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SOLUTIONS FOR THE ORDERED VARIATIONAL INCLUSION PROBLEMS IN BANACH SPACES

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ABSTRACT. In this study, we consider the ordered variational inclusion problems in ordered Banach spaces involving the weak RRD-multivalued mappings. By using the technique of relaxed resolvent operators, we suggest an iterative algorithm and prove the existence of solutions of ordered variational inclusion problems. Also, we prove the convergence of the sequences generated by an iterative algorithm.

KEYWORDS: Ordered variational inclusion problems, Algorithm, Weak-RRD multivalued mappings, Ordered Banach spaces.
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1. Introduction

Most of the problems related to variational inequalities, variational inclusions, complementarity problems and equilibrium problems are solved by the maximal monotone operators and their generalizations such as H-monotonicity, H-accretivity, penalization, regularization and many more in these fields *see*, [1, 2, 6, 7, 9, 13, 25]. Some splitting methods are based on the resolvent operator of the form $[I+\lambda M]^{-1}$, where M is a multivalued monotone mapping, λ is a positive constant and I is an identity mapping.

Generalized nonlinear ordered variational inclusions have wide applications to many fields including for example, mathematical physics, optimization and control theory, nonlinear programming, economics and engineering sciences. Li [18] studied a class of generalized nonlinear ordered variational inequalities in ordered Banach spaces. One year later, the same author [19] studied another class of general nonlinear ordered variational inequalities in the same setting. He continued to his researches in this field by studying a class of nonlinear inclusion problems

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for ordered RME set-valued mappings in ordered Hilbert spaces. The introduction of the concept of ordered (α, λ)-NODM set-valued mappings was first made by Li [21]. He considered a class of nonlinear variational inclusion problems involving (α, λ)-NODM set-valued mappings and proved an existence theorem for solutions of such a class of problems. He defined the resolvent operator associated with an α-NODM set-valued mapping and proved that it is a comparison Lipschitz continuous mapping. Based on the resolvent operator appeared in [21], the author suggested an iterative algorithm and studied the convergence analysis of the sequence generated by his proposed iterative algorithm. With inspiration and motivation from this work, during the past six years, many investigators have shown interest in introducing various kinds of ordered set-valued mappings in the setting of different ordered spaces and defined the resolvent operators associated with them. They used the resolvent operators defined in their papers for solving various classes of ordered variational inequalities/inclusions.

In 2001, Huang and Fang [11] introduced the concepts of generalized m-accretive mapping and studied the properties of resolvent operator with the generalized m-accretive mappings. Essentially, using the resolvent operator techniques, one can show that the variational inclusions are commensurate to the fixed point problems. This equivalent formulation has played a great job in designing some exotic techniques for solving variational inclusions and related optimization problems.

Inspired and motivated by the recent research works, [1]-[30], In this paper, we consider a relaxed resolvent operator $[(I-R)+\lambda M]^{-1}$, where R is a single valued mapping, I is an identity mapping, and show that the relaxed resolvent operator is a comparison mapping with respect to the operator $\oplus$. Finally, we prove the existence of solutions of ordered variational inclusion problems by using the weak-RRD multivalued mappings and also discuss the convergence of an iterative sequences generated by the algorithms.

2. Preliminaries

Let $(X, \|\cdot\|)$ be an ordered Banach spaces whose dual X^* is endowed with the dual norm, denoted also by $\|\cdot\|$. Let d be the metric induced by the norm $\|\cdot\|, 2^X$ (respectively $C(X)$) be the family of nonempty (respectively compact) subsets of X, and $\mathfrak{D}(\cdot,\cdot)$ be the Hausdorff metric on $C(X)$ defined by,

$$\mathfrak{D}(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},$$

where $A, B \in C(X), d(x,B) = \inf_{y\in B} d(x,y)$ and $d(A,y) = \inf_{x\in A} d(x,y)$.

Definition 2.1. Let X be a real Banach space with norm $\|\cdot\|$ and θ be the zero element in X. A nonempty closed convex subset C of X is said to be a cone if

(i) $C + C \subset C$,
(ii) for any $x \in C$ and any $\lambda > 0, \lambda x \in C$.

Also said to be a pointed cone, if for any $x \in C$ and $-x \in C$ then $x = \theta$.

Definition 2.2. [18] Let C be a cone in real Banach space X. A cone C is called normal if there exists a constant $\lambda_{C_N} > 0$ such that for $\theta \le x \le y$, $\|x\| \le \lambda_{C_N}\|y\|$, where λ_{C_N} is a normal constant of C.

Definition 2.3. [29] A relation $\le$ is said to be partial ordered in X, if for any elements $x, y \in X, x \le y$, then $x - y \in C$. The real Banach space X endowed with the ordered relation $\le$ defined by C is called an ordered real Banach space.

Definition 2.4. [29] For arbitrary elements $x, y \in X$, x and y are called comparable to each other, if $x \leq y$ (or $y \leq x$) holds. And denoted by $x \propto y$ for $x \leq y$ and $y \leq x$.

Lemma 2.5. ([18, 23], *Lemma 2.1*) *Let C be a normal cone of an ordered real Banach space X and $\leq$ be a partial ordered relation defined by the cone C. For arbitrary elements $x, y \in X, lub\{x, y\}$ and $glb\{x, y\}$ express the least upper bound and greatest lower bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exists, some binary operations can be defined as follows:*

(i) $x \vee y = lub\{x, y\}$;
(ii) $x \wedge y = glb\{x, y\}$;
(iii) $x \oplus y = (x - y) \vee (y - x)$;
(iv) $x \odot y = (x - y) \wedge (y - x)$;
(v) *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist, $x - y \propto y - x$ and $\theta \leq (x - y) \vee (y - x)$;*
(vi) *If $x \vee y = lub\{x, y\}, x \wedge y = glb\{x, y\}, x \oplus y = (x - y) \vee (y - x), x \odot y = (x - y) \wedge (y - x)$.*

The operators $\vee, \wedge, \oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively. Then here in after relations survive:

(1) $x \oplus y = y \oplus x, x \oplus x = \theta, x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;
(2) *For a real λ, $(\lambda x) \oplus (\lambda y) =| \lambda | (x \oplus y)$;*
(3) $x \odot \theta \leq \theta$, *if* $x \propto \theta$;
(4) $\theta \leq x \oplus y$ *if* $x \propto y$;
(5) *if $x \propto y$ then $x \oplus y = \theta$ if and only if $x = y$;*
(6) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$;
(7) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$;
(8) *let $(x + y) \vee (u + v)$ exist, and if $x \propto u, v$ and $y \propto u, v$, then*
$$(x + y) \oplus (u + v) \leq (x \oplus u + y \oplus v) \wedge (x \oplus v + y \oplus u);$$
(9) *if $x \leq y$ and $u \leq v$, then $x + u \leq y + v$:*
(10) $x \vee y = x + y - (x \wedge y)$;
(11) $\alpha x \oplus \beta x =| \alpha - \beta | x = (\alpha \oplus \beta) | x$, *if* $x \propto \theta$, $\forall x, y, u, v \in X$ *and* $\alpha, \beta, \gamma \in \mathbb{R}$.

Proposition 2.6. [8] *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist $x - y \propto y - x$ and $\theta \leq (x - y) \vee (y - x)$.*

Proposition 2.7. [8] *If for any positive integer n if $x \propto y_n$ and $y_n \longrightarrow y^*$ $(n \longrightarrow \infty)$ then $x \propto y^*$.*

Proposition 2.8. ([23], *Theorem 2.5*) *Let X be a positive Hilbert space and $x, y, z, w \in X$. Then we have the following statements:*

(1) *If $x \leq y$, $\theta \leq z$, then $\langle y, z\rangle \geq \langle x, z\rangle$;*
(2) *If $\theta \leq z$, then $\langle x \vee y, z\rangle \geq \langle x, z\rangle \vee \langle y, z\rangle, \langle x, z\rangle \wedge \langle y, z\rangle \geq \langle x \wedge y, z\rangle$;*
(3) *If $\theta \leq z$, then $\langle x + y, z\rangle \geq \langle x, z\rangle \vee \langle y, z\rangle + \langle x \wedge y, z\rangle$;*
(4) *If $\theta \leq z$, then $\langle x \vee y, z\rangle \geq \langle x, z\rangle + \langle y, z\rangle - \langle x, z\rangle \wedge \langle y, z\rangle$;*
(5) *If $\theta \leq z$, then $\langle x \oplus y, z\rangle \geq \langle x, z\rangle \oplus \langle y, z\rangle$.*

Lemma 2.9. ([17, 19], *Lemma 1.13*) *Let X be an ordered real Banach space and C be a normal cone in X with normal constant λ_{C_N}. Then for arbitrary elements $x, y \in X$, the following relations hold:*

(1) $\|\theta \oplus \theta\| = \|\theta\| = \theta$;

(2) $\|x \wedge y\| \leq \|x\| \wedge \|y\| \leq \|x\| + \|y\|$;
(3) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_{C_N} \|x \oplus y\|$;
(4) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$;
(5) $\lim_{x \longrightarrow x_0} \|A(x) - A(x_0)\| = 0$, *if and only if*

$$\lim_{x \longrightarrow x_0} A(x) \oplus A(x_0) = \theta.$$

Definition 2.10. [18] Let X be a real ordered Banach space and $R : X \longrightarrow X$ be a single valued mapping, Then

(i) R is called comparison, if for each $x, y \in X$, $x \propto y$ then $R(x) \propto R(y)$, $x \propto R(x)$ and $y \propto R(y)$;
(ii) R is called strongly comparison, if R is a comparison and $R(x) \propto R(y)$ if and only if $x \propto y$ for $x, y \in X$.

Definition 2.11. [21] Let X be a real ordered Banach space and $R, B : X \longrightarrow X$ be two single valued mappings, Then R and B is said to be comparable to each other, if for each $x, \in X$, $R(x) \propto B(x)$ (denoted by $R \propto B$). Obviously, if R is comparable, then $R \propto I$, where I is the identity mapping.

Definition 2.12. ([18], Definition 2.10) Let X be a real ordered Banach space and C be the normal cone with normal constant C_{λ_N} in X. A mapping $R : X \longrightarrow X$ is called β-ordered compression mapping if R is a comparison and there exists a constant $0 < \beta < 1$ such that

$$(R(x) \oplus R(y)) \leq \beta(x \oplus y).$$

Lemma 2.13. [17] *Let C be a normal cone in X. If for $x, y \in X$, they can be compared to each other, then the following condition holds:*

$$(x + y) \vee ((-x) + (-y)) \leq (x \vee (-x)) + (y \vee (-y)).$$

Definition 2.14. [20, 23, 24] Let X be a real ordered Banach space and $M : X \longrightarrow 2^X$ be a multivalued mapping. Let $R : X \longrightarrow X$ be a strong comparison and β-ordered compression. Then,

(i) M is called comparison mapping, if for any $v_x \in M(x)$, $x \propto y$, $x \propto v_x$, then for any $v_x \in M(x)$, $v_y \propto M(y)$ and $v_x \propto v_y$, for all $x, y \in X$;
(ii) M is called ordered rectangular mapping, if for each $x, y \in X, v_x \in M(x)$ and $v_y \in M(y)$ such that

$$\langle v_x \odot v_y, -(x \oplus y) \rangle = 0;$$

(iii) M is called γ_R-ordered rectangular mapping with respect to R, if there exists a constant $\gamma_R > 0$, for any $x, y \in X$ there exists $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that

$$\langle v_x \odot v_y, -(R(x) \oplus R(y)) \rangle \geq \gamma_R \|R(x) \oplus R(y)\|^2,$$

where v_x and v_y are said to be γ_R-elements, *respectively*;
(iv) M is called weak comparison mapping with respect to R, if for any $x, y \in X$, $x \propto y$ then there exists $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that $x \propto v_x$, $y \propto v_y$ and $v_x \propto v_y$, where v_x and v_y are said to be weak comparison elements, *respectively*;

(v) M is called λ-weak ordered different comparison mapping with respect to R, if there exists a constant $\lambda > 0$, for any $x, y \in X$ there exists $v_x \in M(R(x))$, $v_y \in M(R(y))$ such that $\lambda(v_x - v_y) \propto (x - y)$, where v_x and v_y are said to be λ-elements, *respectively*;

(vi) A comparison mapping M is called λ-ordered strongly monotonic increasing, if for $x \geq y$ there exists a constant $\lambda > 0$ such that

$$\lambda(v_x - v_y) \geq x - y, \ \forall x, y \in X, v_x \in M(x), v_y \in M(y);$$

(vii) A weak comparison mapping M is said to be a (γ_R, λ)-weak-RRD multivalued mapping with respect to B if M is a γ_R-ordered rectangular and λ-weak ordered different comparison mapping with respect to B and $(R+\lambda M)X = X$ for $\lambda > 0$ and there exists $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that v_x and v_y are (γ_R, λ)-elements, *respectively*.

Remark 2.15. Let X be a real Hilbert space. Then we have the followings:

(i) Every λ-ordered monotone mapping is a λ-weak ordered different comparison mapping.

(ii) The γ_I-ordered rectangular mapping is an ordered rectangular mapping, where I is the identity mapping,

(iii) An ordered RME mapping is a λ-weak RRD-mapping.

Definition 2.16. Let X be an ordered real Banach space. A multivalued mapping $A : X \longrightarrow 2^X$ is said to be $\mathfrak{D}$-Lipschitz continuous if for each $x, y \in X, x \propto y$ there exists a constant $\delta_A > 0$ such that

$$\mathfrak{D}(A(x), A(y)) \leq \delta_A \|x \oplus y\|.$$

Definition 2.17. Let X be an ordered real Banach space. Let $M : X \longrightarrow 2^X$ be a multivalued mapping, $R : X \longrightarrow X$ be a single valued mapping and $I : X \longrightarrow X$ be the identity mapping. Then a weak comparison mapping M is said to be a (γ',λ)-weak RRD multivalued mapping with respect to $(I - R)$, if M is a γ'-ordered rectangular and λ-weak ordered different comparison mapping with respect to $(I-R)$ and $[(I-R)+\lambda M](X) = X$ for $\lambda > 0$ and there exists $v_x \in M((I-R)(x))$ and $v_y \in M((I - R)(y))$ such that v_x and v_y are (γ', λ)-elements, *respectively*.

Example 2.18. Let $X = \mathbb{R}$ and let $R : X \longrightarrow X$ be a mapping defined by

$$R(x) = \frac{x}{2}, \forall x \in X$$

and a multivalued mapping $M : X \longrightarrow 2^X$ is defined by $M(x) = 1$ for $x = 0$ and $M(x) = \{\frac{x}{3}\}$ for $x \neq 0$. Then we can easily check that R is 1-ordered compression and M is $(\frac{1}{7}, 1)$- weak RRD mapping with respect to R.

3. Formulation of Problems

Let X be an ordered Banach space and $A, B, T, G : X \longrightarrow C(X)$ be multivalued mappings. Suppose that $M : X \longrightarrow 2^X$ be a multivalued mapping and $N : X \times X \times X \longrightarrow X$ be a single valued mapping. We consider the following problems for finding $u \in X, v \in A(u), w \in B(u), q \in T(u)$ and $z \in G(u)$ such that

$$\rho \in N(v, w, q) + \tau M(z), \text{ for some } \rho \in X \text{ and } \tau > 0. \tag{3.1}$$

Problem (3.1) is called an ordered variational inclusion problem.

Special Cases:

(i) If $N(v,w,q) \equiv 0$ and G is a single valued mapping, then (3.1) reduces to the following problem of finding $u \in X$ such that

$$\rho \in \tau M(u), \text{ for some } \rho \in X \text{ and } \tau > 0, \tag{3.2}$$

studied by Li *et al.* [23].

(ii) If $B = T \equiv 0$ (zero mappings) and A, G are single valued mappings, then (3.1) reduces to the following problem of finding $u \in X$ such that

$$\rho \in N(u) + \tau M(u), \text{ for some } \rho \in X \text{ and } \tau > 0, \tag{3.3}$$

studied by Li *et al.* [24].

(iii) If $\rho = 0, \tau = 1, T \equiv 0$ (a zero mapping) and G is a single valued mapping, then (3.1) reduces to the following problem of finding $u \in X, v \in A(u), w \in B(u)$ such that

$$0 \in N(v,w) + M(u), \tag{3.4}$$

studied by Verma [30].

(iv) If $\rho = 0, \tau = 1, T \equiv 0$ (a zero mapping) and $N(v,w) = f(v) - p(w)$, then (3.1) reduces to the following problem of finding $u \in X, v \in A(u), w \in B(u), z \in G(u)$ such that

$$0 \in f(v) - p(w) + M(z), \tag{3.5}$$

where $f, p : X \longrightarrow X$ are single valued mappings, studied by Huang [10].

Definition 3.1. ([23], Theorem 3.3) Let X be an ordered Banach space, C be a normal cone with normal constant λ_{C_N} and $M : X \longrightarrow 2^X$ be a weak-RRD-multivalued mapping. Let $I : X \longrightarrow X$ be the identity mapping and $R : X \longrightarrow X$ be a single valued mapping. The relaxed resolvent operator $J_{M,\lambda}^{(I-R)} : X \longrightarrow X$, associated with I, R and M is defined by

$$J_{M,\lambda}^{(I-R)}(x) = [(I-R) + \lambda M]^{-1}(x), \forall x \in X, \tag{3.6}$$

where $\lambda > 0$ is a constant.

Proposition 3.2. ([23], *Theorem 3.3*) *Let X be an ordered Banach space, $R : X \longrightarrow X$ be a β-ordered compression mapping and $M : X \longrightarrow 2^X$ be a multivalued ordered rectangular mapping. Then relaxed resolvent operator $J_{M,\lambda}^{(I-R)} : X \longrightarrow X$ is a single valued for all $\lambda > 0$.*

Proposition 3.3. *Let X be an ordered Banach space and C be a normal cone with normal constant λ_{C_N} in X. Let $\leq$ be an ordering relation defined by the cone C, the operator $\oplus$ be a XOR operator. Let $M : X \longrightarrow 2^X$ be a (γ_R, λ)-weak-RRD-multivalued mapping with respect to $J_{M,\lambda}^{(I-R)}$. Let $R : X \longrightarrow X$ be a single valued mapping and $I : X \longrightarrow X$ be the identity mapping. Then the resolvent operator $J_{M,\lambda}^{(I-R)} : X \longrightarrow X$ is a comparable.*

Proof. Let M be a (γ_R, λ)-weak RRD-multivalued mapping with respect to $J_{M,\lambda}^{(I-R)}$. That is, M is γ_R-ordered rectangular and λ-weak ordered different comparison mapping with respect to $J_{M,\lambda}^{(I-R)}$, so that $x \propto J_{M,\lambda}^{(I-R)}(x)$. For any $x, y \in X$, let $x \propto y$ and let

$$v_x = \frac{1}{\lambda}(x - (I-R)(J_{M,\lambda}^{(I-R)}(x))) \in M(J_{M,\lambda}^{(I-R)}(x)) \tag{3.7}$$

and

$$v_y = \frac{1}{\lambda}(y - (I-R)(J_{M,\lambda}^{(I-R)}(y))) \in M(J_{M,\lambda}^{(I-R)}(y)). \tag{3.8}$$

Using (3.7) and (3.8), we have

$$\begin{aligned} v_x - v_y &= \frac{1}{\lambda}(x - (I-R)(J_{M,\lambda}^{(I-R)}(x))) - \frac{1}{\lambda}(y - (I-R)(J_{M,\lambda}^{(I-R)}(y))) \\ &= \frac{1}{\lambda}(x - y + (I-R)(J_{M,\lambda}^{(I-R)}(y) - J_{M,\lambda}^{(I-R)}(x))). \end{aligned}$$

Since M is a λ-weak ordered different comparison mapping with respect to $J_{M,\lambda}^{(I-R)}$, we have

$$\begin{aligned} \theta &\le \lambda(v_x - v_y) - (x - y) \\ &= (x - y) + (I-R)(J_{M,\lambda}^{(I-R)}(y)) - (I-R)(J_{M,\lambda}^{(I-R)}(x)) - (x - y) \\ &= (I-R)(J_{M,\lambda}^{(I-R)}(y)) - (I-R)(J_{M,\lambda}^{(I-R)}(x)). \end{aligned}$$

If $y \le x$ then $\lambda(v_x - v_y) - (x - y) \in C$ and if $x \le y$ then $(x - y) - \lambda(v_x - v_y) \in C$. Therefore, from Lemma 2.5

$$J_{M,\lambda}^{(I-R)}(x) \propto J_{M,\lambda}^{(I-R)}(y).$$

The proof is completed. □

Definition 3.4. Let X be a real ordered Banach space, C be a normal cone with a normal constant γ_{C_N} in X. A mapping $N : X \times X \times X \longrightarrow X$ is called (μ_N, η_N, ξ_N)-ordered compression mapping, if $x \propto y, u \propto v$ and $p \propto q$, then $N(x, u, p) \propto N(y, v, q)$ and there exist the constants $\mu_N, \eta_N, \xi_N > 0$ such that

$$N(x, u, p) \oplus N(y, v, q) \le \mu_N(x \oplus y) + \eta_N(u \oplus v) + \xi_N(p \oplus q).$$

Lemma 3.5. *Let X be an ordered Banach space and C be a normal cone with normal constant λ_{C_N} in X. Let $\le$ be an ordering relation defined by the cone C. Let $M : X \longrightarrow 2^X$ be a (γ_R, λ)-weak-RRD multivalued mapping with respect to $J_{M,\lambda}^{(I-R)}$. Let $R : X \longrightarrow X$ be a comparison and β-ordered compression mapping and I be the identity mapping. If $\lambda\gamma_R > \beta + 1 > 0$, and $v_x \in M(J_{M,\lambda}^{(I-R)}(x))$ and $v_y \in M(J_{M,\lambda}^{(I-R)}(y))$ are γ_R and λ-elements, respectively. Then relaxed resolvent operator $J_{M,\lambda}^{(I-R)}$ of M is a comparison and*

$$\|J_{M,\lambda}^{(I-R)}(x) \oplus J_{M,\lambda}^{(I-R)}(y)\| \le \frac{1}{\lambda\gamma_R - \beta - 1}\|x \oplus y\|.$$

Proof. Let M be a (γ_R, λ)-weak RRD multivalued mapping with respect to $J_{M,\lambda}^{(I-R)}$. That is M is a γ_R-ordered rectangular and λ-weak ordered different comparison mapping with respect to $J_{M,\lambda}^{(I-R)}$. Then for any $x, y \in X,\ \lambda > 0$, set $u_x = J_{M,\lambda}^{(I-R)}(x)$, $u_y = J_{M,\lambda}^{(I-R)}(y)$ and let

$$v_x = \frac{1}{\lambda}(x - (I-R)(J_{M,\lambda}^{(I-R)}(x))) \in M(J_{M,\lambda}^{(I-R)}(x))$$

and

$$v_y = \frac{1}{\lambda}(y - (I-R)(J_{M,\lambda}^{(I-R)}(y))) \in M(J_{M,\lambda}^{(I-R)}(y)).$$

Since R is β-ordered compression mapping and from Lemma 2.5, we have

$$\begin{aligned} v_x \oplus v_y &= \frac{1}{\lambda}((x - (I-R)(u_x)) \oplus (y - (I-R)(u_y))) \\ &\le \frac{1}{\lambda}(x \oplus y + (I-R)(u_x) \oplus (I-R)(u_y)) \end{aligned}$$

$$\leq \frac{1}{\lambda}(x \oplus y + u_x \oplus u_y + R(u_x) \oplus R(u_y))$$
$$\leq \frac{1}{\lambda}(x \oplus y + u_x \oplus u_y + \beta(u_x \oplus u_y))$$
$$\leq \frac{1}{\lambda}(x \oplus y + (1+\beta)(u_x \oplus u_y)).$$

Since M is γ_R-ordered rectangular mapping with respect to $J^{(I-R)}_{M,\lambda}$, we have

$$\begin{aligned}
\gamma_R\|u_x \oplus u_y\|^2 &\leq \langle v_x \odot v_y, -(u_x \oplus u_y)\rangle \\
&\leq \langle v_x \oplus v_y, u_x \oplus u_y\rangle \\
&\leq \langle \frac{1}{\lambda}(x \oplus y + (1+\beta)(u_x \oplus u_y)), u_x \oplus u_y\rangle \\
&\leq \frac{1}{\lambda}\langle x \oplus y, (u_x \oplus u_y)\rangle + \frac{1+\beta}{\lambda}\langle (u_x \oplus u_y), (u_x \oplus u_y)\rangle \\
&\leq \frac{1}{\lambda}\|x \oplus y\|\|u_x \oplus u_y\| + \frac{1+\beta}{\lambda}\|u_x \oplus u_y\|^2.
\end{aligned}$$

It follows that

$$(\gamma_R - \frac{1+\beta}{\lambda})\|u_x \oplus u_y\| \leq \frac{1}{\lambda}\|x \oplus y\|$$

and consequently, we have

$$\|J^{(I-R)}_{M,\lambda}(x) \oplus J^{(I-R)}_{M,\lambda}(y)\| \leq \frac{1}{\lambda\gamma_R - \beta - 1}\|x \oplus y\|.$$

This completes the proof. □

4. Main Results

In this section, we will show the convergence of the approximation sequences generated by iterative algorithm for finding the solution of problem (3.1).

Algorithm 4.1. *Let $A, B, T, G : X \longrightarrow C(X)$ be the multivalued mappings, $R : X \longrightarrow X$ be a single valued mapping and $I : X \longrightarrow X$ be the identity mapping. Suppose that $N : X \times X \times X \longrightarrow X$ is a single valued mapping and $M : X \longrightarrow C(X)$ is a multivalued mapping. For any given initial $u_0 \in X, v_0 \in A(u_0), w_0 \in B(u_0), q_0 \in T(u_0), z_0 \in G(u_0)$, let*

$$u_1 = u_0 - z_0 + J^{(I-R)}_{M,\lambda}[(I-R)(z_0) + \frac{\lambda}{\tau}(\rho - N(v_0, w_0, q_0))].$$

Since $v_0 \in A(u_0) \in C(X), w_0 \in B(u_0) \in C(X), q_0 \in T(u_0) \in C(X)$ and $z_0 \in G(u_0) \in C(X)$, by Lemma 2.9 there exist $v_1 \in A(u_1) \in C(X), w_1 \in B(u_1) \in C(X), q_1 \in T(u_1) \in C(X)$ and $z_1 \in G(u_1) \in C(X)$ and suppose that $u_0 \propto u_1, v_0 \propto v_1, w_0 \propto w_1, q_0 \propto q_1$ and $z_0 \propto z_1$ such that

$$\begin{aligned}
\|v_1 \oplus v_0\| = \|v_1 - v_0\| &\leq \mathfrak{D}(A(u_1), A(u_0)), \\
\|w_1 \oplus w_0\| = \|w_1 - w_0\| &\leq \mathfrak{D}(B(u_1), B(u_0)), \\
\|q_1 \oplus q_0\| = \|q_1 - q_0\| &\leq \mathfrak{D}(T(u_1), T(u_0)), \\
\|z_1 \oplus z_0\| = \|z_1 - z_0\| &\leq \mathfrak{D}(G(u_1), G(u_0)).
\end{aligned}$$

Continuing the above process inductively, we can define the iterative sequences $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{q_n\}$ and $\{z_n\}$ with the supposition that $u_n \propto u_{n+1}, v_n \propto v_{n+1}, w_n \propto$

$w_{n+1}, q_n \propto q_{n+1}$ and $z_n \propto z_{n+1}$ for all $n \in N$. We define the following iterative schemes:

$$u_{n+1} = u_n - z_n + J_{M,\lambda}^{(I-R)}[(I-R)(z_n) + \frac{\lambda}{\tau}(\rho - N(v_n, w_n, q_n))] \tag{4.1}$$

$$\begin{aligned} &v_{n+1} \in A(u_{n+1}), \|v_{n+1} \oplus v_n\| = \|v_{n+1} - v_n\| \leq \mathfrak{D}(A(u_{n+1}), A(u_n)),\\ &w_{n+1} \in B(u_{n+1}), \|w_{n+1} \oplus w_n\| = \|w_{n+1} - w_n\| \leq \mathfrak{D}(B(u_{n+1}), B(u_n)),\\ &q_{n+1} \in T(u_{n+1}), \|q_{n+1} \oplus q_n\| = \|q_{n+1} - q_n\| \leq \mathfrak{D}(T(u_{n+1}), T(u_n)),\\ &z_{n+1} \in G(u_{n+1}), \|z_{n+1} \oplus z_n\| = \|z_{n+1} - z_n\| \leq \mathfrak{D}(G(u_{n+1}), G(u_n)) \end{aligned} \tag{4.2}$$

where λ, τ are constants and $\rho \in X$.

Now, we convert our problem (3.1) into a fixed point problem.

Lemma 4.2. *Let $u \in X, v \in A(u), w \in B(u), q \in T(u)$ and $z \in G(u)$ be the solution of ordered variational inclusion problems (3.1) involving weak RRD-multivalued mappings if and only if (u, v, w, q, z) satisfies the following relation*

$$u = u - z + J_{M,\lambda}^{(I-R)}[(I-R)(z) + \frac{\lambda}{\tau}(\rho - N(v, w, q))]$$

where

$$J_{M,\lambda}^{(I-R)} = [(I-R) + \lambda M]^{-1}$$

and λ, τ are constants and $\rho \in X$.

Proof. This directly follows from the definition of relaxed resolvent operator $J_{M,\lambda}^{(I-R)}$ and the conditions of comparison mappings, *respectively.* □

Theorem 4.3. *Let X be a real ordered Banach space and C be a normal cone with normal constant λ_{C_N} in X. Let $R : X \longrightarrow X$ be a comparison, β-ordered compression mapping and $I : X \longrightarrow X$ be the identity mapping. Let $N : X \times X \times X \longrightarrow X$ be the (μ_N, η_N, ξ_N)-ordered compression mapping with constants $\mu_N, \eta_N, \xi_N > 0$. Let $A, B, T, G : X \longrightarrow C(X)$ be the multivalued mappings such that A, B, T and G are $\mathfrak{D}$-Lipschitz continuous mapping with constants $\delta_A, \delta_B, \delta_T$ and δ_G, respectively. Suppose that $M : X \longrightarrow C(X)$ is a (γ_R, λ)-weak RRD-multivalued mapping such that the following conditions are satisfied:*

$$\lambda_{C_N}[\tau\lambda\gamma_R(1+\delta_G) + \lambda\mu_N\delta_A + \lambda\eta_N\delta_B + \lambda\xi_N\delta_T] < \lambda\tau\gamma_R + \tau(1+\beta)(\lambda_{C_N} - 1). \tag{4.3}$$

Then the iterative sequences $\{u_n\}, \{v_n\}, \{w_n\}, \{q_n\}$ and $\{z_n\}$ generated by Algorithm 4.1 converge strongly to u, v, w, q and z, respectively and (u, v, w, q, z) is a solution of ordered variational inclusion problems (3.1) involving weak RRD-multivalued mappings, where $u \in X, v \in A(u), w \in B(u), q \in T(u)$ and $z \in G(u)$.

Proof. Let $h(u_n) = [(I-R)(z_n) + \frac{\lambda}{\tau}(\rho - N(v_n, w_n, q_n))]$. Using Algorithm 4.1 and Lemma 2.5, we obtain

$$\begin{aligned} 0 &\leq u_{n+1} \oplus u_n\\ &= (u_n - z_n + J_{M,\lambda}^{(I-R)}(h(u_n))) \oplus (u_{n-1} - z_{n-1} + J_{M,\lambda}^{(I-R)}(h(u_{n-1})))\\ &\leq u_n \oplus u_{n-1} + z_n \oplus z_{n-1} + (J_{M,\lambda}^{(I-R)}(h(u_n)) \oplus J_{M,\lambda}^{(I-R)}(h(u_{n-1}))). \end{aligned} \tag{4.4}$$

Using Definition 2.2, Lemma 3.5 and from (4.4), we have

$$\begin{aligned} &\|u_{n+1} \oplus u_n\|\\ &\leq \lambda_{C_N}\|u_n \oplus u_{n-1} + z_n \oplus z_{n-1} + (J_{M,\lambda}^{(I-R)}(h(u_n)) \oplus J_{M,\lambda}^{(I-R)}(h(u_{n-1})))\| \end{aligned}$$

$$\leq \lambda_{C_N}[\|u_n \oplus u_{n-1}\| + \|z_n \oplus z_{n-1}\| + \|J_{M,\lambda}^{(I-R)}(h(u_n)) \oplus J_{M,\lambda}^{(I-R)}(h(u_{n-1}))\|]$$
$$\leq \lambda_{C_N}[\|u_n \oplus u_{n-1}\| + \mathfrak{D}(G(u_n), G(u_{n-1})) + \|J_{M,\lambda}^{(I-R)}(h(u_n)) \oplus J_{M,\lambda}^{(I-R)}(h(u_{n-1}))\|]$$
$$\leq \lambda_{C_N}[\|u_n \oplus u_{n-1}\| + \delta_G\|u_n \oplus u_{n-1}\| + \frac{1}{\lambda\gamma_R - \beta - 1}\|h(u_n) \oplus h(u_{n-1})\|]. \quad (4.5)$$

Since R is a β-ordered compression mapping and N is a (μ_N, η_N, ξ_N)-ordered compression mappings and A, B, T, G are $\mathfrak{D}$-Lipschitz continuous with respect to the constants $\delta_A, \delta_B, \delta_T$ and δ_G, *respectively*, we have

$$\|h(u_n) \oplus h(u_{n-1})\| = \|[(I-R)(z_n) + \frac{\lambda}{\tau}(\rho - N(v_n, w_n, q_n))]$$

$$\oplus [(I-R)(z_{n-1}) + \frac{\lambda}{\tau}(\rho - N(v_{n-1}, w_{n-1}, q_{n-1}))]\|$$
$$\leq \|(I-R)(z_n) \oplus (I-R)(z_{n-1})\| + \frac{\lambda}{\tau}\|(\rho - N(v_n, w_n, q_n))$$
$$\oplus (\rho - N(v_{n-1}, w_{n-1}, q_{n-1}))\|$$
$$\leq \|z_n \oplus z_{n-1}\| + \|R(z_n) \oplus R(z_{n-1})\| + \frac{\lambda}{\tau}[\mu_N\|v_n \oplus v_{n-1}\|$$
$$+ \eta_N\|w_n \oplus w_{n-1}\| + \xi_N\|q_n \oplus q_{n-1}\|]$$
$$\leq \mathfrak{D}(G(u_n), G(u_{n-1})) + \|R(z_n) \oplus R(z_{n-1})\| + \frac{\lambda}{\tau}[\mu_N\|v_n \oplus v_{n-1}\|$$
$$+ \eta_N\|w_n \oplus w_{n-1}\| + \xi_N\|q_n \oplus q_{n-1}\|]$$
$$\leq \delta_G\|u_n \oplus u_{n-1}\| + \beta\mathfrak{D}(G(u_n), G(u_{n-1})) + \frac{\lambda}{\tau}[\mu_N\mathfrak{D}(A(u_n), A(u_{n-1}))$$
$$+ \eta_N\mathfrak{D}(B(u_n), B(u_{n-1})) + \xi_N\mathfrak{D}(T(u_n), T(u_{n-1}))]$$
$$\leq \delta_G\|u_n \oplus u_{n-1}\| + \beta\delta_G\|u_n \oplus u_{n-1}\| + \frac{\lambda}{\tau}[\mu_N\delta_A\|u_n \oplus u_{n-1}\|$$
$$+ \eta_N\delta_B\|u_n \oplus u_{n-1}\| + \xi_N\delta_T\|u_n \oplus u_{n-1}\|]$$
$$\leq [(1+\beta)\delta_G + \frac{\lambda}{\tau}[\mu_N\delta_A + \eta_N\delta_B + \xi_N\delta_T]]\|u_n \oplus u_{n-1}\|,$$

which implies that

$$\|h(u_n) \oplus h(u_{n-1})\| \leq [(1+\beta)\delta_G + \frac{\lambda}{\tau}[\mu_N\delta_A + \eta_N\delta_B + \xi_N\delta_T]]\|u_n \oplus u_{n-1}\|. \quad (4.6)$$

Using (4.5) and (4.6), we have

$$\|u_{n+1} \oplus u_n\|$$
$$\leq \lambda_{C_N}[1 + \delta_G + \frac{1}{\lambda\gamma_R - \beta - 1}((1+\beta)\delta_G + \frac{\lambda}{\tau}(\mu_N\delta_A + \eta_N\delta_B + \xi_N\delta_T))]\|u_n \oplus u_{n-1}\|.$$

By Lemma 2.9, we have

$$\|u_{n+1} - u_n\| = \|u_{n+1} \oplus u_n\| \leq \lambda_{C_N}[1 + \delta_G + \frac{1}{\lambda\gamma_R - \beta - 1}((1+\beta)\delta_G + \frac{\lambda}{\tau}(\mu_N\delta_A + \eta_N\delta_B$$
$$+ \xi_N\delta_T))]\|u_n \oplus u_{n-1}\|,$$

that is,

$$\|u_{n+1} - u_n\| \leq \Theta\|u_n - u_{n-1}\| \quad (4.7)$$

where

$$\Theta = \lambda_{C_N}[1 + \delta_G + \frac{1}{\lambda\gamma_R - \beta - 1}((1+\beta)\delta_G + \frac{\lambda}{\tau}(\mu_N\delta_A + \eta_N\delta_B + \xi_N\delta_T))].$$

By condition (4.3), we have $0 < \Theta < 1$. Thus $\{u_n\}$ is a Cauchy sequence in X and since X is a complete space, there exists $u \in X$ such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$. From (4.2) and $\mathfrak{D}$-Lipschitz continuity of A, B, T, G, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| &\leq \mathfrak{D}(A(u_{n+1}), A(u_n)) \leq \delta_A\|u_{n+1} - u_n\|,\\
\|w_{n+1} - w_n\| &\leq \mathfrak{D}(B(u_{n+1}), B(u_n)) \leq \delta_B\|u_{n+1} - u_n\|,\\
\|q_{n+1} - q_n\| &\leq \mathfrak{D}(T(u_{n+1}), T(u_n)) \leq \delta_T\|u_{n+1} - u_n\|,\\
\|z_{n+1} - z_n\| &\leq \mathfrak{D}(G(u_{n+1}), G(u_n)) \leq \delta_G\|u_{n+1} - u_n\|. \qquad (4.8)
\end{aligned}$$

It is clear from (4.8) that $\{v_n\}, \{w_n\}, \{q_n\}$ and $\{z_n\}$ are also Cauchy sequences in X and so there exist v, w, q, z in X such that $v_n \longrightarrow v, w_n \longrightarrow w, q_n \longrightarrow q$ and $z_n \longrightarrow z$ as $n \longrightarrow \infty$. By using the continuity of operators $A, B, T, G, J^{(I-R)}_{M,\lambda}$ and Algorithm 4.1, we have

$$u = u - z + J^{(I-R)}_{M,\lambda}[(I-R)z + \frac{\lambda}{\tau}(\rho - N(v, w, q))].$$

From Lemma 4.2, we conclude that (u, v, w, q, z) is a solution of problems (3.1). It remain to show that $v \in A(u), w \in B(u), q \in T(u)$ and $z \in G(u)$. In fact

$$\begin{aligned}
d(v, A(u)) &\leq \|v - v_n\| + d(v_n, A(u))\\
&\leq \|v - v_n\| + \mathfrak{D}(A(u_n), A(u))\\
&\leq \|v - v_n\| + \delta_A\|u_n - u\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Hence $v \in A(u)$. Similarly we can show that $w \in B(u), q \in T(u), z \in G(u)$. This completes the proof. □

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References

1. R. Ahmad, Q. H. Ansari and Salahuddin, A perturbed Ishikawa iterative algorithm for general mixed multivalued mildly nonlinear variational inequalities, Advances in Nonlinear Variational Inequalities, **3(2)**, 2000, 53–64.
2. R. Ahmad, J. Iqbal, S. Ahmed and S. Husain, Solving a variational inclusion problem with its corresponding resolvent equation problem involving XOR-Operation, Nonlinear Functional Analysis and Applications, **24 (3)**, 2019, 565-582.
3. H. Amann, On the number of solutions of nonlinear equations in ordered Banach space, Journal of Functional Analysis, **11**, 1972, 346–384.
4. K. Aoyama, H. Iiduka and W. Takahashi, Weak convergence of an iterative sequence for accretive operator in Banach spaces, Fixed Point Theory and Applications, 2006, Art ID 35390, 2008.
5. I. K. Argyros and Salahuddin, Solution for system of implicit ordered variational inclusions, Transactions of A. Razmadze Mathematical Institute, **173**, 2019, 9–19.
6. F. E. Browder, Nonlinear operator and nonlinear equations of evolution in Banach spaces, In Nonlinear Functional Analysis pp. 1-308, Am. Math. Soc. Rhods Island, New York, 1976.
7. F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, Journal of Mathematical Analysis and Applications, **20**, 1967, 197–228.
8. Y. H. Du, Fixed points of increasing operators in ordered Banach spaces and applications, Applicable Analysis, **38**, 1990, 1–20.
9. Y. P. Fang and N. J. Huang, H-accretive operator and resolvent operator technique for variational inclusions in Banach spaces, Applied Mathematics Letters, **17(6)**, 2004, 647–653.

10. N. J. Huang, A new completely general class of variational inclusions with noncompact valued mappings, Computer Mathematics and Applications, **35(10)**, 1998, 9–14.
11. N. J. Huang and Y. P. Fang, Generalized m-accretive mappings in Banach spaces, Journal of Sichuan University, **38(4)**, 2001, 591–592.
12. T.T. Huong, J. K. Kim and N.T.T. Thuy, Regularization for the problem of finding a solution of a system of nonlinear monotone ill-posed equations in Banach spaces, Journal Korean Mathematical Society, **55(4)**, 2018, 849-875.
13. M. F. Khan and Salahuddin, Generalized co-complementarity problems in p-uniformly smooth Banach spaces, JIPAM, J. Inequality Pure and Applied Mathematics, **7 (2)**, 2006, 1-11, Article ID 66.
14. J. K. Kim, Sensitivity analysis for general nonlinear nonvex set-valued variational inequalities in Banach spaces, Journal of Computational Analysis and Applications, **22(2)**, 2017, 327-335,
15. J. K. Kim and M.I. Bhat, Approximation solvability for a system of nonlinear implicit variational inclusions with H-monotone operators, Demonstratio Mathematica, **51**, 2018, 241-254.
16. J. K. Kim, A. H. Dar and Salahuddin, Existence theorems for the generalized relaxed pseudomonotone variational inequalities, Nonlinear Functional Analysis and Applications, **25(1)**, 2020, 25-34.
17. B. S. Lee and Salahuddin, Fuzzy general nonlinear ordered random variational inequalities in ordered Banach spaces, East Asian Mathematics Journal, **32(4)**, 2016, 685–700.
18. H. G. Li, Approximation solution for general nonlinear ordered variational inequalities and ordered equations in ordered Banach space, Nonlinear Analysis Forum, **13(2)**, 2008, 205–214.
19. H. G. Li, Approximation solution for a new class of general nonlinear ordered variational inequalities and ordered equations in ordered Banach space, Nonlinear Analysis Forum, **14**, 2009, 1–9.
20. H. G. Li, Nonlinear inclusions problems for ordered RME set valued mappings in ordered Hilbert space, Nonlinear Functional Analysis and Applications,**16 (1)**, 2011, 1–8.
21. H. G. Li, A nonlinear inclusion problem involving (α, λ)-NODM set valued mappings in ordered Hilbert space, Applied Mathematics Letters, **25**, 2012, 1384–1388.
22. H. G. Li, L. P. Li and M. M. Jin, A class of nonlinear mixed ordered inclusion problems for ordered (α_A, λ)-ANODM set-valued mappings with strong comparion mapping, Fixed Point Theory and Applications 2014:79 doi.org/10.1186/1687-1812-2014-79.
23. H. G. Li, X. B. Pan, Z. Y. Deng and C. Y. Wang, Solving GNOVI frameworks involving (γ_G, λ)-weak-GRD set valued mappings in positive Hilbert spaces, Fixed Point Theory and Applications, **2014**, 2014, 146.
24. H. G. Li, L. P. Li, J. M. Zheng and M. Jin, Sensitivity analysis for generalized set valued parametric ordered variational inclusion with (α, λ)-NODSM mappings in ordered Banach spaces, Fixed Point Theory and Applications, **2014**, 2014, 122.
25. J. L. Lions and G. Stampacchia, Variational inequalities, Communication in Pure and Applied Mathematics, **20**, 1967, 493–517.
26. Jr. S. B. Nadler, Multi valued contraction mappings, Pacific Journal of Mathematics, **30**, 1969, 475–488.
27. M. Sarfaraz, M. K. Ahmad, A.Kilicman, Approximation solution for system of generalized ordered variational inclusions with $\oplus$ operator in ordered Banach space, Journal of Inequality and Applications 2017:81 doi.org/10.1186/s13660-017-1351-x.
28. S. Saejung, K. W. Chan and P. Yotkaew, Another weak convergence theorem for accretive mappings in Banach spaces, Fixed Point Theory Applications, 2011, Art ID 26 (2011).
29. H. H. Schaefer, Banach lattices and positive operators, Springer, 1974.
30. R. U. Verma, Approximation solvability of a class of nonlinear set valued variational inclusions involving (A, η)-monotone mappings, Journal of Mathematical Analysis and Applications, **377**, 2008, 969–975.

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CHARACTERIZATIONS OF THE BASIC CONSTRAINT QUALIFICATION AND ITS APPLICATIONS

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ABSTRACT. In convex programming, the basic constraint qualification is a necessary and sufficient constraint qualification for the optimality condition. In this paper, we give characterizations of the basic constraint qualification at each feasible solution. By using the result, we give an alternative method for checking up the basic constraint qualification at every feasible point without subdifferentials and normal cones.

KEYWORDS:convex programming, constraint qualification, alternative theorem
AMS Subject Classification: :90C25, 90C30, 90C46

1. Introduction

In this paper, we consider the following convex programming problem:

$$(\mathrm{P}) \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & g_i(x) \le 0 \text{ for each } i \in I, \end{cases}$$

where X is a locally convex Hausdorff topological vector space, I is an arbitrary index set, f is an extended real-valued convex function on X, and g_i is an extended real-valued convex function on X for each $i \in I$. Constraint qualifications are essential in mathematical programming, see [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15] and references therein. In particular, they ensure the existence of Lagrange multipliers or zero duality gap between (P) and its Lagrangian dual problem. These results have played a critical role in the development of convex programming. Additionally, constraint qualifications for the following optimality condition have been studied by many researchers:

$$\exists \lambda \in \mathbb{R}_+^{(I)} \text{ such that } 0 \in \partial f(x_0) + \sum_{i \in I} \lambda_i \partial g_i(x_0),$$

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where $\mathbb{R}_+^{(I)} = \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \geq 0, \{i \in I \mid \lambda_i \neq 0\} : \text{finite}\}$. One of the best-known constraint qualification for the optimality condition is the Slater constraint qualification. It is easy to check whether the Slater constraint qualification holds or not. However, the Slater constraint qualification is often not satisfied for many problems arising in applications. The lack of a constraint qualification can cause both theoretical and numerical difficulties in applications. Recently, it was shown that the basic constraint qualification (BCQ), which was introduced in [3], is a necessary and sufficient constraint qualification for the optimality condition by Li, Ng and Pong, see [8]. To check the BCQ at a feasible point, however, we have to calculate the subdifferential of all g_i and the normal cone of the feasible set at the point. In this point of view, checking up the BCQ at every feasible points is not so easy.

The purpose of this paper is to give characterizations of the basic constraint qualification at each feasible point, and to give an alternative method to checking up the BCQ at every feasible points. The paper is organized as follows. In section 2, we describe our notation and present preliminary results. In section 3, we give characterizations of the basic constraint qualification at each feasible point, and we give an alternative method for checking up the BCQ at every feasible points. Also we remark that alternative results which are generalizations of Farkas' Lemma are given. In section 4, we explain the usefulness of our result obtained in this paper.

2. Preliminaries

In this section, we describe our notation and present preliminary results. Let X be a locally convex Hausdorff topological vector space over the real-field $\mathbb{R}$, let X^* be the continuous dual space of X, and let $\langle v, x\rangle$ denote the value of a functional $v \in X^*$ at $x \in X$. For a subset A^* of X^*, we denote the weak*-closure, the conical hull and the convex hull of A by $\text{cl}A^*$, $\text{cone}A^*$ and $\text{co}A^*$, respectively. Let f be a function from X to $\mathbb{R} \cup \{+\infty\}$. The effective domain of f, denoted by $\text{dom}f$ is defined by

$$\text{dom}f = \{x \in X \mid f(x) < +\infty\},$$

and the epigraph of f, denoted by $\text{epi}f$ is defined by

$$\text{epi}f = \{(x, r) \in X \times \mathbb{R} \mid x \in \text{dom}f, f(x) \leq r\}.$$

The function f is said to be convex, proper and lower semicontinuous (lsc) if $\text{epi}f$ is a convex set, nonempty set and closed set, respectively. When f is a proper lsc convex function, the conjugate function of f, $f^* : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{\langle v, x\rangle - f(x) \mid x \in X\}.$$

The subdifferential of f at $x \in X$, denoted by $\partial f(x)$, is defined by

$$\partial f(x) = \{v \in X^* \mid f(x) + \langle v, y - x\rangle \leq f(y), \forall y \in X\}.$$

For nonempty convex set $A \subseteq X$, the indicator function $\delta_A : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ +\infty & x \notin A. \end{cases}$$

For any $x \in A$, the normal cone of A at x, denoted by $N_A(x)$, is defined by

$$N_A(x) = \{v \in X^* \mid \langle v, y - x\rangle \leq 0, \forall y \in A\}.$$

For proper lsc convex functions g, $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, the infimal convolution of g with h, denoted by $g \oplus h$, is defined by

$$(g \oplus h)(x) = \inf_{x_1+x_2=x} \{g(x_1) + h(x_2)\}.$$

It is well known that if $\mathrm{dom} g \cap \mathrm{dom} h \neq \emptyset$, then

$$(g \oplus h)^* = g^* + h^* \text{ and } (g + h)^* = \mathrm{cl}(g^* \oplus h^*). \tag{2.1}$$

The closure operation in the second equation is superfluous if one of g and h is continuous at some $a \in \mathrm{dom} g \cap \mathrm{dom} h$. Then,

$$\mathrm{epi}(g + h)^* = \mathrm{epi} g^* + \mathrm{epi} h^* \text{ and} \tag{2.2}$$

$$\partial(g + h)(x) = \partial g(x) + \partial h(x), \text{ for each } x \in \mathrm{dom} g \cap \mathrm{dom} h, \tag{2.3}$$

see [2].

We denote by $\mathbb{R}_+^{(I)}$ the space of generalized finite sequences $(\lambda_i)_{i\in I}$ such that $\lambda_i \in \mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geq 0\}$ for each $i \in I$, and with only finitely many λ_i different from zero. Let g_i be an extended real-valued proper lsc convex function on X for each $i \in I$, and let $\lambda \in \mathbb{R}_+^{(I)}$. Assume that each g_i is continuous at least at one point of $\bigcap_{i\in I} \mathrm{dom} g_i$, and $0 \times \infty = 0$. Then

$$\mathrm{epi}\left(\sum_{i\in I} \lambda_i g_i\right)^* = \begin{cases} \sum_{i\in I} \lambda_i \mathrm{epi} g_i^* & \sum_{i\in I} \lambda_i > 0, \\ \{0\} \times \mathbb{R}_+ & \sum_{i\in I} \lambda_i = 0, \end{cases} \tag{2.4}$$

$$\partial \sum_{i\in I} \lambda_i g_i(x) = \sum_{i\in I} \lambda_i \partial g_i(x), \forall x \in \bigcap_{i\in I} \mathrm{dom} g_i. \tag{2.5}$$

Definition 2.1. Let I be an arbitrary index set, g_i an extended real-valued proper lsc convex function on X for each $i \in I$, $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and $\bar{x} \in S$. The family $\{g_i \mid i \in I\}$ is said to satisfy the basic constraint qualification (BCQ) at $\bar{x}$ if

$$N_S(\bar{x}) = \mathrm{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x}),$$

where $I(\bar{x}) = \{i \in I \mid g_i(\bar{x}) = 0\}$.

We introduce the following previous result of the BCQ.

Theorem 2.1. [8] *Let I be an arbitrary index set, g_i an extended real-valued proper lsc convex function on X for each $i \in I$, $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and $\bar{x} \in S$. Then the following statements are equivalent:*

(i) *the family $\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$,*
(ii) *for each real-valued convex function f, $\bar{x}$ is a minimizer of f in S if and only if there exist a finite subset $J \subseteq I(\bar{x})$ and $(\lambda_i)_{i\in J} \in \mathbb{R}_+^J$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i\in J} \lambda_i \partial g_i(\bar{x}).$$

By Theorem 2.1, the BCQ is said to be a necessary and sufficient constraint qualification for the optimality condition.

The following results, a set containment characterization and Fenchel duality, are used in our main theorem.

Theorem 2.2. [6] *Let I be an arbitrary index set, g_i an extended real-valued proper lsc convex function on X for each $i \in I$, $v \in X^*$, and $\alpha \in \mathbb{R}$. Assume that $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$ is nonempty.*

Then the following statements are equivalent:

(i) $\{x \in X \mid g_i(x) \leq 0, \forall i \in I\} \subseteq \{x \in X \mid \langle v, x\rangle \leq \alpha\}$,

(ii) $(v, \alpha) \in \mathrm{clconeco} \bigcup_{i \in I} \mathrm{epi} g_i^*$.

Theorem 2.3. [7] *Let f and g be extended real-valued proper lsc convex functions on X such that $\mathrm{dom} f \cap \mathrm{dom} g \neq \emptyset$. If $\mathrm{epi} f^* + \mathrm{epi} g^*$ is w^*-closed, then*

$$\inf_{x \in X} \{f(x) + g(x)\} = \max_{v \in X^*} \{-f^*(-v) - g^*(v)\}.$$

3. Main Result

Throughout this section, we consider the following convex inequality system:

$$\{g_i(x) \leq 0 \mid i \in I\}$$

where I is an arbitrary index set, and g_i an extended real-valued proper lsc convex function on X for each $i \in I$. Let $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and assume that each g_i is continuous at least at one point of $\bigcap_{i \in I} \mathrm{dom} g_i$. We show the following theorem as our main result.

Theorem 3.1. *Let $\bar{x} \in S = \{x \in X \mid g_i(x) \leq 0 \ \forall i \in I\}$. Then the following statements are equivalent:*

(i) *the family $\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$,*

(ii) *for each real-valued convex function f, $\bar{x}$ is a minimizer of f in S if and only if there exist a finite subset $J \subseteq I(\bar{x})$ and $(\lambda_i)_{i \in J} \in \mathbb{R}^J_+$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i \in J} \lambda_i \partial g_i(\bar{x}),$$

(iii) *the following inclusion holds:*

$$\left\{ v \,\middle|\, (v, \langle v, \bar{x}\rangle) \in \mathrm{clconeco} \bigcup_{i \in I} \mathrm{epi} g_i^* \right\} \subseteq \left\{ v \,\middle|\, (v, \langle v, \bar{x}\rangle) \in \mathrm{coneco} \bigcup_{i \in I} \mathrm{epi} g_i^* \right\},$$

(iv) *for each extended real-valued proper lsc convex function f with $\mathrm{epi} f^* + \mathrm{epi} \delta_S^*$ is w^*-closed, exactly one of the following two statements is true:*

(a) *there exists $x \in X$ such that* $\begin{cases} f(x) < f(\bar{x}), \\ g_i(x) \leq 0, \ \text{for each } i \in I, \end{cases}$

(b) *there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that*

$$\begin{cases} f(x) + \sum_{i \in I} \lambda_i g_i(x) \geq f(\bar{x}) \ \text{for each } x \in X, \\ \lambda_i g_i(\bar{x}) = 0 \ \text{for each } i \in I, \end{cases}$$

(v) *for each $v \in X^*$, exactly one of the following two statements is true:*

(a) *there exists $x \in X$ such that* $\begin{cases} \langle v, x\rangle < \langle v, \bar{x}\rangle, \\ g_i(x) \leq 0 \ \text{for each } i \in I, \end{cases}$

(b) *there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that*

$$\begin{cases} \langle v, x\rangle + \sum_{i \in I} \lambda_i g_i(x) \geq \langle v, \bar{x}\rangle, \ \text{for each } x \in X, \\ \lambda_i g_i(\bar{x}) = 0 \ \text{for each } i \in I. \end{cases}$$

Proof. By Theorem 2.1, (i) and (ii) are equivalent.

We show that (ii) implies (iii). Assume that the statement (ii) holds, and let $v \in X^*$ satisfying $(v, \langle v, \bar{x}\rangle) \in \mathrm{clconeco}\bigcup_{i\in I} \mathrm{epi} g_i^*$. Then, by Theorem 2.2,

$$\{x \in X \mid g_i(x) \leq 0, \forall i \in I\} \subseteq \{x \in X \mid \langle v, x\rangle \leq \langle v, \bar{x}\rangle\}.$$

This shows that $\bar{x}$ is a global minimizer of $-v$ in S. By the statement (ii), there exist a finite subset $J \subseteq I(\bar{x})$ and $(\lambda_i)_{i\in J} \in \mathbb{R}_+^J$, such that

$$0 \in \partial(-v)(\bar{x}) + \sum_{i\in J} \lambda_i \partial g_i(\bar{x}),$$

that is, $v \in \sum_{i\in J} \lambda_i \partial g_i(\bar{x})$. For each $i \in J \subseteq I(\bar{x})$, we show that $w \in \partial g_i(\bar{x})$ if and only if $(w, \langle w, \bar{x}\rangle) \in \mathrm{epi} g_i^*$. Actually,

$$\begin{array}{rcl} w \in \partial g_i(\bar{x}) & \Longleftrightarrow & \forall y \in X, g_i(y) \geq g_i(\bar{x}) + \langle w, y - \bar{x}\rangle \\ & \Longleftrightarrow & \forall y \in X, g_i(y) \geq \langle w, y - \bar{x}\rangle \\ & \Longleftrightarrow & \forall y \in X, \langle w, \bar{x}\rangle \geq \langle w, y\rangle - g_i(y) \\ & \Longleftrightarrow & \langle w, \bar{x}\rangle \geq g_i^*(w) \\ & \Longleftrightarrow & (w, \langle w, \bar{x}\rangle) \in \mathrm{epi} g_i^*. \end{array}$$

Hence,

$$(v, \langle v, \bar{x}\rangle) \in \sum_{i\in J} \lambda_i \mathrm{epi} g_i^* \subseteq \mathrm{coneco} \bigcup_{i\in I} \mathrm{epi} g_i^*.$$

Next, we prove that (iii) implies (iv). Assume that (iii) holds, and let f be an extended real-valued proper lsc convex function with $\mathrm{epi} f^* + \mathrm{epi}\delta_S^*$ is w^*-closed. It is clear that (a) and (b) in (iv) do not hold simultaneously. If (a) does not hold, then for each $x \in S$, $f(x) \geq f(\bar{x})$, that is, $\bar{x}$ is a global minimizer of f in S. By Theorem 2.3,

$$f(\bar{x}) = \min_{x\in S} f(x) = \min_{x\in X}\{f(x) + \delta_S(x)\} = \max_{v\in X^*}\{-f^*(-v) - \delta_S^*(v)\},$$

that is, there exists $v_0 \in X^*$ such that $f(\bar{x}) = -f^*(-v_0) - \delta_S^*(v_0)$. Hence,

$$\begin{array}{rcl} f(\bar{x}) & = & -f^*(-v_0) - \delta_S^*(v_0) \\ & = & -f^*(-v_0) - \sup_{x\in X}\{\langle v_0, x\rangle - \delta_S(x)\} \\ & = & -f^*(-v_0) + \inf_{x\in S} \langle -v_0, x\rangle \\ & \leq & -f^*(-v_0) + \langle -v_0, \bar{x}\rangle \\ & \leq & -(\langle -v_0, \bar{x}\rangle - f(\bar{x})) + \langle -v_0, \bar{x}\rangle \\ & = & f(\bar{x}). \end{array}$$

This shows that $f(\bar{x}) + f^*(-v_0) = \langle -v_0, \bar{x}\rangle$, that is, $-v_0 \in \partial f(\bar{x})$. Additionally, we can see that $\inf_{x\in S} \langle -v_0, x\rangle = \langle -v_0, \bar{x}\rangle$, hence we have

$$\{x \in X \mid g_i(x) \leq 0, \forall i \in I\} \subseteq \{x \in X \mid \langle v_0, x\rangle \leq \langle v_0, \bar{x}\rangle\}.$$

By Theorem 2.2 and the statement (iii),

$$(v_0, \langle v_0, \bar{x}\rangle) \in \mathrm{coneco} \bigcup_{i\in I} \mathrm{epi} g_i^*.$$

Hence, there exist $\lambda \in \mathbb{R}_+^{(I)}$ and $(a_i, b_i) \in \mathrm{epi} g_i^*$ for each $i \in I$ such that

$$(v_0, \langle v_0, \bar{x}\rangle) = \sum_{i\in I} \lambda_i (a_i, b_i).$$

For each $i \in I$ and $x \in X$, $\langle a_i, x\rangle - g_i(x) \leq b_i$. Therefore,

$$\langle v_0, x\rangle - \sum_{i\in I} \lambda_i g_i(x) \leq \langle v_0, \bar{x}\rangle .$$

Since $-v_0 \in \partial f(\bar{x})$, for each $x \in X$,

$$f(\bar{x}) + \langle -v_0, x - \bar{x}\rangle \leq f(x).$$

This shows that

$$f(x) + \sum_{i\in I} \lambda_i g_i(x) \geq f(\bar{x}).$$

Since $\bar{x} \in S$, we can check easily that $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$, hence (b) of (iv) holds.

It is clear that (iv) implies (v).

Finally, we prove that (v) implies (i). Assume that (v) holds. At first, we show the following inclusion:

$$N_S(\bar{x}) \supseteq \text{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x})$$

Actually, let $i \in I(\bar{x})$ and $v \in \partial g_i(\bar{x})$, then for each $x \in S$,

$$\langle v, x - \bar{x}\rangle = g_i(\bar{x}) + \langle v, x - \bar{x}\rangle \leq g_i(x) \leq 0.$$

This shows that $v \in N_S(\bar{x})$, that is, $\partial g_i(\bar{x}) \subseteq N_S(\bar{x})$. Since $N_S(\bar{x})$ is a convex cone, the above inclusion holds. Next, we show the following inclusion:

$$N_S(\bar{x}) \subseteq \text{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x}).$$

Let $v \in N_S(\bar{x})$, then, $\bar{x}$ is a global minimizer of $-v$ in S. Hence, the statement (a) in (v) for $-v$ does not hold. By the statement (b) in (v), there exists $\lambda \in \mathbb{R}^{(I)}_+$ such that for each $x \in X$,

$$\langle -v, x\rangle + \sum_{i\in I} \lambda_i g_i(x) \geq \langle -v, \bar{x}\rangle ,$$

and $\lambda_i g_i(\bar{x}) = 0$ for each $i \in I$. This shows that $(\sum_{i\in I} \lambda_i g_i)^*(v) \leq \langle v, \bar{x}\rangle$. Since $\sum_{i\in I} \lambda_i g_i(\bar{x}) + (\sum_{i\in I} \lambda_i g_i)^*(v) \leq \langle v, \bar{x}\rangle$, we can see that $v \in \partial(\sum_{i\in I} \lambda_i g_i)(\bar{x})$. By the equation (2.5) and the complementarity condition,

$$v \in \partial \left(\sum_{i\in I} \lambda_i g_i\right)(\bar{x}) = \sum_{i\in I} \lambda_i \partial g_i(\bar{x}) \subseteq \text{coneco} \bigcup_{i\in I(\bar{x})} \partial g_i(\bar{x}).$$

This shows that (i) holds. This completes the proof. □

Remark 3.1. By (iii) in Theorem 3.1, an alternative method for checking up the BCQ at every feasible points is given. The method requires a convex cone depents on constraint functions and feasible points, but does not require any subdifferentials and any normal cones, see examples in Section 4.

Remark 3.2. By using Theorem 3.1, we can show Farkas' Lemma. Let $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Put $X = \mathbb{R}^n$, $I = \{1, \ldots, m\}$, $g_i = \langle a_i, \cdot\rangle$ where $a_i = (a_{i1}, \ldots, a_{in})^T$, $i \in I$, and $\bar{x} = 0 \in S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0 \ \forall i \in I\}$. Then, we can see that the statement (iii) of Theorem 3.1 always holds, that is,

$$\left\{ v \in \mathbb{R}^n \,\middle|\, (v, 0) \in \text{clconeco} \bigcup_{i=1}^{m} \text{epi} g_i^* \right\} \subseteq \left\{ v \in \mathbb{R}^n \,\middle|\, (v, 0) \in \text{coneco} \bigcup_{i=1}^{m} \text{epi} g_i^* \right\}.$$

The proof is given as follows: at first, we can see that

$$\text{coneco}\bigcup_{i=1}^{m}\text{epi}g_i^* = \left\{\sum_{i=1}^{m}\lambda_i(a_i,\beta_i)\,\middle|\,\lambda_i\geq 0, \sum_{i=1}^{m}\lambda_i>0, \beta_i\geq 0\right\}\cup\{(0,0)\}$$

and

$$\text{clconeco}\bigcup_{i=1}^{m}\text{epi}g_i^* = \left\{\sum_{i=1}^{m}\lambda_i(a_i,\beta_i)\,\middle|\,\lambda_i\geq 0, \beta_i\geq 0\right\}$$

hold. When $(v,0)=\sum_{i=1}^{m}\lambda_i(a_i,\beta_i)\in \text{clconeco}\bigcup_{i=1}^{m}\text{epi}g_i^*$ for some non-negative λ_i and β_i, it is clear that $\lambda_i\beta_i=0$ for all $i\in I$. If all λ_i are 0 then $v=0$, otherwise $\sum_{i=1}^{m}\lambda_i>0$. Therefore $(v,0)\in\text{coneco}\bigcup_{i=1}^{m}\text{epi}g_i^*$ holds. From Theorem 3.1, the statement (v) holds. When $v=-b$, exactly one of the following two statements is true:

(a) there exists $x\in\mathbb{R}^n$ such that $\langle b,x\rangle>0$ and $Ax\leq 0$,
(b) there exists $y=(y_1,\dots,y_m)\in\mathbb{R}_+^m$ such that $A^Ty=b$.

This is a variation of Farkas' Lemma. From this observation, each (iv) and (v) of Theorem 3.1 can be considered as a kind of alternative results.

4. Examples and Applications

In this section, we explain the usefulness of our results by some examples and applications. At first, we give three examples and we check up the BCQ at every feasible by using the given alternative method.

Example 4.1. Let $g_1:\mathbb{R}\longrightarrow\mathbb{R}$ be a function as follows:

$$g_1(x)=\begin{cases}\frac{1}{2}x^2-x & x\in(-\infty,0],\\ 0 & x\in(0,1),\\ \frac{1}{2}(x-1)^2 & x\in[1,+\infty).\end{cases}$$

Then $S=[0,1]$, and we can calculate the Fenchel conjugate of g_1 as follows:

$$g_1^*(v)=\begin{cases}\frac{1}{2}(v+1)^2 & v\in(-\infty,-1],\\ 0 & v\in(-1,0),\\ \frac{1}{2}v^2+v & v\in[0,+\infty).\end{cases}$$

Furthermore,

$$\{v\in\mathbb{R}\mid (v,vx)\in\text{clconeco epi}g_1^*\}=\begin{cases}(-\infty,0] & x=0,\\ \{0\} & x\in(0,1),\\ [0,+\infty) & x=1,\end{cases}$$

and

$$\{v\in\mathbb{R}\mid (v,vx)\in\text{coneco epi}g_1^*\}=\begin{cases}(-\infty,0] & x=0,\\ \{0\} & x\in(0,1].\end{cases}$$

By Theorem 3.1, the BCQ holds at every point of $[0,1)$, however the BCQ does not hold at 1. By Figure 4.1, it is easy to check whether the BCQ holds or not at every feasible point.

Example 4.2. Let $g_2:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a function as follows:

$$g_2(x_1,x_2)=g(x_1)+g(x_2),$$

where

$$g(t)=\begin{cases}\frac{1}{2}(t+1)^2 & t\in(-\infty,-1],\\ 0 & t\in(-1,1),\\ \frac{1}{2}(t-1)^2 & t\in[1,+\infty).\end{cases}$$

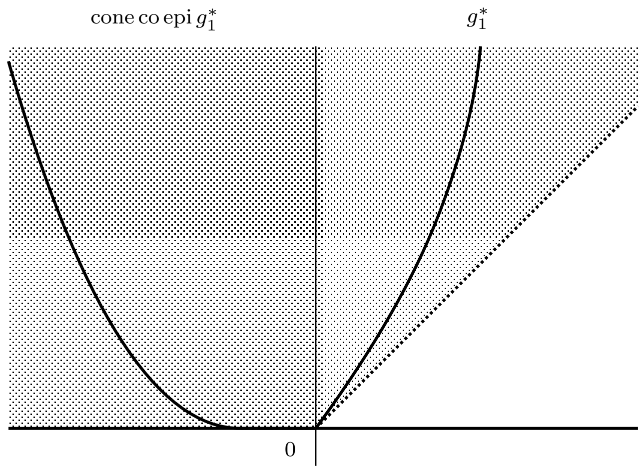

FIGURE 1.

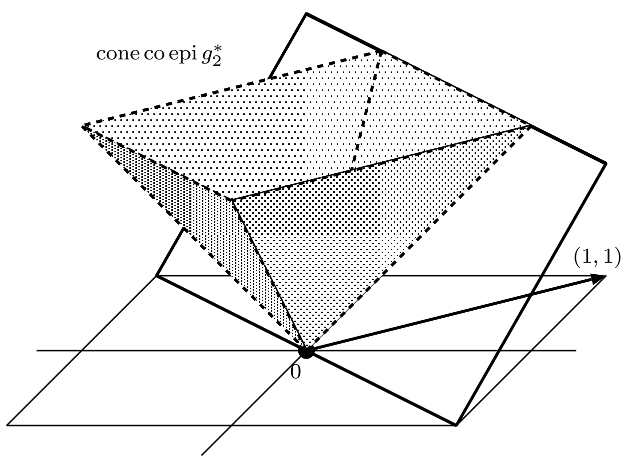

FIGURE 2.

Then, $S = [-1,1]^2$, $g_2^*(v_1, v_2) = \frac{1}{2}v_1^2 + |v_1| + \frac{1}{2}v_2^2 + |v_2|$,

$$\text{coneco epi}g_2^* = \{(v_1, v_2, r) \in \mathbb{R}^3 \mid |v_1| + |v_2| < r\} \cup \{(0,0,0)\},$$

and

$$\text{clconeco epi}g_2^* = \{(v_1, v_2, r) \in \mathbb{R}^3 \mid |v_1| + |v_2| \le r\}.$$

Hence, the BCQ holds at every point in the interior of S, however the BCQ does not hold at every point in the boundary of S by using Theorem 3.1. See Figure 4.2.

Example 4.3. Let $g_3 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function as follows:

$$g_3(x) = \frac{1}{8}(\langle v_0, x\rangle - |\langle v_0, x\rangle|)^2 + \frac{1}{2}(\langle w_0, x\rangle + |\langle w_0, x\rangle|),$$

where v_0, $w_0 \in \mathbb{R}^n \setminus \{0\}$ and $\langle v_0, w_0 \rangle = 0$. Then,

$$S = \{x \in \mathbb{R}^n \mid g_3(x) \le 0\} = \{sv_0 + tw_0 \mid s \ge 0, t \le 0\}$$

and

$$g_3^*(v) = \frac{1}{2}(\langle v_0, v \rangle)^2 + \delta_{\{sv_0 + tw_0 \mid s \le 0, t \in [0,1]\}}(v).$$

Hence,

$$\begin{aligned} \text{coneco epi} g_3^* &= \{(sv_0 + tw_0, r) \in \mathbb{R}^{n+1} \mid r > 0, s \in (-\infty, 0], t \in [0, +\infty)\} \\ &\quad \cup \{(tw_0, r) \in \mathbb{R}^{n+1} \mid r \ge 0, t \in [0, +\infty)\}, \end{aligned}$$

and

$$\text{clconeco epi} g_3^* = \{(sv_0 + tw_0, r) \in \mathbb{R}^{n+1} \mid r \ge 0, s \in (-\infty, 0], t \in [0, +\infty)\}.$$

Therefore, the BCQ holds at every point in the union of the interior of S and $\{\lambda v_0 \mid \lambda > 0\}$, however the BCQ does not hold at every point in $\{tw_0 \mid t \in (-\infty, 0]\}$, by using Theorem 3.1.

When $n \le 2$, as we saw in Example 4.1 and Example 4.2, it is possible to check up the BCQ on the feasible solution S by illustrating coneco$\bigcup_{i \in I}$ epig_i^*. When $n \ge 3$, it is not easy to illustrate coneco$\bigcup_{i \in I}$ epig_i^* in general, but Example 4.3 is a special case in which the BCQ can be checked up without illustrating coneco$\bigcup_{i \in I}$ epig_i^*. When every g_i are sublinear, the BCQ can also be checked up without illustrating coneco$\bigcup_{i \in I}$ epig_i^*, by using just $\partial g_i(0)$, see the following result:

Theorem 4.1. *Let I be an index set, g_i be a real-valued sublinear function on X for each $i \in I$, $S = \{x \in X \mid g_i(x) \le 0, \forall i \in I\}$, and $\bar{x} \in S$. Then the following statements are equivalent:*

(i) *$\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$,*
(ii) *the following inclusion holds:*

$$\{v \mid (v, \langle v, \bar{x} \rangle) \in \text{clconeco} \bigcup_{i \in I} \text{epi}\delta_{\partial g_i(0)}\} \subseteq \{v \mid (v, \langle v, \bar{x} \rangle) \in \text{coneco} \bigcup_{i \in I} \text{epi}\delta_{\partial g_i(0)}\}.$$

Proof. Since g_i is sublinear, we have

$$g_i^* = \delta_{\partial g_i(0)}.$$

By Theorem 3.1, (i) and (ii) are equivalent. □

Example 4.4. Let $g_4 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function as follows:

$$g_4(x) = \|x\| + \langle v_0, x \rangle,$$

where $v_0 \in \mathbb{R}^n$ with $\|v_0\| = 1$ and $n \ge 2$. Then, g_4 is a sublinear function, and $S = \{x \in \mathbb{R}^n \mid g_4(x) \le 0\} = \{tv_0 \mid t \le 0\}$ and the interior of S is empty. We can calculate the subdifferential of g_4 at 0 as follows:

$$\partial g_4(0) = \{v \in \mathbb{R}^n \mid \|v - v_0\| \le 1\}.$$

Additionally, for each $\bar{x} \in S$,

$$\{v \in \mathbb{R}^n \mid (v, \langle v, \bar{x} \rangle) \in \text{coneco epi}\delta_{\partial g_4(0)}\} = \{(0, 0)\}$$

and

$$\{v \in \mathbb{R}^n \mid (v, \langle v, \bar{x} \rangle) \in \text{clconeco epi}\delta_{\partial g_4(0)}\} = \{tv_0 \mid t \le 0\}.$$

Therefore, by Theorem 4.1, the BCQ does not hold at every points in S.

Furthermore, we give the following sufficient condition of the BCQ for a sublinear inequality system:

Theorem 4.2. *Let I be an index set, g_i be a real-valued sublinear function on X for each $i \in I$, $S = \{x \in X \mid g_i(x) \leq 0, \forall i \in I\}$, and assume that S is nonempty. If* $\text{coneco} \bigcup_{i \in I} \partial g_i(0)$ *is w^*-closed, then $\{g_i \mid i \in I\}$ satisfy the BCQ at every points in S.*

Proof. Let $\bar{x} \in S$ and let $v \in X^*$ with $(v, \langle v, \bar{x} \rangle) \in \text{clconeco} \bigcup_{i \in I} \text{epi}\delta_{\partial g_i(0)}$. We may assume that $v \neq 0$, because $(0, 0) \in \text{coneco} \bigcup_{i \in I} \text{epi}\delta_{\partial g_i(0)}$. Then there exists a net $\{(v_\alpha, \beta_\alpha) \mid \alpha \in D\} \subseteq \text{coneco} \bigcup_{i \in I} \text{epi}\delta_{\partial g_i(0)}$ such that

$$(v_\alpha, \beta_\alpha) \longrightarrow (v, \langle v, \bar{x} \rangle).$$

Additionally, for each $\alpha \in D$, there exists $\lambda^\alpha \in \mathbb{R}_+^{(I)}$ and $(x_i^\alpha, \gamma_i^\alpha) \in \text{epi}\delta_{\partial g_i(0)}$ for each $i \in I$ such that

$$(v_\alpha, \beta_\alpha) = \sum_{i \in I} \lambda_i^\alpha (x_i^\alpha, \gamma_i^\alpha).$$

Since $\text{epi}\delta_{\partial g_i(0)} = \partial g_i(0) \times [0, +\infty)$ for each $i \in I$, $v_\alpha \in \text{coneco} \bigcup_{i \in I} \partial g_i(0)$ and $\beta_\alpha \in [0, +\infty)$. This shows that $v \in \text{clconeco} \bigcup_{i \in I} \partial g_i(0)$ and $\langle v, \bar{x} \rangle \in [0, +\infty)$. By the assumption, $v \in \text{coneco} \bigcup_{i \in I} \partial g_i(0)$. Hence there exist $\lambda \in \mathbb{R}_+^{(I)}$ and $v_i \in \partial g_i(0)$ for each $i \in I$ such that

$$v = \sum_{i \in I} \lambda_i v_i.$$

For each $i \in I$,

$$\delta_{\partial g_i(0)}(v_i) = 0 \leq \left\langle \frac{v}{\sum_{i \in I} \lambda_i}, \bar{x} \right\rangle,$$

that is

$$\left(v_i, \left\langle \frac{v}{\sum_{i \in I} \lambda_i}, \bar{x} \right\rangle \right) \in \text{epi}\delta_{\partial g_i(0)}.$$

Therefore

$$(v, \langle v, \bar{x} \rangle) \in \text{coneco} \bigcup_{i \in I} \text{epi}\delta_{\partial g_i(0)}.$$

By Theorem 4.2, $\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$. This completes the proof. □

Example 4.5. Let $g_5 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function as follows:

$$g_5(x) = |\langle v_0, x \rangle|,$$

where $v_0 \in \mathbb{R}^n \setminus \{0\}$. Then, $S = \{x \in \mathbb{R}^n \mid \langle v_0, x \rangle = 0\}$, $\partial g_5(0) = \{tv_0 \mid t \in [-1, 1]\}$, and

$$\text{coneco}\, \partial g_5(0) = \{tv_0 \mid t \in \mathbb{R}\}.$$

Since $\text{coneco}\, \partial g_5(0)$ is closed, the BCQ holds at every points in S by Theorem 4.2.

5. Conclusion

In this paper, we have studied the basic constraint qualification as a sufficient condition for the optimality condition. In Theorem 3.1, we have given equivalent conditions of the BCQ at each feasible solution. Especially, we have given an alternative method for checking up the BCQ at every feasible points without subdifferentials and normal cones at feasible solutions, although the BCQ was defined by using the subdifferentials and the normal cones. We have explained the usefulness of the method to check up the BCQ by using examples in Section 4, and we have applied the main theorem for a sublinear inequality system in Theorem 4.1 and Theorem 4.2.

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References

1. R. I. Boţ, Conjugate Duality in Convex Optimization, Lecture Notes in Economics and Mathematical Systems, Vol. 637, Springer-Verlag Berlin, Heidelberg, 2010
2. M. A. Goberna, V. Jeyakumar and M. A. López, Necessary and sufficient constraint qualifications for solvability of systems of infinite convex inequalities, Nonlinear Anal. 68, 2008, 1184–1194.
3. J. Hiriart-Urruty and C. Lemarechal, Convex Analysis and Minimization Algorithms I., Vol. 305, Springer-Verlag Berlin, Heidelberg, 1993
4. V. Jeyakumar, Constraint qualifications characterizing Lagrangian duality in convex optimization, J. Optim. Theory Appl. 136, 2008, 31–41.
5. V. Jeyakumar, N. Dinh and G. M. Lee, A new closed cone constraint qualification for convex optimization. Research Report AMR 04/8, Department of Applied Mathematics, University of New South Wales, 2004
6. V. Jeyakumar, A. M. Rubinov, B. M. Glover, and Y. Ishizuka, Inequality systems and global optimization, J. Math. Anal. Appl. 202, 1996, 900–919.
7. V. Jeyakumar, A. M. Rubinov and Z. Y. Wu, Generalized Fenchel's conjugation formulas and duality for abstract convex functions, J. Optim. Theory Appl. 132, 2007, 441–458.
8. C. Li, K. F. Ng and T. K. Pong, Constraint qualifications for convex inequality systems with applications in constrained optimization, SIAM J. Optim. 19, 2008, 163–187.
9. S. Suzuki and D. Kuroiwa, On set containment characterization and constraint qualification for quasiconvex programming, J. Optim. Theory Appl. 149, 2011, 554–563.
10. S. Suzuki and D. Kuroiwa, Optimality conditions and the basic constraint qualification for quasiconvex programming, Nonlinear Anal. 74, 2011, 1279–1285.
11. S. Suzuki and D. Kuroiwa, Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming, Nonlinear Anal. 75, 2012, 2851–2858.
12. S. Suzuki and D. Kuroiwa, Necessary and sufficient constraint qualification for surrogate duality, J. Optim. Theory Appl. 152, 2012, 366-367.
13. S. Suzuki and D. Kuroiwa, Some constraint qualifications for quasiconvex vector-valued systems, J. Global Optim. 55, 2013, 539–548.
14. S. Suzuki and D. Kuroiwa, Generators and constraint qualifications for quasiconvex inequality systems, J. Nonlinear Convex Anal. 18, 2017, 2101–2121.
15. S. Suzuki and D. Kuroiwa, Duality Theorems for Separable Convex Programming without Qualifications, J. Optim. Theory Appl. 172, 2017, 669–683.

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HIGH CONVERGENCE ORDER SOLVERS IN BANACH SPACE

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ABSTRACT. The local convergence of an eighth order solver is established using only the first derivative for Banach space valued operators. Earlier studies have used up to the ninth order derivatives, which limit the applicability of the solver. The results are tested using numerical experiments.

KEYWORDS: Banach space, Newton-type, local convergence, Fréchet derivative.
AMS Subject Classification: 65F08, 37F50, 65N12.

1. Introduction

Let $\Omega \subset \mathcal{B}_1$ be nonempty, open, and $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces. One of the greatest challenges in Computational Mathematics is to find a solution x_* of the equation [1, 2, 3, 4, 6, 7, 11, 12, 13, 14, 15, 16, 17]

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where $\mathcal{F} : \Omega \longrightarrow \mathcal{B}_2$ is Fréchet differentiable operator.

In this study, we are concerned with the local convergence of the Newton-type solver given as

$$\begin{aligned} & x_0 \in \Omega, \\ y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ z_n &= x_n - [\frac{1}{4}I + \frac{1}{2}\mathcal{F}'(y_n)^{-1}\mathcal{F}(x_n) + \frac{1}{4}(\mathcal{F}'(y_n)^{-1}\mathcal{F}'(x_n))^2]\mathcal{F}'(x_n)^{-1}\mathcal{F}(y_n) \\ x_{n+1} &= z_n - [\frac{1}{2}I + \frac{1}{2}(\mathcal{F}'(y_n)^{-1}\mathcal{F}'(x_n))^2]\mathcal{F}'(x_n)^{-1}\mathcal{F}(z_n). \end{aligned} \tag{1.2}$$

Methods (1.2) was studied in [6], but for the case $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^k$ (k a natural number). Using conditions on ninth order derivative, and Taylor series(although

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these derivatives do not appear in solver (1.2)), the eighth convergence order was established. The hypotheses on higher order derivatives limit the usage of solver (1.2).

As an academic example: Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define $\mathcal{F}$ on Ω by

$$\mathcal{F}(x) = x^3 \log x^2 + x^5 - x^4$$

Then, we have $x_* = 1$, and

$$\mathcal{F}'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$
$$\mathcal{F}''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$
$$\mathcal{F}'''(x) = 6 \log x^2 + 60x^2 = 24x + 22.$$

Obviously $\mathcal{F}'''(x)$ is not bounded on Ω. So, the convergence of solver (1.2) not guaranteed by the analysis in [6, 7, 8, 9, 11, 15].

Other problems with the usage of solver (1.2) are: no information on how to choose x_0; bounds on $\|x_n - x_*\|$ and information on the location of x_*. All these are addressed in this paper by only using conditions on the first derivative, and in the more general setting of Banach space valued operators. That is how, we expand the applicability of solver (1.2). To avoid the usage of Taylor series and high convergence order derivatives, we rely on the computational order of convergence (COC) or the approximate computational order of convergence (ACOC) [1, 6, 10].

The layout of the rest of the paper includes: the local convergence in Section 2, and the example in Section 3.

2. Ball convergence

We introduce some scalar functions and parameters for the convenience of our convergence analysis of solver (1.2). Let $w_0 : [0, \infty) \longrightarrow [0, \infty)$ be an increasing and continuous function with $w_0(0) = 0$. Suppose that equation

$$w_0(t) = 1 \tag{2.1}$$

has at least one positive solution. Denote by ρ_1 the smallest such solution. Let $w : [0, \rho_1) \longrightarrow [0, \infty)$ and $w_1 : [0, \rho_1) \longrightarrow [0, \infty)$ be increasing and continuous functions with $w(0) = 0$. Define functions ψ_1 and $\bar{\psi}_1$ on the interval $[0, \rho_1)$ by

$$\psi_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)}$$

and

$$\bar{\psi}_1(t) = \psi_1(t) - 1.$$

We have $\bar{\psi}_1(0) = -1$ and $\bar{\psi}_1(t) \longrightarrow \infty$ as $t \longrightarrow \rho_1^-$. The intermediate value theorem assures that equation $\bar{\psi}_1(t) = 0$ has at least one solution in $(0, \rho_1)$. Denote by R_1 the smallest such solution. Suppose that equation

$$w_0(\psi_1(t)t) = 1 \tag{2.2}$$

has at least one positive solution. Denote by ρ_2 the smallest such solution. Set $\rho_0 = \min\{\rho_1, \rho_2\}$. Define functions ψ_2 and $\bar{\psi}_2$ on $[0, \rho_0)$ by

$$\psi_2(t) = \Bigg\{ \frac{\int_0^1 w((1-\theta)\psi_1(t)t)d\theta}{1 - w_0(\psi_1(t)t)} + \frac{(w_0(\psi_1(t)t) + w_0(t)) \int_0^1 w_1(\theta\psi_1(t)t)d\theta}{(1 - w_0(\psi_1(t)t))(1 - w_0(t))}$$

$$+\frac{1}{4}\left[\frac{(w_0(\psi_1(t)t)+w_0(t))^2}{(1-w_0(\psi_1(t)t))^2} + \frac{2(w_0(\psi_1(t)t)+w_0(t))}{1-w_0(\psi_1(t)t)}\right] \times \frac{\int_0^1 w_1(\theta\psi_1(t)t)d\theta}{1-w_0(t)}\Bigg\}$$

and $\bar{\psi}_2(t)=\psi_2(t)-1$. We get $\bar{\psi}_2(0)=-1$ and $\bar{\psi}_2(t)\longrightarrow\infty$ as $t\longrightarrow\rho_0^-$. Denote by R_2 the smallest solution of equation $\bar{\psi}_2(t)=0$ in $(0,\rho_2)$. Suppose that

$$w_0(\psi_3(t)t)=1 \tag{2.3}$$

has at least one positive solution. Denote by ρ_3 the smallest such solution. Set $\rho=\min\{\rho_2,\rho_3\}$. Define functions ψ_3 and $\bar{\psi}_3$ on the interval $[0,\rho)$ by

$$\begin{aligned}\psi_3(t) \quad = \quad & \Bigg\{\frac{\int_0^1 w((1-\theta)\psi_2(t)t)d\theta}{1-w_0(\psi_2(t)t)} \\ & +\frac{(w_0(\psi_2(t)t)+w_0(t))\int_0^1 w_1(\theta\psi_2(t)t)d\theta}{(1-w_0(\psi_2(t)t))(1-w_0(t))} \\ & +\frac{1}{2}\left[\frac{(w_0(\psi_1(t)t)+w_0(t))^2}{(1-w_0(\psi_1(t)t))^2} + \frac{2(w_0(\psi_1(t)t)+w_0(t))w_1(t)}{(1-w_0(t))(1-w_0(\psi_1(t)t))}\right] \\ & \times \frac{\int_0^1 w_1(\theta\psi_2(t)t)d\theta}{1-w_0(t)}\Bigg\}\psi_2(t),\end{aligned}$$

and $\bar{\psi}_3(t)=\psi_3(t)-1$. We get $\bar{\psi}_3(0)=-1$ and $\bar{\psi}_3(t)\longrightarrow\infty$ as $t\longrightarrow\rho^-$. Denote by R_3 smallest solution of equation $\bar{\psi}_3(t)=0$ in $(0,\rho)$. Define a radius of convergence R by

$$R=\min\{R_m\}, m=1,2,3. \tag{2.4}$$

It follows that for each $t\in[0,R)$

$$0 \leq w_0(t)<1, \tag{2.5}$$
$$0 \leq w_0(\psi_1(t)t)<1, \tag{2.6}$$
$$0 \leq w_0(\psi_2(t)t)<1 \tag{2.7}$$

and

$$0\leq\psi_m(t)<1. \tag{2.8}$$

Let $\mathcal{B}(\mathcal{B}_1,\mathcal{B}_2)=\{G:\mathcal{B}_1\longrightarrow\mathcal{B}_2 \text{ be bounded and linear}\}$, $T(x,d)=\{y\in\mathcal{B}_1:\|y-x\|<d;d>0\}$ and $\bar{T}(x,d)=\{y\in\mathcal{B}_1:\|y-x\|\leq d;d>0\}$. We shall use the conditions (C) in the local convergence analysis of solver (1.2) that follows:

- (c1) $\mathcal{F}:\Omega\longrightarrow\mathcal{B}_2$ a continuously differentiable operator in the sense of Fréchet, and there exists $p\in\Omega$ such that $\mathcal{F}(p)=0$, and $\mathcal{F}'(p)^{-1}\in\mathcal{B}(\mathcal{B}_2,\mathcal{B}_1)$.
- (c2) There exists function $w_0:[0,\infty)\longrightarrow[0,\infty)$ continuous, and increasing with $w_0(0)=0$ such that for each $x\in\Omega$

$$\|\mathcal{F}'(p)^{-1}(F'(x)-F'(p))\|\leq w_0(\|x-p\|).$$

Set $\Omega_0=\Omega\cap T(p,\rho_1)$ where ρ_1 is given in (2.1).

(c3) There exist functions $w : [0, \rho_0) \longrightarrow [0, \infty), w_1 : [0, \rho_0) \longrightarrow [0, \infty)$ continuous, and increasing such that for each $x, y \in \Omega_0$

$$\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(y)] - \mathcal{F}'(x))\| \leq w(\|y - x\|)$$

and

$$\|\mathcal{F}'(p)^{-1}\mathcal{F}'(x))\| \leq w_1(\|x - p\|.$$

(c4) $\bar{T}(p, R) \subseteq \Omega$ where R is defined by (2.4) and ρ_1, ρ_2, ρ_3 are given in (2.1)–(2.3), respectively.

(c5) There exists $R_1 \geq R$ such that $\int_0^1 w_0(\theta R_1)d\theta < 1$.
Set $\Omega_1 = \Omega \cap \bar{T}(x_*, R_1)$.

Next, the convergence of solver (1.2) follows using preceding notation and the conditions (C).

Theorem 2.1. *Suppose that the conditions (C) hold. Then, the sequence $\{x_n\}$ starting at $x_0 \in T(p, R) - \{p\}$, and generated by solver (1.2) is well defined, remains in $T(p, R)$ for each $n = 0, 1, 2, \ldots,$ and converges to p. Moreover the following error bounds hold*

$$\|y_n - p\| \leq \psi_1(\|x_n - p\|)\|x_n - p\| \leq \|x_n - p\| < r, \tag{2.9}$$

$$\|z_n - p\| \leq \psi_2(\|x_n - p\|)\|x_n - p\| \leq \|x_n - p\|, \tag{2.10}$$

$$\|x_{n+1} - p\| \leq \psi_3(\|x_n - p\|)\|x_n - p\| \leq \|x_n - p\|, \tag{2.11}$$

where functions ψ_i are given previously and R is defined in (2.4). Furthermore, the limit point p is the only solution of equation $\mathcal{F}(x) = 0$ in the set Ω_1.

Proof. We shall use a mathematical induction based proof. Let $x \in T(p, R) - \{p\}$. By (2.4), (c1) and (c2), we get that

$$\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(p))\| \leq w_0(\|x - p\|) < w_0(R) < 1, \tag{2.12}$$

so by the Banach lemma on invertible operators [15, 16], we have that $\mathcal{F}'(x)^{-1} \in \mathcal{B}(\mathcal{B}_2, \mathcal{B}_1)$, and

$$\|\mathcal{F}'(x)^{-1}\mathcal{F}'(p)\| \leq \frac{1}{1 - w_0(\|x - p\|)}. \tag{2.13}$$

This also shows that y_0 is well defined. Using (2.4), (2.8) (for $m = 1$), (c3), (2.13) and (1.2), we obtain in turn that

$$\begin{aligned} \|y_0 - p\| &\leq \|x_0 - p - \mathcal{F}'(x_0)^{-1}\mathcal{F}'(x_0)\| \\ &\leq \|\mathcal{F}'(x_0)^{-1}(\mathcal{F}(p)\| \\ &\quad \times \| \int_0^1 \mathcal{F}'(p)(\mathcal{F}'(p + \theta(x_0 - p)) - \mathcal{F}'(x_0))(x_0 - p)d\theta\| \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_0 - p\|)d\theta\|x_0 - p\|}{1 - w_0(\|x_0 - p\|)} \\ &= \psi_1(\|x_0 - p\|\|x_0 - p\| \leq \|x_0 - p\| < R, \end{aligned} \tag{2.14}$$

so (2.9) holds for $n = 0$ and $y_0 \in T(p, R)$, so z_0 and x_1 are well defined. By the second substep of solver (1.2) for $n = 0$, we can write by (c1) that

$$\mathcal{F}(x) = \mathcal{F}(x) - \mathcal{F}(p) = \int_0^1 \mathcal{F}'(p + (\theta(x - p))d\theta(x - p). \tag{2.15}$$

Then, by the second condition (c3), and (1.2) we get respectively, that

$$\|\mathcal{F}'(p)^{-1}\mathcal{F}(x)\| \leq \int_0^1 w_1(\theta\|x - p\|)d\theta\|x - p\|, \tag{2.16}$$

and

$$\begin{aligned} z_0 - p &= (y_0 - p - \mathcal{F}'(y_0)^{-1}\mathcal{F}(y_0)) + \mathcal{F}'(y_0)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(y_0))\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_0) \\ &\quad + \frac{1}{4}[(\mathcal{F}'(y_0)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(y_0)))^2 \\ &\quad -2(\mathcal{F}'(y_0)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(y_0))]\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_0). \end{aligned} \tag{2.17}$$

Using (2.4), (2.8) (for $m = 2$), (2.13) (for $x = y_0$), (2.14) and (2.16) (for $x = y_0$) and (2.17), we obtain in turn that

$$\begin{aligned} \|z_0 - p\| &\le \|y_0 - p - \mathcal{F}'(y_0)^{-1}\mathcal{F}(y_0)\| \\ &\quad + \|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(p)\|[\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(y_0) - \mathcal{F}'(p))\| \\ &\quad + \|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\| \\ &\quad \times \|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(p)\|\|\mathcal{F}'(p)^{-1}\mathcal{F}(y_0)\| \\ &\quad + \frac{1}{4}[\|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(p)\|^2(\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\| \\ &\quad + (\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(y_0) - \mathcal{F}'(p))\|)^2 \\ &\quad + 2\|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(p)\|(\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\| \\ &\quad + \|\mathcal{F}'(p)^{-1}(\mathcal{F}'(y_0) - \mathcal{F}'(p))\|)] \\ &\quad \times \|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(p)\|\|\mathcal{F}'(p)^{-1}\mathcal{F}(y_0)\| \\ &\le \left\{ \frac{\int_0^1 w((1-\theta)\|y_0 - p\|)d\theta)}{1 - w_0(\|y_0 - p\|)} \right. \\ &\quad + \left. \frac{\int_0^1 w_1(\theta\|y_0 - p\|)d\theta(w_0(\|y_0 - p\|) + w_0(\|x_0 - p\|))}{(1 - w_0(\|x_0 - p\|))(1 - w_0(\|y_0 - p\|)} \right] \\ &\quad + \frac{1}{4}\left[\frac{(w_0(\|x_0 - p\|) + w_0(\|y_0 - p\|))^2}{(1 - w_0(\|y_0 - p\|))^2} \right. \\ &\quad + \left. \left. \frac{2(w_0(\|x_0 - p\|) + w_0(\|y_0 - p\|))}{1 - w_0(\|y_0 - p\|)} \right] \frac{\int_0^1 w_1(\theta\|y_0 - p\|)d\theta}{1 - w_0(\|x_0 - p\|)} \right\} \|y_0 - p\| \\ &\le \psi_2(\|x_0 - p\|)\|x_0 - p\| \le \|x_0 - p\| < R, \end{aligned} \tag{2.18}$$

which shows (2.11) for $n = 0$, and $z_0 \in T(p, R)$. Moreover, by the third substep of solver (1.2) for $n = 0$, we have that

$$\begin{aligned} x_1 - p &= (z_0 - p - \mathcal{F}'(z_0)^{-1}\mathcal{F}(z_0)) + (\mathcal{F}'(z_0)^{-1} - \mathcal{F}'(x_0)^{-1})\mathcal{F}(z_0) \\ &\quad + \frac{1}{2}[I - (\mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0))^2]\mathcal{F}'(x_0)^{-1}\mathcal{F}(z_0) \\ &= (z_0 - p - \mathcal{F}'(z_0)^{-1}\mathcal{F}(z_0)) \\ &\quad + \mathcal{F}'(z_0)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\mathcal{F}'(x_0)^{-1}\mathcal{F}(z_0) \\ &\quad + \frac{1}{2}[(I - \mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0))^2 + 2(I - \mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0))\mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_0)] \\ &\quad \times \mathcal{F}'(x_0)^{-1}\mathcal{F}(z_0). \end{aligned} \tag{2.19}$$

Using (2.4), (2.8) (for $m = 3$), (2.13) (for $x = y_0, z_0$), (2.16) (for $x = z_0$), (2.18) and (2.20), we get in turn that

$$\begin{aligned} \|x_1 - p\| &\le \|z_0 - p - \mathcal{F}'(z_0)^{-1}\mathcal{F}(z_0)\| \\ &\quad + \|\mathcal{F}'(z_0)^{-1}\mathcal{F}'(p)\|[\|\mathcal{F}'(p)^{-1}(\mathcal{F}'(z_0) - \mathcal{F}'(p))\| \\ &\quad + \|\mathcal{F}'(p)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(p))\|] \end{aligned}$$

$$
\begin{aligned}
& \times\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(p)\|\|\mathcal{F}'(p)^{-1}\mathcal{F}'(z_0)\| \\
& +\frac{1}{2}[\|(\mathcal{F}'(y_0)^{-1}(\mathcal{F}'(y_0)-\mathcal{F}'(x_0))\|^2 \\
& +2\|\mathcal{F}'(y_0)^{-1}(\mathcal{F}'(y_0)-\mathcal{F}'(x_0))\|\|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(p)\| \\
& \times\|\mathcal{F}'(p)^{-1}\mathcal{F}'(x_0)\|] \\
& \times\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(p)\|\|\mathcal{F}'(p)^{-1}\mathcal{F}(z_0)\| \\
\leq & \left\{\frac{\int_0^1 w((1-\theta)\|z_0-p\|)d\theta}{1-w_0(\|z_0-p\|)}\right. \\
& +\frac{(w_0(\|z_0-p\|)+w_0(\|x_0-p\|))\int_0^1 w_1(\theta\|z_0-p\|)d\theta}{(1-w_0(\|z_0-p\|)(1-w_0(\|x_0-p\|))} \\
& \frac{1}{2}\left[\frac{(w_0(\|x_0-p\|)+w_0(\|y_0-p\|))^2}{(1-w_0(\|y_0-p\|))^2}\right. \\
& \frac{2(w_0(\|x_0-p\|)+w_0(\|y_0-p\|))}{1-w_0(\|x_0-p\|)} \\
& \left.\times\frac{w_1(\|z_0-p\|)}{1-w_0(\|y_0-p\|)}\right] \\
& \left.\frac{\int_0^1 w_1(\theta\|z_0-p\|)d\theta}{1-w_0(\|x_0-p\|)}\right\}\|z_0-p\| \\
\leq & \psi_3(\|x_0-p\|)\|x_0-p\| \leq \|x_0-p\| < R, \qquad (2.20)
\end{aligned}
$$

so (2.11) holds for $n=0$ and $x_1 \in T(p,R)$. The induction for (2.11) is completed, if x_m, y_m, z_m, x_{m+1} replace x_0, y_0, z_0, x_1 in the preceding estimates. Then,in view of the estimate

$$\|x_{m+1}-p\| \leq \lambda\|x_m-p\| \leq \|x_m-p\| < R, \qquad (2.21)$$

where $\lambda = \psi_3(\|x_0-p\|) \in [0,1)$, we deduce that $x_{m+1} \in T(p,R)$, and $\lim_{m\longrightarrow\infty} x_m = p$. Further for the uniqueness part, let $p_* \in \Omega_1$ with $\mathcal{F}(p_*) = 0$. Define $G = \int_0^1 \mathcal{F}'(p+\theta(p_*-p))d\theta$. Then, using (c5), we get

$$\|\mathcal{F}'(p)^{-1}(G-\mathcal{F}'(p))\| \leq \int_0^1 w_0(\theta\|p_*-p\|)d\theta \leq \int_0^1 w_0(\theta R)d\theta < 1,$$

so G^{-1} exists, and from

$$0 = \mathcal{F}(p) - \mathcal{F}(p_*) = G(p-p_*),$$

we derive $p = p_*$.

□

Remark 2.1. (a) In the case when $w_0(t) = L_0 t, w(t) = Lt$ and $\Omega_0 = \Omega$, the radius $\rho_A = \frac{2}{2L_0+L}$ was obtained by Argyros in [4] as the convergence radius for Newton's solver under condition (2.7)-(2.9). Notice that the convergence radius for Newton's solver given independently by Rheinboldt [16] and Traub [17] is given by

$$\rho_{TR} = \frac{2}{3L} < \rho_A.$$

As an example, let us consider the function $F(x) = e^x - 1$. Then $\alpha^* = 0$. Set $\Omega = B(0,1)$. Then, we have that $L_0 = e-1 < L = e$, so $\rho_{TR} = 0.24252961 < \rho_A = 0.324947231$.

(b) The local results can be used for projection solvers such as Arnoldi's solver, the generalized minimum residual solver(GMREM), the generalized conjugate solver(GCM) for combined Newton/finite projection solvers and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2, 3, 4].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [2, 3, 10, 13]:

$$F'(x) = P(F(x)),$$

where $P : \mathcal{B}_2 \longrightarrow \mathcal{B}_2$ is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that solvers (1.2) is not changing when we use the conditions of the preceding Theorem instead of the stronger conditions used in [15]. Moreover, we can compute the computational order of convergence (COC) defined as

$$\xi = \ln\left(\frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|}\right) / \ln\left(\frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|}\right)$$

or the approximate computational order of convergence (ACOC) [5, 6]

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence, but not higher order derivatives are used.

3. Numerical example

We present the following example to test the convergence criteria.

Example 3.1. Let $\mathcal{B}_1 = \mathcal{B}_1 = \mathbb{R}^3$, $\Omega = U(0,1), x_* = (0,0,0)^T$ and define $\mathcal{F}$ on Ω by

$$\mathcal{F}(x) = \mathcal{F}(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3)^T. \tag{3.1}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$\mathcal{F}'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows $x_* = (0,0,0)^T$ and since $\mathcal{F}'(x_*) = diag(1,1,1)$, we get by conditions (H) $w_0(t) = (e-1)t$, $w(t) = e^{\frac{1}{e-1}}t, w_1(t) = e^{\frac{1}{e-1}}$.

$$R_1 = 0.382692,\ R_2 = 0.227598, R_3 = 169362.$$

Example 3.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1], \Omega = \bar{U}(0,1)$. Define function F on Ω by

$$F(w)(x) = w(x) - 5\int_0^1 x\theta w(\theta)^3 d\theta.$$

Then, the Fréchet-derivative is given by

$$F'(w(\xi))(x) = \xi(x) - 15\int_0^1 x\theta w(\theta)^2 \xi(\theta) d\theta, \text{ for each } \xi \in \Omega.$$

Then, we have that $x^* = 0, w_0(t) = L_0 t, w(t) = Lt, w_1(t) = 2, L_0 = 7.5 < L = 15$. Then, the radius of convergence are given by

$$R_1 = 0.0667,\ R_2 = 0.0395822 = \rho,\ R_3 = 0.0297337.$$

Example 3.3. Returning back to the motivational example given at the introduction of this study, we can choose $w_0(t) = w(t) = 96.662907t$, $w_1(t) = 1.0631$. Then, the radius of convergence are given by

$$R_1 = 0.00689682,\ R_2 = 0.00457799,\ R_3 = 1 = 0.00378481.$$

References

1. I. K. Argyros, J. A. Ezquerro, J. M. Gutiérrez, M.A. Herńandez, S. Hilout, On the semilocal convergence of efficient Chebyshev-Secant -type solvers, J. Comput. Appl. Math., 235(2011), 3195-3206.
2. I. K. Argyros, S. George, N. Thapa, Mathematical modeling for the solution of equations and systems of equations with applications, Volume-I, Nova Publishes, NY, 2018.
3. I. K. Argyros, S. George, N. Thapa, Mathematical modeling for the solution of equations and systems of equations with applications, Volume-II, Nova Publishes, NY, 2018.
4. I. K. Argyros and S. Hilout, Weaker conditions for the convergence of Newtons solver. Journal of Complexity, 28(3):364-387, 2012.
5. A. Cordero, J. L. Hueso, E. Martínez, J. R. Torregrosa, A modified Newton-Jarratt's composition, Numer. Algor., 55, (2010), 87-99.
6. A. Cardero, H. Arora, Improved Newton-like method for solving systems of equations, SeMA, 74 (2016), 1-7.
7. M. Grau-Sánchez, M. Noguera, S. Amat, On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative solvers, J. Comput. Appl. Math., 237,(2013), 363-372.
8. T. Lotfi, P. Bakhtiari, A. Cordero, K. Mahdiani, J. R. Torregrosa, Some new efficient multipoint iterative solvers for solving nonlinear systems of equations, Int. J. Comput. Math., 92, (2015), 1921-1934.
9. K. Madhu, D. K. R. Babajee, J. Jayaraman, An improvement to double-step Newton solver and its multi-step version for solving system of nonlinear equations and its applications, Numer. Algor., 74,(2017), 593-607.
10. A. A. Magreñán and I. K .Argyros, Improved convergence analysis for Newton-like solvers. Numerical Algorithms, 71(4):811-826, 2016.
11. A. A. Magreñán, A.Cordero, J. M. Gutiérrez, and J. R. Torregrosa, Real qualitative behavior of a fourth-order family of iterative solvers by using the convergence plane. Mathematics and Computers in Simulation, 105:49-61, 2014.
12. A. A. Magreñán and I. K. Argyros, Two-step Newton solvers. Journal of Complexity, 30(4):533-553, 2014.
13. A. M. Osrowski, Solution of equations and systems of equations, Academic Press, New York, 1960.
14. F.A. Potra and V. Pták, Nondiscrete induction and iterative processes, volume 103. Pitman Advanced Publishing Program, 1984.
15. J. R. Sharma, H. Arora, Improved Newton-like solvers for solving systems of nonlinear equations, SeMA J., 74,2(2017), 147-163.
16. W.C. Rheinboldt,An adaptive continuation process for solving systems of nonlinear equations, In: Mathematical models and numerical solvers (A.N.Tikhonov et al. eds.) pub.3, (1977), 129-142 Banach Center, Warsaw Poland.
17. J.F.Traub, Iterative solvers for the solution of equations, AMS Chelsea Publishing, 1982.

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SOME FIXED POINT THEOREMS OF HARDY-ROGER CONTRACTION IN COMPLEX VALUED B-METRIC SPACES

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ABSTRACT. The aim of this paper is to prove the existence and uniqueness of a fixed point of a mapping satisfying the Hardy-Rogers contraction in complex valued b-metric space, we have obtained some fixed point theorems in complex-valued b-metric spaces. This work is generalized and improved some results of Hasanah [5], and well known results in the literature.

KEYWORDS: b-metric space, complex valued b-metric space, Hardy-Rogers contraction, fixed point.
AMS Subject Classification: :46C05, 47D03, 47H09, 47H10, 47H20.

1. Introduction

The axiomatic development of a metric space was essentially carried out by French mathematician Frechet in the year 1906 [4]. After the Banach contraction principle, because of various applications. Many mathematics used the Banach contractive principle to study an existence and uniqueness of fixed points. Banach fixed point theorem in a complete metric space introduced by Banach [2], because it was generalized in many spaces. In 1973, Hardy and Rogers [6], define the generalzed Kannan contraction and prove some fixed point theorem in metric space. In 2011 Azam et.al [1], introduced the notion of complex valued metric space and established sufficient conditions for the existence of common fixed point of a pair of mappings satisfying a contractive condition. In 2012, Sintunavarat and Kumam [10] introduced new spaces called the complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. One year later, Sintunavarat, Cho and Kumam, [11] established the existence of fixed point theorems under the contractive condition in complex valued metric spaces, they introduce the concepts of a C-Cauchy sequence and C-complete in

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complex-valued metric spaces and establish the existence of common fixed point theorems in C-complete complex-valued metric spaces. In 2015, Jleli and Samet [7] introduced a very interesting concept of a generalized metric space, which covers different well-known metric structures including classical metric spaces, b-metric spaces, dislocated metric spaces, modular spaces, and so on.

In 2017, Hasanah [5], study the existence and proved the uniqueness of fixed point of some contractive condition in complete complex valued b-metric spaces.

Motivate by Hasanah [5] and Hardy and Rogers [6], we introduce the Hardy-Rogers contraction it has generalized than the contractive condition of [5], and then we proved the existence and uniqueness of fixed point in complete complex valued b-metric space.

2. Preliminaries

In this section, we suppose some definitions and define the definition of b-metric space in the complex plane, and suppose some lemmas for study in this work.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty]$ is called a metric space if for $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a metric space, and d is called a matric on X.

Next, we provide the definition of b-metric space, this space is generalized than metric space.

Definition 2.2. [3] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.
The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

We give some example for b-metric space.

Example 2.3. Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

Since in real numbers which has completeness property, order is not welll-defined in complex numbers. Before giving the definition of complex valued metric spaces and complex valued b-metric spaces, we define partial order in complex numbers (see [8]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order $\precsim$ on $\mathbb{C}$ as follows;

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

This means that we would have $z_1 \precsim z_2$ if and only if one of the following conditions holds:
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
If one of the conditions (ii), (iii), and (iv) holds, then we write $z_1 \precnsim z_2$. Particulaly, we have $z_1 \prec z_2$ if the condition (iv) is satisfied.

Remark 2.4. We can easily check the following:
(i) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \precsim z_2$ then $az_1 \precsim bz_2, \forall z_1, z_2 \in \mathbb{C}$.
(ii) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$.
(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.
(iv) If $z \in \mathbb{C}, a, b \in \mathbb{R}$ and $a \leq b$, then $az \precsim bz$.

A b-metric on a b-metric sapce is a funcion having real value. Based on the definition of partial order on complex number, real valued b-metric can be generalized into complex valued b-metric as folllows.

Definition 2.5. [1] Let X be a nonmpty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \precsim d(x, y)$
(ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, z) \precsim d(x, y) + d(y, z)$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Next, we give the definition of complex valued b-metric space.

Definition 2.6. [9] Let X be a nonmpty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if, for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \precsim d(x, y)$
(ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$,
(iv) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.
The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is complex valued metric space is defined in Definition 2.5.

For Definition 2.6, we can suppose some example of complex valued b-metric space.

Example 2.7. Let $X = \mathbb{C}$. Define the mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbb{C}, d)$ is complex valued b-metriic space with $s = 2$.

Definition 2.8. [9] Let (X, d) be a complex valued b-metric space.
(i) A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = y \in Y : d(x, y) \prec r \subseteq A$.
(ii) A point $x \in X$ is called limit point of a set A if for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - x) \neq \emptyset$
(iii) A subset $A \subseteq X$ is open if each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is closed if each limit point of A is contained in A.

For study this work we suppose the definition of convergent sequence, Cauchy sequence and complete complex space.

Definition 2.9. [9] Let (X, d) be complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) $\{x_n\}$ is convergent to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x) \prec r$. Thus x is the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is said to be Cauchy sequence if for ever $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.

(iii) If for every Cauchy sequence in$X$ is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Finally, we give some lemmas for proof the main theorems.

Lemma 2.10. [9] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.11. [9] *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

3. Main Results

In this section we give some comditions and prove the existence theorem and unique fixed point of Hardy-Rogers contraction in complete complex valued b-metric space.

Theorem 3.1. *Let (X, d) be a complete complex valued b-metric space with constant $s \geq 1$, and let $T : X \to X$ be a mapping with satisfying Hardy-Rogers contraction, that is*

$$d(Tx, Ty) \precsim \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 d(y, Tx) + \lambda_5 d(x, Ty)$$

for all $x, y \in X$ and λ_i are nonegative real number with $\Sigma_{i=1}^{5} \lambda_i \in [0, \frac{1}{s})$ and $\lambda_4 \leq \frac{\lambda_5}{2s-1}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ from $T : X \to X$, we have there exists $x_1 \in X$ such that $x_1 = Tx_0$. From $x_1 \in X$, there exists $x_2 \in X$ such that $x_2 = Tx_1$. By induction of this process, we have the sequence $\{x_n\} \subseteq X$ such that,

$$x_n = Tx_{n-1} = T^n x_0, \forall n \in \mathbb{N}.$$

Note that for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_{n+2}, x_{n+1}) &= d(Tx_{n+2-1}, Tx_{n+1-1}) \\
&= d(Tx_{n+1}, Tx_n) \\
&\precsim \lambda_1 d(x_{n+1}, x_n) + \lambda_2 d(x_{n+1}, Tx_{n+1}) + \lambda_3 d(x_n, Tx_n) \\
&\quad + \lambda_4 d(x_n, Tx_{n+1}) + \lambda_5 d(x_{n+1}, Tx_n) \\
&\precsim \lambda_1 d(x_{n+1}, x_n) + \lambda_2 d(x_{n+1}, x_{n+2}) + \lambda_3 d(x_n, x_{n+1}) \\
&\quad + \lambda_4 d(x_n, x_{n+2}) + \lambda_5 d(x_{n+1}, x_{n+1}) \\
&\precsim \lambda_1 d(x_{n+1}, x_n) + \lambda_2 d(x_{n+1}, x_{n+2}) + \lambda_3 d(x_n, x_{n+1}) \\
&\quad + \lambda_4 s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + 0.
\end{aligned}$$

$$\begin{aligned}
(1 - (\lambda_2 + \lambda_4 s)) d(x_{n+2}, x_{n+1}) &\precsim \lambda_1 d(x_{n+1}, x_n) + \lambda_3 d(x_{n+1}, x_n) + \lambda_4 s d(x_n, x_{n+1}) \\
d(x_{n+2}, x_{n+1}) &\precsim \frac{\lambda_1 + \lambda_3 + \lambda_4 s}{1 - (\lambda_2 + \lambda_4 s)} d(x_{n+1}, x_n).
\end{aligned}$$

If we take $\gamma = \frac{\lambda_1+\lambda_3+\lambda_4 s}{1-(\lambda_2+\lambda_4 s)}$ and continuing this process, then we have

$$d(x_{n+2}, x_{n+1}) \precsim \gamma d(x_{n+1}, x_n).$$

It follows that,

$$d(x_{n+1}, x_n) \precsim \gamma d(x_n, x_{n-1})$$

and

$$d(x_n, x_{n-1}) \precsim \gamma d(x_{n-1}, x_{n-2})$$
$$\vdots$$
$$d(x_{n+1}, x_n) \precsim \gamma^n d(x_1, x_0),$$

for all $n \in \mathbb{N}$. Hence, $d(x_{n+2}, x_{n+1}) \precsim \gamma^{n+1} d(x_1, x_0)$. For $m \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_n, x_{n+m}) &\precsim s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})] \\
&\precsim sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+m}) \\
&\precsim sd(x_n, x_{n+1}) + s(s[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+m})]) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2(s[d(x_{n+2}, x_{n+3}) \\
&\quad +d(x_{n+3}, x_{n+m})]) \\
&\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3} \\
&\quad + \cdots + s^m d(x_{n+m-1}, x_{n+m}) \\
&\precsim s\gamma^n d(x_0, x_1) + s^2\gamma^{n+1} d(x_0, x_1) + s^3\gamma^{n+2} d(x_0, x_1) \\
&\quad + \cdots + s^m \gamma^{n+m-1} d(x_0, x_1) \\
&\precsim s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}].
\end{aligned}$$

It follows that

$$d(x_n, x_{n+m}) \precsim s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}].$$

By Remark 2.4, taking absolute value on both sides, we have

$$\begin{aligned}
|d(x_n, x_{n+m})| &\leq |s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]| \\
&\leq |s\gamma^n||d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]| \\
&= s\gamma^n |d(x_0, x_1)|[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}].
\end{aligned}$$

Since, $\Sigma_{i=1}^5 \lambda_i \in [0, \frac{1}{s})$ for $s \geq 1$ and $\lambda_4 \leq \frac{\lambda_5}{2s-1}$ then $\gamma < 1$ and $s\gamma < 1$. Since $d(x_0, x_1) \in \mathbb{C}$ and $[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]$ exists, taking limit $n \to \infty$ we have $\gamma^n \to 0$. This implies $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$. By Lemma 2.11, the sequence $\{x_n\}$ is a Cauchy sequence in X. Since, X is a complete complex valued b-metric space then $\{x_n\}$ is a convergent sequence. It follows that $\{x_n\}$ converges to u for some $u \in X$. Next, we can show that u is a fixed point of T. Consider,

$$\begin{aligned}
d(u, Tu) &\precsim s[d(u, x_n) + d(x_n, Tu)] \\
&= s[d(u, x_n) + d(Tx_{n-1}, Tu)] \\
&\precsim s[d(u, x_n) + \lambda_1 d(x_{n-1}, u) + \lambda_2 d(x_{n-1}, Tx_{n-1}) + \lambda_3 d(u, Tu) \\
&\quad +\lambda_4 d(u, Tx_{n-1}) + \lambda_5 d(x_{n-1}, Tu)] \\
(1 - s\lambda_3) d(u, Tu) &\precsim s[d(u, x_n) + \lambda_1 d(x_{n-1}, u) + \lambda_2 d(x_{n-1}, x_n) \\
&\quad +\lambda_4 d(u, x_n) + \lambda_5 d(x_{n-1}, Tu)].
\end{aligned}$$

From Remark 2.4, taking absolute value on both sides, we have

$$\begin{aligned}
|(1 - s\lambda_3) d(u, Tu)| &\leq |s[d(u, x_n) + \lambda_1 d(x_{n-1}, u) + \lambda_2 d(x_{n-1}, x_n) + \lambda_4 d(u, x_n) \\
&\quad +\lambda_5 d(x_{n-1}, Tu)]|
\end{aligned}$$

$$\begin{aligned}
&\leq |s||[d(u,x_n)+\lambda_1 d(x_{n-1},u)+\lambda_2 d(x_{n-1},x_n)+\lambda_4 d(u,x_n) \\
&\quad +\lambda_5 d(x_{n-1},Tu)]| \\
&\leq s[|d(u,x_n)|+|\lambda_1 d(x_{n-1},u)|+|\lambda_2 d(x_{n-1},x_n)|+|\lambda_4 d(u,x_n)| \\
&\quad +|\lambda_5 d(x_{n-1},Tu)|] \\
(1-s\lambda_3)|d(u,Tu)| &\leq s[|d(u,x_n)|+\lambda_1|d(x_{n-1},u)|+\lambda_2|d(x_{n-1},x_n)|+\lambda_4|d(u,x_n)| \\
&\quad +\lambda_5|d(x_{n-1},Tu)|].
\end{aligned}$$

Taking $n \to \infty$ implies $|d(x_n,u)| \to 0$, $|d(x_{n-1},u)| \to 0$. From $\{x_n\}$ is Cauchy sequence in X we have $|d(x_{n-1},x_n)| \to 0$ as $n \to \infty$, then

$$(1-s\lambda_3)|d(u,Tu)| \leq s\lambda_5|d(u,Tu)|.$$

It follows that $(1-s\lambda_3-\lambda_5)|d(u,Tu)| \leq 0$. From $\Sigma_{i=1}^5\lambda_i \in [0,\frac{1}{s})$. Thus $(1-s\lambda_3-\lambda_5) > 0$ and then $|d(u,Tu)| = 0$. Hence $u = Tu$. Therefore u is a fixed point of T.

Finally, we show the uniqueness of the fixed point of T. We assume that there are two fixed points of T which are $x = Tx$ and $y = Ty$. Thus,

$$\begin{aligned}
d(x,y) &= d(Tx,Ty) \\
&\precsim \lambda_1 d(x,y)+\lambda_2 d(x,Tx)+\lambda_3 d(y,Ty)+\lambda_4 d(y,Tx)+\lambda_5 d(x,Ty) \\
&\precsim \lambda_1 d(x,y)+\lambda_2 d(x,x)+\lambda_3 d(y,y)+\lambda_4 d(y,Tx)+\lambda_5 d(x,Ty) \\
&\precsim \lambda_1 d(x,y)+\lambda_4 d(y,x)+\lambda_5 d(x,y) \\
&\precsim (\lambda_1+\lambda_4+\lambda_5)d(x,y).
\end{aligned}$$

By Remark 2.4, taking the absolute value on both sides, we have

$$\begin{aligned}
|d(x,y)| &\leq |(\lambda_1+\lambda_4+\lambda_5)d(x,y)| \\
&\leq (\lambda_1+\lambda_4+\lambda_5)|d(x,y)|.
\end{aligned}$$

From, $\Sigma_{i=1}^5\lambda_i \in [0,\frac{1}{s})$. Then $\lambda_1+\lambda_4+\lambda_5 < 1$, this implies that $|d(x,y)| = 0$. Hence $x = y$. This completes the proof. □

From Theorem 3.1, we have some corollary, as follows:

Corollary 3.2. *Let (X,d) be a complete complex valued b-metric space with constant $s \geq 1$ and let $T : X \to X$ be a function with the following*

$$d(Tx,Ty) \precsim ad(x,Tx)+bd(y,Ty)+cd(x,y), \forall x,y \in X$$

where $a, b,$ and c are nonnegative real numbers and satisfies $s(a+b+c) < 1$. Then T has a uniqe fixed point.

Proof. We put $\lambda_4 = \lambda_5 = 0$, $a = \lambda_2$, $b = \lambda_3$ and $c = \lambda_1$. By theorem 3.1, T has a unique fixed point. This complete the proof. □

Corollary 3.3. *Let (X,d) be a complete complex valued b-metric space with constant $s \geq 1$ and let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \precsim \alpha d(x,Ty)+\beta d(y,Tx)$$

for every $x,y \in X$, where α,β are nonnegative real numbers with $\alpha+\beta < \frac{1}{s}$ and $\beta < \frac{\alpha}{2s-1}$. Then T has a fixed point in X.

Proof. We put $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\alpha = \lambda_5$ and $\beta = \lambda_4$. By theorem 3.1, T has a unique fixed point. This complete the proof. □

Next, we can applied Theorem 3.1 to prove the following theorem.

Theorem 3.4. *Let (X,d) be a complete complex valued b-metric space, with the constant $s \geq 1$. Let $T : X \to X$ be a mapping with satisfying*

$$d(T^n x, T^n y) \precsim \lambda_1 d(x,y) + \lambda_2 d(x, T^n x) + \lambda_3 d(y, T^n y) + \lambda_4 d(y, T^n x) + \lambda_5 d(x, T^n y)$$

for all $x, y \in X$ and λ_i are nonegative real number $\Sigma_{i=1}^5 \lambda_i \in [0, \frac{1}{s})$ and $\lambda_4 \leq \frac{\lambda_5}{2s-1}$. Then T has a unique fixed point.

Proof. Suppose $S = T^n$, by Theorem 3.1, there exists a fixed point u of S, such that

$$Su = u.$$

Thus $T^n u = u$. We obtain that

$$\begin{aligned} d(Tu,u) &= d(T(T^n u), T^n u) \\ &= d(T^n(Tu), T^n u) \\ &\precsim \lambda_1 d(Tu,u) + \lambda_2 d(Tu, T^n(Tu)) + \lambda_3 d(u, T^n u) \\ &\quad + \lambda_4 d(u, T^n(Tu)) + \lambda_5 d(Tu, T^n u) \\ &= \lambda_1 d(Tu,u) + \lambda_2 d(Tu, T(T^n u)) + \lambda_3 d(u,u) \\ &\quad + \lambda_4 d(u, T(T^n u)) + \lambda_5 d(Tu,u) \\ &= \lambda_1 d(Tu,u) + \lambda_2 d(Tu,Tu) + \lambda_3 d(u,u) + \lambda_4 d(u,Tu) \\ &\quad + \lambda_5 d(Tu,u) \end{aligned}$$

$\therefore (1 - \lambda_1 - \lambda_4 - \lambda_5) d(Tu,u) \precsim 0.$
By Remark 2.4, taking absolute value on both side, we have

$$(1 - \lambda_1 - \lambda_4 - \lambda_5)|d(Tu,u)| \leq 0.$$

From $\Sigma_{i=1}^5 \lambda_i < 1$, $(1 - \lambda_1 - \lambda_4 - \lambda_5) > 0$, then $|d(Tu,u)| = 0$. It follows that, $Tu = u$, hence u is a fixed point of T, and then $Tu = u = T^n u$.
Finally, we show that u is a unique fixed point of T. Let v be a fixed point of T, we must show that $u = v$. We see that,

$$\begin{aligned} d(u,v) &= d(T^n u, T^n v) \\ &\precsim \lambda_1 d(u,v) + \lambda_2 d(u, T^n u) + \lambda_3 d(v, T^n u) + \lambda_4 d(v, T^n u) \\ &\quad + \lambda_5 d(u, T^n v) \\ &= \lambda_1 d(u,v) + \lambda_2 d(u,u) + \lambda_3 d(v,v) + \lambda_4 d(v,u) \\ &\quad + \lambda_5 d(u,v) \end{aligned}$$

$\therefore (1 - \lambda_1 - \lambda_4 - \lambda_5) d(u,v) \precsim 0.$
By Remark 2.4, taking absolute value on both side, we have

$$(1 - \lambda_1 - \lambda_4 - \lambda_5)|d(u,v)| \leq 0.$$

Since $\Sigma_{i=1}^5 \lambda_i < 1$, $(1 - \lambda_1 - \lambda_4 - \lambda_5) > 0$, then $|d(u,v)| = 0$. It follows that $u = v$. Therefore, u is a unique fixed point of T. This complete the proof. □

From Thorem 3.4, we can reduce to the following corollary, as follows:

Corollary 3.5. *Let (X,d) be a complete complex valued b-metric space with the constant $s \geq 1$. Let $T : X \to X$ be a mapping (for some fixed n) satisfying*

$$d(T^n x, T^n y) \precsim a d(x, T^n x) + b d(y, T^n y) + c d(x,y)$$

for all $x, y \in X$ where a, b, c are nonnegative real number with $s(a+b+c) < 1$. Then T has a uniqe fixed point in X.

Proof. We put $\lambda_4 = \lambda_5 = 0$, $a = \lambda_2$, $b = \lambda_3$ and $c = \lambda_1$ By theorem 3.1, T has a unique fixed point. This complete the proof. □

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References

1. A. Azam, F. Brain and M. Khan, Common fixed point theorems in complex valued metric space. Numer.Funct.Anal. Optim. 32(3)(2011), 243-253.
2. S. Banach, Sur operations dans les ensembles abstraitsetleur application auxequationsintegrals. Fund. Math. 3(1922), 133-181.
3. I. A. Bakhtin: The contraction principle in quasimetric spaces, Functional Annalysis, vol. 30, (1989), 26-37.
4. M. Frechet, Sur quelques points du calcul functional.RendicontidelCircoloMathematico di-Palermo.22(1)(1906), 1-72.
5. D. Hasanah, Fixed point Theorem in Complex-valued b-metric space. Cauchy-Journal Matlematika Murni Dan Aplikasi. 4(4)(2017), 138-145.
6. G. E. Hardy and T. D. Rogers, A generalization of a fixed point theorem of Reich, Canad. Math. Bull., 16(2)(1973), 201-206.
7. M. Jleli and B. Samet, A generalized metric space and related fixed point theorems, Fixed Point Theory Appl. 33(2015).
8. A. A. Mukheimer, ”Some common fixed point theorems in complex valued b-metric spaces,” Hindawi Publishing Coorporation. The Scientific World Journal, vol. 2014, (2014).
9. K. Rao, P. Swamy and J. Prasad: A common fixed point theorems in complex valued b-metric spaces, Bulletin of Mathematics and Statistics Research , vol. 1, no. 1, (2013).
10. W. Sintunavarat and P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, Journal of Inequalities and Applications volume 2012, Article number: 84 (2012).
11. W. Sintunavarat, Y.J. Cho and P. Kumam, Urysohn integral equations approach by common fixed points in complex valued metric spaces, Advances in Difference Equations, 2013, 2013:49.

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GENERALIZED g-TYPE EXPONENTIAL VECTOR VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. In this works, we introduce a class of generalized g-type exponential vector variational inequality problems in Euclidean spaces and define α_g-relaxed exponentially (τ, μ)-monotone mapping. By utilizing KKM-mapping and Nadler's theorem with α_g-relaxed exponentially (τ, μ)-monotone mapping, we prove that the existence theorems of generalized g-type exponential vector variational inequality problems.

KEYWORDS:Generalized g-type exponential vector variational inequality problems, α_g-relaxed exponentially (τ, μ)-monotone mapping, KKM-mappings, Nadler's Theorem.
AMS Subject Classification: 49J40, 47H06, 47H09,47J20.

1. Introduction

The theory of variational inequalities was introduced by Stampacchia [18], provides a very powerful tools for studying problems arising in fluid mechanics, optimization, transportation, economics, contact problems in elasticity and other branches of physics, *for examples*, free boundary value problems can be studied effectively in the framework of variational inequalities, moving boundary value problems can be characterized by a class of variational inequalities, the traffic assignment problem is a variational inequality problem.

Many real-life problems associated with decision sciences involving multiple objectives or criteria within the treatment. Very often, these objectives and criteria square measure in conflict. Consequently, makes an attempt to model these phenomena by one objective or criterion function have provided only rough models, that are far away from reality. We believe that vector variational inequality problems could also be utilized in this respect, as innovative, powerful and unified modeling while not losing the vector nature of the problems, [1, 2, 3, 10, 13, 20].

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Inspired by recent research works [5, 9, 11, 12, 16, 17, 19, 21], in this article, we introduce a generalized g-type exponential vector variational inequality problems in $\mathbb{R}^n$-space and defined a class of α_g-relaxed exponential (τ, μ)-monotone mappings. We prove that the existence of generalized g-type exponential vector variational inequality problems by utilizing KKM-mapping and Nadler's Theorem.

The rest of this work is organized as follows. In section 2, we mathematically state the generalized g-type exponential vector variational inequality problems and discussed some concepts and remarks. In section 3, we present the main results of our paper and some corollaries to be discussed.

2. Preliminaries

Let $Y = \mathbb{R}^n$ be an Euclidean space and C be a nonempty subset of Y. C is called a cone if $\lambda C \subset C$, for any $\lambda \geq 0$. Further, the cone C is called convex cone if $C + C \subset C$ and C is pointed cone if C is cone and $C \cap \{-C\} = \{\mathbf{0}\}$, where $\mathbf{0}$ indicate a zero vector. C is said to be proper cone, if $C \neq Y$. Now, we consider $C \subseteq Y$ is a pointed closed convex cone with $intC \neq \emptyset$ with apex at origin, where $intC$ is a set of interior points of C. Then, C induces a vector ordering in Y as follows:

(i) $x \leq_C y \Leftrightarrow y - x \in C$;
(ii) $x \not\leq_C y \Leftrightarrow y - x \notin C$;
(iii) $x \leq_{intC} y \Leftrightarrow y - x \in intC$;
(iv) $x \not\leq_{intC} y \Leftrightarrow y - x \notin intC$.

Assume that (Y, C) is an ordered space with the ordering of Y defined by a set C and ordering relation " $\leq_C$ " is a partial order. Then, we have

(i) $x \not\leq_C y \Leftrightarrow x + z \not\leq_C y + z$, for any $x, y, z \in Y$;
(ii) $x \not\leq_C y \Leftrightarrow \lambda x \not\leq_C \lambda y$, for any $\lambda \geq 0$.

Let $K \subseteq X$ be a nonempty closed convex subset of an Euclidean space $X = \mathbb{R}^n$ and (Y, C) be an ordered space induces by the closed convex pointed cone C whose apex at origin with $intC \neq \emptyset$.

Lemma 2.1. [6, 7] *Let (Y, C) be an ordered space induced by the pointed closed convex cone C with $intC \neq \emptyset$. Then for any $x, y, z \in Y$, the following relation hold:*

$$z \not\leq_{intC} x \geq_C y \Rightarrow z \not\leq_{intC} y;$$
$$z \not\geq_{intC} x \leq_C y \Rightarrow z \not\geq_{intC} y.$$

Definition 2.2. A mapping $f : X \longrightarrow Y$ is said to be:

(i) C-convex on X if

$$f(tx + (1-t)y) \leq_C tf(x) + (1-t)f(y),\ \forall x, y \in X, t \in [0, 1];$$

(ii) affine if for any $x_i \in K$ and $\lambda_i \geq 0,\ (1 \leq i \leq n)$ with $\sum_{i=1}^n \lambda_i = 1$, we have

$$f(\sum_{i=1}^n \lambda_i x_i) = \sum_{i=1}^n \lambda_i f(x_i).$$

Definition 2.3. [4] A mapping $f : K \longrightarrow Y$ is said to be completely continuous if for any sequence $\{x_n\} \in K$, $x_n \rightharpoonup x_0 \in K$ weakly, then $f(x_n) \longrightarrow f(x_0)$.

Definition 2.4. Let $f : K \longrightarrow 2^X$ be a set valued mapping. Then, f is said to be KKM-mapping if for any $\{y_1, y_2, \cdots, y_n\}$ of K, we have

$$co\{y_1, y_2, \cdots, y_n\} \subset \bigcup_{i=1}^{n} f(y_i),$$

where $co\{y_1, y_2, \cdots, y_n\}$ denotes the convex hull of $y_1, y_2, \cdots, y_n$.

Lemma 2.5. [8] *Let M be a nonempty subset of Hausdorff topological vector space X and let $f : M \longrightarrow 2^X$ be KKM-mapping. If $f(y)$ is a closed in X for all $y \in M$ and compact for some $y \in M$, then*

$$\bigcap_{y \in M} f(y) \neq \emptyset.$$

Lemma 2.6. [15] *Let E be a normed vector space and* H *be the Hausdorff metric on the collection $CB(E)$ of all closed bounded subsets of E, induced by a metric d in terms of $d(x, y) = \|x - y\|$ which is defined by*

$$\mathrm{H}(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\},$$

for $A, B \in CB(E)$. If A and B are compact subset in E, then for each $x \in A$, there exists $y \in B$ such that

$$\|x - y\| \leq \mathrm{H}(A, B).$$

Definition 2.7. Let $\mu : X \times X \longrightarrow X$ be a mapping and $Q : K \longrightarrow L(X, Y)$ be a single valued mapping, where $L(X, Y)$ is a space of all continuous linear mappings from X to Y. Let $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact set valued mapping, then

(i) Q is said to be μ-hemicontinuous if

$$\lim_{t \longrightarrow 0^+} \langle Q(x + t(y - x)), \mu(y, x)\rangle = \langle Qx, \mu(y, x)\rangle, \text{ for each } x, y \in K.$$

(ii) T is said to be H-hemicontinuous, if for any given $x, y \in K$, the mapping $t \longrightarrow \mathrm{H}(T(x + t(y - x)), Tx)$ is continuous at 0^+, where H is the Hausdorff metric defined on $CB(L(X, Y))$.

Definition 2.8. A function $f : X \longrightarrow R$ is said to be

(i) lower semicontinuous at $x_0 \in X$ if

$$f(x_0) \leq \liminf_{n} f(x_n)$$

for any sequence $\{x_n\} \subset X$ such that $\{x_n\}$ converges to x_0;

(ii) weakly upper semicontinuous at $x_0 \in X$ if

$$f(x_0) \geq \limsup_{n} f(x_n)$$

for any sequence $\{x_n\} \subset X$ such that $\{x_n\}$ converges to x_0 weakly.

Let $K \subseteq X$ be a nonempty closed convex subset of an Euclidean space $\mathbb{R}^n$ and (Y, C) be an ordered Euclidean space induced by a closed convex pointed cone C whose apex at origin with $intC \neq \emptyset$. Let $\tau \in R$ be a nonzero real number, $\mu : K \times K \longrightarrow X$, $g : K \longrightarrow K$, $f : K \times K \longrightarrow Y$ and $Q : L(X, Y) \longrightarrow L(X, Y)$ be the mappings, where $L(X, Y)$ be the space of all continuous linear mappings from X to Y. Let $T : K \longrightarrow 2^{L(X,Y)}$ be a vector set valued mapping. Then, generalized *g-type exponential vector variational inequality problems* is to find $x \in K$, $u \in T(x)$ such that

$$\langle Qu, \frac{1}{\tau}(e^{(\tau\mu(y,g(x)))} - \mathbf{1})\rangle + f(g(x), y) \nleq_{intC} 0, \ \forall y \in K, \tag{2.1}$$

here $\mathbf{1}$ is not a real number, because we deal with a vector in $\mathbb{R}^n$.

Definition 2.9. Let $Q : L(X,Y) \longrightarrow L(X,Y)$ be single-valued mapping. A multivalued mapping $T : K \longrightarrow 2^{L(X,Y)}$ with compact valued is said to be α_g-relaxed exponentially (τ, μ)-monotone with respect to Q and g, if for each pair of points $x, y \in K$, we have

$$\langle Qu - Qv, \frac{1}{\tau}(e^{(\tau\mu(x,g(y)))} - \mathbf{1})\rangle \geq_C \alpha_g(x - y), \ \forall u \in T(x), v \in T(y) \tag{2.2}$$

where $\alpha_g : X \longrightarrow Y$ with $\alpha_g(tx) = t^q \alpha_g(x)$ for all $t > 0$ and $x \in X$, where $q > 1$ is a real number.

Remark 2.10. (i) Assume that Q is an identity mapping and $T : K \longrightarrow L(X,Y)$ is single-valued mapping in (2.2), then T is said to be α_g-relaxed exponentially (τ, μ)-monotone for every pair of points $x, y \in K$, such that

$$\langle Tx - Ty, \frac{1}{\tau}(e^{(\tau\mu(x,g(y)))} - \mathbf{1})\rangle \geq_C \alpha_g(x - y) \tag{2.3}$$

where $\alpha_g : X \longrightarrow Y$ with $\alpha_g(tx) = t^q \alpha_g(x)$ for all $t > 0$ and $x \in X$, where $q > 1$ is a real number.

(ii) If $g \equiv I$ is an identity mapping in (2.3), then T is said to be α-relaxed exponentially (τ, μ)-monotone, for each pair of points $x, y \in K$, such that

$$\langle Tx - Ty, \frac{1}{\tau}(e^{(\tau\mu(x,y))} - \mathbf{1})\rangle \geq_C \alpha(x - y), \tag{2.4}$$

studied in [14].

(iii) If $\alpha \equiv 0$, then (2.4) is said to be exponentially (τ, μ)-monotone, for each pair of points $x, y \in K$, such that

$$\langle Tx - Ty, \frac{1}{\tau}(e^{(\tau\mu(x,y))} - \mathbf{1})\rangle \geq_C 0. \tag{2.5}$$

(iv) If $g \equiv I$ is an identity mapping, then Definition 2.9 becomes an α-relaxed exponentially (τ, μ)-monotone with respect to Q, for each pair of points $x, y \in K$, such that

$$\langle Qu - Qv, \frac{1}{\tau}(e^{(\tau\mu(x,y))} - \mathbf{1})\rangle \geq_C \alpha(x - y), \ \forall u \in T(x), v \in T(y). \tag{2.6}$$

(v) If $\alpha \equiv 0$, then (2.6) is said to be exponentially (τ, μ)-monotone with respect to Q, for each pair of points $x, y \in K$, such that

$$\langle Qu - Qv, \frac{1}{\tau}(e^{(\tau\mu(x,y))} - \mathbf{1})\rangle \geq_C 0, \ \forall u \in T(x), v \in T(y). \tag{2.7}$$

3. Main Results

Theorem 3.1. *Let $g : K \longrightarrow K$ be a single-valued mapping and $\mu : K \times K \longrightarrow X$ be affine in the first variable with $\mu(x, g(x)) = 0$. Let $f : K \times K \longrightarrow Y$ be affine in second variable with condition $f(g(x), x) = 0$ for all $x \in K$. Let $Q : L(X,Y) \longrightarrow L(X,Y)$ be a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact valued mapping, which is* H*-hemicontinuous and α_g-relaxed exponentially (τ, μ)-monotone with respect to Q and g. Then, the following two statements (a) and (b) are equivalent:*

(a) *there exists* $\bar{x} \in K$, $\bar{u} \in T(\bar{x})$ *such that*

$$\langle Q\bar{u}\,,\ \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) \nleq_{intC} 0,\ \forall y \in K,$$

(b) *there exists* $\bar{x} \in K$ *such that*

$$\langle Qv\,,\ \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) \nleq_{intC} \alpha_g(y-\bar{x})\ \forall y \in K, v \in T(y).$$

Proof. Assume that the statement (a) is true, then there exists $\bar{x} \in K$, $\bar{u} \in T(\bar{x})$ such that

$$\langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) \nleq_{intC} 0, \forall y \in K. \tag{3.1}$$

Since T is α_g-relaxed exponentially (τ,μ)-monotone with respect to Q and g, we have

$$\begin{aligned}
&\langle Qv - Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) \geq_C \alpha_g(y-\bar{x}) + f(g(\bar{x}),y), \forall y \in K, v \in T(y),\\
&\Rightarrow \langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) \geq_C \langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle\\
&\qquad + \alpha_g(y-\bar{x}) + f(g(\bar{x}),y), \forall y \in K, v \in T(y),\\
&\Rightarrow \langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) - \alpha_g(y-\bar{x}) \geq_C \langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle\\
&\qquad + f(g(\bar{x}),y), \forall y \in K, v \in T(y).
\end{aligned} \tag{3.2}$$

Utilizing (3.1), (3.2) and Lemma 2.1, we obtain

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) \nleq_{intC} \alpha_g(y-\bar{x}), \forall y \in K, v \in T(y).$$

Conversely, assume that the statement (b) is true, then there exists $\bar{x} \in K$ such that

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) \nleq_{intC} \alpha_g(y-\bar{x}), \forall y \in K, v \in T(y). \tag{3.3}$$

For any $y \in K$, let $y_t = ty + (1-t)\bar{x},\ t \in (0,1]$, $y_t \in K$ and K is convex. Let for all $v_t \in T(y_t)$, we have from (3.3),

$$\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y_t,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y_t) \nleq_{intC} \alpha_g(y_t-\bar{x}) = t^q\alpha_g(y-\bar{x}). \tag{3.4}$$

Now

$$\begin{aligned}
&\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y_t,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y_t)\\
&= \langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(ty+(1-t)\bar{x},g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),ty+(1-t)\bar{x})\\
&= \langle Qv_t, \frac{1}{\tau}(e^{(\tau t\mu(y,g(\bar{x}))+(1-t)\tau\mu(\bar{x},g(\bar{x})))}-\mathbf{1})\rangle + tf(g(\bar{x}),y) + (1-t)f(g(\bar{x}),\bar{x})\\
&\leq_C \langle Qv_t, \frac{1}{\tau}(t(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1}) + (1-t)(e^{(\tau\mu(\bar{x},g(\bar{x})))}-\mathbf{1}))\rangle + tf(g(\bar{x}),y)\\
&= t\{\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y)\}.
\end{aligned} \tag{3.5}$$

Utilizing (3.4), (3.5) and Lemma 2.1, we obtain

$$\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))}-\mathbf{1})\rangle + f(g(\bar{x}),y) \nleq_{intC} t^{q-1}\alpha_g(y-\bar{x}). \tag{3.6}$$

Seeing as $T(y_t)$ and $T(\bar{x})$ are compact, from Lemma 2.6, for each $v_t \in T(y_t)$, there exists $u_t \in T(\bar{x})$ such that

$$\|v_t - u_t\| \leq \mathrm{H}(T(y_t), T(\bar{x})). \tag{3.7}$$

Since $T(\bar{x})$ is compact, without loss of generality, we may possibly assume that

$$u_t \longrightarrow \bar{u} \in T(\bar{x}) \text{ as } t \longrightarrow 0^+.$$

Furthermore, T is H-hemicontinuous, thus it follows that

$$\mathrm{H}(T(y_t), T(\bar{x})) \longrightarrow 0 \text{ as } t \longrightarrow 0^+.$$

Now from (3.7), we have

$$\begin{aligned}\|v_t - \bar{u}\| &\leq \|v_t - u_t\| + \|u_t - \bar{u}\| \\ &\leq \mathrm{H}(T(y_t), T(x_t)) + \|u_t - \bar{u}\| \longrightarrow 0 \text{ as } t \longrightarrow 0^+.\end{aligned} \tag{3.8}$$

As Q is continuous, let $t \longrightarrow 0^+$, we have

$$\begin{aligned}&\|\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle - t^{q-1}\alpha_g(y-\bar{x}) - \langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle\| \\ &\leq \|\langle Qv_t - Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle\| + \|t^{q-1}\alpha_g(y-\bar{x})\| \\ &\leq \frac{1}{\tau}\|Qv_t - Q\bar{u}\|\|e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1}\| + t^{q-1}\|\alpha_g(y-\bar{x})\| \longrightarrow 0 \ \text{ as } t \longrightarrow 0^+.\end{aligned} \tag{3.9}$$

From (3.4), we get

$$\langle Qv_t, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) - t^{q-1}\alpha_g(y-\bar{x}) \in Y \setminus (-intC).$$

Since $Y \setminus (-intC)$ is closed, therefore from (3.9) we have

$$\begin{aligned}&\langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \in Y \setminus (-intC) \\ \Longrightarrow &\langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \not\leq_{intC} 0, \ \forall y \in K,\end{aligned}$$

the proof is completed. ☐

Theorem 3.2. *Let $g : K \longrightarrow K$ be a single-valued mapping, $\mu : K \times K \longrightarrow X$ be affine in the first variable with $\mu(x, g(x)) = 0$ for $x \in K$ and continuous in both variable. Let $f : K \times K \longrightarrow Y$ be affine in second variable with the condition $f(g(x), x) = 0$ for $x \in K$. Let $\alpha_g : X \longrightarrow Y$ be weakly lower semicontinuous with respect to g. Let $Q : L(X, Y) \longrightarrow L(X, Y)$ be a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be nonempty compact valued mapping, which is* H*-hemicontinuous and α_g-relaxed exponentially (τ, μ)-monotone with respect to Q and g. Then* (2.1) *is solvable, that is, there exist $x \in K$, $u \in T(x)$ such that*

$$\langle Qu, \frac{1}{\tau}(e^{(\tau\mu(y,g(x)))} - \mathbf{1})\rangle + f(g(x), y) \not\leq_{intC} 0, \ \forall y \in K.$$

Proof. Consider the set valued mapping $F : K \longrightarrow 2^X$ such that

$$F(y) = \{x \in K : \langle Qu, \frac{1}{\tau}(e^{(\tau\mu(y,g(x)))} - \mathbf{1})\rangle + f(g(x), y) \not\leq_{intC} 0, \forall u \in T(x)\}, y \in K.$$

First, we claim that F is a KKM mapping. If F is not a KKM-mapping, then there exists $(x_1, x_2, \cdots, x_m) \subset K$ such that

$$co\{x_1, x_2, \cdots, x_m\} \not\subseteq \bigcup_{i=1}^{m} F(x_i),$$

there exists at least $x \in co\{x_1, x_2, \cdots, x_m\}, x = \sum_{i=1}^m t_i x_i$, where $t_i \geq 0, i = 1, 2 \cdots, m$, $\sum_{i=1}^m t_i = 1$, but $x \notin \bigcup_{i=1}^m F(x_i)$. From the construction of F, for any $u \in T(x)$, we have

$$\langle Qu, \frac{1}{\tau}(e^{(\tau\mu(x_i, g(x)))} - \mathbf{1})\rangle + f(g(x), x_i) \leq_{intC} 0, \text{ for } i = 1, 2, \cdots, m. \tag{3.10}$$

From (3.10), since μ is affine in first variable and f is affine with respect to second variable, it follows that

$$\begin{aligned}
0 &= \langle Qu, \frac{1}{\tau}(e^{(\tau\mu(x, g(x)))} - \mathbf{1})\rangle + f(g(x), x) \\
&= \langle Qu, \frac{1}{\tau}(e^{(\tau\mu(\sum_{i=1}^m t_i x_i, g(x)))} - \mathbf{1})\rangle + f(g(x), \sum_{i=1}^m t_i x_i) \\
&= \langle Qu, \frac{1}{\tau}(e^{(\sum_{i=1}^m t_i \tau\mu(x_i, g(x)))} - \mathbf{1})\rangle + \sum_{i=1}^m t_i f(g(x), x_i) \\
&\leq_C \langle Qu, \frac{1}{\tau}\sum_{i=1}^m t_i(e^{(\tau\mu(x_i, g(x)))} - \mathbf{1})\rangle + \sum_{i=1}^m t_i f(g(x), x_i) \\
&= \sum_{i=1}^m t_i\{\langle Qu, \frac{1}{\tau}(e^{(\tau\mu(x_i, g(x)))} - \mathbf{1})\rangle + f(g(x), x_i)\} \\
&\leq_{intC} 0,
\end{aligned}$$

this show that $0 \in intC$, which is a contradiction that the fact C is proper. Hence F is KKM-mapping. Define another set valued mapping $G : K \longrightarrow 2^X$ such that

$$G(y) = \{x \in K : \langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y, g(x)))} - \mathbf{1})\rangle + f(g(x), y) \nleq_{intC} \alpha_g(y - x), \forall v \in T(y)\}, y \in K.$$

Since by Theorem 3.1, we have $F(y) \equiv G(y)$ for all $y \in K$. This implies that G is also KKM-mapping.

We claim that for each $y \in K, G(y) \subset K$ is closed in the weak topology of X. Let us suppose that $\bar{x} \in \overline{G(y)}^w$, the weak closure of $G(y)$. Since X is reflexive, there is a sequence $\{x_n\}$ in $G(y)$ such that $\{x_n\}$ converges weakly to $\bar{x} \in K$. Then for each $v \in T(y)$, we have

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y, g(x_n)))} - \mathbf{1})\rangle + f(g(x_n), y) \nleq_{intC} \alpha_g(y - x_n)$$

$$\Rightarrow \ \langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y, g(x_n)))} - \mathbf{1})\rangle + f(g(x_n), y) - \alpha_g(y - x_n) \in Y \setminus (-intC).$$

Since Qv, f and g are continuous, $Y \setminus (-intC)$ is closed, α_g is weakly lower semi-continuous, therefore the sequence

$$\{\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y, g(x_n)))} - \mathbf{1})\rangle + f(g(x_n), y) - \alpha_g(y - x_n)\}$$

converges to

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y, g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) - \alpha_g(y - \bar{x})$$

and

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y, g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) - \alpha_g(y - \bar{x}) \in Y \setminus (-intC).$$

Therefore

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y, g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \nleq_{intC} \alpha_g(y - \bar{x}).$$

Hence $\bar{x} \in G(y)$. This confirm $G(y)$ is weakly closed for all $y \in K$. Furthermore, X is reflexive and $K \subset X$ is a nonempty closed convex and bounded. Therefore, K is weakly compact subset of X and so $G(y)$ is also weakly compact. Therefore, from Lemma 2.5, it follows

$$\bigcap_{y\in K} G(y) \neq \emptyset.$$

There exists $\bar{x} \in K$ such that

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \not\leq_{intC} \alpha_g(y - \bar{x}), \forall y \in K, v \in T(y).$$

Hence, we conclude that from Theorem 3.1, there exists $\bar{x} \in K, \bar{u} \in T(\bar{x})$ such that

$$\langle Q\bar{u}, \frac{1}{\tau}(e^{(\tau\mu(y,g(\bar{x})))} - \mathbf{1})\rangle + f(g(\bar{x}), y) \not\leq_{intC} 0, \ \forall y \in K$$

the proof is completed. □

Theorem 3.3. *Let $g : K \longrightarrow K$ be a single-valued mapping and $\mu : K \times K \longrightarrow X$ be affine in the first variable with $\mu(x, g(x)) = 0$ for all $x \in K$. Let $f : K \times K \longrightarrow Y$ be a continuous mapping and affine in the second variable with the condition $f(g(x), x) = 0$ for all $x \in K$. Let $\alpha_g : X \longrightarrow Y$ be weakly lower semicontinuous. Let $Q : L(X, Y) \longrightarrow L(X, Y)$ be a mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact valued mapping, which is* H*-hemicontinuous and α_g-relaxed exponentially (τ, μ)-monotone with respect to Q and g. There exists $r > 0$ such that*

$$\langle Qv, \frac{1}{\tau}(e^{(\tau\mu(0,g(y)))} - \mathbf{1})\rangle + f(g(y), 0) \leq_{intC} 0, \ \forall y \in K, v \in T(y) \text{ with } \|y\| = r. \tag{3.11}$$

Then (2.1) *is solvable.*

Proof. For $r > 0$, assume that $K_r = \{x \in X, \|x\| \leq r\}$. From Theorem 3.2, we know that (2.1) is solvable over K_r, *i.e.*, there exists $x_r \in K \bigcap K_r$ and $u_r \in T(x_r)$ such that

$$\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu(y,g(x_r)))} - \mathbf{1})\rangle + f(g(x_r), y) \not\leq_{intC} 0, \ \forall y \in K \bigcap K_r. \tag{3.12}$$

Putting $y = 0$ in (3.12) we have

$$\langle Qu_r, \frac{1}{\tau}(e^{\tau\mu(0,g(x_r))} - \mathbf{1})\rangle + f(g(x_r), 0) \not\leq_{intC} 0. \tag{3.13}$$

If $\|x_r\| = r$ for all r, then it is contradicts to (3.11). Hence $r > \|x_r\|$. For any $z \in K$, let us choose $t \in (0, 1)$ small enough such that $(1 - t)x_r + tz \in K \bigcap K_r$. Putting $y = (1 - t)x_r + tz$ in (3.12), we get

$$\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu((1-t)x_r+tz,g(x_r)))} - \mathbf{1})\rangle + f(g(x_r), (1 - t)x_r + tz) \not\leq_{intC} 0. \tag{3.14}$$

Since μ is affine in the first variable and f is affine with respect to the second variable, we have

$$\begin{aligned}
&\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu((1-t)x_r+tz,g(x_r)))} - \mathbf{1})\rangle + f(g(x_r), (1 - t)x_r + tz) \\
&= \langle Qu_r, \frac{1}{\tau}(e^{((1-t)\tau\mu(x_r,g(x_r))+t\tau\mu(z,g(x_r)))} - \mathbf{1})\rangle + tf(g(x_r), z) \\
&\leq_C \langle Qu_r, \frac{1}{\tau}(1 - t)(e^{(\tau\mu(x_r,g(x_r)))} - \mathbf{1}) + \frac{1}{\tau}t(e^{(\tau\mu(z,g(x_r)))} - \mathbf{1})\rangle + tf(g(x_r), z) \\
&= t\{\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu(z,g(x_r)))} - \mathbf{1})\rangle + f(g(x_r), z)\}.
\end{aligned} \tag{3.15}$$

Hence from (3.14), (3.15) and Lemma 2.1, we get

$$\langle Qu_r, \frac{1}{\tau}(e^{(\tau\mu(z,g(x_r)))}-\mathbf{1})\rangle + f(g(x_r),z) \not\leq_{intC} 0,\ \forall z \in K. \tag{3.16}$$

Therefore, (2.1) is solvable and proof is completed. □

We note that, if $g = I$ is an identity mapping, then we have following corollary:

Corollary 3.1. *Let $\mu : K \times K \longrightarrow X$ be an affine in the first variable with $\mu(x,x) = 0$ for all $x \in K$ and $f : K \times K \longrightarrow Y$ be C-convex in the second variable with the condition $f(x,x) = 0$ for all $x \in K$. Let $Q : L(X,Y) \longrightarrow L(X,Y)$ be a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact valued mapping, which is* H*-hemicontinuous and α-relaxed exponentially (τ,μ)-monotone with respect to Q. Then the following two statements (a) and (b) are equivalent:*

(a) *there exists $\bar{x} \in K$, $\bar{u} \in T(\bar{x})$ such that*

$$\langle Q\bar{u}\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle + f(\bar{x},y) \not\leq_{intC} 0,\ \forall y \in K,$$

(b) *there exists $\bar{x} \in K$ such that*

$$\langle Qv\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle + f(\bar{x},y) \not\leq_{intC} \alpha(y-\bar{x})\ \forall y \in K, v \in T(y).$$

We note that, if $g = Q = I$ are identity mapping, then we have following corollary:

Corollary 3.2. *Let $\mu : K \times K \longrightarrow X$ be an affine in the first variable with $\mu(x,x) = 0$ for all $x \in K$ and $f : K \times K \longrightarrow Y$ be C-convex in the second variable with the condition $f(x,x) = 0$ for all $x \in K$. Let $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact valued mapping, which is* H*-hemicontinuous and α-relaxed exponentially (τ,μ)-monotone. Then the following two statements (a) and (b) are equivalent:*

(a) *there exists $\bar{x} \in K$, $\bar{u} \in T(\bar{x})$ such that*

$$\langle \bar{u}\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle + f(\bar{x},y) \not\leq_{intC} 0,\ \forall y \in K,$$

(b) *there exists $\bar{x} \in K$ such that*

$$\langle v\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle + f(\bar{x},y) \not\leq_{intC} \alpha(y-\bar{x})\ \forall y \in K, v \in T(y).$$

We note that, if T is a single valued mapping and $f(x,y) \equiv 0$, a zero mapping, then Corollary 3.2 reduces to the following:

Corollary 3.3. *Let $\mu : K \times K \longrightarrow X$ be an affine in the first variable with $\mu(x,x) = 0$ for all $x \in K$ and $T : X \longrightarrow L(X,Y)$ be α-relaxed exponentially (τ,μ)-monotone. Then the following two statements (a) and (b) are equivalent:*

(a) *there exists $\bar{x} \in K$ such that*

$$\langle \bar{x}\,, \frac{1}{\tau}(e^{(\tau\eta(y,\bar{x}))}-\mathbf{1})\rangle \not\leq_{intC} 0,\ \forall y \in K,$$

(b) *there exists $\bar{x} \in K$ such that*

$$\langle Tz\,, \frac{1}{\tau}(e^{(\tau\mu(y,\bar{x}))}-\mathbf{1})\rangle \not\leq_{intC} \alpha(y-\bar{x})\ \forall y, z \in K.$$

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References

1. R. P. Agarwal, M. K. Ahmad and Salahuddin, Hybrid type generalized multivalued vector complementarity problems, Ukrainian Mathematics Journal, **65**(1), 6-20, 2013, ISSN: 1027-3190.
2. G. A. Anastassiou and Salahuddin, Weakly set valued generalized vector variational inequalities, Journal Computational Analysis and Applications, **15**(4), 2013, 622–632.
3. L. Brouwer, Zur invarianz des n-dimensional gebietes. Mathematical Annllen, **71**(3), 1912, 305–313.
4. S. S. Chang, Salahuddin, C. F. Wen and X. R. Wang, On the existence problem of solutions to a class of fuzzy mixed exponential vector variational inequalities, Journal Nonlinear Science and Applications, **11**, 2018, 916–926.
5. S. S. Chang, Salahuddin, X. R. Wang, L. C. Zhao and J. F. Tang, Generalized (η, g, ϕ)-mixedvector equilibrium problems in fuzzy events, Communications on Applied Nonlinear Analysis, **27** (4), 2020, 49 – 64.
6. L. C. Ceng and J. C. Yao, On generalized variational-like inequalities with generalized monotone multivalued mappings, Applied Mathematics Letters, **22**(3), 2009, 428–434.
7. G. Y. Chen, Existence of solutions for a vector variational inequalities: An extension of Hartman-Stampacchia theorems, Journal of Optimization Theory and Applications, **74**(3), 1992, 445–456.
8. Ky Fan, A generalization of tychonoff's fixed point theorem, Mathematica Annllen **142**, 1961, 305–310.
9. Y. P. Fang and N. J. Huang, Variational-like inequalities with generalized monotone mappings in Banach spaces, Journal of Optimization Theory and Applications, **118**(2), 2003, 327–338.
10. F. Giannessi, Vector variational inequalities and vector equilibria, Mathematical Theories, Nonconvex Optimization and its Applications, **38** Kluwer Academic, Dordrecht, 2000.
11. P. Q. Khanh and N. H. Quan, Generic stability and essential components of generalized KKM points and applications, Journal of Optimization Theory and Applications, **148**, 2011, 488-504.
12. B. S. Lee, M. F. Khan and Salahuddin, Hybrid-type set-valued variational-like inequalities in Reflexive Banach spaces, Journal of Appled Mathematics and Informatics, **27**(5-6), 2009, 1371–1379.
13. B. S. Lee and Salahuddin, Minty lemma for inverted vector variational inequalities, Optimization, **66**(3), 2017, 351–359.
14. N. K. Mahto and R. N. Mohapatra, Existence results of a generalized mixed exponential type vector variational like inequalities, Mathematical Computing, Springer Nature, 2017, 209–220.
15. S. B. Nadler, Multi-valued contraction mappings, Pacific Journal of Mathematics, **30**(2), 1969, 475–488.
16. Salahuddin, General set valued vector variational inequality problems, Communication of Optimization Theory, **2017** , Article ID 13, 2017, 1–16.
17. Salahuddin, Iterative method for non-stationary mixed variational inequalities, Discontinuity, Nonlinearity and Complexity, **9** (4), 2020, 647–655.
18. G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, C. R. Academic Science Paris Ser I Math. **258**, 1964, 4413–4416.
19. R. U. Verma, A. Jayswal and S. Choudhury, Exponential type vector variational-like inequalities and vector optimization problems with exponential type invexities, Journal of Applied Mathematic and Computation, **45**(1-2), 2014, 87–97.
20. G. Wang, S. S. Chang and Salahuddin, On the existence theorems of solutions for generalized vector variational inequalities, Journal of Inequality and Application, **2015**, 2015, DOI 10.1186/s13660-015-0856-4.
21. K. Q. Wu and N. J. Huang, Vector variational-like inequalities with relaxed $\eta - \alpha$-Pseudomonotone mappings in Banach spaces, Journal of Mathematical Inequality, **1**, 2007, 281–290.

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REMARKS ON THE BETTER ADMISSIBLE MULTIMAPS

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ABSTRACT. For a quite long period, we investigated the better admissible class $\mathfrak{B}$ of multimaps on abstract convex spaces. In a paper of Liu et al. [1] in 2010, an extended class $\mathfrak{B}^+$ is introduced and fixed point theorems for maps in such class are proved. As a consequence, they deduce fixed point theorems on abstract convex Φ-spaces. However, we note that $\mathfrak{B} = \mathfrak{B}^+$ and all results in [1] are known by the present author.

KEYWORDS: Abstract convex space; KKM theory; Better admissible class of multimaps; Klee approximable set
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1. Introduction

In 1929, Knaster, Kuratowski and Mazurkiewicz (simply, KKM) obtained the so-called KKM theorem from the weak Sperner lemma and applied it to a new proof of the Brouwer fixed point theorem. Later in 1961, Ky Fan extended the KKM theorem to any topological vector spaces and applied it to various results including the Schauder fixed point theorem.

Since then there have appeared a large number of works devoting applications of the KKM theorem. In 1992, such research field was called the KKM theory by ourselves, and since 2006 the theory has been extended to abstract convex spaces by the present author.

Note that, in the KKM theory, a large number of results were obtained on various classes of topological spaces having abstract convex structure called the (partial) KKM spaces and of multimap classes such as acyclic maps, admissible class $\mathfrak{A}_c^\kappa$, better admissible class $\mathfrak{B}$, and the KKM classes $\mathfrak{KC}$, $\mathfrak{KO}$. Such research is initiated by ourselves and followed by several hundreds of other authors.

In 2010, in a work of Liu, Zhang and Tan [1], a better admissible class $\mathfrak{B}^+$ is introduced and a new fixed point theorem for better admissible multimap is proved on abstract convex spaces. As a consequence, they claimed to deduce a new fixed

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point theorem on abstract convex Φ-spaces. They also claimed that their main results generalize some recent work due to Lassonde, Kakutani, Browder, and Park, without giving any justification.

In the present paper, we show that all of the results in [1] are already known by ourselves.

2. Abstract convex spaces

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D. Multimaps are also called simply maps.

Definition 2.1. [6-8] Let E be a topological space, D a nonempty set, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Examples of abstract convex spaces are given, for example, in [6-8]

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \quad \text{or} \quad \Gamma_A \subset F^+G(A) \text{ for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ to a set Z is called *a* $\mathfrak{K}$-*map* and we say that F belongs to *the KKM family* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$, that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property.

The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$, that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

In our previous works, we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to the (partial) KKM principle. See [6-8] and the references therein.

We obtained the following diagram for subclasses of abstract convex spaces $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ Horvath space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space

$$\Longrightarrow \text{KKM space} \Longrightarrow \text{Partial KKM space}$$
$$\Longrightarrow \text{Abstract convex space.}$$

Recall that any simplex is a KKM space by the KKM theorem and its open-valued version, and that any convex subset of a t.v.s. is a KKM space by the proof of the 1961 KKM Lemma of Ky Fan; see [6]. For other subclasses of (partial) KKM spaces in the diagram, all proofs were well-established in the literature; see [6-8].

Recall also that, as subfamilies of the KKM classes $\mathfrak{KC}$ and $\mathfrak{KO}$, we investigated the better admissible class $\mathfrak{B}$ and the admissible class $\mathfrak{A}_c^\kappa$; see [6-8].

In fact, the authors of [1] formulated our concepts in [2-5] as follows:

Definition 2.4. Let $(X, D; \Gamma)$ be an abstract convex space and Y a topological space. A *better admissible class* $\mathfrak{B}$ of multimaps from X into Y is defined as follows. A multimap $F : X \multimap Y$ belongs to $\mathfrak{B}(X, Y)$ if for any $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, there exists a map $\varphi_N : \Delta_n \longrightarrow \Gamma_N$, and for any continuous function $f : F(\Gamma_N) \longrightarrow \Delta_n$, the composition

$$f \circ F|_{\Gamma_N} \circ \varphi_N : \Delta_n \longrightarrow \Delta_n$$

has a fixed point.

Motivated by the work of the present author, Liu et al. [1] defined *the better admissible class* $\mathfrak{B}^+$ of multimaps on abstract convex space as follows:

Definition 2.5. Let $(X, D; \Gamma)$ be an abstract convex space and Y a topological space. We define a class $\mathfrak{B}^+$ of multimaps from X into Y as follows. A multimap $F : X \multimap Y$ belongs to $\mathfrak{B}^+(X, Y)$ if for any $N \in \langle D \rangle$ with the cardinality $|N| = n+1$, there is a map $G \in \mathfrak{B}(\Gamma_N, Y)$ such that $G(x) \subset F(x)$ for each $x \in \Gamma_N$.

Note that Γ_N can be replaced by the compact set $\varphi_N(\Delta_n)$.

Here let us call F is *an extension* of G and we note the following:

Proposition 2.6. *Every extension of* $\mathfrak{B}$*-maps is also a* $\mathfrak{B}$*-map, that is,* $\mathfrak{B}^+ = \mathfrak{B}$.

Proof. As in Definition 2.5, since F is an extension of some $G \in \mathfrak{B}(\Gamma_N, Y)$, we have

$$f \circ G|_{\Gamma_N} \circ \varphi \subset f \circ F|_{\Gamma_N} \circ \varphi : \Delta_n \multimap \Delta_n.$$

Since $f \circ G|_{\Gamma_N} \circ \varphi$ has a fixed point, so does $f \circ F|_{\Gamma_N} \circ \varphi$. Hence $F \in \mathfrak{B}$. Q.E.D.

3. Fixed point theorems on abstract convex uniform space

In our previous work [5], we introduced the following concepts:

Definition 3.1. An *abstract convex uniform space* $(X, D; \Gamma; \mathcal{U})$ is an abstract convex space such that $(X, \mathcal{U})$ is a uniform space with a basis $\mathcal{U}$ of the uniformity consisting of symmetric entourages. For each $U \in \mathcal{U}$, let $U[x] = \{x \in X : (x, x) \in U\}$ be the U-ball around a given element $x \in X$. For $U \in \mathcal{U}$, a point $x \in X$ is called a *U-fixed point* of a map $F : X \multimap X$ if $F(x) \cap U[x] \neq \emptyset$. The map F is said to have *the almost fixed point property* if it has a U-fixed point for any $U \in \mathcal{U}$.

Definition 3.2.. For an abstract convex uniform space $(E, D; \Gamma; \mathcal{U})$, a subset X of E is said to be *admissible* (in the sense of Klee) if, for each nonempty compact subset K of X and for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \longrightarrow X$ satisfying

(1) $(x, h(x)) \in U$ for all $x \in K$;

(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$; and

(3) there exist continuous functions $p : K \longrightarrow \Delta_n$ and $\phi_N : \Delta_n \longrightarrow \Gamma_N$ with $|N| = n + 1$ such that $h = \phi_N \circ p$.

This definition was given in [5] in 2009, and as [1, Definition 3.2] in 2010.

Definition 3.3.. Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \longrightarrow E$ satisfying conditions (1)-(3) in the preceding definition. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \Gamma_N \subset X$ for some $N \in \langle D \rangle$ in condition (2).

This definition was given in [5] in 2009, and as [1, Definition 3.3] in 2010.

Theorem 3.4. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, $X \subset Y$ subsets of E, and $F : Y \multimap Y$ a map such that $F|_X \in \mathfrak{B}(X, Y)$ and $F(X)$ is Klee approximable into X. Then F has the almost fixed point property.*

Further if $(E, \mathcal{U})$ is Hausdorff, F is closed, and $\overline{F(X)}$ is compact in Y, then F has a fixed point $x_0 \in Y$ (that is, $x_0 \in F(x_0)$).

This was given as [5, Theorem 8.3] in 2009, and, for $\mathfrak{B}^+$, as [1, Theorem 3.1] in 2010.

Theorem 3.5. *Let $(X, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space and $F \in \mathfrak{B}(X, X)$ a multimap such that $F(X)$ is Klee approximable. Then F has the almost fixed point property.*

Further if F is closed and compact, then F has a fixed point $x_0 \in X$.

This was given as [5, Theorem 8.4] in 2009, and, for $\mathfrak{B}^+$, as [1, Corollary 3.1] in 2010.

Theorem 3.6. *Let $(X, D; \Gamma; \mathcal{U})$ be an admissible abstract convex uniform space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

This was given as [5, Theorem 8.5] in 2009, and, for $\mathfrak{B}^+$, as [1, Theorem 3.2] in 2010.

Corollary 3.7. *Let $(X, D; \Gamma; \mathcal{U})$ be a compact admissible abstract convex uniform space. Then any map $F \in \mathfrak{A}_c^\kappa(X, X)$ has a fixed point.*

This was given as [5, Corollary 8.6] in 2009, and, for $\mathfrak{B}^+$, as [1, Corollary 3.2] in 2010.

4. Fixed point theorems on abstract convex Φ-space

In this section, we begin with the following.

Definition 4.1. For a given abstract convex space $(E, D; \Gamma)$ and a topological space X, a map $H : X \multimap E$ is called a Φ-*map* (or a *Fan-Browder map*) if there exists a map $G : X \multimap D$ such that

(i) for each $x \in X$, $\mathrm{co}_\Gamma G(x) \subset H(x)$ [that is, $H(x)$ is Γ-convex relative to $G(x)$]; and

(ii) $X = \bigcup\{\mathrm{Int}\, G^-(y) \mid y \in D\}$.

Definition 4.2. In $(E, D; \Gamma; \mathcal{U})$, a subset Z of E is called a Φ-*set* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : Z \multimap E$ such that $\mathrm{Gr}(H) \subset U$. If E itself is a Φ-set, then it is called a Φ-*space*.

These definitions were given in [5, Section 5] in 2009, and as Definitions 4.1 and 4.2 in [1] in 2010.

Proposition 4.3. *Every locally convex subset Y of a convex subset X of a t.v.s. E is a Φ-subset of X.*

This was given as [4, Proposition 7.1] in 2008. and as [1. Lemma 4.1] in 2010.

Now we have the following fixed point theorem:

Theorem 4.4. *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, and $F \in \mathfrak{KC}(E, E)$ be a compact map. If $\overline{F(E)}$ is a Φ-set, then F has the almost fixed point property. Further if $(E, \mathcal{U})$ is separated and if F is closed, then it has a fixed point.*

This is a combined form of [3, Theorem 12 and Corollary 12.1] in 2008 and reduces to [1, Theorem 4.1] for $\mathfrak{B}^+$.

Corollary 4.5. *Let X be a nonempty convex subset of a Hausdorff t.v.s. Then any compact closed map $F \in \mathfrak{B}(X, X)$ such that $\overline{F(X)}$ is locally convex has a fixed point.*

This was given as [4, Corollary 9.11] in 2008, and as [1, Corollary 4.1] in 2010 for $\mathfrak{B}^+$.

We have the following in [2]:

Theorem 4.6. *Let $(X, D; \Gamma; \mathcal{U})$ be a Hausdorff Φ-space. Then any compact closed map $F \in \mathfrak{B}(X, X)$ has a fixed point.*

This was given as [2, Theorem 4.6] in 2000 and [4, Theorem 9.12] in 2008, and as [1, Corollary 4.2] in 2010 for $\mathfrak{B}^+$.

Remark 4.7. It should be noted that no references of ourselves in this paper appeared in [1]. Moreover, the authors of [1] claimed that their main results generalize some recent work due to Lassonde, Kakutani, Browder, and Park without giving any details or justifications.

References

[1] Liu, X., Zhang, Y. and Tan, R. *Fixed point theorems for better admissible multimaps on abstract convex spaces*, Appl. Math.–J. Chinese Univ. **25**(1) (2010) 55–62.

[2] Park, S. *Fixed points of better admissible multimaps on generalized convex spaces*, J. Korean Math. Soc. **37** (2000), 885–899.

[3] Park, S. *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45**(1) (2008), 1–27.

[4] Park, S. *A survey on fixed point theorems in generalized convex spaces, II*, Nonlinear Analysis and Convex Analysis, RIMS Kôkyûroku, Kyoto Univ. **1611** (2008), 76–85.

[5] Park, S. *Fixed point theory of multimaps in abstract convex uniform spaces*, Nonlinear Anal. TMA **71** (2009), 2468–2480.

[6] Park, S. *A history of the KKM Theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **56**(2) (2017) 1–51.

[7] Park, S. *From simplices to abstract convex spaces — A brief history of the KKM theory*, Results in Nonlinear Analysis **1**(1) (2018) 1–12.

[8] Park, S. *Extending the realm of Horvath spaces*, J. Nonlinear Convex Anal. **20**(8) (2019) 1609–1621.