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SOLUTIONS OF HIGHER ORDER SINGULAR NONLINEAR $(l-1,1)$ CONJUGATE-TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH LOWER-UPPER SOLUTION

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ABSTRACT. Using the monotone iterative technique coupled with method of lower and upper solutions, we establish the existence and uniqueness of solutions for higher order singular nonlinear $(l-1,1)$ conjugate-type fractional differential equation with one nonlocal term.

KEYWORDS: Monotone iterative technique, lower-upper solutions, integral boundary conditions, fractional differential equations, existence and uniqueness solution, Banach contraction principle.
AMS Subject Classification: 34A08, 34B10, 34B16.

1. Introduction

Fractional differential equations are the generalization of ordinary differential equations to non-integer order. This generalization has interesting applications in various fields of chemistry, physics, mechanics, economics, electrodynamics etc. [4, 10]. Boundary value problems [BVP] of fractional differential equations have widespread attention and some attractive results obtained [1, 7, 9, 19] recently. Monotone iterative technique plays an important role to obtain existence of solutions of nonlinear fractional differential equations [5]. This technique is used to obtain the solutions of nonlinear initial value problems [6], boundary value problems [2, 14, 19, 21]. Existence and uniqueness of solutions of Riemann-Liouville fractional

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differential equations with integral boundary conditions is obtained by Nanware et. al. [8]. Sun and Zhao [12] studied the fractional differential equations with integral boundary conditions using monotone iterative method.

In the recent years, the theory of singular boundary value problems has become an important area of investigation [3, 13, 17, 18]. The existence of solutions by using various methods such as lower and upper solution method and fixed point theorem is proved. In [20] X. Zhang et al. obtained the existence and uniqueness of positive solutions when g has singularities at $r = 0$ and (or) 1 by using monotone iterative method. In 2020 [11] S. Song et al. investigated the existence of extremal solutions by using monotone iterative technique coupled with lower and upper solutions for the problem

$$\begin{cases} -D^{\nu}_{0+}z(r) = g(r, z(r)), \quad r \in [0, 1], \\ z(0) = 0, \qquad z(1) = \displaystyle\int_0^1 z(s)\, d\eta(s), \end{cases}$$

where $1 < \nu < 2$, D^{ν}_{0+} is the Riemann-Liouville fractional derivative and $\eta(r)$ is a positive measure function. Y. Wang et al. [15] studied the positive properties of the green function for the Dirichlet-type problem

$$\begin{cases} -D^{\nu}_{0+}z(r) + az(r) = g(r, z(r)), \quad 0 < r < 1, \\ z(0) = 0, \qquad z(1) = 0, \end{cases}$$

where $1 < \nu < 2$, $a > 0$, D^{ν}_{0+} is the Riemann-Liouville fractional derivative. Y. Wang et al.[16] established the existence of positive solutions for resonant problem.

Inspired by the aforementioned works, in this paper we give some sufficient conditions, under which following problems have extremal solutions

$$\begin{cases} D^{\nu}_{0+}z(r) + g(r, z(r)) = 0, \quad 0 < r < 1, \quad l - 1 < \nu \leq l, \\ z^{(k)}(0) = 0, \quad 0 \leq k \leq l - 2, \quad z(1) = \displaystyle\int_0^1 z(s)\, d\eta(s), \end{cases} \tag{1.1}$$

where, D^{ν}_{0+} is the Riemann-Liouville fractional derivative of order $l \geq 2$, $l \in \mathbb{N}$, g has singularities at $r = 0$ and (or) 1, η is a function of bounded variation and $\int_0^1 z(s) d\eta(s)$ denotes the Riemann-Stieltjes integral of z with respect to η, $d\eta$ can be signed measure . The layout of this paper is as follows: In section 2, we present some basic definitions and lemmas that will be used to prove our main results. Section 3 is devoted to uniqueness of solution to BVP (1.1) by using Banach contraction principle. In Section 4, we developed the monotone iterative method and applied it to obtain existence and uniqueness results for Riemann-Liouville fractional differential equations with integral boundary conditions.

2. Preliminaries

In this section, we present some useful definitions and lemmas that will be used in the next section to attain existence and uniqueness results for the nonlinear of BVP (1.1).

Definition 2.1. [10] The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $z : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\nu}z(r) = \frac{1}{\Gamma(\nu)} \int_0^r (r - s)^{\nu - 1} z(s)\, ds$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. [10] The Riemann-Liouville fractional derivative of order $\nu > 0$ of a function $z : (0,\infty) \to \mathbb{R}$ is given by

$$D^{\nu} z(r) = \frac{1}{\Gamma(l-\nu)} (\frac{d}{dr})^{l} \int_0^r (r-s)^{\nu-l+1} z(s)\, ds$$

where $l \in \mathbb{N}$ as the unique positive integer satisfying $l-1 < \nu \le l$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.3. A function $\dot{x}_0 \in C([0,\,1])$ is called a lower solution of BVP (1.1) if it satisfies

$$\begin{cases} D^{\nu}_{0+}\dot{x}_0(r) + g(r,\dot{x}_0(r)) \ge 0, & 0<r<1, l-1<\nu\le l,\\ \dot{x}_0^{(k)}(0)=0, \quad 0\le k\le l-2, & \dot{x}_0(1) \le \int_0^1 \dot{x}_0(s)\, d\eta(s). \end{cases} \tag{2.1}$$

Definition 2.4. A function $\dot{y}_0 \in C([0,\,1])$ is called a upper solution of BVP (1.1) if it satisfies

$$\begin{cases} D^{\eta}_{0+}\dot{y}_0(r) + g(r,\dot{y}_0(r)) \le 0, & 0<r<1, \quad l-1<\nu\le l,\\ \dot{y}_0^{(k)}(0)=0, \quad 0\le k\le l-2, & \dot{y}_0(1) \ge \int_0^1 \dot{y}_0(s)\, d\eta(s). \end{cases} \tag{2.2}$$

Denote

$$\rho(r) = \frac{\nu-2}{\Gamma(\nu-1)} + \sum_{k=1}^{\infty} \frac{r^k}{\Gamma((k+1)\nu-2)}.$$

It is easy to check that (see [15, 16])

$$\rho(0) = \frac{\nu-2}{\Gamma(\nu-1)} < 0,$$

$$\rho^{'}(r) = \sum_{k=1}^{\infty} \frac{k r^{k-1}}{\Gamma((k+1)\nu-2)} > 0, \text{ on } \quad (0,\infty)$$

and

$$\lim_{r\to+\infty} \rho(r) = +\infty.$$

Therefore, there exist a unique $a^* > 0$ such that $\quad \rho(a^*) = 0$.
Set

$$G_a(r) = r^{\nu-1} E_{\nu,\,\nu}(a r^{\nu}), \quad \text{where} \quad E_{\nu,\,\nu}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma((k+1)\nu)} \tag{2.3}$$

is the Mittag-Leffler function ([4, 10]).
For convenience, we list here the following assumptions.
B1] the parameter a satisfies $a \in (0, a^*]$,
B2] $\eta(r)$ is bounded variation in $(0,1)$ such that $0 < \alpha \le 1$, $\alpha = \int_0^1 G_a(s)\, d\eta(s)$ and $0 \le \zeta_\eta(s) = \int_0^1 H_a(r,s)\, d\eta(s)$, $0 < G_a(1) - \int_0^1 G_a(s)\, d\eta(s)$,
B3] $g \in C((0,1)\times[0,\infty),[0,\infty))$ and

$g(r,u) - g(r,v) \ge -a(u-v)$ for $\dot{x}_0 \le u \le v \le \dot{y}_0$, $r \in (0,1)$.
Set

$$K_a(r,s) = H_a(r,s) + G_a(r) h^*(s)$$

where,

$$h^*(s) = \frac{\zeta_\eta(s)}{G_a(1) - \alpha},$$

$$H_a(r,s) = \frac{1}{G_a(1)} \begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1) & \text{if } 0 \le s \le r \le 1 \\ G_a(r)G_a(1-s) & \text{if } 0 \le r \le s \le 1. \end{cases} \tag{2.4}$$

Lemma 2.5. [15] *Suppose that* $[B1]$ *holds and* $y \in L[0,1]$*. Then the problem*

$$\begin{cases} -D^\nu_{0+} z(r) + az(r) = q(r), & 0 < r < 1, \\ z(0) = 0, & z(1) = 0, \end{cases}$$

has a unique solutions

$$z(r) = \int_0^1 H_a(r,s) q(s)\, ds,$$

where

$$H_a(r,s) = \frac{1}{G_a(1)} \begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1) & \text{if } 0 \le s \le r \le 1 \\ G_a(r)G_a(1-s) & \text{if } 0 \le r \le s \le 1. \end{cases}$$

Lemma 2.6. *Suppose that* $[B1]$, $[B2]$ *hold and* $y \in C([0,1])$*. Then linear fractional boundary value problem*

$$\begin{cases} D^\nu_{0+} z(r) - az(r) + q(r) = 0, & 0 < r < 1, \quad l-1 < \nu \le l, \\ z^{(k)}(0) = 0, \quad 0 \le k \le l-2, & z(1) = \int_0^1 z(s)\, d\eta(s), \end{cases} \tag{2.5}$$

has the following unique solution

$$z(r) = \int_0^1 K_a(r,s) q(s)\, ds.$$

Proof:- First apply I^ν on linear equation (2.5) and using result, see in [4, 10], we get

$$z(r) = -\int_0^r G_a(r-s)q(s)\, ds + C_0 G_a(r) + C_1 G_a^{'}(r) + C_2 G_a^{''}(r) \ldots + C_{l-1} G_a^{(l-1)}(r). \tag{2.6}$$

Since $z(0) = 0$ then $C_{l-1} = 0$. Similarly

$$z^{'}(0) = z^{''}(0) = \ldots = z^{l-2}(0) = 0$$

gives

$$C_1 = C_2 = \ldots = C_{l-2} = 0.$$

Then equation (2.6) becomes

$$z(r) = -\int_0^r G_a(r-s)q(s)\, ds + C_0 G(r).$$

Using $z(1) = \int_0^1 z(s)\, d\eta(s)$, we obtain

$$C_0 = \frac{1}{G_a(1)} \left[\int_0^1 z(s)\, d\eta(s) + \int_0^1 G_a(1-s) q(s)\, ds \right].$$

Hence,

$$z(r) = -\int_0^r G_a(r-s)q(s)\, ds + \frac{G_a(r)}{G_a(1)} \left[\int_0^1 z(s)\, d\eta(s) + \int_0^1 G_a(1-s) q(s)\, ds \right],$$

$$= -\int_0^r G_a(r-s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\int_0^1 z(s)\,d\eta(s) + \frac{G_a(r)}{G_a(1)}\int_0^r G_a(1-s)q(s)\,ds$$
$$+ \frac{G_a(r)}{G_a(1)}\int_r^1 G_a(1-s)q(s)\,ds,$$
$$= \frac{1}{G_a(1)}\int_0^r \left[G_a(r)G_a(1-s) - G_a(1)G_a(r-s)\right] q(s)\,ds$$
$$+ \frac{1}{G_a(1)}\int_r^1 G_a(r)G_a(1-s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\int_0^1 z(s)\,d\eta(s),$$
$$= \int_0^1 H_a(r,s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\int_0^1 z(s)\,d\eta(s).$$

Let,

$$\int_0^1 z(s)\,d\eta(s) = \int_0^1 [\int_0^1 H_a(s,\tau)q(\tau)\,d\tau]d\eta(s) + \int_0^1 \frac{G_a(s)}{G_a(1)}\,d\eta(s)\int_0^1 z(s)\,d\eta(s).$$

Therefore

$$[1 - \frac{1}{G_a(1)}\int_0^1 G_a(s)\,d\eta(s)]\int_0^1 z(s)\,d\eta(s) = \int_0^1 [\int_0^1 H_a(s,\tau)q(\tau)\,d\tau]d\eta(s)$$
$$\int_0^1 z(s)\,d\eta(s) = \frac{1}{[1 - \frac{1}{G_a(1)}\int_0^1 G_a(s)\,d\eta(s)]}\int_0^1\int_0^1 H_a(s,\tau)q(\tau)\,d\tau d\eta(s).$$

Therefore $$z(r) = \int_0^1 H_a(r,s)q(s)\,ds + \frac{G_a(r)}{G_a(1)}\left[\frac{1}{1 - \frac{1}{G_a(1)}\int_0^1 G_a(s)\,d\eta(s)}\right]$$
$$\int_0^1\int_0^1 H_a(s,\tau)q(\tau)\,d\tau d\eta(s),$$
$$= \int_0^1 H_a(r,s)q(s)\,ds + \frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\,d\eta(s)}\int_0^1\int_0^1 H_a(s,\tau)q(\tau)\,d\tau d\eta(s),$$
$$= \int_0^1 \left[H_a(r,s) + \frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\,d\eta(s)}\int_0^1 H_a(s,\tau)d\eta(\tau)\right] q(s)\,ds,$$
$$z(r) = \int_0^1 K_a(r,s)q(s)\,ds.$$

Lemma 2.7. *Suppose* $[B1]$, $[B2]$ *holds,then the function* $K_a(r,s)$ *has the following properties*

(i) $K_a(r,s) > 0 \quad \forall\ r,s \in (0,1)$,
(ii) $\psi_2(s)r^{\nu-1} \le K_a(r,s) \le \psi_1(s)r^{\nu-1}, \quad \forall\ r,s \in (0,1)$

where, $\psi_1(s) = G_a(1-s) + G_a(1)h^*(s), \quad \psi_2(s) = \frac{1}{\Gamma(\nu)}h^*(s).$

Proof:- We need to prove that (2) holds. By equation (2.7)

$$\frac{r^{\nu-1}}{\Gamma(\nu)} \leq G_a(r) = r^{\nu-1}\sum_{k=0}^{\infty}\frac{a^k r^{\nu k}}{\Gamma((k+1)\nu)} \leq r^{\nu-1}G_a(1), \quad r \in (0,1). \tag{2.7}$$

$$G_a^{'}(r) = \sum_{k=0}^{\infty}\frac{a^k r^{(k+1)\nu-2}}{\Gamma((k+1)\nu-1)} > 0, \quad r \in (0,1) \tag{2.8}$$

$$\begin{aligned}
G_a^{''}(r) &= \sum_{k=0}^{\infty}\frac{a^k[(k+1)\nu-2]r^{(k+1)\nu-3}}{\Gamma((k+1)\nu-1)},\\
&= r^{\nu-3}\sum_{k=0}^{\infty}\frac{a^k[(k+1)\nu-2]r^{k\nu}}{\Gamma((k+1)\nu-1)},\\
&= r^{\nu-3}\left[\frac{\nu-2}{\Gamma(\nu-1)} + \sum_{k=1}^{\infty}\frac{a^k r^{k\nu}[(k+1)\nu-2]}{\Gamma((k+1)\nu-1)}\right],\\
&= r^{\nu-3}\left[\frac{\nu-2}{\Gamma(\nu-1)} + \sum_{k=1}^{\infty}\frac{a^k r^{k\nu}}{\Gamma((k+1)\nu-2)}\right],\\
&= r^{\nu-3}\rho(ar^{\nu}) < r^{\nu-3}\rho(a) \leq r^{\nu-3}\rho(a^*) = 0, \quad r \in (0,1),
\end{aligned} \tag{2.9}$$

which implies that $G_a(r)$ is strictly increasing on $(0,1)$ and $G_a^{'}(r)$ is strictly decreasing on $(0,1)$. Therefore by (2.7) we have,

$$\begin{aligned}
K_a(r,s) &= H_a(r,s) + G_a(r)h^*(s),\\
&\leq \frac{G_a(r)G_a(1-s)}{G_a(1)} + G_a(r)h^*(s),\\
&= \left[\frac{G_a(1-s)}{G_a(1)} + h^*(s)\right]G_a(r),\\
&\leq \left[\frac{G_a(1-s)}{G_a(1)} + h^*(s)\right]r^{\nu-1}G_a(1),\\
&= [G_a(1-s) + G_a(1)h^*(s)]r^{\nu-1},\\
&= \psi_1(s)r^{\nu-1},
\end{aligned} \tag{2.10}$$

where, $\psi_1(s) = G_a(1-s) + G_a(1)h^*(s)$.
On the other hand, when $0 \leq r \leq s \leq 1$. Note that $G_a(0) = 0$ and monotonocity of $G_a(r)$, it is clear that

$$G_a(r)G_a(1-s) > 0. \tag{2.11}$$

Hence $H_a(r,s) > 0$ and also by $[B2]$, $K_a(r,s) > 0$ when $0 < r \leq s \leq 1$.
When $0 < s < r < 1$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}[G_a(r)G_a(1-s) - G_a(r-s)G_a(1)] &= -G_a(r)G_a^{'}(1-s) + G_a(1)G_a^{'}(r-s)\\
&\geq [G_a(1) - G_a(r)]G_a^{'}(1-s).
\end{aligned} \tag{2.12}$$

Integrating with respect to s, we obtain

$$G_a(r)G_a(1-s) - G_a(r-s)G_a(1) \geq \int_0^s [G_a(1) - G_a(r)]G_a^{'}(1-\mu)\,d\mu,$$

$$= [G_a(1) - G_a(r)]\left[\frac{G_a(1-\mu)}{-1}\right]_0^s,$$
$$= [G_a(1) - G_a(r)][G_a(1) - G_a(1-s)] > 0. \tag{2.13}$$

Then, by (2.4), (2.11), (2.13), we get

$$H_a(r,s) = G_a(r)G_a(1-s) - G_a(r-s)G_a(1) > 0 \quad r,s \in (0,1).$$

Now,

$$K_a(r,s) = H_a(r,s) + G_a(t)h^*(s) \geq G_a(r)h^*(s),$$
$$\geq \frac{r^{\nu-1}}{\Gamma(\nu)}h^*(s) = \psi_2(s)r^{\nu-1} > 0 \quad r,s \in (0,1)$$

where, $\psi_2(s) = \frac{1}{\Gamma(\nu)}h^*(s)$. Hence the proof.

Lemma 2.8. *For* $0 < r_1 \leq r_2 < 1$

(i) $|G_a(r_2) - G_a(r_1)| < E_{\nu,\nu-1}(a)\,|r_2 - r_1| = G_a(1)|r_2 - r_1|$,
(ii) $|G_a(r_2 - s) - G_a(r_1 - s)| < E_{\nu,\nu-1}(a)\,|r_2 - r_1| = G_a(1)|r_2 - r_1|$,
(iii) $|H_a(r_2,s) - H_a(r_1,s)| < 2[G_a(1)]^2|r_2 - r_1|$,
(iv) $|K_a(r_2,s) - K_a(r_1,s)| \leq \max_{0\leq s\leq 1}|K_a(r_2,s) - K_a(r_1,s)| < G_a(1)[2G_a(1) + |h^*(s)|]|r_2 - r_1|$.

Proof:-

1] $$|G_a(r_2) - G_a(r_1)| = \left|r_2^{\nu-1}E_{\nu,\nu}(ar_2^\nu) - r_1^{\nu-1}E_{\nu,\nu}(ar_1^\nu)\right|,$$
$$= \left|r_2^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k r_2^{\nu k}}{\Gamma((k+1)\nu)} - r_1^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k r_2^{\nu k}}{\Gamma((k+1)\nu)}\right|,$$
$$= \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu)}\left|r_2^{\nu(k+1)-1} - r_1^{\nu(k+1)-1}\right|.$$

Applying mean value theorem, we get

$$r_2^{\nu(k+1)-1} - r_1^{\nu(k+1)-1} < [\nu(k+1)-1](r_2 - r_1).$$

Therefore

$$|G_a(r_2) - G_a(r_1)| < \sum_{k=o}^{\infty}\frac{a^k[\nu(k+1)-1]}{\Gamma((k+1)\nu)}|r_2 - r_1|,$$
$$= \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu-1)}|r_2 - r_1|,$$
$$= E_{\nu,\nu-1}(a)|r_2 - r_1| = G_a(1)|r_2 - r_1|.$$

2] $$|G_a(r_2 - s) - G_a(r_1 - s)| = \left|(r_2 - s)^{\nu-1}E_{\nu,\nu}(a(r_2 - s)^\nu) - (r_1 - s)^{\nu-1}E_{\nu,\nu}(a(r_1 - s)^\nu)\right|,$$
$$= \left|(r_2 - s)^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k(r_2 - s)^{\nu k}}{\Gamma((k+1)\nu)} - (r_1 - s)^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k(r_2 - s)^{\nu k}}{\Gamma((k+1)\nu)}\right|,$$
$$= \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu)}\left|(r_2 - s)^{\nu(k+1)-1} - (r_1 - s)^{\nu(k+1)-1}\right|.$$

Applying mean value theorem, we get

$$(r_2 - s)^{\nu(k+1)-1} - (r_1 - s)^{\nu(k+1)-1} < [\nu(k+1)-1](r_2 - r_1).$$

Therefore

$$|G_a(r_2 - s) - G_a(r_1 - s)| < \sum_{k=o}^{\infty} \frac{a^k}{\Gamma((k+1)\nu - 1)} |r_2 - r_1|$$
$$= E_{\nu,\,\nu-1}(a)|r_2 - r_1| = G_a|r_2 - r_1|$$

3] $$\begin{aligned} |H_a(r_2, s) - H_a(r_1, s)| &= |[G_a(r_2)G_a(1-s) - G_a(1)G_a(r_2 - s)] \\ &\quad - [G_a(r_1)G_a(1-s) - G_a(1)G_a(r_1 - s)]|, \\ &< G_a(1-s)|G_a(r_2) - G_a(r_1)| + G_a(1)|G_a(r_1 - s) - G_a(r_2 - s)|, \\ &< G_a(1-s)E_{\nu,\,\nu}(a)|r_2 - r_1| + G_a(1)E_{\nu,\,\nu}(a)|r_2 - r_1|, \\ &= E_{\nu,\,\nu}(a)[G_a(1-s) + G_a(1)]|r_2 - r_1|, \\ &= G_a(1)[G_a(1-s) + G_a(1)]|r_2 - r_1|, \\ &< 2[G_a(1)]^2|r_2 - r_1|. \end{aligned}$$

4] $$\begin{aligned} |K_a(r_2, s) - K_a(r_1, s)| &\le \max_{0\le s\le 1} |K_a(r_2, s) - K_a(r_1, s)|, \\ &= \max_{0\le s\le 1} |[H_a(t_2, s) - H_a(t_1, s)] + [G_a(t_2) - G_a(r_1)]h^*(s)|, \\ &< 2[G_a(1)]^2|r_2 - r_1| + G_a(1)|r_2 - r_1||h^*(s)|, \\ &= G_a(1)[2G_a(1) + |h^*(s)|]|r_2 - r_1|. \end{aligned}$$

Hence the proof.

3. Main Results

Let $\mathscr{C} = C([0,1])$ be endowed with the norm $\|z\| = \max_{0\le r\le 1} |z(r)|$, then $(\mathscr{C}, \|.\|)$ is a Banach space. Now we define the operator $T : \mathscr{C} \to \mathscr{C}$ by

$$(Tz)(r) = \int_0^1 K_a(r, s)g(s, z(s))\, ds.$$

Theorem 3.1. *Prove $T : \mathscr{C} \to \mathscr{C}$ is uniformly continuous.*

Proof:- The operator $T : \mathscr{C} \to \mathscr{C}$ is continuous in the view of non-negativeness and continuity of $K_a(r, s)$, $H_a(r, s)$ and $g(r, z)$. Let $S \subset \mathscr{C}$ be bounded i.e. $\exists$ a positive constants $M > 0$ such that $\|z\| < M \ \forall\ z \in S$, Let $L^* = \max_{0\le r\le 1} |g(r, z)|$ then by Lemma 2.7 the operator $T : S \to \mathscr{C}$ is bounded uniformly. Now to prove T(S) is equicontinuous.
If $z \in S$, $\quad 0 < r_1 < r_2 < 1$ then

$$\begin{aligned} |(Tz)(r_2) - (Tz)(r_1)| &= \left| \int_0^1 [K_a(r_2, s) - K_a(r_1, s)]g(s, z(s))\, ds \right|, \\ &\le \max_{0\le s\le 1} \int_0^1 |K_a(r_2, s) - K_a(r_1, s)|\, |g(s, z(s))|\, ds, \\ &< L^* G_a(1)|r_2 - r_1| \int_0^1 [2G_a(1) + |h^*(s)|]\, ds, \\ &< L^* G_a(1)|r_2 - r_1|[2G_a(1) + \int_0^1 |h^*(s)|]\, ds. \end{aligned}$$

Then $|(Tz)(r_2) - (Tz)(r_1)| \to 0$ uniformly as $r_1 \to r_2$. This shows that $T(S)$ is equicontinuous on $\mathscr{C}$. Then by Arzela-Ascoli theorem, the operator $T : \mathscr{C} \to \mathscr{C}$ is completely continuous.

Theorem 3.2. *Assume that* $[B1]$, $[B2]$ *holds. Then there exist nonnegative constant* N^* *such that function* g *satisfies* $|g(r,x)-g(r,y)| \leq N^*|x-y|, \quad r \in (0,1), \quad x,y \in \mathscr{C}$ *and let*
$\Lambda = \int_0^1 \psi(s)\,ds$ *then the BVP (1.1) has a unique fixed point.*

Proof:- For any $x, y \in \mathscr{C}, \quad r,s \in (0,1)$ and using Lemma 2.7

$$\begin{aligned}
\|Tx(r) - Ty(r)\| &= \max_{0\leq r\leq 1} |Tx(r) - Ty(r)|\,,\\
&= \max_{0\leq r\leq 1} \left|\int_0^1 K_a(r,s)[g(r,x) - g(r,y)]\,ds\right|,\\
&= \max_{0\leq r\leq 1} \int_0^1 |K_a(r,s)||g(r,x) - g(r,y)|\,ds,\\
&\leq r^{\nu-1}N^*\,\|x-y\| \int_0^1 \psi(s)\,ds,\\
&= r^{\nu-1}N^*\Lambda\,\|u-v\|\,.
\end{aligned}$$

Then by Banach contraction mapping theorem, T has a unique fixed point in $\mathscr{C}$, i.e. the BVP (1.1) has a unique solution. The proof is complete.

4. Monotone Iterative Method

In this section, we develop monotone iterative technique combined with the method of lower-upper solutions and we prove the existence and uniqueness theorem of solution for BVP (1.1). For $\dot{x}_0, \dot{y}_0 \in \mathscr{C}$ with $\dot{x}_0 \leq \dot{y}_0$ for $r \in (0,1)$, we denote

$$\Omega^* = [\dot{x}_0, \dot{y}_0] = \{z \in \mathscr{C} : \dot{x}_0 \leq z(r) \leq \dot{y}_0, r \in (0,1)\}\,.$$

Lemma 4.1. *Assume that* $[B1]$, $[B2]$ *holds and* $z \in \mathscr{C}$ *satisfies*

$$\begin{aligned}
&-D^{\nu}z(r) + az(r) \geq 0,\\
&z^{(k)}(0) = 0, \qquad z(1) \geq \int_0^1 z(s)\,d\eta(s)
\end{aligned} \tag{4.1}$$

then for $r \in (0,1)$, $z(r) \geq 0$.

Proof:- Let $q(r) = -D^{\nu}z(r) + az(r)$ and $d = z(1) - \int_0^1 z(s)\,d\eta(s)$. Then from equation (4.1), we have $q(r) \geq 0$, $d \geq 0$. Then by Lemma 2.6, the problem (2.5) has unique solution which can be expressed as

$$\begin{aligned}
z(r) &= \int_0^1 K_a(r,s)q(s)\,ds,\\
&= \int_0^1 H_a(r,s)q(s)\,ds + \left[\frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\,d\eta(s)}\right]\int_0^1 H_a(r,s)\,d\eta(s)
\end{aligned}$$

where,

$$H_a(r,s) = \frac{1}{G_a(1)}\begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1) & \text{if } 0 \leq s \leq r \leq 1\\ G_a(r)G_a(1-s) & \text{if } 0 \leq r \leq s \leq 1.\end{cases}$$

Then by Lemma 2.7, $H_a(r,s) \geq 0$ and $K_a(r,s) \geq 0$ $\forall\ r,s \in (0,1)$. Hence $z(r) \geq 0$, $\forall\ r,s \in (0,1)$.

Theorem 4.2. *Suppose $[B1]$, $[B2]$, $[B3]$ holds, then there exist monotone iterative sequences $\{\dot{x}_m\}$, $\{\dot{y}_m\} \subset \Omega^*$ such that $\dot{x}_m \to \dot{x}$, $\dot{y}_m \to \dot{y}$ as $m \to \infty$ uniformly in Ω^* and $\dot{x}$, $\dot{y}$ are a minimal and maximal solution of BVP (1.1) in Ω^* respectively.*

Proof:- For $\dot{x}_{m-1}$, $\dot{y}_{m-1} \in \mathscr{C}$, $m \geq 1$, we define two sequences $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ respectively by the relations,

$$\begin{cases} D^{\nu}_{0+}\dot{x}_m(r) - a(\dot{x}_m(r) + \dot{x}_{m-1}(r)) + g(r, \dot{x}_{m-1}(r)) = 0, \quad 0 < r < 1 \\ \dot{x}^{(k)}_m(0) = 0, \qquad \dot{x}_m(1) = \int_0^1 \dot{x}_m(s)\, d\eta(s) \end{cases}$$

and

$$\begin{cases} D^{\nu}_{0+}\dot{y}_m(r) - a(\dot{y}_m(r) + \dot{y}_{m-1}(r)) + g(r, \dot{y}_{m-1}(r)) = 0, \quad 0 < r < 1, \\ \dot{y}^{(k)}_m(0) = 0, \qquad \dot{y}_m(1) = \int_0^1 \dot{y}_m(s)\, d\eta(s). \end{cases}$$

Then by Lemma 2.6, $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ are well defined. Firstly we need to show that $\dot{x}_0(r) \leq \dot{x}_1(r) \leq \dot{y}_1(r) \leq \dot{y}_0(r)$ for any $r \in (0, 1)$.
Set $\dot{p}(r) = \dot{x}_1(r) - \dot{x}_0(r)$ and by definition of $\dot{x}_1(r)$ with lower solution $\dot{x}_0(r)$ we get,

$$\begin{aligned} -D^{\nu}_{0+}\dot{p}(r) + a\dot{p}(r) &= -D^{\nu}_{0+}(\dot{x}_1(r) - \dot{x}_0(r)) + a(\dot{x}_1(r) + \dot{x}_0(r)), \\ &= -D^{\nu}_{0+}\dot{x}_1(r) + a(\dot{x}_1(r) + \dot{x}_0(r)) + D^{\nu}_{0+}\dot{x}_0(r), \\ &\geq -a(\dot{x}_1(r) + \dot{x}_0(r)) + g(r, \dot{x}_0(r)) + a(\dot{x}_1(r) + \dot{x}_0(r)) - g(r, \dot{x}_0(r)) \\ &= 0. \\ Also, \dot{p}^{(k)}(0) &= \dot{x}^{(k)}_1(0) - \dot{x}^{k}_0(0) = 0, \\ \dot{p}(1) &= \dot{x}_1(1) - \dot{x}_0(1) \\ &\geq \int_0^1 \dot{x}_1(s)\, d\eta(s) - \int_0^1 x_0(s)\, d\eta(s) \\ &= \int_0^1 [\dot{x}_1(s) - \dot{x}_0(s)]\, d\eta(s) = \int_0^1 \dot{p}(s)\, d\eta(s). \end{aligned}$$

Then by Lemma 4.1, $\dot{p}(r) \geq 0 \Rightarrow \dot{x}_1(r) \geq \dot{x}_0(r)$, $r \in (0, 1)$.
Now to prove $\dot{y}_1(r) \leq \dot{y}_0(r)$ $\forall r \in (0, 1)$. For this, set $\dot{p}(r) = \dot{y}_1(r) - \dot{y}_0(r)$ and by definition of $\dot{y}_1(r)$ with upper solution $\dot{y}_0(r)$ we get,

$$\begin{aligned} -D^{\nu}_{0+}\dot{p}(r) + a\dot{p}(r) &= -D^{\nu}_{0+}(\dot{y}_1(r) - \dot{y}_0(r)) + a(\dot{y}_1(r) + \dot{y}_0(r)), \\ &= -D^{\nu}_{0+}\dot{y}_1(r) + a(\dot{y}_1(r) + \dot{y}_0(r)) + D^{\nu}_{0+}\dot{y}_0(r), \\ &\leq -a(\dot{y}_1(r) + \dot{y}_0(r)) + g(r, \dot{y}_0(r)) + +a(\dot{y}_1(r) + \dot{y}_0(r)) - g(r, \dot{y}_0(r)) \\ &= 0. \\ Also \dot{p}^{(k)}(0) &= \dot{y}^{(k)}_1(0) - \dot{y}^{k}_0(0) = 0, \\ \dot{p}(1) &= \dot{y}_1(1) - \dot{y}_0(1), \\ &\leq \int_0^1 \dot{y}_1(s)\, d\eta(s) - \int_0^1 \dot{y}_0(s)\, d\eta(s), \\ &= \int_0^1 [\dot{y}_1(s) - \dot{y}_0(s)]\, d\eta(s) = \int_0^1 \dot{p}(s)\, d\eta(s). \end{aligned}$$

Then by Lemma 4.1, $\dot{p}(r) \leq 0 \Rightarrow \dot{y}_1(r) \leq \dot{y}_0(r)$, $\forall r \in (0, 1)$.
Now to prove $\dot{x}_1(r) \leq \dot{y}_1(r)$ $\forall r \in (0, 1)$. Set $\dot{p}(r) = \dot{y}_1(r) - \dot{x}_1(r)$. Then by $[B3]$ and

definition of $\dot{x}_1(r)$, $\dot{y}_1(r)$, we get

$$\begin{aligned}
-D^{\nu}_{0+}\dot{p}(r) &= -D^{\nu}_{0+}[\dot{y}_1(r)-\dot{x}_1(r)],\\
&= -D^{\nu}_{0+}\dot{y}_1(r)-[-D^{\nu}_{0+}\dot{x}_1(r)],\\
&= g(r,\dot{y}_0(r))-a[\dot{y}_1(r)-\dot{y}_0(r)]-[g(r,\dot{x}_0(r))-a[\dot{x}_1(r)-\dot{x}_0(r)]],\\
&= [g(r,\dot{y}_0(r))-g(r,\dot{x}_0(r))]-a[\dot{y}_1(r)-\dot{y}_0(r)]+a[\dot{x}_1(r)-\dot{x}_0(r)],\\
&\geq -a(\dot{y}_0(r)-\dot{x}_0(r))-a[\dot{y}_1(r)-\dot{y}_0(r)]+a[\dot{x}_1(r)-\dot{x}_0(r)],\\
&= -a(\dot{y}_1(r)-\dot{x}_1(r)) = -a\dot{p}(r).
\end{aligned}$$

$$Also \dot{p}^{(k)}(0) = \dot{y}_1^{(k)}(0)-\dot{x}_1^{(k)}(0)=0,$$

$$\begin{aligned}
\dot{p}(1) = \dot{y}_1(1)-\dot{x}_1(1) &= \int_0^1 \dot{y}_1(s)\,d\eta(s) - \int_0^1 \dot{y}_1(s)\,d\eta(s),\\
&= \int_0^1 \dot{p}(s)\,d\eta(s).
\end{aligned}$$

Then by Lemma 4.1, $\dot{p}(r)\geq 0 \Rightarrow \dot{x}_1(r)\leq \dot{y}_1(r)\ \forall r\in(0,1)$. Now by mathematical induction method, it is easy to verify that

$$\dot{x}_0(r)\leq \dot{x}_1(r)\leq \dot{x}_2(r)\leq \ldots \leq \dot{x}_m(r)\leq \dot{y}_m(r)\leq \ldots \dot{y}_1(r)\leq \dot{y}_0(r).$$

Thus the sequences $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ are uniformly bounded and monotonically non-decreasing and non-increasing in $\mathscr{C}$. Hence the point-wise limit exist and are given by $\underset{m\to\infty}{lim}\dot{x}_m(r)=\dot{x}(r)$, $\underset{m\to\infty}{lim}\dot{y}_m(r)=\dot{y}(r)$ on $\mathscr{C}$. Next we claim that $\dot{x}(r)$ and $\dot{y}(r)$ are the extremal solutions of BVP (1.1). Let $z(r)$ be any solution of BVP (1.1) different from $\dot{x}(r)$ and $\dot{y}(r)$ in Ω^*. So there exist some i such that $\dot{x}_i(r)\leq z(r)\leq \dot{y}_i(r)$, $r\in(0,1)$. Set $\dot{p}_1(r)=z(r)-\dot{x}_{i+1}(r)$. So that, by assumption $[B3]$, we obtain

$$\begin{aligned}
-D^{\nu}\dot{p}_1(r) &= -D^{\nu}z(r)-(-D^{\nu}\dot{x}_{i+1}),\\
&= g(r,z(r))-[g(r,\dot{x}_i(r))-a(\dot{x}_{i+1}(r)-\dot{x}_i(r))],\\
&= [g(r,z(r))-g(r,\dot{x}_i(r))]+a(\dot{x}_{i+1}(r)-\dot{x}_i(r)),\\
&\geq -a(z(r)-\dot{x}_i(r))+a(\dot{x}_{i+1}(r)-\dot{x}_i(r)),\\
&= -a[z(r)-\dot{x}_i(r)-\dot{x}_{i+1}(r)+\dot{x}_i(r)],\\
&= -a(z(r)-\dot{x}_{i+1}(r)) = -a\dot{p}_1(r),\\
\dot{p}_1^{(k)}(0) &= 0,\quad \dot{p}_1(1)=\int_0^1 \dot{p}_1(s)\,d\eta(s).
\end{aligned}$$

Then by Lemma 4.1, $\dot{p}_1(r)\geq 0$ implying that $\dot{x}_{i+1}(r)\leq z(r)$ for all i. Similarly set $\dot{p}_2(r)=\dot{y}_{i+1}(r)-z(r)$ and using $[B3]$ we obtain

$$\begin{aligned}
-D^{\nu}\dot{p}_2(r) &= -D^{\nu}\dot{y}_{i+1}(r)-(-D^{\nu}z(r)),\\
&= [g(r,\dot{y}_i(r))-a(\dot{y}_{i+1}(r)-\dot{y}_i(r))]-g(r,z(r)),\\
&= [g(r,\dot{y}_i(r))-g(r,z(r))]-a(\dot{y}_{i+1}(r)-\dot{y}_i(r)),\\
&\geq -a(\dot{y}_i(r)-z(r))+a(\dot{y}_{i+1}(r)-\dot{y}_i(r)),\\
&= -a[\dot{y}_i(r)-z(r)-\dot{y}_{i+1}(r)+\dot{y}_i(r)],\\
&= -a(\dot{y}_{i+1}(r)-z(r)) = -a\dot{p}_2(r),\\
\dot{p}_2^{(k)}(0) &= 0,\quad \dot{p}_2(1)=\int_0^1 \dot{p}_2(s)\,d\eta(s).
\end{aligned}$$

Then by Lemma 4.1, $\dot{p}_2(r) \geq 0$ implying that $z(r) \leq \dot{y}_{i+1}(r)$ for all i. Hence $\dot{x}_{i+1}(r) \leq z(r) \leq \dot{y}_{i+1}(r)$, $r \in (0,1)$. Since $\dot{x}_0(r) \leq z(r) \leq \dot{y}_0(r)$ on $\mathscr{C}$. Hence by induction method, it follows that $\dot{x}_i(r) \leq z(r) \leq \dot{y}_i(r)$ for all i. Taking limit as $i \to \infty$, it follows that $\dot{x}(r) \leq z(r) \leq \dot{y}(r)$ on $\mathscr{C}$. Thus the functions $\dot{x}(r)$, $\dot{y}(r)$ are the extremal solutions of the BVP (1.1). The proof is complete.

Next we prove uniqueness of solutions of the BVP (1.1).

Theorem 4.3. *Assume that,*

(i) $[B_1]$, $[B_2]$, $[B_3]$ *holds,*

(ii) *there exists* $a > 0$ *such that the function* g *satisfies the condition*

$$g(r, v) - g(r, v^*) \leq a(v - v^*) \tag{4.2}$$

for $\dot{x}_0 \leq v \leq v^* \leq \dot{y}_0, r \in (0,1)$.

Then the BVP (1.1) has a unique solution in Ω^*.

Proof:- We know $\dot{x}(r) \leq \dot{y}(r)$ on $\mathscr{C}$. It is sufficient to prove that $\dot{x}(r) \geq \dot{y}(r)$. Consider $\dot{p}(r) = \dot{y}(r) - \dot{x}(r)$. Then we have

$$\begin{aligned} -D^\nu \dot{p}(r) &= -D^\nu \dot{y}(r) - (-D^\nu \dot{x}(r)), \\ &= g(r, \dot{y}(r)) - g(r, \dot{x}(r)), \\ &\leq -a(\dot{y}(r) - \dot{x}(r)) = -a\dot{p}(r) \end{aligned}$$

and

$$\dot{p}^{(k)}(0) = 0, \quad \dot{p}(1) = \int_0^1 \dot{p}(s)\, d\eta(s).$$

By Lemma 4.1, $\dot{p}(r) \leq 0$ implying that $\dot{y}(r) \leq \dot{x}(r)$. Hence $\dot{x}(r) = \dot{y}(r)$ is the unique solution of BVP (1.1).

5. Conclusion

By implementing Banach contraction mapping theorem, it is shown that the mapping T has a unique fixed point in $\mathscr{C}$. Monotone iterative sequences $\{\dot{x}_m\}$ and $\{\dot{y}_m\}$ converging uniformly to $\dot{x}(r)$ and $\dot{y}(r)$ as $m \to \infty$ respectively are constructed. Monotone technique developed is applied to prove that $\dot{x}(r)$, $\dot{y}(r)$ are minimal and maximal solutions of problem (1.1) in Ω^*. Uniqueness of solutions of the nonlinear problem (1.1) with integral boundary conditions is also obtained.

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APPLICATIONS OF GENERALIZED ZORN'S LEMMA

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ABSTRACT. In the present article, by applying our 2013 Metatheorem and the Brøndsted-Jachymski Principle, we obtain various forms generalizations of Zorn's Lemma and their applications. Such examples are our version of the Zermelo fixed point theorem, equivalent formulations of the Caristi fixed point theorem, and Jachymski's 2003 theorem on equivalent conditions when fixed point sets are same to periodic point sets.

KEYWORDS: fixed point theorem, preorder, metric space, fixed point, stationary point, maximal element

AMS Subject Classification: 03E04, 03E25, 06A06, 06A75, 47H10, 54E35, 54H25, 58E30, 65K10

1. Introduction

There are several fields in the fixed point theory. *Analytical* fixed point theory is originated from Brouwer in 1912 and concerns mainly with topological vector spaces. *Metric* fixed point theory is originated from Banach in 1922 and deals with generalizations of contractions and nonexpansive maps. *Topological* fixed point theory relates mainly originated works of Lefschetz, Nielsen, and Reidemeister.

Now the *Ordered* fixed point theory began by Zermelo [36](1908) implicitly and was developed mainly by Knaster [16](1928), Zorn [37](1935), Bourbaki [3](1949-50), Tarski [30, 31](1949, 1955), Ekeland [10, 11](1972,1974), Caristi [7](1976), Brézis-Browder [4](1976), Takahashi [29](1991), and many others. Moreover, in 1985-86 [18, 19], we discovered a Metatheorem stating that any maximum elements in ordered sets can be fixed points, stationary points, collectively fixed points, collectively stationary points, and conversely. Consequently, Ordered fixed point theory is a rich source of information on fixed points of families of multimaps on ordered sets.

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Recently in 2022 [20, 21, 22, 24], we obtained an extended form of Metatheorem and applied it to a large number of known or new results. Moreover in 2022 [24], we found the Brøndsted-Jachymski Principle on ordered sets showing the equalities of maximal elements, fixed point sets and periodic point sets of progressive selfmaps. Later we rearranged the order of statements to the 2023 Metatheorem in [25], which will be the basis of future study in various fields of mathematics.

In the present article, by applying Metatheorem and its particular Brøndsted-Jachymski Principle, we obtain various forms of Zorn's Lemma and their applications.

In Section 2, we introduce our 2023 Metatheorem and the Brøndsted-Jachymski Principle. Section 3 devotes various types of generalizations of Zorn's Lemma. We show their important applications in Sections 4-6. In Section 4, we introduce our version of the Zermelo fixed point theorem and its usefulness. Section 5 devotes to equivalent formulations of the Caristi fixed point theorem. In Section 6, we improve Jachymski's 2003 theorem [13] on equivalent conditions when fixed point sets are same to periodic point sets. Finally, Section 7 devotes to some conclusion.

2. Our Metatheorem and The Brøndsted-Jachymski Principle

In order to give some equivalents of the Ekeland variational principle, we introduced a metatheorem in 1985-86 [18, 19] on equivalent statements in the Ordered fixed point theory. Later we found some more additional equivalent statements and, consequently, we obtain an extended version of the metatheorem in 2022 [20-22,24] as follows in [25]:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x,y\in X$. Then the following are equivalent:*

(α) *There exists an element $v\in A$ such that $G(v,w)$ for any $w\in X\backslash\{v\}$.*

($\beta1$) *If $f:A\longrightarrow X$ is a map such that for any $x\in A$ with $x\neq f(x)$, there exists a $y\in X\setminus\{x\}$ satisfying $\neg G(x,y)$, then f has a fixed element $v\in A$, that is, $v=f(v)$.*

($\beta2$) *If $\mathfrak{F}$ is a family of maps $f:A\longrightarrow X$ such that for any $x\in A$ with $x\neq f(x)$, there exists a $y\in X\{x\}$ satisfying $\neg G(x,y)$, then $\mathfrak{F}$ has a common fixed element $v\in A$, that is, $v=f(v)$ for all $f\in\mathfrak{F}$.*

($\gamma1$) *If $f:A\longrightarrow X$ is a map such that $\neg G(x,f(x))$ for any $x\in A$, then f has a fixed element $v\in A$, that is, $v=f(v)$.*

($\gamma2$) *If $\mathfrak{F}$ is a family of maps $f:A\longrightarrow X$ satisfying $\neg G(x,f(x))$ for all $x\in A$ with $x\neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v\in A$, that is, $v=f(v)$ for all $f\in\mathfrak{F}$.*

($\delta1$) *If $F:A\multimap X$ is a multimap such that, for any $x\in A\setminus F(x)$ there exists $y\in X\backslash\backslash\{x\}$ satisfying $\neg G(x,y)$, then F has a fixed element $v\in A$, that is, $v\in F(v)$.*

($\delta2$) *Let $\mathfrak{F}$ be a family of multimaps $F:A\multimap X$ such that, for any $x\in A\setminus F(x)$ there exists $y\in X\setminus\{x\}$ satisfying $\neg G(x,y)$. Then $\mathfrak{F}$ has a common fixed element $v\in A$, that is, $v\in F(v)$ for all $F\in\mathfrak{F}$.*

($\epsilon1$) *If $F:A\multimap X$ is a multimap satisfying $\neg G(x,y)$ for any $x\in A$ and any $y\in F(x)\multimap\{x\}$, then F has a stationary element $v\in A$, that is, $\{v\}=F(v)$.*

($\epsilon2$) *If $\mathfrak{F}$ is a family of multimaps $F:A\multimap X$ such that $\neg G(x,y)$ holds for any $x\in A$ and any $y\in F(x)\setminus\{x\}$, then $\mathfrak{F}$ has a common stationary element $v\in A$, that is, $\{v\}=F(v)$ for all $F\in\mathfrak{F}$.*

($\zeta 1$) *If a multimap $F : A \multimap X$ satisfy, for all $x \in A$ with $F(x) \neq \emptyset$, there exists $y \in X \setminus \{x\}$ such that $\neg G(x,y)$ holds, then there exists $v \in A$ such that $F(v) = \emptyset$.*

($\zeta 2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for all $x \in A$ with $F(x) \neq \emptyset$, there exists $y \in X \setminus \{x\}$ such that $\neg G(x,y)$ holds. Then there exists $v \in A$ such that $F(v) = \emptyset$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A \setminus Y$ there exists a $z \in X \setminus \{x\}$ satisfying $\neg G(x,z)$, then there exists a $v \in A \cap Y$.*

Here, $\neg$ denotes the negation. We give the proof for completeness.

Proof. Note that each of $(\beta), (\gamma), (\epsilon)$ implies (δ). and that $(\beta 2) - (\zeta 2)$ imply $(\beta 1) - (\zeta 1)$, respectively. We adopt our previous proof for $(\alpha) \Longrightarrow (\gamma 1)$ as follows:

$(\alpha) \Longrightarrow (\delta 1)$: Suppose $v \notin F(v)$ in $(\delta 1)$. Then there exists a $y \in X \setminus \{v\}$ satisfying $\neg G(v,y)$. This contradicts (α).

$(\delta 1) \Longrightarrow (\beta 1)$: Clear.

$(\beta 1) \Longrightarrow (\gamma 1)$: Clear.

We prove $(\gamma 1) \Longrightarrow (\alpha)$ as follows:

$(\gamma 1) \Longrightarrow (\epsilon 1)$: Suppose F has no stationary element, that is, $F(x) \setminus \{x\} \neq \emptyset$ for any $x \in A$. Choose a choice function f on $\{F(x) \setminus \{x\} : x \in A\}$. Then f has no fixed element by its definition. However, $\neg G(x, f(x))$ for any $x \in A$. Therefore, by $(\gamma 1)$, f has a fixed element, a contradiction.

$(\epsilon 1) \Longrightarrow (\gamma 2)$: Define a multimap $F : A \multimap X$ by $F(x) := \{f(x) : f \in \mathfrak{F}\} \neq \emptyset$ for all $x \in A$. Since $\neg G(x, f(x))$ for any $x \in A$ and any $f \in \mathfrak{F}$, by $(\epsilon 1)$, F has a stationary element $v \in A$, which is a common fixed element of $\mathfrak{F}$.

$(\gamma 2) \Longrightarrow (\alpha)$: Suppose that for any $x \in A$, there exists a $y \in X \setminus \{x\}$ satisfying $\neg G(x,y)$. Choose $f(x)$ to be one of such y. Then $f : A \longrightarrow X$ has no fixed element by its definition. However, $\neg G(x, f(x))$ for all $x \in A$. Let $\mathfrak{F} = \{f\}$. By $(\gamma 2)$, f has a fixed element, a contradiction.

Consequently, we showed equivalency of $(\alpha) - (\gamma 2)$.

We show that $(\alpha) \Longleftrightarrow (\epsilon 2)$ as follows:

$(\alpha) + (\epsilon 1) \Longrightarrow (\epsilon 2)$: By (α), there exists a $v \in A$ such that $G(v,w)$ for all $w \in X \setminus \{v\}$. For each $F \in \mathfrak{F}$, by $(\epsilon 1)$, we have a $v_F \in A$ such that $\{v_F\} = F(v_F)$. Suppose $v \neq v_F$. Then $G(v, v_F)$ holds by (α) and $\neg G(v, v_F)$ holds by assumption on $(\epsilon 2)$. This is a contradiction. Therefore $v = v_F$ for all $F \in \mathfrak{F}$.

$(\epsilon 2) \Longrightarrow (\epsilon 1) \Longrightarrow (\alpha)$: Already shown.

$(\alpha) \Longrightarrow (\zeta 2)$: By (α) there exists $v \in A$ such that $G(v,x)$ holds for all $x \in X \setminus \{v\}$. Suppose to the contrary, there exists $F \in \mathfrak{F}$ such that $F(v) \neq \emptyset$. By hypothesis, there exists $w \in X$ with $w \neq v$ and $\neg G(v,w)$ holds. Therefore it leads a contradiction and $F(v) = \emptyset$ for all $F \in \mathfrak{F}$.

$(\zeta 2) \Longrightarrow (\alpha)$: Suppose that, for each $x \in A$, there exists $y \in X \setminus \{x\}$ such that $\neg G(x,y)$ holds. For each $x \in A$, define a multimap $F : A \multimap X \setminus \{x\}$ by

$$F(x) = \{y \in X : \neg G(x,y)\} \neq \emptyset \text{ for all } x \in A.$$

Then, by $(\zeta 2)$, there exists $v \in A$ such that $F(v) = \emptyset$. This is a contradiction.

$(\alpha) \Longrightarrow (\eta)$: By (α), there exists a $v \in A$ such that $G(v,w)$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in A \cap Y$.

$(\eta) \Longrightarrow (\alpha)$: For all $x \in A$, let

$$A(x) := \{y \in X : x \neq y, \ \neg G(x,y)\}.$$

Choose $Y = \{x \in X : A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the hypothesis of (η) is satisfied. Therefore, by (η), there exists a $v \in A \cap Y$. Hence $A(v) = \emptyset$; that is, $G(v,w)$ for all $w \neq v$. Hence (α) holds.

This completes our proof. □

Remark 2.1. All of the elements v's in Metatheorem are same as we have seen in the proof. We adopted the Axiom of Choice in $(\gamma 1) \Longrightarrow (\epsilon 1)$.

Example 2.2. Khamsi [15]: Let A be an abstract set partially ordered by $\prec$. We will say that $a \in A$ is a minimal element of A if and only if $b \prec a$ implies $b = a$. The concept of minimal element is crucial in the proofs given for Caristi's fixed point theorem.

(K) *Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent.*
(1) *A contains a minimal element.*
(2) *Any multimap T defined on A, such that for any $x \in A$ there exists $y \in T(x)$ with $y \prec x$, has a fixed point, i.e., there exists a in A such that $a \in T(a)$.*

This follows from Metatheorem (α) and $(\delta 1)$.

For a partially ordered set $(X, \preccurlyeq)$ and a map $f : X \longrightarrow X$, we define

Max$(\preccurlyeq)$: the set of maximal elements;
Fix(f) : the set of fixed points of f;
Per(f) : the set of periodic points $x \in X$; that is, $x = f^n(x)$ for some $n \in \mathbb{N}$.

In our previous work [25], we established the following based on Brøndsted [5] in 1976 and Jachymski [13] in 2003:

Brøndsted-Jachymski Principle. *Let $(X, \preccurlyeq)$ be a partially ordered set and $f : X \longrightarrow X$ be a progressive map (that is, $x \preccurlyeq f(x)$ for all $x \in X$). Then X admits a maximal element $v \in X$ if and only if v is a fixed point of f if and only if v is a periodic point, that is,*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(f) = \mathrm{Per}(f).$$

This is a particular form of Metatheorem and not claiming the non-emptiness of three sets. We noticed that, in most applications of this principle, the existence of a maximal element or a fixed point is achieved by the upper bound of a chain in $(X, \preccurlyeq)$ as we can see examples in Section 4.

3. Generalized Zorn's Lemma

The following is a useful consequence of Metatheorem as in [25] without listing $(\beta 1) - (\zeta 1)$.

Theorem 3.1. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, let $A = S(x_0) = \{y \in X : x_0 \preccurlyeq y\}$ have an upper bound (resp. $A = T(x_0) = \{z \in X : z \preccurlyeq x_0\}$ have a lower bound) $v \in A$.*

Then the following equivalent statements hold:

(α) *There exists a maximal (resp. minimal) element $v \in A$ such that $v \not\preccurlyeq w$ (resp. $w \not\preccurlyeq v$) for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \setminus \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ satisfying $x \preceq f(x)$ (resp. $f(x) \preceq x$) for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A \setminus F(x)$ there exists $y \in X \setminus \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$). Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $x \preceq y$ (resp. $y \preceq x$) holds for any $x \in A$ and any $y \in F(x) \setminus \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.*

(ζ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for all $x \in A$ with $F(x) \neq \emptyset$, there exists $y \in X \setminus \{x\}$ such that $x \preceq y$ (resp. $y \preceq x$) holds. Then there exists $v \in A$ such that $F(v) = \emptyset$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A \setminus Y$ there exists a $z \in X \setminus \{x\}$ satisfying $x \preceq z$ (resp. $z \preceq x$), then there exists a $v \in A \cap Y$.*

Proof. (α) Since A has an upper bound $v \in A$, for each $x \in A$, we have $x_0 \preccurlyeq x \preccurlyeq v$. If $v \preccurlyeq w$ for some $w \in X$, then $w \in S(x_0) = A$ and $w \preccurlyeq v$. Since $(X, \preccurlyeq)$ is partially ordered, we have $w = v$. Hence v is maximal. Therefore, the maximal case of (α) holds. Similarly, the minimal case of (α) also holds.

Let $G(x, y)$ be $x \not\preceq y$ (resp, $y \not\preceq x$). Then (α)$-$(η) are equivalent by Metatheorem. □

Remark 3.2. (1) All the elements v's in Theorem 3.1 are same as we have seen in the proof of Metatheorem.

(2) Note that (α) $\Longleftrightarrow$ (γ1) is a new proof of the Brøndsted-Jachymski Principle.

(3) Theorem 3.1 improves the Abian-Brown fixed point theorem, the Tarski-Kantorovitch theorem, and Zorn's lemma. See [25].

(4) In Sections 4 and 5, we can see some other theorems which are closely related to Theorem 3.1

A partially ordered set $(X, \preccurlyeq)$ is said to be *inductive* (*complete*, resp.) if every non-empty chain in X has an upper bound (a least upper bound, resp.).

From Theorem 3.1, we have the following:

Corollary 3.3. *Let $(X, \preccurlyeq)$ be a partially ordered set satisfying one of the following:*

(a) *all nonempty chain in X has an upper bound ($\Leftrightarrow$ X is inductive),*
(b) *all nonempty chain in X has a least upper bound ($\Leftrightarrow$ X is complete),*
(c) *all nonempty well-ordered subset of X has an upper bound,*
(d) *all nonempty well-ordered subset of X has a least upper bound,*

Then the equivalent statements in Theorem 3.1 for the maximum case hold.

From the Brøndsted-Jachymski Principle and Corollary 3.3, we have the following generalization of Zorn's lemma:

Corollary 3.4. *Let $(X, \preccurlyeq)$ be a partially ordered set satisfying one of (a)–(d) in Corollary 3.3. If $f : X \longrightarrow X$ is progressive, then we have*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(f) = \mathrm{Per}(f) \neq \emptyset.$$

Example 3.5. For complete partially ordered sets, particular results of Corollary 3.4 are known as follows; see Kang ([14], p.20):

Tarski [30] and Davis [8] proved that the completeness of a lattice is equivalent to the existence of fixed points of increasing selfmap. And Tascović [32] proved that a partially ordered set is complete iff every progressive selfmap has a fixed point.

Smarzenski [26] obtained a result related to (γ1). Smithson [27, 28] obtained some fixed point theorems for a partially ordered space satisfying (d) and multimaps satisfying (δ1).

Example 3.6. Recall that Tasković [32] showed that Zorn's lemma is equivalent to the following Theorem 3.1(γ):

(T) *Let $\mathcal{F}$ be a family of self-maps defined on a partially ordered set A such that $x \preccurlyeq f(x)$ [resp. $f(x) \preccurlyeq x$] for all $x \in A$ and all $f \in \mathcal{F}$. If each chain in A has an upper bound (resp. lower bound), then the family $\mathcal{F}$ has a common fixed point.*

4. Extending Zermelo's theorem

The following is known the Zermelo fixed point theorem by Dunford-Schwartz ([9], p.5, Theorem I.2.5.) :

Theorem 4.1. (Zermelo) *Let $(P, \preccurlyeq)$ be a partially ordered set in which every chain has a supremum. Assume that $f : P \longrightarrow P$ is such that f is progressive, that is,*

$$p \preccurlyeq f(p) \text{ for all } p \in P.$$

Then f has a fixed point.

Amann [1] derived several fixed point theorems from Theorem 4.1. For example, Tarski's fixed point theorem, fixed point theorems for condensing maps and nonexpansive maps.

Jachymski [13] noted: "The above theorem attributed to Zermelo although it does not appear *explicitly* in any of his papers. However, a proof of it can be derived from Zermelo's proof [36] of the well-ordering principle. This observation is due to Bourbaki [3], who was the first to formulate the theorem in the above form. (Actually, Bourbaki used well-ordered subsets of P instead of chains so his assumption is formally weaker than that of Theorem 4.1. However it is more convenient for us to work with chains as will be seen in the sequel. The proof of Zermelo's theorem does not depend on the Axiom of Choice (AC). If, however, we allow the use of Zorn's Lemma, then the proof is straightforward; moreover, the assumption on $(P, \preccurlyeq)$ can be weakened then to 'every chain has an upper bound'. This is Kneser's [17] fixed point theorem which turns out to be equivalent to the AC as shown by Abian [1]. In the literature, Zermelo's Theorem is sometimes called the Bourbaki-Kneser theorem (cf. Zeidler [35]. p.504)."

Recently Toyoda [33] also noted: "The Zermelo fixed point theorem is also known as the Bourbaki fixed point theorem or the Bourbaki-Kneser fixed point theorem. It implies the Caristi fixed point theorem, the Bernstein-Cantor-Schröder theorem, the Ekeland variational principle, the Takahashi minimization theorem, and others. Moreover, under the Axiom of Choice, it implies Zorn's Lemma."

We have generalizations of the Zermelo theorem from Section 3, see Theorems 3.1(γ1), 3.3(γ1), Corollaries 3.4(γ1), 3.5 for single-valued case and their multi-valued versions in Section 3. Therefore, many generalizations of Zorn's Lemma also extends the Zermelo theorem.

As an example, from Theorem 3.1(α) and (γ1), we have the following:

Theorem 4.2. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, let $A = S(x_0) = \{y \in X : x_0 \preccurlyeq y\}$ has an upper bound. If $f : A \longrightarrow X$ is a map such that $x \preccurlyeq f(x)$ for any $x \in A$, then*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(f) = \mathrm{Per}(f) \neq \emptyset.$$

Recall that Theorem 3.1(γ1) follows from (α) under the Axiom of Choice.

From Theorem 3.1(α) and (γ1), we have the following:

Theorem 4.3. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, $\varphi : X \longrightarrow X$ a map, and let $B = \{\varphi^n(x_0) \in X : n \in \mathbb{N}\}$ have upper bounds and $A = B \cup \{$its upper bounds$\}$ such that $x \preccurlyeq \varphi(x)$ for all $x \in A$. Then we have*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(\varphi) = \mathrm{Per}(\varphi) \neq \emptyset.$$

5. Equivalent formulations of Caristi theorem

In this section, we consider a particular case of Theorem 3.1 as follows:

Theorem 5.1. *Let $(X, \preccurlyeq)$ be a partially ordered metric space, and a function $\varphi : X \longrightarrow \mathbb{R}^+$ be lower semicontinuous such that*

$$x \preccurlyeq y \quad \text{iff} \quad d(x, y) \leq \varphi(x) - \varphi(y) \quad \text{for} \quad x, y \in X.$$

Let $x_0 \in X$ and $A = S(x_0) = \{y \in X : x_0 \preccurlyeq y\}$ have an upper bound.

Then the following equivalent statements $(\alpha) - (\eta)$ of Theorem 3.1 hold.

(α) *There exists a maximal element $v \in A$, that is, $d(v, w) > \varphi(v) - \varphi(w)$ for any $w \in X \setminus \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \setminus \{x\}$ satisfying $d(x, y) \leq \varphi(x) - \varphi(y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ satisfying $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that, for any $x \in A \setminus F(x)$, there exists $y \in X \setminus \{x\}$ satisfying $d(x, y) \leq \varphi(x) - \varphi(y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$ holds for any $x \in A$ and any $y \in F(x) \setminus \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F_i(v)$ for all $F \in \mathfrak{F}$.*

(ζ) *Let $\mathfrak{F}$ be a family of multimaps $T : A \multimap X$ such that, for all $x \in A$ with $T(x) \neq \emptyset$, there exists $y \in X \setminus \{x\}$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$ holds. Then there exists $v \in A$ such that $T(v) = \emptyset$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A \setminus Y$ there exists a $z \in X \setminus \{x\}$ such that $d(x, z) \leq \varphi(x) - \varphi(z)$, then there exists an element $v \in A \cap Y$.*

From Theorem 5.1 we have equivalent formulations of the Caristi theorem as follows:

Theorem 5.2. *Let (X,d) be a complete metric space, and a function $\varphi : X \longrightarrow \mathbb{R}^+$ be lower semicontinuous such that*

$$x \preccurlyeq y \quad \text{iff} \quad d(x,y) \leq \varphi(x) - \varphi(y) \quad \text{for} \quad x, y \in X.$$

Then the equivalent statements $(\alpha) - (\eta)$ of Theorem 5.1 hold where we include

$(\gamma 1)$ (Caristi) *If $f : X \longrightarrow X$ is a map such that $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ for any $x \in X$, then f has a fixed and periodic element $v \in X$, that is, $v = f(v)$.*

Proof. Since $(\gamma 1)$ holds by the Caristi fixed point theorem and (α) holds by Brunner [**?**] in 1987, so do the others. This completes our proof. □

There are possibly dual equivalent formulations of the Caristi theorem.

6. Jachymski's 2003 Theorem

In this article, we introduced many examples of maps $f : X \longrightarrow X$ satisfying $\text{Per}(f) = \text{Fix}(f) \neq \emptyset$. Such sets X can have more rich properties by applying the following main theorem of Jachymski ([13], Theorem 2):

Theorem 6.1. *Let X be a nonempty abstract set and $T : X \longrightarrow X$. The following statements are equivalent:*

(a) $\text{Per}(T) = \text{Fix}(T) \neq \emptyset$.

(b) (Zermelo) *There exists a partial ordering $\preccurlyeq$ such that every chain in $(X, \preccurlyeq)$ has a supremum and T is progressive with respect to $\preccurlyeq$.*

(c) (Caristi) *There exists a complete metric d and a lower semicontinuous function $\varphi : X \longrightarrow \mathbb{R}^+$ such that T satisfies the Caristi condition.*

(d) *There exists a complete metric d and a d-Lipschitzian function $\varphi : X \longrightarrow \mathbb{R}^+$ such that T satisfies the Caristi condition and T is nonexpansive with respect to d; i.e.*

$$d(Tx, Ty) \leq d(x,y) \quad for \ all \quad x, y \in X.$$

(e) (Hicks-Rhoades) *For each $\alpha \in (0,1)$, there exists a complete metric d such that T is nonexpansive with respect to d and*

$$d(Tx, T^2x) \leq \alpha d(x, Tx) \quad for \ all \quad x \in X.$$

(f) *There exists a complete metric d such that T is continuous with respect to d and for each $x \in X$, the sequence $(T^n x)_{n=1}^{\infty}$ is convergent (the limit may depend on x).*

(g) *There exists a partition of X, $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, such that all the sets X_γ are nonempty, T-invariant and pairwise disjoint, and for all $\gamma \in \Gamma$, $T|_{X_\gamma}$ has a unique periodic point.*

(h) *For each $\alpha \in (0,1)$, there exists a partition of X, $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, and complete metrics d_γ on X_γ such that all the sets X_γ are nonempty; T-invariant and pairwise disjoint; and*

$$d_\gamma(Tx, Ty) \leq \alpha d_\gamma(x,y) \quad for \ all \ x, y \in X.$$

Remark 6.2. ([13]) Implication (a) $\Longrightarrow$ (b) is a converse to Zermelo's theorem. Implication (a) $\Longrightarrow$ (c) is a reciprocal to Caristi's theorem; in fact, a stronger result, (a) $\Longrightarrow$ (d) can be obtained here. Implication (a) $\Longrightarrow$ (e) is a converse to a fixed point theorem of Hicks-Rhoades. Finally (a) $\Longrightarrow$ (f) answers a question posed by Matkowski.

Comments 6.3. Each of (a)–(h) seems to be order theoretic fixed point theorems. For them, we state our own comments.

- (a) This could be $\mathrm{Fix}(T) = \mathrm{Per}(T) = \mathrm{Max}(\preccurlyeq) \neq \emptyset$ by defining $\preccurlyeq$ on X.
- (b) Zermelo's theorem is improved in Section 4 and has many equivalents there. Note that its conclusion should be as above in (a).
- (c) Caristi's theorem is improved by Theorem 5.2 and its conclusion should be as in (a).
- (d) This is a variant of Caristi's theorem and its conclusion should be as in (a).
- (e) Here nonexpansiveness is redundant in view of Theorem H(iv) in [25].

7. Conclusion

Zermelo's fixed point theorem suggested in 1904 and 1908 more than one hundred years ago, Kuratowski-Zorn's Lemma or Zorn's Lemma in 1935, Bourbaki's fixed point theorem in 1949-50, and some other classical results are all improved in the present article. Moreover, their equivalent formulations based on our 2023 Metatheorem and the Brøndsted-Jachymski Principle should be reflected in the most of classical works on Ordered fixed point theory.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our Metatheorem. Some of such theorems can be seen in our previous works [20-25] and the present article. Therefore, Metatheorem is a machine to expand our knowledge easily. In this article we presented relatively old and well-known examples.

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EQUILIBRIUM POINT OF A SUPPLY CHAIN NETWORK COMPRISING DISASTER RELIEF MODEL VIA VARIATIONAL INEQUALITY PROBLEM

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ABSTRACT. In this paper, the supply chain network equilibrium model which composites of the disaster relief part is constructed. In such model, we consider five tiers of decision makers: manufacturers who produce the products for sale and donation, retailers, demand market who can purchase products, freight service providers who transport the relief items (or the products) to demand points and, finally, the demand points. The behavior of all decision makers is considered by using the variational inequality formulation. Furthermore, the qualitative properties of such model are studied. Finally, we give some numerical supply chain examples of such model.

KEYWORDS: supply chain network, equilibrium model, variational inequality, qualitative property.
AMS Subject Classification: : 49J40, 91B50.

1. Introduction

Mathematical modeling of supply chain is a model in economic which contains many important and interesting researches. Supply chain studies are both industrial and academia. The supply chain is the coordination of organizations, people, activities, information and resource related to production and transportation products from suppliers to customers. It can be seen that, in supply chain, there will be a competitive. Since it has multi decision makers, who are able to make independent decisions. As a result, competitive supply chain networks and supply chain equilibrium models and others are examined and studied. Another interesting thing is supply chain management that researchers present in the mathematical models

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of supply chain networks and analyze in the process such as production, transportation, purchase products, etc. In addition, the optimization of their models is considered such as supplier selection, distribution network design, production coordination, and inventory management, etc. Firstly, the content of the supply chain network consist of multiple manufacturers and retailers, with the manufacturer producing the same product while the retailer purchases the product and sells it to the customers in the market. The study and development of research continues to be ongoing. In 2002, A. Nagurney et. al. [12] proposed the concept of supply chain network equilibrium by mentioning manufacturing enterprises, retailers in supply chain competition, the purchasing behavior of consumers and considered by using the variational inequality and establishing the supply chain network equilibrium model. Later, many authors developed this concept in various fields until now, see [3, 6, 19]. But not only that, in 2016, A. Nagurney [10] studied the freight service provision for disaster relief on the concept of network model by establishing the competitive freight service provision network model for disaster relief. The results which obtained from the research was the equilibrium shipment and price patterns in the freight service provision sector. Freight service provision network model describes the network model related to the humanitarian organizations, who must use the freight service providers for transporting supplier to demand point or victims. It is seen that efficient transportation is essential to humanitarian operations and disaster relief. At the same time, it is well known that the cost of the shipping is a secondary concern. However, the principles of study focus on the center of decision-makers and the competition which involves the freight service provision. But the survival of relief organizations is essential and, at the same time, aims to reduce and save victims' relief. But it all depend on smart financial and budget management, thus resulting in the efficient use of the service is necessary. Larger humanitarian organizations may have their own freight forwarding services. But they do not have the financial resources to maintain the freight fleets. So, they have to purchase the service. The freight service providers want to maximize the profit while the humanitarian organizations have no profit. They also have to compete with others in order to gain business. Hence, their behavior is different from humanitarian organizations that not only requires the responsible use of the source of funds, but under pressure to deliver relief items to respond to disasters in a timely manner. According to study [10], the disaster relief supply chain continues to be developed, see [11, 14].

On the other hand, the variational inequality is well known that it is the powerful tool for using in industry, finance, economics, social and pure and applied science. In the supply chain, the variational inequality plays an important role because of having a form that is easy to apply and consider. There have also been many studies on the variational inequality problem for a long time, then we have many researches on such problem and can use the knowledge of the variational inequality for further application and development. Most of the times, the variational inequality in supply chain will be used for finding the optimization and equilibrium of problem by introducing the variational inequality associated with a supply chain network and considering the variational inequality formulations of equilibrium conditions to obtain the quantitative properties of that equilibrium pattern, see [10, 12, 15] and the references therein.

With all of the above in this article, authors are interested the network model which developed from [10] of A. Nagurney and [12] of A. Nagurney et.al. In our network model, we consider the manufacturers who want to produce products for sale in the demand market and still want to produce to help victims as well. In this concept, we were inspired by the epidemic situation (COVID-19) which is the deadliest pandemic of the world today and the affect people, economics and social activities around the world. But the most interesting thing, there is a shortage of medical devices such as PPEs, respirators and devices that protect themselves from COVID-19 such as masks and alcohol gels, etc. For this reason, in order for the public to protect themselves initially, government agencies therefore encourage and seek cooperation for people to wear masks and use alcohol gels all the time. But there might be some people who are poor or in a remote area making it impossible to access these devices. The producers therefore allocate relief for such people. As a result of this situation, people have a great demand for such products (or relief items), resulting in a shortage. This is due to the higher demand, but the amount of production remains the same. In addition, in a part of the manufacturer, if the manufacturers have donated the products to the demand points, then we are interested that they can take the cost of the donation to tax-deductible, which is the income that comes back from the donation. For this reason, we are interested in exploring a supply chain that has both sales and donations in a network to determine the equilibrium network model of such supply chain network.

The rest of this paper organized as follows. In section 2, the fundamental concept of our work is proposed. In section 3, we construct the supply chain network comprising disaster relief and propose the variational inequality of each behavior in such network model and consider the variational inequality problem. The qualitative property of the variational inequality problem is considered. The existence and uniqueness of the solution of the variational inequality problem are established. Finally, the algorithm of the network model which is constructed and used for numerical example to understand in the content and network model is shown in section 4 and 5 and the conclusion of results of this paper is proposed in section 6.

2. Preliminaries

Firstly, we will recall the variational inequality problem which was introduced by Kinderlehrer and Stampacchia in 1964 in [4] as follows: Determine $X^* \in K$ such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in K. \tag{2.1}$$

where X and $F(X)$ are an n-dimensional vector with F is a continuous function from K to $\mathbb{R}^n, K$ is closed and convex, and $\langle \cdot, \cdot \rangle$ denotes the inner product in n-dimensional Euclidean space.

Next, we will present some properties which are used for considering the existence and uniqueness of the solution of the variational inequality problem as follows: throughout of this paper, we let K be a closed and convex set.

Definition 2.1. A mapping $F : K \to \mathbb{R}^n$ is said to be a monotone if

$$\langle F(X') - F(X''), X' - X'' \rangle \geq 0 \tag{2.2}$$

for all $X', X'' \in K$.

Definition 2.2. A mapping $F : K \to \mathbb{R}^n$ is said to be a strictly monotone if for any two $X', X'' \in K$ and $X' \neq X''$ such that

$$\langle F(X') - F(X''), X' - X'' \rangle > 0. \tag{2.3}$$

By the condition of the strictly monotone, we can obtain the existence and uniqueness of the solution of the variational inequality as the following theorem.

Theorem 2.3. *Suppose that F is a strictly monotone mapping on K. Then, the solution of* (2.1) *is unique.*

Definition 2.4. A mapping $F : K \to \mathbb{R}^n$ is said to be a Lipschitz continuous if there exists a real number $L > 0$ such that

$$\|F(X') - F(X'')\| \leq L\|X' - X''\| \tag{2.4}$$

for all $X', X'' \in K$.

For the concept of the Lipschitz continuous will be used to guarantee that an algorithm which is constructed by projected method converges which will be presented again in section 4.

3. Main Results

In the above concepts of the variational inequality (2.1), now we will present some example which used the variational inequality problem to consider the equilibrium model and guarantee the existence unique solution of the variational inequality problem.

By the inequality (2.1), since $F : K \to \mathbb{R}^n$ is a mapping and $X \in K$, which K is a closed and convex set in $\mathbb{R}^n$. So, if we have $X \in K$, and

$$F(X) = (F^1(X), F^2(X), F^3(X), F^4(X), F^5(X)),$$

where F^i is functions depend on X with $i = 1, 2, 3, 4, 5$, which can formulate in standard form (2.1). Then, we can be solve (find X^*) the unique solution of the problem by using the previous concepts of the variational inequality problem.

The following article, we will propose a supply chain network model which can be formulate in a variational inequality problem and used the concepts of variational inequality for considering the equilibrium model.

In this model, we consider M manufacturers who are involved in the production of product for sale and donation which is denoted by a typical m, the N retailers who can be purchased product from manufacturers and transported to K demand markets. The retailers will be denoted by a typical n, and the demand markets will be denoted by a typical k. In another way, M manufacturers will be delivered some products for donation by L competing freight service providers which denote a typical freight service providers by l, and the manufacturers are interested to deliver the relief items to the O demand points, a typical demand points by o. The structure of the supply chain network model comprising disaster relief is depicted in Figure 1.

The following article we will consider the behavior of all desition makers by starting with the profit maximization for manufacture i.

3.1. Behavior of the manufacturers.

Since each manufacturer will produce the products for sale and donation. A manufacturer ships the product to the retailers with the amount of product shipped between manufacturer m and retailer n which denoted by q_{mn}. In another way, a manufacturer still wants to ship the product to the demand points in areas that are scarce and we let q^m_{lo} denotes the amount of the relief items, which has in stock and has prepositioned, that m contracts

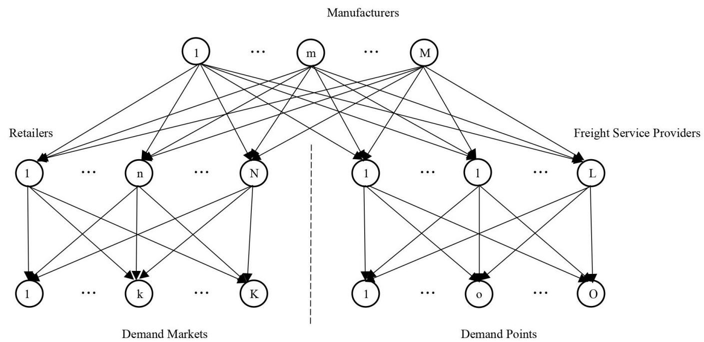

Figure 1. Network structure

with freight service provider l to deliver to demand point o. So, we let q^1_m denotes the nonnegative production output of the product for sale and q^2_m denotes the nonnegative production output of the product for donation by manufacturers m. We group the product shipments between the manufacturers and the retailers into the column vector $Q^1 \in \mathbb{R}^{MN}_+$. We group the relief item shipments of each manufacturers m into the vector $Q^2 \in \mathbb{R}^{MLO}_+$. We assume that each manufacturers m is faced with a production cost function f_m, which can depend on the entire vector of production outputs, that is,

$$f_m = f_m(Q^1, Q^2) \tag{3.1}$$

for all m. A transaction cost denotes by c_{mn}, which is given by

$$c_{mn} = c_{mn}(q_{mn}) \tag{3.2}$$

for all m, n, that the transaction cost includes the cost of shipping the product. Moreover, each manufacturers are faced with a total cost $\hat{c}_{ml}$ associated with transacting with freight service provider l. This cost includes the cost associated with handling the product until pickup by provider l and interacting with provider l. Observe that the cost associated with a manufacturer in transacting with a freight service provider can depend not only on its own shipments associated with the freight service provider but on those of other manufacturers and the same or other freight service providers. The freight service providers guarantee delivery of the disaster relief items in a timely fashion, given what is known about the disaster landscape, and charge accordingly. So, the quantity produced by manufacturer m must satisfy the following conservation of flow equation:

$$q_m = \sum_{n=1}^{N} q_{mn} + \sum_{o=1}^{O} \sum_{l=1}^{L} q^m_{lo} \tag{3.3}$$

This states that the quantity produced by manufacturer m; q_m is equal to the sum of the quantities shipped from the manufacturer to all retailers and all demand points. Next, the price charged of the product by manufacturer m to retailer n (supply price) is denoted by ρ^*_{1mn} and the per unit price that freight service provider l charges m for transport to o is denoted by ρ^{m*}_{lo}. Moreover, in the donation of items, the donor can bring the cost of the donation to a tax deduction. Then, it can be seen that when the product manufacturer donates their stuff, they receive a refund from that donation. The amount of refunded will depend on the manufacturer's tax rate levied on the manufacturer's net income. We denote the tax rate levied on the net yield of manufacturer m as t_m with $t_m \in [0,1)$ and denote the price of product which produce by manufacturer m to donate into all of demand points as ρ_m^*, where this price does not exceed the lowest price the manufacturer m sells to the retailers, that is, $\rho_m^* \leq \rho_{1mn}, \forall n = 1, 2, \cdots, N$. So, these prices which are revealed once the supply chain network equilibrium model for disaster relief is solved.

The total costs, incurred by a manufacturer m, are equal to the sum of their production cost plus the total transaction costs for sale and for donation plus the price charges by freight service provider. The revenue of a manufacturer m is equal to the price that the manufacturer charges for the product (which the retailers are willing to pay) times the total quantity purchased of product from the manufacturer by all the retail outlets and the income received from tax breaks. By the conservation of flow equations (3.3), we can express the criterion of profit maximization for manufacturer m as

$$\text{Maximize} \quad \sum_{n=1}^{N} \rho_{1mn}^* q_{mn} + t_m \sum_{l=1}^{L} \sum_{o=1}^{O} \rho_m^* q_{lo}^m - f_m(Q^1, Q^2) - \sum_{n=1}^{N} c_{mn}(q_{mn}) - \sum_{l=1}^{L} \sum_{o=1}^{O} \rho_{lo}^{m^*} q_{lo}^m - \sum_{l=1}^{L} \hat{c}_{ml}(q_{lo}^m) \tag{3.4}$$

subject to:

$$q_{mn} \geq 0 \quad \text{for} \quad n = 1, 2, \cdots, N, \tag{3.5}$$

$$q_{lo}^m \geq 0 \quad \text{for} \quad l = 1, 2, \cdots, L; o = 1, 2, \cdots, O, \tag{3.6}$$

$$q_o = \frac{\sum_{m=1}^{M} \sum_{l=1}^{L} \sum_{o=1}^{O} q_{lo}^m}{O} \tag{3.7}$$

where q_o is the number of products that received at the demand point o.

Assume that the manufacturers compete in a noncooperative fashion. The production cost functions, the transaction cost functions and the total cost associated with freight service provider for each manufacturer are continuously differentiable and convex. Given that the governing equilibrium concept underlying noncooperative behavior is that of Cournot [1] and Nash [7, 8] which states that each manufacturer will determine their optimal production quantity and shipments. The optimality conditions for all manufacturers can be expressed as the following variational inequality (see Nagurney [9]) : Determine $(Q^{1^*}, Q^{2^*}) \in \mathbb{R}_+^{MN+MLO}$ satisfying

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \left[\sum_{s=1}^{M} \frac{\partial f_s(Q^{1^*}, Q^{2^*})}{\partial q_{mn}} + \frac{\partial c_{mn}(q_{mn}^*)}{\partial q_{mn}} - \rho_{1mn}^* \right] \times [q_{mn} - q_{mn}^*] + \sum_{m=1}^{M} \sum_{l=1}^{L} \left[\sum_{s=1}^{M} \frac{\partial f_s(Q^{1^*}, Q^{2^*})}{\partial q_{lo}^m} + \frac{\partial \hat{c}_{ml}(q_{lo}^{m^*})}{\partial q_{lo}^m} + \rho_{lo}^{m^*} - t_m \rho_m^* \right] \times [q_{lo}^m - q_{lo}^{m*}] \geq 0 \tag{3.8}$$

for all $Q^1 \in \mathbb{R}_+^{MN}, Q^2 \in \mathbb{R}_+^{MLO}$.

The optimality conditions as expressed (3.8) have an economic interpretation that, in the part for sale, a manufacturer will ship a positive amount of the product to a retailer, if the price that the retailer is willing to pay for the product is precisely equal to the manufacturer's marginal production and transaction costs associated with that retailer. But if the manufacturer's marginal production and transaction costs exceed the price what the retailer is willing to pay for the product, then the flow on the link will be zero. In the part for donation, if a manufacturer will ship a positive amount of the product to a demand point. Then, the amount of refund from the tax deduction exceed the total cost of a manufacturer for donation the product to demand point: but if not so, a manufacturer will be also ship a positive amount of the product to a retailer, where the price charged by manufacturer to retailer, ρ^*_{1mn}, is greater than the manufacturer's marginal production and transaction costs and, moreover, the profit for sale product of a manufacturer is greater than the total cost of a manufacturer for donation the product to demand point, that is, the sum of the first term of inequality (3.8) is greater than the sum of the second term of such inequality.

3.2. **Behavior of the retailers.** The retailers are involved in transactions both the manufacturers and the consumers which the retailers wish to obtain the product for their retail outlets as well as with the consumers, who are ultimate purchasers of the product.

A retailer n is faced with a handling cost, which may include the display and storage cost associated with the product, and this cost is denoted by c_n. In the simplest case, we would have that c_n is a function of $\sum_{m=1}^{M} q_{mn}$, that is, the handling cost of a retailer is a function of how much of the product which they have obtained from the various manufacturers. Now, we let the function to depend on the amounts of the product which held by other retailers and we can write

$$c_n = c_n(Q^1) \tag{3.9}$$

for all n. Next, we will denote ρ^*_{2n} for a price with the product at their retail outlet for retailer n, which this price in the model will also be endogenously determined and denotes $\tilde{q}_{nk}$ for the amount of the product, which is purchased by the consumer k from the retailer n. Normally, the retailers want the maximization of profit. Then, the optimization problem of a retailer n is given by

$$\text{Maximize} \quad \rho^*_{2n} \sum_{k=1}^{K} \tilde{q}_{nk} - c_n(Q^1) - \sum_{m=1}^{M} \rho^*_{1mn} q_{mn} \tag{3.10}$$

subject to:

$$\sum_{k=1}^{K} \tilde{q}_{nk} \leq \sum_{m=1}^{M} q_{mn} \tag{3.11}$$

and the nonnegative constraints: $q_{mn} \geq 0$ and $\tilde{q}_{nk} \geq 0$ for all m and k. Objective function (3.10) express that the difference between the revenues minus the handling cost and the payout to the manufacturers should be maximized and constraint (3.11) expresses that consumers cannot purchase more from a retailer than what held in stock.

Now, we consider the optimality conditions of the retailers assuming that each retailer is faced with the optimization problem (3.10) subject to (3.11), and assume that the variables is nonnegativity. Here, we also assume that the retailers compete in a noncooperative manner given the actions of the other retailers so that each retailer maximizes his profits. Note that, at this point, we consider that retailers seek to determine not only the optimal amounts purchased by the consumers from their specific retail outlet but, also, the amount that they wish to obtain from the manufacturers. Hence, there is a monitor that receives data from manufacturers, retailers and consumers to consider and determine the optimal amount and price that manufacturers sell to retailers, amount and price that retailers sell to consumers, and the price that consumers are willing to purchase so that all parties are satisfied in trading. The monitor must be the one who holds the information as well and is the most confidential so that other party does not know each other's information. Therefore, in equilibrium, all the shipments between the tiers of network agents will have to coincide. Assume that the handling cost for each retailer is continuous and convex. The optimality conditions for all the retailers coincide with the solution of the variational inequality: Determine $(Q^{1^*}, Q^{3^*}, \gamma^*) \in \mathbb{R}_+^{MN+NK+N}$ satisfying

$$\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\frac{\partial c_n(Q^{1^*})}{\partial q_{mn}} + \rho_{1mn}^* - \gamma_n^*\right] \times [q_{mn} - q_{mn}^*] + \sum_{n=1}^{N}\sum_{k=1}^{K}[-\rho_{2n}^* + \gamma_n^*] \times [\tilde{q}_{nk} - \tilde{q}_{nk}^*]$$
$$+ \sum_{n=1}^{N}\left[\sum_{m=1}^{M} q_{mn}^* - \sum_{k=1}^{K}\tilde{q}_{nk}^*\right] \times [\gamma_n - \gamma_n^*] \geq 0 \tag{3.12}$$

for all $(Q^1, Q^3, \gamma) \in \mathbb{R}_+^{MN+NK+N}$, where γ_n is the term of the Lagrange multiplier associated with constraint (3.11) for retailer n, γ is the N-dimensional column vector of all the multipliers, and Q^3 is the group of the product flows between the retailer's and the demand markets in the NK-dimensional vector.

We now have the economic interpretation of the retailers, optimality conditions. From the second term in inequality (3.12), we see that, if consumers at demand market k purchase the product from a particular retailer n,(that is, $\tilde{q}_{nk}^*$ is positive) then the price charged by retailer n, ρ_{2n}^*, is precisely equal to γ_n^*, which in the third term in the inequality, serves as the price to clear the market from retailer n. Note that, from the second term (3.12), if no product is sold by a particular retailer, then the price associated with holding the product can exceed the price charged to the consumers. Furthermore, from the first term in inequality (3.12), we can infer that, if a manufacturer transacts with a retailer resulting in a positive flow of the product between the two, then the price γ_n^* is precisely equal to the retailer n's payment to the manufacturer, ρ_{1mn}^*, plus its marginal cost of handling the product from the retailer.

3.3. **Behavior of the consumers at the demand markets.** In this section, we will describe the consumers located at the demand markets. We are interested in deciding on the consumer's product consumption, where consumers have not only the price charged for the product by the retailers, but also the transaction cost to obtain the product.

We recall $\tilde{q}_{nk}$ denotes the amount of the product which is purchased from retailer n by consumers at demand market k and let $\tilde{c}_{nk}$ denotes the transaction cost associated with obtaining the product by consumers at demand market k from retailer n and assume that the transaction cost is continuous and positive. We can write the general form as follows

$$\tilde{c}_{nk} = \tilde{c}_{nk}(Q^3) \tag{3.13}$$

for all n, k. Further, we let ρ_{3k} denotes the price of the product at demand market k and d_k denotes the demand for the product at demand market k and assume the continuous demand functions as follows

$$d_k = d_k(\rho_3) \tag{3.14}$$

for all k, where ρ_3 is the K-dimensional column vector of demand market prices. By (3.14), the demand for consumers for the product at a demand market depends not only on the price of the product at that demand market but also on the prices of the product at the other demand markets. Thus, consumers at a demand market also compete with consumers at other demand markets.

Next, we will discuss the equilibrium conditions between consumers at demand market k and retailer n that is the price of product at demand market k is relative to the consumers who take the price charged by the retailers for the product for retailer n plus the transaction cost associated with obtaining the product, in making their consumption decisions, and there is still a relationship between the demand for the product at demand market k and the amount of the product which is purchased by the consumers from the retailers. Then, we can write the equilibrium conditions for consumers at demand market k as follows: for all retailers $n, n = 1, 2, \cdots, N$,

$$\rho_{2n}^{*} + \tilde{c}_{nk}(Q^{3^*}) \begin{cases} = \rho_{3k}^{*} & \text{if} \quad \tilde{q}_{nk}^{*} > 0, \\ \geq \rho_{3k}^{*} & \text{if} \quad \tilde{q}_{nk}^{*} = 0 \end{cases} \tag{3.15}$$

and

$$d_k(\rho_3^*) \begin{cases} = \sum_{n=1}^{N} \tilde{q}_{nk}^{*} & \text{if} \quad \rho_{3k}^{*} > 0, \\ \leq \sum_{n=1}^{N} \tilde{q}_{nk}^{*} & \text{if} \quad \rho_{3k}^{*} = 0. \end{cases} \tag{3.16}$$

In equilibrium price, Condition (3.15) state that if the consumers at demand market k purchase the product from retailer n, then the price charged by retailer for the product plus the transaction cost does not exceed the price that the consumers are willing to pay for the product and Conditions (3.16) state that if the consumers who are willing to pay for the product at the demand market is positive, then the quantities purchased of the product from the retailers will be precisely equal to the demand for that product at the demand market. This condition correspond to the well-known spatial price equilibrium conditions ([9] and the references therein).

In equilibrium on the conditions (3.15) and (3.16) will have to hold for all demand markets k, and these can also be expressed as a variational inequality problem given by: Determine $(Q^{3*}, \rho_3^*) \in \mathbb{R}_+^{NK+K}$ such that

$$\sum_{n=1}^{N}\sum_{k=1}^{K}\left[\rho_{2n}^{*} + \tilde{c}_{nk}(Q^{3^*}) - \rho_{3k}^{*}\right] \times [\tilde{q}_{nk} - \tilde{q}_{nk}^{*}] + \sum_{k=1}^{K}\left[\sum_{n=1}^{N}\tilde{q}_{nk}^{*} - d_k(\rho_3^*)\right] \times [\rho_{3k} - \rho_{3k}^{*}] \geq 0 \tag{3.17}$$

for all $(Q^3, \rho_3) \in \mathbb{R}_+^{NK+K}$.

3.4. **Behavior of the freight service providers.** Since the freight service providers want to be profit-maximizers, thus they have to cover their costs. The cost associated with freight service providers delivering the relief items from manufacturers to demand points. We let c_{mo}^{l} for the cost associated with freight service provider l delivering the relief items from manufacturer m to demand point o, given by

$$c_{mo}^{l} = c_{mo}^{l}(Q^2) \tag{3.18}$$

for all $l = 1, 2, \cdots, L$. Assume that the cost functions of the freight service providers are continuously differentiable and convex. Note that the cost functions in (3.18) depend on the freight service provider's shipment quantities and those of the other freight service providers because there may be congestion, competition for labor, etc. In the other hand, the revenue of a freight service provider is equal to the per unit price that a freight service provider charges a manufacturer for transport to demand points times the amount of the relief item that a manufacturer contracts with a freight service provider to deliver to demand points. Therefore, the optimization problem of freight service provider $l; l = 1, 2, \cdots, L$, where each freight service providers requires the maximized profits, is illustrated in the following:

$$\text{Maximize} \quad \sum_{m=1}^{M}\sum_{o=1}^{O}\rho_{lo}^{m^*} q_{lo}^{m} - \sum_{m=1}^{M}\sum_{o=1}^{O} c_{mo}^{l}(Q^2) \tag{3.19}$$

subject to :

$$q_{lo}^{m} \geq 0, \tag{3.20}$$

for $o = 1, 2, \cdots, O$ and $m = 1, 2, \ldots, M$.

We assume that the freight service providers $l; l = 1, 2, \cdots, L$, are a noncooperative competition for the disaster relief items and seek to maximize its profits. The optimality conditions of all freight service providers holding simultaneously must satisfy the variational inequality problem: Determine $Q^{2^*} \in \mathbb{R}_+^{MLO}$ such that:

$$\sum_{l=1}^{L}\left(\sum_{m=1}^{M}\sum_{o=1}^{O}\left(\sum_{x=1}^{M}\sum_{y=1}^{O}\frac{\partial c_{xy}^{l}(Q^{2^*})}{\partial q_{lo}^{m}} - \rho_{lo}^{m^*}\right)\right) \times \left[q_{lo}^{m} - q_{lo}^{m^*}\right] \geq 0 \tag{3.21}$$

for all $q_{lo}^{m} \in \mathbb{R}_+^{MLO}$.

The economic interpretation of the freight service providers in optimality conditions is that if a freight service provider transacts with a manufacturers to a demand point resulting in a positive flow of the product between three tiers, then the price $\rho_{lo}^{m^*}$ is precisely equal to its marginal cost of handling the product by a freight service provider from a manufacturer to a demand point.

All of previous behavior, we have the network equilibrium conditions for the supply chain network comprising disaster relief model as follows.

Definition 3.1. [Supply Chain Network Comprising Disaster Relief Equilibrium] A supply chain network comprising disaster relief equilibrium is one which the product flows between the distinct tiers of the decision-makers coincide and the product flows and prices satisfy the sum of the optimality conditions (3.8), (3.12), (3.17) and (3.21).

We now present the variational inequality formulation of the supply chain network comprising disaster relief equilibrium conditions and then discuss how to find the equilibrium prices.

Theorem 3.2. *The equilibrium conditions governing the supply chain network comprising disaster relief model with competition are equivalent to the solution of the variational inequality problem given by determine* $(Q^{1^*}, Q^{2^*}, Q^{3^*}, \gamma^*, \rho_3^*) \in K$ *satisfying*

$$\begin{aligned}
&\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\sum_{s=1}^{M}\frac{\partial f_s(Q^{1^*},Q^{2^*})}{\partial q_{mn}} + \frac{\partial c_{mn}(q_{mn}^*)}{\partial q_{mn}} + \frac{\partial c_n(Q^{1^*})}{\partial q_{mn}} - \gamma_n^*\right] \times [q_{mn} - q_{mn}^*] \\
&+ \sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\sum_{s=1}^{M}\frac{\partial f_s(Q^{1^*},Q^{2^*})}{\partial q_{lo}^{m}} + \frac{\partial \hat{c}_{ml}(q_{lo}^{m^*})}{\partial q_{lo}^{m}} + \sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^{l}(Q^{2^*})}{\partial q_{lo}^{m}} - t_m\rho_m^*\right] \times [q_{lo}^{m} - q_{lo}^{m^*}] \\
&+ \sum_{n=1}^{N}\sum_{k=1}^{K}\left[\tilde{c}_{nk}(Q^{3^*}) + \gamma_n^* - \rho_{3k}^*\right] \times [\tilde{q}_{nk} - \tilde{q}_{nk}^*] + \sum_{n=1}^{N}\left[\sum_{m=1}^{M} q_{mn}^* - \sum_{k=1}^{K}\tilde{q}_{nk}^*\right] \times [\gamma_n - \gamma_n^*] \\
&+ \sum_{k=1}^{K}\left[\sum_{n=1}^{N}\tilde{q}_{nk}^* - d_k(\rho_3^*)\right] \times [\rho_{3k} - \rho_{3k}^*] \geq 0
\end{aligned} \tag{3.22}$$

for all $(Q^1, Q^2, Q^3, \gamma, \rho_3) \in K$, *where*
$K = \{(Q^1, Q^2, Q^3, \gamma, \rho_3) | (Q^1, Q^2, Q^3, \gamma, \rho_3) \in \mathbb{R}_+^{MN+MLO+NK+N+K}\}$.

Proof. The summation of (3.8), (3.12), (3.17) and (3.21). This imply that (3.22).
In conversely, we will consider that the solution to variational inequality (3.22) satisfies the sum of inequalities (3.8), (3.12), (3.17) and (3.21), that is the equilibrium according to Definition 3.1. In the first term of the inequality (3.22) add $-\rho_{1mn}^* + \rho_{1mn}^*$ and in the second term add $-\rho_{lo}^{m^*} + \rho_{lo}^{m^*}$ and, in the third term add $-\rho_{2n}^* + \rho_{2n}^*$. This implies that

$$\begin{aligned}
&\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\sum_{s=1}^{M}\frac{\partial f_s(Q^{1^*},Q^{2^*})}{\partial q_{mn}} + \frac{\partial c_{mn}(q_{mn}^*)}{\partial q_{mn}} + \frac{\partial c_n(Q^{1^*})}{\partial q_{mn}} - \gamma_n^* - \rho_{1mn}^* + \rho_{1mn}^*\right] \times [q_{mn} - q_{mn}^*] \\
+ \quad &\sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\sum_{s=1}^{M}\frac{\partial f_s(Q^{1^*},Q^{2^*})}{\partial q_{lo}^{m}} + \frac{\partial \hat{c}_{ml}(q_{lo}^{m^*})}{\partial q_{lo}^{m}} + \sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^{l}(Q^{2^*})}{\partial q_{lo}^{m}} - t_m\rho_m^* - \rho_{lo}^{m^*} + \rho_{lo}^{m^*}\right] \times [q_{lo}^{m} - q_{lo}^{m^*}] \\
+ \quad &\sum_{n=1}^{N}\sum_{k=1}^{K}\left[\tilde{c}_{nk}(Q^{3^*}) + \gamma_n^* - \rho_{3k}^* - \rho_{2n}^* + \rho_{2n}^*\right] \times [\tilde{q}_{nk} - \tilde{q}_{nk}^*] + \sum_{n=1}^{N}\left[\sum_{m=1}^{M} q_{mn}^* - \sum_{k=1}^{K}\tilde{q}_{nk}^*\right] \times [\gamma_n - \gamma_n^*]
\end{aligned}$$

$$+ \quad \sum_{k=1}^{K}\left[\sum_{n=1}^{N}\tilde{q}_{nk}^{*}-d_k(\rho_3^*)\right]\times[\rho_{3k}-\rho_{3k}^{*}]\geq 0.$$

Thus,

$$\begin{aligned}
&\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\sum_{s=1}^{M}\frac{\partial f_s(Q^{1^*},Q^{2^*})}{\partial q_{mn}}+\frac{\partial c_{mn}(q_{mn}^*)}{\partial q_{mn}}-\rho_{1mn}^*\right]\times[q_{mn}-q_{mn}^*]\\
&+\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\frac{\partial c_n(Q^{1^*})}{\partial q_{mn}}-\gamma_n^*+\rho_{1mn}^*\right]\times[q_{mn}-q_{mn}^*]\\
&+\sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\sum_{s=1}^{M}\frac{\partial f_s(Q^{1^*},Q^{2^*})}{\partial q_{lo}^m}+\frac{\partial \hat{c}_{ml}(q_{lo}^{m^*})}{\partial q_{lo}^m}+\rho_{lo}^{m^*}-t_m\rho_m^*\right]\times[q_{lo}^m-q_{lo}^{m^*}]\\
&+\sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^l(Q^{2^*})}{\partial q_{lo}^m}-\rho_{lo}^{m^*}\right]\times[q_{lo}^m-q_{lo}^{m^*}]\\
&+\sum_{n=1}^{N}\sum_{k=1}^{K}\left[\tilde{c}_{nk}(Q^{3^*})+\rho_{2n}^*-\rho_{3k}^*\right]\times[\tilde{q}_{nk}-\tilde{q}_{nk}^*]\\
&+\sum_{n=1}^{N}\sum_{k=1}^{K}[\gamma_n^*-\rho_{2n}^*]\times[\tilde{q}_{nk}-\tilde{q}_{nk}^*]+\sum_{n=1}^{N}\left[\sum_{m=1}^{M}q_{mn}^*-\sum_{k=1}^{K}\tilde{q}_{nk}^*\right]\times[\gamma_n-\gamma_n^*]\\
&+\sum_{k=1}^{K}\left[\sum_{n=1}^{N}\tilde{q}_{nk}^*-d_k(\rho_3^*)\right]\times[\rho_{3k}-\rho_{3k}^*]\geq 0. \qquad (3.23)
\end{aligned}$$

We see that (3.23) is equivalent to the price and shipment pattern satisfying the sum of inequalities (3.8), (3.12), (3.17) and (3.21). This completes the proof. □

From the above theorem, if we can define $X \equiv (Q^1, Q^2, Q^3, \gamma, \rho_3) \in K$ and $F(X) \equiv (F^1(X), F^2(X), F^3(X), F^4(X), F^5(X))$ where $F^1(X)$ consists of components F_{mn}, with

$$F_{mn}(X) \equiv \sum_{s=1}^{M}\frac{\partial f_s(Q^1,Q^2)}{\partial q_{mn}}+\frac{\partial c_{mn}(q_{mn})}{\partial q_{mn}}+\frac{\partial c_n(Q^1)}{\partial q_{mn}}-\gamma_n,$$

for $m = 1, 2, \cdots, M; n = 1, 2, \cdots, N$, $F^2(X)$ consists of components F_{lo}^m with

$$F_{lo}^m(X) \equiv \sum_{s=1}^{M}\frac{\partial f_s(Q^1,Q^2)}{\partial q_{lo}^m}+\frac{\partial \hat{c}_{ml}(q_{lo}^m)}{\partial q_{lo}^m}+\sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^l(Q^2)}{\partial q_{lo}^m}-t_m\rho_m,$$

for $m = 1, 2, \cdots, M; l = 1, 2, \cdots, L, o = 1, 2, \cdots, O$, $F^3(X)$ consists of components F_{nk} with

$$F_{nk}(X) \equiv \tilde{c}_{nk}(Q^3)+\gamma_n-\rho_{3k}$$

for $n = 1, 2, \cdots, N, k = 1, 2, \cdots, K$, $F^4(X)$ consists of components F_n with

$$F_n(X) \equiv \sum_{m=1}^{M}q_{mn}-\sum_{k=1}^{K}\tilde{q}_{nk}$$

for $n = 1, 2, \cdots, N$, $F^5(X)$ consists of components F_k with

$$F_k(X) \equiv \sum_{n=1}^{N}\tilde{q}_{nk}-d_k(\rho_3)$$

for $k = 1, 2, \cdots, K$. Then, the above functions can be formulated in the standard form (2.1), that is, the variational inequality (3.22) takes on (2.1).

The following article, we will consider the examination of qualitative properties of the equilibrium pattern, that is, the solution to the variational inequality (3.22), equivalently, (2.1).

Since the feasible set underlying the variational inequality problem (3.22) is not compact. So, we can define a rather weak condition to guarantee the existence of a solution pattern. Let

$$K_b = \{(Q^1, Q^2, Q^3, \gamma, \rho_3)|0 \le Q^1 \le b_1, 0 \le Q^2 \le b_2, 0 \le Q^3 \le b_3, 0 \le \gamma \le b_4, 0 \le \rho_3 \le b_5\}, \tag{3.24}$$

which b is a positive scalar and $Q^1 \le b_1, Q^2 \le b_2, Q^3 \le b_3, \gamma \le b_4, \rho_3 \le b_5$, this mean that $q_{mn} \le b_1, q^m_{lo} \le b_2, q_{nk} \le b_3, \gamma_n \le b_4$ and $\rho_{3k} \le b_5$ for all m, n, k, l, o. Then, K_b is a bounded, closed convex subset of $R^{MN+MLO+NK+N+K}$. Since K_b is compact and F is continuous. Therefore, we have the following variational inequality:

$$\langle F(X_b), X - X_b\rangle \ge 0, \quad \text{for all} \quad X \in K_b \tag{3.25}$$

admits at least one solution $X^*_b \equiv (Q^{1b^*}, Q^{2b^*}, Q^{3b^*}, \gamma^{b^*}, \rho_3^{b^*}) \in K_b$. So, by D. Kinderleher and G. Stampacchia [4], we obtain the following theorem.

Theorem 3.3. *Variational inequality (2.1) admits a solution if and only if there exists $b > 0$ such that variational inequality (3.25) admits a solution in K_b with*

$$Q^{1b^*} < b_1, Q^{2b^*} < b_2, Q^{3b^*} < b_3, \gamma^{b^*} < b_4, \rho_3^{b^*} < b_5. \tag{3.26}$$

The following proposition is presented for guarantee the existence of solution of the variational inequality problem in Theorem 3.3 which is proved in the same way of Nagurney and Zhao [15].

Proposition 3.4. *Suppose that there exist positive constants S, T and $W > 0$ such that:*

$$\sum_{s=1}^{M} \frac{\partial f_s(Q^{1^*}, Q^{2^*})}{\partial q^*_{mn}} + \frac{\partial c_{mn}(q^*_{mn})}{\partial q^*_{mn}} + \frac{\partial c_n(Q^{1^*})}{\partial q^*_{mn}} \ge S; \quad \textit{for all } Q^{1^*}, Q^{2^*}, \ \textit{with } q^*_{mn} \ge T,$$
$$\textit{or } \tilde{q}^*_{nk} \ge T \quad \textit{for all } m, n, k \tag{3.27}$$

$$d_k(\rho^*_3) \le T; \qquad \textit{for all} \quad \rho_3 \quad \textit{with} \quad \rho^*_{3k} \ge W, \qquad \textit{for all} \quad k, \tag{3.28}$$

$$\sum_{m=1}^{M} q^*_{mn} - \sum_{k=1}^{K} \tilde{q}^*_{nk} > 0 \quad \textit{with} \quad q^*_{mn} \ge T \quad \textit{or} \quad \tilde{q}^*_{nk} \ge T \quad \textit{for all} \quad m, n, k. \tag{3.29}$$

Then variational inequality (2.1) admits at least one solution.

Proof Choose $b_1 = b_2 = b_3 = b_4 = b > T$ and $b_5 > T_1$ where $T_1 = \max_{nk, Q^3<b, \gamma<b}\{\rho_{2n} + c_{nk}(Q^3)\}$. If we can prove that:

$$Q^{1b^*} < b, Q^{2b^*} < b, Q^{3b^*} < b, \gamma^{b^*} < b, \rho^*_{3k} < b_5,$$

then, by Theorem 3.3, we obtain the existence of the solution (2.1).

(i.) We will show that $q^*_{mn} < b$ and $\tilde{q}^*_{nk} < b$ for all m, n, k.
Assume that there exist x, y, z such that $q^*_{xy} = b > T$ or $\tilde{q}^*_{yz} = b > T$. From assumption (3.27), we have

$$S \le \sum_{s=1}^{M} \frac{\partial f_s(Q^{1^*}, Q^{2^*})}{\partial q^*_{mn}} + \frac{\partial c_{mn}(q^*_{mn})}{\partial q^*_{mn}} + \frac{\partial c_n(Q^{1^*})}{\partial q^*_{mn}} \le \gamma^*_n.$$

Since if $\tilde{q}^*_{nk} > 0$ then (3.15), we have $\rho^*_{2n} + \tilde{c}_{nk}(Q^{3^*}) \le \rho^*_{3k}$ and $\gamma^*_n = \rho^*_{2n}$. Then, $\rho^*_{3k} > 0$. By (3.16) , imply that

$$\sum_{n=1}^{N} \tilde{q}^*_{nk} - d_k(\rho^*_3) \le 0.$$

Hence $d_k(\rho_3^*) > T$. This is a contradiction with the assumption (3.28). Therefore, $q_{mn}^* < b$ and $\tilde{q}_{nk}^* < b$ for all m, n, k.

(ii.) We will show that $q_{lo}^{m^*} < b$. Since (3.7), we know that $\sum_{m=1}^{M} \sum_{l=1}^{L} \sum_{o=1}^{O} q_{lo}^{m} = Oq_o$. Then, q_{lo}^{m} is bounded. So, there exists $b_2 > 0$ such that $q_{lo}^{m^*} < b_2 = b$.

(iii.) We will show that $\gamma^* < b$.
Assume that there exists d such that

$$\gamma_d^* = b > T$$

This imply that

$$\sum_{m=1}^{M} q_{md}^* \leq \sum_{k=1}^{K} \tilde{q}_{dk}^*$$

By (3.11), we have

$$\sum_{m=1}^{M} q_{md}^* = \sum_{k=1}^{K} \tilde{q}_{dk}^*,$$

which is a contradiction with the assumption (3.29). Therefore, $\gamma^* < b$.

(iv.) We will show that $\rho_{3k}^* < b_5$.
Since $\tilde{q}_{nk}^* < b$ which b is a positive scalar. We have $\rho_{2n}^* + \tilde{c}_{nk}(Q^{3^*}) \geq \rho_{3k}^*$. This imply

$$\rho_{3k}^* \leq T_1 < b_5.$$

The proof is complete.

Under the conditions in Proposition 3.4, it is possible to construct b_1, b_2, b_3, b_4 and b_5 large enough so that the restricted variational inequality (3.25) will satisfy the boundedness condition (3.26) and, thus, existence of a solution to the original variational inequality problem according to Theorem 3.3 will hold.

From the assumptions (3.27), (3.28) and (3.29), it is reasonable from an economic point of view. If a large number of products are shipped between a manufacturer and a retailer, then we expect that the marginal production cost plus the marginal transaction cost plus the marginal handling cost exceed a positive lower bound. If the demand market price at a demand market is high, we can expect that the demand for the product is low at that demand market and below upper bound. Finally, if a large number of products are shipped between a manufacturer and a retailer and between a retailer and a demand market, then we expect that the shipment between a manufacturer and a retailer exceed the shipment between a retailer and a demand market.

Next, we will consider the monotonicity properties of the function F in variational inequality (3.22). Then, we recall the definition of an additive production cost functions introduced in Zhang and Nagurney [18] for considering in the qualitative properties.

Definition 3.5. Suppose that, for each manufacture m, the production cost f_m is additive, that is,

$$f_m(q) = f_m^1(q_m) + f_m^2(\bar{q}_m), \tag{3.30}$$

which $f_m^1(q_m)$ is the internal production cost that depend on the manufacturer's own output level q_m, which may include the production operation and the facility maintenance, etc., and $f_m^2(\bar{q}_m)$, is the interdependent part of the production cost that is a function of all the other manufacturers' output levels $\bar{q}_m = (q_1, \ldots, q_{m-1}, q_{m+1}, \ldots, q_m)$ and reflects the impact of the other manufacturers' production patterns on manufacturer m's cost. This interdependent part of the production cost may describe the competition for the resources, consumption of the homogeneous raw materials, etc.

Here, we will consider the qualitative properties of the function F that enters the variational inequality problem and the uniqueness of the equilibrium pattern. So, firstly, the monotonicity and Lipschitz continuity of F is presented. Moreover, the subsequent section, this concept is used for proving the convergence of the algorithmic scheme.

Lemma 3.6. *Suppose that the production cost functions $f_m, m = 1, 2, \ldots, M$, are additive, as defined in Definition 3.5, and $f_m^1, m = 1, 2, \ldots, M$, are convex functions. If the $c_{mn}, c_n, \hat{c}_{ml}$ and c_{mo}^l functions are convex, the $\tilde{c}_{nk}$ functions are monotone increasing, and the d_k functions are monotone decreasing functions of the generalized prices, for all m, n, k, l, o. Then the vector function F that enters the variational inequality (3.22) is monotone mapping.*

Proof. Let $X' \equiv (Q^{1'}, Q^{2'}, Q^{3'}, \gamma', \rho_3')$ and $X'' \equiv (Q^{1''}, Q^{2''}, Q^{3''}, \gamma'', \rho_3'')$ with $X', X'' \in K$. Then,

$$
\begin{aligned}
&\langle F(X') - F(X''), X' - X'' \rangle \\
= &\sum_{m=1}^{M}\sum_{n=1}^{N}\Big(\sum_{s=1}^{M}\frac{\partial f_s(Q^{1'},Q^{2'})}{\partial q'_{mn}} + \frac{\partial c_{mn}(q'_{mn})}{\partial q'_{mn}} + \frac{\partial c_n(Q^{1'})}{\partial q'_{mn}} - \gamma'_n - \sum_{s=1}^{M}\frac{\partial f_s(Q^{1''},Q^{2''})}{\partial q''_{mn}} \\
&\qquad - \frac{\partial c_{mn}(q''_{mn})}{\partial q''_{mn}} - \frac{\partial c_n(Q^{1''})}{\partial q''_{mn}} + \gamma''_n\Big) \times [q'_{mn} - q''_{mn}] \\
&+ \sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\Big(\sum_{s=1}^{M}\frac{\partial f_s(Q^{1'},Q^{2'})}{\partial q_{lo}^{m'}} + \frac{\partial \hat{c}_{ml}(q_{lo}^{m'})}{\partial q_{lo}^{m'}} + \sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^l(Q^{2'})}{\partial q_{lo}^{m'}} - t_m\rho_m \\
&\qquad - \sum_{s=1}^{M}\frac{\partial f_s(Q^{1''},Q^{2''})}{\partial q_{lo}^{m''}} - \frac{\partial \hat{c}_{ml}(q_{lo}^{m''})}{\partial q_{lo}^{m''}} - \sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^l(Q^{2''})}{\partial q_{lo}^{m''}} + t_m\rho_m\Big) \times [q_{lo}^{m'} - q_{lo}^{m''}] \\
&+ \sum_{n=1}^{N}\sum_{k=1}^{K}[\tilde{c}_{nk}(Q^{3'}) + \gamma'_n - \rho'_{3k} - \tilde{c}_{nk}(Q^{3''}) - \gamma''_n + \rho''_{3k}] \times [\tilde{q}'_{nk} - \tilde{q}''_{nk}] \\
&+ \sum_{n=1}^{N}\left[\sum_{m=1}^{M}q'_{mn} - \sum_{k=1}^{K}\tilde{q}'_{nk} - \sum_{m=1}^{M}q''_{mn} + \sum_{k=1}^{K}\tilde{q}''_{nk}\right] \times [\gamma'_n - \gamma''_n] \\
&+ \sum_{k=1}^{K}\left[\sum_{n=1}^{N}\tilde{q}'_{nk} - d_k(\rho'_3) - \sum_{n=1}^{N}\tilde{q}''_{nk} + d_k(\rho''_3)\right] \times [\rho'_{3k} - \rho''_{3k}] \\
= &\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\sum_{s=1}^{M}\frac{\partial f_s(Q^{1'},Q^{2'})}{\partial q'_{mn}} - \sum_{s=1}^{M}\frac{\partial f_s(Q^{1''},Q^{2''})}{\partial q''_{mn}}\right] \times [q'_{mn} - q''_{mn}] \\
&+ \sum_{m=1}^{M}\sum_{n=1}^{N}\left[\frac{\partial c_{mn}(q'_{mn})}{\partial q'_{mn}} - \frac{\partial c_{mn}(q''_{mn})}{\partial q''_{mn}}\right] \times [q'_{mn} - q''_{mn}] \\
&+ \sum_{m=1}^{M}\sum_{n=1}^{N}\left[\frac{\partial c_n(Q^{1'})}{\partial q'_{mn}} - \frac{\partial c_n(Q^{1''})}{\partial q''_{mn}}\right] \times [q'_{mn} - q''_{mn}] \\
&+ \sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\sum_{s=1}^{M}\frac{\partial f_s(Q^{1'},Q^{2'})}{\partial q_{lo}^{m'}} - \sum_{s=1}^{M}\frac{\partial f_s(Q^{1''},Q^{2''})}{\partial q_{lo}^{m''}}\right] \times [q_{lo}^{m'} - q_{lo}^{m''}] \\
&+ \sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\frac{\partial \hat{c}_{ml}(q_{lo}^{m'})}{\partial q_{lo}^{m'}} - \frac{\partial \hat{c}_{ml}(q_{lo}^{m''})}{\partial q_{lo}^{m''}}\right] \times [q_{lo}^{m'} - q_{lo}^{m''}] \\
&+ \sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^l(Q^{2'})}{\partial q_{lo}^{m'}} - \sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^l(Q^{2''})}{\partial q_{lo}^{m''}}\right] \times [q_{lo}^{m'} - q_{lo}^{m''}] \\
&+ \sum_{n=1}^{N}\sum_{k=1}^{K}[\tilde{c}_{nk}(Q^{3'}) - \tilde{c}_{nk}(Q^{3''})] \times [\tilde{q}'_{nk} - \tilde{q}''_{nk}] \\
&+ \sum_{k=1}^{K}\left[-d_k(\rho'_3) + d_k(\rho''_3)\right] \times [\rho'_{3k} - \rho''_{3k}]
\end{aligned}
$$

$$= \quad (I)+(II)+(III)+(IV)+(V)+(VI)+(VII)+(VIII). \tag{3.31}$$

Since f_m for all m are additive and f_m^1 are convex functions, one has

$$(I)=\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\sum_{s=1}^{M}\left(\frac{\partial f_s(Q^{1'},Q^{2'})}{\partial q'_{mn}}-\frac{\partial f_s(Q^{1''},Q^{2''})}{\partial q''_{mn}}\right)\right]\times[q'_{mn}-q''_{mn}]\geq 0$$

and

$$(IV)=\sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\sum_{s=1}^{M}\left(\frac{\partial f_s(Q^{1'},Q^{2'})}{\partial q_{lo}^{m'}}-\frac{\partial f_s(Q^{1''},Q^{2''})}{\partial q_{lo}^{m''}}\right)\right]\times[q_{lo}^{m'}-q_{lo}^{m''}]\geq 0.$$

The convexity of $c_{mn}, c_n, \hat{c}_{ml}$ and c_{mo}^l for all m,n,l,o, we have

$$(II) \quad = \quad \sum_{m=1}^{M}\sum_{n=1}^{N}\left[\frac{\partial c_{mn}(q'_{mn})}{\partial q'_{mn}}-\frac{\partial c_{mn}(q''_{mn})}{\partial q''_{mn}}\right]\times[q'_{mn}-q''_{mn}]\geq 0,$$

$$(III) \quad = \quad \sum_{m=1}^{M}\sum_{n=1}^{N}\left[\frac{\partial c_n(Q^{1'})}{\partial q'_{mn}}-\frac{\partial c_n(Q^{1''})}{\partial q''_{mn}}\right]\times[q'_{mn}-q''_{mn}]\geq 0,$$

$$(V) \quad = \quad \sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\frac{\partial \hat{c}_{ml}(q_{lo}^{m'})}{\partial q_{lo}^{m'}}-\frac{\partial \hat{c}_{ml}(q_{lo}^{m''})}{\partial q_{lo}^{m''}}\right]\times[q_{lo}^{m'}-q_{lo}^{m''}]\geq 0$$

and

$$(VI)=\sum_{m=1}^{M}\sum_{l=1}^{L}\sum_{o=1}^{O}\left[\sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^{l}(Q^{2'})}{\partial q_{lo}^{m'}}-\sum_{x=1}^{L}\sum_{y=1}^{O}\frac{\partial c_{xy}^{l}(Q^{2''})}{\partial q_{lo}^{m''}}\right]\times[q_{lo}^{m'}-q_{lo}^{m''}]\geq 0.$$

Since $\tilde{c}_{nk}$ for all n,k are assumed to be monotone increasing, and d_k for all k are assumed to be monotone decreasing, we have

$$(VII)=\sum_{n=1}^{N}\sum_{k=1}^{K}[\tilde{c}_{nk}(Q^{3'})-\tilde{c}_{nk}(Q^{3''})]\times[\tilde{q}'_{nk}-\tilde{q}''_{nk}]\geq 0$$

and

$$(VIII)=\sum_{k=1}^{K}\left[-d_k(\rho'_3)+d_k(\rho''_3)\right]\times[\rho'_{3k}-\rho''_{3k}]\geq 0.$$

We see that the right hand side of (3.31) is nonnegative. The proof is completed. □

Lemma 3.7. *Assume that the conditions of Lemma 3.6 hold. In addition, suppose that one of the five families of convex functions $f_m^1, c_{mn}, c_n, \hat{c}_{ml}$ and c_{mo}^l for all $m=1,\ldots,M, n=1,\ldots,N, l=1,\ldots,L, o=1,\ldots L$ is a family of strictly convex functions. Suppose that $\tilde{c}_{nk}, n=1,\ldots,N, k=1,\ldots,K$ and $d_k, k=1,\ldots,K$ are strictly monotone. Then, the vector function F that enters the variational inequality (3.22) is strictly monotone, with respect to (Q^1,Q^2,Q^3,ρ_3).*

Proof. For any two distinct $(Q^{1'},Q^{2'},Q^{3'},\rho'_3),(Q^{1''},Q^{2''},Q^{3''},\rho''_3)$, we must have at least one of the following four cases:

(i) $Q^{1'}\neq Q^{1''}$,
(ii) $Q^{2'}\neq Q^{2''}$,
(iii) $Q^{3'}\neq Q^{3''}$,
(iv) $\rho'_3\neq\rho''_3$.

Under the condition of the theorem, if (i) holds true, then, at the right-hand side of (3.31), at least one of $(I),(II),(III)$ and (IV) is positive. If (ii) is true, then at least one of $(I),(IV),(V)$ and (VI) is positive. If (iii) is true, then (VII) is positive. In case of $(iv),(VIII)$ is positive. Hence, we can conclude that the right hand side of (3.31) is greater than zero. The proof is completed. □

Lemma 3.7 has an important implication for the uniqueness of product shipments, Q^1, the relief item shipments, Q^2, the retailer shipments, Q^3 and the prices at the demand markets, ρ_3, at the equilibrium. We note also that no guarantee of a unique $\gamma_n, n = 1, \ldots, N$, can be generally expected at the equilibrium.

Theorem 3.8. *Under the conditions of Lemma 3.7, there is a unique product shipment pattern Q^{1^*}, a unique relief item shipment pattern Q^{2^*}, a unique retail shipment (consumption) pattern Q^{3^*}, and a unique demand price vector ρ_3^* satisfying the equilibrium conditions of the supply chain. In other words, if the variational inequality (3.22) admits a solution, that should be the only solution in Q^1, Q^2, Q^3 and ρ_3.*

Proof. Since the result of Lemma 3.7, we have the strict monotonicity of the vector function F that enters the variational inequality (3.22) and uniqueness follows. By the standard variational inequality theory [4], this theorem holds. □

The following lemma, if the function F that enters the variational inequality problem (3.22) has some conditions then we can show that F is Lipschitz continuous.

Lemma 3.9. *The function F that enters the variational inequality problem (3.22) is Lipschitz continuous under the following conditions:*

(1.) *Each $f_m, m = 1, \ldots, M$, is additive and has a bounded second-order derivative;*
(2.) *$c_{mn}, c_n, \hat{c}_{ml}$ and c^l_{mo} have bounded second-order derivatives for all m, n, l, o;*
(3.) *$\tilde{c}_{nk}$ and d_k have bounded first-order derivatives for all n, k.*

Proof. The result is direct by applying a mean-value theorem from calculus to the vector function F that enters the variational inequality problem (3.22). Since

$$F'(x) = (F^{1'}(X), F^{2'}(X), F^{3'}(X), F^{4'}(X), F^{5'}(X)),$$

where

$$\begin{aligned} F^{1'}(X) &\equiv \sum_{s=1}^{M} \frac{\partial^2 f_s(Q^{1*}, Q^{2*})}{\partial^2 q_{mn}} + \frac{\partial^2 c_{mn}(q^*_{mn})}{\partial^2 q_{mn}} + \frac{\partial^2 c_n(Q^{1*})}{\partial^2 q_{mn}} \\ F^{2'}(X) &\equiv \sum_{s=1}^{M} \frac{\partial^2 f_s(Q^{1*}, Q^{2*})}{\partial^2 q^m_{lo}} + \frac{\partial^2 \hat{c}_{ml}(q^{m*}_{lo})}{\partial^2 q^m_{lo}} + \sum_{x=1}^{L}\sum_{y=1}^{O} \frac{\partial^2 c^l_{xy}(Q^{2*})}{\partial^2 q^m_{lo}} \\ F^{3'}(X) &\equiv \frac{\partial \tilde{c}_{nk}(Q^{3*})}{\partial q_{nk}}; \quad F^{4'}(X) \equiv 0; \quad F^{5'}(X) \equiv -\frac{\partial d_k(\rho_3^*)}{\partial \rho_{3k}}. \end{aligned}$$

By the assumption (1.) – (3.), we have there exists L such that $\|F'(x)\| \leq L$. The proof is completed. □

4. THE ALGORITHM

In this section, an algorithm is presented which can be applied to solve a variational inequality problem that was proposed in the above article. The algorithm is the modified projection method of Korpelevich [5] and is guaranteed to converge which provided that the function F that enters the variational inequality is monotone and Lipschitz continuous (and that a solution exists). Then, the algorithm for our supply chain network model comprising disaster relief as follows, where $\Im$ denotes an iteration counter:

Modified projection method for the solution of variational inequality (3.22).

Step 0. Initialization Set $X_0 \equiv (Q^1_0, Q^2_0, Q^3_0, \gamma_0, \rho^0_{3k}) \in \mathcal{K}$. Let $\Im = 1$ and set α such that $0 < \alpha \leq \frac{1}{L}$, where L is the Lipschitz constant for the problem.

Step 1. Computation Compute $X^{\Im} \equiv (Q^1_{\Im}, Q^2_{\Im}, Q^3_{\Im}, \gamma_{\Im}, \rho^{\Im}_{3k}) \in \mathcal{K}$ by solving the variational inequality subproblem:

$$\left\langle X^{\Im} + (\alpha F(X^{\Im-1}) - X^{\Im-1}), X - X^{\Im-1} \right\rangle \geq 0, \tag{4.1}$$

for all $X \in \mathcal{K}$.

Step 2. Convergence verification If $|Q^1_{\Im} - Q^1_{\Im-1}| \leq \epsilon, |Q^2_{\Im} - Q^2_{\Im-1}| \leq \epsilon, |Q^3_{\Im} - Q^3_{\Im-1}| \leq \epsilon, |\gamma_{\Im} - \gamma_{\Im-1}| \leq \epsilon, |\rho^{\Im}_{3k} - \rho^{\Im-1}_{3k}| \leq \epsilon$ for all $m = 1, 2, \cdots, M, n = 1, 2, \cdots, N, k = 1, 2, \cdots, K, l = 1, 2, \cdots, L, o = 1, 2, \cdots, O$ with $\epsilon > 0$, a pre-specified tolerance, then stop; otherwise, set $\Im := \Im + 1$, and go to Step 1.

Note that the variational inequality subproblem (4.1) can be solved explicitly and in closed form since the feasible set is that of the nonnegative orthant. Indeed, they yield subproblems in the Q^1, Q^2, Q^3, γ_n and ρ_{3k} variables for all n, k.

Next, we state the convergence result for the modified projection method in this model.

Theorem 4.1. *Assume that the function that enters the variational inequality* (3.22) *(or* (2.1)*) satisfies the conditions in Lemma 3.6 and Lipschitz continuous of F. Then the modified projection method described above converges to the solution of the variational inequality* (3.22)*.*

Proof. According to Korprlevich [5], the modified projected method converges to the solution of the variational inequality problem of the form (2.1). We provided that the function F that enters the variational inequality is monotone and Lipschitz continuous and that a solution exists. □

5. NUMERICAL EXAMPLES

In this section, we apply the modified projection method to several numerical examples. The modified projection method was implemented in SCILAB and the computer system used was a ASUS located at the Pibulsongkham Rajabhat University at Phitsanulok, Thailand. The convergence criterion used was that the absolute value of the product flows and prices between two successive iterations differed by no more than 10^{-4}. For the examples, α was set to 0.005 in the algorithm. The numerical examples had the network structure depicted in Figure 2 and consisted of two manufacturers, two retailers, two freight service providers, two demand markets and two demand points. The concept of this research was inspired by the paper of [12] and [10].

Example 5.1. In this example, we consider the supply chain network in Figure 2. There

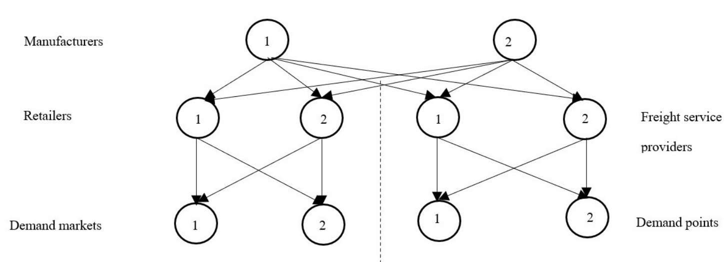

FIGURE 2. Example 6.1

is two manufacturers which each manufacturers transacted some products for sale to two retailers and donation to two freight service providers and then each retailer sent the product to two demand markets and each freight service provider sent the product to two demand point. Here, each manufacturer wish to ship the relief items to demand points 100 items, this implies $q_1 = q_2 = 100$. The data for the first example were constructed for easy interpretation purposes. The production cost functions for the manufacturers were given by:

$$f_1(Q^1,Q^2) = 0.05(q_1^1+q_1^2)^2 + 0.2q_1^1, f_2(Q^1,Q^2) = 0.05(q_2^1+q_2^2)^2 + 0.2q_2^2.$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers were given by:

$$c_{11}(q_{11}) = 0.5(q_{11})^2 + 3.5q_{11}, \qquad c_{12}(q_{12}) = 0.5(q_{12})^2 + 3.5q_{12},$$
$$c_{21}(q_{21}) = 0.5(q_{21})^2 + 3.5q_{21}, \qquad c_{22}(q_{22}) = 0.5(q_{22})^2 + 3.5q_{22}.$$

The handling costs of the retailers, in turn, were given by:

$$c_1(Q^1) = 0.5\left(\sum_{m=1}^{2} q_{m1}\right)^2, \quad c_2(Q^1) = 0.5\left(\sum_{m=1}^{2} q_{m2}\right)^2.$$

The transaction costs of the freight service providers associated with transacting with the manufacturers were given by:

$$\hat{c}_{11} = 2.5(q_{11}^1+q_{12}^1), \qquad \hat{c}_{12} = 2.25(q_{21}^1+q_{22}^1),$$
$$\hat{c}_{21} = 2.5(q_{11}^2+q_{12}^2), \qquad \hat{c}_{22} = 2.25(q_{21}^2+q_{22}^2).$$

The total cost associated with freight service provider $1, \hat{c}_{11}$, is higher than that for freight service provider $2, \hat{c}_{12}$, since it does not have as much experience with the former provider and the transfer cost is higher per unit.

The freight service provider total costs are as follows: For freight service provider 1:

$$c_{11}^1 = 0.01\left(q_{11}^1\right)^2 + 9.67q_{11}^1, \quad c_{12}^1 = 0.1\left(q_{12}^1\right)^2 + 14.88q_{12}^1,$$
$$c_{21}^1 = 0.1\left(q_{11}^2\right)^2 + 9.67q_{11}^2, \quad c_{22}^1 = 0.1\left(q_{12}^2\right)^2 + 14.88q_{12}^2,$$

and for freight service provider 2:

$$c_{11}^2 = 0.1\left(q_{21}^1\right)^2 + 9.67q_{21}^1, \quad c_{12}^2 = 0.1\left(q_{22}^1\right)^2 + 14.88q_{22}^1,$$
$$c_{21}^2 = 0.1\left(q_{21}^2\right)^2 + 9.67q_{21}^2, \quad c_{22}^2 = 0.01\left(q_{22}^2\right)^2 + 14.88q_{22}^2.$$

The transaction costs between the retailers and the consumers at the demand markets were given by:

$$\tilde{c}_{11}(Q^3) = \tilde{q}_{11}+5, \quad \tilde{c}_{12}(Q^3) = \tilde{q}_{12}+5, \quad \tilde{c}_{21}(Q^3) = \tilde{q}_{21}+5, \quad \tilde{c}_{22}(Q^3) = \tilde{q}_{22}+5.$$

The demand functions at the demand markets were:

$$d_1(\rho_3) = -2\rho_{31} - 1.5\rho_{32} + 1000, \quad d_2(\rho_3) = -2\rho_{32} - 1.5\rho_{31} + 1000.$$

Next, assume that the income tax rate as follows.

Net income	Income tax rate
$0 \le x \le 5000$	Tax exemption
$5000 < x \le 10000$	5%
$10000 < x \le 15000$	10%
$x \ge 15000$	20%

where x denotes for the amount of net income. Assume that the product cost price for donation of the manufacturer m, (ρ_m^*), is equal to the minimum of the price which the manufacture m sells the product to the retailers.

Therefore, we computed equilibrium solution via the projection method as follows: The projection method converged in 708 iterations, the product shipments between the two manufacturers and two retailers were:

$$Q^{1^*}: \quad q_{11}^* = 50; q_{12}^* = 53; \quad q_{21}^* = 55; q_{22}^* = 58.$$

The product shipments between the two manufacturers and two demand points via two freight service providers were:

$$Q^{2^*} \quad : \quad q_{11}^{1^*} = 74, \quad q_{12}^{1^*} = 6, \quad q_{21}^{1^*} = 9, \quad q_{22}^{1^*} = 7,$$
$$q_{11}^{2^*} = 8, \quad q_{12}^{2^*} = 7, \quad q_{21}^{2^*} = 10, \quad q_{22}^{2^*} = 80.$$

By the above results, we see that the most of products are shipped to the demand point 1 by the manufacturer 1 via the freight service provider 1 and, also the most of products are shipped to the demand point 2 by the manufacturer 2 via the freight service provider 2 because the shipping cost is cheaper. The product shipments between the two retailers and two demand markets were:

$$Q^{3^*} \quad : \quad \tilde{q}_{11}^* = 62; \tilde{q}_{12}^* = 60; \tilde{q}_{21}^* = 59; \tilde{q}_{22}^* = 57.$$

The vector γ^*, which was equal to the prices charged by the retailers ρ_2^*, were:

$$\gamma^* \quad : \quad \gamma_1^* = 185.32; \gamma_2^* = 191.69.$$

The demand prices at demand markets were:

$$\rho_3^* \quad : \quad \rho_{31}^* = 254.96; \rho_{32}^* = 249.96.$$

In this example, we can interpret, in equilibrium, that if the manufacturer 1 and 2 sell the products to the retailers and demand points in above results then the profit of the manufacturer 1 is $1,645.16$ and the profit of the manufacturer 2 is 1638.455. If the retailers sell the products to the demand markets in above results then the profit of the retailer 1 is 9000.04 and the profit of the retailer 2 is 7184.34. For the freight service provider, if they sent the product from the manufacturers to the demand points in above results then the profit of the freight service provider 1 is 69.66 and the profit of the freight service provider 2 is 77.

Next, the following example, we want to consider that if the manufacturers need to deliver the product to the demand points more than 100 items then we obtain the results as follows.

Example 5.2. In this example, we assume as all Example 5.1, except that each manufacturer wish to ship the product (relief items) 200 items. So, we obtain the results as:

The projection method converged in 1092 iterations, the product shipments between the two manufacturers and two retailers were:

$$Q^{1*} : \quad q_{11}^* = 54; q_{12}^* = 54; q_{21}^* = 55; q_{22}^* = 54.$$

The product shipments between the two manufacturers and two demand points via two freight service providers were:

$$Q^{2*} : \quad q_{11}^{1^*} = 154, q_{12}^{1^*} = 15, q_{21}^{1^*} = 17, q_{22}^{1^*} = 17, q_{11}^{2^*} = 14, q_{12}^{2^*} = 14, q_{21}^{2^*} = 16, q_{22}^{2^*} = 154.$$

Observe that, in the same way Example 5.1, the most of products are shipped to the demand point 1 by the manufacturer 1 via the freight service provider 1 and, also the most of products are shipped to the demand point 2 by the manufacturer 2 via the freight service provider 2. The product shipments between the two retailers and two demand markets were:

$$Q^{3*} : \quad \tilde{q}_{11}^* = \tilde{q}_{12}^* = 53; \tilde{q}_{21}^* = \tilde{q}_{22}^* = 54.$$

The vector γ^*, which was equal to the prices charged by the retailers ρ_2^*, were:

$$\gamma^* : \quad \gamma_1^* = 197.25; \gamma_2^* = 195.75.$$

The demand prices at demand markets were:

$$\rho_3^* : \quad \rho_{31}^* = 254.56; \rho_{32}^* = 255.75.$$

In this example, we have the following results: if the manufacturer 1 and 2 sell the products to the retailers and demand points in above results then the profit of the manufacturer 1 is $1,780.38$ and the profit of the manufacturer 2 is $1,211.11$. If the retailers sell the products to the demand markets in above results then the profit of the retailer 1 is $10,612.8$ and the profit of the retailer 2 is $5,751$. For the freight service provider, if they sent the product from the manufacturers to the demand points in above results then the profit of the freight service provider 1 is 298.86 and the profit of the freight service provider 2 is 320.56. Moreover, comparing the results with Example 5.1, we can see that the trading of products between manufacturers, retailers and consumers is almost no different from Example 5.1, but the difference is clearly in the delivery of donations. In this case, it can be seen that all freight service providers earn more profits than in Example 5.1 due to sending more products to donate and getting more tax relief.

For the following example, we are interested that if all conditions are the same as Example 5.2 except for the part of the retailers which there is a retailer.

Example 5.3. In this example, we are interested that all assumption is the same Example 5.2, except it has not the retailer 2 see Figure 3. So, we obtain the results as:

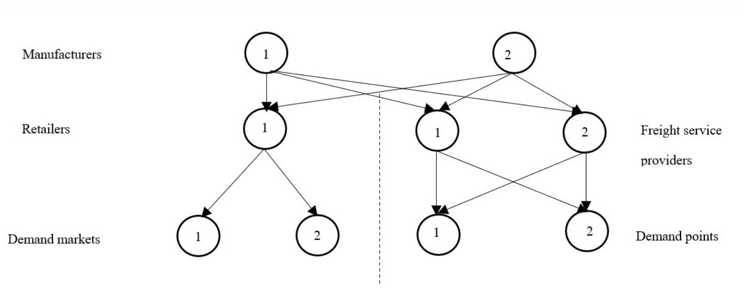

FIGURE 3. Example 5.3

The projection method converged in 832 iterations, the product shipments between the two manufacturers and a retailer were:

$$Q^{1^*}: \quad q_{11}^* = q_{21}^* = 57.$$

The product shipments between the two manufacturers and two demand points via two freight service providers were the same as Example 1,

$$Q^{2^*}: \quad q_{11}^{1^*} = 154, q_{12}^{1^*} = 15, q_{21}^{1^*} = 17, q_{22}^{1^*} = 17, q_{11}^{2^*} = 14, q_{12}^{2^*} = 14, q_{21}^{2^*} = 16, q_{22}^{2^*} = 154.$$

The product shipments between a retailer and two demand markets were:

$$Q^{3^*}: \quad \tilde{q}_{11}^* = 63; \tilde{q}_{12}^* = 64.$$

The vector γ^*, which was equal to the prices charged by the retailers ρ_2^*, were:

$$\gamma^*: \quad \gamma_1^* = 204.23.$$

The demand prices at demand markets were:

$$\rho_3^*: \quad \rho_{31}^* = 268.51; \rho_{32}^* = 266.71.$$

We have the following results: if the manufacturer 1 and 2 sell the products to the retailers and demand points in above results then the profit of the manufacturer 1 is -17.13 and the profit of the manufacturer 2 is -653.61. If the retailers sell the products to the demand markets in above results then the profit of the retailer 1 is $9,595.31$. For the freight service provider, if they sent the product from the manufacturers to the demand points in above results then the profit of the freight service provider 1 is 298.86 and the profit of the freight service provider 2 is 320.56. Therefore, comparing the results with Example 5.2, we can see that the manufacturers sent the products to the demand point in the same of Example 5.2 and must sell more products to retailers than Example 5.2 and also the retailers sell more products to consumers than Example 5.2. If we consider the profit of all parties, it can be seen that the manufactures loses while other part have the same profit.

The following example, we will show that if Example 5.2 remains a demand market then the results as follows.

Example 5.4. If all of condition as the same Example 5.2, except this example has a demand market (Figure 4) and let

$$d_1(\rho_3) = -2\rho_{31} + 1000.$$

Then we have the following results. The projection method converged in 241 iterations,

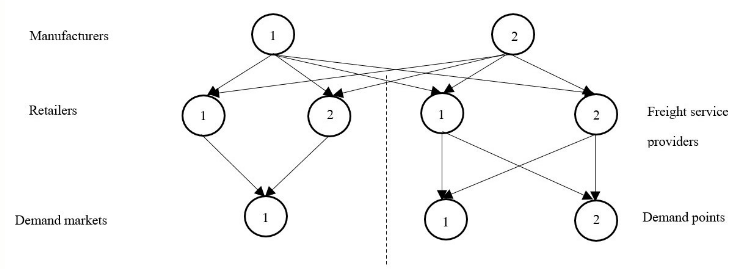

FIGURE 4. Example 5.4

the product shipments between the two manufacturers and two retailers were:

$$Q^{1^*}: \quad q_{11}^* = q_{12}^* = 65; q_{21}^* = q_{22}^* = 66.$$

The product shipments between the two manufacturers and two demand points via two freight service providers were:

$$Q^{2^*}: \quad q_{11}^{1^*} = 153, q_{12}^{1^*} = 15, q_{21}^{1^*} = 17, q_{22}^{1^*} = 16, q_{11}^{2^*} = 15, q_{12}^{2^*} = 14, q_{21}^{2^*} = 16, q_{22}^{2^*} = 154.$$

The product shipments between the retailers and a demand market were:

$$Q^{3^*}: \quad \tilde{q}_{11}^* = \tilde{q}_{21}^* = 131.$$

The vector γ^*, which was equal to the prices charged by the retailers ρ_2^*, were:

$$\gamma^*: \quad \gamma_1^* = 233.13; \gamma_2^* = 233.03.$$

The demand prices at demand markets were:

$$\rho_3^*: \quad \rho_{31}^* = 369.06.$$

We have the following results: if the manufacturer 1 and 2 sell the products to the retailers and demand points in above results then the profit of the manufacturer 1 is $3,920.25$ and the profit of the manufacturer 2 is $3,452.10$. If the retailers sell the products to the demand markets in above results then the profit of the retailer 1 is $8,572.24$ and the profit of the retailer 2 is $8,557.83$. For the freight service provider, if they sent the product from the manufacturers to the demand points in above results then the profit of the freight service provider 1 is 298.69 and the profit of the freight service provider 2 is 314.66. Therefore, comparing the results with Example 5.2, we can see that the manufacturers and retailers have to sell products in the lager quantities and also the price charged by the retailers and demand prices increase. But at the same time, the profits of all parties have also increased. For the donation part, it is not much different from Example 5.2.

The following example, we will show that if Example 5.1 remains a freight service provider then the results as follows.

Example 5.5. If all of condition as the same Example 5.1, except this example has a freight service provider (Figure 5). Then we have the following results. The projection method converged in 755 iterations, the product shipments between the two manufacturers and two retailers were:

$$Q^{1^*}: \quad q_{11}^* = q_{12}^* = 48; q_{21}^* = 60; q_{22}^* = 62.$$

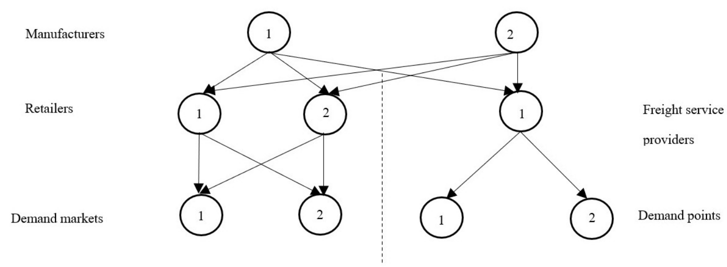

FIGURE 5. Example 5.5

The product shipments between the two manufacturers and two demand points via two freight service providers were:

$$Q^{2^*}: \quad q_{11}^{1^*} = 99, q_{12}^{1^*} = 16, q_{11}^{2^*} = 2, q_{12}^{2^*} = 85.$$

The product shipments between the retailers and a demand market were:

$$Q^{3^*}: \quad \tilde{q}_{11}^* = \tilde{q}_{12}^* = 61; \tilde{q}_{21}^* = \tilde{q}_{22}^* = 59.$$

The vector γ^*, which was equal to the prices charged by the retailers ρ_2^*, were:

$$\gamma^*: \quad \gamma_1^* = 189.91; \gamma_2^* = 192.23.$$

The demand prices at demand markets were:

$$\rho_3^*: \quad \rho_{31}^* = \rho_{32}^* = 251.39.$$

We have the following results: if the manufacturer 1 and 2 sell the products to the retailers and demand points in above results then the profit of the manufacturer 1 is $4,596.87$ and the profit of the manufacturer 2 is $1,490.05$. If the retailers sell the products to the demand markets in above results then the profit of the retailer 1 is $7,266.62$ and the profit of the retailer 2 is $6,269.94$. For the freight service provider, if they sent the product from the manufacturers to the demand points in above results then the profit of the freight service provider 1 is 846.51. Therefore, comparing the results with Example 5.1, we can see that the manufacturers 1 produce more products for donations and less for sale and the manufacturers 2 produce more products for sale and less for donations. In the margins, only the manufacturer 1 and the freight service provider 1 gain more profit.

Remark 5.6. By all of the previous examples, we see that if the network has a demand market (Example 5.4) then all of the parties have more profitable, but also have to produce more and consumers are willing to pay more.

6. CONCLUSION

In this paper, we present the generalized supply chain network, that is, the supply chain network comprising disaster relief which is a combination between a supply chain network and a competitive freight service provider network. Firstly, we proposed a model which was satisfy our supply chain network and considered the behavior of the manufacturers, where the manufacturers want to sell the product and donate as well, the retailers, the demand markets and the freight service provider. So, we obtain the supply chain network comprising disaster relief equilibrium model and the variational inequality which was equivalent to such supply chain network. The existence and uniqueness of the solution of the variational inequality was proposed. Finally, the algorithm which was a tool for using the computing our example was presented and some examples were presented for illustrative in the above articles.

7. Acknowledgements

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