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CLOSEDNESS OF THE OPTIMAL SOLUTION SETS FOR GENERAL VECTOR ALPHA OPTIMIZATION PROBLEMS

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ABSTRACT. The aim of the paper is to study the closedness of the optimal solution sets for general vector alpha optimization problems in Hausdorff locally convex topological vector spaces. Firstly, we present the relationships between the optimal solution sets of primal and dual general vector alpha optimization problems. Secondly, making use of the upper semicontinuity of a set-valued mapping, we discuss the results on closedness of the optimal solution sets for general vector alpha optimization problems in infinite-dimensional spaces.

KEYWORDS: Dual and primal general vector alpha optimization problems; Optimal solution sets; Upper $C-$ continuous set-valued mapping; Hausdorff locally convex topological vector spaces.
AMS Subject Classification: 90C29, 90C46, 49K27

1. Introduction

It is well known that the closedness, upper (lower) semicontinuity and connectedness or contractibility of optimal solution sets in set-valued optimization problems play an important role in the theory of set-valued analysis and applied analysis (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 17] and the references therein). In recent years, Gong [5] studied the connectedness and path connectedness of efficient solution sets of vector equilibrium problems using the scalarization results; Gong and Yao [6, 7] discussed the results about the lower semicontinuity and connectedness of the efficient solution sets for parametric generalized systems which was introduced by Ding and Park [4] with monotone bifunction in real locally convex Hausdorff topological vector spaces; Khanh and Luu [9] obtained the result on the upper semicontinuity of solution set of quasivariational inequalities in Hausdorff topological vector spaces;

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Khanh and Anh [10] investigated the Holder continuity of solution to parametric multivalued vector equilibrium problems in metric linear spaces; Wu and Wu [17] have discussed the characterization of solution sets of a general convex program on a normed vector space using the Gateaux differentiable.

On characterizations of the solution sets for general alpha vector optimization problems have been extensively investigated in recent years because of their fields of applications (see, e.g., [11, 12, 13, 14, 15, 16] and the references therein). For example, Lin and Tan [11, 12] introduced and studied the solution existence results for the systems of quasivariational inclusion problems of type I and related problems in infinite dimensional spaces. On using the upper and lower semicontinuity of set-valued mappings, Tan [15, 16] together with Su [14] have received the result on existences of solution of generalized systems.

However, so far as we known, there are no results in the literature on the closedness of the efficient solutions for dual and primal general vector alpha optimization problems in Hausdorff locally convex topological vector spaces. The purpose of the article is to discuss the closedness for efficient solutions of this problems.

The organization of this paper is as follows. In Section 2, we recall some basic concepts and related properties. Section 3 is devoted to the relationships between the optimal solution sets of dual and primal general vector alpha optimization problems in Hausdorff locally convex topological vector spaces. In this section, the closedness of optimal solution sets plays a central role in this paper. In Section 4, we make a conclusion to emphasis the obtained results again.

2. Preliminaries

Throughout this paper, let X and Y be two Hausdorff locally convex topological vector spaces in which Y be partially ordered by a convex cone C. We recall that C is a cone if $tc \in C$ for every $c \in C$ and every nonnegative number t. C is said to be a convex set if for any $c, d \in C$, the line segment $[c, d] = \{tc + (1-t)d : 0 \leq t \leq 1\}$ belongs to C. If C is a convex set then a cone C is called a convex cone. If C is a closed and convex set then a cone C is called a closed and convex cone. We set $l(C) := C \cap (-C)$. In this case, if $l(C) = \{0\}$ then a cone C is called a pointed cone. We denote D instead of a nonempty subset of X, and $F : D \rightrightarrows Y$ stands for a set-valued mapping F from D into Y. The domain and the graph of F are defined respectively by

$$\begin{aligned} &\operatorname{dom} F = \{x \in D \,:\, F(x) \neq \emptyset\}, \\ &\operatorname{graph} F = \{(x, y) \in D \times Y \,:\, x \in \operatorname{dom} F,\, y \in F(x)\}. \end{aligned}$$

For $A \subset X$, we denote as usual by $\text{int} A, cl\, A$ intead of the interior and the closure of A, respectively. The set of Ideal, Pareto, Proper and Weak minimal points of A with respect to C is denoted respectively as

$$IMin(A|C),\;\; PMin(A|C),\;\; PrMin(A|C) \text{ and } WMin(A|C).$$

The set of Ideal, Pareto, Proper and Weak maximal points of A with respect to C is denoted respectively as

$$IMax(A|C),\;\; PMax(A|C),\;\; PrMax(A|C) \text{ and } WMax(A|C).$$

The concepts of Ideal, Pareto, Proper and Weak minimal and maximal points can be found in Luc [13].

In this paper, the primal general vector alpha optimization problems corresponding to D, F and C (to short, $(GVOP)_{\alpha,\min}$) are defined as follows: finding $\overline{x} \in D$ such that

$$F(\overline{x}) \cap \alpha Min(F(D)|C) \neq \emptyset.$$

The set of such points $\overline{x}$ is said to be a solutions set of $(GVOP)_{\alpha,\min}$ which is denoted by $\alpha S_{\min}(D, F, C)$. The elements of $\alpha Min(F(D)|C)$ are called alpha optimal values of $(GVOP)_{\alpha,\min}$, where $\alpha = I$, $\alpha = P$, $\alpha = Pr$ and $\alpha = W$ instead of the case of Ideal, Pareto, Proper and Weak efficient points, respectively.

The dual general vector alpha optimization problems corresponding to D, F and C of problem $(GVOP)_{\alpha,\min}$, which is denoted by $(GVOP)_{\alpha,\max}$, can be defined as follows: finding $\overline{x} \in D$ such that

$$F(\overline{x}) \cap \alpha Max(F(D)|C) \neq \emptyset.$$

The set of such points $\overline{x}$ is said to be a solutions set of $(GVOP)_{\alpha,\max}$ which is denoted by $\alpha S_{\max}(D, F, C)$. The elements of $\alpha Max(F(D)|C)$ are called alpha optimal values of $(GVOP)_{\alpha,\max}$. The set D is sometimes called the set of alternatives and $F(D)$ is the set of outcomes.

We next recall the following definitions which will be needed in the paper.

Definition 2.1. ([13]) Let A be a nonempty subset of Y. We say that

(i) $x \in A$ is an ideal efficient (or ideal minimal) point of A with respect to C if $y - x \in C$ for every $y \in A$.
The set of ideal minimal points of A is denoted by $IMin(A|C)$.

(ii) $x \in A$ is an efficient (or Pareto-minimal, or nondominated) point of A with respect to C if there is no $y \in A$ with $x - y \in C \setminus l(C)$, where $l(C) := C \cap (-C)$.
The set of efficient points of A is denoted by $PMin(A|C)$.

(iii) $x \in A$ is a (global) proper efficient point of A with respect to C if there exists a convex cone $\widetilde{C}$ which is not the whole space and contains $C \setminus l(C)$ in its interior such that

$$x \in PMin(A|\widetilde{C}).$$

The set of proper efficient points of A is denoted by $PrMin(A|C)$.

(iv) Supposing that int$C$ is nonempty, $x \in A$ is a weak efficient point of A with respect to C if

$$x \in PMin(A|\operatorname{int} C \cup \{0\}).$$

The set of weak efficient points of A is denoted by $WMin(A|C)$.

The concepts of upper and lower semicontinuity with a set-valued mapping play an important role in the paper.

Definition 2.2. ([15, 16]) Let $F : D \rightrightarrows Y$ be a set-valued mapping.

(i) F is said to be upper $C-$ continuous in $\overline{x} \in \operatorname{dom} F$ if for any neighborhood V of the origin in Y, there exists a neighborhood U of $\overline{x}$ such that

$$F(x) \subset F(\overline{x}) + V + C \quad \forall\, x \in U \cap \operatorname{dom} F.$$

(ii) F is said to be lower $C-$ continuous in $\overline{x} \in \operatorname{dom} F$ if for any neighborhood V of the origin in Y, there exists a neighborhood U of $\overline{x}$ such that

$$F(\overline{x}) \subset F(x) + V - C \quad \forall\, x \in U \cap \operatorname{dom} F.$$

(iii) If F is upper $C-$ continuous and lower $C-$ continuous in $\overline{x} \in \text{dom}F$ simultaneously, we say that F is $C-$ continuous in $\overline{x}$.
(iv) If F is upper (resp. lower) $C-$ continuous in any points of $\overline{x} \in \text{dom}F$, we say that F is upper (resp., lower) $C-$ continuous on D.

Let $\emptyset \neq A \subset Y$, $C \subset Y$ be a convex cone. By making use of the concepts in Definition 2.1, we receive the equivalences of efficiency, which can be stated as follows.

Proposition 2.3. *([13]) A equivalent definition of efficiency:*

(i) *$x \in IMin(A|C)$ if and only if $x \in A$ and $A \subset x + C$.*
(ii) *$x \in IMax(A|C)$ if and only if $x \in A$ and $A \subset x - C$.*
(iii) *$x \in PMin(A|C)$ if and only if $A \cap (x - C) \subset x + l(C)$, or equivalently, when C is pointed, $x \in PMin(A|C)$ if and only if $A \cap (x - C) = \{x\}$.*
(iv) *When C is not the whole space, $x \in WMin(A|C)$ if and only if $A \cap (x - intC) = \emptyset$, or equivalently, there is no $y \in A$ such that $x - y \in \{0\} \cup intC$ and not $y - x \in \{0\} \cup intC$.*

It can be easily seen that the following equalities hold

$$\alpha Min(A| - C) = \alpha Max(A|C),$$

$$\alpha Max(A| - C) = \alpha Min(A|C),$$

where α is one of the notions I, P, Pr and W. Moreover, it follows from Proposition 2.2 in Luc [13] that the following inclusions are true:

$$PrMin(A|C) \subset PMin(A|C) \subset WMin(A|C).$$

If, in addition, $IMin(A|C) \neq \emptyset$ then

$$PMin(A|C) = IMin(A|C).$$

Finally, the strict convexity of a set-valued mapping will be provided.

Definition 2.4. ([13]) Let D be a convex subset of domF with $F : D \rightrightarrows Y$. We say that

(i) F is called strictly $C-$ convex on D, when int$C \neq \emptyset$, if for $x_1, x_2 \in D,\ x_1 \neq x_2,\ t \in (0, 1)$, i.e. $0 < t < 1$,

$$F(tx_1 + (1 - t)x_2) \subset tF(x_1) + (1 - t)F(x_2) - \text{int}C.$$

(ii) F is called strictly $C-$ quasiconvex on a nonempty convex subset $D \subset X$, when int$C \neq \emptyset$, if for $y \in Y,\ x_1, x_2 \in D,\ x_1 \neq x_2,\ t \in (0, 1)$, i.e. $0 < t < 1$,

$$F(x_1), F(x_2) \subset y - C \text{ implies } F(tx_1 + (1 - t)x_2) \subset y - \text{int}C.$$

3. Closedness of the optimal solution sets for problems $(GVOP)_{\alpha,\min}$ and $(GVOP)_{\alpha,\max}$

In this section, we discuss the closedness and relationships between the optimal solution sets of dual and primal general vector alpha optimization problems in Hausdorff locally convex topological vector spaces corresponding to D, F and C, where α is one of the qualifications: Pareto, Proper, Ideal and Weak.

Proposition 3.1. *Let $\alpha S_{\min}(D,F,C)$ be the solution set of $(GVOP)_{\alpha,\min}$, where α is one of the notions I, P, Pr and W. We have the following assertions hold.*

(i) $IS_{\min}(D,F,C) \subset PS_{\min}(D,F,C)$. *Moreover, if $IMin(F(D)|C) \neq \emptyset$ then*
$$IS_{\min}(D,F,C) = PS_{\min}(D,F,C),$$
and it is has at most a solution whenever C is pointed.

(ii) $PrS_{\min}(D,F,C) \subset PS_{\min}(D,F,C) \subset WS_{\min}(D,F,C)$.

Proof. Case (i): Let us assume that x be a solution of $(GVOP)_{I,\min}$, which yields that
$$F(x) \cap IMin(F(D)|C) \neq \emptyset.$$
By definitions, it can be easily seen that
$$F(x) \cap PMin(F(D)|C) \neq \emptyset.$$
Therefore, the vector x is an optimal solution of $(GVOP)_{P,\min}$. Making use of Proposition 2.2 [13] in the case $IMin(F(D)|C) \neq \emptyset$, and we obtain the result as required.

Case (ii): It is evident that
$$\begin{aligned} F(x) \cap PrMin(F(D)|C) &\subset F(x) \cap PMin(F(D)|C) \\ &\subset F(x) \cap WMin(F(D)|C) \quad \forall x \in D. \end{aligned}$$
Consequently,
$$PrS_{\min}(D,F,C) \subset PS_{\min}(D,F,C) \subset WS_{\min}(D,F,C),$$
which proves the claim. □

Proposition 3.2. *Let $\alpha S_{\max}(D,F,C)$ be the solution set of $(GVOP)_{\alpha,\max}$, where α is one of the notions I, P, Pr and W. We have the following assertions hold.*

(i) $IS_{\max}(D,F,C) \subset PS_{\max}(D,F,C)$. *Moreover, if $IMax(F(D)|C) \neq \emptyset$ then*
$$IS_{\max}(D,F,C) = PS_{\max}(D,F,C),$$
and it is has at most a solution whenever C is pointed.

(ii) $PrS_{\max}(D,F,C) \subset PS_{\max}(D,F,C) \subset WS_{\max}(D,F,C)$.

Proof. Case (i): Let x be a solution of $(GVOP)_{I,\max}$, which means that
$$F(x) \cap IMax(F(D)|C) \neq \emptyset.$$
By definitions, it is not hard to see that
$$F(x) \cap PMax(F(D)|C) \neq \emptyset.$$
Thus the vector x is a solution of problem $(GVOP)_{P,\max}$. If, in addition, $IMax(F(D)|C) \neq \emptyset$, taking into account of Proposition 2.2 [13], we arrive at the desired conclusion.

Case (ii): It is evident that

$$F(x) \cap PrMax(F(D)|C) \subset F(x) \cap PMax(F(D)|C)$$
$$\subset F(x) \cap WMax(F(D)|C) \ \ \forall x \in D.$$

Therefore,

$$PrS_{\max}(D, F, C) \subset PS_{\max}(D, F, C) \subset WS_{\max}(D, F, C),$$

as was to be shown. □

Proposition 3.3. *Let D be a nonempty convex subset in X and the set-valued mapping $F : D \rightrightarrows Y$ be either strictly $C-$ convex or strictly $C-$ quasiconvex on D. Assume, furthermore, that $F(x)$ is convex set for all $x \in D$. Then*

$$PS_{\min}(D, F, C) = WS_{\min}(D, F, C).$$

If, in addition, $IMin(F(D)|C) \neq \emptyset$, then

$$IS_{\min}(D, F, C) = WS_{\min}(D, F, C),$$

and it is has at most a solution whenever C is pointed.

Proof. Making use of the result obtained in Proposition 3.1 (ii), it suffices to prove that

$$WS_{\min}(D, F, C) \subset PS_{\min}(D, F, C).$$

Take arbitrary $x \in WS_{min}(D, F, C)$ and prove that $x \in PS_{\min}(D, F, C)$. In fact, we assume to the contrary, that $x \notin PS_{\min}(D, F, C)$. By definition, one finds an element $y \in D$ such that

$$F(y) \subset F(x) - C \setminus \{0\}.$$

It is well known that

$$C \setminus \{0\} \subset C, \quad C \setminus \{0\} + C \subset C,$$
$$\operatorname{int} C \subset C, \qquad \operatorname{int}C + C = \operatorname{int}C,$$
$$\frac{1}{2}F(x) + \frac{1}{2}F(x) = F(x) \ \ \forall x \in D.$$

We set

$$z = \frac{1}{2}x + \frac{1}{2}y.$$

Since D is convex set, it ensures that $z \in D$. Using the definition of strictly $C-$ quasiconvexity on D and the set $F(x)$ convex, it follows that

$$\begin{aligned} F(z) &\subset \frac{1}{2}F(x) + \frac{1}{2}F(y) - \operatorname{int}C \\ &\subset \frac{1}{2}F(x) + \frac{1}{2}F(x) - \frac{1}{2}C \setminus \{0\} - \operatorname{int}C \\ &\subset \frac{1}{2}F(x) + \frac{1}{2}F(x) - C - \operatorname{int}C \\ &\subset F(x) - \operatorname{int}C, \end{aligned}$$

which contradicting the condition $x \in WS_{min}(D, F, C)$. So, we have the following equality

$$PS_{\min}(D, F, C) = WS_{\min}(D, F, C).$$

The last case is due to preceding Proposition 3.1 and we get the required conclusion. □

Remark 3.4. It is worth noting that the results obtained in Proposition 3.3 are still holds for the senses

$$PS_{\max}(D,F,C) = WS_{\max}(D,F,C)$$

and

$$IS_{\max}(D,F,C) = WS_{\max}(D,F,C),$$

if the set-valued mapping F is strictly $(-C)-$ convex or strictly $(-C)-$ quasiconvex on D.

Theorem 3.1. *Let D be a nonempty closed subset in X and the set-valued mapping $F : D \rightrightarrows Y$ be upper $C-$ continuous on D and assumming, in addition, that C be a closed convex cone in Y and $F(x)$ compact for any $x \in D$. Then $IS_{\min}(D,F,C)$ is closed.*

Proof. Assume to the contrary, that there exists sequence $(x_\alpha)_\alpha \subset IS_{\min}(D,F,C)$ such that

$$x_\alpha \to \overline{x}, \tag{3.1}$$

where $\overline{x} \notin IS_{min}(D,F,C)$. From the initial assumption, we have that D is a closed subset in X and $(x_\alpha)_\alpha \subset D$, and so, it follows that $\overline{x} \in D$. It is well known that F is upper $C-$ continuous on D, which yields that F is also upper $C-$ continuous at $\overline{x}$. Making use of Definition 2.2, for any open convex neighborhood V of the origin in Y, there exists a neighborhood U of $\overline{x}$ in X such that

$$F(x) \subset F(\overline{x}) + \frac{1}{2}V + C \ \ \forall x \in U \cap \mathrm{dom}F. \tag{3.2}$$

It follows from (3.1) that there exists $\alpha_0 > 0$ such that

$$x_\alpha \in U \cap \mathrm{dom}F \ \text{ for every } \ \alpha > \alpha_0.$$

From (3.2) we obtain the following inclusion

$$F(x_\alpha) \subset F(\overline{x}) + \frac{1}{2}V + C \ \text{ for every } \ \alpha > \alpha_0. \tag{3.3}$$

We arbitrarily take $\alpha > \alpha_0$. It is clear that F is upper $C-$ continuous at x_α. For the preceding open convex neighborhood V, there exists a neighborhood U_α of x_α satisfying

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(x_\alpha) + \frac{1}{2}V + C. \tag{3.4}$$

Since V is convex, it holds that

$$\frac{1}{2}V + \frac{1}{2}V \subset V.$$

Combining (3.3)-(3.4) yields that

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(\overline{x}) + V + C. \tag{3.5}$$

By the initial hypotheses, $F(\overline{x})$ is a compact set, C is a closed cone and V is an open neighborhood arbitrarily, and thus, it follows from (3.5) that

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(\overline{x}) + C. \tag{3.6}$$

Let us see that

$$F(x_\alpha) \cap IMin(F(D)|C) = \emptyset \ \ \forall \alpha > \alpha_0.$$

In fact, if it was not so, then there would exists an element $y_\alpha \in F(x_\alpha)$ with $\alpha > \alpha_0$ such that

$$F(D) \subset y_\alpha + C.$$

Because $\alpha > \alpha_0$, it follows from (3.6) that

$$y_\alpha \in F(\overline{x}) + C.$$

One finds an element $c_\alpha \in C$ such that $y_\alpha - c_\alpha \in F(\overline{x})$. On the other hand, for any $\alpha > \alpha_0$,

$$\begin{aligned} F(D) &\subset \big(y_\alpha - c_\alpha\big) + c_\alpha + C \\ &\subset \big(y_\alpha - c_\alpha\big) + C + C \\ &= \big(y_\alpha - c_\alpha\big) + C. \end{aligned}$$

By virtue of Definition 2.1 together with the fact that $F(\overline{x}) \subset F(D)$, one obtains

$$y_\alpha - c_\alpha \in F(\overline{x}) \cap IMin(F(D)|C) \ \ \forall\, \alpha > \alpha_0,$$

this means that $\overline{x} \in IS_{\min}(D, F, C)$, this is a contradiction. We thus will be allowed to say that the following relation is fulfilled

$$F(x_\alpha) \cap IMin(F(D)|C) = \emptyset \ \ \forall\, \alpha > \alpha_0,$$

it means that for any $\alpha > \alpha_0$, the vector x_α does not belong to the solution set $IS_{\min}(D, F, C)$, which conflicts with the initial assumptions. So the optimal solution set $IS_{\min}(D, F, C)$ is closed, and we get the desired conclusion. □

Proposition 3.5. *Let D be a nonempty closed subset in X and the set-valued mapping $F : D \rightrightarrows Y$ be upper $(-C)-$ continuous on D and assumming, in addition, that C be a closed convex cone in Y and $F(x)$ compact for any $x \in D$. Then $IS_{\max}(D, F, C)$ is closed.*

Proof. We take $Q = -C$, then Q is a closed convex cone in Y and the set-valued mapping F is upper $Q-$ continuous on D. By using the obtained result in Theorem 3.1, we deduce that $IS_{\min}(D, F, Q)$ is closed. Therefore, the optimal solution set $IS_{max}(D, F, C)$ is also closed because the following equality holds

$$IS_{\max}(D, F, C) = IS_{\min}(D, F, Q),$$

which completing the proof. □

Theorem 3.2. *Let D be a nonempty closed subset in X and the set-valued mapping $F : D \rightrightarrows Y$ be upper $l(C)-$ continuous on D and assumming, in addition, that C be a closed convex cone in Y and $F(x)$ compact for any $x \in D$. Then $PS_{\min}(D, F, C)$ and $PS_{\max}(D, F, C)$ are closed.*

Proof. We prove only the case $PS_{\min}(D, F, C)$ is closed because the closedness of $PS_{\max}(D, F, C)$ is similarly proceeded. In fact, suppose to the contrary, that there exists sequence $(x_\alpha)_\alpha \subset PS_{\min}(D, F, C)$ such that

$$x_\alpha \to \overline{x},$$

where $\overline{x} \notin PS_{\min}(D, F, C)$. Arguing similarly as for proving Theorem 3.1, there exist neighborhoods U_α $(\alpha > \alpha_0)$ of x_α satisfying

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(\overline{x}) + l(C). \tag{3.7}$$

We next have to show that

$$F(x_\alpha) \cap PMin(F(D)|C) = \emptyset \ \ \forall\, \alpha > \alpha_0. \tag{3.8}$$

Indeed, if (3.8) does not hold, it means that there is at least an element $z_\alpha \in F(x_\alpha)$ with $\alpha > \alpha_0$ such that

$$F(D) \cap (z_\alpha - C) \subset z_\alpha + l(C) \;\; \forall\, \alpha > \alpha_0.$$

It should be noted here that for every $\alpha > \alpha_0$, by using the proof of Theorem 3.1, we obtain $x_\alpha \in U_\alpha$, and moreover it leads to the following result holds

$$z_\alpha \in F(U_\alpha \cap \mathrm{dom}F).$$

This along with (3.7) lead to there exists $c_\alpha \in l(C)$ with $\alpha > \alpha_0$ satisfying

$$z_\alpha - c_\alpha \in F(\overline{x}).$$

It can be seen that

$$\begin{aligned}
&\mathrm{int}C \subset C, \;\; t\,\mathrm{int}C = \mathrm{int}C, \;\; tC = C, \;\; \forall\, t > 0,\\
&\mathrm{int}C + C = \mathrm{int}C, \;\; (-\mathrm{int}C) + (-C) \subset -(C + C) = -C,\\
&C + C \subset C \;\text{ implies }\; l(C) + l(C) \subset l(C).
\end{aligned}$$

We thus have the following relations

$$\begin{aligned}
F(D) \cap (z_\alpha - c_\alpha - C) &\subset F(D) \cap (z_\alpha - C - C)\\
&\subset F(D) \cap (z_\alpha - C) \subset z_\alpha + l(C)\\
&= z_\alpha - c_\alpha + c_\alpha + l(C)\\
&\subset z_\alpha - c_\alpha + l(C) + l(C)\\
&= z_\alpha - c_\alpha + l(C) \;\; \forall\, \alpha > \alpha_0.
\end{aligned}$$

We set

$$y_\alpha = z_\alpha - c_\alpha.$$

Obviously,

$$y_\alpha \in F(\overline{x}) \cap PMin(F(D)|C) \;\; \forall\, \alpha > \alpha_0.$$

So we conclude that $\overline{x}$ being an optimal solution of problem $(GVOP)_{P,\min}$, which conflicts with the fact that $\overline{x} \notin PS_{\min}(D, F, C)$. Therefore, the optimal solution set of problem $PS_{\min}(D, F, C)$ is closed in X, which completes the proof. □

Theorem 3.3. *Under the assumptions of Theorem 3.1. We have the following assertions hold.*

(i) *If $IS_{\min}(D, F, C) \neq \emptyset$ then $PS_{\min}(D, F, C)$ is closed.*
(ii) *If F is upper $(-C)-$ continuous on D and $IS_{\max}(D, F, C) \neq \emptyset$ then $PS_{\max}(D, F, C)$ is closed.*

Proof. By reasons of similarly, we prove only case (i). In fact, we may assume that the optimal solution set $IS_{\min}(D, F, C) \neq \emptyset$, then it is plain that

$$PS_{\min}(D, F, C) = IS_{\min}(D, F, C).$$

Adapting the result obtained in Theorem 3.1, we conclude that $PS_{\min}(D, F, C)$ is closed and the claim follows. □

Theorem 3.4. *Let D be a nonempty closed subset in X, the set-valued mapping $F : D \rightrightarrows Y$ and assumming, in addition, that C be a closed convex cone with its interior nonempty and be not the whole space in Y and $F(x)$ compact for any $x \in D$. Then*

(i) *$WS_{\min}(D, F, C)$ is closed if F is upper $C-$ continuous on D .*
(ii) *$WS_{\max}(D, F, C)$ is closed if F is upper $(-C)-$ continuous on D.*

Proof. We proof only case (i) by reasons of similarly. Assume to the contrary, that there exists sequence $(x_\alpha)_\alpha \subset WS_{\min}(D,F,C)$ and $\overline{x} \notin WS_{\min}(D,F,C)$ such that

$$x_\alpha \to \overline{x}.$$

Repeat the proof of preceding Theorem 3.1, then one finds $\alpha_0 > 0$, U_α is a neighborhood of x_α such that

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(\overline{x}) + C \ \ \forall\, \alpha > \alpha_0. \tag{3.9}$$

It is not difficult to check that

$$F(x_\alpha) \cap PMin(F(D)|C) = \emptyset \ \ \forall\, \alpha > \alpha_0. \tag{3.10}$$

Indeed, if (3.10) does not hold, then there exists at least an element $w_\alpha \in F(x_\alpha)$ with $\alpha > \alpha_0$ such that

$$w_\alpha \in PMin(F(D)|C).$$

Because C is not the whole space, making use of Proposition 2.3 in Luc [13] to deduce that

$$F(D) \cap \Big(w_\alpha - \mathrm{int}C\Big) = \emptyset \ \ \forall\, \alpha > \alpha_0.$$

Note that for every $\alpha > \alpha_0$, then $x_\alpha \in U_\alpha$, which leads to the following result holds

$$w_\alpha \in F(U_\alpha \cap \mathrm{dom}F).$$

Together this with (3.9), it yields that there exists $c_\alpha \in C$ with $\alpha > \alpha_0$ satisfying

$$w_\alpha - c_\alpha \in F(\overline{x}).$$

Since C is a convex cone with its interior nonempty, it yields that the following equality holds

$$C + \mathrm{int}C = \mathrm{int}C.$$

Consequently,

$$w_\alpha - c_\alpha - \mathrm{int}C \subset w_\alpha - \mathrm{int}C.$$

Therefore,

$$w_\alpha - c_\alpha \in F(\overline{x}) \cap WMin(F(D)|C) \ \ \forall\, \alpha > \alpha_0,$$

which means that

$$\overline{x} \in WS_{\min}(D,F,C),$$

contradicting the fact that $\overline{x}$ is not an optimal solution of problem $(GVOP)_{W,\min}$. So the condition (3.10) holds, which leads to x_α with $\alpha > \alpha_0$ does not being optimal solutions of $(GVOP)_{W,min}$, a contradiction. From here we will be allowed to conclude that the optimal solution set $WS_{\min}(D,F,C)$ is closed and this completes the proof. □

Theorem 3.5. *Let D be a nonempty closed subset in X, the set-valued mapping $F : D \rightrightarrows Y$ and assumming, in addition, that C be a closed convex cone with its interior nonempty, $C \setminus l(C)$ be open, C be not the whole space in Y and $F(x)$ compact for any $x \in D$. Then*

(i) *$PrS_{\min}(D,F,C)$ is closed if F is upper $C-$ continuous on D and the problem $(GVOP)_{I,\min}$ has solution.*
(ii) *$PrS_{\max}(D,F,C)$ is closed if F is upper $(-C)-$ continuous on D and the problem $(GVOP)_{I,\max}$ has solution.*

Proof. Case (i): We take arbitrary sequence $(x_\alpha)_\alpha \subset PrS_{\min}(D,F,C)$ such that

$$x_\alpha \to \overline{x} \in X.$$

Since $x_\alpha \in D$ for every $\alpha \geq 1$ and the set D is closed, one gets $\overline{x} \in D$. By the initial assumption it yields that the problem $(GVOP)_{I,min}$ has solution. On one hand, it follows from Theorem 3.3 that the optimal solution set $PS_{min}(D,F,C)$ is closed. Making use of Proposition 3.1 to deduce that the following result holds

$$(x_\alpha)_\alpha \subset PS_{\min}(D,F,C).$$

Consequently,

$$\overline{x} \in PS_{\min}(D,F,C).$$

By definitions, we get

$$F(\overline{x}) \cap PMin(F(D)|C) \neq \emptyset.$$

Taking $\overline{y} \in F(\overline{x})$ such that

$$F(D) \cap (\overline{y} - C) \subset \{\overline{y}\} + l(C).$$

By picking

$$K = l(C) \cup \text{int}C.$$

Then K is a convex cone which is not the whole space and contains $C \setminus l(C)$ in its interior. In fact, we get

$$C \setminus l(C) = \text{int}C \subset \text{int}\Big(l(C) \cup \text{int}C\Big) = \text{int}K.$$

On the other hand, it is evident that

$$F(D) \cap (\overline{y} - K) \subset \{\overline{y}\} + l(K),$$

which yields that

$$\overline{y} \in F(\overline{x}) \cap PMin(F(D)|K).$$

Consequently,

$$\overline{x} \in PrS_{\min}(D,F,C),$$

which completing the proof of case (i).

Case (ii): Arguing similarly as for the proof of case (i), where a cone C is replaced by a cone $-C$, we also arrive at the conclusion. ☐

To close this paper, we give an example illustrating Theorem 3.5.

Example 3.6. Let $X = \mathbb{R}^2 = \{x = (x_1, x_2) : x_1 \in \mathbb{R}, \ \ x_2 \in \mathbb{R}\}$, $Y = \mathbb{R}$, $D = [-1,0] \times [-1,0] \subset \mathbb{R}^2$, $C = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. We consider the set-valued mapping $F : D \rightrightarrows \mathbb{R}$ is defined by

$$F(x_1, x_2) = \{x_1 + x_2\} \ \ (\forall\, (x_1, x_2) \in D).$$

It can be easily seen that the cone $C \neq Y$ is closed and convex with $\text{int}C = \mathbb{R}_{++}$ (where $\mathbb{R}_{++} := \text{int}\mathbb{R}_+$) and the cone $C \setminus l(C) = \mathbb{R}_{++}$ is open. Notice that for any $x \in D$, the set $F(x)$ is compact. Let us see that F be upper $C-$ continuous on D. In fact, take arbitrary $\overline{x} := (\overline{x}_1, \overline{x}_2) \in D$ and $\epsilon > 0$, define the neighborhood U of $\overline{x}$ by

$$U = \Big\{(x_1, x_2) \in \mathbb{R}^2 \ : \ (x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2 \leq (\frac{\epsilon}{2})^2\Big\}.$$

For every $(x_1, x_2) \in U \cap D$ (note that $D = \text{dom}F$), we obtain the following system

$$\begin{cases} (x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2 \leq (\frac{\epsilon}{2})^2 \\ x_1 + x_2 \leq 0. \end{cases}$$

We have that

$$F(x_1,x_2) \subset F(\overline{x}_1,\overline{x}_2) + (-\epsilon,\epsilon) + \mathbb{R}_+. \tag{3.11}$$

Indeed, (3.11) is equivalent to

$$x_1 + x_2 \in \overline{x}_1 + \overline{x}_2 + (-\epsilon,\epsilon) + \mathbb{R}_+,$$

or equivalently,

$$x_1 + x_2 - \overline{x}_1 - \overline{x}_2 \in (-\epsilon,+\infty).$$

Hence,

$$x_1 + x_2 - \overline{x}_1 - \overline{x}_2 > -\epsilon. \tag{3.12}$$

It is well-known that

$$\begin{aligned} |x_1 + x_2| &\le |x_1 + x_2 - \overline{x}_1 - \overline{x}_2| + |\overline{x}_1 + \overline{x}_2| \\ &\le \sqrt{2\big((x_1-\overline{x}_1)^2 + (x_2-\overline{x}_2)^2\big)} + |\overline{x}_1 + \overline{x}_2| \\ &< \epsilon + |\overline{x}_1 + \overline{x}_2|. \end{aligned}$$

So, (3.12) is fulfilled, which means that the set-valued mapping F is upper $C-$ continuous on D. We next check that $IS_{\min}(D,F,C) \neq \emptyset$. Indeed, we first pick $(x_1,x_2) \in D$, i.e., $x_i \in [-1,0]$ for $i = 1,2$, and one second considers $F(x_1,x_2) \cap IMin(F(D)|C) \neq \emptyset$. By definitions, $x_1 + x_2 \in F(D)$ and $F(D) \subset x_1 + x_2 + \mathbb{R}_+$. By directly calculating,

$$F(D) = \bigcup_{(x_1,x_2)\in D} F(x_1,x_2) = \Big[-2,0\Big].$$

Thus,

$$\begin{cases} x_1 + x_2 \ge -2 \\ x_1 + x_2 \le -2. \end{cases}$$

This system is equivalent to $x_1 + x_2 = -2$, but $x_1 \ge -1$ and $x_2 \ge -1$, which leads to $x_1 = x_2 = -1$. So,

$$IS_{\min}(D,F,C) = \{(-1,-1)\} \neq \emptyset.$$

Thanks to Theorem 3.5 that the solution sets $PrS_{\min}(D,F,C)$ and $PrS_{\max}(D,F,-C)$ are closed. In fact, in this setting, it holds that $PS_{\min}(D,F,C) = \{(-1,-1)\}$ and further, it follows from Proposition 3.1 that

$$PrS_{\min}(D,F,C) \subset \{(-1,-1)\}.$$

We have to show that $(-1,-1) \in PrS_{\min}(D,F,C)$. In fact, we define a convex cone $\widetilde{C} = \mathbb{R}_+ = [0,+\infty)$. It is obvious that $\widetilde{C}$ is not the whole space $Y = \mathbb{R}$ and contains $C \setminus l(C) = \text{int}\mathbb{R}_+$ in its interior such that $(-1,-1) \in PS_{\min}(D,F,\widetilde{C})$, and so, $PrS_{\min}(D,F,C) = \{(-1,-1)\}$, which means that it is a closed set. Similarly, if we take $C = -\mathbb{R}_+$ then $PrS_{\max}(D,F,C) = \{(-1,-1)\}$ is a closed set, as it was checked.

We close this paper by making some comparisons between the results obtained in the paper and the existing one in the literature.

Remark 3.7. As far as we know, there have not been results on closedness of the optimal solution sets for general vector alpha optimization problems in Hausdorff locally convex topological vector spaces involving the upper (lower) C-continuity of set-valued mapping. The differences between our result in the paper with the well-known results of Cheraghi et al. [1], Farajzadeh et al. [2] and Farajzadeh and Shafie [3] are as follows. We study in this paper the relationships between the optimal solution sets of primal and dual general vector alpha optimization problems in which the closedness of optimal solution sets for the same plays a central role, while Cheraghi et al. [1] derived a link between subdifferential and Fréchet differential with $\epsilon-$generalized weak subdifferential and provided a necessary and sufficient condition for achieving a global minimum of a $\epsilon-$generalized weak subdifferential function; Farajzadeh et al. [2] formulated the relationship between the nonsmooth variational-like inequalities and vector optimization problems involving the existence of solution; Farajzadeh and Shafie [3] obtained some existence theorems of the solution of the system of vector quasi-equilibrium problems for a family of multivalued mappings in the setting of topological order spaces.

4. Conclusion

In this paper, we have shown that the optimal solution sets of dual and primal general vector alpha optimization problems in Hausdorff locally convex topological vector spaces are closed. In addition, some the relationships between the optimal solution sets of these problems are also obtained well.

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References

1. P. Cheraghi, A.P. Farajzadeh, S. Suantai, On Optimization via epsilon-Generalized Weak Subdifferentials, Thai Journal of Mathematics, 16 (1), 2018, 147-164.
2. A. Farajzadeh, M.A. Noor, K.I. Noor, Vector nonsmooth variational-like inequalities and optimization problems, Nonlinear Analysis: Theory, Methods & Applications, 71 (7-8), 2009, 3471-3476.
3. A.P. Farajzadeh, A. Shafie, On system of vector quasi-equilibrium problems for multivalued mappings, Kragujevac Journal of Mathematics, 42 (3), 2018, 357-369.
4. X. P. Ding and J. Y. Park, Generalized vector equilibrium problems in generalized convex spaces, J. Optim Theory Appl., 120, 2004, 327-353.
5. X. H. Gong, Connectedness of the solution sets and scalarization for vector equilibrium problems, J. Optim Theory Appl., 133, 2007, 151-161.
6. X. H. Gong and J. C. Yao, Lower semicontinuity of the set of efficient solutions for generalized systems, J. Optim Theory Appl., 138, 2008, 197-205.
7. X. H. Gong and J. C. Yao, Connectedness of the set of efficient solutions for generalized systems, J. Optim Theory Appl., 138, 2008, 189-196.
8. V. Jeyakumar and X. Q. Yang, On characterizing the solution sets of pseudolinear programs, J. Optim Theory Appl., 87, 1995, 747-755.
9. P. Q. Khanh and L. M. Luu, Upper semicontinuity of the solution set to parametric vector quasivariational inequalities, J. Global. Optim., 32, 2005, 569-560.
10. P. Q. Khanh and L. Q. Anh, On the Holder continuity of solution to parametric multivalued vector equilibrium problems, J. Math. Anal. Appl., 321, 2006, 308-315.
11. L. J. Lin and N. X. Tan, On systems of quasivariational inclusion problems of type I and related problems, Vietnam J. Math., 34, 2006, 423-440.
12. L. J. Lin and N. X. Tan, On quasivariational inclusion problems of type I and related problems, J. Global Optim., 39, 2007, 393-407.

13. D. T. Luc, Theory of vector optimization, Lect. Notes in Eco. and Math. System, Springer Verlag, Berlin, Germany, Vol. 319, 1989.
14. T.V. Su, On the upper $C-$ continuity on the solution set of general vector alpha optimization problems in infinite-dimensional spaces, Commun. Optim. Theory, Vol. 2018, 2018, 1-11.
15. N. X. Tan, On the existence of solutions of quasivariational inclusion problems, J. Optim. Theory Appl., 123, 2004, 619-638.
16. N. X. Tan, Quasi-equilibrium problems and fixed point theorems of separately l.s.c and u.s.c mappings, Numer. Funct. Anal. Optim., (online 2018), DOI: 10.1080/01630563.2017.1360346.
17. Z. L. Wu and S. Y. Wu, Characterizations of the solution sets of comvex programs and variational inequality problems, J. Optim. Theory Appl., 130 (2), 2006, 339-358.

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APPROXIMATION OF SOLUTIONS OF SPLIT INVERSE PROBLEM FOR MULTI-VALUED DEMI-CONTRACTIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. Let H_1 and H_2 be real Hilbert spaces and $A_j : H_1 \longrightarrow H_2$, $1 \leq j \leq r$ be bounded linear linear operators, $U_i : H_1 \longrightarrow 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \longrightarrow 2^{H_2}$, $1 \leq j \leq r$ be multi-valued demi-contractive operators. An iterative scheme is constructed and shown to converge weakly to a solution of generalized split common fixed points problem (GSCFPP). Under additional mild condition, the scheme is shown to converge strongly to a solution of GSCFPP. Moreover, our scheme is of special interest.

KEYWORDS: Fixed Point; Multivalued Demi-Contractive Mappings; Split Inverse Problem.
AMS Subject Classification: 47H04, 470H10.

1. Introduction

Let X and Y be two real Banach spaces. A split inverse problem is to find a point $x^* \in X$ that solves IP_1 such that $y^* = Ax^* \in Y$ solves IP_2, where IP_1 and IP_2 are two inverse problems. A simple generalization of inverse problem is split convex feasibility problem (SCFP) which was introduced in 1994 by Censor and Elfving [18] in finite dimensional Hilbert spaces for modelling inverse problems arising from signal detection, computer temography, image recovery and radiation therapy treatment planning (see, e.g.,[5], [16], [19] and [18]). The (SCFP) is formulated as follows:

$$\text{find a point } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

where H_1, H_2 are real Hilbert spaces, $A : H_1 \to H_2$ bounded linear operator, and $C \subseteq H_1$, $Q \subseteq H_2$ are non-empty, closed and convex sets.

In what follows we denote the solution set of the (SCFP) by

$$\Gamma \equiv \Gamma(U, A) := \{y \in C : Ay \in Q\}. \tag{1.2}$$

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In 2002, Byrne in [5] proved that x^* is a solution to (1.2) if and only if it is a fixed point of

$$P_C(I - rA^*(I - P_Q)A),$$

where A^* is the adjoint operator of A, P_C and P_D are the metric projections from H_1 onto C and from H_2 onto Q, respectively, and $r > 0$ is a positive constant. Indeed, this can be easily shown using characterization of projection mapping. Censor and Segal proposed in [21], the following algorithm to solve (1.2)
Algorithm: see [[21], Algorithm 2].
let $x^* \in H_1 := \mathbb{R}^n$ be arbitrary and for $k \in \mathbb{N}$ let

$$x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k), \tag{1.3}$$

where $\gamma \in (0, \frac{2}{L})$, L being the spectral radius of the operator A^*A and I is the identity operator.

In 2010, Moudafi [32] proved the following result for approximation of solution of SCFP involving demicontractive mappings. Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at 0. If $\Gamma \neq \emptyset$, then any sequence $\{x_k\}$ generated by

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U(u_k), \ k \geq 0, \tag{1.4}$$

where $u_k = x_k + \gamma A^*(T - I)Ax_k$, $\gamma \in (0, \frac{1-\mu}{\lambda})$, λ being the spectral radius of the operator A^*A and $\alpha_k \in (0, 1)$,
converges weakly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu}{L}\big)$ and $\alpha_k \in \big(\delta, 1 - \beta - \delta\big)$ for small enough $\delta > 0$.

Recently, inspired and motivated by the result of Moudafi [32], Tang *et al*. [42] proposed a cyclic algorthm (Algorithm 2 below) to solve the SCFP for demi-contractive operators $\{U_i\}_{i=1}^p$ and $\{T_j\}_{j=1}^r$. Then they proved that the sequence generated by the proposed algorithm converges weakly to the solution of SCFP. Their work extends those of Moudafi [32], Censor and Segal [21] and others.
Algorithm 2: [42]
Let $x_0 \in H_1$ be arbitrary and let the sequence $\{x_k\}$ be defined by

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U_{i(k)}(u_k), \ k \geq 0, \tag{1.5}$$

where $u_k = x_k + \gamma A^*(T_{j(k)} - I)Ax_k$, $i(k) = k(mod\ p) + 1$ and $j(k) = k(mod\ r) + 1$, $\gamma \in (0, \frac{1-\mu}{\lambda})$, λ being the spectral radius of the operator A^*A and $\alpha_k \in (0, 1)$.

Very recently, in [25], Gibali proved the following strong convergence result for demicontractive operators; Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at 0 and that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle U(q) - q, \sigma\rangle \geq 0 \ \forall \ q \in H_1, \\ \langle A^*(T - I)Ay, \sigma\rangle \geq 0 \ \forall \ y \in H_1. \end{cases} \tag{1.6}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U(u_k), \ k \geq 0, \tag{1.7}$$

where $u_k = x_k + \gamma A^*(T-I)Ax_k$, $\gamma \in (0, \frac{1-\mu}{\lambda})$, λ being the spectral radius of the operator A^*A and $\alpha_k \in (0,1)$,
converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \left(0, \frac{1-\mu}{L}\right)$ and $\alpha_k \in \left(\delta, 1-\beta-\delta\right)$ for small enough $\delta > 0$.

Motivated by the works of Moudafi [32], A. Gibali [25], Censor and Segal [21], it is our purpose in this paper to solve a general split common fixed points problem formulated as follows:

$$\text{Find a point } x^* \in C := \cap_{i=1}^{n} C_i \text{ such that } A_j x^* \in Q_j, \tag{1.8}$$

where $A_j : H_1 \to H_2$ are bounded linear operators, $C_i = Fix(U_i)$, $1 \le i \le n$ and $Q_j = Fix(T_j)$, $1 \le j \le r$ with $U_i : H_1 \to H_1$ and $T_j : H_2 \to H_2$ multi-valued demi-contractive operators (with constants β_i, $1 \le i \le n$ and μ_j, $1 \le j \le r$, respectively).

2. Preliminaries

We begin with the following definitions and lemmas.

Definition 2.1. Let $T : H \to H$ be an operator and $D \subseteq H$ and $F(T) = \{x \in K : x = Tx\}$.

- The operator T is called nonexpansive, if $\forall x, y \in D$
$$\|Tx - Ty\| \le \|x - y\| \tag{2.1}$$
- T is called quasi-nonexpansive, if $\forall (x,q) \in D \times F(T)$
$$\|Tx - q\| \le \|x - q\| \tag{2.2}$$
- T is called k-strictly pseudo-contractive (see e.g., [28]), if there exists $k \in [0,1)$ such that $\forall (x,y) \in D$
$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \tag{2.3}$$
- T is called demi-contractive (see e.g., [3, 20, 27]), if there exists $\beta \in [0,1)$ such that $\forall (x,q) \in D \times Fix(T)$
$$\|Tx - q\|^2 \le \|x - q\|^2 + \beta\|x - Tx\|^2 \tag{2.4}$$

Definition 2.2. Let H be a real Hilbert space, an operator T is called demiclosed at $q \in H$ (see e.g., [2]), if
for any sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_k \rightharpoonup x^*$ and $Tx_k \to q$, we have $Tx^* = q$.

Definition 2.3. Let H be a real Hilbert space. The map $D : 2^H \times 2^H \longrightarrow \mathbb{R}^+$ defined by

$$\begin{aligned} D(A,B) \quad &= max\{\sup_{y\in A} d(y,B), \sup_{x \in B} d(x,A)\} \text{ for all } A, B \in 2^H, \\ where \quad & d(y,B) := \inf_{x\in A} d(x,y), \end{aligned}$$

is called Pompeiu-Hausedorff distance.

Remark 1. In general, the map D is not a metric. However, it becomes a metric if it is defined on a set of closed and bounded subsets of H.

Let $T : H \to 2^H$ be a multi-valued mapping. An element $x^* \in H$ is said to be a fixed point of T if $x^* \in Tx^*$. We denote by $F(T)$ the fixed points set of T i.e.,

$$F(T) := \{x \in H : x \in Tx\}. \tag{2.5}$$

Definition 2.4. Let H be a real Hilbert space and $CB(H)$ be a set of closed and bounded subsets of H. $T : H \to 2^{CB(H)}$ be a multi-valued mapping. Then, T is said to be demi-closed at zero if for any sequence $\{x_k\} \subset H$ with $x_k \rightharpoonup x^*$, and $d(x_k, Tx_k) \longrightarrow 0$, we have $x^* \in Tx^*$.

Definition 2.5. Let H be a real Hilbert space.

- A multi-valued mapping $T : \mathcal{D}(T) \subseteq H \to 2^{CB(H)}$ is said to be nonexpansive (see e.g., [22]), if
$$D(Tx, Ty) \leq \|x - y\| \ \forall \ x, y \in \mathcal{D}(T) \tag{2.6}$$
- The mapping $T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and
$$D(Tx, Tx^*) \leq \|x - x^*\| \ \forall \ x \in \mathcal{D}(T), \ x^* \in F(T). \tag{2.7}$$
- The mapping $T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be k-strictly pseudo-contractive if there exists there exists a constant $k \in [0,1]$ such that for all $u \in Tx$, $v \in Ty$
$$(D(Tx, Ty))^2 \leq \|x - y\|^2 + k\|x - y - (u - v)\|^2; \text{ and} \tag{2.8}$$
- $T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be demi-contractive if $F(T) \neq \emptyset$ and there exists a constant $k \in [0,1]$ such that for all $x \in \mathcal{D}(T)$, $u \in Tx$
$$(D(Tx, \{y\}))^2 \leq \|x - y\|^2 + k\|x - u\|^2. \tag{2.9}$$

The class of demi-contractive operators is a very important generalization of non-expansive operators Also some operators that arise in optimization problems are of demi-contractive type. See for example, Chidume and Maruster [11].

It is obvious that, the class of multi-valued quasi-nonexpansive is properly contained in the class of multi-valued demi-contractive operators. Indeed, consider the following example:

Example 1. (see e.g., [8]) Let $H = \mathbb{R}$ with the usual metric. Define $T : \mathbb{R} \to \mathbb{R}$ by
$$Tx = \begin{cases} [-3x, -\frac{5x}{2}], & x \in [0, \infty), \\ [-\frac{5x}{2}, -3x], & x \in (-\infty, 0]. \end{cases} \tag{2.10}$$

We have that $F(T) = \{0\}$ and T is a multi-valued demi-contractive mapping which is not quasi-nonexpansive. In fact, for each $x \in (-\infty, 0) \cup (0, \infty)$, we have
$$\begin{aligned} \big(D(Tx, T0)\big)^2 &= |-3x - 0|^2 \\ &= 9|x - 0|^2, \end{aligned}$$
which implies that T is not quasi-nonexpansive.
Also, we have that
$$\begin{aligned} \big(d(x, Tx)\big)^2 &= |x - (-\frac{5x}{2})|^2 \\ &= \frac{49}{4}|x|^2. \end{aligned}$$
Thus,
$$\begin{aligned} \big(D(Tx, T0)\big)^2 &= |x - 0|^2 + 8|x - 0|^2 \\ &= |x - 0|^2 + \frac{32}{49}\big(d(x, Tx)\big)^2. \end{aligned}$$
Therefore, T is a demi-contractive mapping with constant $k = \frac{32}{49} \in (0, 1)$.

Lemma 2.6. *Let $(X, \langle \cdot, \cdot \rangle)$ be an IPS. Then for any $x, y \in X$, and $\alpha \in [0,1]$ the following inequality holds:*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 - \alpha(1-\alpha)\|x-y\|^2 + (1-\alpha)\|y\|^2 \tag{2.11}$$

Lemma 2.7. *(see,e.g., [10]) Let $A, B \in CB(X)$ and $a \in A$. For every $\gamma > 0$, there exixts $b \in B$ such that*

$$d(a,b) \leq D(A,B) + \gamma. \tag{2.12}$$

Lemma 2.8. *(see,e.g., [10]) Let X be a reflexive real Banach space and $A, B \in CB(X)$. Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that*

$$\|a - b\| \leq D(A,B). \tag{2.13}$$

Lemma 2.9. *(see,e.g., [12]) Let E be a normed linear space, $B_1 \in CB(E)$ and $x_0 \in E$ arbitrary. Then the following hold;*

$$D(\{x_0\}, B_1) = \sup_{b_1 \in B_1} \|x_0 - b_1\|$$

Lemma 2.10. ***(Opial's lemma)*** *Let H be a real Hilbert space and $\{x_k\}$ a sequence in H such that there exists a nonempty set $\Gamma \subset H$ satisfying the following;*

i) *For every $y \in \Gamma$, $\lim \|x_k - y\|$ exists.*
ii) *Any weak-cluster point of the sequence x_k belong to Γ.*
Then, there exists $\bar{x} \in \Gamma$ such that $\{x_k\}$ converges weakly to $\bar{x}$.

Lemma 2.11. *Let $T : \mathcal{D}(T) \subseteq H \to 2^H$ be a demi-contractive, then*

$$\langle x - u, x - p \rangle \geq \frac{1-\beta}{2}\|x-u\|^2 \ \forall u \in Tx. \tag{2.14}$$

Proof. Definition of T gives

$$\begin{aligned} (D(Tx,p))^2 &\leq \|x-p\|^2 + \beta\|x-u\|^2 \ \forall u \in Tx \\ D(Tx,p) &\leq \sqrt{\|x-p\|^2 + \beta\|x-u\|^2} \ \forall u \in Tx \end{aligned}$$

We have by lemma (2.9) that $D(Tx,p) = \sup_{u \in Tx} \|u - p\|$.
Using this result we get

$$-\beta\|x-u\|^2 \leq \|x-p\|^2 - \|u-p\|^2 \ \forall u \in Tx \ . \ . \ . \ (i)$$

We observe that $2\langle x-u, x-p \rangle = \|x-u\|^2 + \|x-p\|^2 - \|u-p\|^2$,
this implies $\|x-p\|^2 - \|u-p\|^2 = 2\langle x-u, x-p\rangle - \|x-u\|^2$.
Using this in (i) we have

$$-\beta\|x-u\|^2 \leq 2\langle x-u, x-p\rangle - \|x-u\|^2,$$

hence,

$$\frac{1-\beta}{2}\|x-u\|^2 \leq \langle x-u, x-p\rangle \ \forall u \in Tx,$$

which completes the proof. □

3. Main Result

We now prove weak and strong convergence for our proposed iterative scheme. However, we begin with the following lemma.

3.1. **Weak Convergence Result.**

$$\begin{cases} q_k = x_k + \gamma \sum_{j=1}^{r} A_j^*(b_{j,k} - A_j x_k), \text{ where } b_{j,k} \in T_j(A_j x_k)\ \forall 1 \le j \le r \\ x_{k+1} = (1-\alpha_k)q_k + \frac{\alpha_k}{n}\sum_{i=1}^{n} u_{i,k}, \text{ where } u_{i,k} \in U_i(q_k)\ \forall\ 1 \le i \le n, \end{cases} \tag{3.1}$$

where U_i and T_j are multi-valued demi-contractive for each $1 \le i \le n$, $1 \le j \le r$, respectively, $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ with μ_{max} the maximum of demi-contractive constants of U_i and L being the spectral radius of the operator A^*A and $\alpha_k \in (0,1)$.

Lemma 3.1. *Let $A_j : H_1 \to H_2$, $1 \le j \le r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \le i \le n$ and $T_j : H_2 \to 2^{H_2}$, $1 \le j \le r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i)$ and nonempty $Fix(U_i) = C_i$ and $Fix(T_j) = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i$ and j and $\forall x \in H_1$, $y \in H_2$. Then for arbitrary $x_0 \in H_1$, the sequence $\{x_k\}$ generated by algorithm (3.1) is Féjer monotone with respect to Γ, that is for every $x \in \Gamma$,*

$$\|x_{k+1} - x\| \le \|x_k - x\|\ \forall\ k \in \mathbb{N},$$

provided that $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ and $\alpha_k \in (0,1)$.

Proof. Set $L := sup_{\substack{1\le i\le n \\ 1\le j\le r}} A_i^* A_j$, $\mu_{max} := sup_{1\le i\le n} U_i$, $\beta_{max} := sup_{1\le j\le r} T_j$; where U_i and T_j are demi-contractive constants of U_i and T_j, respectively.
Let $p \in \Gamma$ then from (3.1), we have

$$\begin{aligned}
\|x_{k+1} - p\|^2 &= \|(1-\alpha_k)q_k + \frac{\alpha_k}{n}\sum_{i=1}^{n} u_{i,k} - p\|^2 \\
&= \|q_k - p + \frac{\alpha_k}{n}\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2 \\
&= \|q_k - p\|^2 + 2\frac{\alpha_k}{n}\langle q_k - p, \sum_{i=1}^{n}(u_{i,k} - q_k)\rangle \\
&+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2 \\
&= \|q_k - p\|^2 + 2\frac{\alpha_k}{n}\sum_{i=1}^{n}\langle u_{i,k} - q_k, q_k - p\rangle \\
&+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2 \\
&= \|q_k - p\|^2 - 2\frac{\alpha_k}{n}\sum_{i=1}^{n}\langle q_k - u_{i,k}, q_k - p\rangle \\
&+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2
\end{aligned}$$

Using lemma (2.11), we have

$$\begin{aligned}
&\le \|q_k - p\|^2 - \frac{\alpha_k}{n}\sum_{i=1}^{n}(1-\beta_i)\|q_k - u_{i,k}\|^2 \\
&+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2
\end{aligned}$$

$$\begin{aligned} &\leq \|q_k - p\|^2 - \frac{\alpha_k}{n}(1-\beta_{max_i})\sum_{i=1}^{n}\|q_k - u_{i,k}\|^2 \\ &+ \frac{\alpha_k^2}{n}\sum_{i=1}^{n}\|(u_{i,k} - q_k)\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq \|q_k - p\|^2 - \frac{\alpha_k}{n}(1-\beta_{max})\sum_{i=1}^{n}\|q_k - u_{i,k}\|^2 \\ &+ \frac{\alpha_k^2}{n}\sum_{i=1}^{n}\|(u_{i,k} - q_k)\|^2 \\ &= \|q_k - p\|^2 \\ &- \frac{\alpha_k}{n}((1-\beta_{max}) - \alpha_k)\sum_{i=1}^{n}\|u_{i,k} - q_k\|^2 \quad \ldots (3.0.1) \end{aligned}$$

Also from (3.1), we obtain

$$\begin{aligned} \|q_k - p\|^2 &= \|x_k - p + \gamma\sum_{j=1}^{r}A_j^*(b_{j,k} - A_j x_k)\|^2 \\ &= \|x_k - p\|^2 + 2\gamma\sum_{j=1}^{r}\langle x_k - p, A_j^*(b_{j,k} - A_j x_k)\rangle \\ &+ \gamma^2\|\sum_{j=1}^{r}(b_{j,k} - A_j x_k)\|^2 \\ &= \|x_k - p\|^2 - 2\gamma\sum_{j=1}^{r}\langle A_j x_k - A_j p, A_j x_k - b_{j,k})\rangle \\ &+ \gamma^2\|\sum_{j=1}^{r}(b_{j,k} - A_j x_k)\|^2 \end{aligned}$$

Using lemma (2.11), we get

$$\begin{aligned} &\leq \|x_k - p\|^2 - \gamma\sum_{j=1}^{r}(1-\mu_j)\|b_{j,k} - A_j x_k\|^2 \\ &+ \gamma^2 rL\|b_{j,k} - A_j x_k\|^2 \end{aligned}$$

Hence,

$$\begin{aligned} \|q_k - p\|^2 &\leq \|x_k - p\|^2 - \gamma\sum_{j=1}^{r}(1-\mu_{max})\|b_{j,k} - A_j x_k\|^2 \\ &+ \gamma^2 rL\|b_{j,k} - A_j x_k\|^2 \\ &\leq \|x_k - p\|^2 - \gamma(1-\mu_{max})\sum_{j=1}^{r}\|b_{j,k} - A_j x_k\|^2 \\ &+ \gamma^2 rL\|b_{j,k} - A_j x_k\|^2 \end{aligned}$$

$$\leq \|x_k - p\|^2 - \gamma((1-\mu_{max}) - \gamma rL)\textstyle\sum_{j=1}^{r}\|b_{j,k} - A_j x_k\|^2.$$

Substituting this in (3.0.1) we have,

$$\begin{aligned}\|x_{k+1}-p\|^2 &\leq \|x_k-p\|^2-\gamma((1-\mu_{max})-\gamma rL)\sum_{j=1}^{r}\|b_{j,k}-A_jx_k\|^2\\ &- \frac{\alpha_k}{n}((1-\beta_{max})-\alpha_k)\sum_{i=1}^{n}\|u_{i,k}-q_k\|^2 \ . \ . \ . \ (3.0.2)\\ &\leq \|x_k-p\|^2\end{aligned}$$

provided $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ and $\alpha_k \in \left(0, 1-\beta_{max}\right)$.
Hence, $\{x_k\}$ is Féjer monotone. □

Let $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i)$ and nonempty $Fix(U_i) = C_i$ and $Fix(T_j) = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i \ and \ j$ and $\forall x \in H_1,\ y \in H_2$.
If $\Gamma \neq \emptyset$, then any sequence $\{x_k\}$ generated by algorithm (3.1) converges weakly to a split common fixed point $x^* \in \Gamma$, provided that $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ and $\alpha_k \in \left(\delta, 1-\beta_{max}-\delta\right)$ for small enough $\delta > 0$.

Proof. From (3.0.2), we obtained that $\{\|x_k - p\|\}$ is monotone decreasing thus, $\{x_k\}$ is bounded and $\lim \|x_k - p\|$ exists say, y^*.
Since $\{x_k\}$ is bounded, we have that there exists $\{x_{k_v}\}$ such that

$$\begin{aligned}&x_{k_v} \rightharpoonup x^* \ as \ v \to \infty, \text{ which implies that}\\ &A_jx_{k_v} \longrightarrow A_jx^* \ as \ v \to \infty, \text{ and thus}\\ &A_jx_{k_v} \rightharpoonup A_jx^* \ . \ . \ . \ (3.0.3)\end{aligned}$$

From (3.0.2) also, we have

$$\begin{aligned}&\lim \|b_{j,k} - A_jx_k\| = 0 \ as \ k \to \infty,\\ &\text{which implies that } d(T_j(A_jx_k), A_jx_k) \leq \|b_{j,k} - A_jx_k\| \longrightarrow 0 \ \forall 1 \leq j \leq r,\\ &\text{then, } d(T_j(A_jx_k), A_jx_k) \longrightarrow 0,\\ &\text{thus, } d(T_j(A_jx_{k_v}), A_jx_{k_v}) \longrightarrow 0 \ \forall 1 \leq j \leq r \ . \ . \ . \ (3.0.4)\end{aligned}$$

Since $(T_j - I)$ is demi-closed at 0, we have from (3.0.3) and (3.0.4) that

$$\begin{aligned}A_jx^* &\in T_j(A_jx^*)\\ \Rightarrow A_jx^* &\in F(T_j) \ \forall 1 \leq j \leq r\end{aligned}$$

We also have that

$$q_{k_v} = x_{k_v} + \gamma\sum_{j=1}^{r}A_j^*(b_{j,k} - A_jx_{k_v})$$

Therefore,

$$q_{k_v} \longrightarrow x^* \ . \ . \ . \ (3.0.5)$$

From (3.0.2), we have $\|u_{i,k} - q_k\| \longrightarrow 0$ as $k \longrightarrow 0$,

$$\begin{aligned}&\text{this implies that } d(U_i(q_k), q_k) \leq \|u_{i,k} - q_k\| \ \forall 1 \leq i \leq n,\\ &\text{then, } d(U_i(q_k), q_k) \longrightarrow 0 \ \forall 1 \leq i \leq n,\\ &\text{hence, } d(U_i(q_{k_v}), q_{k_v}) \longrightarrow 0 \ \forall 1 \leq i \leq n.\end{aligned}$$

This together with (3.0.5) imply that $x^* \in U_i(x^*)$ with implies that $x^* \in F(U_i)$ for each $i = 1, 2, ..., n$,
hence, $x^* \in \cap_{i=1}^n F(U_i)$ and $A_j x^* \in F(T_j)$ for each $j = 1, 2, ..., r$. Hence, $x^* \in \Gamma$.
We have shown for any subsequence $\{x_{k_v}\}$ of $\{x_k\}$ such that $x_{k_v} \rightharpoonup x^*$ that $x^* \in \Gamma$.
Thus, applying Opial's lemma we have that there exists $x^{**} \in \Gamma$ such that the sequence $x_k \rightharpoonup x^{**}$.
Hence, weak convergence for $\{x_k\}$ is established. □

We now prove strong convergence for our iterative scheme.

3.2. **Strong Convergence Result.** Let H_1 and H_2 be two real Hilbert spaces and $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i) = C_i$ and $T_j(p) = \{p\}$ for all $p \in F(T_j) = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i = 1, 2, ..., n$ *and* $j = 1, 2, ..., r$ and $\forall x \in H_1,\ y \in H_2$.
Suppose that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u_i - q, \sigma\rangle \geq 0 \ \forall\ 1 \leq i \leq n,\ u_i \in U_i(q) \ and \ q \in H_1, \\ \langle A_j^*(b_j - A_j y), \sigma\rangle \geq 0 \ \forall\ 1 \leq j \leq r,\ b_j \in T_j(A_j y) \ and \ y \in H_1. \end{cases} \tag{3.2}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by (3.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu_{max}}{rL}\big)$ and $\alpha_k \in \big(\delta, 1 - \beta_{max} - \delta\big)$ for small enough $\delta > 0$.

Proof. Let $x^* \in \Gamma$ and choose $x_0 \in H_1$ such that

$$\langle x_0 - x^*, \sigma\rangle > 0,$$

then there exists $\epsilon > 0$ such that

$$\langle x_0 - x^*, \sigma\rangle \geq \epsilon \|x_0 - x^*\|^2.$$

We now proof by induction that

$$\langle x_{k+1} - x^*, \sigma\rangle \geq \epsilon \|x_{k+1} - x^*\|^2 \ \forall\ k \geq 0. \tag{3.3}$$

Indeed, assume it holds up to some $k \geq 0$, then

$$\begin{aligned} \langle x_{k+1} - x^*, \sigma\rangle &= \langle x_{k+1} - x_k + x_k - x^*, \sigma\rangle \\ &= \langle x_{k+1} - x_k, \sigma\rangle + \langle x_k - x^*, \sigma\rangle \\ &= \langle \gamma \sum_{j=1}^r A_j^*(b_{j,k} - A_j x_k) + \frac{\alpha_k}{n} \sum_{i=1}^n (u_{i,k} - q_k), \sigma\rangle \\ &+ \langle x_k - x^*, \sigma\rangle \\ &= \gamma \sum_{j=1}^r \langle A_j^*(b_{j,k} - A_j x_k), \sigma\rangle + \frac{\alpha_k}{n} \sum_{i=1}^n \langle (u_{i,k} - q_k), \sigma\rangle \\ &+ \langle x_k - x^*, \sigma\rangle. \end{aligned}$$

Since $\gamma > 0$, $\alpha_k > 0$ and by (3.1) we get

$$\langle x_{k+1} - x^*, \sigma\rangle \geq \langle x_k - x^*, \sigma\rangle$$

by the induction assumption we have that

$$\langle x_{k+1} - x^*, \sigma\rangle \geq \epsilon \|x_k - x^*\|^2,$$

by lemma (3.1) the sequence $\{x_k\}$ generated by algorithm (3.1) is Féjer monotone with respect to Γ, so that

$$\langle x_{k+1} - x^*, \sigma\rangle \geq \epsilon\|x_{k+1} - x^*\|^2.$$

Therefore, (3.2) holds for all $k \geq 0$.
By theorem (3.3) we have

$$\begin{aligned} &x_k \rightharpoonup x^*, \text{ so that} \\ &\langle g, x_k\rangle \longrightarrow \langle g, x^*\rangle \ \forall\ g \in H_1. \end{aligned}$$

In particular, for $g = \sigma \in H_1$ we get

$$\langle \sigma, x_k\rangle \longrightarrow \langle \sigma, x^*\rangle which implies \langle \sigma, x_k - x^*\rangle \longrightarrow 0 \ as\ k \longrightarrow +\infty.$$

From (3.2) we have

$$\|x_k - x^*\|^2 \leq \frac{1}{\epsilon}\langle x_k - x^*, \sigma\rangle \longrightarrow 0 \ as\ k \longrightarrow +\infty.$$

$$\text{Thus} \quad \|x_k - x^*\|^2 \longrightarrow 0 \ as\ k \longrightarrow +\infty.$$

Consequently, $\|x_k - x^*\| \longrightarrow 0\ as\ k \longrightarrow +\infty$; and hence $x_k \longrightarrow x^* \in \Gamma$ This completes the proof. □

The following corollary is a special case of theorem (3.3) when $i = j = 1$

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be multi-valued demi-contractive (with constants β, μ, respectively) such that $U(p) = \{p\}$ for all $p \in F(U) = C$ and $T(p) = \{p\}$ for all $p \in F(T) = Q$ with $U(x)$ and $T(y)$ closed and bounded $\forall x \in H_1,\ y \in H_2$.*
Assume that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u - q, \sigma\rangle \geq 0 \ \forall\ u \in U(q) \ and\ q \in H_1, \\ \langle A^*(b - Ay), \sigma\rangle \geq 0 \ \forall\ b \in T(Ay) \ and\ y \in H_1. \end{cases} \tag{3.4}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) *converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu}{L}\big)$ and $\alpha_k \in \big(\delta, 1 - \beta - \delta\big)$ for small enough $\delta > 0$.*

The following result generalizes theorem of Moudafi [32] which is a special case of theorem (3.3) where $n = r = 1$, and U and T are single-valued demi-contractive.

Corollary 3.3. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at* 0 *and that there exists $\sigma \neq 0 \in H_1$, such that*

$$\begin{cases} \langle U(q) - q, \sigma\rangle \geq 0 \ \forall\ q \in H_1, \\ \langle A^*(T - I)Ay, \sigma\rangle \geq 0 \ \forall\ y \in H_1. \end{cases} \tag{3.5}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) *converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu}{L}\big)$ and $\alpha_k \in \big(\delta, 1 - \beta - \delta\big)$ for small enough $\delta > 0$.*

Corollary 3.4. *Let H_1 and H_2 be two real Hilbert spaces and $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued quasi-nonexpansive such that $U_i(p) = \{p\}$ for all*

$p \in F(U_i) = C_i$ and $T_j(p) = \{p\}$ for all $p \in F(T_j = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i = 1, 2, ..., n$ and $j = 1, 2, ..., r$ and $\forall x \in H_1,\ y \in H_2$.
Suppose that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u_i - q, \sigma\rangle \geq 0 \ \forall\ 1 \leq i \leq n,\ u_i \in U_i(q) \text{ and } q \in H_1, \\ \langle A_j^*(b_j - A_j y), \sigma\rangle \geq 0 \ \forall\ 1 \leq j \leq r,\ b_j \in T_j(A_j y) \text{ and } y \in H_1. \end{cases} \tag{3.6}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu_{max}}{rL}\big)$ and $\alpha_k \in \big(\delta, 1-\beta_{max}-\delta\big)$ for small enough $\delta > 0$.

3.3. **Numerical Examples.** In order to illustrate numerical application, we consider a special case of our scheme for $i = j = 1$ and $H_1 = H_2 = \mathbb{R}^3$.

All computations in this section were performed using python 3.5.2 terminal based on linux running 64-bit. The first 100 iterations of our scheme are presented in Table 1, and relationship between $\|x - x^*\|$ - values and number of iterations are given in Figure 1, where $x^* = 0 \in \Gamma$.
Now, for $x_0 = (1, 1, 1) \in \mathbb{R}^3$, $\gamma = 0.2$, $\alpha_k = 1 - \alpha_k = 0.5$, $\forall k \geq 1$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ T = \begin{bmatrix} \sqrt{\frac{3}{20}} & \sqrt{\frac{1}{20}} & 0 \\ \sqrt{\frac{1}{20}} & \sqrt{\frac{3}{20}} & \sqrt{\frac{3}{10}} \\ 0 & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{10}} \end{bmatrix},\ \text{and } U = \begin{bmatrix} \sqrt{\frac{1}{10}} & 0 & \sqrt{\frac{3}{10}} \\ 0 & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{10}} \\ \sqrt{\frac{3}{20}} & 0 & \sqrt{\frac{3}{20}} \end{bmatrix}$$

we have the following iterations for $k = 100$.

Iterations	$\|x - x^*\|$
10	$1.09e^{-01}$
20	$7.00e^{-03}$
30	$4.00e^{-04}$
40	$3.37e^{-05}$
50	$2.30e^{-06}$
60	$1.54e^{-07}$
70	$1.04e^{-08}$
80	$6.10e^{-10}$
90	$4.72e^{-11}$
100	$3.20e^{-12}$

Table 1. The first 100 iterations generated by (3.1.6).

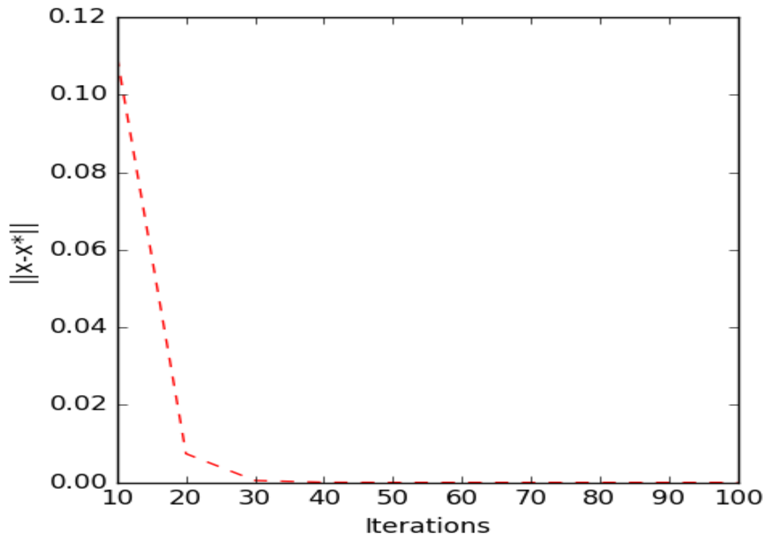

Figure 1. Relationship between $\|x - x^*\|$ - values and number of iterations.

4. Conclusion

In this paper, we have established the approximation of solution of general split inverse problem for multi-valued demi-contractive mappings in Hilbert spaces. Moreover, our result generalises many results in the literature. More precisely, theorem 3.3 generalises theorem 3.8 of [25]. Finally, lemma 2.11 and 3.1 are of special interest.

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References

1. O.A. Boikanyo; *A strongly convergent algorithm for the split common fixed point problem*, Appl. Math. Comput. **265** (2015), 844-853.
2. F. E. Browder; *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201-255.
3. F. E. Browder and W. V. Petryshyn; *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 297-228.
4. C. Byrne; *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Probl. **20**, (2004) 103-120.
5. C. Byrne; *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Prob. **18** (2002), 441-453. http://dx.doi.org/10.1088/0266-5611/18/2/310.
6. C. Byrne; *Iterative projection onto convex sets using multiple Bregman distances*, Inverse Probl. **15**, (1999) 1295-1313.
7. C. Byrne, Y. Censor, A. Gibali and S. Reich; *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), 759-775.

8. C.E. Chidume *et al.*; *The multiple-sets split equality fixed point problem for countable families of multi-valued demi-contractive mappings*, Int. J. Math. Anal. Vol. **9** (2015), 453-469. http://dx.doi.org/10.12988/ijma.2015.412209.
9. C.E. Chidume *et al.*; *The split equality fixed point problem for demi-contractive mappings*, (Accepted to appear 2014): J. Nonlinear Anal. and Opt., Theory and Applications.
10. C. E. Chidume, C. O. Chidume, N. Djitté, and M. S. Minjibir; *Convergence Theorems for Fixed Points of Multi-valued Strictly-Pseudocontractive Mappings in Hilbert Spaces*, (2012, to appear). http://dx.doi.org/10.1155/2013/629468.
11. C.E. Chidume and S. Maruster; *Iterative methods for the computation of fixed points of demi-contractive mappings*, J. Comput. Appl. Math. Vol. **234** (2010), 861-882. http://dx.doi.org/10.1016/j.cam.2010.01.050.
12. C.E. Chidume, M.E. Okpala; *On a general class of Multi-valued Strictly-Pseudocontractive Mappings*, J. of Nonlinear Anal. and Optim. Vol. **5** (2014), 07-20.
13. A. Cegielski; *General method for solving the split common fixed point problem*, J. Optim. Theory Appl. **165** (2015), 385-404.
14. Y. Censor *et al.*; *On the effectiveness of projection methods for convex feasibility problems with linear inequality constraints*, Comput. Optim. Appl. (2011, accepted for publication). doi:10.1007/s10589-011-9401-7. http://arxiv.org/abs/0912.4367.
15. Y. Censor, M.D. Altschule, W.D. Powlis; *On the use of Cimminos simultaneous projections method for computing a solution of the inverse problem in radiation therapy treatment planning*, Inverse Probl. **4**, (1988) 607-623.
16. Y. Censor, T. Bortfeld, B. Martin, A. Trofimov; *A unified approach for inversion problems in intensity-modulated radiation therapy*, Phys. Med. Biol. **51**, (2006) 2353-2365. http://dx.doi.org/10.1088/0031-9155/51/10/001.
17. Y. Censor, R. Davidi,G.T. Herman; *Perturbation resilience and superiorization of iterative algorithms*, Inverse Probl. **26**, (2010) 065008 17 pp.
18. Y. Censor, T. Elfving; *A multiprojection algorithms using Bregman projection in a product space*, Numer. Algorithms. **8**, (1994) 221-239. http://dx.doi.org/10.1007/bf02142692.
19. Y. Censor, T. Elfving, N. Kopf, T. Bortfeld; *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Probl. **21**, (2005) 2071-2084.
20. Y. Censor, A. Gibali and S. Reich; *Algorithm for the split variational inequality problem*, Numer. Algorithms **59** (2012), 301-323.
21. Y. Censor, A. Segal; *The split common fixed point problem for directed operators*, J. Convex Anal. **16**, (2009) 587-600.
22. S. S. Chang *et al.*; *The split feasibility problem for quasi-nonexpansive multi-valued mappings and total asymptotically strict pseudo-contractive mappings*, Appl. Math. Comput. Vol. **219** (2013), 10416-10424. http://dx.doi.org/10.1016/j.amc.2013.04.020.
23. H. Cui, F. Wang; *Iterative methods for the split common fixed point problem in Hilbert spaces*, Fixed Point Theory Appl. (2014), 1-8.
24. Y. Dang, Y. Gao; *The strong convergence of a KM-CQ-like algorithm for a split feasibility problem*, Inverse Probl. **27**, (2011) 015007.
25. A. Gibali; *A new Split Inverse Problem and an application to least intensity feasible solutions*, J. funct. Anal. and Appl. **2** (2017), 243-258.
26. A. Gibali, K.H. Kúfer, P. Súass; *Reformulating the pascoletti-Serafini problem as a bi-level optimization problem*, Contemp. Math. **636**, (2015) 121-129.
27. T. L. Hicks and J. R. Kubicek; *On the Mann iteration process in Hilbert space*, J. Math. Anal. Appl. **59** (1977), 498-504.
28. T. Jifang; *Split common fixed point problem for a family of quasi-nonexpansive multi-valued mappings and total asymptotically strict pseudo-contractive mapping*, Acta Mathematicae Applicatae Sinica **37** (2014), 119-126.
29. P. Kraikaew, S. Saejung; *On split common fixed point problems*, J. Math Anal. Appl. **415** (2014), 513-524.
30. A. Moudafi; *A note on the split common fixed-point problem for quasi-nonexpansive operators*, Nonlinear Anal. **74** (2011), 4083-4087.
31. A. Moudafi;*A relax CQ algorithm for convex feasibility problems*, Nonlinear Anal., **79** (2013), 117-121. http://dx.doi.org/10.1016/j.na.2012.11.013.
32. A. Moudafi; *The split common fixed point problem for demi-contractive mappings*, Inverse Probl. **26**, (2010) 1-6.
33. A. Moudafi; *Split monotone variational inclusions*, J. Optim. Theory Appl. **150** (2011), 275-283.

34. A. Moudafi, Eman Al-Shemas; *Simultaneous iterative methods for split equality problem*, Transactions on Mathematical Programming and Applications, **1** (2013), 1-11.
35. St. Maruster; *Quasi-nonexpansivity and the convex feasibility problem*, An. Univ. Al. **15** (2005), 47-56.
36. St. Maruster, and C. Popirlan; *On the Mann-type iteration and convex feasibility problem*, J. Comput. Appl. Math. **212** (2008), 390-396.
37. S. Maruster and C. Popirlan; *On the Mann-type iteration and convex feasibility problem*, J. Comput. Appl. Math. **212** (2008), 390-396. http://dx.doi.org/10.1016/j.cam.2006.12.012
38. E. Masad and S. Reich; *A note on the multiple-set split convex feasibility problem in Hilbert space*, J. Non. Con. Anal. **8**, (2007) 367-371.
39. M. O. Osilike and A. Udomene; *Demiclosedness principle and convergence theorems for stictly pseudocontractive mappings of the Browder-Petryshyn type*, J. Math. Anal. Appl. **256** (2001), 431-445.
40. R. J. Palta and T. R. Mackie, Eds; *Intensity modulated radiation therapy: the state of art*, in Medical Physics Monograph, **29**, Medical Physics Publishing, Madison, Wis, USA, (2003).
41. B. Qu, N. Xiu; *A note on the CQ algorithm for the split feasibility problem*, Inverse Probl. **21**, (2005) 1655-1666.
42. Y. C. Tang, J.G. Peng, L.W. Liu; *A cyclic algorithm for the split common fixed point problem of demi-contractive mappings in Hilbert spaces*, Math. Model. Anal. **17**, (2012) 457-466.
43. Y. Shehu, P. Cholamjiak; *Another look at the split common fixed point problem for demi-contractive operators*, (Accepted to appear 2015): Inverse Probl. (2015) 1-18.
44. F. Schopfer, T. Schuster, A.K. Louis; *An iterative regularization method for the solution of the split feasibility problem in Banach spaces*, Inverse Probl. **21**, (2005) 1655-1666.

E. Prajisha and P. Shaini, J. Nonlinear Anal. Optim. Vol. 11(1) (2020), 29-40

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COUPLED COINCIDENCE POINT THEOREMS OF MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. In this paper, we introduce a new generalized weakly contractive condition involving expressions of Kannan type contraction and establish coupled coincidence point and coupled common fixed point theorems of a pair of mappings satisfying the new contractive condition.

KEYWORDS: Coupled coincidence point; Coupled common fixed point; Mixed g- monotone property; Partially ordered set.
AMS Subject Classification: 47H10, 54F05

1. Introduction

Nowadays, fixed point techniques are widely applied in many branches of mathematics, especially in nonlinear analysis. One of the most important theorems in this regard is the fundamental theorem in metric fixed point theory, known as Banach contraction principle, which guarantees the existence and uniqueness of fixed point of contraction mappings (a mapping $T : X \to X$ is called a contraction if there exists a constant $c \in [0, 1)$ such that $d(T(x), T(y)) \leq c \cdot d(x, y),\ \forall\ x, y \in X$) defined on a complete metric space. There are many generalizations and extensions of this important result in literature (see, for example [8, 9, 11, 12, 18]). One of the notable extensions of this into partially ordered metric space is done by Ran and Reurings [16]. Further, a lot of research work is done in this line, including the results of Nieto and Lopez [14, 15]. By weakening the condition on contraction, Alber et al. [1] introduced weakly contractive maps and generalized the Banach contraction principle in Hilbert spaces. Afterwards Rhodes [18] obtained a fixed point theorem for weakly contractive maps in complete metric spaces. Followed by this, fixed points of weakly contractive maps and generalized weakly contractive maps are studied.

It is very clear that contraction maps are continuous, so the Banach contraction

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principle is applicable only for continuous functions. But Kannan [12] established a fixed point theorem for functions satisfying contraction condition called Kannan contraction, which need not be continuous.

In 2006, Gnana Bhaskar and Lakshmikantham followed the method of Nieto and Lopez, to weaken the contraction condition by considering a partial order on the metric space, and established coupled fixed point theorems of mixed monotone mappings on partially ordered complete metric space. Thereafter several research work dealing with coupled fixed point theorems are carried out. In 2009, Lakshmikantham and Ciric introduced a new concept called mixed g- monotone mapping and established coupled coincidence point and coupled common fixed point theorems for a mapping had a g and mixed g- monotone property. Also in 2011, Berinde [2] extended the result of Gnana Bhaskar and Lakshmikantham for mixed monotone mappings by weakening the contractive condition as follows:

$$d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) \leq k\cdot[d(x,u)+d(y,v)]\ \forall\ x \geqslant u,\ y \leqslant v$$

Followed by this, several authors have done research in coupled, coupled coincidence and coupled common fixed points of mappings satisfying various contractive type conditions [3, 4, 5, 6, 7]. In 2011, Choudhary et al. [6] established the existence of coupled coincidence points for pairs of mappings g and mixed g- monotone mappings, which are compatible and satisfying the following contractive type condition:

$$\psi(d(F(x,y),F(u,v))) \leq \psi(\max\{d(gx,gu),d(gy,gv)\}) - \phi(\max\{d(gx,gu),d(gy,gv)\}) \tag{1.1}$$

for all $x,y,u,v \in X$ for which $gx \leqslant gu$ and $gy \geqslant gv$, where ψ, ϕ are two control functions satisfying different conditions.

Inspired by the contractive type conditions defined by Berinde [2] and Choudhary et al. [6] and by incorporating the expressions of Kannan type contraction, we have introduced a new contractive type condition. In this paper, we have proved coupled coincidence point and coupled common fixed point theorems for pairs of mappings satisfying the newly introduced contractive condition under the settings of complete metric spaces.

2. Preliminaries

Some useful definitions are given in this section.

Definition 2.1. [13] Let $(X,\leqslant)$ be a partially ordered set and $F: X\times X \longrightarrow X$ and $g: X \longrightarrow X$. We say F has the mixed g- monotone property if F is monotone g- non-decreasing in its first argument and is monotone g- non-increasing in its second argument, that is, for any $x,y \in X$

$x_1,x_2 \in X,\ g(x_1) \leqslant g(x_2) \Longrightarrow F(x_1,y) \leqslant F(x_2,y)$ and

$y_1,y_2 \in X,\ g(y_1) \leqslant g(y_2) \Longrightarrow F(x,y_1) \geqslant F(x,y_2)$.

Definition 2.2. [5] Let (X,d) be a metric space, $F: X\times X \longrightarrow X$ and $g: X \longrightarrow X$ be two mappings. Mappings F and g are said to be compatible if

$$\lim_{n\longrightarrow\infty} d(g(F(x_n,y_n)),F(g(x_n),g(y_n))) = 0,\ \textit{and}$$
$$\lim_{n\longrightarrow\infty} d(g(F(y_n,x_n)),F(g(y_n),g(x_n))) = 0$$

hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\longrightarrow\infty} F(x_n,y_n) = \lim_{n\longrightarrow\infty} g(x_n) = x$ and $\lim_{n\longrightarrow\infty} F(y_n,x_n) = \lim_{n\longrightarrow\infty} g(y_n) = y$ for some $x,y \in X$ are satisfied.

Definition 2.3. [10] An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.4. [13] An element $(x, y) \in X \times X$ is said to be a coupled coincidence point of the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 2.5. [13] An element $(x, y) \in X \times X$ is said to be a coupled common fixed point of the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$.

Definition 2.6. [17] A function $f : X \to [0, \infty)$, where X is a metric space, is called lower semi continuous, if for all $x \in X$ and $\{x_n\} \subseteq X$ with $\lim_{n\to\infty} x_n = x$, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

3. Main Results

In this section, we prove five coupled coincidence point theorems for pairs of mapping g and mixed g- monotone mappings. The first three theorems discuss the existence of coupled coincidence points. One of the results assures the uniqueness of coupled common fixed point and in the last theorem we give an additional condition by which the components of coupled coincidence points are proved to be the same. Throughout this paper let

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) |\ \psi \text{ is continuous, monotone increasing and } \psi(t) = 0 \Leftrightarrow t = 0\} \tag{3.1}$$

and

$$\Phi = \{\phi : [0, \infty) \to [0, \infty) |\ \phi \text{ is lower semi continuous and } \phi(t) = 0 \ \Leftrightarrow \ t = 0\} \tag{3.2}$$

Let $(X, \leqslant)$ be a partial ordered set. Define a partial order $\preceq$ on $X \times X$ as: $(x, y) \preceq (u, v) \ \Leftrightarrow \ x \leqslant u \ \text{ and } \ y \geqslant v, \ \forall\, x, y, u, v \in X$.

Theorem 3.1. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous, compatible functions with $F(X \times X) \subseteq g(X)$, F satisfying the mixed g- monotone property and for all $x, y, u, v \in X$ with $g(x) \leqslant g(u)$, $g(y) \geqslant g(v)$:*

$$\psi[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))] \leq \psi[M(x, y, u, v)] - \phi[M(x, y, u, v)] \tag{3.3}$$

where $\psi \in \Psi$, $\phi \in \Phi$ and
$M(x,y,u,v) = \max\{d(g(x), F(x, y)) + d(g(y), F(y, x)), d(g(u), F(u, v)) + d(g(v), F(v, u))\}$.
If there exist $x_0, y_0 \in X$ with $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. Given $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$.
Since $F(X \times X) \subseteq g(X)$ and F satisfies the mixed g- monotone property, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$ with $g(x_n) \leqslant g(x_{n+1})$ and $g(y_{n+1}) \leqslant g(y_n)$, for $n = 0, 1, 2, \cdots$
If for some $n \in \mathbb{N}$, $g(x_n) = g(x_{n+1})$ and $g(y_{n+1}) = g(y_n)$ then the proof is complete. Otherwise we will proceed as follows:
Since $g(x_n) \leqslant g(x_{n+1})$ and $g(y_{n+1}) \leqslant g(y_n)$, for $n = 0, 1, 2, \cdots$, consider for all $n \in \mathbb{N}$,

$$\psi[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))]$$

$$
\begin{aligned}
&= \psi[d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))] \\
&\leq \psi[\max\{d(g(x_{n-1}), F(x_{n-1}, y_{n-1})) + d(g(y_{n-1}), F(y_{n-1}, x_{n-1})), \\
&\qquad\qquad d(g(x_n), F(x_n, y_n)) + d(g(y_n), F(y_n, x_n))\}] \\
&\quad -\phi[\max\{d(g(x_{n-1}), F(x_{n-1}, y_{n-1})) + d(g(y_{n-1}), F(y_{n-1}, x_{n-1})), \\
&\qquad\qquad d(g(x_n), F(x_n, y_n)) + d(g(y_n), F(y_n, x_n))\}] \\
&= \psi[\max\{d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n)), d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))\}] \\
&- \phi[\max\{d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n)), d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))\}] \quad (3.4)
\end{aligned}
$$

Suppose that for some $m \in \mathbb{N}$
$d(g(x_{m-1}), g(x_m)) + d(g(y_{m-1}), g(y_m)) \leq d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))$.
Now by (3.4) we have

$$
\begin{aligned}
\psi[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] &\leq \psi[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] \\
&\quad -\phi[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))] \\
&< \psi[d(g(x_m), g(x_{m+1})) + d(g(y_m), g(y_{m+1}))]
\end{aligned}
$$

which is a contradiction.
Therefore for all $n \in \mathbb{N}$,
$d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) < d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n))$.
Thus $\{d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))\}$ is a decreasing sequence of nonnegative reals, so there exists a $\delta \geq 0$ such that

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})) = \delta$$

Assume that $\delta > 0$.
By taking the upper limit on both sides of (3.4) we get

$$
\begin{aligned}
\psi[\delta] &\leq \psi[\delta] - \phi[\delta] \\
&< \psi[\delta]
\end{aligned}
$$

which is a contradiction. Therefore $\delta = 0$.
Next, we prove that both $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in X.
We have $g(x_n) \leqslant g(x_{n+1})$ and $g(y_{n+1}) \leqslant g(y_n)$, for $n = 0, 1, 2, \cdots$.
Now consider for $n > m$,

$$
\begin{aligned}
&\psi[d(g(x_m), g(x_n)) + d(g(y_m), g(y_n))] \\
&= \psi[d(F(x_{m-1}, y_{m-1}), F(x_{n-1}, y_{n-1})) + d(F(y_{m-1}, x_{m-1}), F(y_{n-1}, x_{n-1}))] \\
&\leq \psi[\max\{d(g(x_{m-1}), F(x_{m-1}, y_{m-1})) + d(g(y_{m-1}), F(y_{m-1}, x_{m-1})), \\
&\qquad\qquad d(g(x_{n-1}), F(x_{n-1}, y_{n-1})) + d(g(y_{n-1}), F(y_{n-1}, x_{n-1}))\}] \\
&\quad -\phi[\max\{d(g(x_{m-1}), F(x_{m-1}, y_{m-1})) + d(g(y_{m-1}), F(y_{m-1}, x_{m-1})), \\
&\qquad\qquad d(g(x_{n-1}), F(x_{n-1}, y_{n-1})) + d(g(y_{n-1}), F(y_{n-1}, x_{n-1}))] \\
&= \psi[\max\{d(g(x_{m-1}), g(x_m)) + d(g(y_{m-1}), g(y_m)), d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n))\}] \\
&\quad -\phi[\max\{d(g(x_{m-1}), g(x_m)) + d(g(y_{m-1}), g(y_m)), d(g(x_{n-1}), g(x_n)) + d(g(y_{n-1}), g(y_n))\}]
\end{aligned}
$$

By taking the upper limit as $n, m \to \infty$ on both sides we get,

$$\lim_{n,m\to\infty} \psi[d(g(x_m), g(x_n)) + d(g(y_m), g(y_n))] = 0$$

Thus both $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in X.
Since X is a complete metric space there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x \quad \text{and} \quad \lim_{n\to\infty} g(y_n) = y \qquad (3.5)$$

Since $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$ we have,

$$\lim_{n\to\infty} F(x_n, y_n) = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = y \tag{3.6}$$

Since F and g are compatible we have,

$$\begin{aligned} \lim_{n\longrightarrow\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) &= 0, \text{ and} \\ \lim_{n\longrightarrow\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) &= 0 \end{aligned}$$

Now, by the continuity of F and g we have, $F(x, y) = g(x)$ and $F(y, x) = g(y)$. Thus the proof. □

Corollary 3.2. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous, compatible functions with $F(X \times X) \subseteq g(X)$, F satisfying the mixed g- monotone property and for all $x, y, u, v \in X$ with $g(x) \leqslant g(u)$, $g(y) \geqslant g(v)$:*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq M(x, y, u, v) - \phi[M(x, y, u, v)]$$

where $\phi \in \Phi$ and
$M(x,y,u,v) = \max\{d(g(x), F(x,y)) + d(g(y), F(y,x)), d(g(u), F(u,v)) + d(g(v), F(v,u))\}$.
If there exist $x_0, y_0 \in X$ with $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. By taking ψ as the identity function on $[0, \infty)$ in Theorem 3.1, we get the result. □

Corollary 3.3. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous, compatible functions with $F(X \times X) \subseteq g(X)$, F satisfying the mixed g- monotone property and for all $x, y, u, v \in X$ with $g(x) \leqslant g(u)$, $g(y) \geqslant g(v)$:*

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &\quad \leq k \cdot \max\{d(g(x), F(x, y)) + d(g(y), F(y, x)), d(g(u), F(u, v)) + d(g(v), F(v, u))\} \end{aligned}$$

If there exist $x_0, y_0 \in X$ with $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. By taking $\phi(p) = (1 - k)p$, for $p \in [0, \infty)$ in Corollary 3.2, we get the result. □

Corollary 3.4. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose $F : X \times X \to X$ be continuous function with F satisfying the mixed monotone property and for all $x, y, u, v \in X$ with $x \leqslant u$, $y \geqslant v$:*

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &\qquad \leq k \cdot \max\{d(x, F(x, y)) + d(y, F(y, x)), d(u, F(u, v)) + d(v, F(v, u))\} \end{aligned}$$

If there exist $x_0, y_0 \in X$ with $x_0 \leqslant F(x_0, y_0)$ and $y_0 \geqslant F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. By considering g as the identity function on X in Corollary 3.3, we get the result. □

The following theorem guarantees the existence of coupled coincidence points of F and g in which F need not be continuous.

Theorem 3.5. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose that X has the following properties:*

(i) *if an increasing sequence* $\{x_n\}$ *converges to* x *then* $x_n \leqslant x, \ \forall n$
(ii) *if a decreasing sequence* $\{y_n\}$ *converges to* y *then* $y \leqslant y_n, \ \forall n$.

Let $F : X \times X \to X$ *and* $g : X \to X$ *be compatible functions with* $F(X \times X) \subseteq g(X)$, g *an order preserving, continuous function and* F *satisfying the mixed* g- *monotone property and* F *and* g *satisfy the following:*
For all $x, y, u, v \in X$ *with* $g(x) \leqslant g(u), \ g(y) \geqslant g(v)$

$$\psi[d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))] \leq \psi[M(x,y,u,v)] - \phi[M(x,y,u,v)]$$

where $\psi \in \Psi$, $\phi \in \Phi$ *and*
$M(x,y,u,v) = \max\{d(g(x), F(x,y)) + d(g(y), F(y,x)), d(g(u), F(u,v)) + d(g(v), F(v,u))\}$.
If there exist $x_0, y_0 \in X$ *with* $g(x_0) \leqslant F(x_0,y_0)$ *and* $g(y_0) \geqslant F(y_0,x_0)$, *then there exist* $x, y \in X$ *such that* $g(x) = F(x,y)$ *and* $g(y) = F(y,x)$.

Proof. Following as in Theorem 3.1 we can have, an increasing sequence $\{g(x_n)\}$ and a decreasing sequence $\{g(y_n)\}$ defined as $g(x_{n+1}) = F(x_n, y_n)$, $g(y_{n+1}) = F(y_n, x_n)$ such that

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y$$

By the hypothesis we have, $g(x_n) \leqslant x$ and $y \leqslant g(y_n), \ \forall n \in \mathbb{N}$
Since g is order preserving, we get $g(g(x_n)) \leqslant g(x)$ and $g(y) \leqslant g(g(y_n)), \ \forall n \in \mathbb{N}$.
Since g is continuous and F and g are compatible we have

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(g(x_n), g(y_n)) \\ \text{and } g(y) &= \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(g(y_n), g(x_n)) \end{aligned}$$

Suppose $F(x,y) \neq g(x)$ or $F(y,x) \neq g(y)$.
Since $g(g(x_n)) \leqslant g(x)$ and $g(y) \leqslant g(g(y_n)), \ \forall n \in \mathbb{N}$, we have

$$\begin{aligned} &\psi[d(F(g(x_n), g(y_n)), F(x,y)) + d(F(g(y_n), g(x_n)), F(y,x))] \\ &\quad \leq \ \psi[\max\{d(g(g(x_n)), F(g(x_n), g(y_n))) + d(g(g(y_n)), F(g(y_n), g(x_n))), \\ &\qquad\qquad d(g(x), F(x,y)) + d(g(y), F(y,x))\}] \\ &\qquad -\phi[\max\{d(g(g(x_n)), F(g(x_n), g(y_n))) + d(g(g(y_n)), F(g(y_n), g(x_n))), \\ &\qquad\qquad d(g(x), F(x,y)) + d(g(y), F(y,x))\}] \end{aligned}$$

By taking the upper limit on both sides we get

$$\begin{aligned} \psi[d(g(x), F(x,y)) + d(g(y), F(y,x))] &\leq \psi[d(g(x), F(x,y)) + d(g(y), F(y,x))] \\ &\quad -\phi[d(g(x), F(x,y)) + d(g(y), F(y,x))] \\ &< \psi[d(g(x), F(x,y)) + d(g(y), F(y,x))] \end{aligned}$$

which is a contradiction. Thus $g(x) = F(x,y)$ and $g(y) = F(y,x)$.
Hence the proof. □

In the following theorem we omit the completeness of the underlying space X and the compatibility and continuity conditions of the functions F and g assumed in Theorem 3.1. The following theorem guarantees the existence of coupled coincidence points of F and g.

Theorem 3.6. *Let* $(X, d, \leqslant)$ *be a partially ordered metric space and* X *has the following property:*

(i) *if an increasing sequence* $\{x_n\}$ *converges to* x *then* $x_n \leqslant x, \ \forall n$
(ii) *if a decreasing sequence* $\{y_n\}$ *converges to* y *then* $y \leqslant y_n, \ \forall n$.

Let $F : X \times X \to X$ and $g : X \to X$ be two functions with $F(X \times X) \subseteq g(X)$ and F satisfying the mixed g- monotone property and for all $x, y, u, v \in X$ with $g(x) \leqslant g(u),\ g(y) \geqslant g(v)$:

$$\psi[d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))] \leq \psi[M(x,y,u,v)] - \phi[M(x,y,u,v)]$$

where $\psi \in \Psi$, $\phi \in \Phi$ and
$M(x,y,u,v) = \max\{d(g(x),F(x,y)) + d(g(y),F(y,x)), d(g(u),F(u,v)) + d(g(v),F(v,u))\}$.
Suppose $g(X)$ is a complete subspace of X and if there exist $x_0, y_0 \in X$ with $g(x_0) \leqslant F(x_0,y_0)$ and $g(y_0) \geqslant F(y_0,x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Proof. Following as in Theorem 3.1, we get an increasing Cauchy sequence $\{g(x_n)\}$ and a decreasing Cauchy sequence $\{g(y_n)\}$ in X defined as $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$.
Since $g(X)$ is a complete subspace of X, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = g(x) \text{ and } \lim_{n\to\infty} g(y_n) = g(y)$$

By the hypothesis we have, $g(x_n) \leqslant g(x)$ and $g(y) \leqslant g(y_n),\ \forall n \in \mathbb{N}$.
Suppose $F(x,y) \neq g(x)$ or $F(y,x) \neq g(y)$.
Now consider,

$$\begin{aligned}
&\psi[d(F(x_n,y_n),F(x,y)) + d(F(y_n,x_n),F(y,x))] \\
&\quad\leq \psi[\max\{d(g(x_n),g(x_{n+1})) + d(g(y_n),g(y_{n+1})), d(g(x),F(x,y)) + d(g(y),F(y,x))\}] \\
&\qquad -\phi[\max\{d(g(x_n),g(x_{n+1})) + d(g(y_n),g(y_{n+1})), d(g(x),F(x,y)) + d(g(y),F(y,x))\}]
\end{aligned}$$

Taking the upper limit on both sides we get

$$\begin{aligned}
\psi[d(g(x),F(x,y)) + d(g(y),F(y,x))] &\leq \psi[d(g(x),F(x,y)) + d(g(y),F(y,x))] \\
&\quad -\phi[d(g(x),F(x,y)) + d(g(y),F(y,x))] \\
&< \psi[d(g(x),F(x,y)) + d(g(y),F(y,x))]
\end{aligned}$$

which is a contradiction.
Thus $g(x) = F(x,y)$ and $g(y) = F(y,x)$. □

Theorem 3.7. *In addition to the hypothesis of Theorem 3.1 suppose that for any $(x,y),(u,v) \in X \times X$ there exist $(\alpha,\beta) \in X \times X$ such that $(g(\alpha),g(\beta))$ is comparable to $(F(\alpha,\beta),F(\beta,\alpha))$ and $(F(\alpha,\beta),F(\beta,\alpha))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$, then F and g have a unique coupled common fixed point.*

Proof. Theorem 3.1 ensures that the set of all coupled coincidence points of F and g is nonempty.
Let $(x,y),(u,v) \in X \times X$ be any two coupled coincidence points of F and g.
That is, $g(x) = F(x,y),\ g(y) = F(y,x)$ and $g(u) = F(u,v),\ g(v) = F(v,u)$.
First we shall prove that

$$g(x) = g(u),\ g(y) = g(v) \tag{3.7}$$

By the hypothesis there exist $(\alpha,\beta) \in X \times X$ such that $(g(\alpha),g(\beta))$ is comparable to $(F(\alpha,\beta),F(\beta,\alpha))$.
Following as in Theorem 3.1 we can construct an increasing, converging sequence $\{g(\alpha_n)\}$ and a decreasing, converging sequence $\{g(\beta_n)\}$ where $g(\alpha_{n+1}) = F(\alpha_n,\beta_n)$ and $g(\beta_{n+1}) = F(\beta_n,\alpha_n)$, $n \in \mathbb{N} \cup \{0\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$.
By the hypothesis $(F(\alpha,\beta),F(\beta,\alpha))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$.
Since (x,y) and (u,v) are coupled coincidence points of F and g and using the mixed

g- monotone property of F we get $(g(\alpha_n), g(\beta_n))$ is comparable to both $(g(x), g(y))$ and $(g(u), g(v))$.
Consider,

$$\begin{aligned}
&\psi[d(g(x), g(\alpha_{n+1})) + d(g(y), g(\beta_{n+1}))]\\
&= \psi[d(F(x,y), F(\alpha_n, \beta_n)) + d(F(y,x), F(\beta_n, \alpha_n))]\\
&\leq \psi[\max\{d(g(x), F(x,y)) + d(g(y), F(y,x)), d(g(\alpha_n), F(\alpha_n, \beta_n)) + d(g(\beta_n), F(\beta_n, \alpha_n))\}]\\
&\quad -\phi[\max\{d(g(x), F(x,y)) + d(g(y), F(y,x)), d(g(\alpha_n), F(\alpha_n, \beta_n)) + d(g(\beta_n), F(\beta_n, \alpha_n))\}]\\
&= \psi[d(g(\alpha_n), F(\alpha_n, \beta_n)) + d(g(\beta_n), F(\beta_n, \alpha_n))] - \phi[d(g(\alpha_n), F(\alpha_n, \beta_n)) + d(g(\beta_n), F(\beta_n, \alpha_n))]
\end{aligned}$$

Taking the upper limit on both sides as $n \to \infty$ we get
$\lim_{n\to\infty} \{\psi[d(g(x), g(\alpha_{n+1})) + d(g(y), g(\beta_{n+1}))]\} = 0$
Similarly, it can be proved that $\lim_{n\to\infty} \{\psi[d(g(u), g(\alpha_{n+1})) + d(g(v), g(\beta_{n+1}))]\} = 0$
Thus $g(x) = g(u)$ and $g(y) = g(v)$.
That is, for any two coupled coincidence points (x, y) and (u, v) of F and g, $g(x) = g(u)$ and $g(y) = g(v)$.
Let $\gamma = g(x)$ and $\delta = g(y)$.
Since (x, y) is a coupled coincidence point of F and g we have
$F(x, y) = \gamma$ and $F(y, x) = \delta$
Since F and g are compatible we have
$g(\gamma) = F(\gamma, \delta)$ and $g(\delta) = F(\delta, \gamma)$.
That is, (γ, δ) is a coupled coincidence point of F and g.
Therefore $g(\gamma) = g(x) = \gamma$ and $g(\delta) = g(y) = \delta$
Therefore (γ, δ) is a coupled common fixed point of F and g.
The uniqueness of coupled common fixed point of F and g follows from (3.7). □

Theorem 3.8. *In addition to the hypothesis of Theorem 3.1, suppose that $g(x_0)$ and $g(y_0)$ are comparable, then $x = y$.*

Proof. Without loss of generality assume that $g(x_0) \leqslant g(y_0)$.
By following Theorem 3.1 we get $\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} g(y_n) = y$
where $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$.
By the mixed g- monotone property of F, it can be easily verified that $g(x_n) \leqslant g(y_n),\ \forall n \in \mathbb{N}$. Now consider,

$$\begin{aligned}
&\psi[d(F(x_n, y_n), F(y_n, x_n)) + d(F(y_n, x_n), F(x_n, y_n))]\\
&\leq \psi[\max\{d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})), d(g(y_n), g(y_{n+1})) + d(g(x_n), g(x_{n+1}))\}]\\
&\quad -\phi[\max\{d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})), d(g(y_n), g(y_{n+1})) + d(g(x_n), g(x_{n+1}))\}]
\end{aligned}$$

By taking the upper limit as $n \to \infty$ on both sides we get, $\psi[d(x, y) + d(y, x)] = 0$
Thus $x = y$. □

The following example illustrates Theorem 3.5.

Example 3.9. Let $X = [0, 1]$ with the usual order $\leq$ and the usual metric $d(x, y) = |x - y|,\ \forall\ x, y \in X$.
Clearly X is a partially ordered complete metric space satisfying the two properties assumed in Theorem 3.5.
Let $g : X \to X$ and $F : X \times X \to X$ be defined as

$$g(x) = \frac{4}{5}x \quad \text{and} \quad F(x, y) = \begin{cases} 0 & \text{if } x \in [0, \frac{6}{7}) \\ \frac{1}{35} & \text{if } x \in [\frac{6}{7}, 1] \end{cases}$$

It can be seen that F and g are compatible mappings and F is a mixed g- monotone mapping.
Here $F(X \times X) \subseteq g(X)$, g is continuous and order preserving mapping on X.
Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be defined as $\psi(x) = x^2$ and $\phi(x) = \dfrac{400}{529}x^2$, then F and g satisfy the contractive type condition (3.3).
If all $x, y, u, v \in X$ satisfying $g(x) \leq g(u)$, and $g(y) \geq g(v)$, belong to either $[0, \frac{6}{7})$ or $[\frac{6}{7}, 1]$ then the contractive type condition (3.3) is obvious. In the remaining possible cases for the values of $x, y, u, v \in X$ satisfying $g(x) \leq g(u)$ and $g(y) \geq g(v)$, we consider three different cases and verify the validity of the contractive type condition (3.3), which will cover the remaining cases.
Case 1: When $x, v \in [0, \frac{6}{7})$ and $u, y \in [\frac{6}{7}, 1]$

$$\begin{aligned}
d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) &= \left|0 - \frac{1}{35}\right| + \left|\frac{1}{35} - 0\right| \\
&= \frac{2}{35} \qquad (3.8) \\
d(g(x), F(x,y)) + d(g(y), F(y,x)) &= \left|\frac{4}{5}x - 0\right| + \left|\frac{4}{5}y - \frac{1}{35}\right| \\
&= \frac{4}{5}x + \left|\frac{4}{5}y - \frac{1}{35}\right| \\
&\geq \left|\frac{4}{5} \cdot \frac{6}{7} - \frac{1}{35}\right| \\
&= \frac{23}{35} \qquad (3.9) \\
d(g(u), F(u,v)) + d(g(v), F(v,u)) &= \left|\frac{4}{5}u - \frac{1}{35}\right| + \left|\frac{4}{5}v - 0\right| \\
&= \left|\frac{4}{5}u - \frac{1}{35}\right| + \frac{4}{5}v \\
&\geq \left|\frac{4}{5} \cdot \frac{6}{7} - \frac{1}{35}\right| \\
&= \frac{23}{35} \qquad (3.10)
\end{aligned}$$

By (3.9) and (3.10) we have,

$$\begin{aligned}
M(x,y,u,v) &= \max\{d(g(x), F(x,y)) + d(g(y), F(y,x)), d(g(u), F(u,v)) + d(g(v), F(v,u))\} \\
&\geq \frac{23}{35} \qquad (3.11)
\end{aligned}$$

Now, by (3.8) and (3.11) we have,

$$\begin{aligned}
\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) &= \frac{4}{1225} \\
\psi(M(x,y,u,v)) - \phi(M(x,y,u,v)) &= \frac{129}{529} \cdot M(x,y,u,v)^2 \\
&\geq \frac{129}{529} \cdot \frac{529}{1225} \\
&= \frac{129}{1225}
\end{aligned}$$

Therefore,

$$\psi(d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))) \leq \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))$$

Case 2: $x, u, v \in [0, \frac{6}{7})$ and $y \in [\frac{6}{7}, 1]$

$$\begin{aligned} d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) &= |0-0| + \left|\frac{1}{35} - 0\right| \\ &= \frac{1}{35} \qquad (3.12) \\ d(g(x),F(x,y)) + d(g(y),F(y,x)) &= \left|\frac{4}{5}x - 0\right| + \left|\frac{4}{5}y - \frac{1}{35}\right| \\ &= \frac{4}{5}x + \left|\frac{4}{5}y - \frac{1}{35}\right| \\ &\geq \left|\frac{4}{5} \cdot \frac{6}{7} - \frac{1}{35}\right| \\ &= \frac{23}{35} \qquad (3.13) \\ d(g(u),F(u,v)) + d(g(v),F(v,u)) &= \left|\frac{4}{5}u - 0\right| + \left|\frac{4}{5}v - 0\right| \\ &= \frac{4}{5}(u+v) \qquad (3.14) \end{aligned}$$

By (3.13) and (3.14) we have,

$$\begin{aligned} M(x,y,u,v) &= \max\{d(g(x),F(x,y))+d(g(y),F(y,x)), d(g(u),F(u,v))+d(g(v),F(v,u))\} \\ &\geq \frac{23}{35} \qquad (3.15) \end{aligned}$$

Now by (3.12) and (3.15) we have,

$$\begin{aligned} \psi(d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))) &= \frac{1}{1225} \\ \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)) &= \frac{129}{529} \cdot M(x,y,u,v)^2 \\ &\geq \frac{129}{529} \cdot \frac{529}{1225} \\ &= \frac{129}{1225} \end{aligned}$$

Therefore

$$\psi(d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))) \leq \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))$$

Case 3: $x \in [0, \frac{6}{7})$ and $y, u, v \in [\frac{6}{7}, 1]$

$$\begin{aligned} d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) &= \left|0 - \frac{1}{35}\right| + \left|\frac{1}{35} - \frac{1}{35}\right| \\ &= \frac{1}{35} \qquad (3.16) \\ d(g(x),F(x,y)) + d(g(y),F(y,x)) &= \left|\frac{4}{5}x - 0\right| + \left|\frac{4}{5}y - \frac{1}{35}\right| \\ &= \frac{4}{5}x + \left|\frac{4}{5}y - \frac{1}{35}\right| \\ &\geq \left|\frac{4}{5} \cdot \frac{6}{7} - \frac{1}{35}\right| \\ &= \frac{23}{35} \qquad (3.17) \\ d(g(u),F(u,v)) + d(g(v),F(v,u)) &= \left|\frac{4}{5}u - \frac{1}{35}\right| + \left|\frac{4}{5}v - \frac{1}{35}\right| \end{aligned}$$

$$\geq 2\left|\frac{4}{5}\cdot\frac{6}{7}-\frac{1}{35}\right| \tag{3.18}$$
$$= \frac{2\cdot 23}{35} \tag{3.19}$$

By (3.17) and (3.19) we have,

$$\begin{aligned} M(x,y,u,v) &= \max\{d(g(x),F(x,y))+d(g(y),F(y,x)), d(g(u),F(u,v))+d(g(v),F(v,u))\} \\ &\geq \frac{2\cdot 23}{35} \end{aligned} \tag{3.20}$$

Now by (3.16) and (3.20) we have,

$$\begin{aligned} \psi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))) &= \frac{1}{1225} \\ \psi(M(x,y,u,v))-\phi(M(x,y,u,v)) &= \frac{129}{529}\cdot M(x,y,u,v)^2 \\ &\geq \frac{129}{529}\cdot\frac{4\cdot 529}{1225} \\ &= \frac{516}{1225} \end{aligned}$$

Therefore

$$\psi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))) \leq \psi(M(x,y,u,v))-\phi(M(x,y,u,v))$$

Here $(0,0)$ is the only coupled common fixed point of F and g.

Remark 3.10. The above example also illustrates that the contractive type conditions (1.1) and (3.3) are independent.

For, take $x=y=v=\frac{6}{7}-\epsilon$ and $u=\frac{6}{7}$ where $0<\epsilon\leq\frac{6}{7}$.
Now

$$\psi(d(F(x,y),F(u,v))) = \psi\Big(\Big|0-\frac{1}{35}\Big|\Big) = \psi\Big(\frac{1}{35}\Big)$$

$$\begin{aligned} &\psi(\max(d(gx,gu),d(gy,gv)))-\phi(\max(d(gx,gu),d(gy,gv))) \\ &= \psi\Big(\Big|\frac{4}{5}\Big(\frac{6}{7}-\epsilon\Big)-\frac{4}{5}\cdot\frac{6}{7}\Big|\Big)-\phi\Big(\Big|\frac{4}{5}\Big(\frac{6}{7}-\epsilon\Big)-\frac{4}{5}\cdot\frac{6}{7}\Big|\Big) \\ &= \psi\Big(\frac{4}{5}\cdot\epsilon\Big)-\phi\Big(\frac{4}{5}\cdot\epsilon\Big) \end{aligned}$$

Since ψ and ϕ in (1.1) are continuous and $\psi^{-1}\{0\}=\{0\}$ and $\phi^{-1}\{0\}=\{0\}$ we have as $\epsilon\to 0$,
$\psi(\max(d(gx,gu),d(gy,gv)))-\phi(\max(d(gx,gu),d(gy,gv)))\to 0$
but $\psi(d(F(x,y),F(u,v)))=\psi(\frac{1}{35})>0$ for all $\epsilon>0$.
Thus F and g does not satisfy the contractive type condition (1.1).

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References

1. Y.I. Alber, S. Guerre-Delabriere, Principle of Weakly Contractive Maps in Hilbert Spaces. In: Gohberg I., Lyubich Y. (eds) New Results in Operator Theory and Its Applications. Operator Theory: Advances and Applications, vol 98. Birkhäuser, Basel (1997)
2. V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 74(18), 7347 - 7355 (2011)
3. V. Berinde, Coupled fixed point theorems for ϕ- contractive mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal. 75, 3218 - 3228 (2011)
4. V. Berinde, Coupled coincidence point theorems for mixed monotone nonlinear operators, Comput. Math. Appl. 64, 1770 - 1777 (2012)
5. B. S. Choudhury, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal. 73, 2524-2531 (2010)
6. B. S. Choudhury, N. Metiya, A. Kundu, Coupled coincidence point theorems in ordered metric spaces, Ann. Univ. Ferrara 57, 1-16 (2011)
7. B. S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory and Applications 2013, 2013:152
8. B. K. Dass, S. Gupta, An extension of Banach contraction principle through rational expression, Indian J. Pure Appl. Math. 6 (1975), 1455 - 1458.
9. M. Geraghty, On contractive mappings. Proc. Am. Math. Soc. 40, 604 – 608 (1973)
10. T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis 65 (2006) 1379 - 1393.
11. D. S. Jaggi, Some unique fixed point theorems, Indian J. Pure Appl. Math. 8 (1977), 223 - 230.
12. R. Kannan, Some results of fixed points, Bull. Calcutta Math. Soc. 60 (1968) 405 – 408.
13. V. Lakshmikantham, Ljubomir Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis 70 (2009) 4341 - 4349
14. J.J. Nieto, R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223 - 239.
15. J.J. Nieto, R.R. Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Mathematica Sinica, English Series 23 (12) (2007) 2205 - 2212.
16. A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proceedings of The American Mathematical Society 132 (2003) 1435 - 1443.
17. S. Radenović, Z. Kadelburg, D. Jandrlić, A. Jandrlić, Some results on weakly contractive maps, Bulletin of the Iranian Mathematical Society, 38(3), (2012) 625-645.
18. B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal. TMA 47 (4) (2001) 2683 – 2693.

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ITERATIVE SCHEME FOR FIXED POINT PROBLEM OF ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS AND SPLIT EQUILIBRIUM PROBLEM IN HILBERT SPACES

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ABSTRACT. The main objective of this work is to modify the sequence $\{x_n\}$ of the explicit projection algorithm of asymptotically nonexpansive semigroups. We prove the strong convergence theorem of a sequence $\{x_n\}$ to the common fixed point of asymptotically nonexpansive semigroups and the solutions of split equilibrium problems. Our main results extended and improved the results of Pei Zhou and Gou-Jie Zhao [17] and many authors.

KEYWORDS: asymptotically nonexpansive mappings; asymptotically nonexpansive semigroup; fixed point; split equilibrium problem.
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1. Introduction

$\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C\times C \longrightarrow \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by EP.

The split equilibrium problem was introduced by Moudafi [12], he considers the following pair of equilibrium problems in different spaces. Let H_1 and H_2 be two real Hilbert spaces, let $F_1 : C\times C \to \mathbb{R}$ and $F_2 : Q\times Q \to \mathbb{R}$ be nonlinear bifunctions and let $A : H_1 \to H_2$ be a bounded linear operator which C and Q are closed convex subsets of H_1 and H_2, respectively. Then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C. \tag{1.2}$$

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and such that

$$y^* \in Ax^* \in Q, \quad F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{1.3}$$

The solution set of SEP (1.2)-(1.3) is denote by $\Omega = \{p \in \mathrm{EP}(F_1) : Ap \in \mathrm{EP}(F_2)\}$.

Recall that, a mapping $T : C \to C$ and a self mapping f of C is a contraction if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for some $\alpha \in (0, 1)$ and T is a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [5] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. Recall also that a one-parameter family $\mathcal{T} = \{T(t)|0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitian semigroup on C (see, e. g., [15]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$

(ii) $T(s + t)(x) = T(s)T(t), s, t \geq 0, x \in C$

(iii) for each $x \in C$, the maps $t \mapsto T(t)x$ is continuous on $[0, \infty)$

(iv) there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|, x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

In 2010, Tian [16] introduced the following general iterative scheme for finding an element of set of solutions to the fixed point of nonexpansive mapping in a Hilbert space. Define the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)Tx_n, \tag{1.4}$$

where B is $k-$Lipscitzian and $\eta-$strongly monotone operator. Then he prove that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ gererate by (1.4) converges strongly to the unique solution $x^* \in Fix(T)$ of variational inequality

$$\langle(\gamma f - \mu B)x^*, x - x^*\rangle \leq 0, \forall x \in F_{ix}(T). \tag{1.5}$$

In 2011, Ceng et al. [4] added the metric project to the method of Tian (1.4) and studied the following explicit iterative scheme to find fixed points:

$$x_{n+1} = P_C \left[\alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)Tx_n\right]. \tag{1.6}$$

They prove the strong converge of $\{x_n\}$ to a fixed point $x^* \in Fix(T)$ of the same variational in equality (1.5).

In 2008, Plubtieng and Punpaeng [13] introduced the following implicit iterative algorithm to prove a strong convergence theorem for fixed point problem with nonexpansive semigroup:

$$x_n = \alpha_n f(x_n) + (I - \alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \tag{1.7}$$

where x_n is a continuous net and s_n is a positive real divergent net.

In 2014, Kazmi and Rizvi [8] studied the following implcit iterative algorithm. Under some asummptions, they obtain some strong convergence theorem for EP(1.1) and the fixed point problem:

$$u_n = T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n),$$

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \tag{1.8}$$

where s_n and r_n are the continuous nets in $(0,1)$.

In the same year, Zhou and Zhao [17] introduce an explicit iterative scheme for finding a common element of the set of solutions SEP and fixed point for a nonexpansive semigroup in real Hilbert spaces. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{aligned} u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} &= P_C\left[\alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right]. \end{aligned} \tag{1.9}$$

Under suitable conditions, some strong convergence theorems for approximating to these common elements are proved.

Next, we studies some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup for motivation of this work.

Example 1.1. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{1}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{1}{1+2s})x - (\frac{1}{1+2s})y\| = (\frac{1}{1+2s})\|x - y\|,$$

then we have $\mathcal{T}$ is nonexpansive semigroup. If $L_s = 1$ we have $\limsup_{s\to\infty} L_s = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup.

Example 1.2. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{2+2s}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{2+2s}{1+2s})x - (\frac{2+2s}{1+2s})y\| = (\frac{2+2s}{1+2s})\|x - y\|,$$

put $L_s = (\frac{2+2s}{1+2s})$ we have $\limsup_{s\to\infty} L_s = \limsup_{s\to\infty}(\frac{2+2s}{1+2s}) = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup. If we let $s = 1$ we have $\frac{2+2s}{1+2s} = \frac{4}{3} \not< 1$, then $\mathcal{T}$ is not necessary nonexpansive semigroup.

From above example we see that a mapping $\mathcal{T}$ is a nonexpansive semigroup then $\mathcal{T}$ is asymptotically nonexpansive semigroup. But $\mathcal{T}$ is an asymptotically nonexpansive semigroup is not necessary nonexpansive semigroup.

Inspired and motivate by above and [17], the purpose of this paper to introduce an explicit iterative scheme for finding a common element of the set of solutions SEP and fixed point for an asymptotically nonexpansive semigroup in real Hilbert spaces.

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Lemma 2.1. *Let H be a real Hilbert space, then the following hold:*
(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H;$
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, t \in [0,1], \forall x, y \in H.$
(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H.$

Let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$

foe all $y \in C$, such P_C is called the metric projection from H into C. We know that P_C is nonexpansive. It is also known that $P_C x \in C$ and

$$\langle x - P_C x, P_C x - z\rangle \geq 0, \quad \forall x \in H, z \in C. \tag{2.1}$$

It is easy to see that (2.1) is equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|P_C x - z\|^2, \quad \forall x \in H, z \in C. \tag{2.2}$$

Let $B : C \to H$ be a nonlinear mapping. Recall the following definitions.

Definition 2.2. B is said to be

(i) monotone if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C, \tag{2.3}$$

(ii) strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C, \tag{2.4}$$

for such a case, B is said to be $\alpha-$strongly monotone,

(iii) $\alpha-$inverse strongly monotone($\alpha-$ism) if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C, \tag{2.5}$$

(iv) $k-$Lipschitz continuous if exists a constant $k \geq 0$ such that

$$\|Bx - By\| \leq k\|x - y\|, \quad \forall x, y \in C. \tag{2.6}$$

Remark 2.3. Let $\mathcal{F} = \mu B - \gamma f$, where B is a $k-$Lipschitz and $\eta-$strongly monotone operator on H with $k > 0$ and f is a Lipschitz mapping on H with coefficient $L > 0$, $0 < \gamma \leq \mu\eta/L$. It is a simple matter to see that the operator $\mathcal{F}$ is $(\mu\eta - \gamma L)-$strongly monotone over H; that is

$$\langle \mathcal{F}x - \mathcal{F}y, x - y\rangle \geq (\mu\eta - \gamma L)\|x - y\|^2, \quad \forall x, y \in H, \tag{2.7}$$

Lemma 2.4. [6] *Let T be a nonexpansive mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever the sequence of x_n is weakly convergent to x and $(I - T)x_n$ is strongly convergent to y, then $(I - T)x = y$.*

Lemma 2.5. [10] *Assume that A is a strongly positive linear bounded operator on Hilbert space H with coefficient $\tau > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\tau$.*

Lemma 2.6. [7] *Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, If $\{x_n\}$ is a sequence in C satisfying the properties:*
(i) $x_n \rightharpoonup z$; and
(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$,
then $z \in Fix(\mathcal{T})$.

Lemma 2.7. [7] *Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, then for any $u \geq 0$,*

$$\limsup_{u\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)xds - T(u)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.8. [9] *Let T be an asymptotically nonexpansive mapping defined on a bounded convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, then $x \in F(T)$.*

Lemma 2.9. [11] *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|$$

for all $n \geq 1$, then $x_n \to q$.

Definition 2.10. [12] A mapping $T : H \to H$ is said to be averaged if it can be written as the average of the identity mapping and a nonexpansive mapping; that is,

$$T = (1 - \epsilon)I + \epsilon S, \tag{2.8}$$

where $\epsilon \in (0, 1), S : H \to H$ is nonexpansive, and I is the identity operator on H.

Proposition 2.11. [12]

(i) *If $T = (1 - \epsilon)S + \epsilon V$, where $S : H \to H$ is averaged, $V : H \to H$ is nonexpansive, and $\epsilon \in (0, 1)$, then T is averaged.*
(ii) *The composite of finite many averaged mappings is averaged.*
(iii) *If T is $\nu - ism$, then for $\gamma > 0, \gamma T$ is $(\nu/\gamma) - ism$.*
(iv) *T is averaged if and only if its complement $I - T$ is $\nu - ism$ for some $\nu > \frac{1}{2}$.*

Assumption 2.12. [1] *For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:*

($A1$) *$F(x, x) = 0$ for all $x \in C$;*
($A2$) *F is monotone, that is $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;*
($A3$) *for each $x, y \in C$,*

$$\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y); \tag{2.9}$$

($A4$) *for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.*

Lemma 2.13. [2] *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying ($A1$)($A4$). Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(x, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.10}$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r^F(x) = \left\{ z \in C : F(x, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}, \tag{2.11}$$

for all $x \in H$. Then the following hold:

(i) *T_r^F is single valued;*
(ii) *T_r^F is firmly nonexpansive; that is, for any $x, y \in H$*

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.12}$$

(iii) *$F(T_r^F) = \mathrm{EP}(F)$;*
(iv) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 2.14. [3] *Let C be a nonempty closed convex subset of a Hilbert space H, and let $F : C \times C \to \mathbb{R}$ be a bifunction. Let $x \in C$ and $r_1, r_2 \in (0, \infty)$. Then*

$$\left\|T_{r_1}^F x - T_{r_2}^F x\right\| \leq \left|1 - \frac{r_2}{r_1}\right| \left(\left\|T_{r_1}^F x\right\| + \|x\|\right). \tag{2.13}$$

Lemma 2.15. [14] *Assume that* $\{a_n\}, \{b_n\}, \{c_n\}$ *are sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-c_n)a_n + b_n, n \geq 0$$

where $\{a_n\}$ *is asequence in* $(0,1)$ *and* $\{b_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\Sigma_{n=0}^{\infty} c_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{b_n}{c_n} \leq 0$ *or* $\Sigma_{n=0}^{\infty}|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Let $f : H_1 \to H_1$ be a contractive mapping with constant $\beta \in (0,1)$ and let $A : H_1 \to H_2, B : H_1 \to H_1$ be a $\eta-$strongly monotone and $\theta-$Lipschitzian with $\theta > 0, \eta > 0$. In this work, we may assume that $0 < \mu < \frac{2\eta}{\theta^2}$ and $0 < \gamma < \mu(\eta - \frac{\mu\theta^2}{2})/\beta = \frac{\tau}{\beta}$. Let $\Im = \{T(s) : 0 \leq s < \infty\}$ be an aymptotically nonexpansive semigroup on C such that $\Gamma = F(\Im) \cap \Omega \neq \emptyset$. Assume $\{r_n\}$ and $\{s_n\}$ are the continuous nets of positive real numbers such that $\lim_{n\to 0} r_n = r > 0$ and $\lim_{n\to 0} s_n = +\infty$.

In this section, we introduce the following explicit iterative scheme that the nets $\{u_n\}$ and $\{x_n\}$ are generated by

$$\begin{aligned} u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} &= P_C\left[\alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right], \end{aligned} \tag{3.1}$$

where $P_C : H_1 \to C, \delta \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A.

We prove the strong convergence of $\{u_n\}$ and $\{x_n\}$ to a fixed point $x^* \in F(\Im)$ which solve the following variational inequality:

$$\langle(\mu F - \gamma g)x^*, x^* - \overline{x}\rangle \leq 0, \forall \overline{x} \in \Gamma = F(\Im) \cap \Omega. \tag{3.2}$$

In the sequel, we denote by $\{y_n\}$ the sequence defined by

$$y_n = \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds. \tag{3.3}$$

Theorem 3.1. *Let* H_1 *and* H_2 *be two real Hilbert spaces and let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty closed subsets. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Assume that* $F_1 : C \times C \to \mathbb{R}$ *and* $F_2 : Q \times Q \to \mathbb{R}$ *are the bifunctions satisfying Assumption 2.12 and* F_2 *is upper semicontinuous in the first argument. Let the sequence* $\{u_n\}$ *and* $\{x_n\}$ *be generated by (3.1), and suppose that the sequence* $\{\alpha_n\}$ *satisfies the following conditions:*

(i) $\alpha_n \in (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\Sigma_{n=0}^{\infty}\alpha_n = 0$;
(iii) *either* $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

where $\widetilde{s_n} = \frac{1}{s_n}\int_0^{s_n} L_s^T ds \to 1$ *as* $n \to \infty$. *Then the sequence* $\{u_n\}$ *and* $\{x_n\}$ *converge strongly to* $x^* \in \Gamma = F(\Im) \cap \Omega$, *where* $x^* = P_\Gamma(I - \mu B + \gamma f)x^*$, *which is the unique solution of the variational inequality (3.2).*

Proof. For $\alpha_n \in (0,1)$ and $\forall x \in H_1$, define a mapping $G : H_1 \to H_2$ by

$$Gx = P_C\left[\alpha_n \gamma f(x) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(x + \delta A^*(T_{r_n}^{F_2} - I)Ax)ds\right]. \tag{3.4}$$

From Lemma 2.13 we easily know that $T_{r_n}^{F_1}$ and $T_{r_n}^{F_2}$ both are firmly nonexpansive mappings and are averaged operators. From Proposition 2.11, we can obtain that the operator $(I + \delta A^*(T_{r_n}^{F_2} - I)A)$ is averaged and hence nonexpansive. Following Lemma 2.14 and $\forall x, y \in H_1$, we get

$$\begin{aligned}
\|Gx - Gy\| &= \|P_C\left[\alpha_n \gamma f(x) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(x + \delta A^*(T_{r_n}^{F_2} - I)Ax)ds\right] \\
&- P_C\left[\alpha_n \gamma f(y) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(y + \delta A^*(T_{r_n}^{F_2} - I)Ay)ds\right]\| \\
&\leq \|\left[\alpha_n \gamma f(x) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(x + \delta A^*(T_{r_n}^{F_2} - I)Ax)ds\right] \\
&- \left[\alpha_n \gamma f(y) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(y + \delta A^*(T_{r_n}^{F_2} - I)Ay)ds\right]\| \\
&\leq \alpha_n\gamma\|f(x) - f(y)\| \\
&\quad +(1-\alpha_n\tau)\|\frac{1}{s_n}\int_0^{s_n} [T(s)(T_{r_n}^{F_1}(x + \delta A^*(T_{r_n}^{F_2} - I)Ax)) \\
&\quad -T(s)(T_{r_n}^{F_1}(y + \delta A^*(T_{r_n}^{F_2} - I)Ay))ds]\| \\
&\leq \alpha_n\gamma\|f(x) - f(y)\| \\
&\quad +(1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} \|T(s)(T_{r_n}^{F_1}(x + \delta A^*(T_{r_n}^{F_2} - I)Ax) \\
&\quad -T(s)(T_{r_n}^{F_1}(y + \delta A^*(T_{r_n}^{F_2} - I)Ay))\|ds \\
&\leq \alpha_n\gamma\|f(x) - f(y)\| \\
&\quad +(1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T\|T_{r_n}^{F_1}(x + \delta A^*(T_{r_n}^{F_2} - I)Ax) \\
&\quad -T_{r_n}^{F_1}(y + \delta A^*(T_{r_n}^{F_2} - I)Ay))\|ds \\
&\leq \alpha_n\gamma\|f(x) - f(y)\| + (1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T\,\|x-y\|\,ds \\
&\leq \alpha_n\gamma\|f(x) - f(y)\| + (1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T ds\,\|x-y\| \\
&\leq \alpha_n\gamma\beta\|x-y\| + (1-\alpha_n\tau)\widetilde{s_n}\|x-y\| \\
&= (1-\alpha_n(\tau\widetilde{s_n} - \gamma\beta))\|x-y\|. \qquad (3.5)
\end{aligned}$$

Since $\gamma < \frac{\tau}{\beta}$ and $\alpha_n \in (0,1)$ then $(1-\alpha_n(\tau\widetilde{s_n} - \gamma\beta)) < 1$, it follows that G is contraction, by Banach contraction principle, there exists a unique a fixed point x^*. Next, we proved that $\{u_n\}, \{x_n\}$ are bounded. Let $p \in \Gamma = F(S) \cap \Omega$, we obtain that $p = T_{r_n}^{F_1}p$ and $p = T_{r_n}^{F_2}Ap$ and $p = T(s)p$. From (3.1), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{F_1}(I + \delta A^*(T_{r_n}^{F_2} - I)A)x_n - p\|^2 \\
&= \|T_{r_n}^{F_1}(I + \delta A^*(T_{r_n}^{F_2} - I)A)x_n - T_{r_n}^{F_1}p\|^2 \\
&\leq \|x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n - p\|^2 \\
&\leq \|x_n - p\|^2 + \|\delta A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 + 2\delta\langle x_n - p, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 + \delta^2\langle (T_{r_n}^{F_2} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\quad + 2\delta\langle A(x_n - p), (T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 + L\delta^2\langle (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\quad + 2\delta\langle A(x_n - p) + (T_{r_n}^{F_2} - I)Ax_n - (T_{r_n}^{F_2} - I)Ax_n, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 + L\delta^2\|(T_{r_n}^{F_2} - I)Ax_n\|^2 \\
&\quad + 2\delta\left\{\langle T_{r_n}^{F_2}Ax_n - Ap, (T_{r_n}^{F_2} - I)Ax_n\rangle - \|(T_{r_n}^{F_2} - I)Ax_n\|^2\right\} \\
&\leq \|x_n - p\|^2 + L\delta^2\|(T_{r_n}^{F_2} - I)Ax_n\|^2 \\
&\quad + 2\delta\left\{\frac{1}{2}\|(T_{r_n}^{F_2} - I)Ax_n\|^2 - \|(T_{r_n}^{F_2} - I)Ax_n\|^2\right\} \\
&\leq \|x_n - p\|^2 + L\delta^2\|(T_{r_n}^{F_2} - I)Ax_n\|^2 - \delta\|(T_{r_n}^{F_2} - I)Ax_n\|^2 \\
&= \|x_n - p\|^2 + \delta(L\delta - 1)\|(T_{r_n}^{F_2} - I)Ax_n\|^2. \qquad (3.6)
\end{aligned}$$

Since $\delta \in (0, 1/L)$, we have

$$\|u_n - p\| \leq \|x_n - p\|. \qquad (3.7)$$

Put $y_n = \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds$, it follows that

$$\begin{aligned}
\|y_n - p\| &= \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&\leq \frac{1}{s_n}\left\|\int_0^{s_n}(T(s)u_n - T(s)p)ds\right\| \\
&\leq \|u_n - p\| \leq \|x_n - p\|. \qquad (3.8)
\end{aligned}$$

And we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|P_C\left[\alpha_n\gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right] - p\right\| \\
&\leq \left\|\alpha_n\gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&= \left\|\alpha_n(\gamma f(x_n) - \mu Bp) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - (I - \mu\alpha_n B)p\right\| \qquad (3.9) \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1 - \alpha_n\tau)\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1 - \alpha_n\tau)\frac{1}{s_n}\int_0^{s_n}\|T(s)u_n - T(s)p\|\, ds \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1 - \alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T\,\|u_n - p\|\, ds \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1 - \alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T ds\,\|u_n - p\| \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1 - \alpha_n\tau)\widetilde{s_n}\,\|u_n - p\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-\mu Bp\|+(1-\alpha_n\tau)\widetilde{s_n}\,\|u_n-p\| \\
&\leq \alpha_n\gamma\beta\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Bp\|+(1-\alpha_n\tau)\widetilde{s_n}\,\|x_n-p\| \\
&\leq [\widetilde{s_n}-\alpha_n(\tau\widetilde{s_n}-\gamma\beta)]\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Bp\| \qquad (3.10)
\end{aligned}
$$

Since $\{\widetilde{s_n}-\alpha_n(\tau\widetilde{s_n}-\gamma\beta)\}$ is convergence sequence of real number then it is a bounded dequence, we have $K\in\mathbb{R}$ such that

$$
\|x_{n+1}-p\|\leq K\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Bp\|, \qquad (3.11)
$$

we have $\{x_n\}$ is bounded and therefore $\{u_n\},\{y_n\}$ and $\{f(x_n)\}$ are bounded. From (3.10), $\{\|x_n-p\|\}$ is bounded and decreasing sequence, hence $\lim_{n\to\infty}\|x_n-p\|$ exists.

Next, we claim that $\lim_{n\to\infty}\|x_n-u_n\|=0$. From (3.9), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \left\|\alpha_n(\gamma f(x_n)-\mu Bp)+(I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds-(I-\mu\alpha_n B)p\right\|^2 \\
&\leq (1-\alpha_n\tau)^2\left\|\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds-p\right\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-\gamma f(p)+\gamma f(p)-\mu Bp,x_n-p\rangle \\
&\leq (1-\alpha_n\tau)^2\|u_n-p\|^2+2\alpha_n\gamma\beta\|x_n-p\| \\
&\quad +2\alpha_n\langle\gamma f(p)-\mu Bp,x_n-p\rangle \\
&\leq \|u_n-p\|^2+\alpha_n\tau^2\|x_n-p\|^2+2\alpha_n\gamma\beta\|x_n-p\| \\
&\quad +2\alpha_n\|\gamma f(p)-\mu Bp\|\|x_n-p\| \\
&\leq \|x_n-p\|^2+\delta(L\delta-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2+\alpha_n\tau^2\|x_n-p\|^2 \\
&\quad +2\alpha_n\gamma\beta\|x_n-p\|+2\alpha_n\|\gamma f(p)-\mu Bp\|\|x_n-p\|. \qquad (3.12)
\end{aligned}
$$

From (3.12), we obtain

$$
\begin{aligned}
\delta(1-L\delta)\|(T_{r_n}^{F_2}-I)Ax_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\alpha_n(\tau^2\|x_n-p\|^2+2\gamma\beta\|x_n-p\| \\
&\qquad +2\|\gamma f(p)-\mu Bp\|\|x_n-p\|). \qquad (3.13)
\end{aligned}
$$

Since $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n-p\|$ exists, $\lim_{n\to\infty}\alpha_n=0$ and $\delta(1-L\delta)>0$, we obtain that

$$
\lim_{n\to\infty}\|(T_{r_n}^{F_2}-I)Ax_n\|=0. \qquad (3.14)
$$

From (3.1), we have

$$
\begin{aligned}
\|u_n-p\|^2 &= \left\|T_{r_n}^{F_1}(I+\delta A^*(T_{r_n}^{F_2}-I)A)x_n-p\right\|^2 \\
&= \left\|T_{r_n}^{F_1}(I+\delta A^*(T_{r_n}^{F_2}-I)A)x_n-T_{r_n}^{F_1}p\right\|^2 \\
&\leq \langle u_n-p,x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n-p\rangle \\
&= \frac{1}{2}\{\|u_n-p\|^2+\|x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n-p\|^2 \\
&\quad -\|u_n-p-[x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n-p]\|^2\} \qquad (3.15) \\
&\leq \frac{1}{2}\left\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n-\delta A^*(T_{r_n}^{F_2}-I)Ax_n\|^2\right\} \\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2-\delta\|A^*(T_{r_n}^{F_2}-I)Ax_n\|^2 \\
&\quad +2\delta\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|\}.
\end{aligned}
$$

Hence, we obtain

$$\begin{aligned} \|u_n-p\|^2 &= \|x_n-p\|^2-\|u_n-x_n\|^2-\delta\|A^*(T_{r_n}^{F_2}-I)Ax_n\|^2 \\ &\quad +2\delta\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\| \\ &\leq \|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|. \end{aligned} \tag{3.16}$$

It follows from (3.12) and (3.16) that

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \|u_n-p\|^2+\alpha_n\tau^2\|x_n-p\|^2+2\alpha_n\gamma\beta\|x_n-p\| \\ &\quad +2\alpha_n\|\gamma f(p)-\mu Bp\|\|x_n-p\| \\ &\leq \|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\| \\ &\quad +\alpha_n\tau^2\|x_n-p\|^2 \\ &\quad +2\alpha_n\gamma\beta\|x_n-p\|+2\alpha_n\|\gamma f(p)-\mu Bp\|\|x_n-p\| \qquad (3.17) \\ &= \|x_n-p\|^2-\|u_n-x_n\|^2+2\delta\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\| \\ &\quad +\alpha_n\tau^2M_1, \end{aligned}$$

where $M_1=\tau^2\|x_n-p\|^2+2\gamma\beta\|x_n-p\|+2\|\gamma f(p)-\mu Bp\|\|x_n-p\|$. From (3.17), we obtain

$$\|u_n-x_n\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2++2\delta\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|+\alpha_n\tau^2M_1 \tag{3.18}$$

From (3.18), (3.14), $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n-p\|$ exists $\lim_{n\to\infty}\alpha_n=0$ and $\delta>0$, we obtain

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.19}$$

Next, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From (1.4) and Lemma 2.14, we have

$$\begin{aligned} \|u_n-u_{n+1}\| &= \|T_{r_n}^{F_1}(I+\delta A^*(T_{r_n}^{F_2}-I)A)x_n-T_{r_n}^{F_1}(I+\delta A^*(T_{r_{n-1}}^{F_2}-I)A)x_{n-1}\| \\ &\leq \|(x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n)-(x_{n-1}+\delta A^*(T_{r_{n-1}}^{F_2}-I)Ax_{n-1})\| \\ &\quad +\left|1-\frac{r_{n-1}}{r_n}\right|\|T_{r_n}^{F_1}(x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n) \\ &\qquad -(x_{n-1}+\delta A^*(T_{r_{n-1}}^{F_2}-I)Ax_{n-1})\| \\ &\leq \|x_n-x_{n-1}-\delta A^*A(x_n-x_{n-1})\| \\ &\quad +\delta\|A\|\|T_{r_{n-1}}^{F_2}Ax_n-T_{r_{n-1}}^{F_2}Ax_{n-1}\| \\ &\quad +\left|1-\frac{r_{n-1}}{r_n}\right|\|T_{r_n}^{F_1}(x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n) \\ &\qquad -(x_{n-1}+\delta A^*(T_{r_{n-1}}^{F_2}-I)Ax_{n-1})\| \\ &\leq \left(\|x_n-x_{n-1}\|^2-2\delta\|A(x_n-x_{n-1})\|^2+\delta^2\|A\|^4\|x_n-x_{n-1}\|^2\right)^{\frac{1}{2}} \\ &\quad +\delta\|A\|\left(\|Ax_n(x_n-x_{n-1})\|+\left|1-\frac{r_{n-1}}{r_n}\right|\|T_{r_n}^{F_2}Ax_n-Ax_{n-1}\|\right) \\ &\quad +\left|1-\frac{r_{n-1}}{r_n}\right|\|T_{r_n}^{F_1}(x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n) \\ &\qquad -(x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n)\| \\ &\leq \left(1-2\delta\|A\|^2+\delta^2\|A\|^4\right)^{\frac{1}{2}}\|x_n-x_{n-1}\| \\ &\quad +\delta\|A\|^2\left(\|x_n-x_{n-1}\|+\|T_{r_n}^{F_2}Ax_n-Ax_{n-1}\|\right) \end{aligned}$$

$$
\begin{aligned}
& + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\
& \quad - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
\leq \; & (1 - \delta\|A\|^2)\|x_n - x_{n-1}\| + \delta\|A\|^2(\|x_n - x_{n-1}\| \\
& \quad + |1 - \delta\|A\| \frac{r_{n-1}}{r_n}|\|T_{r_n}^{F_2}Ax_n - Ax_{n-1}\|) \\
& + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\
& \quad - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
= \; & \|x_n - x_{n-1}\| + \delta\|A\||1 - \frac{r_{n-1}}{r_n}|\|T_{r_n}^{F_2}Ax_n - Ax_{n-1}\| \\
& + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\
& \quad - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
= \; & \|x_n - x_{n-1}\| + \delta\|A\||1 - \frac{r_{n-1}}{r_n}|(\delta\|A\|\varepsilon_n + \xi_n), \qquad (3.20)
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_n &= \|T_{r_n}^{F_2}Ax_n - Ax_{n-1}\| \\
\xi_n &= \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\|. \qquad (3.21)
\end{aligned}
$$

From (3.3), we obtain

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\| \\
&\leq \|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_{n-1}ds\| \\
&\quad + \|\frac{1}{s_n}\int_0^{s_n} T(s)u_{n-1}ds - \frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\| \\
&\leq \frac{1}{s_n}\int_0^{s_n} \|T(s)(u_n - u_{n-1})\|ds \\
&\quad + \|\frac{1}{s_n}\int_0^{s_n} T(s)u_{n-1}ds - \frac{s_1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\| \\
&\leq \widetilde{s_n}\|u_n - u_{n-1}\| + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \left\|\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| \\
&\quad + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\|. \qquad (3.22)
\end{aligned}
$$

From (3.20) and (3.22), we obtain

$$
\begin{aligned}
\|y_n - y_{n-1}\| &\leq \|x_n - x_{n-1}\| + \delta\|A\||1 - \frac{r_{n-1}}{r_n}|(\delta\|A\|\varepsilon_n + \xi_n) \\
&+ \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \left\|\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\|. \qquad (3.23)
\end{aligned}
$$

From (3.1) again, we obtain

$$
\|x_{n+1} - x_n\| = \|P_C[\alpha_n\gamma f(x_n) + (I - \mu\alpha_n B)y_n]
$$

$$
\begin{aligned}
& \quad - P_C\left[\alpha_{n-1}\gamma f(x_{n-1}) + (I-\mu\alpha_{n-1}B)y_{n-1}\right]\| \\
\leq\ & \|(\alpha_n\gamma f(x_n) + (I-\mu\alpha_n B)y_n) - (\alpha_{n-1}\gamma f(x_{n-1}) \\
& \quad\quad + (I-\mu\alpha_{n-1}B)y_{n-1})\| \\
=\ & \|(\alpha_n\gamma(f(x_n) - f(x_{n-1})) + \gamma(\alpha_n - \alpha_{n-1})f(x_{n-1}) \\
& \quad + (I-\mu\alpha_n B)(y_n - y_{n-1}) + \mu(\alpha_n - \alpha_{n-1})y_{n-1})\| \\
\leq\ & \alpha_n\gamma\beta\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
& \quad + (I-\alpha_n\tau)\|y_n - y_{n-1}\| + \mu|\alpha_n - \alpha_{n-1}|\|y_{n-1})\| \\
\leq\ & \alpha_n\gamma\beta\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
& +(I-\alpha_n\tau)(\|x_n - x_{n-1}\| + \delta\|A\||1 - \frac{r_{n-1}}{r_n}|(\delta\|A\|\varepsilon_n + \xi_n) \\
& + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right|\left\|\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\|) \\
& +\mu|\alpha_n - \alpha_{n-1}|\|y_{n-1}\| \\
=\ & (1-\alpha_n(\tau - \gamma\beta))(\|x_n - x_{n-1}\| \\
& \quad + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\||1 - \frac{r_{n-1}}{r_n}|(\delta\|A\|\varepsilon_n + \xi_n)) \\
& + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right|\left\|\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\| \\
& +\mu|\alpha_n - \alpha_{n-1}|\|y_{n-1}\| \\
\leq\ & (1-\alpha_n(\tau - \gamma\beta))\|x_n - x_{n-1}\| \\
& +M_2(\gamma|\alpha_n - \alpha_{n-1}| + \left|1 - \frac{r_{n-1}}{r_n}\right| + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| + \left|\frac{1}{s_{n-1}}\right| \\
& \quad + \mu|\alpha_n - \alpha_{n-1}|),
\end{aligned} \tag{3.24}
$$

where

$$
M_2 = \max\left\{\sup_{n\leq 1}(\delta\|A\|\varepsilon_n + \xi_n), \sup_{n\leq 1}\left(\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\|\right), \sup_{n\leq 1}\|y_{n-1}\|\right\}. \tag{3.25}
$$

Since $\{x_n\}, \{u_n\}$ and $\{y_n\}$ are bounded, we have $\{Ax_n\}$ and $\{T(s)u_{n-1}\}$ are bounded. Then $M_2 < \infty$.

It follows from condition $(1)-(3)$ we have $\lim_{n\to\infty} r_n = r > 0$, $\lim_{n\to\infty} s_n = +\infty$ and Lemma 2.15, we obtain that

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.26}
$$

Next, we claim that $\lim_{n\to\infty}\|T(s)x_n - x_n\| = 0$. From (3.1) and (3.3), we obtain

$$
\begin{aligned}
\|x_{n+1} - y_n\| &\leq \left\|P_C\left[\alpha_n\gamma f(x_n) + (I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right] - P_C y_n\right\| \\
&\leq \left\|\alpha_n\gamma f(x_n) + (I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - y_n\right\| \\
&\leq \alpha_n\left\|\gamma f(x_n) - \mu B\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|.
\end{aligned} \tag{3.27}
$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\{x_n\}, \{u_n\}$ are bounded, we have

$$
\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \tag{3.28}
$$

From (3.26) and (3.28), we get

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|, \tag{3.29}$$

it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.30}$$

On the other hand, from (3.1), we have

$$\begin{aligned}\|T(s)x_n - x_n\| &= \left\|T(s)x_n - T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &\quad + \left\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &\quad + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - x_n\right\| \\ &\leq \left\|x_n - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &\quad + \left\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\ &\quad + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - x_n\right\| \\ &\leq 2\|x_n - y_n\| + \left\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|,\end{aligned} \tag{3.31}$$

So without loss of generality, we assume that $\Im = \{T(s) : 0 \leq s < +\infty\}$ is asymptotically nonexpansive semigroup on C, and from Lemma 2, we have

$$\lim_{n\to\infty} \left\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| = 0. \tag{3.32}$$

It follows from (3.30), (3.31) and (3.32), we have

$$\lim_{n\to\infty} \|T(s)x_n - x_n\| = 0. \tag{3.33}$$

Next, we claim that there exists a common fixed point of $EP(F_1) \cap EP(F_2)$.

Since $\{x_n\}$ is bounded on Hilbert space, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some $z \in X$. From (3.19), $y_{n_i} \rightharpoonup z$. Now, we show that $z \in EP(F_1)$. From (3.1) and $(A2)$, for any $y \in H$, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n) \tag{3.34}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i}}{-} x_{n_i} r_{n_i}\right\rangle \geq F_1(y, u_{n_i}). \tag{3.35}$$

Since $\frac{u_{n_i}}{-} x_{n_i} r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup z$, from $(A1)$, it follows that $0 \geq F_1(y, z)$ for all $y \in H$. For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)z$, then we get $0 \geq F_1(y_t, z)$. From $(A1)$ and $(A2)$, we have

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, z) \leq tF_1(y_t, y) \tag{3.36}$$

and hence $0 \leq F_1(y_t, y)$. From $(A3)$, we have $0 \leq F_1(z, y)$ for all $y \in H$. Therefor, $z \in EP(F_1)$.

Since $x_{n_i} \rightharpoonup z$ and A is a bounded linear operator, we obtain $Ax_{n_i} \rightharpoonup Az$. Let $v_{n_j} = Ax_{n_j} - T^{F_2}_{r_{n_j}} x_{n_j}$. It follows from (3.14), we have $\lim_{n\to\infty} v_{n_j} = 0$ and $Ax_{n_j} - v_{n_j} = T^{F_2}_{r_{n_j}} x_{n_j}$. Then from Lemma 2.13, we get

$$F_2(Ax_{n_j} - v_{n_j}, y) + \frac{1}{r_{n_j}} \left\langle y - (Ax_{n_j} - v_{n_j}), (Ax_{n_j} - v_{n_j}) - Ax_{n_j} \right\rangle \geq 0, \forall y \in Q. \tag{3.37}$$

Since F_2 is upper semicontinuous in the first argument, and $\limsup_{n\to\infty} r_n = r > 0$, we taking $j \to \infty$, we have

$$F_2(Az - v_{n_j}, y) \geq 0, \forall y \in Q \tag{3.38}$$

that is $Az \in EP(F_2)$ and hence $z \in \Omega$.

Next, we claim that $\langle (\mu F - \gamma f)x^*, x^* - \overline{x} \rangle \leq 0, \forall \overline{x} \in \Gamma = F(S) \cap \Omega$. From (3.1), putting

$$z_n = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \tag{3.39}$$

we can observe that

$$x_{n+1} = P_C z_n \quad = \quad P_C z_n - z_n + \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \tag{3.40}$$

it follows that

$$(\mu B - \gamma f)x_n \quad = \quad \frac{1}{\alpha_n}(P_C z_n - z_n) + \frac{1}{\alpha_n}(x_n - x_{n+1}) + \frac{1}{\alpha_n}(I - \mu\aleph_n B)(y_n - x_n). \tag{3.41}$$

Hence, for each $p \in \Gamma = F(S) \cap \Omega$, we obtain that

$$\begin{aligned} \langle (\mu B - \gamma f)x_n, x_n - p \rangle &= \frac{1}{\alpha_n}\langle P_C z_n - z_n, x_n - p \rangle + \frac{1}{\alpha_n}\langle x_n - x_{n+1}, x_n - p \rangle \\ &\quad + \frac{1}{\alpha_n}\langle (I - \mu\aleph_n B)(y_n - x_n), x_n - p \rangle \\ &= \frac{1}{\alpha_n}\langle P_C z_n - z_n, x_n - p \rangle + \frac{1}{\alpha_n}\langle x_n - x_{n+1}, x_n - p \rangle \\ &\quad + \frac{1}{\alpha_n}\langle y_n - x_n, x_n - p \rangle + \frac{1}{\alpha_n}\langle By_n - Bx_n, x_n - p \rangle. \end{aligned} \tag{3.42}$$

From (3.42) taking limit $n \to \infty$, we have $By_n - Bx_n \to Bx^* - Bx^* = 0, y_n - x_n \to 0$ and $P_C z_n - z_n \to P_C x^* - x^* = 0$, we have

$$\langle (\mu B - \gamma f)x_n, x_n - p \rangle \leq 0, \tag{3.43}$$

which implies that $z = P_\Gamma(I - \mu B + \gamma f)$.

Next, we claim that $z \in \Gamma = F(S) \cap \Omega$. From (3.1), we have $x_{n+1} = P_C z_n$, and for $x^* \in \Gamma$, we have

$$\begin{aligned} x_{n+1} - x^* &= P_C z_n - z_n + z_n - x^* \\ &= P_C z_n - z_n + \alpha_n(\gamma f(x_n) - \mu Bx^*) + (I - \mu\alpha_n B)y_n - (I - \mu\alpha_n B)x^*. \end{aligned} \tag{3.44}$$

Since P_C is the metric projection from H_1 onto C, we obtain

$$\langle P_C z_n - z_n, P_C z_n - x^* \rangle \leq 0. \tag{3.45}$$

It follows from (3.44) and (3.45), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle P_C z_n - z_n, x_{n+1}-x^*\rangle + \alpha_n\langle(\gamma f(x_n)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +\langle (I-\mu\alpha_n B)(y_n - x^*), x_{n+1}-x^*\rangle \\
&\leq \alpha_n\langle(\gamma f(x_n)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +\langle (I-\mu\alpha_n B)(y_n - x^*), x_{n+1}-x^*\rangle \\
&\leq \alpha_n\gamma\langle f(x_n)-f(x^*), x_{n+1}-x^*\rangle \\
&\quad +\alpha_n\langle \gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +\langle (I-\mu\alpha_n B)(y_n - x^*), x_{n+1}-x^*\rangle \\
&\leq \alpha_n\gamma\beta\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +(1-\alpha_n\tau)\|y_n-x^*\|\|x_{n+1}-x^*\| \\
&\leq \alpha_n\gamma\beta\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +(1-\alpha_n\tau)\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\leq (1-\alpha_n(\tau-\gamma\beta))\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad +\alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\leq \frac{(1-\alpha_n(\tau-\gamma\beta))}{2}\left(\|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2\right) \\
&\quad +\alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle, \qquad (3.46)
\end{aligned}$$

it implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \frac{(1-\alpha_n(\tau-\gamma\beta))}{(1+\alpha_n(\tau-\gamma\beta))}\|x_n-x^*\|^2 \\
&\quad +\frac{2\alpha_n}{(1+\alpha_n(\tau-\gamma\beta))}\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\leq (1-\alpha_n(\tau-\gamma\beta))\|x_n-x^*\|^2 \\
&\quad +\frac{2\alpha_n}{(1+\alpha_n(\tau-\gamma\beta))}\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\leq (1-a_n)\|x_n-x^*\|^2 + \alpha_n b_n, \qquad (3.47)
\end{aligned}$$

where

$$\begin{aligned}
a_n &= \alpha_n(\tau-\gamma\beta), \\
b_n &= \frac{2}{(1+\alpha_n(\tau-\gamma\beta))}\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle. \qquad (3.48)
\end{aligned}$$

We see that $\Sigma_{n=0}^{\infty} = +\infty$ and $\limsup_{n\to\infty} b_n \leq 0$. From Lemma 2.15, we have $x_n \to x^*$. This completes the proof. □

Corollary 3.2. [17] *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 2.12 and F_2 is upper semicontinuous in the first argument. Let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $\Gamma = F(\Im) \cap \Omega \neq \emptyset$. Let the sequence $\{u_n\}$ and $\{x_n\}$ be generated by (3.1), and suppose that the sequence $\{\alpha_n\}$ satisfies the following conditions:*

(i) $\alpha_n \in (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$*;*
(ii) $\Sigma_{n=0}^{\infty}\alpha_n = 0$*;*
(iii) *either* $\Sigma_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

where $\widetilde{s_n} = \frac{1}{s_n}\int_0^{s_n} L_s^T ds \to 1$ *as* $n \to \infty$. *Then the sequence* $\{u_n\}$ *and* $\{x_n\}$ *generated by (3.1) converge strongly to* $x^* \in \Gamma = F(\Im) \cap \Omega$, *where* $x^* = P_\Gamma(I - \mu B + \gamma f)x^*$, *which is the unique solution of the variational inequality (3.2).*

Proof. From example 1.1 and example 1.2, we see that a nonexpansive semigroup is $\mathcal{T}$ is asymptotically nonexpansive semigroup. Then this theorem cover by theorem 3.1. □

Corollary 3.3. *Let* H_1 *and* H_2 *be two real Hilbert spaces and let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty closed subsets. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Assume that* $F_1 : C \times C \to \mathbb{R}$ *and* $F_2 : Q \times Q \to \mathbb{R}$ *are the bifunctions satisfying Assumption 2.12 and* F_2 *is upper semicontinuous in the first argument. Let the sequence* $\{u_n\}$ *and* $\{x_n\}$ *be generated by are generated by*

$$\begin{aligned} u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \end{aligned} \tag{3.49}$$

the sequence $\{\alpha_n\}$ *satisfies the following conditions:*

(i) $\alpha_n \in (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\Sigma_{n=0}^{\infty}\alpha_n = 0$;
(iii) *either* $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

where $\widetilde{s_n} = \frac{1}{s_n}\int_0^{s_n} L_s^T ds \to 1$ *as* $n \to \infty$. *Then the sequence* $\{u_n\}$ *and* $\{x_n\}$ *converge strongly to* $x^* \in \Gamma = F(\Im) \cap \Omega$, *where* $x^* = P_\Gamma(I - \mu B + \gamma f)x^*$, *which is the unique solution of the variational inequality (3.2).*

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References

1. E. Blum and W. Oettli, From optimization and variational inequalities to equilibriumproblems, *The Mathematics Student*, 63(1–4)(1994) 123-145.
2. P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *Journal of Nonlinear and Convex Analysis*, 6(1)(2005) 117-136.
3. V. Colao, G. L. Acedo, and G. Marino, An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings, *Nonlinear Analysis: Theory, Methods and Applications*, 71(7-8)(2009) 2708-2715.
4. L. C. Ceng, Q.H. Ansari, and J. C. Yao, Some iterativemethods for finding fixed points and for solving constrained convex minimization problems, *Nonlinear Analysis: Theory, Methods and Applications*, 74(16)(2011) 5286-5302.
5. K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, 35 (1972) 171-174.
6. K. Goebel and W. A. Kink, Topics on Metric Fixed Point Theory, Gambridge University Press, Gambridge, UK, 1990.
7. T. H. Kim, H. K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, *Nonlinear. Anal.*, 64 (2006) 1140-1152.
8. K. R. Kazmi and S. H. Rizvi, Implicit iterative method for approximating a common solution of split equilibriumproblem and fixed point problem for a nonexpansive semigroup, *Arab Journal of Mathematical Sciences*, 20(1)(2014) 57-75.
9. P. K. Lin, K. K. Tan and H. K. Xu, Demiclosedness principle and asymptotic behavior for asymptotically nonexpansive mappings, *Nonlinear. Anal.*, 24(1995) 929?946.

10. G. Marino and H. k Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, 318(2006) 43 - 52.
11. C. Martinez-Yanes, H. K. Xu, Strong convergence of the CQ method for fixed point processes, *Nonlinear Anal.*, 64(2006), 2400-2411.
12. A. Moudafi, Split monotone variational inclusions, *Journal of Optimization Theory and Applications*, 150(2)(2011) 275-283.
13. S. Plubtieng and R. Punpaeng, Fixed-point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces, *Mathematical and Computer Modelling*, 48(1-2)(2008) 279-286.
14. H. K. Xu, Iterative algorithms for nonlinear operators, *Journal of the London Mathematical Society*, 66(1)(2002) 240-256.
15. H. K. Xu, Strong asymptotic behavior of almost-orbits of nonlinear semigroups, *Nonlinear. Anal.*, 46 (2001) 135-151.
16. M. Tian, A general iterative algorithm for nonexpansive mappings in Hilbert spaces, *Nonlinear Analysis*, 73(3)(2010) 689-694.
17. P. Zhou and G. J. Zhao, Explicit Scheme for Fixed Point Problem for Nonexpansive Semigroup and Split Equilibrium Problem in Hilbert Space, *Journal of Applied Mathematics*, (2014), Article ID 858679, 12 pages.

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INEXACT PROXIMAL POINT ALGORITHM FOR MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT. The main aim of this article is to present an inexact proximal point algorithm for constrained multiobjective optimization problems under the locally Lipschitz condition of the cost function. Convergence analysis of the considered method, Fritz-John necessary optimality condition of ϵ-quasi weakly Pareto solution in terms of Clarke subdifferential is derived. The suitable conditions to guarantee that the accumulation points of the generated sequences are Pareto-Clarke critical points are provided.

KEYWORDS:Multiobjective optimization; Quasi-convex functions; Lipschitz continuous function; Clarke subdifferential; Pareto-Clarke critical point.
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1. Introduction

With the development of optimization theory, multiobjective optimization problems have increasingly received much attentions, and have been greatly applied to management, decision-making disciplines, resource planning, engineering, the design of aircraft control systems and so on, see, for example, [30, 31]. In multiobjective optimization, one considers optimization problems with several conflicting objective functions. It is usually hard to find an optimal solution that satisfies all objectives from the mathematical point of view (i.e., there is no ideal minimizer), but we obtain a set of alternatives with different trade-offs, called efficient solutions.

The multiobjective optimization problem is considering the following context: For $I = \{1, 2, ..., m\}$, we put $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_j \geq 0, j \in I\}$, and $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m : x_j > 0, j \in I\}$. For $y, z \in \mathbb{R}^m$, ($z \succeq y$ or $y \preceq z$) means that $z - y \in \mathbb{R}^m_+$, and ($z \succ y$

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or $y \prec z$) means that $z - y \in \mathbb{R}^m_{++}$. By using these relations, we consider the efficient solution concepts of the (constrained) multiobjective minimization problem

$$\min_{x \in C} F(x), \tag{1.1}$$

where $C \subset \mathbb{R}^n$ is the constrained set and $F : \mathbb{R}^n \to \mathbb{R}^m$ is the objective mapping.

There are a number of works that pay attention to the methods for finding efficient solutions of multiobjective optimization problem (1.1). Such as, in 2007, Ceng and Yao [9] developed both an absolute and a relative version of approximate proximal point algorithm. They considered the approximate proximal method via the subproblems of finding weakly efficient points for suitable regularizations of the original mapping. Later, in 2015, Papa Quiroz et al. [29] proposed an inexact proximal point method of constrained multiobjective problems involving locally Lipschitz quasiconvex objective functions. They used proximal distances and assumed that the function is also bounded from below, lower semicontinuous for convergence analysis of the method. They proved that the sequence generated by the proposed method converges to a stationary point of the problem. After that, in 2018, João Carlos de O. Souza [33] studied the convergence of exact and inexact versions of the proximal point method with a generalized regularization function in Hadamard manifolds for solving scalar and vectorial optimization problems involving Lipschitz functions. In 2018, Bento et al. [5] considered the exact proximal point method of the constrained nonsmooth multiobjective optimization problem. They used non-scalarization approach for convergence analysis of the method, where the first order optimality condition of the problem is replaced by a necessary condition for weak Pareto points of a multiobjective problem. For more information on the related works in this direction, ones may see [1, 4, 5, 6, 7, 16, 17, 33]) and the references therein.

In this paper, our interest is to consider an inexact proximal point method for solving the multiobjective optimization problem (1.1).

Using the same technique as in Bento et al. [5], we propose an inexact proximal point algorithm for constrained nonsmooth multiobjective optimization problem. In terms of Clarke subdifferential, we introduce Fritz-John optimality condition of an ϵ-quasi weak Pareto solution, which we use for convergence analysis of our method. We also show that our proposed algorithm is well defined and the sequence achieved by the proposed algorithm converges to a Pareto-Clarke critical point. For a convex objective function F, we obtain the convergence to a weak Pareto solution of the problem.

2. Preliminaries

In this section, we present some basic concepts and results that are of fundamental importance for the development of our work.

The domain of f, denoted by $\operatorname{dom} f$, is the subset of $\mathbb{R}^n$ on which f has a finite valued. A function f is said to be proper when $\operatorname{dom} f \neq \emptyset$. We denote the closed unit ball in $\mathbb{R}^n$ by $\mathbb{B}_{\mathbb{R}^n}$. We say that a scalar valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz at $x \in \mathbb{R}^n$ if there exist a neighborhood U of x and a positive real number L such that

$$|f(z) - f(y)| \leq L\|z - y\|, \quad \forall z, y \in U.$$

A vector valued function $F : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz if all components of F are locally Lipschitz.

Next, we recall some concepts of Clarke directional derivative.

The Clarke directional derivative of a proper locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $x \in \mathbb{R}^n$ in the direction of $d \in \mathbb{R}^n$ is denoted by $f^\diamond(x, d)$, and is defined as

$$f^\diamond(x, d) = \lim_{t \to 0} \sup_{y \to x} \frac{f(y + td) - f(y)}{t}.$$

Now, we recall some concepts involving locally Lipschitz functions and nonconvex constrained sets.

Let $C \subset \mathbb{R}^n$ be a nonempty and closed set. We denote the distance function $d : \mathbb{R}^n \to \mathbb{R}$ of a point $x \in \mathbb{R}^n$ to a set $C \subset \mathbb{R}^n$ as

$$d_C(x) := \inf\{\|x - c\| : c \in C\}. \tag{2.1}$$

We say that a point $x \in C$ is a Pareto-Clarke critical point of F in C if, for any element $v \in T_C(x)$, there exists $i = 1, ..., m$ such that

$$f_i^\diamond(x, v) \geq 0, \tag{2.2}$$

where f_i is the ith component of F and $T_C(x) := \{v \in \mathbb{R}^n : d_C^\diamond(x, v) = 0\}$ denotes the set of all tangent vectors to C at x. As mentioned in [10], page 11, a vector v belongs to $T_C(x)$ if and only if it satisfies the following property: for every sequence $\{x^k\}$ in C converging to x and every sequence t_k in $(0, \infty)$ converging to 0, there is a sequence v^k converging to v such that $x^k + t_k v^k$ belongs to C for all k. The normal cone is the one obtained from tangent cone $T_C(x)$ by polarity. Therefore, the normal cone $N_C(x)$ to C at x is as follows:

$$N_C(x) := \{\varsigma \in \mathbb{R}^n : \langle \varsigma, v\rangle \leq 0, \forall v \in T_C(x)\},$$

see [5]. If C is convex, $N_C(x)$ coincides with normal cones in the sense of convex analysis; (see [10], Proposition 2.4.4).

Now, we remind some basic concepts and properties of multiobjective optimization, which can be found in [24].

A sequence $\{x^k\} \subset \mathbb{R}^m$ is called a decreasing sequence if $x^p \prec x^k$ for $k < p$. A point $\bar{x}$ is said to be an infimum of $\{x^k\}$, if there is no x such that $x \preceq \bar{x}$ and $x \preceq x^k$ satisfying $\bar{x} \preceq x^k$, for all $k \in \mathbb{N}$.

Next, we recall some definitions of optimal solutions and approximate optimal solutions of multiobjective function $F : \mathbb{R}^n \to \mathbb{R}^m$.

Consider a nonempty subset $C \subset \mathbb{R}^n$ and $\epsilon := (\epsilon_1, ..., \epsilon_m) \in \mathbb{R}^m_+$, a point $x^* \in C$ is called

(i) a weak Pareto solution of problem (1.1) if there exists no $x \in C$ such that $f_i(x) < f_i(x^*)$, for all $i \in \{1, ..., m\}$.
(ii) an ϵ-weak Pareto solution of problem (1.1) if there exists no $x \in C$ such that $f_i(x) + \epsilon_i < f_i(x^*)$, for all $i \in \{1, ..., m\}$.
(iii) an ϵ-quasi weak Pareto solution of problem (1.1) if there is no $x \in C$ such that $f_i(x) + \epsilon_i \|x - x^*\| < f_i(x^*)$, for all $i \in \{1, ..., m\}$.

We denote the set of weak Pareto, ϵ-weak Pareto and ϵ-quasi weak Pareto solutions of problem (1.1) by $\arg\min_w\{F(x)|x \in C\}$, $\arg\min_{\epsilon w}\{F(x)|x \in C\}$ and $\arg\min_{\epsilon q-w}\{F(x)|x \in C\}$, respectively. For the detail, see [13] and [25].

Remark 2.1. It is apparent that, if $\epsilon = 0$, then the notions of an ϵ-weakly Pareto solution and an ϵ-weakly quasi Pareto solution defined above coincide with the usual one of a weak Pareto solution. Also, for the case, $\epsilon \neq 0$, it is easy to see that, $\arg\min_w\{F(x)|x \in C\} \subset \arg\min_{\epsilon w}\{F(x)|x \in C\}$ and $\arg\min_w\{F(x)|x \in C\} \subset \arg\min_{\epsilon q-w}\{F(x)|x \in C\}$. While, the sets $\arg\min_{\epsilon w}\{F(x)|x \in C\}$ and $\arg\min_{\epsilon q-w}\{F(x)|x \in C\}$ might be two different sets. For detail, see [13].

Now, we remind Clarke subdifferential concept of scalar and vector functions. The Clarke subdifferential of scalar valued function $f : \mathbb{R}^n \to \mathbb{R}$ at x, denoted by $\partial f(x)$, is defined as

$$\partial f(x) := \{w \in \mathbb{R}^n : \langle w, d\rangle \leq f^\diamond(x, d), \quad \forall d \in \mathbb{R}^n\},$$

see Clarke [11].
The Clarke subdifferential of $F : \mathbb{R}^n \to \mathbb{R}^m$ at $x \in \mathbb{R}^n$, denoted by $\partial F(x)$, is defined as

$$\partial F(x) := \{U \in \mathbb{R}^{m\times n} : U^T d \preceq F^\diamond(x, d), \quad \forall d \in \mathbb{R}^n\},$$

where $F^\diamond(x, d) := \{f_1^\diamond(x, d), ..., f_m^\diamond(x, d)\}$.

Proposition 2.2. *([11], Proposition 1.4)* $f^\diamond(x; v) = \max\{\xi \cdot v : \xi \in \partial f(x)\}$.

Remark 2.3. It is noted in [5] that, combining (2.2) with Proposition 2.2, we have the following alternative definition: a point $x \in \mathbb{R}^n$ is a Pareto-Clarke critical point of F in C if, for any $v \in T_C(x)$, there exist $i \in \{1, ..., m\}$ and $\xi \in \partial f_i(x)$ such that $\langle \xi, v\rangle \geq 0$. Thus, if x is not a Pareto-Clarke critical point of F in C, there exists $v \in T_C(x)$ such that $Uv \prec 0$, $\forall U \in \partial F(x)$.

The necessary condition for a point to be a Pareto-Clarke critical point of a vector-valued function can be found in Bento et al. ([5] Lemma 1), and is given below.

Proposition 2.4. [5] *Let $w \in \mathbb{R}^m_+ \setminus \{0\}$ and assume that C is closed and nonempty set. If $-U^T w \in N_C(x)$ for some $U \in \partial F(x)$, then x is a Pareto-Clarke critical point of F.*

For the nonconvex case, a formula for the Clarke sudifferential of the distance function (2.1) defined in Burke, Ferris and Qian [3] is as follows:

Proposition 2.5. [3] *Let $C \subset \mathbb{R}^m$ be a nonempty and closed set: If $x \in C$, then*

$$\partial d_C(x) \subset \mathbb{B}[0, 1] \cap N_C(x), \tag{2.3}$$

where $\mathbb{B}[0, 1]$ denotes the closed unit ball in $\mathbb{R}^m$.

Now, we recall some basic definitions of multiobjective functions.
Consider a vector function $F : \mathbb{R}^n \to \mathbb{R}^m$, we say that
i) F is called $\mathbb{R}^m_+$-convex if, for every $x, y \in \mathbb{R}^n$, the following condition holds:

$$F((1 - t)x + ty) \preceq (1 - t)F(x) + tF(y), \quad \forall t \in [0, 1].$$

ii) F is called $\mathbb{R}^m_+$-quasiconvex if, for every $x, y \in \mathbb{R}^n$, the following condition holds:

$$F((1 - t)x + ty) \preceq \max\{F(x), F(y)\}, \quad \forall t \in [0, 1],$$

where the maximum is considered coordinate by coordinate.

Remark 2.6. A vector function $F : \mathbb{R}^n \to \mathbb{R}^m$ is convex (resp. quasi-convex) iff F is componentwise convex (resp. quasi-convex), see Definition 6.2 and Corollary 6.6 of [24], pages 29, 31, respectively.

Next propositions will be useful in the following section.

Proposition 2.7. *([34], Theorem 3.2.1) Let $C \subset \mathbb{R}^n$ be a nonempty set and $f : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function on $\mathbb{R}^n$ with constant L. If $\bar{x}$ is a minimizer for the constrained minimization problem,*

$$\min f(x), \quad x \in C, \tag{2.4}$$

and $\tau \geq L$, then $\bar{x}$ is also a minimizer for the unconstrained minimization problem

$$\min\{f(x) + \tau d_C(x)\}, \quad x \in \mathbb{R}^n. \tag{2.5}$$

If $\tau > L$ and C is a closed set, then the converse assertion is also true: Any minimizer $\bar{x}$ for the unconstrained problem (2.5) is also a minimizer for the constrained problem (2.4).

Proposition 2.8. *([1], Proposition 2.6.1) Let $C = \mathbb{R}^n$ and $\hat{x}$ be a Pareto-Clarke critical point of a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$. If F is $\mathbb{R}^m_+$-convex, then $\hat{x}$ is a weak Pareto solution of the problem (1.1).*

Proposition 2.9. *([28], Proposition 5.3(ii)) For a function $f : \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ with modulus $l > 0$, it holds that*

$$\|x^*\| \leq l, \quad \forall x^* \in \partial f(\bar{x}). \tag{2.6}$$

Proposition 2.10. *([28], Theorem 5.10) Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions at $\bar{x} \in \mathbb{R}^n$, then*

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x}). \tag{2.7}$$

Proposition 2.11. [10] *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, i = 1, 2, ..., m$, be locally Lipschitz function at $x \in \mathbb{R}^n$ for all $i = \{1, ..., m\}$. Then, the function $f(x) := \max\{f_i(x) | i \in \{1, ..., m\}\}$ is also locally Lipschitz at x and*

$$\partial f(x) \subset \bigcup \left\{ \partial(\sum_{i=1}^{m} \lambda_i f_i)(x) | \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1, \lambda_i[f_i(x) - f(x)] = 0 \right\}.$$

Proposition 2.12. *([2], Theorem 2.1) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper quasiconvex locally Lipschitz function on $\mathbb{R}^n$. If $x^* \in \partial f(x)$ such that $\langle x^*, \hat{x} - x\rangle > 0$, then $f(x) \leq f(\hat{x})$.*

The next definition and result will be useful for the existence of the set of minimizers of a vector function which can be found in [24].

Definition 2.13. [24] A subset A of $\mathbb{R}^m$ is said to be $\mathbb{R}^m_+$-complete, if any decreasing sequence of A is bounded by an element of A, i.e., whenever $\{x^k\} \subset A$ is a decreasing sequence, then there exists $x \in A$ such that $x \preceq x^k$, for all $k \geq 0$.

Proposition 2.14. *([24], Lemma 3.5) If $A \subset \mathbb{R}^m$ is closed, has a lower bound (i.e., $\exists$ some $a \in A$ such that, for all $x \in A$, $a \preceq x$), then A is $\mathbb{R}^m_+$-complete.*

Proposition 2.15. *([24], Theorem 3.3) Consider the multiobjective problem (1.1). Then, $\arg\min\{F(x) | x \in C\}$ is nonempty iff $F(C)$ has a $\mathbb{R}^m_+$-complete section.*

3. Necessary optimality condition

In this section, we consider multiobjective optimization problem (1.1) of finding the quasi-weak Pareto point of a vector valued function F subject to the following constrained set

$$C := \{x \in D | g_j(x) \leq 0, \ j = 1, ..., p\},$$

where $D \subset \mathbb{R}^n$ is a nonempty and closed set, and $g_j : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function. We provide necessary conditions for a point $x^* \in C$ to be an ϵ-quasi weak Pareto solution associated to the problem (1.1).

Proposition 3.1. *Let* $x^* \in \arg\min_{\epsilon q-w}\{F(x)|x \in C\}$. *Then, there exist* $t_i \geq 0$ *and* $\mu_j \geq 0$ *for* $i \in \{1,...,m\}$ *and* $j \in \{1,...,p\}$ *with* $\sum_{i=1}^m t_i + \sum_{j=1}^p \mu_j = 1$ *and* $\tau > 0$ *such that*

$$0 \in \sum_{i=1}^m t_i \partial f_i(x^*) + \sum_{j=1}^p \mu_j \partial g_j(x^*) + \sum_{i=1}^m t_i \epsilon_i \mathbb{B}_{x^*} + \tau \partial d_D(x^*),$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $\epsilon_i \in \mathbb{R}^m_+$ *for* $i \in \{1,...,m\}$ *and* $\mathbb{B}_{x^*}$ *denotes the closed unit ball of* x^*.

Proof. For each $x \in C$, put $\Psi(x) = \max_{\substack{i\in\{1,\dots,m\}\\ j\in\{1,\dots,p\}}}\{f_i(x) - f_i(x^*) + \epsilon_i\|x - x^*\|, g_j(x)\}$. Observe that $\Psi(x^*) = 0$.
Next, since x^* is an ϵ-quasi weak Pareto optimal point, then there is no $x \in C$ such that

$$f_i(x) + \epsilon_i\|x - x^*\| < f_i(x^*), \quad \forall i \in \{1,...,m\}. \tag{3.1}$$

It can be easily verified that $0 \leq \Psi(x)$, which infers that for all $x \in C$, we have

$$\Psi(x^*) = \inf_{x\in C} \Psi(x).$$

It follows that x^* is also a minimizer to the constrained optimization problem

$$\min_{x\in C} \Psi(x).$$

Proposition 2.11 and locally Lipschitz properties of functions f_i and g_j imply that the function Ψ is also locally Lipschitz around x^*. Let L be a locally Lipschitz constant of Ψ at x^* and $\tau \geq L$, then applying the Proposition 2.7 to the last problem, we obtain

$$0 \in \partial(\Psi(x^*) + \tau d_D(x^*)). \tag{3.2}$$

Also, the sum rule (2.7) implies that

$$0 \in \partial\Psi(x^*) + \tau\partial d_D(x^*). \tag{3.3}$$

Now, by Proposition 2.11 and invoking the sum rule (2.7) applied to the Ψ, there exist non-negative real numbers $t_i \geq 0$ and $\mu_j \geq 0$ such that $\sum_{i=1}^m t_i + \sum_{j=1}^p \mu_j = 1$ and

$$\partial\Psi(x^*) \subset \left\{\sum_{i=1}^m t_i \partial f_i(x^*) + \sum_{i=1}^m t_i \epsilon_i B_{x^*} + \sum_{j=1}^p \mu_j \partial g_j(x^*)\right\}. \tag{3.4}$$

and the desired result follows by combining (3.3) with (3.4). □

4. Inexact Proximal Point Algorithm

In this section, we consider $C \subset \mathbb{R}^n$ a nonempty and closed set and $F : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz function.

Next, we consider the inexact proximal point algorithm, for obtaining a Pareto-Clarke critical point of F in C. Take a bounded sequence of positive real numbers $\{\lambda_k\}$, and a sequence $\{e^k\} \subset \mathbb{R}^m_{++}$ such that $\|e^k\| = 1$, for all $k \in \mathbb{N}$. The method generates the sequence $\{x^k\} \in C$ as follows.

4.1. **Algorithm.**

INITIALIZATION: Choose an arbitrary initial point

$$x^1 \in C. \tag{4.1}$$

STOPPING CRITERION: Given x^k, if x^k is a Pareto-Clarke critical point, then stop. Otherwise go to the iterative step.

ITERATIVE STEP: Take the next iterate $x^{k+1} \in C$ as y such that there exists $\epsilon^k \in \mathbb{R}^m_+$ satisfying

$$y \in \arg\min_{\epsilon^k q-w}\{F(x) + \frac{\lambda_k}{2}\|x - x^k\|^2 e^k | x \in \Omega_k\}, \tag{4.2}$$

$$\epsilon^k \preceq \sigma_k \frac{\lambda_k}{2}\|y - x^k\| e^k, \tag{4.3}$$

where $\Omega_k := \{x \in C | F(x) \preceq F(x^k)\}$ and $\{\sigma_k\} \subset [0, 1)$.

From now on, we will assume that $0 \prec F$.

4.2. **Existence of iterates.**

Proposition 4.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Then, the sequence $\{x^k\}$, generated by Algorithm 4.1, is well defined.*

Proof. We proceed by induction: It holds for $k = 1$, due to (4.1). Assume that x^k exists and define

$$F_k(x) := F(x) + \frac{\lambda_k}{2}\|x - x^k\|^2 e^k.$$

Since $x^k \in \Omega_k$, we have, $F_k(\Omega_k) \neq \emptyset$. By assumption on F, that is $0 \prec F$, we get, $0 \prec F_k(x)$. Now, let $\{y^p\} \subset F_k(\Omega_k)$ such that $y^p \to y$. Since $y^p \in F_k(\Omega_k)$, there exists $z^p \in \Omega_k$ satisfying $y^p = F_k(z^p)$, for any p. We claim that $\{z^p\}$ is bounded, if not, then there is $\{p_j\} \subset \{p\}$ such that $z^{p_j} \to \infty$ as $j \to \infty$, then coercivity of F_k infers that $\|F_k(z^{p_j})\| \to +\infty$ as $j \to \infty$. On the other hand, $\|F_k(z^p)\| \to \|y\|$ because $y^p = F_k(z^p)$ and $y^p \to y$, which is a contradiction. Hence, we proved that $\{z^p\}$ is a bounded sequence. Subsequently, there are $\{z^{p_j}\} \subset \{z^p\}$ and $z \in \mathbb{R}^n$ such that $z^{p_j} \to z$ as $j \to \infty$. Moreover, by the continuity of F, we know that Ω_k is a closed set. Hence, $z \in \Omega_k$. Applying continuity of F_k and using uniqueness of limit, we can assert that $y \in F_k(\Omega_k)$. This proves $F_k(\Omega_k)$ is closed.

Subsequently, by Proposition 2.14 and property of $\mathbb{R}^m_+$ that all decreasing sequences having lower bound converges to its infimum, we know that $F_k(\Omega_k)$ is $\mathbb{R}^m_+$-complete. Thus, Proposition 2.15 infers that

$$\arg\min_w\{F_k(x) | x \in \Omega_k\}$$

is not empty. Therefore, by Remark 2.1, it follows that $\arg\min_{\epsilon^k q-w}\{F_k(x) | x \in \Omega_k\} \neq \emptyset$. □

Remark 4.2. Note that if Algorithm 4.1 terminates after finite number of iterations, then it terminates at a Pareto-Clarke critical point.

4.3. **Convergence Analysis.**

In this section, first we present some results which play an important role in our subsequent considerations. Then, we show that the sequence generated by our algorithm converges to a Pareto-Clarke critical point.

Proposition 4.3. *For all $k \in \mathbb{N}$, there exists $A_k \in \mathbb{R}^{m\times n}$, $\alpha^k, \beta^k \in \mathbb{R}^m_+$, $\tau_k > 0$ and $w^k \in \mathbb{R}^n$ such that*

$$A_k^T(\alpha^k + \beta^k) + \lambda_{k-1}\langle e^{k-1}, \alpha^k\rangle(x^k - x^{k-1}) + \langle \epsilon^{k-1}, \alpha^k\rangle v^k + \tau_k w^k = 0, \tag{4.4}$$

where $v^k \in \mathbb{B}_{x^k}$, $w^k \in \mathbb{B}[0,1] \cap N_C(x^k)$ *and* $\sum_{i=1}^m (\alpha_i^k + \beta_i^k) = 1$, $\forall k \in \mathbb{N}$.

Proof. For every k, consider the functions

$$W_k(x) := F(x) - F(x^k), \text{ and } F_k(x) := F(x) + \frac{\lambda_k}{2}\|x - x^k\|^2 e^k.$$

As, F and $\|x - x^k\|^2$ are locally Lipschitz, the coordinate functions $(W_k)_i(.) := F(.) - F(x^k)$ and $(F_k)_i(.) := F(.) + \frac{\lambda_k}{2}\|. - x^k\|^2 e^k$, $i \in \{1, ..., m\}$, of $W_k(x)$ and $F_k(x)$, respectively, are also locally Lipschitz.

Since x^k is an ϵ-quasi weak Pareto solution for

$$\min F_{k-1}(x) \text{ such that } W_{k-1}(x) \preceq 0,$$

hence the desired result follows by applying Proposition 3.1, for each $k \in \mathbb{N}$ fixed with f_i and g_j by F_{k-1} and W_{k-1}, respectively, and taking into account that, from Proposition 2.5, we have

$$\partial d_C(x^k) \subset \mathbb{B}[0,1] \cap N_C(x^k), \ \forall k \in \mathbb{N}.$$

In this case, $A_k^T = [u_1^k ....u_m^k]$, where $u_i^k \in \partial f_i(x^k)$ with $i \in \{1, ..., m\}$, $\alpha^k = (\alpha_1^k, ..., \alpha_m^k)^T$ and $\beta^k = (\beta_1^k, ..., \beta_m^k)^T$. □

Proposition 4.4. *If there exists* $k \in \mathbb{N}$ *such that* $x^{k+1} = x^k$*, then* x^k *is a Pareto-Clarke critical point of* F*.*

Proof. Suppose that for any $k \in \mathbb{N}$, $x^{k+1} = x^k$, which implies that $\epsilon^k = 0$. Then by Proposition 4.3, we obtain

$$A_{k+1}^T(\alpha^{k+1} + \beta^{k+1}) + \tau_k w^{k+1} = 0, \tag{4.5}$$

which infers that

$$-A_{k+1}^T(\alpha^{k+1} + \beta^{k+1}) \in N_C(x^{k+1}). \tag{4.6}$$

Since $\sum_{i=1}^m (\alpha_i^{k+1} + \beta_i^{k+1}) = 1$, we can say that $(\alpha^{k+1} + \beta^{k+1}) \in \mathbb{R}_+^m \setminus \{0\}$. Moreover, $A_{k+1} \in \partial F(x^{k+1})$, then using Proposition 2.4, we obtain the desired result. □

Proposition 4.5. *Let* $k_0 \in \mathbb{N}$ *be such that* $\alpha^{k_0} = 0$*. Then* x^{k_0} *is a Pareto-Clarke critical point of* F*.*

Proof. If there exists $k_0 \in \mathbb{N}$ such that $\alpha_{k_0} = 0$ then, from (4.4), we have

$$A_{k_0}^T \beta^{k_0} + \tau_{k_0} w^{k_0} = 0, \tag{4.7}$$

where $\tau_{k_0} > 0$, $w^{k_0} \in N_C(x^{k_0})$. Since $A_{k_0} \in \partial F(x^{k_0})$ and $\beta^{k_0} \in \mathbb{R}_+^m \setminus \{0\}$, the desired result follows by using Proposition 2.4. □

From now on, we will assume the sequences $\{\lambda_k\}$, $\{\epsilon^k\}$ and $\{x^k\}$ are infinite sequences generated by Algorithm 4.1, then $\alpha^k \neq 0$ and $x^{k+1} \neq x^k$, in view of Proposition 4.4 and 4.5, respectively.
Next we prove that every cluster point of x^k, if any, is Pareto-Clarke critical point.

Theorem 4.1. *Assume that there exist scalars* $a, b, c, d \in \mathbb{R}_{++}$ *such that* $a \leq \lambda_k \leq b$, $c \leq e_i^k \leq d$, $\sigma_k \leq d < 1$*, for all* $k \in \mathbb{N}$ *and* $i \in \{1, ..., m\}$*. Then, every cluster point of* $\{x^k\}$*, if any, is a Pareto-Clarke critical point of* F*.*

Proof. Since

$$x^{k+1} \in \arg\min_{\epsilon^k q - w}\{F(x) + \frac{\lambda_k}{2}\|x - x^k\|^2 e^k | x \in \Omega_k\},$$

we have

$$\max_{1\leq i\leq m}\{f_i(x^k)-f_i(x^{k+1})+\epsilon_i^k\|x^k-x^{k+1}\|-\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_i^k\}\geq 0.$$

Hence for any k, there exists some index $i_0:=i_0(k)\in\{1,...,m\}$, where the maximum in the last inequality is attained. Thus,

$$f_{i_0}(x^k)-f_{i_0}(x^{k+1})+\epsilon_{i_0}^k\|x^k-x^{k+1}\|-\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k\geq 0,$$

which provides us

$$\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k-\epsilon_{i_0}^k\|x^k-x^{k+1}\|\leq f_{i_0}(x^k)-f_{i_0}(x^{k+1}).$$

By (4.3) and boundedness assumption of $\{\lambda_k\}$ and $\{e^k\}$, we obtain

$$\begin{aligned}f_{i_0}(x^k)-f_{i_0}(x^{k+1})&\geq\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k-\epsilon_{i_0}^k\|x^{k+1}-x^k\|\\&\geq\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k-\sigma_k\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k\\&\geq(1-\sigma_k)\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k.\end{aligned}$$

Then, from the boundedness of $\{\lambda_k\}$, $\{\epsilon^k\}$ and $\{\sigma_k\}$, we obtain

$$(1-d)\frac{ac}{2}\|x^{k+1}-x^k\|^2\leq f_{i_0}(x^k)-f_{i_0}(x^{k+1}). \tag{4.8}$$

Combining (4.2) with the definition of Ω_k, it follows that $\{F(x^k)\}$ is nonincreasing sequence, and by assumption on F, i.e, $0\prec F$, we have that $\{F(x^k)\}$ is a convergent sequence. Hence, by taking $k\to+\infty$ on (4.8), we get

$$\lim_{k\to+\infty}(x^{k+1}-x^k)=0. \tag{4.9}$$

Take $\bar{x}$ as a cluster point of $\{x^k\}$, then there exists subsequence $\{x^{k_j}\}$ of $\{x^k\}$ converging to $\bar{x}$. Therefore, by applying Proposition 4.3 for the sequence $\{x^{k_j}\}$, we have that there exist sequences $A_{k_j+1}\in\partial F(x^{k_j+1})$, $\alpha^{k_j+1},\beta^{k_j+1}\in\mathbb{R}^m_+$ and $v^{k_j+1}\in B_{x^{k_j+1}}$ such that

$$A^T_{k_j+1}(\alpha^{k_j+1}+\beta^{k_j+1})+\lambda_{k_j}\langle e^{k_j},\alpha^{k_j+1}\rangle(x^{k_j+1}-x^{k_j})+\langle\epsilon^{k_j},\alpha^{k_j+1}\rangle v^{k_j+1}+\tau_{k_j+1}w^{k_j+1}=0, \tag{4.10}$$

where $\sum_{i=1}^m(\alpha_i^{k_j+1}+\beta_i^{k_j+1})=1$ and $w^{k_j+1}\in N_C(x^{k_j+1})$.

From the convergence of $\{x^{k_j}\}$, we obtain that $\{x^{k_j}\}$ is bounded. By locally Lipschitz property of F, it follows by (2.6) that their subgradients are bounded. So from the above conditions, the sequences A_{k_j}, v^{k_j}, α^{k_j}, β^{k_j}, w^{k_j} are bounded. Thus, equality (4.10) implies that τ_{k_j} is also bounded. Now, without loss of generality, we may assume that the sequences A_{k_j}, v^{k_j}, α^{k_j}, β^{k_j}, w^{k_j} and τ_{k_j} converge to $\bar{A}$, $\bar{v}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{w}$ and $\bar{\tau}$ respectively. Also, since $\lambda_{k_j}\langle e^{k_j},\alpha^{k_j+1}\rangle$ is bounded, then by letting k_j goes to infinity in (4.10), we obtain

$$\bar{A}^T(\bar{\alpha}+\bar{\beta})+\bar{\tau}\bar{w}=0. \tag{4.11}$$

Since $\bar{w}\in N_C(\bar{x})$, $(\bar{\alpha}+\bar{\beta})\in\mathbb{R}^m_+\setminus\{0\}$, $\bar{A}\in\partial F(\bar{x})$, it follows from (4.11) that

$$-\bar{A}^T(\bar{\alpha}+\bar{\beta})\in N_C(\bar{x}),$$

and this together with Proposition 2.4, enables us to say that $\bar{x}$ is a Pareto-Clarke critical point of F. This completes the proof. □

Next, we will present full covergence theorem of proposed Algorithm 4.1. The following definition and lemma will be useful in our proof.

Definition 4.6. Let $\Omega \subset \mathbb{R}^n$ be a nonempty set. A sequence $\{z^k\} \subset \Omega$ is said to be Fejér convergent to a nonempty set Ω iff, for all $z \in \Omega$,

$$\|z^{k+1} - z\|^2 \leq \|z^k - z\|^2 + \vartheta_k, \quad k = 0, 1, ...$$

where $\{\vartheta_k\} \subset (0, \infty)$ satisfies $\sum_{k=1}^{\infty} \vartheta_k < \infty$.

The following result on Fejér convergence is well known.

Lemma 4.7. [15] *Let $\Omega \subset \mathbb{R}^n$ be a nonempty set and $\{z^k\} \subset \Omega$ be a Fejér convergent sequence to Ω, then:*

- *The sequence $\{z^k\}$ is bounded.*
- *If a cluster point $\bar{z}$ of $\{z^k\}$ belongs to Ω, the whole sequence $\{z^k\}$ converges to $\bar{z}$ as k goes to $+\infty$.*

Now, we will consider that $F : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-quasiconvex, C is convex set, and the following well-known assumption.
$\mathcal{H}\mathbf{1}$: The set $(F(x^0) - \mathbb{R}^m_+) \cap F(C)$ is $\mathbb{R}^m_+$-complete.

Theorem 4.2. *Assume that $\mathcal{H}\mathbf{1}$ holds true and $\sum_{k=0}^{+\infty} \sigma_k < +\infty$. Then, the sequence $\{x^k\}$ generated by the Algorithm 4.1, converges to a Pareto-Clarke critical point of F.*

Proof. Define

$$E := \bigcap_{k=0}^{+\infty} \Omega_k.$$

Assumption $\mathcal{H}\mathbf{1}$ implies that E is nonempty. Take $x^* \in E$, which infers that $x^* \in \Omega_k$ for $k \in \mathbb{N}$. It is easy to see that:

$$\|x^k - x^*\|^2 = \|x^{k+1} - x^*\|^2 + \|x^k - x^{k+1}\|^2 + 2\langle x^k - x^{k+1}, x^{k+1} - x^*\rangle, \ \forall k \in \mathbb{N}. \quad (4.12)$$

Following the steps of the proof of Theorem 4.1,

$$\lambda_k \langle e^k, \alpha^{k+1}\rangle (x^k - x^{k+1}) = A^T_{k+1}(\alpha^{k+1} + \beta^{k+1}) + \langle \epsilon^k, \alpha^{k+1}\rangle v^{k+1} + \tau_{k+1} w^{k+1}, \ \forall k \in \mathbb{N}. \quad (4.13)$$

Now, combining (4.12) with (4.13), we get

$$\begin{aligned} &\frac{\lambda_k b_k}{2}\left(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \|x^k - x^{k+1}\|^2\right) \\ &= \left\langle A^T_{k+1}(\alpha^{k+1} + \beta^{k+1}) + \langle \epsilon^k, \alpha^{k+1}\rangle v^{k+1} + \tau_{k+1} w^{k+1}, x^{k+1} - x^* \right\rangle \\ &= \sum_{i=1}^m (\alpha_i^{k+1} + \beta_i^{k+1})\langle u_i^{k+1}, x^{k+1} - x^*\rangle + \sum_{i=1}^m \alpha_i^{k+1} \epsilon_i^k \langle v^{k+1}, x^{k+1} - x^*\rangle \\ &+ \tau_{k+1}\langle w^{k+1}, x^{k+1} - x^*\rangle, \end{aligned} \quad (4.14)$$

where $b_k = \langle e^k, \alpha^{k+1}\rangle$, $u_i^{k+1} \in \partial f_i(x^{k+1}), \forall k \in \mathbb{N}$ and $i \in \{1, ..., m\}$. Since F is $\mathbb{R}^m_+$-quasiconvex function, in particular, f_i is quasiconvex for each $i \in \{1, ..., m\}$. As $x^* \in \Omega_k$ and $u_i^{k+1} \in \partial f_i(x^{k+1})$, it follows by Proposition 2.12 that

$$\sum_{i=1}^m (\alpha_i^{k+1} + \beta_i^{k+1})\langle u_i^{k+1}, x^{k+1} - x^*\rangle \geq 0, \ \forall k \in \mathbb{N}. \quad (4.15)$$

As C is a convex set, $w^{k+1} \in N_C(x^{k+1})$ together with $\tau_{k+1} > 0$ and characterization of convex normal cone imply that

$$\tau_{k+1}\langle w^{k+1}, x^{k+1} - x^*\rangle \geq 0, \ \forall k \in \mathbb{N}. \tag{4.16}$$

By combining the inequalities (4.15), (4.16) with (4.14), we obtain

$$\begin{aligned} \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \|x^k - x^{k+1}\|^2 &\geq -\frac{2}{\lambda_k b_k}\sum_{i=1}^m \alpha_i^{k+1}\epsilon_i^k\langle v^{k+1}, x^{k+1} - x^*\rangle \\ &\geq -\sigma_k\|x^{k+1} - x^k\|\|x^* - x^{k+1}\|, \ \forall k \in \mathbb{N}. \end{aligned} \tag{4.17}$$

As, $r + s \geq 2\sqrt{rs}$ holds for $r, s \geq 0$, taking $s := \|x^{k+1} - x^k\|$ and $r := \|x^* - x^{k+1}\|$, we obtain

$$\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \|x^k - x^{k+1}\|^2 \geq -\frac{\sigma_k}{2}\left[\|x^{k+1} - x^k\|^2 + \|x^* - x^{k+1}\|^2\right], \ \forall k \in \mathbb{N}.$$

Thus, we get

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \left(\frac{1}{1-\sigma_k}\right)\|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 \\ &\leq \left(1 + \frac{\sigma_k}{1-\sigma_k}\right)\|x^k - x^*\|^2, \ \forall k \in \mathbb{N}. \end{aligned} \tag{4.18}$$

Since $\sum_{k=0}^{\infty} \sigma_k^2 < +\infty$, it follows that

$$K_0 := \sum_{k=k_0}^{+\infty} \frac{2\sigma_k^2}{1 - 2\sigma_k^2} < +\infty \ \text{ and } \ K_1 := \prod_{j=k_0}^{+\infty}\left(1 + \frac{2\sigma_j^2}{1-2\sigma_j^2}\right) < +\infty.$$

By (4.18), observe that for all $k \geq k_0$

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \left(1 + \frac{2\sigma_k^2}{1-2\sigma_k^2}\right)\|x^k - x^*\|^2 \\ &\leq \left(1 + \frac{2\sigma_{k-1}^2}{1-2\sigma_{k-1}^2}\right)\left(1 + \frac{2\sigma_k^2}{1-2\sigma_k^2}\right)\|x^{k-1} - x^*\|^2 \\ &\ \ \vdots \\ &\leq \prod_{j=k_0}^{k}\left(1 + \frac{2\sigma_j^2}{1-2\sigma_j^2}\right)\|x^{k_0} - x^*\|^2 \\ &\leq \prod_{j=k_0}^{\infty}\left(1 + \frac{2\sigma_j^2}{1-2\sigma_j^2}\right)\|x^{k_0} - x^*\|^2 \\ &= K_1\|x^{k_0} - x^*\|^2. \end{aligned}$$

This shows that $\{x^k\}$ is bounded. Then (4.18) becomes

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \frac{2\sigma_k^2}{1-2\sigma_k^2}K^2, \ \forall k \in \mathbb{N}. \tag{4.19}$$

where $K = \sup_k \|x^k - x^*\|$. Take $\eta_k = \frac{2\sigma_k^2}{1-2\sigma_k^2}K^2$. Since $\eta_k > 0$ and $\sum_{k=1}^{\infty} \eta_k < +\infty$, we obtain that $\{x^k\}$ is quasi-Fejér convergent to E and boundedness of $\{x^k\}$ implies that the sequence $\{x^k\}$ has a cluster point $\bar{x}$. Since Theorem 4.1 implies that $\bar{x} \in E$. Therefore using Lemma 4.7 with $U = E$, we conclude that the whole sequence $\{x^k\}$ converges to $\bar{x}$ as k goes to $+\infty$, where $\bar{x}$ is a Pareto-Clarke critical point of F. □

Corollary 4.8. *If $C = \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-convex and locally Lipschitz function, then the sequence $\{x^k\}$ converges to a weak Pareto optimal point of F.*

Proof. It is immediate from Proposition 2.8. □

5. Conclusion

Bento et al. [5] proposed an exact proximal point method for nonconvex and non-differentiable constrained multiobjective optimization problems. Later, Bento et al. [6] extended the above work in the Riemannian context. Furthermore, for full convergence analysis, they assumed that the objective function is convex. After that Lucas Vidal de [23] proposed and analyzed an inexact version of proximal point method presented by Bento et al. [6]. They also derived the Fritz John necessary optimality condition in terms of Mordukovich subdifferential for convergence analysis of the algorithm.
In this article, we developed an inexact version of proximal point method of Bento et al. [5]. In terms of Clarke subdifferential, we introduced Fritz-John necessary optimality condition of ϵ-quasi weakly Pareto solution, which we apply for convergence analysis of our proposed method. We also presented that the proposed method is well defined and under some suitable conditions the sequence attained by our proposed method converges to a Pareto-Clarke critical point. The newly proposed inexact proximal point algorithm is important because of its practical point of view. Notice that, the proximal point method is a conceptual algorithm, and its computational performance strongly depends on the method used to solve the subproblems. Hence, in practice computations introduce numerical errors in order to solve the auxiliary minimization problems and these methods usually provide only approximate solutions of the subproblems. Clearly, it is very important, from the view of practice, to study the asymptotic behavior of iterations of the algorithm in the presence of computational errors.

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References

1. Apolinario, H.C., Papa Quiroz, E.A., Oliveria, P.R.: A scalarization proximal point method for quasiconvex multiobjective minimization. J. Glob. Optim. 64, 79–96 (2016)
2. Aussel, D.: Subdifferential properties of quasiconvex and pseudoconvex functions: unified approach. J. Optim. Theory Appl. 97(1), 29–45 (1998)
3. Burke, J. V., Ferris, M. C., Qian M.: On the Clarke subdifferential of the distance function of a closed set, J. Math. Anal. Appl., 166, 199–213 (1992)
4. Branke, J., Deb, K., Miettinen, K., Slowinski, R. eds.: Practical approaches to multiobjective optimization, Dagstuhl Seminar, Dagstuhl, Wadern, Germany, 06501, (2007)
5. Bento, G.D.C., Cruz Neto, J.X., López, G., Soubeyran, A., Souza, J.C.O.,: The proximal point method for locally Lipschitz functions in multiobjective optimization with application to the compromise problem. SIAM Journal on Optimization, 28(2), pp.1104-1120 (2018)
6. Bento, G.D.C., da Cruz Neto, J.X. and de Meireles, L.V.: Proximal Point Method for Locally Lipschitz Functions in Multiobjective Optimization of Hadamard Manifolds. Journal of Optimization Theory and Applications, 179(1), pp.37-52 (2018)
7. Burachik, R. S., Kaya, C. Y., Rizvi, M.M.: A new scalarization technique to approximate Pareto fronts of problems with disconnected feasible sets, J. Optim. Theory Appl., 162, 428-446 (2014)
8. Bolte, J., Daniilidis, A., Lewis, A., Shiota, M.: Clarke subgradients of stratifiable functions. SIAM J. Optim. 18, 556–572 (2007)
9. Ceng, Lu-Chuan, and Jen-Chih Yao. "Approximate proximal methods in vector optimization." European Journal of Operational Research 183(1), 1-19 (2007)
10. Clarke, H.F.: Optimization and Nonsmooth Analysis, Classics in Applied Mathematics. SIAM, New York (1983)

11. Clarke, F. H.: Generalized gradients and applications, Trans. Amer. Math. Soc., 205, 247–262 (1975)
12. Custodio, A.L., Madeira, J.F.A., Vaz, A.I.F., Vicente, L.N.: Direct multisearch for multiobjective optimization. SIAM J. Optim. 21, 1109–1140 (2011)
13. Chuong, T.D., Kim, D.S.: Approximate solutions of multiobjective optimization problems. Positivity 20, 187–207 (2016)
14. Chuasuk, P., A. Farajzadeh, and A. Kaewcharoen. An iterative algorithm for solving split feasibility problems and fixed point problems in p-uniformly convex and smooth Banach spaces, Journal of Computational Analysis and Applications, 49 (2020)
15. Combettes, P. L.: Quasi-Fejerian analysis of some optimization algorithms, in Inherently Parallel Algorithms in Feasibility and Optimization and their Applications, vol. 8 of Studies in Computational Mathematics, pp. 115–152, North-Holland, Amsterdam, (2001)
16. Durea M., Strugariu, R.: Some remarks on proximal point algorithm in scalar and vectorial cases, Nonlinear Funct. Anal. Appl., 15, 307-319 (2010)
17. Eichfelder, G.: An adaptive scalarization method in multiobjective optimization, SIAM J. Optim., 19, 1694-1718 (2009)
18. Gal, T., Hanne, T.: On the development and future aspects of vector optimization and MCDM. A tutorial. In: Multicriteria Anal., (Coimbra, 1994), 130-145. Springer, Berlin (1997)
19. Gromicho, J.: Quasiconvex Optimization and Location Theory. Kluwer Academic Publishers, Dordrecht, The Netherlands (1998)
20. Gutiérrez, C., Jiménez, B., Novo, V.: ϵ-Pareto optimality conditions for convex multiobjective programming via max function. Numer. Funct. Anal. Optim. 27, 57–70 (2006)
21. Ioffe, A.D., Tihomirov, V.M.: Theory of extremal problems. North-Holland Publishing Co., Amsterdam-New York (1979)
22. Khonchaliew, M.; Farajzadeh, A.; Petrot, N. Shrinking Extragradient Method for Pseudomonotone Equilibrium Problems and Quasi-Nonexpansive Mappings, 11, 480, Symmetry (2019)
23. Meireles, Lucas Vidal de.: Proximal point methods for multiobjective optimization in riemannian manifolds (PhD Thesis), (2019).
24. Luc, D.T.: Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin (1989)
25. Miettinen, K.M.: Nonlinear Multiobjective Optimization. Kluwer Academic Publishers, Boston (1999)
26. Mordukhovich, B.S., Shao, Y.H.: Nonsmooth sequential analysis in Asplund spaces. Trans. Amer. Math. Soc. 348(4), 1235–1280 (1996)
27. Mas-Colell, A., Whinston, M.D., Green, J.R.: Microeconomic Theory. Oxford University Press, New York, NY, USA (1995)
28. Penot J. P. Circa-Subdifferentials, Clarke Subdifferentials. In: Calculus Without Derivatives. Graduate Texts in Mathematics, vol 266. Springer, New York, NY (2013)
29. Quiroz, E.P., Ramirez, L.M. and Oliveira, P.R.: An inexact proximal method for quasiconvex minimization. European Journal of Operational Research, 246(3), pp.721-729 (2015)
30. Schy, A., Giesy, D.P.: Multicriteria optimization techniques for design of aircraft control systems. In: Stadler, W. (ed.) Multieriteria Optimization in Engineering and in the Sciences. Plenum Press, New York (1988)
31. Stadler, W.: Multicriteria Optimization in Engineering and in the Sciences. Plenum Press, New York (1988)
32. Son, T.Q., Van Tuyen, N. and Wen, C.F.: Optimality conditions for approximate Pareto solutions of a nonsmooth vector optimization problem with an infinite number of constraints arXiv preprint arXiv:1808.10100 (2018)
33. Souza, J.C.D.O.,: Proximal Point Methods for Lipschitz Functions on Hadamard Manifolds: Scalar and Vectorial Cases. Journal of Optimization Theory and Applications, 179(3), pp.745-760 (2018)
34. Vinter, R.B.: Optimal Control. Birkhauser, Basel (2000)

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CONVERGENCE THEOREMS OF MONOTONE (α, β)-NONEXPANSIVE MAPPINGS FOR NORMAL-S ITERATION IN ORDERED BANACH SPACES WITH CONVERGENCE ANALYSIS

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ABSTRACT. In this work, we prove some theorems of existence of fixed points for a monotone (α, β)-nonexpansive mapping in a uniformly convex ordered Banach space. Also, we prove some weak and strong convergence theorems of normal-S iteration under some control condition. Finally, we give two numerical examples to illustrate the main result in this paper.

KEYWORDS:Ordered Banach space; fixed point; monotone (α, β)-nonexpansive mapping; normal S-iteration

AMS Subject Classification: :47H07; 47H10.

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1. Introduction

Let E be an ordered Banach space with the partial order $\leq$. A mapping $T : E \to E$ said to be *monotone* if $Tx \leq Ty$ for all $x, y \in E$ with $x \leq y$ and *monotone nonexpansive* if T is monotone and

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in E$ with $x \leq y$.

In 2015, Dehaish and Khamsi [1] consider Mann's iteration $\{x_n\}$ for a monotone nonexpansive mapping $T : C \to C$ defined by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n,$$

for each $n \geq 1$, where $\{\beta_n\}$ in $(0, 1)$ for finding some order fixed points of monotone nonexpansive mappings in uniformly convex ordered Banach spaces for prove some weak convergence theorems. The results of Dehaish and Khamsi , they gave the control condition $\{\beta_n\}$ in $[a, b]$ with $a > 0$ and $b < 1$, but their results do not entail $\beta_n = \frac{1}{n+1}$

Thus, to improve the results mentioned above, in 2016, Song et al. [2] they proved some weak convergence theorems of Mann's iteration satisfies the following condition:

$$\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty.$$

Clearly, this control condition $\{\beta_n\}$ contains $\beta_n = \frac{1}{n+1}$ as a special case.

In 2016, Song et al. [3] considered the convergence theorems of Mann's iteration for a monotone α-nonexpansive mapping T in an ordered Banach space E.

In 2017, Muangchoo-in et al. [4] introduced the notion of a monotone (α, β)-nonexpansive mapping T in an ordered Banach space E and proved some existence theorems of fixed points by using the assumption $\liminf_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0$. and some weak and strong convergence theorems of Ishikawa type iteration as follows are obtained :

$$\begin{cases} y_n = (1 - s_n)x_n + s_n Tx_n, \\ x_{n+1} = (1 - s_n)x_n + s_n T(y_n) \end{cases} \tag{1.1}$$

for each $n \geq 1$, where $\{s_n\}$ is the sequences in $[0, 1]$. Under the control condition

$$\liminf_{n \longrightarrow \infty} s_n(1 - s_n) > 0 \textbf{ or } \limsup_{n \longrightarrow \infty} s_n(1 - s_n) > 0.$$

In 2013, Sahu, D.R. [5] introduced Normal S-iteration process defined as follows : For C a convex subset of normed space X and a non-linear mapping T of C into itself, for each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} y_n = (1 - s_n)x_n + s_n Tx_n, \\ x_{n+1} = T(y_n) \end{cases} \tag{1.2}$$

for each $n \geq 1$, where $\{s_n\}$ is the sequences in $(0, 1)$.

Motivated by the results mentioned above, in this paper, we show some existence of a fixed point of a monotone (α, β)-nonexpansive mapping in ordered Banach spaces by do not use the condition $\liminf_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0$. And we prove some weak and strong convergence theorems of Normal S-iteration for a monotone (α, β)-nonexpansive mapping under the condition

$$\limsup_{n \longrightarrow \infty} s_n(1 - s_n) > 0, \quad \liminf_{n \longrightarrow \infty} s_n(1 - s_n) > 0.$$

Finally, we give a numerical example to illustrate the main result in this paper.

2. Preliminaries

Let P be a closed and convex cone of a real Banach space E. A *partial order* "$\leq$" with respect to P in E is defined as follows:

$$x \leq y \ (x < y) \text{ if and only if } \ y - x \in P \ (y - x \in \mathring{P}),$$

for all $x, y \in E$, where $\mathring{P}$ is the interior of P.

In this paper, let E be a Banach space with the norm $\|\cdot\|$ and the partial order $\leq$. Let $F(T) = \{x \in E : Tx = x\}$ denote the set of all fixed points of a mapping T. Also, we assume that the order intervals are convex and closed. Recall that an order interval is any of the subsets

$$[x, \rightarrow) = \{p \in E; x \leq p\} \ \text{ or } \ (\leftarrow, x] = \{p \in E; p \leq x\}$$

for any $a \in C$. An *order interval* $[x, y]$ for all $x, y \in E$ is given by

$$[x, y] = [x, \rightarrow) \cap (\leftarrow, y] = \{z \in E : x \leq z \leq y\}. \tag{2.1}$$

Then the convexity of the order interval $[x, y]$ implies that

$$x \leq tx + (1-t)y \leq y, \tag{2.2}$$

for all $x, y \in E$ with $x \leq y$.

A Banach space E is said to be:

(1) *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$;
(2) *uniformly convex* if, for all $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$.

The following inequality was shown by Xu [6] in a uniformly convex Banach space E, which is known as *Xu's inequality.*

Lemma 2.1. [6] *For any real numbers $q > 1$ and $r > 0$, a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \longrightarrow [0, +\infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^q \leq t\|x\|^q + (1-t)\|y\|^q - \omega(q, t)g(\|x - y\|), \tag{2.3}$$

for all $x, y \in B_r(0) = \{x \in E; \|x\| \leq r\}$ and $t \in [0, 1]$, where $\omega(q, t) = t^q(1-t) + t(1-t)^q$.

In particular, take $q = 2$ and $t = \frac{1}{2}$,

$$\Big\|\frac{x+y}{2}\Big\|^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}g(\|x - y\|). \tag{2.4}$$

Lemma 2.2. [7] *Let K be a nonempty closed convex subset of a reflexive Banach space E. Assume that $\rho : K \longrightarrow R$ is a proper convex lower semi-continuous and coercive function. Then the function ρ attains its minimum on K, that is, there exists $x \in K$ such that*

$$\rho(x) = \inf_{y \in K} \rho(y).$$

Lemma 2.3. [8] *A Banach space E is said to satisfy Opial's condition if, whenever any sequence $\{x_n\}$ in E converges weakly to a point x,*

$$\limsup_{n \longrightarrow \infty} ||x_n - x|| < \limsup_{n \longrightarrow \infty} ||x_n - y||$$

for any $y \in E$ such that $y \neq x$.

Definition 2.4. [4] Let K be a nonempty closed subset of an ordered Banach space $(E, \leq)$. A mapping $T : K \to K$ is said to be :

(1) *monotone (α,β)-nonexpansive* if T is monotone and, for some $\alpha,\beta<1$,
$\|Tx-Ty\|^2 \le \alpha\|Tx-y\|^2+\beta\|Ty-x\|^2+(1-(\alpha+\beta))\|x-y\|^2$,
for all $x,y\in K$ with $x\le y$, which is equivalent to

$$\|Tx-Ty\|^2 \le \|x-y\|^2+\frac{\alpha+\beta}{1-\beta}\|Tx-x\|^2 + \frac{2}{1-\beta}\|Tx-x\|\left[\|x-y\|+\|Tx-Ty\|\right]. \quad (2.5)$$

(2) *monotone quasi-nonexpansive* if T is monotone, $F(T)\neq\emptyset$ and $\|Tx-p\|\le \|x-p\|$
for all $p\in F(T)$ and $x\in K$ with $x\le$ p or $p\le x$.

Remark 2.5. If $\beta=\alpha$, then (α,β)-nonexpansive is α-nonexpansive mapping.

3. Main Results

3.1. The existence of fixed points.

We denote

$$F_{\le}(T)=\{p\in F(T): p\le x_1\},\quad F_{\ge}(T)=\{p\in F(T): x_1\le p\}.$$

Note that, since the partial order $\le$ is defined by the closed convex cone P, it is obvious that both $F_{\le}(T)$ and $F_{\ge}(T)$ are closed convex.

Now, we introduce the following lemma to find fixed points of a monotone (α,β)-nonexpansive mapping in Banach space E:

Lemma 3.1. *Let K be a nonempty closed and convex subset of a Banach space $(E,\le)$. Let $T: K\to K$ be a monotone mapping and assume that the sequence $\{x_n\}$ defined by Normal S-iteration (1.2) and $x_1\le Tx_1$ (or $Tx_1\le x_1$). Then we have*

(1) *$x_n\le y_n\le x_{n+1}$ (or $x_n\ge y_n\ge x_{n+1}$);*
(2) *$x_n\le x$ (or$x\le x_n$) for all $n\le 1$ if $\{x_n\}$ weakly converges to a point $x\in K$.*

Proof. (1) Let $k_1,k_2\in K$ such that $k_1\le k_2$. Then we have

$$k_1\le(1-\alpha)k_1+\alpha k_2\le k_2$$

for all $\alpha\in[0,1]$ since order intervals are convex. By the assumption, we have $x_1\le Tx_1$ and so the inequality is true for $n=1$. Assume that $x_k\le Tx_k$ for $k\ge 2$. We will show that $x_{k+1}\le Tx_{k+1}$ by convexity and monotonicity, we have

$$x_k\le(1-s_k)x_k+s_kTx_k=y_k\le Tx_k,$$

i.e., $x_k\le y_k\le Tx_k\le Ty_k=x_{k=1}$. since $y_k\le x_{k+1}$ by T is monotone then $Ty_k=x_{k+1}\le Tx_{k+1}$. By induction, we can conclude that $x_n\le Tx_n$ is true for all $n\ge 1$.
Now we have $x_n\le Tx_n$ for all $n\ge1$ by convexity

$$x_n\le(1-s_n)x_n+s_nTx_n=y_n\le Tx_n,$$

since T is monotonicity $x_n\le y_n$ then $Tx_n\le Ty_n$, that is $x_n\le y_n\le Tx_n\le Ty_n=x_{n+1}$. Hence, we conclude that $x_n\le y_n\le x_{n+1}$
On the other hand, if we assume $Tx_1\le x_1$, then we can show that $x_n\ge y_n\ge x_{n+1}$

(2) From Dehaish and Khamsi [1, Lemma 3.1]), we have the conclusion. This completes the proof. □

Next, we show some existence theorems of fixed points of monotone (α,β)-nonexpansive mappings in a uniformly convex ordered Banach space $(E,\le)$.

Theorem 3.1. *Let K be a nonempty and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume $x_1 \leq Tx_1$ and the sequence $\{x_n\}$ defined by Normal S-iteration (1.2) is bounded with $x_n \leq w$ for some $w \in K$. Then $F_{\geq}(T) \neq \emptyset$.*

Proof. From Lemma 3.1, we have $x_1 \leq \cdots \leq x_n \leq x_{n+1}$. Let $C_n = \{z \in K : x_n \leq z\}$ for all $n \geq 1$. Then C_n is closed convex and $w \in C_n$. So C_n is nonempty. Let $C^* = \bigcap_{n=1}^{\infty} C_n$. Then C^* is a nonempty and closed convex subset of K. Since $\{x_n\}$ is bounded, we can define a function $\rho : C^* \longrightarrow [0, +\infty)$ by

$$\rho(z) = \limsup_{n \longrightarrow \infty} \|x_n - z\|^2,$$

for all $z \in C^*$. it follows from Lemma 2.2 that, there exists $z^* \in C^*$ such that

$$\rho(z^*) = \inf_{z \in C^*} \rho(z). \tag{3.1}$$

By the definition of C^*, we have

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots \leq z^*.$$

Since T is monotone, it follows from Lemma 3.1 that

$$x_n \leq Tx_{n+1} \leq Tz^*,$$

for each $k \geq 1$, which means that $Tz^* \in C^*$ and hence $\frac{z^* + Tz^*}{2} \in C^*$. Thus, by (3.1), we have

$$\rho(z^*) \leq \rho\Big(\frac{z^* + Tz^*}{2}\Big), \quad \rho(z^*) \leq \rho(Tz^*). \tag{3.2}$$

On the other hand, it follows from Definition 2.4 that

$$\begin{aligned}\|Tx_n - Tz^*\|^2 &\leq \|x_n - z^*\|^2 + \frac{\alpha + \beta}{1 - \beta}\|Tx_n - x_n\|^2 \\ &\quad + \frac{2}{1 - \beta}\|Tx_n - x_n\|\big[|\alpha|\|x_n - z^*\| + |\beta|\|Tx_n - Tz^*\|\big].\end{aligned}$$

Since the sequence $\{x_n\}$ is bounded and $\liminf_{k \longrightarrow \infty} \|x_n - Tx_n\| = 0$, we have

$$\|Tx_n - Tz^*\|^2 \leq \|x_n - z^*\|^2,$$

and then

$$\limsup_{k \longrightarrow \infty} \|Tx_n - Tz^*\|^2 \leq \limsup_{k \longrightarrow \infty} \|x_n - z^*\|^2. \tag{3.3}$$

Thus, using (3.3), we have

$$\begin{aligned}\rho(Tz^*) &= \limsup_{k \longrightarrow \infty} \|x_n - Tz^*\|^2 \\ &= \limsup_{k \longrightarrow \infty} \|Tx_n - Tz^*\|^2 \\ &\leq \limsup_{k \longrightarrow \infty} [\|x_n - z^*\|^2 \\ &= \rho(z^*).\end{aligned} \tag{3.4}$$

Now, we show that $z^* = Tz^*$. From Lemma 2.1 with $q = 2$ and $t = \frac{1}{2}$ and (3.4) that is,

$$\begin{aligned}\rho\Big(\frac{z^* + Tz^*}{2}\Big) &= \limsup_{k \longrightarrow \infty} \Big\|x_n - \frac{z^* + Tz^*}{2}\Big\|^2 \\ &= \limsup_{k \longrightarrow \infty} \Big\|\frac{x_n - z^*}{2} + \frac{x_n - Tz^*}{2}\Big\|^2\end{aligned}$$

$$\leq \limsup_{k\longrightarrow\infty}\Big(\frac{1}{2}\|x_n - z^*\|^2 + \frac{1}{2}\|x_n - Tz^*\|^2 - \frac{1}{4}g(\|z^* - Tz^*\|)\Big)$$
$$\leq \frac{1}{2}\rho(z^*) + \frac{1}{2}\rho(Tz^*) - \frac{1}{4}g(\|z^* - Tz^*\|)$$
$$= \rho(z^*) - \frac{1}{4}g(\|z^* - Tz^*\|).$$

By Lemma 2.1, we have

$$\frac{1}{4}g(\|z^* - Tz^*\|) \leq \rho(z^*) - \rho\Big(\frac{z^* + Tz^*}{2}\Big) \leq 0.$$

Thus we have $g(\|z^* - Tz^*\|) = 0$ and so $z^* = Tz^*$ by the property of g. □

Theorem 3.2. *Let K be a nonempty and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume $Tx_1 \leq x_1$ and the sequence $\{x_n\}$ defined by Normal S-iteration (1.2) is bounded with $w \leq x_n$ for some $w \in K$ Then $F_{\leq}(T) \neq \emptyset$.*

Proof. the proof same Theorem 3.1, by let $x_{n+1} \leq x_n \leq \cdots \leq x_1$. □

3.2. **The convergence of Normal S-iteration.** In this section, we prove some convergence theorems of Normal S-iteration for a monotone (α, β)-nonexpansive mapping in an ordered Banach space E.

Theorem 3.3. *Let K be a nonempty and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$) and $F_{\geq}(T) \neq \emptyset$ (or $F_{\leq}(T) \neq \emptyset$). Then we have*

1. *the sequence $\{x_n\}$ is bounded;*
2. *$\|x_{n+1} - p\| \leq \|x_n - p\|$ and $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists for all $p \in F_{\geq}(T) \neq \emptyset$ (or $F_{\leq}(T) \neq \emptyset$);*
3. *$\liminf_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$ provided $\limsup_{n\longrightarrow\infty} s_n(1 - s_n) > 0$;*
4. *$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$ provided $\liminf_{n\longrightarrow\infty} s_n(1 - s_n) > 0$.*

Proof. Without loss of generality, we assume that $x_1 \leq p \in F_{\geq}(T) \neq \emptyset$. Now, we claim $x_n \leq p$ for all $n \geq 1$. In fact, a mapping T is monotone, we have $x_1 \leq Tx_1 \leq Tp = p$ and $x_1 \leq y_1 \leq Tx_1 \leq p$ then we have $y_1 \leq p$. Again from T is monotone, then $Ty_1 \leq Tp = p$ from $x_1 \leq Ty_1$. By convex we can get $x_2 \leq p$, and so $x_1 \leq x_2 \leq p$. Suppose that $x_k \leq p$ for some $k \geq 2$. Then $Tx_k \leq Tp = p$ by monotonicity, from the condition (1) of Lemma 3.1 we have $x_k \leq y_k \leq Tx_k \leq Ty_k$ and $x_k \leq y_k \leq Tx_k \leq p$. Since $y_k \leq p$ then $Ty_k \leq Tp = p$. And $x_k \leq Ty_k$ by convexity

$$x_k \leq (1 - s_k)x_k + s_k Ty_k = x_{k+1} \leq Ty_k.$$

That is, we get $x_{k+1} \leq p$. Hence we conclude $x_n \leq p$ for all $n \leq 1$.
It follows from Lemma 3.1 that $\|Tx_n - p\| \leq \|x_n - p\|$ for all $n \geq 1$ and so

$$\begin{aligned}\|y_n - p\| &= \|(1 - s_n)x_n + s_n Tx_n - p\|\\ &\leq (1 - s_n)\|x_n - p\| + s_n\|T(x_n) - p\|\\ &\leq (1 - s_n)\|x_n - p\| + s_n\|x_n - p\|\\ &= \|x_n - p\|.\end{aligned}$$

Consequently, we have

$$\|x_{n+1} - p\| = \|T(y_n) - p\|$$

$$\begin{aligned} &\le \|y_n - p\| \\ &\le \|x_n - p\| \\ &\cdots \\ &\le \|x_1 - p\|. \end{aligned}$$

Then the sequence $\{\|x_n - p\|\}$ is non-increasing and bounded and hence the conclusions (1) and (2) hold.

Now, we show that the conclusion (3) and (4) hold. From Lemma 2.1 with $q = 2$, $t = s_n$ and Lemma 3.1 it follows that,

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|Ty_n - p\|^2 \\ &= \|y_n - p\|^2 \\ &\le \|(1 - s_n)(x_n - p) + s_n(Tx_n - p)\|^2 \\ &\le (1 - s_n)\|x_n - p\|^2 + s_n\|x_n - p\|^2 - s_n(1 - s_n)g(\|x_n - Tx_n\|) \\ &= \|x_n - p\|^2 - s_n(1 - s_n)g(\|x_n - Tx_n\|) \end{aligned}$$

which implies that

$$s_n(1 - s_n)g(\|x_n - Tx_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Then it follows from the conclusion (2) that

$$\limsup_{n \longrightarrow \infty} s_n(1 - s_n)g(\|x_n - Tx_n\|) = 0.$$

From the conclusion (3), since $\limsup_{n \longrightarrow \infty} s_n(1 - s_n) > 0$,

$$\left(\limsup_{n \longrightarrow \infty} s_n(1 - s_n)\right)\left(\liminf_{n \longrightarrow \infty} g(\|x_n - Tx_n\|)\right) \le \limsup_{n \longrightarrow \infty} s_n(1 - s_n)g(\|x_n - Tx_n\|),$$

we have

$$\liminf_{n \longrightarrow \infty} g(\|x_n - Tx_n\|) = 0.$$

Hence we have

$$\liminf_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0,$$

by the properties of g. From the conclusion (4), since $\liminf_{n \longrightarrow \infty} s_n(1 - s_n) > 0$,

$$\left(\liminf_{n \longrightarrow \infty} s_n(1 - s_n)\right)\left(\limsup_{n \longrightarrow \infty} g(\|x_n - Tx_n\|)\right) \le \limsup_{n \longrightarrow \infty} s_n(1 - s_n)g(\|x_n - Tx_n\|),$$

we have

$$\lim_{n \longrightarrow \infty} g(\|x_n - Tx_n\|) = \limsup_{n \longrightarrow \infty} g(\|x_n - Tx_n\|) = 0.$$

Hence we have

$$\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0,$$

by the properties of g. □

Theorem 3.4. *Let K be a nonempty and closed convex subset of a uniformly convex ordered Banach space $(E, \le)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that E satisfies Opial's condition and the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $x_1 \le Tx_1$ (or $Tx_1 \le x_1$). If $F_{\ge}(T) \ne \emptyset$ (or $F_{\le}(T) \ne \emptyset$) and $\liminf_{n \longrightarrow \infty} s_n(1 - s_n) > 0$, then the sequence $\{x_n\}$ converges weakly to a fixed point z of T.*

Proof. It follows from Theorem 3.3 that $\{x_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a point $z \in K$. From Lemma 3.1, it follows that $x_1 \leq x_{n_k} \leq z$ (or $z \leq x_{n_k} \leq x_n$) for all $k \geq 1$.
From Definition 2.4 that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2 + \frac{2}{1-\beta}\|Tx-x\|\big[|\alpha|\|x-y\| + |\beta|\|Tx - Ty\|\big].$$

Since the sequence $\{x_n\}$ is bounded and $\lim_{k\longrightarrow\infty}\|x_{n_k} - Tx_{n_k}\| = 0$, we have

$$\limsup_{k\longrightarrow\infty}\|Tx_{n_k} - Tz\|^2 \leq \limsup_{k\longrightarrow\infty}\|x_{n_k} - z\|^2$$

and hence

$$\limsup_{k\longrightarrow\infty}\|Tx_{n_k} - Tz\| \leq \limsup_{k\longrightarrow\infty}\|x_{n_k} - z\|. \tag{3.5}$$

Now, we prove that $z = Tz$. In fact, suppose that $z \neq Tz$. Then, by (3.5) and Opial's condition, we have

$$\begin{aligned}\limsup_{k\longrightarrow\infty}\|x_{n_k} - z\| &\leq \limsup_{k\longrightarrow\infty}\|x_{n_k} - Tz\| \\ &\leq \limsup_{k\longrightarrow\infty}(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tz\|) \\ &\leq \limsup_{k\longrightarrow\infty}\|x_{n_k} - z\|,\end{aligned}$$

which is a contraction. This implies that $z \in F_{\geq}(T)$ (or $z \in F_{\leq}(T)$). Using the conclusion (2) of Theorem 3.3, $\lim_{n\longrightarrow\infty}\|x_n - z\|$ exists.

Now, we show that the sequence $\{x_n\}$ converge weakly to the point z. Suppose that this does not hold. Then there exists a subsequence $\{x_{n_j}\}$ to converge weakly to a point $x \in K$ and $z \neq x$. Similarly, we must have $x = Tx$ and $\lim_{n\longrightarrow\infty}\|x_n - x\|$ exists. It follows from Opial's condition that

$$\lim_{n\longrightarrow\infty}\|x_n - z\| < \lim_{n\longrightarrow\infty}\|x_n - x\| = \limsup_{j\longrightarrow\infty}\|x_{n_j} - x\| < \lim_{n\longrightarrow\infty}\|x_n - z\|,$$

which is a contradiction and hence we get $x = z$. □

Theorem 3.5. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α,β)-nonexpansive mapping. Assume the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $x_1 \leq Tx_1$. If $\limsup_{n\longrightarrow\infty} s_n(1-s_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F_{\geq}(T)$.*

Proof. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p \in K$. From Lemma 3.1, it follows that $x_1 \leq x_{n_k} \leq p$ for all $k \geq 1$. By Theorem 3.1, we have $F_{\geq}(T) \neq \emptyset$ and it follows from Theorem 3.3 that $\{x_n\}$ is bounded and

$$\liminf_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0.$$

Assume that

$$\liminf_{k\longrightarrow\infty}\|x_{n_k} - Tx_{n_k}\| = 0.$$

From Definition 2.4 that

$$\|Tx - Tp\|^2 \leq \|x-p\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2$$

$$+\frac{2}{1-\beta}\|Tx-x\|\big[|\alpha|\|x-p\|+|\beta|\|Tx-Tp\|\big].$$

Since the sequence $\{x_{n_k}\}$ is bounded and

$$\lim_{k\longrightarrow\infty}\|x_{n_k}-p\|=0,\quad \lim_{k\longrightarrow\infty}\|x_{n_k}-Tx_{n_k}\|=0,$$

we have

$$\limsup_{k\longrightarrow\infty}\|Tx_{n_k}-Tp\|^2\le 0$$

and hence

$$\lim_{k\longrightarrow\infty}\|Tx_{n_k}-Tp\|=0. \tag{3.6}$$

Therefore, we have

$$\limsup_{k\longrightarrow\infty}\|x_{n_k}-Tp\|\le\limsup_{k\longrightarrow\infty}(\|x_{n_k}-Tx_{n_k}\|+\|Tx_{n_k}-Tp\|)=0$$

and so $\lim_{k\longrightarrow\infty}\|x_{n_k}-Tp\|=0$, which implies that $p\in F_{\ge}(T)$. Using the conclusion (2) of Theorem 3.3, $\lim_{k\longrightarrow\infty}\|x_{n_k}-p\|$ exists and so $\lim_{k\longrightarrow\infty}\|x_n-p\|=0$. □

Theorem 3.6. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E,\le)$ and a mapping $T:K\to K$ be a monotone (α,β)-nonexpansive mapping. Assume the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $x_1\le Tx_1$. If $\liminf_{n\longrightarrow\infty}s_n(1-s_n)>0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p\in F_{\ge}(T)$.*

Proof. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p\in K$. From Lemma 3.1, it follows that $x_1\le x_{n_k}\le p$ for all $k\ge 1$. By Theorem 3.1, we have $F_{\ge}(T)\ne\emptyset$ and it follows from Theorem 3.3 that $\{x_n\}$ is bounded and

$$\liminf_{n\longrightarrow\infty}\|x_n-Tx_n\|=0.$$

Without loss of generality, we can assume that

$$\liminf_{k\longrightarrow\infty}\|x_{n_k}-Tx_{n_k}\|=0.$$

From Definition 2.4 that

$$\|Tx-Tp\|^2\le\|x-p\|^2+\frac{\alpha+\beta}{1-\beta}\|Tx-x\|^2$$
$$+\frac{2}{1-\beta}\|Tx-x\|\big[|\alpha|\|x-p\|+|\beta|\|Tx-Tp\|\big].$$

Since the sequence $\{x_{n_k}\}$ is bounded and

$$\lim_{k\longrightarrow\infty}\|x_{n_k}-p\|=0,\quad \lim_{k\longrightarrow\infty}\|x_{n_k}-Tx_{n_k}\|=0,$$

we have

$$\liminf_{k\longrightarrow\infty}\|Tx_{n_k}-Tp\|^2\le 0$$

and hence

$$\lim_{k\longrightarrow\infty}\|Tx_{n_k}-Tp\|=0. \tag{3.7}$$

Therefore, we have

$$\liminf_{k\longrightarrow\infty}\|x_{n_k}-Tp\|\le\liminf_{k\longrightarrow\infty}(\|x_{n_k}-Tx_{n_k}\|+\|Tx_{n_k}-Tp\|)=0$$

and so $\lim_{k\longrightarrow\infty}\|x_{n_k}-Tp\|=0$, which implies that $p\in F_{\ge}(T)$. Using the conclusion (2) of Theorem 3.3, $\lim_{k\longrightarrow\infty}\|x_{n_k}-p\|$ exists and so $\lim_{k\longrightarrow\infty}\|x_n-p\|=0$. □

Similarly, the following theorem can be proved:

Theorem 3.7. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $Tx_1 \leq x_1$. If either $\liminf_{n\longrightarrow\infty} s_n(1 - s_n) > 0$ or $\limsup_{n\longrightarrow\infty} s_n(1 - s_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F_{\leq}(T)$.*

From Theorem 3.5, we have the following:

3.3. **The numerical examples.** Now, we give two numerical examples to illustrate the following examples, by we add Normal-S iteration for compare with Mann's iteration and Ishiwaka's iteration of [4] in the first example. And the last, we show the example between Mann's iteration and Normal-S iteration.

Example 3.2. Let $T : [0, 1] \to [0, 1]$ be a mapping defined by

$$Tx = \begin{cases} 0.25 & if \quad x \neq 1, \\ 0.5 & if \quad x = 1. \end{cases}$$

for any $x \in [0, 1]$. Then T is a $(0.8, 0.2)$-nonexpansive mapping. Define the sequences $s_n = \frac{1}{4} + \frac{1}{n^2}$ for each $n \geq 1$, then $\limsup_{n\to\infty} s_n(1-s_n) > 0$. Then all the conditions of Theorem 3.5 are satisfied. Also, 0.25 is a fixed point of T.

TABLE 1. The convergent step of $\{x_n\}$ for Example with $s_n = \frac{1}{4} + \frac{1}{n^2}$

Number of iterations	Sequence of Mann	Sequence of Ishikawa	Sequence of Normal-S
1	0.5000000	0.5000000	0.5000000
2	0.1875000	0.3294046	0.2500000
4	0.2300347	0.2518192	0.2500000
6	0.2402544	0.2502132	0.2500000
8	0.2448648	0.2500322	0.2500000
10	0.2472181	0.2500053	0.2500000
12	0.2484731	0.2500009	0.2500000
14	0.2491557	0.2500001	0.2500000
16	0.2495311	0.2500000	0.2500000

Example 3.3. Let $T_1 : [-1.5, -1] \to [-1.5, -1]$ or $T_2 : [1, 1.5] \to [1, 1.5]$ be the mappings defined by

$$Tx = arctan(5x).$$

The fixed points of mappings T_1 and T_2 are -1.4320322 and 1.4320322 respectively.

It is easy to see that T is monotone. Next we will show that T is a $(0.9, 0.1)$-nonexpansive mapping. By using Matlab R2015b software, we get

$$\min_{x,y\in[1,1.5]} \big\{0.9\|arctan(5x) - y\|^2 + 0.1\|arctan(5y) - x\|^2 + (1 - 0.9 - 0.1)\|x - y\|^2 - \|arctan(5x) - arctan(5y)\|^2\big\} = 4.37 \cdot 10^{-0.6} > 0.$$

then implies that

$$\|arctan(5x)-arctan(5y)\|^2 \leq 0.9\|arctan(5x)-y\|^2+0.1\|arctan(5y)-x\|^2+(1-0.9-0.1)\|x-y\|^2$$

for all $x, y \in [1, 1.5]$. And it is true for all $x, y \in [-1.5, -1]$ too. Therefore T is a monotone $(0.9, 0.1)$-nonexpansive mapping.

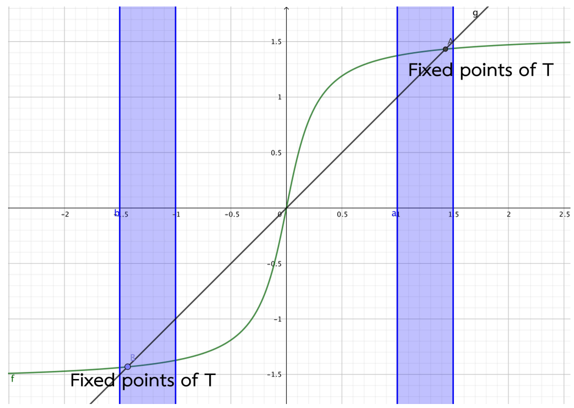

FIGURE 1. The fixed points of T are -1.4320322 and 1.4320322

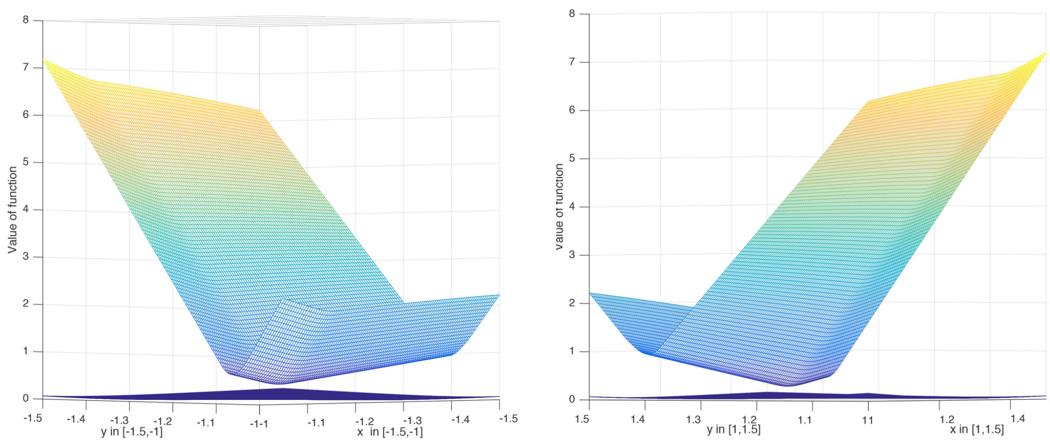

FIGURE 2. The value of mappings T_1 and T_2

Next we show the numerical solution of T, the numerical solution of this example is presented in Table 2.

Note that, if we set $x = 1.5$, $y = 1$ and $\alpha = \beta = 0.9$ then, the mapping T is not α-nonexpansive mapping.

From observing the numerical behavior, if we choose x_0 nearly is the solution then the sequence convergence is fast. Next we will show the convergent behavior of $\{s_n\}$ for iterative comparison between Mann's iteration, Ishikawa's iteration and normal-S iteration. by fixing $x_0 = 1.2$ and using three groups of sequences s_n for $n \geq 1$ are :

(i) $s_n = \frac{1}{4} + \frac{1}{n^k}, k \in \{0.01, 2, 5\}$;

(ii) $s_n = \frac{1}{4} + \frac{1}{\log^k(n+1)}, k \in \{0.01, 2, 5\}$;

(iii) $s_n = \frac{1}{4} + \frac{\log^k(n+1)}{n+2}, k \in \{0.01, 2, 5\}$:

All these sequences satisfy all condition of convergence theorems, Next figures describe the convergent behavior of three situations for value k.

TABLE 2. The convergent step of $\{x_n\}$ for Example 3.3 with $s_n = \frac{1}{4} + \frac{1}{n^2}$

Number of Iterations	Sequence value of Mann		Sequence value of Normal S	
	$x_0 = 1.2$	$x_0 = -1.3$	$x_0 = 1.2$	$x_0 = -1.3$
1	1.2000000	-1.3000000	1.2000000	-1.3000000
2	1.4570595	-1.4476837	1.4343860	-1.4335135
3	1.4457227	-1.4405986	1.4321554	-1.4321098
4	1.4412475	-1.4377994	1.4320401	-1.4320372
5	1.4386414	-1.4361689	1.4320327	-1.4320325
6	1.4369073	-1.4350837	1.4320322	-1.4320322
7	1.4356822	-1.4343169	1.4320322	-1.4320322
8	1.4347894	-1.4337581	1.4320322	-1.4320322
9	1.4341269	-1.4333435	1.4320322	-1.4320322
10	1.4336299	-1.4330323	1.4320322	-1.4320322

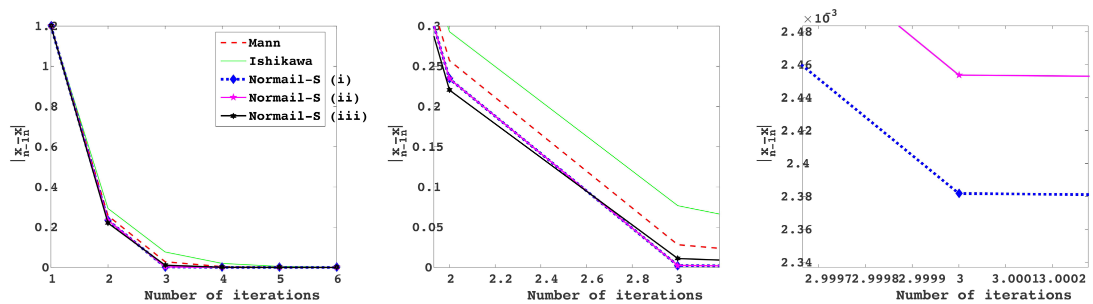

FIGURE 3. The behavior of sequence by fixing $k = 0.01$

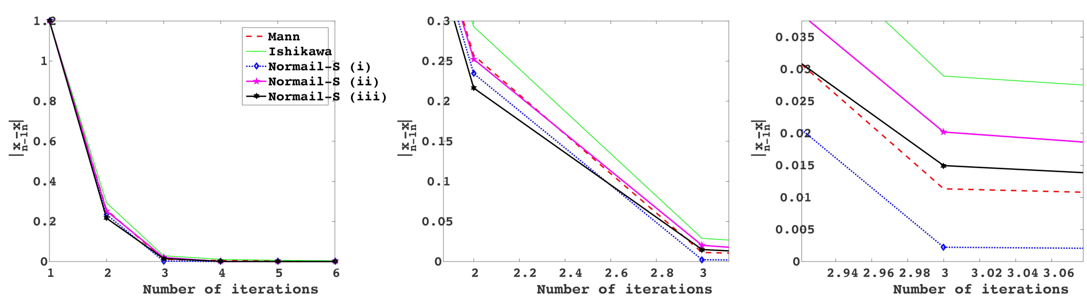

FIGURE 4. The behavior of sequence by fixing $k = 2$

The last figure describes the convergent behaviour for comparison k in three groups

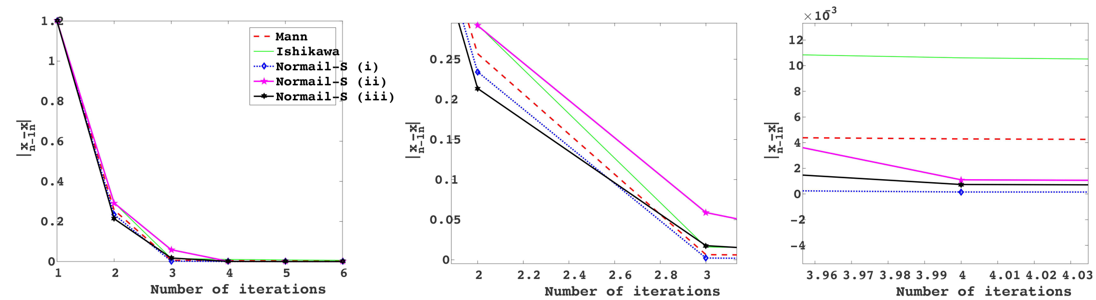

FIGURE 5. The behavior of sequence by fixing $k = 5$

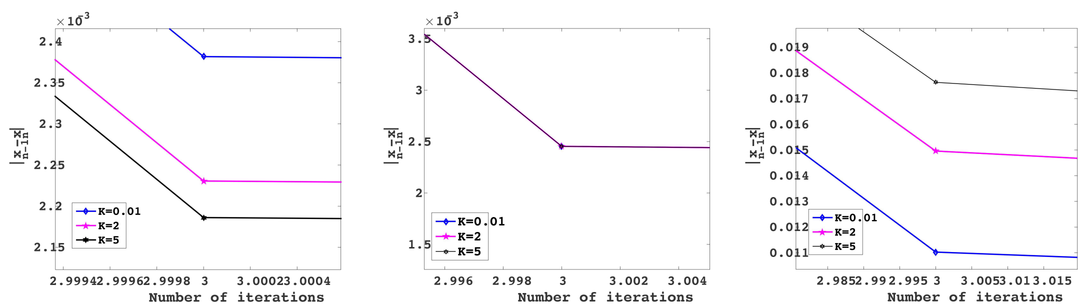

FIGURE 6. The convergent behaviour of each k for cases of group(i), group(ii) and group(iii)

4. Conclusion

We get the results about the convergence theorems of monotone (α, β)-nonexpansive mapping for the sequence $\{x_n\}$ is defined by normal-S iteration. In part of numerical, we give the examples for show the convergent behavior of sequence $\{s_n\}$ of normal-S iteration (in Figure 3 4 5 6)

5. Acknowledgements

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References

1. B. A. B. Dehaish and M. A. Khamsi, Mann iteration process for monotone nonexpansive mappings, Fixed Point Theory Appl. 177 (2015).

2. Y. Song, K. Promluang, P. Kumam and Y. J. Cho, Fixed point theorems and iterative approximations for monotone nonexpansive mappings in ordered Banach spaces, Fixed Point Theorem Appl. 73 (2016).
3. Y. Song, P. Kumam and Y. J. Cho, Some convergence theorems of the Mann iteration for monotone α-nonexpansive mappings, Appl. Math. Comput. (2016), 74 – 82.
4. K. Muangchoo-in, D. Thongtha, P. Kumam and Y. J. Cho, Fixed point theorems and convergence theorems for monotone (α, β)-nonexpansive mappings in ordered Banach spaces, Creat. Math. and Inf. 26 (2017), 163 – 180.
5. D.R. Sahu, Application of theS-iteration process to constrained minimization problem and split feasibility problem, Fixed Point Theory. 12 (2013), 187 – 204.
6. H. K. Xu, Inequality in Banach spaces with applications, Nonlinear Anal. 16 (1991), 1127 – 1138.
7. W. Takahashi, Nonlinear Functional Analysis–Fixed Point Theory and its Applications, Proceedings of the International Conference. (2000).
8. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings in Banach spaces, Bull. Amer. Math. Soc. 73 (1967), 591 – 597.
9. W. R. Mann, Mean value methods in iteration, Proceedings of the American Mathematical Society. 4 (1954), 15 – 26.
10. S. Ishikawa, Fixed points and iterations of non-expansive mappings in Banach spaces, Proceedings of the American Mathematical Society. 59 (1976), 65 – 71.
11. F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings relate to maximal monotone operators in Banach spaces, Arch. Math. 91 (2008), 166 – 177.
12. E. Naraghirad, N. C. Wong and J. C. Yao, Approximating fixed points of α-nonexpansive mappings in uniformly convex Banach spaces and Cat(0), Fixed point Theory Appl. 57 (2013).
13. Y. Shehu and P. Cholamjiak, Another look at the split common fixed point problem for demicontractive operators, RACSAM. 110 (2016), 201-218.
14. W. Cholamjiak, P. Cholamjiak and S. Suantai, Convergence of iterative schemes for solving fixed point of multi-valued nonselfmappings and equilibrium problems, J. Nonlinear Sci. Appl. 8 (2015), 1245-1256.
15. P. Cholamjiak and S. Suantai, Weak convergence theorems for a countable family of strict pseudocontractions in Banach spaces, Fixed Point Theory Appl. 2010, 632137 (2010), doi:10.1155/2010/632137.
16. P. Cholamjiak and S. Suantai, Viscosity approximation methods for a nonexpansive semigroup in Banach spaces with gauge functions, J. Glob. Optim. 54 (2012), 185-197.
17. S. Chandok, Common Fixed Point Theorems for Generalized Contractions Mappings, Thai Journal of Mathematics. 16 (2018), 305-314.
18. M. Wattanataweekul and C. Klanarong Convergence Theorems for Common Fixed Points of Two G-Nonexpansive Mappings in a Banach Space with a Directed Graph, Thai Journal of Mathematics. 16 (2018), 503-516.