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Contact

Narin Petrot (narinp@nu.ac.th)
Center of Excellence in Nonlinear Analysis and Optimization,
Department of Mathematics, Faculty of Science,
Naresuan University, Phitsanulok, 65000, Thailand.

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FROM *KKM* CLASS TO $\mathfrak{KC}$ CLASS OF MULTIMAPS

SEHIE PARK*[1]

[1] The National Academy of Sciences, Republic of Korea; Seoul 06579 and Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea

ABSTRACT. This is to give a short history of our multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ in the KKM theory. We show that many authors adopted or imitated the inadequate definition of the KKM class due to Chang and Yen in 1996. We list such works in chronological order and also introduce other works which extended the class properly. Our study on such history will improve the KKM theory in the new millennium.

KEYWORDS:Abstract convex space, KKM theorem, KKM class of multimaps, $\mathfrak{A}_c^\kappa$, $\mathfrak{B}$, $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$

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1. Introduction

The KKM theory, first called by the author in 1992, is the study on applications of equivalent formulations or generalizations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz in 1929. The KKM theorem is one of the most well-known and important existence principles and provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Since the theorem and its many equivalent formulations or extensions are powerful tools in showing the existence of solutions of a lot of problems in pure and applied mathematics, many scholars have been studying its further extensions and applications.

The KKM theory was first devoted to convex subsets of topological vector spaces mainly by Ky Fan and Granas, and later to the so-called convex spaces by Lassonde, to c-spaces by Horvath and others, to G-convex spaces mainly by the present author. Since then a large number of authors introduced imitations, modifications, and generalizations of G-convex spaces and published hundreds of papers. Motivated by this, in 2006-09, we proposed new concepts of abstract convex spaces and partial

* *Corresponding author.*
Email address : park35@snu.ac.kr; sehiepark@gmail.com.
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KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory properly.

Now the KKM theory becomes the study of abstract convex spaces due to ourselves in 2006 and we obtained a large number of new results in such frame. For the history of the KKM theory, see our previous article [Park 2017]. Moreover, applications of the KKM theory to equilibrium theory, variational inequalities, best approximations, economic theory, and many others can be seen in the references therein.

More early, motivated our works, Chang and Yen [1996] introduced the so-called KKM class of multimaps. According to Google Scholar in 2022, more than 171 papers have quoted the paper, and we noticed that most of these papers adopted or imitated the inadequate definition of the KKM class. This is mainly because such authors were ignorant of the fact that *the KKM theorem also holds for open-valued KKM maps*. This was discovered by W. K. Kim [1987] and Shih and Tan [1987]. Motivated by this, in order to improve the KKM theory, we replaced the KKM class by the modern classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ of multimaps.

This survey article is to give a history of the multimap classes KKM and $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ in the KKM theory, and to let the readers know that the usage of KKM class is not adequate. We show that many authors adopted or imitated the inadequate definition of the KKM class due to Chang and Yen [1996]. We list such works in chronological order and also introduce other works which extended the class properly. Our study on such history will improve the KKM theory in the new millennium.

Section 2 deals with preliminaries on abstract convex spaces. In Section 3, we introduce a short history of our multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$. Section 4 deals with the literature on the KKM class due to Chang and Yen [1996] and many of its modifications or imitations appeared mainly in the new millennium. In Section 5, we introduce some articles concerned with our multimap classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$. Finally, Section 6 deals with some conclusion.

2. Preliminary on abstract convex spaces

The following is given by Park [2006], where $\langle D\rangle$ denotes the collection of nonempty finite subsets of a set D:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if, for any $A \in \langle X \cap D\rangle$, we have $\Gamma_A \subset X$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Later we always assumed that E is a topological space in an abstract convex space $(E, D; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called *a* $\mathfrak{K}$*-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E,Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{−map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E,Z) \subset \mathfrak{KC}(E,Z) \cap \mathfrak{KO}(E,Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation $\mathrm{KKM}(E,Z)$ instead of $\mathfrak{KC}(E,Z)$.

From now on, in this section, we give examples of abstract convex spaces $(E,D;G)$ in the chronological order. For the most of the examples, we have $1_E \in \mathfrak{KC}(E,E)$ [or $1_E \in \mathfrak{KO}(E,E)$].

1. If $E = \Delta_n$ is an n-simplex, D is the set of its vertices, $\Gamma = \mathrm{co}$ is the convex hull operation, then the celebrated KKM principle [1929] says that $1_E \in \mathfrak{KC}(E,E)$. In this case, note that $1_E \notin \mathfrak{K}(E,E)$. A simple example for $n = 1$ is as follows: Let $\Delta_1 := [0,1]$, $D := \{0,1\}$, and $G(0) := [0,\frac{1}{2}]$, $G(1) := (\frac{1}{2},1]$. Then G is a KKM map, but $G(0) \cap G(1) = \emptyset$.
2. If D is a nonempty subset of a topological vector space E (not necessarily Hausdorff), Fan's KKM lemma [1961] says that $1_E \in \mathfrak{KC}(E,E)$.
3. Let E be a topological vector space with a neighborhood system $\mathcal{V}$ of its origin. A subset X of E is said to be *almost convex* [Jeng et al. 2006] if for any $V \in \mathcal{V}$ and for any finite subset $A := \{x_1, x_2, \cdots, x_n\}$ of X, there exists a subset $B := \{y_1, y_2, \cdots, y_n\}$ of X such that $y_i - x_i \in V$ for each $i = 1,2,\cdots,n$ and $\mathrm{co}\, B \subset X$. By choosing $\Gamma_A := B$ for each $A \in \langle X\rangle$, $(X;\Gamma)$ becomes a G-convex space and hence an abstract convex space.
4. If X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology, then $(X,D;\Gamma)$ becomes a *convex space* generalizing the one due to Lassonde [1983]. Note that any convex subset of a t.v.s. is a convex space, but not conversely. For a convex space (X, co), Lassonde showed that $1_X \in \mathfrak{KC}(X,X)$.
5. In the same year, Kim [1989] and Shih and Tan [1989] showed that $1_E \in \mathfrak{KO}(E,E)$ when E is an n-simplex. Therefore, in general, we have
$$\mathfrak{K}(E,E) \subsetneq \mathfrak{KC}(E,E) \cap \mathfrak{KO}(E,E).$$
6. A well-known subclass of G-convex spaces due to Horvath [1987-1993] can be generalized as follows: A G-convex space $(X,D;\Gamma)$ is called a *c-space* (or an *H-space*) if each Γ_A is ω-connected (that is, n-connected for all $n \geq 0$) and $\Gamma_A \subset \Gamma_B$ for $A \subset B$ in $\langle D\rangle$. For a c-space (X,Γ), Horvath showed that $1_X \in \mathfrak{KC}(X,X)$. In particular, Khamsi [1996] obtained $1_X \in \mathfrak{KC}(X,X)$ for a hyperconvex metric space X.
7. In early 1990's, the author [1993] introduced the admissible class $\mathfrak{A}_c^\kappa(X,Y)$ of multimaps $X \multimap Y$ between topological spaces and showed that this class is contained in the class $\mathfrak{KC}(X,Y)$ when X is a convex space and Y is a Hausdorff space [1994]. Motivated by this, Chang and Yen [1996] defined the KKM class of maps on convex subsets of topological vector spaces, and further, Chang et al. [1999] extended the KKM-class to S-KKM class. On the other hand, the author extended the $\mathfrak{A}_c^\kappa$-class to the 'better' admissible $\mathfrak{B}$-class on convex spaces, supplied a large number of examples, and showed that, in the class of compact closed multimaps from convex spaces

to Hausdorff spaces, two subclasses $\mathfrak{B}$ and $\mathfrak{KC}$ coincide [1997]. Moreover, H. Kim [2005] showed that two classes KKM and s-KKM of multimaps from a convex space into a topological space are identical whenever s is surjective [this is the only case S-KKM is slightly meaningful].

8. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \longrightarrow \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

   Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. It is possible to assume $\Gamma(A) = \phi_A(\Delta_n)$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$. In case $X \supset D$, the G-convex space is denoted by $(X \supset D; \Gamma)$.

   For details on G-convex spaces, see Park [2000-2003] and Park et al. [1993-2005], where basic theory was extensively developed and lots of examples of G-convex spaces were given.

9. For a G-convex space $(X, D; \Gamma)$ and a topological space Z, we defined the classes $\mathfrak{K}, \mathfrak{KC}, \mathfrak{KO}$ of multimaps $F : X \multimap Z$, and showed that $1_X \in \mathfrak{KC}(X, X) \cap \mathfrak{KO}(X, X)$. Moreover, we noted that if $F : X \longrightarrow Z$ is a continuous single-valued map or if $F : X \multimap Z$ has a continuous selection, then $F \in \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$. Furthermore, for a Hausdorff space Z, it is shown that $\mathfrak{A}_c^\kappa(X, Z) \subset \mathfrak{KC}(X, Z) \cap \mathfrak{KO}(X, Z)$ by H. Kim and the author [2005].

10. Usually, a *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see Sortan [1984], where the bibliography lists 283 papers. For any subset $X \subset E$, its $\mathcal{C}C$-*convex hull* is defined and denoted by $\mathrm{Co}_{\mathcal{C}} X := \bigcap\{Y \in \mathcal{C} \mid X \subset Y\}$. We say that X is *$\mathcal{C}$-convex* if $X = \mathrm{Co}_{\mathcal{C}} X$. Now we can consider the map $\Gamma : \langle E \rangle \multimap E$ given by $\Gamma_A := \mathrm{Co}_{\mathcal{C}} A$. Then $(E, \mathcal{C})$ becomes our abstract convex space $(E; \Gamma)$.

    Notice that our abstract convex space $(E \supset D; \Gamma)$ becomes a convexity space $(E, \mathcal{C})$ for the family $\mathcal{C}$ of all Γ-convex subsets of E.

11. For any metric space (M, d), Amini et al. [2005] introduced a convexity structure similar to the one for hyperconvex metric space; see Khamsi [1996]. They defined an $\mathcal{NR}$-metric space (M, d) and showed that, for any subadmissible subset X of M, $1_X \in \mathfrak{KC}(X, X)$ holds. Here, subadmissible subsets are simply Γ-convex subsets.

    Recall that, for a G-convex space $(X, D; \Gamma)$ and a Hausdorff space Y, Park and Kim [1997] showed that an acyclic map $F : X \multimap Y$ or, more generally, a map $F \in \mathfrak{A}_c^\kappa(X, Y)$ belongs to the class $\mathfrak{KC}$. Amini et al. [2005] repeatedly claimed that they obtained this result in 2005. More early in Park [1994], the result was obtained for convex spaces and this is the origin of the study of the so-called KKM-class of multimaps.

12. Imitating the original definition of S-KKM maps of Chang et al. [1999], Amini et al. [2007] defined the S-KKM class for a classical convexity space $(X, \mathcal{C})$ with a nonempty set Z and a topological space Y as follows: If $S : Z \multimap X$, $F : X \multimap Y$, and $G : Z \multimap Y$ are three multimaps satisfying

$$F(\mathrm{Co}_{\mathcal{C}}(S(A)) \subset G(A) \quad \text{for each} \quad A \in \langle Z \rangle,$$

then G is called a $\mathcal{C}$-S-KKM map with respect to F. If the map $F : X \multimap Y$ satisfies the requirement that for any $\mathcal{C}$-S-KKM map G with respect to F, the family $\{\overline{G(z)} \mid z \in Z\}$ has the finite intersection property, then F is said to have the S-KKM property with respect to $\mathcal{C}$. Amini et al. defined

$$S\text{-KKM}_{\mathcal{C}}(Z, X, Y) := \{F : X \multimap Y \mid F \text{ has the } S\text{-KKM property with respect to } \mathcal{C}\}.$$

It should be noted that, by putting $\Gamma_A := \mathrm{Co}_{\mathcal{C}}(S(A))$ for each $A \in \langle Z \rangle$, S-KKM$_{\mathcal{C}}(Z, X, Y)$ becomes simply $\mathfrak{KC}(X, Y)$. Therefore, it should be eliminate the S-KKM class.

13. There were many imitations, modifications, or fake extensions of G-convex spaces like the so-called L-spaces, spaces having property (H), M-spaces, MC-spaces, FC-space, GFC-space, simplicial spaces, FWC-spaces, etc. They were all destroyed now.

3. Multimap classes in the KKM theory

We already introduced certain broad classes $\mathfrak{KO}$ and $\mathfrak{KC}$ of maps in several papers; see Park [2018]. Note that $\mathfrak{KC}$ includes the KKM class introduced by Chang and Yen [1996] as a special case. With these concepts, some coincidence theorems and fixed point theorems were proved in abstract convex spaces by ourselves; see Park [2018].

Subclasses of multimaps in the KKM theory were appeared as follows:

1929 identity function —- KKM
1961 identity function —- Fan
1989 continuous function —- Park
1991 acyclic map —- Shioji, Park
1993 admissible map $\mathfrak{A}_c$ —- Park
1994 admissible map $\mathfrak{A}_c^\kappa$ —- Park
1996 the class *KKM* —- Chang and Yen
1997 better admissible map $\mathfrak{B}$ —- Park
1997 KKM family $\mathfrak{K}$ —- Park
1998 $\mathfrak{B}^\kappa$ —- Park
2003 $\mathfrak{KC}$, $\mathfrak{KO}$ —- Park
2004 $\mathfrak{B}^p$ —- Park

Recall that W. K. Kim [1987] and Shih and Tan [1987] discovered that the well-known KKM theorem in 1929 also holds for open-valued multimaps. This open-valued KKM theorem has been generalized to various types of abstract convex spaces by the present author. Consequently, we introduced multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$ in 2003.

Mutual relations of the classes $\mathfrak{A}_c^\kappa$, $\mathfrak{B}$, and $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ depend on the nature of the related abstract convex spaces; see the previous works of Park in 2006–2021.

Even after we defined these classes, many authors imitated or adopted the class *KKM* for two decades. Most of their results are mere copies of the classical ones; see Park [2021].

4. On the KKM class of multimaps

In this section, we introduce several ones among 171 papers stated in Google Scholar in 2021. Key statements in each paper are quoted in their original expressions, and, in most cases, certain comments by the present author are added.

Chang and Yen [1996] — JMAA203

Assume that X is a convex subset of a linear space and Y is a topological space. If $S, T : X \longrightarrow 2^Y$ are two set-valued mappings such that $T(coA) \subset SA$ for each finite subset A of X, then we call S a generalized KKM mapping w.r.t. T, where coA denotes the convex hull of A. Let $T : X \longrightarrow 2^Y$ be a set-valued mapping such that if $S : X \longrightarrow 2^Y$ is a generalized KKM mapping w.r.t. T then the family $\overline{Sx} : x \in X\}$ has the finite intersection property (where $\overline{Sx}$ denotes the closure of Sx), then we say that T has the KKM property. Denote KKM$(X, Y) = \{T : X \longrightarrow 2^Y \mid T$ has the KKM property$\}$

Remark 4.1. Generalized KKM mappings were first introduced by Park [1989], and followed by some others.

Comments: Note that the KKM class contains the admissible class $\mathfrak{A}_c^\kappa$ due to Park. Later, we denoted KKM class by $\mathfrak{K}$ in [Park 1997] and, extended it to the classes $\mathfrak{KC}$ and $\mathfrak{KO}$ for abstract convex spaces; see Section 3 of the present paper.

Chang-Yen's paper has 171 citations (see Google Scholar in 2022) and many peoples still use their obsolete generalized KKM classes. In Section 2 we gave some examples in the new millennium.

Chang-Yen's definition of generalized KKM mapping seems to be not elegant, and works only for closed-valued maps. Recall that more early there have appeared open-valued KKM maps in 1987. This is why we defined the classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ later.

Lin, Ko, and Park [1998] — Discuss. Math. Diff. Incl. 18

In this paper, a set-valued map with G-KKM property is defined and a minimax theorem for set-valued maps with G-KKM property on G-convex space is established. As a consequence of these results we verify coincidence theorem for set-valued maps with G-KKM property on G-convex spaces. Finally, we apply our results to the best approximation problem and fixed point problem.

Chang, Huang, Jeng, and Kuo [1999] — JMAA229

Definition 4.2. Let X be a nonempty set, Y a nonempty convex set of a linear space, and Z a topological space. If $S : X \longrightarrow 2^Y$, $T : Y \longrightarrow 2^Z$, and $F : X \longrightarrow 2^Z$ are three multifunctions satisfying

$$T(coS(A)) \subseteq F(A)$$

for any $A \in \langle X \rangle$, then F is called a generalized S-KKM mapping with respect to T. If the multifunction $T : Y \longrightarrow 2^Z$ satisfies the requirement that for any generalized S-KKM mapping F with respect to T the family $\{\overline{Fx} : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The class S-KKM(X, Y, Z) is defined to be the set $\{T : Y \longrightarrow 2^Z : T$ has the S-KKM property$\}$

Comments: Note that F can be assumed closed-valued, and no consideration on open-valued mappings is given.

Agarwal and O'Regan [2002] — DSA21

We also discuss KKM maps in this paper. Here again X is a convex subset of a Hausdorff topological vector space and Y a topological space. If $S,\ T : X \longrightarrow 2^Y$ are two set valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say S is a generalized KKM map w.r.t. T. T is said to have the KKM property if for any generalized KKM w.r.t. T map $S : X \longrightarrow 2^Y$, the family

$$\{\overline{S(x)} : x \in X\}$$

has the finite intersection property. We let

$$KKM(X,Y) = \{T : X \longrightarrow 2^Y : T \text{ has the KKM property}\}.$$

Comments: The Hausdorffness is redundant and S can be assumed closed-valued, and no consideration on open-valued mappings is given. From now on we will not repeat such comments.

Lin, Ansari, and Wu [2003] — JOTA117

Let X be a convex space, and let Y be a Hausdorff topological space. If $S,\ T : X \longrightarrow 2^Y$ are multivalued maps such that

$$T(coN) \subseteq S(N), \text{ for each } N \in \langle X\rangle,$$

then S is said to be generalized KKM mapping w.r.t. T (Chang-Yen [1996]). The multivalued map $T : X \longrightarrow 2^Y$ is said to have the KKM property (Chang-Yen [1996]) if $S : X \longrightarrow 2^Y$ is a generalized KKM mapping w.r.t. T such that the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property.

Chen, Chang, and Yen [2004] — JKMS41

We generalized the KKM property to the following form for a nearly-convex set X. Assume that X is a nearly-convex subset of a linear space and Y is a topological space. If $T,\ S : X \longrightarrow 2^Y$ are two set-valued mapping such that $T(coA \cap X)\overline{S(A)}$ for each finite subset A of X, then we call S a generalized KKM mapping with respect to T, where $co(A)$ denotes the convex hull of A. Let $T : X \longrightarrow 2^Y$ be a set-valued KKM mapping with respect to T then the family $\overline{Sx} : x \in X\}$ has the finite intersection property (where $\overline{Sx}$ denotes the closure of Sx), then we say that T has the KKM property. Denote

$$KKM(X.Y) = \{T : X \longrightarrow 2^Y \mid T\text{has the KKM property}\}$$

Remark 4.3. Generalized KKM mappings were first introduced by Park [1989], and followed by some others.

Comments: Note that the above is the same one to Chang-Yen [1996] just replacing a convex subset by a nearly-convex subset.

Shahzad [2004] — NA56

Definition 4.4. Let X be a convex subset of a Hausdorff topological vector space and Y a topological space. If $S,\ T : X \longrightarrow 2^Y$ are two set-valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say that S is a generalized KKM map w.r.t. T. The map $T : X \longrightarrow 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. T map S, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. We let

$$KKM(X,Y) = \{T : X \longrightarrow 2^Y : T \text{ has the KKM property}\}.$$

Comments: Similarly, the author defined generalized S-KKM map w.r.t. some T.

Zafarani [2004] — Liège73

Chang and Yen [1996] made a systematic study of the class of the KKM mappings: Let X be a nonempty convex subset of a topological vector space and Y a topological space. If $G : X \longrightarrow 2^Y,\ F : X \longrightarrow 2^Y$ are two multivalued maps such that for any $A \in \langle X\rangle$, $F(co(A)) \subseteq G(A)$, then G is said to be a generalized KKM mapping respect to F. Let $F : X \longrightarrow 2^Y$ be a multivalued mapping such that if $G : X \longrightarrow 2^Y$ is a generalized KKM mapping with respect to F, then the family $\{clG(x) : x \in X\}$ has the finite intersection property. In this case, we say that F has the KKM property. We define $KKM(X,Y) = \{F : X \longrightarrow 2^Y : T$ has the KKM property$\}$

Comments: Many results on the KKM theory on various types of spaces are introduced. Zafarani's Γ-convex spaces are motivated by our G-convex spaces and the same to our original abstract convex spaces in 2006. But he did not establish any theory on his spaces. He adopted a KKM theorem, a very particular form of our KKM theorems. Many terms in this paper are obsolete; for example, the KKM class, the S-KKM class, and the generalized S-KKM class belong to our $\mathfrak{KC}$ class in our abstract convex space theory. Zafarani introduced NR-metric spaces, which generalize hyperconvex metric spaces and are G-convex spaces.

Amini, Fakhar, and Zafarani [2005] — NA60

Let (M,d) be a metric space and X a subadmissible subset of M. A multifunction $G : X \multimap M$ is called a KKM mapping, if for each $A \in \langle X\rangle X,\ co(A) \subset G(A)$. More generally, if Y is a topological space and $G : X \multimap Y,\ F : X \multimap Y$ are two multifunctions such that for any $A \in \langle X\rangle,\ F(co(A)) \subseteq G(A)$, then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \multimap Y$ satisfies the requirement that for any generalized KKM mapping $G : X \multimap Y$ with respect to F the family $\{cl\, G(x) : x \in X\}$ has the finite intersection property, then F is said to have the KKM property. We define

$$KKM(X,Y) := \{F : X \multimap Y : F \text{ has the KKM property}\}$$

.

Comments: On the surface, this is a very nice paper. However, the authors adopted inadequate terminology of Chang-Yen. For example, the class of KKM type mappings is $\mathfrak{KC}$ in our works.

Fakhar and Zafarani [2005] — JOTA126

Let X be a convex subset of a t.v.s. E and let Y be a topological space. If $\Gamma : X \longrightarrow 2^Y,\ T : X \longrightarrow 2^Y$ are two multivalued mappings such that, for any $A \in \langle X\rangle,\ T(\mathrm{co}A) \subseteq \Gamma(A)$, then Γ is said to be a generalized KKM mapping with respect to T. Let $T : X \longrightarrow 2^Y$ be a multivalued mapping such that, if $\Gamma : X \longrightarrow 2^Y$ is a generalized KKM mapping with respect to T, then the family $\{\mathrm{cl}\Gamma(x) : x \in X\}$ has the finite intersection property; in this case, we say that T has the KKM property. Denote

$$KKM(X,Y) := \{T : X \longrightarrow 2^Y : T \text{ has the KKM property}\}.$$

Fakhar and Zafarani [2005a] — Belgium 12

Let $(X,D;\Gamma)$ be a G-convex space and Y be a topological space. A multivalued map $F : D \longrightarrow 2^X$ is called a KKM map if for each $A \in \langle D\rangle,\ \Gamma(A) \subset \bigcup_{x\in A} F(x)$. More generally if $G : D \longrightarrow 2^Y,\ F : X \longrightarrow 2^Y$ are two multivalued maps such that for any $A \in \langle D\rangle,\ F(\Gamma(A)) \subseteq G(A)$, then G is said to be a generalized KKM mapping with respect to F. Let $F : X \longrightarrow 2^Y$ be a multivalued mapping such that if $G : D \longrightarrow 2^Y$ is a generalized KKM mapping with respect to F, then the family

$\{clG(x) : x \in D\}$ has the finite intersection property. In this case we say that F has the KKM property. We define

$$\mathfrak{K}(X, Y) := \{F : X \longrightarrow 2^Y : F \text{ has the generalized KKM property}\}.$$

When X is a convex subset of a topological vector space, the class $\mathfrak{K}(X, Y)$ was introduced and studied by Chang and Yen [1996]. This concept is further extended to G-convex spaces by Lin, Ko, and Park [1998].

Comments: The notation $\mathfrak{K}$ was originally introduced by Park in 1997.

H. Kim [2005] — NA63

Let X be a convex subset of a vector space and Y a topological space. In 1996, Chang and Yen defined the following: A multimap $T : X \multimap Y$ is said to have the KKM property if, for any map $F : X \multimap Y$ with closed values satisfying

$T(coN) \subset F(N)$ for all $N \in \langle X \rangle X$,

the family $\{F(x)\}_{x\in X}$ has the finite intersection property. We denote KKM$(X, Y) := \{T : X \multimap Y \mid T$ has the KKM property$\}$.

Comments: This is the first paper assuming closed-valued multimaps F in KKM.

Chen [2006] —JMAA323

Comments: Motivated by Amimi-Fakhar-Zafarni [2005], Chen established some fixed point theorems with domain as a nearly-subadmissible subset of a complete metric space (M, d) for a k-set contraction map, which does not need to be a compact map. He also deduces a generalization of the approximate fixed point theorem for the lower semicontinuous mappings on a metric space.

He defines a generalized KKM mapping with respect to T, a slightly modified definition of Chang-Yen [1996].

The main result is the following fixed point theorem for the k-set contraction.

Theorem 4.1. *Let (M, d) be a complete metric space and X be a nonempty bounded nearly subadmissible subset of M. If $T \in KKM(X, X)$ is a k-set contraction, $0 < k < 1$ and closed with $T(X) \subset X$, then T has a fixed point in X.*

Since the KKM families $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ were introduced in 2003, $KKM(X, X)$ can be replaced by more correct $\mathfrak{KC}$.

Recall that any article adopting or imitating the *KKM* class of Chang-Yen [1996] is obsolete.

Jeng, Hsu, and Huang [2006]— JMAA319

Similar to Chang-Yen [1996] we now extend the concept of generalized KKM mapping in the following manner.

Definition 4.5. Suppose X and Y are two nonempty subsets of a linear space E, and $T,\ F : X \multimap Y$. We say that F is a generalized KKM mapping with respect to T if for any $A = \{x_1, \dots, x_n\} \in \langle X \rangle$ there is $B = \{y_1, \dots, y_n\} \in \langle X \rangle$ satisfying

(a) co$(B) \subseteq X$, and

(b) $T(co\{y_i : i \in I\}) \subseteq \bigcup_{i\in I} F(x_i)$ for any nonempty subset I of $\{1, \cdots, n\}$.

Definition 4.6. Let X and Y be two nonempty subsets of a topological vector space E. If a multifunction $T : X \multimap Y$ satisfies that for any generalized KKM mapping $F : X \multimap Y$ with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the KKM property. The class

KKM(X, Y) is defined to be the set

$$\{T : X \multimap Y : T \text{ has the KKM property}\}.$$

Kuo, Huang, Jeng, and Shih [2006]—FPTA2006

The concept of S-KKM property of Chang et al. [1999] can be extended to G-convex spaces.

Definition 4.7. Let X be a nonempty set, $(Y, D; \Gamma)$ a G-convex space and Z a topological space. If $S : X \multimap D,\ T : Y \multimap Z$ and $F : X \multimap Z$ are three multimaps satisfying

$$T(\Gamma_{S(A)}) \subseteq F(A)$$

for any $A \in \langle X \rangle$, then F is called a S-KKM mapping with respect to T. If the multimap $T : Y \multimap Z$ satisfies that for any S-KKM mapping F with respect to T, the family $\{\overline{F}(x) : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. The class S-KKM(X, Y, Z) is defined to be the set $\{T : X \multimap Y : T$ has the S-KKM property$\}$.

When $D = Y$ is a nonempty convex subset of a linear space with $\Gamma_B = co(B)$ for $B \in \langle Y \rangle$, the S-KKM(X, Y, Z) is just that as in Chang et al. [1999].

Comments: Now the S-KKM class is obsolete.

Shahzad [2006]—Simon Stevin

Definition 4.8. Let X be a nonempty set, Y a nonempty convex subset of a Hausdorff topological vector space and Z a topological space. If $S : X \longrightarrow 2^Y,\ T : Y \longrightarrow 2^Z,\ F : X \longrightarrow 2^Z$ are three set-valued maps such that $T(co(S(A))) \subseteq F(A)$ for each nonempty finite subset A of X , then F is called a generalized S-KKM map w.r.t. T . If the map $T : X \longrightarrow 2^Z$ is such that for any generalized S-KKM w.r.t. T map F, the family

$$\{\overline{F(x)} : x \in X\}$$

has the finite intersection property, then F is said to have the S-KKM property. The class

$$S - KKM(X, Y, Z) = \{T : Y \longrightarrow 2^Z : T\text{has the S-KKM property}\}.$$

Comments: Now the S-KKM class is obsolete.

Amini, Fakhar, and Zafarani [2007] — NA66

Like the work of Chang et al. [1999], we introduce the family of multifunctions with the S-KKM property as follows. Let Z be a nonempty set, $(X, \mathcal{C})$ an abstract convex space, and Y a topological space. If $S : Z \multimap X,\ F : X \multimap Y$ and $G : X \multimap Y$ are three multifunctions satisfying

$$F(co_{\mathcal{C}}(S(A))) \subseteq \bigcup_{x \in A} G(x)$$

for each $A \in \langle Z \rangle$, then G is called a $\mathcal{C}$-S-KKM mapping with respect to F. If the multifunction $F : X \multimap Y$ satisfies the requirement that for any $\mathcal{C}$-S-KKM mapping G with respect to F, the family $\{clG(x) : x \in X\}$ has the finite intersection property, then F is said to have the S-KKM property with respect to $\mathcal{C}$. We define

$$S - KKM_{\mathcal{C}}(Z, X, Y) := \{F : X \multimap Y : F \text{ has } S\text{-KKM property with respect to } \mathcal{C}\}.$$

Comments: In this paper, we notice the following: (1) Here abstract convex spaces mean spaces having the routine convexity structure, (2) Chang-Yen's KKM class

[1996] should be replaced by the $\mathfrak{KC}$ class. (3) The S-KKM class of Chang et al. [1999] is simply a $\mathfrak{KC}$ class.

Chen [2007] — Sci. Math. Jpn. 2007

A G-convex space $(X, D; \Gamma)$, where D is a nonempty subset of X, is called an L-convex space in this paper.

Definition 4.9. Let X be an L-convex space, Y a topological space such that for each $N \in \langle X \rangle$ with $|N| = n+1$, there exists a continuous mapping $\psi_N : \Delta_N \longrightarrow X$. If $T, F : X \longrightarrow 2^Y$ are two set-valued function satisfying that $T(\psi_N(\Delta_N)) \subset F(N)$ for each $N \in \langle X \rangle$ with $|N| = n+1$, then F is said to be a generalized R_{ψ_N}-KKM mapping with respect to T and ψ_N. Moreover, if the set-valued function $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized R_{ψ_N}-KKM mapping with respect to T and ψ_N the family $\{\overline{Fx} \mid x \in X\}$ has the finite intersection property, then T is said to have the R_{ψ_N}-KKM property. The class R_{ψ_N}-$KKM(X, Y)$ is defined to be the set $\{T : X \longrightarrow 2^Y \mid T$ has the R_{ψ_N}-KKM property$\}$. (* This ψ_N may be different from the ϕ_N of the definition for the L-convex space.)

Comments: In the above definition of a class R_{ψ_N}-$KKM(X, Y)$, the role of G-convex spaces (its author's L-convex spaces) is not clear. This remark also works all of Theorems 4–20 in this paper. Moreover, Theorems 21-23 in this paper are concerned with G-convex spaces and follow easily from the known results in the G-convex space theory.

Chen and Chang [2007] — JMAA329

Abstract: We first establish a fixed point theorem for a k-set contraction map on the family KKM(X, X), which not needs to be a compact map. Next, we establish the matching theorems, coincidence theorems and minimax theorems on the family KKM(X, Y) and the Φ-mapping.

Al-Thagafi and Shahzad [2008] — FPTA2008

Let $T : A \longrightarrow 2^B$. We say that (e) T is an $\mathfrak{A}_c^\kappa$-multimap if for every compact set K in A, there exists an $\mathfrak{A}_c$-multimap $f : K2^B$ such that $f(x) \subseteq T(x)$ for each $x \in K$, (f) T is a **K**-multimap (or Kakutani multimap) if T is upper semicontinuous with compact and convex values, (g) $S : A \longrightarrow 2^B$ is a generalized **KKM**-multimap with respect to T if $T(coD) \subseteq S(D)$ for each finite subset D of A, (h) T has the **KKM** property if, whenever $S : A \longrightarrow 2^B$ is a generalized **KKM** multimap w.r.t. T, the family $\{\overline{S(x)} : x \in A\}$ has the finite intersection property; (i) T is a **PK**-multimap if there exists a multimap $g : A \longrightarrow 2^B$ satisfying $A = \bigcup\{\operatorname{int} g^{-1}(y) : y \in B\}$ and $\operatorname{co}(g(x)) \subseteq T(x)$ for every $x \in A$.

Balaj [2008]—NA68

Definition 4.10. (See Chang-Yen [1996]) Let X be a convex subset of a topological vector space and Y be a topological space. If $S,\ T : X \multimap Y$ are two maps such that

$$T(coA) \subseteq S(A) \text{ for each nonempty finite subset } A \text{ of } X,$$

then S is said to be *generalized* KKM *w.r.t.* T. The map $T : X \multimap Y$ is said to have the KKM *property* if for each $S : X \multimap Y$ which is a generalized KKM map w.r.t. T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property.

We denote by KKM(X, Y) the family of maps having the KKM property.

Comments: The author still follows the work of Chang-Yen [1996].

Chang, Chen, and Huang [2008]—TJM12

Recently, Amini, Fakhar and Zafarani [2005] introduced the class KKM(X, Y) in metric space, and get some results about fixed point theorems and matching theorem. In this work, we use the conception of J. C. Jeng, H. C. Hsu and Y. Y. Huang [2006] to define the KKM family on metric space. We establish a generalized KKM theorem in a hyperconvex metric space, and then we use this theorem to get a fixed point theorem, the matching theorem, the coincidence theorem, minimax inequality theorems and the variational inequality theorems.

Suppose X is a bounded subset of a metric space (M, d). Then the admissible hull of X is defined by

$$ad(X) = \bigcap \{B \subset M : B \text{ is a closed ball in } M \text{ such that } X \subset B\}$$

Definition 4.11. Let X be a metric space, Y be a nonempty set, and Z be a hyperconvex metric space. If $T : X \longrightarrow 2^Z,\ F : Y \longrightarrow 2^Z$ are two set-valued mappings satisfying that for each $\{y_1, y_2, \ldots, y_n\} \in \langle Y \rangle$, there exists $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ such that $T(ad(\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\})) \subset \bigcup_{j=1}^{k} F(y_{i_j})$, for all $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, then F is called a generalized KKM mapping with respect to T. If the set-valued mapping $T : X \longrightarrow 2^Z$ satisfies the requirement that for any generalized KKM mapping $F : Y \longrightarrow 2^Z$ with respect to T, the family $\{\overline{F(y)} : y \in Y\}$ has the finite intersection property, then T is said to have the KKM property. We denote

$$KKM(X, Z) = \{T : X \longrightarrow 2^Z \,|\, T \text{ has the KKM property}\}.$$

Chang, Chen, and Peng [2008] — NA69

Definition 4.12. Let (M, d) be a metric space. A subset X of M is called admissible if it is an intersection of closed balls in M. The collection of all admissible subsets in M is denoted by $\mathcal{A}(M)$. The smallest admissible set containing a bounded subset X of M is called the admissible hull of X and denoted by ad(X). So

$$ad(X) = \bigcap_{x \in M} B(x, r_x(X)),$$

where $B(x, r_x(X))$ is the closed ball centered at x with radius $r_x(X) \geq 0$ and $r_x(X) = \sup\{d(x, y) : y \in X\}$.

Definition 4.13. Let M be a metric space and $X \subset M$. A set-valued mapping $F : X \longrightarrow 2^M$ is called a KKM map if

$$ad(\{x_1, x_2, \ldots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i),$$

for any $x_1, x_2, \ldots, x_n \in X$.

Definition 4.14. Let X be a metric space and Y a topological space. If $F, T : X \longrightarrow 2^M$ are two set-valued mappings such that for each $A \in \langle X \rangle,\ T(\text{ad}(A)) \subset F(A)$, then F is called a generalized KKM mapping with respect to T. If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping $F : X \longrightarrow 2^Y$ with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has finite intersection property, then T is said to have the KKM property. The class KKM(X, Y) is defined as the set $\{T : X \longrightarrow 2^Y \mid T \text{ has the KKM property}\}$.

Chen and Chang [2008] — NA69

We first define the generalized ${}_g$KKM mapping and the family ${}_g$KKM(X, Y).

Definition 4.15. Let X be a metric space, and Y a nonempty set. If $F : Y \longrightarrow 2^X$ is a set-valued mapping satisfying that for each $\{y_1, y_2, \ldots, y_n\} \in \langle Y \rangle$, there exists $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ such that $\text{ad}\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \subset \bigcup_{j=1}^k F(y_{i_j})$, for all $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, then F is called a generalized ${}_g$KKM mapping.

Definition 4.16. Let X be a metric space, Z a nonempty set, and Y a topological space. If $T : X \longrightarrow 2^Y$, $F : Z \longrightarrow 2^Y$ are two set-valued mappings satisfying that for each $\{z_1, z_2, \ldots, z_n\} \in \langle Z \rangle$, there exists $\{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ such that $T(\text{ad}(\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\})) \subset \bigcup_{j=1}^k F(z_{i_j})$, for all $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, then F is called a generalized ${}_g$KKM mapping with respect to T . If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized ${}_g$KKM mapping $F : Y \longrightarrow 2^Y$ with respect to T, the family $\{overline F(z) : z \in Z\}$ has the finite-intersection property, then T is said to have the ${}_g$KKM property. We denote

$$ {}_g\text{KKM}(X, Y) = \{T : X \longrightarrow 2^Y \mid T \text{ has the } {}_g\text{KKM property}\}. $$

Agarwal, Balaj, and O'Regan [2009] — Appl.Anal.88

We introduce the concept of a family of set-valued mappings generalized KKM w.r.t. other family of set-valued mappings. We then prove that if X is a nonempty compact convex subset of a locally convex Hausdorff topological vector space and $\mathcal{T}$ and $\mathcal{S}$ are two families of self set-valued mappings of X such that $\mathcal{S}$ is generalized KKM w.r.t. $\mathcal{T}$, under some natural conditions, the set-valued mappings $S \in \mathcal{S}$ have a fixed point. Other common fixed point theorems and minimax inequalities of Ky Fan type are obtained as applications.

The following close concept is due to Lin and Chang [1998]: if X is a nonempty set, Y is a convex subset of a vector space and S, $T : X \multimap Y$ are two set-valued mappings, S is called a T-KKM mapping if $\text{co}(\bigcup_{n=1}^n T(x_i)) \subseteq \bigcup_{n=1}^n S(x_i)$, for any nonempty finite subset $\{x_1, \ldots, x_n\}$ of X. Inspired by these concepts we introduce a new one, concerning two families of set-valued mappings.

Definition 4.17. Let X be a nonempty set, Y be a convex subset of a vector space and $\mathcal{T}$ and $\mathcal{S}$ are two families of set-valued mappings with nonempty values from X into Y. We say that $\mathcal{S}$ is *generalized KKM* w.r.t. $\mathcal{T}$ if for any nonempty finite subfamily $\{S_1, \ldots, S_n\}$ of $\mathcal{S}$ there exist $T_1, \ldots, T_n \in \mathcal{T}$ such that $\text{co}(\bigcup_{i \in I} T_i(x)) \subseteq \bigcup_{i \in I} S_i(x))$, for each nonempty subset I of $\{1, \ldots, n\}$ and for all $x \in X$.

Remark 4.18. If Y is a convex subset of a topological vector space and $\mathcal{S}$ is generalized KKM w.r.t. $\mathcal{T}$, then for each $x \in X$, $\{\overline{S(x)} : S \in \mathcal{S}\}$ has the finite intersection property.

Comments: No consideration on open-valued maps is given.

Amini-Harandi, Farajzadeh, O'Regan, and Agarwal [2009] — NFAA14

Definition 4.19. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$ F(\Gamma(A)) \subseteq G(A), \text{ for all } A \in \langle X \rangle, $$

then G is called a *KKM* map with respect to F. A *KKM* map $G : D \multimap Z$ is a KKM map with respect to the identity map 1_E. A multimap $F : E \multimap Z$ is said to have the KKM property if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{\overline{G(x)}\}_{x \in X}$ has the finite intersection property. We denote

$$ KKM(E, Z) := \{F : E \multimap Z : F \text{ has the KKM property}\}. $$

Comments: The authors adopt abstract convex (uniform) spaces due to Park, but still imitate the KKM class of Chang-Yen [1996].

Chen [2009] — NA71

The generalized KKM property on a convex subset of a Hausdorff topological vector space that was introduced by Chang and Yen [1996], we now extended this class KKM(X, Y) to be the class KKM$^*(X, Y)$ for the almost convex set X.

Definition 4.20. Let X be a nonempty almost convex subset of a topological vector space E, and Y a topological space. If $T,\ F : X \longrightarrow 2^Y$ are two set-valued mappings such that for each finite subset A of X and every neighborhood V of the origin 0 of E, there exists a convex-inducing mapping $h_{A,V} : A \longrightarrow X$ such that $T(co(h_{A,V}(A))) \subset F(A)$, then we call F a generalized KKM* mapping with respect to T.

If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM* mapping $F : X \longrightarrow 2^Y$ with respect to T, the family $\{\overline{Fx} : x \in X\}$ has the finite intersection property, then T is said to have the KKM* property. Denote

$$KKM^*(X, Y) = \{T : X \longrightarrow 2^Y \mid T \text{ has the KKM}^* \text{ property }\}.$$

Comments: Chang-Yen's class KKM(X, Y) is extended to the class KKM$^*(X, Y)$ for the almost convex set X. No consideration on open-valued maps is given.

Chen and Chang [2009] — FPTA2009

In 1996, Chang and Yen introduced the family KKM(X, Y) on the topological vector spaces and got results about fixed point theorems, coincidence theorems, and its applications on this family. Later, Amini et al. [2005] introduced the following concept of the KKM(X, Y) property on a subadmissible subset of a metric space (M, d).

Let X be an nonempty subadmissible subset of a metric space (M, d), and let Y a topological space. If $T,\ F : X \longrightarrow 2^Y$ are two set-valued mappings such that for any $A \in \langle X \rangle X,\ T((A)) \subset F(A)$, then F is called a generalized KKM mapping with respect to T. If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping F with respect to T, the family $\{overline F(x) : x \in X\}$ has finite intersection property, then T is said to have the KKM property. The class KKM(X, Y) is denoted to be the set $\{T : X \longrightarrow 2^Y : T$ has the KKM property$\}$.

Chen, Chang, and Chung [2009] — TJM13

Chang and Yen [1996] introduced the family KKM(X, Y), and got some results about fixed point theorems, coincidence theorems and some applications on this family. In this paper, we establish some coincidence theorems, generalized variational inequality theorems and minimax inequality theorems for the family KKM$^*(X, Y)$ and the generalized Φ-mapping on a nonconvex set.

Definition 4.21. Let X be a nonempty almost-convex subset of a topological vector space E, and Y a topological space. If $T,\ F : X \longrightarrow 2^Y$ are two set-valued mappings such that for each finite subset A of X and every neighborhood V of the origin 0 of E, there exists a convex-inducing mapping $h_{A,V} : A \longrightarrow X$ such that $T(\bar{(}h_{A,V}(A))) \subset F(A)$, then we call F a generalized KKM* mapping with respect to T.

If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM* mapping $F : X \longrightarrow 2^Y$ with respect to T, the family $\{\overline{Fx} : x \in X\}$ has the finite intersection property, then T is said to have the KKM* property. Denote

$$KKM^*(X,Y) = \{T : X \longrightarrow 2^Y \mid T \text{ has the KKM}^* \text{ property}\}.$$

Chang, Chen, and Chen [2010] — NA72

Abstract: We first define the family 2-$_g$KKM(X,Y) in a hyperconvex metric space, and then we get a 2-$_g$KKM theorem and a fixed point theorem without compactness assumption. Next, by using the 2-$_g$KKM theorem, we get the matching theorems, coincidence theorems, variational inequality theorems and minimax inequality theorems.

Definition 4.22. Let X be a metric space, Z a nonempty set, and Y a topological space. If $T : X \longrightarrow 2^Y,\ F : Z \longrightarrow 2^Y$ are two set-valued mappings satisfying that for each $\{z_1, z_2, \dots, z_n\} \in \langle Z\rangle$, there exists $\{x_1, x_2, \dots, x_n\} \in \langle X\rangle$ such that $T(ad(\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\})) \subset \bigcup_{j=1}^k F(z_{i_j})$, for all $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$, then F is called a generalized $_g$KKM mapping with respect to T . If the set-valued mapping $T : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized $_g$KKM mapping $F : Y \longrightarrow 2^Y$ with respect to T, the family $\{\overline{F(z)} : z \in Z\}$ has the finite intersection property, then T is said to have the $_g$KKM property. We denote

$${}_gKKM(X,Y) = \{T : X \longrightarrow 2^Y \mid T \text{ has the } {}_g\text{KKM property}\}$$

Chen, Chang, and Huang [2010] — AML23

Abstract: We use the conception of the abstract convexity to define the almost S-KKM$_c$ mapping, al-S-KKM$_c(X,Y,Z)$ family, and almost Φ-spaces. In the setting of the almost Φ-spaces, we establish some new fixed point theorems for the al-S-KKM$_c$ type set-valued mapping which is a generalized set contraction mapping. Our results generalize the results of Amini et al. [2007].

Khamsi and Hussain [2010] — NA73

Let M be a metric type space and X a subadmissible subset of M. A multifunction $G : X \longrightarrow 2^M$ is called a KKM mapping, if for each $A \in \langle X\rangle$, we have $\overline{A} \subset G(A) = \bigcup\{G(a), a \in A\}$. More generally, if Y is a topological space and $G : X \longrightarrow 2^Y,\ F : X \longrightarrow 2^Y$ are two multifunctions such that for any $A \in \langle X\rangle$, we gave $F(\bar{(A)}) \subset G(A)$, then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping $G : X \longrightarrow 2^Y$ with respect to F the family $\{\overline{G(x)}, x \in X\}$ has the finite intersection property, then F is said to have the KKM property. We define

$$KKM(X,Y) = \{F : X \longrightarrow 2^Y,\ F \text{ has the KKM property}\}$$

Turkoglua, Abuloha, and Abdeljawad [2010] — NA72

Let (M,d) be a cone metric space and X a subadmissible subset of M. A multifunction $G : X \longrightarrow 2^M$, is called a KKM mapping, if for each $A \in \langle X\rangle,\ Co(A) \subset G(A)$.

If Y is a topological space and $G : X \longrightarrow 2^Y,\ F : X \longrightarrow 2^Y$ are two multifunctions such that for every $A \in \langle X\rangle,\ F(Co(A)) \subset G(A)$, then G is called a generalized KKM mapping with respect to F. If the multifunction $F : X \longrightarrow 2^Y$ satisfies the requirement that for any generalized KKM mapping $G : X \longrightarrow 2^Y$ with respect to F, the family $\{cl(G(x) : x \in X)\}$ has the finite intersection property (f.i.p.), then

F is said to have the KKM property. We define

$$KKM(X,Y) = \{F : X \longrightarrow 2^Y : F \text{ has the KKM property}\}.$$

Balaj and Coroianu [2011] — BKMS48

Assume that X is a convex subset of a vector space and Y is a topological space. If $S,\ T : X \multimap Y$ are two set-valued mappings such that $T(coA) \subseteq S(A)$ for each nonempty finite subset A of X, then we say that S is a KKM mapping with respect to T. A set-valued mapping $T : X \multimap Y$ is said to have the KKM property if for any $S : X \multimap Y$, KKM mapping with respect to T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. Denote $\mathbf{KKM}(X,Y) = \{T : X \multimap Y : T$ has the KKM property$\}$.

Cho, Delavar, Mohammadzadeh, and Roohi [2011] — JIneqAppl2011

Let Y be a nonempty set, Z be a minimal space, $s : Y \longrightarrow D$ be a function and (X, D, Γ) be an abstract convex space. Let $T : X \multimap Z$ and $F : Y \multimap Z$ be two multimaps. we say that F is generalized s-KKM with respect to T if

$$T(\Gamma(s(A))) \subseteq F(A) \text{ for any } A \in \langle Y\rangle.$$

Comments: Minimal spaces can be made into topological spaces.

Chaipunya and Kumam [2013] — JIA2013

Definition 4.23. Let M be a circular metric space, X be a subadmissible subset of M and Y be a topological space. Let $F,\ G : X \multimap Y$ be two multivalued maps. If for each $A \in \langle X\rangle$ we have $F(ad(A)) \subset G(A)$, then G is said to be a generalized KKM map with respect to F.

Definition 4.24. Let M be a circular metric space, X be a subadmissible subset of M and Y be a topological space. A multivalued map $F : X \multimap Y$ is said to satisfy the KKM property if for any generalized KKM map $G : X \multimap Y$ with respect to F, the family $\{\overline{G(x)} : x \in X\}$ has the finite intersection property. In general, we write

$$KKM(X,Y) := \{F : X \multimap Y : F \text{ satisfy the KKM property}\}.$$

Fakhar, Lotfipour, and Zafarani [2013] — JGO55

We assume that X is a convex space, Y a Hausdorff topological space and Z a Hausdorff topological vector space. . . . Suppose that $K \subseteq X$ and $S : K \rightrightarrows X$ is a set-valued map, then S is called to be a KKM map if

$$convA \subseteq \bigcup_{x\in A} S(x), \text{ for each } A \in \langle K\rangle.$$

Let $T,\ H : X \rightrightarrows Y$ be set-valued maps. The set-valued map H is said to be a generalized KKM map with respect to (w.r.t.) T if $T(\text{conv}A) \subseteq H(A)$, for each $A \in \langle X\rangle$. The set-valued map T has the KKM property if the following statement is satisfied:

If $S : X \rightrightarrows Y$ is a generalized KKM map w.r.t. T, then the family $\{\text{cl}S(x) : x \in X\}$ has the finite intersection property. The family of all set-valued maps $T : X \rightrightarrows Y$ having the KKM property is denoted by $\text{KKM}(X,Y)$. The class $\text{KKM}(X,Y)$ was introduced and studied by Chang and Yen [1996].

Tang and Zhang [2014] — AAA2014

Definition 4.25. Let X be a nonempty set, Y a nonempty convex subset of a linear space, and Z a topological space, and let $S : X \longrightarrow 2^Y,\ T : Y \longrightarrow 2^Z$, and F :

$X \longrightarrow 2^Z$ be three multivalued mappings. F is said to be a $\mathfrak{A}$-KKM mapping with respect to T if, for any $\{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists $y_i \in S(x_i)(i = 0, 1, \ldots, n)$, such that, for any $\{y_{i_0}, \ldots, y_{i_k}\} \subset \{y_0, \ldots, y_n\}$, one has

$$T(co\{y_{i_0}, \ldots, y_{i_k}\}) \subset \bigcup_{j=0}^{k} F(x_{i_j}).$$

The multivalued mapping $T : Y \longrightarrow 2^Z$ is said to have the $\mathfrak{A}$-KKM property, if, for any $\mathfrak{A}$-KKM mapping F with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property. Let the set $\{T : T$ has the $\mathfrak{A}$-KKM property$\}$ be denoted by $\mathfrak{A}$-$KKM(X, Y, Z)$.

Comments: What is the role of $\mathfrak{A}$? Simply F could be closed-valued. Moreover, we can consider the case F is open-valued.

S. Huang [2021] — JNCA22(6)

Let M be a connected Riemannian manifold endowed with a Riemannian metric.

Definition 4.26. Let X and Z be nonempty sets, Y a convex set in M and $T : Y \longrightarrow 2^Z$ a multifunction. A multifunction $S : X \longrightarrow 2^Z$ is a generalized KKM mapping with respect to T if for any $\{x_1, \ldots, x_n\} \in \langle X \rangle$, there is $\{y_1, \ldots, y_n\} \in \langle Y \rangle$ such that

$$T(co\{y_i : i \in I\}) \subset \bigcup \{S(x_i) : i \in I\}, \quad \forall I \subset \{1, \ldots, n\},$$

If Z is a topological space, T is said to have the KKM property if for any generalized KKM mapping $S : X \longrightarrow 2^Z$ with respect to T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. The collection of all multifunctions $T : Y \longrightarrow 2^Z$ with KKM property is denoted by KKM(X, Y, Z).

Final Remark for Section 4. We can add up more and more examples on various types of modifications, imitations, or fake generalizations of G-convex spaces. But we will stop here.

5. On the classes $\mathfrak{KC}$ and $\mathfrak{KO}$ of multimaps

In 1987, W. K. Kim [1987] and Shih and Tan [1987] independently discovered that the KKM theorem also holds for open-valued multimaps; see also Lassonde [1990]. Consequently, all papers in Section 4 and many others can be modified to corresponding open-valued versions.

Moreover, the KKM class of multimaps can be extended to the $\mathfrak{KC}$-class of multimaps, and new $\mathfrak{KO}$-class and $\mathfrak{K}$-class are derived for abstract convex spaces due to Park in the new millennium.

In this section, we introduce articles concerning such new classes and related topics.

Park [1997] — NA30

Chang and Yen [1996] extended the class $\mathfrak{A}_c^\kappa$ to multimap class *KKM* having the KKM property and obtained some generalized results in the KKM theory and fixed point theory. We improve their definition as follows:

Let (X, D) be a convex space, Y a Hausdorff space, and $T : X \multimap Y$. We say that T has *the KKM property* provided that the family $\{Sx : x \in D\}$ has the finite intersection property whenever $S : D \multimap Y$ has closed values and $T(coN) \subset S(N)$

for each $N \in \langle D \rangle$. Let

$$T \in \mathfrak{K}(X,Y) \Longleftrightarrow T : X \multimap Y \text{ has the KKM property.}$$

We will denote their class by $\mathfrak{K}$.

Let X be a convex space and Y a Hausdorff space. In this paper, we define a new "better¡± admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X,Y) \Longleftrightarrow F : X \multimap Y$ such that, for any polytope P in X and any continuous map $f : F(P) \longrightarrow P$, $f(F|_P)$ has a fixed point.

Our new class contains the admissible class $\mathfrak{A}_c^\kappa$ due to the author and generalizes closed maps in KKM due to Chang and Yen [1996].

Comments: In this paper, we first used the notations $\mathfrak{K}$ and $\mathfrak{B}$. Later we changed their meanings. Note that our $\mathfrak{K}$ seems to be better than the KKM of Chang and Yen [1996].

Park [1997a] — MSR Hot-Line 1(9)

We give general Schauder type fixed point theorems for compact multimaps in the 'better' admissible class $\mathfrak{B}$ defined on admissible convex subsets (in the sense of Klee) of a topological vector space not necessarily locally convex. Our new theorems subsume a large number of particular forms, and generalize them in terms of the involving spaces and the multimaps as well. We apply our new results to condensing maps.

Park [1998] — JKMS35(4)

We give general fixed point theorems for compact multimaps in the 'better' admissible class $\mathfrak{B}^\kappa$ defined on admissible convex subsets (in the sense of Klee) of a topological vector space not necessarily locally convex. Those theorems are used to obtain results for Φ-condensing maps. Our new theorems subsume more than seventy known or possible particular forms, and generalize them in terms of the involving spaces and the multimaps as well. Further topics closely related to our new theorems are discussed and some related problems are given in the last section.

In 1998, we obtained the following:

Theorem 5.1. *Let E be a Hausdorff t.v.s. and X an admissible* (*in the sense of Klee*) *convex subset of E. Then any compact closed map $F \in \mathfrak{B}(X,X)$ has a fixed point.*

In [1998], it was shown that Theorem subsumes more than sixty known or possible particular cases and generalizes them in terms of the involving spaces and multimaps as well. Later, further examples of maps in the class $\mathfrak{B}$ were known.

Park [2003]— JNCA4

Let $(X,D;\Gamma)$ be a G-convex space and Y a topological space. A multimap $F : X \multimap Y$ is said to have the *KKM property* if, for any map $G : D \multimap Y$ with closed [open] values satisfying

$$F(\Gamma_A) \subset G(A) \text{ for all } A \in \langle D \rangle,$$

Some authors use the notation KKM(X,Y). Note that $1_X \in \mathfrak{K}(X,X)$. $\cdots$

From now on, $\mathfrak{KC}$ denote the class $\mathfrak{K}$ for closed-valued maps G, and $\mathfrak{KO}$ for open-valued maps G.

Park [2006] — NAF11

We introduce basic results in the KKM theory on abstract convex spaces and the map classes $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$. We study the nature of Kakutani type maps, $\mathfrak{B}$-maps, and $\mathfrak{KC}$-maps in G-convex spaces; and show that generalizations of the key results in known works are consequences of the G-convex space theory and the new abstract convex space theory.

Park [2007] — NAF12

We study the mutual relations among multimap classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ on abstract or generalized convex spaces. We show also that the examples given by Jeng, Huang, and Zhang [2002] can be used to deduce more examples of $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps. Finally, some historical remarks on classes $\mathfrak{KC}$, $\mathfrak{KO}$, and $\mathfrak{B}$ are added.

Park [2007a] — NAF12(2)

Our principal aim is to introduce basic results in the KKM theory on abstract convex spaces and the map class $\mathfrak{K}$ as in Park [2008b]. These are applied to simplify various modifications of the concept of generalized convex spaces. We discuss the nature of these modifications and criticize recently appeared so-called generalizations of our previous works due to other authors.

In Section 2, we introduce our new abstract convex spaces, KKM maps, and the map class $\mathfrak{KC}$ [or $\mathfrak{KO}$] in [2008b], and, in Section 3, a few basic theorems in our KKM theory for those spaces given there. Section 4 deals with KKM type theorems for G-convex spaces, which are shown to be easily deduced from our new results on abstract convex spaces. Sections 5-8 are devoted to various modifications of G-convex spaces and KKM type maps appeared in the 21st century. We show that most of them are mere modifications without having any proper example or any applicability. Such modifications are, for examples, L-spaces, generalized R-KKM maps, pseudo H-spaces, and others.

Park [2008b] — JKMS45(1)

Definition 5.1. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \text{ for all} A \in \langle D \rangle$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{K}$*-map* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$*-map* is defined for closed-valued maps G, and a $\mathfrak{KO}$*-map* for open-valued maps G. Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation KKM(E, Z) instead of $\mathfrak{KC}(E, Z)$.

Park [2010] — Tamkang41(1)

Abstract: Recent results in the KKM theory on abstract convex spaces and the related multimap classes $\mathfrak{KC}$ and $\mathfrak{KO}$ are applied to deduce generalizations of results on KKM maps in metric spaces in Amini et al. [2007] and generalized KKM theorems on hyperconvex metric spaces in Chang et al. in [2008, 2008a].

Yang, Huang, and Lee [2011] — TJM15(1)

Park [2006] introduced a new concept of abstract convex spaces which include convex subsets of topological vector spaces, convex spaces, C-spaces and G-convex spaces as special cases. Park [2006] also introduced certain broad classes $\mathfrak{RO}$ and $\mathfrak{RC}$ of maps (having the KKM property). The class $\mathfrak{RC}(X, Y)$ includes the well-known class KKM(X, Y) introduced by Chang and Yen [1996] as a special case. With these new concepts, he obtained some coincidence theorems and fixed point theorems in abstract convex spaces. Very recently, Park [2008a-d] further studied KKM theory in abstract convex spaces with applications to fixed points, maximal elements, equilibria problems and other problems.

Comments: $\mathfrak{R}$ should be $\mathfrak{K}$.

Yang and Huang [2012] — BKMS49(6)

A coincidence theorem for a compact $\mathfrak{RC}$-map is proved in an abstract convex space. Several more general coincidence theorems for noncompact $\mathfrak{RC}$-maps are derived in abstract convex spaces. Some examples are given to illustrate our coincidence theorems. As applications, an alternative theorem concerning the existence of maximal elements, an alternative theorem concerning equilibrium problems and a minimax inequality for three functions are proved in abstract convex spaces."

Recently, Park [2006] introduced a new concept of abstract convex spaces which include convex subsets of topological vector spaces, convex spaces, C-spaces and G-convex spaces as special cases. Park [2006] also introduced certain broad classes $\mathfrak{RO}$ and $\mathfrak{RC}$ of maps (having the KKM property), which includes the well-known class $KKM(X, Y)$ introduced by Chang and Yen [1996] as a special case. With these new concepts, some coincidence theorems and fixed point. theorems were proved in abstract convex spaces by Park [2006]. Very recently, Park [2008a-d] further studied KKM theory in abstract convex spaces with applications to fixed points, maximal elements, equilibria problems and other problems. It is noted that, in the KKM theory, there have appeared a number of coincidence theorems with many significant applications.

Lu and Hu [2013] — JFSA2013

The main purpose of this paper is to establish a new collectively fixed point theorem in noncompact abstract convex spaces. As applications of this theorem, we obtain some new existence theorems of equilibria for generalized abstract economies in noncompact abstract convex spaces.

Comments: The authors followed our works faithfully, but $\mathfrak{K}$, $\mathfrak{KC}$, $\mathfrak{KO}$ are denoted as $\mathfrak{R}$, $\mathfrak{RC}$, $\mathfrak{RO}$.

Lu, Zhang, and Li [2021] — AIMS Math.6(11)

Abstract: In this paper, by using the KKM theory and the properties of Γ-convexity and $\mathfrak{RC}$-mapping, we investigate the existence of collectively fixed points for a family with a finite number of set-valued mappings on the product space of noncompact abstract convex spaces.

6. Conclusion

In this paper, we followed the original sources faithfully.

Some one said that the progress of mathematics often follows a standard path: the discovery of a new theorem, followed by a systematic exploration of that theorem.

(1) Two standard ways of exploring theorems are by weakening the hypotheses and strengthening the conclusion. Here new hypothesis should be carefully and properly chosen.

(2) Another way is to find similar situation or similar hypotheses to known results and follow the same conclusion. Here we see certain parallelism.

Note that most of papers mentioned in Section 4 are of such types (1) or (2) and variants of the same ones for G-convex spaces.

The papers listed in Section 5 are relatively new and mainly related abstract convex spaces. However some of their authors misused $\mathfrak{R}$ instead of traditional $\mathfrak{K}$.

This survey is to help the authors to improve their works for the current KKM theory.

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HOMOCLINIC TRANSITION TO CHAOS IN THE DUFFING OSCILLATOR DRIVEN BY PERIODIC PIECEWISE LINEAR FORCES

S. VALLIPRIYATHARSINI[1], A. Z. BAZEERA[1], V. CHINNATHAMBI*[1] AND S. RAJASEKAR[2]

[1] Department of Physics, Sadakathullah Appa College, Tirunelveli 627 011, Tamilnadu, India
[2] School of Physics, Bharathidasan University, Tiruchirapalli 620 024, Tamilnadu, India.

ABSTRACT. We have applied the Melnikov criterion to examine a homoclinic bifurcation and transition to chaos in the Duffing oscillator driven by different forms of periodic piecewise linear forces. The periodic piecewise linear forces considered are triangular, hat, trapezium, quadratic and rectangular types of forces. For all the forces, an analytical threshold condition for the homoclinic transition to chaos is derived using Melnikov method and Melnikov threshold curves are drawn in a parameter space. Using the Melnikov threshold curves, we have found a critical forcing amplitude f_c above which the system may behave chaotically. We have analyzed both analytically and numerically the homoclinic transition to chaos in the Duffing system with ϵ-parametric force also. The predictions from Melnikov method have been further verified numerically by integrating the governing equation and finding areas of chaotic behaviour.

KEYWORDS:Duffing oscillator, Melnikov criterion, Piecewise linear force, Homoclinic bifurcation, Chaos.
AMS Subject Classification: :37D45, 34C37, 34D10, 34A08, 37J20, 37C29.

1. Introduction

The forced Duffing oscillator is a seminal system for the study of chaotic dynamics and development of analytical and experimental techniques for nonlinear systems [1-5]. The problem of its nonlinear dynamics has attracted researchers from various fields of research across natural science and physics [6-8], mathematics [9,10], mechanical engineering [11-13] and electrical engineering [14-16]. Melnikov method is a powerful analytical tool to provide an approximate criterion for the occurrence of hetero/homoclinic chaos in a wide class of dynamical systems [17]. It is also

* *Corresponding author.*
Email address : veerchinnathambi@gmail.com.
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an effective approach to detect chaotic dynamics and to analyze near homoclinic motion with deterministic or random perturbation. Usually this method is applied explicitly to systems which possesses homoclinic orbits in multi-well potential like Duffing double-well or pendulum systems [18,19]. Recently, this method has been applied to certain nonlinear systems to predict the occurrence of horseshoe chaos [20-25].

In recent years there has been a great deal of interest on the study of effect of various periodic forces in certain linear and nonlinearly damped systems [26-30]. In the present paper, we study both analytically and numerically the effect of various periodic piecewise linear forces in the Duffing oscillator equation

$$\ddot{x} + \alpha\,\dot{x}\, - \omega_0^2 x + \beta x^3 = F(t), \tag{1.1}$$

where α is the damping coefficient, ω_0 is the natural frequency and β is the constant parameter which plays the role of nonlinear parameter. $F(t)$ is an external time dependent periodic driving force. Recently, Baber Ahmad [31] studied both analytically and numerically the stabilization of the pendulum motion with different forms of periodic piecewise linear forces. Our objective here is to explore the possibility of homoclinic transition to chaos in Eq.(1.1) using both analytical and numerical techniques. In the present analysis, we use Melnikov analytical method to study the influence of different forms of periodic piecewise linear forces.

The paper is organized in the following way. In Section 2, we obtain the Melnikov threshold condition for the transverse intersections of homoclinic orbits for the system (1.1) separately for each of the above periodic piecewise linear forces. In Section 3, we analyze the homoclinic transition to chaos by plotting the Melnikov threshold curves in $(f-\omega)$ parameter plane where f and ω are the amplitude and frequency of the external periodic forces and numerically measuring the time τ_M elapsed between two successive transverse intersections for all the forces. We verify the analytical prediction with the numerically calculated critical values of f at which the transverse intersections of stable and unstable manifolds of the saddle occur. The Melnikov threshold value is also compared with the onset of asymptotic chaos wherever possible. In Section 4 we analyze the homoclinic transition to chaos in the system (1.1) driven by an ϵ-parametric control force. Finally Section 5 contains the concluding remarks.

2. Calculation of Melnikov function

We consider the perturbed Duffing equation with periodic piecewise linear forces in the form

$$\dot{x} = y \tag{2.1a}$$

$$\dot{y} = \omega_0^2 x - \beta x^3 + \epsilon[-\alpha\,\dot{x}\, + F(t)] \tag{2.1b}$$

where ϵ is a small parameter. The periodic piecewise linear forces of our interest are triangular, hat, trapezium, quadratic and rectangular types of forces. Figure 1 depicts the different forms of periodic piecewise linear forces.

The unperturbed system ($\epsilon = 0$) with the potential and Hamiltonian functions are given by

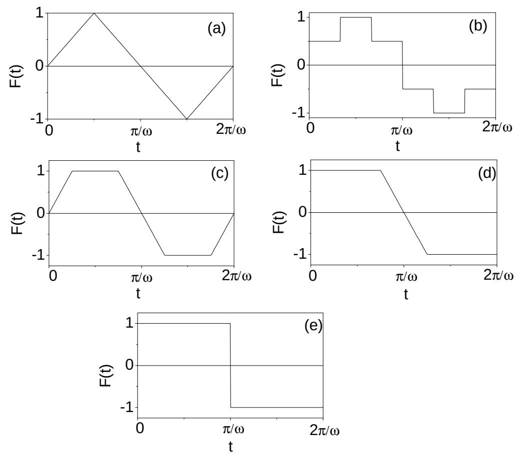

FIGURE 1. Form of various periodic piecewise linear forces (a) Triangular wave (b) Hat wave (c) Trapezium wave (d) Quadratic wave and (e) Rectangular wave. For all the forces period is $2\pi/\omega$, $\omega = 1$ and amplitude f is 1.0.

$$V(x) = -\frac{1}{2}\,\omega_0^2 x^2 + \frac{1}{4}\,\beta x^4. \tag{2.2}$$

$$H(x, y) = \frac{1}{2}\,y^2 - \frac{1}{2}\,\omega_0^2 x^2 + \frac{1}{4}\,\beta x^4. \tag{2.3}$$

Depending on the set of parameters, it can be considered at least three physically interesting situations where the potential is single-well, double-well and double hump-well. Throughout this paper, our analysis is of the double-well case. The unperturbed system has three fixed points. The origin is a saddle $(x^*, y^*) = (0, 0)$ and the other two fixed points are elliptic $(x^*, y^*) = (\pm\sqrt{\omega_0^2/\beta}, 0)$. The saddle point is connected to itself by two homoclinic orbits. The two homoclinic orbits connecting the saddle to itself are given by

$$W^{\pm}(x_h(\tau), y_h(\tau)) = \left(\pm\sqrt{\frac{2\,\omega_0^2}{\beta}}\ \mathrm{sech}\sqrt{\omega_0^2}\tau, \mp\omega_0^2\sqrt{\frac{2}{\beta}}\ \mathrm{sech}\sqrt{\omega_0^2}\tau\ \tanh\sqrt{\omega_0^2}\tau\right) \tag{2.4}$$

where $\tau = t - t_0$. The phase space motion of Eq.(2.1) is illustrated in Fig.2. The homoclinic orbits are indicated in it.

The Melnikov theory [9,12,17,32] allows us to calculate the Melnikov function $M(t_0)$ for a class of perturbed system for which homoclinic or heteroclinic orbit is known either analytically or numerically. $M(t_0)$ is proportional to the distance between the stable manifold (W_s) and unstable manifold (W_u) of a saddle. When the stable and unstable manifolds are separated then the sign of $M(t_0)$ always remain same. $M(t_0)$ oscillates when (W_s) and (W_u) intersects transversely (*horseshoe dynamics*). A zero of $M(t_0)$ corresponds to tangential intersections. The occurrence of transverse intersections implies that the Poincaré map of the system has the so called *horseshoe chaos*.

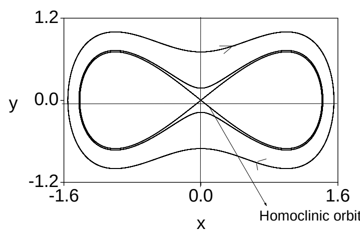

FIGURE 2. Phase trajectories of the unperturbed Duffing oscillator. The analytical expression for the homoclinic orbits is given by Eq.(2.4).

For Eq.(2.1), the Melnikov function is

$$M^{\pm}(t_0) = \int_{-\infty}^{+\infty} y_h \left[-\alpha\, y_h + F(\tau + t_0)\right]\, d\tau \tag{2.5}$$

In the following, we calculate the Melnikov function for the system (Eq.(2.1)) with different forms periodic piecewise linear forces. The period of all the forces are set to $T = 2\pi/\omega$.

2.1. **Triangular Type Force.** For the system (2.1) driven by Triangular type force,

$$F(t) = \begin{cases} \frac{4ft}{T}, & 0 \le t < \frac{T}{4} \\ -\frac{4ft}{T} + 2f, & \frac{T}{4} \le t < \frac{3T}{4} \\ \frac{4ft}{T} - 4f, & \frac{3T}{4} \le t < T, \end{cases} \tag{2.6}$$

where t is taken as mod $(2\pi/\omega)$. Its Fourier series is

$$F(t) = \frac{8f}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} \sin(2n-1)\omega t}{(2n-1)^2}. \tag{2.7}$$

Using Eq.(2.7), the Melnikov function is worked out as

$$M^{\pm}(t_0) \quad = \quad A \pm f \sum_{n=1}^{\infty} B_n \, \cos(2n-1)\omega t_0 \tag{2.8a}$$

where,

$$A \quad = \quad -\frac{4}{3}\frac{\alpha(\omega_0^2)^{3/2}}{\beta}, \tag{2.8b}$$

$$B_n \quad = \quad \frac{8\sqrt{2}\omega}{\pi\sqrt{\beta}} \frac{(-1)^{n+1}}{(2n-1)} \,\mathrm{sech}\left[\frac{(2n-1)\pi\omega}{2\sqrt{\omega_0^2}}\right] \tag{2.8c}$$

2.2. **Hat Type Force.** For the system (2.1) driven by Hat type force,

$$F(t) = \begin{cases} \frac{f}{2}, & 0 \le t < \frac{T}{6} \\ f, & \frac{T}{6} \le t < \frac{T}{3} \\ \frac{f}{2}, & \frac{T}{3} \le t < \frac{T}{2} \\ \frac{-f}{2}, & \frac{T}{2} \le t < \frac{2T}{3} \\ -f, & \frac{2T}{3} \le t < \frac{5T}{6} \\ \frac{-f}{2}, & \frac{5T}{6} \le t < T \end{cases} \tag{2.9}$$

Its Fourier series is

$$F(t) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \left(1 - \cos n\pi + 2\cos\frac{n\pi}{3}\right) \sin n\omega t \tag{2.10}$$

Using Eq.(2.10), the Melnikov function is worked out to be

$$\begin{aligned} M^{\pm}(t_0) &= A \mp f \sum_{n=1}^{\infty} C_n\, n\cos n\omega t_0 \pm f \sum_{n=1}^{\infty} D_n \left[\cos(n\pi + n\omega t_0) - \cos(n\pi - n\omega t_0)\right] \\ &\quad \mp f \sum_{n=1}^{\infty} E_n \left[\cos(n\pi/3 + n\omega t_0) - \cos(n\pi/3 - n\omega t_0)\right] \end{aligned} \tag{2.11a}$$

where,

$$C_n = \sqrt{2/\beta}\,\omega \text{ sech}\left[\frac{n\pi\omega}{2\sqrt{\omega_0^2}}\right] \tag{2.11b}$$

$$D_n = \frac{1}{\sqrt{2\beta}}\,\omega \text{ sech}\left[\frac{n\pi\omega}{2\sqrt{\omega_0^2}}\right] \tag{2.11c}$$

$$E_n = 2\sqrt{2\beta}\,\omega \text{ sech}\left[\frac{n\pi\omega}{2\sqrt{\omega_0^2}}\right] \tag{2.11d}$$

2.3. **Trapezium Type Force.** The mathematical representation for the Trapezium type force is

$$F(t) = \begin{cases} \frac{8ft}{T}, & 0 \le t < \frac{T}{8} \\ f, & \frac{T}{8} \le t < \frac{3T}{8} \\ \frac{8f}{T}(\frac{T}{2} - t), & \frac{3T}{8} \le t < \frac{5T}{8} \\ -f, & \frac{5T}{8} \le t < \frac{7T}{8} \\ \frac{8f(t-T)}{T}, & \frac{7T}{8} \le t < T. \end{cases} \tag{2.12}$$

Its Fourier series is

$$F(t) = \frac{16}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{4}\right) \sin n\omega t \tag{2.13}$$

Using Eq.(2.13), the Melnikov function is worked out to be

$$M^{\pm}(t_0) = A \mp f \sum_{n=1}^{\infty} F_n \left[\sin(n\pi/4 + n\omega t_0) - \sin(n\pi/4 - n\omega t_0)\right] \tag{2.14a}$$

where,

$$F_n = \frac{8\sqrt{2/\beta}\,\omega}{\pi} \frac{1}{n} \text{ sech}\left[\frac{n\pi\omega}{2\sqrt{\omega_0^2}}\right]. \tag{2.14b}$$

2.4. **Quadratic Type Force.** The mathematical expresssion for the Quadratic type force is

$$F(t) = \begin{cases} f, & 0 \le t < \frac{3T}{8} \\ \frac{8f}{T}(\frac{T}{2} - t), & \frac{3T}{8} \le t < \frac{5T}{8} \\ -f, & \frac{5T}{8} \le t < T. \end{cases} \tag{2.15}$$

Its Fourier series is

$$F(t) = \sum_{n=0}^{\infty} \left(\frac{2f}{n\pi} + \frac{8f}{n^2\pi^2} \sin\left(\frac{n\pi}{4}\right)\right) \sin n\omega t \tag{2.16}$$

Using Eq.(2.16), the Melnikov function is worked out to be

$$\begin{aligned} M^{\pm}(t_0) &= A \mp f \sum_{n=1}^{\infty} G_n \cos n\omega t_0 \mp \\ & \qquad f \sum_{n=1}^{\infty} H_n \left[\sin(n\pi/4 + n\omega t_0) - \sin(n\pi/4 - n\omega t_0)\right] \end{aligned} \tag{2.17a}$$

where,

$$G_n = 2\sqrt{2/\beta}\,\omega \operatorname{sech}\left[\frac{n\pi\omega}{2\sqrt{\omega_0^2}}\right] \tag{2.17b}$$

$$H_n = \frac{4\sqrt{2/\beta}\,\omega}{\pi} \frac{1}{n} \operatorname{sech}\left[\frac{n\pi\omega}{2\sqrt{\omega_0^2}}\right]. \tag{2.17c}$$

2.5. **Rectangular Type Force.** The mathematical representation for the Rectangular type force is

$$F(t) = \begin{cases} f, & 0 \le t < \frac{T}{2} \\ -f, & \frac{T}{2} \le t < T. \end{cases} \tag{2.18}$$

Its Fourier series is

$$F(t) = \frac{4f}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n-1)} \sin(2n-1)\omega t \tag{2.19}$$

Using Eq.(2.19), the Melnikov function is worked out to be

$$M^{\pm}(t_0) = A \mp f \sum_{n=0}^{\infty} I_n \cos(2n-1)\omega t_0 \tag{2.20a}$$

where,

$$I_n = 4\sqrt{2/\beta}\,\omega \operatorname{sech}\left(\frac{(2n-1)\pi\omega}{2\sqrt{\omega_0^2}}\right). \tag{2.20b}$$

From Eqs.(2.8), (2.11), (2.14), (2.17) and (2.20) we can obtain the condition for transverse intersections of stable manifolds ($W_s^{\pm}$) and unstable manifolds ($W_u^{\pm}$), that is, $M^{\pm}(t_0)$ to change sign at some t_0.

3. Homoclinic Transition to Chaos

In this section we analyze the occurrence of homoclinic transition to chaos in the system (2.1) driven by different forms of periodic piecewise forces. For our study we fix the values of the parameters in Eq.(2.1) as $\alpha = 0.5, \beta = 1.0, \omega_0^2 = 1.0$ and $\omega = 1.0$. We consider sufficiently large number of terms, say, 100 terms in the summation of equation for $M(t_0)$. Figure 3 shows the plot of f_M versus n, the number of terms in the summation (Eq.(2.8)) for the triangular type force. f_M converses to a constant value with increase in n. For $n > 5$, the variation of f_M is negligible. Similar result is found for other types of periodic piecewise linear forces also. Hence in our numerical calculation we fix $n = 50$. First we analyze the occurrence of homoclinic transition to chaos numerically by measuring the time τ_M elapsed between two successive change in the sign of $M(t_0)$. For a fixed value of f, t_0 is varied from 0 to $200T$, where $T = 2\pi/\omega$ is the period of the driving force. If the sign of $M(t_0)$ remains the same in this time interval then there is no zero of $M(t_0)$ and τ_M is assumed to infinity. τ_M can be determined from the Eqs.

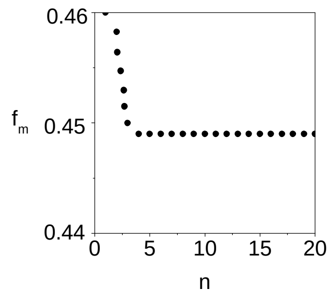

FIGURE 3. f_{M} versus n, the number of terms in the summation in Eq.(2.8a), for $\alpha = 0.5$, $\beta = 1.0$, $\omega_0^2 = 1.0$ and $\omega = 1$, when the system (2.1) is driven by the triangular type force. Variation in f_M converges to a constant value with increase in n.

(2.8),(2.11),(2.14),(2.17) and (2.20) for triangular, hat, trapezium, quadratic and rectangular types of forces. τ_M is calculated for a range of amplitude of the driving force. The value of f at which first intersection time of $M(t_0)$ changes sign and thereby giving finite τ_M is the Melnikov threshold value for homoclinic transition to chaos. Figure 4 shows the variation of $1/\tau_M$ versus f for the system (2.1) driven by different forms of periodic piecewise linear forces. Continuous curve represents the inverse of first intersection time ($1/\tau_{M^+}$) of stable and unstable branches of homoclinic orbits W^+. Dashed curve corresponds to the orbits of W^-. Homoclinic transition to chaos does not occur when $1/\tau_M$ is zero and it occurs in the region when $1/\tau_M > 0$.

In Fig.4(a), when the system (2.1) driven by triangular type force, both $1/\tau_{M^+}$ and $1/\tau_{M^-}$ are zero (that is $1/\tau_{M^\pm}$ are infinity) in the interval $0 < f < 0.45707$. This implies that homoclinic transition to chaos does not occur for $f < f_M^\pm = 0.45707$. For $f > f_M^\pm$=0.45707, both $M^+(t_0)$ and $M^-(t_0)$ oscillate and hence $1/\tau_M$ are nonzero. This implies that homoclinic transition to chaos is possible in this region. The Melnikov threshold values of the other types of periodic piecewise linear forces, namely, hat, trapezium, quadratic and rectangular types of forces are $f_M^\pm = 0.29206, 0.31680, 0.42605$ and 0.27948. From these values, it is observed that, among the five forces, f_M is maximum for the triangular type force and is minimum for the rectangular type force. Thus the onset of homoclinic transition to chaos can be either delayed or advanced by an appropriate choice of periodic piecewise linear forces.

Then we analyze the occurrence of homoclinic transition to chaos by plotting the threshold curves in the $(f - \omega)$ parameters plane also. The threshold curves for homoclinic transition to chaos in the (f, ω) plane for the different forms of periodic piecewise linear forces are shown in Fig.5. In the parameter region below the threshold curve no transverse intersection of stable and unstable manifolds of saddle occurs and above the threshold curve the transverse intersection of stable and unstable manifolds of the saddle occur. Just above the Melnikov threshold curve onset of cross-well chaos is expected. This figure clearly illustrates the effect of various periodic piecewise forces.

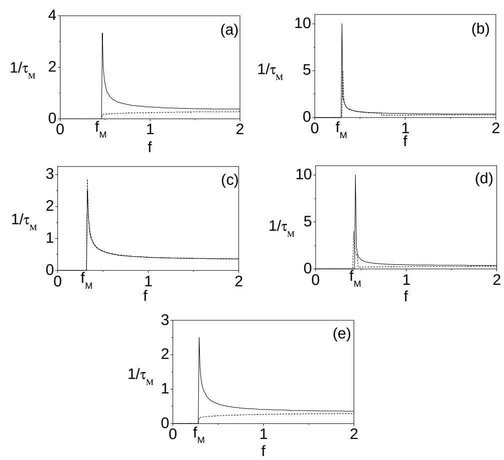

FIGURE 4. Variation of $1/\tau_M$ versus f for the system (2.1) driven by (a) triangular, (b) hat, (c) trapezium, (d) quadratic and (e) rectangular types of forces. The values of the other parameters are $\alpha = 0.5$, $\beta = 1$, $\omega_0^2 = 1$ and $\omega = 1$. Solid curve is for positive sign in $M(t_0)$ and dashed curve is for negative sign in $M(t_0)$.

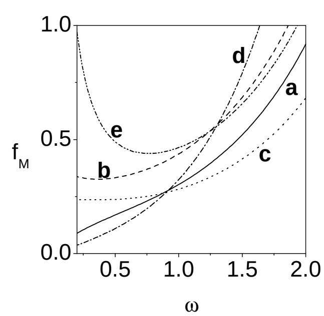

FIGURE 5. Melnikov threshold curves for horseshoe chaos in the $(f-\omega)$ plane for the system (2.1) driven by the forces (a) triangular, (b) hat, (c) trapezium, (d) quadratic and (e) rectangular types of forces. The values of the other parameters in Eq.(2.1) are $\alpha = 0.5$, $\beta = 1$ and $\omega_0^2 = 1$.

We have verified the analytical prediction by direct simulation of the system (1.1). In Fig.6 we plotted the orbits of the saddle for two values of the forces - one for $f < f_M$ and another for $f > f_M$. For clarity, only part of the manifolds are shown. The unstable manifolds are obtained by numerically integrating the Eq.(1.1) by the fourth-order Runge-Kutta method in the forward time for a set of 900 initial conditions chosen around the perturbed saddle point. The stable manifolds are obtained by integrating the the Eq.(1.1) in reverse time. In the left side subplots (Figs.6a,6c,6e,6g and 6i), $f < f_M$ the stable and unstable manifolds are well separated. That is homoclinic transition to chaos does not occurs in this region. In the right side subplots (Figs.6b,6d,6f,6h and 6j) for $f > f_M$, we can clearly

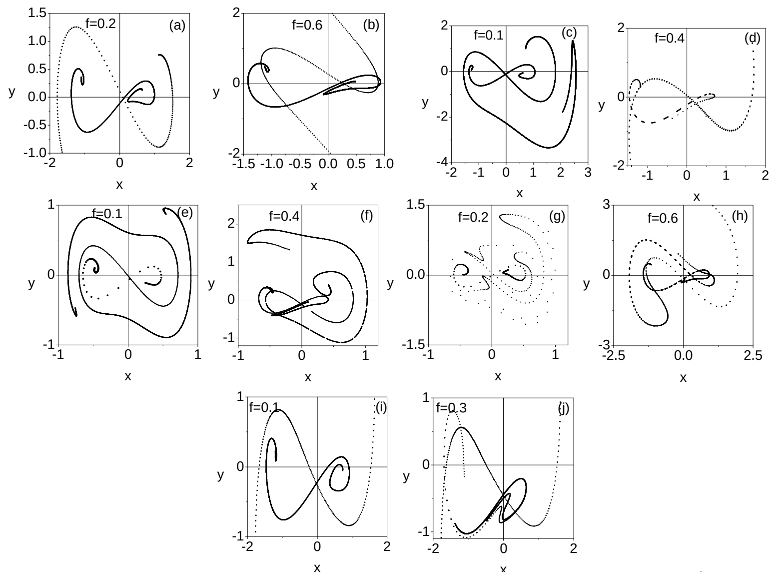

FIGURE 6. Numerically computed stable and unstable manifolds of the saddle fixed point of the system (2.1). The system is driven by (a-b) triangular, (c-d) hat, (e-f) trapezium, (g-h) quadratic and (i-j) rectangular types of forces. The values of the other parameters are $\alpha = 0.5$, $\beta = 1$, $\omega_0^2 = 1$ and $\omega = 1$. Left side subplots (Figs.6a,6c,6e,6g and 6i) are for $f < f_M$ while the right side subplots (Figs.6b,6d,6f,6h and 6j) are for $f > f_M$.

notice the transverse intersections of orbits at certain places. That is homoclinic transition to chaos is possible in this region.

4. Homoclinic Transition to Chaos due to an ϵ-Parametric Control Force

In the previous section, all the above results can be considered as nonparametric control force. In this section, an ϵ-parametric control force is defined one of the periodic piecewise linear force. This ϵ-parametric force with $0 < \epsilon < 1$ is given by (similar to quadratic type force (Eq.(2.15)),

$$F(t) = \begin{cases} f, & 0 \le t < \frac{1-\epsilon}{2}\, T \\ \frac{f}{\epsilon}(\frac{-2}{T}t + 1), & \frac{1-\epsilon}{2}T \le t < \frac{1+\epsilon}{2}\, T \\ -f, & \frac{1+\epsilon}{2}T \le t < T. \end{cases} \tag{4.1}$$

and is illustrated in Fig.7 with different values of ϵ, namely, $\epsilon = 0.17, 0.5, 0.7$ and 0.9. Its Fourier series is

$$F(t) = \sum_{n=1}^{\infty} \left(\frac{2f}{n\pi} + \frac{2f}{\epsilon\, n^2\pi^2}\, \sin(\epsilon n\pi) \right) \sin n\omega t \tag{4.2}$$

Using Eq.(4.2), the Melnikov function is worked out to be

$$M^{\pm}(t_0) \quad = \quad A \mp f \sum_{n=1}^{\infty} J_n \cos n\omega t_0 \ \mp$$

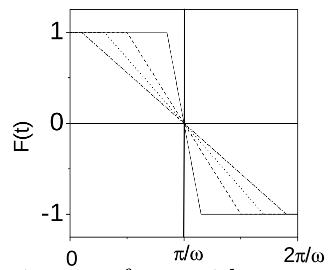

FIGURE 7. Form of Quadratic type force with $\epsilon = 0.17$(solid curve), $\epsilon = 0.5$(dashed curve), $\epsilon = 0.7$(dotted curve) and $\epsilon = 0.9$(dashed dot curve).

$$f \sum_{n=1}^{\infty} K_n \left[\sin(\epsilon n\pi + n\omega t_0) - \sin(\epsilon n\pi - n\omega t_0)\right] \tag{4.3a}$$

where,

$$J_n = 2\sqrt{2/\beta}\,\omega \operatorname{sech}\left[\frac{n\pi\omega}{2\sqrt{\omega_0^2}}\right] \tag{4.3b}$$

$$K_n = \frac{\sqrt{2/\beta}}{\epsilon n\pi}\,\omega \operatorname{sech}\left[\frac{n\pi\omega}{2\sqrt{\omega_0^2}}\right]. \tag{4.3c}$$

and the value of A is given in Eq.(2.8(b)). The occurrence of homoclinic transition

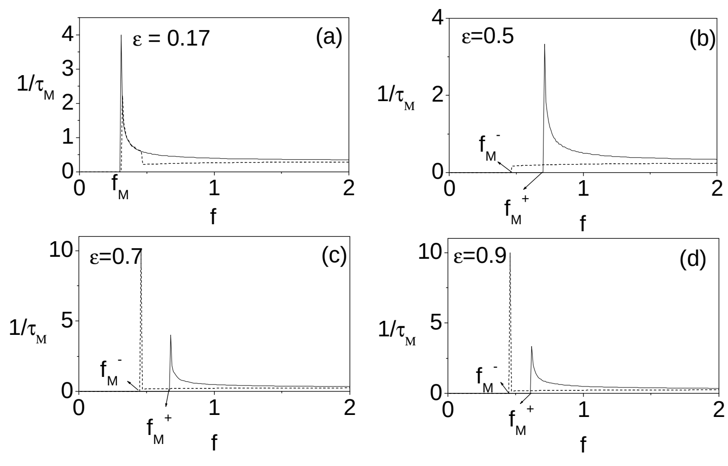

FIGURE 8. Variation of $1/\tau_M$ versus f for the system (2.1) driven by the $\epsilon-$ parametric force for four values of ϵ, namely, $\epsilon = 0.17, 0.5, 0.7, 0.9$. The values of the other parameters are $\alpha = 0.5$, $\beta = 1$, $\omega_0^2 = 1$ and $\omega = 1$. Solid curve is for positive sign in $M(t_0)$ and dashed curve is for negative sign in $M(t_0)$.

to chaos can be studied numerically measuring the time τ_M elapsed between two successive transverse intersections. τ_M can be determined from Eq.(4.3). Figure 8 shows the variation of $1/\tau_{M^\pm}$ versus f for four values of ϵ, namely, $\epsilon = 0.17, 0.5, 0.7$ and 0.9. Continuous curve represents the inverse of first intersection time $(1/\tau_M)$ of stable and unstable branches of homoclinic orbits W^+. Dashed curve corresponds to the orbits of W^-. Homoclinic transition to chaos does not occur when $1/\tau$ is zero and it occurs in the region $1/\tau > 0$. In Fig.8(a), for $\epsilon = 0.17$, both $1/\tau_{M^+}$ and $1/\tau_{M^-}$ are zero in the interval $0 < f < 0.29736$. This implies that homoclinic transition to chaos does not occur for $f < f_M^\pm = 0.29736$. For $f > f_M^\pm = 0.29736$, both $M^+(t_0)$ and $M^-(t_0)$ oscillate and hence $1/\tau_{M^\pm}$ are nonzero. This implies that homoclinic transition to chaos is possible in this region. For $\epsilon = 0.5$, $1/\tau_{M^+}$ and $1/\tau_{M^-}$ are zero for $f < f_M^+ = 0.70023$ and $f < f_M^- = 0.46077$ respectively. For f values in the interval $f_M^- < f < f_M^+$, $M_-(t_0)$ alone oscillate which implies that the homoclinic transition to chaos can take place only in the region $x < 0$. For $f \geq f^+$, homoclinic transition can occur in both regions $x < 0$ and $x > 0$. The Melnikov threshold values for $\epsilon = 0.7$ we find $f_M^+ = 0.66667$ and $f_M^- = 0.44897$; for $\epsilon = 0.9$ $f_M^+ = 0.60714$ and $f_M^- = 0.45068$.

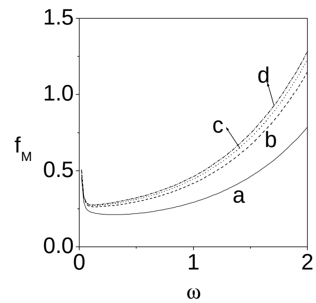

FIGURE 9. Melnikov threshold curves for horseshoe chaos in the $(f - \omega)$ plane for the system (2.1) driven by an $\epsilon-$ parametric force for four values of ϵ, namely, (a) $\epsilon = 0.17$, (b) $\epsilon = 0.5$, (c) $\epsilon = 0.7$ and (d) $\epsilon = 0.9$. The values of the other parameters in Eq.(2.1) are $\alpha = 0.5$, $\beta = 1$ and $\omega_0^2 = 1$.

The threshold curve for homoclinic transition to chaos in the (f, ω) plane for four values of ϵ, namely, $\epsilon = 0.17, 0.5, 0.7, 0.9$ is shown in Fig.9. Above the threshold curve, the system can transit to chaotic motions and below the threshold curve the system shows periodic behaviour. The results are also verified in Fig.10. The numerically computed W^s and W^u of the sadddle in the Poincar*é* map for $\epsilon = 0.5$ is shown in Fig.10. Transverse intersections of stable and unstable branches of both the homoclinic orbits are seen in Fig.10(c) for $f = 0.8$ (which is above the threshold value $f_M^+ = 0.70023$). For $\epsilon = 0.6$ (which is in between f_M^- and f_M^+), we see the transverse intersections of W_s^- and W_s^+ orbits alone at two places, which is clearly evident in Fig.10(b). The stable and unstable orbits are well separated in Fig.10(a) for $f = 0.2$ (which is below f_M^-). Homoclinic transition to chaos does not occur in this region. Similarly, we can verified the analytical results for other values of ϵ, namely, $\epsilon = 0.17, 0.7$ and 0.9. For different values of ϵ, the Melnikov threshold values of the system (2.1) driven by ϵ-parametric force is given in table 1. From table 1 we observed that the Melnikov threshold value of rectangular force falls

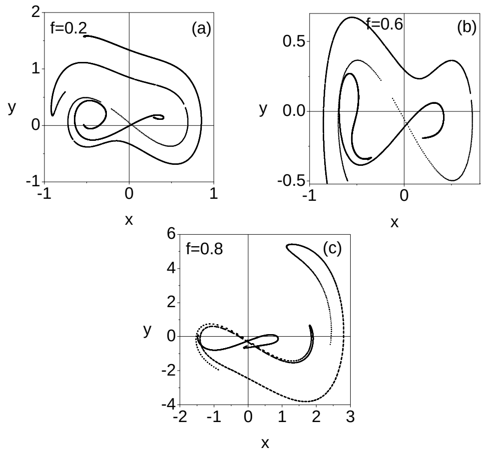

FIGURE 10. Numerically computed stable and unstable manifolds of the saddle fixed point of the system (2.1) driven by an $\epsilon-$ parametric force for three values of f with $\epsilon = 0.5$. The values of the other parameters are $\alpha = 0.5$, $\beta = 1$, $\omega_0^2 = 1$ and $\omega = 1$.

between $\epsilon = 0.2$ and $\epsilon = 0.1$. More clearly the parametric force with $\epsilon = 0.17...$ gives the Melnikov threshold value $(f_M^{\pm})$ almost equals with rectangular force. For $\epsilon \to 0$, ϵ-parametric force becomes quadratic type force.

TABLE 1. Melnikov threshold values of the system (2.1) with $\epsilon-$ parametric force for few values of ϵ with $\alpha = 0.5$, $\omega_0^2 = 1$, $\beta = 1.0$, and $\omega = 1.0$

	Melnikov Threshold values $(f_M^{\pm})$	
$\epsilon-$ values	f_M^-	f_M^+
0.1	0.17517	0.17517
0.17	0.29736	0.29736
0.2	0.17971	0.34977
0.3	0.44274	0.51077
0.4	0.46088	0.62415
0.5	0.46077	0.70023
0.6	0.46315	0.70805
0.7	0.44898	0.66667
0.8	0.44615	0.62302
0.9	0.45068	0.60714

5. CONCLUSION

The Melnikov method is sensitive to a global homoclinic bifurcation and gives a necessary condition for the occurrence of horseshoe chaos . Applying this method, we obtained the Melnikov threshold condition for onset of homoclinic transition to chaos that is transverse intersections of stable and unstable branches of homoclinic orbits. Threshold curves are drawn in a parameter space. These curves separating the chaotic and non-chaotic regions are obtained. Among the five periodic piecewise linear forces, f_M is maximum for the triangular type force. Also we have analyzed the occurrence of homoclinic transition to chaos in the Duffing system with an $\epsilon-$ parametric control force. The analytical results have been confirmed by numerical simulation. Numerical investigations including the computation of stable and unstable manifolds saddle and threshold curves are used to detect homoclinic transition to chaos. With the good agreement obtained between analytical and numerical predictions, we emphasize that the Melnikov analysis can be successfully used to predict the onset of chaos in the presence of weak periodic perturbation.

It is important to study the effect of nonlinear fractional damping in Duffing oscillator driven by periodic piecewise linear forces such as triangular, hat, trapezium, quadratic and rectangular types of forces. These will be investigated in future.

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A THREE LAYER SUPPLY CHAIN COORDINATION POLICIES FOR PRICE SENSITIVE AND EXPONENTIALLY DECLINING DEMAND WITH RECOMMENDED RETAIL PRICE BY MANUFACTURER

A. R. NIGWAL*[1], U. K. KHEDLEKAR[2] AND R. P. S. CHANDEL[3]

[1]Department of Mathematics, Ujjain Engineering Collage, Ujjain M.P. India
[2]Department of Mathematics and Statistics, Dr. Harisingh Gour Vishwavidyalaya, Sagar M.P. India (A Central University)
[3]Department of Mathematics, Model Science College Jabalpur M.P. India

ABSTRACT. The article develops an integrated supply chain coordination for multi-channel and multi-echelon supply, in which a single manufacturer, multiple non-competitive distributors, and non-competitive retailers are work together as members of the supply chain. The formulation of this model is based on deterministic exponential decreasing and price-dependent demand on the retailer's end. We formulated the model in two different scenarios, first, one is decentralized,d and the second one is centralized. The integrated profit function has been derived for each supply chain member, incorporating sharing holding costs among the distributors and retailers. We optimized selling price, economical order quantity, wholesale price, and profits for every echelon supply chain member in the finite and certain time horizon for decentralized and centralized scenarios respectively. Finally, we have done sensitivity analysis for some key parameters to examine their influence on the model's outputs. On the basis of numerical studies, we have also proposed managerial insights.

KEYWORDS:Inventory, holding cost, net profit, multi-channel multi-echelon supply chain, coordination.
AMS Subject Classification: :90B05, 90B30, 90B50.

1. Introduction

Due to globalization of market, growing of business competition, growth of population, awareness of consumer and legislative pressure have encouraged business industrialist and organizations to work together with their up stream, down stream members and customers. Furthermore better coordination among all upstream and

* *Corresponding author.*
Email address : arnw@rediffmail.com, arnwnigwal@gmail.com.
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downstream members, make the performance of entire supply chain is effective and efficient. Consequently collaboration and coordination among all echelon supply chain members are very important for an efficient supply chain. Otherwise due to the lack of coordination among each members they would optimizes its own local objectives ignoring impact on whole supply chain, may causes of lower profit.

In the previous decades, a several number of articles have been developed on supply chain management. Some relevant articles are mentioned here. Parlar and weng [17]applied the quantity discount policy in a supplier relationship with considering linear demand of buyer. They analyzed that quantity discount scheme can be very useful for obtaining more profit. Weng [30] presented a model in which they determine optimal lot size, optimal quantity discount policies. Also they analyze the effects of quantity discount on increasing demand of consumer. For development of models, generally price demand relationship are used by authors but Lau and Lau[2] developed a inventory model by considering different demand curve functions and investigate the effects on the single echelon supply chain system along with multi echelon system.

Most of the previously published literatures had adopted trade credit policy among suppliers and retailers only but Huang [29] adopted trade credit policy among not only suppliers and retailers but also retailers and customers in their supply chain model. Change *et al.*[8] developed an inventory model for deteriorating items assuming that a suppliers offers to purchaser a permission of delay in payment when the purchased order quantity is large. Chunge and Liao[13] developed a supply chain model based on trade credit period for exponential deteriorating items in which they assumed the condition that the supplier offers permission of delay in payment which is depends on order quantity.

Karim And Suzuki [18] provided a literature review on warranty claim data analysis in following topics:

- **(1):** Age based claims analysis,
- **(2):** Aggregated warranty claim analysis,
- **(3):** Two dimensional warranty cost analysis,
- **(4):** Warranty cost analysis etc.

Li and Liu, [12] developed optimal supply chain coordination using optimal quantity discount scheme considering probabilistic demand of single product in multiple time interval. Ding and Chen [9] developed a three layer supply chain model for short life cycle product. They highlighted, the coordination issues of three layer supply chain and suggested that three layer supply chain can be fully coordinated with certain contract of revenue sharing among manufacturer and supplier as long as supplier and retailers. Cachon and Lariviere [10] provided a two layer supply chain model with revenue sharing contracts. In this study it has been assumed that retailer's have to pay not only a wholesale price per unit of product to supplier but also pay a fixed percentage of revenue.

Crook Russel and Combs [26] suggested in their inventory model that collaborative environment in supply chain management create much better platform for each supply chain members to grow. They analyzed in this study that, how a weak member is benefited from strong a member in collaborative supply chain management.

Jain *et al.*[11] developed literature review on supply chain management and focused some issues on supply chain management. They gave a classification of more than 5889 published articles and try to find the status of literature on supply chain management. Kadadevaramath *et al.*[24] developed three layer supply chain coordination model by using four particle swarm optimization algorithm. They optimize their objectives by using the following various limitations:

(1): Ordering capacity of vendor
(2): Production capacity of plant
(3): Demand depends on various parameters etc.

Barron *et al.* [14] modified the model of Kadadevaramath *et al.*[24]. In this study optimality can be optimized by using integer linear programing solver technique in place of particle swarm optimization algorithm Kadadevaramath*et al.*[24]. Further they removed the following limitations of Kadadevaramath *et al.*[24] model:

(1): single product model is converted into multiple products model,
(2): single time interval is converted into multiple interval.

Barron*et al.*[15] proposed a vendor managed EOQ inventory model for multi products in which they considered multiple restrictions for optimizing total cost. It is more advance in the following three aspects than previously published works:

(1): The total cost is less than recently research work,
(2): The number of evaluations of the total cost function is less than recently research work,
(3): Computational time is less than recently research work.

Daya *et al.*[16] developed a three stage supply chain model, which formed by single supplier single manufacturer and multiple retailers. In this study they proposed a derivative free solution procedure to derive a optimal solution considering all inputs are constant. They optimized setup cost, holding cost, raw material cost and ordering cost along with the profits of each echelon member. Sarkar and Majumdar [7] developed integrated supply chain coordination for vendor and buyer, based on the following two different approaches:

(1): demand is a function of lead time which depends on probability distribution,
(2): demand is free from lead time.

They optimizes lead time and ordering cost for buyer and reorder point and setup cost for vendor. They also suggested that discrete investment gives better results instead of continuous investment and it may be reduce the setup cost. Modak *et al.*[20] presented two layer dual-channel supply chain, incorporating social responsibility in two different scenario first one is centralize and another one is decentralize. The development of this study is based on the following two different approaches:

(1): price dependent retail demand function,
(2): price dependent e-tail demand function.

After investigation they suggested in the centralized scenario model outputs are better than decentralized.

Pal *et al.*[5] proposed three layer production inventory model considering with three stage credit policy in which supplier provides the certain credit period to manufacturer, manufacturer a provide certain credit period to retailer and retailer also offers credit period to customers. They optimized replenishment lot size, and production rate for manufacturer. Sana [25] presented a three stage supply chain production inventory model which contains, a supplier, a manufacturer and a retailer. During the production he assume that perfect and imperfect both items are produced. They optimized production rate and replenishment rate per unit time for maximization of average profit. Pal *et al.*[6] developed perfect and imperfect three layer production inventory model consisting supplier, manufacturer and retailer. They assumed that the imperfect products are reworkable and rework process is started after end of regular production. They optimized order size of raw material, production rate per unit, production cost per unit and lead time.

Zhao and Chen[23] focused on the pricing strategies of a two-echelon supply chain for single manufacturer and two retailers. They developed price decision model considering the sensitivity of the retail quantity to the wholesale price of manufacturer and sales prices of the retailers. Khedlekar *et al.*[27] developed a production inventory model for deteriorating items. For this they designed two cases, first one is production without disruption and another one is production with disruption system allowing with shortage. Khedlekar *et al.*[28] developed continuous two layer supply chain inventory model by considering price and stock dependent demand for deteriorating items. Revenue sharing on preservation technology are also considered by authors.

Modak *et al.*[21] proposed a two layer supply chain formed by single manufacturer and single retailer for single product. They consider demand function as a function of quality, warranty, and sales price of the product. They optimized profit functions of the manufacturer and retailer under two the scenarios, centralized and decentralized. Nigwal *et al.*[4] developed a three layer multi channel reverse supply chain inventory model for used product in which a single remanufacturer multi-collector and multi-retailer work together as supply chain members. Gupta *et al.*[19] developed an imperfect production inventory model in which they consider imperfect production with and without disruption allowing with and without shortage. Modak *et al.*[22] developed a multi-channel, multi-echelon inventory model for single product incorporating single manufacturer more than one retailers and distributors as the members of the chain. The profit functions of each members have been formulated and optimized. The formulation of profit functions are based on demand of retailer's end.

In this model we considered a three layer multi-channel and multi-echelon supply chain model consisting a single manufacturer, more than one distributors and retailers. It is shown in the Figure (1). At starting the manufacturer provides the fixed lot size of the products to k^{th} (k=1 2...n) distributors and k^{th} (k=1 2...n) distributors supplies the products to jk^{th} (j=1 2 3....n_k), (k=1 2....n) retailers, where each retailer is associated to a certain distributor according to the geographical conditions. Since requisition of products is to be made at retailer's end therefor the total demand of all retailers is fulfilled by all distributors and the total demand of all distributors are fulfilled by the single manufacturer. Manufacturer and distributors assimilates EOQ delivery policy. In this paper we considered random order cycle time for manufacturer which is equally applicable for all distributors as well as all retailers.

The objective of this research is to find optimal retail price, initial order size for retailers in decentralized and centralized situation considering retailer's price sensitive and time dependent demand with sharing holding cost. We will also determine which coordination policy can be adopted that maximize model outputs.

2. Notations and Assumptions

Following notations are used in this model.

p^m : Maximum retail price determined by manufacturer,

D^r_{jk} : jk^{th}retailer's demand (per unit time) depends on retailer's price and time t in decentralized policy,

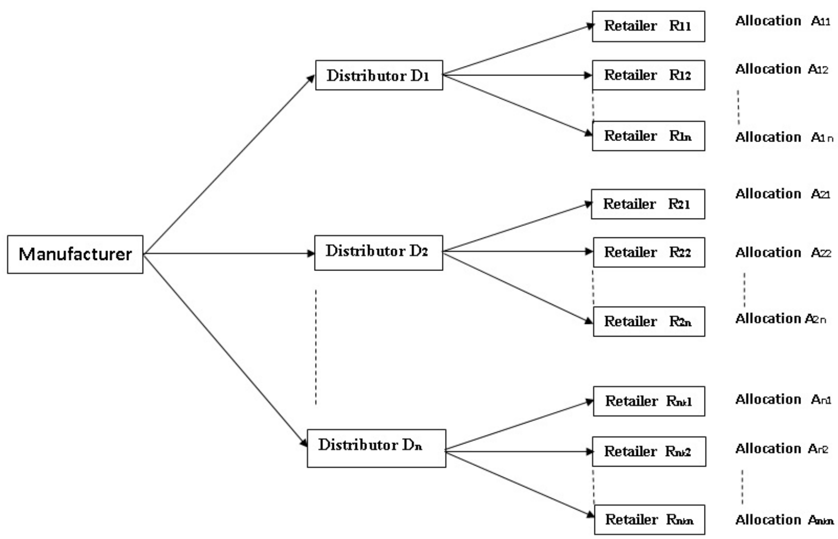

Figure 1. Supply Chain Distribution Network

D_k^d : k^{th} distributors's demand (per unit time) in decentralized policy,
D^m : Manufacture's demand (per unit time) in decentralized policy,
p_{jk}^r : Retailing price per unit product of jk^{th} retailer in decentralized policy,
p_{jk}^{rc} : Retailing price per unit product of jk^{th} retailer in centralized policy,
w_k^d : k^{th} Distributor's wholesale price per unit product in decentralized policy,
w^m : Manufacturer's wholesale price per unit product in decentralized policy,
c : Production cost per unit product,
NP_{jk}^r : Net profit of jk^{th} retailer in decentralized policy,
NP_k^d : Net profit of k^{th} distributor in decentralized policy,
NP^m : Net profit of manufacturer in decentralized policy,
n : Total number of distributors,
n^r : Total number of retailers,
NP^c : Net profit of whole channel in centralized policy,
β : Difference coefficient of $(p_{jk}^r - p^m)$ which may be positive or negative,
η : Price sensitive factor of demand function,
T : Total time horizon,
Q_{jk}^r : Initial demand of jk^{th} retailer,
Q_k^d : Initial demand of k^{th} distributor,
Q^m : Initial demand of manufacturer,
λ : Sharing coefficient of holding cost,
h : Holding cost per unit per unit time.

Assumptions: The following assumptions are made in this model

- Demand of product in the market is D_{jk}^r at the rate per unit time t; where $D_{jk}^r = a_{jk}e^{-\alpha t} - \eta p_{jk}^r + \beta(p^m - p_{jk}^r)$, is nonnegative exponential function of t and p_{jk}^r, where a_{jk} is demand scale parameter, β is difference coefficient of p^m and p_{jk}^r, $\alpha > 0$, $a_{jk} > 0$, $\beta > 0$, $\eta > 0$, and $0 \leq t \leq T$,
- Holding cost is constant and it is shared by retailers and distributors,
- The lead time is zero, and replenishment rate is infinite, however the planning horizon is finite,

- $a_k = \sum_{j=1}^{n_k} a_{jk}$ and $a = \sum_{j=1}^{n_k} \sum_{j=1}^{n} a_{jk}$,
- There is no competitive environment between retailers and distributors because each retailer's and distributors are associated according different geographical areas.
- We used the forward and backward substitution method to find the optimal decision variables.

3.

The study has been developed under the following two situations:

3.1. **Decentralized Policy.** In this scenario the all supply chain members are independent to take their decisions to optimize their objectives and manufacturer is a leader of supply chain. Therefore, firstly manufacturer announce the wholesale price of product, and letter distributors and retailers optimize their decision variables. Formulation of model is based on deterministic demand of retailer's end. Therefore firstly proposed model of retailer could be formulated as

3.1.1. *Mathematical Model for Retailers.* Since manufacturer manufactures the product, he absolutely knows all those cost which are related to the production. Therefore manufacturer can lead the supply chain of the product and also determine the maximum retail price at which the product is expected to be sold. This retail price of the product is called manufacturer's determined retail price (MDRP). The MDRP generally printed on the packet or tag of the product. It can be easily searched by the customer. In generally according to the market conditions consumers are satisfied or dissatisfied with MDRP.Initially we assume that the certain lot of product is distributed by manufacturer to n distributors d_1, d_2, d_3,...d_n. Distributors d_1, d_2, d_3,...d_n supply certain lot of the products to n_1 n_2 n_3...n_k retailers respectively. As per assumptions $jk^{th}t$ retailer receives the stock, at time t, $0 \le t \le T$. The rate of changes in the jk^{th} retailer's inventory level is balanced by demand. At any time t the following nonlinear equation may represent the inventory status of jk^{th} retailer

$$\begin{aligned} \frac{dI^r_{jk}(t)}{dt} &= -D^r_{jk}, \quad \text{where} \quad 0 \le t \le T. \\ &= -\left(a_{jk}e^{-\alpha t} - \eta p^r_{jk} + \beta(p^m - p^r_{jk})\right), \end{aligned} \tag{3.1}$$

where $j = 1, 2, 3, ..., n_k$ and $k = 1, 2, 3, ..., n$, with boundary condition $I^r_{jk}(t)$ =0, at $t = T$. solution of equation (3.1) gives

$$I^r_{jk}(t) = \frac{a_{jk}}{\alpha}(e^{-\alpha t} - e^{-\alpha T}) + (\eta + \beta)p^r_{jk}(t - T) + \beta p^m(T - t) \tag{3.2}$$

The initial inventory level $I^r_{jk}(0)$ for jk^{th} retailer at time $t = 0$, where $t \in [0, T]$ is

$$I^r_{jk}(0) = Q^r_{jk} = \frac{a_{jk}}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta)p^r_{jk}T + \beta p^m T \tag{3.3}$$

The sales revenue SR^r_{jk} in replenishment time period $[0, T]$ can be formulated as

$$SR^r_{jk} = \int_0^T p^r_{jk} D^r_{jk} dt$$

$$SR^r_{jk} = p^r_{jk}\left(\frac{a_{jk}}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta)p^r_{jk}T + \beta P^m T\right) \tag{3.4}$$

Purchase cost PC^r_{jk} of jk^{th}retailer can be formulated as

$$PC^r_{jk} = \int_0^T w^d_k D^r_{jk} d$$

$$PC^r_{jk} = w^d_k \left(\frac{a_{jk}}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta)p^r_{jk}T + \beta p^m T \right) \tag{3.5}$$

The inventory holding cost IHC^r_{jk} per unit of per unit time of jk^{th}retailer is

$$IHC^r_{jk} = h \int_0^T I^r_{jk}(t)dt$$

$$IHC^r_{jk} = h \int_0^T \frac{a_{jk}}{\alpha}(e^{-\alpha t} - e^{-\alpha T}) + (\eta + \beta)p^r_{jk}(t - T) + \beta p^m(T - t)dt \tag{3.6}$$

$$IHC^r_{jk} = h \left(\frac{a_{jk}}{\alpha^2}(1 - e^{-\alpha t} - T\alpha e^{-\alpha T}) - (\eta + \beta)p^r_{jk}\frac{T^2}{2} + \beta p^m \frac{T^2}{2} \right) \tag{3.7}$$

The net profit of jk^{th} retailer must be after subtraction of purchasing cost and sharing holding costs from sales revenue. Hence the net profit function NP^r_{jk} of jk^{th} retailer is

$$\begin{aligned} NP^r_{jk} = {} & (p^r_{jk} - w^d_k)\left[\frac{a_{jk}}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta)p^r_{jk}T + \beta p^m T\right] \\ & - h\lambda\left[\frac{a_{jk}}{\alpha^2}(1 - e^{-\alpha t} - T\alpha e^{-\alpha T}) - (\eta + \beta)p^r_{jk}\frac{T^2}{2} + \beta p^m \frac{T^2}{2}\right] \end{aligned} \tag{3.8}$$

According to the Taylor's theorem for small value of α the exponential function e^{-at} can be approximated by $1 - \alpha T + \frac{\alpha^2 T^2}{2}$ i.e $e^{-\alpha T} \approx 1 - \alpha T + \frac{\alpha^2 T^2}{2}$. Substituting the approximated value into the equation (3.8) we have

$$\begin{aligned} NP^r_{jk} = {} & (p^r_{jk} - w^d_k)\left[a_{jk}\left(T - \frac{\alpha T^2}{2}\right) - (\eta + \beta)p^r_{jk}T + \beta p^m T\right] \\ & - h\lambda\left[\frac{a_{jk}}{2}(T^2 - \alpha T^3) - (\eta + \beta)p^r_{jk}\frac{T^2}{2} + \beta p^m \frac{T^2}{2}\right] \end{aligned} \tag{3.9}$$

Proposition 3.1. *The optimal selling price p^{r*}_{jk} of jk^{th} retailer associated with k^{th} distributor's wholesale price w^d_k is p^{r*}_{jk} and where*

$$p^{r*}_{jk} = \frac{w^d_k}{2} + \frac{\beta p^m}{2(\eta + \beta)} + \frac{\lambda h T}{4} + \frac{a_{jk(1-e^{-\alpha T})}}{2\alpha(\eta + \beta)T} \tag{3.10}$$

Proof. At an optimal point, NP^r_{jk}, $\frac{\partial NP^r_{jk}}{\partial p^r_{jk}}$ must vanish i.e.

$$-(p^r_{jk} - w^d_k)(\eta + \beta)T + \left[\frac{a_{jk}}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta)p^r_{jk}T + \beta p^m T\right] + h\lambda(\eta + \beta)\frac{T^2}{2} = 0 \tag{3.11}$$

optimal value of p^r_{jk} is given by following equation

$$p^r_{jk} = \frac{w^d_k}{2} + \frac{\beta p^m}{2(\eta + \beta)} + \frac{\lambda h T}{4} + \frac{a_{jk}(1 - e^{-\alpha T})}{2\alpha(\eta + \beta)T} \tag{3.12}$$

□

Proposition 3.2. *NP^r_{jk} shows concavity in p^r_{jk} if $\eta > 0$ and $\beta > 0$.*

Proof. Second order partial derivatives of NP^r_{jk} in p^r_{jk} is

$$\frac{\partial^2 NP^r_{jk}}{\partial p^{r2}_{jk}} = -2(\eta + \beta)T. \tag{3.13}$$

Hence NP^r_{jk} is a concave function in p^r_{jk} if $\eta > 0$ and $\beta > 0$.
By using backward substitution method the optimal demand of the product at jk^{th} (j=1 2 3 ...n_k ,k=1 2 3 ...n) retailer's fag end is

$$D^{r*}_{jk} = a_{jk}e^{-\alpha T} - \frac{c(\eta+\beta)}{8} - \frac{(1-e^{-\alpha T})}{2\alpha T}\left(\frac{a}{4n^r} + \frac{a_k}{2n_j} + a_{jk}\right) + \frac{15\beta p^m}{8} \tag{3.14}$$

□

3.1.2. *Mathematical Model for Distributors.* There are n^{th} distributors d_1, d_2, d_3,...d_n and demand at k^{th} distributor's end is the sum of all jk^{th} retailer's demand. Hence the demand of k^{th} distributors can be written as

$$D^d_k = \sum_{j=1}^{n_k} D^r_{jk} = a_k e^{-\alpha t} - (\eta+\beta)\sum_{j=1}^{n_k} p^r_{jk} + n_j\beta p^m$$

Therefor the rate of changes in the k^{th} distributor's inventory is balanced by demand. At any time t following nonlinear equation represents the inventory status of k^{th} distributor

$$\begin{aligned} \frac{dI^d_k(t)}{dt} &= -D^d_k \quad \text{where} \quad 0 \le t \le T, \\ \frac{dI^d_k(t)}{dt} &= -\left(a_k e^{-\alpha t} - (\eta+\beta)\sum_{j=1}^{n_k} p^r_{jk} + n_j\beta p^m\right) \text{where} \quad 0 \le t \le T. \end{aligned} \tag{3.15}$$

with boundary condition $I_m(t_s) = 0$, at $t = T$. Now we derived the net profit function for k^{th} distributor during a time interval of length $[0, T]$. The net profit function for k^{th} distributor must be after subtraction of purchasing cost and sharing holding costs from sales revenue. The solution of equation (3.15) gives

$$I^d_k(t) = \frac{a_k}{\alpha}(e^{-\alpha t} - e^{-\alpha T}) + (\eta+\beta)\sum_{j=1}^{n_k} p^r_{jk}(t-T) + \beta p^m n_j(T-t) \tag{3.16}$$

At the initial time $t = 0$ the inventory level for k^{th} retailer is, where $t \in [0, T]$

$$I^d_k(0) = Q^d_k = \frac{a_k}{\alpha}(1 - e^{-\alpha T}) - (\eta+\beta)\sum_{j=1}^{n_k} p^r_{jk}T + \beta p^m n_j T \tag{3.17}$$

The sales revenue SR^d_k in the replenishment time period $[0, T]$ can be formulated as

$$SR^d_k = \int_0^T w^d_k D^d_k dt$$

$$SR^d_k = w^d_k\left(\frac{a_k}{\alpha}(1 - e^{-\alpha T}) - (\eta+\beta)\sum_{j=1}^{n_k} p^r_{jk}T + \beta p^m T\eta_j\right) \tag{3.18}$$

Purchase cost of k^{th} distributor in the interval [0, T] is

$$PC^d_k = \int_0^T w^m D^d_k dt$$

$$PC^r_{jk} = w^m\left(\frac{a_k}{\alpha}(1 - e^{-\alpha T}) - (\eta+\beta)\sum_{j=1}^{n_k} p^r_{jk}T + \beta p^m T\eta_j\right) \tag{3.19}$$

The inventory holding cost IHC^r_{jk} per unit per unit time is

$$IHC^d_k = h\int_0^T I^d_k(t)dt$$

$$IHC^d_k = h\left(\frac{a_k}{\alpha^2}(1-e^{-\alpha t}-T\alpha e^{-\alpha T})-(\eta+\beta)\sum_{j=1}^{n_k}p^r_{jk}\frac{T^2}{2}+\beta p^m\eta_j\frac{T^2}{2}\right) \tag{3.20}$$

Hence the net profit function for k^{th} distributor per unit time is

$$\begin{aligned} NP^d_k = (w^d_k - w^m)&\left[\frac{a_k}{\alpha}(1-e^{-\alpha T})-(\eta+\beta)\sum_{j=1}^{n_k}p^r_{jk}T+\beta p^m T\eta_j\right] \\ &- h(1-\lambda)\left[\frac{a_k}{\alpha^2}(1-e^{-\alpha t}-T\alpha e^{-\alpha T})-(\eta+\beta)\sum_{j=1}^{n_k}p^r_{jk}\frac{T^2}{2}+\beta p^m\eta_j\frac{T^2}{2}\right] \end{aligned} \tag{3.21}$$

where p^r_{jk} is given by (3.10)

Proposition 3.3. *The optimal wholesale price of k^{th} distributor associated with manufacturer's wholesale price w^m is w^{d*}_k, where*

$$w^{d*}_k = \frac{w^m}{2}+\frac{a_k(1-e^{-\alpha T})}{2\alpha(\eta+\beta)Tn_j}-\frac{\lambda hT}{2}+\frac{\beta p^m}{2(\eta+\beta)}+\frac{Th}{4} \tag{3.22}$$

Proof. Partial differentiation of equation (3.21) gives

$$\begin{aligned} \frac{\partial NP^d_k}{\partial w^d_k} =& -(w^d_k-w^m)n_k(\eta+\beta)\frac{T}{2}+\frac{a_k}{2\alpha}(1-e^{-\alpha T})-w^d_k n_k(\eta+\beta)\frac{T}{2}+\beta p^m n_k\frac{T}{2} \\ &-(\eta+\beta)\lambda h n_k\frac{T^2}{4}+h(1-\lambda)n_k(\eta+\beta)\frac{T^2}{4} \end{aligned} \tag{3.23}$$

If w^{d*}_k is an optimal value of w^d_k then $\frac{\partial NP^d_k}{\partial w^d_k}=0$ i.e.

$$\begin{aligned} &-(w^d_k-w^m)n_k(\eta+\beta)\frac{T}{2}+\frac{a_k}{2\alpha}(1-e^{-\alpha T})-w^d_k n_k(\eta+\beta)\frac{T}{2}+\beta p^m n_k\frac{T}{2} \\ &-(\eta+\beta)\lambda h n_k\frac{T^2}{4}+h(1-\lambda)n_k(\eta+\beta)\frac{T^2}{4}=0 \end{aligned} \tag{3.24}$$

solution of equation (3.24) gives

$$w^{d*}_k = \frac{w^m}{2}+\frac{a_k(1-e^{-\alpha T})}{2\alpha(\eta+\beta)Tn_k}-\frac{\lambda hT}{2}+\frac{\beta p^m}{2(\eta+\beta)}+\frac{Th}{4} \tag{3.25}$$

for optimality of NP^d_k at point $w^d_k = w^{d*}_k$, we have $\frac{\partial^2 NP^d_k}{\partial {w^d_k}^2} = -n_j(\eta+\beta)\frac{T}{2}$ for $\beta > 0$ and $\eta > 0$. Hence the optimal values of NP^d_k exists at w^{d*}_k □

3.1.3. *Mathematical Model for Manufacturer.* Manufacturer provides the initial lot of product to all distributors according to their demands. Therefore demand of product at manufacturer end is equal to the sum of all k^{th} distributor's demand. Hence the demand of manufacturer can be written as

$$D^m = \sum_{k=1}^{n} D^d_k = ae^{-\alpha t}-(\eta+\beta)\sum_{j=1}^{n_k}\sum_{k=1}^{n}p^r_{jk}+n^r\beta p^m$$

Hence the rate of changes in the manufacturer's inventory is balanced by demand of all distributors. At any time T the following nonlinear equation may represent the inventory status:

$$\begin{aligned} \frac{dI_k^d(t)}{dt} &= -D^m \quad \text{where} \quad 0 \le t \le T, \\ \frac{dI_k^d(t)}{dt} &= -\left(ae^{-\alpha t} - (\eta + \beta) \sum_{j=1}^{n_k} \sum_{k=1}^{n} p_{jk}^r + n^r \beta P_m \right) \text{where} \quad 0 \le t \le T. \end{aligned} \tag{3.26}$$

with boundary condition $I_m(t_s) = 0$, at $t = T$. The solution of equation (3.26) gives

$$I^m(t) = \frac{a}{\alpha}(e^{-\alpha t} - e^{-\alpha T}) + (\eta + \beta) \sum_{j=1}^{n_k} \sum_{k=1}^{n} p_{jk}^r (t - T) + \beta p^m n^r (T - t) \tag{3.27}$$

The initial inventory level for manufacturer at time $t = 0$, where $t \in [0, T]$ is

$$I^m(0) = Q^m = \frac{a}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta) \sum_{j=1}^{n_k} \sum_{k=1}^{n} p_{jk}^r T + \beta p^m n^r T \tag{3.28}$$

Now we derived the net profit function of manufacturer during a time interval of length $[0, T]$. The net profit function of manufacturer after can be obtain, after subtraction of production cost per unit from sales revenue. The sales revenue of manufacturer in the replenishment time period $[0, T]$ can be formulated as

$$SR^m = \int_0^T w^m D^m dt$$

$$SR^m = w^m \left(\frac{a}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta) \sum_{j=1}^{n_j} \sum_{k=1}^{n} p_{jk}^r T + \beta p^m T n^r \right) \tag{3.29}$$

Manufacturing cost of product for manufacturer is

$$PC^d = c \int_0^T D^m dt$$

$$PC^m = c \left(\frac{a_k}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta) \sum_{j=1}^{n_k} \sum_{k=1}^{n} p_{jk}^r T + \beta p^m T n^r \right) \tag{3.30}$$

Hence the net profit function NP^m of manufacturer is

$$NP^m = (w^m - c) \left[\frac{a}{\alpha}(1 - e^{-\alpha T}) - (\eta + \beta) \sum_{j=1}^{n_k} \sum_{k=1}^{n} p_{jk}^r T + \beta p^m T n^r \right] \tag{3.31}$$

where p_{jk}^r is given by (3.10)

Proposition 3.4. *The optimal wholesales price of manufacturer associated with production cost of unit product is* w^{m*}, *where*

$$w^{m*} = \frac{c}{2} + \frac{a(1 - e^{-\alpha T})}{2\alpha(\eta + \beta) T n^r} \quad - \frac{hT}{4} + \frac{\beta p^m}{2(\eta + \beta)} \tag{3.32}$$

Proof. Partial differentiation of equation (3.31) gives

$$\frac{\partial NP^m}{\partial w^m} = -(w^m - c)(\eta+\beta)n^r\frac{T}{4} + \frac{a(1-e^{-\alpha T})}{4\alpha} - (\eta+\beta)w^m n^r\frac{T}{4} + n^r\beta p^m\frac{T}{4} - (\eta+\beta)hn^r\frac{T^2}{8} \tag{3.33}$$

If w^{m*} is an optimal value of w^m then $\frac{\partial NP^m}{\partial w^m} = 0$ i.e.

$$-(w^m - c)(\eta+\beta)n^r\frac{T}{4} + \frac{a(1-e^{-\alpha T})}{4\alpha} - (\eta+\beta)w^m n^r\frac{T}{4} + n^r\beta p^m\frac{T}{4} - (\eta+\beta)hn^r\frac{T^2}{8} = 0 \tag{3.34}$$

where p^r_{jk} is given by (3.10), equation (3.34) yields

$$w^{m*} = \frac{c}{2} + \frac{a(1-e^{-\alpha T})}{2\alpha(\eta+\beta)Tn^r} - \frac{hT}{4} + \frac{\beta p^m}{2(\eta+\beta)} \tag{3.35}$$

for optimality of NP^m at point $w^m = w^{m*}$, we have $\frac{\partial^2 NP^d_{kk}}{\partial w^{d^2}_k} = -n^r(\eta+\beta)\frac{T}{2}$, for $\beta > 0$ and $\eta > 0$. Hence optimal profit NP^m exists at w^{m*} □

Proposition 3.5. *If p^m is an optimum suggested price and w^{m*} is a wholesale price given by manufacturer, also w^{d*}_k is an optimum wholesales price given by distributors , then optimal selling price is given by*

$$(i)\ p^{r*}_{jk} = \frac{c}{8} + \frac{a(1-e^{-\alpha T})}{2\alpha(\eta+\beta)}\left(\frac{a}{4n^r} + \frac{a_k}{2n_k} + a_{jk}\right) + \frac{7\beta p^m}{8(\eta+\beta)}, \tag{3.36}$$

where j=1 2 3....n_k, and k=1 2 3....n,

$$(ii)\ w^{d*}_k - w^{m*} > 0,$$

where k=1 2 3....n,

$$(iii)\ p^{r*}_{jk} - w^{d*}_k > 0$$

Proof. (i) Substituting the values of w^{d*}_k and w^{m*} from equations (3.25) and (3.35) respectively into the equation (3.12) we get p^{r*}_{jk} in terms of T and other parameters, which obvious. (ii) It is obvious from equation (3.12) and equation (3.25) and model stability. (iii) It is also obvious from part (i) and equation (3.25). □

3.2. **Centralized Policy.** In the centralized scenario all supply chain members work together as a single unit and cooperate to each other. In this scenario only manufacturer can take all decisions about supply chain and which are equally applicable on all supply chain members. The mathematical model can be formulated as following

3.2.1. *Mathematical Model.* In this scenario manufacturer is a leader of whole supply chain and he is a single decision maker, therefore he can take all decisions to optimize profit of whole chain. If p^{rc}_{jk} is a retail price of jk^{th} retailer, w^d_j is a whole sale price of k^{th} distributor, w^m is a whole sale price of manufacturer, c is a manufacturing cost, IHC^r_{jk} is a holding cost of jk^{th} retailer and IHC^d_k a is holding cost of k^{th} distributor, then the profit function is

$$\begin{aligned} NP^c &= \sum_{j=1}^{n_k}\sum_{k=1}^{n}\left[\left(p_{jk}^{rc} - w_j^d\right)D_{jk}^r - \lambda(IHC_{jk}^r)\right] \\ &+ \sum_{k=1}^{n}\left[\left(w_k^r - w^m\right)D_k^d - (1-\lambda)IHC_k^d\right] + (w^m - c) \\ &= \sum_{j=1}^{n_k}\sum_{k=1}^{n}\left[\left(p_{jk}^{rc} - c\right)D_{jk}^r - \lambda(IHC_{jk}^r)\right] \end{aligned}$$

$$\begin{aligned} NP^c = &\sum_{j=1}^{n_k}\sum_{k=1}^{n}\left(p_{jk}^{rc} - c\right)\left(\frac{a_{jk}}{\alpha}(1-e^{-\alpha T}) - (\eta+\beta)p_{jk}^{rc}T + \beta p^m T\right) \\ &- \sum_{j=1}^{n_k}\sum_{k=1}^{n} h\left(\frac{a_{jk}}{\alpha^2}(1-e^{-\alpha t} - T\alpha e^{-\alpha T}) - (\eta+\beta)p_{jk}^{rc}\frac{T^2}{2} + \beta p^m\frac{T^2}{2}\right) \end{aligned} \tag{3.37}$$

By using backward substitution method the optimal demand of the products at jk^{th} (j=1 2 3 ...n_k ,k=1 2 3 ...n) retailer's fag end is as

$$D_{jk}^{rc} = a_{jk}e^{-\alpha T} - \frac{c(\eta+\beta)}{2} - \frac{a_{jk}(1-e^{-\alpha T})}{2\alpha T} + \frac{\beta p^m}{2} + \frac{(\eta+\beta)Th}{4} \tag{3.38}$$

Proposition 3.6. *In the centralized scenario the optimal selling price of jk^{th} retailer associated with manufacturing cost is p_{jk}^{rc*}, where*

$$p_{jk}^{rc*} = \frac{c(\eta+\beta)T + \frac{a_{jk}}{\alpha}(1-e^{-\alpha T}) + \beta p^m T - (\eta+\beta)\frac{T^2}{2}}{2(\eta+\beta)Th} \tag{3.39}$$

Proof. Partial differentiation of equation (3.37) gives

$$\begin{aligned} \frac{\partial NP^c}{\partial p_{jk}^{r*}} = &\sum_{j=1}^{n_k}\sum_{k=1}^{n}\left[-\left(p_{jk}^r - c\right)(\eta+\beta)T + \left(\frac{a_{jk}}{\alpha}(1-e^{-\alpha T}) - (\eta+\beta)p_{jk}^r T + \beta p^m T\right)\right] \\ &- \sum_{j=1}^{n_k}\sum_{k=1}^{n}(\eta+\beta)\frac{T^2}{2}h \end{aligned} \tag{3.40}$$

If p_{jk}^{rc*} is an optimal value of p_{jk}^{rc} then $\frac{\partial NP^c}{\partial p_{jk}^{r*}} = 0$
i.e.

$$\begin{aligned} &\left(p_{jk}^r - c\right)(\eta+\beta)T - \left(\frac{a_{jk}}{\alpha}(1-e^{-\alpha T}) - (\eta+\beta)p_{jk}^r T + \beta p^m T\right) \\ &+ (\eta+\beta)\frac{T^2}{2}h = 0 \end{aligned} \tag{3.41}$$

solution of equation (3.41) gives

$$p_{jk}^{rc*} = \frac{c(\eta+\beta)T + \frac{a_{jk}}{\alpha}(1-e^{-\alpha T}) + \beta p^m T - (\eta+\beta)\frac{T^2}{2}h}{2(\eta+\beta)T} \tag{3.42}$$

for optimality of NP^c at point p_{jk}^{rc}=p_{jk}^{rc*}, we have
$-2n^r(\eta+\beta)T$, for $\eta > 0$ and $\beta > 0$,
Hence optimum value of NP^c exists at p_{jk}^{rc*}. □

3.3. **Numerical example.** For numerically illustration of this supply chain model we have assumed that the supply chain is formed by a manufacturer M, two distributors (D_1, D_2) and four retailers (R_{11},R_{12}, R_{21} and R_{22}). According to the Figure (1), each retailer is associated with certain distributors. A manufacturer has to provide certain quantity of product and distributors have to provide certain quantity of product to respective retailers. We consider the following data set, the demand scale parameters at each retailer's end are a_{11} =75, a_{12} = 73, a_{21} = 74, a_{22} = 76 units, manufacturer determined maximum retail price is p^m=275 price coefficient parameter is $\eta = 0.1$, difference coefficient of retail price and suggested price is $\beta = 1.5$, production cost is c =150, shape parameter is $\alpha = 0.002$ and random time is T=1.05. The model outputs are given in the following table:

Table 1: Decentralized Policy

Optimal	R_{11}	R_{12}	R_{21}	R_{22}	D_1	D_2	M
Price	285.11	284.48	284.80	285.80	265.58	265.89	227.15
Demand	31	30	30	31	61	61	122
EOL	32	31	34	33	-	-	-
Profit	640.40	600.10	610	650.68	2480.54	2522.15	10002
Total profit				17515.87			

Table 2: Centralized Policy

Optimal	R_{11}	R_{12}	R_{21}	R_{22}	D_1	D_2	M
Price	227.31	226.69	227	227.62	-	-	-
Demand	-	-	-	-	-	-	493
Profit	-	-	-	-	-	-	40050.14
Total profit				40050.14			

3.4. **Sensitivity Analysis.** Through the analysis of table 1 and 2 shows that, in the decentralized policy retail price of product is comparatively higher than the centralized policy but due to less demand of products, total profit of whole supply chain is more less than the centralized policy.

Proposition 3.7. *All profits are as follows with respect to basic demand of product* $\frac{\partial NP^r_{jk}}{\partial a_{jk}} > 0$, $\frac{\partial NP^d_k}{\partial a_k} > 0$, $\frac{\partial NP^m}{\partial a} > 0$, *and* $\frac{\partial NP^c}{\partial a_{jk}} > 0$,

Intuitively, all supply chain member's profit in both policy shows incremental property with respect to basic demand when retailing price and suggested retail price are constant. It is shown in the table 3.

Table 3: Sensitive analysis with base demand parameter

	changes	NP^r_{11}	NP^r_{12}	NP^r_{21}	NP^r_{22}	NP^d_1	NP^d_2	NP^m	NP^c
	-15%	585.3	546.8	556.3	595.2	2263.8	2303.6	8697.2	36566
a	-5%	621.0	580.4	590.2	630.2	2400.6	2441.5	9220.4	38765
	5%	661.1	620.1	630.2	621.4	2561.8	2604.0	9837.1	41356
	15%	698.0	655.9	666.3	708.6	2707.2	2750.6	10393.1	43692

Proposition 3.8. *Behavior of each profit function with respect to* α *are as follow* $\frac{\partial NP^r_{jk}}{\partial \alpha} < 0$, $\frac{\partial NP^d_k}{\partial \alpha} < 0$, $\frac{\partial NP^m}{\partial \alpha} < 0$, *and* $\frac{\partial NP^c}{\partial \alpha} < 0$,

Proposition 3.8 states the impact of shape parameter on each supply chain members, when selling price and suggested retail price are constant, then profit of each supply chain member decreases, as α increases.

Table 4: Sensitive analysis with base scale parameter

	changes	NP^r_{11}	NP^r_{12}	NP^r_{21}	NP^r_{22}	NP^d_1	NP^d_2	NP^m	NP^c
α	-15%	640.4	600.1	610.0	650.7	2480.6	2522.2	9526.6	40051
	-5%	640.5	600.2	610.0	650.8	2480.8	2522.4	9527.2	40053
	5%	640.4	600.1	610.0	650.7	2480.5	2522.0	9526.0	40048
	15%	640.3	600.1	610.0	650.7	2480.3	2521.2	9526.4	40046

Proposition 3.9. *Partial derivative all profits with respect to* η *are as follow* $\frac{\partial NP^r_{jk}}{\partial \eta} < 0$, $\frac{\partial NP^d_k}{\partial \eta} < 0$, $\frac{\partial NP^m}{\partial \eta} < 0$, *and* $\frac{\partial NP^c}{\partial \eta} < 0$,

Proposition 3.9 states the influence of the parameter η, which measure the sensitivity of consumers to the retailing price of product, profit of each supply chain member is decreases as η increases.

Table 5: Sensitive analysis with η

	changes	NP^r_{11}	NP^r_{12}	NP^r_{21}	NP^r_{22}	NP^d_1	NP^d_2	NP^m	NP^c
η	-15%	658.2	617.1	627.2	668.6	2550.0	2592.4	9792.5	41168
	-5%	646.3	605.8	615.8	656.6	2503.5	2545.4	9614.3	40420
	5%	634.5	594.5	604.4	644.8	2457.7	2499.1	9439.0	39683
	15%	547.6	511.4	520.3	556.9	2117.5	2155.1	8135.6	34205

Proposition 3.10. *Partial derivatives of all profits with respect to* β *are as follow* $\frac{\partial NP^r_{jk}}{\partial \beta} > 0$, $\frac{\partial NP^d_k}{\partial \beta} > 0$, $\frac{\partial NP^m}{\partial \beta} > 0$, *and* $\frac{\partial NP^c}{\partial \beta} > 0$,

Proposition 3.10 shows the influence of coefficient of difference between manufacturer determined retail price and actual selling price β of the product. Increment of β increases the profit of all supply chain members.

Table 6: Sensitive analysis with β

	changes	NP^r_{11}	NP^r_{12}	NP^r_{21}	NP^r_{22}	NP^d_1	NP^d_2	NP^m	NP^c
β	-15%	586.9	545.4	555.6	597.5	2216.5	2259.5	8702.7	36596
	-5%	622.4	581.8	591.8	632.8	2407.9	2449.8	9256.2	38891
	5%	658.4	618.5	628.4	668.7	2553.6	2594.9	9803.8	41215
	15%	694.9	655.5	665.3	704.9	2700.6	2741.4	10362.7	43562

Proposition 3.11. *Behavior of each profit function with respect to* p^m *are as follow* $\frac{\partial NP^r_{jk}}{\partial p^m} > 0$, $\frac{\partial NP^d_k}{\partial p^m} > 0$, $\frac{\partial NP^m}{\partial p^m} > 0$, *and* $\frac{\partial NP^c}{\partial p^m} > 0$,

For certain data set proposition shows that profits of all supply chain member increases as manufacturer determined selling price p^m increases which is shown in the following table.

Table 7: Sensitive analysis with p^m

	changes	NP^r_{11}	NP^r_{12}	NP^r_{21}	NP^r_{22}	NP^d_1	NP^d_2	NP^m	NP^c
p^m	275	640.4	600.1	610.0	650.6	2480.54	2522.15	10002	40050
	280	679.4	637.9	648.1	690.0	2634.15	2677.00	10619	42519
	285	719.6	676.8	687.3	730.4	2792.37	2836.47	11254	45061
	290	760.9	716.9	727.7	772.1	2955.21	3000.56	11908	47678

4. Conclusion

We have developed an integrated multi-channel and multi-echelon supply chain coordination policy for two different scenarios, in which first one is decentralized and second one is centralized scenario. The model follows the exponential time declining, price sensitive and manufacturer determined retail price dependent demand, incorporating sharing holding cost among retailers and distributors. Particularly, the manufacturer who act as stackelberg leader of whole chain, decides wholesale and suggested retail price of product, according to their goal and expenditure. On the basis of manufacturer's decision we optimized the retail price, wholesale price of distributors, initial order quantity for retailers, distributors and manufacturer, optimal profits of each supply members in certain finite time horizon. Model may be applicable on those products which are well established in the market and have high holding cost as long as non fluctuated demand with time.

Management should follow the following suggestions for beneficial purposes (i) Keep balance between retail price and suggested retail price, because profits of all supply chain members show positive behavior with suggested retail price. But increment of suggested retail price may causes reducing demand. (ii) Keep always $p^m > p^r_{jk}$ i.e $\beta > 0$, because profits of all supply chain members show positive behavior with respect to β. (iii) Proposition 3.8 shows the profit of all supply chain members are sensitive with retailing price, therefore management should make better strategies before making the changes in retail price. (iv) Managerial insights of study is that firstly management should collect all information about demand of product with the help of retailers and then announce the whole sale price and manufacturer determined retail price. Observation of model outputs shows that management should make a contractual policy for better coordination among all supply chain members because in the centralized scenario model outputs are better than the decentralized scenario. One can be extended this model by incorporating stockout situation at retailers end. One can be extend this model by incorporating probabilistic demand or discrete demand and also one can extend this model by incorporating variable holding cost. One can be extended this model by incorporating setup cost dependent suggested retail price by manufacturer.

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ON EXISTENCE OF SOLUTION OF IMPLICIT VECTOR EQUILIBRIUM PROBLEMS FOR TRIFUNCTION

TIRTH RAM*[1] AND ANU KUMARI KHANNA [2]

*1 Department of Mathematics, University of Jammu, Jammu-180006 India

[2] Department of Mathematics, University of Jammu, Jammu-180006 India

ABSTRACT. In this work, we introduce and study an extended version of implicit vector equilibrium problems for trifunction in real Hausdorff topological vector spaces. We prove some new existence results for the solution of these problems by using KKM lemma in real Hausdorff topological vector spaces. Some special cases are also discussed.

KEYWORDS: Equilibrium problem, upper semicontinuity, KKM-map, trifunction.
AMS Subject Classification: 49J40, 47H04, 52A07.

1. Introduction

The area of an equilibrium theory is dynamic and has been experiencing an explosive growth in both theory and applications, as a consequences of research techniques and problems drawn from various fields. This theory of equilibrium problem is being intensively studied by Blum and Oettli [2], where they proposed it as a generalization of optimization and variational inequality problem to study a wide class of problems. It has been extended to vector equilibrium problems, vector optimization problems and vector saddle point problems, see [2, 6, 8, 12]. In 2005, Kazmi and Raouf [10] introduced a class of operator equilibrium problems and from this there are plentiful problems for equilibrium problems with operator solutions, see for example [7, 10, 14, 15, 16, 17, 18] and the references therein. Implicit vector equilibrium problem is a generalization of implicit vector variational inequality, for more detail, we refer to [1, 3, 4, 13].

In 1929, Knaster, Kuratowski and Mazurkiewicz [11] established the well-known KKM theory, which is one of the few areas among the subjects from nonlinear analysis that could provide an easy and convenient forms and tools for the study of problems from applied sciences, such as economics, optimization and game theory.

* *Corresponding author.*
Email address : tir1ram2@yahoo.com, anukhanna4j@gmail.com.
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We have employed KKM theorems to prove the existence of solutions of implicit vector equilibrium problems.
Throughout this paper, Let Z be an ordered topological vector space with an ordering cone C in Z. Note that the cone C in Z defines a partial ordering $\leq_C$ as follows:

$$x \leq y \;\Leftrightarrow\; y - x \in C,\; \forall x, y \in Z,$$
$$x \nleq y \;\Leftrightarrow\; y - x \notin C,\; \forall x, y \in Z.$$

If the $intC \neq \phi$, then the weak ordering in Z is defined as follows:

$$x < y \;\Leftrightarrow\; y - x \in intC,\; \forall x, y \in Z,$$
$$x \nless y \;\Leftrightarrow\; y - x \notin intC,\; \forall x, y \in Z.$$

Now we will work under the following settings:

Let X, Y and Z be real Hausdorff topological vector spaces, and let $K \subseteq X$ and $D \subseteq Y$ be nonempty set. Let C be a closed and convex cone in Z such that $intC \neq \phi$. Let $S : K \to 2^K$ and $T : K \to 2^D$ be set-valued mappings. In this paper, we consider the following implicit vector equilibrium problem:

Find $x^* \in K,\; y \in T(x^*)$ such that

$$F(h(x^*), y, u) \notin -intC,\; \forall u \in S(x^*), \tag{1.1}$$

where $F : K \times D \times K \to Z$ be a trifunction and $h : K \to K$ be a map from K into itself.

Throughout the paper, 2^X denotes the set of all nonempty subsets of X.

Some special cases

(i) If $h : K \to K$ is the identity map on K, then (1.1) reduces to the problem of finding $x^* \in K$ such that

$$F(x^*, y, u) \notin -intC, \text{ for all } y \in T(x^*), u \in S(x^*), \tag{1.2}$$

which is called the implicit vector variational inequality studied by Chiang et al. [3].

(ii) If $S(x) = K,\; \forall x \in K$, then (1.2) reduces to the problem of finding $x^* \in K$ such that

$$F(x^*, y, u) \notin -intC, \text{ for all } u \in K,$$

which is extensively studied in [4].

(iii) If $Z = \mathbb{R},\; C = [0, \infty)$, and X as well as Y are finite dimensional spaces, then (1.2) reduces to implicit variational inequality studied by Cubiotti and Yao [5].

(iv) If $Z = \mathbb{R},\; C = [0, \infty)$, and $Y = X^*$, the topological dual of X and

$$F(x^*, y, u) = \langle y, u - x^* \rangle,$$

where $\langle ., . \rangle$ is the dual pairing between X^* and X, then (1.2) reduces to generalized quasi variational inequality studied by Shih and Tan [19], and Yao [20].

The main aim of this paper is to study the existence of solution of implicit vector equilibrium problems for trifunction in real Hausdorff topological vector spaces by using KKM -lemma. In section- 2, we recall some necessary definitions and results which are needed in the latter section. Some new existence results for the solution of implicit vector equilibrium problems for trifunction have been established in section-3.

2. Preliminaries

Now we give some definitions and preliminary results needed in the next sections.

Definition 2.1. A set-valued map $T : K \to 2^Y$ is called a KKM-map, if for every finite subset $\{x_1, x_2, ..., x_n\}$ of K, $co\{x_1, x_2, ...x_n\} \subseteq \bigcup_{i=1}^{n} T(x_i)$, where $co\{x_1, x_2, ...x_n\}$ denotes the convex hull of the set $\{x_1, x_2, ...x_n\}$.

Definition 2.2. A set-valued map $T : X \to 2^Y$ is called upper semicontinuous (for short, u.s.c) at $x_0 \in X$, if for any net $\{x_\lambda\}$ in X such that $x_\lambda \to x_0$ and for any net $\{y_\lambda\}$ in Y with $y_\lambda \in T(x_\lambda)$ such that $y_\lambda \to y_0$ in Y, we have $y_0 \in T(x_0)$. T is called upper semicontinuous on X if it is upper semicontinuous at every point of X.

To prove the existence results for the solutions of problem (1.1), we shall use the following lemmas:

Lemma 2.3. [9] *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Let $T : K \to 2^X$ be a KKM-map, such that for any $y \in K$, $T(y)$ is closed and $T(y^*)$ is contained in a compact set $B \subseteq X$ for some $y^* \in K$. Then, there exist $x^* \in B$ such that $x^* \in T(y)$, for all $y \in K$, that is, $\bigcap_{y \in K} T(y) \neq \phi$.*

Lemma 2.4. [13] *Let (Z, C) be an ordered topological vector space with a closed and convex cone C. Then for any $x, y, z \in Z$, we have*

(i) $x - y \in -intC$ *and* $x \notin -intC \implies y \notin -intC$.
(ii) $x + y \in -C$ *and* $x + z \notin -intC \implies z - y \notin -intC$.
(iii) $x + z - y \notin -intC$ *and* $-y \in -C \implies x + z \notin -intC$.
(iv) $x + y \notin -intC$ *and* $y - z \in -C \implies x + z \notin -intC$.

3. Existence Results

In this section, we prove some new existence results for the solutions of implicit vector equilibrium problem for trifunction.

Theorem 3.1. *Let $K \subseteq X$ be a nonempty convex set and $D \subseteq Y$ be a nonempty set. Let C be a closed and convex cone in Z such that $intC \neq \phi$. Let $S : K \to 2^K$ and $T : K \to 2^D$ be continuous set-valued mappings. Let $h : K \to K$ be a continuous mapping. Let $F : K \times D \times K \to Z$ be a continuous mapping with respect to the first argument. Suppose that the following assumptions holds:*

(1) *the map $W : K \to 2^Z$ defined by $W(x) = Z \setminus \{-intC\}$, $\forall x \in K$ is upper semicontinuous on K,*
(2) *there exists a set-valued map $G : K \times D \times K \to Z$ such that*
   (i) *$G(h(x), y, x) \notin -intC$, for all $x \in K, y \in T(x)$,*
   (ii) *$G(h(x), y, u) - F(h(x), y, u) \notin -intC$, for all $x \in K, y \in T(x), u \in S(x)$,*
   (iii) *$\{u \in K : G(h(x), y, u) \in -intC\}$ is convex, for all $x \in K, y \in T(x), u \in S(x)$,*

(3) *furthermore, suppose that there exists a nonempty compact and convex subset M of K such that for each $x \in K \setminus M$, $y \in T(x)$, there exists $u \in M$ such that $F(h(x), y, u) \in -intC$.*

Then, there exists $x^* \in K$, $y \in T(x)$ such that $F(h(x^*), y, u) \notin -intC$, for all $u \in S(x^*)$.

Proof. For each $u \in K$, define a set-valued map $P : K \to 2^M$ as

$$P(u) = \{x \in M : F(h(x), y, u) \notin -intC,\ \forall y \in T(x),\ \forall u \in S(x)\}.$$

We first prove that $P(u)$ is closed,for all $u \in K$. For this, let $\{x_\alpha\}$ be a net in $P(u)$ such that $x_\alpha \to x$. Then $x \in M$ (as M is compact). It follows from $x_\alpha \in P(u)$ that

$$F(h(x_\alpha), y, u) \notin -intC,\ \forall y \in T(x_\alpha),\ \forall u \in S(x_\alpha).$$

$$\text{So, } F(h(x_\alpha), y, u) \in W(x_\alpha) = Z \setminus \{-intC\},\ \forall y \in T(x_\alpha),\ \forall u \in S(x_\alpha).$$

Again, since $F(x, y, u)$ is continuous with respect to x and h, S, T are also continuous, we have

$$F(h(x_\alpha), y, u) \to F(h(x), y, u).$$

Therefore by the upper semicontinuity of W, we have

$$F(h(x), y, u) \in W(x),\ \forall y \in T(x),\ \forall u \in S(x).$$

$$\text{Therefore, } F(h(x), y, u) \notin -intC,\ \forall y \in T(x),\ \forall u \in S(x).$$

Hence $P(u)$ is closed, for all $u \in K$.
Next, we will show that

$$\bigcap_{u \in K} P(u) \neq \phi.$$

Since M is compact, it is sufficient to show that the family $\{P(u)\}_{u \in K}$ has the finite intersection property. For this, let $\{u_1, u_2, \cdots, u_n\}$ be a finite subset of K. Set $N = co[M \cup \{u_1, u_2, \cdots, u_n\}]$. Clearly, N is compact and convex subset of K. Next, for each $u \in K$, we define two set-valued mappings, $T_1, T_2 : K \to 2^N$ as follows:

$$T_1(u) = \{x \in N : F(h(x), y, u) \notin -intC,\ \forall y \in T(x),\ \forall u \in S(x)\}$$

and

$$T_2(u) = \{x \in N : G(h(x), y, u) \notin -intC,\ \forall y \in T(x),\ \forall u \in S(x)\}.$$

By assumption (i), (ii) of (2), we have

$$G(h(u), y, u) \notin -intC$$

and

$$G(h(u), y, u) - F(h(u), y, u) \in -intC.$$

It follows from Lemma 2.4(i), $F(h(u), y, u) \notin -intC$. and so $T_1(u) \neq \phi$.
Since $T_1(u)$ is a closed subset of a compact set N. Therefore $T_1(u)$ is compact. Now we will show that T_2 is a KKM-map. Suppose there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of N and $\lambda_i \geq 0$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \lambda_i = 1$ such that

$$\bar{x} = \sum_{i=1}^n \lambda_i x_i \notin \bigcup_{j=1}^n T_2(x_j).$$

Then $G(h(\bar{x}), y, x_j) \in -intC$, $j = 1, 2, \cdots, n$.
From assumption (2)(iii), we have

$$G(h(\bar{x}), y, \bar{x}) \in -intC,$$

which is a contradiction to (2)(i). Hence T_2 is a KKM-map.
From assumption (2)(ii) and Lemma 2.4(i), we have

$$T_2(u) \subseteq T_1(u) \ , \forall u \in K.$$

Infact, $x \in T_2(u)$ implies $G(h(x), y, u) \notin -intC$ and by assumption (2)(ii), we have,

$$G(h(x), y, u) - F(h(x), y, u) \in -intC$$

$$\text{or } F(h(x), y, u) \notin -intC \text{ and hence } x \in T_1(u).$$

So, T_1 is also a KKM -map.

From Lemma 2.3, there exists $x^* \in N$ such that $x^* \in T_1(u),$ for all $u \in K$.

This implies that there exists $x^* \in N$ such that

$$F(h(x^*), y, u) \notin -intC.$$

Therefore by assumption (3), we have $x^* \in M$ and moreover $x^* \in P(u_i)\ , i = 1, 2, \cdots, n$. Hence $\{P(u)\}_{u \in K}$ has the finite intersection property.This completes the proof. □

Corollary 3.2. *Let $K \subseteq X$ be a nonempty convex set and $D \subseteq Y$ be a nonempty set. Let C be a closed and convex cone in Z such that $intC \neq \phi$. Let $S : K \to 2^K$ and $T : K \to 2^D$ be a continuous set-valued mapping. Let $F : K \times D \times K \to Z$ be continuous mappings with respect to the first argument. Suppose that the following assumptions holds:*

(1) *the map $W : K \to 2^Z$ defined by $W(x) = Z \setminus \{-intC\}$, for all $x \in K$ is upper semicontinuous on K,*
(2) *there exists a set-valued map $G : K \times D \times K \to Z$ such that*
 (i) *$G(x, y, x) \notin -intC,$ for all $x \in K, y \in T(x)$,*
 (ii) *$G(x, y, u) - F(x, y, u) \notin -intC,$ for all $x \in K, y \in T(x), u \in S(x)$,*
 (iii) *$\{u \in K : G(x, y, u) \in -intC\}$ is convex, for all $x \in K, y \in T(x), u \in S(x)$,*
(3) *furthermore, suppose that there exists a nonempty compact and convex subset M of K such that for each $x \in K \setminus M,\ y \in T(x)$, there exists $u \in M$ such that $F(x, y, u) \in -intC$.*

Then there exists $x^* \in K,\ y \in T(x)$ such that

$$F(x^*, y, u) \notin -intC, \text{ for all } u \in S(x^*).$$

Proof. If $h : K \to K$ be the identity map in the above Theorem 3.1, then it can be easily checked that all the assumptions of Theorem 3.1 are satisfied. □

Remark 3.3. The above corollary gives the existence results for the solution of implicit vector equilibrium problem for trifunction in Chiang et al. [3] without the compactness of K, closedness of D if we replace assumptions (i)-(vi) in [3] by the hypotheses of above Corollary 3.2 of this paper.

4. Conclusion

In this work, implicit vector equilibrium problems for trifunction in real Hausdorff topological vector space is considered, and established some existence results for the solution of the problems by using KKM-lemma. Some special cases have also been discussed to show that our results are generalization of several authors.

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