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GENERALIZED NONLOCAL BOUNDARY CONDITION FOR FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATION VIA HILFER FRACTIONAL DERIVATIVE

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ABSTRACT. Fractional calculus has been very popular due to the application in real world problems. This paper aimed to investigate the existence, uniqueness and Ulam stability for nonlinear fractional pantograph differential equations with generalized nonlocal boundary conditions involving Hilfer fractional derivative. The analysis were done through Banach and Kranoselskii's fixed point theorems. Finally, example are given to illustrate the theoretical results.

KEYWORDS: Pantograph differential equation, Ulam stability, Fixed point theorems, Nonlocal condition, Hilfer-fractional derivative.
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1. Introduction

Fractional calculus is a generalized of ordinary differentiation and integration to arbitrary non-integer order. Fractional differential equations (FDE's) have picked up significance during the past decades due to its applicability in science and engineering. The primary concept of a fractional derivative was introduced in a letter written to Guillaume del' Hopital by Gottfried Wilhelm Leibniz in 1695 [28]. Because of the history effects associated with the dynamics of the models, non-integer order derivatives have been shown to be useful in modeling various phenomena. Non-integer order derivatives have been effectively utilized to describe physical processes in medicine, physics, image processing, optimization, electrodynamics, nanotechnology, biotechnology, engineering and many more fields, see [5, 17, 24, 16, 12, 44, 37] and the references cited therein.

Zhou et al. in [45], considered the presence of mild solutions for FDE's with Caputo fractional derivative. By applying the Laplace transform and probability density function, they gave a reasonable definition of mild solution. Utilizing the same strategy, Zhou et al. [46], gave a definition of mild solution for FDEs with Riemann Liouville fractional derivative. On the other hand, Hilfer proposed a generalized Rieman-Liouville fractional derivative, Hilfer fractional derivative, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. For case and details, see [33, 23, 32, 2, 1, 25, 21, 35, 36, 3, 30, 34, 6] and references therein.

An uncommon kind of delay differential equations is so called pantograph equations. It occurs in different fields of pure and applied mathematics, for examples, electrodynamics, control systems, number theory, probability, and quantum mechanics. Many researchers have studied the pantograph-type delay differential equation using analytical and numerical techniques [13, 18, 19, 26, 26, 38, 42, 41, 7, 8]. As of late, stability of FDE's has pulled in expanding interest due to it's applications in solving real life problems such as economics, biology and optimization. Different types of stability such as Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability has been given much attention for FDE's which involves different types of operators, see [1, 14, 22, 27, 31, 39, 9, 21, 10, 20, 11]. For example, in [18], Balachandran et al. established the existence of solutions of abstract fractional pantograph equations with different types of initial conditions of the form:

$$\begin{cases} {}^C D^{\alpha}_{0+} z(t) = f(t, z(t), z(\gamma t)), \quad t \in J = [0, a], \quad 0 < \alpha < 1, \quad 0 < \gamma < 1, \\ z(0) = z_0, \end{cases} \tag{1.1}$$

where ${}^C D^{\alpha}_{0+}(\cdot)$ is the Caputo fractional derivative of order α and $f : J \times X \times X \to X$ is a continuous function. Vivek et al. [43] extended the results of [18] to differential equations involving Hilfer fractional derivative.

Vivek et al. [40], considered an implicit fractional differential equations with nonlocal condition described by:

$$\begin{cases} D^{\alpha,\beta}_{0^+} z(t) = f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)), & t \in [0, T], \\ I^{1-\gamma}_{0^+} z(0) = \sum_{i=1}^{m} c_i z(\eta_i), & \alpha \le \gamma = \alpha + \beta - \alpha\beta, \quad \eta_i \in (0, T), \end{cases} \tag{1.2}$$

where $D^{\alpha,\beta}_{0^+}(\cdot)$ is the Hilfer fractional derivative of order $(0 < \alpha < 1)$ and type $0 \le \beta \le 1$, $I^{1-\gamma}_{0^+}(\cdot)$ is the Riemann-Liouville fractional integral of order $1 - \gamma$. The existence and uniqueness results were proved by Schaefer fixed point theorem and Banach's Contraction principle. Moreover, the authors addressed the stability analysis via Gronwall's lemma. Recently, Asawasamrit et al. [15] investigated the existence of solutions to nonlocal boundary value problems for fractional differential equations which involves Hilfer fractional derivative

$$\begin{cases} D_{a^+}^{\alpha,\beta} z(t) = f(t, z(t)), \quad t \in [a, b], \quad 1 < \alpha < 2, \quad 0 \leq \beta \leq 1, \\ z(a) = 0, \quad z(b) = \displaystyle\sum_{i=1}^{m} c_i I_{a^+}^{\gamma_i} z(\eta_i), \quad c_i \in \mathbb{R}, \quad \gamma_i > 0, \quad \eta_i \in [a, b], \end{cases} \tag{1.3}$$

where $D_{a^+}^{\alpha,\beta}(\cdot)$ is the Hilfer fractional derivative of order α and type β, $I_{a^+}^{\gamma_i}(\cdot)$ is the Riemann-Liouville fractional integral of order γ_i and $i = 1, \dots, m$. Using different types of fixed point theorems, the authors proved the existence and uniqueness results.

Motivated by the aforementioned discussions, this manuscript investigates the existence and uniqueness of the solutions of nonlinear fractional pantograph differential equations (NFPDE):

$$D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t), x(\lambda t)), \ t \in J = [0, b], \ 1 < \alpha < 2, \ 0 \leq \beta \leq 1, \ 0 < \lambda < 1, \tag{1.4}$$

$$x(0) = 0, \ x(b) = \sum_{i=1}^{m} c_i x(\tau_i) + \sum_{j=1}^{k} d_j I_{0^+}^{\rho_j} x(\delta_j), \ \rho_j > 0, \ \tau_i, \delta_j \in [0, b], \tag{1.5}$$

where $D_{0^+}^{\alpha,\beta}(\cdot)$ is the Hilfer fractional derivative of order α and type β, $I_{0^+}^{\rho_j}(\cdot)$ is the Riemann-Liouville fractional integral of order $\rho_j > 0$. $\tau_i, d_j \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, k$ and $f : J \times \mathbb{R}^2 \to \mathbb{R}$, is a given continuous functions. Moreover, two different types of Ulam stability are investigated.

Remark 1.1. We note that the application of nonlocal condition: $\displaystyle\sum_{i=1}^{m} c_i x(\tau_i) + \sum_{j=1}^{k} d_j I_{0^+}^{\rho_j} x(\delta_j)$, in physical problems yields an excellent results than the initial condition $x(b) = x_b$ [4]. In addition,

- If $d_j = 0$, the generalized nonlocal condition reduces to multipoint nonlocal condition [15, 42, 40].
- If $c_i = 0$, the generalized nonlocal condition reduces to nonlocal Riemann-Liouville integral condition [21].
- If $\rho_j \to 1$ and $c_i = 0$, reduces to nonlocal integral condition.

The outline of the paper is as follows: In Section **??**, we give some prerequisite definitions and results concerning Hilfer fractional operator. In Section **??**, we derived the equivalence between the proposed problem and Volterra integral equation. The existence and uniqueness of the solution of NFPDE are investigated. Stability analysis in the frame of Ulam-Hyers and generalized Ulam-Hyers stable are proved. Example are given to demonstrate the theoretical results. Finally, conclusions part of the paper are given in Section 5

2. Preliminaries

In this section, we recall some preliminaries facts, lemmas and definitions with respect to fractional operators and Hilfer differential equation [29].

Let $J=[0,b]$ $(-\infty<0<b<\infty)$ be a finite interval of $\mathbb{R}$ and $C[0,b]$ be the space of continuous function on $[0,b]$. Let $X=\mathcal{C}([0,b],\mathbb{R})$ denotes the Banach space of all continuous from $[0,b]$ to $\mathbb{R}$ endowed with the norm defined by

$$\|x\|=\max_{t\in[0,b]}|x(t)|.$$

Definition 2.1. [29] The left-sided Riemann-Liouville fractional integral of order $\alpha\in\mathbb{R}^+$ of a function f is defined by

$$I^{\alpha}_{0^+}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\quad t>0,\quad \alpha>0, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes Gamma function.

Definition 2.2. [29] Let $\alpha\in\mathbb{R}^+,\ n\in\mathbb{N}$ and $f\in C([0,b],\mathbb{R})$. The operator

$$^{RL}D^{\alpha}_{0^+}f(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-\alpha-1}f(s)ds, & t>0,\ n-1<\alpha<n,\\ \frac{d^n}{dt^n}f(t), & \alpha=n,\end{cases} \tag{2.2}$$

is called left-sided Riemann-Liouville fractional derivative of order α of a function f.

Definition 2.3. [29] Suppose $\alpha\in\mathbb{R}^+,\ n\in\mathbb{N}$ and $f\in C^n[0,b]$. The Caputo fractional derivative of order $(n-1<\alpha<n)$ of a function f is given by

$$^{C}\mathcal{D}^{\alpha}_{0^+}f(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_{0^+}^t(t-s)^{n-\alpha-1}\left(\frac{d}{dt}f\right)^n(s)ds, & t>0,\\ \frac{d^n}{dt^n}f(t), & \alpha=n,\end{cases} \tag{2.3}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.4. [29] Let $n-1<\alpha<n$ and $0\le\beta\le1$, with $n\in\mathbb{N}$. The left-sided Hilfer fractional derivative of order α and type β of a function f is defined by

$$\left(D^{\alpha,\beta}_{0^+}f\right)(t)=I^{\beta(n-\alpha)}_{0^+}\left[\mathcal{D}^n\left(I^{(1-\beta)(n-\alpha)}_{0^+}f\right)\right](t), \tag{2.4}$$

where $\mathcal{D}^n=\left(\frac{d}{dt}\right)^n$ and I is the Riemann-Liouville fractional integral defined in equation equations (2.1).

in particular, if $n=2$, Definition 2.4 is equivalent with

$$\left(D^{\alpha,\beta}_{0^+}f\right)(t)=I^{\beta(2-\alpha)}_{0^+}\left[\mathcal{D}^2\left(I^{(1-\beta)(2-\alpha)}_{0^+}f\right)\right](t). \tag{2.5}$$

Thus, throughout this manuscript, we discuss the case where $n=2,\ 1<\alpha<2$, $0\le\beta\le1$ and $\gamma=\alpha+2\beta-\alpha\beta$.

Remark 2.5. It's worth to mention that:

- The derivative is considered as an interpolator between the Riemann-Liouville and Caputo fractional derivatives since

$$D^{\alpha,\beta}_{0^+}f(t)=\begin{cases}D^{\alpha}_{0^+}f(t), & \beta=0,\\ I^{n-\alpha}_{0^+}\mathcal{D}^nf(t), & \beta=1.\end{cases} \tag{2.6}$$

Next, we recall some properties of Hilfer derivative and integral operators.

Lemma 2.6. [29] *Let $\alpha,\beta\in\mathbb{C}$ such that $Re(\alpha)\ge0$ and $Re(\beta)>0$, then there exists,*

$$\left(I^{\alpha}_{0^+}s^{\beta-1}\right)(t)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}t^{\beta+\alpha-1}$$

and

$$\left(D^{\alpha}_{0+}s^{\alpha-1}\right)(t) = 0, \quad 0 < \alpha < 1.$$

Lemma 2.7. [29] *Let $Re(\alpha) > 0$ and $Re(\beta) > 0$. If $f \in L^1(J)$, for each $t \in [0, b]$, then the following properties holds:*

$$\left(I^{\alpha}_{0+}I^{\beta}_{0+}f\right)(t) = \left(I^{\alpha+\beta}_{0+}f\right)(t)$$

and

$$\left(D^{\alpha}_{0+}I^{\alpha}_{0+}f\right)(t) = f(t).$$

Lemma 2.8. [29] *Let $Re(\alpha) > 0$, $n = -[-Re(\alpha)]$, $f \in L_1(0, b)$ and $(I^{\alpha}_{0+}f)(t) \in AC^n[0, b]$, then,*

$$(I^{\alpha}_{0+}D^{\alpha}_{0+}f)(t) = f(t) - \sum_{j=1}^{n}\frac{t^{\alpha-j}}{\Gamma(\alpha - j + 1)}(I^{j-\alpha}_{0+}f)(0). \tag{2.7}$$

Furthermore, if $1 < \alpha < 2$, we get

$$(I^{\alpha}_{0+}D^{\alpha}_{0+}f)(t) = f(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(I^{1-\alpha}_{0+}f)(0) + \frac{t^{\alpha-2}}{\Gamma(\alpha + 1)}(I^{2-\alpha}_{0+}f)(0). \tag{2.8}$$

Theorem 2.1. ***(Krasnoselskii's fixed point theorem)*** *Let $\mathcal{B}$ be a nonempty bounded closed convex subset of a Banach space X. Let $T_1, T_2 : \mathcal{B} \to X$ be two continuous operators satisfying:*
(i) $T_1x + T_2y \in \mathcal{B}$ whenever $x, y \in \mathcal{B}$;
(ii) T_1 is compact and continuous;
(iii) T_2 is contraction mapping;
then, there exist $u \in \mathcal{B}$ such that $u = T_1u + T_2u$.

Theorem 2.2. ***(Contraction Mapping Principle)*** *Let X be a Banach space, $\mathcal{N} \subset X$ be closed and $T : \mathcal{N} \to \mathcal{N}$ a contraction mapping i.e*

$$\|Tx - Ty\| \leq k\|x - y\|, \text{ for all } x, y \in \mathcal{N} \text{ and } k \in (0, 1),$$

then $\mathcal{N}$ has a unique fixed point.

For shortness of notation, we take I^{α}_{0+} and I^{α}_{0+} as I^{α} and D^{α} respectively.

3. Main Results

This section presents the uniformity connecting NFPDE (1.4) − (1.5) and the Volterra integral equation. In addition, the existence and uniqueness of solutions of NFPDE (1.4) − (1.5) were prove using Banach and Kransnoselkii's fixed point theorems.

Lemma 3.1. *Let $1 < \alpha < 2$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + 2\beta - \alpha\beta$, and let $f : J \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f \in \mathcal{C}([J, \mathbb{R}])$ for any $x \in \mathcal{C}([J, \mathbb{R}])$. A function $x \in \mathcal{C}([J, \mathbb{R}])$ is a solution of problem (1.4) − (1.5) if and only if x satisfies the Volterra integral equation:*

$$\begin{aligned} x(t) =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Bigg(I^{\alpha}f(t, x(t), x(\lambda t))(b) - \sum_{i=1}^{m}c_iI^{\alpha}f(t, x(t), x(\lambda t))(\tau_i) \\ &+ \sum_{j=1}^{k}d_jI^{\alpha+\rho_j}f(t, x(t), x(\lambda t))(\delta_j)\Bigg) + I^{\alpha}f(t, x(t), x(\lambda t)), \end{aligned} \tag{3.1}$$

where

$$\Lambda = \frac{1}{\Gamma(\gamma)} \sum_{i=1}^{m} c_i \tau_i^{\gamma-1} + \sum_{j=1}^{k} \frac{d_j}{\Gamma(\gamma+\rho_j)} \delta_i^{\rho_j+\gamma-1} + \frac{b^{\gamma-1}}{\Gamma(\gamma)} \neq 0. \tag{3.2}$$

Proof. Suppose $x \in \mathcal{C}([J,\mathbb{R}])$ satisfies problem (1.4) − (1.5), then we show that x is also a satisfies the integral equation (3.1). Indeed, setting $(I^{2-\gamma,\rho}x)(0) = e_1$, $(I^{1-\gamma,\rho}x)(0) = e_2$, and applying definition 2.4 and Lemma 2.8, yields

$$x(t) = \frac{e_2}{\Gamma(\gamma)} t^{\gamma-1} + \frac{e_1}{\Gamma(\gamma)} t^{\gamma-2} + I^{\alpha} f(t, x(t), x(\lambda t)). \tag{3.3}$$

From the first boundary condition of equation (1.5), we can see that $e_1 = 0$, which implies

$$x(t) = \frac{e_2}{\Gamma(\gamma)} t^{\gamma-1} + I^{\alpha} f(t, x(t), x(\lambda t)). \tag{3.4}$$

Substituting $t = \tau_i$ and multiplying both sides by c_i in (3.4), give

$$c_i x(\tau_i) = \frac{c_i e_2}{\Gamma(\gamma)} \tau_i^{\gamma-1} + c_i I^{\alpha} f(t, x(t), x(\lambda t))(\tau_i), \tag{3.5}$$

which implies

$$\sum_{i=1}^{m} c_i x(\tau_i) = \frac{e_2}{\Gamma(\gamma)} \sum_{i=1}^{m} c_i \tau_i^{\gamma-1} + \sum_{i=1}^{m} c_i I^{\alpha} f(t, x(t), x(\lambda t))(\tau_i). \tag{3.6}$$

Now, putting $t = \delta_j$ and multiplying through by d_j in (3.4), we have

$$d_j x(\delta_j) = \frac{d_j e_2}{\Gamma(\gamma)} \delta_j^{\gamma-1} + d_j I^{\alpha} f(t, x(t), x(\lambda t))(\delta_j). \tag{3.7}$$

Applying I^{ρ_j} to both sides of (3.7) and using Lemma 2.6, we get

$$d_j I^{\rho_j} x(\delta_j) = \frac{d_j e_2}{\Gamma(\gamma+\rho_j)} \delta_j^{\gamma+\rho_j-1} + d_j I^{\alpha+\rho_j} f(t, x(t), x(\lambda t))(\delta_j). \tag{3.8}$$

Thus,

$$\sum_{j=1}^{m} d_j I^{\rho_j} x(\delta_j) = \sum_{j=1}^{m} \frac{d_j e_2}{\Gamma(\gamma+\rho_j)} \delta_j^{\gamma+\rho_j-1} + \sum_{j=1}^{m} d_j I^{\alpha+\rho_j} f(t, x(t), x(\lambda t))(\delta_j). \tag{3.9}$$

From the second boundary condition: $x(b) = \sum_{i=1}^{m} c_i x(\tau_i) + \sum_{j=1}^{k} d_j I^{\rho_j} x(\delta_j)$ and in view of equations (3.6) and (3.9), we obtain

$$\begin{aligned} \sum_{i=1}^{m} c_i x(\tau_i) + \sum_{j=1}^{k} d_j I^{\rho_j} x(\delta_j) =& \frac{e_2}{\Gamma(\gamma)} \sum_{i=1}^{m} c_i \tau_i^{\gamma-1} + \sum_{i=1}^{m} c_i I^{\alpha} f(t, x(t), x(\lambda t))(\tau_i) \\ &+ \sum_{j=1}^{m} \frac{d_j e_2}{\Gamma(\gamma+\delta_j)} \delta_j^{\gamma+\rho_j-1} + \sum_{j=1}^{m} d_j I^{\alpha+\rho_j} f(t, x(t), x(\lambda t))(\delta_j). \end{aligned} \tag{3.10}$$

It follows from (3.4), that

$$x(b) = \frac{e_2}{\Gamma(\gamma)} b^{\gamma-1} + I^{\alpha} f(t, x(t), x(\lambda t))(b). \tag{3.11}$$

In view of equations (3.10) and (3.11), we have

$$\frac{e_2}{\Gamma(\gamma)}b^{\gamma-1}+I^{\alpha}f(t,x(t),x(\lambda t))(b)=\frac{e_2}{\Gamma(\gamma)}\sum_{i=1}^{m}c_i\tau_i^{\gamma-1}+\sum_{i=1}^{m}c_iI^{\alpha}f(t,x(t),x(\lambda t))(\tau_i)$$
$$+\sum_{j=1}^{m}\frac{d_je_2}{\Gamma(\gamma+\rho_j)}\delta_j^{\gamma+\rho_j-1}+\sum_{j=1}^{m}d_jI^{\alpha+\rho_j}f(t,x(t),x(\lambda t))(\delta_j). \tag{3.12}$$

Hence,

$$e_2=\frac{1}{\Lambda}\Bigg(I^{\alpha}f(t,x(t),x(\lambda t))(b)+\sum_{i=1}^{m}c_iI^{\alpha}f(t,x(t),x(\lambda t))(\tau_i)$$
$$+\sum_{j=1}^{m}d_jI^{\alpha+\delta_j}f(t,x(t),x(\lambda t))(\delta_j)\Bigg). \tag{3.13}$$

Therefore, by substituting equation (3.13) in (3.4), the result follows. The converse follows directly. Hence the proof is completed. □

Let us denote

$$\phi=\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^{\alpha}+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)}+\frac{b^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.14}$$

3.1. **Existence result via Kransnoselskii's fixed point theorem.**
In this subsection, we investigate the existence of solution of problem (1.4) – (1.5) with helps of Kransnoselkii's fixed point theorem 2.1. Thus, followings hypotheses are needed.
(H_1) Let $f:J\times\mathbb{R}^2\to\mathbb{R}$ be a function such that $f\in C[0,b]$ for any $x\in C[0,b]$. For all $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in J$ there exist a constants $K>0$ such that

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\le K(|u-\bar{u}|+|v-\bar{v}|).$$

(H_2) There exist $\theta\in\mathcal{C}([0,b],\mathbb{R})$ such that

$$|f(t,x(s),x(\gamma s))|\le\theta(t)$$

for each $t\in J$
(H_3) Suppose that

$$K\eta<\frac{1}{2}, \tag{3.15}$$

where

$$\eta=\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^{\alpha}+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)}. \tag{3.16}$$

Theorem 3.1. *Let $1<\alpha<2$, $0\le\beta\le1$ and $\gamma=\alpha+2\beta-\alpha\beta$. Suppose that the hypotheses $(H_1)-(H_3)$ are satisfied, then the problem (1.4) – (1.5) has at least one solution on J.*

Proof. Setting $\|\theta\| = \sup_{t\in J}|\theta(t)|$ and choosing $k \geq \phi\|\theta\|$ where ϕ is defined as in equation (3.14) and construct a closed convex set $x \in \mathcal{B}_k = \{x \in \mathcal{X} : \|x\| \leq k\}$. Define the operators T_1 and T_2 on $\mathcal{B}_k$ as follows

$$\begin{aligned} T_1x(t) =& I^\alpha f(s, x(s), x(\lambda s))(t), \text{ for all } t \in [0, b].\\ T_2x(t) =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Bigg[I^\alpha f(s, x(s), x(\lambda s))(b) - \sum_{i=1}^{m} c_i I^\alpha f(s, x(s), x(\lambda s))(\tau_i)\\ &+ \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} f(s, x(s), x(\lambda s))(\delta_j)\Bigg], \text{ for all } t \in [0, b]. \end{aligned}$$

We give the prove in the following steps.
Step 1. We show that $T_1x + T_2x \in \mathcal{B}_k$.
Thus, for any $x, y \in \mathcal{B}_k$, yields

$$\begin{aligned} |(T_1x(t) + T_2y(t))| \leq& \sup_{t\in J}\Bigg\{I^\alpha|f(s, x(s), x(\lambda s))|(t) + \frac{t^{\gamma-1}}{|\Lambda\Gamma(\gamma)|}I^\alpha|f(s, y(s), y(\lambda s))|(b)\\ &+\frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|c_i|I^\alpha|f(s, y(s), y(\lambda s))|(\tau_i)\\ &+\frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}|d_j|I^{\alpha+\rho_j}|f(s, y(s), y(\lambda s))|(\delta_j)\Bigg\}\\ \leq& \|\theta\|\Bigg(\frac{b^\alpha}{\Gamma(\alpha+1)} + \frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^\alpha\\ &+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)}\Bigg)\\ \leq& \phi\|\theta\|\\ \leq& k < \infty. \end{aligned} \tag{3.17}$$

Step 2. We show that, the operator T_2 is contractive.
Let $x, y \in \mathcal{C}([J, \mathbb{R}])$ and $t \in J$, then

$$\begin{aligned} |(T_2x(t) + T_2y(t))| \leq& \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}I^\alpha|f(s, x(s), x(\lambda s)) - f(s, y(s), y(\lambda s))|(b)\\ &+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|c_i|I^\alpha|f(s, x(s), x(\lambda s)) - f(s, y(s), y(\lambda s))|(\tau_i)\\ &+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}|d_j|I^{\alpha+\rho_j}|f(s, x(s), x(\lambda s)) - f(s, y(s), y(\lambda s))|(\delta_j)\\ \leq& 2K\Bigg(\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^\alpha \end{aligned} \tag{3.18}$$

$$+\frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)}\Bigg)\|x-y\| \\ \leq 2K\eta\|x-y\|. \tag{3.19}$$

Hence, it follows from (3.15), that T_2 is a contraction map.
Step 3. We show that the operator T_1 is continuous and compact.
Indeed, since f is continuous this implies that T_1 is also continuous and for any $x \in C[0,b]$, we get

$$\|T_1x\| \leq \frac{b^\alpha}{\Gamma(\alpha+1)}\|\theta\|,$$

which shows that the operator T_1 is uniformly bounded on $\mathcal{B}_k$. Finally, we shows that T_1 is compact.
Denoting $\sup_{(t,x)\in J\times\mathcal{B}_k}|f(t,x(t),x(\lambda t))| = f^* < \infty$. Thus, for any $0 < t_1 < t_2 < T$ gives

$$\begin{aligned}|(T_1x)(t_2)-(T_1x)(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{t-1}-(t_1-s)^{\alpha-1}]|f(s,x(s),x(\lambda s))|ds \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,x(s),x(\lambda s))|ds \\ &\leq \frac{f^*}{\Gamma(\alpha+1)}\Big((t_2-t_1)^\alpha+t_2^\alpha-t_1^\alpha\Big) \\ &\to 0\end{aligned} \tag{3.20}$$

as $t_2 \to t_1$. As a consequence of Arzela-Ascoli theorem, implies that the operator T_1 is compact on $\mathcal{B}_\kappa$. Thus, by Theorem 2.1, problem (1.4) – (1.5) has at least one solution on J. □

3.2. **Uniqueness result via Banach contraction principle.**

Now, we prove the uniqueness of problem (1.4) – (1.5) by means of Banach contraction principle.

Theorem 3.2. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha+\beta-\alpha\beta$. Suppose that assumption (H_1) holds such that $2K\phi < 1$, where ϕ is defined by (3.14). Then if there exist a solution of problem (1.4) – (1.5) is unique on J.*

Proof. Define the operator $T : X \to X$ by

$$\begin{aligned}(Tx)(t) =&\frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\Bigg[I^\alpha f(s,x(s),x(\lambda s))(b)-\sum_{i=1}^{m}c_iI^\alpha f(s,x(s),x(\lambda s))(\tau_i) \\ &+\sum_{j=1}^{k}d_jI^{\alpha+\rho_j}f(s,x(s),x(\lambda s))(\delta_j)\Bigg]+I^\alpha f(s,x(s),x(\lambda s))(t),\end{aligned} \tag{3.21}$$

then, clearly the operator T is well defined. It enough to show that the operator T has a fixed point which is a solution of problem (1.4) – (1.5).
Let, $\mathcal{N} = \sup_{t\in J}|f(t,0,0)| < \infty$ and setting $\kappa \geq \frac{\mathcal{N}\phi}{1-2K\phi}$. It suffices to show that $T\mathcal{B}_\kappa \subset \mathcal{B}_\kappa$, where $x \in \mathcal{B}_\kappa = \{x \in C[0,b] : \|x\| \leq \kappa\}$.

Indeed, for any $x \in \mathcal{B}_\kappa$, we have

$$
\begin{aligned}
|(Tx)(t)| &\le \sup_{t\in J}\Bigg\{ \frac{t^{\gamma-1}}{|\Lambda\Gamma(\gamma)|} I^{\alpha}|f(s,x(s),x(\lambda s))|(b) + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|c_i|I^{\alpha}|f(s,x(s),x(\lambda s))|(\tau_i) \\
&\quad + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}|d_j|I^{\alpha+\rho_j}|f(s,x(s),x(\lambda s))|(\delta_j) + I^{\alpha}|f(s,x(s),x(\lambda s))|(t)\Bigg\} \\
&\le \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}\big(|f(s,x(s),x(\lambda s)) - f(s,0,0)| + |f(s,0,0)|\big)(b)
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
&+ \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|c_i|I^{\alpha}\big(|f(s,x(s),x(\lambda s)) - f(s,0,0)| + |f(s,0,0)|\big)(\tau_i) \\
&+ \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}|d_j||I^{\alpha+\rho_j}\big(|f(s,x(s),x(\lambda s)) - f(s,0,0)| + |f(s,0,0)|\big)(\delta_j) \\
&+ I^{\alpha}\big(|f(s,x(s),x(\lambda s)) - f(s,0,0)| + |f(s,0,0)|\big)(t) \\
\le& \Big(2K\|x\| + \mathcal{N}\Big)\Bigg\{ \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}(b) + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|c_i|I^{\alpha}(\tau_i) \\
&+ \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}|d_j||I^{\alpha+\rho_j}(\delta_j) + I^{\alpha}(t)\Bigg\} \\
\le& \Big(2K\|x\| + \mathcal{N}\Big)\Bigg\{ \frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^{\alpha} \\
&+ \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)} + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\Bigg\} \\
\le& \phi\big(2K\kappa + \mathcal{N}\big) \\
\le& \kappa.
\end{aligned} \tag{3.23}
$$

This shows that, $T\mathcal{B}_\kappa \subset \mathcal{B}_\kappa$.
Now, for any $x_1, x_2 \in \mathcal{X}$ and each $t \in J$, yields

$$
\begin{aligned}
&|((Tx_1)(t) - (Tx_2)(t))| \\
&\qquad \leq \frac{b^{\gamma-1}}{|\Lambda\Gamma(\gamma)|} I^\alpha |f(s, x_1(s), x_1(\lambda s)) - f(s, x_2(s), x_2(\lambda s))|(b) \\
&\qquad + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^\alpha |f(s, x_1(s), x_1(\lambda s)) - f(s, x_2(s), x_2(\lambda s))|(\tau_i) \\
&\qquad + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| I^{\alpha+\rho_j} |f(s, x_1(s), x_1(\lambda s)) - f(s, x_2(s), x_2(\lambda s))|(\delta_j) \\
&\qquad + I^\alpha |f(s, x_1(s), x_1(\lambda s)) - f(s, x_2(s), x_2(\lambda s))|(t) \\
&\qquad \leq 2K \left(\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} \sum_{i=1}^{m} |c_i| \tau_i^\alpha \right. \\
&\qquad \left. + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} \frac{|d_j| \delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)} + \frac{b^\alpha}{\Gamma(\alpha+1)} \right) \|x_1 - x_2\| \\
&\qquad \leq 2K\phi \|x_1 - x_2\|.
\end{aligned}
\tag{3.24}
$$

Therefore, it follows that the operator T is a contraction mapping. Thus, Theorem 2.2, guarantee the existence of a unique solution of problem (1.4) – (1.5) on J. □

4. Ulam-Hyers stability

In this section, the Ulam-Hyers and generalized Ulam-Hyers stability for NFPDE (1.4)–(1.5) are investigate. Thus, before we prove the theorem we need the following definitions, remark and lemma which are important in this section.

Definition 4.1. The NFPDE (1.4) – (1.5) is said to be Ulam-Hyers stable if there exists a real constant $\psi > 0$ such that for all $\epsilon > 0$ and for every solution $y \in C([0, b], \mathbb{R})$ of the inequality

$$|D^{\alpha,\beta} y(t) - f(t, y(t), y(\lambda t))| \leq \epsilon, \quad t \in J, \tag{4.1}$$

there exists a solution $x \in C([0, b], \mathbb{R})$ of the problem (1.4) – (1.5) with

$$|y(t) - x(t)| \leq \psi\epsilon, \quad t \in J. \tag{4.2}$$

Definition 4.2. The NFPDE (1.4) – (1.5) is said to be generalized Ulam-Hyers stable if there is $\nu_f \in (\mathbb{R}^+, \mathbb{R}^+)$ and $\nu_f(0) = 0$ such that for every solution $y \in C([0, b], \mathbb{R})$ of problem (1.4) – (1.5) there exists a solution $x \in C([0, b], \mathbb{R})$ of the problem (1.4) – (1.5) such that:

$$|y(t) - x(t)| \leq \nu_f(\epsilon), \quad t \in J, \tag{4.3}$$

holds.

Remark 4.3. A function $y \in C([0, b], \mathbb{R})$ is a solution of (1.4) – (1.5) if and only if there exists a function $h \in C([0, b], \mathbb{R})$ (which depends on y) such that

- $|g(t)| < \epsilon, \quad t \in J.$
- $D^{\alpha,\beta} y(t) = f(t, y(t), y(\lambda t)) + h(t) \quad t \in J.$

It follows from Remark 4.3, that

$$
\begin{aligned}
y(t) =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[I^{\alpha}f(s,y(s),y(\lambda s))(b) - \sum_{i=1}^{m} c_i I^{\alpha} f(s,y(s),y(\lambda s))(\tau_i)\right. \\
&\left. + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} f(s,y(s),y(\lambda s))(\delta_j)\right] + I^{\alpha}f(s,y(s),y(\lambda s))(t) \\
&+ \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[I^{\alpha}g(b) - \sum_{i=1}^{m} c_i I^{\alpha} g(\tau_i) + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} g(\delta_j)\right] + I^{\alpha}g(t),
\end{aligned} \tag{4.4}
$$

is the solution of the following equation:

$$
D^{\alpha,\beta}y(t) = w(t,y(t),y(\lambda t)) + h(t), \quad t \in J. \tag{4.5}
$$

Lemma 4.4. *Let $1 < \alpha < 2$ and $0 \le \beta \le 1$. If $y \in C([0,b],\mathbb{R})$ is a solution of problem (1.4) − (1.5), then y is a solution of the following integral inequality:*

$$
|y(t) - B_y - I^{\alpha}f(s,y(s),y(\lambda s))(t)| \le \phi\epsilon, \tag{4.6}
$$

where

$$
\begin{aligned}
B_y =& \frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[I^{\alpha}f(s,y(s),y(\lambda s))(b) - \sum_{i=1}^{m} c_i I^{\alpha} f(s,y(s),y(\lambda s))(\tau_i)\right. \\
&\left. + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} f(s,y(s),y(\lambda s))(\delta_j)\right].
\end{aligned}
$$

Proof. Indeed, from Remark 4.3 and equation (4.4), that

$$
\begin{aligned}
|y(t) - B_y - I^{\alpha}f(s,y(s),y(\lambda s))(t)| &= \left|\frac{t^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[I^{\alpha}g(b) - \sum_{i=1}^{m} c_i I^{\alpha}g(\tau_i)\right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{k} d_j I^{\alpha+\rho_j} g(\delta_j)\right] + I^{\alpha}g(t)\right| \\
&\le I^{\alpha}|g(t)| + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}|g(b)| + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} c_i I^{\alpha}|g(\tau_i)| \\
&\quad + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} d_j I^{\alpha+\rho_j}|g(\delta_j)| \\
&\le \epsilon\left[\frac{b^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}\sum_{i=1}^{m}|c_i|\tau_i^{\alpha}\right. \\
&\quad \left. + \frac{b^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{j=1}^{k}\frac{|d_j|\delta_j^{\alpha+\rho_j}}{\Gamma(\alpha+\rho_j+1)} + \frac{b^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&= \phi\epsilon.
\end{aligned} \tag{4.7}
$$

□

Theorem 4.1. *Suppose that the assumption (H_1) holds with $K\phi < \frac{1}{2}$, then the NFPDE (1.4) − (1.5) is Ulam-Hyers stable on J and accordingly generalized Ulam-Hyers stable.*

Proof. Let $y \in C([0,b],\mathbb{R})$ be the solution of the inequality (4.1) and $x \in C([0,b],\mathbb{R})$ be the unique solution of problem (1.4) − (1.5). Thus,

$$\begin{aligned}
&|y(t)-x(t)| = \\
&\Bigg| y(t) - \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha} f(s,x(s),x(\lambda s))(b) + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^{\alpha} f(s,x(s),x(\lambda s))(\tau_i) \\
&- \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| I^{\alpha+\rho_j} f(s,x(s),x(\lambda s))(\delta_j) - I^{\alpha} f(s,x(s),x(\lambda s))(t) \Bigg| \\
&\le \Big| y(t) - B_y - I^{\alpha} f(s,y(s),y(\lambda s))(t) \Big| \\
&\quad + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |c_i| I^{\alpha} \big| f(s,y(s),y(\lambda s)) - f(s,x(s),x(\lambda s)) \big| (\tau_i) \\
&\quad + \frac{t^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{j=1}^{k} |d_j| I^{\alpha+\rho_j} \big| f(s,y(s),y(\lambda s)) - f(s,x(s),x(\lambda s)) \big| (\delta_j) \\
&\quad + I^{\alpha} \big| f(s,y(s),y(\lambda s)) - f(s,x(s),x(\lambda s)) \big| (t) \\
&\le \epsilon\phi + 2K\phi |y(t)-x(t)|,
\end{aligned} \tag{4.8}$$

which implies that

$$|y(t)-x(t)| \le \frac{\epsilon\phi}{1-2K\phi}. \tag{4.9}$$

Therefore,

$$|y(t)-x(t)| \le \psi\epsilon, \tag{4.10}$$

where

$$\psi = \frac{\phi}{1-2K\phi},$$

such that $K\phi < \frac{1}{2}$. Hence, we conclude that the NFPDE(1.4) − (1.5) is Ulam-Hyers stable. Moreover, setting $\nu_f(\epsilon) = \psi\epsilon$ such that $\nu_f(0) = 0$, the NFPDE (1.4) − (1.5) is generalized Ulam-Hyers stable. □

Example 4.5. Consider NFPDE of the form:

$$\begin{cases} D^{\frac{6}{5},\frac{1}{5}} x(t) = \frac{1}{10^{t+3}\left(1+|x(t)|+|x(\frac{1}{6}t)|\right)}, & t \in J = [0,1], \\ x(0) = 0, \quad x(1) = \frac{1}{3}x(\frac{1}{3}) - \frac{1}{2}x(\frac{1}{2}) + \frac{1}{4} I^{\frac{1}{4}} x(\frac{1}{4}). \end{cases} \tag{4.11}$$

By comparing (1.4) − (1.5) with (4.11), we obtain the followings:
$\alpha = \frac{6}{5}$, $\beta = \frac{1}{5}$, $\gamma = \frac{1}{35}$, $\lambda = \frac{1}{6}$, $b = 1$, $c_1 = \frac{1}{3}$, $c_2 = \frac{-1}{2}$, $\tau_1 = \frac{1}{3}$, $\tau_2 = \frac{1}{2}$, $d_1 = \frac{1}{4}$, $\rho_1 = \frac{1}{4}$ and $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is a function defined by

$$f(t,u,v) = \frac{1}{10^{t+3}\left(1+|u|+|v|\right)}, \quad t \in J, \quad u,v \in \mathbb{R}.$$

Clearly, the function f is continuous and for all $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$,

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{10^3}\left(|u-\bar{u}| + |v-\bar{v}|\right).$$

Thus, assumption (H_1) is satisfies with $K = \frac{1}{10^3}$. Hence, by simple calculation, we obtain $|\Lambda| \approx 0.7578$ and $\phi \approx 2.3206$.
So,

$$2K\phi = \frac{2}{10^3} \times 2.3206 < 1.$$

Thus, it follows from Theorem 3.14 that problem (1.4) − (1.5) has a unique solution on J, since all the assumptions are satisfied.
In addition, $K\phi = \frac{1}{10^3} \times 2.3206 < \frac{1}{2}$. Thus, by Theorem 4.1, problem (1.4) − (1.5) is both Ulam-Hyers and generalized Ulam-Hyers stable on J.

5. Conclusions

We investigate the existence and uniqueness of solutions for problem (1.4) − (1.5) by employed the techniques of Banach and Kransnoselkii's fixed point theorems. We also establish the uniformity between generalized problem (1.4) − (1.5) and the Volterra integral equation. Ulam-Hyers and generalized Ulam-Hyers stability of solutions to (1.4) − (1.5) using the classical calculus approach are established. Finally, as an application example were given to illustrate the main results.

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COMMON FIXED POINT RESULTS FOR GENERALIZED CYCLIC CONTRACTION PAIRS INVOLVING CONTROL FUNCTIONS ON PARTIAL METRIC SPACES

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ABSTRACT. In this paper, our purpose is to establish some coincidence point and common fixed point results for a pair of self mappings satisfying some generalized cyclic contraction type conditions involving a control function with two variables in partial metric spaces. Moreover, we provide some examples to analyze and illustrate our main results.

KEYWORDS: partial metric, cyclic contraction, 0-completeness, common fixed point.
AMS Subject Classification: 54H25, 47H10.

1. Introduction

In 1994, Matthews [18] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks and proved the well known Banach Contraction Principle in this setting. Complete partial metric space is a useful framework to model several complex problems in theory of computation. The works of [5, 6, 8, 11, 9, 10, 19, 20, 21] are viable and have opened new avenues for applications in different fields of mathematics and applied sciences. It is interesting to note that in partial metric spaces, self distance of an arbitrary point need not be equal to zero. Matthews [18] introduced a class of open p-balls in partial metric spaces which generates a T_0 topology on X. This facilitated the initiation of open and closed sets, neighbourhoods and other allied notions in partial metric spaces. Recently, many authors studied fixed points of cyclic mappings in several spaces. In 2003, Kirk et al.[17] introduced the notion of cyclic mappings and proved some fixed point theorems for these mappings. Some results for cyclic contractions in partial metric spaces have been obtained in [4, 1, 7, 14, 15]. In 2013, Shatanawi et al.[24]

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proved some common fixed point theorem with the help of control functions, namely, altering distance functions due to Khan et al.[16]. After that, several generalized control functions were used to obtain fixed point results in various spaces. Motivated by the works in [12, 22, 25], we will prove some coincidence points and common fixed point results for a pair of self mappings satisfying some generalized cyclic contraction type conditions involving a control function with two variables in partial metric spaces. Our results extend and unify several existing results in the literature. Finally, we give some examples to justify the validity of our results.

2. Some Basic Concepts

In this section, we begin with some basic facts and properties of partial metric spaces.

Definition 2.1. [18] A partial metric on a nonempty set X is a function $p : X\times X \to \mathbb{R}^+$ such that for all x, y, $z \in X$:

(p_1) $p(x,x) = p(y,y) = p(x,y) \Longleftrightarrow x = y$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a partial metric space.

It is clear that if $p(x,y) = 0$, then from (p_1) and (p_2), it follows that $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Example 2.2. [18] Let $X = [0,\infty)$ and $p(x,y) = max\,\{x,\, y\}$, for all x, $y \in X$. Then (X,p) is a partial metric space.

Example 2.3. [18] Let $X = \{[a,b] : a,\, b \in \mathbb{R},\, a \leq b\}$ and $p([a,b],[c,d]) = max\,\{b,\, d\} - min\,\{a,\, c\}$. Then (X,p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X,\, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Theorem 2.4. *If $U \in \tau_p$ and $x \in U$, then there exists $r > 0$ such that $B_p(x,r) \subseteq U$.*

Proof. Since U is an open set containing x, there exists an open p-ball, say $B_p(y,\epsilon)$ such that $x \in B_p(y,\epsilon) \subseteq U$. Then $p(x,y) < p(y,y) + \epsilon$. Let us choose $0 < r < p(y,y) - p(x,y) + \epsilon$ and consider the open p-ball $B_p(x,r)$. Then it is easy to verify that $B_p(x,r) \subseteq B_p(y,\epsilon) \subseteq U$. □

Remark 2.5. Let (X,p) be a partial metric space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(w.r.t.) τ_p if and only if $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$.

Let $x_n \to x$ w.r.t. τ_p and $\epsilon > 0$. Then there exists a natural number n_0 such that $x_n \in B_p(x,\epsilon)$ for all $n \geq n_0$. This gives that $p(x_n,x) - p(x,x) < \epsilon$ for all $n \geq n_0$. Since $p(x_n,x) - p(x,x) \geq 0$, it follows that $\mid p(x_n,x) - p(x,x) \mid < \epsilon$ for all $n \geq n_0$. This proves that $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$.

Conversely, suppose that $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$. We shall show that $x_n \to x$ w.r.t. τ_p. Let $U \in \tau_p$ and $x \in U$. Then there exists $\epsilon > 0$ such that $x \in B_p(x,\epsilon) \subseteq U$.

By hypotheses, it follows that

$$\lim_{n\to\infty}(p(x_n, x) - p(x, x)) = 0.$$

So, there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x) - p(x, x) < \epsilon$ for all $n \geq n_0$. This ensures that $x_n \in B_p(x, \epsilon)$ for all $n \geq n_0$ and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_p on X.

Definition 2.6. [18] Let (X, p) be a partial metric space and let (x_n) be a sequence in X. Then

- (i) (x_n) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. This will be denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.
- (ii) (x_n) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
- (iii) (X, p) is said to be complete if every Cauchy sequence (x_n) in X converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Definition 2.7. [23] A sequence (x_n) in (X, p) is called 0-Cauchy if

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$.

It is easy to verify that every closed subset of a 0-complete partial metric space is 0-complete.

Lemma 2.8. *Let (X, p) be a partial metric space.*

- (a) *(see [3, 13]) If $p(x_n, z) \to p(z, z) = 0$ as $n \to \infty$, then $p(x_n, y) \to p(z, y)$ as $n \to \infty$ for each $y \in X$.*
- (b) *(see [23]) If (X, p) is complete, then it is 0-complete.*

The converse assertion of (b) may not hold, in general. The following example supports the above remark.

Example 2.9. [23] The space $X = [0, \infty) \cap \mathbb{Q}$ with the partial metric $p(x, y) = max\,\{x,\, y\}$ is 0-complete, but it is not complete. Moreover, the sequence (x_n) with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, p), but it is not a 0-Cauchy sequence.

Definition 2.10. [17] Let X be a nonempty set, $q \in \mathbb{N}$, and let $f : X \to X$ be a self-mapping. Then $X = \cup_{i=1}^{q} A_i$ is a cyclic representation of X with respect to f if

- (a) $A_i,\ i = 1, 2, \cdots, q$ are nonempty subsets of X;
- (b) $f(A_1) \subseteq A_2,\ f(A_2) \subseteq A_3, \cdots, f(A_{q-1}) \subseteq A_q,\ f(A_q) \subseteq A_1$.

Definition 2.11. [2] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.12. [13] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \textit{ whenever } Sx = Tx.$$

Proposition 2.13. [2] *Let S and T be weakly compatible self maps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

3. Main Results

In this section, we will prove some coincidence point and common fixed point theorems for a pair of self mappings defined on a 0-complete partial metric space and satisfying a generalized contraction type condition involving a control function of two variables. In 2013, Nashine et al.[22] introduced a class of generalized control functions as follows:

Let Φ denote the class of all functions $\varphi : [0,\infty)^2 \to [0,\infty)$ satisfying the following conditions:

(a) φ is lower semicontinuous;
(b) $\varphi(s,t) = 0$ if and only if $s = t = 0$.

We begin with the following theorem.

Theorem 3.1. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$ and A_1, A_2, $\cdots$, A_q be nonempty subsets of X. Suppose the mappings f, $g : X \to X$ are such that $g(A_1)$, $g(A_2)$, $\cdots$, $g(A_q)$ are closed subsets of (X,p) and satisfy the following conditions:*

($C1$) *$f(A_i) \subseteq g(A_{i+1})$ for $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$;*
($C2$) *there exists $\varphi \in \Phi$ such that*

$$p(fx,fy) \le M(gx,gy) - \varphi(p(gx,gy), p(gx,fx))$$

for any $(gx,gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$ with $A_{q+1} = A_1$, where $M(gx,gy) = max\{p(gx,gy), p(gx,fx), p(gy,fy), \frac{p(gx,fy)+p(fx,gy)}{2}\}$.

Then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. Let $Y = \cup_{i=1}^{q} A_i$ and $x_0 \in Y$ be arbitrary. Then there exists $i_0 \in \{1, 2, \cdots, q\}$ such that $x_0 \in A_{i_0}$. Since $f(A_{i_0}) \subseteq g(A_{i_0+1})$, there exists $x_1 \in A_{i_0+1}$ such that $gx_1 = fx_0$. Continuing this process, we can construct a sequence (x_n) such that $gx_n = fx_{n-1},\ n = 1, 2, 3, \cdots$, where $x_n \in A_{i_0+n}$ and $A_{q+k} = A_k$.

If $p(gx_n, gx_{n+1}) = 0$ for some $n \in \mathbb{N}$, then $gx_n = gx_{n+1} = fx_n$ and hence gx_{n+1} is a point of coincidence of f and g.

Without loss of generality, we may assume that

$$p(gx_n, gx_{n+1}) > 0,\ \forall n \in \mathbb{N}.$$

Therefore,

$$\varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) > 0,\ \forall n \in \mathbb{N}. \tag{3.1}$$

We note that for all $n \in \mathbb{N}$, there exists $i \in \{1, 2, \cdots, q\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$ and so, $(gx_n, gx_{n+1}) \in g(A_i) \times g(A_{i+1})$. By using condition ($C2$), we obtain

$$\begin{aligned} p(gx_{n+1}, gx_{n+2}) &= p(fx_n, fx_{n+1}) \\ &\le M(gx_n, gx_{n+1}) - \varphi(p(gx_n, gx_{n+1}), p(gx_n, fx_n)) \\ &= M(gx_n, gx_{n+1}) - \varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})), \end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
M(gx_n, gx_{n+1}) &= max \left\{ \begin{array}{l} p(gx_n, gx_{n+1}), p(gx_n, fx_n), p(gx_{n+1}, fx_{n+1}), \\ \frac{p(gx_n, fx_{n+1}) + p(fx_n, gx_{n+1})}{2} \end{array} \right\} \\
&= max \left\{ \begin{array}{l} p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2}), \\ \frac{p(gx_n, gx_{n+2}) + p(gx_{n+1}, gx_{n+1})}{2} \end{array} \right\} \\
&\leq max \left\{ \begin{array}{l} p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2}), \\ \frac{p(gx_n, gx_{n+1}) + p(gx_{n+1}, gx_{n+2})}{2} \end{array} \right\} \\
&= max\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2})\}.
\end{aligned}
$$

Thus, we obtain from condition (3.2) that

$$
\begin{aligned}
p(gx_{n+1}, gx_{n+2}) \leq\ & max\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2})\} \\
& -\varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})). \qquad (3.3)
\end{aligned}
$$

If $max\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2})\} = p(gx_{n+1}, gx_{n+2})$, then by using condition (3.1), we get

$$
\begin{aligned}
p(gx_{n+1}, gx_{n+2}) &\leq p(gx_{n+1}, gx_{n+2}) - \varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) \\
&< p(gx_{n+1}, gx_{n+2}),
\end{aligned}
$$

which is a contradiction.
Therefore, $max\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2})\} = p(gx_n, gx_{n+1})$.

Thus, condition (3.3) reduces to

$$
\begin{aligned}
p(gx_{n+1}, gx_{n+2}) &\leq p(gx_n, gx_{n+1}) - \varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) \\
&< p(gx_n, gx_{n+1}). \qquad (3.4)
\end{aligned}
$$

This shows that $(p(gx_n, gx_{n+1}))$ is a nonincreasing sequence of positive real numbers. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(gx_n, gx_{n+1}) = r. \qquad (3.5)$$

Taking the upper limit as $n \to \infty$ in (3.4) and using condition (3.5) and lower semicontinuity of φ, we get

$$
\begin{aligned}
r &\leq r - \liminf_{n\to\infty} \varphi(p(gx_n, gx_{n+1}), p(gx_n, gx_{n+1})) \\
&\leq r - \varphi(r, r),
\end{aligned}
$$

which implies that $\varphi(r, r) = 0$ and hence $r = 0$.
Therefore,

$$\lim_{n\to\infty} p(gx_n, gx_{n+1}) = 0. \qquad (3.6)$$

We now show that (gx_n) is 0-Cauchy in $g(Y)$.

If possible, suppose that (gx_n) is not a 0-Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences (gx_{m_i}) and (gx_{n_i}) of (gx_n) such that n_i is the smallest positive integer satisfying

$$p(gx_{m_i}, gx_{n_i}) \geq \epsilon \ for \ n_i > m_i > i. \qquad (3.7)$$

So, it must be the case that

$$p(gx_{m_i}, gx_{n_i-1}) < \epsilon. \qquad (3.8)$$

Using conditions (3.7), (3.8) and (p_4), we obtain

$$\begin{aligned}\epsilon &\leq p(gx_{m_i}, gx_{n_i})\\ &\leq p(gx_{m_i}, gx_{n_i-1}) + p(gx_{n_i-1}, gx_{n_i}) - p(gx_{n_i-1}, gx_{n_i-1})\\ &< \epsilon + p(gx_{n_i-1}, gx_{n_i}).\end{aligned}$$

This gives that

$$\epsilon \leq p(gx_{m_i}, gx_{n_i}) < \epsilon + p(gx_{n_i-1}, gx_{n_i}).$$

Passing to the limit as $i \to \infty$ and using condition (3.6), we have

$$\lim_{i\to\infty} p(gx_{m_i}, gx_{n_i}) = \epsilon. \tag{3.9}$$

We observe that for all i, there exists $r_i \in \{1, 2, \cdots, q\}$ such that $n_i - m_i + r_i \equiv 1[q]$. Then $x_{m_i-r_i}$ (for large i, $m_i > r_i$) and x_{n_i} lie in different adjacently labelled sets A_j and A_{j+1} for certain $j \in \{1, 2, \cdots, q\}$ where $A_{q+1} = A_1$. So, $(gx_{m_i-r_i}, gx_{n_i}) \in g(A_j) \times g(A_{j+1})$.

By using condition $(C2)$, we get

$$\begin{aligned}p(gx_{m_i-r_i+1}, gx_{n_i+1}) &= p(fx_{m_i-r_i}, fx_{n_i})\\ &\leq M(gx_{m_i-r_i}, gx_{n_i})\\ &\quad -\varphi(p(gx_{m_i-r_i}, gx_{n_i}), p(gx_{m_i-r_i}, fx_{m_i-r_i})),\end{aligned} \tag{3.10}$$

where

$$M(gx_{m_i-r_i}, gx_{n_i}) = max\left\{\begin{array}{l} p(gx_{m_i-r_i}, gx_{n_i}), p(gx_{m_i-r_i}, fx_{m_i-r_i}),\\ \\ p(gx_{n_i}, fx_{n_i}), \frac{p(gx_{m_i-r_i}, fx_{n_i}) + p(fx_{m_i-r_i}, gx_{n_i})}{2}\end{array}\right\}. \tag{3.11}$$

We now compute that $\lim_{i\to\infty} p(gx_{m_i-r_i}, gx_{m_i}) = 0$. By repeated use of (p_4), we get

$$\begin{aligned}p(gx_{m_i-r_i}, gx_{m_i}) &\leq \sum_{l=0}^{r_i-1} p(gx_{m_i-r_i+l}, gx_{m_i-r_i+l+1})\\ &\leq \sum_{l=0}^{q-1} p(gx_{m_i-r_i+l}, gx_{m_i-r_i+l+1}).\end{aligned}$$

Taking the limit as $i \to \infty$ and using condition (3.6), it follows that

$$\lim_{i\to\infty} p(gx_{m_i-r_i}, gx_{m_i}) = 0. \tag{3.12}$$

Using (p_4), we have

$$\begin{aligned}p(gx_{m_i-r_i}, gx_{n_i}) &\leq p(gx_{m_i-r_i}, gx_{m_i}) + p(gx_{m_i}, gx_{n_i}) - p(gx_{m_i}, gx_{m_i})\\ &\leq p(gx_{m_i-r_i}, gx_{m_i}) + p(gx_{m_i}, gx_{n_i}).\end{aligned}$$

Taking the upper limit as $i \to \infty$ and using conditions (3.9) and (3.12), we get

$$\limsup_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) \leq \epsilon. \tag{3.13}$$

Again,

$$\begin{aligned}\epsilon &\leq p(gx_{m_i}, gx_{n_i})\\ &\leq p(gx_{n_i}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}) - p(gx_{m_i-r_i}, gx_{m_i-r_i})\\ &\leq p(gx_{n_i}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}).\end{aligned}$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.12) and (3.13), we obtain

$$\epsilon \leq \limsup_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) = \epsilon.$$

By an argument similar to that used above, we can prove that

$$\liminf_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i}) = \epsilon. \tag{3.14}$$

By (p_4), we have

$$\begin{aligned} p(gx_{m_i-r_i}, gx_{n_i+1}) &\leq p(gx_{m_i-r_i}, gx_{n_i}) + p(gx_{n_i}, gx_{n_i+1}) - p(gx_{n_i}, gx_{n_i}) \\ &\leq p(gx_{m_i-r_i}, gx_{n_i}) + p(gx_{n_i}, gx_{n_i+1}). \end{aligned}$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.6) and (3.14), we obtain

$$\limsup_{i\to\infty} p(gx_{m_i-r_i}, gx_{n_i+1}) \leq \epsilon. \tag{3.15}$$

Moreover,

$$\begin{aligned} \epsilon &\leq p(gx_{m_i}, gx_{n_i}) \\ &\leq p(gx_{n_i}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}) \\ &\leq p(gx_{n_i}, gx_{n_i+1}) + p(gx_{n_i+1}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}). \end{aligned}$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.6), (3.12) and (3.15), we get

$$\epsilon \leq \limsup_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i}) = \epsilon.$$

Similarly, we can show that

$$\liminf_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i}) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i}) = \epsilon. \tag{3.16}$$

Furthermore, by (p_4), we have

$$p(gx_{n_i}, gx_{m_i-r_i+1}) \leq p(gx_{m_i-r_i}, gx_{m_i-r_i+1}) + p(gx_{m_i-r_i}, gx_{n_i}).$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.6) and (3.14), we obtain

$$\limsup_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) \leq \epsilon. \tag{3.17}$$

Now,

$$\begin{aligned} \epsilon &\leq p(gx_{m_i}, gx_{n_i}) \\ &\leq p(gx_{n_i}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}) \\ &\leq p(gx_{n_i}, gx_{m_i-r_i+1}) + p(gx_{m_i-r_i+1}, gx_{m_i-r_i}) + p(gx_{m_i-r_i}, gx_{m_i}). \end{aligned}$$

Passing to the upper limit as $i \to \infty$ and using conditions (3.6), (3.12) and (3.17), we get

$$\epsilon \leq \limsup_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) = \epsilon.$$

By an argument similar to that used above, we can show that

$$\liminf_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p(gx_{n_i}, gx_{m_i-r_i+1}) = \epsilon. \tag{3.18}$$

Similarly, we have

$$\lim_{i\to\infty} p(gx_{n_i+1}, gx_{m_i-r_i+1}) = \epsilon. \tag{3.19}$$

Taking the limit as $i \to \infty$ in (3.11) and using conditions (3.6), (3.14), (3.16), (3.18), we have

$$\lim_{i\to\infty} M(gx_{m_i-r_i}, gx_{n_i}) = \epsilon. \tag{3.20}$$

Passing to the upper limit as $i \to \infty$ in (3.10) and using conditions (3.19), (3.20) and lower semicontunuity of the function φ, we get

$$\begin{aligned} \epsilon &\leq \epsilon - \liminf_{i\to\infty} \varphi(p(gx_{m_i-r_i}, gx_{n_i}), p(gx_{m_i-r_i}, fx_{m_i-r_i})) \\ &\leq \epsilon - \varphi(\epsilon, 0), \end{aligned}$$

which implies that $\varphi(\epsilon, 0) = 0$, a contradiction, since $\epsilon > 0$. This proves that (gx_n) is a 0-Cauchy sequence in $g(Y)$. As $g(Y) = \cup_{i=1}^{q} g(A_i)$, it follows that $g(Y)$ is a closed subset of the 0-complete partial metric space (X, p) and hence $g(Y)$ is 0-complete. So, (gx_n) converges to some point $u \in g(Y)$ such that $p(u, u) = 0$. Therefore,

$$\lim_{n\to\infty} p(gx_n, u) = p(u, u) = 0. \tag{3.21}$$

We shall prove that $u \in \cap_{i=1}^{q} g(A_i)$.

As $x_0 \in A_{i_0}$, by $(C1)$, it follows that the sequence $(gx_{nq})_{n\geq 0} \subseteq g(A_{i_0})$. Since $g(A_{i_0})$ is closed, condition (3.21) ensures that $u \in g(A_{i_0})$. Again, by $(C1)$, we get $(gx_{nq+1})_{n\geq 0} \subseteq g(A_{i_0+1})$, where $A_{q+k} = A_k$. Proceeding as above, we obtain that $u \in g(A_{i_0+1})$. Continuing in this way, we get

$$u \in \cap_{i=1}^{q} g(A_i). \tag{3.22}$$

Now we shall show that u is a point of coincidence of f and g.

Indeed, since $u \in g(Y)$, there exists $t \in Y$ such that $u = gt$. Now, if $x_n \in A_i$ for some $i \in \{1, 2, \cdots, q\}$, then $(gt, gx_n) = (u, gx_n) \in g(A_{i-1}) \times g(A_i)$ where $A_0 = A_q$, because $u \in \cap_{i=1}^{q} g(A_i)$. By applying $(C2)$, we obtain that for all $n \in \mathbb{N}$,

$$\begin{aligned} p(ft, gx_{n+1}) &= p(ft, fx_n) \\ &\leq M(gt, gx_n) - \varphi(p(gt, gx_n), p(gt, ft)), \end{aligned} \tag{3.23}$$

where

$$M(gt, gx_n) = max\,\{p(gt, gx_n), (gt, ft), p(gx_n, fx_n), \frac{p(gt, fx_n) + p(ft, gx_n)}{2}\}.$$

By Lemma 2.8, we get

$$\lim_{n\to\infty} M(gt, gx_n) = max\{0, p(gt, ft), 0, \frac{p(ft, gt)}{2}\} = p(gt, ft).$$

Taking the upper limit as $n \to \infty$ in (3.23) and using Lemma 2.8 and lower semicontinuity of the function φ, it follows that

$$\begin{aligned} p(ft, gt) &\leq p(gt, ft) - \liminf_{n\to\infty} \varphi(p(gt, gx_n), p(gt, ft)) \\ &\leq p(gt, ft) - \varphi(0, p(gt, ft)), \end{aligned}$$

which implies that $\varphi(0, p(gt, ft)) = 0$ and hence $p(gt, ft) = 0$, that is, $gt = ft = u$.

Therefore, u is a point of coincidence of f and g such that $u \in \cap_{i=1}^q g(A_i)$ and $p(u, u) = 0$.

For uniqueness, we assume that there is another point of coincidence v of f and g such that $v \in \cap_{i=1}^q g(A_i)$ and $p(v, v) = 0$. By supposition, there exists $x \in X$ satisfying $v = gx = fx$. Taking $u \in g(A_i)$, $v \in g(A_{i+1})$ and applying $(C2)$, we have

$$\begin{aligned} p(u, v) &= p(ft, fx) \\ &\leq max\{p(gt, gx), p(gt, ft), p(gx, fx), \frac{p(gt, fx) + p(ft, gx)}{2}\} \\ &\quad -\varphi(p(gt, gx), p(gt, ft)) \\ &= max\{p(u, v), p(u, u), p(v, v), \frac{p(u, v) + p(u, v)}{2}\} \\ &\quad -\varphi(p(u, v), p(u, u)) \\ &= p(u, v) - \varphi(p(u, v), 0). \end{aligned}$$

This gives that $\varphi(p(u, v), 0) = 0$ and hence $p(u, v) = 0$, that is, $u = v$. Thus, f and g have a unique point of coincidence $u \in \cap_{i=1}^q g(A_i)$ and $p(u, u) = 0$.

If f and g are weakly compatible, then by Proposition 2.13, f and g have a unique common fixed point in $\cap_{i=1}^q g(A_i)$. □

Corollary 3.2. *Let (X, p) be a 0-complete partial metric space and let f, $g : X \to X$ be self mappings. Suppose that $f(X) \subseteq g(X)$ and $g(X)$ is a closed subset of (X, p). If there exists $\varphi \in \Phi$ such that*

$$p(fx, fy) \leq M(gx, gy) - \varphi(p(gx, gy), p(gx, fx))$$

for all $x, y \in X$, then f and g have a unique point of coincidence u in $g(X)$ such that $p(u, u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from Theorem 3.1 by taking $A_1 = A_2 = \cdots = A_q = X$. □

Corollary 3.3. *Let (X, p) be a 0-complete partial metric space and let $f : X \to X$ be a self mapping. Suppose there exists $\varphi \in \Phi$ such that*

$$p(fx, fy) \leq M(x, y) - \varphi(p(x, y), p(x, fx))$$

for all $x, y \in X$, where $M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{p(x,fy)+p(y,fx)}{2}\}$. Then f has a unique fixed point u in X such that $p(u, u) = 0$.

Proof. The proof follows from Theorem 3.1 by taking $A_1 = A_2 = \cdots = A_q = X$ and $g = I$, the identity map on X. □

Corollary 3.4. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$ and A_1, A_2, $\cdots$, A_q be nonempty subsets of X. Suppose the mappings f, $g : X \to X$ are such that $g(A_1)$, $g(A_2)$, $\cdots$, $g(A_q)$ are closed subsets of (X,p) and satisfy the following conditions:*

$(C1)$ $f(A_i) \subseteq g(A_{i+1})$ for $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$;
$(C3)$ there exists $r \in [0,1)$ such that

$$p(fx, fy) \leq r\, max\,\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{p(gx, fy) + p(fx, gy)}{2}\}$$

for any $(gx, gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$ with $A_{q+1} = A_1$.

Then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. From $(C3)$, we get

$$\begin{aligned} p(fx, fy) &\leq r\, max\,\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{p(gx, fy) + p(fx, gy)}{2}\} \\ &= M(gx, gy) - (1-r)\, M(gx, gy) \\ &\leq M(gx, gy) - (1-r)\, max\,\{p(gx, gy), p(gx, fx)\} \\ &= M(gx, gy) - \varphi(p(gx, gy), p(gx, fx)), \end{aligned}$$

where $\varphi(s,t) = (1-r)\, max\,\{s, t\}$, $\forall s, t \in [0, \infty)$. Obviously, $\varphi \in \Phi$. The result now follows from Theorem 3.1 by considering $\varphi(s,t) = (1-r)\, max\,\{s, t\}$, $\forall s, t \in [0, \infty)$. □

Corollary 3.5. *Let (X,p) be a 0-complete partial metric space and $f : X \to X$ be a mapping. If there exists $r \in [0,1)$ such that*

$$p(fx, fy) \leq r\, max\,\{p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(fx, y)}{2}\} \tag{3.24}$$

for all x, $y \in X$, then f has a unique fixed point u in X with $p(u,u) = 0$.

Proof. Condition (3.24) gives that

$$\begin{aligned} p(fx, fy) &\leq r\, max\,\{p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(fx, y)}{2}\} \\ &= M(x, y) - (1-r)\, M(x, y) \\ &\leq M(x, y) - (1-r)\, max\,\{p(x, y), p(x, fx)\} \\ &= M(x, y) - \varphi(p(x, y), p(x, fx)), \end{aligned}$$

where $\varphi(s,t) = (1-r)\, max\,\{s, t\}$, $\forall s, t \in [0, \infty)$. The result follows from Theorem 3.1 by taking $A_1 = A_2 = \cdots = A_q = X$, $g = I$ and $\varphi(s,t) = (1-r)\, max\,\{s, t\}$, $\forall s, t \in [0, \infty)$. □

Corollary 3.6. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$ and A_1, A_2, $\cdots$, A_q be nonempty subsets of X. Suppose the mappings f, $g : X \to X$ are such that $g(A_1)$, $g(A_2)$, $\cdots$, $g(A_q)$ are closed subsets of (X,p) and satisfy the following conditions:*

$(C1)$ $f(A_i) \subseteq g(A_{i+1})$ for $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$;
$(C4)$ there exist α, β, γ, $\delta \geq 0$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that

$$p(fx, fy) \leq \alpha\, p(gx, gy) + \beta\, p(gx, fx) + \gamma\, p(gy, fy) + \delta\, (p(gx, fy) + p(fx, gy))$$

for any $(gx, gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$ with $A_{q+1} = A_1$.

Then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u)=0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. From condition $(C4)$, we obtain

$$\begin{aligned} p(fx,fy) &\leq \alpha\, p(gx,gy)+\beta\, p(gx,fx)+\gamma\, p(gy,fy)+\delta\,(p(gx,fy)+p(fx,gy)) \\ &\leq (\alpha+\beta+\gamma+2\delta)\, M(gx,gy) \\ &= r\, M(gx,gy), \end{aligned}$$

where $r=(\alpha+\beta+\gamma+2\delta)\in[0,1)$. Thus, condition $(C3)$ holds true and Corollary 3.4 can be applied to obtain the desired result. ☐

Corollary 3.7. *Let (X,p) be a 0-complete partial metric space. Suppose the mapping $f:X\to X$ satisfies the following condition:*

$$p(fx,fy)\leq M(x,y)-\frac{p(x,y)+p(x,fx)}{2+p(x,y)+p(x,fx)}$$

for all $x,y\in X$. Then f has a unique fixed point u in X such that $p(u,u)=0$.

Proof. The proof follows from Theorem 3.1 by taking $A_1=A_2=\cdots=A_q=X$, $g=I$ and $\varphi(s,t)=\frac{s+t}{2+s+t}$, $\forall s,\, t\in[0,\infty)$. ☐

Remark 3.8. Taking $g=I$ in Theorem 3.1, we obtain Theorem 13[22]. As a special case of Corollary 3.6, we obtain several important fixed point results in partial metric spaces including Matthews version of Banach contraction theorem [18].

Next we present our second main theorem.

Theorem 3.9. *Let (X,p) be a 0-complete partial metric space and let $f,T:X\to X$ be mappings. Suppose there exists $\varphi\in\Phi$ such that*

$$p(fx,Ty)\leq N(x,y)-\varphi(p(x,y),\frac{p(x,fx)+p(y,Ty)}{2}) \tag{3.25}$$

for all $x,y\in X$, where $N(x,y)=max\left\{p(x,y),p(x,fx),p(y,Ty),\frac{p(x,Ty)+p(y,fx)}{2}\right\}$. Then f and T have a unique common fixed point u in X with $p(u,u)=0$.

Proof. We first prove that u is a fixed point of T if and only if u is a fixed point of f with $p(u,u)=0$.

Suppose that u is a fixed point of T, that is, $Tu=u$. Then, by using condition (3.25), we obtain

$$\begin{aligned} p(fu,u) &= p(fu,Tu) \\ &\leq N(u,u)-\varphi(p(u,u),\frac{p(u,fu)+p(u,Tu)}{2}), \end{aligned}$$

where

$$\begin{aligned} N(u,u) &= max\left\{p(u,u),p(u,fu),p(u,Tu),\frac{p(u,Tu)+p(u,fu)}{2}\right\} \\ &= max\left\{p(u,u),p(u,fu),\frac{p(u,u)+p(u,fu)}{2}\right\} \\ &= max\left\{p(u,u),p(u,fu)\right\} \\ &= p(u,fu). \end{aligned}$$

Therefore,

$$p(fu,u) \le p(u,fu) - \varphi(p(u,u), \frac{p(u,fu)+p(u,u)}{2}),$$

which implies that $\varphi(p(u,u), \frac{p(u,fu)+p(u,u)}{2}) = 0$. This gives that $\frac{p(u,fu)+p(u,u)}{2} = p(u,u) = 0$, that is, $p(u,fu) = 0$ and hence $fu = u$ with $p(u,u) = 0$.

By an argument similar to that used above, we can show that if u is a fixed point of f, then u is also a fixed point of T with $p(u,u) = 0$.

Let $x_0 \in X$ be arbitrary. We can construct a sequence (x_n) in X such that

$$x_n = \begin{cases} fx_{n-1}, \ if \ n \ is \ odd, \\ \\ Tx_{n-1}, \ if \ n \ is \ even. \end{cases}$$

We assume that $x_n \neq x_{n-1}$ for every $n \in \mathbb{N}$. If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_{2n} = fx_{2n}$ and hence x_{2n} is a fixed point of f. By our previous discussion, it follows that x_{2n} is also a fixed point of T. So, x_{2n} becomes a common fixed point of f and T. The case $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N} \cup \{0\}$ can be treated similarly to achieve our goal. Therefore, $p(x_n, x_{n-1}) > 0$, $\forall n \in \mathbb{N}$ and hence

$$\varphi(p(x_n, x_{n-1}), \frac{p(x_n, x_{n-1}) + p(x_{m+1}, x_m)}{2}) > 0, \forall n, \ m \in \mathbb{N}. \tag{3.26}$$

We now show that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.

By using condition (3.25), we obtain

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &= p(fx_{2n}, Tx_{2n+1}) \\ &\le N(x_{2n}, x_{2n+1}) \\ &\quad -\varphi(p(x_{2n}, x_{2n+1}), \frac{p(x_{2n}, fx_{2n}) + p(x_{2n+1}, Tx_{2n+1})}{2}), \end{aligned}$$

where

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= max \begin{Bmatrix} p(x_{2n}, x_{2n+1}), p(x_{2n}, fx_{2n}), p(x_{2n+1}, Tx_{2n+1}), \\ \frac{p(x_{2n}, Tx_{2n+1}) + p(x_{2n+1}, fx_{2n})}{2} \end{Bmatrix} \\ &= max \begin{Bmatrix} p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \\ \frac{p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})}{2} \end{Bmatrix} \\ &\le max \begin{Bmatrix} p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \\ \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2} \end{Bmatrix} \\ &= max\,\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

Therefore,

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &\le max\,\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} \\ &\quad -\varphi(p(x_{2n}, x_{2n+1}), \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2}). \end{aligned} \tag{3.27}$$

If $max\,\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} = p(x_{2n+1}, x_{2n+2})$, then by using (3.26), we obtain from condition (3.27) that

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &\leq p(x_{2n+1}, x_{2n+2}) \\ &\quad -\varphi(p(x_{2n}, x_{2n+1}), \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2}) \\ &< p(x_{2n+1}, x_{2n+2}), \end{aligned}$$

which is a contradiction. Therefore,

$$max\,\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} = p(x_{2n}, x_{2n+1}).$$

Thus, condition (3.27) becomes

$$\begin{aligned} p(x_{2n+1}, x_{2n+2}) &\leq p(x_{2n}, x_{2n+1}) \\ &\quad -\varphi(p(x_{2n}, x_{2n+1}), \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2}) \\ &< p(x_{2n}, x_{2n+1}). \end{aligned} \tag{3.28}$$

Similarly, we can show that

$$\begin{aligned} p(x_{2n}, x_{2n+1}) &\leq p(x_{2n-1}, x_{2n}) - \varphi(p(x_{2n-1}, x_{2n}), \frac{p(x_{2n+1}, x_{2n}) + p(x_{2n-1}, x_{2n})}{2}) \\ &< p(x_{2n-1}, x_{2n}). \end{aligned} \tag{3.29}$$

Combining conditions (3.28) and (3.29), we get

$$\begin{aligned} p(x_n, x_{n+1}) &\leq p(x_{n-1}, x_n) - \varphi(p(x_{n-1}, x_n), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}) \\ &< p(x_{n-1}, x_n), \ \forall n \in \mathbb{N}. \end{aligned} \tag{3.30}$$

Thus, $(p(x_n, x_{n+1}))$ is a nonincreasing sequence of positive numbers. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{3.31}$$

Taking the upper limit as $n \to \infty$ in (3.30) and using (3.31) and lower semicontinuity of φ, we obtain

$$\begin{aligned} r &\leq r - \liminf_{n\to\infty} \varphi(p(x_{n-1}, x_n), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}) \\ &\leq r - \varphi(r, r), \end{aligned}$$

which implies that $\varphi(r, r) = 0$ and hence $r = 0$. Therefore,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.32}$$

We shall show that (x_n) is a 0-Cauchy sequence in X.

It is sufficient to show that (x_{2n}) is a 0-Cauchy sequence. If possible, suppose that (x_{2n}) is not a 0-Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find two subsequences (x_{2m_i}) and (x_{2n_i}) of (x_{2n}) such that n_i is the smallest positive integer for which

$$p(x_{2m_i}, x_{2n_i}) \geq \epsilon \ for \ n_i > m_i > i. \tag{3.33}$$

This implies that

$$p(x_{2m_i}, x_{2n_i-2}) < \epsilon. \tag{3.34}$$

By repeated use of (p_4) and by condition (3.34), we have

$$\begin{aligned} p(x_{2n_i+1}, x_{2m_i}) &\leq p(x_{2n_i+1}, x_{2n_i}) + p(x_{2n_i}, x_{2m_i}) - p(x_{2n_i}, x_{2n_i}) \\ &\leq p(x_{2n_i+1}, x_{2n_i}) + p(x_{2n_i}, x_{2n_i-1}) \\ &\quad + p(x_{2n_i-1}, x_{2n_i-2}) + p(x_{2n_i-2}, x_{2m_i}) \\ &< p(x_{2n_i+1}, x_{2n_i}) + p(x_{2n_i}, x_{2n_i-1}) \\ &\quad + p(x_{2n_i-1}, x_{2n_i-2}) + \epsilon. \end{aligned}$$

Passing to the upper limit as $i \to \infty$, we get

$$\limsup_{n\to\infty} p(x_{2n_i+1}, x_{2m_i}) \leq \epsilon.$$

From (3.33), we get

$$\epsilon \leq p(x_{2m_i}, x_{2n_i}) \leq p(x_{2m_i}, x_{2n_i+1}) + p(x_{2n_i+1}, x_{2n_i}).$$

Taking the upper limit as $i \to \infty$, we have

$$\epsilon \leq \limsup_{i\to\infty} p(x_{2m_i}, x_{2n_i+1}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(x_{2m_i}, x_{2n_i+1}) = \epsilon.$$

Similarly, $\liminf\limits_{i\to\infty} p(x_{2m_i}, x_{2n_i+1}) = \epsilon$. Therefore,

$$\lim_{i\to\infty} p(x_{2m_i}, x_{2n_i+1}) = \epsilon. \tag{3.35}$$

Again,

$$\begin{aligned} p(x_{2n_i}, x_{2m_i-1}) &\leq p(x_{2n_i}, x_{2n_i-1}) + p(x_{2n_i-1}, x_{2n_i-2}) \\ &\quad + p(x_{2n_i-2}, x_{2m_i}) + p(x_{2m_i}, x_{2m_i-1}) \\ &< \epsilon + p(x_{2n_i}, x_{2n_i-1}) + p(x_{2n_i-1}, x_{2n_i-2}) + p(x_{2m_i}, x_{2m_i-1}). \end{aligned}$$

Passing to the upper limit as $i \to \infty$, we obtain

$$\limsup_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) \leq \epsilon. \tag{3.36}$$

Also,

$$\epsilon \leq p(x_{2n_i}, x_{2m_i}) \leq p(x_{2n_i}, x_{2m_i-1}) + p(x_{2m_i-1}, x_{2m_i}).$$

Taking the upper limit as $i \to \infty$ and using conditions (3.32) and (3.36), we get

$$\epsilon \leq \limsup_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) \leq \epsilon.$$

Thus,

$$\limsup_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) = \epsilon.$$

Similarly, we can obtain

$$\liminf_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) = \epsilon.$$

Therefore,

$$\lim_{i\to\infty} p(x_{2n_i}, x_{2m_i-1}) = \epsilon. \tag{3.37}$$

By an argument similar to that used above, we can obtain

$$\lim_{i\to\infty} p(x_{2n_i}, x_{2m_i}) = \epsilon \tag{3.38}$$

and

$$\lim_{i\to\infty} p(x_{2n_i+1}, x_{2m_i-1}) = \epsilon. \tag{3.39}$$

By using condition (3.25), we have

$$\begin{aligned} p(x_{2n_i+1}, x_{2m_i}) &= p(fx_{2n_i}, Tx_{2m_i-1}) \\ &\le N(x_{2n_i}, x_{2m_i-1}) \\ &\quad -\varphi(p(x_{2n_i}, x_{2m_i-1}), \frac{p(x_{2n_i}, fx_{2n_i}) + p(x_{2m_i-1}, x_{2m_i})}{2}), \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} N(x_{2n_i}, x_{2m_i-1}) &= max \left\{ \begin{array}{l} p(x_{2n_i}, x_{2m_i-1}), p(x_{2n_i}, fx_{2n_i}), p(x_{2m_i-1}, Tx_{2m_i-1}), \\ \frac{p(x_{2n_i}, Tx_{2m_i-1}) + p(x_{2m_i-1}, fx_{2n_i})}{2} \end{array} \right\} \\ &= max \left\{ \begin{array}{l} p(x_{2n_i}, x_{2m_i-1}), p(x_{2n_i}, x_{2n_i+1}), p(x_{2m_i-1}, x_{2m_i}), \\ \frac{p(x_{2n_i}, x_{2m_i}) + p(x_{2m_i-1}, x_{2n_i+1})}{2} \end{array} \right\}. \end{aligned} \tag{3.41}$$

Taking the limit as $i \to \infty$ in (3.41) and using conditions (3.32), (3.37), (3.38), (3.39), we get

$$\lim_{i\to\infty} N(x_{2n_i}, x_{2m_i-1}) = max\,\{\epsilon,\ 0,\ 0,\ \frac{\epsilon+\epsilon}{2}\} = \epsilon. \tag{3.42}$$

Passing to the upper limit as $i \to \infty$ in (3.40) and using conditions (3.32), (3.35), (3.37), (3.42) and lower semicontinuity of φ, we get

$$\begin{aligned} \epsilon &= \limsup_{i\to\infty} p(x_{2n_i+1}, x_{2m_i}) \\ &\le \limsup_{i\to\infty} N(x_{2n_i}, x_{2m_i-1}) \\ &\quad - \liminf_{i\to\infty} \varphi(p(x_{2n_i}, x_{2m_i-1}), \frac{p(x_{2n_i}, x_{2n_i+1}) + p(x_{2m_i-1}, x_{2m_i})}{2}) \\ &\le \epsilon - \varphi(\epsilon, 0), \end{aligned}$$

which implies that $\varphi(\epsilon, 0) = 0$ and hence $\epsilon = 0$, a contradiction. Therefore, (x_n) is a 0-Cauchy sequence in X. Since (X, p) is 0-complete, there exists $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = p(u, u) = 0$. This ensures that $\lim_{n\to\infty} p(x_{2n}, u) = p(u, u) = 0$ and $\lim_{n\to\infty} p(x_{2n+1}, u) = p(u, u) = 0$. Moreover, by Lemma 2.8, $\lim_{n\to\infty} p(x_{2n}, Tu) = p(u, Tu)$ and $\lim_{n\to\infty} p(x_{2n+1}, Tu) = p(u, Tu)$.

By using condition (3.25), we obtain

$$\begin{aligned} p(x_{2n+1}, Tu) &= p(fx_{2n}, Tu) \\ &\le N(x_{2n}, u)) - \varphi(p(x_{2n}, u), \frac{p(x_{2n}, fx_{2n}) + p(u, Tu)}{2}), \end{aligned} \tag{3.43}$$

where

$$\begin{aligned} N(x_{2n}, u) &= max \left\{ \begin{array}{l} p(x_{2n}, u), p(x_{2n}, fx_{2n}), p(u, Tu), \\ \frac{p(x_{2n}, Tu) + p(u, fx_{2n})}{2} \end{array} \right\} \\ &= max \left\{ \begin{array}{l} p(x_{2n}, u), p(x_{2n}, x_{2n+1}), p(u, Tu), \\ \frac{p(x_{2n}, Tu) + p(u, x_{2n+1})}{2} \end{array} \right\} \\ &\to p(u, Tu)\ as\ n \to \infty. \end{aligned}$$

Taking the upper limit as $n \to \infty$ in (3.43), we have

$$\begin{aligned} p(u,Tu) &\leq p(u,Tu) - \liminf_{i\to\infty} \varphi(p(x_{2n},u), \frac{p(x_{2n},x_{2n+1}) + p(u,Tu)}{2}) \\ &\leq p(u,Tu) - \varphi(0, \frac{1}{2}\, p(u,Tu)), \end{aligned}$$

which gives that $\varphi(0, \frac{1}{2}\, p(u,Tu)) = 0$. This assures that $p(u,Tu) = 0$ and hence $Tu = u$. By our previous discussion, u is also a fixed point of f. Therefore, u is a common fixed point of f and T with $p(u,u) = 0$.

For uniqueness, let v be another common fixed point of f and T in X with $p(v,v) = 0$. By applying condition (3.25), we get

$$p(u,v) = p(fu,Tv) \leq N(u,v) - \varphi(p(u,v), \frac{p(u,fu) + p(v,Tv)}{2}), \tag{3.44}$$

where

$$\begin{aligned} N(u,v) &= max\left\{p(u,v), p(u,fu), p(v,Tv)\}, \frac{p(u,Tv) + p(v,fu)}{2}\right\} \\ &= max\{p(u,v), 0, 0, p(u,v)\} \\ &= p(u,v). \end{aligned}$$

Thus, condition (3.44) becomes

$$p(u,v) \leq p(u,v) - \varphi(p(u,v), 0),$$

which implies that $\varphi(p(u,v), 0) = 0$ and hence $p(u,v) = 0$, that is, $u = v$. Therefore, f and T have a unique common fixed point in X. □

Corollary 3.10. *Let (X,p) be a 0-complete partial metric space and let the mappings $f, T : X \to X$ be such that*

$$p(fx,Ty) \leq r\, max\left\{p(x,y), p(x,fx), p(y,Ty), \frac{p(x,Ty) + p(y,fx)}{2}\right\} \tag{3.45}$$

for all $x, y \in X$, where $r \in [0,1)$ is a constant. Then f and T have a unique common fixed point u in X with $p(u,u) = 0$.

Proof. From condition (3.45), we have

$$\begin{aligned} p(fx,Ty) &\leq r\, max\{p(x,y), p(x,fx), p(y,Ty), \frac{p(x,Ty) + p(y,fx)}{2}\} \\ &= N(x,y) - (1-r)\, N(x,y) \\ &\leq N(x,y) - (1-r)\, max\{p(x,y), \frac{p(x,fx) + p(y,Ty)}{2}\} \\ &= N(x,y) - \varphi(p(x,y), \frac{p(x,fx) + p(y,Ty)}{2}), \end{aligned}$$

where $\varphi(s,t) = (1-r)\, max\{s,t\}$, $\forall s,\, t \in [0,\infty)$. Obviously, $\varphi \in \Phi$. Now applying Theorem 3.9 we can obtain the desired result. □

Corollary 3.11. *Let (X,p) be a 0-complete partial metric space and let $f : X \to X$ be a mapping. Suppose there exists $\varphi \in \Phi$ such that*

$$p(fx,fy) \leq N'(x,y) - \varphi(p(x,y), \frac{p(x,fx) + p(y,fy)}{2})$$

for all $x, y \in X$, where $N'(x,y) = max\left\{p(x,y), p(x,fx), p(y,fy), \frac{p(x,fy)+p(y,fx)}{2}\right\}$. Then f has a unique fixed point u in X with $p(u,u) = 0$.

Proof. The proof follows from Theorem 3.9 by considering $T = f$. □

Corollary 3.12. *Let (X, p) be a 0-complete partial metric space and let the mappings $f, T : X \to X$ be such that*

$$p(fx, Ty) \leq \alpha\, p(x, y) + \beta\, p(x, fx) + \gamma\, p(y, Ty) + \delta\, (p(x, Ty) + p(y, fx)) \tag{3.46}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\delta < 1$. Then f and T have a unique common fixed point u in X with $p(u, u) = 0$.

Proof. From condition (3.46), we obtain

$$\begin{aligned} p(fx, Ty) &\leq \alpha\, p(x, y) + \beta\, p(x, fx) + \gamma\, p(y, Ty) + \delta\, (p(x, Ty) + p(y, fx)) \\ &\leq (\alpha + \beta + \gamma + 2\delta)\, N(x, y) \\ &= r\, N(x, y) \\ &= N(x, y) - (1 - r)\, N(x, y) \\ &\leq N(x, y) - (1 - r)\, max\{p(x, y), \frac{p(x, fx) + p(y, Ty)}{2}\} \\ &= N(x, y) - \varphi(p(x, y), \frac{p(x, fx) + p(y, Ty)}{2}), \end{aligned}$$

where $r = (\alpha + \beta + \gamma + 2\delta) \in [0, 1)$ and $\varphi(s, t) = (1 - r)\, max\{s, t\},\ \forall s, t \in [0, \infty)$. Now applying Theorem 3.9, we can obtain the desired result. □

Corollary 3.13. *Let (X, p) be a 0-complete partial metric space and let the mappings $f, T : X \to X$ be such that*

$$p(fx, Ty) \leq N(x, y) - \frac{p(x, y) + \frac{1}{2}\,(p(x, fx) + p(y, Ty))}{2 + p(x, y) + \frac{1}{2}\,(p(x, fx) + p(y, Ty))}$$

for all $x, y \in X$. Then f and T have a unique common fixed point u in X with $p(u, u) = 0$.

Proof. The proof follows from Theorem 3.9 by taking $\varphi : [0, \infty)^2 \to [0, \infty)$ as $\varphi(s, t) = \frac{s+t}{2+s+t}$. □

Remark 3.14. The results of this study are obtained under the weaker assumption that the underlying partial metric space is 0-complete. However, they also valid if the space is complete.

Finally, we give some examples to justify the validity of our main results.

Example 3.15. Let $X = \{[1 - 3^{-n}, 1] : n \in \mathbb{N}\} \cup \{[1, 1 + 3^{-n}] : n \in \mathbb{N}\} \cup \{\{1\}\}$, where $\{1\} = [1, 1]$. We define $p : X \times X \to \mathbb{R}^+$ by $p([a, b], [c, d]) = max\,\{b, d\} - min\,\{a, c\}$. Then (X, p) is a 0-complete partial metric space. Let $A_1 = \{[1-3^{-n}, 1] : n \in \mathbb{N}\} \cup \{\{1\}\}$ and $A_2 = \{[1, 1 + 3^{-n}] : n \in \mathbb{N}\} \cup \{\{1\}\}$. Obviously, $X = A_1 \cup A_2$. Define mappings $f, g : X \to X$ by

$$fx = \begin{cases} [1, 1 + 3^{-(n+2)}], & if\ \ x = [1 - 3^{-n}, 1], \\ [1 - 3^{-(n+2)}, 1], & if\ \ x = [1, 1 + 3^{-n}], \\ \{1\}, & if\ \ x = \{1\} \end{cases}$$

and

$$gx = \begin{cases} [1-3^{-(n+1)}, 1], \quad if \;\; x = [1-3^{-n}, 1], \\ [1, 1+3^{-(n+1)}], \;\; if \;\; x = [1, 1+3^{-n}], \\ \{1\}, \quad if \;\; x = \{1\}. \end{cases}$$

Then, $f(A_1) \subseteq g(A_2)$, $f(A_2) \subseteq g(A_1)$ and $g(A_1)$, $g(A_2)$ are closed subsets of (X,p). Thus, condition $(C1)$ holds true. We now verify condition $(C2)$ with the control function $\varphi : [0,\infty)^2 \to [0,\infty)$ given by $\varphi(s,t) = \frac{1}{2}\, max\,\{s,\, t\}$. We now consider the following cases:

Case-I: $x = [1-3^{-n}, 1] \in A_1$, $y = [1, 1+3^{-k}] \in A_2$, $n, k \in \mathbb{N}$ with $n < k$.

In this case, we have $3^{-k} < 3^{-n}$ and $3^{-k} \le 3^{-(n+1)}$. Then,

$$p(fx,fy) = p([1,1+3^{-(n+2)}],[1-3^{-(k+2)},1]) = \frac{1}{9}\,(3^{-n}+3^{-k}) < \frac{2}{9}.\,3^{-n}.$$

$$\begin{aligned} p(gx,gy) &= p([1-3^{-(n+1)},1],[1,1+3^{-(k+1)}]) = 3^{-(k+1)} + 3^{-(n+1)} \\ &= \frac{1}{3}.\,3^{-k} + 3^{-(n+1)} \le (\frac{1}{3}+1)\,3^{-(n+1)} = \frac{4}{9}.\,3^{-n}. \end{aligned}$$

$$p(gx,fx) = p([1-3^{-(n+1)},1],[1,1+3^{-(n+2)}]) = 3^{-(n+2)} + 3^{-(n+1)} = \frac{4}{9}.\,3^{-n}.$$

$$p(gy,fy) = p([1,1+3^{-(k+1)}],[1-3^{-(k+2)},1]) = 3^{-(k+1)} + 3^{-(k+2)} < \frac{4}{9}.\,3^{-n}.$$

$$p(gx,fy) = p([1-3^{-(n+1)},1],[1-3^{-(k+2)},1]) = 3^{-(n+1)} = \frac{1}{3}.\,3^{-n}.$$

$$p(fx,gy) = p([1,1+3^{-(n+2)}],[1,1+3^{-(k+1)}]) = 3^{-(n+2)} = \frac{1}{9}.\,3^{-n}.$$

Now,

$$\frac{p(gx,fy)+p(fx,gy)}{2} = \frac{1}{2}(\frac{1}{3}.\,3^{-n} + \frac{1}{9}.\,3^{-n}) = \frac{2}{9}.\,3^{-n} < \frac{4}{9}.\,3^{-n}.$$

Thus, $M(gx,gy) = \frac{4}{9}.\,3^{-n}$ and

$$\varphi(p(gx,gy),p(gx,fx)) = \frac{1}{2}\,max\,\{p(gx,gy),p(gx,fx)\} = \frac{2}{9}.\,3^{-n}.$$

Therefore,

$$p(fx,fy) < \frac{2}{9}.\,3^{-n} = M(gx,gy) - \varphi(p(gx,gy),p(gx,fx)).$$

Case-II: $x = [1-3^{-n}, 1] \in A_1$, $y = [1, 1+3^{-k}] \in A_2$, $n, k \in \mathbb{N}$ with $n > k$.

In this case, we have $3^{-k} > 3^{-n}$ and $3^{-n} \le 3^{-(k+1)}$. Then, $p(fx,fy) < \frac{2}{9}.\,3^{-k}$, $p(gx,gy) = \frac{1}{3}\,(3^{-k}+3^{-n}) \le \frac{4}{9}.\,3^{-k}$, $p(gx,fx) = \frac{4}{9}.\,3^{-n}$, $p(gy,fy) = \frac{4}{9}.\,3^{-k}$ and $p(gx,fy) = 3^{-(k+2)} = \frac{1}{9}.\,3^{-k}$, $p(fx,gy) = 3^{-(k+1)} = \frac{1}{3}.\,3^{-k}$. So, $\frac{p(gx,fy)+p(fx,gy)}{2} = \frac{2}{9}.\,3^{-k}$.

Moreover, we note that $p(gx,gy) = \frac{1}{3}\,(3^{-k}+3^{-n}) > \frac{2}{3}.\,3^{-n} > \frac{4}{9}.\,3^{-n} = p(gx,fx)$.

Thus, $M(gx,gy) = \frac{4}{9}.\,3^{-k}$ and

$$\varphi(p(gx,gy),p(gx,fx)) = \frac{1}{2}\,max\,\{p(gx,gy),p(gx,fx)\} = \frac{1}{2}\,p(gx,gy) = \frac{1}{6}\,(3^{-k}+3^{-n}).$$

Therefore,

$$\begin{aligned} M(gx,gy)-\varphi(p(gx,gy),p(gx,fx)) &= \frac{4}{9}.3^{-k}-\frac{1}{6}\left(3^{-k}+3^{-n}\right) \\ &\geq \frac{4}{9}.3^{-k}-\frac{1}{6}.3^{-k}-\frac{1}{6}.3^{-(k+1)} \\ &= \frac{2}{9}.3^{-k} \\ &> p(fx,fy). \end{aligned}$$

Case-III: $x=[1-3^{-n},1]\in A_1,\ y=[1,1+3^{-k}]\in A_2,\ n,\,k\in\mathbb{N}$ with $n=k$.

Then,
$p(fx,fy)=\frac{2}{9}.3^{-n}$, $p(gx,gy)=\frac{2}{3}.3^{-n}$, $p(gx,fx)=\frac{4}{9}.3^{-n}$, $p(gy,fy)=\frac{4}{9}.3^{-n}$ and $p(gx,fy)=\frac{1}{3}.3^{-n}$, $p(fx,gy)=\frac{1}{3}.3^{-n}$. So, $\frac{p(gx,fy)+p(fx,gy)}{2}=\frac{1}{3}.3^{-n}$.

Thus, $M(gx,gy)=\frac{4}{9}.3^{-n}$ and

$$\varphi(p(gx,gy),p(gx,fx))=\frac{1}{2}\,max\left\{p(gx,gy),p(gx,fx)\right\}=\frac{2}{9}\,3^{-n}.$$

Therefore,

$$p(fx,fy)=\frac{2}{9}.3^{-n}=M(gx,gy)-\varphi(p(gx,gy),p(gx,fx)).$$

Case-IV: $x=[1-3^{-n},1]\in A_1$, $n\in\mathbb{N}$, $y=\{1\}\in A_2$.

Then,

$$p(fx,fy)=p([1,1+3^{-(n+2)}],\{1\})=3^{-(n+2)}=\frac{1}{9}.3^{-n}.$$

$$p(gx,gy)=p([1-3^{-(n+1)},1],\{1\})=3^{-(n+1)}=\frac{1}{3}.3^{-n}.$$

$$p(gx,fx)=p([1-3^{-(n+1)},1],[1,1+3^{-(n+2)}])=3^{-(n+2)}+3^{-(n+1)}=\frac{4}{9}.3^{-n}.$$

$p(gy,fy)=p(\{1\},\{1\})=0$, $p(gx,fy)=p([1-3^{-(n+1)},1],\{1\})=\frac{1}{3}.3^{-n}$, $p(fx,gy)=p([1,1+3^{-(n+2)}],\{1\})=\frac{1}{9}.3^{-n}$. Thus, $M(gx,gy)=\frac{4}{9}.3^{-n}$ and

$$\varphi(p(gx,gy),p(gx,fx))=\frac{1}{2}\,max\left\{p(gx,gy),p(gx,fx)\right\}=\frac{2}{9}\,3^{-n}.$$

Therefore,

$$p(fx,fy)=\frac{1}{9}.3^{-n}<\frac{2}{9}.3^{-n}=M(gx,gy)-\varphi(p(gx,gy),p(gx,fx)).$$

Case-V: $x=\{1\}\in A_1$, $y=[1,1+3^{-n}]\in A_2$, $n\in\mathbb{N}$.

In this case, we have

$$p(fx,fy)=\frac{1}{9}.3^{-n},\ M(gx,gy)=\frac{4}{9}.3^{-n},\ \varphi(p(gx,gy),p(gx,fx))=\frac{1}{6}.3^{-n}.$$

Therefore,

$$p(fx,fy)=\frac{1}{9}.3^{-n}<\frac{5}{18}.3^{-n}=M(gx,gy)-\varphi(p(gx,gy),p(gx,fx)).$$

Case-VI: $x=y=\{1\}$ is trivial.

The other possibility is treated similarly. Moreover, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.1 and $\{1\}$ is the unique common fixed point of f and g in $g(A_1) \cap g(A_2)$ with $p(\{1\}, \{1\}) = 0$.

The following example supports our Theorem 3.9.

Example 3.16. Let $X = [0, 1]$ and define $p : X \times X \to \mathbb{R}^+$ by $p(x, y) = max\,\{x,\, y\}$ for all $x,\, y \in X$. Then (X, p) is a 0-complete partial metric space. Let $f,\, T : X \to X$ be defined by

$$fx = \frac{x^2}{1+x} \text{ and } Tx = \frac{x^2}{2+x}.$$

Define $\varphi : [0, \infty)^2 \to [0, \infty)$ by $\varphi(s, t) = \frac{1}{2}\, max\,\{s,\, t\}$.

We now verify condition (3.25) for all $x,\, y \in X$.

Case-I: $x, y \in X$ with $y \leq x$.
Then,

$$p(fx, Ty) = max\,\{\frac{x^2}{1+x}, \frac{y^2}{2+y}\} \leq max\,\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\} = \frac{x^2}{1+x} \leq \frac{x}{2},$$

$$\begin{aligned} N(x, y) &= max\left\{p(x, y), p(x, fx), p(y, Ty), \frac{p(x, Ty) + p(y, fx)}{2}\right\} \\ &= max\left\{x,\, x,\, y,\, \frac{x + max\,\{y,\, \frac{x^2}{1+x}\}}{2}\right\} \\ &= x \end{aligned}$$

and

$$\varphi(p(x, y), \frac{p(x, fx) + p(y, Ty)}{2}) = \varphi(x, \frac{x+y}{2}) = \frac{1}{2}x.$$

Thus,

$$p(fx, Ty) \leq \frac{x}{2} = N(x, y) - \varphi(p(x, y), \frac{p(x, fx) + p(y, Ty)}{2}).$$

Case-II: $x, y \in X$ with $x \leq y$.
This case can be treated in a similar way to that of Case-I and we compute $p(fx, Ty) \leq \frac{y}{2}$, $N(x, y) = y$, $\varphi(p(x, y), \frac{p(x,fx)+p(y,Ty)}{2}) = \frac{y}{2}$. Thus,

$$p(fx, Ty) \leq \frac{y}{2} = N(x, y) - \varphi(p(x, y), \frac{p(x, fx) + p(y, Ty)}{2}).$$

Thus, we have all the conditions of Theorem 3.9 and 0 is the unique common fixed point of f and T in X with $p(0, 0) = 0$.

4. Conclusion

Matthews [18] exploited the idea of fixed points of contractive mappings in partial metric spaces. In recent investigations, the study of fixed point theory involving a control function takes a vital role in many aspects. In this paper, we used control functions to obtain some coincidence points and common fixed point results in partial metric spaces. Significance of this study lies in the fact that the results are obtained under the weaker assumption that the underlying partial metric space is 0-complete. However, they also valid if the space is complete.

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APPROXIMATING FIXED POINTS OF THE NEW SP*-ITERATION FOR GENERALIZED $\alpha-$NONEXPANSIVE MAPPINGS IN $CAT(0)$ SPACES

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ABSTRACT. In this paper, we study the convergence of the newly defined SP*-iteration to fixed point for the generalized $\alpha-$nonexpansive mappings in $CAT(0)$ spaces. Our results improve and extend some recently results in the literature of fixed point theory in $CAT(0)$ spaces.

KEYWORDS: Fixed point, iteration process, $\Delta-$convergence, $CAT(0)$ space, generalized $\alpha-$nonexpansive mappings.
AMS Subject Classification: 47H09; 47H10.

1. Introduction

The existence of a fixed point is very important in several areas of mathematics and other sciences. The numerous numbers of researchers attracted in these direction and developed iterative process has been investigated to approximate fixed point for not only nonexpansive mapping, but also for some wider class of nonexpansive mappings. This is an active area of research, several well known scientists in the world paid and still pay attention to the qualitative study of iteration methods. The well-known Banach contraction theorem use Picard iteration process [25] for approximation of fixed point. Some of the well-known iterative processes are those of Mann [21], Ishikawa [13], Noor [23] , SP-Iteration [26], Picard Normal S-iteration [14] and so on.

It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces, any convex subset of a Euclidian space $\mathbb{R}^n$ with the induced metric, the complex Hilbert ball with a hyperbolic metric and many others. For

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discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [4]. Burago et al. [5] contains a somewhat more elementary treatment, and Gromov [12] a deeper study. Fixed point theory in $CAT(0)$ space has been first studied by Kirk (see [15],[16]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. On the other hand, we know that every Banach space is a $CAT(0)$ space. Since then the fixed point theory in CAT(0) has been rapidly developed and much papers a appeared.(see [6],[7],[8],[9],[10],[15],[16],[17],[18]).

Recently, Kirk and Panyanak [18] used the concept of $\Delta-$convergence introduced by Lim [20] to prove on the $CAT(0)$ space analogs of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [6] obtained $\Delta-$convergence theorems for the Picard, Mann and Ishikawa iteration processes for nonexpansive mappings in the $CAT(0)$ space. In addition, the convergence results for generalized nonexpansive mappings are obtained by using different iteration processes in $CAT(0)$ spaces (see [2], [27]).

In the sequel, we need the following definitions and useful lemmas to prove our main results of this paper.

Lemma 1.1. [6] *Let X be a $CAT(0)$ space.*

(i) *For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that $d(x,z) = td(x,y)$ and $d(y,z) = (1-t)d(x,y)$. (I) We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (I).*

(ii) *For $x, y \in X$ and $t \in [0,1]$, we have $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$.*

Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a $CAT(0)$ space X. For $x \in X$, set $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(K,\{x_n\}) = \inf_n \{r(x,\{x_n\}) : x \in K\}$ and the asymptotic center of x_n relative to K is the set $A(K,\{x_n\}) = \{x \in K : r(x,\{x_n\}) = r(K,\{x_n\})\}$. It is known that, in a $CAT(0)$ space, $A(K,\{x_n\})$ consists of exactly one point; please, see [9], Proposition 7.

We now recall the definition of Δ-convergence and weak convergence in $CAT(0)$ space.

Definition 1.2. ([20],[18]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to $\Delta-$converge to $x \in X$ if x is the unique asymptotic center of u_n for every subsequence $\{x_n\}$ of $\{x_n\}$.In this case we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x is the $\Delta-$limit of $\{x_n\}$.

Lemma 1.3. *([18]) Given $\{x_n\} \in X$ such that $\{x_n\}$, $\Delta-$converges to x and given $y \in X$ with $y \neq x$, then $\limsup_{n\to\infty} d(x_n,x) < \limsup_{n\to\infty} d(x_n,y)$.*

In a Banach space the above condition is known as the Opial property.

Let (X,d) be a metric space and K a nonempty subset of X. Let $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$ and we denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. X is a complete $CAT(0)$ space, K is a nonempty convex subset of X and $T : K \to K$ is a mapping. T is called nonexpansive if for each $x, y \in K$, $d(Tx,Ty) \leq d(x,y)$.

Lemma 1.4. *([18]) Every bounded sequence in a complete $CAT(0)$ space always has a $\Delta-$convergent subsequence.*

Lemma 1.5. *([8]) Let K be closed convex subset of a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in K. Then asymptotic center of $\{x_n\}$ is in K.*

Lemma 1.6. [19] *Suppose that X is a complete $CAT(0)$ space and $x \in X$. Let T be a mapping on K. $0 < k \leq t_n \leq m < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\limsup_{n\to\infty} d(t_n x_n \oplus (1-t_n)y_n, x) = r$ hold for $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

A number of extensions and generalizations of nonexpansive mappings have been considered by many mathematicians, see [[1],[11] [24], [29]], in recent years. In 2008, Suzuki [29] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called (C) *condition*. Let K be a nonempty convex subset of a Banach space X, a mapping $T : K \to K$ is satisfy condition (C) if for all $x, y \in K$, $\frac{1}{2}d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq d(x, y)$. Suzuki [29] showed that the mapping satisfying *condition* (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In 2011, Aoyama and Kohsaka [1] introduced the class of α-nonexpansive mappings in the setting of Banach spaces and obtained some fixed point results for such mappings.

In 2017, Pant and Shukla [24] introduced a new type of nonexpansive mappings called generalized α-nonexpansive mappings and obtain a number of existence and convergence theorems.This new class of nonlinear mappings properly contains non-expansive, Suzuki-type generalized nonexpansive mappings and partially extends firmly nonexpansive and α-nonexpansive mappings.

In what follows, we give the following definition and lemma to be used in main results.

Definition 1.7. [24]. A mapping $T : K \to K$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0, 1)$ and for each $x, y \in K$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha d(Tx, y) + \alpha d(Ty, x) + (1 - 2\alpha)d(x, y). \tag{1.1}$$

The next simple examples can show these facts. We see that T is a generalized α-nonexpansive mappings but does not satisfy condition (C).

Example 1.8. Let $K = [0, 6]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define a mapping $T : K \to K$ by

$$Tx = \begin{cases} 0, & x \neq 6 \\ 3, & x = 6 \end{cases}$$

For $x \in (3, 4]$ and $y = 6$, $\frac{1}{2}d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) = 3 > 6 - x = d(x, y)$. Thus T does not satisfy Suzuki's condition (C). However, T is a generalized $\alpha-$nonexpansive mapping with $\alpha \geq \frac{1}{3}$.

Example 1.9. Let $M = \{(0, 0), (3, 0), (0, 6), (6, 0), (6, 7), (7, 6)\}$ is a subset of $\mathbb{R}^2$. Define a norm $\|.\|$ by $\|(x_1, x_2)\| = |x_1| + |x_2|$. Then $(M, \|.\|)$ is a Banach space.

Define a mapping $T : M \to M$ by $T(0, 0) = (0, 0)$, $T(3, 0) = (0, 0)$, $T(0, 6) = (0, 0)$, $T(6, 0) = (3, 0)$, $T(6, 7) = (6, 0)$, $T(7, 6) = (0, 6)$.

We note that for $\alpha \geq \frac{1}{4}$,

$$\|Tx - Ty\| \leq \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\|,$$

if $(x,y) \neq ((6,7),(7,6))$. In case of $x=(6,7)$ and $y=(7,6)$, we have

$$\frac{1}{2}\|x-Tx\| = \frac{1}{2}\|y-Ty\| = \frac{7}{2} > 2 = \|x-y\|.$$

Thus, T is a generalized $\alpha-$nonexpansive mapping. However, for $x=(6,7)$ and $y=(7,6)$,

$$\begin{aligned} \|Tx-Ty\|^2 &= 144 > 90\alpha+4 && (1.2)\\ &= 49\alpha+49\alpha+(1-2\alpha).4 && (1.3)\\ &= \alpha\|Tx-y\|^2+\alpha\|Ty-x\|^2+(1-2\alpha)\|x-y\|^2. && (1.4) \end{aligned}$$

Thus, T is not an $\alpha-$nonexpansive mapping for any $\alpha<1$. Further, for $x=(6,0)$ and $y=(7,6)$

$$\frac{1}{2}\|x-Tx\| = \frac{3}{2} < 7 = \|x-y\| \quad but, \|Tx-Ty\| = 9 > 7 = \|x-y\|. \tag{1.5}$$

Thus, T does not satisfy Suzuki's condition (C).

Now we give the following well-known facts about generalized α-nonexpansive mapping, which can be found in [24].

Lemma 1.10. (1) *If T is a generalized α -nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*

(2) *If T is a generalized α -nonexpansive mapping, then $F(T)$ is closed. Moreover if X is strictly convex and K is convex, then $F(T)$ is also convex.*

(3) *If T is a generalized α -nonexpansive mapping, then for each $x,y\in K$, for each $x,y\in K$,*

$$d(x,Ty) \leq (\frac{3+\alpha}{1-\alpha})d(Tx,x)+d(x,y).$$

(4) *If X has Opial property, T is a generalized α -nonexpansive mapping, $\{x_n\}$ converges weakly to a point z and $\lim_{n\to\infty} d(x_n,Tx_n)=0$, then $z\in F(T)$.*

2. The new iteration process

Let X be a real Banach space and K be a nonempty subset of X, and $T:K\to K$ be a mapping. We have $\{a_n\},\{b_n\}$ and $\{c_n\}$ real sequences in $[0,1]$. Recently, Phuengrattana and Suantai ([26]) defined the SP-iteration as follows:

$$\begin{cases} z_n = (1-c_n)x_n + c_nTx_n,\\ y_n = (1-b_n)z_n + b_nTz_n,\\ x_{n+1} = (1-a_n)y_n + a_nTy_n, \forall n\in\mathbb{N}, \end{cases} \tag{2.1}$$

where $x_1\in K$. They showed that the Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges better than the others for the class of continuous and nondecreasing functions. In 2014, Kadioglu and Yildirim [14] introduced Picard Normal S-iteration process and they established that the rate of convergence of the Picard Normal S-iteration process is faster than other fixed point iteration process that was in existence then. The Picard Normal S-iteration[14] as follows:

$$\begin{cases} z_n = (1-b_n)x_n + b_nTx_n,\\ y_n = (1-a_n)z_n + a_nTz_n,\\ x_{n+1} = Ty_n, \forall n\in\mathbb{N}, \end{cases} \tag{2.2}$$

where $x_1 \in K$. Let's note here that some fixed points properties and demiclosedness principle for generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces are studied(see [22]). They further established strong and Δ-convergence theorems of Picard Normal S-iteration scheme generated by (2.2) for the generalized α-nonexpansive mappings in the frame work of uniformly convex hyperbolic spaces.

Motivated by above, in this paper, we introduce a new iteration called as SP*-iteration scheme:for arbitrary $x_1 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = T((1-c_n)x_n + c_nTx_n), \\ y_n = T((1-b_n)z_n + b_nTz_n), \\ x_{n+1} = T((1-a_n)y_n + a_nTy_n), \forall n \in \mathbb{N}, \end{cases} \tag{2.3}$$

First we give a useful definition that is used to determine the faster iteration which converge to the same point. The following definition about the rate of convergence is given by [3].

Definition 2.1. [3] Let α_n and β_n be two sequences of positive numbers that converge to a and b, respectively. Assume that there exists

$$\ell = \lim_{n\to\infty} \frac{|\alpha_n - a|}{|\beta_n - b|}. \tag{2.4}$$

(1) If $\ell = 0$, then it can be said that α_n converges to a faster than β_n converges to b.
(2) If $0 < \ell < \infty$, then it can be said that α_n and β_n have the same rate of convergence.

Theorem 2.2. *Let K be a nonempty closed convex subset of a norm space X. A mapping $T : K \to K$ is contraction with contraction factor $\theta \in (0,1)$ and $p \in F(T)$. Let $\{u_n\}$ defined by the iteration (2.2) and $\{x_n\}$ defined by the iteration (2.3), where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are real sequences in $[\varsigma, 1-\varsigma]$ for all $n \in \mathbb{N}$ and for some ς in $[0,1]$. Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our new iteration defined by (2.3) faster than (2.2).*

Proof. As proved in [14],

$$\|u_{n+1} - p\| \le \theta^n[(1-\varsigma(1-\theta))^2]^n\|u_1 - p\|, \tag{2.5}$$

for all $n \in \mathbb{N}$. Let

$$k_n = \theta^n[(1-\varsigma(1-\theta))^2]^n\|u_1 - p\|. \tag{2.6}$$

It follows from (2.3), we have,

$$\begin{aligned} \|z_n - p\| &= \|T((1-c_n)x_n + c_nTx_n) - p\| \\ &\le \theta[\|(1-c_n)(x_n - p) + c_n(Tx_n - p)\|] \\ &\le \theta[(1-c_n)\|x_n - p\| + c_n\theta\|x_n - p\|] \\ &= \theta[1 - c_n(1-\theta)]\|x_n - p\|. \end{aligned} \tag{2.7}$$

Similarly, using (2.3), we get

$$\begin{aligned} \|y_n - p\| &= \|T((1-b_n)z_n + b_nTz_n) - p\| \\ &\le \theta[\|(1-b_n)(z_n - p) + b_n(Tz_n - p)\|] \\ &\le \theta[(1-b_n)\|z_n - p\| + b_n\theta\|z_n - p\|] \\ &\le \theta^2[1 - b_n(1-\theta)][1 - c_n(1-\theta)]\|x_n - p\|. \end{aligned} \tag{2.8}$$

Again using (2.3), we get

$$\begin{aligned}\|x_{n+1}-p\| &= \|T((1-a_n)y_n+a_nTy_n)-p\|\\ &\le \theta[\|(1-a_n)(y_n-p)+a_n(Ty_n-p)\|]\\ &\le \theta[(1-a_n)\|y_n-p\|+a_n\theta\|y_n-p\|]\\ &\le \theta^3[(1-a_n(1-\theta)][1-b_n(1-\theta)][1-c_n(1-\theta)]\|x_n-p\|.\end{aligned} \tag{2.9}$$

Hence, we get

$$\|x_{n+1}-p\| \le \theta^{3n}[(1-\varsigma(1-\theta))^3]^n\|x_1-p\|. \tag{2.10}$$

Let

$$m_n = \theta^{3n}[(1-\varsigma(1-\theta))^3]^n\|x_1-p\|. \tag{2.11}$$

Then, we get

$$\begin{aligned}\frac{m_n}{k_n} &= \frac{\theta^{3n}[(1-\varsigma(1-\theta))^3]^n\|x_1-p\|}{\theta^n[(1-\varsigma(1-\theta))^2]^n\|u_1-p\|}\\ &\to 0 \quad as \quad n\to\infty.\end{aligned} \tag{2.12}$$

Consequently $\{x_n\}$ converges faster than $\{u_n\}$. □

In order to show numerically that our new iteration (SP*-iteration) process (2.3) have a good speed of convergence comparatively to (2.1) and (2.2) , we consider the following example.

Example 2.3. Let us define a function $T : [0, 10) \to [0, 10)$ by $T(x) = \sqrt{2x+3}$. Then clearly T is a contraction map. Let $\{a_n\} = 0.75, \{b_n\} = 0.75, \{c_n\} = 0.75$ $\forall n \in \mathbb{N}$. 3 is the fixed point of T. The iterative values for initial value $x_1 = 4$ are given in Table 1. The efficiency of new iteration process is clear. We can see that our new iteration process (2.3) have a good speed of convergence comparatively to (2.1) and (2.2) iteration processes.

In this paper, we apply SP*-iteration (2.3) in a $CAT(0)$ space for generalized nonexpansive mappings as follows

$$\begin{cases} z_n = T((1-c_n)x_n \oplus c_nTx_n),\\ y_n = T((1-b_n)z_n \oplus b_nTz_n),\\ x_{n+1} = T((1-a_n)y_n \oplus a_nTy_n) \forall n \in \mathbb{N}, \end{cases} \tag{2.13}$$

where K is a nonempty closed convex subset of a $CAT(0)$ space, $x_1 \in K$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\} \in [0, 1]$.

Inspired and motivated by these facts, in this paper, we consider generalized α-nonexpansive mappings. Further we prove some convergence theorems of a new iterative process generated by (2.13) to fixed point for generalized α-nonexpansive mappings in $CAT(0)$ spaces.

3. Convergence of SP*-iteration process for generalized $\alpha-$nonexpansive mappings

Lemma 3.1. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X , T be a generalized $\alpha-$nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_1 \in K$, let $\{x_n\}$ be a sequence generated by (2.13) with $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ real sequences in $[0, 1]$, then $\lim_{n\to\infty} d(x_n, p)$ exits for any $p \in F(T)$.*

TABLE 1. Sequences generated by SP-iteration, Picard Normal S-iteration and SP*-iteration processes for mapping T of Example 2.1.

	SP-iteration	Picard Normal S-iteration	New iteration(SP*-iteration)
x_1	4	4	4
x_2	3, 119517509335370	3, 079156197588850	3, 004353238197250
x_3	3, 014854024750290	3, 006567594269410	3, 000020148061540
x_4	3, 001855414595090	3, 000547099976860	3, 000000093277940
x_5	3, 000231905910340	3, 000045590279090	3, 000000000431840
x_6	3, 000028987912010	3, 000003799180300	3, 000000000002000
x_7	3, 000003623483900	3, 000000316598290	3, 000000000000010
x_8	3, 000000452935410	3, 000000026383190	3, 000000000000000
x_9	3, 000000056616920	3, 000000002198600	3, 000000000000000
x_{10}	3, 000000007077120	3, 000000000183220	3, 000000000000000
x_{11}	3, 000000000884640	3, 000000000015270	3, 000000000000000
x_{12}	3, 000000000110580	3, 000000000001270	3, 000000000000000
x_{13}	3, 000000000013820	3, 000000000000110	3, 000000000000000
x_{14}	3, 000000000001730	3, 000000000000010	3, 000000000000000
x_{15}	3, 000000000000220	3, 000000000000000	3, 000000000000000
x_{16}	3, 000000000000030	3, 000000000000000	3, 000000000000000
x_{17}	3, 000000000000000	3, 000000000000000	3, 000000000000000

Proof. For any $p \in F(T)$, and $x \in K$, since for T a generalized $\alpha-$nonexpansive mapping, $\frac{1}{2}d(p,Tp) = 0 \leq d(p,x)$ implies that

$$\begin{aligned} d(Tp,Tx) &\leq \alpha d(Tp,x) + \alpha d(Tx,p) + (1-2\alpha)d(p,x) \\ &\leq \alpha d(Tp,x) + \alpha d(Tp,Tx) + (1-2\alpha)d(p,x) \qquad (3.1) \end{aligned}$$

$$\begin{aligned} (1-\alpha)d(Tp,Tx) &\leq \alpha d(Tp,x) + (1-2\alpha)d(p,x) \\ &= (1-\alpha)d(p,x) \qquad (3.2) \end{aligned}$$

Thus, $d(Tp,Tx) \leq d(p,x)$. Then we show that T is a quasi-nonexpansive mapping. Now, using (2.13) and Lemma 1.10(1), we have,

$$\begin{aligned} d(z_n,p) &= d(T((1-c_n)x_n \oplus c_nTx_n),p) \qquad (3.3) \\ &\leq d((1-c_n)x_n \oplus c_nTx_n,p) \\ &\leq (1-c_n)d(x_n,p) + c_nd(x_n,p) \\ &\leq d(x_n,p). \end{aligned}$$

Using (2.13), (3.3) and Lemma 1.10(1), we get

$$\begin{aligned} d(y_n,p) &= d(T((1-b_n)z_n \oplus b_nTz_n),p) \qquad (3.4) \\ &\leq d((1-b_n)z_n \oplus b_nTz_n,p) \\ &\leq (1-b_n)d(z_n,p) + b_nd(z_n,p) \\ &\leq d(z_n,p) \\ &\leq d(x_n,p). \end{aligned}$$

By using (2.13) , (3.4) and Lemma 1.10(1), we get

$$\begin{aligned} d(x_{n+1},p) &= d(T((1-a_n)y_n \oplus a_nTy_n),p) \qquad (3.5) \\ &\leq d((1-a_n)y_n \oplus a_nTy_n,p) \end{aligned}$$

$$
\begin{aligned}
&\leq (1-a_n)d(y_n,p)+a_nd(Ty_n,p)\\
&\leq (1-a_n)d(x_n,p)+a_nd(y_n,p)\\
&\leq (1-a_n)d(x_n,p)+a_nd(x_n,p)\\
&\leq d(x_n,p)
\end{aligned}
$$

This implies that $\{d(x_n-p)\}$ is bounded and non-increasing for all $p \in F(T)$. It follows that $\lim_{n\to\infty} d(x_n,p)$ exits. □

Theorem 3.2. *Let K be a nonempty closed convex subset of a complete $CAT(0)$ space X, T be a generalized $\alpha-$nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_1 \in K$, let $\{x_n\}$ be a sequence in K defined by (2.13) with $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ real sequences in $[0,1]$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,Tx_n)=0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n,p)$ exits and $\{x_n\}$ is bounded. Put $\lim_{n\to\infty} d(x_n,p)=r$. From (3.3) and (3.4), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(x_n,p)=r \tag{3.6}$$

and

$$\limsup_{n\to\infty} d(y_n,p) \leq \limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(x_n,p)=r. \tag{3.7}$$

Next,

$$\limsup_{n\to\infty} d(Tx_n,p) \leq \limsup_{n\to\infty} d(x_n,p)=r. \tag{3.8}$$

On the other hand,

$$
\begin{aligned}
d(x_{n+1}-p) &= d(T((1-a_n)y_n \oplus a_nTy_n,p))\\
&\leq (1-a_n)d(y_n,p)+a_nd(Ty_n,p)\\
&\leq (1-a_n)d(y_n,p)+a_nd(y_n,p)\\
&\leq d(y_n,p).
\end{aligned}
\tag{3.9}
$$

So we can get $d(x_{n+1},p) \leq d(y_n,p)$. Therefore $r \leq \liminf_{n\to\infty} d(y_n,p)$. Thus we have $r=\lim_{n\to\infty} d(y_n,p)$. Now

$$
\begin{aligned}
r=\lim_{n\to\infty} d(y_n,p) &\leq \lim_{n\to\infty} d(z_n,p)\\
&= \lim_{n\to\infty} d(T((1-c_n)x_n \oplus c_nTx_n),p)\\
&\leq \lim_{n\to\infty} d((1-c_n)x_n \oplus c_nTx_n,p)\\
&\leq \lim_{n\to\infty} (1-c_n)d(x_n,p)+c_nd(Tx_n,p)\\
&\leq \lim_{n\to\infty} (1-c_n)d(x_n,p)+c_nd(x_n,p)\\
&\leq \lim_{n\to\infty} d(x_n,p)=r.
\end{aligned}
\tag{3.10}
$$

Hence, we get

$$\lim_{n\to\infty} d((1-c_n)x_n \oplus c_nTx_n,p)=r. \tag{3.11}$$

Thus by Lemma 1.6, we have

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0. \tag{3.12}$$

Conversely, suppose that $\{x_n\}$ is bounded $\lim_{n\to\infty} d(x_n,Tx_n)=0$. Let $p \in A(K,\{x_n\})$. By Lemma 1.10(3), we have,

$$\begin{aligned} r(Tp,\{x_n\}) = \limsup_{n\to\infty} d(x_n,Tp) &\leq \limsup_{n\to\infty} \frac{3+\alpha}{1-\alpha} d(Tx_n,x_n) + d(x_n,p) + d(p,Tp) \\ &\leq \limsup_{n\to\infty} d(x_n,p) = r(p,\{x_n\}) \end{aligned}$$

This implies that for $Tp = p \in A(K,\{x_n\})$. Since X is complete $CAT(0)$ then $A(K,\{x_n\})$ is singleton, hence $Tp = p$. This completes the proof. □

Now , we prove the $\Delta-$convergence theorem of a iterative process generated by (2.13) in $CAT(0)$ spaces.

Theorem 3.3. *Let X, K, T and $\{x_n\}$ be as in Theorem 3.2 with $F(T) \neq \emptyset$. Then x_n, $\Delta-$converges to a fixed point of T.*

Proof. Theorem 3.2 guarantees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n) = 0$. Let $W_\Delta(x_n) = \bigcup A(\{u_n\})$; where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$: We claim that $W_\Delta(x_n) \subseteq F(T)$. Let $u \in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = u$. By Lemma 1.4 and Lemma 1.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta \lim_{n\to\infty} v_n = v \in K$. Since $\lim_{n\to\infty} d(v_n,Tv_n) = 0$ and T is a generalized α-nonexpansive mapping, then, we have $d(v_n,Tv) \leq \frac{3+\alpha}{1-\alpha} d(Tv_n,v_n) + d(v_n,v)$. By taking lim sup and using Opial property, we obtain $v \in F(T)$. Now, we claim that $u = v$. Assume on contrary, that $u \neq v$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n,v)$ exists and by the uniqueness of asymptotic centers, then we have

$$\begin{aligned} \lim_{n\to\infty} d(v_n,v) &< \lim_{n\to\infty} d(v_n,u) \\ &\leq \lim_{n\to\infty} d(u_n,u) \\ &< \lim_{n\to\infty} d(u_n,v) \\ &= \lim_{n\to\infty} d(x_n,v) \\ &= \lim_{n\to\infty} d(v_n,v), \end{aligned}$$

which is contraction. Thus $u = v \in F(T)$ and $W_\Delta(x_n) \subseteq F(T)$. To show that $\{x_n\}$, $\Delta-$converges to a fixed point of T, we show that $W(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 1.4 and Lemma 1.5, there exists a subsequence $\{v_n\}$ of u_n such that $\Delta \lim_{n\to\infty} v_n = v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, we claim that $x = v$. If not, then existence $\lim_{n\to\infty} d(x_n,v)$ and uniqueness of asymptotic centers imply that

$$\begin{aligned} \lim_{n\to\infty} d(v_n,v) &< \lim_{n\to\infty} d(v_n,x) \\ &\leq \lim_{n\to\infty} d(x_n,x) \\ &< \lim_{n\to\infty} d(x_n,v) \\ &= \lim_{n\to\infty} d(v_n,v). \end{aligned}$$

This is a contradiction and hence $x = v \in F(T)$. Therefore, $W_\Delta(x_n) = x$. □

In the next result, we prove the strong convergence theorem as follows.

Theorem 3.4. *Let T be a generalized $\alpha-$nonexpansive mapping on a compact convex subset K of a complete $CAT(0)$ space X. $\{x_n\}$ be as in Theorem 3.2 with $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Teorem 3.2, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since K is compact, by Lemma 1.4, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in K$ such that $\{x_{n_k}\}$ converges p. By Lemma 1.10, we have $d(x_{n_k}, Tp) \leq \frac{3+\alpha}{1-\alpha} d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p)$ for all $k \geq 0$. Then $\{x_{n_k}\}$ converges Tp. This implies $Tp = p$. Since T is quasinonexpansive , we have $d(x_{n+1}, p) \leq d(x_n, p)$for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ converges strongly to p. □

Finally, we briefly discuss the strong convergence theorem using condition (A) introduced by Senter and Dotson[28] in $CAT(0)$ space X as follows.

Theorem 3.5. *Let X, K, T and $\{x_n\}$ be as in Theorem 3.2 with $F(T) \neq \emptyset$. Also if, for T satisfies condition (A), then $\{x_n\}$ defined by (2.13) converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists and so $\lim_{n\to\infty} d(x_n, F(T))$. Also by Theorem 3.2, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from condition (A) that $\lim_{n\to\infty} f(d(x_n, F(T)) \leq \lim_{n\to\infty} d(x_n, Tx_n)$. That is, $\lim_{n\to\infty} f(d(x_n, F(T)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus , we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_k\} \subset F(T)$ such that $d(x_{n_k}, y_k) < \frac{1}{2^k}$ for all $k \in \mathbb{N}$. We can easily show that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, therefore $p \in F(T)$ and $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(x_n, p)$ exists, we have that $x_n \to p$. The proof is completed. □

4. Conclusions

We study the convergence of the newly defined SP*-iteration process (2.13) to fixed for the generalized $\alpha-$nonexpensive mappings in nonlinear $CAT(0)$ spaces. These results presented in this paper extend and generalize some works for $CAT(0)$ space in the literature.

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FIXED POINT THEOREMS IN PARTIAL b-METRIC-LIKE SPACES

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ABSTRACT. In this paper, we introduce an interesting extension of the b-metric space called b-metric-like space. We investigate some contraction mapping in partial b-metric-like space and prove the existence of fixed point of this mapping in partial b-metric-like space under some conditions.

KEYWORDS: Partial b-metric-like space, Partial b-metric space, Cauchy sequence, Fixed point.
AMS Subject Classification: :46C05, 47H09, 47H10,

1. Introduction

In 1920, Banach [4] introduced a Banach Contraction Principle.

Theorem 1.1. *Let (X, d) be a complete metric space and let T be a contraction on X, there exists $r \in [0, 1)$ such that*

$$d(Tx, Ty) \leq rd(x, y), \forall x, y \in X.$$

Then T has a unique fixed point.

In recent years, many scholars have proposed a series of new concepts of contraction mapping and new fixed point theorems [5, 6, 7, 8, 9, 10].

In 1993, Bakhtin [5] introduced the concept of b-metric space which is a generalization of metric space. He proved the famous Banach Contraction Principle in the b-metric space, also see [6]. In 1994, S.G. Matthews [7] introduced the concept of partial metric space and proved the Banach Contraction Principle in the partial metric space.

In 2013, the notion of b-metric-like spaces were introduced by Alghamdi [8] and some fixed point theorems were studied in such spaces.

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Definition 1.2. Let X be a nonempty set, $s \geq 1$ be a given real number and let $b : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y, z \in X$, the following conditions hold:

$(Pb1)$ $x = y$ if and only if $b(x,x) = b(x,y) = b(y,y)$;
$(Pb2)$ $b(x,x) \leq b(x,y)$;
$(Pb3)$ $b(x,y) = b(y,x)$;
$(Pb4)$ $b(x,y) \leq s[b(x,z) + b(z,y)] - b(z,z)$.

Then the pair (X, b) is called a partial b-metric space. The number s is called the coefficient of (X, b, s).

In 2014, S. Satish [9] introduced the concept of partial b-metric space and the fixed point theorem of Banach Contraction Principle and Kannan type mapping was proved in partial b-metric space. In 2018, J. Zhou, D. Zheng and G. Zhang [10] proved some fixed point theorem for C-contractive mapping and Meir-Keeler mapping in partial b-metric space which generalized and extended the result of S.K. Chatterjea [6] and S. Satish [9], respectively.

In this paper, we introduce a new definition for a partial b-metric-like space and the fixed point theorem for C-contractive mapping and Meir-Keeler mapping was proved in partial b-metric-like space. The new results can be viewed as some unified forms of the previous results. That is, some fixed point theorem in partial b-metric space considered and studied by J. Zhou, D. Zheng and G. Zhang.

2. Preliminaries

The following concepts and results are needed for the results.

Definition 2.1. Let X be a nonempty set, $s \geq 1$ be a given real number and let $P_L : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y, z \in X$, the following conditions hold:

(Pb_L1) if $P_L(x,x) = P_L(x,y) = P_L(y,y)$, then $x = y$;
(Pb_L2) $P_L(x,x) \leq P_L(x,y)$;
(Pb_L3) $P_L(x,y) = P_L(y,x)$;
(Pb_L4) $P_L(x,y) \leq s[P_L(x,z) + P_L(z,y)] - P_L(z,z)$.

Then the pair (X, P_L) is called a partial b-metric-like space. The number s is called the coefficient of (X, P_L, s).

Remark 2.2. It is clear that every partial b-metric space is a partial b-metric-like space with the zero self-distance. However, the converse of this fact need not hold.

Remark 2.3. In a partial b-metric space (X, b, s), if $x, y \in X$ and $b(x,y) = 0$, then $x = y$ but the converse may not be true.

Remark 2.4. ([10]) It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

Definition 2.5. Let (X, P_L, s) be a partial b-metric-like space. Let x_n be any sequence in X and $x \in X$. Then

(i) the x_n sequence is said to be convergent and converges to x if $\lim_{n \longrightarrow \infty} P_L(x_n, x)$ exists and is finite.
(ii) the x_n sequence is said to be Cauchy sequence in (X, P_L, s) if $\lim_{n,m \longrightarrow \infty} P_L(x_n, x_m)$ exists and is finite.

(iii) (X, P_L, s) is said to be a complete partial b-metric-like space if for every Cauchy sequence x_n in X there exists $x \in X$ such that

$$\lim_{n,m\longrightarrow\infty} P_L(x_n, x_m) = \lim_{n\longrightarrow\infty} P_L(x_n, x) = P_L(x, x).$$

3. Main Results

In this section, we shall prove the existence of fixed point in partial b-metric-like space under some conditions.

Theorem 3.1. *Let (X, P_L, s) be a complete partial b-metric-like space with coefficient $s \geq 1$ and $f : X \longrightarrow X$ be a mapping satisfying the following condition: for $x, y \in X$*

$$P_L(fx, fy) \leq \lambda \left[P_L(x, fy) + P_L(y, fx)\right] \tag{3.1}$$

where $\lambda \in [0, \frac{1}{2s})$. Then f has unique fixed point $z \in X$ and $P_L(z, z) = 0$.

Proof. First we prove the existence of fixed point. Let $x_n = f^n x_0$ and $P_{L_n} = P_L(x_n, x_{n+1})$, where x_0 is arbitrary point of X.

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of f. Therefore, we can suppose $x_{n+1} \neq x_n,\ P_{L_n} > 0$ for each $n \in \mathbb{N}$, from (3.1) and definition of partial b-metric-like space. Consider

$$\begin{aligned}
P_{L_n} =& P_L(x_n, x_{n+1}) \\
=& P_L(fx_{n-1}, fx_n) \\
\leq& \lambda \left[P_L(x_{n-1}, fx_n) + P_L(x_n, fx_{n-1})\right] \\
=& \lambda \left[P_L(x_{n-1}, x_{n+1}) + P_L(x_n, x_n)\right] \\
\leq& \lambda \left[s \left[P_L(x_{n-1}, x_n) + P_L(x_n, x_{n+1})\right] - P_L(x_n, x_n) + P_L(x_n, x_n)\right] \\
=& \lambda \left[s \left[P_L(x_{n-1}, x_n) + P_L(x_n, x_{n+1})\right]\right] \\
=& \lambda s \left[P_{L_{n-1}} + P_{L_n}\right].
\end{aligned}$$

Let $\mu = \lambda s$ and $\lambda \in [0, \frac{1}{2s})$. Then $P_{L_n} \leq \mu[P_{L_{n+1}} + P_{L_n}]$, where $\mu \in [0, \frac{1}{2})$.
Therefore, $P_{L_n} \leq \alpha P_{L_{n-1}}$ where $\alpha = \frac{\mu}{1-\mu} < 1$. On repeating this process we obtain

$$P_{L_n} \leq \alpha^n b_0.$$

Hence $\lim_{n\longrightarrow\infty} P_{L_n} = 0$. Next, we shall show that x_n is a Cauchy sequence in X. Let $P_{L_n} = P_L(x_n, x_m)$, from (3.1) and (Pb_L4) that for $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned}
P_L(x_n, x_m) =& P_L(f^n x_0,\ f^m x_0) \\
\leq& \lambda \left[P_L(x_{n+1}, fx_{m-1}) + P_L(x_{m-1}, fx_{n-1})\right] \\
=& \lambda \left[P_L(x_{n-1}, x_m) + P_L(x_{m-1}, x_n)\right] \\
\leq& \lambda\ \big[s \left[P_L(x_{n-1}, x_n) + P_L(x_n, x_m)\right] - P_L(x_n, x_n) \\
&\quad + s \left[P_L(x_{m-1}, x_m) + P_L(x_n, x_m)\right] - P_L(x_m, x_m)\big] \\
=& \lambda s P_L(x_{n-1}, x_n) + 2\lambda s P_L(x_n, x_m) + \lambda s P_L(x_{m-1}, x_m).
\end{aligned}$$

Since (3.1) and (Pb_L4), we have $P_L(x_n, x_m) \leq \beta[P_{L_{n-1}} + P_{L_{m-1}}]$ where $\beta = \frac{\lambda s}{1-2\lambda s}$.
Therefore, $\{x_n\}$ is a Cauchy sequence in X and $\lim_{n,m\longrightarrow\infty} P_L(x_n, x_m) = 0$.

By the completeness of X, there exists $z \in X$ such that

$$\lim_{n\longrightarrow\infty} P_L(x_n, z) = \lim_{n,m\longrightarrow\infty} P_L(x_n, x_m) = P_L(z, z) = 0.$$

Now, we shall prove that z is a fixed point of f. Let $d_n = P_L(fx_n, fu)$ and $P_{L_n} = (z, fx_n)$ for each $n \in \mathbb{N}$. Consider, for $n \in \mathbb{N}$,

$$\begin{aligned} d_n =& P_L(fx_n, fz) \\ \leq& \lambda \left[P_L(x_n, fz) + P_L(fx_n, z)\right] \\ =& \lambda \left[P_L(fx_{n-1}, fz) + P_L(fx_n, z)\right] \\ =& \lambda(d_{n-1} + \ P_{L_n}). \end{aligned}$$

We take upper limit on both sides to the above inequality,

$$\limsup_{n\longrightarrow\infty} d_n \leq \limsup_{n\longrightarrow\infty} d_{n-1} + \limsup_{n\longrightarrow\infty} P_{L_n}.$$

Since $\lim_{n\longrightarrow\infty} P_{L_n} = 0$, we have $\limsup_{n\longrightarrow\infty} d_n = 0$. Thus $\lim_{n\longrightarrow\infty} d_n = 0$.
That is,

$$\begin{aligned} d_n =& P_L(fx_n, fz) \\ \leq& \lambda \left[P_L(x_n, fz) + P_L(fx_n, z)\right] \\ =& \lambda \left[P_L(fx_{n-1}, fz) + P_L(fx_n, z)\right] \\ =& \lambda(d_{n-1} + \ P_{L_n}). \end{aligned}$$

We take limit on both sides to the inequality, then $P_L(z, fz) \leq 0$. Hence $fz = z$. Therefore z is a fixed point of f.

Next, we prove unique fixed point. Let $z, v \in X$ be two distinct fixed points of f, that is, $z = fz \neq fv = v$. Then, we have $P_L(z, z) = P_L(v, v)$. Since (3.1), we have

$$\begin{aligned} P_L(z, v) =& P_L(fz, fv) \\ \leq& \lambda[P_L(z, fv) + P_L(v, fz)] \\ =& \lambda[P_L(z, v) + P_L(v, z)] \\ =& 2\lambda P_L(z, v) \\ <& \frac{1}{s} P_L(z, v), \end{aligned}$$

a contradiction. Thus, we have $z = v$.

Next, we prove that $z \in X$ is the fixed point of f, that is $fz = z$. From (3.1), we obtain

$$\begin{aligned} P_L(z, z) =& P_L(fz, fz) \\ \leq& \lambda[P_L(z, fz) + P_L(z, fz)] \\ =& \lambda[P_L(z, z) + P_L(z, z)] \\ =& 2\lambda P_L(z, z) \\ <& \frac{1}{s} P_L(z, z) \\ <& P_L(z, z). \end{aligned}$$

It is a contradiction. Hence, we have $P_L(z, z) = 0$. □

If (X, P_L, s) is a partial b-metric space and $P_L = b$, then Theorem 3.1 reduces to the following result.

Corollary 3.2. *([10]) Let (X, b, s) be a complete partial b-metric space with coefficient $s \geq 1$ and $f : X \longrightarrow X$ be a mapping satisfying the following condition:*

$$b(fx, fy) \leq \lambda[b(x, fy) + b(y, fx)] \ \ x, y \in X,$$

where $\lambda \in [0, \frac{1}{2s})$. *Then* f *has unique fixed point* $z \in X$ *and* $b(z,z) = 0$.

If (X, b, s) is a b-metric space in Corollary 3.2, then we have the following corollary.

Corollary 3.3. *([9]) Let* (X, b, s) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $f : X \longrightarrow X$ *be a mapping satisfying the following condition:*

$$b(fx, fy) \leq \lambda[b(x, fy) + b(y, fx)] \quad x, y \in X,$$

where $\lambda \in [0, \frac{1}{2s})$. *Then* f *has unique fixed point* $z \in X$ *and* $b(z,z) = 0$.

Theorem 3.4. *Let* (X, P_L, s) *be a complete partial b-metric-like space with coefficient* $s > 1$ *and* $f : X \longrightarrow X$ *be a mapping satisfying the following condition:*
for each $\varepsilon > 0$ *there exist* $\delta > 0$ *such that*

$$\varepsilon \leq P_L(x, z) < \varepsilon + \delta \Rightarrow sP_L(fx, fz) < \varepsilon. \tag{3.2}$$

Then f *has a unique fixed point* $z \in X$ *and* $P_L(z,z) = 0$.

Proof. By (3.2), for all $x, y \in X$ and $x \neq y$,

$$sP_L(fx, fy) < \ P_L(x, y). \tag{3.3}$$

Let $x_0 \in X$. We can choose sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n = f^2 x_{n-1} = \cdots = f^{n+1} x_0,$$

for $n = 0, 1, 2, 3 \cdots$.
If $x_{n+1} = x_n$ for all $n \in \mathbb{N}$, then f have a fixed point.

Let $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By the inequality (3.3) with $x = x_{n-1}$ and $y = x_n$, we obtain

$$sP_L(x_n, x_{n+1}) < P_L(x_{n-1}, x_n).$$

For $s > 1$, $\{P_L(x_n, x_{n+1})\}$ is a decrease sequence, it is easy to see that

$$\lim_{n \longrightarrow \infty} P_L(x_n, x_{n+1}) = 0.$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence in X. We can choose an N sufficiently large such that when $n > N$,

$$P_L(x_n, x_{n+1}) \ \frac{\varepsilon - \frac{\varepsilon}{s}}{s + s^2}.$$

Let $K(x_N, \varepsilon) = \{y \in X : P_L(y, x_N) \leq \varepsilon\}$.
If $x_m \in K(x_N, \varepsilon)$ with $m > N$, then $x_m \neq x_N$. Making use of the inequality $P_L(x, y) \leq s\,[P_L(x, z) + P_L(z, y)] - P_L(z, z)$, we obtain that

$$\begin{aligned}
P_L(f^2 x_m, x_N) \leq & s\left[P_L(f^2 x_m, f^2 x_N) + P_L(f^2 x_N, x_N)\right] - P_L(f^2 x_N, f^2 x_N) \\
\leq & s\left[P_L(f^2 x_m, f^2 x_N) + P_L(f^2 x_N, x_N)\right] \\
\leq & s\left[\frac{1}{s} P_L(x_{m+1}, x_{N+1}) + P_L(f^2 x_N, x_N)\right] \\
\leq & P_L(x_{m+1}, x_{N+1}) + s^2\left[P_L(x_{N+2}, x_{N+1}) + P_L(x_{N+1}, x_N)\right] - sP_L(x_N, x_N) \\
\leq & \frac{1}{s} P_L(x_m, x_N) + \left(s + s^2\right) P_L(x_{N+1}, x_N) \\
\leq & \frac{\varepsilon}{s} + \left(s + s^2\right)\left(\frac{\varepsilon - \frac{\varepsilon}{s}}{s + s^2}\right) \\
= & \varepsilon.
\end{aligned}$$

Therefore $f^2 x_m \in K(x_N, \varepsilon)$. That is f^2 maps $K(x_N, \varepsilon)$ into itself.
Since $x_{N+1} \in K(x_N, \varepsilon)$, we have $x_{N+3}, x_{N+5} \in K(x_N, \varepsilon)$. By

$$P_L(x,y) \leq s\,[P_L(x,z) + P_L(z,y)] - P_L(z,z),$$

we get

$$\begin{aligned} P_L(x_{N+2}, x_N) \leq & s\,[P_L(x_{N+2}, x_{N+1}) + P_L(x_{N+1}, x_N)] - P_L(x_{N+1}, x_{N+1}) \\ \leq & s P_L(x_{N+1}, x_N) + s\left[\frac{1}{s} P_L(x_N, x_{N+1})\right] \\ \leq & s\left(\frac{\varepsilon - \frac{\varepsilon}{s}}{s+s^2} + \frac{\varepsilon - \frac{\varepsilon}{s}}{s+s^2}\right) \\ < & \varepsilon. \end{aligned}$$

Hence $x_{N+2} \in K(x_N, \varepsilon)$. Similarly, $x_{N+4}, x_{N+6} \in K(x_N, \varepsilon)$.
This implies that $\{x_n : n \geq N\} \subset K(x_N, \varepsilon)$. Since $x_n,\ x_m \in K(x_N, \varepsilon)$, for $n > m > N$, we get

$$\begin{aligned} P_L(x_n, x_m) \leq & s\,[P_L(x_n, x_N) + P_L(x_N, x_m)] - P_L(x_N, x_N) \\ \leq & 2s\varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X and $\lim_{n,m \longrightarrow \infty} P_L(x_n, x_m) = 0$.
By completeness of X there exists $z \in X$, such that

$$\lim_{n \longrightarrow \infty} P_L(x_n, z) = \lim_{n,m \longrightarrow \infty} P_L(x_n, x_m) = P_L(z,z) = 0. \tag{3.4}$$

To show that, z is a fixed point of f. We must prove that $fz = z$.
By (3.4) and $P_L(x,y) \leq s\,[P_L(x,z) + P_L(z,y)] - P_L(z,z)$, We have

$$\begin{aligned} P_L(fz, z) \leq & s\,[P_L(z, fx_n) + P_L(fx_n, fz)] - P_L(fx_n, fx_n) \\ \leq & s\left[P_L(z, x_{n+1}) + \frac{1}{s} P_L(x_n, z)\right] \\ = & s P_L(z, x_{n+1}) + P_L(x_n, z). \end{aligned}$$

Passing to limit as $n \longrightarrow \infty$, we obtain

$$P_L(fz, z) \leq 0.$$

Hence $fz = z$, so z is a fixed point of f.
We want to show that $P_L(z,z) = 0$. Suppose that $P_L(z,z) > 0$. From (3.4), we can get

$$P_L(z,z) = P_L(fz, fz) \leq \frac{1}{s} P_L(z,z) < P_L(z,z),$$

a contradiction. Therefore $P_L(z,z) = 0$.

Next, we prove unique fixed point. Let $z, v \in X$ be two distinct fixed points of f, that is, $z = fz \neq fv = v$. Since (3.3), we have

$$\begin{aligned} P_L(z,v) = & P_L(fz, fv) \\ < & \frac{1}{s} P_L(z,v) \\ < & P_L(z,v), \end{aligned}$$

a contradiction. Thus $z = v$.
Therefore z is a fixed point of f and it is unique fixed point of f. □

If (X, P_L, s) is a partial b-metric space and $P_L = b$, then Theorem 3.4 reduces to the following result.

Corollary 3.5. *Let (X, b, s) be a complete partial b-metric space with coefficient $s > 1$and $f : X \longrightarrow X$ be a mapping satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq b(x, z) < \varepsilon + \delta \Rightarrow sb(fx, fz) < \varepsilon.$$

Then f has a unique fixed point $z \in X$ and $b(z, z) = 0$.

If (X, b, s) is a b-metric space in Corollary 3.5, then we have the following corollary.

Corollary 3.6. *Let (X, b, s) be a complete b-metric space with coefficient $s > 1$ and $f : X \longrightarrow X$ be a mapping satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq b(x, z) < \varepsilon + \delta \Rightarrow sb(fx, fz) < \varepsilon.$$

Then f has a unique fixed point $z \in X$ and $b(z, z) = 0$.

4. Conclusion

We have introduced a new extension of the concept of partial b-metric space, called a partial b-metric-like space. Furthermore, we proved some fixed point results for these C-contractive mappings. One can easily extend these results to some fixed point theorem in partial b-metric space (see [10]).

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APPROXIMATION SOLVABILITY OF A PERTURBED MANN ITERATIVE ALGORITHM WITH ERRORS FOR A SYSTEM OF MIXED VARIATIONAL INCLUSIONS INVOLVING $\oplus$ OPERATION

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ABSTRACT. In this paper, we consider a new resolvent operator associated with XOR-NODSM mappings and give some of its fascinating properties supported by a well constructed example. As an application, we introduce and study a system of general mixed variational inclusions involving $\oplus$ operation in ordered Hilbert spaces. Further, we propose a perturbed Mann Iterative Algorithm with errors for approximating the solution of this class of problems. Our results can be complemented as the refinement and generalization of the corresponding results of recent works.

KEYWORDS: Variational inclusion, $\oplus$-operation, Resolvent operator, Algorithm, Convergence.
AMS Subject Classification: 47H05, 47H10, 47J25, 49J40.

1. Introduction

It is now a well known fact that variational inequality theory, introduced and studied by Stampacchia [27] and Fischera [12] in early 1960's, has been instrumental in the study of potential theory, elasticity, mathematical programming, network economics, transportation research and regional sciences.

Variational inclusions as the generalization of variational inequalities have been widely studied in recent years. An important aspect in the theory of variational inequalities is the existence of solution and development of efficient and implementable iterative algorithms. Among different methods for solving variational inclusion problems, resolvent operator technique has been widely used. The applications of the resolvent operator technique have been explored and improved

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recently, for instance, Fang and Huang [10] introduced a class of H-monotone operators and defined the associated class of resolvent operators which extended the classes of resolvent operators associated with η-subdifferential operators of Ding and Lou [8] and maximal monotone operators of Huang and Fang [14]. For more details, we refer [7,8,10,11,13,16,17] and the references therein.

In 2001, Huang and Fang [13] introduced the generalized m-accretive mappings and defined the resolvent operator for such class of mappings in Banach spaces. Since then a number of researchers investigated several classes of generalized m-accretive mappings such as H-accretive, $H(\cdot,\cdot)$-accretive, (H,η)-accretive and (A,η)-accretive mappings, see for example [4,6–8,11,13,16–18].

XOR operation, that is $\oplus$ operation, is a binary operation and behaves the same way as that of the ADD operation. This operation enjoys some nice properties such as commutativity, associativity and that every element under this operation is self-inverse. In Boolean algebra, it is the same as addition modulo(2). XOR is a logical operation that is true if and only if its arguments differ. XOR operator finds its applications in generating pseudo-random numbers, detecting errors in digital communications, etc. Until now several researchers have used XOR operation and its allied forms for solving some classes of variational inequalities and variational inclusions, see for example [1,2,20,21,23,25].

Motivated and inspired by the above, in this paper, using the concept of XOR-NODSM mappings involving $\oplus$ operation and the new resolvent operator technique associated with XOR-NODSM mappings, we introduce and study a system of general mixed variational inclusions involving $\oplus$ operation in ordered positive Hilbet spaces and construct a new iterative algorithm with errors for this system of variational inclusions. Some properties of the associated resolvent operator have also been discussed by invoking $\oplus$ and $\odot$ operations supported by a well constructed example. Finally, we discuss the approximation solvability of the system considered. Our results improve and generalize the corresponding results of recent works, see for example [1,3–11,13–23,25,26].

2. Preliminaries

Let C be a cone with partial ordering " $\leq$ ". An ordered Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ is called positive if $0 \leq x$ and $0 \leq y$, then $0 \leq \langle x, y\rangle$ holds. Throughout the paper, $\mathcal{H}_p$ is assumed to be a real ordered positive Hilbert space. We denote the family of all nonempty (respectively, compact) subsets of $\mathcal{H}_p$ by $2^{\mathcal{H}_p}$ (respectively, $C^*(\mathcal{H}_p)$). The metric induced by the norm is denoted by d and the Hausdörff metric on $C^*(\mathcal{H}_p)$ by $\mathcal{D}(\cdot,\cdot)$.

Now, we recall some known definitions and results which are important to achieve the goal of this paper.

Definition 2.1 ([9]). A nonempty closed convex subset C of $\mathcal{H}_p$ is said to be a cone if:

(i) for any $x \in C$ and any $\lambda > 0, \lambda x \in C$;
(ii) $x \in C$ and $-x \in C$, then $x = 0$.

Definition 2.2 ([26]). A cone C is called a normal cone if and only if there exists a constant $\lambda_N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda_N \|y\|$, $\forall\ x, y \in \mathcal{H}_p$, where λ_N is called the normal constant of C.

Definition 2.3. For any $x, y \in \mathcal{H}_p, x \leq y$ if and only if $y - x \in C$.

The relation $\leq$ is a partial ordered relation in $\mathcal{H}_p$. The real Hilbert space $\mathcal{H}_p$ endowed with the ordered relation $\leq$ defined by C is called an ordered real Hilbert space.

Definition 2.4 ([26]). Let $x, y \in \mathcal{H}_p$, then x and y are said to be comparable to each other if either $x \leq y$ or $y \leq x$ holds and is denoted by $x \propto y$.

Definition 2.5 ([26]). For any $x, y \in \mathcal{H}_p$, $lub\{x, y\}$ denotes the least upper bound and $glb\{x, y\}$ denotes the greatest lower bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, then some binary operations are given below:

(i) $x \vee y = lub\{x, y\}$;
(ii) $x \wedge y = glb\{x, y\}$;
(iii) $x \oplus y = (x - y) \vee (y - x)$;
(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee, \wedge, \oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Lemma 2.6 ([9]). *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist such that $(x - y) \propto (y - x)$ and $0 \leq (x - y) \vee (y - x)$.*

Lemma 2.7 ([9]). *For any natural number n, $x \propto y_n$ and $y_n \longrightarrow y^*$ as $n \longrightarrow \infty$, then $x \propto y^*$.*

Proposition 2.8 ([21, 22]). *Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then the following relations hold for all $x, y, u, v, w \in \mathcal{H}_p$ and $\alpha, \beta, \lambda \in \mathbb{R}$:*

(i) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;
(ii) $x \propto 0$, *then* $-x \oplus 0 \leq x \leq x \oplus 0$;
(iii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
(iv) $0 \leq x \oplus y$, *if* $x \propto y$;
(v) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$;
(vi) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$;
(vii) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$;
(viii) *if x, y and w are comparable to each other, then* $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$;
(ix) $\alpha x \oplus \beta x = |\alpha - \beta| x = (\alpha \oplus \beta)x$, *if* $x \propto 0$.

Proposition 2.9 ([9]). *Let C be a normal cone in $\mathcal{H}_p$ with constant λ_N, then for each $x, y \in \mathcal{H}_p$, the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0$;
(ii) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$;
(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_N \|x \oplus y\|$;
(iv) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 2.10 ([21]). Let $F : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be a single-valued mapping, then:

(i) F is said to be comparison mapping, if for each $x, y \in \mathcal{H}_p$, $x \propto y$ then $F(x) \propto F(y)$, $x \propto F(x)$ and $y \propto F(y)$;
(ii) F is said to be strongly comparison mapping, if F is a comparison mapping and $F(x) \propto F(y)$ if and only if $x \propto y$, $\forall\, x, y \in \mathcal{H}_p$.

Definition 2.11 ([20]). A single-valued mapping $F : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is said to be β-ordered compression mapping if F is a comparison mapping and

$$F(x) \oplus F(y) \leq k(x \oplus y), \text{ for } 0 < k < 1.$$

Definition 2.12 ([20]). Let $A, B : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ and $H : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be single-valued mappings. Then for all $x, y \in \mathcal{H}_p$:

(i) H is said to be t_1-ordered compression mapping in the first argument, if

$$H(x,\cdot)\oplus H(y,\cdot)\le t_1(x\oplus y),\ 0<t_1<1;$$

(ii) H is said to be t_2-ordered compression mapping in the second argument, if

$$H(\cdot,x)\oplus H(\cdot,y)\le t_2(x\oplus y),\ 0<t_2<1;$$

(iii) H is said to be k_1-ordered compression mapping with respect to A, if

$$H(A(x),\cdot)\oplus H(A(y),\cdot)\le k_1(x\oplus y),\ 0<k_1<1;$$

(iv) H is said to be k_2-ordered compression mapping with respect to B, if

$$H(\cdot,B(x))\oplus H(\cdot,B(y))\le k_2(x\oplus y),\ 0<k_2<1.$$

Definition 2.13 ([20]). A single-valued mapping $F:\mathcal{H}_p\longrightarrow\mathcal{H}_p$ is said to be Lipschitz-type-continuous if there exists a constant $\beta>0$ such that

$$\|F(x)\oplus F(y)\|\le\beta\|x\oplus y\|,\ \forall\ x,y\in\mathcal{H}_p.$$

Definition 2.14 ([20]). A set-valued mapping $T:\mathcal{H}_p\longrightarrow C^*(\mathcal{H}_p)$ is said to be $\mathcal{D}$-Lipschitz-type-continuous if for all $x,y\in\mathcal{H}_p$, $x\propto y$, there exists a constant $\gamma>0$ such that

$$\mathcal{D}\big(T(x),T(y)\big)\le\gamma\|x\oplus y\|.$$

Definition 2.15 ([21]). Let $M:\mathcal{H}_p\longrightarrow 2^{\mathcal{H}_p}$ be a set-valued mapping. Then:

(i) M is said to be a comparison mapping if for any $v_x\in M(x), x\propto v_x$, and if $x\propto y$, then for $v_x\in M(x)$ and $v_y\in M(y), v_x\propto v_y,\ \forall\ x,y\in\mathcal{H}_p$;

(ii) A comparison mapping M is said to be α-non-ordinary difference mapping if there exists a constant $\alpha>0$ such that

$$(v_x\oplus v_y)\oplus\alpha(x\oplus y)=0\text{ holds},\ \forall\ x,y\in\mathcal{H}_p,v_x\in M(x),v_y\in M(y);$$

(iii) A comparison mapping M is said to be θ-ordered rectangular if there exists a constant $\theta>0$ such that

$$\langle v_x\odot v_y,-(x\oplus y)\rangle\ge\theta\|x\oplus y\|^2\text{ holds},\ \forall\ x,y\in\mathcal{H}_p,v_x\in M(x),v_y\in M(y);$$

(iv) A comparison mapping M is said to be ρ-XOR-ordered strongly monotone compression mapping if for $x\propto y$, there exists a constant $\rho>0$ such that

$$\rho(v_x\oplus v_y)\ge x\oplus y,\ \forall\ x,y\in\mathcal{H}_p,v_x\in M(x),v_y\in M(y).$$

Definition 2.16 ([20]). Let $A,B:\mathcal{H}_p\longrightarrow\mathcal{H}_p$ and $H:\mathcal{H}_p\times\mathcal{H}_p\longrightarrow\mathcal{H}_p$ be single-valued mappings, then H is said to be:

(i) mixed comparison mapping with respect to A and B, if for each $x,y\in\mathcal{H}_p$, $x\propto y$, then $H(A(x),B(x))\propto H(A(y),B(y))$, $x\propto H(A(x),B(x))$ and $y\propto H(A(y),B(y))$;

(ii) mixed strongly comparison mapping with respect to A and B, if for each $x,y\in\mathcal{H}_p$, $H(A(x),B(x))\propto H(A(y),B(y))$ if and only if $x\propto y$.

Definition 2.17. Let $A,B:\mathcal{H}_p\longrightarrow\mathcal{H}_p$ and $H:\mathcal{H}_p\times\mathcal{H}_p\longrightarrow\mathcal{H}_p$ be single-valued mappings such that $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B. Then, a set-valued comparison mapping $M:\mathcal{H}_p\times\mathcal{H}_p\longrightarrow 2^{\mathcal{H}_p}$ is said to be (α,ρ)-XOR-NODSM if:

(i) M is an α-non-ordinary difference mapping and ρ-XOR-ordered strongly monotone compression mapping;

(ii) $\left[H(A,B)\oplus\rho M(\cdot,\zeta)\right](\mathcal{H}_p)=\mathcal{H}_p$, for some fixed $\zeta\in\mathcal{H}_p$.

Definition 2.18. Let $M : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be an (α, ρ)-XOR-NODSM mapping. Then, for fixed $\zeta \in \mathcal{H}_p$, the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)} : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is defined as:

$$\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) = [H(A,B) \oplus \rho M(\cdot,\zeta)]^{-1}(x), \ \forall \, x \in \mathcal{H}_p. \tag{2.1}$$

Now, we discuss some properties of the generalized resolvent operator.

Proposition 2.19. *Let $A, B : \mathcal{H}_p \longrightarrow \mathcal{H}_p, H : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be single-valued mappings such that $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B. Let $M : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ is the set-valued θ-ordered rectangular mapping with $\rho\theta > |k_1 - k_2|$. Then the resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)} : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is single-valued.*

Proof. For any given $u \in \mathcal{H}_p$ and $\rho > 0$, let $x, y = [H(A,B) \oplus \rho M(\cdot,\zeta)]^{-1}(u)$. Then,

$$v_x = \frac{1}{\rho}\left[u \oplus H\big(A(x), B(x)\big)\right] \in M(x,\zeta) \text{ and } v_y = \frac{1}{\rho}\left[u \oplus H\big(A(y), B(y)\big)\right] \in M(y,\zeta).$$

Using (i) and (ii) of Proposition 2.8, we have

$$\begin{aligned}
v_x \odot v_y &= \frac{1}{\rho}\left[u \oplus H\big(A(x), B(x)\big)\right] \odot \frac{1}{\rho}\left[u \oplus H\big(A(y), B(y)\big)\right] \\
&= \frac{1}{\rho}\left\{\left[u \oplus H\big(A(x), B(x)\big)\right] \odot \left[u \oplus H\big(A(y), B(y)\big)\right]\right\} \\
&= -\frac{1}{\rho}\left\{\left[u \oplus H\big(A(x), B(x)\big)\right] \oplus \left[u \oplus H\big(A(y), B(y)\big)\right]\right\} \\
&= -\frac{1}{\rho}\left\{(u \oplus u) \oplus \left[H\big(A(x), B(x)\big) \oplus H\big(A(y), B(y)\big)\right]\right\} \\
&= -\frac{1}{\rho}\left\{0 \oplus \left[H\big(A(x), B(x)\big) \oplus H\big(A(y), B(y)\big)\right]\right\} \\
&\le -\frac{1}{\rho}\left[H\big(A(x), B(x)\big) \oplus H\big(A(y), B(y)\big)\right] \\
&\le -\frac{1}{\rho}\left\{\left[H\big(A(x), B(x)\big) \oplus H\big(A(x), B(y)\big)\right]\right. \\
&\qquad \left. \oplus \left[H\big(A(x), B(y)\big) \oplus H\big(A(y), B(y)\big)\right]\right\}.
\end{aligned} \tag{2.2}$$

Since, M is θ-ordered rectangular mapping, $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B and using (2.2), we have

$$\begin{aligned}
\theta\|x \oplus y\|^2 &\le \langle v_x \odot v_y, -(x \oplus y)\rangle \\
&\le \Big\langle -\frac{1}{\rho}\left\{\left[H\big(A(x), B(x)\big) \oplus H\big(A(x), B(y)\big)\right]\right. \\
&\qquad \left. \oplus \left[H\big(A(x), B(y)\big) \oplus H\big(A(y), B(y)\big)\right]\right\}, -(x \oplus y)\Big\rangle \\
&\le \frac{1}{\rho}\left\{\Big\langle H\big(A(x), B(x)\big) \oplus H\big(A(x), B(y)\big), x \oplus y\Big\rangle\right. \\
&\qquad \left. \oplus \Big\langle H\big(A(x), B(y)\big) \oplus H\big(A(y), B(y)\big), x \oplus y\Big\rangle\right\} \\
&\le \frac{1}{\rho}\langle k_1(x \oplus y), x \oplus y\rangle \oplus \langle k_2(x \oplus y), x \oplus y\rangle \\
&\le \frac{|k_1 - k_2|}{\rho}\|x \oplus y\|^2.
\end{aligned}$$

i.e.,

$$\left(\theta - \frac{|k_1 - k_2|}{\rho}\right) \|x \oplus y\|^2 \leq 0, \text{ for } \theta > \frac{|k_1 - k_2|}{\rho},$$

which shows that $\|x \oplus y\| = 0$, which implies $x \oplus y = 0$.

Therefore, $x = y$, that is the resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is single-valued for $\rho\theta > |k_1 - k_2|$. □

Proposition 2.20. *Let $M : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be an (α, ρ)-XOR-NODSM set-valued mapping such that $H(\cdot, \cdot)$ is mixed strongly comparison mapping with respect to A and B. Then the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is a comparison mapping.*

Proof. Since M is (α, ρ)-XOR-NODSM set-valued mapping, therefore M is α-non-ordinary difference as well as ρ-XOR-ordered strongly monotone compression mapping.

For any $x, y \in \mathcal{H}_p$, let $x \propto y$,

$$v_x^* = \frac{1}{\rho}\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x), \zeta\right) \tag{2.3}$$

$$\text{and } v_y^* = \frac{1}{\rho}\left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y), \zeta\right). \tag{2.4}$$

Since M is ρ-XOR-ordered strongly monotone compression mapping, therefore in view of (2.3) and (2.4), we have

$$\begin{aligned}
(x \oplus y) &\leq \rho(v_x^* \oplus v_y^*) \\
&\leq \frac{\rho}{\rho}\Big\{\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \\
&\quad \oplus \left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\Big\} \\
&\leq (x \oplus y) \oplus \Big[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \\
&\quad \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\Big].
\end{aligned}$$

Thus,

$$0 \leq H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)$$

$$\begin{aligned}
0 \leq &\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right] \\
&\vee \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right].
\end{aligned}$$

It follows that either

$$0 \leq \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]$$

or

$$0 \leq \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right) - H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right]$$

This implies

$$H(A, B)\big(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\big) \propto H(A, B)\big(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\big).$$

Since $H(\cdot,\cdot)$ is mixed strongly comparison mapping with respect to A, B, it follows that, $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \propto \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)$, thereby showing that the resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}$ is a comparison mapping. □

Proposition 2.21. *Let the mappings A, B, H, M be same as defined in Proposition 2.19, then the generalized resolvent operator $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)} : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is $\frac{1}{\rho\theta-(k_1+k_2)}$ -Lipschitz-type-continuous for $\rho\theta > (k_1 + k_2)$, i.e.,*

$$\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\| \le \frac{1}{\rho\theta-(k_1+k_2)}\|x \oplus y\|, \ \forall\ x, y \in \mathcal{H}_p.$$

Proof. For any $x, y \in \mathcal{H}_p$, let $x \propto y$,

$$v^*_x = \frac{1}{\rho}\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x), \zeta\right)$$

$$\text{and } v^*_y = \frac{1}{\rho}\left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right] \in M\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y), \zeta\right).$$

$$\begin{aligned}\text{Now,}\quad v^*_x \oplus v^*_y &= \frac{1}{\rho}\left\{\left[x \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right]\right.\\ &\quad\left.\oplus \left[y \oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\}\\ &= \frac{1}{\rho}\left\{(x \oplus y) \oplus \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\\ &\quad\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\}. \end{aligned} \tag{2.5}$$

Since $M(\cdot,\zeta)$ is θ-ordered rectangular mapping and using (2.5), for any $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \in M(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x), \zeta)$ and $\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y) \in M(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y), \zeta)$, we have

$$\begin{aligned}&\theta\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2\\ &\le \left\langle v^*_x \odot v^*_x, -(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y))\right\rangle\\ &\le \left\langle v^*_x \oplus v^*_x, \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\rangle\\ &= \frac{1}{\rho}\left\langle (x \oplus y) \oplus \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\\ &\quad\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right], \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\rangle\\ &\le \frac{1}{\rho}\left\{\left\|(x \oplus y) \oplus \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\right.\\ &\quad\left.\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\| \left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\right\}\\ &\le \frac{1}{\rho}\left\{\left\|(x \oplus y) - \left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\right.\\ &\quad\left.\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\right\| \left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\right\}\\ &\le \frac{1}{\rho}\left\{\left[\|x \oplus y\| + \left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right.\right.\right.\\ &\quad\left.\left.\left.\oplus\ H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right), B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\|\right] \left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\right\}\\ &\le \frac{1}{\rho}\Big\{\|x \oplus y\| \left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x) \oplus \mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\end{aligned}$$

$$+\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\| \left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\Bigg\}. \tag{2.6}$$

Since $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B, we have

$$\begin{aligned}
&\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\|\\
&=\Bigg\|\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right]\\
&\quad\oplus\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\Bigg\|\\
&\le\Bigg\|\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right]\\
&\quad-\left[H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right]\Bigg\|\\
&\le\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\right\|\\
&\quad+\left\|H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\right)\right)\oplus H\left(A\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right),B\left(\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right)\right)\right\|\\
&\le(k_1+k_2)\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|.
\end{aligned}$$

Thus, from (2.6), we have

$$\begin{aligned}
\theta\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2 &\le \frac{1}{\rho}\|x\oplus y\|\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\\
&\quad+\frac{k_1+k_2}{\rho}\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|^2.
\end{aligned}$$

This implies,

$$\left\|\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(x)\oplus\mathcal{R}^{H(A,B)}_{\rho,M(\cdot,\zeta)}(y)\right\|\le\frac{1}{\rho\theta-(k_1+k_2)}\|x\oplus y\|,\ \forall\, x,y\in\mathcal{H}_p.$$

This completes the proof. □

In support of Propositions 2.19-2.21, we present the following example.

Example 2.22. Let $\mathcal{H}_p=[0,\infty)\times[0,\infty)$ with the usual inner product and norm and let $C=[0,1]\times[0,1]$ be a normal cone. Let $A,B:\mathcal{H}_p\longrightarrow\mathcal{H}_p$ and $H:\mathcal{H}_p\times\mathcal{H}_p\longrightarrow\mathcal{H}_p$ be defined by

$$A(x)=\left(\frac{x_1}{9}+3,\frac{x_2}{9}+6\right),\quad B(x)=\left(\frac{x_1}{3}+1,\frac{x_2}{3}+2\right)$$

$$\text{and}\qquad H\left(A(x),B(x)\right)=\frac{A(x)}{3}\oplus B(x),\ \forall\, x=(x_1,x_2)\in\mathcal{H}_p.$$

For $x=(x_1,x_2),y=(y_1,y_2)\in\mathcal{H}_p, x\propto y$, we have

$$\begin{aligned}
H\left(A(x),u\right)\oplus H\left(A(y),u\right)&=\left(\frac{A(x)}{3}\oplus u\right)\oplus\left(\frac{A(y)}{3}\oplus u\right)\\
&=\frac{1}{3}\left(A(x)\oplus A(y)\right)\\
&=\frac{1}{3}\left[\left(A(x)-A(y)\right)\vee\left(A(y)-A(x)\right)\right]\\
&=\frac{1}{3}\left[\left\{\left(\frac{x_1}{9}+3,\frac{x_2}{9}+6\right)-\left(\frac{y_1}{9}+3,\frac{y_2}{9}+6\right)\right\}\right.
\end{aligned}$$

$$\vee \left\{\left(\frac{y_1}{9}+3, \frac{y_2}{9}+6\right)-\left(\frac{x_1}{9}+3, \frac{x_2}{9}+6\right)\right\}\Big]$$
$$=\frac{1}{3}\left[\left(\frac{x_1-y_1}{9}, \frac{x_2-y_2}{9}\right) \vee\left(\frac{y_1-x_1}{9}, \frac{y_2-x_2}{9}\right)\right]$$
$$=\frac{1}{27}[(x-y) \vee(y-x)]$$
$$=\frac{1}{27}(x \oplus y)$$
$$\leq \frac{1}{24}(x \oplus y).$$

Hence, H is $\frac{1}{24}$-ordered compression mapping with respect to A. Similarly, we can show that H is $\frac{1}{2}$-ordered compression mapping with respect to B.

Suppose that the set-valued mapping $M: \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be defined by

$$M(x)=\big\{(3x_1, 3x_2)\big\},\ \forall\ x=(x_1, x_2) \in \mathcal{H}_p.$$

It can be easily verified that M is a comparison mapping, 1-XOR-ordered strongly monotone compression mapping and 3-non-ordinary difference mapping. Further, it is clear that for $\rho=1$, $\big[H(A, B)+\rho M\big](\mathcal{H}_p)=\mathcal{H}_p$. Hence, M is an $(3,1)$-XOR-NODSM strongly monotone compression mapping.

Let $\upsilon_x=(3x_1, 3x_2) \in M(x)$ and $\upsilon_y=(3y_1, 3y_2) \in M(y)$, then

$$\begin{aligned}
\big\langle \upsilon_x \odot \upsilon_y,-(x \oplus y)\big\rangle &= \big\langle \upsilon_x \oplus \upsilon_y, x \oplus y\big\rangle \\
&= \big\langle 3x \oplus 3y, x \oplus y\big\rangle \\
&= 3\big\langle x \oplus y, x \oplus y\big\rangle \\
&= 3\|x \oplus y\|^2 \\
&\geq 2\|x \oplus y\|^2.
\end{aligned}$$

i.e.,

$$\big\langle \upsilon_x \odot \upsilon_y,-(x \oplus y)\big\rangle \ \geq 2\ \|x \oplus y\|^2,\ \forall\ x, y \in \mathcal{H}_p.$$

Thus, M is 2-ordered rectangular comparison mapping.

The resolvent operator defined by (2.1) is given by

$$\mathcal{R}_{\rho, M}^{H(A, B)}(x)=\left(\frac{27 x_1}{73}, \frac{27 x_2}{73}\right),\ \forall\ x=(x_1, x_2) \in \mathcal{H}_p.$$

It is easy to verify that the resolvent operator defined above is comparison and single-valued mapping.

Further,

$$\begin{aligned}
\left\|\mathcal{R}_{\rho, M}^{H(A, B)}(x) \oplus \mathcal{R}_{\rho, M}^{H(A, B)}(y)\right\| &= \left\|\frac{27 x}{73} \oplus \frac{27 y}{73}\right\| \\
&= \frac{27}{73}\|x \oplus y\| \\
&\leq \frac{24}{35}\|x \oplus y\|.
\end{aligned}$$

i.e.,

$$\left\|\mathcal{R}_{\rho, M}^{H(A, B)}(x) \oplus \mathcal{R}_{\rho, M}^{H(A, B)}(y)\right\| \leq \frac{24}{35}\|x \oplus y\|,\ \forall\ x, y \in \mathcal{H}_p.$$

This shows that the resolvent operator is $\mathcal{R}_{\rho, M}^{H(A, B)}$ is $\frac{24}{35}$-Lipschitz-type-continuous.

3. Formulation of the Problem and Existence of Solution

Let $\mathcal{H}_p$ be a real ordered positive Hilbert space. For each $i = 1, 2$, let $A, B, g_i, p_i, G_i : \mathcal{H}_p \longrightarrow \mathcal{H}_p, F_i, H : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be single-valued mappings and $S, T : \mathcal{H}_p \longrightarrow C^*(\mathcal{H}_p), M_i : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be set-valued mappings. Then, for fixed $\zeta, \zeta' \in \mathcal{H}_p$, we consider the following generalized system of mixed variational inclusion problem (in short, GSMVIP):

For any $\omega_1, \omega_2 \in \mathcal{H}_p$, find $x, y \in \mathcal{H}_p, u \in S(x), v \in T(x), u' \in S(y), v' \in T(y)$ such that

$$\begin{cases} \omega_1 \in F_1\big((g_1 - p_1)(u), G_1(v)\big) \oplus M_1\big((g_1 - p_1)(x), \zeta\big) \\ \omega_2 \in F_2\big(G_2(v'), (g_2 - p_2)(u')\big) \oplus M_2\big(\zeta', (g_2 - p_2)(y)\big). \end{cases} \tag{3.1}$$

For suitable choices of mappings and the underlying space $\mathcal{H}_p$, GSMVIP (3.1) encompasses several classes of variational inclusions including those involving involving XOR-operation as special cases, see for example [1, 2, 4, 21, 23, 25] and the related references cited therein.

Lemma 3.1. *The generalized system of mixed variational inclusion problem (3.1) admits a solution (x, y, u, v, u', v') where $x, y \in \mathcal{H}_p, u \in S(x), v \in T(x), u' \in S(y), v' \in T(y)$ if and only if it satisfies the following equations:*

$$(g_1 - p_1)(x) = \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\big[\rho_1 F_1\big((g_1 - p_1)(u), G_1(v)\big) \oplus H(A, B)(g_1 - p_1)(x) \oplus \rho_1\omega_1\big] \tag{3.2}$$

$$(g_2 - p_2)(y) = \mathcal{R}^{H(A,B)}_{\rho_2, M_2(\zeta',\cdot)}\big[\rho_2 F_2\big(G_2(v'), (g_2 - p_2)(u')\big) \oplus H(A, B)(g_2 - p_2)(y) \oplus \rho_2\omega_2\big], \tag{3.3}$$

where $\mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)} = \big[H(A, B) \oplus \rho_1 M_1(\cdot, \zeta)\big]^{-1}$, $\mathcal{R}^{H(A,B)}_{\rho_2, M_2(\zeta',\cdot)} = \big[H(A, B) \oplus \rho_2 M_2(\zeta', \cdot)\big]^{-1}$ *and* $\rho_1, \rho_2 > 0$.

Proof. Using the definition of the generalized resolvent operator and suppose

$$(g_1 - p_1)(x) = \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)}\big[\rho_1 F_1\big((g_1 - p_1)(u), G_1(v)\big) \oplus H(A, B)(g_1 - p_1)(x) \oplus \rho_1\omega_1\big].$$

Then,

$$\begin{aligned} (g_1 - p_1)(x) = \big[H(A, B) \oplus \rho_1 M_1(\cdot, \zeta)\big]^{-1}\big[&\rho_1 F_1\big((g_1 - p_1)(u), G_1(v)\big) \\ &\oplus H(A, B)(g_1 - p_1)(x) \oplus \rho_1\omega_1\big] \\ \Rightarrow H(A, B)(g_1 - p_1)(x) \oplus \rho_1 M_1\left((g_1 - p_1)(x), \zeta\right)& \\ \ni \rho_1 F_1\big((g_1 - p_1)(u), G_1(v)\big) &\oplus H(A, B)(g_1 - p_1)(x) \oplus \rho_1\omega_1. \end{aligned}$$

Which gives $\omega_1 \in F_1\big((g_1 - p_1)(u), G_1(v)\big) \oplus M_1\big((g_1 - p_1)(x), \zeta\big)$, the first inclusion of the GSMVIP (3.1). Similarly, we can get the second inclusion from equation (3.3). □

4. Iterative Algorithm and Convergence Analysis

Based on Lemma 3.1 and Nadler's Theorem [24], we establish the following iterative algorithm to approximate the solution of GSMVIP (3.1).

Iterative Algorithm 4.1. Step 1. For any $\omega_1, \omega_2 \in \mathcal{H}_p$ and $\rho_1, \rho_2 > 0$, choose $x_0, y_0 \in \mathcal{H}_p, u_0 \in S(x_0), v_0 \in T(x_0), u'_0 \in S(y_0)$ and $v'_0 \in T(y_0)$.
Step 2. Let

$$\begin{aligned}
&(g_1 - p_1)x_{n+1} \\
&= \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)} \Big\{ \rho_1 F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus H(A,B)(g_1 - p_1)(x_n) \oplus \rho_1\omega_1 \oplus e_n \Big\}, \\
&(g_2 - p_2)y_{n+1} \\
&= \mathcal{R}^{H(A,B)}_{\rho_2, M_2(\zeta',\cdot)} \Big\{ \rho_2 F_2\big(G_2(v'_n), (g_2 - p_2)(u'_n)\big) \oplus H(A,B)(g_2 - p_2)(y_n) \oplus \rho_2\omega_2 \oplus e'_n \Big\}.
\end{aligned}$$

Step 3. Choose $u_{n+1} \in S(x_{n+1}), v_{n+1} \in T(x_{n+1}), u'_{n+1} \in S(y_{n+1})$ and $v'_{n+1} \in T(y_{n+1})$ such that

$$\begin{cases}
\|u_{n+1} \oplus u_n\| \le \|u_{n+1} - u_n\| \le \left(1 + (1+n)^{-1}\right) \mathcal{D}\left(S(x_{n+1}), S(x_n)\right), \\
\|v_{n+1} \oplus v_n\| \le \|v_{n+1} - v_n\| \le \left(1 + (1+n)^{-1}\right) \mathcal{D}\left(T(x_{n+1}), T(x_n)\right), \\
\|u'_{n+1} \oplus u'_n\| \le \|u'_{n+1} - u'_n\| \le \left(1 + (1+n)^{-1}\right) \mathcal{D}\left(S(y_{n+1}), S(y_n)\right), \\
\|v'_{n+1} \oplus v'_n\| \le \|v'_{n+1} - v'_n\| \le \left(1 + (1+n)^{-1}\right) \mathcal{D}\left(T(y_{n+1}), T(y_n)\right).
\end{cases} \tag{4.1}$$

Step 4. Choose errors $\{e_n\}, \{e'_n\} \subset \mathcal{H}_p$ to take into account the possible inexact computations such that, for all $\nu_1, \nu_2 \in (0,1)$

$$\sum_{j=1}^{\infty} \|e_j \oplus e_{j-1}\| \nu_1^{-j} < \infty, \ \sum_{j=1}^{\infty} \|e'_j \oplus e'_{j-1}\| \nu_2^{-j} < \infty, \lim_{n\to\infty} e_n = 0, \lim_{n\to\infty} e'_n = 0.$$

Step 5. If $u_{n+1} \in S(x_{n+1}), v_{n+1} \in T(x_{n+1}), u'_{n+1} \in S(y_{n+1})$ and $v'_{n+1} \in T(y_{n+1})$ satisfy (4.1) to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to Step 2.

Next, we prove the following theorem which ensures the convergence of iterative sequences generated by the Iterative Algorithm 4.1.

Theorem 4.2. *Let $C \subset \mathcal{H}_p$be a normal cone with constant λ_N. For $i = 1, 2$, let $A, B, g_i, p_i, G_i : \mathcal{H}_p \longrightarrow \mathcal{H}_p, F_i, H : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ be single-valued mappings such that:*

(i) *$H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B;*
(ii) *F_1 is τ_1-Lipschitz-type continuous with respect to $(g_1 - p_1)$ in the first argument and σ_1-Lipschitz-type-continuous with respect to G_1 in the second argument;*
(iii) *F_2 is τ_2-Lipschitz-type continuous with respect to $(g_2 - p_2)$ in the second argument and σ_2-Lipschitz-type-continuous with respect to G_2 in the first argument;*
(iv) *$(g_i - p_i)$ is r_i-Lipschitz-type-continuous and $(g_i - p_i \oplus I)$ is δ_i-Lipschitz-type-continuous, where I is the Identity mapping.*

Also, let $S, T : \mathcal{H}_p \longrightarrow C^*(\mathcal{H}_p)$ and $M_1, M_2 : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow 2^{\mathcal{H}_p}$ be set-valued mappings such that:

(i) *M_i is (α_i, ρ_i)-XOR-NODSM and θ_i-ordered rectangular mapping, repectively, for $i = 1, 2$;*
(ii) *S is γ_1-$\mathcal{D}$-Lipschitz-type-continuous and T is γ_2-$\mathcal{D}$-Lipschitz-type-continuous.*

If $x_{n+1} \propto x_n, y_{n+1} \propto y_n, (g_1 - p_1)(x_{n+1}) \propto (g_1 - p_1)(x_n), (g_2 - p_2)(y_{n+1}) \propto (g_2 - p_2)(y_n)$*, for* $n = 0, 1, 2, \cdots$ *and the following conditions are satisfied:*

$$0 < \varphi = \frac{\lambda_N |\rho_1| (\tau_1 \gamma_1 + \sigma_1 \gamma_2) + r_1 \lambda_N (k_1 + k_2)}{(1 - \delta_1) [\rho_1 \theta_1 - (k_1 + k_2)]} < 1, \tag{4.2}$$

$$0 < \vartheta = \frac{\lambda_N |\rho_2| (\sigma_2 \gamma_2 + \tau_2 \gamma_1) + r_2 \lambda_N (k_1 + k_2)}{(1 - \delta_2) [\rho_2 \theta_2 - (k_1 + k_2)]} < 1. \tag{4.3}$$

Then GSMVIP (3.1) has a solution (x, y, u, v, u', v')*, where* $x, y \in \mathcal{H}_p, u \in S(x), v \in T(x), u' \in S(y), v' \in T(y)$*. Also, the sequences* $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{u'_n\}, \{v'_n\}$ *generated by the Iterative Algorithm 4.1 converge strongly to* x, y, u, v, u', v'*, respectively.*

Proof. By Algorithm 4.1 and Proposition 2.8, we have

$$\begin{aligned}
0 &\leq (g_1 - p_1)x_{n+1} \oplus (g_1 - p_1)x_n \\
&= \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)} \Big\{ \rho_1 F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus H(A,B)(g_1 - p_1)(x_n) \oplus \rho_1 \omega_1 \oplus e_n \Big\} \\
&\quad \oplus \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)} \Big\{ \rho_1 F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \oplus H(A,B)(g_1 - p_1)(x_{n-1}) \\
&\quad \oplus \rho_1 \omega_1 \oplus e_{n-1} \Big\}.
\end{aligned}$$

Now, using Proposition 2.9 and Lipschitz-type-continuity of the generalized resolvent operator, we have

$$\begin{aligned}
&\big\| (g_1 - p_1)x_{n+1} \oplus (g_1 - p_1)x_n \big\| \\
&\leq \lambda_N \Big\| \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)} \Big\{ \rho_1 F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus H(A,B)(g_1 - p_1)(x_n) \oplus \rho_1 \omega_1 \oplus e_n \Big\} \\
&\quad \oplus \mathcal{R}^{H(A,B)}_{\rho_1, M_1(\cdot,\zeta)} \Big\{ \rho_1 F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \oplus H(A,B)(g_1 - p_1)(x_{n-1}) \oplus \rho_1 \omega_1 \\
&\quad \oplus e_{n-1} \Big\} \Big\| \\
&\leq \frac{\lambda_N}{\rho_1 \theta_1 - (k_1 + k_2)} \big\| \big[\rho_1 F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus H(A,B)(g_1 - p_1)(x_n) \oplus \rho_1 \omega_1 \oplus e_n \big] \\
&\quad \oplus \big[\rho_1 F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \oplus H(A,B)(g_1 - p_1)(x_{n-1}) \oplus \rho_1 \omega_1 \oplus e_{n-1} \big] \big\| \\
&\leq \frac{\lambda_N |\rho_1|}{\rho_1 \theta_1 - (k_1 + k_2)} \big\| F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \big\| \\
&\quad + \frac{\lambda_N}{\rho_1 \theta_1 - (k_1 + k_2)} \| H(A,B)(g_1 - p_1)(x_n) \oplus H(A,B)(g_1 - p_1)(x_{n-1}) \| \\
&\quad + \frac{\lambda_N}{\rho_1 \theta_1 - (k_1 + k_2)} \| e_n \oplus e_{n-1} \|.
\end{aligned} \tag{4.4}$$

Since, XOR operator is associative, $F_1 : \mathcal{H}_p \times \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is τ_1-Lipschitz-type-continuous with respect to $(g_1 - p_1)$ in first argument and σ_1-Lipschitz-type-continuous with respect to G_1 in second argument and S, T are γ_1, γ_2-$\mathcal{D}$-Lipschitz-type-continuous, respectively, therefore in view of Algorithm 4.1, we have

$$\begin{aligned}
&\big\| F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \big\| \\
&\qquad \leq \big\| F_1\big((g_1 - p_1)(u_n), G_1(v_n)\big) \oplus F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_n)\big) \big\| \\
&\qquad\quad + \big\| F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_n)\big) \oplus F_1\big((g_1 - p_1)(u_{n-1}), G_1(v_{n-1})\big) \big\| \\
&\qquad \leq \tau_1 \| u_n \oplus u_{n-1} \| + \sigma_1 \| v_n \oplus v_{n-1} \| \\
&\qquad \leq \tau_1 \| u_n - u_{n-1} \| + \sigma_1 \| v_n - v_{n-1} \|
\end{aligned}$$

$$\begin{aligned}
&\leq \tau_1\left(1+n^{-1}\right)\mathcal{D}\left(S(x_n),S(x_{n-1})\right)+\sigma_1\left(1+n^{-1}\right)\mathcal{D}\left(T(x_n),T(x_{n-1})\right)\\
&\leq \tau_1\gamma_1\left(1+n^{-1}\right)\|x_n-x_{n-1}\|+\sigma_1\gamma_2\left(1+n^{-1}\right)\|x_n-x_{n-1}\|\\
&=\left[(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)\right]\|x_n-x_{n-1}\|. \qquad (4.5)
\end{aligned}$$

Since $H(\cdot,\cdot)$ is k_1-ordered compression mapping with respect to A and k_2-ordered compression mapping with respect to B and (g_1-p_1) is r_1-Lipschitz-type-continuous, we have

$$\begin{aligned}
&\|H(A,B)(g_1-p_1)(x_n)\oplus H(A,B)(g_1-p_1)(x_{n-1})\|\\
&\leq \Big\|\big[H\big(A(g_1-p_1)(x_n),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\big]\\
&\quad\oplus\big[H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_{n-1})\big)\big]\Big\|\\
&\leq \Big\|\big[H\big(A(g_1-p_1)(x_n),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\big]\\
&\quad-\big[H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_{n-1})\big)\big]\Big\|\\
&\leq \Big\|H\big(A(g_1-p_1)(x_n),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\Big\|\\
&\quad+\Big\|H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_n)\big)\oplus H\big(A(g_1-p_1)(x_{n-1}),B(g_1-p_1)(x_{n-1})\big)\Big\|\\
&\leq k_1\|(g_1-p_1)(x_n)\oplus(g_1-p_1)(x_{n-1})\|+k_2\|(g_1-p_1)(x_n)\oplus(g_1-p_1)(x_{n-1})\|\\
&=(k_1+k_2)\|(g_1-p_1)(x_n)\oplus(g_1-p_1)(x_{n-1})\|\\
&\leq r_1(k_1+k_2)\|x_n\oplus x_{n-1}\|\\
&\leq r_1(k_1+k_2)\|x_n-x_{n-1}\|. \qquad (4.6)
\end{aligned}$$

Using (4.5) and (4.6) in (4.4), we have

$$\begin{aligned}
&\big\|(g_1-p_1)x_{n+1}\oplus(g_1-p_1)x_n\big\|\\
&\quad\leq \frac{\lambda_N|\rho_1|}{\rho_1\theta_1-(k_1+k_2)}\left[(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)\right]\|x_n-x_{n-1}\|\\
&\qquad+\frac{\lambda_N}{\rho_1\theta_1-(k_1+k_2)}r_1(k_1+k_2)\|x_n-x_{n-1}\|+\frac{\lambda_N}{\rho_1\theta_1-(k_1+k_2)}\|e_n\oplus e_{n-1}\|\\
&\quad=\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)+r_1\lambda_N(k_1+k_2)}{\rho_1\theta_1-(k_1+k_2)}\|x_n-x_{n-1}\|\\
&\qquad+\frac{\lambda_N}{\rho_1\theta_1-(k_1+k_2)}\|e_n\oplus e_{n-1}\|. \qquad (4.7)
\end{aligned}$$

Since $(g_1-p_1\oplus I)$ is δ_1-Lipschitz-type-continuous mapping and in view of (4.7), we have

$$\begin{aligned}
&\|x_{n+1}\oplus x_n\|\\
&\quad=\|[(g_1-p_1)x_{n+1}\oplus(g_1-p_1)x_n]\oplus[(g_1-p_1)x_{n+1}\oplus x_{n+1}\oplus(g_1-p_1)x_n\oplus x_n]\|\\
&\quad\leq\|[(g_1-p_1)x_{n+1}\oplus(g_1-p_1)x_n]-[(g_1-p_1)x_{n+1}\oplus x_{n+1}\oplus(g_1-p_1)x_n\oplus x_n]\|\\
&\quad\leq\|(g_1-p_1)x_{n+1}\oplus(g_1-p_1)x_n\|+\|(g_1-p_1\oplus I)x_{n+1}\oplus(g_1-p_1\oplus I)x_n\|\\
&\quad\leq\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)+r_1\lambda_N(k_1+k_2)}{\rho_1\theta_1-(k_1+k_2)}\|x_n-x_{n-1}\|\\
&\qquad+\frac{\lambda_N}{\rho_1\theta_1-(k_1+k_2)}\|e_n\oplus e_{n-1}\|+\delta_1\|x_{n+1}\oplus x_n\|.
\end{aligned}$$

This implies,

$$\|x_{n+1}\oplus x_n\|\leq\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)+r_1\lambda_N(k_1+k_2)}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}\|x_n-x_{n-1}\|$$

$$+\frac{\lambda_N}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}\left\|e_n \oplus e_{n-1}\right\|.$$

Since $x_{n+1} \propto x_n,\ n=0,1,2,\cdots$, we have

$$\|x_{n+1}-x_n\| \leq \varphi_n\|x_n-x_{n-1}\|+\eta\left\|e_n \oplus e_{n-1}\right\|, \tag{4.8}$$

where

$$\varphi_n=\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)\left(1+n^{-1}\right)+r_1\lambda_N(k_1+k_2)}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}, \eta=\frac{\lambda_N}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}.$$

Let

$$\varphi=\frac{\lambda_N|\rho_1|(\tau_1\gamma_1+\sigma_1\gamma_2)+r_1\lambda_N(k_1+k_2)}{(1-\delta_1)\left[\rho_1\theta_1-(k_1+k_2)\right]}.$$

It is clear that $\varphi_n \longrightarrow \varphi$ as $n \longrightarrow \infty$. By (4.2), we know that $0<\varphi<1$ and hence there exists $n_0>0$ and $\varphi_0 \in (0,1)$ such that $\varphi_n \leq \varphi_0$ for all $n \geq n_0$. Therefore, by (4.8), we have

$$\|x_{n+1}-x_n\| \leq \varphi_0\|x_n-x_{n-1}\|+\eta\left\|e_n \oplus e_{n-1}\right\|,\ \forall\ n \geq n_0. \tag{4.9}$$

(4.9) implies that

$$\|x_{n+1}-x_n\| \leq \varphi_0^{n-n_0}\|x_{n_0+1}-x_{n_0}\|+\eta\sum_{j=1}^{n-n_0}\varphi_0^{j-1}t_{n-(j-1)}, \tag{4.10}$$

where $t_n=\|e_n \oplus e_{n-1}\|$ for all $n \geq n_0$. Hence, for any $m \geq n > n_0$, we have

$$\begin{aligned}\|x_m-x_n\| &\leq \sum_{k=n}^{m-1}\|x_{k+1}-x_k\| \\ &\leq \sum_{k=n}^{m-1}\varphi_0^{k-n_0}\|x_{n_0+1}-x_{n_0}\|+\eta\sum_{k=n}^{m-1}\varphi_0^{k}\left[\sum_{j=1}^{k-n_0}\frac{t_{k-(j-1)}}{\varphi_0^{k-(j-1)}}\right].\end{aligned}$$

Since $\sum_{j=1}^{\infty}\|e_j \oplus e_{j-1}\|\nu_1^{-j}<\infty,\ \forall\ \nu_1 \in (0,1)$ and $0<\varphi_0<1$, it follows that $\|x_m-x_n\| \to 0$ as $n \longrightarrow \infty$, and so $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}_p$. Thus, there exists $x \in \mathcal{H}_p$ such that $x_n \to u$, as $n \to \infty$. Similarly, we can show $\{y_n\}$ to be a Cauchy sequence in $\mathcal{H}_p$ and thus there exists $y \in \mathcal{H}_p$ such that $y_n \to y$, thanks to the completeness of $\mathcal{H}_p$.

By Algorithm 4.1 and the $\mathcal{D}$-Lipschitz continuity of S and T, we have

$$\left\{\begin{array}{l}\|u_{n+1} \oplus u_n\| \leq \|u_{n+1}-u_n\| \leq \left(1+(1+n)^{-1}\right)\gamma_1\|x_{n+1}-x_n\|, \\ \|v_{n+1} \oplus v_n\| \leq \|v_{n+1}-v_n\| \leq \left(1+(1+n)^{-1}\right)\gamma_2\|x_{n+1}-x_n\|, \\ \|u'_{n+1} \oplus u'_n\| \leq \|u'_{n+1}-u'_n\| \leq \left(1+(1+n)^{-1}\right)\gamma_1\|y_{n+1}-y_n\|, \\ \|v'_{n+1} \oplus v'_n\| \leq \|v'_{n+1}-v'_n\| \leq \left(1+(1+n)^{-1}\right)\gamma_2\|y_{n+1}-y_n\|.\end{array}\right. \tag{4.11}$$

It follows that $\{u_n\},\{v_n\},\{u'_n\}$ and $\{v'_n\}$ are all Cauchy sequences. Thus, there exists $u,v,u',v' \in \mathcal{H}_p$ such that $u_n \to u, v_n \to v, u'_n \to u'$ and $v'_n \to v'$ as $n \to \infty$.

Now, we show that $u \in S(x), v \in T(x), u' \in S(y)$ and $v' \in T(y)$. Since $u_n \in S(x_n)$, we have

$$\begin{aligned}d(u,S(x)) &\leq \|u \oplus u_n\|+d(u_n,S(x)) \\ &\leq \|u-u_n\|+\|u_n \oplus S(x)\| \\ &\leq \|u-u_n\|+\mathcal{D}\left(S(x_n),S(x)\right) \\ &\leq \|u-u_n\|+\gamma_1\|x_n-x\| \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Since $S(x)$ is closed, it follows that $u \in S(x)$. Similarly, we can show that $v \in T(x), u' \in S(y)$ and $v' \in T(y)$. Thus in view of Lemma 3.1 we conclude that (x, y, u, v, u', v'), such that $x, y \in \mathcal{H}_p, u \in S(x), v \in T(x), u' \in S(y), v' \in T(y)$, is a solution of GSMVIP (3.1). This completes the proof. □

5. Conclusion

The results presented in this paper generalizes many known results in the literature. The class of XOR-NODSM mappings involving ⊕ operation is much wider and more general than those of (A, η)-accretive operator, (H, η)-monotone operator as already discussed by many researchers. The resolvent operator associated with XOR-NODSM mappings can be further exploited to solve different classes of variational inclusions and related systems in the setting of Banach and semi-inner product spaces considered in, see for example [1–11, 13–23, 25, 26].

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