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วารสาร มรภ.กพ. วิทยาศาสตร์ คณิตศาสตร์ และเทคโนโลยี
Journal of KPRU Science Mathematics and Technology

Evaluation of Time-Dependent Displacement of Particle Bound in Anharmonics Oscillator Potential Perturb by Electrostatic External Force via Heisenberg Picture Method

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Abstract

We consider the time-dependent electrostatic force anharmonic oscillator potential in one-dimension. Here, we derive the existing Heisenberg equations of motion from Newton's second law. We can be used the time-dependent Hamiltonian operator for the time-dependent electrostatic force anharmonic oscillator system. We use the principal of Wronskian method solve for solution of the expectation value of displacement operator for particle bound in anharmonic oscillator potential system. The behavior of the expectation value of displacement operator is wave oscillate depend on the parameter linear frequency, the initial charge, the initial electric, the parameter σ and μ.

Keywords : Displacement operator, Time-dependent external force, Heisenberg equation of motion

INTRODUCTION

Having established the observational meaning of the variable we have called momentum operator ($\widehat{P}$), we can find the relation between newtonian mechaincs and the theory we have developed here. We have already studied the quantum-mechanical analog to Newton's first law; here is the second. The problem is to evaluate $d\langle P\rangle/dt$ and this requires the commutator $[\widehat{H},\widehat{P}]$. It is easily shown to be equal to $[\widehat{H},\widehat{P}] = i\hbar\partial V/\partial x$ so that

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Received : 27/08/2024 Revised : 18/11/2024 Accepted : 13/12/2024

$$\frac{d}{dt}\langle P\rangle = -\langle\frac{\partial V}{\partial x}\rangle \quad (1)$$

Thus there exists a relation among expectation values which exactly parallels Newton's second law expressed in terms of the potential energy. It is the most and it encourages us to look at Newtonian physics as a series of relationships among the expected results of measurements. Equation (1) are known as Ehrenfest's theorems or Heisenberg equation of motion (Mark, 2014) [6].

Russell Akridge will consider several model of transition amplitudes and probabilities for the harmonic oscillator potential with a time-dependent forcing function proportional to cosine beginning at time zero are evaluated to ground state order using time-dependent perturbation theory [1]. Next, Cordero-Soto Ricardo create model of the classical equations of motion for particle bounded harmonic oscillator potential perturbation the time-dependent damped oscillations are derived for the corresponding expectation values of the time-dependent position operator (Cordero-Soto, et.al., 2009) [4]. Currently, Castanos L.O. and Zuniga-Segundo A. show that the classical time-dependent forced harmonic oscillator with constant mass, linear frequency and the use of elementary properties of the coherent states simplifies the description of the system and in determining the solution in the RWA (Castanos & Zuniga-Segundo, 2019) [10]. The purpose of this paper, we will calculate the expectation value of displacement operator under the time-dependent external damping force anharmonic oscillator potential. The scheme of the article is as follows. In section materials and method, we illustrate evaluation of the expectation value of displacement operator of particle bounded in anharmonics oscillator potential under time-dependent electrostatic external force. In section results, we can be plot graph relation between the expectation value of displacement operator and time. The last section contains our conclusions.

MATERIALS AND METHODS

Evaluation of time-dependent displacement operator of particle bound in anharmonics oscillator potential under time-dependent external force

The behavior of an oscillator subjected to a time-dependent force is of importance in may contexts. When the oscillations are those of a small massive system, such as a molecule, the force can often be approximate as being constant over the dimensions of the unforced motions, and is described by adding the potential $-\lambda\widehat{q}_d + \alpha_a\xi(t)\widehat{P}_{lin}$ to the Hamiltonian.

Let $\widehat{P}_{lin}$ and $\widehat{q}_d$ be the canonical linear momentum and position operators (displacement

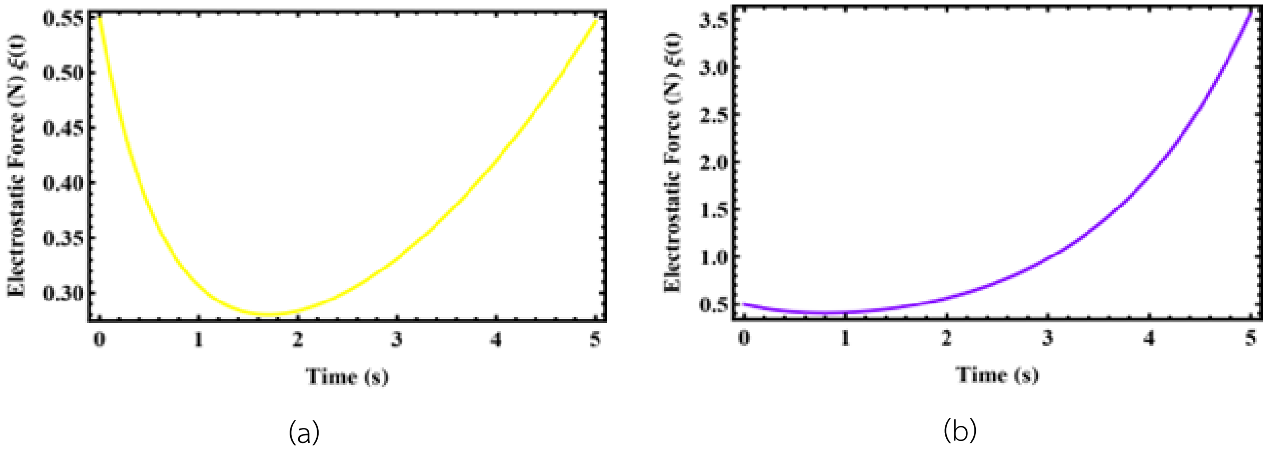


Fig. 1: Plots of the electrostatic external force as a function of time showing particle bound state $(2\sigma < \mu)$ (a) and particle unbound state $(2\sigma > \mu)$ (b)

operator) of a particle of mass m moving in one dimension under the influence of a Static linear restoring force and anharmonic oscillator potential, i.e. a potential energy proportional to $\widehat{q}_d^2$, of a strength such that the angular frequency is ω_f . The Hamiltonian, in the Heisenberg picture (Jussi & Jukka, 2020) [11], [12], [14], [16], is thus

$$\widehat{H}_F(q_d, P_{lin}, t) = \frac{\widehat{P}_{lin}^2}{2m} + \frac{1}{2}k\widehat{q}_d^2 - \lambda\widehat{q}_d + \alpha_a\xi(t)\widehat{P}_{lin}, \tag{2}$$

where $k = m\omega_f^2$ is the spring constant, λ are positive real constants, α_a are positive real constants, $\xi(t)$ is the time-dependent electrostatic force (Kurt & Tung-Mow, 2004), (Sang, 2003)[3], (Andrews, 2010)[2] in figure 1. The equation of motion of position and linear momentum operators in the Heisenberg picture are (Shigeji et.al. 2014)[7], (Moonsri et.al., 2020) [12] $\frac{d}{dt}\langle\widehat{q}_d(t)\rangle = \frac{1}{i\hbar}\langle[\widehat{q}_d, \widehat{H}_F(t)]\rangle$ and $\frac{d}{dt}\langle\widehat{P}_{lin}(t)\rangle = \frac{1}{i\hbar}\langle[\widehat{P}_{lin}(t), \widehat{H}_F(t)]\rangle$ The Heisenberg equations of motion the case of the expectation value of the position operator are :

$$\begin{aligned}\frac{d}{dt}\langle\widehat{q}_d(t)\rangle &= \frac{1}{i\hbar}\langle[\widehat{q}_d, \widehat{H}_F(t)]\rangle \\ &= \frac{1}{i\hbar}\langle[\frac{d}{dt}, (\frac{\widehat{P}_{lin}^2}{2m} + \frac{1}{2}k\widehat{q}_d^2 - \lambda\widehat{q}_d + \alpha_a\xi(t)\widehat{P}_{lin})]\rangle \\ &= \frac{1}{i\hbar}\langle(\frac{1}{2m}[\widehat{q}_d(t), \widehat{P}_{lin}^2] + \frac{1}{2}m\omega_f^2[\widehat{q}_d(t), \widehat{q}_d^2(t)] - \lambda[\widehat{q}_d(t), \widehat{q}_d(t)] + \alpha_a[\widehat{q}_d(t), \xi(t)\widehat{P}_{lin}])\rangle\end{aligned} \tag{3}$$

Substituting the commutator of $[\widehat{q}_d(t), \widehat{q}_d^2(t)] = 0, [\widehat{q}_d(t), \widehat{q}_d(t)] = 0$ and $[\widehat{q}_d(t), \widehat{P}_{lin}^2] = 2i\hbar\widehat{P}_{lin}, [\widehat{q}_d(t), \widehat{P}_{lin}] = i\hbar$ into The Heisenberg equations of motion equation (3) yield

$$\frac{d}{dt}\langle\widehat{q}_d(t)\rangle = \frac{\langle\widehat{P}_{lin}(t)\rangle}{m} + \alpha_a\xi(t) \tag{4}$$

These are the famous first-order linear ordinary differential equation. The Heisenberg equations of motion the case of the expectation value of the linear momentum operator are :

$$\begin{aligned}\frac{d}{dt}\langle\widehat{P}_{lin}(t)\rangle &= \frac{1}{i\hbar}\langle[\widehat{P}_{lin}(t),\widehat{H}_F(t)]\rangle \\ &= \frac{1}{i\hbar}\langle[\widehat{P}_{lin}(t),\frac{\widehat{P}^2_{lin}}{2m}+\frac{1}{2}k\widehat{q}^2_d-\lambda\widehat{q}_d+\alpha_a\xi(t)\widehat{P}_{lin}]\rangle \\ &= \frac{1}{i\hbar}\langle(\frac{1}{2m}[\widehat{P}_{lin}(t),\widehat{P}^2_{lin}]+\frac{1}{2}m\omega_f^2[\widehat{P}_{lin}(t),\widehat{q}^2_d]-\lambda[\widehat{P}_{lin}(t),\widehat{q}_d(t)]+\alpha_a\xi(t)[\widehat{P}_{lin}(t),\widehat{P}_{lin}(t)])\rangle\end{aligned} \tag{5}$$

Substituting the relationship commutator of $[\widehat{P}_{lin},\widehat{P}^2_{lin}]=0, [\widehat{P}_{lin},\widehat{q}^2_d(t)]=-2i\hbar\widehat{q}_d, [\widehat{P}_{lin},\widehat{q}_d(t)]=-i\hbar, [\widehat{P}_{lin},\widehat{P}_{lin}]$ into the Heisenberg equation of motion in case of the linear momentum operator equation (5), we have

$$\frac{d}{dt}\langle\widehat{P}_{lin}(t)\rangle = \lambda - m\omega_f^2\langle\widehat{q}_d(t)\rangle \tag{6}$$

These are the famous first-order linear ordinary differential equation. Differentiating equation (4) with respect to time gives us

$$\frac{d^2}{dt^2}\langle\widehat{q}_d(t)\rangle = \frac{1}{m}\frac{d}{dt}\langle\widehat{P}_{lin}(t)\rangle + \alpha_a\frac{d}{dt}\xi(t) \tag{7}$$

Substituting Equation (6) into equation (7), we can rewrite equation (7) as

$$\begin{aligned}\frac{d^2}{dt^2}\langle\widehat{q}_d(t)\rangle &= \frac{1}{m}(\lambda - m\omega_f^2\langle\widehat{q}_d(t)\rangle) + \alpha_a\frac{d}{dt}\xi(t) \\ \frac{d^2}{dt^2}\langle\widehat{q}_d(t)\rangle + \omega_f^2\langle\widehat{q}_d(t)\rangle &= \frac{\lambda}{m} + \alpha_a\frac{d}{dt}\xi(t)\end{aligned} \tag{8}$$

We now define of the time-dependent electrostatic external force to give $\xi(t) = q_0E_0e^{-\mu t}\cosh^2(\sigma t)$ (Russell 1995), where q_0 is the electron charge, μ is the positive damping constant, E_0 is the initial electrostatic force (Chew et. al., 2016), (Chew et. al., 2019), and σ are positive real constant. Sketch $\xi(t)$ as a function of time, we have With this definition, the first-derivative term becomes

$$\begin{aligned} \frac{d}{dt}\xi(t) &= \frac{d}{dt}(q_0E_0e^{-\mu t}\cosh^2(\sigma t)) = \frac{q_0E_0}{2}(\frac{d}{dt}(e^{-\mu t}\cosh(2\sigma t)) + \frac{de^{-\mu t}}{dt}) \\ \frac{d}{dt}\xi(t) &= \frac{q_0E_0}{2}(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t}) \end{aligned} \quad (9)$$

Substituting equation (9) into equation (8), we can rewrite Equation (8) as

$$\frac{d^2}{dt^2}\langle\widehat{q}_d(t)\rangle + \omega_f^2\langle\widehat{q}_d(t)\rangle = \frac{\lambda}{m} + \frac{q_0\alpha_aE_0}{2}(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t}) \quad (10)$$

This is a non-homogeneous, second-order, linear differential equation. The solution of equation (10) is given by the sum of two parts according to the following theorem: If $\langle\widehat{q}_d^p(t)\rangle$ is a particular solution of an non-homogeneous differential equation and the complementary function $\langle\widehat{q}_d^c(t)\rangle$ is the solution of corresponding homogeneous equation (that is, equation (10) with the right side equal to zero), then $\langle\widehat{q}_d(t)\rangle = \langle\widehat{q}_d^c(t)\rangle + \langle\widehat{q}_d^p(t)\rangle$ is also a solution of the non-homogeneous differential equation. $\langle\widehat{q}_d^c(t)\rangle$ is the solution of the homogeneous differential equation (Nouredine, 2001)[13].

$$\frac{d^2}{dt^2}\langle\widehat{q}_d(t)\rangle + \omega_f^2\langle\widehat{q}_d(t)\rangle = 0 \quad (11)$$

Solve the solution by using the auxiliary equation $\gamma^2 + \omega_f^2 = 0, \gamma = \pm i\omega_f$. We obtain an alternative solution;

$$\langle\widehat{q}_d^c(t)\rangle = C_1e^{i\omega_f t} + C_2e^{-i\omega_f t} = C_1(\cos(\omega_f t) + i\sin(\omega_f t)) + C_2(\cos(\omega_f t) + i\sin(\omega_f t))$$
$$= (C_1 + C_2)\cos(\omega_f t) + (iC_1 - iC_2)\sin(\omega_f t) = A\cos(\omega_f t) + B\sin(\omega_f t)$$

With this definition the parameter becomes $A = A\cos(\phi)$ and $B = A\sin(\phi)$. We can rewrite the complementary function $\langle\widehat{q}_d^c(t)\rangle$ as

$$\begin{aligned} \langle\widehat{q}_d^c(t)\rangle &= A\cos(\omega_f t)\cos(\phi) + A\sin(\omega_f t)\sin(\phi), \\ \langle\widehat{q}_d^c(t)\rangle &= A\cos(\omega_f t - \phi) \end{aligned} \quad (12)$$

where A is the amplitude of the oscillator, ϕ is a positive real constant, called the initial phase angle. According to equation (10), the applied time-dependent electrostatic force $(\xi(t))$ varies sinusoidally, so we expect the resulting steady-state $\langle\widehat{q}_d^p(t)\rangle$ to vary sinusoidally. With this definition the parameter becomes $\langle\widehat{q}_d^{c1}(t)\rangle = \cos(\omega_f t), \langle\widehat{q}_d^{c2}(t)\rangle = \sin(\omega_f t)$, and the variable function Wronskian method are $f(t) = \frac{\lambda}{m} + \frac{q_0\alpha_0E_0}{2}(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t})$ [12]. To take care of this situation, we must have a particular solution of the form

$$\langle \widehat{q}_d^p(t)\rangle = \langle \widehat{q}_d^{c1}(t)\rangle u_p^1(t) + \langle \widehat{q}_d^{c2}(t)\rangle u_p^2(t) \tag{13}$$

Then, suppose the particular solution $\langle \widehat{q}_d^p(t)\rangle$ of equation (10) from the $\xi(t)$ by using Wronskian's method find the trigonometry solution by using this form $\langle \widehat{q}_d^{c1}(t)\rangle = \cos(\omega_f t), \langle \widehat{q}_d^{c1}(t)\rangle' = -\omega_f \sin(\omega_f t), \widehat{q}_d^{c2} = \sin(\omega_f t(t)), \langle \widehat{q}_d^{c2}(t)\rangle' = \omega_f \cos(\omega_f t)$ and substitute in Wronskian's method, we get

$$W = \begin{vmatrix} \langle \widehat{q}_d^{c1}(t)\rangle & \langle \widehat{q}_d^{c2}(t)\rangle \\ \langle \widehat{q}_d^{c1}(t)\rangle' & \langle \widehat{q}_d^{c2}(t)\rangle' \end{vmatrix} = \begin{vmatrix} \cos(\omega_f t) & \sin(\omega_f t) \\ -\omega_f \sin(\omega_f t) & \omega_f \cos(\omega_f t) \end{vmatrix} = \omega_f$$

Consider the first Wronskian's method defined as W_1 , we satisfy the solution as

$$W_1 = \begin{vmatrix} 0 & \langle \widehat{q}_d^{c2}(t)\rangle \\ f(t) & \langle \widehat{q}_d^{c2}(t)\rangle' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \sin(\omega_f t) \\ \frac{\lambda}{m} + \frac{q_0\alpha_0 E_0}{2}(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t}) & \omega_f \cos(\omega_f t) \end{vmatrix}$$

$$W_1 = -\sin(\omega_f t)(\frac{\lambda}{m} + \frac{q_0\alpha_0 E_0}{2}(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t})) \tag{14}$$

So we set the parameter of into equation (14) as

$$W_1 = -\sin(\omega_f t)(\frac{\lambda}{m} + \delta(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t})) \tag{15}$$

We may write

$$\begin{aligned} u_p^1(t) &= \int_0^t \frac{W_1}{W}dt = -\frac{1}{\omega_f}\int_0^t \sin(\omega_f t)(\frac{\lambda}{m} + \delta(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t}))dt \\ u_p^1(t) &= -\frac{1}{\omega_f}(\frac{\lambda}{m}\int_0^t \sin(\omega_f t))dt + 2\sigma\delta\int_0^t e^{-\mu t}\sinh(2\sigma t)\sin(\omega_f t)dt \\ &\quad -\mu\int_0^t e^{-\mu t}\cosh(2\sigma t)\sin(\omega_f t)dt - int_0^t \mu e^{-\mu t}\sin(\omega_f t)dt \end{aligned} \tag{16}$$

Substituting $e^{-\mu t}\sinh(2\sigma t) = (e^{\beta t} - e^{-\tau t})/2, e^{-\mu t}\cosh(2\sigma t) = (e^{\beta t} + e^{-\tau t})/2$ into equation (16), we can rewrite equation (16) as

$$u_p^1(t) = -\frac{1}{\omega_f}(\frac{\lambda}{m}\int_0^t \sin(\omega_f t)dt + \delta\kappa_1 \int_0^t e^{\beta t}\sin(\omega_f t)dt - \delta\kappa_2 \int_0^t e^{-\tau t}\sin(\omega_f t)dt - \int_0^t \mu e^{-\mu t}\sin(\omega_f t)dt) \tag{17}$$

where we set the parameter $\beta = 2\sigma - \mu, \tau = 2\sigma + \mu$. The first integral function, the second integral function, the third integral function, and the fourth integral function can be found by evaluation of the right-hand side of equation (17) by using the integration by part technique. We can rewrite equation (17) as

$$u_p^1(t) = -\frac{1}{\omega_f}\Big[\frac{\lambda}{m}(1-\cos(\omega_f t)) + \Big(\frac{\delta\kappa_1 e^{\beta t}(\beta\sin(\omega_f t) - \omega_f\cos(\omega_f t)) + \delta\kappa_1\omega_f}{(\beta^2+\omega_f^2)}\Big) + \Big(\frac{\delta\kappa_2 e^{-\tau t}(\tau\sin(\omega_f t) + \omega_f\cos(\omega_f t)) - \delta\kappa_2\omega_f}{(\tau^2+\omega_f^2)}\Big) + \Big(\frac{\mu e^{-\mu t}(\mu\sin(\omega_f t) + \omega_f\cos(\omega_f t)) - \mu\omega_f}{(\mu^2+\omega_f^2)}\Big)\Big] \tag{18}$$

Consider the second Wronskian's method defined as W_2 , we satisfy the solution as

$$W_2 = \begin{vmatrix} \langle \widehat{q}_d^{c1}(t)\rangle & 0 \\ \langle \widehat{q}_d^{c1}(t)\rangle' & f(t) \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\omega_f t) & 0 \\ -\omega_f\sin(\omega_f t) & \frac{\lambda}{m} + \frac{q_0\alpha_0 E_0}{2}(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t}) \end{vmatrix}$$

$$W_2 = \cos(\omega_f t)\left(\frac{\lambda}{m} + \delta(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t})\right) \tag{19}$$

We may write

$$u_p^2(t) = \int_0^t \frac{W_2}{W}dt = -\frac{1}{\omega_f}\int_0^t \cos(\omega_f t)(\frac{\lambda}{m} + \delta(2\sigma e^{-\mu t}\sinh(2\sigma t) - \mu e^{-\mu t}\cosh(2\sigma t) - \mu e^{-\mu t}))dt \tag{20}$$

Substituting $e^{-\mu t}\sinh(2\sigma t) = (e^{\beta t} - e^{-\tau t})/2, e^{-\mu t}\cosh(2\sigma t) = (e^{\beta t} + e^{-\tau t})/2$ into equation (20), we can be solved equation (20) by using the integration by part technique, we get

$$u_p^2(t) = -\frac{1}{\omega_f}\Big[\frac{\lambda}{m}(\sin(\omega_f t)) + \Big(\frac{\delta\kappa_1 e^{\beta t}(\beta\cos(\omega_f t) - \omega_f\sin(\omega_f t)) + \delta\kappa_1\beta}{(\beta^2+\omega_f^2)}\Big) + \Big(\frac{\delta\kappa_2 e^{-\tau t}(\tau\cos(\omega_f t) + \omega_f\sin(\omega_f t)) - \delta\kappa_2\tau}{(\tau^2+\omega_f^2)}\Big) + \Big(\frac{\mu e^{-\mu t}(\mu\cos(\omega_f t) + \omega_f\sin(\omega_f t)) - \mu^2}{(\mu^2+\omega_f^2)}\Big)\Big] \tag{21}$$

Substituting $\langle\widehat{q}_d^{c1}(t)\rangle = \cos(\omega_f t), \langle\widehat{q}_d^{c2}(t)\rangle = \sin(\omega_f t)$ and equation (18) and equation (21) into equation (13), thus a particular solution of the non-homogeneous differential equation of equation (10) as $\langle\widehat{q}_d^{p}\rangle = \langle\widehat{q}_d^{c1}\rangle u_p^1(t) + \langle\widehat{q}_d^{c2}\rangle u_p^2(t)$

$$\begin{aligned}\langle\widehat{q}_d^{p}(t)\rangle = &\frac{\sin(\omega_f t)}{\omega_f}\Big[\frac{\lambda}{m}\sin(\omega_f t) + \Big(\frac{\delta\kappa_1 e^{\beta t}(\beta\cos(\omega_f t) + \omega_f\sin(\omega_f t)) - \delta\kappa_1\beta}{(\beta^2+\omega_f^2)}\Big)\\ &+ \Big(\frac{\delta\kappa_2 e^{-\tau t}(\tau\cos(\omega_f t) - \omega_f\sin(\omega_f t)) - \delta\kappa_2\tau}{(\tau^2+\omega_f^2)}\Big) + \Big(\frac{\mu e^{-\mu t}(\mu\cos(\omega_f t) - \omega_f\sin(\omega_f t)) - \mu^2}{(\mu^2+\omega_f^2)}\Big)\Big]\\ &- \frac{\cos(\omega_f t)}{\omega_f}\Big[\frac{\lambda}{m}(1-\cos(\omega_f t)) + \Big(\frac{\delta\kappa_1 e^{\beta t}(\beta\sin(\omega_f t) - \omega_f\cos(\omega_f t)) + \delta\kappa_1\beta}{(\beta^2+\omega_f^2)}\Big)\\ &+ \Big(\frac{\delta\kappa_2 e^{-\tau t}(\tau\sin(\omega_f t) + \omega_f\cos(\omega_f t)) - \delta\kappa_2\omega_f}{(\tau^2+\omega_f^2)}\Big) + \Big(\frac{\mu e^{-\mu t}(\mu\sin(\omega_f t) + \omega_f\cos(\omega_f t)) - \mu\omega_f}{(\mu^2+\omega_f^2)}\Big)\Big]\end{aligned} \tag{22}$$

Then, we will get the general solution in the non-homogeneous second-order linear differential equation for equation (10),

$$\begin{aligned}\langle\widehat{q}_d(t)\rangle = &A\cos(\omega_f t - \phi) + \frac{\sin(\omega_f t)}{\omega_f}\Big[\frac{\lambda}{m}\sin(\omega_f t) + \Big(\frac{\delta\kappa_1 e^{\beta t}(\beta\cos(\omega_f t) + \omega_f\sin(\omega_f t)) - \delta\kappa_1\beta}{(\beta^2+\omega_f^2)}\Big)\\ &+ \Big(\frac{\delta\kappa_2 e^{-\tau t}(\tau\cos(\omega_f t) - \omega_f\sin(\omega_f t)) - \delta\kappa_2\tau}{(\tau^2+\omega_f^2)}\Big) + \Big(\frac{\mu e^{-\mu t}(\mu\cos(\omega_f t) - \omega_f\sin(\omega_f t)) - \mu^2}{(\mu^2+\omega_f^2)}\Big)\Big]\\ &- \frac{\cos(\omega_f t)}{\omega_f}\Big[\frac{\lambda}{m}(1-\cos(\omega_f t)) + \Big(\frac{\delta\kappa_1 e^{\beta t}(\beta\sin(\omega_f t) - \omega_f\cos(\omega_f t)) + \delta\kappa_1\beta}{(\beta^2+\omega_f^2)}\Big)\\ &+ \Big(\frac{\delta\kappa_2 e^{-\tau t}(\tau\sin(\omega_f t) + \omega_f\cos(\omega_f t)) - \delta\kappa_2\omega_f}{(\tau^2+\omega_f^2)}\Big) + \Big(\frac{\mu e^{-\mu t}(\mu\sin(\omega_f t) + \omega_f\cos(\omega_f t)) - \mu\omega_f}{(\mu^2+\omega_f^2)}\Big)\Big]\end{aligned} \tag{23}$$

The equation (23) is called the expectation value of the displacement or the expectation value of the position of particle bound anharmonic oscillator potential by perturbed time-dependent external force $(\xi(t))$. Putting this into program the mathematica for plot graph. (Moonsri et. al., 2020)

Case 1: We can explain behavior of numerical for the expectation value of the displacement of the particle bounded in the anharmonic oscillator potential $(2\sigma < \mu)$. We can show the relationship between default variables, dependent variables, and control variables in the table 1.

Case 2: We can explain behavior of numerical for the expectation value of the displacement of the particle unbounded in the anharmonic oscillator potential . We can show the relationship between default variables, dependent variables, and control variables in the table 2.

Table 1: Representation the relationship between default variables, dependent variables, and control variables in case of $(2\sigma < \mu)$

Dependent variable	Control variable	Independent variables
The expectation value of time-dependent displacement $\langle \widehat{q}_d(t) \rangle$ In case of $(2\sigma < \mu)$	$f = 0.12Hz, \sigma = 0.3, m = 1, \mu = 0.9,$ $\phi = 0rad, E_0 = 5N/C, \lambda = 0.5,$ $A = 0.5, \alpha_a = 0.3$	$q_0 = 0.3C, q_0 = 1.0C, q_0 = 1.7C,$ $q_0 = 2.4C$
	$f = 0.12Hz, \sigma = 0.3, m = 1, \mu = 0.9,$ $\phi = 0rad, E_0 = 5N/C, q_0 = 0.5C,$ $A = 0.5, \alpha_a = 0.3$	$\lambda_1 = 0.2C, \lambda_2 = 0.3C, \lambda_3 = 0.4C,$ $\lambda_4 = 0.5C$
	$f = 0.12Hz, \sigma = 0.3, m = 1, \mu = 0.9,$ $\phi = 0rad, \lambda = 0.5, q_0 = 0.5C,$ $A = 0.5, \alpha_a = 0.3$	$E_0 = 1.0N/C, E_0 = 4.0N/C,$ $E_0 = 7.0N/C, E_0 = 10.0N/C,$
	$E_0 = 5N/C, \sigma = 0.3, m = 1, \mu = 0.9,$ $\phi = 0rad, \lambda = 0.5, q_0 = 0.5C,$ $A = 0.5, \alpha_a = 0.3$	$f = 0.10Hz, f = 0.11Hz,$ $f = 0.12Hz, f = 0.13Hz,$
	$E_0 = 5N/C, \sigma = 0.3, m = 1, \mu = 0.9,$ $f = 0.12Hz, \lambda = 0.5, q_0 = 0.5C,$ $A = 0.5, \alpha_a = 0.3$	$\phi = 0rad, \phi = \pi/8rad, \phi = \pi/6rad,$ $\phi = 4rad,$
	$E_0 = 5N/C, \sigma = 0.3, m = 1, \mu = 0.9,$ $\phi = 0rad, \lambda = 0.5, q_0 = 0.5C,$ $A = 0.5, f = 0.12Hz,$	$\alpha_a = 0.1\alpha_a = 0.3\alpha_a = 0.5$ $\alpha_a = 0.7$

RESULTS

We can explain behavior of numerical and result for the expectation value of the displacement oscillation in equation (23). These the expectation value of the displacement are illustrated in Figure 2 to Figure 7. Next, we can be illustrated plot graph relationship between of the expectation value of the displacement and time in the Figure 2 and Figure 4 ($(2\sigma < \mu)$), where we can vary the parameter electric charge (Figure 2(a)), the parameter . (Figure 2(b)) respectively.

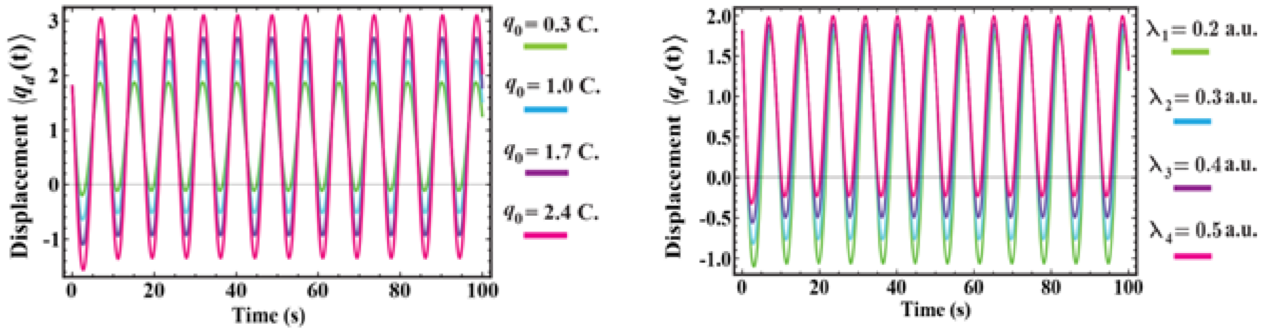
We can be illustrated plot graph relationship between of the expectation value of the displacement and time in the Figure 2(a) and Figure 2(b) ($(2\sigma < \mu)$), where we can vary the parameter

Table 2: Representation the relationship between default variables, dependent variables, and control variables in case of $(2\sigma > \mu)$

<table>
<tr><th>Dependent variable</th><th>Control variable</th><th>Independent variables</th></tr>
<tr><td rowspan="6">The expectation value of time-dependent displacement $\langle \widehat{q}_d(t) \rangle$ In case of $(2\sigma < \mu)$</td><td>$f = 0.12Hz, \sigma = 0.3, m = 1, \mu = 0.9,$
$\phi = 0rad, E_0 = 5N/C, \lambda = 0.5,$
$A = 0.5, \alpha_a = 0.3$</td><td>$q_0 = 0.3C, q_0 = 1.0C, q_0 = 1.7C,$
$q_0 = 2.4C$</td></tr>
<tr><td>$f = 0.12Hz, \sigma = 0.3, m = 1, \mu = 0.9,$
$\phi = 0rad, E_0 = 5N/C, q_0 = 0.5C,$
$A = 0.5, \alpha_a = 0.3$</td><td>$\lambda_1 = 0.2C, \lambda_2 = 0.3C, \lambda_3 = 0.4C,$
$\lambda_4 = 0.5C$</td></tr>
<tr><td>$f = 0.12Hz, \sigma = 0.3, m = 1, \mu = 0.9,$
$\phi = 0rad, \lambda = 0.5, q_0 = 0.5C,$
$A = 0.5, \alpha_a = 0.3$</td><td>$E_0 = 1.0N/C, E_0 = 4.0N/C,$
$E_0 = 7.0N/C, E_0 = 10.0N/C,$</td></tr>
<tr><td>$E_0 = 5N/C, \sigma = 0.3, m = 1, \mu = 0.9,$
$\phi = 0rad, \lambda = 0.5, q_0 = 0.5C,$
$A = 0.5, \alpha_a = 0.3$</td><td>$f = 0.10Hz, f = 0.11Hz,$
$f = 0.12Hz, f = 0.13Hz,$</td></tr>
<tr><td>$E_0 = 5N/C, \sigma = 0.3, m = 1, \mu = 0.9,$
$f = 0.12Hz, \lambda = 0.5, q_0 = 0.5C,$
$A = 0.5, \alpha_a = 0.3$</td><td>$\phi = 0rad, \phi = \pi/8rad, \phi = \pi/6rad,$
$\phi = 4rad,$</td></tr>
<tr><td>$E_0 = 5N/C, \sigma = 0.3, m = 1, \mu = 0.9,$
$\phi = 0rad, \lambda = 0.5, q_0 = 0.5C,$
$A = 0.5, f = 0.12Hz,$</td><td>$\alpha_a = 0.1\alpha_a = 0.3\alpha_a = 0.5$
$\alpha_a = 0.7$</td></tr>
</table>

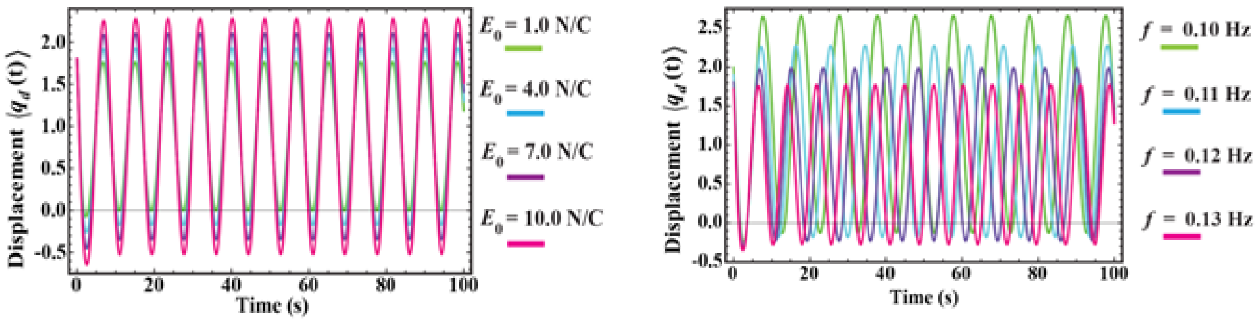
the parameter electric charge (q_0) (Figure 2(a)), the parameter . (Figure 2(b)) respectively. From Figure 2(a), the light green color solid line is the electric charge $(q_0) = 0.3C$. The light blue color solid line is the electric charge $(q_0) = 0.1C$. The purple color solid line is the electric charge $(q_0) = 1.7C$. The pink color solid line is the electric charge $(q_0) = 2.4C$. The expectation value of displacement for particle is oscillation because the particles are bounded by potential energy and the electrostatic external force as a function of time at very time. From Figure 2(b), the solid light green, light blue, purple, and pink line, they are for the parameter $\lambda_1 = 0.2a.u., \lambda_2 = 0.3a.u., \lambda_3 = 0.4a.u.$, and $\lambda_5 = 0.5a.u.$, respectively. The graphs of the expectation value of displacement for particle has a shifted axis because of the electrostatic external force acting on the particle.

From Figure 3(a), the light green color solid line is the electrostatic value $E_0 = 1.0N/C$



(a) (b)

Fig. 2: Graphs of the expectation value of displacement. (a) The graphs illustrate the expectation value of displacement in case of vary the parameter electric charge. (b) The graphs illustrate the expectation value of displacement in case of vary the parameter λ .

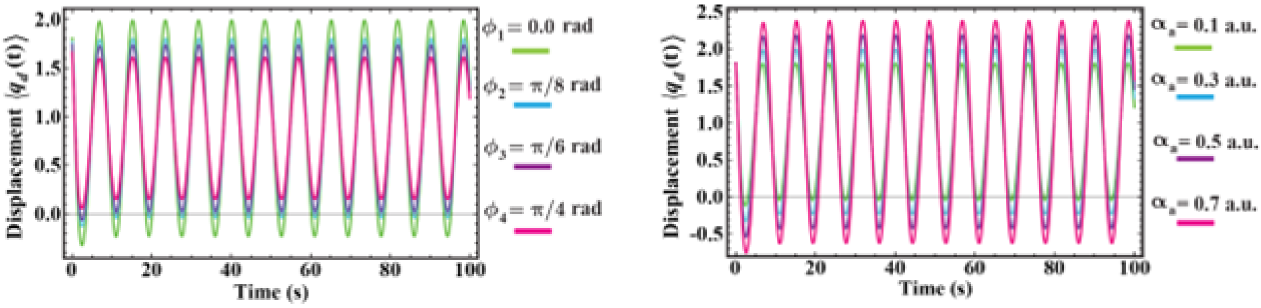


(a) (b)

Fig. 3: Illustration of the relative parameter between the expectation value of displacement and time in equation (23) versus the electrostatic value E_0 (a) and the linear frequency value (f)(b) .

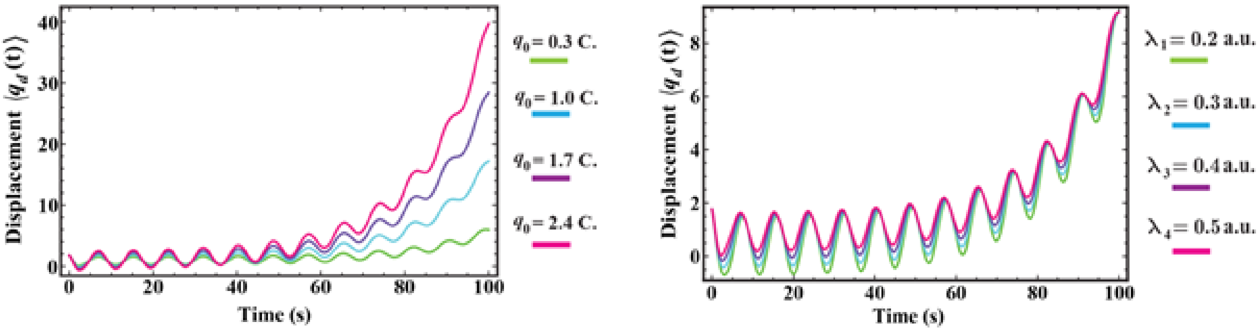
. The light blue color solid line is the electrostatic value $E_0 = 4.0N/C$. The purple color solid line is the electrostatic value $E_0 = 7.0N/C$. The pink color solid line is the electrostatic value $E_0 = 10.0N/C$. From Figure 3(b), the light green color solid line is the linear frequency value $f = 0.1Hz$. The light blue color solid line is the linear frequency value $f = 0.11Hz$. The purple color solid line is the linear frequency value $f = 0.12Hz$. The pink color solid line is the linear frequency value $f = 0.13Hz$.

From Figure 4(a), the light green color solid line is the initial phase angle value $\phi_1 = 0rad$. The light blue color solid line is the initial phase angle value $\phi_2 = \pi/8rad$. The purple color solid line is the initial phase angle value $\phi_3 = \pi/6rad$. The pink color solid line is the initial phase angle value $\phi_4 = \pi/4rad$. From Figure 4(b), the light green color solid line is the parameter value $\alpha_a = 0.1a.u.$. The light blue color solid line is the parameter value $\alpha_a = 0.3a.u.$. The purple color



(a) (b)

Fig. 4: Plots of the expectation value of displacement in equation (23) as a function of time showing versus the initial phase angle value (ϕ) (a) and the α_a parameter value (b)



(a) (b)

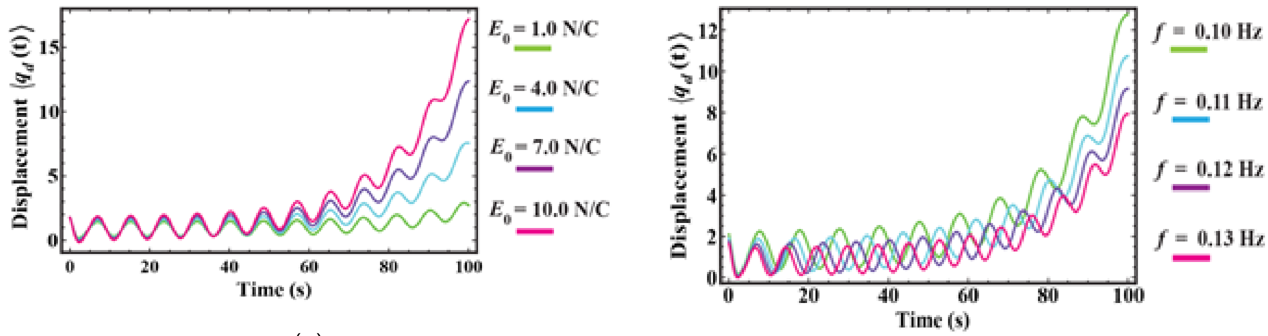
Fig. 5: Illustration of the relative parameter between the expectation value of displacement and time in equation (23) versus the electric charge value (q_0) (a) and the (λ) parameter value (b)

solid line is the parameter value $\alpha_a = 0.5a.u.$. The pink color solid line is the parameter value $\alpha_a = 0.7a.u.$.

Finally, we can be illustrated plot graph relationship between of the expectation value of displacement and time in the Figure 5 to Figure 7 in case of (($2\sigma > \mu$)), where we can vary the electric charge value (q_0) (Figure 5(a)), the parameter (λ) (Figure 5(b)), the electrostatic value (E_0) (Figure 6(a)) and the linear frequency value (f) (Figure 6(b)) respectively.

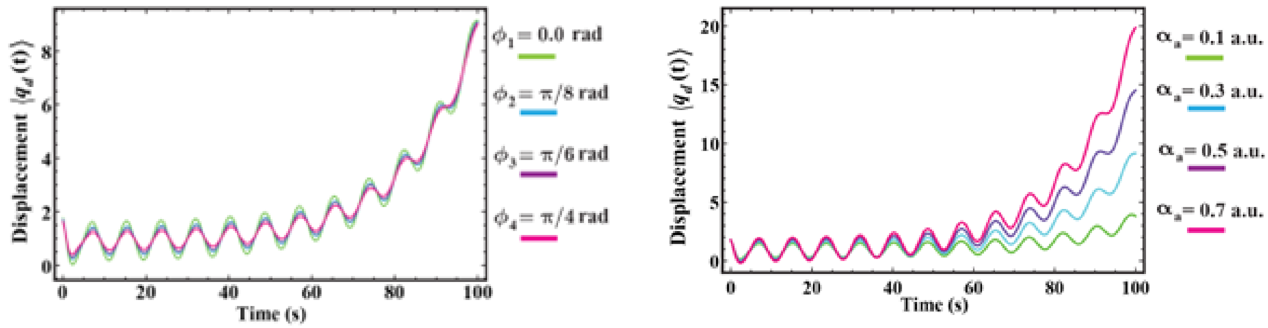
From Figure 5(a), the light green color solid line is the electric charge $(q_0) = 0.3C$. The light blue color solid line is the electric charge $(q_0) = 1.0C$. The purple color solid line is the electric charge $(q_0) = 1.7C$. The pink color solid line is the electric charge $(q_0) = 2.4C$. From Figure 5(b), the light green color solid line is the parameter $\lambda_1 = 0.2a.u.$. The light blue color solid line is the parameter $\lambda_2 = 0.3a.u.$. The purple color solid line is the parameter $\lambda_3 = 0.4a.u.$.

From Figure 6(a), the light green color solid line is the electrostatic value $E_0 = 1.0N/C$



(a) (b)

Fig. 6: Illustration of the relative parameter between the expectation value of displacement and time in equation (23) versus the electrostatic value E_0 (a) and the linear frequency value (f)(b)



(a) (b)

Fig. 7: Illustration of plots of the graph expectation value of displacement in equation (23) as a function of time versus the initial phase angle value ϕ (a) and the α_a parameter value (b).

. The light blue color solid line is the electrostatic value $E_0 = 4.0N/C$. The purple color solid line is the electrostatic value $E_0 = 7.0N/C$. The pink color solid line is the electrostatic value $E_0 = 10.0N/C$. From Figure 6(b), the light green color solid line is the linear frequency value $f = 0.1Hz$. The light blue color solid line is the linear frequency value $f = 0.11Hz$. The purple color solid line is the linear frequency value $f = 0.12Hz$. The pink color solid line is the linear frequency value $f = 0.13Hz$.

From Figure 7(a), the light green color solid line is the initial phase angle value $\phi_1 = 0rad$. The light blue color solid line is the initial phase angle value $\phi_2 = \pi/8rad$. The purple color solid line is the initial phase angle value $\phi_3 = \pi/6rad$. The pink color solid line is the initial phase angle value $\phi_4 = \pi/4rad$. From Figure 7(b), the light green color solid line is the parameter value $\alpha_a = 0.1a.u.$. The light blue color solid line is the parameter value $\alpha_a = 0.3a.u.$. The purple color solid line is the parameter value $\alpha_a = 0.5a.u.$. The pink color solid line is the parameter value

$\alpha_a = 0.7a.u.$.

DISCUSSION

The expectation value of displacement of solution in equation (23) are plotted in Figure 2 - Figure 4 in case of $((2\sigma > \mu)$). From Figure 2(a), if higher the electric charge (q_0) parameter affect increasing value amplitude of the graph expectation value of displacement but wavelength of the expectation value of displacement invariable value. From Figure 2(b), if higher the λ parameter affect increasing value amplitude of the graph expectation value of displacement (-). Thus the effect of the interaction Heisenberg picture is that the product of the electric charge (q_0) parameter and the λ parameter and the expectation value of displacement is function of time and denote the change for particle anharmonic oscillator under influence of the time-dependent electrostatic damping force anharmonic oscillator. The physical meaning of this can be seen in Figure 3(a) the electrostatic value E_0 can increase with increase the expectation value of displacement of particle bound into the anharmonic oscillator potential. The physical meaning of this can be seen in Figure 3(b) the linear frequency (f_q) can increase with decrease the expectation value of displacement and wavelength of particle bound into the anharmonic oscillator potential. The behavior of the expectation value of displacement of solution in equation (23) is wave-Sine function oscillate under influence of the time-dependent electrostatic force $(\xi(t) = qE_0e^{-\mu t}\cosh^2(\sigma t))$. From Figure 4(a), if higher the initial phase angle affect decreasing value amplitude of the expectation value displacement of particle bound into the anharmonic oscillator potential but wavelength of the expectation value of displacement invariable value. From Figure 4(b), if higher the (α_a) parameter affect increasing value amplitude of the expectation value displacement. Thus the effect of the interaction Heisenberg picture is that the product of the initial phase angle, the (α_a) parameter and the expectation value of displacement denote the change for particle anharmonic oscillator potential under influence of the time-dependent external electrostatic force $\xi(t) = qE_0e^{-\mu t}\cosh^2(\sigma t)$.

The expectation value of displacement of solution in equation (23) are Illustrated in Figure 5 to Figure 7 in case of $((2\sigma > \mu))$. From Figure 5(a), if higher the initial electric charge parameter (q) affect increasing value amplitude of the expectation value of displacement of particle unbound into the anharmonic oscillator potential. From Figure 5(b), if higher the (λ) parameter value affect increasing value amplitude of the expectation value of displacement of particle unbound into the anharmonic oscillator potential. From Figure 6(a), if higher the electrostatic value affect increasing value amplitude of the expectation value of displacement of particle unbound into the anharmonic oscillator potential. From Figure 6(a), if higher the linear frequency value affect decreasing value amplitude of the expectation value of displacement of particle unbound into the anharmonic oscillator potential. The physical meaning of this can be seen in Figure 7(a) the initial phase angle parameter (ϕ) can increase with decrease the expectation value of displacement and wavelength of particle unbound into the anharmonic oscillator potential. The physical meaning of this can be seen in Figure 7(b) the (α_a) parameter value can increase with supplement the expectation value of displacement of particle unbound into the anharmonic oscillator potential.

CONCLUSIONS

We can evaluate the expectation value of the displacement $(\langle\widehat{q}_d\rangle)$ of solution in equation (23) under influencer of the time-dependent electrostatic force $(\xi(t))$. Using the Heisenberg picture method and the principal of Wronskian method we showed that the treatment of the anharmonic oscillator potential system is simplified and that the expectation value of the displacement $(\langle\widehat{q}_d\rangle)$ of the system can be evaluated very easily. The behavior of the expectation value of the displacement $(\langle\widehat{q}_d\rangle)$ for the anharmonic oscillator potential system is wave oscillate (function of Sine or Cosine) $(2\sigma < \mu)$ bounded state with that depend on the parameter $f, \sigma, q_0, \mu, \phi, E_0$ and A . The initial electric charge q_0 , the (λ) parameter value, the electrostatic value E_0 and the (α_a) parameter value is directly proportional to the expectation value of the displacement. The linear frequency value f , the initial phase angle ϕ is inversely proportional to the expectation value of the displacement.

The deportation of the expectation value of the displacement $(\langle\widehat{q}_d\rangle)$ for the anharmonic oscillator potential system is wave oscillate increase exponential function $(2\sigma > \mu)$ unbounded state with that ancillary the parameter $f, \sigma, q_0, \mu, \phi, E_0$ and A. We can apply the expectation value of the displacement result to the particular case of evaluation the propagator from the Newtonian dynamics of $(\langle\widehat{q}_d\rangle)$. The initial electric charge q_0, the (λ) parameter value, the electrostatic value E_0 and the (α_a) parameter value is directly proportional to the expectation value of the displacement. The linear frequency value f , the initial phase angle ϕ is inversely proportional to the expectation value of the displacement.

ACKNOWLEDGMENT

We would like to thank the Phetchabun Rajabhat University, Department of Science Education. And also Faculty of Science and Technology, Phetchabun Rajabhat University for partial support.

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