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FOMITE FACTORS AND SILENT SPREAD: A VSEIQCR STUDY OF VIRAL DISEASES

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ABSTRACT. This study investigates the impact of both detected and undetected viral cases, alongside environmental pathogens, on infection transmission dynamics. A VSEIQCR model is formulated and refined to analyze the study and to assess the basic reproduction number using the next-generation matrix method. The findings reveal a rapid escalation in viral cases, correlating with the rise in undetected cases. The study suggests that identifying and isolating individuals exposed to or infected by the virus, whether detected or undetected, is deemed imperative for curtailing disease transmission. Additionally, The study emphasizes the role of fomites in infection spread. It stands out for its innovative approach, examining the interconnections among vaccination, quarantine, and contamination strategies within a cohesive research framework, thereby setting a precedent in the field.

Key words: Epidemic model, Basic reproduction number, Next-generation method, Environment pathogens (Fomites).

AMS Subject Classification: : 93C10, 93C35

1. Introduction

Some viral diseases can spread through the presence of saliva in the environment [5]. On January 30, 2020, in response to the recommendations of the Emergency Committee, the Director-General of the World Health Organization (WHO) declared the outbreak of COVID-19 a Public Health Emergency of International Concern (PHEIC) [20]. Due to its worldwide spread, the WHO [5] declared it a pandemic on March 11, 2020. COVID-19, caused by Severe Acute Respiratory Syndrome Coronavirus-2 (SARS-CoV-2), first emerged in China in December 2019 [1], [11],

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[20]. Contact tracing and quarantine are key strategies adopted by India to control transmission and mortality [5]. However, when interviewing individuals infected with COVID-19 for contact tracing, some contacts may be omitted due to recall bias. These missed cases, which remain asymptomatic throughout the incubation period (2-14 days), increase the risk of involuntary transmission to the community [20]. As a result, it is essential to investigate their role in the spread of the disease. Many research articles have attempted to understand the dynamics of transmitting this disease with the help of mathematical modeling. Kermack and McKendrick [6], [7], [8] framed the initial SIR (Susceptible, infected, recovered) compartmental model to study the dynamics of a disease. Mandal et al. [9] have designed an SEIR (Susceptible, Exposed, Infected, Recovered) model to prevent or delay local outbreaks by imposing travel restrictions in India from countries affected by COVID-19. Since these studies focused on the risk of disease through direct transmission between humans, the impact of undetected cases on infection risk within the community remains uncertain. Yang and Wang [20] modeled taking into account the level of pathogens in the reservoir of the environment and their role in the spread of the disease.

Choi and Ki [1] developed a SEIHR (Susceptible, Exposed, Infected, Hospitalized, Recovered) model and estimated the basic reproduction number by the number of confirmed cases reported in Korea.

Sujata and Sumanta [12] studied the impact of the undetected infected persons on the transmission dynamics of COVID-19 for the period 22 March 2020 to 4 May 2020.

Recent studies have advanced the understanding of epidemic models by exploring various incidence rates and treatment functions. Sharma and Sharma [14] investigate the stability of an SIR model incorporating an alert class and modified saturated incidence rate, revealing critical insights into disease dynamics and treatment efficacy. Building on this, Umdekar, Sharma, and Sharma [15] extend the analysis to an SEIR model with similar modifications, highlighting its implications for epidemic control strategies. Additionally, Sharma and Sharma [16] provide a detailed study of an SIQR model with Holling type–II incidence rate, contributing to the broader understanding of model variations and their impacts on disease spread. In 2024, Soni et. al [18] present a comprehensive analysis of prevention strategies for epidemic control using a SEIQHRV (Susceptible, Exposed, Infected, Quarantined, Hospitalized, Recovered, vaccinated) model.

The paper's organization is as follows: Section 2 elaborates on the Methodology, describing the assumptions and notations employed in constructing the model. It also presents the Formulation of the model through diagrams and differential equations. In Section 3, the basic reproduction number of the model is estimated using the next-generation method. Successive sections, namely 4, 5, and 6, delve into the numerical results, main results, and conclusions, respectively.

2. METHODOLOGY

Drawing from prior work by Sujata and Sumanta [12], we expand our model by introducing quarantine and contaminated compartments as innovative components. Employing the next-generation matrix method, we calculate the basic reproduction number. Subsequently, we conduct simulations using MATLAB software to analyze the role of parameters and variables in controlling viral diseases.

2.1. **ASSUMPTION AND NOTATION.** We make the following assumptions to make our model more realistic.

1. The population distribution is homogeneous so that there are equal chances to contract and propagate the disease.

2. The Entire population is divided into various compartments of the model.

3. Each compartment has some specific property.

4. Susceptible people may become ill after coming in contact with exposed, mild infected, or severely infected people. Also, they may be ill due to contact with the containment surface or area.

5. A part of the susceptible population gets vaccinated and another part does not need to be vaccinated due to inbuilt natural immunity within them.

6. A part of the vaccinated population enters into the susceptible compartment again due to the loss of temporary immunity and another part of the vaccinated population enters into the recovered class due to permanent immunity.

7. Exposed populations is further divided into two infected compartments named as a mildly infected compartment (I_1) and severely infected compartment (I_2). Those in the exposed compartment are asymptomatic carriers and can spread the disease.

8. Population of both infected compartments has equal chances of recovery at a rate γ without the need for any kind of treatment due to the development of natural immunity during the disease period.

9. Populations of both infected compartments get treatment at a rate of σ_1 and σ_2 respectively.

10. Someone in the recovered compartment developed permanent immunity, and was never to be infected again.

11. Quarantined individuals are eligible for treatment and permanent recovery and enter into the recovered compartment at δ rate.

12. Depending upon the severity of infection, exposed or infected individuals can contaminate a non-infected environment that may surge the number of virulent pathogens in the atmosphere.

The following notations were used to build the model:

1. V: Vaccinated population
2. S: Susceptible population
3. E: Exposed population
4. I_1: Infected population, detected through appropriate testing
5. I_2: Undetected Infected population
6. Q: Quarantine population
7. C: Environmental reservoir of the pathogen (i.e. fomites contaminated with coronavirus)
8. R: Recovered population
9. β_e : Rate of transmission between exposed and susceptible persons
10. β_{i_1} : Rate of transmission between susceptible and detected infected persons
11. β_{i_2} : Rate of transmission between susceptible and undetected infected persons
12. β_c : Rate of transmission environment (fomites) to human
13. π : *Influx rate in the population*
14. μ : Natural death rate in the population
15. γ : Rate of the recovery from disease
16. ω : Rate of the death due to disease
17. α^{-1} : Period of incubation
18. β : Percentage of the undetected infected persons
19. σ_1 : Rate of transmission from infected to quarantine compartment

20. σ_2 : Rate of transmission from undetected infected to quarantine compartment
21. η: Rate of removal of the coronavirus from the atmosphere
22. ξ_1: Contribution of exposed persons to the container of the pathogens in the environment
23. ξ_2: Contribution of detected infected persons to the container of the pathogens in the environment
24. ξ_3: Contribution of undetected infected persons to the container of the pathogens in the environment
25. δ: Rate of transmission from quarantine to recovered compartment
26. ψ : Percentage of the susceptible persons who are vaccinated
27. λ : Percentage of the vaccinated persons whose immunity is temporary
28. ρ : Rate at which a susceptible person becomes vaccinated
29. κ : Rate at which vaccinated person lose their immunity

2.2. **FORMULATION OF THE MODEL.** In our VSEIQCR model (See Figure 1), we distributed the total human populations into seven compartments- vaccinated (V), susceptible (S),exposed (E), infected detected I_1, infected undetected I_2, quarantined (Q) and recovered (R). Now, we have introduced an additional compartment (C) for the environmental container of the coronavirus pathogen, which contributes to the spread of the infection.

$$\frac{dS}{\mathrm{dt}} = \pi - \beta_e SE - \beta_{i_1} SI_1 - \beta_{i_2} SI_2 - \beta_c SC - (\mu+\rho)S + \lambda\kappa V$$

$$\frac{dE}{\mathrm{dt}} = \beta_e SE + \beta_{i_1} SI_1 + \beta_{i_2} SI_2 + \beta_c SC - (\mu+\alpha)\, E$$

$$\frac{dI_1}{\mathrm{dt}} = \alpha\beta E - (\mu+\omega+\gamma+\sigma_1)\, I_1$$

$$\frac{dI_2}{\mathrm{dt}} = \alpha\,(1-\beta)\, E - (\mu+\omega+\gamma+\sigma_2)\, I_2$$

$$\frac{\mathrm{dC}}{\mathrm{dt}} = \xi_1 E + \xi_2 I_1 + \xi_3 I_2 - \eta C$$

$$\frac{\mathrm{dQ}}{\mathrm{dt}} = \sigma_1 I_1 + \sigma_2 I_2 - (\mu+\delta)\, Q$$

$$\frac{\mathrm{dR}}{\mathrm{dt}} = (I_1+I_2)\,\gamma + \delta Q + \rho\,(1-\psi)\, S + \kappa\,(1-\lambda)\, V - \mu R$$

$$\frac{\mathrm{dV}}{\mathrm{dt}} = \rho\psi S - (\mu+\kappa)\, V$$

Obviously, the system of equations has a disease-free equilibrium, $X_{\mathrm{DFE}} = \left(\frac{\pi}{\mu+\rho}\ ,\ 0,\ 0,0,0,0,0,0\right).$

3. THE BASIC REPRODUCTION NUMBER

The basic reproduction number is the measurement of a disease's potential spread. It represents the average number of secondary infections caused by a single infectious person in a completely susceptible population. This number indicates whether a disease will die out or persist in the population. Specifically, $R_0 < 1$ implies that the disease will eventually die out, while $R_0 \geq 1$ suggests that the disease will continue to affect the population over time. Soni et al. [17] investigate the basic reproduction number R_0 and herd immunity for COVID-19 in India, emphasizing their critical

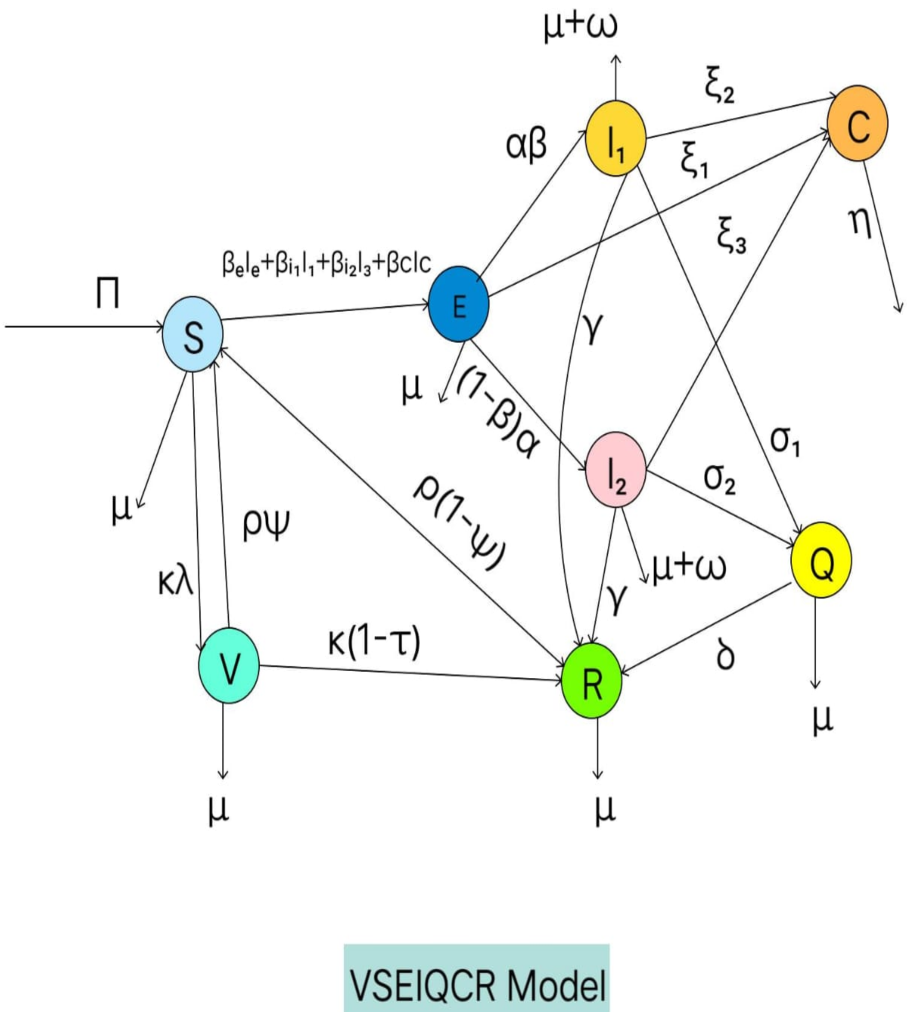

FIGURE 1. VSEIQCR Model

relationship. The study highlights how R_0 , a measure of disease transmission potential, directly influences the threshold for achieving herd immunity.
We derive this number with the help of next-generation method given by Van den Driessche [19]. This method separates the compartments into infected compartment (E, I_1, I_2, C) and uninfected compartments (V, S, Q, R). x and y denote the vector of variables in the infected and the non-infected compartments i.e. $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ where $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$ represent the $E, I_1, I_2, C, S, Q, R, V$ compartments respectively. The dynamical system of equations may be written as:

$$x_1^{'} = F_1(x, y) = \beta_e SE + \beta_{i1} SI_1 + \beta_{i2} SI_2 + \beta_c SC - (\mu + \alpha) E,$$

$$x_2^{'} = F_2(x, y) = \alpha\beta E - (\mu + \omega + \gamma + \sigma_1) I_1,$$

$$x_3^{'} = F_3\,(x,y) = \ \alpha\,(1-\beta)\,E - (\mu+\omega+\gamma+\sigma_2)\,I_2,$$

$$x_4^{'} = F_4\,(x,y) = \xi_1 E + \xi_2 I_1 + \xi_3 I_2 - \eta C,$$

$$y_1^{'} = \ G_1\,(x,y) = \pi - \beta_e SE - \beta_{i1} SI_1 - \beta_{i2} SI_2 - \beta_c SC - \mu S - (\mu+\rho)S + \lambda\kappa V,$$

$$y_2^{'} = \ \sigma_1 I_1 + \sigma_2 I_2 - (\mu+\delta)\,Q,$$

$$y_3^{'} = (I_1+I_2)\,\gamma + \delta Q + \rho\,(1-\psi)\,S + \kappa\,(1-\lambda)\,V - \mu R,$$
$$y_4^{'} = \rho\psi S - (\mu+\kappa)\,V,$$

Now, we divide the infection compartments right hand side as shown below:

$X_i^{'} = M_i\,(x,y) - N_i\,(x,y)\ \ \forall\ i = 1,\ 2,\ 3,\ 4.$ where $M_i\,(x,y)$ is the rate of new infection in the compartment x_i ($\forall\ i = 1,\ 2,\ 3,\ 4$) and $N_i\,(x,y)$ represent the other transitory terms of infected compartment.

$$\begin{aligned}
&M_1\,(x,y) = \beta_e SE + \beta_{i1} SI_1 + \beta_{i2} SI_2 + \beta_c SC - (\mu+\alpha)\,E, && N_1\,(x,y) = 0,\\
&M_2\,(x,y) = 0, && N_2\,(x,y) = -\alpha\beta E + (\mu+\omega+\gamma+\sigma_1)\,I_1,\\
&M_3\,(x,y) = 0, && N_3\,(x,y) = -\alpha\,(1-\beta)\,E + (\mu+\omega+\gamma+\sigma_2)\,I_2,\\
&M_4\,(x,y) = 0, && N_4\,(x,y) = -\xi_1 E - \xi_2 I_1 - \xi_3 I_2 + \eta C.
\end{aligned}$$

The linearized system of infected compartments may be written as:

$$x_i^{'} = (F-T)x,$$

where, F and T are the infections and the transition matrices respectively.

$$F = \left[\frac{\partial M_i}{\partial x_j}\right],$$
$$T = \left[\frac{\partial N_i}{\partial x_j}\right].$$

which arise from linearizing the system around the disease-free equilibrium.

$$F = \begin{bmatrix} \beta_e S_0 & \beta_{i_1} S_0 & -\beta_{i_2} S_0 & \beta_c S_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$T = \begin{bmatrix} \alpha+\mu & 0 & 0 & 0 \\ -\alpha\beta & \mu+\omega+\gamma+\sigma_1 & 0 & 0 \\ -\alpha(1-\beta) & 0 & \mu+\omega+\gamma+\sigma_2 & 0 \\ -\xi_1 & -\xi_2 & -\xi_3 & \eta \end{bmatrix}.$$

The next-generation matrix is defined as

$$D = FT^{-1}.$$

The basic reproduction number $\mathbb{R}_0$ for the model (1) is determined by the spectral radius of the next-generation matrix D and is given by: $\mathbb{R}_0 = R_1 + R_2 + R_3 + R_4$, where

$$R_1 = \frac{\beta_e \pi}{(\mu+\alpha)(\mu+\rho)},$$

$$R_2 = \frac{\beta_{i1}\alpha\beta\pi}{(\mu+\alpha)(\mu+\rho)(\mu+\omega+\gamma+\sigma_1)},$$

$$R_3 = \frac{\beta_{i2}\alpha(1-\beta)\pi}{(\mu+\alpha)(\mu+\rho)(\mu+\omega+\gamma+\sigma_2)},$$

$$R_4 = \frac{\beta_c \pi \left\{\alpha\beta(\mu+\omega+\gamma+\sigma_2)\xi_2 + \alpha\xi_3(1-\beta)(\mu+\omega+\gamma+\sigma_1) + \xi_1(\mu+\omega+\gamma+\sigma_1)(\mu+\omega+\gamma+\sigma_2)\right\}}{(\mu+\alpha)(\mu+\omega+\gamma+\sigma_1)(\mu+\omega+\gamma+\sigma_2)(\mu+\rho)\eta}$$

,
where $R_1,\ R_2,\ R_3$, *and* R_4 *provide the evaluation of the risk of disease by pathways S to E, S to* I_1, *S to* I_2 *compartment and from environment to human respectively.*

4. LOCAL STABILITY AT DISEASE FREE EQUILIBRIUM

The Jacobean matrix of the model is given by:

$$J = \begin{bmatrix} -\Delta_1-\rho & -\beta_e & -\beta_{i_1} & -\beta_{i_2} & -\beta_c & 0 & 0 & \lambda\kappa \\ -\Delta_1 & -(\mu+\alpha) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha\beta & -(\mu+\omega+\gamma+\sigma_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha(1-\beta) & 0 & -(\mu+\omega+\gamma+\sigma_2) & 0 & 0 & 0 & 0 \\ 0 & \xi_1 & \xi_2 & \xi_3 & -\eta & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & \sigma_2 & 0 & -(\mu+\delta) & 0 & 0 \\ \rho(1-\psi) & 0 & \gamma & \gamma & 0 & \delta & -\mu & \kappa(1-\lambda) \\ \rho\psi & 0 & 0 & 0 & 0 & 0 & 0 & -(\mu+\kappa) \end{bmatrix},$$

where $\Delta_1 = \beta_e E + \beta_{i1} I_1 + \beta_{i2} I_2 + \beta_c C$.
At the point X_{DFE} the Jacobean matrix of the model is given by:

$$J_{X_{DFE}} = \begin{bmatrix} -\rho & -\beta_e & -\beta_{i_1} & -\beta_{i_2} & -\beta_c & 0 & 0 & \lambda\kappa \\ 0 & -(\mu+\alpha) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha\beta & -(\mu+\omega+\gamma+\sigma_1) & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha(1-\beta) & 0 & -(\mu+\omega+\gamma+\sigma_2) & 0 & 0 & 0 & 0 \\ 0 & \xi_1 & \xi_2 & \xi_3 & -\eta & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & \sigma_2 & 0 & -(\mu+\delta) & 0 & 0 \\ \rho(1-\psi) & 0 & \gamma & \gamma & 0 & \delta & -\mu & \kappa(1-\lambda) \\ \rho\psi & 0 & 0 & 0 & 0 & 0 & 0 & -(\mu+\kappa) \end{bmatrix}$$

Using MATLAB software and the parameter values outlined in the table 1, we evaluated the eigenvalues at the disease-free equilibrium X_{DFE}.

Our analysis revealed that all eigenvalues at this equilibrium have negative real parts (i.e. $-0.73, -0.7997, -0.9, -0.8543, -0.2, -1.515, -1.015, -7.7$). According to the Routh-Hurwitz criterion, this result confirms local asymptotic stability when the basic reproductive number $\mathbb{R}_0$ is less than 1, and indicates instability when it exceeds 1.

5. GLOBAL STABILITY AT DISEASE FREE EQUILIBRIUM

To investigate the global stability of the disease-free equilibrium (DFE) in our model, we conducted a numerical simulation based on the set of differential equations governing the system. The value of $\mathbb{R}_0$ depends on each of its component. Given the provided parameter values in the table 1, we compute the basic reproduction number $\mathbb{R}_0$ and its components $\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3, and \mathbb{R}_4$. $\mathbb{R}_1 = 1.2483 \times 10^{-6}, \mathbb{R}_2 = 5.7468 \times 10^{-7}, \mathbb{R}_3 = 1.1020 \times 10^{-7}, \mathbb{R}_4 = 2.9339 \times 10^{-8}$. Thus, the basic reproduction number is: $\mathbb{R}_0 = 1.9625 \times 10^{-6}$. This value of $\mathbb{R}_0$ being much less than 1 confirms that the disease-free equilibrium (DFE) is globally stable, the disease will not spread in the population, and the system will return to the DFE over time. We use the values in table 1 with $S_0 = 70,000$, $E_0 = 50,000$, $I_{1_0} = 3,0,000$, $I_{2_0} = 4,0,000$, $C_0 = 5000$, $Q_0 = 25,000$, $V_0 = 30,000$, and $R_0 = 50,000$ to perform a simulation for the Disease-Free Equilibrium (DFE), as shown in figure 2.

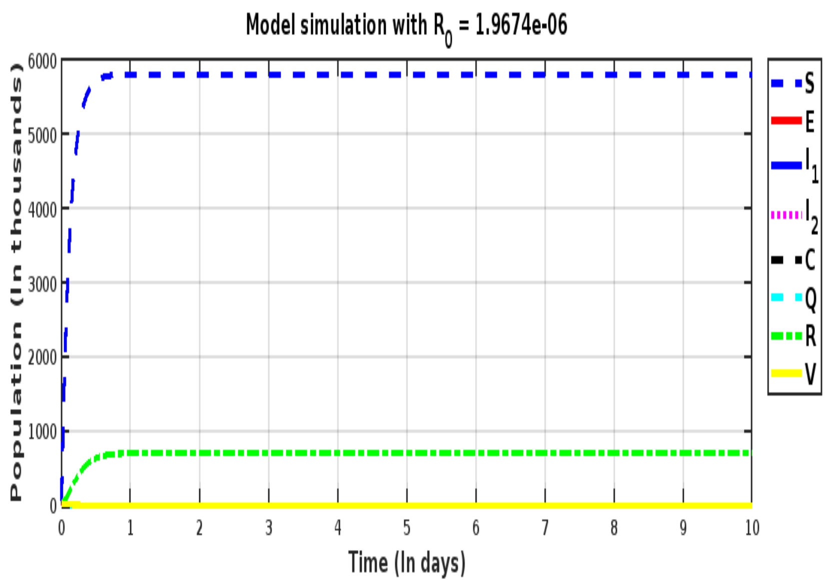

FIGURE 2. DFE

The simulation results, depicted in figure 2, demonstrate the temporal evolution of each compartment within the model. Over the course of the simulation:

1. Susceptible Population (S): The susceptible population stabilizes near the value $\frac{\pi}{\mu+\rho}$, consistent with the DFE. This behavior indicates that the introduction of disease-related perturbations does not significantly deplete the susceptible population, maintaining its stability.
2. Exposed (E), Infected I_1 and I_2: The exposed and infected populations, both I_1 and I_2, consistently approach zero as time progresses. This outcome suggests that the infection does not sustain itself within the population and tends to die out, leading to a return to the disease-free state.
3. Contaminated Carrier (C) and Quarantined (Q): Similar to the exposed and infected populations, the carrier and quarantined compartments also trend towards zero. This further reinforces the notion that the disease cannot persist within the population under the given parameters.

4. Recovered (R) and Vaccinated (V): The recovered and vaccinated populations stabilize at levels that do not interfere with the overall disease dynamics, thus supporting the DFE stability.

These results collectively confirm the global stability of the disease-free equilibrium within the context of our model. The simulations reveal that irrespective of initial conditions, the system invariably returns to the DFE over time, thereby affirming the robustness of this equilibrium. The observed stability is consistent across all simulated scenarios, indicating that the model effectively captures the mechanisms necessary for disease eradication under the given parameters.

Furthermore,to establish global stability at the DFE, we propose the following Lyapunov function:

$$V(S,E,I_1,I_2,C,Q,R,V) = \frac{1}{2}\left(\frac{S-S^*}{S^*}\right)^2 + \frac{E}{\mu+\alpha} + \frac{I_1}{\mu+\omega+\gamma+\sigma_1} + \frac{I_2}{\mu+\omega+\gamma+\sigma_2}$$

where $S^* = \frac{\pi}{\mu+\rho}$.

Time Derivative of the Lyapunov Function. The time derivative of $V(S,E,I_1,I_2,C,Q,R,V)$ is computed as follows:

$$\frac{dV}{dt} = \frac{\partial V}{\partial S}\frac{dS}{dt} + \frac{\partial V}{\partial E}\frac{dE}{dt} + \frac{\partial V}{\partial I_1}\frac{dI_1}{dt} + \frac{\partial V}{\partial I_2}\frac{dI_2}{dt}$$

where the partial derivatives are:

$$\frac{\partial V}{\partial S} = \frac{S-S^*}{S^{*2}}$$

$$\frac{\partial V}{\partial E} = \frac{1}{\mu+\alpha}, \quad \frac{\partial V}{\partial I_1} = \frac{1}{\mu+\omega+\gamma+\sigma_1}, \quad \frac{\partial V}{\partial I_2} = \frac{1}{\mu+\omega+\gamma+\sigma_2}$$

Substituting the derivatives from the system:

$$\begin{aligned}\frac{dV}{dt} = \frac{S-S^*}{S^{*2}}\left(\pi-\mu S-\rho S+\lambda\kappa V-\beta_e SE-\beta_{i1}SI_1-\beta_{i2}SI_2-\beta_c SC\right) \\ +\frac{\beta_e SE+\beta_{i1}SI_1+\beta_{i2}SI_2+\beta_c SC-(\mu+\alpha)E}{\mu+\alpha} \\ +\frac{\alpha\beta E-(\mu+\omega+\gamma+\sigma_1)I_1}{\mu+\omega+\gamma+\sigma_1} \\ +\frac{\alpha(1-\beta)E-(\mu+\omega+\gamma+\sigma_2)I_2}{\mu+\omega+\gamma+\sigma_2}\end{aligned}$$

Given the structure of $\frac{dV}{dt}$, the following conclusions can be drawn:

Near the DFE: Since $S \approx S^*$ and all other compartments E, I_1, I_2 are small or zero, $V(S,E,I_1,I_2,C,Q,R,V)$ is positive and $\frac{dV}{dt} \leq 0$. This indicates that the Lyapunov function does not increase over time, with equality only at the DFE.

Global Behavior: If $\frac{dV}{dt}$ is negative semi-definite, and the only equilibrium where V is minimized is the DFE, then the system will asymptotically approach the DFE regardless of the initial conditions, provided they are in the feasible region. To demonstrate that $\frac{dV}{dt}$ is negative semi-definite and that the only equilibrium where V is minimized is the Disease-Free Equilibrium (DFE), we analyze the sign of each term in the expression for $\frac{dV}{dt}$.

First Term:

$$\frac{S - S^*}{S^{*2}} \left(\pi - \mu S - \rho S + \lambda\kappa V - \beta_e SE - \beta_{i1} SI_1 - \beta_{i2} SI_2 - \beta_c SC\right)$$

$S - S^*$ changes sign depending on whether $S > S^*$ or $S < S^*$. The expression inside the parentheses represents the change in S over time. In the DFE, $S = S^*$ and the terms involving infected compartments (E, I_1, I_2, C) are zero.

Second Term:

$$\frac{\beta_e SE + \beta_{i1} SI_1 + \beta_{i2} SI_2 + \beta_c SC - (\mu + \alpha)E}{\mu + \alpha}$$

This term represents the change in E. At DFE, $E = 0$ and the entire term becomes zero. - Outside DFE, this term is positive when there is transmission but negative when the removal rate $(\mu + \alpha)$ dominates.

Third Term:

$$\frac{\alpha\beta E - (\mu + \omega + \gamma + \sigma_1)I_1}{\mu + \omega + \gamma + \sigma_1}$$

At DFE, $I_1 = 0$, $E = 0$, so this term is zero. Otherwise, this term can be positive or negative depending on the balance between infection (first term) and removal (second term).

Fourth Term:

$$\frac{\alpha(1 - \beta)E - (\mu + \omega + \gamma + \sigma_2)I_2}{\mu + \omega + \gamma + \sigma_2}$$

Similar analysis to the third term. At DFE, $I_2 = 0$, $E = 0$, and the term is zero.

At the DFE, where $E = 0$, $I_1 = 0$, $I_2 = 0$, $C = 0$, $Q = 0$, and $S = S^*$:

$$\frac{dV}{dt} = 0$$

For non-DFE equilibria, $\frac{dV}{dt} \leq 0$. The term $\frac{S-S^*}{S^{*2}}$ multiplied by the expression involving infections is negative because infections reduce susceptible individuals. Similarly, the terms involving E, I_1, and I_2 become negative when infected compartments are non-zero, reflecting the progression of disease and recovery or removal of infected individuals.

$\frac{dV}{dt}$ is negative semi-definite, indicating that V does not increase over time and decreases whenever there are infected individuals. The only point where $\frac{dV}{dt} = 0$ and V reaches its minimum is at the Disease-Free Equilibrium (DFE), where all compartments except S are zero. This demonstrates that the system stabilizes at the DFE, where no infection persists.

We find that the entire population will eventually consist only of susceptible, vaccinated, or recovered individuals and the Lyapunov function ensures that even if the disease initially spreads through the population, the natural dynamics of the system will drive the population back to the disease-free equilibrium, where the disease cannot persist. Thus, this analysis guarantees that, under the given model and assumptions, the disease will not become endemic or persist in the long run; instead, it will fade out, leaving the population disease-free. Therefore, the disease-free equilibrium $X_{\text{DFE}} = \left(\frac{\pi}{\mu+\rho}, 0, 0, 0, 0, 0, 0, 0\right)$ is globally asymptotically stable.

6. NUMERICAL RESULTS

We estimated the basic reproduction number for the period from 16 January 2021 to 28 February 2021. After starting of the vaccination program in India, we eagerly want to know the impact of undetected infected persons on the transmission dynamics of COVID-19. As of 16 January 2020, the cumulative confirmed cases, death cases and recovered cases were 10558637, 151720 and 10196056 respectively, whereas on 28 February 2020, these data became 11111978, 156603 and 10784401 respectively. Because, those who are exposed and Undetected tend to live in the community, they can spread the disease at the same rate [13].

Also, the family members impacted by COVID-19 may survive in the environment from a couple of hours to several days [20]. In this study, this value was taken as 5 hours and, consequently, the elimination rate of virus is 0.2 per day. We estimated the recovery rate (γ) as follows- $\frac{\text{Difference of cumulative recovered}}{\text{Difference of cumulative confirmed}} * \frac{1}{\text{Average recovery time in days}} = \frac{10784401-10196056}{11111978-10558637} * \frac{1}{14} = 0.075 \; per \; day$. As of 1 January 2021, the total estimated population of India was 1,390,537,387 [3].

TABLE 1. **Parameter Estimates**

S.No.	Parameter	Estimated Value	Sources
1	π	47964 per day	[3]
2	β_e	$0.25 \times 0.1231 \times 10^{-7}$	[13]
3	β_{i_1}	$0.25 \times 0.5944 \times 0.1231 \times 10^{-7}$	[13]
4	β_{i_2}	$0.25 \times 0.1231 \times 10^{-7}$	[13]
5	β_c	1.03×10^{-8}	[13]
6	μ	7.344 per day	[4]
7	γ_1, γ_2	0.075 per day	Estimated
8	ω	0.01	Estimated
9	α	7 days	[20]
10	β	0.9	[13]
11	σ_1, σ_2	$0.7, 0.2$	Assumed
12	η	0.2	[12]
13	ξ_1	0.001	[13]
14	ξ_2	0.000398	[13]
15	ξ_3	0.001	[13]
16	δ	0.1243	[10]
17	ψ	0.008	Assumed
18	λ	0.05	Assumed
19	ρ	0.9	Assumed
20	κ	0.07	Assumed

In the below table 2, we estimated the contribution of exposed individuals, undetected infected individuals, detected infected individuals, and contaminated pathogens in the final reproduction number. We vary the values of the transmission rate (ρ) and percentage of undetected infected individuals $(1-\beta)$ simultaneously, to check the impact of undetected infected and vaccinated individuals on the transmission of epidemic COVID-19.

As $(1-\beta)$ increases from 0.1 to 0.4, the overall risk R_0 (which is the sum of all individual risks) slightly increases. This trend suggests that as a smaller proportion of the population remains immune (i.e., $(1-\beta)$ increases), the total risk of disease

TABLE 2. **Results for Different Values of ρ**

S.No.	$(1-\beta)$	R_1	R_2	R_3	R_4	R_0
When $\rho = 0.9$						
1	0.1	1.2483e-06	5.7503e-07	1.1453e-07	2.9553e-08	1.9674e-06
2	0.2	1.2483e-06	5.1113e-07	2.2907e-07	3.0589e-08	2.0191e-06
3	0.3	1.2483e-06	4.4724e-07	3.4360e-07	3.1625e-08	2.0707e-06
4	0.4	1.2483e-06	3.8335e-07	4.5814e-07	3.2661e-08	2.1224e-06
When $\rho = 0.8$						
1	0.1	1.2636e-06	5.8209e-07	1.1594e-07	2.9916e-08	1.9915e-06
2	0.2	1.2636e-06	5.1741e-07	2.3188e-07	3.0965e-08	2.0438e-06
3	0.3	1.2636e-06	4.5273e-07	3.4782e-07	3.2013e-08	2.0962e-06
4	0.4	1.2636e-06	3.8806e-07	4.6376e-07	3.3062e-08	2.1485e-06

transmission also increases. The risk via the pathway from susceptible to exposed individuals (R_1) is the largest contributor to the total risk R_0, highlighting that the initial exposure is the most critical phase in disease spread.

The risks associated with the pathways to detected infected individuals (R_2), undetected infected individuals (R_3), and environmental transmission (R_4) are smaller, but they do increase as $(1-\beta)$ increases. Comparing $\rho = 0.9$ and $\rho = 0.8$, the risks R_1, R_2, R_3, and R_4 are slightly higher when $\rho = 0.8$ than when $\rho = 0.9$. This indicates that a lower ρ leads to an increased overall risk of disease transmission.

The data show that reducing the proportion of the immune population ($(1-\beta)$) results in a higher overall risk of disease transmission. The most significant risk occurs at the initial stage (from susceptible to exposed individuals). Hence, strategies to minimize disease transmission should focus on enhancing immunity and maintaining or improving vaccination.

Additionally, the risk R_2 associated with detected infected individuals (I_1) is consistently higher than R_3 associated with undetected infected individuals (I_2) across all values of $(1-\beta)$ and both ρ values, suggesting that detected infected individuals pose a greater risk to disease spread, possibly due to more interactions or higher infectiousness. The smaller increase in R_3 suggests that undetected infected individuals contribute less to transmission risk relative to detected cases.

We can easily observe that in both the cases (when $\rho = 0.9$ or when $\rho = 0.8$) whenever the number of undetected infected individuals is 0.1, 0.2, and 0.3 percent, the dominant part of disease risk is detected infected compartment, whereas, whenever its value is 0.4 percent the undetected infected individuals compartment contribution dominantly in the risk of disease spread. In this context, the contribution of $\mathbb{R}_4$ in the total basic reproduction number cannot be neglected.

As $(1-\beta)$ increases, R_4 shows a slight increase, suggesting that as more individuals become susceptible, fomite transmission risk also rises, though less significantly than other pathways. When ρ decreases, R_4 values increase slightly, similar to other risks, but still remain minor compared to direct transmission routes. This highlights that while environmental hygiene and disinfection are important, the primary focus for controlling disease spread should be on direct transmission routes and improving immunity in the population.

6.1. **EFFECT OF PARAMETERS ON INFECTED COMPARTMENT.** We set $\beta_e = 0.3 \times 10^{-5}$, $\beta_{i_1} = 0.3 \times 10^{-2}$, $\beta_{i_2} = 0.3 \times 10^{-2}$, and $\beta_c = 0.5 \times 10^{-6}$, while keeping the remaining parameters the same as listed in Table 1. This setup is used to perform a simulation to assess the effect of a particular parameter on various infected compartments. For the simulation, we initialize with $S_0 = 70,000$, $E_0 = 50,000$, $I_{1_0} = 3,0,000$, $I_{2_0} = 4,0,000$, $C_0 = 5000$, $Q_0 = 25,000$, $V_0 = 30,000$, and $R_0 = 50,000$ to perform simulation.

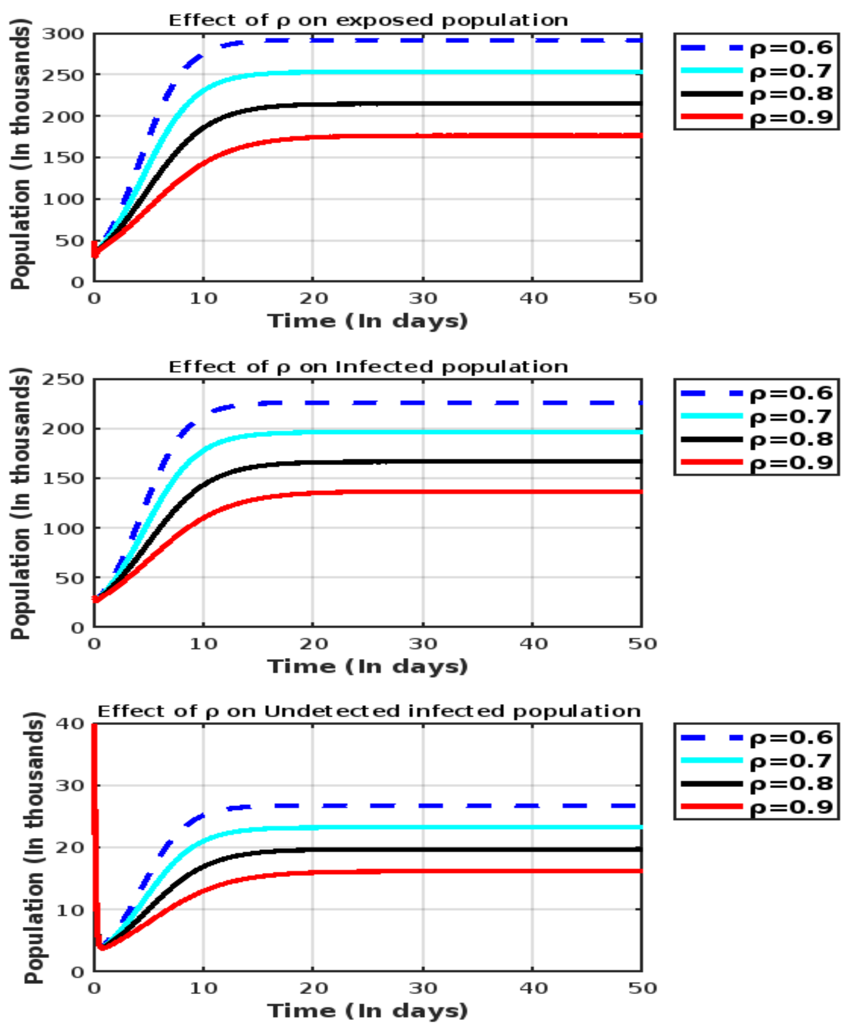

FIGURE 3. Effect of ρ

In Figure 3, as the vaccination rate ρ of susceptible individuals increases, all the curves corresponding to the infected compartments show a decline in their populations.

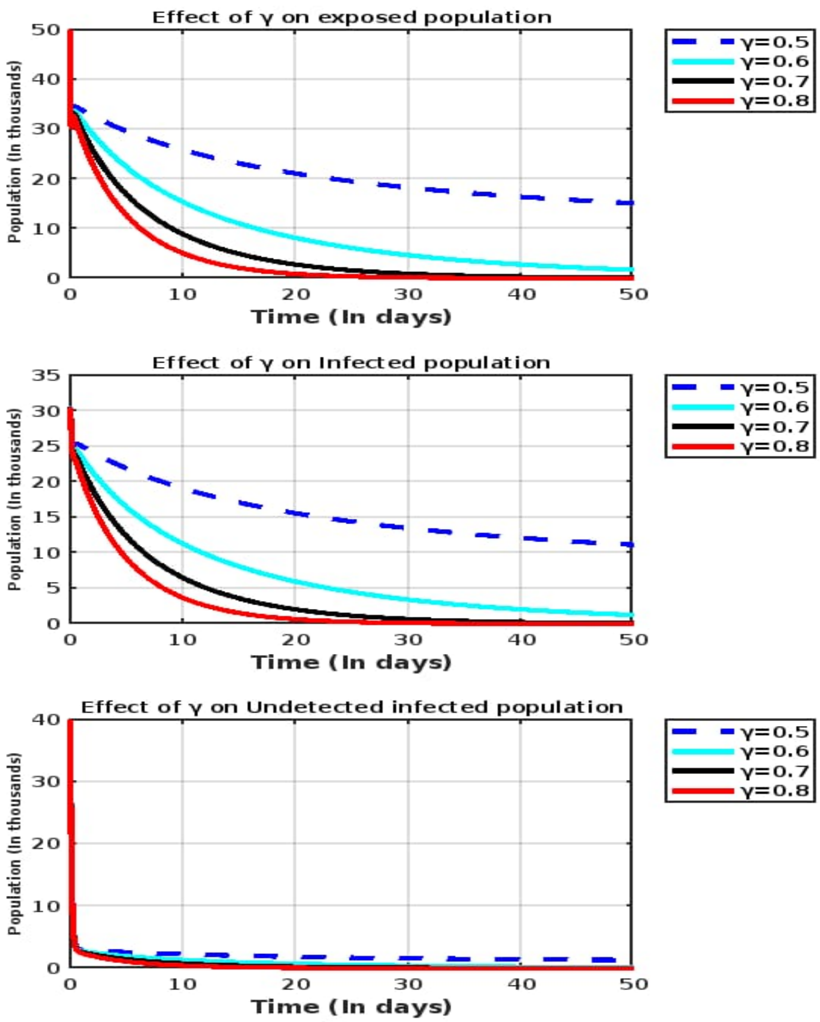

FIGURE 4. Effect of γ

In Figure 4, as the recovery rate of infected individuals γ increases, all the curves corresponding to the infected compartments show a decline in their populations.

In Figure 5, as the percentage of undetected infected populations β decreases, all the curves corresponding to the infected compartments show a decline in their populations.

In Figure 6, as the quarantine rate σ_1 of infected individuals increases, all the curves corresponding to the infected compartments show a decline in their populations.

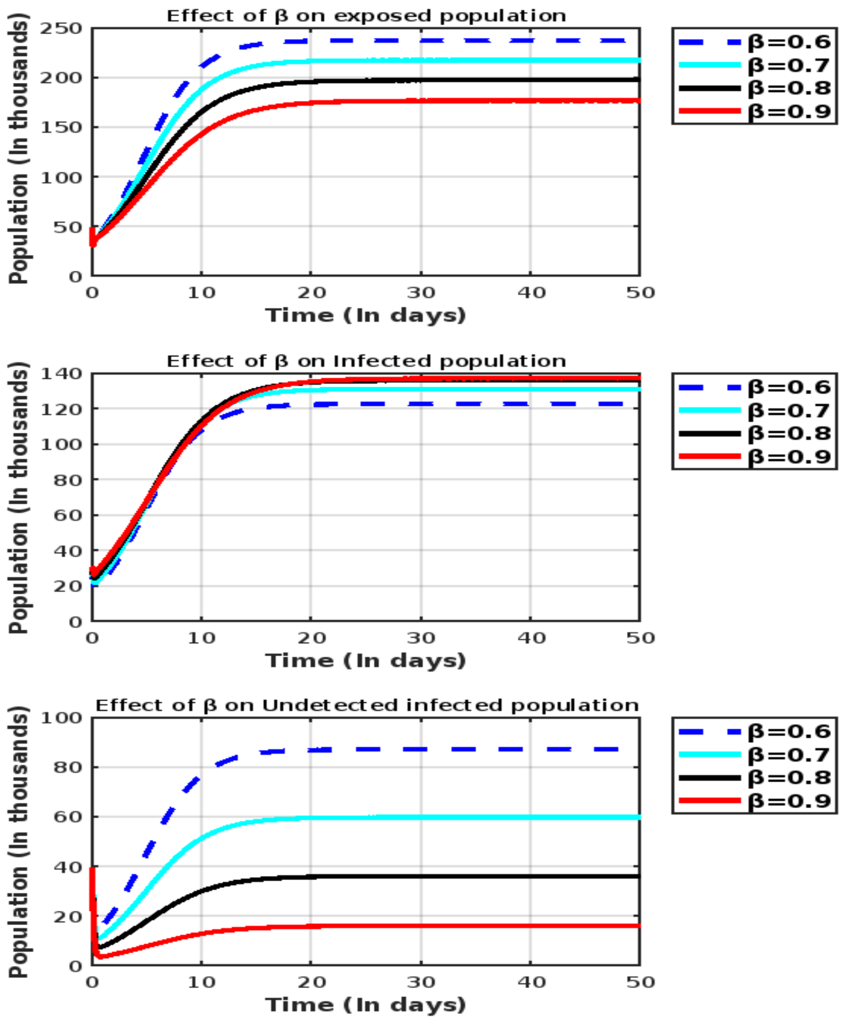

FIGURE 5. Effect of β

In Figure 7, as the quarantine rate σ_2 of undetected infected individuals increases, all the curves corresponding to the infected compartments show a decline in their populations.

In Figure 8, as the transmission rate β_{i_1} decreases, all the curves corresponding to the infected compartments show a decline in their populations.

In Figure 9, as the transmission rate β_{i_2} decreases, all the curves corresponding to the infected compartments display a decline in their populations.

In Figure 10, as the transmission rate β_e decreases, all the curves corresponding to the infected compartments display a decline in the corresponding populations.

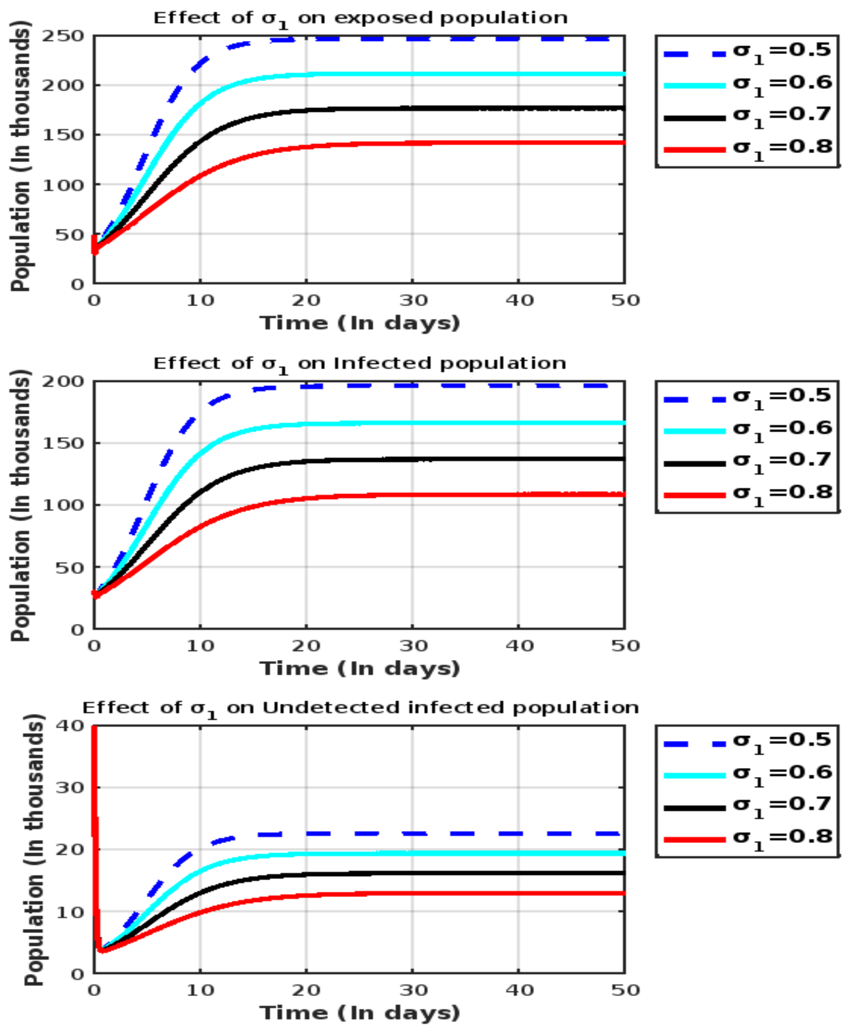

FIGURE 6. Effect of σ_1

7. MAIN RESULTS

According to the World Health Organization (WHO) [5], individuals susceptible to infection may contract the virus through contact with contaminated objects or surfaces, known as fomites. Before cleansing their hands, these individuals may inadvertently touch their eyes, mouth, or nose, thereby facilitating transmission. Therefore, the role of fomites in COVID-19 transmission is a crucial aspect addressed in our study. Mitigation strategies include frequent hand washing and sanitization, as well as the consistent use of masks as preventive measures.

Moreover, exposed and undetected infected individuals contribute to the spread of COVID-19 unknowingly. Thus, it is important to detect, isolate, and treat them.

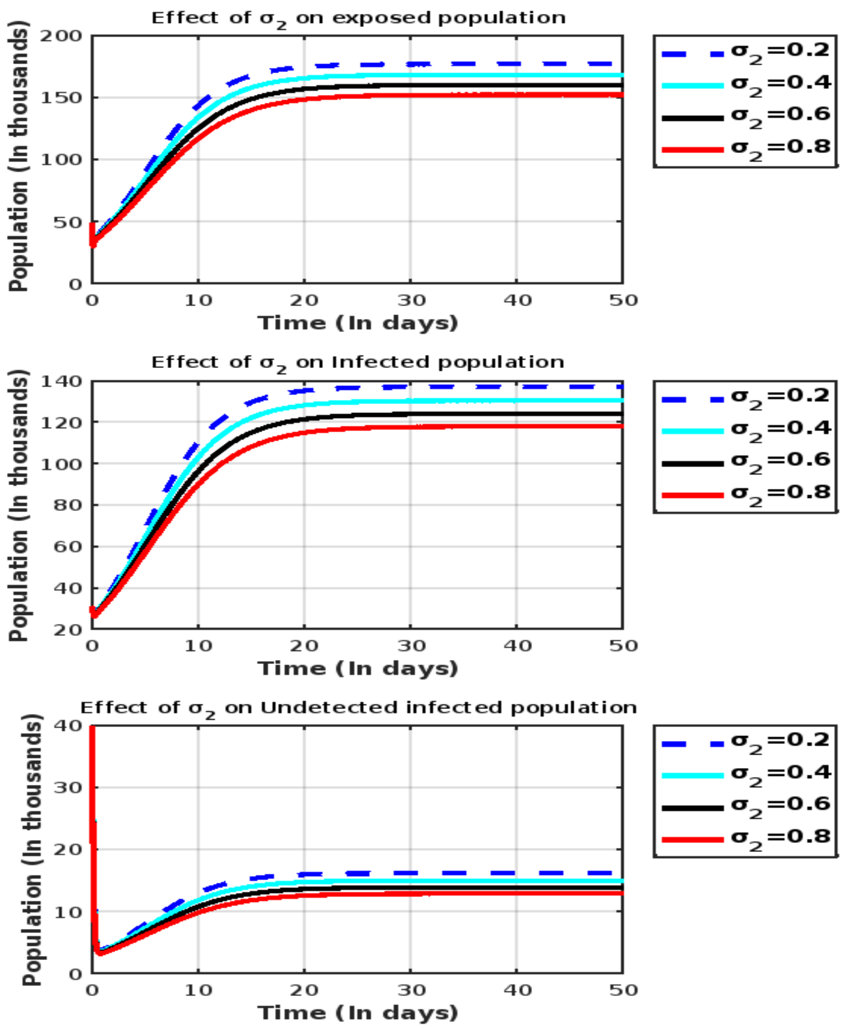

FIGURE 7. Effect of σ_2

This underscores the significance of tracing and testing to prevent the spread of the disease. Even though vaccines are available to control and reduce the risk of infection, undetected infected individuals still significantly contribute to the basic reproduction number. Hence, detecting and isolating these individuals can help prevent the spread of infection.

Furthermore, it is necessary to decrease the risk of spread by increasing the percentage of vaccinated people. Some individuals may lose their immunity due to a low immune response against the disease, so it is important for people to complete both vaccine doses. They should follow the guidelines issued by the Government of India and should not neglect preventive precautions. Our study may be the first to simultaneously include fomites, quarantine, undetected infected individuals,

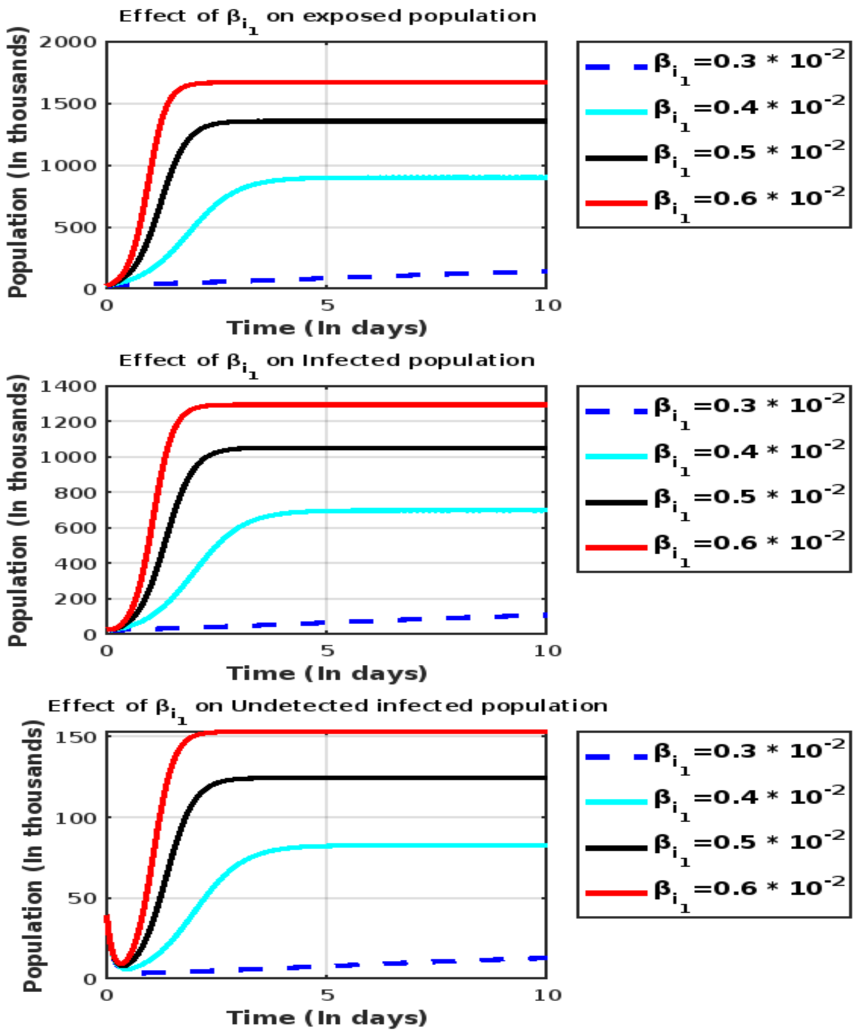

FIGURE 8. Effect of β_{i_1}

and vaccinated individuals in a mathematical model of COVID-19 transmission within the population. The estimation of the basic reproduction number is based on parameter values selected from other relevant studies; therefore, our findings may differ from the original values.

8. CONCLUSIONS

The risk of COVID-19 spread increases with the rise in undetected infected cases. Additionally, carelessness and lack of awareness regarding fomite transmission significantly contribute to the persistence of the disease within the population. Since undetected

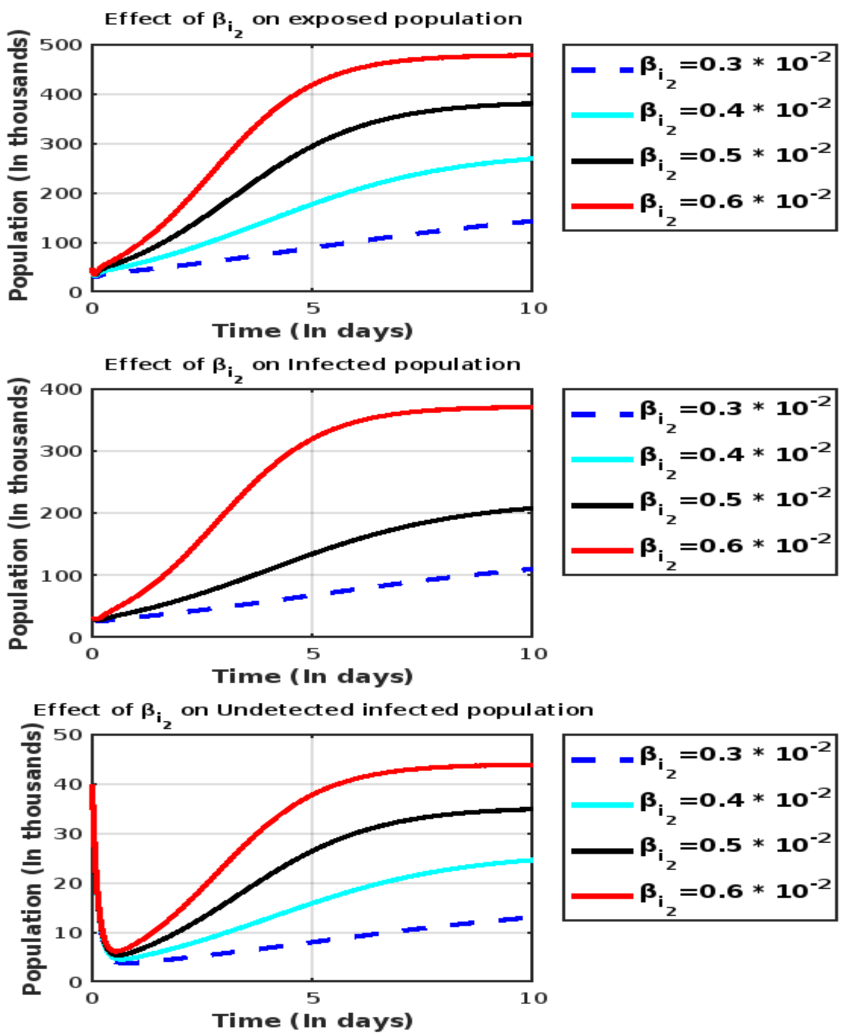

FIGURE 9. Effect of β_{i2}

infected and exposed individuals unknowingly spread the virus, it is crucial to identify, isolate, and treat them to prevent further transmission.

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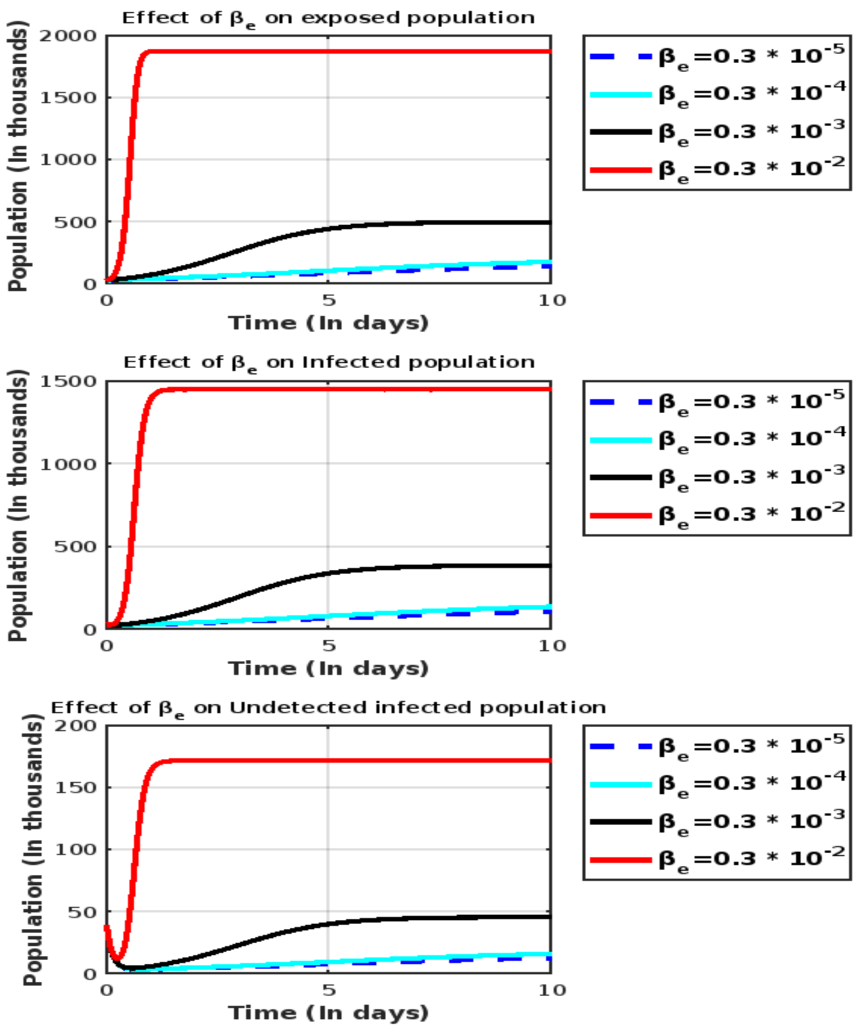

FIGURE 10. Effect of β_e

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CONVERGENCE BEHAVIOR OF MODIFIED-BERNSTEIN-KANTROVINCH-STANCU OPERATORS

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ABSTRACT. The study introduces a Kantrovinch-Stancu type modification of the modified-Bernstein operator, examining its convergence properties for Hölder's class of functions. It evaluates the rate of convergence through the modulus of continuity and Peetre's K-functional, providing insights into the efficiency of the proposed operators. Additionally, the research establishes a Vornovskaya type asymptotic result and investigates weighted approximation with polynomial growth, shedding light on the behavior of approximations under varying conditions. To illustrate the convergence behavior empirically, the study employs MATLAB software to present numerical examples, offering tangible evidence of the theoretical findings. Through this comprehensive analysis, the study contributes to understanding the performance and applicability of the Kantrovinch-Stancu modification in approximation theory, with implications for various fields relying on function approximation techniques.

KEYWORDS: Modulus of continuity, Kantrovinch operator, Bernstein operator, Moment estimates.
AMS Subject Classification: 41A10, 41A25.

1. Introduction and Preliminaries

Positive linear operators are widely used in various fields of science and engineering. This widely spread area provides us the key tools for exploring the Computer-aided geometric designs, signal processing, image compression, data analysis, numerical analysis, and solution to ordinary and partial differential equations that

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arises in mathematical modeling of real word phenomena. A very famous polynomial in this regards, was studied by Bernstein [1] and the Bernstein operator for every bounded function $\psi \in C[0,1]$, $n \geq 1$ and $t \in [0,1]$ is defined as

$$B_n(\psi;t) = \sum_{i=0}^{n} p_{n,i}(t)\psi\left(\frac{i}{n}\right),$$

and $p_{n,i}(t) = \binom{n}{i} t^{i-1}(1-t)^{n-i-1}$ is Bernstein basis function. Usta [2] presented a new modification for $\psi \in C[0,1],\ n \in \mathbb{N},\ t \in (0,1)$ as

$$\mathcal{B}_n(\psi;t) = \sum_{k=0}^{n} \binom{n}{k} (k-nt)^2 t^{k-1}(1-t)^{n-k-1}\psi\left(\frac{k}{n}\right). \tag{1.1}$$

Recently, Sofyahoğlu [3] introduced a parametric generalization of (1.1). Thereafter, different modification of the above operator have become interest to many researchers. For more details on parametric generalizations, we refer the readers to [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Kantrovinch [14] introduced a modification involving integral for the class of Lebesgue integrable functions on [0,1] given by

$$K_n(\psi;t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \psi(u)du, \tag{1.2}$$

where $t \in (0,1)$. Recently, [15] introduced some approximation properties of Szász-Kantorovich type operators allied with d-symmetric d-orthogonal Brenke type polynomials. Also, [16] considered bivariate Summation-integral type hybrid operators and studied their approximation behavior. For applications point of view, refer [17, 18, 19, 20, 21].

The motivation behind the study stems from the need to enhance function approximation techniques, particularly for functions within Hölder's class. Traditional Bernstein operators, while effective, may not always offer optimal convergence rates for diverse functions. By introducing a Kantrovinch-Stancu type modification, this research aims to improve approximation efficiency. Investigating convergence properties through modulus of continuity and Peetre's K-functional provides a deeper understanding of how these new operators perform. The practical application of these theoretical insights, supported by MATLAB simulations, underscores the relevance of this work in advancing approximation theory and its applications across various fields that rely on accurate function representation.

We now introduce Kantrovinch-Stancu modification of the operator given by equation (1.1) based on Stancu parameters $0 \leq \alpha_1 \leq \alpha_2$, as follows:

$$\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t) = (n+\alpha_2)\sum_{k=0}^{n} \binom{n}{k} (k-nt)^2 t^{k-1}(1-t)^{n-k-1}\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}} \nu(u)du,\ t \in (0,1). \tag{1.3}$$

2. Moment Estimation

Using the preliminaries, we can prove the following identities for Modified-Bernstein-Kantrovinch-Stancu operators :

Lemma 2.1. *(see [2]) The modified-Bernstein operators $\mathcal{B}_n(.;t)$, for $n \in \mathbb{N}$, satisfy the following identities:*

(i) $\mathcal{B}_n(1;t) = 1$;

(ii) $\mathcal{B}_n(y;t) = \left(\frac{n-2}{n}\right)t + \frac{1}{n}$;
(iii) $\mathcal{B}_n(y^2;t) = \left(\frac{n^2-7n+6}{n^2}\right)t^2 + \left(\frac{5n-6}{n^2}\right)t + \frac{1}{n^2}$;
(iv) $\mathcal{B}_n(y^3;t) = \left(\frac{n^3-15n^2+38n-24}{n^3}\right)t^3 + 12\left(\frac{n^2-4n+3}{n^3}\right)t^2 + \left(\frac{13n-14}{n^3}\right)t + \frac{1}{n^3}$.

Lemma 2.2. *For $n \in \mathbb{N}$ the operator $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu(y);t)$ satisfies the followings:*

(i) $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(1;t) = 1$;
(ii) $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y;t) = \left(\frac{n-2}{n+\alpha_2}\right)t + \frac{1}{n+\alpha_2}(\frac{3}{2} + \alpha_1)$;
(iii) $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^2;t) = \frac{1}{(n+\alpha_2)^2}\{(n^2-7n+6)t^2 + (6n-8+2\alpha_1(n-2))t + (\alpha_1 + 1)(\alpha_2+2) + \frac{1}{3}\}$.

Proof. Using the linear property of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)$, we've

$$\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y;t) = \frac{n}{n+\alpha_2}\mathcal{B}_{n,a}(y;t) + \frac{1}{n+\alpha_2}(\alpha_1 + \frac{1}{2})\mathcal{B}_{n,a}(1;t).$$

By using preliminaries, we can see part (2) is true. In a similar manner, we can prove other parts of above result. □

Let us denote the r^{th} order moment of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^r;t)$ by $\gamma_{n,r}^{(\alpha_1,\alpha_2)}(t)$.

Lemma 2.3. *For $n \in \mathbb{N}$, the r^{th} $(r = 1, 2, 4)$ ordered moments of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(.;t)$ are given by*

(i) $\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t) = -\left(\frac{2+\alpha_1}{n+\alpha_2}\right)t + \frac{1}{n+\alpha_2}\left(\alpha_1 + \frac{3}{2}\right)$;
(ii) $\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t) = \frac{1}{(n+\alpha_2)^2}\{(-3n+6+\alpha_2^2+4\alpha_2)t^2 + (3n-8-2\alpha_1\alpha_2-4\alpha_1 - 3\alpha_2)t + (\alpha_1+1)(\alpha_2) + \frac{1}{3}\}$;

Proof. Using the linear property of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(.;t)$ and lemma (2.2), above lemma can be derived easily. □

Corollary 2.4. *For $n \in \mathbb{N}$, operator $\mathcal{K}_n(\alpha_1,\alpha_2)(.;t)$ satisfies the followings:*

(i) $\lim_{n\longrightarrow\infty} n\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t);t) = -(2+\alpha_2)t + \left(\alpha_1 + \frac{3}{2}\right)\frac{3}{2}$;
(ii) $\lim_{n\longrightarrow\infty} n\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t) = 3t(1-t)$.

3. Approximation Properties of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(.;t)$

3.1. **Local Approximation.**

Theorem 3.1. *Let $\nu \in C(0,1)$, then*

$$\lim_{n\longrightarrow\infty} \mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t) = \nu(t),$$

uniformly on $(0,1)$.

Proof. Using lemma (2.2), we have

$$\lim_{n\longrightarrow\infty} \mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^k;t) = t^k;\ (k = 0,\ 1,\ 2),$$

uniformly on (0,1). The required result is immediately given by Korovkin type theorem [22]. □

3.2. **Rate of Convergence.** For $\nu \in C(0,1)$, the modulus of continuity of ν is defined as

$$\omega(\nu,\zeta) = \sup_{|y-t|\leq\zeta}\left\{\sup_{t\in(0,1)}|\nu(y)-\nu(t)|\right\}.$$

Also from [23], we can write

$$|\nu(y)-\nu(t)| \leq \left(1+\frac{(y-t)^2}{\zeta^2}\right)\omega(f,\zeta).$$

By [24], ∃ a constant $M>0$ such that

$$K(\nu;\zeta) \leq M\omega_2(\nu,\sqrt{\zeta}),\ \zeta>0, \tag{3.1}$$

where Peetre's functional $K(\nu;\zeta)$ is given by

$$K(\nu;\zeta) = \inf_{f\in C^2[0,1]}\{\|\nu-f\|+\zeta\,\|f''\|\},\ \zeta>0,$$

with $C^2[0,1]=\{\nu\in C[0,1]:\nu',\nu''\in C[0,1]\}$ and

$$\omega_2(\nu,\sqrt{\eta}) = \sup_{0<|h|<\sqrt{\eta}}\left\{\sup_{t,t+2h\in(0,1)}|\nu(t+2h)-2\nu(t+h)+\nu(t)|\right\}$$

is the second ordered modulus of continuity of ν on (0,1).

Theorem 3.2. *Let $t\in(0,1)$ and $\nu\in C[0,1]$. Then we have*

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| \leq 2\omega\left(\nu,\sqrt{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)}\right),$$

where $\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)=\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)$, is the second ordered central moment of nth proposed operator.

Proof. For $\nu\in C[0,1]$, we obtain

$$\begin{aligned}\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| &= (n+\alpha_2)\sum_{k=0}^{n}p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}}|\nu(y)-\nu(t)|\,dy\\ &\leq (n+\alpha_2)\sum_{k=0}^{n}p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}}\left(1+\frac{(y-t)^2}{\zeta^2}\right)\omega(f,\zeta)dy\\ &= \left(1+\frac{1}{\zeta^2}\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)\right)\omega(\nu,\zeta).\end{aligned}$$

By taking $\zeta^2=\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)$, we reach the required result. □

Next, we define Hölder's class of functions for $\alpha\in(0,1]$ as follows:

$$\mathcal{H}_\alpha(0,1)=\{\nu\in C(0,1):|\nu(y)-\nu(t)|\leq M_\nu\,|y-t|^\alpha;\ y,t\in(0,1)\}.$$

The following theorem gives the rate of convergence for Hölder's class of functions:

Theorem 3.3. *Let $t\in(0,1)$ and $\nu\in\mathcal{H}_\alpha(0,1)$. Then we have*

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| \leq M\left(\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)\right)^{\frac{\alpha}{2}},$$

where $\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)$ is the second ordered central moment of n^{th} proposed operator.

Proof. For $\nu \in \mathcal{H}_\alpha(0,1)$, consider

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| = n\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}} |\nu(y)-\nu(t)|\,dy.$$

On applying Hölder's inequality with $p=\frac{2}{\alpha},\ q=\frac{2}{2-\alpha}$ twice, we are led to

$$\begin{aligned}\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| &\leq \left\{n\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}} |\nu(y)-\nu(t)|^{\frac{2}{\alpha}}\,dy\right\}^{\frac{\alpha}{2}}\\ &\leq M\left\{n\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}} |y-t|^2\,dy\right\}^{\frac{\alpha}{2}}\\ &= M\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)^{\frac{\alpha}{2}},\end{aligned}$$

which completes the result. □

Theorem 3.4. *Let $\nu \in C[0,1]$ and $t \in (0,1)$. Then for all $n \in \mathbb{N}$, $\exists$ a positive constant M such that*

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| \leq M\omega_2\left(\nu;\frac{1}{2}\sqrt{\frac{1}{2}\left\{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)+\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)^2\right\}}\right)+2\omega\left(\nu,\left|\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right|\right).$$

Proof. Firstly, we define an auxiliary operator

$$A_n^{(\alpha_1,\alpha_2)}(\psi;t)=\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\psi;t)-\psi\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right)+\psi(t). \quad (3.2)$$

Then, we have $A_n^{(\alpha_1,\alpha_2)}(1;t)=1$ and $A_n^{(\alpha_1,\alpha_2)}(y-t;t)=0$. Now Taylor's expansion for $\psi \in C^2[0,1]$ is given by

$$\psi(y)=\psi(t)+(y-t)\psi'(t)+\int_t^y (y-u)\psi''(u)du,\ t\in(0,1).$$

Applying auxiliary operator to both sides of above expansion, we obtain

$$\begin{aligned}A_n^{(\alpha_1,\alpha_2)}(\psi;t)-\psi(t) &= \mathcal{K}_n^{(\alpha_1,\alpha_2)}\left(\int_t^y (y-u)g''(u)du;t\right)\\ &\quad -\int_t^{\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)}\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)-u\right)\psi''(u)du.\end{aligned} \quad (3.3)$$

Now,

$$\left|\int_t^y (y-u)\psi''(u)du\right| \leq \frac{1}{2}\|\psi''\|(y-t)^2$$

and

$$\begin{aligned}&\left|\int_t^{\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)}\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)-u\right)\psi''(u)du\right|\\ &\qquad\leq \frac{1}{2}\|\psi''\|\left(\frac{-2-\alpha_2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right)^2\\ &\qquad= \frac{1}{2}\|\psi''\|\left(\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right)^2.\end{aligned}$$

Rewriting equation (3.3), we obtain

$$\left|A_n^{(\alpha_1,\alpha_2)}(\psi;t)-\psi(t)\right| \leq \frac{1}{2}\|\psi''\|\,\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)+\frac{1}{2}\|\psi''\|\left(\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right)^2$$
$$=\frac{1}{2}\|\psi''\|\left\{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)+\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)^2\right\}. \tag{3.4}$$

Also,

$$\left|A_n^{(\alpha_1,\alpha_2)}(\psi;t)\right| \leq 3\|\psi\|. \tag{3.5}$$

In the view of equations (3.4) and (3.5), we get

$$\begin{aligned}\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| &= \Big|\, A_n^{(\alpha_1,\alpha_2)}(\nu;t)+\nu\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right)-\nu(t)-\nu(t)+\psi(t)\\ &\quad -\psi(t)+A_n^{(\alpha_1,\alpha_2)}(\psi;t)-A_n^{(\alpha_1,\alpha_2)}(\psi;t)\,\Big|\\ &\leq \left|A_n^{(\alpha_1,\alpha_2)}(\nu-\psi;t)-(\nu-\psi)(t)\right|\\ &\quad+\left|A_n^{(\alpha_1,\alpha_2)}(\psi;t)-\psi(t)\right|+\left|\nu\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right)-\nu(t)\right|\\ &\leq 4\|\nu-\psi\|+\frac{1}{2}\|\psi''\|\left\{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)+\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)^2\right\}\\ &\quad+\omega(\nu,\zeta)\left(1+\frac{1}{\zeta}\left|\frac{-2-\alpha_2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right|\right).\end{aligned}$$

Taking infimum to RHS of above equation over $\psi \in C^2[0,1]$ and $\zeta = \left|\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right|$, we are led to

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| \leq 4K\left(\nu;\frac{1}{8}\left\{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)+\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)^2\right\}\right)+2\omega\left(\nu,\left|\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right|\right).$$

We reach the required result immediately by using equation (3.1). □

3.3. **Voronovskaya-type Asymptotic Result.** In this subsection, we derive an asymptotic formula for the proposed operator as follows:

Theorem 3.5. *Let $\nu \in C^2[0,1]$. and $t \in (0,1)$. Then, we have*

$$\lim_{n\longrightarrow\infty} n(\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t))=\left\{(-2-\alpha_2)t+\left(\alpha_1+\frac{3}{2}\right)\right\}\nu'(t)+\frac{3}{2}t(1-t)\nu''(t).$$

Proof. From Peano form of remainder of Taylor's expansion, we can write

$$\nu(y)=\nu(t)+(y-t)\nu'(t)+\frac{1}{2}(y-t)^2\nu''(t)+(y-t)^2\epsilon(y,t), \tag{3.6}$$

where $\epsilon(y,t)=\frac{\nu''(z)-\nu''(t)}{2}$ for some z lying between t and y. Also, $\lim_{y\longrightarrow t}\epsilon(y,t)=0$. Now, operating the equation (3.6) by $\mathcal{K}_n(.;t)$, we get

$$\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)=\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t);t)\nu'(t)+\frac{1}{2}\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)\nu''(t)$$
$$+\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\epsilon(y,t)(y-t)^2;t).$$

Using corollary (2.4) and Cauchy-Schwartz inequality, we can deduce

$$\begin{aligned}\lim_{n\longrightarrow\infty} n(\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)) &= \nu'(t)\lim_{n\longrightarrow\infty} n\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t);t)\\ &+\frac{1}{2}\nu''(t)\lim_{n\longrightarrow\infty} n\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)\\ &+\lim_{n\longrightarrow\infty}\big(n\mathcal{K}_n^{(\alpha_1,\alpha_2)}\big((y-t)^2\epsilon(y,t);t\big)\big)\\ &\le\left\{(-2-\alpha_2)t+\left(\alpha_1+\frac{3}{2}\right)\right\}\nu'(t)+\frac{3}{2}t(1-t)\nu''(t)\\ &+\lim_{n\longrightarrow\infty}\sqrt{n^2\mathcal{K}_n^{(\alpha_1,\alpha_2)}\big((y-t)^4;t\big)}\sqrt{\mathcal{K}_n^{(\alpha_1,\alpha_2)}\big(\epsilon^2(y,t);t\big)}.\end{aligned} \tag{3.7}$$

By theorem (3.1), we have

$$\lim_{n\longrightarrow\infty}\mathcal{K}_n^{(\alpha_1,\alpha_2)}\big(\epsilon^2(y,t);t\big)=\epsilon^2(t,t)=0.$$

Using above equation in (3.7), we are led to the required result. □

3.4. **Weighted Approximation.** Consider a weight function $\sigma(t)=1+t^2$ on (0,1). Let $B_\sigma(0,1)$ denotes the space of all functions φ on (0,1) such that

$$|\varphi(t)|\le M_\varphi\sigma(t)$$

and $C_\sigma(0,1)$ be the subspace of all continuous functions in $B_\sigma(0,1)$ endowed with norm $\|.\|_\sigma$ given by

$$\|\varphi\|_\sigma=\sup_{t\in(0,1)}\frac{\varphi(t)}{\sigma(t)}.$$

Next, we prove an inequality and convergence for the operator $\mathcal{K}_n(.;t)$ in weighted space as follows:

Lemma 3.1. *Let $\nu\in C_\sigma(0,1)$. Then following inequality holds for $\mathcal{K}_n(\nu;t)$*

$$\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)\right\|_\sigma\le M\|\nu\|_\sigma.$$

Proof. By using definition of proposed operator, we may write

$$\begin{aligned}\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)\right\|_\sigma &= \sup_{t\in(0,1)}\frac{\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)\right|}{\sigma(t)}\\ &\le\|\nu\|_\sigma\sup_{t\in(0,1)}\frac{n}{1+t^2}\sum_{k=0}^{n}b_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}}(1+u^2)du\\ &=\|\nu\|_\sigma\sup_{t\in(0,1)}\frac{1}{1+t^2}\{1+\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^2;t)\}\le M\|\nu\|_\sigma.\end{aligned}$$

□

Theorem 3.6. *For $\nu\in C_\sigma(0,1)$, the newly modified operator $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(.;t)$ satisfies*

$$\lim_{n\longrightarrow\infty}\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right\|_\sigma=0.$$

Proof. From lemma (2.2), we obtain

$$\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y;t)-t\right\|_\sigma=\sup_{t\in(0,1)}\frac{\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y;t)-t\right|}{1+t^2}\le\frac{1}{n+\alpha_2}\left|\alpha_1-\alpha_2-\frac{1}{2}\right|.$$

Also,

$$\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^2;t)-t^2\right\|_\sigma = \sup_{t\in(0,1)} \frac{\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right|}{1+t^2}$$
$$\leq \frac{1}{(n+\alpha_2)^2}\left\{n(2\alpha_1-2\alpha_2-1)+\alpha_1^2-\alpha_2^2-\alpha_1+\frac{1}{3}\right\}.$$

Thus, in limiting condition, we can write

$$\lim_{n\longrightarrow\infty}\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^j;t)-t^j\right\| = 0;\ j=0,1,2.$$

Then, the weighted convergence holds for all $\nu \in C_\sigma(0,1)$ from the results given by Gadjiev [25]. □

4. Graphical Analysis

Now, we introduce some simulation results in order to substantiate the convergence behavior of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\psi;t)$ for continuous function ψ by using MATLAB.

To test the approximation behavior of newly defined operators, let us consider a polynomial function $\psi(t)=t^3-t^2+\frac{t}{10}+0.1$. As the new sequence of operators is defined on $(0,1)$, so for that we will consider approximation over equally spaced grids in $[0.0005, 0.9995]$. Figure (1) and (2) shows the approximation and error in the approximation by proposed operator to $\psi(t)$ respectively for $n=20, 50$ and 100 at $\alpha_1=\alpha_2=0$.

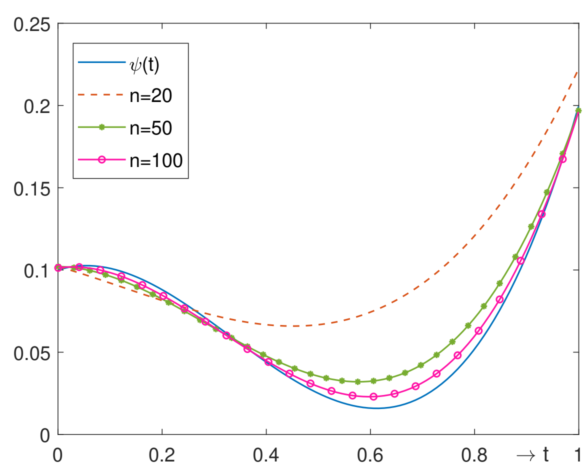

Figure 1. Approximation by proposed operator $\mathcal{K}_n^{(0,0)}(\psi;t)$ to ψ at different values of n.

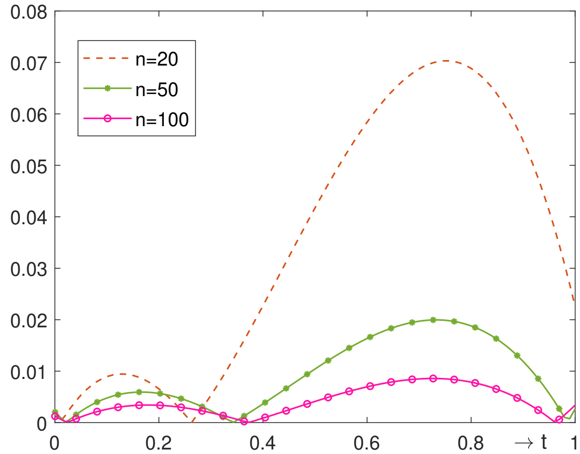

FIGURE 2. Error in the approximation by proposed operator $\mathcal{K}_n^{(0,0)}(\psi;t)$ to ψ at different values of n.

5. Conclusion

In this manuscript, we presented modified-Bernstein-Kantrovinch-Stancu operators. We discussed their rate of convergence, asymptotic formula, and weighted approximation of these operators with polynomial growth. Also, we included some numerical simulations in order to test the newly defined operators.

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A COMPARATIVE STUDY OF LAPLACE DECOMPOSITION METHOD AND VARIATIONAL ITERATION METHOD FOR SOLVING NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this study, we compare the Laplace decomposition approach to the variational iteration method. This research focuses on comparing methodologies to resolve integro-differential equations that are nonlinear.The result show how practical and successful these methods are. We compare the results to four cases to assess the solution's correctness.

KEYWORDS:Nonlinear Integro-Differential Equations, Laplace decomposition method, Variational iteration method.

AMS Subject Classification: 44A10,45D05,45J05.

1. Introduction

An Integro-Differential Equation is one that includes both the integral and derivative of unknown functions. Solving Integro-differential Equations is critical in science and engineering [1, 2]. In many scientific and technical domains, complicated physical processes are described by means of nonlinear problems. Nonlinear phenomena can be seen in a wide range of scientific domains, including chemical kinetics, solid state physics, fluid dynamics, mathematical biology and plasma physics. Numerous physical processes ,including the formation of glass, heat transmission, diffusion in general, diffusion of neutrons and coexistence of biological species with varying rates of generation involve the use of Integro-differential equations without linearity [1]. Integro-differential equations that are not linear fall into two categories: nonlinear Volterra equations and others nonlinear Fredholm equations. In this paper, we look at two successful approaches regarding the resolution of Volterra

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integro-differential equations that are not linear: LDM and VIM. The following is one kind of Volterra integro-differential equation that is not linear:

$$\frac{\mathbf{d^j v}}{\mathbf{dx^j}} = g(x) + \int_0^x K(x,t)G(v(t))dt, \tag{1.1}$$

where $G(v(t))$ function that is nonlinear of $v(t)$.

The present paper has the following structure.We define LDM and VIM in part 2, show the comparison results with four instances in section 3, and provide a conclusion in section 4.

2. Methods description

2.1. **Laplace Decomposition Method.** Combining the Adomian Decomposition and Laplace Transform techniques are also referred to as the Laplace Decomposition method (LDM).This methods main benefits is its ability to find a nonlinear equation's precise or approximate solution [3]. Differential equations can be successfully solved using the Laplace Decomposition method (LDM),which was initially presented by Suheil A. Khuri [4, 5]. When equation (1.1) is run through both sides using the Laplace transform, the result is

$$\begin{aligned} &s^j \mathbf{L}\{v(x)\} - s^{j-1}v(0) - s^{j-2}v'(0) - ... - v^{j-1}(0) \\ &= \mathbf{L}\{g(x)\} + \mathbf{L}\{K(x-t)\} + \mathbf{L}\{G(v(t))\} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \mathbf{L}\{v(x)\} = \frac{1}{s}v(0) + v(0) + \frac{1}{s^2}v'(0) + ... + \frac{1}{s^j}v^{(j-1)}(0) \\ + \frac{1}{s^j}\mathbf{L}\{g(x)\} + \frac{1}{s^j}\mathbf{L}\{K(x-t)\}\mathbf{L}\{G(v(t))\} \end{aligned} \tag{2.2}$$

In order to accomplish this, the linear expression v(x) on the left is first expressed using an endless succession of parts provided by,

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \tag{2.3}$$

recursively find the components $v_n(x), n \geq 0$.

For treating the non-linear component $G(v(x))$, the Adomian polynomial shall be embodied by an endless series, A_n we apply the Adomian polynomial get around its difficulties [1, 7, 8] in the format,

$$G(v(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{2.4}$$

where,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{j=0}^{n} \lambda^j v_j\right)_{\lambda=0, n=0,1,2,...}$$

is obtained for all forms of nonlinearity types. (2.3) and (2.4) into (2.2) result in

$$\mathbf{L}\left(\sum_{j=0}^{n} v_n(x)\right) = \frac{1}{s}v(0) + \frac{1}{s^2}v^{'}(0) + ... + \frac{1}{s^j}v^{(j-1)}(0) + \frac{1}{s^j}\mathbf{L}\{g(x)\} + \frac{1}{s^j}\mathbf{L}\{K(x-t)\}\mathbf{L}\left(\sum_{n=0}^{\infty} A_n(x)\right), \tag{2.5}$$

with the Adomian decomposition approach, the recursive connection listed below can be used

$$\mathbf{L}\{v_0(x)\} = \frac{1}{s}v(0) + \frac{1}{s^2}v^{'}(0) + ... + \frac{1}{s^j}v^{(j-1)}(0) + \frac{1}{s^j}\mathbf{L}\{g(x)\}, \tag{2.6}$$

and

$$\mathbf{L}\{v(x)\} = \frac{1}{s^j}\mathbf{L}\{K(x-t)\}\mathbf{L}\{A_n(x)\}, n \geqslant 1. \tag{2.7}$$

When the first portion of (2.6) is subjected to the inverse Laplace transform $\boldsymbol{v_0(\mathbf{x})}$ is obtained which defined $\boldsymbol{A_0}$.Consequently, by using second portion of (2.7) the components of equation (2.3) will be fully determined.

2.2. **Variational Iteration Method.** Ji-Huan He developed the Variational iteration technique (VIM) [9, 10].If there is a closed form solution, VIM offers quickly converging successive approximations of the precise answer. Without requiring any special limitations, the VIM manages both linear and nonlinear issues are treated similarly [1]. It is necessary to specify the starting conditions in order to fully determine the precise solution. For the equation for integro-differential that is not linear (1.1) the correction functional is,

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\psi)\left[v_n^{(j)}(\psi) - f(\psi) - \int_0^{\psi}[K(\psi,r)G(\tilde{v}_n(r))dr]d\psi\right]. \tag{2.8}$$

There are two key phases involved in using the Variational iteration method. Prior to anything else, the Lagrange multiplier λ [11, 12, 13] must be found. This can be done best by utilizing a constrained variation and integration by parts. Either a function or constant can be the Lagrange multiplier λ. After λ has been established, the following approximations $v_{(n+1)}(x)$, for $n \geqslant 0$ of the answer $v(x)$, should be computed using an iteration formula that is not constrained in any way. Any selected function can serve as the zeroth approximation v_0. However ,for the selective zeroth approximation v_0, it is preferable to utilize the initial values $v(0), v^{'}(0), \cdots$

$$\begin{aligned}
&v^{'} + g\left(v(\psi), v^{'}(\psi)\right) = 0,\ \lambda = -1,\\
&v_0(x) = v(0),\ \ for\ first\ order\ v_n^{'}\\
&v^{''} + g\left(v(\psi), v^{'}(\psi), v^{''}(\psi)\right) = 0, \lambda = \psi - x\\
&v_0(x) = v(0) + xv^{'}(0), \text{ for second order } v_n^{''},\\
&v^{'''} + g\left(v(\psi), v^{'}(\psi), v^{''}(\psi), v^{'''}(\psi)\right) = 0,\ \lambda = -\frac{1}{2!}(\psi - x)^2,\\
&v_0(x) = v(0) + xv^{'}(0) + \frac{1}{2!}x^2v^{''}(0), \text{ for third order } v_n^{'''},
\end{aligned} \tag{2.9}$$

So on. As a consequence, the answer is provided by

$$v(x) = \lim_{n\longrightarrow\infty} v_n(x). \tag{2.10}$$

3. Main Result

Example 3.1. Take the integro-differential equation that is nonlinear,

$$\frac{dv}{dx} = \frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x.x - \frac{3}{e^x} - \frac{1}{4e^{2x}} + \int_0^x (x-t)\, v^2(t)\, dt, v(0) = 2, \tag{3.1}$$

Using Laplace Decomposition Method. Using the provided initial condition and the Laplace transforms of equation (3.1) ,we have

$$sv(s) = 2 + \frac{9}{4s} - \frac{5}{2s^2} - \frac{1}{s^3} - \frac{3}{s+1} - \frac{1}{4(s+2)} + \frac{1}{s^2} L\left\{v^2(x)\right\},$$

$$v(s) = \frac{2}{s} + \frac{9}{s(4s)} - \frac{5}{s(2s^2)} - \frac{1}{s(s^3)} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)} + \frac{1}{s^3} L\left\{v^2(x)\right\} \tag{3.2}$$

Using the reverse Laplace transformation of the equation (3.2),we get

$$v(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \dots + L^{-1}\left[\frac{1}{s^3} L \ \left\{v^2(x)\right\}\right] \tag{3.3}$$

The solution is decomposed as an infinite sum and nonlinear term by Adomian polynomial as given below

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \quad and \quad v^2(x) = \sum_{n=0}^{\infty} A_n \tag{3.4}$$

substitute equation (3.4) into equation (3.3) we get ,

$$\sum_{n=0}^{\infty} v_n(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \dots + L^{-1}\left[\frac{1}{s^3} L\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{3.5}$$

When we compare the equation above's two sides, we obtain

$$v_0(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \dots$$

$$v_1(x) = L^{-1}\left[\frac{1}{s^3} L[A_0]\right],$$

$$v_2(x) = L^{-1}\left[\frac{1}{s^3} L[A_1]\right],$$

$$\vdots$$

where, $A_0 = v_0^2$, $A_1 = 2v_0v_1$, $A_2 = 2v_0v_2 + v_1^2 \cdots$. and so on we get the following recursive relation

$$v_0(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \dots,$$

$$v_1(x) = \frac{2}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{20}x^5 + \dots.$$

$$\vdots$$

According to (3.4), the series solution is supplied by,

$$v(x) = 2 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots,$$

that arrives to the precise solution,

$$v(x) = 1 + e^{-x}$$

This is the precise solution to equation (3.1).

Using Variational Iteration Technique.

$$v_{n+1}(x) = v_n(x) - \int_0^x \left[v'_n(t) - \frac{9}{4} + \frac{5}{2}t + \frac{1}{2}t^2 + 3e^{-t} + \frac{1}{4}e^{-2t} - \int_0^t \left((t-r)\, v_n^2(r) \right) dr \right] dt \tag{3.6}$$

In the case of the first-order integro-differential equation, we utilized λ= -1. Using the above initial condition let's choose $v_0(x) = v(0) = 2$.Following are the consecutive estimations obtained by including the correction functional with this selection.

$$v_0(x) = 2,$$
$$v_1(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \ldots,$$
$$v_2(x) = 2 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + 5\frac{x^4}{4!} - \frac{x^5}{5!} + \ldots,$$
$$\vdots$$

further approximations follow in this manner.

Admittedly, the VIM uses

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x) \,. \tag{3.7}$$

This provides a precise solution by,

$$v(x) = 1 + e^{-x}.$$

We validated through substitution.

Example 3.2. Take the integro-differential equation that is nonlinear

$$\frac{dv}{dx} = 1 - \frac{1}{3}e^{x} + \frac{1}{3}e^{-2x} + \int_0^x e^{x-t} v^2(t)\, dt,\ v(0) = 0. \tag{3.8}$$

Using Laplace Decomposition Method. Using the provided initial condition and the Laplace transforms of equation (3.8) ,we have

$$sv(s) = \frac{1}{s} - \frac{1}{3(s-1)} + \frac{1}{3(s+2)} + \frac{1}{s-1} L\left[v^2(x)\right]$$

$$v(s) = \frac{1}{s.s} - \frac{1}{3s(s-1)} + \frac{1}{3s(s+2)} + \frac{1}{s(s-1)} L\left[v^2(x)\right] \tag{3.9}$$

Using the reverse Laplace transformation of the equation (3.9),we get,

$$v(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{1}{8}x^4 - \ldots + L^{-1}\left[\frac{1}{S(S-1)} L\left[v^2(x)\right]\right] \tag{3.10}$$

The solution is decomposed as an infinite sum and nonlinear term by Adomian polynomial as given below

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \; and \qquad v^2(x) = \sum_{n=0}^{\infty} A_n \tag{3.11}$$

substitute equation (3.11) into equation (3.10) we get ,

$$\sum_{n=0}^{\infty} v_n(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{1}{8}x^4 - ... + L^{-1}\left[\frac{1}{s(s-1)} L\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{3.12}$$

When we compare the equation above's two sides, we obtain

$$\begin{aligned} v_0(x) &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{1}{8}x^4 - \dots \\ v_1(x) &= L^{-1}\left[\frac{1}{s(s-1)} L[A_0]\right], \\ v_2(x) &= L^{-1}\left[\frac{1}{s(s-1)} L[A_1]\right], \\ &\vdots \end{aligned}$$

where, $A_0 = v_0^2$, $A_1 = 2v_0v_1$, $A_2 = 2v_0v_2 + v_1^2$ .... and so on
We get the following recursive relation ,

$$\begin{aligned} v_0(x) &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{1}{8}x^4 - \dots \\ v_1(x) &= \frac{x^4}{12} - \frac{x^5}{30} + \frac{x^6}{72} - \frac{x^7}{126} + \dots \\ &\vdots \end{aligned}$$

According to (3.11), the series solution is supplied by,

$$v(x) = 1 - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\right).$$

that arrives to the precise solution,

$$v(x) = 1 - e^{-x}.$$

Using Variational Iteration Technique. For (3.8), the correction functional is provided by

$$v_{n+1}(x) = v_n(x) \quad - \int_0^x \left[v'_n(t) - 1 + \frac{1}{3}e^t - \frac{1}{3}e^{-2t} - \int_0^t \left(e^{t-r} v_n^2(r)\right) dr\right] dt \tag{3.13}$$

In the case of the first-order integro-differential equation, we utilized λ= -1. Using the above initial condition let's choose $v_0(x) = v(0) = 0$.Following are the consecutive estimations obtained by including the correction functional with this selection.

$$\begin{aligned} v_0(x) &= 0, \\ v_1(x) &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{8} + \dots, \end{aligned}$$

$$v_2(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots,$$
$$\vdots$$

further approximations follow in this manner.
Admittedly, the VIM uses,

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x) \,, \tag{3.14}$$

This provides a precise solution by,

$$v(x) = 1 - e^{-x}.$$

We validated through substitution.

Example 3.3. Take the integro-differential equation that is nonlinear,

$$\frac{dv}{dx} = -1 + \int_0^x (x-t)\, v^2(t)\, dt, \qquad v(0) = 0 \tag{3.15}$$

Using Laplace Decomposition Method:
Using the provided initial condition and the Laplace transforms of equation (3.15) ,we have

$$sv(s) = -\frac{1}{s} + \frac{1}{s^2} L\left[v^2(x)\right] \tag{3.16}$$

$$v(s) = -\frac{1}{s.s} + \frac{1}{s.s^2} L\left[v^2(x)\right] \tag{3.17}$$

Using the reverse Laplace transformation of the equation (3.17),we get,

$$v(x) = -x + L^{-1}\left[\frac{1}{s^3} L\left[v^2(x)\right]\right]. \tag{3.18}$$

The solution is decomposed as an infinite sum and nonlinear term by Adomian polynomial as given below

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \ and \qquad v^2(x) = \sum_{n=0}^{\infty} A_n \tag{3.19}$$

substitute equation (3.19) into equation (3.18) we get ,

$$\sum_{n=0}^{\infty} v_n(x) = -x + L^{-1}\left[\frac{1}{s^3} L\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{3.20}$$

When we compare the equation above's two sides, we obtain

$$v_0(x) = -x,$$
$$v_1(x) = L^{-1}\left[\frac{1}{s^3} L[A_0]\right],$$
$$v_2(x) = L^{-1}\left[\frac{1}{s^3} L[A_1]\right]$$
$$\vdots$$

Where, $A_0 = v_0^2$, $A_1 = 2v_0v_1$, $A_2 = 2v_0v_2 + v_1^2$ .... and so on
We get the following recursive relation ,

$$v_0 = -x,$$

$$v_1 = \frac{x^5}{60},$$

$$v_2 = \frac{-x^9}{15120},$$

$$\vdots$$

According to (3.19), the series solution is supplied by,

$$v(x) = -x + \frac{x^5}{60} - \frac{x^9}{15120} + \ldots$$

Using Variational Iteration Technique:
For (3.15), the correction functional is provided by,

$$v_{n+1}(x) = v_n(x) - \int_0^x \left[v'_n(t) + 1 - \int_0^t \big((t-r)\, v_n^2(r)\big) dr \right] dt \tag{3.21}$$

In the case of the first-order integro-differential equation, we utilized λ= -1.
Using the above initial condition let's choose $v_0(x) = v(0) = 0$.Following are the consecutive
estimations obtained by including the correction functional with this selection.

$$\begin{aligned} v_0(x) &= 0 \\ v_1(x) &= -x, \\ v_2(x) &= -x + \frac{x^5}{60}, \\ v_3(x) &= -x + \frac{x^5}{15} - \frac{x^9}{15120} + \ldots \\ &\vdots \end{aligned} \tag{3.22}$$

further approximations follow in this manner.
Admittedly, the VIM uses

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x) \ , \tag{3.23}$$

This gives solution

$$v(x) = -x + \frac{x^5}{60} - \frac{x^9}{15120} + \ldots$$

We validated through substitution.

Example 3.4. Take the integro-differential equation that is nonlinear,

$$\frac{dv}{dx} = x + \int_0^x v^2(t)\, dt, \qquad v(0) = 0 \tag{3.24}$$

Here kernel $K(x,t) = 1$

Using Laplace Decomposition Method. Using the provided initial condition and the Laplace transforms of equation (3.24),we have

$$sv\left(s\right)=\frac{1}{s^2}+\frac{1}{s}L\left[v^2\left(x\right)\right] \tag{3.25}$$

$$v\left(s\right)=\frac{1}{s.s^2}+\frac{1}{s.s}L\left[v^2\left(x\right)\right]. \tag{3.26}$$

Using the reverse Laplace transformation of the equation (3.26),we get,

$$v\left(x\right)=\frac{x^2}{2}+L^{-1}\left[\frac{1}{s^2}L\left[v^2\left(x\right)\right]\right]. \tag{3.27}$$

The solution is decomposed as an infinite sum and nonlinear term by Adomian polynomial as given below

$$v\left(x\right)=\sum_{n=0}^{\infty}v_n\left(x\right) \quad and \qquad v^2\left(x\right)=\sum_{n=0}^{\infty}A_n \tag{3.28}$$

substitute equation (3.28) into equation (3.27) we get ,

$$\sum_{n=0}^{\infty}v_n\left(x\right)=\frac{x^2}{2}+L^{-1}\left[\frac{1}{s^2}L\left[\sum_{n=0}^{\infty}A_n\right]\right]. \tag{3.29}$$

When we compare the equation above's two sides, we obtain

$$\begin{aligned}
v_0\left(x\right)&=\frac{x^2}{2},\\
v_1\left(x\right)&=L^{-1}\left[\frac{1}{s^2}L\left[A_0\right]\right],\\
v_2\left(x\right)&=L^{-1}\left[\frac{1}{s^2}L\left[A_1\right]\right],\\
&\;\;.\\
&\;\;.\\
&\;\;.
\end{aligned}$$

Where, $A_0=v_0^2$, $A_1=2v_0v_1$, $A_2=2v_0v_2+v_1^2\ldots.$ and so on We get the following recursive relation

$$\begin{aligned}
v_0&=\frac{x^2}{2},\\
v_1&=\frac{x^6}{120},\\
v_2&=\frac{x^{10}}{10080},\\
&\;\;.\\
&\;\;.\\
&\;\;.
\end{aligned}$$

According to (3.28), the series solution is supplied by,

$$v\left(x\right)=\frac{x^2}{2}+\frac{x^6}{120}+\frac{x^{10}}{10080}+\ldots$$

Using Variational Iteration Tecnique:

For (3.24), the correction functional is provided by,

$$v_{n+1}\left(x\right)=v_n\left(x\right)-\int_0^x\left[v'_n\left(t\right)-t-\int_0^t\left(v_n^2\left(r\right)\right)dr\right]dt \tag{3.30}$$

In the case of the first-order integro-differential equation, we utilized λ= -1.

Using the above initial condition let's choose $v_0(x) = v(0) = 0$.Following are the consecutive estimations obtained by including the correction functional with this selection.

$$\begin{aligned} v_0(x) &= 0, \\ v_1(x) &= \frac{x^2}{2}, \\ v_2(x) &= \frac{x^2}{2} + \frac{x^6}{120}, \\ v_3(x) &= \frac{x^2}{2} + \frac{x^6}{120} + \frac{x^{10}}{10800} + \dots, \\ &\vdots \end{aligned} \tag{3.31}$$

further approximations follow in this manner. Admittedly, the VIM uses

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x) \;, \tag{3.32}$$

This gives solution

$$v(x) = \frac{x^2}{2} + \frac{x^6}{120} + \frac{x^{10}}{10800} + \dots.$$

We validated through substitution.

4. Conclusion

This work presents the successful application of Lagrangian multiplier (VIM) and Lagrangian differentiation (LDM) techniques for solving integro-differential nonlinear equations. Both methods yield approximations with greater accuracy or closed forms of solutions when available. The LDM is a powerful tool that can deal with both nonlinear and linear integro-differential equations, and for nonlinear operators, the VIM does not have any specific criteria, such as linearization or Adomian polynomials. While VIM requires the evaluation of the Lagrangian multiplier λ, both methods yield the same solution for the aforementioned examples. These two methods are strong and righteous. Based on the comparison of these two powerful methods, therefore, it may be said that VIM is simpler for finding the nonlinear integro-differential equations.

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GENERALIZED α-ψ-φ-F-CONTRACTIVE MAPPINGS IN QUASI-b-METRIC-LIKE SPACES

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ABSTRACT. In this paper, we introduce some new generalized mappings in quasi-b-metric-like spaces and establish some fixed point theorems with concrete examples. Our results generalize fixed point results in the literature.

KEYWORDS: Fixed point, Quasi-b-metric-like space, Generalized α-ψ-Suzuki-contractive mapping, C-class function.
AMS Subject Classification: 47H10, 54H25.

1. Introduction

In an attempt to generalize Banach's fixed point theorem, Czerwik [7] in 1993 introduced b-metric space as a generalization of metric spaces. Later, many authors proved existence of fixed points for generalized contractions under b-metric space setting. Similiarly, the notion of metric-like space was introduced by Harandi[8] in 2012 under which many fixed point results were proved. In 2014, Ansari [2] introduced the concept of C-class functions which covers a large class of contractive conditions, and many researchers derived results using C-class functions. Recently, Afshari *et al.* [1] proved some fixed point results for generalized α-ψ-Suzuki-contractions in quasi-b-metric-like spaces. In this paper some fixed point results are derived for generalized α-ψ-Suzuki-contractions in quasi-b-metric-like spaces via C-class functions.

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2. Preliminaries

Definition 2.1. [6] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u, v) = 0 \implies u = v$, for all $u, v \in X$;
(ii) $d(u, u) = 0$, for all $u \in X$;
(iii) $d(u, v) = d(v, u)$, for all $u, v \in X$;
(iv) $d(u, v) \leq s[d(u, w) + d(w, v)]$, for all $u, v, w \in X$.

Then, d is a b-metric on X and the pair (X, d) is called a b-metric space, and s is its coefficient (see [5, 17] for more information on b-metric spaces).

If the conditions (i), (iii) and (iv) in Definition 2.1 are satisfied, then the space (X, d) is called a b-metric-like space. See [13] for more information on fixed points for some mappings in b-metric-like spaces.

Remark 2.2. Every b-metric space is a b-metric-like space, but the converse is not true.

Definition 2.3. [15] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u, v) = d(v, u) = 0 \iff u = v$, for all $u, v \in X$;
(ii) $d(u, v) \leq s[d(u, w) + d(w, v)]$, for all $u, v, w \in X$.

Then, d is a quasi-b-metric on X and the pair (X, d) is called a quasi-b-metric space.

Definition 2.4. [12] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u, v) = d(v, u) = 0 \implies u = v$, for all $u, v \in X$;
(ii) $d(u, v) \leq s[d(u, w) + d(w, v)]$, for all $u, v, w \in X$.

Then the pair (X, d) is called a quasi-b-metric-like space (or a dislocated quasi-b-metric space).

Remark 2.5. All b-metric-like spaces and quasi-b-metric spaces are obviously quasi-b-metric-like spaces, but the converse is not true.

See [9] for a generalization of b-metric-like spaces.

Example 2.6. Let $X = \{a_1, a_2, a_3\}$ be any set of three distinct elements.

Define $d : X \times X \longrightarrow [0, \infty)$ by
$$d(u, v) = \begin{cases} 0 & \text{if } (u, v) = (a_3, a_3); \\ 2 & \text{if } (u, v) \in \{(a_1, a_1), (a_2, a_1)\}; \\ 0.5 & \text{if } (u, v) \in (a_1, a_2); \\ 0.25 & \text{otherwise.} \end{cases}$$

Then (X, d) is a quasi-b-metric-like space with coefficient $s = 4$. Since $d(a_1, a_2) \neq d(a_2, a_1)$, it is clear that (X, d) is not a b-metric-like space; and since $d(a_1, a_1) \neq 0$, and $d(a_2, a_2) \neq 0$, it is also clear that (X, d) is not a quasi-b-metric space.

Definition 2.7. [1] Let (X, d) be a quasi-b-metric-like space. Let $\{u_n\}$ be a sequence in X and $u \in X$. The sequence $\{u_n\}$ converges to u if $\lim_{n \longrightarrow \infty} d(u_n, u) = d(u, u) = \lim_{n \longrightarrow \infty} d(u, u_n)$.

Definition 2.8. [1] Let (X, d) be a quasi-b-metric-like space. A sequence $\{u_n\}$ in X is said to be a left-Cauchy (respectively, right-Cauchy) sequence if $\lim_{n > m \longrightarrow \infty} d(u_n, u_m)$ (respectively, if $\lim_{m > n \longrightarrow \infty} d(u_n, u_m)$) exists and is finite. A sequence $\{u_n\}$ is said to be Cauchy if it is left-Cauchy and right-Cauchy.

Definition 2.9. [1] Let (X,d) be a quasi-b-metric-like space. We say that

(i) (X,d) is left-complete if each left-Cauchy sequence in X is convergent;
(ii) (X,d) is right-complete if each right-Cauchy sequence in X is convergent;
(iii) (X,d) is complete if and only if each Cauchy sequence in X is convergent.

Definition 2.10. [1] Let (X,d) be a quasi-b-metric-like space. A mapping $T: X \longrightarrow X$ is continuous if for any sequence $\{u_n\}$ in X converging to $u \in X$, the sequence $\{Tu_n\}$ converges to Tu.

For $s \geq 1$, let Ψ_s be the family of functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;
(ii) $\forall t > 0$, $\sum\limits_{n=1}^{\infty} s^n \psi^n(t)$ is finite, where ψ^n is the n^{th} iterate of ψ.

It is clear that if $\psi \in \Psi_s$, then $\psi(t) < t$, for all $t > 0$. For $s \geq 1$, we have $\psi^n(t) \leq s^n\psi^n(t)$, and since $\sum\limits_{n=1}^{\infty} s^n\psi^n(t) < \infty$, by comparison test, $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$, and so we can conclude that $\Psi_s \subseteq \Psi_1$.

Samet *et al.* [14] introduced the concept of α-admissible mappings as follows.

Definition 2.11. [14] Let $\alpha : X \times X \longrightarrow [0,\infty)$ be a function and $T : X \longrightarrow X$ be a mapping. Then T is α-admissible if $\alpha(u,v) \geq 1$ implies $\alpha(Tu,Tv) \geq 1$.

Afshari *et al.* [1] introduced the concepts of right-α-orbital admissible mappings and left-α-orbital admissible mappings.

Definition 2.12. [1] Let $\alpha : X \times X \longrightarrow [0,\infty)$ be a function and $T : X \longrightarrow X$ be a mapping.

(i) T is right-α-orbital admissible if $\alpha(u,Tu) \geq 1 \implies \alpha(Tu,T^2u) \geq 1$.
(ii) T is left-α-orbital admissible if $\alpha(Tu,u) \geq 1 \implies \alpha(T^2u,Tu) \geq 1$.
(iii) T is α-orbital admissible if T is both right-α-admissible and left-α-admissible.

The notion of α-ψ-contractive mappings was defined by Samet [14] in the following way.

Definition 2.13. [14] Let (X,d) be a metric space and $T : X \longrightarrow X$ be a given mapping. Then T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \longrightarrow [0,\infty)$ and $\psi \in \Psi_1$ such that $\alpha(u,v)d(Tu,Tv) \leq \psi(d(u,v))$, for all $u,v \in X$.

In 2008, Suzuki [16] proved the following theorem as a generalization of Banach contraction principle that characterizes metric completeness in which $\theta : [0,1) \longrightarrow (\frac{1}{2},1]$ is a nondecreasing function defined by $\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ (1+r)^{-1} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$

Theorem 2.14. [16] *Let (X,d) be a complete metric space. Then every mapping T on X satisfying the following:*

$\exists\, r \in [0,1)$ *such that* $\forall\, u,v \in X$, $\theta(r)d(u,Tu) \leq d(u,v) \implies d(Tu,Tv) \leq rd(u,v)$,

has a unique fixed point.

Using Suzuki method, Afshari *et al.* [1] proved some fixed point results for generalized α-ψ-Suzuki contractive mappings in the setting of quasi-b-metric-like spaces as follows.

Definition 2.15. [1] Let (X,d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-Suzuki-contractive mapping of type A if there exist $\alpha : X \times X \longrightarrow [0,\infty)$, $\psi \in \Psi_s$ and $r \in [0,1)$ such that

(i) $\forall\, u,v \in X, \theta(r)d(u,Tu) \leq d(u,v)$ implies $\alpha(u,v)d(Tu,Tv) \leq \psi(M(u,v))$;
(ii) $\forall\, u,v \in X, \theta(r)d(Tu,u) \leq d(v,u)$ implies $\alpha(v,u)d(Tv,Tu) \leq \psi(M'(u,v))$,

where

$$M(u,v) = \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)}{2s}\right\},$$
$$M'(u,v) = \max\left\{d(v,u), d(Tu,u), d(Tv,v), \frac{d(Tv,u)}{2s}\right\}.$$

Example 2.16. [1] Let $X = [-1,1]$ and let $T : X \longrightarrow X$ be defined by $T(u) = u/2$. Define $d : X \times X \longrightarrow [0,\infty)$ by $d(u,v) = |u-v|^2 + 3u^2 + 2v^2$. Then (X,d) is a quasi-b-metric-like space and T is an α-ψ-Suzuki-contractive mapping of type A.

Theorem 2.17. [1] *Let (X,d) be a complete quasi-b-metric-like space and $T : X \longrightarrow X$ be an α-ψ-Suzuki-contractive mapping of type A. Suppose also that T is α-orbital admissible, continuous and there exists $u_0 \in X$ such that $\alpha(u_0,Tu_0) \geq 1$ and $\alpha(Tu_0,u_0) \geq 1$. Then T has a fixed point $u \in X$ and $d(u,u) = 0$.*

The following is the definition of a C-class function introduced by Ansari [2]. Many researchers then developed fixed point results and best proximity results using C-class functions. For example, see [3, 4, 10].

Definition 2.18. [2] A continuous function $F : [0,\infty)^2 \to \mathbb{R}$ is called a C-class function if for any $p,q \in [0,\infty)$, the following conditions hold:
(1) $F(p,q) \leq p$;
(2) $F(p,q) = p$ implies that either $p = 0$ or $q = 0$.
The family of all C-class functions is denoted by $\mathcal{C}$.

Example 2.19. [2] The following are some C-class functions:

(i) $F(p,q) = p - q$, for all $p,q \in [0,\infty)$.
(ii) $F(p,q) = mp$, for all $p,q \in [0,\infty)$ and $m \in (0,1)$.
(iii) $F(p,q) = \frac{p}{(1+q)^r}$, for all $p,q \in [0,\infty)$ and $r \in (0,\infty)$.
(iv) $F(p,q) = \log(q + a^p)/(1+q)$, for all $p,q \in [0,\infty)$ and $a > 1$.

Definition 2.20. [11] An ultra altering distance function is a continuous, nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, for $t > 0$ and $\varphi(0) = 0$. The set of all ultra altering distance functions is denoted by Φ_U.

3. Main results

The following definition is proposed in this paper.

Definition 3.1. Let (X,d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-φ-F-contractive mapping of type A if there exist $\alpha : X \times X \longrightarrow [0,\infty)$, $\psi \in \Psi_s$, $\varphi \in \Phi_U$, $F \in \mathcal{C}$ and $r \in [0,1)$ such that the following are satisfied:

(A1) $\forall u,v \in X, \theta(r)d(u,Tu) \leq d(u,v) \implies$
$\alpha(u,v)\; M_A(u,v) \leq F\left(\psi(d(u,v)), \varphi(d(u,v))\right)$;

(A2) $\forall u,v \in X, \theta(r)d(Tu,u) \leq d(v,u) \implies$
$\alpha(v,u)\; M_A(u,v) \leq F\left(\psi(d(v,u)), \varphi(d(v,u))\right)$,

where $M_A(u,v) = \max\{d(u,Tv), d(v,Tv), d(v,Tu), d(Tv,v)\}$.

We have now our first main result.

Lemma 3.2. *Let (X,d) be a complete quasi-b-metric-like space with coefficient s and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type A. If T is α-orbital admissible, and if there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$, then $\lim_{n\to\infty} d(u_n, u_{n+1}) = \lim_{n\to\infty} d(u_{n+1}, u_n) = 0$, where $u_k = T^k u_0$, for $k \in \mathbb{N}$.*

Proof. If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then the proof is complete. If not, then $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Since T is right-α-orbital admissible, it can be derived that $\alpha(u_0, u_1) = \alpha(u_0, Tu_0) \geq 1 \implies \alpha(Tu_0, Tu_1) = \alpha(u_1, u_2) \geq 1$. Then by induction we get that

$$\alpha(u_{n-1}, u_n) \geq 1, \forall n \in \mathbb{N}. \tag{3.1}$$

Similarly, since T is left-α-orbital admissible, it can also be derived that $\alpha(u_1, u_0) = \alpha(Tu_0, u_0) \geq 1 \implies \alpha(Tu_1, Tu_0) = \alpha(u_2, u_1) \geq 1$.

Inductively, we get that

$$\alpha(u_n, u_{n-1}) \geq 1, \forall n \in \mathbb{N}. \tag{3.2}$$

Since T is an α-ψ-φ-F-contractive mapping of type A, by taking $u = u_{n-1}$ and $v = u_n$ in (A1) of Definition 3.1, we find that $\theta(r)d(u_{n-1}, Tu_{n-1}) \leq d(u_{n-1}, u_n)$ implies

$$\begin{aligned}
d(u_n, u_{n+1}) &\leq \alpha(u_{n-1}, u_n) d(u_n, u_{n+1}) \text{ by using (3.1)} \\
&\leq \alpha(u_{n-1}, u_n) \max\{d(u_{n-1}, u_{n+1}), d(u_n, u_{n+1}), d(u_n, u_n), d(u_{n+1}, u_n)\} \\
&= \alpha(u_{n-1}, u_n) \max\{d(u_{n-1}, Tu_n), d(u_n, Tu_n), d(u_n, Tu_{n-1}), d(Tu_n, u_n)\} \\
&= \alpha(u_{n-1}, u_n) M_A(u_{n-1}, u_n) \\
&\leq F(\psi(d(u_{n-1}, u_n)), \varphi(d(u_{n-1}, u_n))) \\
&\leq \psi(d(u_{n-1}, u_n)) \\
&< d(u_{n-1}, u_n).
\end{aligned}$$

Therefore, $d(u_n, u_{n+1}) \leq \psi(d(u_{n-1}, u_n))$ and $d(u_n, u_{n+1}) < d(u_{n-1}, u_n)$, for all $n \in \mathbb{N}$. Since $d(u_n, u_{n+1}) \leq \psi(d(u_{n-1}, u_n))$ for all $n \in \mathbb{N}$, inductively, we get $d(u_n, u_{n+1}) \leq \psi^n(d(u_0, u_1))$ for all $n \in \mathbb{N}$. Therefore, $\lim_{n\longrightarrow\infty} d(u_n, u_{n+1}) \leq \lim_{n\longrightarrow\infty} \psi^n(d(u_0, u_1)) = 0$, since $\psi \in \Psi_1$. Thus

$$\lim_{n\longrightarrow\infty} d(u_n, u_{n+1}) = 0.$$

Similarly, by taking $u = u_{n-1}$ and $v = u_n$ in (A2) of Definition 3.1, we find that $\theta(r)d(Tu_{n-1}, u_{n-1}) \leq d(u_n, u_{n-1})$ implies

$$\begin{aligned}
d(u_{n+1}, u_n) &\leq \alpha(u_n, u_{n-1}) d(u_{n+1}, u_n) \text{ by using (3.2)} \\
&\leq \alpha(u_n, u_{n-1}) \max\{d(u_{n-1}, u_{n+1}), d(u_n, u_{n+1}), d(u_n, u_n), d(u_{n+1}, u_n)\} \\
&= \alpha(u_n, u_{n-1}) \max\{d(u_{n-1}, Tu_n), d(u_n, Tu_n), d(u_n, Tu_{n-1}), d(Tu_n, u_n)\} \\
&= \alpha(u_n, u_{n-1}) M_A(u_{n-1}, u_n) \\
&\leq F(\psi(d(u_n, u_{n-1})), \varphi(d(u_n, u_{n-1}))) \\
&\leq \psi(d(u_n, u_{n-1})) \\
&< d(u_n, u_{n-1}).
\end{aligned}$$

Therefore, $d(u_{n+1}, u_n) \leq \psi(d(u_n, u_{n-1}))$ and $d(u_{n+1}, u_n) < d(u_n, u_{n-1})$, $\forall n \in \mathbb{N}$. Since $d(u_{n+1}, u_n) \leq \psi(d(u_n, u_{n-1}))$, for all $n \in \mathbb{N}$, inductively, we get that $d(u_{n+1}, u_n) \leq \psi^n(d(u_1, u_0))$, for all $n \in \mathbb{N}$.

Therefore, $\lim\limits_{n\longrightarrow\infty} d(u_{n+1},u_n) \leq \lim\limits_{n\longrightarrow\infty} \psi^n(d(u_1,u_0)) = 0.$

$$\text{Thus, } \lim_{n\longrightarrow\infty} d(u_{n+1},u_n) = 0.$$

This completes the proof. □

Theorem 3.3. *Let (X,d) be a complete quasi-b-metric-like space with coefficient s, and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type A and continuous. If T is α-orbital admissible, and if there exists $u_0 \in X$ such that $\alpha(u_0,Tu_0) \geq 1$ and $\alpha(Tu_0,u_0) \geq 1$, then there exists an element $u \in X$ which is a fixed point of T and $d(u,u) = 0$.*

Proof. We have $\lim\limits_{n\longrightarrow\infty} d(u_{n+1},u_n) = 0$ and $\lim\limits_{n\longrightarrow\infty} d(u_n,u_{n+1}) = 0$, from Lemma 3.2. Now, we prove that the sequence $\{u_n\}$ is Cauchy. For $k \in \mathbb{N}$, we have

$$\begin{aligned} d(u_n,u_{n+k}) &\leq sd(u_n,u_{n+1}) + s^2 d(u_{n+1},u_{n+2}) \cdots + s^k d(u_{n+k-1},u_{n+k}) \\ &\leq \sum_{p=n}^{n+k-1} s^{p-n+1}\psi^p(d(u_0,u_1)) \\ &\leq \sum_{p=n}^{\infty} s^p\psi^p(d(u_0,u_1)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Therefore, $\{u_n\}$ is right-Cauchy.
Similarly, $\{u_n\}$ is left-Cauchy, since we have

$$\begin{aligned} d(u_n,u_{n+k}) &\leq sd(u_{n+k},u_{n+k-1}) + s^2 d(u_{n+k-1},u_{n+k-2}) \cdots + s^k d(u_{n+1},u_n) \\ &\leq \sum_{p=n}^{n+k-1} s^{n+k-p}\psi^p(d(u_1,u_0)) \\ &\leq \sum_{p=n}^{\infty} s^p\psi^p(d(u_1,u_0)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Consequently, $\{u_n\}$ is Cauchy in (X,d) since it is both right-Cauchy and left-Cauchy. Since X is complete, there exists $u \in X$ such that

$$d(u,u) = \lim_{n,m\to\infty} d(u_m,u_n) = \lim_{n\to\infty} d(u_n,u) = \lim_{n\to\infty} d(u,u_n) = \lim_{n,m\to\infty} d(u_n,u_m) = 0.$$

By the continuity of T,

$$u = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} Tu_n = Tu.$$

This completes the proof. □

We provide the following example as an illustration of Theorem 3.3.

Example 3.4. Let $X = \{0,1,2\}$ and define $d : X \times X \longrightarrow [0,\infty)$ by

$$d(u,v) = \begin{cases} 0 & \text{if } (u,v) \in \{(1,1),(2,2)\}; \\ 1 & \text{if } (u,v) = (0,1); \\ 2 & \text{if } (u,v) \in \{(0,0),(1,0)\}; \\ \frac{1}{4} & \text{elsewhere.} \end{cases}$$

Then, (X,d) is a quasi-b-metric-like space with coefficient $s = 4$.

Define $T : X \longrightarrow X$ by $Tu = \begin{cases} 0 & \text{if } u = 0; \\ 2 & \text{if } u \in \{1,2\}. \end{cases}$

Let

- $\alpha : X \times X \longrightarrow [0,\infty)$ be defined by $\alpha(u,v) = \begin{cases} 1 & \text{if } (u,v) = (2,2); \\ \frac{1}{128} & \text{elsewhere,} \end{cases}$
- $\psi \in \Psi_4$ be defined by $\psi(t) = \frac{t}{8}$, $\forall t \geq 0$,
- $\varphi \in \Phi_U$ be defined by $\varphi(t) = t$, $\forall t \geq 0$,
- $F \in \mathcal{C}$ be defined by $F(p,q) = \frac{p}{2}$, $\forall p, q \in [0,\infty)$, and
- $r = 0$.

Then, T becomes a generalized α-ψ-φ-F-contractive mapping of type A. Here, all the conditions of Theorem 3.3 are satisfied, and 2 is a fixed point of T and $d(2,2) = 0$.

Now, let us define a generalized α-ψ-φ-F-contractive mapping of type B.

Definition 3.5. Let (X,d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-φ-F-contractive mapping of type B if there exist $\alpha : X \times X \longrightarrow [0,\infty)$, $\psi \in \Psi_s$, $\varphi \in \Phi_U$, $F \in \mathcal{C}$ and $r \in [0,1)$ such that the following conditions are satisfied:

(B1) $\forall\, u, v \in X, \theta(r)d(u,Tu) \leq d(u,v) \implies$
$\alpha(u,v)d(Tu,Tv) \leq F(\psi(M_B(u,v)), \varphi(M_B(u,v)))$;

(B2) $\forall\, u, v \in X, \theta(r)d(Tu,u) \leq d(v,u) \implies$
$\alpha(v,u)\ d(Tv,Tu) \leq F\left(\psi(M'_B(u,v)), \varphi(M'_B(u,v))\right)$,

where

$$M_B(u,v) = \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)}{2s}\right\},$$
$$M'_B(u,v) = \max\left\{d(v,u), d(Tu,u), d(Tv,v), \frac{d(Tv,u)}{2s}\right\}.$$

Lemma 3.6. *Let (X,d) be a complete quasi-b-metric-like space and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type B. If T is α-orbital admissible and if there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$, then $\lim_{n\to\infty} d(u_n, u_{n+1}) = \lim_{n\to\infty} d(u_{n+1}, u_n) = 0$, where $u_k = T^k u_0$, for $k \in \mathbb{N}$.*

Proof. If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then the proof is complete. If not, then $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Then by Lemma 3.2, $d(u_{n-1}, u_n) \geq 1$ and $d(u_n, u_{n-1}) \geq 1$ for all $n \in \mathbb{N}$. Since T is an α-ψ-φ-F-contractive mapping of type B, by taking $u = u_{n-1}$ and $v = u_n$ in (B1) of Definition 3.5, we find that $\theta(r)d(u_{n-1}, Tu_{n-1}) \leq d(u_{n-1}, u_n)$ implies

$$\begin{aligned}
d(u_n, u_{n+1}) &\leq \alpha(u_{n-1}, u_n)d(u_n, u_{n+1}) \text{ by (3.1)} \\
&= \alpha(u_{n-1}, u_n)d(Tu_{n-1}, Tu_n) \\
&\leq F(\psi(M_B(u_{n-1}, u_n)), \varphi(M(u_{n-1}, u_n))) \\
&\leq \psi(M_B(u_{n-1}, u_n)) \\
&= \psi\left(\max\left\{d(u_{n-1}, u_n), d(u_{n-1}, u_n), d(u_n, u_{n+1}), \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1})}{2s}\right\}\right) \\
&= \psi\left(\max\left\{d(u_{n-1}, u_n), d(u_n, u_{n+1}), \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1})}{2}\right\}\right) \\
&= \psi\left(\max\left\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\right\}\right).
\end{aligned}$$

$$\text{Thus, } d(u_n, u_{n+1}) \leq \psi\left(\max\left\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\right\}\right). \tag{3.3}$$

If $\max\{d(u_{n-1},u_n), d(u_n,u_{n+1})\} = d(u_n,u_{n+1})$, then (3.3) implies that $d(u_n,u_{n+1}) \leq \psi(d(u_n,u_{n+1})) < d(u_n,u_{n+1})$, which is a contradiction.
So $\max\{d(u_{n-1},u_n), d(u_n,u_{n+1})\} = d(u_{n-1},u_n)$. Then we have

$$d(u_n,u_{n+1}) \leq \psi(d(u_{n-1},u_n)) < d(u_{n-1},u_n), \forall n \in \mathbb{N}.$$

Therefore, $\{d(u_n,u_{n+1})\}$ is a decreasing sequence and $d(u_n,u_{n+1}) \leq \psi(d(u_{n-1},u_n))$, for all $n \in \mathbb{N}$. Then inductively we get that

$$d(u_n,u_{n+1}) \leq \psi^n(d(u_0,u_1)), \forall n \in \mathbb{N}.$$

Hence $\lim_{n\longrightarrow\infty} d(u_n,u_{n+1}) \leq \lim_{n\longrightarrow\infty} \psi^n(d(u_0,u_1)) = 0$, since $\psi \in \Psi_1$. So

$$\lim_{n\longrightarrow\infty} d(u_n,u_{n+1}) = 0.$$

Similarly, by taking $u = u_{n-1}$ and $v = u_n$ in (B2) of Definition 3.5, we find that $\theta(r)d(Tu_{n-1},u_{n-1}) \leq d(u_n,u_{n-1})$ implies

$$\begin{aligned}
d(u_{n+1},u_n) &\leq \alpha(u_n,u_{n-1})d(u_{n+1},u_n) \quad \text{by using (3.2)} \\
&= \alpha(u_n,u_{n-1})d(Tu_n,Tu_{n-1}) \\
&\leq F(\psi(M'_B(u_{n-1},u_n)), \varphi(M'(u_{n-1},u_n))) \\
&\leq \psi(M'_B(u_{n-1},u_n)) \\
&= \psi\left(\max\left\{d(u_n,u_{n-1}), d(u_n,u_{n-1}), d(u_{n+1},u_n), \frac{d(u_n,u_{n-1}) + d(u_{n+1},u_n)}{2s}\right\}\right) \\
&= \psi\left(\max\left\{d(u_n,u_{n-1}), d(u_{n+1},u_n), \frac{d(u_n,u_{n-1}) + d(u_{n+1},u_n)}{2}\right\}\right) \\
&= \psi\left(\max\left\{d(u_n,u_{n-1}), d(u_{n+1},u_n)\right\}\right).
\end{aligned}$$

$$\text{So, } d(u_{n+1},u_n) \leq \psi\left(\max\left\{d(u_n,u_{n-1}), d(u_{n+1},u_n)\right\}\right). \tag{3.4}$$

If $\max\{d(u_n,u_{n-1}), d(u_{n+1},u_n)\} = d(u_{n+1},u_n)$, then (3.4) implies that $d(u_{n+1},u_n) \leq \psi(d(u_{n+1},u_n)) < d(u_{n+1},u_n)$, which is a contradiction.

Therefore, $\max\{d(u_n,u_{n-1}), d(u_{n+1},u_n)\} = d(u_n,u_{n-1})$. Thus we have $d(u_{n+1},u_n) \leq \psi(d(u_n,u_{n-1})) < d(u_n,u_{n-1})$ for all $n \in \mathbb{N}$. Therefore, $\{d(u_{n+1},u_n)\}$ is a decreasing sequence and $d(u_{n+1},u_n) \leq \psi(d(u_n,u_{n-1}))$, for all $n \in \mathbb{N}$. Then inductively we get that

$$d(u_{n+1},u_n) \leq \psi^n(d(u_1,u_0)), \forall n \in \mathbb{N}.$$

Therefore, $\lim_{n\longrightarrow\infty} d(u_{n+1},u_n) \leq \lim_{n\longrightarrow\infty} \psi^n(d(u_1,u_0)) = 0$, since $\psi \in \Psi_1$. So

$$\lim_{n\longrightarrow\infty} d(u_{n+1},u_n) = 0.$$

This completes the proof. □

The following theorem can easily be proved as that of Theorem 3.3.

Theorem 3.7. *Let (X,d) be a complete quasi-b-metric-like space and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type B and continuous. If T is α-orbital admissible and if there exists $u_0 \in X$ such that $\alpha(u_0,Tu_0) \geq 1$ and $\alpha(Tu_0,u_0) \geq 1$, then there exists an element $u \in X$ which is a fixed point of T and $d(u,u) = 0$.*

We illustrate Theorem 3.7 with the following examples.

Example 3.8. The function $T : X \longrightarrow X$ defined on the quasi-b-metric-like space $X = \{0, 1, 2\}$ given in Example 3.4 is also a generalized α-ψ-φ-F-contractive mapping of type B, for the same α, ψ, φ, F and r given in Example 3.4. Here, all the conditions of Theorem 3.7 are satisfied, and 2 is a fixed point of T and $d(2, 2) = 0$.

Example 3.9. Let $T : X \longrightarrow X$ be the same function defined on the quasi-b-metric-like space $X = \{0, 1, 2\}$ given in Example 3.4.
Let

- $\alpha : X \times X \longrightarrow [0, \infty)$ be defined by $\alpha(u, v) = \begin{cases} 1 & \text{if } (u, v) = (2, 2); \\ \frac{1}{16} & \text{elsewhere,} \end{cases}$
- $\psi \in \Psi_4$ be defined by $\psi(t) = \frac{t}{8}$, $\forall t \geq 0$,
- $\varphi \in \Phi_U$ be defined by $\varphi(t) = t$, $\forall t \geq 0$,
- $F \in \mathcal{C}$ be defined by $F(p, q) = \frac{p}{2}$, $\forall p, q \in [0, \infty)$, and
- $r = 0$.

Then, T becomes a generalized α-ψ-φ-F-contractive mapping of type B, and not of type A. Here, all the conditions of Theorem 3.7 are satisfied, and 2 is a fixed point of T and $d(2, 2) = 0$.

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CONVERGENCE THEOREMS FOR OPERATORS WITH PROPERTY (E) IN $CAT(0)$ SPACES

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ABSTRACT. In this paper, we study the convergence of the SP*-iteration process to fixed point for operators with property (E) in $CAT(0)$ spaces. We also prove the stability of the SP*-iteration process in $CAT(0)$ spaces. Our results improve and extend some recently results in the literature of fixed point theory in $CAT(0)$ spaces.

KEYWORDS:Fixed point, iteration process, stability, Δ-convergence, $CAT(0)$ space, Garcia-Falset mapping.
AMS Subject Classification:47H09; 47H10.

1. Introduction

It is essential for many fields of study, including mathematics, to have fixed points. The conditions under which maps have solutions are given by fixed point results. In particular, fixed point methods have been applied in many fields, such as informatics, biology, chemistry, economics, and engineering. Determining the precise value of the intended fixed point is a crucial and ultimately the last step in solving the problem, but determining its existence is a crucial initial step. Using an iterative procedure is one of the best ways to obtain the intended fixed point. A number of researchers have recently shown interest in these areas and have developed iterative procedures that have been investigated to estimate fixed points for a larger class of nonexpansive mappings as well as for nonexpansive mappings. The existence of a fixed point is very important in several areas of mathematics and other sciences. The numerous numbers of researchers attracted in these direction and developed iterative process has been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings. This is an active area of research, several well known scientists in the world paid and still pay attention to the qualitative study of iteration methods. The well-known

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Banach contraction theorem use Picard iteration process [28] for approximation of fixed point. Some of the well-known iterative processes are those of Mann [24], Ishikawa [17], Noor [25], SP-iteration [29], Picard Normal S-iteration [18] and so on. Let $\mathcal{X}$ be a real Banach space and $\mathcal{M}$ be a nonempty subset of $\mathcal{X}$, and $\mathcal{G}:\mathcal{M}\to\mathcal{M}$ be a mapping. We have $\{\tau_n\},\{\sigma_n\}$ and $\{\kappa_n\}$ real sequences in $[0,1]$. Recently, Phuengrattana and Suantai ([29]) defined the SP-iteration as follows:

$$\begin{cases} z_n = (1-\kappa_n)u_n + \kappa_n \mathcal{G}u_n, \\ v_n = (1-\sigma_n)z_n + \sigma_n \mathcal{G}z_n, \\ u_{n+1} = (1-\tau_n)v_n + \tau_n \mathcal{G}v_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $u_1 \in \mathcal{M}$. They showed that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges better than the others for the class of continuous and nondecreasing functions. In 2014, Kadıoglu and Yıldırım [18] introduced Picard Normal S-iteration process and they established that the rate of convergence of the Picard Normal S-iteration process is faster than other fixed point iteration process that was in existence then. The Picard Normal S-iteration [18] as follows:

$$\begin{cases} z_n = (1-\sigma_n)u_n + \sigma_n \mathcal{G}u_n, \\ v_n = (1-\tau_n)z_n + \tau_n \mathcal{G}z_n, \\ u_{n+1} = \mathcal{G}v_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $u_1 \in \mathcal{M}$.

In 2021, Temir and Korkut [35] introduced SP*-iteration process and they established that the rate of convergence of the SP*-iteration scheme is faster than above iteration processes. Now we give SP*-iteration process:for arbitrary $u_1 \in \mathcal{M}$ construct a sequence $\{u_n\}$ by

$$\begin{cases} z_n = \mathcal{G}((1-\kappa_n)u_n + \kappa_n \mathcal{G}u_n), \\ v_n = \mathcal{G}((1-\sigma_n)z_n + \sigma_n \mathcal{G}z_n), \\ u_{n+1} = \mathcal{G}((1-\tau_n)v_n + \tau_n \mathcal{G}v_n), \forall n \in \mathbb{N}. \end{cases} \tag{1.3}$$

Some generalizations of nonexpansive mappings and the study of related fixed point theorems have been intensively carried out over past decades [1, 4, 14, 26, 27, 33, 34, 36, 37]. A class of generalized nonexpansive mappings (in short GNMs) on a nonempty subset $\mathcal{M}$ of a Banach space $\mathcal{X}$ has been defined by Suzuki [33]. Such mappings were referred to as belonging to the class of mappings satisfying condition (C) (also referred as Suzuki GNM), which properly includes the class of nonexpansive mappings. Every self-mapping $\mathcal{G}$ on $\mathcal{M}$ providing condition (C) has an almost fixed point sequence for a nonempty bounded and convex subset $\mathcal{M}$. Two new classes of GNMs that are wider than those providing the condition (C) were presented in 2011 by Garcia-Falset et al. [14], while retaining their fixed point properties. The resulting property was called condition (E) (in the sequel, the class of mappings satisfying condition (E) will be referred to as Garcia-Falset-generalized nonexpansive mappings or Garcia-Falset mappings).

In this paper, we apply SP*-iteration (1.3) for operators with property (E) in the context of $CAT(0)$ space as follows

$$\begin{cases} z_n = \mathcal{G}((1-\kappa_n)u_n \oplus \kappa_n \mathcal{G}u_n), \\ v_n = \mathcal{G}((1-\sigma_n)z_n \oplus \sigma_n \mathcal{G}z_n), \\ u_{n+1} = \mathcal{G}((1-\tau_n)v_n \oplus \tau_n \mathcal{G}v_n), \forall n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\mathcal{M}$ is a nonempty closed convex subset of a $CAT(0)$ space, $u_1 \in \mathcal{M}$, $\{\tau_n\}$, $\{\sigma_n\}$ and $\{\kappa_n\} \in [0,1]$.

Inspired and motivated by these facts, in this paper, we prove some convergence theorems of SP*-iterative process generated by (1.4) to fixed point of operators with Property (E) in $CAT(0)$ spaces. In 2021, Temir and Korkut [35] introduced the iterative process generated by (1.4) (SP*-iteration process) and they established that the rate of convergence of the SP*-iteration process is faster than the SP-iteration process and the Picard Normal S-iteration process. Since only the convergence analysis of the SP*-iterative process was studied in [35], we also prove the stability of the SP*-iterative process in this study. In addition, we provide an example that satisfies condition (E) but the mapping is neither a generalized α-nonexpansive mapping nor does it satisfy condition (C).

2. Preliminaries

First we present some basic concepts and definitions.

Let $\mathcal{G}$ be a self-mapping defined on a nonempty subset of a $CAT(0)$ space. A point $u \in \mathcal{M}$ is called a fixed point of $\mathcal{G}$ if $\mathcal{G}u = u$ and we denote by $Fix(\mathcal{G})$ the set of fixed points of $\mathcal{G}$, that is, $Fix(\mathcal{G}) = \{u \in \mathcal{M} : \mathcal{G}u = u\}$. A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ is called contraction if there exists $\theta \in [0, 1)$ such that

$$d(\mathcal{G}u, \mathcal{G}v) \leq \theta d(u, v),$$

for all $u, v \in \mathcal{M}$. If $\theta = 1$ in inequality above, then $\mathcal{G}$ is said to be a nonexpansive mapping.

Definition 2.1. A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ satisfies condition (C) on $\mathcal{M}$ if for all $u, v \in \mathcal{M}$, $\frac{1}{2}d(u, \mathcal{G}u) \leq d(u, v) \Rightarrow d(\mathcal{G}u, \mathcal{G}v) \leq d(u, v)$.

Suzuki [33] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In 2017, Pant and Shukla [26] introduced a new type of nonexpansive mappings called generalized α-nonexpansive mappings and obtain a number of existence and convergence theorems. This new class of nonlinear mappings properly contains nonexpansive, Suzuki-type GNMs and partially extends firmly nonexpansive and α-nonexpansive mappings.

Definition 2.2. A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $u, v \in \mathcal{M}$,

$$\frac{1}{2}d(u, \mathcal{G}u) \leq d(u, v) \text{ implies } d(\mathcal{G}u, \mathcal{G}v) \leq \alpha d(\mathcal{G}u, v) + \alpha d(\mathcal{G}v, u) + (1 - 2\alpha)d(u, v).$$

Recently, Garcia-Falset et al. [14] studied GNMs satisfying condition (E) that have a weaker property than Suzuki GNMs.

Definition 2.3. A mapping $\mathcal{G} : \mathcal{M} \to \mathcal{X}$ satisfies condition (E_μ) on $\mathcal{M}$, if there exists $\mu \geq 1$ such that

$$d(u, \mathcal{G}v) \leq \mu d(u, \mathcal{G}u) + d(u, v)$$

for all $u, v \in \mathcal{M}$.

Moreover, it is said that $\mathcal{G}$ satisfies condition (E) on $\mathcal{M}$, whenever $\mathcal{G}$ satisfies condition (E_μ), for some $\mu \geq 1$. It is clearly seen that if $\mathcal{G} : \mathcal{M} \to \mathcal{X}$ is nonexpansive, then it satisfies condition (E_1) and from Lemma 7 in [33] we know that if $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ satisfies condition (C) on $\mathcal{M}$, then $\mathcal{G}$ satisfies condition (E_3) (see [14]). By Lemma 5.2 in [26], if $\mathcal{G} : \mathcal{M} \to \mathcal{X}$ is a generalized α-nonexpansive mapping, then it satisfies condition (E) on $\mathcal{M}$; see [26] for a proof. Therefore, the class of generalized α-nonexpansive mappings is subordinated to the class of mappings satisfying condition (E). Proposition 1 in [14], we know also that if $\mathcal{G}:\mathcal{M}\to\mathcal{X}$ a mapping which satisfies condition (E) on $\mathcal{M}$ has some fixed point, then $\mathcal{G}$ is quasi-nonexpansive. Example 2 that is in [14] shows the converse is not true.

It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces, any convex subset of a Euclidian space $\mathbb{R}^n$ with the induced metric, the complex Hilbert ball with a hyperbolic metric and many others. For discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [6]. Burago et al. [8] contains a somewhat more elementary treatment, and Gromov [15] a deeper study. Fixed point theory in $CAT(0)$ space has been first studied by Kirk (see [19],[20]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. On the other hand, we know that every Banach space is a $CAT(0)$ space. Since then the fixed point theory in CAT(0) has been rapidly developed and much papers a appeared.(see [9],[10],[11],[12],[13],[19],[20],[21], [22]).

Recently, Kirk and Panyanak [22] used the concept of $\Delta-$convergence introduced by Lim [23] to prove on the $CAT(0)$ space analogs of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [9] obtained $\Delta-$convergence theorems for the Picard, Mann and Ishikawa iteration processes for nonexpansive mappings in the $CAT(0)$ space. In addition, the convergence results for generalized nonexpansive mappings are obtained by using different iteration processes in $CAT(0)$ spaces (see [2], [3], [30], [31]).

If u, v_1, v_2 are points of a $CAT(0)$ spaces, and and if v_0 is the midpoint of the segment $[y_1, y_2]$ then the $CAT(0)$ inequality implies

$$d^2(u,v_0)\leq \frac{1}{2}d^2(u,v_1)+\frac{1}{2}d^2(u,v_2)-\frac{1}{4}d^2(v_1,v_2).$$

This is the (CN) inequality of Bruhat and Tits [7]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality ([[6], p. 163]).

In the sequel, we need the following definitions and useful lemmas to prove our main results of this paper.

Lemma 2.4. *([9]) Let $\mathcal{X}$ be a $CAT(0)$ space.*

(i) For $u,v\in\mathcal{X}$ and $t\in[0,1]$, there exists a unique point $z\in[u,v]$ such that $d(u,z)=td(u,v)$ and $d(v,z)=(1-t)d(u,v)$.

(ii) For $u,v\in\mathcal{X}$ and $t\in[0,1]$, we have $d((1-t)u\oplus tv,z)\leq(1-t)d(u,z)+td(v,z)$.

Let $\{u_n\}$ be a bounded sequence in a closed convex subset $\mathcal{M}$ of a $CAT(0)$ space $\mathcal{X}$. For $x\in\mathcal{X}$, set $r(u,\{u_n\})=\limsup_{n\to\infty} d(u,u_n)$. The asymptotic radius $r(\{u_n\})$ of $\{u_n\}$ is given by $r(\mathcal{M},\{u_n\})=\inf_n\{r(u,\{u_n\}):u\in\mathcal{M}\}$ and the asymptotic center of u_n relative to K is the set $A(\mathcal{M},\{u_n\})=\{u\in\mathcal{M}:r(x,\{u_n\})=r(\mathcal{M},\{u_n\})\}$. It is known that, in a $CAT(0)$ space, $A(\mathcal{M},\{u_n\})$ consists of exactly one point; see [12], Proposition 7.

We now recall the definition of Δ-convergence and weak convergence in $CAT(0)$ space.

Definition 2.5. ([22],[23]) A sequence $\{u_n\}$ in a $CAT(0)$ space $\mathcal{X}$ is said to Δ-converge to $u\in\mathcal{X}$ if u is the unique asymptotic center of every subsequence $\{u_n\}$. In this case we write $\Delta-\lim_{n\to\infty} u_n=u$ and call u is the $\Delta-$limit of $\{u_n\}$.

Lemma 2.6. *([22]) Given $\{u_n\}\in X$ such that $\{u_n\}$, Δ-converges to u and given $v\in\mathcal{X}$ with $v\neq u$, then $\limsup_{n\to\infty} d(u_n,u)<\limsup_{n\to\infty} d(u_n,v)$.*

Lemma 2.7. *([22]) Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence.*

Lemma 2.8. *([11]) Let $\mathcal{M}$ be closed convex subset of a complete $CAT(0)$ space and $\{u_n\}$ be a bounded sequence in $\mathcal{M}$. Then asymptotic center of $\{u_n\}$ is in $\mathcal{M}$.*

Next, Harder and Hicks [16] introduced the following definition of $\mathcal{G}$-stability :

Definition 2.9. ([16]) Let $\{t_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $\mathcal{M}$. Then , an iteration process

$$t_{n+1} = f(\mathcal{G}, t_n),\ \ for\ n = 1, 2, ...$$

is said to be $\mathcal{G}$-stable or stable with respect to $\mathcal{G}$ for some function f, converging to fixed point p, if $\epsilon_n = d(t_{n+1}, f(\mathcal{G}, t_n))$ for $n = 1, 2, ...,$ we have $\lim\limits_{n\longrightarrow\infty} \epsilon_n = 0 \Leftrightarrow \lim\limits_{n\longrightarrow\infty} t_n = p$.

In what follows, we shall make use of the following well-known lemma.

Lemma 2.10. *([5]) Let $\{\epsilon_n\}$ and $\{u_n\}$ be sequences of positive real numbers satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,$$

$n \in \mathbb{N}$ and $\delta \in [0, 1)$. If $\lim\limits_{n\to\infty} \epsilon_n = 0$ then $\lim\limits_{n\to\infty} u_n = 0$.

3. Stability of SP*-iteration process

In this section, we prove that the SP*-iteration process defined by (1.4) is stable. First, we prove the following strong convergence theorem.

Theorem 3.1. *Let $\mathcal{M}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, $\mathcal{G}$ be a contraction mapping with $Fix(\mathcal{G}) \neq \emptyset$. For arbitrary chosen $u_1 \in \mathcal{M}$, $\{u_n\}$ be a sequence generated by (1.4) with real sequences $\{\tau_n\}, \{\sigma_n\}$ and $\{\kappa_n\} \in [0, 1]$ with $\sum_{n=1}^{\infty} \tau_n = \infty$. Then $\{u_n\}_{n=1}^{\infty}$ converges strongly to an unique fixed point of $\mathcal{G}$.*

Proof. We will prove that $u_n \to p$ as $n \to \infty$ from (1.4), we have,

$$\begin{aligned} d(z_n, p) &= d(\mathcal{G}((1-\kappa_n)u_n \oplus \kappa_n \mathcal{G}u_n), p) \\ &\leq \theta[(1-\kappa_n)d(u_n, p) + \kappa_n \theta d(u_n, p)] \\ &= \theta[1-\kappa_n(1-\theta)]d(u_n, p). \end{aligned} \tag{3.1}$$

Similarly, from (1.4) and (3.1), we get

$$\begin{aligned} d(v_n, p) &= d(\mathcal{G}((1-\sigma_n)z_n \oplus \sigma_n \mathcal{G}z_n), p) \\ &\leq \theta[(1-\sigma_n)d(z_n, p) + \sigma_n d(\mathcal{G}z_n, p)] \\ &\leq \theta[(1-\sigma_n)d(z_n, p) + \sigma_n \theta d(z_n, p)] \\ &= \theta[(1-\sigma_n(1-\theta))d(z_n, p)] \\ &\leq \theta^2[(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(u_n, p). \end{aligned} \tag{3.2}$$

From (1.4) and (3.2) , we get

$$\begin{aligned} d(u_{n+1}, p) &= d(\mathcal{G}((1-\tau_n)v_n \oplus \tau_n \mathcal{G}v_n), p) \\ &\leq \theta[(1-\tau_n)d(v_n, p) + \tau_n d(\mathcal{G}v_n, p)] \\ &\leq \theta[(1-\tau_n)d(v_n, p) + \tau_n \theta d(v_n, p)] \\ &= \theta[1-\tau_n(1-\theta)d(v_n, p)] \\ &\leq \theta^3[(1-\tau_n(1-\theta))(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(u_n, p) \end{aligned}$$

Considering that $\{\tau_n\}, \{\sigma_n\}$ and $\{\kappa_n\} \in [0,1]$, $\theta \in [0,1)$, and rearranging the above inequality, we get

$$d(u_{n+1},p) \leq \theta^3[1-\tau_n(1-\theta)]d(u_n,p)$$

By induction, we get

$$\begin{array}{rcl} d(u_n,p) & \leq & \theta^3[1-\tau_{n-1}(1-\theta)]d(u_{n-1},p) \\ & . & \\ & . & \\ & . & \\ d(u_2,p) & \leq & \theta^3[1-\tau_1(1-\theta)]d(u_1,p). \end{array}$$

Therefore, we obtain

$$d(u_{n+1},p) \quad \leq \quad \theta^{3n}\prod_{k=1}^{n}[1-\tau_k(1-\theta)]d(u_1,p),$$

$\theta < 1$ and $\tau_k \in [0,1]$ for $k = 1,2,....$ Then we have $[1-\tau_k(1-\theta)] \leq 1$ for $k = 1,2,....$ So, we know that $1-u \leq e^{-u}$ for all $u \in [0,1]$. Hence we have

$$d(u_{n+1},p) \leq \theta^{3n} e^{-(1-\theta)\sum_{k=1}^{n}\tau_k} d(u_1,p). \tag{3.3}$$

Taking the limit of both sides of the above inequality , $u_n \to p$ as $n \to \infty$. □

Now we prove that the iteration defined by (1.4) is stable with respect to $\mathcal{G}$.

Theorem 3.2. *Suppose that all conditions of Theorem 3.1 hold. Then the iteration process (1.4) is $\mathcal{G}$-stable.*

Proof. Let $\{t_n\}$ be any arbitrary sequence in $\mathcal{M}$. $t_{n+1} = f(\mathcal{G}, t_n)$ is the sequence generated by (1.4) and $\epsilon_n = d(t_{n+1}, f(\mathcal{G}, t_n))$ for $n = 1,2,...$, in which

$$\begin{cases} r_n = \mathcal{G}((1-\kappa_n)t_n \oplus \kappa_n \mathcal{G} t_n), \\ s_n = \mathcal{G}((1-\sigma_n)r_n \oplus \sigma_n \mathcal{G} r_n), \\ t_{n+1} = \mathcal{G}((1-\tau_n)s_n \oplus \tau_n \mathcal{G} s_n), \forall n \in \mathbb{N}. \end{cases}$$

We have to prove that $\lim\limits_{n\longrightarrow\infty} \epsilon_n = 0 \Leftrightarrow \lim\limits_{n\longrightarrow\infty} t_n = p$.

Suppose $\lim\limits_{n\longrightarrow\infty} \epsilon_n = 0$. We prove that $\lim\limits_{n\longrightarrow\infty} t_n = p$:

$$\begin{array}{rcl} d(t_{n+1},p) & \leq & d(t_{n+1}, f(\mathcal{G},t_n)) + d(f(\mathcal{G},t_n),p) \\ & \leq & \epsilon_n + \theta[1-\tau_n(1-\theta)d(s_n,p)] \end{array} \tag{3.4}$$

and

$$\begin{array}{rcl} d(s_n,p) & = & d(\mathcal{G}((1-\sigma_n)r_n \oplus \sigma_n \mathcal{G} r_n),p) \\ & \leq & \theta[(1-\sigma_n)d(r_n,p) + \sigma_n d(\mathcal{G} r_n,p)] \\ & \leq & \theta[(1-\sigma_n)d(r_n,p) + \sigma_n \theta d(r_n,p)] \\ & = & \theta[(1-\sigma_n(1-\theta))]d(r_n,p) \end{array} \tag{3.5}$$

and

$$\begin{array}{rcl} d(r_n,p) & = & d(\mathcal{G}((1-\kappa_n)t_n \oplus \kappa_n \mathcal{G} t_n),p) \\ & \leq & \theta[(1-\kappa_n)d(t_n,p) + \kappa_n \theta d(t_n,p)] \\ & = & \theta[1-\kappa_n(1-\theta)]d(t_n,p). \end{array} \tag{3.6}$$

Substituting (3.6) in (3.5), we obtain

$$d(s_n, p) \leq \theta^2[(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(t_n, p). \tag{3.7}$$

Substituting (3.7) in (3.4), we get

$$d(t_{n+1}, p) \leq \epsilon_n + \theta^3[(1-\tau_n(1-\theta))(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(t_n, p).$$

Since $\{\tau_n\}, \{\sigma_n\}$ and $\{\kappa_n\} \in [0,1]$, $\theta \in [0,1)$ and $\theta^3[(1-\tau_n(1-\theta))(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))] < 1$, we can easily seen that all conditions of Lemma 2.10 are fulfilled by above inequality. Hence by Lemma 2.10 we get $\lim_{n\longrightarrow\infty} t_n = p$.

Conversely, let $\lim_{n\longrightarrow\infty} t_n = p$, we have

$$\begin{aligned}
\epsilon_n &= d(t_{n+1}, f(\mathcal{G}, t_n)) \\
&\leq d(t_{n+1}, p) + d(f(\mathcal{G}, t_n), p) \\
&\leq d(t_{n+1}, p) + \theta^3[(1-\tau_n(1-\theta))(1-\sigma_n(1-\theta))(1-\kappa_n(1-\theta))]d(t_n, p).
\end{aligned}$$

By taking the limit as $n \to \infty$ in the above inequality we have $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Hence (1.4) is stable with respect to $\mathcal{G}$. □

4. Convergence of SP*-iteration process for operators with property (E)

Lemma 4.1. *Let $\mathcal{M}$ be a nonempty closed convex subset of a complete $CAT(0)$ space $\mathcal{X}$, $\mathcal{G}$ be a mapping satisfying condition (E) with $Fix(\mathcal{G}) \neq \emptyset$. For arbitrary chosen $x_1 \in \mathcal{M}$, let $\{u_n\}$ be a sequence generated by (1.4) with $\{\tau_n\}$, $\{\sigma_n\}$ and $\{\kappa_n\}$ real sequences in $[0,1]$. Assume that $\liminf_{n\to\infty}(1-\kappa_n)\kappa_n > 0, \liminf_{n\to\infty}(1-\sigma_n)\sigma_n > 0$ and $\liminf_{n\to\infty}(1-\tau_n)\tau_n > 0$. Then $Fix(\mathcal{G}) \neq \emptyset$ if and only if $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(u_n, \mathcal{G}u_n) = 0$.*

Proof. Assume that $Fix(\mathcal{G}) \neq \emptyset$. $\mathcal{G}$ is a quasi-nonexpansive because $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ is a Garcia-Falset GNM. Using (1.4), for any $p \in Fix(\mathcal{G})$, because of $\mathcal{G}$ quasi-nonexpansive mapping, then we have

$$\begin{aligned}
d^2(z_n, p) &= d^2(\mathcal{G}((1-\kappa_n)u_n \oplus \kappa_n\mathcal{G}u_n), p) \\
&\leq d^2((1-\kappa_n)u_n \oplus \kappa_n\mathcal{G}u_n, p) \\
&\leq (1-\kappa_n)d^2(u_n, p) + \kappa_n d^2(\mathcal{G}u_n, p) - (1-\kappa_n)\kappa_n d^2(\mathcal{G}u_n, u_n) \\
&\leq d^2(u_n, p) - (1-\kappa_n)\kappa_n d^2(\mathcal{G}u_n, u_n) \leq d^2(u_n, p).
\end{aligned} \tag{4.1}$$

Using (1.4) and (4.1), we get

$$\begin{aligned}
d^2(v_n, p) &= d^2(\mathcal{G}((1-\sigma_n)z_n \oplus \sigma_n\mathcal{G}z_n), p) \\
&\leq d^2((1-\sigma_n)z_n \oplus \sigma_n\mathcal{G}z_n, p) \\
&\leq (1-\sigma_n)d^2(z_n, p) + \sigma_n d^2(\mathcal{G}z_n, p) - (1-\sigma_n)\sigma_n d^2(\mathcal{G}z_n, z_n) \\
&\leq d^2(z_n, p) - (1-\sigma_n)\sigma_n d^2(\mathcal{G}z_n, z_n) \\
&\leq d^2(z_n, p) \leq d^2(u_n, p).
\end{aligned} \tag{4.2}$$

By using (1.4) and (4.2), we get

$$\begin{aligned}
d^2(u_{n+1}, p) &= d^2(\mathcal{G}((1-\tau_n)v_n \oplus \tau_n\mathcal{G}v_n), p) \\
&\leq d^2((1-\tau_n)v_n \oplus \tau_n\mathcal{G}v_n, p) \\
&\leq (1-\tau_n)d^2(v_n, p) + \tau_n d^2(\mathcal{G}v_n, p) - (1-\tau_n)\tau_n d^2(\mathcal{G}v_n, v_n)
\end{aligned} \tag{4.3}$$

$$\begin{aligned} &\leq d^2(v_n,p) - (1-\tau_n)\tau_n d^2(\mathcal{G}v_n, v_n) \\ &\leq d^2(v_n,p) \leq d^2(u_n,p). \end{aligned}$$

This implies that $\{d(u_n,p)\}$ is bounded and non-increasing for all $p \in Fix(\mathcal{G})$. Put $\lim_{n\to\infty} d(u_n,p) = c$. From (4.1) and (4.2), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(u_n,p) = c$$

and

$$\limsup_{n\to\infty} d(v_n,p) \leq \limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(u_n,p) = c.$$

From (4.3), we can get $d(u_{n+1},p) \leq d(v_n,p)$. Therefore $c \leq \liminf_{n\to\infty} d(v_n,p)$. Thus we have $c = \lim_{n\to\infty} d(v_n,p)$. Next

$$c = \lim_{n\to\infty} d(v_n,p) \leq \lim_{n\to\infty} d(z_n,p) \leq \lim_{n\to\infty} d(u_n,p) = c.$$

Now, using (4.1), we know that

$$d^2(z_n,p) \leq d^2(u_n,p) - (1-\kappa_n)\kappa_n d^2(\mathcal{G}u_n,u_n).$$

Thus

$$(1-\kappa_n)\kappa_n d^2(\mathcal{G}u_n,u_n) \leq d^2(u_n,p) - d^2(z_n,p)$$

so that

$$d^2(\mathcal{G}u_n,u_n) \leq \frac{1}{(1-\kappa_n)\kappa_n}[d^2(u_n,p) - d^2(z_n,p)].$$

We have

$$\lim_{n\to\infty} d^2(\mathcal{G}u_n,u_n) \leq 0.$$

Hence $\lim_{n\to\infty} d(\mathcal{G}u_n,u_n) = 0$.

Conversely, suppose that $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(u_n,\mathcal{G}u_n) = 0$. Let $p \in A(\mathcal{M},\{u_n\})$. Then we have,

$$\begin{aligned} r(\mathcal{G}p,\{u_n\}) = \limsup_{n\to\infty} d(u_n,\mathcal{G}p) &\leq \limsup_{n\to\infty} \mu d(\mathcal{G}u_n,u_n) + \limsup_{n\to\infty} d(u_n,p) \\ &= \limsup_{n\to\infty} d(u_n,p) = r(p,\{u_n\}). \end{aligned}$$

This implies that for $\mathcal{G}p = p \in A(\mathcal{M},\{u_n\})$. Since $\mathcal{X}$ is complete $CAT(0)$ then $A(\mathcal{M},\{u_n\})$ is singleton, hence $\mathcal{G}p = p$. This completes the proof. □

Now , we prove the Δ-convergence theorem of a iterative process generated by (1.4) in $CAT(0)$ spaces.

Theorem 4.2. *Let $\mathcal{X},\mathcal{M},\mathcal{G}$ and $\{u_n\}$ be as in Lemma 4.1 with $Fix(\mathcal{G}) \neq \emptyset$. Then u_n, Δ-converges to a fixed point of $\mathcal{G}$.*

Proof. Lemma 4.1 guarantees that the sequence $\{u_n\}$ is bounded and $\lim_{n\to\infty} d(\mathcal{G}u_n,u_n) = 0$. Let $W_\Delta(u_n) = \bigcup A(\{\omega_n\})$; where the union is taken over all subsequences $\{\omega_n\}$ of $\{u_n\}$: We claim that $W_\Delta(u_n) \subseteq Fix(\mathcal{G})$. Let $\omega \in W_\Delta(u_n)$. Then, there exists a subsequence $\{\omega_n\}$ of $\{u_n\}$ such that $A(\{\omega_n\}) = \omega$. Since $\mathcal{G}$ is a mapping with condition (E), we obtain $d(\omega_n,\mathcal{G}\omega) \leq \mu d(\omega_n,\mathcal{G}\omega_n) + d(\omega_n,\omega)$. Using this last inequality and fact that $\lim_{n\to\infty} d(\omega_n,\mathcal{G}\omega_n) = 0$, taking limsup on both sides implies that $\limsup_{n\to\infty} d(\omega_n,\mathcal{G}\omega) \leq \limsup_{n\to\infty} d(\omega_n,\omega)$. Hence $r(\mathcal{G}\omega,\{\omega_n\}) \leq r(\omega,\{\omega_n\})$. However, ω is the unique asymptotic center of $\{\omega_n\}$, which implies that $\omega = \mathcal{G}\omega$, that is, $\omega \in Fix(\mathcal{G})$.

By Lemma 2.7 and Lemma 2.8, there exists a subsequence $\{\zeta_n\}$ of $\{\omega_n\}$ such that $\Delta - \lim_{n\to\infty} \zeta_n = \zeta \in \mathcal{G}$. Since $\lim_{n\to\infty} d(\zeta_n, \mathcal{G}\zeta_n) = 0$ and $\mathcal{G}$ is a Garcia-Falset mapping, then, we have

$$d(\zeta_n, \mathcal{G}\zeta) \leq \mu d(\mathcal{G}\zeta_n, \zeta_n) + d(\zeta_n, \zeta).$$

By taking limsup and using Opial property, we obtain $\zeta \in Fix(\mathcal{G})$. Now, we claim that $\omega = \zeta$. Assume on contrary, that $\omega \neq \zeta$. By Lemma 4.1, $\lim_{n\to\infty} d(u_n, \zeta)$ exists and by the uniqueness of asymptotic centers, then we have

$$\begin{aligned} \lim_{n\to\infty} d(\zeta_n, \zeta) &< \lim_{n\to\infty} d(\zeta_n, \omega) \leq \lim_{n\to\infty} d(\omega_n, \omega) \\ &< \lim_{n\to\infty} d(\omega_n, \zeta) \ = \lim_{n\to\infty} d(\omega_n, \zeta) \\ &= \lim_{n\to\infty} d(\zeta_n, \zeta), \end{aligned}$$

which is contradiction. Thus $\omega = \zeta \in Fix(\mathcal{G})$ and $W_\Delta(\omega_n) \subseteq Fix(\mathcal{G})$. To show that $\{\omega_n\}$, Δ-converges to a fixed point of $\mathcal{G}$, we show that $W\Delta(u_n)$ consists of exactly one point. By Lemma 2.7 and Lemma 2.8, there exists a subsequence $\{\zeta_n\}$ of ω_n such that $\Delta - \lim_{n\to\infty} \zeta_n = \zeta \in \mathcal{M}$. Let $A(\{\omega_n\}) = \{\omega\}$ and $A(\{\omega_n\}) = \{\rho\}$. We have already seen that $\omega = \zeta$ and $\zeta \in Fix(\mathcal{G})$. Finally, we claim that $\rho = \zeta$. If not, then existence $\lim_{n\to\infty} d(u_n, \zeta)$ and uniqueness of asymptotic centers imply that

$$\begin{aligned} \lim_{n\to\infty} d(\zeta_n, \zeta) &< \lim_{n\to\infty} d(\zeta_n, \rho) \leq \lim_{n\to\infty} d(\omega_n, \rho) \\ &< \lim_{n\to\infty} d(\omega_n, \zeta) \ = \lim_{n\to\infty} d(\zeta_n, \zeta). \end{aligned}$$

This is a contradiction and hence $\rho = \zeta \in Fix(\mathcal{G})$. Therefore, $W_\Delta(\omega_n) = \rho$. In conclusion $W_\Delta(\omega_n)$ is a singleton and unique element is a fixed point of $\mathcal{G}$. This proves Δ-convergence of u_n. □

In the next result, we prove the strong convergence theorem as follows.

Theorem 4.3. *Let $\mathcal{X}, \mathcal{M}, \mathcal{G}$ and $\{u_n\}$ be as in Lemma 4.1 with $Fix(\mathcal{G}) \neq \emptyset$ such that $\mathcal{M}$ is compact subset of $\mathcal{X}$. Then $\{u_n\}$ converges strongly to a fixed point of $\mathcal{G}$.*

Proof. By Lemma 4.1, we have $\lim_{n\to\infty} d(u_n, \mathcal{G}u_n) = 0$. Since $\mathcal{M}$ is compact, by Lemma 2.7, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $p \in \mathcal{M}$ such that $\{u_{n_k}\}$ converges p. Then we have $d(u_{n_k}, \mathcal{G}p) \leq \mu d(\mathcal{G}u_{n_k}, u_{n_k}) + d(u_{n_k}, p)$ for all $k \geq 1$. So $\{u_{n_k}\}$ converges $\mathcal{G}p$. This implies $\mathcal{G}p = p$. Since $\mathcal{G}$ is quasi-nonexpansive , we have $d(u_{n+1}, p) \leq d(u_n, p)$ for all $n \in \mathbb{N}$. Therefore $\{u_n\}$ converges strongly to p. □

Finally, we briefly discuss the strong convergence theorem using condition (I) introduced by Senter and Dotson[32] in $CAT(0)$ space $\mathcal{X}$ as follows.

Theorem 4.4. *Let $\mathcal{G}$ be a Garcia-Falset mapping on a nonempty closed convex subset $\mathcal{M}$ of a complete $CAT(0)$ space $\mathcal{X}$. $\{u_n\}$ be as in Lemma 4.1 with $Fix(\mathcal{G}) \neq \emptyset$. Also if, for $\mathcal{G}$ satisfies condition (I), then $\{u_n\}$ defined by (1.4) converges strongly to a fixed point of $(\mathcal{G})$.*

Proof. By Lemma 4.1, we have $\lim_{n\to\infty} d(u_n, p)$ exists and so $\lim_{n\to\infty} d(u_n, Fix(\mathcal{G}))$. Also by Lemma 4.1, $\lim_{n\to\infty} d(u_n, \mathcal{G}u_n) = 0$.

It follows from condition (I) that $\lim_{n\to\infty} f(d(u_n, Fix(\mathcal{G})) \leq \lim_{n\to\infty} d(u_n, \mathcal{G}u_n)$. That is, $\lim_{n\to\infty} f(d(u_n, Fix(\mathcal{G})) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have $\lim_{n\to\infty} d(u_n, Fix(\mathcal{G})) = 0$. Thus , we have a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $\{y_k\} \subset Fix(\mathcal{G})$ such that $d(x_{n_k}, y_k) < \frac{1}{2^k}$ for all $k \in \mathbb{N}$. We can easily show that $\{y_k\}$ is a Cauchy sequence in $Fix(\mathcal{G})$ and so it converges to a point p. Since $Fix(\mathcal{G})$ is closed, therefore $p \in Fix(\mathcal{G})$ and $\{u_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(u_n, p)$ exists, we have that $u_n \to p$. Thus the proof is completed. □

Next, we give the following example satisfying condition (E), but it is neither a generalized α-nonexpansive mapping nor does it satisfy condition (C).

Example 4.5. Let $\mathcal{X} = \mathbb{R}$ be a $CAT(0)$ space and $\mathcal{M} = [0, 1]$ be a closed convex subset of $\mathbb{R}$ endowed with the usual norm. Define a mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ by $\mathcal{G}u = \begin{cases} 0, & 0 \le u < \frac{1}{100} \\ \frac{2u}{3}, & \frac{1}{100} \le u \le 1. \end{cases}$ In order to see that $\mathcal{G}$ satisfies condition (E_3) on $[0, 1]$, we consider the following cases:

(i) $u \in [0, \frac{1}{100})$ and $v \in [0, \frac{1}{100})$. Then we have

$$d(u, \mathcal{G}v) = |u - 0| = |u| = d(u, \mathcal{G}u) \le \mu d(u, \mathcal{G}u) + d(u, v).$$

So, $\mathcal{G}$ satisfies condition (E_1).

(ii) $u \in [\frac{1}{100}, 1]$ and $v \in [\frac{1}{100}, 1]$. Then we have

$$\begin{aligned} d(u, \mathcal{G}v) &= \left|u - \frac{2v}{3}\right| = \left|\frac{3u - 2v}{3}\right| \\ &= \left|\frac{u}{3} + \frac{2u}{3} - \frac{2v}{3}\right| \\ &= \frac{u}{3} + \frac{2}{3}|u - v|. \end{aligned}$$

Turning to the right side of the inequality in Definition 2.3,

$$\mu d(u, \mathcal{G}u) + d(u, v) = \mu \left|u - \frac{2u}{3}\right| + |u - v|.$$

If we choose the admissible parameter $\mu = 1$, the mapping will satisfy condition (E).

(iii) $u \in [\frac{1}{100}, 1]$ and $v \in [0, \frac{1}{100})$, which leads to $d(u, \mathcal{G}u) = \left|u - \frac{2u}{3}\right|$. Evaluating condition (E) for this case, we have

$$\begin{aligned} d(u, \mathcal{G}v) &= |u - 0| \le \frac{3u}{3} + |u - v| \\ &= 3(\frac{u}{3}) + d(u, v) = 3d(u, \mathcal{G}u) + d(u, v). \end{aligned}$$

So, if we choose the admissible parameter $\mu = 3$, then the mapping will prove to have condition (E). Taking the maximum value of μ, we conclude that $\mathcal{G}$ satisfies (E_3) with $\mathcal{G}(0) = 0$ fixed point.

Now, let us prove that $\mathcal{G}$ is not a generalized α-nonexpansive mapping. We shall take $u = \frac{1}{150}$ and $v = \frac{1}{100}$. It follows that

$$\frac{1}{2}d(u, \mathcal{G}u) = \frac{1}{2}\left|\frac{1}{150} - 0\right| = \frac{1}{300} = \left|\frac{1}{150} - \frac{1}{100}\right| = |u - v|.$$

If we consider the left side of the inequality in Definition 2.2,

$$d(\mathcal{G}u, \mathcal{G}v) = \left|0 - \frac{2}{3}\frac{1}{100}\right| = \frac{1}{150}.$$

Turning to the right side of the inequality in Definition 2.2, for $\alpha \in [0,1)$,

$$\begin{aligned}
& \alpha d(\mathcal{G}u, v) + \alpha d(\mathcal{G}v, u) + (1-2\alpha) d(u,v) \\
= \quad & \alpha \left| \frac{1}{150} - \frac{2}{3}\frac{1}{100} \right| + \alpha \left| 0 - \frac{1}{150} \right| + (1-2\alpha) \left| \frac{1}{150} - \frac{1}{100} \right| \\
= \quad & 0 + \frac{\alpha}{100} + \frac{1}{300} - \frac{\alpha}{300} \\
= \quad & \frac{3\alpha}{300} + \frac{1}{300} - \frac{2\alpha}{300} \\
= \quad & \frac{\alpha}{300} + \frac{1}{300} = (\alpha+1)\frac{1}{300}.
\end{aligned}$$

So, for $\alpha \in [0,1)$, the implications fails to be satisfied, which leads to the conclusion that $\mathcal{G}$ is not a generalized α-nonexpansive mapping.

In order to we show that $\mathcal{G}$ does not satisfy condition (C), we take also $u = \frac{1}{150}$ and $v = \frac{1}{100}$. Then we have

$$\frac{1}{2} d(u, \mathcal{G}u) = \frac{1}{2} \left| \frac{1}{150} - 0 \right| = \frac{1}{300} = \left| \frac{1}{150} - \frac{1}{100} \right| = |u - v| = d(u,v).$$

If we apply the inequality in Definition 2.1, we get

$$d(\mathcal{G}u, \mathcal{G}v) = \left| 0 - \frac{2}{3}\frac{1}{100} \right| = \frac{1}{150} > \frac{1}{300} = d(u,v).$$

Thus $\mathcal{G}$ does not satisfy condition (C).

5. Conclusions

We get some results on the strong and Δ-convergence of SP*-iteration process (1.4) in given [35] for the mapping with Property (E) in nonlinear $CAT(0)$ spaces. The result herein complements the some results of [14, 36, 37] from linear setting to $CAT(0)$ spaces. We also prove the stability of SP*-iteration process generated by (1.4) in given [35] in this paper. In addition, we give an illustrative numerical example that satisfies condition (E). As seen in Example 4.5, the mapping is neither a generalized αnonexpansive mapping nor does it satisfy condition (C). Further, in future studies, iteration process can be developed and iteration that converges faster than prominent iterations can be presented.

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NEW APPLICATIONS OF THE METATHEOREM IN ORDERED FIXED POINT THEORY

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In memory of 65 years with Gyoung

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ABSTRACT. Our aim in this paper is to find new applications of our long-standing 2023 Metatheorem. In fact, a certain particular form of Metatheorem on fixed point theorems characterizes metric completeness. Moreover, classical theorems due to Banach, Rus-Hicks-Rhoades, Nadler, Covitz-Nadler, Oettli-Théra, Edelstein, Turinici, Tasković, Khamsi and ourselves are equivalently formulated or improved by applying Metatheorem.

KEYWORDS: Quasi-metric space, fixed point, RHR contraction principle, orbitally complete, orbitally continuous.
AMS Subject Classification: 06A75, 47H10, 54E35, 54H25, 58E30, 65K10.

1. Introduction

Our Metatheorem in Ordered Fixed Point Theory has a long history and many applications. Its more than one hundred applications produce old and new theorems and clarify mutual relations among them. One of the main applications of them is closely related to the Banach contraction principle — the origin of Metric Fixed Point Theory.

Let (X,d) be a metric space. A Banach contraction $T : X \longrightarrow X$ is a map satisfying

$$d(Tx,Ty) \leq \alpha\, d(x,y) \quad \text{for all } x,y \in X$$

with some $\alpha \in [0,1)$. There have been appeared thousands of articles related to the Banach contraction. It is well-known that the Banach contraction does not characterize the metric completeness.

Recently, we introduced the Rus-Hicks-Rhoades (RHR) map $T : X \longrightarrow X$ [31], [7] satisfying

$$d(Tx,T^2x) \leq \alpha\, d(x,Tx) \quad \text{for all } x \in X$$

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with some $\alpha \in [0,1)$. See our recent works [27], [28], [30]. The RHR maps are also known as graphic contractions, iterative contractions, weakly contractions, or Banach mappings; see Berinde et al. [2],[3]. Moreover, it is recently known that well-known metric fixed point theorems related to the RHR maps hold for quasi-metric spaces (without assuming the symmetry); see [28], [30].

Our aim in the present paper is to find new applications of our Metatheorem. In fact, certain classical theorems due to Banach, Rus-Hicks-Rhoades [26], [27], [29], Nadler [11], Covitz-Nadler [5], Oettli-Théra [12], Edelstein [6], Turinici [34], [35], Tasković [33], and Khamsi [10] are equivalently formulated or improved by applying our Metatheorem. Especially, the completeness of quasi-metric spaces are equivalent to several fixed point or other theorems due to Rus-Hicks-Rhoades, Nadler, Covitz-Nadler, Oettli-Théra, and others.

This paper is organized as follows: Section 2 is to introduce our long-standing Metatheorem. In Section 3, basic terminology on quasi-metric spaces are given as preliminaries. Section 4 is to introduce our recent versions of the Rus-Hicks-Rhoades (RHR) contraction principle and the new Banach contraction principle. Section 5 devotes to a certain particular form of Metatheorem on fixed point theorems which characterizes metric completeness. In Sections 6-10, several theorems due to Edelstein, Turinici, Tasković, and Khamsi are equivalently formulated by applying our Metatheorem. Finally, Section 11 is for the epilogue.

In this paper, multimaps are always non-empty valued.

2. Our 2023 Metatheorem

Our Metatheorem has a long history. We obtained the following form called the new 2023 Metatheorem in [19],[24],[26]:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x,y \in X$. Then the following are equivalent:*

(α) *There exists an element $v \in A$ such that the negation of $G(v,w)$ holds for any $w \in X\backslash\{v\}$.*

($\beta 1$) *If $f : A \longrightarrow X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*

($\beta 2$) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

($\gamma 1$) *If $f : A \longrightarrow X$ is a map such that $G(x,fx)$ for any $x \in A$ with $x \neq fx$, then f has a fixed element $v \in A$, that is, $v = fv$.*

($\gamma 2$) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ satisfying $G(x,fx)$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

($\delta 1$) *If $F : A \multimap X$ is a multimap such that, for any $x \in A\backslash Fx$ there exists $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then F has a fixed element $v \in A$, that is, $v \in Fv$.*

($\delta 2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A\backslash Fx$ there exists $y \in X\backslash\{x\}$ satisfying $G(x,y)$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*

($\epsilon 1$) *If $F : A \multimap X$ is a multimap satisfying $G(x,y)$ for any $x \in A$ and any $y \in Fx\backslash\{x\}$, then F has a stationary element $v \in A$, that is, $\{v\} = Fv$.*

(ϵ2) If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $G(x, y)$ holds for any $x \in A$ and any $y \in Fx \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.

(η) If Y is a subset of X such that for each $x \in A \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $G(x, z)$, then there exists a $v \in A \cap Y$.

For the proof, see Park [19], [24], [26]. Each item in Metatheorem has a long history. Especially, (η) is originated from Oettli-Théra [12].

This Metatheorem guarantees the truth of all items when one of them is true. Since 1985, we have shown nearly one hundred cases of such situation. See [13]–[26].

3. Quasi-metric Spaces

It is well-known that some key-results in Metric Fixed Point Theory hold for quasi-metric spaces. For example, Banach contraction principle, Nadler or Covitz-Nadler fixed point theorem, Ekeland variational principle, Caristi fixed point theorem, Takahashi minimization principle, and many others.

We recall the following:

Definition 3.1. A *quasi-metric* on a nonempty set X is a function $\delta : X \times X \longrightarrow \mathbb{R}^+ = [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $\delta(x, y) = \delta(y, x) = 0 \Longleftrightarrow x = y$;
(b) (triangle inequality) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

A *metric* on a set X is a quasi-metric satisfying

(c) (symmetry) $\delta(x, y) = \delta(y, x)$ for all $x, y \in X$.

The convergence and completeness in a quasi-metric space (X, δ) are defined as follows:

Definition 3.2. ([1], [8])
(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n \longrightarrow \infty} \delta(x_n, x) = \lim_{n \longrightarrow \infty} \delta(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $n > m > N$.
(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $m > n > N$.
(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 3.3. ([1], [8])
(1) (X, δ) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, δ) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, δ) is *complete* if every Cauchy sequence in X is convergent.

Definition 3.4. Let (X, δ) be a quasi-metric space and $T : X \longrightarrow X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, Tx, \cdots, T^n x, \cdots\}.$$

The space X is said to be *T-orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\longrightarrow\infty} T^n x = x_0 \Longrightarrow \lim_{n\longrightarrow\infty} T^{n+1} x = T x_0$$

for any $x \in X$.

4. The Rus-Hicks-Rhoades Contraction Principle

For quasi-metric spaces (X, δ), simply δ is not symmetric.

Definition. The *orbit* of a selfmap $T : X \longrightarrow X$ at $x \in X$ is the set $O(x,T) = \{T^n x : n = 0, 1, 2, \dots\}$. The space X is said to be *T-orbitally complete* if every (right)-Cauchy sequence in $O(x,T)$ is convergent in X. A selfmap T of X is said to be *T-orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\longrightarrow\infty} T^n(x) = x_0 \implies \lim_{n\longrightarrow\infty} T^{n+1}(x) = T(x_0)$$

for any $x \in X$.

The following in Park [27], [28], [30] is called the Rus-Hicks-Rhoades (RHR) Contraction Principle:

Theorem P. *Let (X, δ) be a quasi-metric space and let $T : X \longrightarrow X$ be an RHR map; that is,*

$$\delta(T(x), T^2(x)) \le \alpha\, \delta(x, T(x)) \text{ for every } x \in X,$$

where $0 \le \alpha < 1$.

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\longrightarrow\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \le \frac{\alpha^n}{1-\alpha}\delta(x, T(x)), \quad n = 1, 2, \cdots,$$

$$\delta(T^n(x), x_0) \le \frac{\alpha}{1-\alpha}\delta(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) *x_0 is a fixed point of T, and, equivalently,*

(iii) *$T : X \longrightarrow X$ is orbitally continuous at $x_0 \in X$.*

This was proved in [30] by analyzing a typical proof of the Banach Contraction Principle.

For the condition: there exists $0 < \alpha < 1$ such that $d(T(x), T^2(x)) \le \alpha \cdot d(x, T(x))$, for all $x \in X$, we meet the following names: graphic contraction, iterative contraction, weakly contraction, Banach mapping, etc.

Moreover, the following consequence of Theorem P in Park [30] extends the usual Banach Contraction Principle:

Theorem Q. *Let (X, δ) be a quasi-metric space and let $T : X \longrightarrow X$ be an improved Banach contraction, that is, for each $x \in X$, there exists a $y \in X$ such that*

$$\delta(T(x), T(y)) \le \alpha\, \delta(x, y) \text{ where } 0 \le \alpha < 1.$$

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\longrightarrow\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha}\delta(x, T(x)), \quad n = 1, 2, \cdots,$$
$$\delta(T^n(x), x_0) \leq \frac{\alpha}{1-\alpha}\delta(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) x_0 *is the unique fixed point of* T *(equivalently,* $T : X \longrightarrow X$ *is orbitally continuous at* $x_0 \in X$*).*

The Banach Contraction Principle appeared in thousands of publications should be corrected as in Theorem Q.

We began our study on RHR maps in [27] and [28]. Later we found a large number of examples of RHR maps in [29], [30], where we showed a large number of metric fixed point theorems can be extended or improved.

5. Completeness of Quasi-metric Spaces

In our previous work [30], we obtained the following RHR theorem:

Theorem H(γ1)**.** *Let* (X, δ) *be a quasi-metric space,* $0 < \alpha < 1$*, and* $f : X \longrightarrow X$ *be a map satisfying*

$$\delta(f(x), f^2(x)) \leq \alpha\, \delta(x, f(x)) \text{ for all } x \in X\backslash\{f(x)\}.$$

Then f *has a fixed point* $v \in X$ *if and only if* X *is f-orbitally complete.*

Let (X, δ) be a quasi-metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A, B) = \max\{\sup\{\delta(a, B) : a \in A\},\ \sup\{\delta(b, A) : b \in B\}\},$$

where $\delta(a, B) = \inf\{\delta(a, b) : b \in B\}$. Then H is called a generalized Hausdorff quasi-metric since it may have infinite values.

Recently, as a basis of Ordered Fixed Point Theory [19], [24], [26], we obtained the 2023 Metatheorem and Theorem H including Nadler's fixed point theorem [11] in 1969 and its extended version by Covitz-Nadler [5] in 1970.

From Theorem H(γ1) and Metatheorem, we have the following new version:

Theorem H. ([24], [26], [30]) *Let* (X, δ) *be a quasi-metric space and* $0 < r < 1$*. Then the following statements are equivalent:*

(0) (X, δ) *is complete.*

(α) *For a multimap* $T : X \longrightarrow \mathrm{Cl}(X)$*, there exists an element* $v \in X$ *such that* $H(Tv, Tw) > r\, \delta(v, w)$ *for any* $w \in X\backslash\{v\}$*.*

(β) *If* $\mathfrak{F}$ *is a family of maps* $f : X \longrightarrow X$ *such that, for any* $x \in X\backslash\{fx\}$*, there exists a* $y \in X\backslash\{x\}$ *satisfying* $\delta(fx, fy) \leq r\, \delta(x, y)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v = fv$ *for all* $f \in \mathfrak{F}$*.*

(γ) *If* $\mathfrak{F}$ *is a family of maps* $f : X \longrightarrow X$ *satisfying* $\delta(fx, f^2x) \leq r\, d(x, fx)$ *for all* $x \in X\backslash\{fx\}$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v = fv$ *for all* $f \in \mathfrak{F}$*.*

(δ) *Let* $\mathfrak{F}$ *be a family of multimaps* $T : X \longrightarrow \mathrm{Cl}(X)$ *such that, for any* $x \in X\backslash Tx$*, there exists* $y \in X\backslash\{x\}$ *satisfying* $H(Tx, Ty) \leq r\, \delta(x, y)$*. Then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v \in Tv$ *for all* $T \in \mathfrak{F}$*.*

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $T : X \longrightarrow \mathrm{Cl}(X)$ *satisfying* $H(Tx, Ty) \leq r\, \delta(x, y)$ *for all* $x \in X$ *and any* $y \in Tx\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in X$*, that is,* $\{v\} = Tv$ *for all* $T \in \mathfrak{F}$*.*

(η) *If* Y *is a subset of* X *such that for each* $x \in X \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *satisfying* $H(Tx, Tz) \leq r\, \delta(x, z)$ *for a* $T : X \longrightarrow \mathrm{Cl}(X)$*, then there exists a* $v \in X \cap Y = Y$*.*

Proof. The equivalency (α)-(η) follows from Metatheorem. When $\mathfrak{F}$ is a singleton, (β)-(ϵ) are denoted by ($\beta1$)-($\epsilon1$), respectively. They are also logically equivalent to (α)-(η) by Metatheorem. Note that ($\gamma1$) follows from Theorem H($\gamma1$). The equivalency of (0) and ($\gamma1$) is given in [29], [30]. Then Theorem H holds. □

Remark 5.1. (1) The completeness in (0) can be replaced by f-orbitally or T-orbitally completeness according to the corresponding situation.

(2) ($\beta1$) properly extends the Banach contraction principle.

(3) ($\gamma1$) is the Rus-Hicks-Rhoades theorem and equivalent to (0).

(4) Further, ($\delta1$) and ($\epsilon1$) extend the well-known theorems of Nadler [11] and Covitz-Nadler [5] on multi-valued contraction.

(5) Actually, the proof of Theorem H covers the corresponding ones of Banach, Rus [31], Hicks-Rhoades [7], Nadler [11], Covitz-Nadler [5], and Oettli-Théra [12].

(6) There are a large number of characterizations of metric completeness. It is well-known that the Banach contraction does not characterize. However, so does its slight generalized form ($\beta1$) and the RHR map in ($\gamma1$).

We have a single-valued version of Theorem H(α) as follows:

Theorem H($\alpha1$). *Let* (X, δ) *be a quasi-metric space,* $f : X \longrightarrow X$ *a map and* $0 < r < 1$*. Then* X *is* f*-orbitally complete if and only if there exists an element* $v \in X$ *such that* $\delta(fv, fw) > r\, \delta(v, w)$ *for any* $w \in X \backslash \{v\}$*.*

This is also equivalent to all items in Theorem H. In some sense, this shows that the Banach contraction principle does not characterize the metric completeness. But so does the RHR theorem or Theorem H($\gamma1$).

6. Edelstein [6] in 1962

In this section, we apply Metatheorem to a particular situation when $f : X \longrightarrow X$ is a map and $G(x, y)$ means $\delta(x, fx) \leq \delta(y, fy)$ for $x, y \in X$.

Definition 6.1. A map $f : X \longrightarrow X$ on a quasi-metric space (X, δ) is said to be *contractive* if

$$\delta(fx, fy) < \delta(x, y)$$

for all $x, y \in X$ with $x \neq y$.

We recall the well-known Edelstein fixed point theorem:

Theorem 6.2. (Edelstein) *Let* (X, d) *be a compact metric space and* $f : X \longrightarrow X$ *be a contractive map. Then* f *has a unique fixed point* $v \in X$*, and moreover, for each* $x \in X$*, we have* $\lim_{n \longrightarrow \infty} f^n(x) = v$*.*

Motivated by Theorem 6.2, we have the following from our Metatheorem:

Theorem 6.3. *Let* (X, δ) *be a compact quasi-metric space. Then the following statements are equivalent:*

(α) *For a map* $f : X \longrightarrow X$*, there exists a point* $v \in X$ *such that* $\delta(fv, fw) \geq \delta(v, w)$ *for any* $w \in X \backslash \{v\}$*.*

(β1) *For a map $f : X \longrightarrow X$ such that, for any $x \in X$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(fx, fy) < \delta(x, y)$, then f has a fixed point $v \in X$, that is, $v = fv$.*

(β2) *If $\mathfrak{F}$ is a family of maps $f : X \longrightarrow X$ such that, for any $x \in X$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(fx, fy) < \delta(x, y)$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(γ1) *If $f : X \longrightarrow X$ is a map such that, for any $x \in X$ satisfying $\delta(fx, f^2x) < \delta(x, fx)$ for all $x \in X$ with $x \neq fx$, then f has a fixed point $v \in X$, that is, $v = fv$.*

(γ2) *If $\mathfrak{F}$ is a family of maps $f : X \longrightarrow X$ satisfying $\delta(fx, f^2x) < \delta(x, fx)$ for all $x \in X$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(δ1) *If $T : X \longrightarrow \mathrm{Cl}(X)$ is a multimap such that for any $x \in X \backslash Tx$ there exists a $y \in X \backslash \{x\}$ satisfying $H(Tx, Ty) < \delta(x, y)$, then T has a fixed point $v \in X$, that is, $v \in T(v)$.*

(δ2) *If $\mathfrak{F}$ is a family of multimaps $T : X \longrightarrow \mathrm{Cl}(X)$ such that for any $x \in X \backslash Tx$ there exists a $y \in X \backslash \{x\}$ satisfying $H(Tx, Ty) < \delta(x, y)$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v \in Tv$ for all $T \in \mathfrak{F}$.*

(ϵ1) *If $T : X \longrightarrow \mathrm{Cl}(X)$ is a multimap such that $H(Tx, Ty) < \delta(x, y)$ holds for any $x \in X$ and any $y \in Tx \backslash \{x\}$, then T has a stationary point $v \in X$, that is, $\{v\} = Tv$.*

(ϵ2) *If $\mathfrak{F}$ is a family of multimaps $T : X \longrightarrow \mathrm{Cl}(X)$ such that $H(Tx, Ty) < \delta(x, y)$ holds for any $x \in X$ and any $y \in Tx \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary point $v \in X$, that is, $\{v\} = Tv$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in X \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $H(Tx, Tz) < \delta(x, z)$ for a multimap $T : X \longrightarrow \mathrm{Cl}(X)$, then there exists a $v \in X \cap Y = Y$.*

PROOF. Equivalency follows from Metatheorem. □

Remark 6.4. (1) Theorem 6.3 means the equivalency of the items (α)-(η). Therefore, each items are conjecture.

(2) Each item implies the Edelstein Theorem 6.2. This is clear for (β1), (γ1), (δ1), and (ϵ1),

(3) In case f is continuous in (α), all (α), (β1), (γ1), (δ1), and (ϵ1) are true. In fact, let a map $\varphi : X \longrightarrow \mathbb{R}^+$ by putting

$$\varphi(x) = \delta(x, fx), \quad x \in X.$$

Then φ is continuous and bounded below, so it has a minimum value at a point $v \in X$. Hence (α) holds. Moreover, (β1)–(η) also hold by Metatheorem.

From Theorem 6.3, we can deduce several fixed point theorems on a compact quasi-metric space (X, δ) extending the Edelstein Theorem 6.1.

For example, we have the following:

Theorem 6.5. (β1) *If $f : X \longrightarrow X$ is a continuous map such that for any $x \in X$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(x, fx) > d(y, fy)$, then f has a fixed point $v \in X$, that is, $v = fv$.*

($\gamma 2$) *If* $\mathfrak{F}$ *is a family of continuous maps* $f : X \longrightarrow X$ *satisfying* $d(x, fx) > d(fx, f^2x)$ *for all* $x \in X$ *with* $x \neq fx$, *then* $\mathfrak{F}$ *has a common fixed point* $v \in X$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

($\epsilon 1$) *If* $T : X \multimap X$ *is a multimap and* $f : X \longrightarrow X$ *is a continuous selection of* T *such that* $d(x, fx) > d(y, fy)$ *holds for any* $x \in X$ *and any* $y \in T(x) \backslash \{x\}$, *then* T *has a stationary point* $v \in X$, *that is,* $\{v\} = T(v)$.

Recently, Kirk and Shahzad raised one open question on Edelstein's fixed point theorem. In 2018, Suzuki [32] gave a negative answer to this question, and extended Edelstein's theorem to semimetric spaces.

7. Turinici [34] in 1980

Turinici's main result ([34], Theorem 3.1) is as follows:

Theorem 7.1. *Let* (X, d) *be a metric space, and* $\preccurlyeq$ *an ordering on* X *such that*

(1) $\preccurlyeq$ *is a closed ordering on* X,
(2) (X, d) *is a* $\preccurlyeq$*-asymptotic metric space, and*
(3) (X, d) *is a* $\preccurlyeq$*-complete metric space.*

Then, for every $x \in X$ *there is a maximal element* $z \in X$ *such that* $x \preccurlyeq z$.

This can be applied to our Metatheorem as follows:

Theorem 7.2. *Let* (X, d) *be a metric space, and* $\preccurlyeq$ *an ordering on* X *satisfying* (1)-(3). *Let* $z \in X$ *and* $A := \{x \in X : z \preccurlyeq x\}$.

Then the following equivalent statements hold:

(α) *There exists an element* $v \in A$ *such that* $w \prec v$ *for any* $w \in X \backslash \{v\}$.

(β) *If* $\mathfrak{F}$ *is a family of maps* $f : A \longrightarrow X$ *such that for any* $x \in A$ *with* $x \neq fx$, *there exists a* $y \in X \backslash \{x\}$ *satisfying* $x \preccurlyeq y$), *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(γ) *If* $\mathfrak{F}$ *is a family of maps* $f : A \longrightarrow X$ *satisfying* $x \preccurlyeq fx$ *for all* $x \in A$ *with* $x \neq fx$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(δ) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : A \multimap X$ *such that, for any* $x \in A \backslash Fx$ *there exists* $y \in X \backslash \{x\}$ *satisfying* $x \preccurlyeq y$. *Then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v \in Fv$ *for all* $F \in \mathfrak{F}$.

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $F : A \multimap X$ *such that* $x \preccurlyeq y$ *holds for any* $x \in A$ *and any* $y \in Fx \backslash \{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in A$, *that is,* $\{v\} = Fv$ *for all* $F \in \mathfrak{F}$.

(η) *If* Y *is a subset of* X *such that for each* $x \in A \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *satisfying* $x \preccurlyeq y$, *then there exists a* $v \in A \cap Y$.

Proof. Under the hypothesis, the conclusion of ([34], Theorem 3.1) is "for every $x \in X$, there is a maximal element $z \in X$." Replacing (x, z) by (z, v), we obtain α. The equivalency is obtained from Metatheorem, where $G(x, y)$ is replaced by $x \preccurlyeq y$. □

Note that (α) and ($\gamma 1$) are [34], Theorems 3.1 and 3.2, respectively.

8. Tasković [33] in 1986

Recall that Tasković [33] showed that Zorn's lemma is equivalent to the following:

Theorem 8.1. *Let $\mathfrak{F}$ be a family of selfmaps defined on a partially ordered set A such that $x \leq fx$ (resp. $fx \leq x$), for all $x \in A$ and all $f \in \mathfrak{F}$. If each chain in A has an upper bound (resp. lower bound), then the family $\mathfrak{F}$ has a common fixed point.*

This can be applied to the following:

Theorem 8.2. *Let A be a partially ordered set such that each chain in A has an upper bound. Then the following equivalent statements hold:*

(α) *There exists an element $v \in A$ such that $w \prec v$ for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow A$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in A \backslash \{x\}$ satisfying $x \preccurlyeq y$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow A$ satisfying $x \preccurlyeq fx$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap A$ such that, for any $x \in A \backslash Fx$ there exists $y \in A \backslash \{x\}$ satisfying $x \preccurlyeq y$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap A$ such that $x \preccurlyeq y$ holds for any $x \in A$ and any $y \in Fx \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of A such that for each $x \in A \backslash Y$ there exists a $z \in A \backslash \{x\}$ satisfying $x \preccurlyeq y$, then there exists a $v \in Y$.*

Proof. Note that (α) is a form of Zorn's lemma and (γ) is the theorem due to Tasković. Therefore Theorem 7.2 holds by Metatheorem. □

Other true statements (β1)-(ϵ1) can be also obtained.

9. Khamsi [10] in 2009

In [10], Khamsi gave a characterization of the existence of minimal elements in partially ordered sets in terms of fixed point of multimaps.

Let A be an abstract set partially ordered by $\prec$. We will say that $a \in A$ is a minimal element of A if and only if $b \prec a$ implies $b = a$. The concept of minimal element is crucial in the proofs given to Caristi's fixed point theorem.

The following is [10], Theorem 1:

Theorem 9.1. *Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent.*

(1) *A contains a minimal element,*

(2) *Any multimap T defined on A such that for any $x \in A$, there exists $y \in Tx$ with $y \prec x$, has a fixed point, i.e there exists $a \in A$ such that $a \in Ta$.*

According to our method in the present paper, Theorem 9.1 can be extended as follows:

Theorem 9.2. *Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent:*

(α) *There exists an element $v \in A$ such that $w \prec v$ for any $w \in A \backslash \{v\}$.*

(β) If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow A$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in A\backslash\{x\}$ satisfying $x \preccurlyeq y$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.

(γ) If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow A$ satisfying $x \preccurlyeq fx$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.

(δ) Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap A$ such that, for any $x \in A\backslash Fx$ there exists $y \in A\backslash\{x\}$ satisfying $x \preccurlyeq y$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.

(ϵ) If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $x \preccurlyeq y$ holds for any $x \in A$ and any $y \in Fx\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.

(η) If Y is a subset of A such that for each $x \in A\backslash Y$ there exists a $z \in A\backslash\{x\}$ satisfying $x \preccurlyeq y$, then there exists a $v \in Y$.

Proof. Let $G(x, y)$ means $x \prec y$. Then Theorem 8.2 follows from Metatheorem. □

Note that (α) and ($\delta 1$) are (1) and (2) of Theorem 9.1. Therefore Theorem 9.2 extends Theorem 9.1.

In what follows Khamsi assumes that $\eta : [0, \infty) \longrightarrow [0, \infty)$ is nondecreasing, continuous, such that there exist $c > 0$ and $\delta_0 > 0$ such that for any $t \in [0, \delta_0]$ we have $\eta(t) \geq c\,t$. Under these assumptions we have the following result.

Theorem 9.3. *Let M be a complete metric space. Define the relation $\prec$ by*

$$x \prec y \iff \eta(d(x, y)) \leq \phi(y) - \phi(x)$$

where η and ϕ satisfy all the above assumptions.
Then the following equivalent statement hold:

(α) $(M, \prec)$ has a minimal element x_*, i.e. if $x \prec x_*$ then we must have $x = x_*$.

($\gamma 1$) If $f : M \longrightarrow M$ is a map such that $fx \prec x$ for any $x \in X$, then f has a fixed element $v \in A$, that is, $v = fv$.

($\delta 1$) If $F : M \multimap M$ is a multimap such that, for any $x \in M\backslash Fx$ there exists $y \in X\backslash\{x\}$ satisfying $y < x$, then F has a fixed element $v \in M$, that is, $v \in Fv$.

Note that (α) - ($\delta 1$) are due to Khamsi ([10], Theorems 2-4), respectively. Applying our Metatheorem, we can make some more as for ($\beta 2$)-($\epsilon 2$) and (η).

10. Turinici [35] in 2009

In [35], some pseudometric versions of the Brézis-Browder ordering principle [4] are discussed. An application of these facts to equilibrium points is also included.

Let $(M, \preccurlyeq)$ be a quasi-ordered structure; and $x \mapsto \varphi(x)$ stand for a function between M and $R_+ \cup \{\infty\} = [0, \infty]$. The following is ([35], Proposition 1):

Proposition 10.1. ([35]) *Assume*

(1a) *$(M, \preccurlyeq)$ is sequentially inductive: each ascending sequence has an upper bound (modulo $(\preccurlyeq)$),*

(1b) *ψ is $(\preccurlyeq)$-decreasing $(x \preccurlyeq y \Longrightarrow \psi(x) \geq \psi(y))$, and*

(2a) *$(M, \preccurlyeq)$ is almost regular (modulo φ):*

$$\forall x \in M,\ \forall \varepsilon > 0,\ \exists y = y(x, \varepsilon) \succcurlyeq x \text{ with } \varphi(y) \leq \varepsilon.$$

Let $u \in M$ and $A = \{y \in M : u \preccurlyeq y\}$. Then there exists $v \in A$ with $\varphi(v) = 0$ (hence v is $(\preccurlyeq, \varphi)$- maximal).

From Proposition 10.1 and our Metatheorem, we have the following extended version:

Theorem 10.2. *Under the hypothesis of Proposition 9.1, the following equivalent statements hold:*

(α) *There exists an element $v \in A$ such that $w \prec v$ for any $w \in M\backslash\{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow M$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in M\backslash\{x\}$ satisfying $x \preccurlyeq y$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow M$ satisfying $x \preccurlyeq fx$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap M$ such that, for any $x \in A\backslash Fx$ there exists $y \in M\backslash\{x\}$ satisfying $x \preccurlyeq y$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap M$ such that $x \preccurlyeq y$ holds for any $x \in A$ and any $y \in Fx\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of M such that for each $x \in A\backslash Y$ there exists a $z \in M\backslash\{x\}$ satisfying $x \preccurlyeq y$, then there exists a $v \in A \cap Y$.*

Turinici stated that the following ordering principle ([35], Proposition 2) is then available (cf. Kang and Park [13]):

Proposition 10.3. *Assume that $(M, \preccurlyeq)$ is sequentially inductive and weakly regular (modulo d). Then, for each $u \in M$, there exists a $(\preccurlyeq, d)$- maximal $v \in M$ with $u \preccurlyeq v$.*

In [35], Propositions 2, 3, 4, 5 and Theorems 2, 3, 4 are all maximality statements and can be also equivalently formulated by Metatheorem.

11. Conclusion

In this paper, by applying Metatheorem, we obtain equivalent forms of some known theorems. Most of them are new and useful as the original theorems. Therefore our Metatheorem is the way to lead new truth from the equivalent old one. This method was already applied almost one hundred times by the author in [13]-[30].

Recall that there are many articles characterizing metric completeness. Recall that the Banach contraction does not characterize the completeness. In the present article, we introduced a surprising result. Theorem H shows that certain general forms of theorems of Banach, Nadler, Covitz-Nadler, and others characterize completeness of quasi-metric spaces. More precisely, the Rus-Hicks-Rhoades theorem is the one of such theorems.

In our previous works [27]–[30], we concentrated the study of RHR maps and found the close relation between such maps and completeness.

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