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Truncated Three-Parameter Lindley Distribution for Lifetime Data

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Abstract

In this research, a three-parameter Lindley distribution is improved by truncating from both left and right sides. The truncated three-parameter Lindley distribution is proposed about the probability density function, cumulative distribution function, survival function, hazard function, moments and parameter estimation. The developed distribution and the baseline distribution are applied 4 real lifetime datasets and are compared the model performances. The numerical experiments reveal that the truncated three-parameter Lindley distribution gives the lowest values of the negative log-likelihood (-Log(L)) and the Akaike information criterion (AIC). Moreover, the developed distribution provides a large p-values in Kolmogorov-Smirnov test (K-S test).

Keywords: Three-parameter Lindley distribution; Lifetime data; Truncated distribution

1. Introduction

Lindley distribution, proposed by Lindley [1], is one way of modelling to explain lifetime data in term of system or mechanism. Lindley distribution is continuously developed to increase the model performance, for example, Shanker and Mishra [2, 3] represented the probability density function (P.D.F) and cumulative distribution function (C.D.F) of two-parameter and quasi two-parameter Lindley distributions, respectively.

A three-parameter Lindley distribution (TLD), proposed by Shanker et al. [4], contains two-parameter gamma distribution and one parameter exponential and Lindley distributions are specific case. It has been applied for modelling lifetime data. The P.D.F and the C.D.F are given as

$$f(x;\theta,\alpha,\beta)=\frac{\theta^2}{\alpha\theta+\beta}(\alpha+\beta x)e^{-\theta x}, \tag{1a}$$

$$F(x;\theta,\alpha,\beta)=1-\left[1+\frac{\theta\beta x}{\theta\alpha+\beta}\right]e^{-\theta x}, \tag{1b}$$

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where $x > 0, \theta > 0, \beta > 0$ and $\theta\alpha + \beta > 0$. The P.D.F. and C.D.F. plots of the three-parameter Lindley distribution for different values of θ, α and β are represented in Fig. 1.

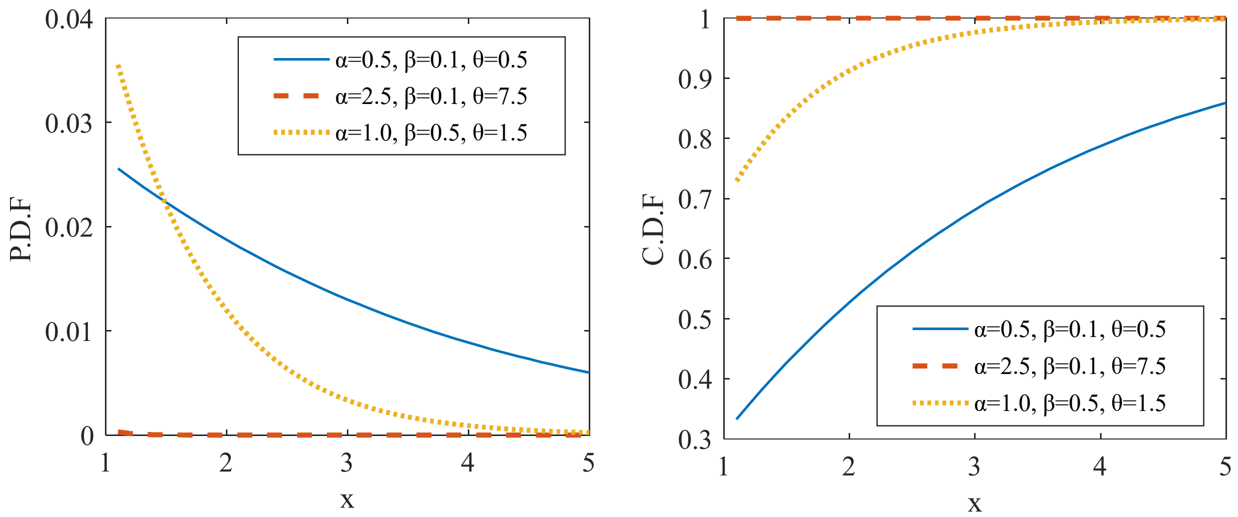

Fig. 1 P.D.F and C.D.F plots of TLD for different values of parameters θ, α and β

The three-parameter Lindley distribution is applied for some lifetime data, for example, Shanker et al. [4] applied for the survival times of guinea pigs infected with virulent tubercle bacilli, AI-Omari et al. [5] proposed using the three-parameter Lindley distribution for economic design of acceptance sampling plans for truncated life tests.

Truncated distribution is one of the famous techniques to increase a robustness of an estimator from a parent distribution. The truncated distribution on a smaller interval provides a new distribution than a parent distribution. Johnson et al. [6] derived a truncated normal distribution from the standard normal distribution. For others distributions, especially, lifetime distribution, there are many researchers proposed truncated lifetime distributions. For example, Savenkov [7] proposed a truncated the Weibull distribution which is to be utilized to evaluate the theoretical capacity factor of potential wind (or wave) energy. It was found that a truncated form of Weibull distribution is used for modeling the effect in the cut-in wind speed or cut-in wave height of a power generator. Aryuyuen [8] introduced a truncated two-parameter Lindley distribution and its utilization. The truncated two-parameter Lindley distribution provides a reasonable and better implementation of the real data. Zaninetti [9] applied a truncated two-parameter Lindley distribution of the luminosity function of the Sloan Digital Sky Survey (SDSS) galaxies and to the photometric maximum of the 2MASS Redshift Survey (2MRS) galaxies.

To increase the robustness of the estimator, a truncated three-Lindley distribution is proposed in this study. It is presented in term of probability density function, cumulative distribution function, survival function (S(x)), hazard function (H(x)), parameter estimation and application for 4 lifetime data.

2. Truncating Distribution

Let $X \in (-\infty, \infty)$, a continuous random variable, be a parent distribution with the parameter Θ. The P.D.F. and C.D.F of X have infinite support and are within $[a,b]$, $-\infty < a \le x \le b < \infty$. The P.D.F of the truncated distribution where $-\infty < a \le x \le b < \infty$ is given as equation (2) [10]:

$$t(x;\Theta,a,b) = \frac{f(x;\Theta)}{F(b;\Theta) - F(a;\Theta)} \;\; ; -\infty < a \le x \le b < \infty, \tag{2}$$

where $f(x) = \begin{cases} t(x) \; ; \; a \le x \le b \\ 0 \quad ; \text{otherwise} \end{cases}$, $t\left(x \middle| a \le x \le b\right)$, $t(x) \ge 0$, for all x.

Since $\int_a^b t\left(x \middle| a \le x \le b\right) dx$, then,

$$\int_a^b t\left(x \middle| a \le x \le b\right) dx = \frac{1}{F(b) - F(a)} \int_a^b f(x) dx$$

$$= \frac{1}{F(b) - F(a)} \left[F(x) \right]_{x=a}^{x=b}$$

$$= \frac{1}{F(b) - F(a)} \left[F(b) - F(a) \right] = 1$$

The parent distribution can be truncated both left and right sides or be truncated only one side. In this study, the three-parameter Lindley distribution is presented truncation for both sides that is called a doubly truncated distribution.

3. Method

The three-parameter Lindley distribution is developed by truncation both left and right sides. The P.D.F and C.D.F. of the developed distribution are shown in section 3.1 and 3.2. Then, its statistical properties are presented in term of survival and hazard functions, moments and parameter estimation using maximum likelihood estimation as shown in section 3.3 to 3.5, respectively.

3.1 The P.D.F of Truncated Three- Parameter Lindley Distribution (TTLD)

Let $X \sim$ TTLD $(\theta, \alpha, \beta, a, b)$ be distributed as the TTLD random variable with the parameter θ, α, β, a and b, the P.D.F is shown in equation (3).

$$t(x|a \le x \le b) = \frac{\theta^2 (\alpha + \beta x) e^{-\theta x}}{(\theta\alpha + \beta + \theta\beta a) e^{-\theta a} - (\theta\alpha + \beta + \theta\beta b) e^{-\theta b}} \tag{3}$$

and $t(x) = 0$ otherwise.

Proof. Let X ~TLD (θ,α,β) and X be called the TTLD on interval [a, b] with parameter θ,α and β respectively. $t(x|a \le x \le b)$ has the property as:

$$\int_a^b t(x|a \le x \le b)dx = \int_a^b \left(\frac{\theta^2(\alpha+\beta x)e^{-\theta x}}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta b)e^{-\theta b}} \right) dx$$

$$= \frac{1}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta b)e^{-\theta b}} \int_a^b \theta^2(\alpha+\beta x)e^{-\theta x}dx$$

$$= \frac{1}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta b)e^{-\theta b}} \int_a^b \theta^2(\alpha+\beta x)e^{-\theta x}dx$$

$$= \frac{1}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta b)e^{-\theta b}} \left[-(\theta\alpha+\beta+\theta\beta x)e^{-\theta x} \right]_a^b$$

$$= \frac{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta b)e^{-\theta b}}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta b)e^{-\theta b}} = 1$$

3.2 The C.D.F of TTLD

The C.D.F of TTLD is given by equation (4).

$$T(x|a \le x \le b) = \frac{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta x)e^{-\theta x}}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta b)e^{-\theta b}} \qquad (4)$$

The P.D.F and C.D.F plots of TTLD are shown in Fig. 2. Based on a comparison of P.D.F plots in Fig. 1 and Fig. 2, the TTLD gives a higher value of P.D.F than TLD for the same parameters θ,α and β.

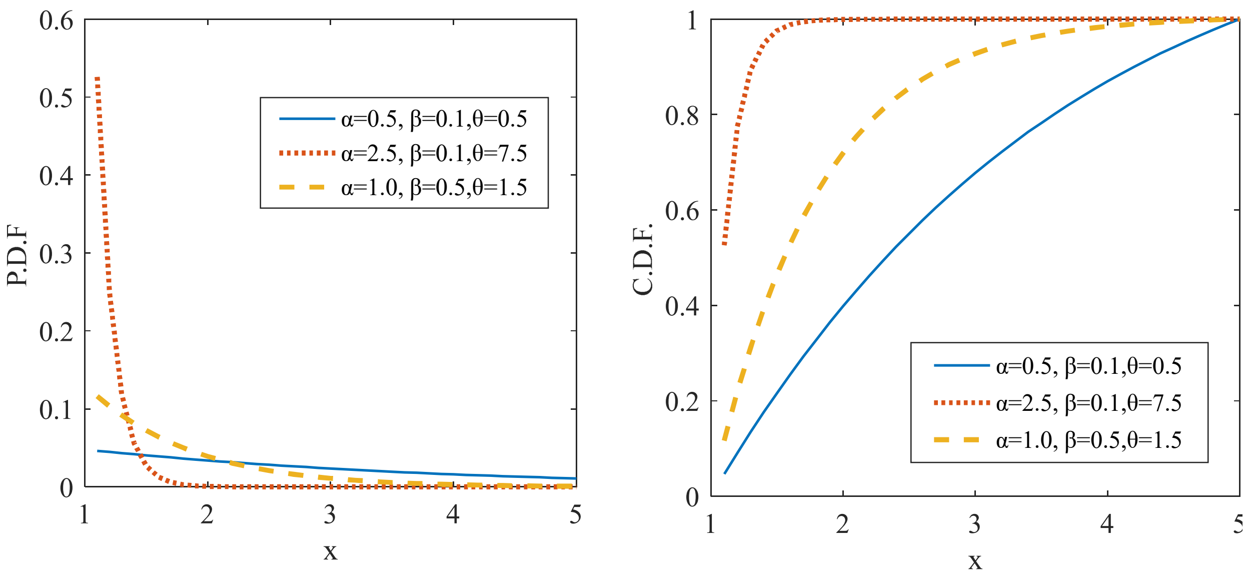

Fig. 2 P.D.F and C.D.F plots of TTLD on [1.0,5.0]

3.3 Survival and hazard functions

S(*x*), a survival function, is the feasibility that a matter survives longer than time *x*. It is written as:

$$S(x)=P(X>0)=1-F(x)\quad=\frac{(\theta\alpha+\beta+\theta\beta x)e^{-\theta x}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}} \tag{5}$$

H(*x*), a hazard function, is the ratio of *t*(*x*) to S(*x*), given by

$$H(x)=\frac{t(x)}{S(x)}\quad=\frac{\theta^2(\alpha+\beta x)e^{-\theta x}}{(\theta\alpha+\beta+\theta\beta x)e^{-\theta x}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}} \tag{6}$$

The S(x) and H(x) plots of TTLD are shown in Fig. 3. Fig. 3 shows that the S(x) plots are decreasing. The H(x) plots are similar to bathtub curve and it trends to increase.

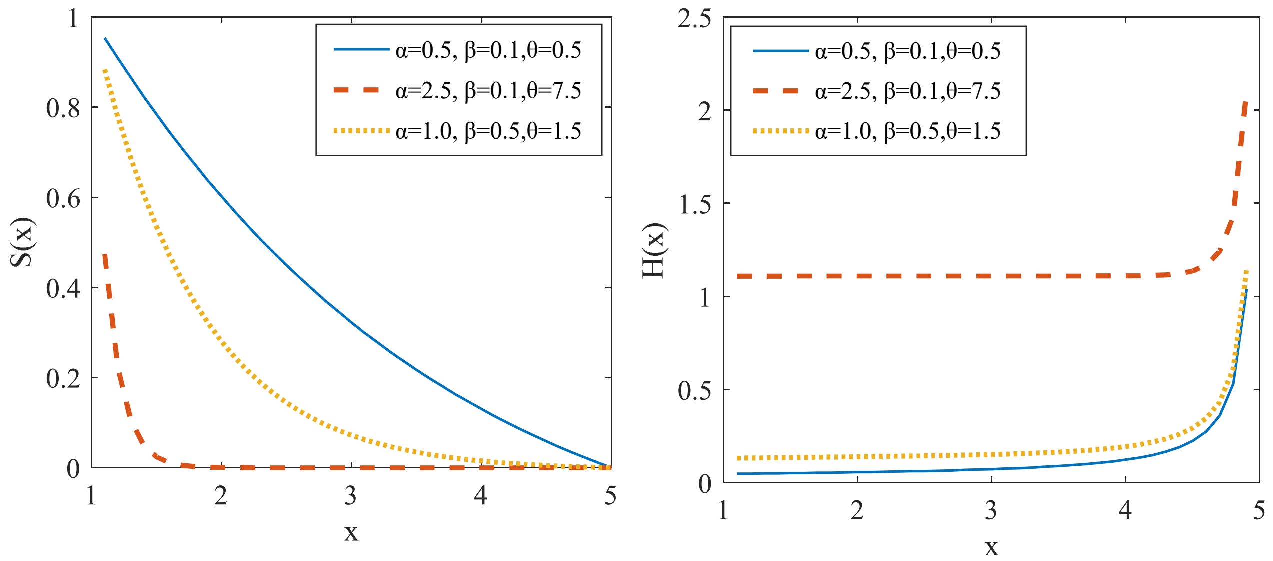

Fig. 3 S(x) and H(x) plots of TTLD on [1.0, 5.0]

3.4 Moments

Moments of the proposed distribution provide some characteristics of TTLD such as mean, variance, skewness, kurtosis, etc. Some moments are shown through moments generating function that as:

$$\mu'(k)=E(X^k)=\int_x x^k f(x)dx.,$$

$$\mu'(k)=E(X^k)=\frac{\theta^2}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}}\int_a^b\left((\alpha+\beta x)x^k e^{-\theta x}\right)dx,$$

$$=\frac{\theta^2}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}}\left[\frac{\alpha}{\theta}\int_a^b\left(x^k e^{-\theta x}\right)d(\theta x)+\frac{\beta}{\theta}\int_a^b\left(x^{k+1}e^{-\theta x}\right)d(\theta x)\right]. \tag{7}$$

By using the gramma function $\Gamma(k,b)=\int_0^b x^{k-1}e^{-x}dx$, the equation (7) is written as equation (8).

$$\mu'(k) = E\left(X^k\right)$$

$$= \frac{\theta^2}{(\theta\alpha+\beta+\theta\beta a)e^{-\theta a} - (\theta\alpha+\beta+\theta\beta b)e^{-\theta b}}\left[\frac{\alpha}{\theta^{k+1}}\int_a^b\left((\theta x)^k e^{-\theta x}\right)d(\theta x) + \frac{\beta}{\theta^{k+2}}\int_a^b\left((\theta x)^{k+1} e^{-\theta x}\right)d(\theta x)\right]$$

$$= \frac{(\alpha\theta[\Gamma(k+1,b)-\Gamma(k+1,a)]+\beta[\Gamma(k+2,b)-\Gamma(k+2,a)])}{\left((\theta\alpha+\beta+\theta\beta a)e^{-\theta a}-(\theta\alpha+\beta+\theta\beta b)\,e^{-\theta b}\right)\theta^k} \tag{8}$$

E(X) and V(X) are revealed in equation (9) and (10).

$$\mu'(1) = E\left(X^1\right) = \frac{(\alpha\theta[\Gamma(2,b)-\Gamma(2,a)]+\beta[\Gamma(3,b)-\Gamma(3,a)])}{\theta\left((\theta\alpha+\beta+\theta\beta\, a)e^{-\theta a}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}\right)} \tag{9}$$

$$\mu'(2) = E\left(X^2\right) = \frac{(\alpha\theta[\Gamma(3,b)-\Gamma(3,a)]+\beta[\Gamma(4,b)-\Gamma(4,a)])}{\theta^2\left((\theta\alpha+\beta+\theta\beta a)e^{-\theta a}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}\right)}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{(\alpha\theta[\Gamma(3,b)-\Gamma(3,a)]+\beta[\Gamma(4,b)-\Gamma(4,a)])}{\theta^2\left((\theta\alpha+\beta+\theta\beta a)e^{-\theta a}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}\right)} - \left[\frac{(\alpha\theta[\Gamma(2,b)-\Gamma(2,a)]+\beta[\Gamma(3,b)-\Gamma(3,a)])}{\theta\left((\theta\alpha+\beta+\theta\beta\, a)e^{-\theta a}-(\theta\alpha+\beta+\theta\beta b)e^{-\theta b}\right)}\right]^2 \tag{10}$$

3.5 Parameter Estimation

The parameter estimation of TTLD is answered using maximum likelihood estimation (MLE). Let $\tilde{x} = (x_1, x_2, ..., x_n)$ be a random sample with size *n*. X ~TTLD $(\theta,\alpha,\beta,a,b)$ is independent and identically distributed. The likelihood function of TTLD ($L(\Theta|\tilde{x})$) is as equation (11),

$$L(\Theta|\tilde{x}) = \prod_{i=1}^{n}\frac{\theta^2(\alpha+\beta x_i)e^{-\theta x_i}}{e^{-\theta a}(\alpha\theta+\beta+\beta\theta a)-e^{-\theta b}(\alpha\theta+\beta+\beta\theta b)}. \tag{11}$$

The log-likelihood function of X based on $\tilde{x}$ is given as

$$log\left(L\left(\alpha,\beta,\theta,a,b|\tilde{x}\right)\right) = LL(\Theta|\tilde{x}),$$

$$LL(\Theta|\tilde{x}) = n\,log\left(\frac{\theta^2}{e^{-\theta a}(\alpha\theta+\beta+\beta\theta a)-e^{-\theta b}(\alpha\theta+\beta+\beta\theta b)}\right) + \sum_{i=1}^{n}\left(log(\alpha+\beta x_i)\right) - \sum_{i=1}^{n}(\theta x_i)$$

The MLEs of α, β and θ can be solved using equation (12), (13) and (14), respectively.

$$\frac{\partial LL(\Theta|\tilde{x})}{\partial\alpha} = \sum_{i=1}^{n}\left(\frac{1}{\alpha+\beta x_i}\right) - \frac{n\left(\theta e^{-a\theta}-\theta e^{-b\theta}\right)}{(\beta+\alpha\theta+\beta\theta a)e^{-a\theta}-(\beta+\alpha\theta+\beta\theta b)e^{-b\theta}} \tag{12}$$

$$\frac{\partial LL(\Theta|\tilde{x})}{\partial \beta} = \sum_{i=1}^{n}\left(\frac{x_i}{\alpha + \beta x_i}\right) - \frac{n\left((a\theta+1)e^{-a\theta} - (b\theta+1)e^{-b\theta}\right)}{(\beta+\alpha\theta+\beta\theta a)e^{-a\theta} - (\beta+\alpha\theta+\beta\theta b)e^{-b\theta}} \tag{13}$$

$$\frac{\partial LL(\Theta|\tilde{x})}{\partial \theta} = \frac{n}{\theta^2}\left((\beta+\alpha\theta+\beta\theta a)e^{-a\theta} - (\beta+\alpha\theta+\beta\theta b)e^{-b\theta}\right)\left(\frac{2\theta}{\left((\beta+\alpha\theta+\beta\theta a)e^{-a\theta} - (\beta+\alpha\theta+\beta\theta b)e^{-b\theta}\right)}\right.$$

$$\left. - \frac{\theta^2\left((\alpha+\beta a)e^{-a\theta} - (\alpha+\beta b)e^{-b\theta} - a(\beta+\alpha\theta+\beta\theta a)e^{-a\theta} + b(\beta+\alpha\theta+\beta\theta b)e^{-b\theta}\right)}{\left((\beta+\alpha\theta+\beta\theta a)e^{-a\theta} - (\beta+\alpha\theta+\beta\theta b)e^{-b\theta}\right)^2}\right) - \sum_{i=1}^{n} x_i \tag{14}$$

Sine equation (12), (13) and (14) are nonlinear algebraic equation and are not in closed-form expression, the parameters $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ are answered using Newton-Raphson method.

4. Applications

The baseline distribution and the proposed distribution are applied for 4 lifetime datasets. The performance of the TLD and the TTLD are considered using *–log* (*L*), Akaike's information criterion (AIC) and Kolmogorov-Smirnov test (K-S test) as equation (15) and (16).

$$AIC = -2\,log(L) + 2p\ , \tag{15}$$

$$K - S = \sup_x \left| F_n(x) - F_0(x) \right|, \tag{16}$$

where $F_n(x)$ is the empirical cumulative probability, $F_0(x)$ is the theoretical cumulative distribution function evaluated at x, n is a number of samples and p is a number of parameters. An appropriate distribution will provide the lowest values of *–log* (*L*) and AIC. Moreover, the model gives a large p-values for K-S test (>0.05), the model corresponds the real dataset. The details of the 4 lifetime datasets with descriptive statistics are as:

Dataset 1: The number of cycles to failure for 25 100-cm specimens of yarn, tested at a particular strain level [11]

15	20	38	42	61	76	86	98	121	146	149	157	175
176	180	180	198	220	224	251	264	282	321	325	653	

Dataset 2: The number of million revolutions before failure for each of the 23 deep groove ball bearings in the life tests [11].

17.88	28.92	33.00	41.52	42.12	45.60	48.80	51.84	51.96	54.12	55.56	67.80
68.44	68.64	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Dataset 3: The strength values of 63 pieces of 1.5 cm glass fibers measured at the National Physical Laboratory, England. Unfortunately, the units of measurements are not given in the paper, and they are taken from Smith & Naylor [12].

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81	2.00
0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01
0.77	1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24
0.81	1.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66	1.70	1.77	1.84	0.84
1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89		

Dataset 4: The strength (mega Pascal) values of 31 pieces of glass of the aircraft window [13].

18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77	26.78
27.05	27.67	29.9	31.11	33.2	33.73	33.76	33.89	34.76	35.75	35.91	36.98	37.08
37.09	39.58	44.045	45.29	45.381								

The descriptive statistics of the 4 datasets are shown in Table 1. Based on the results of table1, there is a rather high variation for dataset 1 and 2 and a low variation for dataset 3 and 4 the skewness coefficient (S.C.) of the dataset 1 and 2 are greater than +1 that mean the both datasets are highly positive skewed distribution [14]. The kurtosis coefficients of the dataset are between -0.622 to 5.662.

Table 1 Mean, standard deviation (S.D.), skewness coefficient (S.C.) and kurtosis coefficient (K.C.) of the 4 datasets

Dataset	Mean	S.D.	S.C.	K.C.
1	178.3200	133.8008	1.842	5.662
2	72.2296	37.4804	1.009	0.929
3	1.5068	0.3241	-0.922	1.103
4	30.8114	7.2534	0.426	-0.622

The numerical experiments are shown in Table 2. It reveals –log (L), AIC and K-S test of TLD and TTLD for 4 lifetime datasets. TTLD is a consistently better fit than TLD because most of the datasets have the lower –log (L) and AIC. In addition, the TTLD gives a larger p-value (p-value > 0.05) for K-S test except dataset 3. Both TLD and TTLD do not fit for dataset 3 because they provide a small p-value (p-value < 0.05) for the K-S test. It is possible that dataset 3, a very small standard deviation and negatively skewed distribution, is suitable for the other distributions. For dataset 4, the TTLD corresponds the dataset, but TLD does not correspond the dataset because the TTLD has a large p-value (p-value = 0.7467) for the K-S test while TLD has a small p-value (p-value < 0.05) for the K-S test.

Table 2 Parameter estimation and statistic values of TLD and TTLD based on lifetime datasets

Dataset	Parameter	Parameter Estimation TLD	Parameter Estimation TTLD	$-\log(L)$ TLD	$-\log(L)$ TTLD	AIC TLD	AIC TTLD	K-S (p-value) TLD	K-S (p-value) TTLD
1	θ	0.0109	0.01096	152.5336	152.0522	311.0672	314.1044	0.1135 (0.8688)	0.1126 (0.8742)
	α	1.43×10^{-4}	1.71×10^{-4}						
	β	4.28×10^{-4}	5.12×10^{-4}						
	a	-	15						
	b	-	653						
2	θ	0.0274399	0.0250	115.6319	112.2088	237.2638	234.4176	0.1899 (0.3348)	0.1274 (0.8041)
	α	0.0000041	0.0003						
	β	0.0000081	0.0001						
	a	-	17.88						
	b	-	173.40						
3	θ	1.3663	0.1497	64.6158	30.0305	135.2316	70.0610	0.3785 (1.6×10^{-8})	0.2442 (8.6×10^{-4})
	α	6.36×10^{-6}	0.0001						
	β	1.56×10^{-4}	0.0005						
	a	-	0.55						
	b	-	2.24						
4	θ	0.06154	0.05352	127.6618	101.0968	261.3237	212.1937	0.3702 (<0.01)	0.1169 (0.7467)
	α	0.00037	0.00014						
	β	0.00020	0.000075						
	a	-	18.83						
	b	-	45.3810						

5. Conclusion

In this study, the three-parameter Lindley distribution is developed by truncating from both left and right sides. The truncated three- parameter Lindley distribution is presented with its statistical properties such as a probability density function, cumulative distribution function, survival function, hazard function, moments and parameter estimation. In addition, the improved distribution is applied for 4 real lifetime datasets. The model performances are compared using the negative log-likelihood $-\log(L)$, the Akaike information criterion (AIC) and the Kolmogorov-Smirnov test (K-S test). The results reveal that the developed distribution provides the model which is a better fit than the baseline distribution because most of the datasets has a smaller $-\log(L)$ and AIC. In addition, the proposed distribution gives a large p-values for K-S test.

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