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COMMON FIXED POINT THEOREMS FOR ASYMPTOTIC REGULARITY IN GENERALIZED b-METRIC SPACES

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ABSTRACT. In this paper, we introduce the concept of a b-metric-like space, which is an intriguing extension of the b-metric space, and it encompasses certain sufficient conditions for the existence of a common fixed point. The discoveries presented here build upon and broaden previous research in this area..

KEYWORDS:b-metric-like space, Common fixed point, Asymptotic regularity.
AMS Subject Classification: :46C05, 47H09, 47H10,

1. Introduction

In 1920, Banach [1] introduced the Banach contraction principle, which has long been one of the most essential methods for approximating solutions to nonlinear problems. Several authors have expanded and developed it in various disciplines due to its usefulness in various branches.

The Banach contraction principle states

Theorem 1.1. [1] *Let (X, d) be a complete metric space and let f be a contraction on X, there exists $M \in [0, 1)$ such that*

$$d\left(fx, fy\right) \leq Md\left(x, y\right), \forall x, y \in X.$$

Then f has a unique fixed point.

In recent years, many scholars have proposed a series of new concepts of contraction mapping and new fixed point theorems [2, 3, 4, 5, 6, 7]. In 1993, Bakhtin [2] introduced the concept of b-metric space which is a generalization of metric space. He proved the famous Banach Contraction Principle in the b-metric space, also see [3].

In 2013, the concept of a b-metric-like space was introduced first by Alghamdi

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[5]. For some fixed point results on b-metric-like spaces, see [8] and [9].

In 2019, Bisht and Singh [10] obtain the existence of common fixed point theorems for mappings satisfying Lipschitz–Kannan type condition.

Theorem 1.2. [10] *If (X, d) is a complete metric space and $f, g : X \longrightarrow X$. Suppose that f is asymptotically regular with respect to g and there exist $M \in [0, 1)$ and $K \in [0, \infty)$ satisfying*

$$d(fx, fy) \leq Md(gx, gy) + K\{d(fx, gx) + d(fy, gy)\}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

In 2020, Arunchai, Mungchai and Thala [11] propose the common fixed point theorems for asymptotic regularity on b-metric spaces. The results presented in the paper improve and extend some previous results.

Theorem 1.3. [11] *If (X, b) is a complete b-metric space and $f, g : X \longrightarrow X$. Suppose that f is asymptotically regular with respect to g and there exist $M \in [0, 1)$ and $K \in [0, \infty)$ satisfying*

$$b(fx, fy) \leq Mb(gx, gy) + K\{b(fx, gx) + b(fy, gy)\}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

In this paper, we introduce common fixed point theorems for asymptotic regularity in generalized b-metric spaces, which is a fascinating extension of the b-metric space and contains some sufficient conditions for the presence of a common fixed point. The findings here improve and expand upon prior research.

2. Preliminaries

The following concepts and results are needed for the results.

Definition 2.1. [5] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $b_l : X \times X \longrightarrow \mathbb{R}^+$ is a b-metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_l1) $b_l(x, y) = 0$ implies $x = y$
(b_l2) $b_l(x, y) = b_l(y, x)$
(b_l3) $b_l(x, z) \leq s[b_l(x, y) + b_l(y, z)]$

A b-metric-like space is a pair (X, b_l) such that X is a nonempty set and b_l is a b-metric-like on X. The number s is called the coefficient of (X, b_l).

Definition 2.2. [5] Let (X, b_l) be a b-metric-like space with coefficient s , $\{x_n\}$ be any sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ is said to be convergent to x with respect to τ_{b_l} if $\lim_{n \longrightarrow \infty} b_l(x_n, x) = b_l(x, x)$.
(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, b_l), if $\lim_{n,m \longrightarrow \infty} b_l(x_n, x_m)$ exists and is finite.
(iii) (X, b_l) is said to be a complete b-metric-like space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n,m \longrightarrow \infty} b_l(x_n, x_m) = \lim_{n \longrightarrow \infty} b_l(x_n, x) = b_l(x, x).$$

Note that in a b-metric-like space the limit of a convergent sequence may not be unique.

Definition 2.3. [12] Let f and g be self-mapings of a set X (*i.e.*, $f, g : X \longrightarrow X$). If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The set of coincidence points of f and g are denoted by $C(f, g)$ and the set point of coincidences of f and g are denoted by $PC(f, g)$. If $w = x$ then x is a common fixed point of f and g and the set of common fixed points is denoted by $F(f, g)$.

Definition 2.4. [12] Let f and g be self-mapings of a set X (*i.e.*, $f, g : X \longrightarrow X$). Then f and g are called weakly compatible if they commute at every coincidence point,*i.e.*, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Definition 2.5. Let f and g be two self-mappings of a b-metric-like space (X, b_l). Then f and g are called compatible if $\lim\limits_{n\longrightarrow\infty} b_l(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = z$$

for some $t \in X$.

Definition 2.6. Let f and g be two self-mappings of a b-metric-like space (X, b_l). Then f is asymptotically regular with respect to g at $x_0 \in X$, if there exists a sequence $\{x_n\}$ in X such that $gx_{n+1} = fx_n$, for $n = 0, 1, 2, ,$ and

$$\lim_{n\longrightarrow\infty} b_l(gx_{n+1}, gx_{n+2}) = 0.$$

Definition 2.7. Let f and g be two self-mappings of a b-metric-like space (X, b_l) and let $\{x_n\}$ be a sequence in X such that $fx_n = gx_{n+1}$. Then the set $O(x_0, f, g) = \{fx_n : n = 0, 1, 2, ...\}$ is called the (f, g)-orbit at x_0 and g is called (f, g)-orbitally continuous if $\lim\limits_{n\longrightarrow\infty} fx_n = z$ implies $\lim\limits_{n\longrightarrow\infty} gfx_n = gz$ or f is called (f, g)-orbitally continuous if $\lim\limits_{n\longrightarrow\infty} fx_n = z$ implies $\lim\limits_{n\longrightarrow\infty} ffx_n = fz$. We say f and g are orbitally continuous if f is (f, g)-orbitally continuous and g is (f, g)-orbitally continuous.

3. Main Results

In this section, we shall prove the existence of common fixed point in b-metric-like space under some conditions.

Theorem 3.1. *If (X, b_l) is a complete b-metric-like space and $f, g : X \longrightarrow X$. Suppose that f is asymptotically regular with respect to g and there exist $M \in [0, 1)$ and $K \in [0, \infty)$ satisfying*

$$b_l(fx, fy) \leq Mb_l(gx, gy) + K\{b_l(fx, gx) + b_l(fy, gy)\} \tag{3.1}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

Proof. Since f is asymptotically regular with respect to g at $x_0 \in x$, there exists a sequence $\{y_n\} \in X$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim\limits_{n\longrightarrow\infty} b_l(gx_{n+1}, gx_{n+2}) = \lim\limits_{n\longrightarrow\infty} b_l(y_n, y_{n+1}) = 0$. To show that $\{y_n\} \in$ is a Cauchy sequence. Using (3.1), for any n and any $p > 0$,

$$\begin{aligned} b_l(fx_{n+p}, fx_n) &= b_l(y_{n+p}, y_n) \\ &\leq s[b_l(y_{n+p}, y_{n+p+1}) + b_l(y_{n+p+1}, y_n)] \end{aligned}$$

$$\begin{aligned}
&\leq sb_l(y_{n+p}, y_{n+p+1}) + sb_l(y_{n+p+1}, y_n) \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2[b_l(y_{n+p+1}, y_{n+1}) + b_l(y_{n+1}, y_n)] \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2 b_l(y_{n+p+1}, y_{n+1}) + s^2 b_l(y_{n+1}, y_n) \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2\Big[Mb_l(y_{n+p}, y_n) + K\{b_l(y_{n+p+1}, y_{n+p}) \\
&\quad + b_l(y_{n+1}, y_n)\}\Big] + s^2 b_l(y_{n+1}, y_n) \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2 Mb_l(y_{n+p}, y_n) + s^2 Kb_l(y_{n+p+1}, y_{n+p}) \\
&\quad + s^2 Kb_l(y_{n+1}, y_n) + s^2 b_l(y_{n+1}, y_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
b_l(y_{n+p}, y_n) - s^2 Mb_l(y_{n+p}, y_n) \leq\ & sb_l(y_{n+p}, y_{n+p+1}) + s^2 Kb_l(y_{n+p+1}, y_{n+p}) \\
& + s^2 Kb_l(y_{n+1}, y_n) + s^2 b_l(y_{n+1}, y_n)
\end{aligned}$$

and so,

$$(1 - s^2 M)b_l(y_{n+p}, y_n) \leq (s + s^2 K)b_l(y_{n+p}, y_{n+p+1}) + (s^2 + s^2 K)b_l(y_{n+1}, y_n).$$

Then,

$$b_l(y_{n+p}, y_n) \leq \frac{(s + s^2 K)}{(1 - s^2 M)} b_l(y_{n+p}, y_{n+p+1}) + \frac{(s^2 + s^2 K)}{(1 - s^2 M)} b_l(y_{n+1}, y_n).$$

By f is asymptotic regularity with respect to g, we get that $\lim_{n\longrightarrow\infty} b_l(y_{n+p}, y_n) = 0$. Since $\lim_{m,n\longrightarrow\infty} b_l(y_m, y_n) = 0$ exists and finite, so $\{y_n\}$ is a Cauchy sequence. Since (X, b_l) is a complete b-metric-like space, we have $\{y_n\} \in X$ converges to $z \in X$ so that

$$\lim_{n\longrightarrow\infty} b_l(y_n, z) = b_l(z, z) = \lim_{m,n\longrightarrow\infty} b_l(y_m, y_n) = 0.$$

Hence,

$$\lim_{n\longrightarrow\infty} b_l(y_n, z) = \lim_{n\longrightarrow\infty} b_l(fx_n, z) = \lim_{n\longrightarrow\infty} b_l(gx_{n+1}, z) = 0.$$

Thus,

$$\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_{n+1} = z.$$

Since f and g are (f, g)-orbitally continuous, we get

$$\lim_{n\longrightarrow\infty} ffx_n = \lim_{n\longrightarrow\infty} fgx_{n+1} = fz$$

and

$$\lim_{n\longrightarrow\infty} gfx_n = \lim_{n\longrightarrow\infty} ggx_{n+1} = gz.$$

Since f and g are compatible, we obtain that $\lim_{n\longrightarrow\infty} b_l(fgx_{n+1}, gfx_n) = 0$.
Thus,

$$fz = \lim_{n\longrightarrow\infty} fgx_{n+1} = \lim_{n\longrightarrow\infty} gfx_n = gz.$$

Hence $fz = gz$ so $z \in C(f, g)$. That is $C(f, g) \neq \emptyset$.
By compatibility of f and g, we have $gfz = fgz = ffz = ggz$ and (f, g)-orbitally continuous of f and g implies that $b_l(fz, fz) = 0$.
Using (3.1), we obtain

$$\begin{aligned}
b_l(fz, ffz) &\leq Mb_l(gz, gfz) + K\{b_l(fz, gz) + b_l(ffz, gfz)\} \\
&= Mb_l(fz, ffz) + K\{b_l(fz, fz) + b_l(ffz, ffz)\}.
\end{aligned}$$

So,

$$b_l(fz, ffz) \leq Mb_l(fz, ffz).$$

Hence,

$$b_l(fz, ffz) - Mb_l(fz, ffz) \leq 0$$

and then,

$$(1 - M)b_l(fz, ffz) \leq 0.$$

Therefore,

$$b_l(fz, ffz) = 0.$$

Hence $fz = ffz = gfz$. This implies that fz is a common fixed point of f and g.

Suppose that fz and sz are common fixed point of f and g. This implies that $fz = ffz = gfz$ and $sz = fsz = gsz$ .To show that $fz = sz$.
Using (3.1) , we obtain that

$$\begin{aligned} b_l(fz, sz) &= b_l(ffz, fsz) \\ &\leq Mb_l(gfz, gsz) + K\{b_l(ffz, gfz) + b_l(fsz, gsz)\} \\ &\leq Mb_l(ffz, fsz) + K\{b_l(ffz, ffz) + b_l(fsz, fsz)\}. \end{aligned}$$

So,

$$b_l(ffz, fsz) \leq Mb_l(ffz, fsz).$$

Hence,

$$b_l(ffz, fsz) - Mb_l(ffz, fsz) \leq 0$$

and then,

$$(1 - M)b_l(ffz, fsz) \leq 0.$$

Therefore,

$$b_l(ffz, fsz) = 0.$$

Hence $fz = ffz = fsz = sz$. That is f and g have a unique common fixed point. ☐

Corollary 3.2. *If (X, b_l) is a complete b-metric-like space and $f, g : X \longrightarrow X$. Suppose that f is asymptotically regular with respect to g and there exist $K \in [0, \infty)$ satisfying*

$$b_l(fx, fy) \leq K\{b_l(fx, gx) + b_l(fy, gy)\} \tag{3.2}$$

for all $x, y \in X$. Further, suppose that f and g are (f, g)-orbitally continuous and compatible. Then $C(f, g) \neq \emptyset$ and f and g have a unique common fixed point.

Proof. Since f is asymptotically regular with respect to g at $x_0 \in x$, there exists a sequence $\{y_n\} \in X$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim\limits_{n \longrightarrow \infty} b_l(gx_{n+1}, gx_{n+2}) = \lim\limits_{n \longrightarrow \infty} b_l(y_n, y_{n+1}) = 0$. To show that $\{y_n\} \in$ is a Cauchy sequence. By (3.2), for any n and any $p > 0$,

$$\begin{aligned} b_l(fx_{n+p}, fx_n) &= b_l(y_{n+p}, y_n) \\ &\leq s[b_l(y_{n+p}, y_{n+p+1}) + b_l(y_{n+p+1}, y_n)] \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + sb_l(y_{n+p+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2[b_l(y_{n+p+1}, y_{n+1}) + b_l(y_{n+1}, y_n)] \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2b_l(y_{n+p+1}, y_{n+1}) + s^2b_l(y_{n+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2[K\{b_l(y_{n+p+1}, y_{n+p}) + b_l(y_{n+1}, y_n)\}] \\ &+ s^2b_l(y_{n+1}, y_n) \\ &\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2Kb_l(y_{n+p+1}, y_{n+p}) + s^2Kb_l(y_{n+1}, y_n) \\ &+ s^2b_l(y_{n+1}, y_n) \\ &\leq (s + s^2K)b_l(y_{n+p}, y_{n+p+1}) + (s^2K + s^2)b_l(y_{n+1}, y_n). \end{aligned}$$

Since f is asymptotic regularity with respect to g, we have $\lim_{n\longrightarrow\infty} b_l(y_{n+p}, y_n) = 0$. By $\lim_{m,n\longrightarrow\infty} b_l(y_m, y_n) = 0$ exists and finite, $\{y_n\}$ is a Cauchy sequence. Since (X, b_l) is a complete b-metric-like space, we obtain $\{y_n\} \in X$ converges to $z \in X$ so that

$$\lim_{n\longrightarrow\infty} b_l(y_n, z) = b_l(z, z) = \lim_{m,n\longrightarrow\infty} b_l(y_m, y_n) = 0.$$

Hence,

$$\lim_{n\longrightarrow\infty} b_l(y_n, z) = \lim_{n\longrightarrow\infty} b_l(fx_n, z) = \lim_{n\longrightarrow\infty} b_l(gx_{n+1}, z) = 0.$$

Therefore,

$$\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_{n+1} = z.$$

By f and g are (f, g)-orbitally continuous, we get

$$\lim_{n\longrightarrow\infty} ffx_n = \lim_{n\longrightarrow\infty} fgx_{n+1} = fz$$

and

$$\lim_{n\longrightarrow\infty} gfx_n = \lim_{n\longrightarrow\infty} ggx_{n+1} = gz.$$

Since f and g are compatible, we have $\lim_{n\longrightarrow\infty} b_l(fgx_{n+1}, gfx_n) = 0$.
Thus,

$$fz = \lim_{n\longrightarrow\infty} fgx_{n+1} = \lim_{n\longrightarrow\infty} gfx_n = gz.$$

Hence $fz = gz$ such that $z \in C(f, g)$, so $C(f, g) \neq \emptyset$.
By compatibility of f and g, we have $gfz = fgz = ffz = ggz$ and (f, g)-orbitally continuous of f and g implies that $b_l(fz, fz) = 0$.
Using (3.2), we obtain that

$$\begin{aligned} b_l(fz, ffz) &\leq K\{b_l(fz, gz) + b_l(ffz, gfz)\} \\ &= K\{b_l(fz, fz) + b_l(ffz, ffz)\}. \end{aligned}$$

So,

$$b_l(fz, ffz) \leq 0.$$

Thus,

$$b_l(fz, ffz) = 0.$$

Hence $fz = ffz = gfz$. This implies that fz is a common fixed point of f and g.

Suppose that fz and sz are common fixed point of f and g implies that $fz = ffz = gfz$ and $sz = fsz = gsz$ .To show that $fz = sz$.
Using (3.2) , we have

$$\begin{aligned} b_l(fz, sz) &= b_l(ffz, fsz) \\ &\leq K\{b_l(ffz, gfz) + b_l(fsz, gsz)\} \\ &\leq K\{b_l(ffz, ffz) + b_l(fsz, fsz)\}. \end{aligned}$$

So,

$$b_l(fz, sz) \leq 0.$$

Thus,

$$b_l(fz, sz) = 0.$$

Hence $fz = sz$. This implies that f and g have a unique common fixed point. □

In the next theorem, we relax the condition of orbital continuity for a pair of mappings considered in Theorem 3.1, while also relaxing compatibility by introducing the minimal non-commuting notion, i.e., non-trivial weak compatibility.

Theorem 3.3. *If (X, b_l) is b-metric-like space and f and g be self-mappings on an arbitrary non-empty set Y with values in a b-metric-like space X. Suppose that f is asymptotically regular with respect to g and gY is a complete subset of X for $M, K \in [0,1)$ satisfying*

$$b_l(fx, fy) \leq Mb_l(gx, gy) + K\{b_l(fx, gx) + b_l(fy, gy)\}$$

for all $x, y \in Y$. Then $C(f,g) \neq \emptyset$ Moreover, if $Y = X$, then f and g have a unique common fixed point provided f and g are non-trivially weakly compatible.

Proof. By f is asymptotically regular with respect to g at $x_0 \in x$, there exists a sequence $\{y_n\} \in X$ in X such that $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gx_{n+2}) = \lim_{n \longrightarrow \infty} b_l(y_n, y_{n+1}) = 0$. To show that $\{y_n\} \in$ is a Cauchy sequence. Using (3.1), for any n and any $p > 0$,

$$\begin{aligned}
b_l(fx_{n+p}, fx_n) &= b_l(y_{n+p}, y_n) \\
&\leq s[b_l(y_{n+p}, y_{n+p+1}) + b_l(y_{n+p+1}, y_n)] \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + sb_l(y_{n+p+1}, y_n) \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2[b_l(y_{n+p+1}, y_{n+1}) + b_l(y_{n+1}, y_n)] \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2 b_l(y_{n+p+1}, y_{n+1}) + s^2 b_l(y_{n+1}, y_n) \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2 \Big[Mb_l(y_{n+p}, y_n) + K\{b_l(y_{n+p+1}, y_{n+p}) \\
&\quad + b_l(y_{n+1}, y_n)\}\Big] + s^2 b_l(y_{n+1}, y_n) \\
&\leq sb_l(y_{n+p}, y_{n+p+1}) + s^2 Mb_l(y_{n+p}, y_n) + s^2 Kb_l(y_{n+p+1}, y_{n+p}) \\
&\quad + s^2 Kb_l(y_{n+1}, y_n) + s^2 b_l(y_{n+1}, y_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
b_l(y_{n+p}, y_n) - s^2 Mb_l(y_{n+p}, y_n) \leq\ & sb_l(y_{n+p}, y_{n+p+1}) + s^2 Kb_l(y_{n+p+1}, y_{n+p}) \\
& + s^2 Kb_l(y_{n+1}, y_n) + s^2 b_l(y_{n+1}, y_n)
\end{aligned}$$

and then,

$$(1 - s^2 M)b_l(y_{n+p}, y_n) \leq (s + s^2 K)b_l(y_{n+p}, y_{n+p+1}) + (s^2 + s^2 K)b_l(y_{n+1}, y_n).$$

Hence,

$$b_l(y_{n+p}, y_n) \leq \frac{(s + s^2 K)}{(1 - s^2 M)} b_l(y_{n+p}, y_{n+p+1}) + \frac{(s^2 + s^2 K)}{(1 - s^2 M)} b_l(y_{n+1}, y_n).$$

By f is asymptotic regularity with respect to g, we have $\lim_{n \longrightarrow \infty} b_l(y_{n+p}, y_n) = 0$. Since $\lim_{m,n \longrightarrow \infty} b_l(y_m, y_n) = 0$ exists and finite, so $\{y_n\}$ is a Cauchy sequence in gY. By gY is a complete subset of X and $y_n = fx_n = gx_{n+1}$is a Cauchy sequence in gY, there exists some $z \in X$ such that

$$\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gz) = b_l(gz, gz) = \lim_{m,n \longrightarrow \infty} b_l(gx_{n+1+m}, gx_{n+1}) = 0.$$

Hence,

$$\lim_{n \longrightarrow \infty} b_l(gx_{n+1}, gz) = \lim_{n \longrightarrow \infty} b_l(fx_n, gz) = 0.$$

Therefore,

$$\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_{n+1} = gz.$$

Using (3.1), we obtain

$$b_l(fx_n, fz) \leq Mb_l(gx_n, gz) + K\{b_l(fx_n, gx_n) + b_l(fz, gz)\} = Kb_l(fz, gz).$$

Thus,

$$b_l(gz, fz) = b_l(fx_n, fz) \leq Kb_l(fz, gz)$$

and so,

$$b_l(gz, fz) - Kb_l(fz, gz) \leq 0.$$

Hence,

$$(1-K)b_l(fz, gz) \leq 0.$$

Therefore,

$$b_l(fz, gz) = 0.$$

Hence $fz = gz$ such that $z \in C(f, g)$. That is $C(f, g) \neq \emptyset$.
Since $Y = X$ and f and g are non-trivially weakly compatible, we have $gfz = fgz$.
Moreover, implies that $gfz = fgz = ffz = ggz$.
Using (3.1), we obtain

$$\begin{aligned} b_l(fz, ffz) &\leq Mb_l(gz, gfz) + K\{b_l(fz, gz) + b_l(ffz, gfz)\} \\ &= Mb_l(fz, ffz) + K\{b_l(fz, fz) + b_l(ffz, ffz)\}. \end{aligned}$$

Thus,

$$b_l(fz, ffz) \leq Mb_l(fz, ffz).$$

Hence,

$$b_l(fz, ffz) - Mb_l(fz, ffz) \leq 0$$

and so,

$$(1-M)b_l(fz, ffz) \leq 0.$$

Therefore,

$$b_l(fz, ffz) = 0.$$

Hence $fz = ffz = gfz$. This implies that fz is a common fixed point of f and g.

Suppose that fz and sz are common fixed point of f and g, we get that $fz = ffz = gfz$ and $sz = fsz = gsz$. To show that $fz = sz$.
Using (3.1) , we have

$$\begin{aligned} b_l(fz, sz) &= b_l(ffz, fsz) \\ &\leq Mb_l(gfz, gsz) + K\{b_l(ffz, gfz) + b_l(fsz, gsz)\} \\ &\leq Mb_l(ffz, fsz) + K\{b_l(ffz, ffz) + b_l(fsz, fsz)\}. \end{aligned}$$

Thus,

$$b_l(ffz, fsz) \leq Mb_l(ffz, fsz).$$

Hence,

$$b_l(ffz, fsz) - Mb_l(ffz, fsz) \leq 0$$

and so,

$$(1-M)b_l(ffz, fsz) \leq 0.$$

Therefore,

$$b_l(ffz, fsz) = 0.$$

Hence $fz = ffz = fsz = sz$. This implies that f and g have a unique common fixed point. □

Corollary 3.4. *If (X, b_l) is b-metric-like space and f and g be self-mappings on an arbitrary non-empty set Y with values in a b-metric-like space X. Suppose that f is asymptotically regular with respect to g and gY is a complete subset of X for $M, K \in [0,1)$ satisfying*

$$b_l(fx, fy) \leq Mmax\left\{b_l(gx, gy), b_l(fx, gx), b_l(fy, gy), \frac{b_l(fx, gy) + b_l(fy, gx)}{2}\right\}$$

for all $x, y \in Y$. Then $C(f,g) \neq \emptyset$ Moreover, if $Y = X$, then f and g have a unique common fixed point provided f and g are non-trivially weakly compatible.

Remark 3.5. Let $K = 0$ in Theorem 3.3, we obtain

$$\begin{aligned} b_l(fx, fy) &\leq Mb_l(gx, gy) + K\{b_l(fx, gx) + b_l(fy, gy) \\ &= Mb_l(gx, gy) \\ &\leq Mmax\left\{b_l(gx, gy), b_l(fx, gx), b(fy, gy), \frac{b_l(fx, gy) + b_l(fy, gx)}{2}\right\} \end{aligned}$$

Hence satisfy the condition in Corollary 3.4.

Corollary 3.6. [11] *If (X, b) is b-metric space and f and g be self-mappings on an arbitrary non-empty set Y ith values in a b-metric space X. Suppose that f is asymptotically regular with respect to g and gY is a complete subset of X for $M, K \in [0,1)$ satisfying*

$$b(fx, fy) \leq Mmax\left\{b(gx, gy), b(fx, gx), b(fy, gy), \frac{b(fx, gy) + b(fy, gx)}{2}\right\}$$

for all $x, y \in Y$. Then $C(f,g) \neq \emptyset$ Moreover, if $Y = X$, then f and g have a unique common fixed point provided f and g are non-trivially weakly compatible.

4. Conclusion

We have introduced a new extension of the concept of a b-metric space, termed a b-metric-like space. Furthermore, we have established Common Fixed Point Theorems for Asymptotic Regularity in b-Metric-Like Spaces.

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MODIFIED GERAGHTY TYPE VIA SIMULATION FUNCTIONS

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ABSTRACT. We explore the Geraghty contraction through a simulation function, elucidating certain conditions for the existence and uniqueness of coincidence points for multiclass mappings involving the Geraghty function in metric spaces. The results presented in this work are consistent with those found in existing literature.

KEYWORDS: Geraghty type contraction mapping, simulation function, point of coincidence, common fixed point.
AMS Subject Classification: 47H09: 47H10.

1. Introduction

The field of fixed point theory emerged in the last quarter of the nineteenth century, and it has since been utilized extensively to establish the existence and uniqueness of solutions, particularly for functional equations. A significant contribution to this area is the Banach contraction principle, attributed to Banach [1], which has found widespread application in various contemporary research endeavors [2, 3, 4, 5, 6]. Fixed point theory finds applications across diverse fields such as engineering, economics, and computer science.

Geraghty [22] introduced the Cauchy criteria for convergence of contractive iterations in complete metric spaces, which led to the development of the Geraghty contraction. Subsequently, Khojasteh et al. [21] introduced the concept of $\mathcal{Z}$-contractions, which has been further investigated and summarized by numerous researchers [7, 8, 9, 10, 11, 12, 13, 14, 15]. Fixed point theory offers a rich platform for conducting interesting research.

Let Ω and Ψ be two self-maps defined on a non-empty set Π. If $\eta = \Omega\mu = \Psi\mu$ for some $\mu \in \Pi$, then μ is termed a coincidence point of Ω and Ψ. Consequently, η is referred to as a point of coincidence of Ω and Ψ. Furthermore, η is deemed

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a common fixed point of Ω and Ψ if $\mu = \eta$. A pair (Ω, Ψ) of self-maps is termed weakly compatible if they commute at their coincidence points.

In this article, we modify the Geraghty contraction using a simulation function and investigate the requisite conditions for its existence. We also focus on non-commuting type mappings, which are crucial for establishing the existence of common fixed points and the uniqueness of coincidence points, as well as common fixed points for classes of mappings in complete metric spaces. Finally, we provide an illustrative example to corroborate our theorem.

2. Preliminaries

Definition 2.1. [17] Two self-mappings Ω and Ψ of a metric space (Π, Λ) are compatible if

$$\lim_{n \longrightarrow \infty} \Lambda(\Psi\Omega(\mu_n), \Omega\Psi(\mu_n)) = 0$$

whenever $\{\mu_n\}$ is a sequence in Π such that

$$\lim_{n \longrightarrow \infty} \Omega(\mu_n) = \lim_{n \longrightarrow \infty} \Psi(\mu_n) = t$$

for some $t \in \Pi$.

Theorem 2.1. [18] *Let Ω and Ψ be weakly compatible self-maps defined on a non-empty set Π. If Ω and Ψ have a unique point of coincidence $\eta = \Omega\mu = \Psi\eta$, then η is the unique common fixed point of Ω and Ψ.*

Definition 2.2. [19] Let (Π, Λ) is a metric space and $\Omega, \Psi : \Pi \longrightarrow$ be two mappings. The mappings Ω and Ψ are said to satisfy the common limit in the range of Ψ (shortly, (CLR_Ψ) property) if there exists a sequence $\{\mu_n\}$ in Π such that

$$\lim_{n \longrightarrow \infty} \Omega(\mu_n) = \lim_{n \longrightarrow \infty} \Psi(\mu_n) = \Psi(\mu)$$

for some $\mu \in \Pi$. The importance of (CLR_Ψ)-property ensures that one does not require the closeness of range subspaces.

Lemma 2.3. [20] *Let (Π, Λ) be a metric space and let $\{\mu_n\}$ be a sequence in Π such that $\Lambda(\mu_n, \mu_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. If $\{\mu_n\}$ is not a Cauchy sequence in Π, then there exist $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that $n_k > m_k > k$ and the following sequences tend to ε when $k \longrightarrow \infty$:*

$$\{\Lambda(\mu_{m_k}, \mu_{n_k})\}, \{\Lambda(\mu_{m_k}, \mu_{n_k+1})\}, \{\Lambda(\mu_{m_k-1}, \mu_{n_k})\},$$
$$\{\Lambda(\mu_{m_k-1}, \mu_{n_k+1})\}, \{\Lambda(\mu_{m_k+1}, \mu_{n_k+1})\}.$$

Definition 2.4. [21] A mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

- (ζ_1) $\zeta(0, 0) = 0$;
- (ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
- (ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\lim_{n\to\infty} \zeta(t_n, s_n) < 0$.

Denoted by $\mathcal{Z}$ is the set of all simulation functions.

Example 2.5. [21] The following are some examples of simulation functions.

- (i) $\zeta(t, s) = \alpha s - t$ for all $t, s \in [0, \infty)$, where $\alpha \in [0, 1)$;
- (ii) $\zeta(t, s) = \frac{s}{1+s} - t$ for all $t, s \in [0, \infty)$;
- (iii) $\zeta(t, s) = sf(s) - t$ for all $t, s \in [0, \infty)$, where $f : [0, \infty) \to [0, 1)$ such that $\lim_{t\to c} f(t) < 1$ for all $c > 0$.

Definition 2.6. [21] Let $(\mathit{\Pi}, \mathit{\Lambda})$ be a metric space and $\zeta \in \mathcal{Z}$. A mapping $\mathit{\Omega} : \mathit{\Pi} \longrightarrow \mathit{\Pi}$ is called a $\mathcal{Z}$-contraction with respect to ζ if

$$\zeta(\mathit{\Lambda}(\mathit{\Omega}\mu, \mathit{\Omega}\upsilon), \mathit{\Lambda}(\mu, \upsilon)) \geq 0$$

holds for all $\mu, \upsilon \in \mathit{\Pi}$.

We denote by $\mathcal{F}$ the class of all functions $\beta : [0, \infty) \longrightarrow [0, 1)$satisfying $\beta(t_n) \longrightarrow 1$, implies $t_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 2.7. [22] Let $(\mathit{\Pi}, \mathit{\Lambda})$ be a metric space. A map $\mathit{\Omega} : \mathit{\Pi} \longrightarrow \mathit{\Pi}$ is called Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $\mu, \upsilon \in \mathit{\Pi}$,

$$\mathit{\Lambda}(\mathit{\Omega}\mu, \mathit{\Omega}\upsilon) \leq \beta(\mathit{\Lambda}(\mu, \upsilon))\mathit{\Lambda}(\mu, \upsilon)$$

Theorem 2.2. [22] *Let $(\mathit{\Pi}, \mathit{\Lambda})$ be a complete metric space. Mapping $\mathit{\Omega} : \mathit{\Pi} \longrightarrow \mathit{\Pi}$ is Geraghty contraction. Then $\mathit{\Omega}$ has a fixed point $\mu \in \mathit{\Pi}$, and $\{\mathit{\Omega}^n \mu_1\}$ converges to μ.*

3. Main Results

Theorem 3.1. *Let $(\mathit{\Pi}, \mathit{\Lambda})$ be a complete metric space and $\mathit{\Omega}, \mathit{\Psi} : \mathit{\Pi} \longrightarrow \mathit{\Pi}$ be two self-mappings. Suppose that there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta\left(\mathit{\Lambda}(\mathit{\Omega}\mu, \mathit{\Omega}\upsilon), \beta(\mathit{\Upsilon}_{\mathit{\Psi}\mathit{\Omega}}(\mu, \upsilon))\mathit{\Upsilon}_{\mathit{\Psi}\mathit{\Omega}}(\mu, \upsilon)\right) \geq 0, \tag{3.1}$$

for all $\mu, \upsilon \in \mathit{\Pi}$ with $\mathit{\Psi}\mu \neq \mathit{\Psi}\upsilon$, where $\beta : [0, \infty) \longrightarrow (0, 1)$ and

$$\mathit{\Upsilon}_{\mathit{\Psi}\mathit{\Omega}}(\mu, \upsilon) = \max\left\{\mathit{\Lambda}(\mathit{\Psi}\mu, \mathit{\Psi}\upsilon), \frac{[1 + \mathit{\Lambda}(\mathit{\Psi}\mu, \mathit{\Omega}\mu)]\mathit{\Lambda}(\mathit{\Psi}\upsilon, \mathit{\Omega}\upsilon)}{1 + \mathit{\Lambda}(\mathit{\Psi}\mu, \mathit{\Psi}\upsilon)}\right\}.$$

Suppose that there exists a Picard-Jungck sequence $\{j_n\}$ of $(\mathit{\Omega}, \mathit{\Psi})$. Also assume that, at least, one of the following conditions holds:

(i) *$(\mathit{\Omega}\mathit{\Pi}, \mathit{\Lambda})$ or $(\mathit{\Psi}\mathit{\Pi}, \mathit{\Lambda})$ is complete*;
(ii) *$(\mathit{\Pi}, \mathit{\Lambda})$ is complete, $\mathit{\Psi}$ is continuous, $\mathit{\Omega}$ and $\mathit{\Psi}$ are compatible.*

Then $\mathit{\Omega}$ and $\mathit{\Psi}$ have a unique point of coincidence.

Proof. Firstly, we will show that the point of coincidence of $\mathit{\Omega}$ and $\mathit{\Psi}$ is unique. Suppose that η_1 and η_2 are distinct points of coincidence of $\mathit{\Omega}$ and $\mathit{\Psi}$. It follows that there exist two points θ_1 and $\theta_2\,(\theta_1 \neq \theta_2)$ such that $\mathit{\Omega}\theta_1 = \mathit{\Psi}\theta_1 = \eta_1$ and $\mathit{\Omega}\theta_2 = \mathit{\Psi}\theta_2 = \eta_1$. Then $d(\mathit{\Omega}\theta_1, \mathit{\Omega}\theta_2) > 0$ and using (ζ_2), we obtain

$$0 \leq \zeta\left(\mathit{\Lambda}(\mathit{\Omega}\theta_1, \mathit{\Omega}\theta_2), \beta(\mathit{\Upsilon}_{\mathit{\Psi}\mathit{\Omega}}(\theta_1, \theta_2))\mathit{\Upsilon}_{\mathit{\Psi}\mathit{\Omega}}(\theta_1, \theta_2)\right), \tag{3.2}$$

where

$$\begin{aligned}
\mathit{\Upsilon}_{\mathit{\Psi}\mathit{\Omega}}(\theta_1, \theta_2) &= \max\left\{\mathit{\Lambda}(\mathit{\Psi}\theta_1, \mathit{\Psi}\theta_2), \frac{[1 + \mathit{\Lambda}(\mathit{\Psi}\theta_1, \mathit{\Omega}\theta_1)]\mathit{\Lambda}(\mathit{\Psi}\theta_2, \mathit{\Omega}\theta_2)}{1 + \mathit{\Lambda}(\mathit{\Psi}\theta_1, \mathit{\Psi}\theta_2)}\right\} \\
&= \max\left\{\mathit{\Lambda}(\eta_1, \eta_2), \frac{[1 + \mathit{\Lambda}(\eta_1, \eta_1)]\mathit{\Lambda}(\eta_2, \eta_2)}{1 + \mathit{\Lambda}(\eta_1, \eta_2)}\right\} \\
&= \max\left\{\mathit{\Lambda}(\eta_1, \eta_2), 0\right\} \\
&= \mathit{\Lambda}(\eta_1, \eta_2).
\end{aligned}$$

This together with (3.2) show that

$$\begin{aligned}0 &\leq \zeta\left(\varLambda(\varOmega\theta_1,\varOmega\theta_2),\beta(\varUpsilon_{\varPsi\varOmega}(\theta_1,\theta_2))\varUpsilon_{\varPsi\varOmega}(\theta_1,\theta_2)\right)\\&=\zeta\left(\varLambda(\eta_1,\eta_2),\beta(\varLambda(\eta_1,\eta_2))\varLambda(\eta_1,\eta_2)\right)\\&<\beta(\varLambda(\eta_1,\eta_2))\varLambda(\eta_1,\eta_2)-\varLambda(\eta_1,\eta_2)\\&<\varLambda(\eta_1,\eta_2)-\varLambda(\eta_1,\eta_2)\\&=0\end{aligned}$$

which is a contradiction. Suppose that there is a Picard-Jungck sequence $\{j_n\}$ such that $j_n=\varOmega\mu_n=\varPsi\mu_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$. If $j_m=j_{m+1}$ for some $m\in\mathbb{N}\cup\{0\}$, then $\varPsi\mu_{m+1}=j_m=j_{m+1}=\varOmega\mu_{m+1}$. Hence $\varPsi$ and $\varOmega$ have a coiucidence point μ_{m+1}. Therefore, we assume that $j_n\neq j_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$. Also, $\varLambda(j_{n+1},j_{n+2})>0$ and taking $\mu=\mu_{n+1}$, $\upsilon=\mu_{n+2}$ in (3.1), we get that

$$\zeta\left(\varLambda(\varOmega\mu_{n+1},\varOmega\mu_{n+2}),\beta(\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2}))\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2})\right)\geq 0, \tag{3.3}$$

where

$$\begin{aligned}&\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2})\\&=\max\left\{\varLambda(\varPsi\mu_{n+1},\varPsi\mu_{n+2}),\frac{[1+\varLambda(\varPsi\mu_{n+1},\varOmega\mu_{n+1})]\varLambda(\varPsi\mu_{n+2},\varOmega\mu_{n+2})}{1+\varLambda(\varPsi\mu_{n+1},\varPsi\mu_{n+2})}\right\}\\&=\max\left\{\varLambda(j_n,j_{n+1}),\frac{[1+\varLambda(j_n,j_{n+1})]\varLambda(j_{n+1},j_{n+2})}{1+\varLambda(j_n,j_{n+1})}\right\}.\end{aligned}$$

This together with (3.3) show that

$$\begin{aligned}0 &\leq \zeta\left(\varLambda(\varOmega\mu_{n+1},\varOmega\mu_{n+2}),\beta(\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2}))\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2})\right)\\&=\zeta\left(\varLambda(j_{n+1},j_{n+2}),\beta(\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2}))\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2})\right)\\&<\beta(\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2}))\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2})-\varLambda(j_{n+1},j_{n+2})\\&<\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2})-\varLambda(j_{n+1},j_{n+2}).\end{aligned}\tag{3.4}$$

If $\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2})=\varLambda(j_{n+1},j_{n+2})$, inequality (3.4) gives

$$\varLambda(j_{n+1},j_{n+2})<\varLambda(j_{n+1},j_{n+2})$$

which is a contradiction. Hence, $\varUpsilon_{\varPsi\varOmega}(\mu_{n+1},\mu_{n+2})=\varLambda(j_n,j_{n+1})$. This implies that

$$\varLambda(j_{n+1},j_{n+2})<\varLambda(j_n,j_{n+1})$$

for all $n\in\mathbb{N}\cup\{0\}$. Thus, there exists $\rho>0$ such that $\lim\limits_{n\longrightarrow\infty}\varLambda(j_n,j_{n+1})=\rho$. Assme that $\rho>0$. In this case we get that

$$\frac{\varLambda(j_{n+1},j_{n+2})}{\varLambda(j_n,j_{n+1})}\leq\beta\left(\varLambda(j_n,j_{n+1})\right)<1,$$

taking $n\longrightarrow\infty$, we get $\lim\limits_{n\longrightarrow\infty}\beta\left(\varLambda(j_n,j_{n+1})\right)=1$ which is a contradiction to the fact that $\lim\limits_{n\longrightarrow\infty}\varLambda(j_n,j_{n+1})=\rho>0$. Hence, $\lim\limits_{n\longrightarrow\infty}\varLambda(j_n,j_{n+1})=0$. Next , we will show that $j_n\neq j_m$, whenever $n\neq m$. Assume that $j_n=j_m$ for some $n>m$. Then we can claim that $\mu_{n+1}=\mu_{m+1}$. If $\mu_{n+1}\neq\mu_{m+1}$, then

$$\varOmega\mu_n\neq\varOmega\mu_m\Rightarrow j_n\neq j_m$$

which is obviously impossible . Hence

$$\begin{aligned}\mu_{n+1}=\mu_{m+1}&\Rightarrow\varOmega\mu_{n+1}=\varOmega\mu_{m+1}\\&\Rightarrow j_{n+1}=j_{m+1}.\end{aligned}$$

Then following above, we obtain

$$\begin{aligned}\varLambda(j_{m+1}, j_m) &< \varLambda(j_m, j_{m-1}) \\ &\vdots \\ &< \varLambda(j_{n+1}, j_n) \\ &= \varLambda(j_{m+1}, j_m)\end{aligned}$$

which is a contradiction. Now, we will show that $\{j_n\}$ is a Cauchy sequence. Assume that $\{j_n\}$ is not a Cauchy sequence. Taking $\mu = \mu_{m_k+1}$, $\upsilon = \mu_{n_k+1}$ in (3.1), we get that

$$\zeta\left(\varLambda(\varOmega\mu_{m_k+1}, \varOmega\mu_{n_k+1}, \beta(\varUpsilon_{\varPsi\varOmega}(\mu_{m_k+1}, \mu_{n_k+1}))\varUpsilon_{\varPsi\varOmega}(\mu_{m_k+1}, \mu_{n_k+1})\right) \geq 0, \tag{3.5}$$

where

$$\begin{aligned}&\varUpsilon_{\varPsi\varOmega}(\mu_{m_k+1}, \mu_{n_k+1}) \\ &= \max\left\{\varLambda(\varPsi\mu_{m_k+1}, \varPsi\mu_{n_k+1}), \frac{[1+\varLambda(\varPsi\mu_{m_k+1}, \varOmega\mu_{m_k+1})]\varLambda(\varPsi\mu_{n_k+1}, \varOmega\mu_{n_k+1})}{1+\varLambda(\varPsi\mu_{m_k+1}, \varPsi\mu_{n_k+1})}\right\} \\ &= \max\left\{\varLambda(j_{m_k}, j_{n_k}), \frac{[1+\varLambda(j_{m_k}, j_{m_k+1})]\varLambda(j_{n_k}, j_{n_k+1})}{1+\varLambda(j_{m_k}, j_{n_k})}\right\} \\ &= \varLambda(j_{m_k}, j_{n_k}).\end{aligned}$$

This together with (3.5) show that

$$\begin{aligned}0 &\leq \zeta\left(\varLambda(\varOmega\mu_{m_k+1}, \varOmega\mu_{n_k+1}, \beta(\varUpsilon_{\varPsi\varOmega}(\mu_{m_k+1}, \mu_{n_k+1}))\varUpsilon_{\varPsi\varOmega}(\mu_{m_k+1}, \mu_{n_k+1})\right) \\ &= \zeta(\varLambda(j_{m_k+1}, j_{n_k+1}), \beta(\varLambda(j_{m_k}, j_{n_k}))\varLambda(j_{m_k}, j_{n_k})) \\ &\leq \zeta(\phi_k, \varphi_k)\end{aligned} \tag{3.6}$$

where $0 < \phi_k = \varLambda(j_{m_k+1}, j_{n_k+1})$ and $0 < \varphi_k = \beta(\varLambda(j_{m_k}, j_{n_k}))\varLambda(j_{m_k}, j_{n_k})$. Since the sequence $\{j_n\}$ is not a Cauchy sequence and usiug Lemma 2.3, we have $\{\varLambda(j_{m_k}, j_{n_k})\}$ and $\{\varLambda(j_{m_k+1}, j_{n_k+1})\}$ both the sequence tend to $\varepsilon > 0$ as $k \longrightarrow \infty$. So,

$$\begin{aligned}\phi_k &= \varLambda(j_{m_k+1}, j_{n_k+1}) \\ &\leq \beta(\varLambda(j_{m_k}, j_{n_k}))\varLambda(j_{m_k}, j_{n_k}) \\ &= \varphi_k \\ &< \varLambda(j_{m_k}, j_{n_k})\end{aligned} \tag{3.7}$$

and using the sandwich theorem, $\{\varphi_k\}$, where $\varphi_k = \beta(\varLambda(j_{m_k}, j_{n_k}))\varLambda(j_{m_k}, j_{n_k})) \longrightarrow \varepsilon$ as $k \longrightarrow \infty$. Hence , we have $0 < \phi_k, \varphi_k \longrightarrow \varepsilon$.
Thus,

$$0 \leq \overline{\lim_{k\longrightarrow\infty}} \zeta(\phi_k, \varphi_k) = \overline{\lim_{k\longrightarrow\infty}}(\varphi_k - \phi_k) = \varepsilon - \varepsilon = 0$$

which is a contradiction. Hence, the Picard-Jungck sequence $\{j_n\}$ is a Cauchy sequence. from condition(i), $(\varPsi\varPi, \varLambda)$ is complete, then there exists $\omega \in \varPi$ such that $j_n = \varPsi\mu_{n+1} \longrightarrow \varPsi\omega$ as $n \longrightarrow \infty$ which implies

$$\lim_{n\longrightarrow\infty} \varLambda(\varPsi\mu_{n+1}, \varPsi\omega) = 0. \tag{3.8}$$

We will show that $\varOmega\omega = \varPsi\omega$. Let $\varOmega\omega \neq \varPsi\omega$ and $\varLambda(\varOmega\omega, \varPsi\omega) > \sigma$. From (3.8), there exists $n_0 \in \mathbb{N}$ such that

$$\varLambda(\varOmega\mu_n, \varPsi\omega) < \sigma = \varLambda(\varOmega\omega, \varPsi\omega)$$

for all $n \geq n_0$. So,

$$\varOmega\mu_n \neq \varOmega\omega \Rightarrow \varLambda(\varOmega\mu_n, \varOmega\omega) > 0 \tag{3.9}$$

for all $n \geq n_0$. Now, there dose not exist some $n \geq n_3$

$$\Psi\mu_{n+1} = \Psi\omega.$$

Hence, there exists a partial subsequence $\{\Psi\mu_{t_k}\}$ of $\{\Psi\mu_{n+1}\}$ such that

$$\Psi\mu_{t_k} \neq \omega \tag{3.10}$$

for all $k \in \mathbb{N}$. Let $n_2 \in \mathbb{N}$ be such that $t_{n_2} \geq n_0$. Using (3.9) and (3.10), we have $\Lambda(\Omega\mu_{t_n}, \Omega\omega) > 0$ and $\Lambda(\Psi\mu_{n+1}, \omega) > 0$ for all $n > n_2$. Using (ζ_2), we get

$$\begin{aligned}
0 &\leq \zeta\left(\Lambda(\Omega\omega, \Omega\mu_{t_n}, \beta(\Upsilon_{\Psi\Omega}(\omega, \mu_{n+1}))\Upsilon_{\Psi\Omega}(\omega, \mu_{n+1})\right) \\
&= \zeta\left(\Lambda(\Omega\omega, \Omega\mu_{t_n}, \beta(\Lambda(\Psi\omega, \Psi\mu_{n+1}))\Lambda(\Psi\omega, \Psi\mu_{n+1})\right) \\
&< \beta(\Lambda(\Psi\omega, \Psi\mu_{n+1}))\Lambda(\Psi\omega, \Psi\mu_{n+1}) - \Lambda(\Omega\omega, \Omega\mu_{t_n}) \\
&< \Lambda(\Psi\omega, \Psi\mu_{n+1}) - \Lambda(\Omega\omega, \Omega\mu_{t_n}).
\end{aligned}$$

Taking $n \longrightarrow \infty$, we obtain

$$\begin{aligned}
0 &< \Lambda(\Psi\omega, \Psi\omega) - \Lambda(\Omega\omega, \Psi\omega) \\
&= 0 - \Lambda(\Omega\omega, \Psi\omega).
\end{aligned}$$

This implies that $\eta = \Psi\omega = \Omega\omega$ and η is the unique point coincidence of Ω and Ψ. In the same way, we can show that $\varrho = \Omega\omega = \Psi\omega$ is a unique point of coincidence of Ω and Ψ when $(\Omega\Pi, \Lambda)$ is complete.

From condition(ii), (Π, Λ) is complete, there exists $\omega \in \Pi$ such that $j_n = \Omega\mu_n = \Psi\mu_{n+1} \longrightarrow \omega$ as $n \longrightarrow \infty$. Since Ψ is continuous, we get

$$\lim_{n\longrightarrow\infty} \Psi(\Omega\mu_n) = \Psi\omega \Rightarrow \lim_{n\longrightarrow\infty} \Lambda(\Psi(\Omega\mu_n), \Psi\omega) = 0 \tag{3.11}$$

and

$$\lim_{n\longrightarrow\infty} \Psi(\Psi\mu_{n+1}) = \Psi\omega \Rightarrow \lim_{n\longrightarrow\infty} \Lambda(\Psi(\Psi\mu_{n+1}), \Psi\omega) = 0. \tag{3.12}$$

We claim that $\lim_{n\longrightarrow\infty} \Omega(\Psi\mu_n) = \Omega\omega$. If not, there exists a subsequence $\{\Omega(\Psi\mu_{t_k})\}$ of $\{\Omega(\Psi\mu_n)\}$ such that

$$\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega) > 0 \tag{3.13}$$

for all $k \in \mathbb{N}$. Then there does not exist some $k_1 \in \mathbb{N}$ for all $n > k_1$

$$\Psi(\Psi\mu_{n+1}) = \Psi\omega.$$

Thus, there exists a partial subsequence $\{\Psi(\Psi\mu_{t_r})\}$ of $\{\Psi(\Psi\mu_{n+1})\}$ such that

$$\Psi(\Psi\mu_{t_r}) \neq \Psi\omega \tag{3.14}$$

for all $r \in \mathbb{N}$. Hence, using (3.13) and (3.14), we have $\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega) > 0$ and $\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega) > 0$ for all $k, r \in \mathbb{N}$. Using (ζ_2), we obtain

$$\begin{aligned}
0 &\leq \zeta(\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega), \beta(\Upsilon_{\Psi\Omega}(\Psi\mu_{t_r}, \omega))\Upsilon_{\Psi\Omega}(\Psi\mu_{t_r}, \omega) \\
&= \zeta\left(\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega), \beta(\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega))\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega)\right) \\
&< \beta(\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega))\Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega)) - \Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega) \\
&< \Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega) - \Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega).
\end{aligned}$$

Hence, we have $\Lambda(\Omega(\Psi\mu_{t_k}), \Omega\omega) < \Lambda(\Psi(\Psi\mu_{t_r}), \Psi\omega) \longrightarrow 0$ as $k \longrightarrow \infty$ which is a contradiction. This implies that

$$\lim_{n\longrightarrow\infty} \Lambda(\Omega(\Psi\mu_n), \Omega\omega) = 0. \tag{3.15}$$

Further, since Ω and Ψ are compatible, we have

$$\lim_{n\longrightarrow\infty} \Lambda(\Omega(\Psi\mu_n), \Psi(\Omega\mu_n) = 0. \tag{3.16}$$

Finally, using (3.11), (3.15) aud (3.16), we have

$$\begin{aligned} \Lambda(\Omega\omega,\Psi\omega) &= \Lambda(\Omega\omega,\Omega(\Psi\mu_n)) + \Lambda(\Omega(\Psi\mu_n),\Psi(\Omega\mu_n)) + \Lambda(\Psi(\Omega\mu_n),\Psi\omega) \\ &\Rightarrow \Lambda(\Omega\omega,\Psi\omega) \leq 0 \\ &\Rightarrow \Lambda(\Omega\omega,\Psi\omega) = 0. \end{aligned}$$

This implies that $\varrho = \Psi\omega = \Omega\omega$ and ϱ is the unique point of coincidence of Ω and Ψ. Thus, the mappings Ω and Ψ have a unique point of coincidence. □

Theorem 3.2. *Let $\Omega,\Psi : \Pi \longrightarrow \Pi$ be two self-maps defined on a complete metric space (Π,Λ). Assume there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta\left(\Lambda(\Omega\mu,\Omega\upsilon), \beta(\Upsilon_{\Psi\Omega}(\mu,\upsilon))\Upsilon_{\Psi\Omega}(\mu,\upsilon)\right) \geq 0, \tag{3.17}$$

for all $\mu,\upsilon \in \Pi$ with $\Psi\mu \neq \Psi\upsilon$, where $\beta : [0,\infty) \longrightarrow (0,1)$ and

$$\Upsilon_{\Psi\Omega}(\mu,\upsilon) = \max\left\{\Lambda(\Psi\mu,\Psi\upsilon), \frac{[1+\Lambda(\Psi\mu,\Omega\mu)]\Lambda(\Psi\upsilon,\Omega\upsilon)}{1+\Lambda(\Psi\mu,\Psi\upsilon)}\right\}.$$

Suppose that, there exists a Picard-Jungck sequence $\{\mu_n\}$ of (Ω,Ψ). Also assume that, $(\Omega\Pi,\Lambda)$ or $(\Psi\Pi,\Lambda)$ is complete and Ω and Ψ are weakly compatible. Then Ω and Ψ have a unique common fixed point in Π.

Proof. It follows Theorem 3.1, Ω and Ψ have a unique point of coincidence. Further, since Ω and Ψ are weakly compatible, then according to Theorem 2.1, they have a unique common fixed point in Π. □

Theorem 3.3. *Let $\Omega,\Psi : \Pi \longrightarrow \Pi$ be two self-maps defined on a complete metric space (Π,Λ). Assume there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta\left(\Lambda(\Omega\mu,\Omega\upsilon), \beta(\Upsilon_{\Psi\Omega}(\mu,\upsilon))\Upsilon_{\Psi\Omega}(\mu,\upsilon)\right) \geq 0, \tag{3.18}$$

for all $\mu,\upsilon \in \Pi$ with $\Psi\mu \neq \Psi\upsilon$, where $\beta : [0,\infty) \longrightarrow (0,1)$ and

$$\Upsilon_{\Psi\Omega}(\mu,\upsilon) = \max\left\{\Lambda(\Psi\mu,\Psi\upsilon), \frac{[1+\Lambda(\Psi\mu,\Omega\mu)]\Lambda(\Psi\upsilon,\Omega\upsilon)}{1+\Lambda(\Psi\mu,\Psi\upsilon)}\right\}.$$

Suppose that, there exists a Picard-Jungck sequence $\{\mu_n\}$ of (Ω,Ψ). Also assume that, $(\Omega\Pi,\Lambda)$ or $(\Psi\Pi,\Lambda)$ is complete, Ω and Ψ are satisfy (CLR_g)-property. Then Ω and Ψ have a unique common fixed point in Π.

Proof. Using Ω and Ψ are satisfy (CLR_g)-property in Definition 2.2 and Theorem 3.1. □

Example 3.1. Let $\Pi = \{0,4,5\}$ and $\Lambda : \Pi\times\Pi \longrightarrow [0,\infty)$ be defined by $\Lambda(\mu,\upsilon) = |\mu-\upsilon|$. Define $\Omega,\Psi : \Pi \longrightarrow \Pi$ as

$$\Omega\mu = \begin{pmatrix} 0 & 4 & 5 \\ 4 & 4 & 4 \end{pmatrix} \quad \text{and} \quad \Psi\mu = \begin{pmatrix} 0 & 4 & 5 \\ 5 & 4 & 0 \end{pmatrix}.$$

Suppose $\zeta(t,s) = \dfrac{s}{s+1} - t$, $\beta(t) = \dfrac{1}{1+\dfrac{t}{9}}$ for $t>0$ and $\beta(t) = \dfrac{1}{2}$ for $t=0$.

Case (i): For $\mu = 0$, $\upsilon = 4$. From (3.1), we obtain

$$\begin{aligned} \zeta\left(\Lambda(\Omega 0,\Omega 4), \beta(\Upsilon_{\Psi\Omega}(0,4))\Upsilon_{\Psi\Omega}(0,4)\right) &= \zeta\left(\Lambda(4,4), \beta(\Upsilon_{\Psi\Omega}(0,4))\Upsilon_{\Psi\Omega}(0,4)\right) \\ &= \zeta\left(0, \beta(\Upsilon_{\Psi\Omega}(0,4))\Upsilon_{\Psi\Omega}(0,4)\right), \end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\Upsilon_{\Psi\Omega}(0,4) &= \max\left\{\Lambda(\Psi 0,\Psi 4), \frac{[1+\Lambda(\Psi 0,\Omega 0)]\Lambda(\Psi 4,\Omega 4)}{1+\Lambda(\Psi 0,\Psi 4)}\right\}\\ &= \max\left\{\Lambda(5,4), \frac{[1+\Lambda(5,4)]\Lambda(4,4)}{1+\Lambda(5,4)}\right\}\\ &= \max\{1,0\}\\ &= 1.\end{aligned}$$

This together with (3.19) show that

$$\begin{aligned}\zeta\left(0,\beta(\Upsilon_{\Psi\Omega}(0,4))\Upsilon_{\Psi\Omega}(0,4)\right) &= \zeta\left(0,\beta(1)\cdot 1\right)\\ &= \frac{\beta(1)}{\beta(1)+1}\\ &\geq 0.\end{aligned}$$

Case (ii): For $\mu=0$, $\upsilon=5$. From (3.1), we obtain

$$\begin{aligned}\zeta\left(\Lambda(\Omega 0,\Omega 5),\beta(\Upsilon_{\Psi\Omega}(0,5))\Upsilon_{\Psi\Omega}(0,5)\right) &= \zeta\left(\Lambda(4,4),\beta(\Upsilon_{\Psi\Omega}(0,5))\Upsilon_{\Psi\Omega}(0,5)\right)\\ &= \zeta\left(0,\beta(\Upsilon_{\Psi\Omega}(0,5))\Upsilon_{\Psi\Omega}(0,5)\right),\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\Upsilon_{\Psi\Omega}(0,5) &= \max\left\{\Lambda(\Psi 0,\Psi 5), \frac{[1+\Lambda(\Psi 0,\Omega 0)]\Lambda(\Psi 5,\Omega 5)}{1+\Lambda(\Psi 0,\Psi 5)}\right\}\\ &= \max\left\{\Lambda(5,0), \frac{[1+\Lambda(5,4)]\Lambda(0,4)}{1+\Lambda(5,0)}\right\}\\ &= \max\left\{5,\frac{4}{3}\right\}\\ &= 5.\end{aligned}$$

This together with (3.20) show that

$$\begin{aligned}\zeta\left(0,\beta(\Upsilon_{\Psi\Omega}(0,5))\Upsilon_{\Psi\Omega}(0,5)\right) &= \zeta\left(0,\beta(5)\cdot 5\right)\\ &= \frac{5\beta(5)}{5\beta(5)+1}\\ &\geq 0.\end{aligned}$$

Case (iii): For $\mu=4$, $\upsilon=5$. From (3.1), we obtain

$$\begin{aligned}\zeta\left(\Lambda(\Omega 4,\Omega 5),\beta(\Upsilon_{\Psi\Omega}(4,5))\Upsilon_{\Psi\Omega}(4,5)\right) &= \zeta\left(\Lambda(4,4),\beta(\Upsilon_{\Psi\Omega}(4,5))\Upsilon_{\Psi\Omega}(4,5)\right)\\ &= \zeta\left(0,\beta(\Upsilon_{\Psi\Omega}(4,5))\Upsilon_{\Psi\Omega}(4,5)\right),\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}\Upsilon_{\Psi\Omega}(4,5) &= \max\left\{\Lambda(\Psi 4,\Psi 5), \frac{[1+\Lambda(\Psi 4,\Omega 4)]\Lambda(\Psi 5,\Omega 5)}{1+\Lambda(\Psi 4,\Psi 5)}\right\}\\ &= \max\left\{\Lambda(4,0), \frac{[1+\Lambda(4,4)]\Lambda(0,4)}{1+\Lambda(4,0)}\right\}\\ &= \max\left\{4,\frac{4}{5}\right\}\\ &= 4.\end{aligned}$$

This together with (3.21) show that

$$\begin{aligned}\zeta\left(0, \beta(\Upsilon_{\Psi\Omega}(4,5))\Upsilon_{\Psi\Omega}(4,5)\right) &= \zeta\left(0, \beta(4)\cdot 4\right) \\ &= \frac{4\beta(4)}{4\beta(4)+1} \\ &\geq 0.\end{aligned}$$

Therefore, all the assumptions of Theorem 3.1 are satisfied, and as per its conclusion, Ω and Ψ have a unique point of coincidence $\mu = 4$, making it their unique common fixed point.

4. Conclusion

This paper focuses on investigating the existence and uniqueness of coincidence points and Geraghty-type common fixed points under contractive conditions using simulation functions within the context of complete metric spaces. The obtained results are illustrated with examples to demonstrate their applicability.

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THEOREMS OF THE MINIMIZATION PROBLEM AND FIXED POINT PROBLEM OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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ABSTRACT. In this paper, we propose an iterative algorithm designed to address the minimization and fixed point problems associated with total asymptotically nonexpansive mappings in CAT(0) spaces. We establish strong convergence theorems and $\triangle$-convergence theorems for solving these problems. Furthermore, we apply the key findings to solve the equilibrium problem in CAT(0) spaces.

KEYWORDS minimization problem; fixed point problem; total asymptotically nonexpansive mapping; convergence theorem; CAT(0) space
AMS Subject Classification: 47H09; 47J25

1. Introduction

Let K be a nonempty subset of a CAT(0) space (X, d), and consider the mapping $T : K \to K$. We denote the set of fixed points of T by $F(T) = u \in K : u = Tu$. The study of fixed point theory in CAT(0) spaces was initiated by Kirk [14] in 2003. Kirk demonstrated the existence of a fixed point for a nonexpansive mapping defined on a bounded, closed, and convex subset of a CAT(0) space. Subsequently, numerous authors proposed various iterative schemes to approximate fixed points of nonexpansive mappings in CAT(0) spaces. One such algorithm is the Mann iterative algorithm introduced by He et al. [21] in CAT(κ) spaces, defined as follows:

$$\begin{cases} u_1 \in X, \\ u_{n+1} = \alpha_n u_n \oplus (1-\alpha_n) T u_n, \quad \forall_n \geq 1, \end{cases} \tag{1.1}$$

where α_n is a sequence in $[0, 1]$, and they proved some $\triangle$-convergence theorems of nonexpansive mappings in CAT(κ) spaces for $\kappa \geq 0$. Other iterative algorithms have

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also been proposed to solve this problem, such as the Ishikawa iteration method, S-iteration method, and hybrid-CR three steps iteration methods. For further details, refer to [1, 22, 23, 24, 25, 31, 32, 33, 34, 35].

The proximal point algorithm (PPA), introduced by Martinet [2] in 1970, has attracted significant attention from researchers. Rockafellar further utilized the PPA to solve convex minimization problems in Hilbert spaces. Nevanlinna investigated the minimization problem in Banach spaces using the PPA under suitable conditions [15]. More information on PPA in Hilbert or Banach spaces can be found in the works of Solodov [16], Kamimura [17], Shehu [18], and others.

Recently, many PPA convergence results have been extended from linear to nonlinear spaces. Bačák introduced the PPA in CAT(0) spaces to solve the minimization problem in 2013, which is defined as follows:

$$\begin{cases} u_1 \in X, \\ u_{n+1} = \arg\min_{q \in X} \left[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n) \right] \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where $\lambda_n > 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Cholamjiak et al. [20] proposed the following iteration method in 2015 to solve the minimization and fixed point problems of nonexpansive mappings in CAT(0) spaces:

$$\begin{cases} p_n = \arg\min_{q \in X} \left[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n) \right], \\ y_n = (1-\beta_n) u_n \oplus \beta_n T_1 p_n, \\ u_{n+1} = (1-\alpha_n) T_1 u_n \oplus \alpha_n T_2 y_n, \quad \forall_n \geq 1. \end{cases} \tag{1.3}$$

where $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a, b, \lambda_n \geq \lambda > 0$, f is a proper convex lower semi-continuous function. They obtained a $\triangle$-convergence theorem.

Chang, Yao, Wang, and Qin [4] introduced the iteration method described below in 2016 to solve the minimization and fixed point problems of asymptotically nonexpansive mappings in CAT(0) spaces:

$$\begin{cases} p_n = \arg\min_{q \in K} \left[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n) \right], \\ y_n = \alpha_n u_n \oplus \beta_n T_1^n u_n \oplus \gamma_n T_2^n p_n, \\ x_{n+1} = \delta_n T_2^n u_n \oplus \eta_n S_1^n u_n \oplus \xi_n S_2^n y_n, \quad n \geq 1. \end{cases} \tag{1.4}$$

where $0 < a \leq \alpha_n, \beta_n, \gamma_n, \delta_n, \eta_n, \xi_n < 1, a \in (0,1)$ is a positive constant, $\lambda_n \geq \lambda > 0$, g is a proper convex lower semi-continuous function. They obtained a $\triangle$-convergence result, and when one of the mappings T_1, T_2, S_1 and S_2 has semi-compactness, they established a strong convergence theorem.

Motivated by ongoing research in this area and inspired by Cholamjiak's iteration method and Chang's method, we delve into the minimization and fixed point problems of total asymptotically nonexpansive mappings in CAT(0) spaces in this paper. We introduce a novel algorithm and derive some strong convergence theorems and $\triangle$-convergence theorems by amalgamating the proximal point algorithm with Mann's iterative method. Finally, we apply the key findings to solve the equilibrium problem in CAT(0) spaces.

2. Preliminaries

Let (X,d) be a metric space and $p,q\in X$. A geodesic path joining p to q is an isometry $c:[0,d(p,q)]\to X$ such that $c(0)=p$ and $c(d(p,q))=q$. The image of a geodesic path joining p to q is called a geodesic segment between p and q. When it is unique, this geodesic segment is denoted by $[p,q]$. The metric space (X,d) is said to be a geodesic space, if every two points of X are joined by a geodesic. In this paper, we write $(1-t)p\oplus tq$ for the unique point h in $[p,q]$ such that

$$d(h,p)=td(p,q), d(h,q)=(1-t)d(p,q).$$

A geodesic space (X,d) is called a CAT(0) space, if the geodesic segment connecting two points is unique and the following inequality holds [5]:

$$d^2((1-t)p\oplus tq,h)\leq(1-t)d^2(p,h)+td^2(q,h)-(1-t)d^2(p,q)$$

for all $p,q,h\in X$.

A subset K of a CAT(0) space X is said to be convex if $[p,q]\subseteq K$ for all $p,q\in K$. For more fundamental knowledge about CAT(0) spaces, refer to read [5]-[11].

It is well known that any complete and simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space and the Hilbert ball with the hyperbolic metric [12], Pre-Hilbert space [6], Euclidean building [11] and R-tree [13] are also examples of CAT(0) spaces.

Definition 2.1. Let $T:X\to X$ be a mapping. T is said to be

(i) nonexpansive, if $d(Tp,Tq)\leq d(p,q)$, for any $p,q\in X$.

(ii) asymptotically nonexpansive, if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $k_n\to1$ as $n\to\infty$ such that $d(T^np,T^nq)\leq k_nd(p,q)$, for any $n\geq1$ and any $p,q\in X$.

(iii) total asymptotically nonexpansive, if there exists nonnegative sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\mu_n\to0$, $\nu_n\to0$ and a strictly increasing continuous function $\xi:[0,1)\to[0,\infty)$ with $\xi(0)=0$ such that

$$d(T^np,T^nq)\leq d(p,q)+\nu_n\xi(d(p,q))+\mu_n,\quad\forall_n\geq1,\ p,q\in X.$$

(iv) uniformly L-Lipschitzian, if there exists a constant $L>0$ such that

$$d(T^np,T^nq)\leq Ld(p,q),\quad\forall_n\geq1,\ p,q\in X.$$

Let $\{u_n\}$ be a bounded sequence of a complete CAT(0) space X. Then $A(\{u_n\})=\{u\in X:\limsup_{n\to\infty}d(u,u_n)\leq\limsup_{n\to\infty}d(h,u_n),\forall\,h\in X\}$ is said to be the asymptotic center of $\{u_n\}$. It is known [26] that in a complete CAT(0) space X, the asymptotic center of $\{u_n\}$ consists of exactly one point.

Definition 2.2. [14, 28] A sequence $\{u_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $u\in X$ if u is the unique asymptotic center of any subsequence $\{u_{n_k}\}\subset\{u_n\}$. Symbolically, we write it as $\triangle-\lim_{n\to\infty}u_n=u$.

Lemma 2.3. [27] *Let K be a closed and convex subset of CAT(0) space X and $\{u_n\}$ be a bounded sequence in K. Then $\triangle-\lim_{n\to\infty}u_n=u$ implies that $u_n\to u$ (i.e. $\limsup_{n\to\infty}d(u_n,u)=\inf_{y\in K}\limsup_{n\to\infty}d(u_n,y)$).*

Lemma 2.4. [5] *Let X be a CAT(0) space and $p,q,h\in X$. Then*

(i) $d((1-t)p\oplus tq,h))\leq(1-t)d(p,h)+td(q,h)$, $t\in[0,1]$,

(ii) $d^2((1-t)p\oplus tq,h))\leq(1-t)d^2(p,h)+td^2(q,h)-t(1-t)d^2(p,q)$, $t\in[0,1]$.

Lemma 2.5. [27] *Let $\{u_n\}$ be a bounded sequence of complete CAT(0) space X. Then*

(i) *$\{u_n\}$ has a $\triangle$-convergent subsequence,*
(ii) *the asymptotic center of $\{u_n\} \subset K \subset X$ is in K, where K is nonempty closed and convex.*

Lemma 2.6. [5] *Let $\{u_n\}$ be a bounded sequence of a complete CAT(0) space and $A(\{u_n\}) = \{u\}$. Let $\{u_{n_k}\}$ be an arbitrary subsequence of $\{u_n\}$ and $A(\{u_{n_k}\}) = \{q\}$ If $\lim_{n\to\infty} d(u_n, q)$ exists, then $u = q$.*

Definition 2.7. A function $g : K \to (-\infty, \infty]$ is said to be convex if the following inequality holds

$$g(\lambda p \oplus (1-\lambda)q) \leq \lambda g(p) + (1-\lambda)g(q), \quad \text{for all} \quad p, q \in K, \quad \lambda \in [0,1].$$

Definition 2.8. [29] Let $g : X \to (-\infty, \infty]$ be a proper convex lower semi-continuous function, for all $\lambda > 0$, the Moreau-Yosida resolvent of f in CAT(0) space X is defined by

$$J_\lambda^g := \arg\min_{q\in X}[g(y) + \frac{1}{2\lambda}d^2(q,p)], \quad \forall\, p \in X.$$

It is known that the fixed points set $Fix(J_\lambda^g(p))$ of the resolvent of g is consistent with the set $\arg\min_{q\in X} g(q)$ of minimizers of g, and J_λ^g is a nonexpansive mapping [30].

Lemma 2.9. [30] *Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty]$ be a proper convex lower semi-continuous function. Then*

$$J_\lambda p := J_\mu(\frac{\lambda-\mu}{\lambda}J_\lambda p \oplus \frac{\mu}{\lambda}p), \quad \textit{for all}\ p \in X\ \textit{and}\ \lambda > \mu > 0.$$

Lemma 2.10. [7] *Let (X, d) be a complete CAT(0) space and $g : X \to (-\infty, \infty]$ be a proper convex lower semi-continuous function. Then*

$$\frac{1}{2\lambda}d^2(J_\lambda p, q) - \frac{1}{2\lambda}d^2(p,q) + \frac{1}{2\lambda}d^2(p, J_\lambda p) + g(J_\lambda p) \leq g(q), \ \textit{for all}\ p, q \in X, \lambda > 0.$$

Lemma 2.11. [8] *Let C be a closed and convex subset of complete CAT (0) space X and let $T : K \to X$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping. If $\{u_n\}$ is a bounded sequence in K such that $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ and $\triangle - \lim_{n\to\infty} u_n = u$, then $Tu = u$.*

Lemma 2.12. [9] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ n \geq 1,$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.13. [8] *Let X be a CAT(0) space,$x \in X$ be a given point and $\{a_n\}$ be a sequence in $[b, c]$, and $b, c \in (0,1), 0 < b(1-c) \leq \frac{1}{2}$, let $\{u_n\}$ and $\{p_n\}$ be any sequences in X such that $\limsup_{n\to\infty} d(u_n, u) \leq r$, $\limsup_{n\to\infty} d(p_n, u) \leq r$ and $\lim_{n\to\infty} d((1-a_n)u_n \oplus a_n p_n, u) = r$, for some $r \geq 0$, then $\lim_{n\to\infty} d(u_n, p_n) = 0$.*

3. Main Results

We suppose the following conditions are satisfied:

(1) (X, d) is a complete CAT(0) space.
(2) $K \subset X$ is a nonempty closed convex subset, $T : K \to K$ is a uniformly L-Lipschitzian total asymptotically nonexpansive mapping, $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, and there exists a constant $M > 0$ such that $\xi(r) \le Mr, r \ge 0$.
(3) $g : X \to (-\infty, \infty]$ is a proper convex lower semi-continuous function, $J^g_{\lambda_n} : X \to X$ is the Moreau-Yosida resolvent of $g, \lambda_n \ge \lambda > 0$.
(4) $\{\alpha_n\}$ is a sequence in $[b, c]$, and $b, c \in (0, 1), 0 < b(1 - c) \le \frac{1}{2}$

Theorem 3.1. *Let $(X, d), K, T, g, J^g_{\lambda_n}, \lambda_n, \{\alpha_n\}$ satisfy the above conditions. Let $u_1 \in X$ be chosen arbitrarily and the sequence $\{u_n\}$ be defined as follows:*

$$\begin{cases} p_n = J^g_{\lambda_n}(u_n) = \arg\min_{q \in X}[g(q) + \frac{1}{2\lambda_n} d^2(q, u_n)], \\ u_{n+1} = T^n((1 - \alpha_n)u_n \oplus \alpha_n T^n p_n), \ n \ge 1. \end{cases} \tag{3.1}$$

(I) *If $\Omega = F(T) \cap \arg\min_{q \in X} g(q) \neq \phi$, then $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.*
(II) *In addition, if $\Omega = F(T) \cap \arg\min_{q \in X} g(q) \neq \phi$ and T is semi-compact, then $\{u_n\}$ converges strongly to a point $u \in \Omega$.*

Proof. Now we will demonstrate the conclusion (I). The proof is divided into five steps.

Step 1. Firstly we show that $\{u_n\}$ is bounded.

Let $u^* \in \Omega$, since $J^g_{\lambda_n}$ is a nonexpansive mapping, from (3.1), we have

$$d(p_n, u^*) = d(J^g_{\lambda_n}(u_n), u^*) = d(J^g_{\lambda_n}(u_n), J^g_{\lambda_n}(u^*)) \le d(u_n, u^*), \tag{3.2}$$

and from Lemma 2.4 (i), we can obtain that

$$\begin{aligned} d(x_{n+1}, u^*) &= d(T^n((1 - \alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) \\ &\le d((1 - \alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) \\ &\quad + \nu_n \xi(d((1 - \alpha_n)u_n \oplus \alpha_n T^n p_n, u^*)) + \mu_n \\ &\le (1 + \nu_n M)d((1 - \alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) + \mu_n \\ &\le (1 + \nu_n M)[(1 - \alpha_n)d(u_n, u^*) + \alpha_n d(T^n p_n, u^*)] + \mu_n \\ &\le (1 + \nu_n M)[(1 - \alpha_n)d(u_n, u^*) + \alpha_n (d(p_n, u^*) + \nu_n \xi(d(p_n, u^*)) + \mu_n] + \mu_n \\ &\le (1 + \nu_n M)[(1 + \nu_n M)d(u_n, u^*) + \mu_n] + \mu_n \\ &\le (1 + \nu_n M)^2 d(u_n, u^*) + (2 + \nu_n M)\mu_n. \end{aligned} \tag{3.3}$$

Since $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, it follows from Lemma 2.12 that $\lim_{n\to\infty} d(u_n, u^*)$ exists. This implies that $\{u_n\}$ is bounded. Obviously, $\{p_n\}$ is also bounded.

Step 2. We show that $\lim_{n\to\infty} d(u_n, p_n) = 0$.

By Lemma 2.10, we have

$$\frac{1}{2\lambda_n} d^2(p_n, u^*) - \frac{1}{2\lambda_n} d^2(u_n, u^*) + \frac{1}{2\lambda_n} d^2(u_n, p_n) \le g(u^*) - g(p_n). \tag{3.4}$$

Since $g(u^*) \leq g(p_n)$, from (3.4), we can get

$$d^2(u_n, p_n) \leq d^2(u_n, u^*) - d^2(p_n, u^*). \tag{3.5}$$

Since $\lim_{n\to\infty} d(u_n, u^*)$ exists, without loss of generality, we may assume $\lim_{n\to\infty} d(u_n, u^*) = c \geq 0$ By (3.2), we have

$$\sum_{n\to\infty} d(p_n, u^*) \leq \sum_{n\to\infty} d(u_n, u^*) = c, \tag{3.6}$$

and from (3.3), we can obtain that

$$d(u_n, u^*) \leq \frac{d(u_n, u^*)}{\alpha_n} - \frac{d(u_{n+1}, u^*)}{\alpha_n(1+\nu_n M)} + d(p_n, u^*) + \nu_n \xi(d(p_n, u^*)) + \mu_n + \frac{\mu_n}{\alpha_n(1+\nu_n M)}. \tag{3.7}$$

It follows from $\lim_{n\to\infty} d(u_n, u^*) = c,\ \mu_n \to 0$, and $\nu_n \to 0$ that

$$c = \liminf_{n\to\infty} d(u_n, u^*) \leq \liminf_{n\to\infty} d(p_n, u^*). \tag{3.8}$$

Combining (3.6) and (3.8), we have

$$\lim_{n\to\infty} d(p_n, u^*) = c. \tag{3.9}$$

Thus it follows from (3.5) that

$$\lim_{n\to\infty} d(u_n, p_n) = 0. \tag{3.10}$$

Step 3. We show that

$$\lim_{n\to\infty} d(u_n, T^n p_n) = \lim_{n\to\infty} d(u_n, u_{n+1}) = \lim_{n\to\infty} d(p_n, p_{n+1}) = 0.$$

Since

$$d(T^n p_n, u^*) = d(T^n p_n, T^n u^*) \leq d(p_n, u^*) + \nu_n \xi(d(p_n, u^*)) + \mu_n \tag{3.11}$$

we have

$$\limsup_{n\to\infty} d(T^n p_n, u^*) \leq \limsup_{n\to\infty} d(p_n, u^*) = c. \tag{3.12}$$

Due to (3.3) we have

$$\begin{aligned} c = \lim_{n\to\infty} d(u_{n+1}, u^*) &= \lim_{n\to\infty} d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) \\ &\leq \lim_{n\to\infty} ((1+\nu_n M)^2 d(u_n, u^*) + (2+\nu_n M)\mu_n) \\ &= c. \end{aligned} \tag{3.13}$$

This implies that

$$\lim_{n\to\infty} d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) = c \tag{3.14}$$

and

$$\begin{aligned} &d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) \\ &\leq (1+\nu_n M) d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) + \mu_n \end{aligned}$$

which

$$\begin{aligned} \limsup_{n\longrightarrow\infty} d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), u^*) &\leq \limsup_{n\longrightarrow\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) \\ c &\leq \limsup_{n\longrightarrow\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*). \end{aligned} \tag{3.15}$$

Also, we have

$$\begin{aligned} &d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) \leq (1-\alpha_n) d(u_n, u^*) + \alpha_n d(T^n p_n, u^*) \\ &\limsup_{n\longrightarrow\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) \leq c. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we have

$$\limsup_{n\longrightarrow\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, u^*) = c.$$

Since $\{\alpha_n\}$ is a sequence in $[b,c]$, and $b,c \in (0,1), 0 < b(1-c) \le \frac{1}{2}$, from (3.12), (3.14), (3.15), (3.16) and Lemma 2.13, we have

$$\lim_{n\to\infty} d(u_n, T^n p_n) = 0. \tag{3.17}$$

In addition, we also have

$$\begin{aligned}\lim_{n\to\infty} d(u_{n+1}, T^n p_n) &= \lim_{n\to\infty} (d(T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), T^n p_n) \\ &\le \lim_{n\to\infty} d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, p_n) \\ &\quad + \nu_n \xi(d((1-\alpha_n)u_n \oplus \alpha_n T^n p_n, p_n)) + \mu_n) \\ &= 0.\end{aligned} \tag{3.18}$$

So, from (3.17) and (3.18), we know that

$$\lim_{n\to\infty} d(u_n, u_{n+1}) = 0. \tag{3.19}$$

Since

$$d(p_n, p_{n+1}) \le d(p_n, u_n) + d(u_n, u_{n+1}) + d(u_{n+1}, p_{n+1}),$$

from (3.10) and (3.19), we have

$$\lim_{n\to\infty} d(p_n, p_{n+1}) = 0. \tag{3.20}$$

Step 4. We show that

$$\lim_{n\to\infty} d(p_n, Tp_n) = \lim_{n\to\infty} d(u_n, Tu_n) = \lim_{n\to\infty} d(u_n, J_\lambda^g u_n) = 0.$$

In the view of (3.10), and (3.17), we can obtain that

$$d(p_n, T^n p_n) \le d(p_n, u_n) + d(u_n, T^n p_n) \to 0 \quad (\text{as } n \to \infty). \tag{3.21}$$

Since T is uniformly L-Lipschitzian, combining (3.20) and (3.21), we may get

$$\begin{aligned}d(p_n, Tp_n) &\le d(p_n, p_{n+1}) + d(p_{n+1}, T^{n+1}p_{n+1}) + d(T^{n+1}p_{n+1}, T^{n+1}p_n) \\ &\quad + d(T^{n+1}p_n, Tp_n) \\ &\le (1+L)d(p_n, p_{n+1}) + d(p_{n+1}, T^{n+1}p_{n+1}) + Ld(T^n p_n, p_n) \\ &\to 0 \ (\text{as } n \to \infty).\end{aligned} \tag{3.22}$$

In addition, we also have

$$\begin{aligned}d(p_n, Tp_n) &\le d(u_n, p_n) + d(p_n, Tp_n) + d(Tp_n, Tu_n) \\ &\le (1+L)d(u_n, p_n) + d(p_n, Tp_n) \to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{3.23}$$

It follows from (3.10) and Lemma 2.9 that

$$\begin{aligned}d(J_\lambda^g u_n, u_n) &\le d(J_\lambda^g u_n, J_{\lambda_n}^g(u_n)) + d(p_n, u_n) \\ &\le d(J_\lambda^g u_n, J_\lambda^g((\frac{\lambda_n - \lambda}{\lambda_n})J_{\lambda_n}^g(u_n) \oplus \frac{\lambda}{\lambda_n}u_n)) + d(p_n, u_n) \\ &\le d(u_n, (1 - \frac{\lambda}{\lambda_n})(J_{\lambda_n}^g(u_n) \oplus \frac{\lambda}{\lambda_n}u_n)) + d(p_n, u_n) \\ &\le (1 - \frac{\lambda}{\lambda_n})d(u_n, p_n) + d(p_n, u_n) \to 0 \quad (\text{as } n \to \infty).\end{aligned} \tag{3.24}$$

Step 5. Finally we prove that $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.

Denote $\omega_w(u_n) = \bigcup_{\{u_{n_i}\}\subset\{u_n\}} A(\{u_{n_i}\})$. Let $z \in \omega_w(u_n)$, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $A(\{u_{n_i}\}) = \{z\}$. By Lemma 2.5, there exists a subsequence $\{v_{n_j}\}$ of $\{u_{n_i}\}$ such that $\triangle$ - $\lim_{n\to\infty} v_{n_j} = u$. Because J_λ^g is a nonexpansive mapping, it follows from (3.24), (3.23), (3.22), (3.10) and Lemma 2.11 that $u \in F(J_\lambda^g) \cap F(T)$. This implies that $u \in \Omega$. Since $\lim_{n\to\infty} d(u_n, u^*)$ exists for any $u^* \in \Omega$. Then $\lim_{n\to\infty} d(u_n, u)$ also exists.

Next we prove that $\omega_w(u_n)$ consists of exactly one point. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that $A(\{u_{n_i}\}) = \{z\}$ and $A(\{u_n\}) = \{u\}$. Because $z \in \omega_w(u_n) \subset \Omega$, we know that $z \in \Omega$. Thus, $\lim_{n\to\infty} d(u_n, z)$ exists. By Lemma 2.6, we know that $z = u$. This means that $\omega_w(u_n)$ consists of exactly one point. It follows from Definition 2.2 that $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.

Next we prove the conclusion (II).

From T is semi-compact and $\lim_{n\to\infty} d(p_n, Tp_n) = 0$, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{p_{n_k}\} \to u_*$. It follows from $\lim_{n\to\infty} d(u_n, p_n) = 0$ that the subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges strongly to u_*. Because $\triangle$ - $\lim_{n\to\infty} u_n = u$, then $u_* = u$. Due to $\lim_{n\to\infty} d(u_n, u)$ exists and $\lim_{k\to\infty} d(u_{n_k}, u) = 0$, we know that $\{u_n\}$ converges strongly to a point $u \in \Omega$. The proof is completed. $\square$

Every asymptotically nonexpansive mapping is also a total asymptotically nonexpansive mapping, and every nonexpansive mapping is also a total asymptotically nonexpansive mapping. Therefore, when T is an asymptotically nonexpansive mapping, the following result holds in Theorem 3.1.

Corollary 3.1. *Let $(X, d), K, g, J_{\lambda_n}^g, \lambda_n, \{\alpha_n\}$ be the same as them of Theorem 3.1, $T : K \to K$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ satisfying $\lim_{n\to\infty} k_n = 1$. Let $u_1 \in X$ be chosen arbitrarily and the sequence $\{u_n\}$ be defined as follows:*

$$\begin{cases} p_n = J_{\lambda_n}^g(u_n) = \arg\min_{q\in X}[g(q) + \dfrac{1}{2\lambda_n} d^2(q, u_n)] \\ u_{n+1} = T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), \quad n \geq 1, \end{cases} \tag{3.25}$$

(I) *If $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$, then $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.*

(II) *In addition, if $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$ and T is semi-compact, then $\{u_n\}$ converges strongly to a point $u \in \Omega$.*

In Theorem 3.1, when T is a nonexpansive mapping, the following result holds.

Corollary 3.2. *Let $(X, d), K, g, J_{\lambda_n}^g, \lambda_n, \{\alpha_n\}$ be the same as them of Theorem 3.1, $T : K \to K$ be a nonexpansive mapping. Let $u_1 \in X$ be chosen arbitrarily and the sequence $\{u_n\}$ be defined as follows:*

$$\begin{cases} p_n = J_{\lambda_n}^g(u_n) = \arg\min_{q\in X}[g(q) + \dfrac{1}{2\lambda_n} d^2(q, u_n)] \\ u_{n+1} = T^n((1-\alpha_n)u_n \oplus \alpha_n T^n p_n), \quad n \geq 1, \end{cases} \tag{3.26}$$

(I) *If $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$, then $\{u_n\}$ $\triangle$-convergent to a point $u \in \Omega$.*

(II) *In addition, if $\Omega = F(T) \cap \arg\min_{q\in X} g(q) \neq \phi$ and T is semi-compact, then $\{u_n\}$ converges strongly to a point $u \in \Omega$.*

4. Applications

In this section, we apply the main results to solve equilibrium problem in CAT(0) spaces.

4.1. **Equilibrium problem.** Let (X,d) be a complete CAT(0) space and K be a nonempty closed convex subset of it. Suppose that $F: K\times K \to \mathbb{R}$ is a bifunction, the equilibrium problem $(shortly, EP)$ is to find a point $u^* \in K$ such that

$$F(u^*,q) \geq 0, \quad \forall\, q\in K.$$

Denote the solution set of EP by $(shortly, EP(F))$. In order to solve EP, we need the following assumptions on F :

(i) $F(p,p)=0$ for all $p\in K$;
(ii) $F(p,q)+F(q,p)\leq 0$ for all $p,q\in K$;
(iii) For each $p\in K, q\mapsto F(p,q)$ is convex;
(iv) For each $\overline{p}\in X, r>0$, there exists a compact subset $D_{\overline{p}}\subseteq K$ containing a point $h\in D_{\overline{p}}\subseteq K$ such that

$$F(p,h)-\frac{1}{r}\langle \overrightarrow{ph}, \overrightarrow{p\overline{p}}\rangle <0 \quad \forall\, p\in D_{\overline{p}}\subseteq K.$$

Lemma 4.1. *([10]) Let K be a nonempty closed convex subset of a complete CAT(0) space X and $F: K\times K\to\mathbb{R}$ be a bifunction satisfying (i)-(iv). For any $r>0$ and $p\in X$, define the following resolvent $T_r: X\to K$ of F:*

$$T_r p = \{h\in K: F(h,q)-\frac{1}{r}\langle \overrightarrow{ph}, \overrightarrow{p\overline{p}}\rangle \geq 0,\ \forall\, q\in K\},$$

then, the following conclusions holds

(i) *T_r is a single-valued firmly nonexpansive mapping;*
(ii) *$F(T_r)=EP(F)$;*
(iii) *$EP(F)$ is closed and convex.*

It follows Corollary 3.2 and Lemma 4.1 that the following result holds.

Theorem 4.1. *Let $K\subset X$ be a nonempty closed convex subset of complete CAT(0) space $(X,d), g: X\to(-\infty,\infty]$ be a proper convex lower semi-continuous function, $F: K\times K\to\mathbb{R}$ be a bifunction satisfying (i)-(iv), T_r be the resolvent of F . Let $u_1\in X$ be chosen arbitrarily and the sequence $\{u_n\}$ be defined as follows:*

$$\begin{cases} p_n = J^g_{\lambda_n}(u_n) = \arg\min\limits_{q\in X}[g(q)+\dfrac{1}{2\lambda_n}d^2(q,u_n)] \\ x_{n+1} = T((1-\alpha_n)u_n \oplus \alpha_n T_r p_n), \quad n\geq 1, \end{cases} \tag{4.1}$$

where $\lambda_n\geq\lambda>0$, $\{\alpha_n\}$ be a sequence in $[b,c]$, and $b,c\in(0,1), 0<b(1-c)\leq\dfrac{1}{2}$.

(i) *If $\Omega = F(T_r)\cap\arg\min\limits_{q\in X} g(q)\neq\emptyset$ then $\{u_n\}$ $\triangle$-convergent to a point $u\in\Omega$.*
(ii) *In addition, if $\Omega = F(T_r)\cap\arg\min\limits_{q\in X} g(q)\neq\emptyset$ and T_r is semi-compact, then $\{u_n\}$ converges strongly to a point $u\in\Omega$.*

5. Conclusion

This paper introduces an iterative algorithm aimed at tackling the minimization and fixed point problems arising from total asymptotically nonexpansive mappings in CAT(0) spaces. We provide strong convergence theorems and $\triangle$-convergence theorems to address these problems effectively. Additionally, we demonstrate the applicability of our results by solving the equilibrium problem in CAT(0) spaces.

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