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EPISODES IN METRIC FIXED POINT THEORY RELATED TO F. E. BROWDER

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ABSTRACT. F.E. Browder in 1979 posed a fixed point theorem of great generality and complexity such that a large part of literature on contractive type of maps can be subsumed under an intuitive and simple mode of argument. Immediately after, several researchers found better theorems than his result. Since then only a few authors quote his theorem. However, until recently there have been appeared certain minor papers related Browder's aim. Our aim in this paper is to introduce the contents of them.

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1. Prologue

In 1979, Felix Browder [4] began as follows: "The contraction mapping principle (known under a number of names including those of Picard, Banach and Cacciopoli) has long been one of the simplest and most useful tools in the study of nonlinear problems. During the past two decades, the development of nonlinear functional analysis has taken a diversity of forms and used a great variety of more sophisticated tools and methods. Yet in many of the theories which have been created, from one point of view, these more sophisticated results can be regarded as far-reaching extensions of contraction mapping principle in contexts of richer structure.

Over the same decades, however, there has grown an extensive literature devoted to sharper forms of the contraction mapping principle on its native terrain, i.e. for mappings of complete metric spaces. As can be seen from a recent survey by Rhoades [17] (in which 149 different conditions are analyzed and compared), this literature has reached a point of such scholastic complexity and unreadability that its usefulness is open to serious question."

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Forty-five years later in 2024, the situation became more serious than Browder's time. In fact, there have been appeared several hundreds of contractive type conditions and thousands of new artificial metric type spaces. Note that the present writer never have made any new contractive conditions nor any new artificial spaces in the metric fixed point theory.

Browder [4] made a theorem in order to unify known fixed point theorems until that time. Immediately after [4] appeared, Hegeds̈ and Szilágyi [7] published a very general metric fixed point theorem. Based on their work, the present writer [14] in 1980 classified contractive type conditions in order to improve the survey by Billy E. Rhoades [17]. Without recognizing [7] and [14], Walter [20] improved Browder's result. However, Park [15] did the same work based on [7] and [14]. Since then, there have appeared several papers closely related to Browder's.

In this paper, we recall contents of such articles related to the lost paper of Browder as small episodes in the history of the metric fixed point theory. All quoted papers in this paper are introduced in the chronological order.

Now we begin with the following:

2. Browder [4] in 1979

"It is our purpose in the present discussion to show that a large part of this complexity can be subsumed under an intuitive and simple mode of argument. We present a fixed point theorem of all great generality and complexity which includes all the detailed cases which the author has seen in the literature. We deduce this principle in an easily comprehensible way from a simpler more conceptual principle. The proof of the latter is a slight variant of an earlier proof given for contractive fixed point theorems by the writer" in 1968 and 1976.

Browder gives the following definition:

Definition 1. Let Φ be a function from $\mathbb{R}^+$, the non-negative reals, to $\mathbb{R}^+$. Then Φ is said to be a contractive gauge function if $\Phi(0) = 0$, Φ is non-decreasing from the right (i.e., r_j decreasing and converging to $r > 0$ implies that $\Phi(r_j)$ converges to $\Phi(r)$), and

$$\Phi(r) < r$$

for all $r > 0$.

The following is the main result of Browder:

Theorem 2. *Let M be a complete metric space with metric d, M_0 a subset of M. Let f be a continuous mapping of M into M which carries M_0 into M_0. Let Φ be a contractive gauge function in the sense of Definition 1. For each x in M_0, suppose that there exists a positive integer $n(x)$ and for each y in M_0, three subsets $J_1 = J_1(x, y, n), J_2 = J_2(x, y, n), J_3 = J_3(x, y, n)$ of $\mathbb{N} \times \mathbb{N}$ such that for each $n \geq n(x)$, $y \in M_0$,*

$$d(f^n x, f^n y) \leq \Phi(\max(\sup_{[j,k]\in J_1} d(f^j x, f^k y), \sup_{[r,s]\in J_2} d(f^r x, f^s y), \sup_{[m,t]\in J_3} d(f^m x, f^t y))).$$

Then: f has a fixed point x_0 in M such that for each x in M_0, $f^j x$ converges to x_0 in M as $j \longrightarrow +\infty$.

In order to prove this, Browder needed the following Propositions:

Proposition 2. *Let M be a complete metric space, M_0 a subset of M, f a continuous mapping of M into M which carries M_0 into M_0. Let Φ be a contractive gauge function. Let $O(f, x)$ denote the orbit of x under f, i.e. $O(f, x) = \bigcup_{j\geq 0}\{f^j x\}$.*

Suppose that for each x in M_0, there exists $n(x)$ a positive integer such that $\operatorname{diam} O(f, f^{n(x)}x)) \leq \Phi(\operatorname{diam} O(f,x)))$.

Then f has a fixed point in $\bigcap_{j\geq 0} \operatorname{cl}(O(f, f^j x))$ for each x in M_0.

Proposition 3. *Let M be a complete metric space, M_0 a subset of M, f a continuous mapping of M into M which carries M_0 into M_0. Let Φ be a contractive gauge function as in Definition 1. For any pair $\{x, y\}$ in M, let $O(f, x, y)$ be the orbit of the pair under f, i.e.*

$$O(f, x, y) = \bigcup_{j\geq 0} \{f^j x\} \cup \{f^j y\}.$$

Suppose that for each x in M_0, there exists a positive integer $n(x)$ such that for any pair x, y in M_0,

$$\operatorname{diam}\left(O(f, f^{n(x)+n(y)}(x), f^{n(x)+n(y)}(y))\right) \leq \Phi(\operatorname{diam} O(f, x, y))).$$

Then for each x in M_0, $f^j x$ converges to a fixed point of f in M, and this fixed point is independent of the choice of x in M_0.

In Erratum to this paper in 1981, Browder [5] stated: "In my paper [4], the results as stated are not valid without the additional assumption that all the orbits are bounded (as an assumption which is applied implicitly throughout the proofs). With this additional assumption, the corrected theorems are indeed valid. However, in terms of the original intention of the paper to include all the principal results in the catalogue of contractive fixed point theorems of Rhoades [17], this requires stronger hypotheses to ensure the orbit boundedness from contractivity-type hypotheses. A detailed treatment of such results (as well as of the possibility of dropping continuity hypotheses) is given in the paper by W. Walter [20]."

However, only a few quoted his theorem.

3. Hegedüs and Szilágyi [7] in 1980

Independently to Browder's work, the following appeared:

Theorem HS. *Let X be a complete metric space, and f a selfmap of X such that* $\operatorname{diam}(O(x)) < \infty$ *for any $x \in X$. Suppose that for any x, $y \in X$,*

(HS) *for any $\varepsilon > 0$, there exist a ε_0 and a $\delta > 0$ such that $0 < \varepsilon_0 < \varepsilon$ and*

$$\varepsilon \leq \operatorname{diam}(O(x) < \varepsilon + \delta \quad \textit{implies} \quad d(fx, fy) \leq \varepsilon_0.$$

Then f has a unique fixed point $p \in X$ and $f^n x$ converges to p for each $x \in X$.

For a non-continuous selfmap, this theorem is one of the sharpest result.

4. Kasahara [10] in 1980

Kasahara obtained the following (formulated by Park [15]):

Proposition 2. *Let X be a complete metric space, Y a subset of X, f a continuous selfmap of X such that $fY \subset Y$ and* $\operatorname{diam}(O(x)) < \infty$ *for any $x \in Y$. Suppose that for each $x \in Y$, there exists a positive integer $\sharp x$ such that*

(v) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \operatorname{diam}(O(x)) < \varepsilon + \delta \textit{ implies } \operatorname{diam}(O(f^{\sharp x}x)) < \varepsilon.$$

Then f has a fixed point in $\bigcap_{n\geq 0} \overline{O(f^n x)}$ for each $x \in Y$.

Proposition 3. *Let X be a complete metric space, Y a subset of X, f a continuous selfmap of X such that $fY \subset Y$ and* $\operatorname{diam}(O(x)) < \infty$ *for any $x \in Y$. Suppose that* *for each $x \in Y$, there exists a positive integer $\sharp x$ such that for any pair $x,\ y \in Y$, the following condition holds:*

(vi) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \operatorname{diam}(O(x,y)) < \varepsilon + \delta \text{ implies } \operatorname{diam}(O(f^{\sharp x+\sharp y}x, f^{\sharp x+\sharp y}y)) < \varepsilon.$$

Then for each $x \in Y$, $\{f^n x\}$ converges to a fixed point of f in X, and this fixed point is independent of the choice of $x \in Y$.

Influenced by Hegedüs and Szilágyi [7], Kasahara noticed that his propositions extend the corresponding ones of Browder [4].

Note also that Browder had to assume that each orbit in Y be bounded in his Propositions 2 and 3. In Erratum [5] to his paper, Browder in 1981 agreed this fact.

5. Park [14] in 1980

After the comparative study of Billy Rhoades on contractive conditions [17], there had appeared wider classes of mappings of the form

$$d(fx, fy) < \operatorname{diam}(O(x) \cup O(y)),$$

where f is a selfmap of a metric space (X,d). A point $x \in X$ is said to be regular for f if $\operatorname{diam} O(x) < \infty$.

Given $x,\ y \in X$, let

$$m(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)\} \text{ and}$$

$$\delta(x,y) = \operatorname{diam}\{O(x) \cup O(y)\}.$$

In order to update Rhoades' program, we made a classification of contractive type conditions in [14]. The following is a part of them.

(C) Given $\varepsilon > 0$, there exist $\epsilon_0 < \epsilon$ and $\delta_0 > 0$ such that for any $x,\ y \in X$,

(Cd) $\varepsilon \leq d(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$.
(Cm) $\varepsilon \leq m(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$.
(Cδ) $\varepsilon \leq \delta(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$. (Hegedüs-Szilágyi [7]).

(D) There exists a nondecreasing right continuous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ such that $\phi(t) < t$ for $t > 0$ and, for any $x,\ y \in X$,

(Dd) $d(fx,fy) \leq \phi(d(x,y))$.
(Dm) $d(fx,fy) \leq \phi(m(x,y))$.
(Dδ) $d(fx,fy) \leq \phi(\delta(x,y))$ if $x,\ y$ are regular. (Hegedüs-Szilágyi)

Now we give a theorem for a map satisfying condition (Cδ).

Theorem 2(Cδ). *Let f be a selfmap of a metric space X. Suppose there exists a regular point $u \in X$ such that*
(1) *$O(u)$ has a regular cluster point $p \in X$, and*
(2) *the condition (Cδ) holds on $O(u) \cup O(p)$.*

Then f has a unique fixed point p in $\overline{O(u)}$ and $f^n u \longrightarrow p$.

In [14], a large number of consequences of Theorem 2(Cδ) were given.

6. Walter [20] in 1981

"Throughout this note, X denotes a complete metric space with distance function d, and T is a map from X into itself. Powers of T are defined by $T^0x = x$ and $T^{n+l}x = T(T^n x),\ n \geq 0$. Occasionally, we use the notation $x^k = T^k x$, in particular $x^0 = x,\ x^1 = Tx$, for the sake of brevity. $\cdots$ The letter ϕ denotes a contractive gauge function, i.e. a continuous, increasing function from $\mathbb{R}_+$, the non-negative reals, into $\mathbb{R}_+$ which satisfies $\phi(s) < s$ for $s > 0$."

" We consider two conclusions:

(FA) T has one and only one fixed point $z \in X$.

(SA) The successive approximations converge, i.e. there exists $z \in X$ such that $d(T^k x, z) \longrightarrow 0$ as $k \longrightarrow \infty$ for any $x \in X$.

Let us have

(C4) $\quad d(Tx, Ty) \leq \phi(\operatorname{diam} O(x, y))$ for $x,\ y \in X$.

Theorem 2. (C4) implies (FP, SA), if all orbits are bounded."

Instead of the above, we borrow the following from Kirk-Saliga [11]:

"We state Walter's result below. (The underlying ideas are those of Browder [4].) In this theorem, ϕ denotes a contractive gauge function on a metric space M. This means $\phi : R^+ \longrightarrow R^+$ is continuous, nondecreasing, and satisfies $\phi(s) < s$ for $s > 0$. "

Theorem 4.2. *Let M be a complete metric space and suppose $T : M \longrightarrow M$ has bounded orbits and satisfies the following condition. For each $x,\ y \in M$,*

$$d(T(x), T(y)) \leq \phi(\operatorname{diam}(O(x, y))).$$

Then T has a unique fixed point $z \in M$ and $\lim_{k \longrightarrow \infty} T^k(x) = z$ for each $x \in M$.

Comments: Walter showed that Browder's sophisticated contractive type condition can be replaced by rather concise forms. However, Walter's contractive gauge functions are still partial to those in our next paper. In fact, in terms of Park [14] in 1980, Walter adopted the contractive type condition more strict than the type (D), and Park used the conditions of the type (C) more general than (D). For the comparison of those conditions, see [10] and [14].

7. Park [15] in 1981

Abstract: F.E. Browder [4] posed a fixed point theorem of great generality and complexity such that a large part of literature on contractive type of maps can be subsumed under an intuitive and simple mode of argument. In this paper, we present sharper forms of such fixed point theorems which show that Browder's result can be stated in more general setting and include much more detailed cases than his.

Theorem 1. *Let X be a complete metric space, Y a subset of X, and f a selfmap of X such that $fY \subset Y$ and $\operatorname{diam}(O(x)) < \infty$ for all $x \in Y$. Suppose that for each $x \in Y$ there exists a positive integer $\sharp x$ such that the following condition holds:*

(vii) *for any $\varepsilon > 0$ there exists a $\delta > 0$ and an ε_0 with $0 < \varepsilon_0 < \varepsilon$ such that*

$$x,\ y \in Y,\ \varepsilon \leq \operatorname{diam}(O(x, y)) < \varepsilon + \delta \text{ imply } d(f^m x, f^n y) \leq \varepsilon_0$$

for each $m \geq \sharp x, n \geq \sharp y$.

If f is continuous, then f has a fixed point p in X such that for each $x \in Y$, $\{f^n x\}$ converges to p in X as $n \longrightarrow \infty$.

Theorem 1 is a direct consequences of Proposition 3 in [10], [15] since (vii) implies (vi) there, and extends the main result of [4].

A number of variations of Theorems 1 and HS are possible; see [9].

For non-complete metric spaces Theorems 1 and HS are easily extended as follows:

Theorem 2. *Let f be a selfmap of a metric space X. Suppose that there exists a point $u \in X$ with $\mathrm{diam} O(u) < \infty$ such that*

(1) *$O(u)$ has a cluster point $p \in X$ with $\mathrm{diam} O(p) < \infty$,*
(2) *f is (orbitally) continuous at p, and*
(3) *$Y = O(u, p)$ satisfies the conditions in Theorem 1.*

Then f has a unique fixed point p in $\overline{O(u)}$ and $f^n u \longrightarrow p$ as $n \longrightarrow \infty$.

Then Theorem 2(Cδ) in Park [14] is listed as Theorem 4.

8. Kirk and Saliga [11] in 2000

"Motivated by Browder's elegant unification of numerous diverse contractive conditions, Walter [20] proved a far-reaching extension of Banach's theorem. We use this fact to show that Theorem 4.1 extends to a much wider class of mappings under the additional assumption that the orbits of T are bounded.

Using this fact we obtain the following. (It seems that the condition initially appeared in a paper of Hegedüs [6].)"

Theorem 4.3. *Let M be a complete metric space and suppose $T : M \longrightarrow M$ has bounded orbits and satisfies: there exists $\alpha < 1$ such that for each $x,\ y \in M$,*

$$d(T(x), T(y)) \leq \alpha \,\mathrm{diam}(O(x, y)) \text{ for all } x,\ y \in M.$$

Suppose $\{x_n\} \subset M$ satisfies $\lim_n d(x_n, T(x_n)) = 0$. Then T has a unique fixed point $z \in M$, and $\lim_{c \longrightarrow 0^+} \mathrm{diam}(L_c) = 0$. Moreover, $\lim_n d(x_n, T(x_n)) = 0$ if and only if $\lim_n x_n = z$.

Here, $L_c := \{x \in M : F(x) \leq c\}$ for all $c \geq 0$, where $F(x) := d(x, Tx)$ for every $x \in M$. It was noticed in [3] that F is an r.g.i. on M. We recall (see [2]) that a function $G : M \longrightarrow R$ is said to be a regular-global-inf (r.g.i.) at $x \in M$ if $G(x) > \inf_M(G)$ implies there exist $\varepsilon > 0$ such that $\varepsilon < G(x) - \inf_M(G)$ and a neighborhood N_x of x such that $G(y) > G(x) - \varepsilon$ for each $y \in N_x$. If this condition holds for each $x \in M$, then G is said to be an r.g.i. on M:

To prove Theorem 4.3, the authors have used the preceding result of Walter [20].

In the final Section: By taking $y = T(x)$, one has

$$d(T(x), T^2(x)) \leq \alpha \,\mathrm{diam}(O(x, Tx))) = \alpha \,\mathrm{diam}(O(x)) \text{ for all } x \in M.$$

and this quickly leads to

$$\mathrm{diam}(O(Tx))) = \alpha \,\mathrm{diam}(O(x)) \text{ for all } x \in M.$$

This can be rewritten as

$$\mathrm{diam}(O(x)) \leq (1 - \alpha)^{-1}[\mathrm{diam}(O(x)) - \mathrm{diam}(O(Tx)))] \text{ for all } x \in M.$$

Since $d(x, T(x)) \leq \mathrm{diam}(O(x))$, if the mapping $\varphi : M \longrightarrow R$ defined by setting $\varphi(x) = \mathrm{diam}(O(x))$ is lower semicontinuous, then this condition, much weaker than the one in Theorem 4.3, assures that T has a fixed point by the Caristi fixed point theorem.

9. Akkouchi [1] in 2001

Abstract: We prove that the conclusion of a result of Kirk and Saliga [Theorem 4.3] remain valid for a wide class of contractive gauge functions.

2.1. Let Φ be the set of continuous functions $\phi : R^+ \longrightarrow R^+$ such that ϕ is nondecreasing on R^+ and such that the mapping $x \mapsto x - \phi(x)$ from $[0, +\infty[$ onto $[0, +\infty[$ is strictly increasing. We notice that each element ϕ in Φ is a gauge function and that Φ_1 is strictly contained in Φ. Indeed, we can give examples of elements in $\Phi \setminus \Phi_1$.

The following theorem is the main result of this short communication.

Theorem C. *Let (M, d) be a complete metric space and suppose $T : M \longrightarrow M$ has bounded orbits and satisfies the following condition:*

$$d(Tx, Ty) \leq \phi(\text{diam}(O(x, y))) \ \textit{for all} \ x, y \in M,$$

where $\phi \in \Phi$. Then

(1) *T has a unique fixed point $z \in M$, and* $\lim_{k \longrightarrow +\infty} T^k(x) = z$ *for each* $x \in M$.
(2) $\lim_{c \longrightarrow 0^+} \text{diam}(L_c) = 0$.
(3) *For each sequence* $\{x_n\} \subset M$, $\lim_n d(x_n, Tx_n) = 0$ *if and only if* $\lim_n x_n = z$.
(4) *The map $F : x \mapsto d(x, Tx)$ is an r.g.i. on M.*

Comments: According to our classification [15] in 1980 or Section 5, the contractive condition belongs to (Dδ). Hence it may be extended to (Cδ).

10. Jachymski [8] in 2001

In 1927, Knaster proved a fixed point theorem for increasing — under set-inclusion — mappings, on and to the family of all subsets of a set. In 1939 Tarski extended Knaster's result to increasing mappings on a complete lattice and he gave its applications in set theory and topology, but his result was unpublished until 1955.

"In the sequel we shall show how the Knaster-Tarski fixed point theorem can be used to derive some results of metric fixed point theory. We start with Amann's [A] proof (see also Zeidler [Z, p. 512]) of a fixed point theorem for the so-called diametric contractions. In fact, we shall extend his argument by considering a more general class of mappings: A selfmap f of a bounded metric space is said to be a *diametric φ-contraction* if there is a non-decreasing function $\varphi : R_+ \longrightarrow R_+ (R_+$ denotes the set of all nonnegative reals) such that $\lim_{n \longrightarrow \infty} \varphi^n(t) = 0$ for all $t > 0$ (see Matkowski [M] or Dugundji and Granas [DG, p. 12]) and

$$\text{diam} f(A) \leq \varphi(\text{diam}\, A)$$

for all nonempty, closed and f-invariant subsets A of X.

A more realistic special case here is the following *Walter's contraction* [20], i.e., a mapping f which satisfies the inequality

$$d(fx, fy) \leq \varphi(\text{diam}(O_f(x) \cup O_f(y))) \text{ for all } x,\ y \in X,$$

where $O_f(x) := \{f^{n-1}x : n \in N\}$ is an *orbit* of f at a point x and the function φ is nondecreasing, right continuous and $\varphi(t) < t$ for $t > 0$ (then $\lim_{n \longrightarrow \infty} \varphi^n(t) = 0$; cf. Browder [4]). That such an f satisfies (2.1), follows immediately from the fact that $x \in A$ implies $O_f(x) \subset A$ if A is f-invariant.

Theorem 2.3 *Let (X, d) be a complete bounded metric space and f be a diametric φ-contraction on X. Then f has a unique fixed point.*

Comments: For the references [A, Z, M, DG], see [8]. Jachymski gave a very lengthy proof and some related results. He showed that the Knaster-Tarski fixed point theorem also yields some extensions of the Contraction Principle for mappings on uniform spaces, given by K.-K. Tan, Taraftar, D.H. Tan and Angelov.

11. Proinov [16] in 2006

Proinov unified the Boyd-Wong [BW], Jachymski [J], Matkowski [M] and Meir-Keeler [MK] type contractions (for such references, see [16]) and proved the following interesting generalization of the Banach contraction principle.

Theorem. [16] *Let (E, ρ) be a complete metric space and $f : E \longrightarrow E$ a continuous and asymptotically regular mapping such that*

(1) $\rho(f(x), f(y)) \leq \psi(L(x, y))$ *for all* $x,\ y \in E$,
(2) $\rho(f(x), f(y)) < L(x, y)$ *for all* $x,\ y \in E$ *whenever* $L(x, y) \neq 0$;
where $\mu \geq 0,\ L(x, y) = \rho(x, y) + \mu[\rho(x, f(x)) + \rho(y, f(y))]$, *and* $\psi : [0, \infty) \longrightarrow [0, \infty)$ *is such that for each* $\xi > 0$ *there exists a* $\delta > \xi$ *such that* $\xi < t < \delta$ *implies* $\psi(t) \leq \xi$.

Then f admits a unique fixed point in E. Moreover, if $\mu = 1$ and ψ is continuous with $\psi(t) < t$ for all $t > 0$ then continuity of f is not needed.

Comments: Proinov's theorem seems to be very close to the works of Hegedӧs-Szilágyi [7] with our conditions (Cδ) and (Dδ) in Section 5.

12. Jachymski [9] in 2009

Abstract: We revisit a fixed point theorem for contractions established by Felix Browder in 1968. We show that many definitions of contractive mappings which appeared in the literature after 1968 turn out to be equivalent formulations or even particular cases of Browder's definition. We also discuss the problem of the existence of approximate fixed points of continuous mappings; in particular, we settle it in the affirmative for Browder contractions. Finally, we recall three problems concerning Browder contractions which remain unsolved.

In Section 2 the author is interested in existence of approximate fixed points.

Some further extensions of the Banach Principle and their connections with Browder's theorem is discussed in Section 3. A novelty here is Theorem 13 by means of which a recent result of Branciari for mappings satisfying a contractive condition of integral type can be subsumed under Browder's theorem.

The author close the paper with three questions concerning Browder contractions. They deal with a set-valued version of Browder's theorem (Question 1), stability of successive approximations (Question 2), and continuous dependence of fixed points on parameters (Question 3).

13. Tasković [T] in 2009

Abstract: We prove that a result of Kirk and Saliga [11, Theorem 4.2., p.149] has been for the first time proved before 25 years in Tasković [18, Theorem 1, p.250]. But the authors neglected and ignored this historical fact.

From Introduction: In recent years a great number of papers have presented generalizations of the well-known Banach-Picard contraction principle. Recently, Kirk and Saliga have proved the following statement (see [11, Theorem 4.2., p.149]).

Theorem 1. (Kirk-Saliga [11], Walter [20]) *Let (X, ρ) be a complete metric space and suppose $T : X \longrightarrow X$ has bounded orbits and satisfies the following condition:*

$$\rho[Tx, Ty] \leq \Phi(\mathrm{diam}\{x, y, Tx, Ty, T^2x, T^2y, \dots\})$$

for all $x,\ y \in X$, where $\Phi : \mathbb{R}^0_+ \longrightarrow \mathbb{R}^0_+ := [0, +\infty)$ is a continuous non-decreasing function and satisfies $\Phi(t) < t$ for every $t > 0$. Then T has a unique fixed point $\zeta \in X$ and $\{T^n(a)\}_{n\in\mathbb{N}}$ converges to ζ for every $a \in X$.

In connection with this, in 1980 I have proved the following result of fixed point on metric spaces which has a best long of all known sufficiently conditions (linear and nonlinear) for the existing unique fixed point.

Theorem 2. (Tasković [18]) *Let T be a mapping of a metric space (X, ρ) into itself and let X be T-orbitally complete. Suppose that there exists a function $\varphi : \mathbb{R}^0_+ \longrightarrow \mathbb{R}^0_+ := [0, +\infty)$ satisfying*

$$(\forall t \in \mathbb{R}_+ := (0, +\infty))(\varphi(t) < t \text{ and } \lim sup_{z\longrightarrow t+0}\varphi(z) < t)$$

such that

$$\rho[Tx, Ty] \leq \varphi(\mathrm{diam}\{x, y, Tx, Ty, T^2x, T^2y, \dots\}$$

and $\mathrm{diam}\, O(x) \in \mathbb{R}^0_+$ *for all $x,\ y \in X$.*

Then T has a unique fixed point $\zeta \in X$ and $\{T^n(a)\}_{n\in\mathbb{N}}$ converges to ζ for every $a \in X$.

We notice that Theorem 1 is a very special case of Theorem 2.

14. Kumam, Dung, and Sitytithakerngkiet [12] in 2015

Abstract: We state and prove a generalization of Ćirić fixed point theorems in metric space by using a new generalized quasi-contractive map. These theorems extend other well known fundamental metrical fixed point theorems in the literature (Banach, Kannan, Nadler, Reich, etc.) Moreover, a multi-valued version for generalized quasi-contraction is also established.

Definition 2.4. Let $T : X \longrightarrow X$ be a mapping on metric space X. The mapping T is said to be a *generalized quasi-contraction* iff there exists $q \in [0, 1)$ such that for all $x,\ y \in X$,

$$\begin{aligned} d(Tx, Ty) \leq q \max\{&d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx),\\ &d(x, T^2x), d(Tx, T^2x), d(x, T^2y), d(T^2x, Ty)\end{aligned}$$

Some corollaries and multi-valued versions of them are added.

Comments: The contractive condition implies the Hegedüs condition $d(Tx, Ty) \leq q\, \mathrm{diam}\, \{O_T(x) \cup O_T(y)\}$. Hence the main theorem was already known.

15. Pant and Khantwal [13] in 2023

"We present some new existence results for single and multivalued mappings in metric spaces on very general settings. Some illustrative examples are presented to validate our theorems."

Theorem 2.2. *Suppose (E, ρ) is a metric space. Let $f : E \longrightarrow E$ a mapping such that for some $v_0 \in E$,*

$$\frac{1}{2}\rho(x, f(x)) \leq \rho(x, y) \Longrightarrow \rho(f(x), f(y)) \leq \psi(N(x, y))$$

for all $x,\ y \in \overline{O(v_0, f)}$ *with* $x \neq y$*, where*

$$N(x,y) = \max\{\rho(x,y), \rho(x, f(x)), \rho(y, f(y)), \frac{1}{2}[\rho(y, f(x)) + \rho(x, f(y))]\}.$$

If E *is* f*-orbitally complete then the sequence of iterations* $(f^n(v_0))$ *is Cauchy in* E *and converges to the unique fixed point of* f *in* $\overline{O(v_0, f)}$*.*

The authors added six Corollaries of the same nature.

16. Epilogue

All story in this paper begins from Browder, but no one quotes his theorem (except Jachymski [9] and the present one) and no one practically uses his theorem. However, certain influences of his work still remain from time to time.

In the below, $\Longrightarrow$ means certain influence. Then we have

$$\text{Browder} \Longrightarrow \text{Walter} \Longrightarrow \text{Kirk-Saliga} \Longrightarrow \text{Akkouchi.}$$

$$\text{Hegedüs-Szylági} \Longrightarrow \text{Kasahara} \Longrightarrow \text{Park.}$$

$$\text{Walter} \Longrightarrow \text{Jachymski.}$$

$$\text{Browder} \Longrightarrow \text{Jachymski.}$$

The first and the third rows seem to be obsolete.

Especially, Kirk-Saliga mentioned Hegedüs, but not followed him. Art Kirk was a long time friend of the writer. Jachymski [8] did not mention Hegedüs-Szylági's work. Kasahara made friendship with the writer just before he passed away. The writer met Browder a long time ago at the conferences at Berkeley and Marseilles.

Almost all works mentioned in this paper are obsolete. Nowadays researchers are adopting new terminology like as weaker spaces than metric ones, orbital completeness, orbital continuity of maps, etc.

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BEST PROXIMITY POINT AND FIXED POINT THEOREMS IN COMPLEX VALUED RECTANGULAR b-METRIC SPACES

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ABSTRACT. The aim of this paper, by using the concept of continuity of $\phi : [0,\infty)^2 \to [0,\infty)^2$ which satisfying $\phi(t) \prec t$ and $\phi(0) = 0$ to defined some contraction condition of T introduced by G. Meena [12], we prove the unique best proximity point of A and fixed point of T in complex valued rectangular b-metric space. Our results extend and improve the results of G. Meena [12], and many others.

KEYWORDS: best proximity point, rectangular b-metric spaces, rectangular complex valued b-metric spaces.
AMS Subject Classification: : 46C05, 47D03, 47H09, 47H10, 47H20.

1. Introduction

Fixed Points theorems in metric spaces was introduceed in 1906 [8] by Fréchet. After that, many mathematicians studied and proved the existence theorem of fixed points for use the Banach contraction principle in metric spaces and every generalized metric spaces [6, 7, 13] and [15].

The notion of b-metric spaces was introduced In 1989 by Bakhtin [3]. After, many mathematicians extended the fixed point theorems from metric spaces to b-metric spaces, for example in [1, 2]

In 2000, A. Branciari [5], he give a fixed point theorem related to the contraction mapping principle of Banach and Caccioppoli; here we have considered generalized metric spaces, that is metric spaces with the triangular inequality replaced by similar ones which involve four or more points instead of three.

In 2011, A. Azam, B. Fisher and M. Khan [2] defined the definition of notion of complex valued metric spaces and prove the common fixed point theorems in complex valued metric spaces of a pair of mappings satisfying a contractive condition.

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In the same year, S. Bhatt, S. Chaukiyal and R. C. Dimri [4] proved a common fixed point theorem for weakly compatible maps in complex valued metric spaces without using the notion of continuity.

In 2015, O. Ege [6], introduce complex valued rectangular b-metric spaces. We prove an analogue of Banach contraction principle and prove a different contraction principle with a new condition and a fixed point theorem in this space.

In 2018, G. Meena [12], introduced the best proximity points for non-self mappings between two subsets in the setting of complex valued rectangular metric spaces by using the concept of P-property.

The aim of this paper, we introdued [6, 12] we study and suppose some contractive condition and proved the best proximity point result in b-metric space. Therefore, our results are comprehensive the results if [10].

2. Preliminaries

In this work, we let X be a nonempty set and we recalled some definitions and lemmas for using in section 3.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a metric space, and d is called a metric on X.

Next, we suppose the definition of b-metric space, this space is generalized than metric spaces.

Definition 2.2. [3] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

The following is some example for b-metric spaces.

Example 2.3. [3] Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

In 2000, A. Branciari [5] present the notion of rectangular metric space as follows.

Definition 2.4. [5] Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is called a rectangular metric on X if for any $x, y \in X$ and all distinct points $u, v \in X - \{x, y\}$, it satisfies the following conditions:
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, u) + d(u, v) + d(v, y)$.
In this case, the pair (X, d) is called a **rectangular metric space**.

There is a completeness property in real number but on order relation is not welll-defined in complex numbers. Before giving the definition of complex valued metric spaces and complex-valued b-metric spaces, we define partial order in complex numbers (see [11]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order relation $\precsim$ on $\mathbb{C}$ as follows;

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

This means that we would have $z_1 \precsim z_2$ if and only if one of the following conditions holds:
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
If one of the conditions $(ii), (iii)$, and (iv) holds, then we write $z_1 \prec z_2$. From the above partial order relation we have the following remark.

Remark 2.5. We can easily check the following:
(i) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \precsim z_2$ then $az_1 \precsim bz_2, \forall z_1, z_2 \in \mathbb{C}$.
(ii) If $0 \precsim z_1 \prec z_2$ then $|z_1| < |z_2|$.
(iii) If $z_1 \precsim z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$.
(iv) If $z \in \mathbb{C}$, for $a, b \in \mathbb{R}$ and $a \leq b$, then $az \precsim bz$.

A b-metric on a b-metric sapce is a funcion having real value. Based on the definition of partial order on complex number, real-valued b-metric can be generalized into compleex-valued b-metric as folllows.

Definition 2.6. [2] Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \precsim d(x, y) + d(y, z)$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Next, we give the definition of complex valued b-metric space.

Definition 2.7. [13] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \precsim d(x, y)$
(ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, z) \precsim s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is complex valued metric space which is defined in Definition 2.6. The following example is some example of complex valued b-metric space.

Example 2.8. [13] Let $X = \mathbb{C}$. Define the mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbb{C}, d)$ is complex valued b-metriic space with $s = 2$.

From A. Branciari [5] and [13] we can define the notion of rectangular b-metric space as follows.

Definition 2.9. [6] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued rectangular b-metric on X if for any $x, y \in X$ and all distinct points $u, v \in X - \{x, y\}$, it satisfies the following conditions:
(i) $0 \preccurlyeq d(x, y)$
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, z) \preccurlyeq s[d(x, u) + d(u, v) + d(v, y)]$.
In this case, the pair (X, d) is called a **complex valued rectangular b-metric space**.

Example 2.10. [6] Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \mathbb{Z}^+$ and $d : X \times X \to \mathbb{C}$ defined as follows:

$$g(x, y) = d(y, x)$$

for all $x, y \in X$ and

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2t & \text{if } x, y \in A \\ \frac{t}{2n} & \text{if } x \in A \text{ and } y \in \{2, 3\} \\ t & \text{otherwise,} \end{cases}$$

where $t > 0$ is a constant. Then $X, d)$ is a complex valued rectangular b-metric space with coefficient $s = 2 > 1$.

Definition 2.11. [6] Let (X, d) be a complex valued rectangular b-metric space.
(i) A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

(ii) A point $x \in X$ is called limit point of a set A if for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - x) \neq \emptyset$
(iii) A subset $A \subseteq X$ is open if each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is closed if each limit point of A is contained in A.

Definition 2.12. [6] Let (X, d) be complex valued rectangular b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.
(i) The sequence $\{x_n\}$ is converges to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x) \prec r$. Thus x is the limit of (x_n) and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence if for ever $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.
(iii) If for every Cauchy sequence in$X$ is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 2.13. [6] *Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.14. [6] *Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Definition 2.15. [12] The max function for complex numbers with partial order relation $\preccurlyeq$ is defined as
(i) $\max\{z_1, z_2\} = z_2 \Rightarrow z_1 \preccurlyeq z_2$;

(ii) $z_1 \preccurlyeq \max\{z_1, z_2\} \Rightarrow z_1 \preccurlyeq z_2$ or $z_1 \preccurlyeq z_3$.
On the similar lines Singh et al. [14] defined min function as
(i) $\min\{z_1, z_2\} = z_1 \Rightarrow z_1 \preccurlyeq z_2$;
(ii) $\min\{z_1, z_2\} \preccurlyeq z_3 \Rightarrow z_1 \preccurlyeq z_3$ or $z_2 \preccurlyeq z_3$. Now we introduce the best proximity point and some related concept in complex valued rectangular metric space.

Definition 2.16. [16] Let A and B be two nonempty bounded subsets of a complex valued rectangular b-metric space (X, d). Then $\{d(x, y) : x \in A, y \in B\}$ is always bounded below by $z_0 = 0 + 0i$ and hence $\inf\{d(x, y) : x \in A, y \in B\}$ exists. Here we define

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$
$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

From the above definition, it is clear that for every $x \in A_0$ there exists $y \in B_0$ such that $d(x, y) = d(A, B)$ and conversely, for every $y \in B_0$ there exists $x \in A_0$ such that $d(x, y) = d(A, B)$.

Definition 2.17. [16] Let A and B be two nonempty bounded subsets of a complex valued rectangular b-metric space (X, d) and $T : A \to B$ be a non-self-mapping. A point $x \in A$ is called a best proximity point of T if $d(x, Tx) = d(A, B)$.

The definition of P-property was introduced in [17]. Now we define them in complex valued rectangular b-metric space.

Definition 2.18. [17] Let A and B be two nonempty subsets of a complex valued rectangular b-metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ such that

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

3. Main Results

In this section, we consider the context of Matkowski [9] the function $\phi : [0, \infty)^2 \to [0, \infty)^2$ such that $\phi(t) \prec t$ and $\phi(0) = 0$[where $t = (t_1, t_2) \in [0, \infty)^2$]. We denote Φ the family of function of ϕ.

Theorem 3.1. *Let A and B be two nonempty bounded subsets of a complete complex valued rectangular b-metric space (X, d) with a pair (A, B) satisfies the P-property. Let a continuous mapping $T : A \to B$ with $T(A_0) \subset B_0$, where A_0 is nonempty, if there exists $L > 0$ and a continuous $\phi \in \Phi$, such that*

$$d(Tx, Ty) \preccurlyeq k\phi\Big(\max\Big\{\frac{(d(x,Ty)-d(A,B))(d(y,Tx)-d(A,B))(d(x,Tx)+d(y,Ty)-2d(A,B))}{1+d(x,y)}, d(x, y)\Big\}\Big)$$
$$+L\min\Big\{(d(x, Tx)-d(A, B)), (d(y, Ty)-d(A, B)), (d(x, Ty)-d(A, B)), (d(y, Tx)-d(A, B))\Big\} \tag{3.1}$$

for all $x, y \in X$, where $0 < k < \frac{1}{s} \leq 1$. Then T has a unique best proximity point in A.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subset B_0$ we have $Tx_0 \in B_0$ then there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Again $Tx_1 \in B_0$, then there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

By continuing this process we can form a sequence $\{x_n\}$ in A_0, with

$$d(x_{n+1}, Tx_n) = d(A, B), \ \forall n \in \mathbb{N}.$$

From a pair (A, B) satisfying P-property, we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0-1} = x_{n_0}$, then we have

$$d(x_{n_0}, Tx_{n_0-1}) = d(A, B) = d(x_{n_0-1}, Tx_{n_0-1}). \tag{3.2}$$

This proof is complete.

Assume that $x_{n-1} \neq x_n$, for all $n \in \mathbb{N}$. We replace $x = x_{n-1}$ and $y = x_n$ in (3.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preccurlyeq k\phi\Big(\max\Big\{ \tfrac{(d(x_{n-1},Tx_n)-d(A,B))(d(x_n,Tx_{n-1})-d(A,B))(d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)-2d(A,B))}{1+d(x_{n-1},x_n)}, \\ &\qquad d(x_{n-1}, x_n)\Big\}\Big) \\ &\quad + L\min\Big\{(d(x_{n-1}, Tx_{n-1}) - d(A, B)), (d(x_n, Tx_n) - d(A, B)), \\ &\qquad (d(x_{n-1}, Tx_n) - d(A, B)), (d(x_n, Tx_{n-1}) - d(A, B))\Big\}. \end{aligned}$$

It follows that,

$$d(x_n, x_{n+1}) \preccurlyeq k\phi\left(d(x_{n-1}, x_n)\right).$$

From the definition of ϕ we have

$$d(x_n, x_{n+1}) \preccurlyeq kd(x_{n-1}, x_n).$$

It follows that

$$d(x_n, x_{n+1}) \preccurlyeq kd(x_{n-1}, x_n) \preccurlyeq k^2 d(x_{n-2}, x_{n-1}) \preccurlyeq \cdots \preccurlyeq k^n d(x_0, x_1).$$

For any $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\preccurlyeq s[d(x_n, x_{n-1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\preccurlyeq sd(x_n, x_{n-1}) + sd(x_{n+1}, x_{n+2}) + s^2[d(x_{n+2}, x_{n+3}) \\ &\quad + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_m)] \\ &\preccurlyeq sd(x_n, x_{n-1}) + sd(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) \\ &\quad + s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) + d(x_{n+6}, x_m)] \\ &\preccurlyeq sd(x_n, x_{n-1}) + sd(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) \\ &\quad + s^3 d(x_{n+4}, x_{n+5}) + s^3 d(x_{n+5}, x_{n+6}) + \cdots \\ &\quad + s^{\frac{(m-n-1)}{2}}[d(x_{n+(m-n-3)}, x_{n+(m-n-2)}) + d(x_{n+(m-n-2)}, x_{n+(m-n-1)} \\ &\quad + d(x_{n+(m-n-1)}, x_m)] \\ &\preccurlyeq \Big[sk^n + sk^{n+1} + s^2k^{n+2} + s^2k^{n+3} + s^3k^{n+4} + s^3k^{n+5} + \cdots \\ &\quad + s^{\frac{(m-n-1)}{2}} k^{m-1}\Big] d(x_0, x_1) \\ &\preccurlyeq \Big[(sk)^n + (sk)^{n+1} + (sk)^{n+2} + (sk)^{n+3} + (sk)^{n+4} + (sk)^{n+5} + \cdots + \\ &\quad (sk)^{n+(m-n-1)}\Big] d(x_0, x_1) \\ &= (sk)^n[1 + (sk) + (sk)^2 + (sk)^3 + (sk)^4 + \cdots + (sk)^{m-n-1}] d(x_0, x_1) \\ &\preccurlyeq (sk)^n[1 + (sk) + (sk)^2 + (sk)^3 + (sk)^4 + \cdots] d(x_0, x_1) \\ &= \frac{(sk)^n}{1 - sk} d(x_0, x_1) \to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Thus $\{x_n\}$ is a cauchy sequence in A. Since X is complete, so there exists $u \in X$ such that $x_n \to u \in X$. Since A is closed subset of X, we have $u \in A$. Next, we show that u is a best proximity point in A. Using the rectangular b-metric, we get

$$\begin{aligned} d(u,Tu) &\precsim s[d(u,x_{n+1}) + d(x_{n+1},Tx_n) + d(Tx_n,Tu)] \\ &\precsim sd(u,x_{n+1}) + sd(x_{n+1},Tx_n) \end{aligned}$$
$$+sk\phi\Big(\max\Big\{\tfrac{(d(x_n,Tu)-d(A,B))(d(u,Tx_n)-d(A,B))(d(x_n,Tx_n)+d(u,Tu)-2d(A,B))}{1+d(x_n,u)}, d(x,y)\Big\}\Big)$$
$$+L\min\Big\{(d(x_n,Tx_n)-d(A,B)),(d(u,Tu)-d(A,B)),(d(x_n,Tu)-d(A,B)), (d(u,Tx_n)-d(A,B))\Big\}$$

From (3.3), taking $n \to \infty$, we get

$$d(u,Tu) \precsim d(A,B).$$

Since $u \in A, Tu \in B$ and the definition of $d(A,B)$, it follows that

$$d(u,Tu) = d(A,B).$$

Hence, u is the best proximity point of T.

Finally, we show that u is a unique best proximity point of T. Let $u^* \in A$ is another best proximity point of T. Then

$$d(u^*,Tu^*) = d(A,B).$$

Assume $u \neq u^*$, by using P-property, we have

$$d(u,u^*) = d(Tu,Tu^*)$$
$$\precsim k\phi\Big(\max\Big\{\tfrac{(d(u,Tu^*)-d(A,B))(d(u^*,Tu)-d(A,B))(d(u,Tu)+d(u^*,Tu^*)-2d(A,B))}{1+d(u,u^*)}, d(u,u^*)\Big\}\Big)$$
$$+L\min\Big\{(d(u,Tu)-d(A,B)),(d(u^*,Tu^*)-d(A,B)),(d(u,Tu^*)-d(A,B)), (d(u^*,Tu)-d(A,B))\Big\}$$
$$\begin{aligned} &\precsim k\phi(d(u,u^*)) \\ &\precsim kd(u,u^*). \end{aligned}$$

A contradiction. Hence, $d(u,u^*) = 0$ or $u = u^*$ is a unique best proximity point of T. □

From Theorem 3.1, we have the parallel result with the result of G. Meena [12], as follows.

Corollary 3.2. [12] *Let A and B be two nonempty bounded subsets of a complete complex valued rectangular metric space (X,d) with a pair (A,B) satisfies the P-property. Let a continuous mapping $T : A \to B$ with $T(A_0) \subset B_0$, where A_0 is nonempty, if there exists $L > 0$ and a continuous $\phi \in \Phi$, such that*

$$d(Tx,Ty) \precsim k\phi\Big(\max\Big\{\tfrac{(d(x,Ty)-d(A,B))(d(y,Tx)-d(A,B))(d(x,Tx)+d(y,Ty)-2d(A,B))}{1+d(x,y)}, d(x,y)\Big\}\Big)$$
$$+L\min\Big\{(d(x,Tx)-d(A,B)),(d(y,Ty)-d(A,B)),(d(x,Ty)-d(A,B)), (d(y,Tx)-d(A,B))\Big\},$$

for all $x,y \in X$, where $0 < k < 1$. Then T has a unique best proximity point in A.

Theorem 3.3. *Let (X,d) be a complete complex valued rectangular b-metric space. Let a mapping $T : X \to X$ and a continuous $\phi \in \Phi$, such that*

$$d(Tx,Ty) \precsim k\phi\left(\max\left\{\frac{d(x,Ty)d(y,Tx)(d(x,Tx)+d(y,Ty))}{1+d(x,y)}, d(x,y)\right\}\right)$$

$$+L\min\left\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right\},$$

for all $x, y \in X$, and k is any real number with $0 < k < \frac{1}{s} \leq 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$, from $T : X \to X$ there exists $x_1 \in X$ such that $x_1 = Tx_0$. From $x_1 \in X$ there exists $x_2 \in X$ such that $x_2 = Tx_1$. By the following method we have a sequence $\{x_n\} \subseteq X$ such that $x_{n+1} = Tx_n$. Consider,

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\preccurlyeq k\phi\Big(\max\Big\{\tfrac{d(x_{n-1},Tx_n)d(x_n,Tx_{n-1})(d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n))}{1+d(x_{n-1},x_n)},d(x_{n-1},x_n)\Big\}\Big) \\ &\quad +L\min\Big\{d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,Tx_{n-1}\Big\}. \end{aligned}$$

It follows that,

$$d(x_n,x_{n+1}) \preccurlyeq k\phi\left(d(x_{n-1},x_n)\right).$$

From the definition of ϕ we have

$$d(x_n,x_{n+1}) \preccurlyeq kd(x_{n-1},x_n).$$

It follows that

$$d(x_n,x_{n+1}) \preccurlyeq kd(x_{n-1},x_n) \preccurlyeq k^2d(x_{n-2},x_{n-1}) \preccurlyeq \cdots \preccurlyeq k^nd(x_0,x_1).$$

For any $m > n$, we have

$$\begin{aligned} d(x_n,x_m) &\preccurlyeq s[d(x_n,x_{n-1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_m)] \\ &\preccurlyeq sd(x_n,x_{n-1})+sd(x_{n+1},x_{n+2})+s^2[d(x_{n+2},x_{n+3})+d(x_{n+3},x_{n+4}) \\ &\quad +d(x_{n+4},x_m)] \\ &\preccurlyeq sd(x_n,x_{n-1})+sd(x_{n+1},x_{n+2})+s^2d(x_{n+2},x_{n+3})+s^2d(x_{n+3},x_{n+4}) \\ &\quad +s^3[d(x_{n+4},x_{n+5})+d(x_{n+5},x_{n+6})+d(x_{n+6},x_m)] \\ &\preccurlyeq sd(x_n,x_{n-1})+sd(x_{n+1},x_{n+2})+s^2d(x_{n+2},x_{n+3})+s^2d(x_{n+3},x_{n+4}) \\ &\quad +s^3d(x_{n+4},x_{n+5})+s^3d(x_{n+5},x_{n+6})+\cdots \\ &\quad +s^{\frac{(m-n-1)}{2}}\Big[d(x_{n+(m-n-3)},x_{n+(m-n-2)})+d(x_{n+(m-n-2)},x_{n+(m-n-1)}) \\ &\quad +d(x_{n+(m-n-1)},x_m)\Big] \\ &\preccurlyeq \Big[sk^n+sk^{n+1}+s^2k^{n+2}+s^2k^{n+3}+s^3k^{n+4}+s^3k^{n+5}+\cdots \\ &\quad +s^{\frac{(m-n-1)}{2}}k^{m-1}\Big]d(x_0,x_1) \\ &\preccurlyeq [(sk)^n+(sk)^{n+1}+(sk)^{n+2}+(sk)^{n+3}+(sk)^{n+4}+(sk)^{n+5}+\cdots \\ &\quad +(sk)^{n+(m-n-1)}]d(x_0,x_1) \\ &= (sk)^n[1+(sk)+(sk)^2+(sk)^3+(sk)^4+\cdots+(sk)^{m-n-1}]d(x_0,x_1) \\ &\preccurlyeq (sk)^n[1+(sk)+(sk)^2+(sk)^3+(sk)^4+\cdots]d(x_0,x_1) \\ &= \frac{(sk)^n}{1-sk}d(x_0,x_1) \to 0 \text{ as } m,n\to\infty. \end{aligned}$$

Thus $\{x_n\}$ is a cauchy sequence in A. Since X is complete, so there exists $u \in X$ such that $x_n \to u \in X$. Since A is closed subset of X, we have $u \in A$. Next, we show that u is a fixed point of T. Using the rectangular b-metric, we get

$$\begin{aligned} d(u,Tu) &\preccurlyeq s[d(u,x_{n+1})+d(x_{n+1},Tx_n)+d(Tx_n,Tu)] \\ &\preccurlyeq sd(u,x_{n+1})+sd(x_{n+1},Tx_n) \end{aligned}$$

$$+sk\phi\left(\max\left\{\frac{d(x_n,Tu)d(u,Tx_n)(d(x_n,Tx_n)+d(u,Tu))}{1+d(x_n,u)},d(x,y)\right\}\right)$$
$$+L\min\{d(x_n,Tx_n),d(u,Tu),d(x_n,Tu),d(u,Tx_n)\} \tag{3.3}$$

From (3.3), taking $n\to\infty$, we get

$$d(u,Tu)\precsim skd(u,Tu).$$

Hence, u is fixed point of T.

Finally, we show that u is a unique fixed point of T. Let $u^*\in A$ is anotherfixed point of T. Then $u^*=Tu^*$. Assume $u\neq u^*$, consider

$$\begin{aligned}
d(u,u^*) &= d(Tu,Tu^*)\\
&\precsim k\phi\left(\max\left\{\frac{d(u,Tu^*)d(u^*,Tu)(d(u,Tu)+d(u^*,Tu^*)}{1+d(u,u^*)},d(u,u^*)\right\}\right)\\
&\quad +L\min\{d(u,Tu),d(u^*,Tu^*),d(u,Tu^*),d(u^*,Tu)\}\\
&\precsim k\phi(d(u,u^*))\\
&\precsim kd(u,u^*).
\end{aligned}$$

A contradiction. Hence, $u=u^*$ is a unique fixed point of T. □

Corollary 3.4. [12] *Let (X,d) be a complete complex valued rectangular b-metric space. Let a mapping $T:X\to X$ and a continuous $\phi\in\Phi$, such that*

$$\begin{aligned}
d(Tx,Ty) &\precsim k\phi\left(\max\left\{\frac{d(x,Ty)d(y,Tx)(d(x,Tx)+d(y,Ty))}{1+d(x,y)},d(x,y)\right\}\right)\\
&\quad +L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\},
\end{aligned}$$

for all $x,y\in X$, and k is any real number with $0<k<\frac{1}{s}<1$. Then T has a unique fixed point in X.

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FIXED POINT THEOREM FOR ĆIRIĆ'S QUASICONTRACTION IN GENERALIZED COMPLEX VALUED METRIC SPACES

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ABSTRACT. In this work, we introduce some property of the generalized complex valued metric space and we extend some fixed point results thai is Ćirić's fixed point theorem. Some are recover various complex valued metric space and complex valued b-metric space. Our results extended and improve some results of Mohamed Jleli and Bessem Samet [17].

KEYWORDS: generalized complex valued metric space, Ćirić's quasicontraction.
AMS Subject Classification: 47H10; 54H25; 30L15

1. Introduction

The fixed points theorems has been studied by many mathematicians and fixed Points theorems in metric spaces was introduceed in 1906 [**?**] by Fréchet. In 1922, Banach [6], introduced a fixed point theorem in metric space for contraction mapping.

In recent years, many researcher proved the fixed points theorem in generalizations of metric spaces, see example [9, 3, 20] and references therein. The notion of dislocated metric spaces was introduced in 2000 by Hitzler and Seda [21], see [4]-[18] and references therein.

Very recently, A. Azam, B. Fisher and M. Khan [2] defined the definition of notion of complex valued metric spaces and prove the common fixed point theorems in complex valued metric spaces of a pair of mappings satisfying a contractive condition.

Recently, Jleli and Samet [13], introduce a new concept of generalized metric spaces for which we extend some well-known fixed point results including Banach contraction principle. In 2017, Elkouch and Marhrani [15], proved the existence results for the Kannan contraction in generalized metric space.

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In this paper, motivate by Elkouch and Marhraniwe [15], present a generalized complex valued metric space and prove the relationship between this space with complex valued b-metric space, complex valued dislocated metric space and complex valued metric space. In the final section, we prove the fixed point theorem for a mapping T with satisfying the Ćirić's k-quasicontraction.

2. Preliminaries

In this section, we give some definitions and lemmas for this work.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a **metric space**, and d is called a metric on X.

In 2000, Hitzler and Seda [21], introduce the notion of dislocated metric space as follows.

Definition 2.2. [21] Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a dislocated metric on X if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0 \quad \Rightarrow \quad x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a **dislocated metric space**.

It is easy to show that, the metric space X is dislocated metric space.
Next, we suppose the definition of b-metric space, this space is generalized than metric spaces.

Definition 2.3. [1] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a **b-metric space**. The number $s \geq 1$ is called the coefficient of (X, d).

The following is some example for b-metric spaces.

Example 2.4. [1] Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

In 2017, Elkouch and Marhrani [15] defined a new class of metric space, let X be a nonempty set, and $D : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, define the set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0 \right\}.$$

Definition 2.5. ([15]) A mapping D is called a generalized metric if it satisfies the following conditions
1. For every $(x,y) \in X \times X$, we have

$$D(x,y) = 0 \Rightarrow x = y.$$

2. For every $(x,y) \in X \times X$, we have

$$D(x,y) = D(y,x).$$

3. There exists a real constant $C > 0$ such that for all $(x,y) \in X \times X$ and $\{x_n\} \in C(D,X,x)$, we have

$$D(x,y) \leq C \limsup_{n\to\infty} D(x_n,y).$$

The pair (X,D) is called a **generalized metric space**.

It is not difficult to observe that metric d in Definition 2.1 satisfies all the conditions $(i)-(iii)$ with $C=1$. In 2015 Mohamed Jleli and Bessem Samet [17] prove that any dislocated metric space is is a generalized metric and any b-metric on X is a generalized metric on X.

In this work we will study the generalized metric space in a complex form. Let $\mathbf{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbf{C}$. Define a partial order relation $\preceq$ on $\mathbf{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \preceq z_2$ if one of the followings holds:
(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$.
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$.
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \preceq z_2$ if $z_1 \preceq z_2$ and $z_1 \neq z_2$ i.e. one of (2),(3) and (4) is satisfied and we will write $z_1 \prec z_2$ only (4) is satisfied.

Remark 2.6. We can easily to check the following:
(i) If $a, b \in \mathbf{R}, 0 \leq a \leq b$ and $z_1 \preceq z_2$ then $az_1 \preceq bz_2, \forall z_1, z_2 \in \mathbf{C}$.
(ii) $0 \preceq z_1 \preceq z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \preceq z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.
Azam et al. [2] defined the complex valued metric space in the following way:

Lemma 2.7. *For any $z \in \mathbf{C}$ with $0 \prec z$ then there exists $r \in \mathbf{C}$ with $0 \prec r$ such that $z = r|z|$.*

Proof Let $z \in \mathbf{C}$ with $0 \prec z$. Put $r = \frac{Re(z)}{|z|} + \frac{Im(z)}{|z|}i \succ 0$. It implied that

$$\begin{aligned} z &= Re(z) + Im(z)i \\ &= \frac{Re(z)}{|z|} \cdot |z| + \frac{Im(z)}{|z|} i \cdot |z| \\ &= \left[\frac{Re(z)}{|z|} + \frac{Im(z)}{|z|} i\right] |z| \\ &= r \cdot |z| \end{aligned}$$

This complete the proof.

[2] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbf{C}$ satisfies the following conditions:
$(C1)$ $0 \preceq d(x,y)$, for all $x, y \in X$, and $d(x,y) = 0$ if and only if $x = y$;

$(C2)$ $d(x,y) = d(y,x)$ for all $x,y \in X$;
$(C3)$ $d(x,y) \preceq d(x,z) + d(z,y),$ for all $x,y \in X$.

Then d is called a complex valued metric on X and (X,d) is called a **complex valued metric space**.

Definition 2.8. Let X be a nonempty set. A function $d : X \times X \to \mathbf{C}$ is called a complex valued dislocated metric on X if for $x,y,z \in X$ the following conditions are satisfied.
(i) $d(x,y) = 0 \Rightarrow x = y$;
(ii) $d(x,y) = d(y,x)$;
(iii) $d(x,z) \preceq d(x,y) + d(y,z)$.
The pair (X,d) is called a **complex valued dislocated metric space**.

Definition 2.9. [23] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbf{C}$ is called a complex valued b-metric on X if, for all $x,y,z \in \mathbf{C}$, the following conditions are satisfied:
(i) $0 \preceq d(x,y)$
(ii) $d(x,y) = 0$ if and only if $x = y$,
(iii) $d(x,y) = d(y,x)$,
(iv) $d(x,z) \preceq s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a complex valued b-metric space. We see that if $s = 1$ then (X,d) is complex valued metric space which is defined in Definition **??**. The following example is some example of complex valued b-metric space.

Example 2.10. [23] Let $X = \mathbf{C}$. Define the mapping $d : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ by $d(x,y) = |x-y|^2 + i|x-y|^2$ for all $x,y \in X$. Then $(\mathbf{C},d)$ is complex valued b-metriic space with $s = 2$.

In this work, we consider a nonempty set X, and $D : X \times X \to \mathbf{C}$ be a given mapping. For every $x \in X$, we define the set

$$C(D,X,x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} |D(x_n,x)| = 0 \right\}.$$

Definition 2.11. Let X be a nonempty set, a mapping $D : X \times X \to \mathbf{C}$ is called a generalized complex value metric if it satisfies the following condition
1. For every $x,y \in X$, we have

$$0 \preceq D(x,y).$$

2. For every $x,y \in X$, we have

$$D(x,y) = 0 \Rightarrow x = y.$$

3. For all $x,y \in X$, we have

$$D(x,y) = D(y,x).$$

4. There exists a complex constant $0 \prec r$ such that for all $x,y \in X$ and $\{x_n\} \in C(D,X,x)$, we have

$$D(x,y) \preceq r \limsup_{n\to\infty} |D(x_n,y)|.$$

Then a pair (X,D) is called a **generalized complex valued metric space**.

Definition 2.12. [15] Let (X,D) be a generalized complex valued metric space, let $\{x_n\}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}$ is converge to x in X, if $\{x_n\} \in C(D,X,x)$. We denote by $\lim_{n\to\infty} x_n = x$.

Example 2.13. [12] Let $X = [0,1]$ and let $D : X \times X \to \mathbf{C}$ be the mapping define by for any $x, y \in X$

$$\begin{cases} D(x,y) = (x+y)i; x \neq 0 \text{ and } y \neq 0 \\ D(x,0) = D(0,x) = \frac{x}{2}i \end{cases}$$

Proof Let $x, y \in X$, we have $x \geq 0$ and $y \geq 0$, thus $x + y \geq 0$.
If $D(x,y) = (x+y)i = 0 + (x+y)i \succeq 0 + 0i = 0$.
If $D(x,0) = \frac{x}{2}i = 0 + \frac{x}{2}i \succeq 0 + 0i = 0$.

$$\text{Hence } D(x,y) \succeq 0.$$

If $D(x,y) = 0$, then $(x+y)i = 0$. Hence, $x = 0 = y$.
If $x \neq 0$ and $y \neq 0$, $D(x,y) = (x+y)i = (y+x)i = D(y,x)$ and $D(x,0) = D(0,x)$.
Let $\{x_n\} = \{\frac{(n-1)x}{n}\} \subseteq X$, we see that $\limsup_{n\to\infty} |D(x_n, x)| = 0$ and put $r = i$, then we have

$$D(0,y) = \frac{y}{2}i \text{ and } \limsup_{n\to\infty} |D(x_n,y)| = \limsup_{n\to\infty} \sqrt{(\frac{(n-1)x}{n} + y)^2} = x + y.$$

Hence, $D(0,y) = \frac{y}{2}i \preceq (x+y)i$, and we see that

$$D(x,y) = (x+y)i \text{ and } \limsup_{n\to\infty} |D(x_n,y)| = \limsup_{n\to\infty} \sqrt{(\frac{(n-1)x}{n} + y)^2} = x + y.$$

Hence, $D(x,y) = (x+y)i \preceq r \limsup_{n\to\infty} |D(x_n,y)|$.

Definition 2.14. [15] Let (X, D) be a generalized complex valued metric space. Then a sequence $\{x_n\}$ in X is said to Cauchy sequence in X, if $\lim_{n\to\infty} |D(x_n, x_{n+m})| = 0$.

Definition 2.15. [15] Let (X, D) be a generalized complex valued metric space. If every Cauchy sequence is convergent in X then (X, D) is called a complete complex valued metric space.

Definition 2.16. [19] The max function for complex numbers with partial order relation $\preceq$ is defined as
(i) $\max\{z_1, z_2\} = z_2 \Rightarrow z_1 \preceq z_2$;
(ii) $z_1 \preceq \max\{z_1, z_2\} \Rightarrow z_1 \preceq z_2$ or $z_1 \preceq z_3$.
On the similar lines Singh et al. [22] defined min function as
(i) $\min\{z_1, z_2\} = z_1 \Rightarrow z_1 \preceq z_2$;
(ii) $\min\{z_1, z_2\} \preceq z_3 \Rightarrow z_1 \preceq z_3$ or $z_2 \preceq z_3$. Now we introduce the best proximity point and some related concept in complex valued rectangular metric space.

3. Some Property on Generalized Complex Valued Metric Space

In this section we prove some propositions for use in the main theorem and prove some fixed point theorem in generalized complex valued metric space.

Proposition 3.1. *Let (X,D) be a generalized complex valued metric space. Let $\{x_n\}$ be a sequence in X and $(x,y) \in X \times X$. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.*

Proof Suppose that $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, by Definition 2.12 we have

$$|D(x_n, x)| \to 0, \ |D(x_n, y)| \to 0, \text{ as } n \to \infty.$$

Using the property (4) in Definition 2.11, we have there exists a complex constant $0 \prec r$ such that for all $x, y \in X$ and since $\{x_n\} \in C(D, X, x)$ such that

$$D(x,y) \preceq r \limsup_{n\to\infty} |D(x_n, y)|.$$

Hence, $D(x, y) = 0$. Using property (2) in Definition 2.11, we have $x = y$.

Proposition 3.2. *Any complex valued b-metric space is a generalized complex valued metric space on X.*

Proof Let $\{x_n\} \in C(d, X, x)$. From the Definition 2.9(iv), we have

$$d(x,y) \preceq s[d(x,x_n) + d(x_n,y)].$$

It follows that, from Lemma 2.7, we have there exists $r_1, r_2 \in \mathbf{C}$ with $0 \prec r_1, r_2$ such that

$$\begin{aligned} d(x,x_n) &= r_1|d(x,x_n)| \\ d(x_n,y) &= r_2|d(x_n,y)|. \end{aligned}$$

Then

$$d(x,y) \preceq s[r_1|d(x,x_n)| + r_2|d(x_n,y)|].$$

From $\{x_n\} \in C(d, X, x)$, we have

$$d(x,y) \preceq sr_2 \limsup_{n\to\infty} |d(x_n,y)|].$$

Since, $0 \prec r_2$ and $0 \prec s$ then $r = sr_2 \succ 0$ such that

$$d(x,y) \preceq r \limsup_{n\to\infty} |d(x_n,y)|].$$

Hence (X, d) is a generalized complex valued metric space.

It is not difficult to observe that the complex valued metric d satisfies (1-4) of Definition 2.11 and any complex valued dislocated metric space is generalized complex valued metric space.

4. Ćirić's quasicontraction in Generalized Complex Valued Metric Space

In 1974, Ćirić's [11] introduced a class of self-maps on a metric space (X, d) which satisfy the following condition:

$$d(Sx,Sy) \preceq q \max\{d(x,y), d(x,Sx), d(y,Sy), d(x,Sy), d(y,Sx)\}, \tag{4.1}$$

for every $x, y \in X$ and $0 \le q < 1$. The maps satisfying Condition 4.1 are said to be quasicontractions.

In this section we extend Ćirić's fixed point theorem for quasicontraction is a self-maps on generalized complex valued metric space (X, D) defined by:

$$D(Tx,Ty) \preceq k \max\{D(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\},$$

for every $x, y \in X$ and $k \in (0, 1)$. We say that T is a k-quasicontraction

Proposition 4.1. *Suppose that $T : X \to X$ is a k-quasicontraction for some $k \in (0, 1)$. Then any fixed point $p \in X$ of T satisfies*

$$|D(p,p)| < \infty \Rightarrow D(p,p) = 0.$$

Proof Let $p \in X$ be a fixed point of T such that $|D(p,p)| < \infty$. Since T is a k-quasicontraction for some $k \in (0,1)$, we have

$$\begin{aligned} D(p,p) = D(Tp,Tp) &\preceq k \max\{D(p,p), D(p,Tp), D(p,Tp), D(p,Tp), D(p,Tp)\} \\ &= kD(p,p). \end{aligned}$$

From Remark 2.6(ii), we have

$$|D(p,p)| \leq k|D(p,p)|.$$

Since $k \in (0,1)$, we get $D(p,p) = 0$. This proof is complete.

Next, we suppose that, for every $x \in X$

$$\delta(D,T,x) = \sup\left\{|D(T^i x, T^j x)| : i,j \in \mathbf{N}\right\}.$$

From Proposition 4.1 we have the following result.

Theorem 4.2. *Let (X,D) be a complete generalized complex valued metric space, and let $T : X \to X$ is a k-quasicontraction for some $k \in \left(0, \inf\{1, \frac{1}{|r|}\}\right)$ and there exists element $x_0 \in X$ such that $\delta(D,T,x_0) < \infty$. Then the sequence $\{T^n x_0\}$ converges to some $p \in X$.*

If $D(x_0,Tp) \prec \infty$ and $D(p,Tp) \prec \infty$, then p is a fixed point of T. Moreover, If p' is a fixed point of T in X such that $|D(p,p')| < \infty$ and $|D(p',p')| < \infty$ then $p = p'$.

Proof Let $n \in \mathbf{N}$, for all $i,j \in \mathbf{N}$, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) = D(T(T^{n+i-1}x_0), T(T^{n+j-1}x_0)).$$

By Definition of quasicontraction, we have

$$D(T^{n+i}x_0, T^{n+j}x_0) \preceq k \max\left\{\begin{array}{l} D(T^{n+i-1}x_0, T^{n+j-1}x_0), D(T^{n+i-1}x_0, T^{n+i}x_0), \\ D(T^{n+j-1}x_0, T^{n+j}x_0), D(T^{n+j-1}x_0, T^{n+i}x_0), \\ D(T^{n+i-1}x_0, T^{n+j}x_0) \end{array}\right\}$$

Then we have

$$\delta(D,T,T^n x_0) \leq k\delta(D,T,T^{n-1}x_0).$$

Hence, for any $n \geq 1$, we have

$$\delta(D,T,T^n x_0) \leq k^n \delta(D,T,x_0). \tag{4.2}$$

By (4.2) we see that for any $m,n \in \mathbf{N}$

$$|D(T^n x_0, T^{n+m}x_0)| \leq \delta(D,T,T^n x_0) \leq k^n \delta(D,T,x_0). \tag{4.3}$$

Since $\delta(D,T,x_0) < \infty$ and $k \in (0,1)$, it follows that

$$\lim_{n\to\infty} |D(T^n x_0, T^{n+m}x_0)| = 0.$$

Hence $\{T^n x_0\}$ is a Cauchy sequence. Since (X,D) is complete, there exists a element $p \in X$ such that $\{T^n x_0\}$ convergent to p.

Suppose that $D(x_0,Tp) \prec \infty$ and $D(p,Tp) \prec \infty$. Then for any $m,n \in \mathbf{N}$

$$|D(T^n x_0, T^{n+m}x_0)| \leq k^n \delta(D,T,x_0). \tag{4.4}$$

From (4.3) and the property (4) in Definition 2.11, there exists $0 \prec \gamma$ such that

$$D(p, T^n x_0) \leq r \limsup_{m\to\infty} |D(T^n x_0, T^{n+m}x_0)| \leq \gamma k^n \delta(D,T,x_0), \tag{4.5}$$

for all $n \in \mathbf{N}$. Consider,

$$D(Tx_0,Tp) \preceq k \max\{D(x_0,p), D(x_0,Tx_0), D(p,Tp), D(Tx_0,p), D(x_0,Tp)\}. \tag{4.6}$$

From (4.4), (4.5) and (4.6), we get

$$\begin{aligned} D(Tx_0,Tp) &\preceq k\max\{D(x_0,p),D(x_0,Tx_0),D(p,Tp),rk\delta(D,T,x_0),D(x_0,Tp)\} \\ &= kM_1, \end{aligned} \tag{4.7}$$

where $M_1 = \max\{D(x_0,p),D(x_0,Tx_0),D(p,Tp),rk\delta(D,T,x_0),D(x_0,Tp)\} \prec \infty$. Using the above inequality (4.3), (4.5), (4.7) and Lemma 2.7, we have complex number $0 \prec c_1$ such that

$$\begin{aligned} D(T^2x_0,Tp) &\preceq k\max\{D(Tx_0,p),D(Tx_0,T^2x_0),D(p,Tp),D(T^2x_0,p),D(Tx_0,Tp)\} \\ &\preceq \max\{\gamma k^2\delta(D,T,x_0),c_1k^2|D(Tx_0,T^2x_0)|,kD(p,Tp),\gamma k^2\delta(D,T,x_0),k^2M_1\} \\ &\preceq \max\{\gamma k^2\delta(D,T,x_0),c_1k^2\delta(D,T,x_0),kD(p,Tp),\gamma k^2\delta(D,T,x_0),k^2M_1\} \\ &= \max\{k^2M_2,kD(p,Tp)\}, \end{aligned}$$

where $M_2 = \max\{\gamma\delta(D,T,x_0),c_1\delta(D,T,x_0),M_1\} \prec \infty$. Since, $k < 1$ it follows that

$$kD(T^2x_0,Tp) \prec D(T^2x_0,Tp) \preceq \max\{k^2M_2,kD(p,Tp)\}. \tag{4.8}$$

Again, using the above inequality (4.3), (4.5), (4.7), (4.8) and Lemma 2.7, we have complex number $0 \prec c_2$ such that

$$\begin{aligned} D(T^3x_0,Tp) &\preceq k\max\Big\{D(T^2x_0,p),D(T^2x_0,T^3x_0),D(p,Tp),D(T^2x_0,Tp), \\ &\qquad D(p,T^3x_0)\Big\} \\ &\preceq \max\Big\{\gamma k^3\delta(D,T,x_0),c_2k^3|D(T^2x_0,T^3x_0)|,kD(p,Tp), \\ &\qquad kD(T^2x_0,Tp),\gamma k^3\delta(D,T,x_0)\Big\} \\ &\preceq \max\{k^3M_3,kD(p,Tp)\}, \end{aligned}$$

where $M_3 = \max\{\gamma\delta(D,T,x_0),c_2\delta(D,T,x_0),M_2\} \prec \infty$. Continuing this process, by induction above inequality and Lemma 2.7, we have

$$D(T^nx_0,Tp) \preceq \{k^nM,kD(p,Tp)\}, \tag{4.9}$$

for every $n \geq 1$ and $M = \max\{M_1,M_2,...,M_n\} \prec \infty$. Since $D(x_0,Tp) \prec \infty$ and $D(p,Tp) \prec \infty$, we have

$$\limsup_{n\to\infty}|D(T^nx_0,Tp)| \leq k|D(p,Tp)|. \tag{4.10}$$

By Definition 2.11 (4), there exists $0 \prec r$ such that

$$D(Tp,p) \preceq r\limsup_{n\to\infty}|D(Tp,T^nx_0)|. \tag{4.11}$$

By remark 2.6 (ii) and (4.10), we have

$$|D(Tp,p)| \leq |r|\limsup_{n\to\infty}|D(Tp,T^nx_0)| \leq |r|k|D(p,Tp)|. \tag{4.12}$$

Since $|r|k < 1$, we have $|D(Tp,p)| = 0$ thus $D(Tp,p) = 0$ it follows that p is a fixed point of T, and then

$$D(p,p) = D(Tp,p) = 0. \tag{4.13}$$

If $p^{'}$ is any fixed point of T such that $|D(p,p^{'})| < \infty$ and $|D(p^{'},p^{'})| < \infty$. From Proposition 4.1 we have $D(p^{'},p^{'}) = 0$ and then

$$\begin{aligned} D(p,p^{'}) &= D(Tp,Tp^{'}) \\ &\preceq k\max\Big\{D(p,p^{'}),D(p,Tp),D(p^{'},Tp^{'}),D(Tp,p^{'}),D(p,Tp^{'})\Big\} \end{aligned}$$

$$\preceq \quad k\max\left\{D(p,p^{'}),D(p,p),D(p^{'},p^{'}),D(p,p^{'}),D(p,p^{'})\right\}$$
$$\preceq \quad kD(p,p^{'}).$$

By remark 2.6 (ii), we have

$$|D(p,p^{'})| \leq k|D(p,p^{'})|.$$

Since $k < 1$, we have $|D(p,p^{'})| = 0$ thus $D(p,p^{'}) = 0$. Hence $p = p^{'}$. This proof is complete.

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SOLVABILITY OF EQUATIONS INVOLVING PERTURBATIONS OF m-ACCRETIVE OPERATORS IN BANACH SPACES

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ABSTRACT. It is purpose of this paper to give several results for the solvability of the equation $p \in Ax + Sx$, where A is an m-accretive operator on a Banach space E and S is a mapping on a subset of E, with elementary proofs. We give proofs of them without using degree theory.

KEYWORDS: Nonlinear operator, accretive operator, fixed point.
AMS Subject Classification: 47H14, 47H10

1. Introduction

Let E be a real Banach space, A an m-accretive operator on E, S a mapping of a subset of E into E, and p an element of E. In [5], [6] and [8], the solvability of the equation

$$p \in Au + Su \tag{1.1}$$

for the case that $(I+A)^{-1}$ is compact and S is continuous has been studied. Results in these papers are proved using degree theory. For example, Theorem 1 in [5] is proved using Theorem 6.3.2 in [7] and Theorem 5 in [6] is proved using Theorem 4.4.11 in [7].

It is purpose of this paper to give several results for the solvability of the equation (1.1), with elementary proofs. We give proofs of them without using degree theory.

In Section 3, we introduce a fixed point theorem for a continuous mapping of a closed ball into E; see Proposition 3.1. To use Proposition 3.1 in proofs of main results, we need Propositions 2.2 and 2.3. In Section 4, using Propositions 2.2 and 3.1, we consider the solvability of the equation (1.1) in the case that $(I + A)^{-1}$ is compact and S is continuous. Moreover, in Section 5, using Propositions 2.3 and

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3.1, we consider the equation related with (1.1) for the case that S is compact and the compactness of $(I+A)^{-1}$ is not required.

2. Preliminaries

Throughout this paper, E denotes a real Banach space with norm $\|\cdot\|$, E^* the topological dual of E and $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$. The normalized duality mapping of E is denoted by J, that is, it is a set-valued mapping of E into E^* defined by $Jx = \{x^* \in E^* \mid \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ for $x \in E$.

For each subset C of E, we denote by $\overline{C}$ and ∂C, the closure of C and the boundary of C, respectively. Moreover we denote by $\operatorname{co} C$ the convex hull of C and by $\overline{\operatorname{co}}\, C$ the closed convex hull of C. Mazur's theorem asserts that if a subset C of E is compact, then $\overline{\operatorname{co}}\, C$ is compact; see Theorem 5.2.6 in [2].

Let x be an element of E and r a positive real number. We denote by $B_r(x)$ the open ball with center at x and radius r and by $B_r[x]$ the closed ball with center at x and radius r. To prove our main results, we need the following lemma related with the Minkowski functional associated to $B_r[0]$.

Lemma 2.1. *Let r be a positive real number. Define mappings f and M on E by*

$$f(x) = \frac{r}{\max\{r, \|x\|\}} \quad \text{and} \quad Mx = f(x)x$$

for $x \in E$. Then the following hold:

(1) *For $x \in B_r[0]$, $f(x) = 1$ and $Mx \in B_r[0]$;*
(2) *for $x \notin B_r[0]$, $f(x) \in (0,1)$ and $Mx \in B_r[0]$;*
(3) *f and M are continuous.*

Therefore, f is a continuous mapping of E into $(0,1]$ and M is a continuous mapping of E into $B_r[0]$.

Proof. We only show that the range of M is a subset of $B_r[0]$. If $x \in B_r[0]$, then $Mx = x \in B_r[0]$. If $x \notin B_r[0]$, then $\|Mx\| = \|\frac{r}{\|x\|}x\| = r$. Therefore, for all $x \in E$, we have $Mx \in B_r[0]$. □

Let T be a mapping of a subset C of E into E. We often use $D(T)$ instead of C. T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Nonexpansive mappings are continuous. T is said to be bounded if T maps bounded subsets of C onto bounded sets. Especially, nonexpansive mappings are bounded. T is said to be compact if T is continuous and it maps bounded subsets of C onto relatively compact sets. For continuous mappings, we know Brouwer's fixed point theorem; if T is a continuous mapping of a compact convex subset of an Euclidean space into itself, then T has a fixed point; see Theorem 2.1.11 in [11]. For an elementary proof of the theorem, see [13]. Schauder's fixed point theorem is the following; if T is a continuous mapping of a compact convex subset of a normed space into itself, then T has a fixed point; see Theorem 2.3.7 in [11].

Let A be a set-valued operator on E. The symbols $D(A)$ and $R(A)$ denote the domain and the range of A, respectively, that is, $D(A) = \{x \in E \mid Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Ax \mid x \in D(A)\}$. A is said to be accretive if for $x, y \in D(A)$, $u \in Ax$ and $v \in Ay$, there exists $j \in J(x-y)$ such that $\langle u - v, j\rangle \ge 0$. A is said to be m-accretive if A is accretive and $R(I + \lambda A) = E$ for all $\lambda > 0$.

Let A be an accretive operator on E and λ a positive real number. Since A is accretive, $(I + \lambda A)^{-1}$ is a single-valued mapping of $R(I + \lambda A)$ onto $D(A)$. The mapping $(I+\lambda A)^{-1}$ is called a resolvent of A and it is denoted by J_λ, that is, $J_\lambda x = (I+\lambda A)^{-1}x$ for $x \in R(I+\lambda A)$. Then, since $x \in J_\lambda x + \lambda A J_\lambda x$, we have

$$\frac{x - J_\lambda x}{\lambda} \in AJ_\lambda x.$$

The mapping $\frac{I-J_\lambda}{\lambda}$ is called a Yosida approximation of A and it is denoted by A_λ, that is, $A_\lambda x = \frac{1}{\lambda}(x - J_\lambda x)$ for $x \in R(I+\lambda A)$. Therefore, for $x \in R(I+\lambda A)$, we have $A_\lambda x \in AJ_\lambda x$. Since A is accretive, the resolvent J_λ is nonexpansive. Then, the resolvent is continuous and bounded. For these results, we refer the reader to [12].

In Sections 4 and 5, we use the following.

Proposition 2.2. *Let A be an m-accretive operator on E and S a continuous mapping of $D(S)$ into E such that J_1 is compact and $\overline{D(A)} \subset D(S)$. Let p be an element of E. Define a mapping T on E by*

$$Tx = p + J_1 x - SJ_1 x$$

for $x \in E$. Then T is compact. Moreover if x is a fixed point of T, then $J_1 x$ is a solution of the equation (1.1).

Proof. Since J_1 is continuous, T is continuous. Since J_1 is compact, $\overline{J_1(B_r[0])}$ is compact for $r > 0$. Note that $J_1(B_r[0]) \subset \overline{J_1(B_r[0])} \subset \overline{D(A)} \subset D(S)$. Since S is continuous, $S(\overline{J_1(B_r[0])})$ is compact, that is, $SJ_1(B_r[0])$ is relatively compact. Therefore $T(B_r[0])$ is also relatively compact. Hence T is compact.

If x is a fixed point of T, then, since $p + J_1x - SJ_1x = x$, we have

$$p \quad = \quad x - J_1x + SJ_1x = A_1x + SJ_1x \in AJ_1x + SJ_1x.$$

Hence J_1x is a solution of the equation (1.1). □

Proposition 2.3. *Let A be an m-accretive operator on E and S a compact mapping of $D(S)$ into E such that $D(A) \subset D(S)$. Let p be an element of E and n a positive integer. Define a mapping T_n on E by*

$$T_n x = p - SJ_n(nx)$$

for $x \in E$. Then T_n is compact. Moreover if x_n is a fixed point of T_n, then $J_n(nx_n)$ is a solution of the equation

$$p \in Au + Su + \frac{1}{n}u. \tag{2.1}$$

Proof. Since the resolvent J_n is continuous, T_n is continuous. Since J_n is bounded, $J_n(nB_r[0])$ is bounded for $r > 0$. Then, since S is compact, $SJ_n(nB_r[0])$ is relatively compact. Therefore $T_n(B_r[0])$ is also relatively compact. Hence T_n is compact.

If x_n is a fixed point of T_n, then, since $A_n(nx_n) = x_n - \frac{1}{n}J_n(nx_n)$, we have

$$\begin{aligned} p &= x_n + SJ_n(nx_n) \\ &= A_n(nx_n) + SJ_n(nx_n) + \frac{1}{n}J_n(nx_n) \in AJ_n(nx_n) + SJ_n(nx_n) + \frac{1}{n}J_n(nx_n). \end{aligned}$$

Hence $J_n(nx_n)$ is a solution of the equation (2.1). □

In Section 3, we introduce a fixed point theorem for a continuous mapping of a closed ball into E; see Proposition 3.1. To prove Proposition 3.1, we need the following; see Lemma 1 in [10] and a result of Singbal in [1]. For the sake of completeness, we give a proof.

Lemma 2.4. *Let T be a continuous mapping of a closed ball $B_r[0]$ in E into itself. If $T(B_r[0])$ is relatively compact, then there exists a fixed point of T.*

Proof. Since $T(B_r[0])$ is relatively compact, $D = \overline{T(B_r[0])}$ is compact. For $s > 0$, there exist $x_1, x_2 \dots, x_n \in D$ such that $\{B_s(x_i) \mid i = 1, 2, \dots, n\}$ is a cover of D. For each $i = 1, 2, \dots, n$, we define a continuous mapping d_i on E by

$$d_i(x) = \max\{0, s - \|x - x_i\|\}$$

for $x \in E$. For any i and $x \in E$, $d_i(x)\|x - x_i\| \leq s d_i(x)$ holds because $d_i(x) \neq 0$ implies $\|x - x_i\| < s$. Moreover, for any $x \in D$, there exists i_0 satisfying $\|x - x_{i_0}\| < s$, that is, $d_{i_0}(x) > 0$. Therefore, for any $x \in D$, we have $\sum_{i=1}^n d_i(x) > 0$. If we define a function h_i for each $i = 1, 2, \dots, n$ by

$$h_i(x) = \frac{d_i(x)}{\sum_{i=1}^n d_i(x)}$$

for $x \in D$, then the following hold; for any $x \in D$ and $i = 1, 2, \dots, n$, $0 \leq h_i(x) \leq 1$; for any $x \in D$, $\sum_{i=1}^n h_i(x) = 1$. We consider a continuous mapping T_s on D defined by

$$T_s x = \sum_{i=1}^n h_i(x) x_i$$

for $x \in D$. Then we have

$$\|x - T_s x\| \leq \frac{1}{\sum_{i=1}^n d_i(x)} \sum_{i=1}^n d_i(x)\|x - x_i\| \leq \frac{s}{\sum_{i=1}^n d_i(x)} \sum_{i=1}^n d_i(x) = s$$

for $x \in D$. Since T is continuous, $T_s T$ is a continuous mapping of $\mathrm{co}(\{x_i\}_{i=1}^n)$ into itself. By Brouwer's fixed point theorem, there exists $y \in \mathrm{co}(\{x_i\}_{i=1}^n)$ satisfying $T_s T y = y$. Since $Ty \in D$, we have $\|Ty - T_s T y\| \leq s$, that is, $\|Ty - y\| \leq s$. Therefore, for any $s > 0$, there exists $y \in \mathrm{co}(\{x_i\}_{i=1}^n) \subset B_r[0]$ such that $\|y - Ty\| \leq s$.

By the argument so far, there exists a sequence $\{y_m\} \subset B_r[0]$ satisfying $\|y_m - Ty_m\| \leq \frac{1}{m}$ for all $m = 1, 2, \dots$. That is, $\lim_{m \longrightarrow \infty} \|y_m - Ty_m\| = 0$. Since D is compact, there exists a subsequence $\{y_{m_j}\}$ of $\{y_m\}$ such that $\{Ty_{m_j}\}$ converges to some $z \in D$. Since $\lim_{m \longrightarrow \infty} \|y_m - Ty_m\| = 0$, $\{y_{m_j}\}$ also converges to z. Moreover, since T is continuous and

$$\|z - Tz\| \leq \|z - Ty_{m_j}\| + \|Ty_{m_j} - Tz\|,$$

we have $z = Tz$. □

Remark 2.5. Using Schauder's fixed point theorem, we can prove Lemma 2.4. In fact, since $\overline{T(B_r[0])}$ is compact, the closed convex hull of $\overline{T(B_r[0])}$ is also compact by Mazur's theorem. Then, by Schauder's fixed point theorem, we obtain the conclusion; see Theorem 4.4.10 in [7].

On the other hand, in the proof of Lemma 2.4, we use Brouwer's fixed point theorem. Consider the finite dimensional linear space L spanned by $\{x_i\}_{i=1}^n$, where $x_1, x_2, \dots, x_n$ are the elements in the proof of Lemma 2.4. Since any two Hausdorff linear topologies on L coincide, the relative topology of L induced by E is the Euclidian topology of L. Then we can consider that $\mathrm{co}(\{x_i\}_{i=1}^n)$ is a compact convex subset of the Euclidean space L. By Brouwer's fixed point theorem, the continuous mapping $T_s T$ in the proof of Lemma 2.4 has a fixed point. Therefore the above proof of Lemma 2.4 is a direct proof from Brouwer's fixed point theorem. For the mapping T_s, see Theorem 2 in [9].

3. Fixed point theorems

To prove results in Sections 4 and 5, we need the following fixed point theorem.

Proposition 3.1. *Let T be a continuous mapping of a closed ball $B_r[0]$ into E such that $T(B_r[0])$ is relatively compact. Then, there exists $x \in B_r[0]$ such that*

$$f(Tx)Tx = x,$$

where f is defined as in Lemma 2.1. Moreover the following hold:

(i) *If $Tx \in B_r[0]$, then $Tx = x$;*
(ii) *if $x \in B_r(0)$, then $Tx = x$.*

Proof. Define a mapping V on $B_r[0]$ by $Vx = f(Tx)Tx$ for $x \in B_r[0]$. We know that the mapping M in Lemma 1 is continuous and the range of M is a subset of $B_r[0]$. Then we see that $M(\overline{T(B_r[0])})$ is compact and

$$V(B_r[0]) = MT(B_r[0]) \subset M(\overline{T(B_r[0])}) \subset B_r[0].$$

By Lemma 2.1, V is a continuous mapping of $B_r[0]$ into itself. Since $V(B_r[0])$ is relatively compact, by Lemma 2.4, there exists $x \in B_r[0]$ such that

$$Vx = x.$$

We show (i). If $Tx \in B_r[0]$, then we have $f(Tx) = 1$ by Lemma 2.1. Hence we have $Tx = x$. To prove (ii), suppose that $Tx \notin B_r[0]$. Then, since $x \in B_r(0)$ and $f(Tx)Tx = x$, we have

$$r > \|x\| = \|f(Tx)Tx\| = \left\| \frac{r}{\|Tx\|} Tx \right\| = \frac{r}{\|Tx\|} \|Tx\| = r.$$

This is a contradiction. Therefore $Tx \in B_r[0]$. Hence, by (i), $Tx = x$. ☐

The condition $Tx \in B_r[0]$ in (i) of Proposition 3.1 is related to the condition that $T(\partial B_r[0]) \subset B_r[0]$, which is a condition of Rothe's fixed point theorem; see Theorem 4.2.3 in [11].

Corollary 3.2. *Let T be a compact mapping of a closed ball $B_r[0]$ into E such that $T(\partial B_r[0]) \subset B_r[0]$. Then there exists a fixed point of T.*

Proof. By Proposition 3.1, there exists $x \in B_r[0]$ such that

$$f(Tx)Tx = x,$$

where f is defined as in Lemma 2.1. If $x \in B_r(0)$, then $Tx = x$ by (ii) of Proposition 3.1. If $x \in \partial B_r[0]$, then, since $T(\partial B_r[0]) \subset B_r[0]$, we have $Tx \in B_r[0]$. Hence $Tx = x$ by (i) of Theorem 3.1. ☐

The condition $Tx \in B_r[0]$ in (i) of Proposition 3.1 is related to the following:

$$\text{If } x \in \partial B_r[0] \text{ and } c > 1, \text{ then } Tx \neq cx. \tag{3.1}$$

For (3.1), see Theorem 0.2.3 in [3] and Theorems 4.4.3 and 6.3.2 in [7].

Corollary 3.3. *Let T be a continuous mapping of a closed ball $B_r[0]$ into E such that $T(B_r[0])$ is relatively compact and the condition (3.1) holds. Then there exists a fixed point of T.*

Proof. By Proposition 3.1, there exists $x \in B_r[0]$ such that $f(Tx)Tx = x$, where f is defined as in Lemma 2.1. Since $f(Tx)$ is in $(0, 1]$ by Lemma 2.1, we have $c = \frac{1}{f(Tx)} \in [1, \infty)$ and

$$Tx = cx.$$

If $x \in \partial B_r[0]$, then we have $c = 1$ by the condition (3.1). Hence $Tx = x$. If $x \in B_r(0)$, by (ii) of Proposition 3.1, we have $Tx = x$. □

4. Main results

In this section, we consider the solvability of the equation (1.1) for the case that $(I + A)^{-1}$ is compact and S is continuous. Proposition 3.1 is crucial in the proofs of results in this section.

By Propositions 2.2 and 3.1, we obtain the following. The condition (4.1) is related to the condition $T(\partial B_r[0]) \subset B_r[0]$ in Corollary 3.2.

Theorem 4.1. *Let A be an m-accretive operator on E such that J_1 is compact and S is a continuous mapping of $D(S)$ into E with $\overline{D(A)} \subset D(S)$. Let p be an element of E. Suppose there exists a positive constant r satisfying the following:*

$$\text{If } x \in \partial B_r[0], \text{ then } \|p + J_1 x - SJ_1 x\| \leq r. \tag{4.1}$$

Then the equation (1.1) has a solution.

Proof. Let T be a mapping defined by $Tx = p + J_1 x - SJ_1 x$ for $x \in E$. By Proposition 2.2, T is compact. By Proposition 3.1, there exists $x \in B_r[0]$ such that

$$f(Tx)Tx = x,$$

where f is defined as in Lemma 2.1.

For the case that $x \in B_r(0)$, by (ii) of Proposition 3.1, we have $Tx = x$. For the case that $x \in \partial B_r[0]$, since $\|p + J_1 x - SJ_1 x\| \leq r$, we have $\|Tx\| \leq r$, that is, $Tx \in B_r[0]$. By (i) of Proposition 3.1, we see $Tx = x$. Therefore, in both cases, $Tx = x$ holds. Hence, by Proposition 2.2, (1.1) has a solution. □

Theorem 4.2. *Let A be an m-accretive operator on E such that J_1 is compact and S is a continuous mapping of $D(S)$ into E with $\overline{D(A)} \subset D(S)$. Let p be an element of E. Suppose there exists a positive constant r such that if $x \in \partial B_r[0]$, then there exists $j \in E^*$ satisfying $\langle x, j \rangle > 0$ and*

$$\langle A_1 x - p + SJ_1 x, j \rangle \geq 0.$$

Then the equation (1.1) has a solution.

Proof. Let T be a mapping defined by $Tx = p + J_1 x - SJ_1 x$ for $x \in E$. By Proposition 2.2, T is compact. By Proposition 3.1, there exists $x \in B_r[0]$ such that $f(Tx)Tx = x$, where f is defined as in Lemma 2.1. Since $f(Tx) \in (0, 1]$, by Lemma 2.1, we have $1 \leq c = \frac{1}{f(Tx)}$ and

$$Tx = cx.$$

For the case that $x \in B_r(0)$, by (ii) of Proposition 3.1, we have $Tx = x$. For the case that $x \in \partial B_r[0]$, there exists $j \in E^*$ such that $\langle x, j \rangle > 0$ and $\langle A_1 x - p + SJ_1 x, j \rangle \geq 0$. Since $A_1 x - p + SJ_1 x = x - Tx = x - cx$, we have

$$0 \leq \langle x, j \rangle \langle A_1 x - p + SJ_1 x, j \rangle = \langle x, j \rangle \langle x - cx, j \rangle = (1 - c)\langle x, j \rangle^2.$$

Then, since $\langle x, j \rangle \neq 0$, we have $c \leq 1$. Thus $c = 1$. Therefore, in both cases, $Tx = x$ holds. Hence, by Proposition 2.2, (1.1) has a solution. □

Moreover, we obtain the following.

Theorem 4.3. *Let A be an m-accretive operator on E such that J_1 is compact and S is a continuous mapping of $D(S)$ into E with $\overline{D(A)} \subset D(S)$. Let p be an element of E. Suppose there exist positive constant b and r which satisfy $B_b(0) \cap D(A) \neq \emptyset$, $p \in B_b(0)$,*

$$r > 2b + \sup\{\|Sx\| \mid x \in B_b(0) \cap D(A)\},$$

and the following: If $x \in \partial B_r[0]$ satisfies $J_1 x \notin B_b(0)$, then there exists $j \in E^*$ satisfying $\langle x, j\rangle > 0$ and

$$\langle A_1 x - p + SJ_1 x, j\rangle \geq 0.$$

Then the equation (1.1) has a solution.

Proof. Let T be a mapping defined by $Tx = p + J_1 x - SJ_1 x$ for $x \in E$. By Proposition 2.2, T is compact. By Proposition 3.1, there exists $x \in B_r[0]$ such that $f(Tx)Tx = x$, where f is defined as in Lemma 2.1. Since $f(Tx) \in (0, 1]$, by Lemma 2.1, we see $1 \leq c = \frac{1}{f(Tx)}$ and $Tx = cx$.

For the case that $x \in B_r(0)$, by (ii) of Proposition 3.1, we have $Tx = x$. Next we consider the case that $x \in \partial B_r[0]$. Assume that $J_1 x \in B_b(0)$. Then, by $c \in [1, \infty)$, we have

$$\begin{aligned} r &\leq cr = c\|x\| = \|cx\| = \|p + J_1 x - SJ_1 x\| \\ &\leq \|p\| + \|J_1 x\| + \|SJ_1 x\| < 2b + \sup\{\|Sx\| \mid x \in D(A) \cap B_b(0)\} < r. \end{aligned}$$

This is a contradiction. So, $J_1 x \notin B_b(0)$ holds. From this, there exists $j \in E^*$ satisfying $\langle x, j\rangle > 0$ and $\langle A_1 x - p + SJ_1 x, j\rangle \geq 0$. Then, in the same way as in the proof of Theorem 4.2, we have $c = 1$. Therefore, in both cases, $Tx = x$ holds. Hence, by Proposition 2.2, (1.1) has a solution. □

By Theorem 4.3, we obtain the following. Theorem 2 in [5] is related to the condition (4.2).

Corollary 4.1. *Let A be an m-accretive operator on E such that J_1 is compact and S is a bounded continuous mapping of $\overline{D(A)}$ into E. Let p be an element of E. Suppose there exists a positive constant b which satisfy $J_1 0 \in B_b(0)$, $p \in B_b(0)$ and the following:*

$$\text{If } z \in D(A), \|z\| \geq b,\ y \in Az \text{ and } j \in J(z - J_1 0),\ \text{then } \langle y - p + Sz, j\rangle \geq 0. \quad (4.2)$$

Then the equation (1.1) has a solution.

Proof. Note that, since $J_1 0 \in D(A) \cap B_b(0)$, $D(A) \cap B_b(0) \neq \emptyset$. Since S is bounded, there exists $r > 0$ satisfying the condition $r > 2b + \sup\{\|Sx\| \mid x \in B_b(0) \cap D(A)\}$ in Theorem 4.3. Let x be an element of $\partial B_r[0]$ with $J_1 x \notin B_b(0)$. Set $z = J_1 x$. Obviously, we see $z \in D(A)$, $\|z\| \geq b$ and $A_1 x \in AJ_1 x = Az$. Then, since $A_1 x \in Az$, $A_1 0 \in AJ_1 0$ and A is accretive, there exists $j \in J(z - J_1 0) = J(J_1 x - J_1 0)$ such that

$$\langle A_1 x - A_1 0, j\rangle \geq 0.$$

By (4.2), for such z, $A_1 x$ and j, $\langle A_1 x - p + Sz, j\rangle = \langle A_1 x - p + SJ_1 x, j\rangle \geq 0$ holds. Furthermore, since $J_1 0 \in B_b(0)$ and $J_1 x \notin B_b(0)$, we have

$$\begin{aligned} \langle x, j\rangle &= \langle A_1 x + J_1 x, j\rangle \\ &= \langle A_1 x - A_1 0, j\rangle + \langle A_1 0 + J_1 x, j\rangle \\ &\geq \langle A_1 0 + J_1 x, j\rangle \\ &= \langle J_1 x - J_1 0, j\rangle = \|J_1 x - J_1 0\|^2 > 0. \end{aligned}$$

So, all conditions of Theorem 4.3 are fulfilled. That is, (1.1) has a solution. □

5. Results for the case that S is compact

In this section, we consider

$$p \in \overline{(A+S)(B_c(0) \cap D(A))}, \tag{5.1}$$

where A is an m-accretive operator on E, S is a compact mapping on a subset of E, p is an element of E and c is a positive constant. Theorem 1 in [4] and Theorem 3 in [6] are related to (5.1).

By Propositions 2.3 and 3.1, we obtain the following results.

Theorem 5.1. *Let A be an m-accretive operator on E and S a compact mapping of $D(S)$ into E with $D(A) \subset D(S)$. Let p be an element of E. Suppose there exist positive constants b, r and a positive integer n_0 which satisfy $p \in B_b(0)$, $J_n 0 \in B_b(0)$ for all positive integers n,*

$$r > b + \sup\{\|Sx\| \mid x \in B_{2b}(0) \cap D(A)\},$$

and the following: If $n \geq n_0$, $x \in B_r[0]$ and $J_n(nx) \notin B_{2b}(0)$, then for all $j_n \in J(J_n(nx) - J_n 0)$,

$$\langle A_n(nx) - p + SJ_n(nx), j_n \rangle \geq 0.$$

Then the equation (5.1) has a solution as $c = 2b$.

Proof. Let n be a positive integer with $n \geq n_0$ and T_n a mapping defined by $T_n x = p - SJ_n(nx)$ for $x \in E$. By Proposition 2.3, T_n is compact. By Proposition 3.1, there exists $x_n \in B_r[0]$ such that

$$f(T_n x_n) T_n x_n = x_n,$$

where f is defined as in Lemma 2.1. Since $f(T_n x_n) \in (0, 1]$ by Lemma 2.1, we have $1 \leq c_n = \frac{1}{f(T_n x_n)}$ and

$$T_n x_n = c_n x_n.$$

We show that $J_n(nx_n) \in B_{2b}(0)$. Assume that $J_n(nx_n) \notin B_{2b}(0)$. Since $J_n 0 \in B_b(0)$, we have

$$\|J_n(nx_n) - J_n 0\| \quad \geq \quad \|J_n(nx_n) - J_n 0\| - \|J_n 0\| \geq \|J_n(nx_n)\| - 2\|J_n 0\| > 0.$$

So, we see $\|J_n(nx_n) - J_n 0\| > 0$ and $s = \|J_n(nx_n)\| - 2\|J_n 0\| > 0$.

Since $A_n(nx_n) \in AJ_n(nx_n)$, $A_n 0 \in AJ_n 0$ and A is accretive, there exists $j_n \in J(J_n(nx_n) - J_n 0)$ satisfying $\langle A_n(nx_n) - A_n 0, j_n \rangle \geq 0$. Then, since

$$\left\langle \frac{1}{n} J_n(nx_n) + A_n 0, j_n \right\rangle = \frac{1}{n} \langle J_n(nx_n) - J_n 0, j_n \rangle = \frac{1}{n} \|J_n(nx_n) - J_n 0\|^2$$

and $A_n(nx_n) = \frac{1}{n}(nx_n - J_n(nx_n))$, we see

$$\begin{aligned} \langle x_n, j_n \rangle &= \left\langle A_n(nx_n) + \frac{1}{n} J_n(nx_n), j_n \right\rangle \\ &= \left\langle \frac{1}{n} J_n(nx_n) + A_n 0, j_n \right\rangle + \langle A_n(nx_n) - A_n 0, j_n \rangle \\ &\geq \frac{1}{n} \|J_n(nx_n) - J_n 0\|^2. \end{aligned}$$

So we have $\langle x_n, j_n \rangle \geq \frac{1}{n} \|J_n(nx_n) - J_n 0\|^2 > 0$. Also, we see

$$\begin{aligned} &- \left\langle \frac{1}{n} J_n(nx_n) - \frac{1}{n} J_n 0, j_n \right\rangle - \frac{1}{n} \langle J_n 0, j_n \rangle \\ &\leq -\frac{1}{n} \|J_n(nx_n) - J_n 0\|^2 + \frac{1}{n} \|J_n 0\| \|J_n(nx_n) - J_n 0\| \end{aligned}$$

$$\begin{aligned} &\le -\frac{1}{n}\|J_n(nx_n)-J_n0\|\left(\|J_n(nx_n)-J_n0\|-\|J_n0\|\right) \\ &\le -\frac{s}{n}\|J_n(nx_n)-J_n0\|. \end{aligned}$$

By the argument so far, considering $n \ge n_0$, $x \in B_r[0]$, $J_n(nx) \notin B_{2b}(0)$, and $j_n \in J(J_n(nx_n)-J_n0)$, the following holds:

$$\begin{aligned} 0 &\le \langle A_n(nx_n)-p+SJ_n(nx_n), j_n\rangle \\ &= \langle x_n - p + SJ_n(nx_n), j_n\rangle - \left\langle \frac{1}{n}J_n(nx_n) - \frac{1}{n}J_n0, j_n \right\rangle - \frac{1}{n}\langle J_n0, j_n\rangle \\ &\le (1-c_n)\langle x_n, j_n\rangle - \frac{s}{n}\|J_n(nx_n)-J_n0\|. \end{aligned}$$

Furthermore, since $\|J_n(nx_n)-J_n0\|>0$, $s>0$, $\langle x_n, j_n\rangle > 0$ and $1\le c_n$, we have

$$0 < \frac{s}{n}\|J_n(nx_n)-J_n0\| \le (1-c_n)\langle x_n, j_n\rangle \le 0.$$

This is a contradiction. Hence $J_n(nx_n)\in B_{2b}(0)$ holds.

We show that $x_n \in B_r(0)$. Assume that $x_n \in \partial B_r[0]$. Then, since $J_n(nx_n) \in B_{2b}(0)$ and $1\le c_n$, we have a contradiction:

$$\begin{aligned} r &\le c_n\|x_n\| = \|c_n x_n\| = \|p - SJ_n(nx_n)\| \le \|p\| + \|SJ_n(nx_n)\| \\ &\le b + \|SJ_n(nx_n)\| < r. \end{aligned}$$

Thus, since $x_n \in B_r(0)$, by (ii) of Proposition 3.1, $T_n x_n = x_n$ holds.

Since $n \ge n_0$ is arbitrary, by Proposition 2.3, there is a sequence $\{x_n\}_{n\ge n_0}$ such that $J_n(nx_n)$ is a solution of (2.1) for each $n\ge n_0$, that is,

$$\begin{aligned} p - \frac{1}{n}J_n(nx_n) &= A_n(nx_n) + SJ_n(nx_n) \\ &\in AJ_n(nx_n) + SJ_n(nx_n) \in (A+S)(B_{2b}(0)\cap D(A)) \end{aligned}$$

for all $n \ge n_0$. Since $J_n(nx_n)\in B_{2b}(0)$, we know $\lim_{n\longrightarrow\infty}\left\|\frac{1}{n}J_n(nx_n)\right\| = 0$. Therefore, we have the desired result $p \in \overline{(A+S)(B_{2b}(0)\cap D(A))}$. □

Theorem 5.2. *Let A be an m-accretive operator on E and S a compact mapping of $D(S)$ into E with $D(A)\subset D(S)$. Let p be an element of E. Suppose there exist a positive constant r and a positive integer n_0 which satisfy the following:*

$$\text{If } n\ge n_0 \text{ and } x\in\partial B_r[0], \text{ then } \|p - SJ_n(nx)\| \le r.$$

Then there exists a sequence $\{x_n\}_{n\ge n_0}$ such that $J_n(nx_n)$ is a solution of the equation (2.1) for each $n\ge n_0$. Suppose further $\lim_{n\longrightarrow\infty}\left\|\frac{1}{n}J_n(nx_n)\right\| = 0$. Then, $p\in \overline{(A+S)(D(A))}$ holds.

Proof. Let n be a positive integer with $n \ge n_0$ and T_n a mapping defined by $T_n x = p - SJ_n(nx)$ for $x\in E$. By Proposition 2.3, T_n is compact. By Proposition 3.1, there exists $x_n \in B_r[0]$ such that $f(T_n x_n)T_n x_n = x_n$, where f is defined as in Lemma 2.1.

For the case that $x_n \in B_r(0)$, by (ii) of Proposition 3.1, we have $T_n x_n = x_n$. For the case that $x_n\in\partial B_r[0]$, we know $\|T_n x_n\|\le r$, that is, $T_n x_n \in B_r[0]$. By (i) of Proposition 3.1, we have $T_n x_n = x_n$. In both cases, $T_n x_n = x_n$ holds.

Since $n\ge n_0$ is arbitrary, by Proposition 2.3, there is a sequence $\{x_n\}_{n\ge n_0}$ such that $J_n(nx_n)$ is a solution of (2.1) for each $n \ge n_0$. Since for all $n\ge n_0$,

$$p - \frac{1}{n}J_n(nx_n) = A_n(nx_n) + SJ_n(nx_n) \in (A+S)(D(A)),$$

in the case that $\lim_{n\longrightarrow\infty}\left\|\frac{1}{n}J_n(nx_n)\right\| = 0$, $p\in\overline{(A+S)(D(A))}$ holds. □

Theorem 5.3. *Let A be an m-accretive operator on E and S a compact mapping of $D(S)$ into E with $D(A) \subset D(S)$. Let p be an element of E. Suppose there exist a positive constant r and a positive integer n_0 which satisfy the following: If $n \geq n_0$ and $x \in \partial B_r[0]$, then there exists $j_n \in E^*$ such that $\langle x, j_n \rangle > 0$ and*

$$\langle x - p + SJ_n(nx), j_n \rangle \geq 0.$$

Then there exists a sequence $\{x_n\}_{n \geq n_0}$ such that $J_n(nx_n)$ is a solution of the equation (2.1) for each $n \geq n_0$. Suppose further $\lim_{n \longrightarrow \infty} \left\| \frac{1}{n} J_n(nx_n) \right\| = 0$. Then, $p \in \overline{(A+S)(D(A))}$ holds.

Proof. Let n be a positive integer with $n \geq n_0$ and T_n a mapping defined by $T_n x = p - SJ_n(nx)$ for $x \in E$. By Proposition 2.3, T_n is compact. By Proposition 3.1, there exists $x_n \in B_r[0]$ such that $f(T_n x_n) T_n x_n = x_n$, where f is defined as in Lemma 2.1. Also, by Lemma 2.1, we know $f(T_n x_n) \in (0, 1]$ and $c_n = \frac{1}{f(T_n x_n)} \in [1, \infty)$. So, $T_n x_n = c_n x_n$.

For the case that $x_n \in B_r(0)$, by (ii) of Proposition 3.1, we have $T_n x_n = x_n$. For the case that $x_n \in \partial B_r[0]$, there exists $j_n \in E^*$ such that $\langle x_n, j_n \rangle > 0$ and $\langle x_n - p + SJ_n(nx_n), j_n \rangle \geq 0$. From this, we have

$$0 \leq \langle x_n, j_n \rangle \langle x_n - p + SJ_n(nx_n), j_n \rangle = (1 - c_n) \langle x_n, j_n \rangle^2.$$

Then we see $c_n = 1$ and $T_n x_n = x_n$. In both cases, $T_n x_n = x_n$ holds. The rest of the proof is the same as in the proof of Theorem 5.2. ☐

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