Pathumwan Academic Journal, Vol. 5, No. 12, January - April 2015: 11 - 19

Comparison of Numerical Methods for Solutions of Linear Fredholm Integral Equation of the Second Kind

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Abstract

The purpose of this study is to compare the efficiency of three methods; Trapezoidal rule method, Gauss-Nystrom method, and Hermite series Method for the solution of Linear Fredholm Integral Equation of the Second Kind. The numerical methods are involved in partition of grids and weight function for apporoximation of solutions and simplified to be a system of linear equation. The numerical solutions are illustrated and compared by Absolute Error measurement as errror analysis. Also, one example is discussed and solved by three methods and the results are compared. The findings showed that Gauss-Nystrom method was more efficient than Trapezoidal rule and Hermite series method, which was based on minimum absolute error.

Keywords: Linear Fredholm Integral Equation, Trapezoidal rule, Gauss-Nystrom, Hermite Series, Absolute Error

1. Introduction

Integral equation is an equation formed as unknown function which is a part of the integrand under the integral sign. The remained another part of the integrand is often called the kernel function, two variable functions. Actually, integral equation is derived from differential equation described for natural phenomena such as radioactive energy transfer, oscillation of a string, and forecasting in human population [1-4]. In recent year, some problems and models occurred in several fields especially science and engineering have related to linear Fredholm integral equation of the second kind (LFIESK) [5] which is formed

$$f(x) = u(x) + \int_a^b K(x,t)u(t)dt \qquad (1)$$

where $K(x,t)$ is kernel function, $f(x)$ is given function, and $u(x)$ is the unknown function.

Generally, LFIESK cannot solve for the analytical solution. Most researchers have succeeded in the numerical solution. This requires the most accuracy and efficient for approximated methods. Therefore,

Tutorial Paper **Received 12 February 2015**

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Accepted 30 March 2015

there are several researchers have proposed various methods for solving LFIESK. Doucet et al. [6] solved the integral equation using Markov Chain Monte Carlo and applied the method for solving the rational expectation pricing model. Latter, Yalcinbas and Aynigul [4] proposed the series solution of linear Fredholm integral equations based on Hermite polynomials. In 2014, the numerical method for solving Fredholm integral equations by the four Chebyshev polynomials was offered by Nadir [3]. Moreover, Zhang et al. [1] presented a novel method based on Radial Basis function, Meshless method, for solving the linear integral equations. Although there are several methods showed above, the differences amongst these computation methods are also discussed. The four Chebyshev polynomials [3] for solving integral equation needs transform domain of integration into $[-1,1]$ but the second member of the four Chebyshev polynomials satisfied only for some conditions. Using Markov Chain Monte Carlo Methods [6] for solving integral equation spends a lot of time because the methods are random processes operated under probability. Besides, Meshless method is suitable for solving high dimensional integral equations. That is to say, Meshless method [1] will be efficient with the solution of multi-dimensional integral equations. Also, Hermite series method [4] is the easy method for solution of integral equations with low computation and high accuracy, but it does not exist under the expansion in Hermite series in $-1 \le x,t \le 1$ of function $f(x)$ and $K(x,t)$ which is undefined. Furthermore, two of the efficient methods are Trapezoidal rule method and Gauss-Nystrom method used in general condition of integral equations. However, this paper consider in the comparison of accuracy that is the minimum of absolute error (AE) amongst Trapezoidal rule method, Gauss-Nystrom method, and Hermite series method. Computational example is also provided.

2. Numerical method for solving LFIESK

In this section, three numerical methods for the approximation solution of LFIESK as shown in equation (1); Trapezoidal rule method, Gauss-Nystrom method, and Hermite series method, are presented. Also, the formula of the accuracy, AE, is introduced.

2.1 Trapezoidal rule method

Trapezoidal rule method [5] for integration is one of the methods for solving LFIESK. In other words, the area under the curve can be estimated by Trapezoidal rule. This method bases on construction of n grids divided from the interval $[a,b]$ into sub-grids with equal length, h where $h = \frac{b-a}{n}$, $t_j = a + jh$, $j = 0,1,2,...,N$.

Term of the integration $\int_a^b K(x,t)u(t)dt$ in LFIESK (equation (1)) will be approximated by the Trapezoidal rule as formed

$$\int_a^b K(x,t)u(t)dt = \frac{h}{2}[K(x,t_0)u(t_0) + K(x,t_N)u(t_N)] + h\sum_{j=1}^{N-1} K(x,t_j)u(t_j) \,. \tag{2}$$

Substitute equation (2) into equation (1) to acquire

$$f(x) = u(x) + \frac{h}{2}[K(x,t_0)u(t_0) + K(x,t_N)u(t_N)] + h\sum_{j=1}^{N-1} K(x,t_j)u(t_j) \,. \tag{3}$$

In order to construct the system of linear equation from equation (3), replacing t_i' s to x is assigned. The system of linear equation is as follows

$$f(t_i) = u(t_i) + \frac{h}{2}[K(t_i,t_0)u(t_0) + K(t_i,t_N)u(t_N)] + h\sum_{j=1}^{N-1} K(t_i,t_j)u(t_j) \,. \tag{4}$$

Let $f(t_i) = f_i$, $K(t_i,t_j) = K_{ij}$, and $u(t_i) = u_i$. The rearranged equation (4) becomes the system of linear equation

$$f_i = u_i + \frac{h}{2}[K_{i0}u_0 + K_{iN}u_N] + h\sum_{j=1}^{N-1} K_{ij}u_j \tag{5}$$

Matrix represented by system of $N+1$ linear equations with $N+1$ unknown variables can be formed

$$(I + KW)u = f \tag{6}$$

where $f = [f_i]$, $u = [u_i]^T$, $K = [k_{ij}]$, and $W = diag(\frac{h}{2}, h, ..., h, \frac{h}{2})$.

Equation (6) can be solved easily by the classical approaches in numerical method such as Gaussian elimination method, LU decomposition method, and iterative methods.

2.2 Gauss-Nystrom method

Gauss-Nystrom method [5] bases on choosing the appropriate nodal abscissas and weights for integration to approximate the approximation solution of LFIESK (equation (1)). For this paper, Gauss-Lengedre quadrature is chosen as the nodal abscissas and weights based on Lengedre function for integration. Therefore, not only is determination of nodal abscissas (x_i) the zeros of the Lengedre polynomial that is the orthogonal polynomial with respect to the weight $w(x)=1$ over domain $[-1,1]$, but also weights (W_i) of each nodal abscissas are based on derivatives of the Lengedre polynomial. Nodal abscissas and its weights [2] are as follows:

$$x_i = \cos\frac{\pi(i-0.25)}{n+0.5} \tag{7}$$

$$W_i = \frac{2}{(1-x_i^2)[\frac{d}{dx}p_n(x_i)]^2} \tag{8}$$

where $p_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x^2)^n]$ or Lengedre polynomial

n is an order of Lengedre polynomial.

However, in approximation of the solution of LFIESK (equation (1)) by using this method, the domain of integration can be transformed from $[a,b]$ into $[-1,1]$ by linear transformation,

$$t = a + \frac{b-a}{2}(z+1), \quad -1 \le z \le 1 . \tag{9}$$

Then, LFIESK (equation (1)) becomes

$$f(x) = u(x) + \int_{-1}^{1} k(x,z)u(z)dz \tag{10}$$

where $k(x,z) = \frac{b-a}{2}K(x,z)$.

Similar process with Trapezoidal rule method can be applied to form the system of N linear equations with N unknown variables in matrix form

$$(I+k)u = f . \tag{11}$$

Equation (11) can be solved by matrix algebra to find unknown variables u_i which is the approximation solution $\hat{u}(x)$ of LFIESK.

2.3 Hermite series method

Hermite series method [4] base on an orthogonal continuous polynomials which are called Hermite polyonomials on $[-\infty,\infty]$ with respect to the weight function, $w(x)=e^{-x^2}$. Let $H_n(x)$ be the Hermite polynomial of n degrees where $H_n(x)=2xH_{n-1}(x)-2(n-1)H_{n-2}(x)$ with $H_0(x)=1,\ H_1(x)=2x$. The approximation solution of LFIESK equation (1) $\hat{u}(x)$, the function $f(x)$, and kernel function $K(x,t)$ are assumed as linear combination of Hermite polynomials, Hermite series. Thus,

$$u(x)=\sum_{i=1}^{N} c_i H_i(x)\,, \tag{12}$$

$$f(x)=\sum_{i=1}^{N} f_i H_i(x) \tag{13}$$

and $$K(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N} K_{i,j} H_i(x) H_j(t) \tag{14}$$

where $H_x=[H_0(x)\,H_1(x)\ldots H_N(x)]$

$C=[c_0\,c_1\ldots c_N]^T$.

After that, equation (12-14) is substituted into equation (1) with simplification to matrix form as

$$f=C+K\{\int_{-1}^{1} H_t^T H_t dt\}C \text{ or } f=(I+KQ)C \tag{15}$$

where $Q=\{\int_{-1}^{1} H_t^T H_t dt\}=[q_{ij}];\ i,j=0,1,\ldots,N$

$f=[f_0\ f_1\cdots f_N]^T$

$H_t=[H_0(t)\,H_1(t)\ldots H_N(t)]$

$H_t=[H_0(t)\,H_1(t)\ldots H_N(t)]$

$$K=\begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix}$$

$[K(x,t)]=H_x K H_t^T$.

If $(I+KQ)$ in equation (15) is invertible, then the approximated solution $\hat{u}(x)$ of equation (1) is existent and $C=(I+KQ)^{-1}f$. Continuously, the approximated solution $\hat{u}(x)$ of equation (1) is computed in term of linear combination of unknown coefficients C and Hermite polynomials.

3. Accuracy of solution

In the research, the accuracy to desire of minimum error, absolute error is typically applied. Absolute error is an applied measurement of error, which is the absolute of difference between approximated solution $u(\hat{x})$ and exact solution $u(x)$, defined as

$$AE(x)=|\hat{u}(x)-u(x)|. \tag{16}$$

4. Computational results

In this section, the numerical results, which obtain from one of the examples from Yalcinbas and Aynigul [4], were illustrated. Let us now solve the linear Fredholm integral equation with second kind [4],

$$e^{2x}-x=u(x)-\int_0^1 xe^{-2t}u(t)dt \tag{17}$$

with exact solution $u(x)=e^{2x}$.

The exact solution and approximated solution from are illustrated in Table 1, the error analysis is given in Figure 1, and the comparison of solutions is also shown in Figure 2.

Table 1 Exact and approximated solutions of equation (17)

i	x_i	Exact solution $u(x_i)=e^{2x_i}$	Numerical Method for solving LFIESK					
			Trapezoidal rule method for $N=10$		Hermite series method [4] for $N=10$		Gauss-Nystrom method for $N=4$	
			$\hat{u}(x_i)$	$AE(x_i)$	$\hat{u}(x_i)$	$AE(x_i)$	$\hat{u}(x_i)$	$AE(x_i)$
0	0	1	1	0	1	0	1	0
1	0.1	1.221402758	1.22140	2.758E-06	1.22140	2.758E-06	1.22140	2.758E-06
2	0.2	1.491824698	1.49183	5.302E-06	1.49183	5.302E-06	1.49183	5.302E-06
3	0.3	1.8221188	1.82212	1.2E-06	1.82212	1.2E-06	1.82212	1.2E-06
4	0.4	2.225540928	2.22554	9.28E-07	2.22555	9.072E-06	2.22554	9.28E-07
5	0.5	2.718281828	2.71828	1.828E-06	2.71829	8.172E-06	2.71828	1.828E-06
6	0.6	3.320116923	3.32012	3.077E-06	3.32013	1.3077E-05	3.32012	3.077E-06

i	x_i	Exact solution $u(x_i)=e^{2x_i}$	Numerical Method for solving LFIESK					
			Trapezoidal rule method for $N=10$		Hermite series method [4] for $N=10$		Gauss-Nystrom method for $N=4$	
			$\hat{u}(x_i)$	$AE(x_i)$	$\hat{u}(x_i)$	$AE(x_i)$	$\hat{u}(x_i)$	$AE(x_i)$
7	0.7	4.055199967	4.05520	3.3E-08	4.05521	1.0033E-05	4.05520	3.3E-08
8	0.8	4.953032424	4.95303	2.424E-06	4.95304	7.576E-06	4.95303	2.424E-06
9	0.9	6.049647464	6.04965	2.536E-06	6.04965	2.536E-06	6.04965	2.536E-06
10	1.0	7.389056099	7.38906	3.901E-06	7.38902	3.6099E-05	7.38906	3.901E-06

Figure 1 gives information about comparison of the absolute error on domain $[0,1]$ of LFIESK (equation (17)) between Hermite series method and Trapezoidal rule method. Figure 2 shows the solutions obtained from Trapezoidal rule method, Hermite series method, and Gauss-Nystrom method and exact solution of LFIESK (equation (17)). On the whole, estimated solution can be best fit by Gauss-Nystrom method among these methods which solve LFIESK (equation (17)).

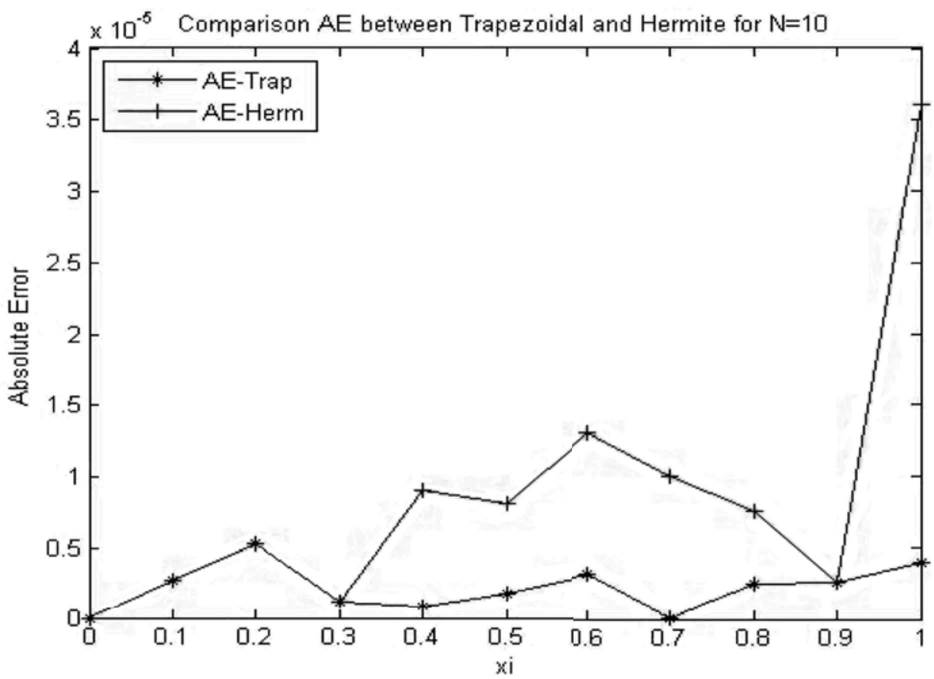

Figure 1 The absolute errors of exact solution between Trapezoidal and Hermite for $N=10$.

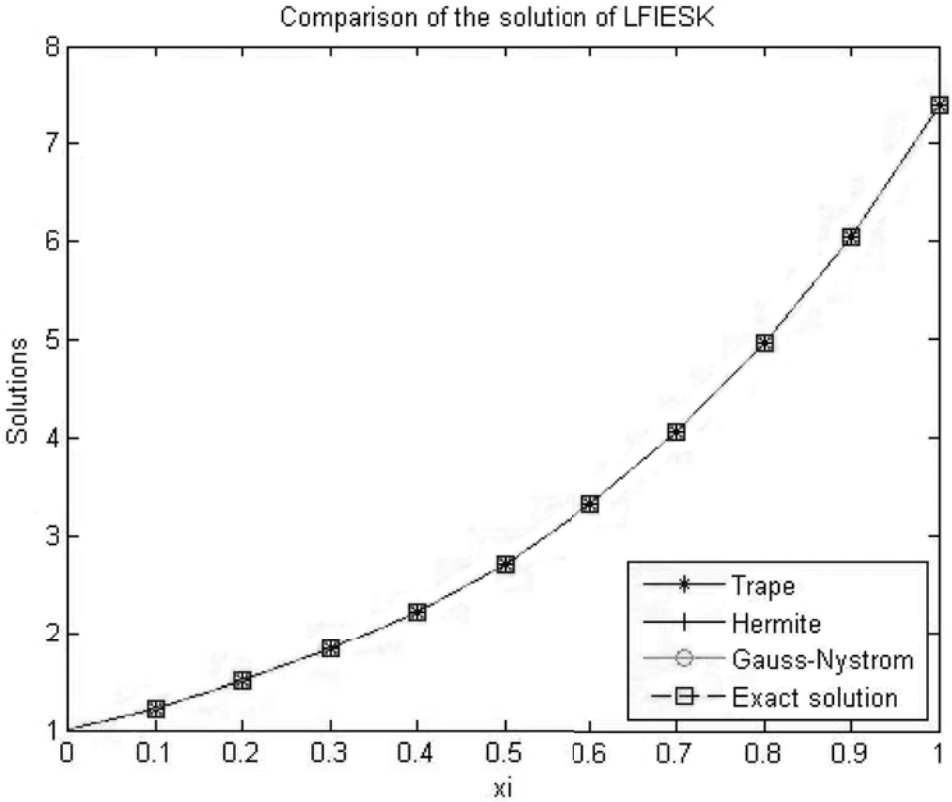

Figure 2 The solutions of LFIESK (equation (17)).

5. Discussion and Conclusion

In this paper, three methods for the solution of LFIESK equation were demonstrated and compared. In order to find the efficient method, the absolute error measurement was applied. Table 1 and Figure 1 illustrate the absolute errors of the solutions of LFIESK (equation (17)). Overall, the absolute error from Hermite series method is higher than Trapezoidal rule method over domain $[0,1]$ for $N = 10$ which provides the minimum absolute error amongst the comparisons for $N = 7 - 10$ [4]. As a result, Trapezoidal rule method is the best of both methods by absolute error measurement. After that, the Trapezoidal rule method was compared with Gauss-Nystrom method. As shown in Table 1, the approximated solutions from Trapezoidal rule method and Gauss-Nystrom method are equal, but the number of N is not equal. That is, Gauss-Nystrom method is successfully computed just $N = 4$ for the equal approximated solutions, while Trapezoidal rule method is computed the equal approximated solution for $N = 10$. Consequently, amongst three methods; namely, Trapezoidal method, Gauss-Nystrom method, and Hermite series method, Gauss-Nystrom method is the best efficient method for the solution of LFIESK (equation (17)) as shown in Figure 2. As discussed previously, Gauss-Nystrom method is a very flexible and potential method, but it is still based on weight function and has also certain limitations. This alternative is to conduct research in the future works and to focus on novel developments.

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