer k with $k \equiv 0, 1 \pmod{3}$, then the Diophantine equation (1) has no non-negative integer solution.

Proof. Assume that $n = 24k + 15$ for some non-negative integer k with $k \equiv 0, 1 \pmod{3}$. Therefore, $n \equiv 3, 6 \pmod{9}$. By Theorem 3(2), we can conclude that (1) has no non-negative integer solution. ■

By Theorem 4 and Theorem 21, we found all solutions of the Diophantine equation (1) for $z = 1$ and some positive integer n with above conditions.

Example 22. The Diophantine equation $9^x + 9^y + 15 = w^2$ has only one non-negative integer solution, i.e., $(x, y, w) = (0, 1, 5)$.

Example 23. The Diophantine equation $9^x + 9^y + 39 = w^2$ only two non-negative integer solutions. That are $(x, y, w) \in \{(0, 1, 7), (0, 2, 11)\}$.

Conclusions

This research demonstrates some conditions for the non-existence of non-negative integer solution of the Diophantine equation $9^x + 9^y + n^z = w^2$, where n is a positive integer and x, y, z, w are non-negative integers, and investigates all solutions of this equation for some positive integer n . However, the results of this work cannot cover all cases. For example, when $n = 18$ and $n = 54$. This unsolved problem remains an interesting topic for further investigation.

Acknowledgements

The authors would like to thank the reviewers for their careful reading of this manuscript and their useful comments. This work was supported by Research and Development Institute and Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

Declaration of conflicting interests

The authors declared that they have no conflicts of interest in the research, authorship, and this article's publication.

References

- Asthana, S. & Singh, M.M. (2020). On the Diophantine equation $3^x + 117^y = z^2$. *GANITA*, 70(2), 43-47.
- Bacani, J. B. & Rabago, J. F. T. (2014). On the Diophantine equation $3^x + 5^y + 7^z = w^2$. *Konuralp Journal of Mathematics*. 2(2), 64-69.
- Burshtein, N. (2020). Solutions of the Diophantine equations $p^x + (p + 1)^y + (p + 2)^z = M^2$ for primes $p \geq 2$ when $1 \leq x, y, z \leq 2$. *Annals of Pure and Applied Mathematics*. 22(1), 41-49.
- Chotchaisthit, S. (2012). On the Diophantine equation $4^x + p^y = z^2$ where p is a prime number. *American Journal of Mathematics and Sciences*. 1(1), 191-193.

- Pandichelvi, V. & Sandhya, P. (2022). Investigation of solutions to an exponential Diophantine equation $p_1^x + p_2^y + p_3^z = M^2$ for prime triplets (p_1, p_2, p_3) . *International Journal of Scientific Research and Engineering Development*. 5(2), 22-31.
- Siraworakun, A. & Tadee, S. (2023). Solutions of the Diophantine equation $p^x + q^y = z^2$, where $p, q \equiv 3 \pmod{4}$. *International Journal of Mathematics and Computer Science*. 18(1): 131-136.
- Sroysang, B. (2014). More on the Diophantine equation $4^x + 10^y = z^2$. *International Journal of Pure and Applied Mathematics*. 91(1). 135-138.
- Tadee, S. (2023). On the Diophantine equation $n^x + 10^y = z^2$. *WSEAS Transactions on Mathematics*. 22, 150-153.
- Tangjai, W. & Chubthaisong, C. (2021). On the Diophantine equation $3^x + p^y = z^2$ where $p \equiv 2 \pmod{3}$. *WSEAS Transactions on Mathematics*. 20, 283-287.