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BALANCED MAPPINGS AND AN ITERATIVE SCHEME IN COMPLETE GEODESIC SPACES

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ABSTRACT. In this paper, we define a balanced mapping by a maximizer of a certain function generated by a finite number of mappings without regard to their order and find its fundamental properties in a complete CAT(1) space. Furthermore, we approximate a fixed point of a balanced mapping which is generated by a finite number of quasinonexpansive and Δ-demiclosed mappings by using Mann's iterative scheme.

KEYWORDS: Common fixed point, CAT(1) space, quasinonexpansive, Mann type, iteration
AMS Subject Classification: 47H09

1. Introduction

In the study of nonlinear analysis, we approximate a fixed point of many kinds of mappings. We focus on a balanced mapping which is generated by a finite number of mappings without regard to their order. Hasegawa and Kimura [2] defined it by proving the following theorem in the setting of complete CAT(0) spaces. We will extend its definition in the setting of complete CAT(1) spaces.

Theorem 1.1. (Hasegawa–Kimura [2]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \dots, N$. Let $\alpha^k \in [0, 1]$ for every $k = 1, 2, \dots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let x be a point of X. Then the set*

$$\operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha^k d(T^k x, y)^2$$

consists of one point.

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We know that there are various kinds of iterative schemes which is effective to find fixed points of nonexpansive mappings. We pay attention to Mann's [3] iterative scheme. A number of authors have proved approximation theorems by using that scheme. Reich [7] proved it in a Banach space. Dhompongsa and Panyanak [1] proved it in a CAT(0) space. Kimura, Saejung, and Yotkaew [4] proved it by using a quainonexpansive and Δ-demiclosed mapping in a CAT(1) space. We particularly note that Hasegawa and Kimura [2] proved the convergence of Mann type iteration by using a balanced mapping.

Theorem 1.2. (Hasegawa–Kimura [2]) *Let X be a complete* CAT(0) *space. Let T^k be a nonexpansive mapping from X to X for every $k = 1, 2, \dots, N$ such that $F = \bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. For a given real number $a \in \left]0, \frac{1}{2}\right]$, let $\left\{\alpha_n^k\right\}, \{\beta_n\} \subset [a, 1-a]$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Define U_n be a mapping from X to X by*

$$U_n x = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha_n^k d(T^k x, y)^2$$

for every $x \in X$ and $n \in \mathbb{N}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in F.

In this paper, we define a balanced mapping in a complete CAT(1) space and prove a convergence theorem of Mann type iteration by using it. Namely, our results are a modified version of the results by Hasegawa and Kimura [2] in a complete CAT(1) space.

2. Preliminaries

Let X be a metric space and $\{x_n\}$ a sequence in X. An element $z \in X$ is said to be an asymptotic center of $\{x_n\} \subset X$ if

$$\limsup_{n\to\infty} d(x_n, z) = \inf_{x \in X} \limsup_{n\to\infty} d(x_n, x).$$

Moreover, we say $\{x_n\}$ Δ-converges to a Δ-limit z if z is the unique asymptotic center of any subsequences of $\{x_n\}$. For $x, y \in X$, a mapping $c : [0, l] \to X$ is called a geodesic if c satisfies

$$c(0) = x, c(l) = y, \text{ and } d(c(u), c(v)) = |u - v|$$

for every $u, v \in [0, l]$. An image of $[x, y]$ of c is called a geodesic segment joining x and y. For $r > 0$, X is said to be an r-geodesic space if for every $x, y \in X$ with $d(x, y) < r$, there exists a geodesic c joining x and y. Moreover, if such a geodesic segment is unique for each pair of points, then X is said to be a uniquely r-geodesic space.

Let X be a uniquely π-geodesic space. For a triangle $\triangle(x, y, z) \subset X$ such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$, let a comparison triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$ in two-dimensional unit sphere $\mathbb{S}^2$ be such that each corresponding edge has the same length as that of the original triangle. X is called a CAT(1) space if every $p, q \in \triangle(x, y, z)$ and their corresponding points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z})$ satisfy that

$$d(p, q) \le d_{\mathbb{S}^2}(\bar{p}, \bar{q}),$$

where $d_{\mathbb{S}^2}$ is the spherical metric on $\mathbb{S}^2$.

Let X be a CAT(1) space. For every $x, y \in X$ with $d(x,y) < \pi$ and $\alpha \in [0,1]$, if $z \in [x,y]$ satisfies that $d(y,z) = \alpha d(x,y)$ and $d(x,z) = (1-\alpha)d(x,y)$, then we denote z by $z = \alpha x \oplus (1-\alpha)y$. A subset $C \subset X$ is called π-convex if $\alpha x \oplus (1-\alpha)y \in C$ for every $x, y \in C$ with $d(x,y) < \pi$ and $\alpha \in [0,1]$.

Let X be a CAT(1) space and let T be a mapping from X to X such that the set $F(T) = \{z \in X : z = Tz\}$ of fixed points of T is not empty. If $d(Tx,p) \le d(x,p)$ for every $x \in X$ and $p \in F(T)$, then we call T a quasinonexpansive mapping.

T is said to be a strongly quasinonexpansive mapping if T is a quasinonexpansive mapping, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ whenever $\{x_n\} \subset X$ satisfies $\sup_{n\in\mathbb{N}} d(x_n,p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(Tx_n,p)) = 1$ for every $p \in F(T)$.

Let X be a CAT(1) space and let T be a mapping from X to X such that $F(T) \neq \emptyset$. T is said to be a Δ-demiclosed mapping if $z \in F(T)$ whenever $\{x_n\}$ Δ-converges to z and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let X be a complete CAT(1) space and let $C \subset X$ be a nonempty closed π-convex subset such that $d(x,C) = \inf_{y\in C} d(x,y) < \pi/2$ for every $x \in X$. Then for every $x \in X$, there exists a unique point $x_0 \in C$ satisfying

$$d(x,x_0) = \inf_{y\in C} d(x,y).$$

We define the metric projection P_C from X onto C by $P_C x = x_0$.

We introduce some lemmas used for our results.

Lemma 2.1. (Kimura and Satô [5]) *Let X be a* CAT(1) *space. For every $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2\pi$ and $\alpha \in [0,1]$, the following inequality holds:*

$$\cos d(x,w) \sin d(y,z) \ge \cos d(x,y) \sin(\alpha d(y,z)) + \cos d(x,z) \sin((1-\alpha)d(y,z)),$$

where $w = \alpha y \oplus (1-\alpha)z$.

Lemma 2.2. (Kimura and Satô [6]) *Let X be a* CAT(1) *space. For every $x, y, z \in X$ with $d(x,y) + d(y,z) + d(z,x) < 2\pi$ and $\alpha \in [0,1]$, the following inequality holds:*

$$\cos d(x,w) \ge \alpha \cos d(x,y) + (1-\alpha) \cos d(x,z),$$

where $w = \alpha y \oplus (1-\alpha)z$.

Lemma 2.3. (Kimura and Satô [6]) *Let X be a* CAT(1) *space and y_0, y_1 and y elements of X such that $d(y_0,y) + d(y_1,y) + d(y_0,y_1) < 2\pi$. Then we have*

$$\cos d\left(\frac{1}{2}y_0 \oplus \frac{1}{2}y_1, y\right) \cos \frac{d(y_0,y_1)}{2} \ge \min\{\cos d(y_0,y), \cos d(y_1,y)\}.$$

3. Balanced mapping in CAT(1) spaces

In this section, we define a balanced mapping and find its fundamental properties in a CAT(1) space. We begin with the following theorem which guarantees that the balanced mapping can be defined as a single-valued mapping.

Theorem 3.1. *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let x^k be a point of X for every $k = 1, 2, \dots, N$. Let $\alpha^k \in [0,1]$ for every $k = 1, 2, \dots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Then the set*

$$\operatorname*{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$$

consists of one point.

Proof. Let $D = \sup_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$ and $\{y_n\}$ a sequence in X such that $\lim_{n\to\infty} \sum_{k=1}^N \alpha^k \cos d(x^k, y_n) = D$. For $m, n \in \mathbb{N}$, from Lemma 2.3, we have

$$\sum_{k=1}^N \alpha^k \cos d\left(x^k, \frac{1}{2}y_n \oplus \frac{1}{2}y_m\right) \cos\frac{d(y_n, y_m)}{2} \geq \sum_{k=1}^N \alpha^k \min\{\cos d(y_n, x^k), \cos d(y_m, x^k)\}.$$

Thus we get

$$\cos\frac{d(y_n, y_m)}{2} \geq \frac{\sum_{k=1}^N \alpha^k \min\{\cos d(y_n, x^k), \cos d(y_m, x^k)\}}{D}.$$

Hence we obtain $\{y_n\}$ is a Cauchy sequence. By the completeness of X, there exists $u = \lim_{n\to\infty} y_n$. From the continuity of the metric, we get $\sum_{k=1}^N \alpha^k \cos d(x^k, u) = \sup_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$. Hence $\operatorname{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$ is nonempty. Let $u, v \in \operatorname{argmax}_{y\in X} \sum_{k=1}^N \cos d(x^k, y)$ and suppose $u \neq v$. By Lemma 2.1, we have

$$\begin{aligned}\sum_{k=1}^N \alpha^k \cos d(x^k, u) \sin d(u, v) &\geq \sum_{k=1}^N \alpha^k \cos d\left(x^k, \frac{1}{2}u \oplus \frac{1}{2}v\right) \sin d(u, v)\\ &\geq \sin\frac{d(u, v)}{2} \sum_{k=1}^N \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).\end{aligned}$$

Dividing by $\sin(d(u, v)/2)$, we get

$$2\cos\frac{d(u, v)}{2} \sum_{k=1}^N \alpha^k \cos d(x^k, u) \geq \sum_{k=1}^N \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).$$

Similarly, we get

$$2\cos\frac{d(u, v)}{2} \sum_{k=1}^N \alpha^k \cos d(x^k, v) \geq \sum_{k=1}^N \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).$$

Therefore, we obtain

$$2\cos\frac{d(u, v)}{2} \sum_{k=1}^N \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)) \geq 2\sum_{k=1}^N \alpha^k (\cos d(x^k, u) + \cos d(x^k, v)).$$

Then we have

$$1 > \cos\frac{d(u, v)}{2} \geq 1,$$

which is a contradiction. Hence we get $u = v$. □

By Theorem 3.1, we know the set $\operatorname{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(x^k, y)$ is a singleton. In what follows, a balanced mapping U from X to X for a sequence $\alpha^1, \alpha^2, \ldots, \alpha^N \in [0, 1]$ and mappings $T^1, T^2, \ldots, T^N$ is defined by

$$Ux = \operatorname*{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(T^k x, y)$$

for every $x \in X$. We prove some basic properties of balanced mappings in this section.

Theorem 3.2. *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive mapping from X to X for every $k = 1, 2, \dots, N$ such that $\bigcap_{k=1}^N F(T^k) \neq \emptyset$. Let $\alpha^k \in \,]0,1[$ for every $k = 1, 2, \dots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then $F(U) = \bigcap_{k=1}^N F(T^k)$.*

Proof. Let $z \in \bigcap_{k=1}^N F(T^k)$. Then we have

$$\begin{aligned} Uz &= \operatorname*{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(T^k z, y) \\ &= \operatorname*{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(z, y) \\ &= \operatorname*{argmax}_{y\in X} \cos d(z, y) \\ &= z. \end{aligned}$$

Hence we get $z \in F(U)$. Let $z \in F(U), w \in \bigcap_{k=1}^N F(T^k)$ and $t \in \,]0,1[$. We may assume that $z \neq w$. From Lemma 2.1, we have

$$\begin{aligned} &\sum_{k=1}^N \alpha^k \cos d(T^k z, z) \sin d(z, w) \\ &\quad \geq \sum_{k=1}^N \alpha^k \cos d(T^k z, tz \oplus (1-t)w) \sin d(z, w) \\ &\quad \geq \sum_{k=1}^N \alpha^k \cos d(T^k z, z) \sin td(z, w) + \sum_{k=1}^N \alpha^k \cos d(T^k z, w) \sin(1-t)d(z, w) \\ &\quad \geq \sum_{k=1}^N \alpha^k \cos d(T^k z, z) \sin td(z, w) + \cos d(z, w) \sin(1-t)d(z, w). \end{aligned}$$

Hence we get

$$2\sum_{k=1}^N \alpha^k \cos d(T^k z, z)(\sin d(z, w) - \sin td(z, w)) \geq \cos d(z, w) \sin(1-t)d(z, w),$$

and it implies that

$$\begin{aligned} &2\sum_{k=1}^N \alpha^k \cos d(T^k z, z) \sin \frac{(1-t)d(z,w)}{2} \cos \frac{(1+t)d(z,w)}{2} \\ &\quad \geq 2\cos d(z, w) \sin \frac{(1-t)d(z,w)}{2} \cos \frac{(1-t)d(z,w)}{2}. \end{aligned}$$

Dividing by $2\sin((1-t)d(z,w)/2)\cos d(z,w)$ and tending $t \to 1$, we get

$$\sum_{k=1}^N \alpha^k \cos d(T^k z, z) \geq 1.$$

Therefore we have $\cos d(T^k z, z) = 1$ for every $k = 1, 2, \dots, N$. Hence we get $z \in \bigcap_{k=1}^N F(T^k)$. □

Lemma 3.1. *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive mapping from X to X for every*

$k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^N F(T^k) \neq \emptyset$. Let $\alpha^k \in [0,1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then we have

$$\sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for every $x \in X$ and $z \in \bigcap_{k=1}^N F(T^k)$.

Proof. Let $z \in \bigcap_{k=1}^N F(T^k)$ and $t \in \,]0,1[$. Then, from Lemma 3.2, we have $z \in F(U)$. We may assume that $Ux \neq z$ since if $Ux = z$, the inequality is obvious true. By Lemma 2.1, we get

$$\begin{aligned}
&\sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \sin d(Ux, z) \\
&\quad \geq \sum_{k=1}^N \alpha^k \cos d(T^k x, tUx \oplus (1-t)z) \sin d(Ux, z) \\
&\quad \geq \sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \sin t d(Ux, z) + \sum_{k=1}^N \alpha^k \cos d(T^k x, z) \sin(1-t) d(Ux, z) \\
&\quad \geq \sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \sin t d(Ux, z) + \sum_{k=1}^N \alpha^k \cos d(x, z) \sin(1-t) d(Ux, z) \\
&\quad = \sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \sin t d(Ux, z) + \cos d(x, z) \sin(1-t) d(Ux, z).
\end{aligned}$$

Hence we obtain

$$\sum_{k=1}^N \alpha^k \cos d(T^k x, Ux)(\sin d(Ux, z) - \sin t d(Ux, z)) \geq \cos d(x, z) \sin(1-t) d(Ux, z),$$

and it implies that

$$\begin{aligned}
&2 \sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \sin \frac{(1-t) d(Ux, z)}{2} \cos \frac{(1+t) d(Ux, z)}{2} \\
&\qquad \geq 2 \cos d(x, z) \sin \frac{(1-t) d(Ux, z)}{2} \cos \frac{(1-t) d(Ux, z)}{2}.
\end{aligned}$$

Dividing by $2 \sin((1-t) d(Ux, z)/2)$ and tending $t \to 1$, we get

$$\sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for $x \in X$. $\square$

Theorem 3.3. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^N F(T^k) \neq \emptyset$. Let $\alpha^k \in [0,1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a quasinonexpansive mapping.*

Proof. From Lemma 3.2, let $z \in F(U) = \bigcap_{k=1}^N F(T^k)$. By Lemma 3.1, we have

$$\sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \geq \cos d(x, z)$$

for $x \in X$. Since $\cos d(T^k x, Ux) \leq 1$, we get

$$\begin{aligned}\cos d(Ux, z) &\geq \sum_{k=1}^N \alpha^k \cos d(T^k x, Ux) \cos d(Ux, z) \\ &\geq \cos d(x, z).\end{aligned}$$

Thus, we obtain

$$d(Ux, z) \leq d(x, z).$$

Hence U is a quasinonexpansive mapping. □

Theorem 3.4. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive and Δ-demiclosed mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^N F(T^k) \neq \emptyset$. Let $\alpha^k \in]0, 1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a Δ-demiclosed mapping.*

Proof. From Lemma 3.2, let $z \in F(U) = \bigcap_{k=1}^N F(T^k)$. Let $\{x_n\} \subset X$ satisfying $d(Ux_n, x_n) \to 0$ and $\{x_n\}$ Δ-converges to $x_0 \in X$. By Lemma 3.1, we have

$$\sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) \cos d(Ux_n, z) \geq \cos d(x_n, z).$$

Then we get

$$\sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) \geq \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)}.$$

Since $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(Ux_n, z)) = 1$, we obtain

$$\lim_{n\to\infty} \sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) = 1.$$

Hence we get $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ for every $k = 1, 2, \ldots, N$. Then we have $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$. Since T^k is a Δ-demiclosed mapping for every $k = 1, 2, \ldots, N$, we obtain $x_0 \in F$. □

Theorem 3.5. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a strongly quasinonexpansive mapping from X to X for every $k = 1, 2, \ldots, N$ such that $\bigcap_{k=1}^N F(T^k) \neq \emptyset$. Let $\alpha^k \in]0, 1[$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^N \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then U is a strongly quasinonexpansive mapping.*

Proof. From Lemma 3.2, let $z \in F(U) = \bigcap_{k=1}^N F(T^k)$. Let $\{x_n\} \subset X$ satisfying $\limsup_{n\to\infty} d(x_n, z) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(Ux_n, z)) = 1$. By Lemma 3.1, we have

$$\sum_{k=1}^N \alpha^k \cos d(T^k x_n, Ux_n) \cos d(Ux_n, z) \geq \cos d(x_n, z).$$

Then we get

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x_n, Ux_n) \geq \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)}.$$

Since $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(Ux_n, z)) = 1$, we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{N} \alpha^k \cos d(T^k x_n, Ux_n) = 1.$$

Hence we get $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ for every $k = 1, 2, \ldots, N$. For any $k = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)} &= \liminf_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)} \\
&\leq \liminf_{n\to\infty} \frac{\cos d(x_n, z)}{\cos(d(Ux_n, T^k x_n) + d(T^k x_n, z))} \\
&= \liminf_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(T^k x_n, z)} \\
&\leq \limsup_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(T^k x_n, z)} \\
&\leq \limsup_{n\to\infty} \frac{\cos d(x_n, z)}{\cos(d(T^k x_n, Ux_n) + d(Ux_n, z))} \\
&= \limsup_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)} \\
&= \lim_{n\to\infty} \frac{\cos d(x_n, z)}{\cos d(Ux_n, z)}.
\end{aligned}$$

Thus we obtain $\lim_{n\to\infty}(\cos d(x_n, z)/\cos d(T^k x_n, z)) = 1$. Since T^k is a strongly quasinonexpansive mapping for every $k = 1, 2, \ldots, N$, we get $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$. Since $\lim_{n\to\infty} d(T^k x_n, Ux_n) = 0$ and $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$, we obtain $\lim_{n\to\infty} d(Ux_n, x_n) = 0$. □

4. An iterative scheme for balanced mappings

In this section, we prove a convergence theorem of a Mann iterative sequence by using a balanced mapping in a complete CAT(1) space.

Lemma 4.1. *Let X be a complete* CAT(1) *space such that $d(x, y) < \pi/2$ for every $x, y \in X$. Let T^k be a mapping from X to X for every $k = 1, 2, \ldots, N$. Let $\alpha^k \in [0, 1]$ for every $k = 1, 2, \ldots, N$ such that $\sum_{k=1}^{N} \alpha^k = 1$. Let U be a balanced mapping for $\{\alpha^k\}$ and $\{T^k\}$. Then we have*

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \geq \frac{\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Uy)}{\cos d(Ux, Uy)}$$

for every $x, y \in X$.

Proof. Let $t \in]0,1[$. We may assume $Ux \neq Uy$. By Lemma 2.1, we have

$$\begin{aligned}
&\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \sin d(Ux, Uy) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, tUx \oplus (1-t)Uy) \sin d(Ux, Uy) \\
&\quad \geq \sum_{k=1}^{N} \alpha^k (\cos d(T^k x, Ux) \sin t d(Ux, Uy) + \cos d(T^k x, Uy) \sin(1-t) d(Ux, Uy)).
\end{aligned}$$

Then we get

$$\begin{aligned}
&2 \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \cos \frac{(1+t)d(Ux,Uy)}{2} \sin \frac{(1-t)d(Ux,Uy)}{2} \\
&\quad \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Uy) \sin(1-t) d(Ux, Uy).
\end{aligned}$$

Dividing by $2\cos((1+t)d(Ux,Uy)/2)\sin((1-t)d(Ux,Uy)/2)$, we get

$$\sum_{k=1}^{N} \alpha^k \cos d(T^k x, Ux) \geq \sum_{k=1}^{N} \alpha^k \cos d(T^k x, Uy) \frac{\cos \dfrac{(1-t)d(Ux,Uy)}{2}}{\cos \dfrac{(1+t)d(Ux,Uy)}{2}}.$$

Tending $t \to 1$, we obtain the desired result. $\square$

Theorem 4.1. *Let X be a complete* CAT(1) *space such that $d(x,y) < \pi/2$ for every $x, y \in X$. Let T^k be a quasinonexpansive and Δ-demiclosed mapping from X to X for every $k = 1, 2, \dots, N$ such that $\bigcap_{k=1}^{N} F(T^k) \neq \emptyset$. For a given real number $a \in]0, 1/2]$, let $\{\alpha_n^k\}, \{\delta_n\} \subset [a, 1-a]$ for every $k = 1, 2, \dots, N$ and $n \in \mathbb{N}$ such that $\sum_{k=1}^{N} \alpha_n^k = 1$. Let U_n be a balanced mapping for $\{\alpha_n^k\}$ and $\{T^k\}$. For a given point $x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \delta_n x_n \oplus (1-\delta_n) U_n x_n$$

for every $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to a point in $\bigcap_{k=1}^{N} F(T^k)$.

Proof. From Lemma 3.2, we know that $F(U_n) = \bigcap_{k=1}^{N} F(T^k)$ for every $n \in \mathbb{N}$. Let $z \in F = F(U_n) = \bigcap_{k=1}^{N} F(T^k)$. From Lemmas 2.2 and 3.3, we have

$$\begin{aligned}
\cos d(x_{n+1}, z) &= \cos d(\delta_n x_n \oplus (1-\delta_n) U_n x_n, z) \\
&\geq \delta_n \cos d(x_n, z) + (1-\delta_n) \cos d(U_n x_n, z) \\
&\geq \cos d(x_n, z).
\end{aligned}$$

Thus, we obtain $d(x_{n+1}, z) \leq d(x_n, z)$ for all $n \in \mathbb{N}$ and there exists

$$D = \lim_{n\to\infty} d(x_n, z) \leq d(x_1, z) < \frac{\pi}{2}.$$

Since $\{\delta_n\} \subset [a, 1-a]$, from Lemma 2.1, we get

$$\begin{aligned}
&\cos d(x_{n+1}, z) \sin d(x_n, U_n x_n) \\
&\quad = \cos d(\delta_n x_n \oplus (1-\delta_n) U_n x_n, z) \sin d(x_n, U_n x_n) \\
&\quad \geq \cos d(x_n, z) \sin \delta_n d(x_n, U_n x_n) + \cos d(U_n x_n, z) \sin(1-\delta_n) d(x_n, U_n x_n)
\end{aligned}$$

$$\begin{aligned}&\geq \cos d(x_n, z)(\sin \delta_n d(x_n, U_n x_n) + \sin(1-\delta_n) d(x_n, U_n x_n))\\&\geq 2\cos d(x_n, z)\sin a d(x_n, U_n x_n).\end{aligned}$$

Putting $E = \lim_{n\to\infty} d(x_n, U_n x_n)$ and tending $n \to \infty$, we get

$$\cos D \sin E \geq 2 \cos D \sin aE.$$

Using elementary calculation, we have $E = 0$, that is,

$$\lim_{n\to\infty} d(x_n, U_n x_n) = 0.$$

We show $\lim_{n\to\infty} d(x_n, T^k x_n) = 0$ for all $k = 1, 2, \ldots, N$. Since $\{x_n\}$ is bounded, it follows that

$$\begin{aligned}D = \lim_{n\to\infty} d(x_n, z) &\leq \lim_{n\to\infty}(d(x_n, U_n x_n) + d(U_n x_n, z))\\&= \lim_{n\to\infty} d(U_n x_n, z)\\&\leq \lim_{n\to\infty} d(x_n, z) = D.\end{aligned}$$

Thus we get $\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(U_n x_n, z) = D$. By Lemma 4.1, we have

$$\begin{aligned}\sum_{k=1}^{N} \alpha_n^k \cos d(T^k x_n, U_n x_n) &\geq \frac{\sum_{k=1}^{N} \alpha_n^k \cos d(T^k x_n, z)}{\cos d(U_n x_n, z)}\\&\geq \frac{\sum_{k=1}^{N} \alpha_n^k \cos d(x_n, z)}{\cos d(U_n x_n, z)}\\&\geq \frac{\cos d(x_n, z)}{\cos d(U_n x_n, z)}.\end{aligned}$$

Since $\alpha_n^k \leq 1 - a < 1$, we obtain $\lim_{n\to\infty} d(T^k x_n, U_n x_n) = 0$ for every $k = 1, 2, \ldots, N$. Since $\lim_{n\to\infty} d(U_n x_n, x_n) = 0$, we also get $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$. Let x_0 be an asymptotic center of $\{x_n\}$ and for every $\{x_{n_k}\} \subset \{x_n\}$, let y be an asymptotic center of $\{x_{n_k}\}$. There exists $\{x_{n_{k_l}}\} \subset \{x_{n_k}\}$ satisfying that $\{x_{n_{k_l}}\}$ Δ-converges to w. Since T^k is Δ-demiclosed and $\lim_{n\to\infty} d(T^k x_n, x_n) = 0$ for every $k = 1, 2, \ldots, N$, we get $w \in F$. Since there exists $\lim_{n\to\infty} d(x_{n_k}, w)$, we have

$$\begin{aligned}\limsup_{k\to\infty} d(x_{n_k}, w) &= \lim_{k\to\infty} d(x_{n_k}, w)\\&= \lim_{l\to\infty} d(x_{n_{k_l}}, w)\\&\leq \limsup_{l\to\infty} d(x_{n_{k_l}}, y)\\&\leq \limsup_{k\to\infty} d(x_{n_k}, y).\end{aligned}$$

Since y is an asymptotic center of $\{x_{n_k}\}$, we obtain $y = w$. Then we have $y \in F$. Hence we get

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, y) &= \lim_{n\to\infty} d(x_n, y)\\&= \lim_{k\to\infty} d(x_{n_k}, y)\\&\leq \limsup_{k\to\infty} d(x_{n_k}, x_0)\\&\leq \limsup_{n\to\infty} d(x_n, x_0).\end{aligned}$$

Since x_0 is an asymptotic center of $\{x_n\}$, we obtain $x_0 = y$. Therefore we obtain $\{x_n\}$ Δ-converges to $x_0 \in F$. □

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A PRACTICAL APPROACH TO OPTIMIZATION

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ABSTRACT. We present a new approach for finding a minimal value of an arbitrary function assuming only its continuity. The process avoids verifying Lagrange- or KKT-conditions. The method enables us to obtain a Brouwer fixed point (of a continuous function mapping from a cube into itself).

KEYWORDS: convex algorithm, optimization, particle swarm optimization, pattern-search.
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1. Introduction

For a given set of continuous functions $f, g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n : \mathcal{C} = \prod_{i=1}^{p}[c_i, d_i] \to \mathbb{R}$, a minimization problem of the form

$$\begin{aligned} &\min_{x\in\mathcal{C}} f(x) \\ \text{subject to } &g_i(x) = 0 \ (i = 1, 2, \ldots, m) \\ &h_i(x) \leq 0 \ (i = 1, 2, \ldots, n). \end{aligned} \tag{1.1}$$

is well known. For the Problem (1.1), f is called the objective function and the equalities (described by g_i) and the inequalities (described by h_i) are called the constraints. We call the set $\mathcal{A} = \{x \in \mathcal{C} : g_i(x) = 0 \ (i = 1, 2, \ldots, m) \text{ and } h_i(x) \leq 0 \ (i = 1, 2, \ldots, n)\}$ the feasible set of Problem (1.1). If $\mathcal{A}$ is not empty, it is compact since it is a zero set of the continuous function F defined below. Consequently, Problem (1.1) always has a solution if $\mathcal{A}$ is not empty. The subject is well understood for convex optimization with Lagrange multipliers and Karush-Kuhn-Tucker conditions are its familiar main tools. It is the purpose of this article to introduce an alternative method in minimizing a function without using the tools mentioned above. The method can be considered as a complement to the "penalty method". It transforms the constrained Problem (1.1) of f into an unconstrained one of a deformation f_t of f. "It also serves as a toolkit using for approximating a result by applying any existing software. We choose to work on some well-known software to find a decreasing sequence $\{f_t(x_n)\}$, namely, particle swarm optimization (PSO), particle-search algorithm, and convex optimization. By testing the method over many kinds of objective functions f, we believe the method is quite practical. It is found that a problem may work well under one software but not under some others. Moreover, the method can be performed to obtain a Brouwer fixed point and applied to a vector optimization.

In computational science, particle swarm optimization (PSO) [12, 13, 14] is the computational method that optimization problem by iteratively trying to improve a candidate solution with regard to a given measure of quality. A basic variant of the PSO algorithm works by having a population (swarm) of candidate solutions (particles). These particles are moved around in the search-space according to a simple formula. The movements of the particles are guided by their own best known position in the search-space. The entire swarm's best known position. When improved positions are being discovered these will then come to guide the movements of the swarm. The process is repeated and by doing so it is hoped, but not guaranteed, that a satisfactory solution will eventually be discovered.

Pattern search algorithm is a family of numerical optimization methods. It finds a sequence of points that approach an optimal point. The value of the objective function either decreases or remains the same from each point in the sequence to the next [1, 2, 8].

Convex optimization is a subfield of mathematical optimization that studies the problem of minimizing convex functions over convex sets. Convex algorithm is a mathematical method of solving convex optimization [4, 5, 7]. The key to the algorithmic success in minimizing convex functions is that these functions exhibit a local to global phenomenon. This local to global phenomenon is that local minimal of convex functions are in fact global minimal.

2. Methodology

Put $G_i = |g_i|$ $(i = 1, 2, \ldots, m)$, $H_i = |h_i| + h_i$ $(i = 1, 2, \ldots, n)$, and $F = \sum_{i=1}^{m} G_i + \sum_{i=1}^{n} H_i$. Clearly, F is continuous and $F(x) = 0$ if and only if x satisfies the constraints of Problem (1.1) (i.e., it lies in the feasible set $\mathcal{A}$). For large numbers K and M, set for $t \in (0, 1)$, $f_t = (1 - t)(f - K) + tMF$.
Since we are going to work on the deformed function f_t for t sufficiently close to 1, we therefore take any existing software available. We select 3 softwares, namely Particle Swarm Optimization, Pattern-Search, and Convex Algorithm. We let K to be large to be certained that the graph of $f - K$ totally lies under the graph of F. As for large M, we try to make it easy for a software to find a decreasing sequence $\{f_t(x_n)\}$. The parameter t getting close to 1 is to making the iteration point x_n being closer to or lying in the feasible set $\mathcal{A}$.

Proposition 2.1. *For any $t \in (0, 1)$ with $f_t > 0$ outside $\mathcal{A}$, x is a minimizer of Problem (1.1) if and only if x is a minimizer of f_t.*

Proof. This is straightforward since $f_t = (1 - t)(f - K)$ on $\mathcal{A}$. □

By the term "minimizer" it is meant to be a minimal element, i.e., a local minimizer.

Algorithm 1 Example code (PAO our Algorithm)

Input Set up problem 1.1
Parameter K, M, t
Output x

$G_i = |g_i|$ $(i = 1, 2, \ldots, m)$
$H_i = |h_i| + h_i$ $(i = 1, 2, \ldots, n)$
$F = \sum_{i=1}^{m} G_i + \sum_{i=1}^{n} H_i$
$f_t = (1 - t)(f - K) + tMF$
$x = \arg\min_{x \in C} f_t(x)$

---

3. Applications

3.1. **Brouwer Fixed Points.** The Brouwer fixed theorem says that any continuous mapping $T = (f_1, \ldots, f_d) : \prod_{i=1}^{d}[a_i, b_i] \to \prod_{i=1}^{d}[a_i, b_i]$ always has a fixed point. See [3, 6, 10, 9] for some new proofs. To find a fixed point of T, set in Problem (1.1), $f(x_1, x_2, \ldots, x_d) = 1$ and $g_i(x_1, x_2, \ldots, x_d) = f_i(x_1, x_2, \ldots, x_d) - x_i$ $(i = 1, 2, \ldots, d)$. (See Example 4.6 and 4.7.)

3.2. **Vector Optimization.** Given continuous mappings $f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n : \mathcal{C} = \prod_{i=1}^{p}[c_i, d_i] \longrightarrow \mathbb{R}$. We need to solve

$$\begin{aligned} &\min_{x \in \mathcal{C}} (f_1(x), f_2(x), \ldots, f_k(x)) \text{ (with respect to an order)} \\ &\text{subject to } g_i(x) = 0 \ (i = 1, 2, \ldots, m) \\ &\qquad h_i(x) \leq 0 \ (i = 1, 2, \ldots, n). \end{aligned} \tag{3.1}$$

We consider the problem of the forms:

(1) $\min_{x\in\mathcal{C}} \sum_{i=1}^{k} f_i(x)$. Set $f = \sum_{i=1}^{k} f_i$ for the objective function in Problem (1.1). (See Example 4.8.)
(2) Finding $x^* = (x_1^*, x_2^*, \dots, x_p^*) \in \mathcal{C}$ such that $f_i(x^*) \le c_i$, where $c_i \le t_i$ for some thresholds $t_i \quad (i = 1, 2, \dots, k)$. To comply with Problem (1.1), we set $f = 1$ as an objective function and additionally define $h_i = f_i - c_i$ $(i = n+1, n+2, \dots, n+k)$. (See Example 4.9.)

In practice, if we only want to find a point x^* with $f(x^*) \le c$ for some assigned number c, Problem (1.1) can read as

$$\begin{aligned} &\min_{x\in\mathcal{C}} 1 \\ \text{subject to } & g_i(x) = 0 \ (i = 1, 2, \dots, m) \\ & h_i(x) \le 0 \ (i = 1, 2, \dots, n) \\ & f(x) - c \le 0. \end{aligned} \tag{3.2}$$

3.3. **Quantiles.** For a distribution function $f : \mathbb{R} \longrightarrow [0, 1]$, a quantile at $\alpha \in [0, 1]$ is defined as $f^{-1}(\alpha) = \inf\{x \in \mathbb{R} : f(x) \ge \alpha\}$. There does not exist a method to extend the concept to multivalued case. We are given a continuous function $f : \prod_{i=1}^{p}[c_i, d_i] \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$. We need to find x^* giving $f(x^*) = c$. We set Problem (1.1) as:

$$\begin{aligned} &\min_{x\in\mathcal{C}} 0 \\ \text{subject to } & f(x) = c. \end{aligned}$$

See Example 4.10.

3.4. **Non-emptiness of the feasible set.** To see if the feasible set is non-empty, we set $f_t = (1-k)(-K) + tMF$ and find a minimizer x_t and see if $x_t \in A$, i.e., $F(x_t) = 0$. Thus, any kind of problems on non-emptiness of sets defined by sets of equations and inequalities can be verified by our method. Consequently, assumptions on non-emptiness in many theorems can be worked out. For examples, non-emptiness of fixed points of mappings assumed in various results.

4. Numerical Examples

We choose $\mathcal{C} = [-10, 10]^p, K = 100, M = 10000$ and $t = 0.95$. We experiment on nine Examples, and record results in three Tables. The Tables display approximate minimizers and constraint validation.

Example 4.1. [11]

$$\begin{aligned} \min_{x\in\mathcal{C}} \quad & x_1^2 + x_1x_2 + x_2^2 - 5x_2 \\ \text{subject to} \quad & x_1 + x_2 = 1 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

Example 4.2. [11]

$$\begin{aligned} \min_{x\in\mathcal{C}} \quad & -(x_1-3)^6 - (x_2-4)^6 \\ \text{subject to} \quad & x_1^2 + x_2^2 \le 25 \\ & x_1 + x_2 \ge 7 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

Example 4.3. [11][Geometric Programming]

$$
\begin{array}{ll}
\min\limits_{x\in\mathcal{C}} & \frac{1}{x_1x_2x_3} + x_1x_2 \\
\text{subject to} & 0.5x_1x_3 + 0.25x_1x_2 \leq 1 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0
\end{array}
$$

Example 4.4. [11]

$$
\begin{array}{ll}
\min\limits_{x\in\mathcal{C}} & \frac{1}{x_1x_2x_3} + x_1x_2 + x_3^7 \\
\text{subject to} & 0.5x_1x_3 + 0.25x_1x_2 \leq 1 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0
\end{array}
$$

Example 4.5.

$$
\begin{array}{ll}
\min\limits_{x\in\mathcal{C}} & 4x_1 + 10x_2 + 15x_3 \\
\text{subject to} & x_1 + 2x_2 + 3x_3 = 3 \\
& 3x_1 + x_2 + 2x_3 = 7.5 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_3 \geq 0
\end{array}
$$

Example 4.6.

$$
\begin{array}{ll}
\min\limits_{x\in\mathcal{C}} & 1 \\
\text{subject to} & 0.5(\cos(x_1 + x_2 - x_3^4x_5))x_4 - x_1 = 0 \\
& 0.1(|x_1x_2 + x_3 - x_5| + x_4^2) - x_2 = 0 \\
& (x_1 + x_3x_4 - (x_2 + x_5)^2)/30 - x_3 = 0 \\
& (x_1 - x_2^2 + x_3 - x_5^2)/12 - x_4 = 0 \\
& (x_1 + x_2 - (x_3 + x_5 + x_4)^2)/40 - x_5 = 0
\end{array}
$$

Example 4.7.

$$
\begin{array}{ll}
\min\limits_{x\in\mathcal{C}} & 1 \\
\text{subject to} & 0.001((x_1 + 3)^2 + (x_2 - 2)^4 + x_3^2 + x_4^2 + x_5) - x_1 = 0 \\
& 0.01(x_1 + (x_2 + 5)^2 + x_3 + x_4 + (x_5 + 2)) - x_2 = 0 \\
& 0.001(x_1^4 + (x_4 - 3)^2 + (x_5 + 2)^2) - x_3 = 0 \\
& 0.001((x_3 - 3)^4 + x_5^2 + x_1^4) - 1 - x_4 = 0 \\
& 0.01(x_1^2 + x_2 + x_3 - (x_5 - 1)^2) - x_5 = 0
\end{array}
$$

Example 4.8. [11]

$$
\begin{array}{ll}
\min\limits_{x\in\mathcal{C}} & (x_1^2 - 5x_1 + 7x_2) + (-x_1^2 - x_2^2) + (x_1 - 1)^2 + (x_2 - 5)^2 \\
\text{subject to} & 3x_1 + 4x_2 = 6 \\
& x_1 + x_2 = 2 \\
& 2x_1 + 3x_2 \leq 6 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{array}
$$

Example 4.9. [11]

$$\begin{array}{ll} \min\limits_{x\in\mathcal{C}} & 1 \\ \text{subject to} & 2x_1 + x_2 \leq 1 \\ & x_1^2 \leq 1 \\ & \sqrt{x_1^2 + x_2^2} - x_1^3 \leq 2 \\ & -x_1^3 + 0.5(-x_2 - x_2^3 + |x_2^3 - x_2|) \leq 0, \quad x_1, x_2 \in \mathbb{R}. \end{array}$$

Example 4.10. Find $f(x_1, \ldots, x_5) = \sin((x_1^2 + x_2^3 - x_3x_4^2) + \cos(x_5^3 - x_1 + x_3^2x_4^3)) + x_1x_2 + x_3^2x_4 + (1-x_2)^2 + (1-x_3)^2 + (1-x_4)^2$ restricted to $x_1^2 + 2x_2^2 - x_5 - 2x_3^2x_4 = 0$, $(1-x_3)^2 - x_2^2x_4^2 + \cos(x_4x_5^3) \leq 2$. Find $x \in A$ for which $f(x) = 19$.

TABLE 1. Particle Swarm Optimization

Example	PSO				
	initial point	value	x	$\max\limits_{x\in\mathcal{C}} \lvert g_i(x)\rvert$	$\max\limits_{x\in\mathcal{C}} h_j(x)$
4.1	-	-4	$(0, 1)$	0	0
4.2	-	-2	$(4, 3)$	–	0
4.3	-	0.6325	$(10, 0.0316, 10)$	-	-0.0316
4.4	-	2.4397	$(0.1141, 9.9975, 0.7715)$	-	-0.1141
4.5	-	12.6	$(2.4, 0.3, 0)$	0	0
4.6	-	1	$(-1.977\times10^{-11}, 1.02\times10^{-12}, -1.067\times 10^{-12}, -2.719\times10^{-11}, 6.04\times10^{-12})$	2.546×10^{-11}	–
4.7	-	1	$(0.018, 0.291, 0.019, -0.921, -0.007)$	1.766×10^{-12}	–
4.8	-	16	$(2, 0)$	0	0
4.9	-	1	$(0.7312, 1.0271)$	–	-0.4654
4.10	-	0	$(-4.03, -1.27, 1.93, 1.77, 6.28)$	0	9.82

Table 2. Pattern-Search Optimization

Example	Pattern-Search				
	initial point	value	x	$\max_{x\in\mathcal{C}} \lvert g_i(x)\rvert$	$\max_{x\in\mathcal{C}} h_j(x)$
4.1	$(1,1)$	-4	$(0,1)$	0	0
4.2	$(1,1)$	-94.3669	$(4.6094, 1.9374)$	–	0.4532
4.3	$(1,1,1)$	0.6325	$(0.6325, 0.5, 10)$	–	-0.5
4.4	$(1,1,1)$	2.4397	$(1.1385, 1, 0.7715, 1)$	–	-0.2762
4.5	$(1,1,1)$	12.6429	$(2.3571, 0, 0.2143)$	1.5259×10^{-5}	0
4.6	$(0,0,0,0,0)$	1	$(0,0,0,0,0)$	0	–
4.7	$(0,0,0,0,0)$	1	$(0.019, 0.029, 1.93, -0.92, -0.007)$	3.978×10^{-6}	–
4.8	(1,1)	18.7777	$(0.6667, 1)$	1.5259^{-5}	-0.6667
4.9	(1,1)	1	$(0,1)$	–	-1
4.10	$(0,0,0,0,0)$	0	$(-2, -2.01, -0.88, 2.13, 8.88)$	0	20.16

Table 3. Convex Algorithm

Example	Convex Algorithm				
	initial point	value	x	$\max_{x\in\mathcal{C}} \lvert g_i(x)\rvert$	$\max_{x\in\mathcal{C}} h_j(x)$
4.1	$(1,1)$	-3.9694	$(0.0076, 0.9924)$	7.3×10^{-9}	-0.0076
4.2	$(1,1)$	-1.2957	$(3.9302, 3.0698)$	–	-7.97×10^{-13}
4.3	$(1,1,1)$	0.6325	$(0.5623, 0.5623, 10)$	–	-0.5623
4.4	$(1,1,1)$	2.4397	$(1.0670, 1.0670, 0.7715)$	–	-0.3038
4.5	$(1,1,1)$	12.6392	$(2.3608, 0.0253, 0.1962)$	0.3167×10^{-7}	-0.0253
4.6	$(0,0,0,0,0)$	1	$(-1.520\times 10^{-10}, 1.283\times 10^{-11}, 1.265\times 10^{-10}, 2.282\times 10^{-10}, -3.341\times 10^{-11})$	2.661×10^{-10}	–
4.7	$(0,0,0,0,0)$	1	$(0.018, 0.291, 0.019, -0.921, -0.007)$	8.413×10^{-9}	–
4.8	(1,1)	16	$(2,0)$	0	0
4.9	(1,1)	1	$(-0.0888, 0.8020)$	–	-0.8013
4.10	$(0,0,0,0,0)$	0	$(-0.73, -0.53, -0.25, -0.95, 1.21)$	0	4.47

5. Discussion

In this paper, we transform a constrained optimization to an unconstrained one. Under our approach, the given objective function f (subjected to some constraints) is replaced by a deformed function f_t (without constraints) for some t. We chose to use some software packages to approximate a minimizer of f_t. We observe that all outcomes approximately satisfy corresponding constraints. Of course, we may obtain different minimizers from different software. It is challenging to construct a new algorithm for finding a global minimizer even for some special cases.

6. Appendix

In this appendix, we give the MATLAB GUI for finding a minimizer by using POA method. The MATLAB GUI of POA method is given in Figures 1, 2 and 3.

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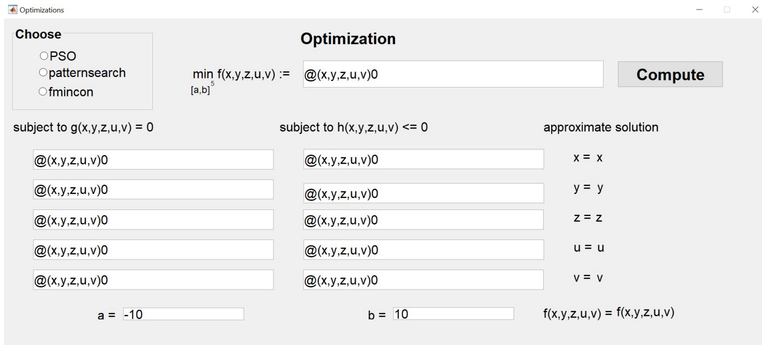

FIGURE 1. MATLAB GUI for PAO method.

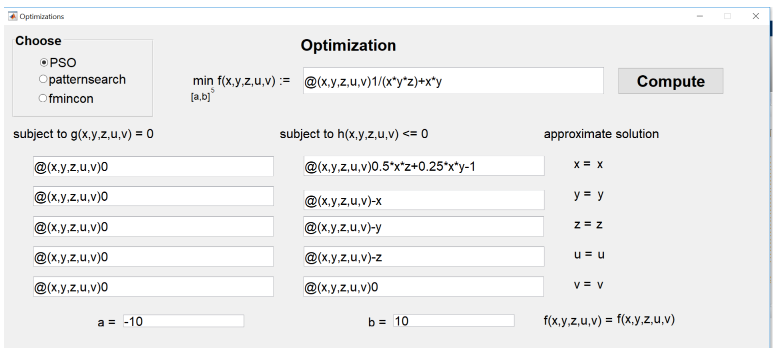

FIGURE 2. In put data.

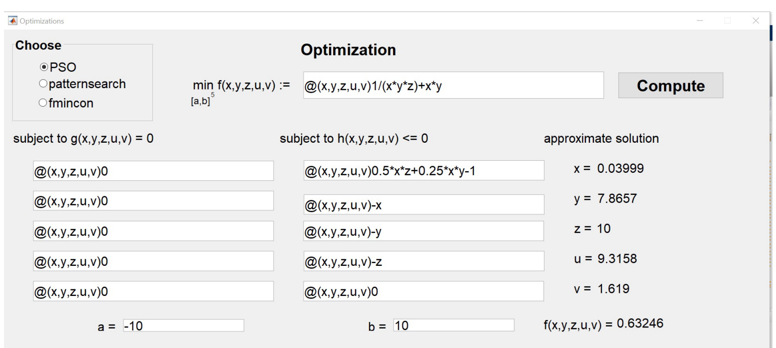

FIGURE 3. Result.

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RADIUS OF THE PERTURBATION OF THE OBJECTIVE FUNCTION PRESERVES THE KKT CONDITION IN CONVEX OPTIMIZATION

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ABSTRACT. The problem to find the maximum radius of the perturbation of the objective function which preserves the KKT condition at a feasible point is studied. The maximum radius of the problem is described, and certain values concerned with the extreme direction of a positive polar cone of the union of the subdifferentials of the active constraint functions at the point are observed.

KEYWORDS: convex optimization problem, KKT optimality condition, the basic constraint qualification, extreme direction
AMS Subject Classification: Primary 90C25; Secondary 90C46

1. Introduction

We study stability of KKT condition for the following convex optimization problem:

$$\text{(P)} \quad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, i \in I \end{array} \tag{1.1}$$

where I is a non-empty index set, $f, g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ are proper convex functions, $i \in I$, and assume that the constraint set $S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ \forall i \in I\}$ is not empty. Under some regularity condition for the family of constraint functions $\{g_i, i \in I\}$, which is called constraint qualification, if $\bar{x}$ is a minimizer of (P), then the KKT condition holds, that is, there exists a finite $J \subset I(\bar{x})$ and

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$\lambda_j \geq 0$, $j \in J$, called KKT multipliers, such that

$$0 \in \partial f(\bar{x}) + \sum_{j \in J} \lambda_j \partial g_j(\bar{x}), \tag{1.2}$$

where $I(\bar{x}) = \{i \in I \mid g_i(\bar{x}) = 0\}$. The most famous constraint qualification is the Slater condition (1.3). Assume that I is a non-empty finite index set, and all g_i are real-valued convex functions. If the Slater condition

$$\exists x_0 \in \mathbb{R}^n \ s.t. \ g_i(x_0) < 0, \ \forall i \in I, \tag{1.3}$$

is satisfied, then the KKT condition holds for any real-valued convex function f. There are many results about constraint qualification for the KKT condition, and the basic constraint qualification (BCQ in short) is called a necessary and sufficient constraint qualification from the following result:

Theorem 1.1 ([4]; cf. [3]). *Let I be a non-empty index set, $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper convex functions, $i \in I$, and assume that $\bar{x} \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ \forall i \in I\}$. Then the following two statements are equivalent:*

(i) *The family $\{g_i : i \in I\}$ satisfies the BCQ at $\bar{x}$, that is,*

$$N_S(\bar{x}) = \operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x})$$

holds,

(ii) *For each convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\bar{x}$ is a minimizer of (1.1) with $\operatorname{cl} S$, the closure of S, in place of S if and only if there exists a finite subset $J \subset I(\bar{x})$ and $\lambda_j \geq 0$, $j \in J$, such that (1.2) holds.*

Constraint qualifications guarantees the KKT condition holds when $\bar{x}$ is a minimizer of (1.1) for every convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. However, the conditions do not always hold. If $\{g_i, i \in I\}$ does not satisfy the BCQ at $\bar{x} \in S$, then there exists a convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\bar{x}$ is a minimizer of (1.1) with $\operatorname{cl} S$ in place of S, but there does not exist a finite subset $J \subset I(\bar{x})$ and $\lambda_j \geq 0$, $j \in J$, such that (1.2) holds. We are interested in when the KKT condition holds at $\bar{x}$ under the luck of such constraint qualifications, for example, see [6].

The purpose of this paper is to consider the following problem under the KKT condition holds but the BCQ does not hold at $\bar{x}$ for given f, g_i in (1.1):

$$\begin{array}{ll} \text{Maximize} & r \geq 0 \\ \text{subject to} & \text{the KKT condition holds at } \bar{x} \text{ in (1.1) with} \\ & \text{the objective function } f + \langle c, \cdot \rangle \text{ whenever } \|c\| \leq r. \end{array} \tag{1.4}$$

The problem is the same to find the maximum radius of the perturbation of the objective function which preserves the KKT condition at $\bar{x}$. The problem is motivated from [5]; under some conditions,

$$\partial f(\bar{x}) \subset \operatorname{int}(\operatorname{cone} \operatorname{co} \{u_1, \cdots, u_n\})$$

holds for any $u_i \in -\partial g_{t_i}(\bar{x})$, $i = 1, \ldots, n$. The paper is organized as follows: In section 2, we describe preliminary results about the notions of extreme point and extreme direction, and give a characterization of extreme direction which is a similar result to extreme point. In section 3, we describe the maximum radius of the problem (1.4), and observe the maximum value and other related values include the notion of extreme direction. Finally we give a conclusion in section 4.

2. Preliminaries

For a convex set $C \subset \mathbb{R}^n$, $x \in C$ is called an extreme point if there does not exist $x_1, x_2 \in C$, $\lambda \in (0,1)$ such that $x_1 \neq x_2$ and $x = (1-\lambda)x_1 + \lambda x_2$, or equivalently, $C \setminus \{x\}$ is convex. Denote $\operatorname{ext} C$ the set of all extreme points of C. For a convex cone $C \subset \mathbb{R}^n$, $x \in C$ is called an extreme direction if $x \neq 0$ and for all $x_1, x_2 \in C$ such that $x = x_1 + x_2$, we have $x_1, x_2 \in \mathbb{R}_+ x$, where $\mathbb{R}_+ x = \{tx \mid t \geq 0\}$. Denote $\operatorname{extd} C$ the set of all extreme directions of C. We obtain a similar result to extreme point for extreme direction as follows:

Proposition 2.1. *Assume that convex cone C is pointed, that is, $C \cap (-C) = \{0\}$. For any $x \in C \setminus \{0\}$, $x \in \operatorname{extd} C$ if and only if $C \setminus \mathbb{R}_+ x$ is convex.*

Proof. Assume that $x \in \operatorname{extd} C$. For any $x_1, x_2 \in C \setminus \mathbb{R}_+ x$ and $\alpha \in (0,1)$, it is clear that $(1-\alpha)x_1 + \alpha x_2 \in C$. If $(1-\alpha)x_1 + \alpha x_2 \in \mathbb{R}_+ x$, then $(1-\alpha)x_1 + \alpha x_2 = tx$ for some $t \geq 0$. If $t > 0$, since

$$\frac{1-\alpha}{t}x_1 + \frac{\alpha}{t}x_2 = x, \quad \frac{1-\alpha}{t}x_1 \in C, \quad \frac{\alpha}{t}x_2 \in C$$

and $x \in \operatorname{extd} C$, then $\frac{1-\alpha}{t}x_1, \frac{\alpha}{t}x_2 \in \mathbb{R}_+ x$ and $x_1, x_2 \in \mathbb{R}_+ x$. This is a contradiction. If $t = 0$, since $(1-\alpha)x_1 + \alpha x_2 = 0$ and C is pointed, then $x_1 = x_2 = 0 \in \mathbb{R}_+ x$, which contradicts to $x_1, x_2 \in C \setminus \mathbb{R}_+ x$.

Conversely, assume that $C \setminus \mathbb{R}_+ x$ is convex. Let $x_1, x_2 \in C$ such that $x = x_1 + x_2$. If $x_1, x_2 \notin \mathbb{R}_+ x$, since $x_1, x_2 \in C \setminus \mathbb{R}_+ x$,

$$x = 2\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \in C \setminus \mathbb{R}_+ x.$$

This contradicts to $x \in \mathbb{R}_+ x$. If one of two is in $\mathbb{R}_+ x$, for example $x_1 \in \mathbb{R}_+ x$, then $x_1 = tx$ for some $t \geq 0$, that is $x_2 = x - x_1 = (1-t)x$. If $1 - t < 0$, since $x_2 \in C \cap (-C)$ and C is pointed, we have $x_2 = 0 \in \mathbb{R}_+ x$ and if $1 - t \geq 0$, then $x_2 \in \mathbb{R}_+ x$. ☐

Remark 2.2. The assumption pointed in this result is essential. Take a non-zero vector x_0 and define $C = \mathbb{R}x_0$. Clearly $x_0 \in C$, C is a convex cone, and $C \setminus \mathbb{R}_+ x_0$ is convex, however x_0 is not any extreme direction of C because $x_0 = 3x_0 + (-2)x_0$, $3x_0 \in C$, $-2x_0 \in C$, but $-2x_0 \notin \mathbb{R}_+ x_0$.

Denote the positive polar cone of $A \subset \mathbb{R}^n$ as $A^+ = \{b \in \mathbb{R}^n \mid \langle b, a \rangle \geq 0, \forall a \in A\}$. Then the following result holds, see [1]:

Proposition 2.3. *Let $D \subset \mathbb{R}^n$ be a closed pointed convex cone. Then $D = (\operatorname{extd} D^+)^+$. Let $D \subset \mathbb{R}^n$ be a closed pointed convex cone with nonempty interior. Then $D = (\operatorname{extd} D^+)^+$.*

3. Main results

Let I be a non-empty index set, $f, g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, be proper convex functions. Assume that the KKT condition (1.2) holds at $\bar{x} \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ \forall i \in I\}$. Define

$$K = -\operatorname{cone} \operatorname{co} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),$$

where $I(\bar{x}) = \{i \in I \mid g_i(\bar{x}) = 0\}$, and $\operatorname{cone} A$ and $\operatorname{co} A$ are the conical and convex hull of $A \subset \mathbb{R}^n$, respectively. Then it is easy to verify that the KKT condition (1.2) holds at $\bar{x}$ if and only if

$$\partial f(\bar{x}) \cap K \neq \emptyset \tag{3.1}$$

holds. At first, we observe problem (1.4) in the following situation:

$$\partial f(\bar{x}) \cap K \neq \emptyset \quad \text{and} \quad \partial f(\bar{x}) \cap \operatorname{int} K = \emptyset. \tag{3.2}$$

In this case, we have the optimal value of problem (1.4) is 0 from the following result:

Theorem 3.1. *Assume that (3.2) holds and* $\operatorname{int} K \neq \emptyset$. *For every* $r > 0$ *there exists* $c \in \mathbb{R}^n$ *such that* $\|c\| \leq r$ *and* $(\partial f(\bar{x}) + c) \cap K = \emptyset$. *The optimal value of problem (1.4) is 0.*

Proof. We give a proof of the first part of this theorem by using the separation theorem. From $\partial f(\bar{x}) \cap \operatorname{int} K = \emptyset$, we can choose non-zero $a \in \mathbb{R}^n$ such that for all $y \in \partial f(\bar{x})$ and $k \in \operatorname{int} K$,

$$\langle a, y \rangle \leq 0 < \langle a, k \rangle .$$

This shows that $0 \leq \langle a, k \rangle$ for each $k \in K$. We may assume that $\|a\| = 1$. Therefore, for every $r > 0$, $c = -ra$ satisfies $\|c\| = r$ and $(\partial f(\bar{x}) + c) \cap K = \emptyset$ holds because

$$\langle a, y + c \rangle = \langle a, y \rangle - r \leq -r < 0$$

for all $y \in \partial f(\bar{x})$. The second part of this theorem is easy to show from the first part of this theorem, the assumption $\partial f(\bar{x}) \cap K \neq \emptyset$, and the fact $\partial(f + \langle c, \cdot \rangle)(\bar{x}) = \partial f(\bar{x}) + c$. □

Next, we observe problem (1.4) in the following situation:

$$\partial f(\bar{x}) \cap \operatorname{int} K \neq \emptyset. \tag{3.3}$$

Theorem 3.2. *Assume that (3.3) holds,* K *is closed and pointed, and* $\partial f(\bar{x})$ *is compact. Then the optimal value of problem (1.4) is equal to*

$$r_0 := \inf_{d \in K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle . \tag{3.4}$$

Moreover if f *is differentiable at* $\bar{x}$, *then*

$$r_0 = \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \langle \nabla f(\bar{x}), d \rangle . \tag{3.5}$$

Proof. It is clear that $r_0 > 0$ from the assumption. In general, the optimal value of problem (1.4) is as follows:

$$v = \sup\{r > 0 \mid (\partial f(\bar{x}) + c) \cap K \neq \emptyset, \forall c \in \mathbb{R}^n(\|c\| \leq r)\}. \tag{3.6}$$

Also $(K^+)^+ = K$ holds because K is a closed pointed convex cone from Proposition 2.3. At first, we show $r_0 \geq v$. For any $r \in \mathbb{R}(0 < r < v)$, there exist r' such that $r < r'$ and for all $c \in \mathbb{R}^n(\|c\| \leq r')$, $(\partial f(\bar{x}) + c) \cap K \neq \emptyset$. Take $y' \in \partial f(\bar{x})$ satisfying $y' + c \in K$. For all $d \in K^+(\|d\| = 1)$, $\langle y' + c, d \rangle \geq 0$, that is

$$\sup_{y \in \partial f(\bar{x})} \langle y, d \rangle \geq \langle y', d \rangle \geq \langle -c, d \rangle .$$

Since c is arbitrary, we have $\sup_{y \in \partial f(\bar{x})} \langle y, d \rangle \geq r' > r$. Also d and r are arbitrary, then

$$r_0 = \inf_{d \in K^+(\|d\|=1)} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle \geq v.$$

On the other hand, we show $(\partial f(\bar{x}) + c) \cap K \neq \emptyset$ for all $c \in \mathbb{R}^n(\|c\| \leq r_0)$. Otherwise, there exists $c_0 \in \mathbb{R}^n(\|c_0\| \leq r_0)$ such that

$$(\partial f(\bar{x}) + c_0) \cap K = \emptyset.$$

Since $\partial f(\bar{x}) + c_0$ is compact convex, by using the strong separation theorem, there exists $d_0 \in K^+ \setminus \{0\}$ such that

$$\sup_{y \in \partial f(\bar{x})} \langle y + c_0, d_0 \rangle < 0.$$

We may assume that $\|d_0\| = 1$, then we have

$$r_0 \leq \sup_{y \in \partial f(\bar{x})} \langle y, d_0 \rangle < \langle -c_0, d_0 \rangle \leq \| - c_0\| \leq r_0,$$

which is a contradiction. Therefore, $(\partial f(\bar{x}) + c) \cap K \neq \emptyset$ for all $c \in \mathbb{R}^n(\|c\| \leq r_0)$ and then we have $r_0 \leq v$.

In general we have

$$r_0 = \inf_{d \in K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle \leq \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle =: r_1. \tag{3.7}$$

Assume that f is differentiable at $\bar{x}$. We show $(\nabla f(\bar{x}) + c) \cap K \neq \emptyset$ for all $c \in \mathbb{R}^n(\|c\| \leq r_1)$. Otherwise, there exists $c_0 \in \mathbb{R}^n(\|c_0\| \leq r_0)$ such that

$$(\nabla f(\bar{x}) + c_0) \cap K = \emptyset.$$

Since $K = (\operatorname{extd} K^+)^+$, there exists $d_0 \in \operatorname{extd} K^+ \setminus \{0\}$ such that

$$\langle \nabla f(\bar{x}) + c_0, d_0 \rangle < 0.$$

In the same way to the above, we have a contradiction. This shows that $r_1 \leq v$ and finally we have $r_1 = v$. □

Remark 3.1. (i) When K is finitely generated, it is well-known that the number of extreme direction is finite except for the difference in length. Therefore, the number of elements of $\{d \in \operatorname{extd} K^+ \mid \|d\| = 1\}$ is finite, but even in this situation, the number of elements of $\{d \in K^+ \mid \|d\| = 1\}$ is infinite. This means that to determine r_1 is easier than to determine r_0.

(ii) The value r_1 is an upper bound of the problem (1.4), but it is not feasible of the problem in general, see Example 3.2. When f is differentiable, r_1 becomes the optimum value of the problem (1.4).

(iii) For any fixed $y \in \partial f(\bar{x}) \cap \operatorname{int} K$, put $r_y = \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \langle y, d \rangle$. In a similar way to the proof, we have $0 < r_y \leq r_0$. Put

$$r_2 := \sup_{y \in \partial f(\bar{x}) \cap \operatorname{int} K} r_y = \sup_{y \in \partial f(\bar{x}) \cap \operatorname{int} K} \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \langle y, d \rangle$$

then r_2 is a lower bound of the problem (1.4). This means that every perturbation of the objective function radius r_2 preserves the KKT condition at $\bar{x}$. But r_2 is not equal to r_0 in general, see Example 3.2.

(iv) The compactness assumption of $\partial f(\bar{x})$ is redundant when f is continuous at $\bar{x}$. In this case, since $\sup_{y \in \partial f(\bar{x})} \langle y, d \rangle = f'(\bar{x}, d)$, which is the directional derivative of f at $\bar{x}$ in direction d defined as $\lim_{t \downarrow 0}(f(\bar{x} + td) - f(\bar{x}))/t$, then the optimal value of (1.4) is equal to:

$$r_0 = \inf_{d \in K^+, \|d\|=1} f'(\bar{x}, d).$$

(v) The closedness assumption of K is redundant when $I(\bar{x})$ is finite and all g_i, $i \in I(\bar{x})$ are differentiable.

Example 3.2. Assume that $I(\bar{x}) = \{1, 2\}$, and $\nabla g_1(\bar{x}) = (-1, 0)$, $\nabla g_2(\bar{x}) = (0, -1)$, Then we can see that

$$K = \operatorname{cone} \operatorname{co} \{(1, 0), (0, 1)\} = K^+,$$

and

$$\operatorname{extd} K^+ = \operatorname{cone}\{(1,0),(0,1)\} \setminus \{(0,0)\}.$$

If $\partial f(\bar{x}) = \operatorname{co}\{(1,0),(0,1)\}$, then

$$r_0 = \inf_{d \in K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle = \frac{1}{\sqrt{2}},$$

$$r_1 = \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \sup_{y \in \partial f(\bar{x})} \langle y, d \rangle = 1,$$

$$r_2 = \sup_{y \in \partial f(\bar{x}) \cap \operatorname{int} K} \inf_{d \in \operatorname{extd} K^+, \|d\|=1} \langle y, d \rangle = \frac{1}{2}.$$

Corollary 3.3. *Assume that I is finite, f and g_i, $i \in I$, are differentiable at $\bar{x}$, and $\nabla f(\bar{x}) \in -\operatorname{int} \operatorname{cone} \operatorname{co}\{\nabla g_i(\bar{x}) \mid i \in I(\bar{x})\}$. Then $\{d \in \operatorname{extd} K^+ \mid \|d\| = 1\}$ is finite and for every $c \in \mathbb{R}^n$ satisfying $\|c\| \leq r_1$, KKT condition (1.2) at $\bar{x}$ in (1.1) with the objective function $f + \langle c, \cdot \rangle$ holds, where*

$$r_1 = \min\{\langle \nabla f(\bar{x}), d \rangle \mid d \in \operatorname{extd} K^+, \|d\| = 1\}.$$

Proof. The convex cone $\operatorname{cone} \operatorname{co}\{\nabla g_i(\bar{x}) \mid i \in I(\bar{x})\}$ is closed because it is finitely generated. □

We give an example of the corollary as follows:

Example 3.4. Assume that $I(\bar{x}) = \{1,2,3,4\}$, and $\nabla g_1(\bar{x}) = (-1,-1,0)$, $\nabla g_2(\bar{x}) = (0,-1,0)$, $\nabla g_3(\bar{x}) = (0,0,-1)$, $\nabla g_4(\bar{x}) = (-1,0,-1)$. Then we can see that

$$K = \operatorname{cone} \operatorname{co}\{(1,1,0),(0,1,0),(0,0,1),(1,0,1)\},$$

$$K^+ = \operatorname{cone} \operatorname{co}\{(1,0,0),(0,1,0),(0,0,1),(-1,1,1)\},$$

and

$$\operatorname{extd} K^+ = \operatorname{cone}\{(1,0,0),(0,1,0),(0,0,1),(-1,1,1)\} \setminus \{(0,0,0)\}.$$

When f is differentiable at $\bar{x}$ and $\nabla f(\bar{x}) = (y_1, y_2, y_3) \in \operatorname{int} K$, then the optimal value of (1.4) is equal to

$$\begin{aligned} r_1 &= \min\{\langle \nabla f(\bar{x}), d \rangle \mid d \in \operatorname{extd} K^+, \|d\| = 1\} \\ &= \min\left\{y_1, y_2, y_3, (-y_1 + y_2 + y_3)/\sqrt{3}\right\}. \end{aligned}$$

4. Conclusion

We have studied the problem to find the maximum radius of the perturbation of the objective function which preserves the KKT condition at a feasible point. The situation changed when the subdifferential of the objective function at the feasible point meets the interior of a convex cone which was generated by the subdifferentials of the active constraint functions at the point, or not. We have described the maximum radius of the problem in each cases, in Theorem 3.1 and Theorem 3.2. Also we observe the maximum value and other related values include the notion of extreme direction, of which a characterization was given in Section 2. Finally we have applied Theorem 3.2 to a differentiable convex minimization problem in which the maximum radius was simply expressed and an example was given.

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COMMON FIXED POINT THEOREMS FOR SOME CONTRACTIVE CONDITION WITH φ-MAPPING IN COMPLEX VALUED B-METRIC SPACES

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ABSTRACT. In this paper, by using the concept of φ-mappings introduced by Mohanta and Maitra [8], we can prove the existence and the uniqueness of common fixed points for some generalized contractive mappings in complex-valued b-metric spaces. Our results extend and improve the results of Tripathi and Dubey [12] and many others.

KEYWORDS:common fixed point, complex-valued metric space, complex valued b-metric space.
AMS Subject Classification: :46C05, 47D03, 47H09, 47H10, 47H20.

1. Introduction

The concept of a metric space was introduced by Frechet in 1906 [7]. Many mathematicians studied the existence and the uniqueness of fixed points by using the Banach contraction principle. The principle was also proved in some generalized metric spaces, see [5].

Fixed point theorems in metric spaces have been studied extensively by many researchers as in [13, 6] and [11]. In 1989, Bakhtin [3] introduced the notion of b-metric spaces. After that, many researchers extended fixed point theorems from metric spaces to b-metric spaces, for example in [1, 2]

In 2011, A. Azam, B. Fisher and M. Khan [2] introduced the notion of complex valued metric spaces and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition. A complex valued metric space is a generalization of the classical metric space. Bhatt et al. [4] have proved a common fixed point theorem for weakly compatible mappings in a complex valued metric space. In 2013, Mohanta and Maitra [8], introduced the concept of common fixed points with φ-mapping in complex valued metric spaces, In 2017, Zada et. al. [14], proved common fixed point theorems in complex valued metric spaces with $(E.A)$ and (CLR) properties.

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The aim of this paper is to introduce some contractive conditions of two mappings by using the concept of φ-mappings and prove the existence and the uniqueness of common fixed points in complex valued b-metric spaces. Therefore, our results are comprehensive the results of [8] and [12].

2. Preliminaries

In this section, we present some definitions and lemmas for using in section 3, and define the definition of b-metric space in the complex plane.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a metric space, and d is called a metric on X.

Next, we suppose the definition of b-metric space, this space is generalized than metric spaces.

Definition 2.2. [3] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

The following is some example for b-metric spaces.

Example 2.3. [3] Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

There is a completeness property in real number but on order relation is not welll-defined in complex numbers. Before giving the definition of complex valued metric spaces and complex-valued b-metric spaces, we define partial order in complex numbers (see [9]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order relation $\preccurlyeq$ on $\mathbb{C}$ as follows;

$$z_1 \preccurlyeq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

This means that we would have $z_1 \preccurlyeq z_2$ if and only if one of the following conditions holds:
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
If one of the conditions $(ii), (iii)$, and (iv) holds, then we write $z_1 \prec z_2$. From the above partial order relation we have the following remark.

Remark 2.4. We can easily check the following:
(i) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \preccurlyeq z_2$ then $az_1 \preccurlyeq bz_2, \forall z_1, z_2 \in \mathbb{C}$.
(ii) If $0 \preccurlyeq z_1 \prec z_2$ then $|z_1| < |z_2|$.
(iii) If $z_1 \preccurlyeq z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$.
(iv) If $z \in \mathbb{C}$, for $a, b \in \mathbb{R}$ and $a \leq b$, then $az \preccurlyeq bz$.

A b-metric on a b-metric sapce is a funcion having real value. Based on the definition of partial order on complex number, real-valued b-metric can be generalized into compleex-valued b-metric as folllows.

Definition 2.5. [2] Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \preccurlyeq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \preccurlyeq d(x, y) + d(y, z)$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Next, we give the definition of complex valued b-metric space.

Definition 2.6. [11] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \preccurlyeq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \preccurlyeq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is complex valued metric space which is defined in Definition 2.5. The following example is some example of complex valued b-metric space.

Example 2.7. [11] Let $X = \mathbb{C}$. Define the mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbb{C}, d)$ is complex valued b-metriic space with $s = 2$.

Definition 2.8. [10] Let (X, d) be a complex valued b-metric space.
(i) A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in Y : d(x, y) \prec r\} \subseteq A.$$

(ii) A point $x \in X$ is called limit point of a set A if for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - x) \neq \emptyset$
(iii) A subset $A \subseteq X$ is open if each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is closed if each limit point of A is contained in A.

Definition 2.9. [10] Let (X, d) be complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.
(i) The sequence $\{x_n\}$ is converges to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x) \prec r$. Thus x is the limit of (x_n) and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence if for ever $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.
(iii) If for every Cauchy sequence in$X$ is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Definition 2.10. [8] Let $P = \{z \in \mathbb{C} : re(z) \geqslant 0 \text{ and } Im(z) \geqslant 0\}$. A nondecreasing mapping $\varphi : P \to P$ is called a φ-mapping if

(i) $\varphi(0) = 0$ and $0 \prec \varphi(z) \prec z$ for $z \in P - \{0\}$;
(ii) $\varphi(z) \prec z$ for every $z \succ 0$;
(iii) $\lim_{n\to\infty} \varphi^n(z) = 0$ for every $z \in P - \{0\}$.

Lemma 2.11. [10] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.12. [10] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

In this section, we define some contraction by using a φ-mapping, and prove the existence and uniqueness of common fixed point theorem in a complete complex valued b-metric space.

Theorem 3.1. *Let (X, d) be a complete complex valued b-metric space and the mappings $S, T : X \to X$ are self mappings satisfying the condition*

$$d(Sx, Ty) \preccurlyeq \varphi[\lambda \frac{d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)}{d(x, Ty) + d(y, Sx)} + \mu d(x, y)] \tag{3.1}$$

for all $x, y \in X$, where $d(x, Ty) + d(y, Sx) \neq 0$ and λ, μ are nonnegative reals with the condition $\lambda + \mu < 1$. If φ is continuous then S and T has a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. We define

$$\begin{aligned} x_{2n+1} &= Sx_{2n} \text{ and} \\ x_{2n+2} &= Tx_{2n+1}, n = 0, 1, 2, 3, ... \end{aligned}$$

By equations (3.1) and (3.2), we consider

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\preccurlyeq \varphi[\lambda \frac{d(x_{2n}, Sx_{2n})d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n+1})d(x_{2n+1}, Sx_{2n})}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})} \\ &\quad + \mu d(x_{2n}, x_{2n+1})] \\ &\preccurlyeq \varphi[\lambda \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})} \\ &\quad + \mu d(x_{2n}, x_{2n+1})]. \end{aligned} \tag{3.2}$$

From $\lambda + \mu < 1$ and (3.2), we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= \varphi[\lambda \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2})}{d(x_{2n}, x_{2n+2})} + \mu d(x_{2n}, x_{2n+1})] \\ &\preccurlyeq \varphi[\lambda d(x_{2n}, x_{2n+1}) + \mu d(x_{2n}, x_{2n+1})] \\ &= \varphi[(\lambda + \mu)d(x_{2n}, x_{2n+1})] \\ &\preccurlyeq \varphi[d(x_{2n}, x_{2n+1})]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= \varphi[\lambda \frac{d(x_{2n-1}, x_{2n})d(x_{2n-1}, x_{2n+1})}{d(x_{2n-1}, x_{2n+1})} + \mu d(x_{2n-1}, x_{2n})] \\ &\preccurlyeq \varphi[\lambda d(x_{2n-1}, x_{2n}) + \mu d(x_{2n-1}, x_{2n})] \\ &= \varphi[(\lambda + \mu)d(x_{2n-1}, x_{2n})] \\ &\preccurlyeq \varphi[d(x_{2n-1}, x_{2n})]. \end{aligned}$$

By mathematical induction, implies that

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &\preccurlyeq \varphi(\varphi(\varphi(\cdots \varphi(d(x_0, x_1))))) \\ &= \varphi^{2n+1} d(x_0, x_1). \end{aligned} \tag{3.3}$$

From (3.3) and Definition 2.10, we conclude that

$$d(x_{n+1}, x_{n+2}) \preccurlyeq \varphi^{n+1} d(x_0, x_1) \tag{3.4}$$

So, for $m > n$ and Definition 2.6, we consider

$$\begin{aligned} d(x_n, x_{n+m}) &\preccurlyeq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})] \\ &\preccurlyeq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+m}) \\ &\preccurlyeq sd(x_n, x_{n+1}) + s[s(d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+m}))] \\ &\preccurlyeq sd(x_n, x_{n+1}) + s^2(d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+m}) \\ &\preccurlyeq sd(x_n, x_{n+1}) + s^2(d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+m}) \end{aligned}$$

$$\begin{aligned}
& +s^{n+m-1}d(x_{n+m-2},x_{n+m-1})+...+s^{m}d(x_{n+m-1},x_{n+m}) \\
\precsim\ & sd(x_n,x_{n+1})+s^2(d(x_{n+1},x_{n+2})+s^3d(x_{n+2},x_{n+3})+s^3d(x_{n+3},x_{n+m}) \\
& +...+s^{n+m-1}d(x_{n+m-2},x_{n+m-1})+s^{m}d(x_{n+m-1},x_{n+m}) \\
\precsim\ & s\varphi^n d(x_0,x_1)+s^2\varphi^{n+1}(d(x_0,x_1)+s^3d(x_0,x_1)+s^3\varphi^{n+2}d(x_0,x_1) \\
& +...+s^{n+m-1}\varphi^{n+m-2}d(x_0,x_1)s^m\varphi^{n+m-1}d(x_0,x_1) \\
=\ & [s\varphi^n+s^2\varphi^{n+1}+s^3+s^3\varphi^{n+2}+s^{n+m-1}\varphi^{n+m-2}+s^m\varphi^{n+m-1}]d(x_0,x_1).
\end{aligned} \tag{3.5}$$

From remark 2.4 (ii), we have

$$|d(x_n,x_{n+m})|\leqslant[s\varphi^n+s^2\varphi^{n+1}+s^3+s^3\varphi^{n+2}+\cdots+s^{n+m-1}\varphi^{n+m-2}+s^m\varphi^{n+m-1}]|d(x_0,x_1)|. \tag{3.6}$$

From (3.4), (3.6) and Taking $n\to\infty$, it follows that $|d(x_n,x_{n+m})|\to\infty$.

By Lemma 2.12, implies that $\{x_n\}$ is a cauchy sequence in X. Since X is complete, there exists $u\in X$ such that $x_n\to u$. Now, we show that u is a fixed point of T and S. Consider,

$$\begin{aligned}
d(Su,x_{2n+2}) &= d(Su,Tx_{2n+1}) \\
&\precsim \varphi(\lambda\frac{d(u,Su)d(u,Tx_{2n+1})+d(x_{2n+1},Tx_{2n+1})d(x_{2n+1},Su)}{d(u,Tx_{2n+1})+d(x_{2n+1},Su)}+\mu d(u,x_{2n+1})) \\
&\precsim \varphi(\lambda\frac{d(u,Su)d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},Su)}{d(u,x_{2n+2})+d(x_{2n+1},Su)}+\mu d(u,x_{2n+1})).
\end{aligned} \tag{3.7}$$

From φ is a continuous, (3.7), $x_n\to u$ as $n\to\infty$ and Definition 2.6 (1), we have

$$\begin{aligned}
d(Su,u) &= \lim_{n\to\infty}d(Su,x_{2n+1}) \\
&\precsim \lim_{n\to\infty}\varphi(\lambda\frac{d(u,Su)d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},Su)}{d(u,x_{2n+2})+d(x_{2n+1},Su)}+\mu d(u,x_{2n+1})) \\
&= \varphi(\lim_{n\to\infty}(\lambda\frac{d(u,Su)d(u,x_{2n+2})+d(x_{2n+1},x_{2n+2})d(x_{2n+1},Su)}{d(u,x_{2n+2})+d(x_{2n+1},Su)}+\mu d(u,x_{2n+1}))) \\
&= \varphi(\lambda\frac{d(u,Su)d(u,u)+d(u,u)d(u,Su)}{d(u,u)+d(u,Su)}+\mu d(u,u)) \\
&= \varphi(0)=0.
\end{aligned} \tag{3.8}$$

Thus $u=Su$. Hence u is a fixed point of S. Next, we show that u is a fixed point of T. Consider,

$$\begin{aligned}
d(x_{2n+1},Tu) &= d(Sx_{2n},Tu) \\
&\precsim \varphi(\lambda\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)+d(u,Tu)d(u,Sx_{2n})}{d(x_{2n},Tu)+d(u,Sx_{2n})}+\mu d(x_{2n},u)) \\
&= \varphi(\lambda\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tu)+d(u,Tu)d(u,x_{2n+1})}{d(x_{2n},Tu)+d(u,x_{2n+1})}+\mu d(x_{2n},u)).
\end{aligned} \tag{3.9}$$

From φ is a continuous, (3.9), $x_n\to u$ as $n\to\infty$ and Definition 2.6 (1), we have

$$\begin{aligned}
d(u,Tu) &= \lim_{n\to\infty}d(x_{2n+1},Tu) \\
&\precsim \lim_{n\to\infty}\varphi(\lambda\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tu)+d(u,Tu)d(u,x_{2n+1})}{d(x_{2n},Tu)+d(u,x_{2n+1})}+\mu d(x_{2n},u)) \\
&= \varphi(\lim_{n\to\infty}(\lambda\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tu)+d(u,Tu)d(u,x_{2n+1})}{d(x_{2n},Tu)+d(u,x_{2n+1})}+\mu d(x_{2n},u))) \\
&= \varphi(\lambda\frac{d(u,u)d(u,Tu)+d(u,Tu)d(u,u)}{d(u,Tu)+d(u,u)}+\mu d(u,u)) \\
&= \varphi(0)=0.
\end{aligned} \tag{3.10}$$

Thus $u = Tu$. Hence u is a fixed point of T. Therefore, u is a common fixed point of S and T. Finally, we prove the uniqueness of common fixed point of S and T. Suppose that v is a common fixed point of S and T. So $Sv = v = Tv$. Now, we show that $u = v$. Assume that $u \neq v$, we consider

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\precsim \varphi(\lambda\frac{d(u,Su)d(u,Tv)+d(v,Tv)d(v,Su)}{d(u,Tv)+d(v,Su)}+\mu d(u,v)) \\ &= \varphi(\lambda\frac{d(u,u)d(u,v)+d(v,v)d(v,u)}{d(u,v)+d(v,u)}+\mu d(u,v)) \\ &= \varphi(\mu d(u,v)). \end{aligned} \tag{3.11}$$

Since $\mu < 1$, we have $\mu d(u,v) < d(u,v)$. By Definition 2.10 (2) and φ is a nondecreasing, we have

$$d(u,v) \precsim \varphi(\mu d(u,v)) \precsim \varphi(d(u,v)) \prec d(u,v). \tag{3.12}$$

From remark 2.4 (ii), taking absolute value of both side, we have

$$|d(u,v)| < |d(u,v)|.$$

It is a contradiction. We can conclude that $u = v$. Therefore u is a uniqueness common fixed point of S and T. □

From Theorem 3.1, we have the parallel result with the result of Dubey et. al [12] as following.

Corollary 3.2. *Let (X,d) be a complete complex valued b-metric space and the mappings $S,T : X \to X$ satisfy the condition*

$$d(Sx,Ty) \precsim \lambda\frac{d(x,Sx)d(x,Ty)+d(y,Ty)d(y,Sx)}{d(x,Ty)+d(y,Sx)}+\mu d(x,y) \tag{3.13}$$

for all $x,y \in X$, where $d(x,Ty)+d(y,Sx) \neq 0$ and λ,μ are nonnegative reals with $\lambda+\mu < 1$. If either S or T is continuous and the pair (S,T) is compatible, then S and T has a unique common fixed point.

Proof. If $\varphi = I$ is an identity mapping, then (3.1) reduces to (3.13) and suppose one of S and T is continuous and the pair (S,T) is compatible, then S and T has a unique common fixed point. This completes the proof. □

Theorem 3.3. *Let (X,d) be a complete complex valued b-metric space and the mappings $S,T : X \to X$ are self mappings satisfying the condition*

$$d(S^n x,T^n y) \precsim \varphi[\lambda\frac{d(x,S^n x)d(x,T^n y)+d(y,T^n y)d(y,S^n x)}{d(x,T^n y)+d(y,S^n x)}+\mu d(x,y)] \tag{3.14}$$

for all $x,y \in X$, $n \geq 1$, where $d(x,Ty)+d(y,Sx) \neq 0$ and λ,μ are nonnegative reals with the condition $\lambda+\mu < 1$. If φ is continuous then S and T has a unique common fixed point.

Proof. Suppose $A = S^n$ and $B = T^n$, by Theorem 3.1, there exists a common fixed point u of A and B, such that

$$Au = u = Bu.$$

Thus $S^n u = u$ and $T^n u = u$. We claim that $Su = u$. Assume that $Su \neq u$, we have

$$\begin{aligned} d(Su,u) &= d(S(S^n u),T^n u) \\ &= d(S^n(Su),T^n u) \\ &\precsim \varphi\left[\lambda\frac{d(Su,S^n(Su))d(Su,T^n u)+d(u,T^n u)d(u,S^n(Su))}{d(Su,T^n u)+d(u,S^n(Su))}+\mu d(Su,u)\right] \\ &= \varphi\left[\lambda\frac{d(Su,S(S^n u))d(Su,T^n u)+d(u,T^n u)d(u,S(S^n u))}{d(Su,T^n u)+d(u,S(S^n u))}+\mu d(Su,u)\right] \\ &= \varphi\left[\lambda\frac{d(Su,Su)d(Su,u)+d(u,u)d(u,Su)}{d(Su,u)+d(u,Su)}+\mu d(Su,u)\right] \end{aligned}$$

$$= \varphi\left[\mu d(Su,u)\right].$$

From Definition 2.10, we have $d(Su,u) \prec \mu d(Su,u)$. A contradiction, because $\mu < 1$. Hence, $Su = u$. Next, we claim that $Tu = u$. Assume that $Tu \neq u$, we have

$$\begin{aligned}
d(u,Tu) &= d(S^n u, T(T^n u)) \\
&= d(S^n u, T^n(Tu)) \\
&\precsim \varphi\left[\lambda \frac{d(u,S^n u)d(u,T^n(Tu)) + d(Tu,T^n(Tu))d(Tu,S^n u)}{d(u,T^n(Tu)) + d(Tu,S^n u)} + \mu d(u,Tu)\right] \\
&= \varphi\left[\lambda \frac{d(u,S^n u)d(u,T(T^n u)) + d(Tu,T(T^n u))d(Tu,S^n u)}{d(u,T(T^n u)) + d(Tu,S^n u)} + \mu d(u,Tu)\right] \\
&= \varphi\left[\lambda \frac{d(u,u)d(u,Tu) + d(Tu,Tu)d(Tu,u)}{d(u,Tu) + d(Tu,u)} + \mu d(u,Tu)\right] \\
&= \varphi\left[\mu d(u,Tu)\right].
\end{aligned}$$

From Definition 2.10, we have $d(u,Tu) \prec \mu d(u,Tu)$. A contradiction, because $\mu < 1$. Hence, $Tu = u$. Hence u is a common fixed point of S and T.

Finally, we show that u is a unique fixed point of S and T. Let v be a common fixed point of S and T, thus $S^n v = v = T^n v$. We must show that $u = v$. Assume that $u \neq v$, we have

$$\begin{aligned}
d(u,v) &= d(S^n u, T^n v) \\
&\precsim \varphi\left[\lambda \frac{d(u,S^n u)d(u,T^n v) + d(v,T^n v)d(v,S^n u)}{d(u,T^n v) + d(v,S^n u)} + \mu d(u,v)\right] \\
&= \varphi\left[\lambda \frac{d(u,u)d(u,v) + d(v,v)d(v,u)}{d(u,v) + d(v,u)} + \mu d(u,v)\right] \\
&= \varphi\left[\mu d(u,v)\right].
\end{aligned}$$

From Definition 2.10, we have $d(u,v) \prec \mu d(u,v)$. A contradiction, because $\mu < 1$. Hence, $u = v$. Therefore, u is a unique common fixed point of S and T. □

Example 3.4. Let $X = \mathbb{C}$. Define a function $d : X \times X \to \mathbb{C}$ such that

$$d(z_1,z_2) = |x_1 - x_2|^2 + i|y_1 - y_2|^2,$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$.

From Example 18 in [9], it implied that (X,d) is a complete complex valued b-metric space with $s = 2$. Now, we define two self-mappings $S, T : X \to X$ as follows:

$$Sz = \begin{cases} 0 & \text{if } a,b \in \mathbb{Q}, \\ 2 & \text{if } a \in \mathbb{Q}^C, b \in \mathbb{Q} \\ 2i & \text{if } a,b \in \mathbb{Q}^C \\ 2+2i & \text{if } a \in \mathbb{Q}, b \in \mathbb{Q}^C \end{cases} \quad \text{and } Tz = \begin{cases} 0 & \text{if } a,b \in \mathbb{Q}, \\ 1 & \text{if } a \in \mathbb{Q}^C, b \in \mathbb{Q} \\ i & \text{if } a,b \in \mathbb{Q}^C \\ 1+i & \text{if } a \in \mathbb{Q}, b \in \mathbb{Q}^C \end{cases}$$

where $z = a + bi \in X$. We see that $S^n z = 0 = T^n z$ for $n > 1$, so

$$d(S^n x, T^n y) = 0 \precsim \lambda \frac{d^2(x,y)}{1 + d(x,y)} + \mu d(y, T^n y) + \rho d(x, S^n x),$$

for all $x, y \in X$ and $\lambda, \mu, \rho \geq 0$ with $2(\lambda + \rho) + \mu < 1$. So all conditions of Theorem 3.3 are satisfied to get a unique common fixed point 0 of S and T.

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REFORMS OF A GENERALIZED KKMF PRINCIPLE

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ABSTRACT. In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano obtained a generalization of Ky Fan's 1984 KKM theorem on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. In this article, we deduce the better abstract versions of such results from a general KKM theorem on abstract convex spaces in our previous works.

KEYWORDS:KKM theorem, Fan's 1961 KKM lemma, 1984 KKM theorem, Fan-Browder fixed point theorem, abstract convex space, (partial) KKM space.
AMS Subject Classification: 47H04, 47H10, 49J27, 49J35, 49J53, 52A01, 54H25.

1. Introduction

The KKM theory, first called by the author in 1992, is the study on applications of any equivalent or extended formulations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz [8] in 1929. The KKM theorem is one of the most well-known and important existence principles and provides the foundations for many of the modern essential results in diverse areas of mathematical sciences.

The KKM theory was originally devoted to convex subsets of topological vector spaces mainly by Ky Fan and Granas, and later to the so-called convex spaces by Lassonde [9], to c-spaces (or H-spaces) by Horvath [7] and others, to generalized convex (G-convex) spaces mainly by the present author. Since 2006, we proposed new concepts of abstract convex spaces and (partial) KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory. Consequently we have obtained a large number of new results in such frame; see [10, 14, 19].

Recall that a milestone on the history of the KKM theory was erected by Fan in 1961 [4]. His 1961 KKM Lemma (or the Fan-KKM theorem or the KKMF principle [2]) extended the KKM theorem to arbitrary topological vector spaces and was

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applied to various problems in his subsequent papers. Moreover, his lemma was extended in 1979 and 1984 [5, 6] to his 1984 KKM theorem with a new coercivity (or compactness) condition for noncompact convex sets; see [10, 14, 19].

In 1993, we introduced generalized convex (G-convex) spaces $(E, D; \Gamma)$ [20] and, in 1998, we derived new concept of them removing the original monotonicity restriction; see [10, 14, 19]. Motivated by our original G-convex spaces in 1993, Ben-El-Mechaiekh, Chebbi, Florenzano, and Llinares [1] in 1998 introduced L-spaces (E, Γ) and claimed incorrectly that G-convex spaces are particular to their L-spaces. Since then a number of authors followed the misconception of [1] and published incorrect obsolete articles even after we established the KKM theory on abstract convex spaces in 2006–2010.

In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano [2] obtained a generalization of Ky Fan's 1984 KKM theorem [6] on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. This type of studies also followed by many others whom may adequately called the L-space theorists.

In our previous work [18] in 2013, we obtained some results on generalizations on those in [2] based on our theory of abstract convex spaces. The present article is a continuation and supplement of [18] and aims to improve some contents of [18] and [2].

Section 2 is devoted to a short history of KKM type theorems. It begins with the original KKM theorem and ends with one of our most general extension in our recent works on abstract convex theory. In Section 3, we introduce some coercing families extending the one in [2] and show that they are complicated forms of a simple consequence of the coercivity due to S. Y. Chang [3] early in 1989. Section 4 deals with reforms or extensions of some results of [2] and [18].

For preliminaries on the KKM theoretic terminology on abstract convex spaces, see our previous work [18, 19].

2. The KKMF principle and generalizations

In 1929, Knaster, Kuratowski, and Mazurkiewicz [8] obtained the following so-called KKM theorem from the weak Sperner lemma in 1928:

Theorem 2.1. *(KKM [8]) Let A_i $(0 \le i \le n)$ be $n+1$ closed subsets of an n-simplex $p_0 p_1 \cdots p_n$. If the inclusion relation*

$$p_{i_0} p_{i_1} \cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0} p_{i_1} \cdots p_{i_k}$ $(0 \le k \le n,\ 0 \le i_0 < i_1 < \cdots < i_k \le n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

Later it was known that this holds also for open subsets instead of closed ones; see [10, 19].

From 1961, Ky Fan showed that the KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone of the history of the KKM theory was erected by Fan [4]. He extended the KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma 2.2. *(Fan [4]) Let X be an arbitrary set in a Hausdorff topological vector space Y. To each $x \in X$, let a closed set $F(x)$ in Y be given such that the following two conditions are satisfied:*

(i) *The convex hull of a finite subset $\{x_1, \cdots, x_n\}$ of X is contained in $\bigcup_{i=1}^n F(x_i)$.*
(ii) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the 1961 KKM Lemma of Ky Fan (or the Fan-KKM theorem or the KKMF principle [2]). Later the Hausdorffness of Y was known to be superfluous.

Moreover, Fan [5, 6] in 1984 introduced a KKM theorem with a more general coercivity (or compactness) condition for noncompact convex sets as follows:

Theorem 2.3. *(Fan [6]) In a Hausdorff topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

From now on, in this Section, *all theorems holds for a topological vector space $E = X$, its nonempty subset D, and $\Gamma : \langle D \rangle \multimap E$ is the convex hull operation.*

The following particular form of Lassonde [[9], Theorem I] for $X = Y$ in 1983 extends the 1984 theorem of Fan:

Theorem 2.4. *(Lassonde [9]) Let D be an arbitrary set in a convex space X, and $F : D \multimap X$ be a multimap having the following properties*

(i) *for each $x \in D$, $F(x)$ is compactly closed in X;*
(ii) *F is a KKM map, that is, for any finite subset $\{x_1, \ldots, x_n\}$ of D,*

$$co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F(x_i);$$

(iii) *For some c-compact subset $L \subset X$, $\bigcap\{F(x) \mid x \in L \cap D\}$ is compact.*

Then $\bigcap\{F(x) | x \in D\} \neq \emptyset$.

In 1989, S. Y. Chang [3] obtained the following theorem which eliminated the concept of c-compact sets in Theorem 2.4

Theorem 2.5. *(Chang [3]) Let D be a nonempty subset of a convex space X and $F : D \multimap X$ be a multimap. Suppose that*

(i) *for each $x \in D$, $F(x)$ is closed in X;*
(ii) *F is a KKM map;*
(iii) *there exist a nonempty compact subset K of X and, for each finite subset N of D, a compact convex subset L_N of X containing N such that*

$$L_N \cap \bigcap\{F(x) \mid x \in L_N \cap D\} \subset K.$$

Then $\bigcap\{F(x) \mid x \in D\} \neq \emptyset$.

For a long period, the present author tried to unify hundreds of generalizations of the KKM type theorems and, finally, obtained the following standard forms in [11, 12]:

Theorem 2.6. *Let $(E, D; G)$ be a partial KKM space [resp. KKM space], and $G : D \multimap E$ be a multimap satisfying*

(1) *G has closed [resp. open] values; and*
(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).*

Then $\{G(y)\}_{y\in D}$ has the finite intersection property. Further, if

(3) *$\bigcap_{y\in M} \overline{G(y)}$ is compact for some $M \in \langle D \rangle$, then we have*

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Actually the first part of Theorem 2.6 is a definition.

Consider the following related four conditions due to Luc et al. in 2010 for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.
(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*)
(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).
(d) G is closed-valued.

From the definition of $\mathfrak{KC}$-maps, we have a whole intersection property of the Fan type under certain 'coercivity' conditions. The following is given in [15, 16, 17, 19]:

Theorem 2.7. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, Z)$, and $G : D \multimap Z$ a multimap such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*
(2) *there exists a nonempty compact subset K of Z such that either*
(i) *$K \sup \bigcap\{\overline{G(y)}|y \in M\}$ for some $M \in \langle D \rangle$; or*
(ii) *for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{(L_N)}$ is compact, and*

$$K \sup \overline{(L_N)} \cap \bigcap\{\overline{G(y)}|y \in D'\}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap \overline{G(y)}|y \in D\} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y)|y \in D\} \neq \emptyset$; and*
(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y)|y \in D\} \neq \emptyset$.*

Remark 2.8. 1. The coercivity (ii) is originated from S. Y. Chang [6] in 1989.
2. Taking $\overline{K}$ instead of K, we may assume K is closed and the closure notations in (i) and (ii) can be erased.
3 In our previous work [16, 17, 19], we showed that a particular form of Theorem 2.7 unifies several important KKM type theorems appeared in history.
4 Many particular forms of Theorem 2.7 have equivalent formulations or lead many KKM theoretic results.

3. Various coercing families

In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano [2] obtained a generalization of Ky Fan's 1984 KKM theorem [6] on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the

Fan-Browder fixed point theorem to multimaps on non-compact convex sets. This type of studies also followed by the L-space theorists.

The following is given by Ben-El-Mechaiekh, Chebbi, and Florenzano [[2], Definition 2.1]:

[A] ([2]) Consider a subset X of a Hausdorff topological vector space E and a topological space Z. A family $\{(D_i, K_i)\}_{i\in I}$ of pairs of sets is said to be *coercing* for a map $F : X \multimap Z$ if and only if:

(i) for each $i \in I$, D_i is contained in a compact convex subset of X, and K_i is a compact subset of Z;

(ii) for each $i, j \in I$, there exists $k \in I$ such that $D_i \cup D_j \subset D_k$;

(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x\in D_k} F(x) \subset K_i$.

If I is a singleton, the family is called a *single* coercing family. Note that $(E \supset X; \mathrm{co})$ is a G-convex space and that (ii) will be shown redundant.

Motivated by [2], we defined the following [18]

[B] Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. We say that a map $G : D \multimap Z$ has a *coercing family* $\{(D_i, K_i)\}_{i\in I}$ if and only if

(1) for each $i \in I$, K_i is a compact subset of Z and $D_i \subset D$ such that, for each $N \in \langle D\rangle$, there exist a compact subset L^i_N of E that is Γ-convex relative to $D_i \cup N$;

(2) for each $i \in I$, there exists $k \in I$ with $\bigcap_{y\in D_k} F(y) \subset K_i$.

Remark 3.1. In [2], it is noted that the condition (2) holds *if and only if* the 'dual' map $\Phi : Z \multimap X$ of F, defined by $\Phi(z) = X \setminus F^{-}(z),\ z \in Z$, verifies

$(2)'$ $\forall i \in I,\ \exists k \in I,\ \forall z \in Z \setminus K_i, \quad \Phi(z) \cap C_k \neq \emptyset.$

In [2], there are given several deep examples of condition $(2)'$ related to an exceptional family, an escaping sequence, an attracting trajectory, and others.

The coercivity [**B**] improves [**A**] as follows:

Proposition 3.2. [**A**] $\Longrightarrow$ [**B**].

Proof. Since each D_i is contained in a compact convex subset of $X \subset E$ by [**A**](i) and E is a Hausdorff topological vector space, for each $N \in \langle X\rangle$, there exists a compact convex subset L^i_N of E containing $D_i \cup N$; see Lassonde [9]. Therefore, Condition [**B**](1) holds. Since [**A**](iii) is the same to [**B**](2), all requirements of [**B**] are satisfied. □

Note that Condition [**A**](ii) is redundant for [**B**].

Let us begin with the following particular form of the condition (ii) in Theorem 2.7 with $sG : D \multimap Z$ instead of $G : D \multimap Z$ [18]

[C] Let $(E, D; \Gamma)$ be an abstract convex space, $G : D \multimap E$ a multimap, Z a topological space, and $s : E \longrightarrow Z$ a continuous map such that

(C) there exists a nonempty compact subset K of Z such that, for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and

$$s(L_N) \cap \bigcap_{y\in D'} sG(y) \subset K.$$

Note that $s \in \mathfrak{KC}(E, Z)$. Condition [**C**] appeared as Condition (I) in [18]. The following corrects [[18], Proposition 3.6]:

Proposition 3.3. [**B**] $\Longrightarrow$ [**C**].

Proof. Let $G : D \multimap E$ and $s : E \longrightarrow Z$ be given in [**C**] and let $F = sG : D \multimap Z$. Choose any $i \in I$ by (B)(1), we have K_i and D_i such that, for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset $L_N := L^i_N$ of E relative to $D' = D_i \cup N$. By [**B**](2), we have a $k \in I$ such that

$$\bigcap_{y\in D_k} F(y) = \bigcap_{y\in D_k} sG(y) \subset K_i.$$

Since i was arbitrary, we may assume $k = i$ and $K = K_i$. Moreover, since $D' = D_k \cup N$, we have

$$\bigcap_{y\in D'} sG(y) \subset K \implies s(L_N) \cap \bigcap_{y\in D'} sG(y) \subset K.$$

Hence the coercivity condition [**C**] holds. □

Note that Conditions [**A**], [**B**], and [**C**] are examples of the coercivity (ii) in Theorem 2.7.

4. Generalization of the KKMF principle

In this section, we deduce generalized better forms of the main theorems in [2]:

Theorem 4.1. *Let $(E, D; \Gamma)$ be a partial KKM space. Suppose that*

(1) *$G : D \multimap E$ is a closed-valued KKM map,*
(2) *the coercivity condition* [**C**] *or* [**B**] *holds for G.*

Then we have $K \cap \bigcap_{y\in D} G(y) \neq \emptyset$.

Proof. We apply Theorem 2.7 with $F = s$.

(1) Since $s^{-1}G$ is a closed-valued KKM map, Condition [**C**] implies $\Gamma_A \subset R(A) = s^{-1}G(A)$ and $s\Gamma_A \subset sR(A) = G(A)$ for all $A \in \langle D\rangle$. Therefore $\overline{G}$ is a KKM map w.r.t. s.

(2) Condition (2) implies (ii) in Theorem 2.7 with $F = s$ and $G = sR$. Therefore, by the case (ii) of Theorem 2.7, we have

$$s(E) \cap K \cap \bigcap_{y\in D} sR(y) \neq \emptyset.$$

This implies the conclusion. □

Corollary 4.2. *Let E be a Hausdorff topological vector space, Y a convex subset of E, X a non-empty subset of Y, and $F : X \multimap Y$ a KKM map with closed (in Y) values. If F admits a coercing family in the sense of* [**A**] *without (ii), then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Proof. Put $E = Y$, $D = X$, and $G = F$ in Theorem 4.1. Since [**A**] $\implies$ [**B**] $\implies$ [**C**], we have the conclusion by Theorem 4.1. □

The main theorem [[2], Theorem 3.1] of [2] is Corollary 4.2 under the assumption of [**A**] and the compactly closed values of F. However, replacing the topology of Y to compactly generated one (as in k-spaces), we may assume F has closed values.

If $D_i = D$ and $K_i = K$ for all $i \in$, D is contained in a compact convex subset of X and K is a compact subset of Y, then Corollary 4.2 reduces to the 1984 KKM theorem of Ky Fan [6] which in turn generalizes the 1961 KKM Lemma of Ky Fan [4]; see Section 2.

Now we formulate Theorem 4.1 to a fixed point theorem:

Theorem 4.3. *Let $(X;\Gamma)$ be a partial KKM space and $\Phi : X \multimap X$ be a map with open fibers and non-empty values. If Φ admits a coercing family* [**B**] *in the sense of Remark 3.1, then the map* $\mathrm{co}_\Gamma\Phi$ *has a fixed point.*

Proof. Suppose that $\mathrm{co}_\Gamma(\Phi)$ has no fixed point, i.e., $x \notin \mathrm{co}_\Gamma(\Phi)(x)$ for all $x \in X$. Define $F : X \multimap X$ by

$$F(x) := \{y \in X \mid x \notin \Phi(y)\}, \quad x \in X.$$

Then F is closed-valued. We claim that F is a KKM map. Suppose that for some $N \in \langle X\rangle$, there exists $z \in \mathrm{co}_\Gamma N$ such that $z \notin F(N)$. Then $N \subset \Phi(z)$ and $z \in \mathrm{co}_\Gamma(\Phi(z))$, which contradicts the assumption that $\mathrm{co}_\Gamma(\Phi)$ has no fixed point. To complete our proof, we remember the coercing family is also a coercing family [**B**] with Remark 3.1. Theorem 4.1 implies $\bigcap_{x\in X} F(x) \neq \emptyset$ which contradicts the fact that Φ has non-empty values. □

Corollary 4.4. *Let X be a non-empty convex subset of a topological vector space E and $\Phi : X \multimap X$ be a map with open fibers (in X) and non-empty values. If Φ admits a coercing family* [**A**]*(without (ii)) in the sense of Remark 3.1, then the map* $\mathrm{conv}(\Phi)$ *has a fixed point.*

Note that [[2], Theorem 3.2] is Corollary 4.4 under the assumption of [**A**] and the compactly open fibers of Φ. However, replacing the topology of Y to compactly generated one (as in k-spaces), we may assume Φ has open fibers. Note that Corollary 4.4 generalize the Fan-Browder fixed point theorem.

As was noted by [2], the results in this section can be used to extend existing results on the solvability of complementarity problems, existence of zero on non-compact domains and existence of equilibria for qualitative games and abstract economies.

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