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I-CONVERGENT TRIPLE DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTION

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ABSTRACT. The main objective of this paper is to introduce classes of I-convergent triple difference sequence spaces, $c^3_{0I}(\Delta, \mathcal{F})$, $c^3_I(\Delta, \mathcal{F})$, $\ell^3_{\infty I}(\Delta, \mathcal{F})$, $M^3_I(\Delta, \mathcal{F})$ and $M^3_{0I}(\Delta, \mathcal{F})$, by using sequence of modulii function $\mathcal{F} = (f_{pqr})$. We also study some algebraic and topological properties of these new sequence spaces.

KEYWORDS: Triple sequence spaces, Difference sequence space, I-convergence, Modulus functions, Ideal, Statistical convergence.
AMS Subject Classification: :40C05; 46A45; 46E30.

1. Introduction

A triple sequence (real or complex) is a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ are the set of natural numbers, real numbers, and complex numbers respectively. We denote by ω''' the class of all complex triple sequence (x_{pqr}), where $p, q, r \in \mathbb{N}$. Then under the coordinate wise addition and scalar multiplication ω''' is a linear space. A triple sequence can be represented by a matrix, in case of double sequences we write in the form of a square. In case of triple sequence it will be in the form of a box in three dimensions.

The different types of notions of triple sequences and their statistical convergence were introduced and investigated initially by Sahiner et. al [19]. Later Debnath et.al [1, 2], Esi et.al [3, 4, 5], Jalal and Malik [11, 12, 13] and many others authors have studied it further and obtained various results. Kizmaz [14] introduced the notion of difference sequence spaces, he defined the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ as follows.

$$Z(\Delta) = \{x = (x_k) \in \omega \colon (\Delta x_k) \in Z\}$$

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for $Z = \ell_\infty$, c and c_0
Where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$
The difference operator on triple sequence is defined as

$$\Delta x_{mnk} = x_{mnk} - x_{(m+1)nk} - x_{m(n+1)k} - x_{mn(k+1)} + x_{(m+1)(n+1)k} + x_{(m+1)n(k+1)} + x_{m(n+1)(k+1)} - x_{(m+1)(n+1)(k+1)}$$

and $\Delta^0_{mnk} = (x_{mnk})$.

Statistical convergence was introduced by Fast [6] and later on it was studied by Fridy [7, 8] from the sequence space point of view and linked it with summability theory. The notion of statistical convergent double sequence was introduced by Mursaleen and Edely [17].

I-convergence is a generalization of the statistical convergence. Kostyrko et. al. [15] introduced the notion of I-convergence of real sequence and studied its several properties. Later Jalal [9, 10], Salat et.al [18] and many other researchers contributed in its study. Tripathy and Goswami [22] extended this concept in probabilistic normed space using triple difference sequences of real numbers. Sahiner and Tripathy [20] studied I-related properties in triple sequence spaces and showed some interesting results. Tripathy [21] extended the concept in I-convergent double sequence and later Kumar [16] obtained some results on I-convergent double sequence. In this paper we have defined I-convergent triple difference sequence spaces, $c^3_{0I}(\Delta, F)$, $c^3_I(\Delta, F)$, $\ell^3_{\infty I}(\Delta, F)$, $M^3_I(\Delta, F)$ and $M^3_{0I}(\Delta, F)$, by using sequence of modulii function $F = (f_{pqr})$ and also studied some algebraic and topological properties of these new sequence spaces.

2. Definitions and preliminaries

Definition 2.1. Let $X \neq \phi$. A class $I \subset 2^X$ (Power set of X) is said to be an ideal in X if the following conditions holds good:

(i) I is additive that is if $A, B \in I$ then $A \cup B \in I$;
(ii) I is hereditary that is if $A \in I$, and $B \subset A$ then $B \in I$.

I is called non-trivial ideal if $X \notin I$

Definition 2.2. [19] A triple sequence (x_{pqr}) is said to be convergent to L in Pringsheim's sense if for every $\epsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that

$$|x_{pqr} - L| < \epsilon \quad \text{whenever} \quad p \geq \mathbf{N}, q \geq \mathbf{N}, r \geq \mathbf{N}$$

and write as $\lim_{p,p,r \to \infty} x_{pqr} = L$.

Note: A triple sequence is convergent in Pringsheim's sense may not be bounded [19].

Example Consider the sequence (x_{pqr}) defined by

$$x_{pqr} = \begin{cases} p + q & \text{for all } p = q \text{ and } r = 1 \\ \frac{1}{p^2 qr} & \text{otherwise} \end{cases}$$

Then $x_{pqr} \to 0$ in Pringsheim's sense but is unbounded.

Definition 2.3. A triple sequence (x_{pqr}) is said to be I-convergence to a number L if for every $\epsilon > 0$,

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr} - L|) \geq \epsilon\} \in I.$$

In this case we write $I - \lim x_{pqr} = L$.

Definition 2.4. A triple sequence (x_{pqr}) is said to be I-null if $L = 0$. In this case we write $I - \lim x_{pqr} = 0$.

Definition 2.5. [19] A triple sequence (x_{pqr}) is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that

$$|x_{pqr} - x_{lmn}| < \epsilon \quad \text{whenever} \quad p \geq l \geq \mathbf{N}, q \geq m \geq \mathbf{N}, r \geq n \geq \mathbf{N}$$

Definition 2.6. A triple sequence (x_{pqr}) is said to be $I-$Cauchy sequence if for every $\epsilon > 0$, there exists $\mathbf{N} \in \mathbb{N}$ such that

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr} - a_{lmn}| \geq \epsilon\} \in I$$

whenever $p \geq l \geq \mathbf{N}, q \geq m \geq \mathbf{N}, r \geq n \geq \mathbf{N}$

Definition 2.7. [19] A triple sequence (x_{pqr}) is said to be bounded if there exists $M > 0$, such that $|x_{pqr}| < M$ for all $p, q, r \in \mathbb{N}$.

Definition 2.8. A triple sequence (x_{pqr}) is said to be $I-$bounded if there exists $M > 0$, such that $\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{pqr}| \geq M\} \in I$ for all $p, q, r \in \mathbb{N}$.

Definition 2.9. A triple sequence space E is said to be solid if $(\alpha_{pqr} x_{pqr}) \in E$ whenever $(x_{pqr}) \in E$ and for all sequences (α_{pqr}) of scalars with $|\alpha_{pqr}| \leq 1$, for all $p, q, r \in \mathbb{N}$.

Definition 2.10. Let E be a triple sequence space and $x = (x_{pqr}) \in E$. Define the set $S(x)$ as

$$S(x) = \left\{ \left(x_{\pi(pqr)} \right) : \pi \text{ is a permutations of } \mathbb{N} \right\}$$

If $S(x) \subseteq E$ for all $x \in E$, then E is said to be symmetric.

Definition 2.11. A triple sequence space E is said to be convergence free if $(y_{pqr}) \in E$ whenever $(x_{pqr}) \in E$ and $x_{pqr} = 0$ implies $y_{pqr} = 0$ for all $p, q, r \in \mathbb{N}$.

Definition 2.12. A triple sequence space E is said to be sequence algebra if $x \cdot y \in E$, whenever $x = (x_{pqr}) \in E$ and $y = (y_{pqr}) \in E$, that is product of any two sequences is also in the space.

Definition 2.13. A function $f : [0, \infty) \to [0, \infty)$ is called a modulus function if it satisfies the following conditions

(i) $f(x) = 0$ if and only if $x = 0$.
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0$ and $y \geq 0$.
(iii) f is increasing.
(iv) f is continuous from the right at 0.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (4) that f is continuous on $[0, \infty)$. Furthermore, from condition (2) we have $f(nx) \leq nf(x)$, for all $n \in \mathbb{N}$, and so
$f(x) = f\big(nx(\frac{1}{n})\big) \leq nf\big(\frac{x}{n}\big)$.
Hence $\frac{1}{n} f(x) \leq f(\frac{x}{n})$ for all $n \in \mathbb{N}$

We now define the following sequence spaces

$$c^3_{0I}(\Delta, F) = \left\{ x \in \omega^{'''} : I - \lim f_{pqr}(|\Delta x_{pqr}|) = 0 \right\}$$

$$c^3_{I}(\Delta, F) = \left\{ x \in \omega^{'''} : I - \lim f_{pqr}(|\Delta x_{pqr} - b|) = 0, \quad \text{for some} \quad b \right\}$$

$$\ell^3_{\infty I}(\Delta, F) = \left\{ x \in \omega^{'''} : \sup_{p,q,r \in \mathbb{N}} f_{pqr}(|\Delta x_{pqr}|) = 0 \right\}$$

$$M^3_I(\Delta, F) = c^3_I(\Delta, F) \cap \ell^3_{\infty I}(\Delta, F)$$

$$M^3_{0I}(\Delta, F) = c^3_{0I}(\Delta, F) \cap \ell^3_{\infty I}(\Delta, F)$$

3. Algebraic and Topological Properties of the new Sequence spaces

Theorem 3.1. *The triple difference sequence spaces* $c^3_{0I}(\Delta, F), c^3_I(\Delta, F), \ell^3_{\infty I}(\Delta, F), M^3_I(\Delta, F)$ *and* $M^3_{0I}(\Delta, F)$ *all are linear for the sequence of modulii* $F = (f_{pqr})$.

Proof. We shall prove it for the sequence space $c^3_I(\Delta, F)$, for the other spaces, it can be established similarly.
Let $x = (x_{pqr}), y = (y_{pqr}) \in c^3_I(\Delta, F)$ and $\alpha, \beta \in \mathbb{R}$ such that $|\alpha| \leq 1$ and $|\beta| \leq 1$, then

$$I - \lim f_{pqr}(|\Delta x_{pqr} - b_1|) = 0, \text{ for some } b_1 \in \mathbb{C}$$
$$I - \lim f_{pqr}(|\Delta y_{pqr} - b_2|) = 0, \text{ for some } b_2 \in \mathbb{C}$$

Now for a given $\epsilon > 0$ we set

$$X_1 = \left\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(|\Delta x_{pqr} - b_1|) > \frac{\epsilon}{2}\right\} \in I \tag{2.1}$$

$$X_2 = \left\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(|\Delta y_{pqr} - b_2|) > \frac{\epsilon}{2}\right\} \in I \tag{2.2}$$

Since f_{pqr} is a modulus function, so it is non-decreasing and convex, hence we get

$$\begin{aligned} f_{pqr}(|(\alpha\Delta x_{pqr} + \beta\Delta y_{pqr}) - (\alpha b_1 + \beta b_2)|) &= f_{pqr}(|(\alpha\Delta x_{pqr} - \alpha b_1) + (\beta\Delta y_{pqr} - \beta b_2)|) \\ &\leq f_{pqr}(|\alpha||\Delta x_{pqr} - b_1|) + f_{pqr}(|\beta||\Delta y_{pqr} - b_2|) \\ &= |\alpha| f_{pqr}(|\Delta x_{pqr} - b_1|) + |\beta| f_{pqr}(|\Delta y_{pqr} - b_2|) \\ &\leq f_{pqr}(|\Delta x_{pqr} - b_1|) + f_{pqr}(|\Delta y_{pqr} - b_2|) \end{aligned}$$

From (2.1) and (2.2) we can write

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(|(\alpha\Delta x_{pqr} + \beta\Delta y_{pqr}) - (\alpha b_1 + \beta b_2)|) > \epsilon\} \subset X_1 \cup X_2$$

Thus $\alpha x + \beta y \in c^3_I(\Delta, F)$
This completes the proof. □

Theorem 3.2. *The triple difference sequence* $x = (x_{pqr}) \in M^3_I(\Delta, F)$ *is* I*-convergent if and only if for every* $\epsilon > 0$ *there exists* $I_\epsilon, J_\epsilon, K_\epsilon \in \mathbb{N}$ *such that*

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(|\Delta x_{pqr} - \Delta x_{I_\epsilon J_\epsilon K_\epsilon}|) \leq \epsilon\} \in M^3_I(\Delta, F)$$

Proof. Let $b = I - \lim \Delta x$. Then we have

$$A_\epsilon = \left\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(|\Delta x_{pqr} - b|) \leq \frac{\epsilon}{2}\right\} \in M^3_I(\Delta, F) \qquad \text{for all}, \epsilon > 0.$$

Next fix $I_\epsilon, J_\epsilon, K_\epsilon \in A_\epsilon$ then we have

$$|\Delta x_{pqr} - \Delta x_{I_\epsilon J_\epsilon K_\epsilon}| \leq |\Delta x_{pqr} - b| + |b - \Delta x_{I_\epsilon J_\epsilon K_\epsilon}| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all, } p, q, r \in A_\epsilon$$

Thus

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(|\Delta x_{pqr} - \Delta x_{I_\epsilon J_\epsilon K_\epsilon}|) \leq \epsilon\} \in M^3_I(\Delta, F)$$

Conversely suppose that

$$\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(|\Delta x_{pqr} - \Delta x_{I_\epsilon J_\epsilon K_\epsilon}|) \leq \epsilon\} \in M^3_I(\Delta, F)$$

we get $\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f_{pqr}(|\Delta x_{pqr} - \Delta x_{I_\epsilon J_\epsilon K_\epsilon}|) \leq \epsilon\} \in M^3_I(\Delta, F)$, for all $\epsilon > 0$.
Then given $\epsilon > 0$ we can find the set

$$B_\epsilon = \{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \Delta x_{pqr} \in [\Delta x_{I_\epsilon J_\epsilon K_\epsilon} - \epsilon\, , \, \Delta x_{I_\epsilon J_\epsilon K_\epsilon} + \epsilon]\} \in M^3_I(\Delta, F)$$

Let $J_\epsilon = [\Delta x_{I_\epsilon J_\epsilon K_\epsilon} - \epsilon\, , \, \Delta x_{I_\epsilon J_\epsilon K_\epsilon} + \epsilon]$ if $\epsilon > 0$ is fixed than $B_\epsilon \in M^3_I(\Delta, F)$ as well as $B_{\frac{\epsilon}{2}} \in M^3_I(\Delta, F)$.

Hence $B_\epsilon \cap B_{\frac{\epsilon}{2}} \in M_I^3(\Delta, F)$
Which gives $J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ that is $\{(p,q,r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \Delta x_{pqr} \in \mathbb{N}\} \in M_I^3(\Delta, F)$
Which implies diam $J \leq$ diam J_ϵ
where the diam of J denotes the the length of interval J.
Now by the principal of induction a sequence of closed interval can be found

$$J_\epsilon = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_s \supseteq \cdots\cdots$$

with the help of the property that diam $I_s \leq \frac{1}{2}$diam I_{s-1}, for $s = 1, 2, \cdots$ and $\{(p,q,r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \Delta x_{pqr} \in I_{pqr}\}$ for $(p,q,r = 1,2,3\cdots)$
Then there exists a $\xi \in \cap I_s$ where $s \in \mathbb{N}$ such that $\xi = I - \lim \Delta x$
So that $f_{pqr}(\xi) = I - \lim f_{pqr}(\Delta x)$ therefore $b = I - \lim f_{pqr}(\Delta x)$.
Hence the proof is complete. □

Theorem 3.3. *The $F = (f_{pqr})$ be a sequence of modulus functions then the inclusions $c_{0I}^3(\Delta, F) \subset c_I^3(\Delta, F) \subset \ell_{\infty I}^3(\Delta, F)$ holds .*

Proof. The inclusion $c_{0I}^3(\Delta, F) \subset c_I^3(\Delta, F)$ is obvious.
We prove $c_I^3(\Delta, F) \subset \ell_{\infty I}^3(\Delta, F)$, let $x = (x_{pqr}) \in c_I^3(\Delta, F)$ then there exists $b \in \mathbb{C}$ such that $I - \lim f_{pqr}(|\Delta x_{pqr} - b|) = 0$,
Which gives $f_{pqr}(|\Delta x_{pqr}|) \leq f_{pqr}(|\Delta x_{pqr} - b|) + f_{pqr}(|b|)$
On taking supremum over p, q and r on both sides gives
$x = (x_{pqr}) \in \ell_{\infty I}^3(\Delta, F)$
Hence the inclusion $c_{0I}^3(\Delta, F) \subset c_I^3(\Delta, F) \subset \ell_{\infty I}^3(\Delta, F)$ holds. □

Theorem 3.4. *The triple difference sequence $c_{0I}^3(\Delta, F)$ and $M_{0I}^3(\Delta, F)$ are solid.*

Proof. We prove the result for $c_{0I}^3(\Delta, F)$.
Consider $x = (x_{pqr}) \in c_{0I}^3(\Delta, F)$, then $I - \lim_{p,q,r} f_{pqr}(|\Delta x_{pqr}|) = 0$
Consider a sequence of scalar (α_{pqr}) such that $|\alpha_{pqr}| \leq 1$ for all $p, q, r \in \mathbb{N}$.
Then we have

$$\begin{aligned} I - \lim_{p,q,r} f_{pqr}(|\Delta \alpha_{pqr}(x_{pqr})|) &\leq I - |\alpha_{pqr}| \lim_{p,q,r} f_{pqr}(|\Delta x_{pqr}|) \\ &\leq I - \lim_{p,q,r} f_{pqr}(|\Delta x_{pqr}|) \\ &= 0 \end{aligned}$$

Hence $I - \lim_{p,q,r} f_{pqr}(|\Delta \alpha_{pqr} x_{pqr}|) = 0$ for all $p, q, r \in \mathbb{N}$
Which gives $(\alpha_{pqr} x_{pqr}) \in c_{0I}^3(\Delta, F)$
Hence the sequence space $c_{0I}^3(\Delta, F)$ is solid.
The result for $M_{0I}^3(\Delta, F)$ can be similarly proved. □

Theorem 3.5. *The triple difference sequence spaces $c_{0I}^3(\Delta, F)$, $c_I^3(\Delta, F)$, $\ell_{\infty I}^3(\Delta, F)$, $M_I^3(\Delta, F)$ and $M_{0I}^3(\Delta, F)$ are sequence algebras.*

Proof. We prove the result for $c_{0I}^3(\Delta, F)$.
Let $x = (x_{pqr}), y = (y_{pqr}) \in c_{0I}^3(\Delta, F)$
Then we have $I - \lim f_{pqr}(|\Delta x_{pqr}|) = 0$ and $I - \lim f_{pqr}(|\Delta y_{pqr}|) = 0$
and $I - \lim f_{pqr}(|\Delta(x_{pqr} \cdot y_{pqr})|) = 0$ as

$$\begin{aligned} \Delta(x_{pqr} \cdot y_{pqr}) =& x_{pqr} \cdot y_{pqr} - x_{(p+1)qr} \cdot y_{(p+1)qr} - x_{p(q+1)r} \cdot y_{p(q+1)r} - x_{pq(r+1)} \cdot y_{pq(r+1)} + \\ & x_{(p+1)(q+1)r} \cdot y_{(p+1)(q+1)r} + x_{(p+1)q(r+1)} \cdot y_{(p+1)q(r+1)} + x_{p(q+1)(r+1)} \cdot \\ & y_{p(q+1)(r+1)} - x_{(p+1)(q+1)(r+1)} \cdot y_{(p+1)(q+1)(r+1)} \end{aligned}$$

It implies that $x \cdot y \in c_{0I}^3(\Delta, F)$
Hence the proof.

The result can be proved for the spaces $c^3_I(\Delta, \mathcal{F})$, $\ell^3_{\infty I}(\Delta, \mathcal{F})$, $M^3_I(\Delta, \mathcal{F})$ and $M^3_{0I}(\Delta, \mathcal{F})$ in the same way. □

Theorem 3.6. *In general the sequence spaces* $c^3_{0I}(\Delta, \mathcal{F})$, $c^3_I(\Delta, \mathcal{F})$ *and* $\ell^3_{\infty I}(\Delta, \mathcal{F})$ *are not convergence free.*

Proof. We prove the result for the sequence space $c^3_I(\Delta, \mathcal{F})$ using an example. Example. Let $I = I_f$ define the triple sequence $x = (x_{pqr})$ as

$$x_{pqr} = \begin{cases} 0 & \text{if } p = q = r \\ 1 & \text{otherwise} \end{cases}$$

Then if $f_{pqr}(x) = x_{pqr}$ $\forall$ $p, q, r \in \mathbb{N}$, we have $x = (x_{pqr}) \in c^3_I(\Delta, \mathcal{F})$.
Now define the sequence $y = y_{pqr}$ as

$$y_{pqr} = \begin{cases} 0 & \text{if } r \text{ is odd , and } \; p, q \in \mathbb{N} \\ lmn & \text{otherwise} \end{cases}$$

Then for $f_{pqr}(x) = x_{pqr}$ $\forall$ $p, q, r \in \mathbb{N}$, it is clear that $y = (y_{pqr}) \notin c^3_I(\Delta, \mathcal{F})$
Hence the sequence spaces $c^3_I(\Delta, \mathcal{F})$ is not convergence free.
The space $c^3_I(\Delta, \mathcal{F})$ and $\ell^3_{\infty I}(\Delta, \mathcal{F})$ are not convergence free in general can be proved in the same fashion. □

Theorem 3.7. *In general the triple difference sequences* $c^3_{0I}(\Delta, \mathcal{F})$ *and* $c^3_I(\Delta, \mathcal{F})$ *are not symmetric if* I *is neither maximal nor* $I = I_f$.

Proof. We prove the result for the sequence space $c^3_{0I}(\Delta, \mathcal{F})$ using an example. Example. Define the triple sequence $x = (x_{pqr})$ as

$$x_{pqr} = \begin{cases} 0 & \text{if } r = 1, \; \text{ for all } p, q \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

Then if $f_{pqr}(x) = x_{pqr}$ $\forall$ $p, q, r \in \mathbb{N}$, we have $x = (x_{pqr}) \in c^3_{0I}(\Delta, \mathcal{F})$.
Now if $x_{\pi(pqr)}$ be a rearrangement of $x = (x_{pqr})$ defined as

$$x_{\pi(pqr)} = \begin{cases} 1 & \text{for } \; p, q, r \text{ even} \in K \\ 0 & \text{otherwise} \end{cases}$$

Then $\{x_{\pi(p,q,r)}\} \notin c^3_{0I}(\Delta, \mathcal{F})$ as $\Delta x_{\pi(pqr)} = 1$
Hence the sequence spaces $c^3_{0I}(\Delta, \mathcal{F})$ is not symmetric in general.
The space $c^3_I(\Delta, \mathcal{F})$ is not symmetric in general can be proved in the same fashion. □

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COMMON FIXED POINTS OF GENERALIZED CO-CYCLIC WEAKLY CONTRACTIVE MAPS

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ABSTRACT. In this paper, we introduce generalized co-cyclic weakly contractive maps and prove the existence of common fixed points in complete metric spaces. We deduce some corollaries from our main results and provide examples in support of our results.

KEYWORDS: cyclic representation, co-cyclic representation, co-cyclic weakly contractive maps, generalized co-cyclic weakly contractive maps.

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1. Introduction

In 1997, Alber and Guerre-Delabriere [2] introduced weakly contractive mappings as a generalization of contraction maps and proved some fixed point results in Hilbert space setting. In 2001, Rhoades [7] extended this concept to Banach spaces. In 2003, Kirk, Srinivasan and Veeramani [6] introduced cyclic contractions and proved fixed point results for not necessarily continuous mappings. In 2013, Harjani, Lopez and Sadarangani [4] proved existence of fixed points of continuous cyclic weakly contractive selfmaps in compact metric spaces. Recently, Alemayehu [1] introduced co-cyclic weakly contractive maps and proved common fixed points results in compact metric spaces.

In this paper, we denote
$\tau = \{\varphi : [0,\infty) \to [0,\infty)/\varphi$ is non-decreasing, $\varphi(0) = 0, \varphi(t) > 0$ for $t > 0\}$, and
$\Phi = \{\varphi : [0,\infty) \to [0,\infty)/\varphi$ is continuous on $[0,\infty)$and $\varphi(t) = 0 \Leftrightarrow t = 0\}$.

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Definition 1.1. [8] Let X be a non-empty set, m a positive integer and $f : X \to X$ a selfmap and $X = \cup_{i=1}^{m} A_i$ is said to be a *cyclic representation of* X *with respect to the map* f if

(i) $A_i,\ i = 1, 2, ..., m$ are non-empty subsets of X
(ii) $f(A_1) \subset A_2, ..., f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Definition 1.2. [1] Let X be a non-empty set, m a positive integer and $T,\ f : X \to X$ be two selfmaps. $X = \cup_{i=1}^{m} A_i$ is said to be a *co-cyclic representation of* X *w.r.t.* T and f if

(i) $A_i,\ i = 1, 2, ..., m$ are non-empty subsets of X
(ii) $T(A_1) \subset f(A_2), ..., T(A_{m-1}) \subset f(A_m)$ and $T(A_m) \subset f(A_1)$.

Here we note that, by taking f as the identity map, we get a cyclic representation of X with respect to the selfmap T introduced by Rus [8].

Definition 1.3. [1] Let (X, d) be a metric space, m a positive integer, $A_1, A_2, ..., A_m$ closed non-empty subsets of X and $X = \cup_{i=1}^{m} A_i$. Let $f, T : X \to X$ be two selfmaps. If

(i) $X = \cup_{i=1}^{m} A_i$ is a co-cyclic representation of X w. r. t. T and f, and
(ii) there exists $\varphi \in \tau$ such that

$$d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy)) \tag{1.1}$$

for any $x \in A_i$ and $y \in A_{i+1}$, where $A_{m+1} = A_1$

then we say that T is a *co-cyclic weakly contractive map w.r.t.* f *with* $\varphi \in \tau$.

Definition 1.4. [5] Two self mappings f and T of a metric space (X, d) are said to be *weakly compatible* if they commute at their coincidence points, i.e., if $fu = Tu$ for $u \in X$ then $fTu = Tfu$.

Remark 1.5. In [1], maps $f,\ T$ satisfying (i) and (ii) of Definition 2.3 are mentioned as 'co-cyclic weak contractions'. But the terminology 'T is a co-cyclic weakly contractive map w. r. t. f' is more appropriate as the inequality (1.1) is indicating 'weakly contractive' property. For more details on weakly contractive maps, we refer [2] and [7].

Alemayehu [1] proved the following theorem in compact metric spaces.

Theorem 1.1. [1] *Let* (X, d) *be a compact metric space and let* $T,\ f : X \to X$ *be two selfmaps. Suppose that* m *a positive integer,* $A_1, A_2, ..., A_m$ *are non-empty subsets of* X*,* $X = \cup_{i=1}^{m} A_i$ *and* T *is a co-cyclic weakly contractive map w. r. t.* f *with* $\varphi \in \tau$*.*
If the pair of operators (f, T) *is weakly compatible on* X*, then* f *and* T *have a unique common fixed point in* X*.*

Unfortunately, the proof of Theorem 2.1 contains many argumental errors. For more details, we refer [3]. A rectified version of this theorem is the following.

Theorem 1.2. [3] *Let* (X, d) *be a compact metric space and let* $T,\ f : X \to X$ *be two selfmaps. Suppose that* m *a positive integer,* $A_1, A_2, ..., A_m$ *are non-empty closed subsets of* X*,* $X = \cup_{i=1}^{m} A_i$ *and* T *is a co-cyclic weakly contractive map w. r. t.* f *with* $\varphi \in \tau$*. If* f *is one-one and* T *and* f *are continuous, then* f *and* T *have a coincidence point in* X*. Further, if the maps* f *and* T *are weakly compatible then* f *and* T *have a unique common fixed point in* X*.*

In Definition 1.3, if (i) holds and (ii) holds with $\varphi \in \Phi$ then we say that T is a co-cyclic weakly contractive map w. r. t. f with $\varphi \in \Phi$.

In Section 2, we prove the existence of common fixed points of a pair of co-cyclic weakly contractive maps with $\varphi \in \Phi$ in complete metric spaces. In Section 3, we define generalized co-cyclic weakly contractive maps w. r. t. f and T by using $\varphi \in \Phi$ and prove the existence of common fixed points in complete metric spaces. In Section 4, we deduce some corollaries from our main results and provide examples in support of our results.

In the following, we prove Theorem 2.2 for the case of complete metric spaces in which the selfmaps f and T are such that T is a co-cyclic weakly contractive map w. r. t. f with $\varphi \in \Phi$.

2. Common fixed points of co-cyclic weakly contractive maps

Theorem 2.1. *Let (X,d) be a complete metric space. Suppose that m is a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X, $X = \cup_{i=1}^{m} A_i$. Let $T,\ f : X \to X$ be two selfmaps. Suppose that T is a co-cyclic weakly contractive map w. r. t. f with $\phi \in \Phi$. If f is one-one and $f(A_i)$ is closed, then there exists $z \in \cap_{i=1}^{m} A_i$ such that z is a coincidence point of f and T.*

Proof. Let $x_0 \in X = \cup_{i=1}^{m} A_i$. Then $x_0 \in A_i$ for some $i \in \{1,2,3,..,m\}$. Then $Tx_0 \in T(A_i) \subset f(A_{i+1})$ and hence $Tx_0 = fx_1 \in f(A_{i+1})$ for some $x_1 \in A_{i+1}$. Now, since $Tx_1 \in T(A_{i+1}) \subset f(A_{i+2})$, we have $Tx_1 = fx_2$ for some $x_2 \in A_{i+2}$. On continuing this process, we get a sequence $\{x_n\} \subset X$ such that

$$Tx_n = fx_{n+1} \text{ for all } n = 1,2,\ldots\ . \tag{2.1}$$

Hence, for each n, there exists a positive integer $i_n \in \{1,2,...,m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$ satisfying

$$Tx_n = fx_{n+1}. \tag{2.2}$$

If there exists $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, then we have $Tx_{n_0+1} = Tx_{n_0} = fx_{n_0+1}$ so that f and T have a coincidence point x_{n_0+1}.
Hence, w. l. g., we assume that $x_n \neq x_{n+1}$ for all $n = 1,2,\ldots$. Then $fx_n \neq fx_{n+1}$ for all n. Further, from the construction of $\{x_n\}$, we have $Tx_n \neq Tx_{n+1}$ for all $n = 1,2,\ldots$.

Now, by (2.2) and since T is a co-cyclic weakly contractive map w. r. t. f with $\varphi \in \Phi$, we have

$$d(fx_n, fx_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq d(fx_{n-1}, fx_n) - \varphi(d(fx_{n-1}, fx_n)) \tag{2.3}$$

for each $n = 1,2,\ldots$. Therefore
$d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n)$ for all $n \geq 1$.
Hence $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence of non-negative reals and hence converges to a limit r (say), $r \geq 0$.
Now, on letting $n \longrightarrow \infty$ in (2.3) and using the continuity of ϕ we have
$r \leq r - \lim\limits_{n \longrightarrow \infty} \varphi(d(fx_{n-1}, fx_n)) = r - \phi(r)$ and hence $\varphi(r) = 0$ so that $r = 0$.
We now prove that $\{fx_n\}$ is a Cauchy sequence in X.
For this purpose, first we show that for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that if $p, q \geq n$ with $p - q \equiv 1 \pmod{m}$, then $d(fx_p, fx_q) < \epsilon$.

If it is false, then there exists an $\epsilon > 0$ such that for each $n \in \mathbb{N}$ we can find sequences $\{p_n\}$ and $\{q_n\}$ such that $p_n > q_n \geq n$ with $p_n - q_n \equiv 1 \pmod{m}$ and $d(fx_{p_n}, fx_{q_n}) \geq \epsilon$.

Now, let n be such that $n > 2m$. Then for $q_n \geq n$ we choose p_n such that p_n is the smallest positive integer greater than q_n satisfying $p_n - q_n \equiv 1 \pmod{m}$ and $d(fx_{q_n}, fx_{p_n}) \geq \epsilon$, which implies that $d(fx_{q_n}, fx_{p_{n-m}}) < \epsilon$.

By using the triangular inequality, we have
$\epsilon \leq d(fx_{q_n}, fx_{p_n})$
$\leq d(fx_{q_n}, fx_{p_{n-m}}) + \Sigma_{i=1}^{m} d(fx_{p_n-i}, fx_{p_n-i+1}) < \epsilon + \Sigma_{i=1}^{m} d(fx_{p_n-i}, fx_{p_n-i+1})$.
On letting $n \longrightarrow \infty$, by using $\lim\limits_{n\longrightarrow\infty} d(fx_n, fx_{n+1}) = 0$ we have

$$\lim_{n\longrightarrow\infty} d(fx_{q_n}, fx_{p_n}) = \epsilon. \tag{2.4}$$

Again, by the triangular inequality, we have
$\epsilon \leq d(fx_{q_n}, fx_{p_n})$
$\leq d(fx_{q_n}, fx_{q_n+1}) + d(fx_{q_n+1}, fx_{p_n+1}) + d(fx_{p_n+1}, fx_{p_n})$
$\leq d(fx_{q_n}, fx_{q_n+1}) + d(fx_{q_n+1}, fx_{q_n}) + d(fx_{q_n}, fx_{p_n}) + d(fx_{p_n}, fx_{p_n+1}) + d(fx_{p_n+1}, fx_{p_n})$
$\leq 2d(fx_{q_n}, fx_{q_n+1}) + d(fx_{q_n}, fx_{p_n}) + 2d(fx_{p_n+1}, fx_{p_n})$
On letting $n \longrightarrow \infty$ and by using (2.4), we have

$$\lim_{n\longrightarrow\infty} d(fx_{q_n+1}, fx_{p_n+1}) = \epsilon. \tag{2.5}$$

In fact, x_{q_n} and x_{p_n} lie in different adjacently labelled sets A_i and A_{i+1}, for $1 \leq i \leq m$. Now by using the inequality (1.1) with $\varphi \in \Phi$ we have
$d(fx_{q_n+1}, fx_{p_n+1}) = d(Tx_{q_n}, Tx_{p_n})$

$$\leq d(fx_{q_n}, fx_{p_n}) - \varphi(d(fx_{q_n}, fx_{p_n})). \tag{2.6}$$

On letting $n \longrightarrow \infty$, by using the continuity property of φ in (2.6) and using (2.4) we have
$\epsilon \leq \epsilon - \phi(\epsilon)$ so that $\epsilon = 0$,
a contradiction. So we conclude that our assumption is wrong. Therefore given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $p, q \geq n_0$ with $p - q \equiv 1 \pmod{m}$ then

$$d(fx_p, fx_q) \leq \frac{\epsilon}{2}. \tag{2.7}$$

Since $\lim\limits_{n\longrightarrow\infty} d(fx_n, fx_{n+1}) = 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(fx_n, fx_{n+1}) \leq \frac{\epsilon}{2m} \tag{2.8}$$

for each $n \geq n_1$.

Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, ..., m\}$ such that $s - r \equiv k \pmod{m}$. We choose $j = m - k + 1$. Then, since $m + 1 \equiv 1 \pmod{m}$, we have $s+j-r = s+(m-k+1)-r = (s-r)+(m+1)-k \equiv 1 \pmod{m}$.
$d(fx_r, fx_s) \leq d(fx_r, fx_{s+j}) + d(fx_{s+j}, fx_{s+j-1}) + ... + d(fx_{s+1}, fx_s)$
$d(fx_r, fx_s) \leq \frac{\epsilon}{2} + (j+1).\frac{\epsilon}{2} \leq \frac{\epsilon}{2} + m.\frac{\epsilon}{2m} = \epsilon$.

Therefore, given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(fx_r, fx_s) \leq \epsilon$ for all $r, s \geq n$. Hence $\{fx_n\}$ is a Cauchy sequence. Since (X, d) is complete, we have $\lim\limits_{n\longrightarrow\infty} fx_n = x$ for some $x \in X$. Since $x_0 \in X = \cup_{i=1}^{m} A_i$ implies $x_0 \in A_i$ for some i and $x_l \in A_{i+l}$ for all $l \in \{1, 2, ..., m\}$. In particular, $x_m \in A_{i+m} = A_i$ and $x_{2m} \in A_i, ..., x_{km} \in A_i$ for all $k = 0, 1, 2, ...$. Since $\{x_{km}\} \subset A_i$, we have $\{f(x_{km})\} \subset f(A_i)$. Since $f(A_i)$ is closed and $\{f(x_{km})\}$ is a subsequence of $\{f(x_n)\}$ we have $fx_{km} \longrightarrow x$ as $k \longrightarrow \infty$ and $x \in f(A_i)$.

We now show that $x \in \cap_{i=1}^{m} f(A_i)$. We have $x_{l+km} \in A_{i+l+km} = A_{i+l}$ for all $l = 1, 2, ..., m$ which implies that $f(x_{l+km}) \in f(A_{i+l})$ for all l. So $i+l \equiv i_0 \pmod m$ for some $i_0 \in \{1, 2, ..., m\}$. Therefore $f(x_{l+km}) \in f(A_{i_0})$. Now $l \in \{1, 2, ..., m\}$ implies $f(x_{l+km}) \longrightarrow x$ as $k \longrightarrow \infty$. Since $f(A_{i_0})$ is closed, we have $x \in f(A_{i_0})$. Note that, for any $i \in \{1, 2, ..., m\}$ we have $\{i + l/l = 1, 2, ..., m\} = \{1, 2, ..., m\}$ under congruent modulo m. Since this is true for any $l \in \{1, 2, ..., m\}$ it follows that $x \in \cap_{i=1}^{m} f(A_i)$. Hence $x \in f(A_i)$ for each $i = 1, 2, ..., m$ so that there exists $z_i \in A_i$ such that $x = fz_i$ for each $i = 1, 2, ..., m$. *i.e.*, $x = fz_1 = fz_2 = ... = fz_m$ for some $z_1 \in A_1, z_2 \in A_2, ..., z_m \in A_m$. Since f is one-one, we have $z_1 = z_2 = ... = z_m = z$ (say). Hence $x = fz,\ z \in \cap_{i=1}^{m} A_i$.

Now we prove that z is a coincidence point of f and T.
By using the inequality (1.1) with $\varphi \in \Phi$, we have

$$\begin{aligned} d(fx_{l+km}, Tz) &= d(Tx_{l+km-1}, Tz) \\ &\le d(fx_{l+km-1}, fz) - \varphi(d(fx_{l+km-1}, fz)), \end{aligned}$$

since $x_{l+km-1} \in A_{l+km-1}$ and $z \in A_{l+km}$.
On letting $k \longrightarrow \infty$, we have

$$d(x, Tz) \le d(x, Tz) - \varphi(d(x, Tz)).$$

Hence $d(fz, Tz) \le d(fz, Tz) - \varphi(d(fz, Tz))$ which implies that $\varphi(d(fz, Tz)) = 0$. Since $\varphi \in \Phi$ we have $fz = Tz$ and z is a coincidence point of f and T in X. □

Theorem 2.2. *In addition to the hypotheses of Theorem* 3.1*, if the maps T and f are weakly compatible then T and f have a unique common fixed point.*

Proof. By Theorem 3.1, we have
$Tz = fz = u$ (say). Since T and f are weakly compatible, we have
$Tu = Tfz = fTz = fu$ implies $Tu = fu$.
Now, we prove that $Tu = u$.
Since $Tz \in X = \cup_{i=1}^{m} A_i$ implies $Tz \in A_i$ for some i and $z \in \cap_{i=1}^{m} A_i$, we have $z \in A_i$ for all $i \in \{1, 2, ..., m\}$.
Now, by the inequality (1.1) with $\varphi \in \Phi$ we have

$$\begin{aligned} d(Tz, TTz) &\le d(fz, fTz) - \varphi(d(fz, fTz)) \\ &\le d(Tz, TTz) - \varphi(d(Tz, TTz)) \end{aligned}$$

so that $Tz = TTz$ and hence $u = Tu = fu$.
Therefore u is a common fixed point of f and T.
We now show that $u \in \cap_{i=1}^{m} A_i$ since $Tu = fu = u$, we have $u \in A_i$ for some i.
Now, $u \in A_i \Rightarrow Tu \in T(A_i)$

$$\begin{aligned} &\Rightarrow Tu \in T(A_i) \subset f(A_{i+1}) \\ &\Rightarrow Tu = fv \in f(A_{i+1}) \text{ for some } v \in A_{i+1}. \end{aligned}$$

Therefore $fu = fv$ for some $v \in A_{i+1}$, since f is one-one we have $u = v \in A_{i+1}$ so that $u \in A_{i+1}$. By repeating the same argument, we get $u \in \cap_{i=1}^{m} A_i$.

In the following, we prove the uniqueness of common fixed point of T and f.
Let y and z be two common fixed points of T and f. Then we have $Ty = fy = y$ and $Tz = fz = z$ and $y, z \in \cap_{i=1}^{m} A_i$.
From the inequality (1.1) with $\varphi \in \Phi$ we have

$$\begin{aligned} d(y, z) &= d(Ty, Tz) \\ &\le d(fy, fz) - \varphi(d(fy, fz)) \\ &\le d(y, z) - \varphi(d(y, z)) \text{ so that } \varphi(d(y, z)) = 0. \end{aligned}$$

Since $\varphi \in \Phi$ it follows that $y = z$. Therefore f and T have a unique common fixed point in X. □

3. Common fixed points of generalized co-cyclic weakly contractive maps

In the following, we introduce generalized co-cyclic weakly contractive maps by using an element $\varphi \in \Phi$.

Definition 3.1. Let (X, d) be a metric space, m a positive integer, $A_1, A_2, ..., A_m$ closed non-empty subsets of X and $X = \cup_{i=1}^m A_i$. Let $f, T : X \to X$ be two selfmaps. If

(i) $\cup_{i=1}^m A_i$ is a co-cyclic representation of X w. r. t. f and T
(ii) there exists $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)) \tag{3.1}$$

for any $x \in A_i$ and $y \in A_{i+1}$, $A_{m+1} = A_1$, where
$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}(d(fx, Ty) + d(Tx, fy))\}$

then we say that T is a *generalized co−cyclic weakly contractive map w. r. t f with* $\phi \in \Phi$.

Theorem 3.1. *Let (X, d) be a complete metric space. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X, $X = \cup_{i=1}^m A_i$. Let T, $f : X \to X$ be two selfmaps. Suppose that T is a generalized co-cyclic weakly contractive map w. r. t. f. If f is one-one and $f(A_i)$ is closed, then there exist $z \in \cap_{i=1}^m A_i$ such that z is a coincidence point of f and T.*

Proof. Let $x_0 \in X = \cup_{i=1}^m A_i$. Then proceeding as in the proof of Theorem 2.1, we obtain a sequence $\{x_n\} \subset X$ satisfying (2.1) and (2.2). Without loss of generality we assume that $x_n \neq x_{n+1}$ for all $n = 1, 2, ...$. Then $fx_n \neq fx_{n+1}$ for all n. Further, from the construction of $\{x_n\}$, we have $Tx_n \neq Tx_{n+1}$ for all $n = 1, 2, ...$.
Now, by applying the inequality (3.1) to the sequence $\{fx_n\}$ we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq M(x_{n-1}, x_n) - \varphi(M(x_{n-1}, x_n)) \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \\ &\qquad \tfrac{1}{2}(d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1}))\} \\ &= \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad \tfrac{1}{2}(d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n))\} \\ &\leq \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad \tfrac{1}{2}(d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}))\} \\ &= \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\} \leq M(x_{n-1}, x_n) \end{aligned}$$

so that
$M(x_{n-1}, x_n) = max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}$.
If $\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\} = d(fx_n, fx_{n+1})$
then, from (3.2) we have
$d(fx_n, fx_{n+1}) \leq d(fx_n, fx_{n+1}) - \varphi(d(fx_n, fx_{n+1})) < d(fx_n, fx_{n+1})$,
a contradiction.
Hence $M(x_{n-1}, x_n) = d(fx_{n-1}, fx_n)$.
Now, from (3.2) we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq d(fx_{n-1}, fx_n) - \varphi(d(fx_{n-1}, fx_n)). \\ &\leq d(fx_{n-1}, fx_n). \end{aligned} \tag{3.3}$$

Therefore $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence of non-negative reals and hence converges to a limit r (say), $r \geq 0$.

Now on letting $n \longrightarrow \infty$ in (3.3), we have
$r \leq r - \phi(r)$, and hence $\phi(r) = 0$ so that $r = 0$.
We now prove that $\{fx_n\}$ is a Cauchy sequence in X.
Here onwards proceeding as in the proof of Theorem 2.1 we get (2.4) and (2.5).
In fact, x_{q_n} and x_{p_n} lie in different adjacently labelled sets A_i and A_{i+1}, for $1 \leq i \leq m$. Now by using the inequality (3.1), we have
$d(fx_{q_n+1}, fx_{p_n+1}) = d(Tx_{q_n}, Tx_{p_n})$

$$\leq M(x_{q_n}, x_{p_n}) - \varphi(M(x_{q_n}, x_{p_n})) \tag{3.4}$$

where

$$\begin{aligned}\epsilon \leq d(fx_{p_n}, fx_{q_n}) \leq M(x_{q_n}, x_{p_n}) &= \max\{d(fx_{q_n}, fx_{p_n}), d(fx_{q_n}, Tx_{q_n}), d(fx_{p_n}, Tx_{p_n}), \\ &\qquad \tfrac{1}{2}(d(fx_{q_n}, Tx_{p_n}) + d(fx_{p_n}, Tx_{q_n}))\} \\ &= \max\{d(fx_{q_n}, fx_{p_n}), d(fx_{q_n}, fx_{q_n+1}), d(fx_{p_n}, fx_{p_n+1}), \\ &\qquad \tfrac{1}{2}(d(fx_{q_n}, fx_{p_n+1}) + d(fx_{p_n}, fx_{q_n+1}))\} \\ &\leq \max\{d(fx_{q_n}, fx_{p_n}), d(fx_{q_n}, fx_{q_n+1}), d(fx_{p_n}, fx_{p_n+1}), \tfrac{1}{2}(d(fx_{q_n}, fx_{p_n}) \\ &\qquad + d(fx_{p_n}, fx_{p_n+1}) + d(fx_{p_n}, fx_{q_n}) + d(fx_{q_n}, fx_{q_n+1}))\} \longrightarrow \epsilon \text{ as } n \longrightarrow \infty\end{aligned}$$

so that $\lim\limits_{n\longrightarrow\infty} M(x_{q_n}, x_{p_n}) = \epsilon$.
Hence, on letting $n \longrightarrow \infty$, using the continuity property of φ in (3.4) and using (2.4) and (2.5) we get that

$$\begin{aligned}\epsilon = \lim_{n\longrightarrow\infty} d(fx_{q_n+1}, fx_{p_n+1}) &= \lim_{n\longrightarrow\infty} M(x_{q_n}, x_{p_n}) - \lim_{n\longrightarrow\infty} \varphi(M(x_{q_n}, x_{p_n})) \\ &= \epsilon - \varphi(\epsilon),\end{aligned}$$

a contradiction.
Therefore, given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $p, q \geq n_0$ with
$p - q \equiv 1 \pmod m$ then

$$d(fx_p, fx_q) \leq \frac{\epsilon}{2}. \tag{3.5}$$

Since $\lim\limits_{n\longrightarrow\infty} d(fx_n, fx_{n+1}) = 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(fx_n, fx_{n+1}) \leq \frac{\epsilon}{2m} \text{ for each } n \geq n_1. \tag{3.6}$$

Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, ..., m\}$ such that $s - r \equiv k \pmod m$. We choose $j = m - k + 1$. Then, since $m + 1 \equiv 1 \pmod m$, we have $s+j-r = s+(m-k+1)-r = (s-r)+(m+1)-k \equiv 1 \pmod m$. Now

$$\begin{aligned}d(fx_r, fx_s) &\leq d(fx_r, fx_{s+j}) + d(fx_{s+j}, fx_{s+j-1}) + ... + d(fx_{s+1}, fx_s) \\ &\leq \tfrac{\epsilon}{2} + (j+1).\tfrac{\epsilon}{2} \leq \tfrac{\epsilon}{2} + m.\tfrac{\epsilon}{2m} = \epsilon.\end{aligned}$$

Therefore, given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(fx_r, fx_s) \leq \epsilon$ for all $r, s \geq n$. Hence $\{fx_n\}$ is a Cauchy sequence. Since (X, d) is complete $\lim\limits_{n\longrightarrow\infty} fx_n = x$ for some $x \in X$. From here onwards, again proceeding as in the proof of Theorem 3.1 we have $x = fz,\ z \in \cap_{i=1}^m A_i$.

Now we prove that z is a coincidence point of f and T.
By using the inequality (3.1), we have
$d(fx_{l+km}, Tz) = d(Tx_{l+km-1}, Tz)$

$$\leq M(x_{l+km-1}, z) - \varphi(M(x_{l+km-1}, z)) \tag{3.7}$$

where

$$\begin{aligned}M(x_{n_k}, z) &= \max\{d(fx_{l+km-1}, fz), d(fx_{l+km-1}, Tx_{l+km-1}), d(fz, Tz), \\ &\qquad \tfrac{1}{2}(d(fx_{l+km-1}, Tz) + d(fz, Tx_{l+km-1}))\} \\ &= \max\{d(fx_{l+km-1}, fz), d(fx_{l+km-1}, fx_{l+km}), d(fz, Tz), \\ &\qquad \tfrac{1}{2}(d(fx_{l+km-1}, Tz) + d(fz, Tx_{l+km}))\},\end{aligned}$$

since $x_{l+km-1} \in A_{l+km-1}$ and $z \in A_{l+km}$.
On letting $k \longrightarrow \infty$, we have
$\lim\limits_{k \longrightarrow \infty} M(x_{n_k}, z) = d(fz, Tz)$.
On letting $k \longrightarrow \infty$ in (3.7), we have
$d(x, Tz) \leq d(x, Tz) - \varphi(d(x, Tz))$.
Hence $d(fz, Tz) \leq d(fz, Tz) - \varphi(d(fz, Tz))$ which implies that $\varphi(d(fz, Tz)) = 0$ so that $fz = Tz$.

□

Theorem 3.2. *In addition to the hypotheses of Theorem* 3.1*, if the maps T and f are weakly compatible then T and f have a unique common fixed point.*

Proof. From the proof of Theorem 3.1 we have $Tz = fz = u$ (say). Since T and f are weakly compatible, we have
$Tu = Tfz = fTz = fu$.
Now we prove that $Tu = u$.
Since $Tz \in X = \cup_{i=1}^{m} A_i$ implies $Tz \in A_i$ for some i and $z \in \cap_{i=1}^{m} A_i$, we have $z \in A_i$ for all $i \in \{1, 2, ..., m\}$.
Now, by the inequality (3.1) we have

$$d(Tz, TTz) \leq M(z, Tz) - \varphi(M(z, Tz)) \tag{3.8}$$

where

$$\begin{aligned} M(z, Tz) &= \max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz), \tfrac{1}{2}(d(fz, TTz) + d(fTz, Tz))\} \\ &= \max\{d(Tz, TTz), d(Tz, Tz), d(TTz, TTz), d(Tz, TTz)\} \\ &= d(Tz, TTz). \end{aligned}$$

From (3.8), we have
$d(Tz, TTz) \leq d(Tz, TTz) - \varphi(d(Tz, TTz))$ so that $Tz = TTz$ and hence $u = Tu = fu$.
Therefore u is a common fixed point of f and T.
We now show that $u \in \cap_{i=1}^{m} A_i$. Since $Tu = fu = u$, we have $u \in A_i$ for some i.

$$\begin{aligned} \text{Now, } u \in A_i &\Rightarrow Tu \in T(A_i) \\ &\Rightarrow Tu \in T(A_i) \subset f(A_{i+1}) \\ &\Rightarrow Tu = fv \in f(A_{i+1}) \text{ for some } v \in A_{i+1}. \end{aligned}$$

Therefore $fu = fv$ for some $v \in A_{i+1}$, since f is one-one we have $u = v \in A_{i+1}$ so that $u \in A_{i+1}$. By repeating the same argument, we get $u \in \cap_{i=1}^{m} A_i$.

Uniqueness of common fixed point of T and f follows from the inequality (3.1) trivially. □

4. Corollaries and Examples

By choosing $f = I_X$ in Theorem 2.1, we have the following corollary.

Corollary 4.1. *Let (X, d) be a complete metric space. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X and $X = \cup_{i=1}^{m} A_i$. If $T : X \to X$ is a mapping such that*

(i) *$\cup_{i=1}^{m} A_i$ is a cyclic representation of X w. r. t. T*
(ii) *there exists $\varphi \in \Phi$ such that $d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$ for any $x \in A_i$ and $y \in A_{i+1}, A_{m+1} = A_1$*

then there exists $z \in \cap_{i=1}^{m} A_i$ such that $Tz = z$.

By choosing $f = I_X$ in Theorem 3.1, we have the following corollary.

Corollary 4.2. *Let (X, d) be a complete metric space. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X and $X = \cup_{i=1}^{m} A_i$. If $T : X \to X$ is a mapping such that*

(i) *$\cup_{i=1}^{m} A_i$ is a cyclic representation of X w. r. t. T*
(ii) *there exists $\varphi \in \Phi$ such that $d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y))$ for any $x \in A_i$ and $y \in A_{i+1}, A_{m+1} = A_1$ where*
$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$

then there exists $z \in \cap_{i=1}^{m} A_i$ such that $Tz = z$.

Example 4.3. Let $X = \mathbb{R}$ with the usual metric. Let $A_1 = (-\infty, 2]$ and $A_2 = [2, \infty)$. We define $T,\ f : X \longrightarrow X$ by $Tx = \frac{2+x}{2}$ and $fx = 6 - 2x$. We define $\varphi : [0, \infty) \longrightarrow [0, \infty)$ by $\varphi(t) = \frac{t}{2+t}, t \geq 0$. Then $\varphi \in \Phi$. Clearly, $X = A_1 \cup A_2$ is co-cyclic representation of X w.r.t. T and f. Now we verify the inequality (1.1) in the following:
For $x \in A_1$ and $y \in A_2$, then $d(Tx, Ty) = |\frac{x}{2} - \frac{y}{2}|$ and $d(fx, fy) = |2x - 2y|$
$d(Tx, Ty) = |\frac{x}{2} - \frac{y}{2}| \leq |2x - 2y| - \varphi(|2x - 2y|) = d(fx, fy) - \varphi(d(fx, fy))$.
Clearly, T and f are weakly compatible and satisfy all the hypotheses of Theorem 2.2 and 2 is the unique common fixed point of T and f and $2 \in A_1 \cap A_2$.

In the following, we provide examples in support of the results obtained in Section 4.

Example 4.4. Let $X = \{0, 2, 3, 5\}$ with the usual metric. Let $A_1 = \{0, 2\}$ and $A_2 = \{2, 3, 5\}$. We define $T,\ f : X \longrightarrow X$ by $T0 = T2 = 2, T3 = 0, T5 = 2;\ f0 = 0, f2 = 2,\ f3 = 5$ and $f5 = 3$. We define $\varphi : [0, \infty) \longrightarrow [0, \infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{1+t} & \text{if } 0 \leq t \leq 5 \\ \frac{5}{6}e^{-(t-5)} & \text{if } 5 \leq t < \infty. \end{cases}$$

Here we observe that $\varphi \in \Phi$. Now we verify the inequality (3.1) in the following:
<u>Case</u> (i): $x = 0$ and $y = 3$
then $d(T0, T3) = 2$ and $M(0, 3) = 5$
$d(Tx, Ty) = d(T0, T3) = 2$
$\leq 5 - \varphi(5) = M(0, 3) - \varphi(M(0, 3)) = M(x, y) - \varphi(M(x, y))$.
<u>Case</u> (ii): $x = 2$ and $y = 3$
then $d(T2, T3) = 2$ and $M(2, 3) = 5$
$d(Tx, Ty) = d(T2, T3) = 2$
$\leq 5 - \varphi(5) = M(2, 3) - \varphi(M(2, 3)) = M(x, y) - \varphi(M(x, y))$.
In the other cases the inequality (3.1) trivially holds.
Clearly, T and f are weakly compatible and satisfy all the hypotheses of Theorem 3.2 and 2 is the unique common fixed point of T and f and $2 \in A_1 \cap A_2$.

If we relax the weakly compatibility property of f and T of Theorem 3.2 then T and f may not have a common fixed point.

Example 4.5. Let $X = \{1, 2, 3, 4\}$ with the usual metric. Let $A_1 = \{1, 2, 3\}$ and $A_2 = \{2, 3, 4\}$. We define $T,\ f : X \longrightarrow X$ by $T1 = 2, T2 = T3 = 3, T4 = 4$;
$f1 = 4, f2 = 3,\ f3 = 2$ and $f4 = 1$. We define $\varphi : [0, \infty) \longrightarrow [0, \infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{2+t} & \text{if } 0 \leq t \leq 4 \\ \frac{2}{3}e^{-(t-4)} & \text{if } 4 \leq t < \infty. \end{cases}$$

Then $\varphi \in \Phi$. Now we verify the inequality (3.1) in the following:
<u>Case</u> (i): $x = 1$ and $y = 2$

then $d(T1, T2) = 1$ and $M(1, 2) = 2$

$$d(Tx, Ty) = d(T1, T2) = 1$$
$$\leq 2 - \varphi(2) = M(1, 2) - \varphi(M(1, 2)) = M(x, y) - \varphi(M(x, y)).$$

Case (ii): $x = 1$ and $y = 3$

then $d(T1, T3) = 1$ and $M(1, 3) = 2$

$$d(Tx, Ty) = d(T1, T3) = 1$$
$$\leq 2 - \varphi(2) = M(1, 3) - \varphi(M(1, 3)) = M(x, y) - \varphi(M(x, y)).$$

Case (iii): $x = 1$ and $y = 4$

then $d(T1, T4) = 2$ and $M(1, 4) = 3$

$$d(Tx, Ty) = d(T1, T4) = 2$$
$$\leq 3 - \varphi(3) = M(1, 4) - \varphi(M(1, 4)) = M(x, y) - \varphi(M(x, y)).$$

Case (iv): $x = 2$ and $y = 3$

In this case, the inequality (3.1) trivially holds.

Case (v): $x = 2$ and $y = 4$

then $d(T2, T4) = 1$ and $M(2, 4) = 3$

$$d(Tx, Ty) = d(T2, T4) = 1$$
$$\leq 3 - \varphi(3) = M(2, 4) - \varphi(M(2, 4)) = M(x, y) - \varphi(M(x, y)).$$

Case (vi): $x = 3$ and $y = 4$

then $d(T3, T4) = 1$ and $M(3, 4) = 3$

$$d(Tx, Ty) = d(T3, T4) = 1$$
$$\leq 3 - \varphi(3) = M(3, 4) - \varphi(M(3, 4)) = M(x, y) - \varphi(M(x, y))$$

Hence 2 is the coincidence point of T and f and $2 \in A_1 \cap A_2$. Here, we note that f and T are not weakly compatible, since $T2 = 3$ and $f2 = 3$ then $T(f(2)) = T(3) = 3$ and $f(T(2)) = f(3) = 2$ so that $T(f(2)) \neq f(T(2))$. Hence f and T satisfy all the hypotheses of Theorem 3.2 except the weakly compatible property of f and T, and we observe that f and T have no common fixed points in X.

Further, we observe that at $x = 1$ and $y = 2$

$d(T1, T2) = 1 \nleq 1 - \varphi(1) = d(f1, f2) - \varphi d(f1, f2)$ for any $\varphi \in \Phi$ and any $\phi \in \tau$. Therefore T is not a co-cyclic weakly contractive map w. r. t. f with any $\varphi \in \tau$. Hence Theorem 1.2 is not applicable.

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SYSTEM OF GENERALIZED HIERARCHICAL VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. In this study, we discussed the solution of a system of generalized hierarchical variational inequality problems in Hilbert spaces by using the concepts of Mainge′s. We also discussed some applications.

KEYWORDS:System of generalized hierarchical variational inequality problems; fixed point problems; r-strongly monotone mappings; Hilbert spaces.
AMS Subject Classification: 49J40, 47H06.

1. Introduction

In various practical problems arising in decision theory, economics theory, game theory portfolio selection, etc. it is required to optimize the ratio of several linear or nonlinear functions to achieve the goal efficiently. The optimization problems are called mathematical functional programming problems or optimal control problems. The study of mathematical functional programming problem has been of great interest in the recent past due to its diversified applications. The variational inequality theory is well known and well developed because of its applications in the diversified area of science, social science, engineering, and commercial management. The variational inequality problems provide a convenient framework for the unified study of the optimal solution in many optimization related fields. Several numerical methods has been developed for solving variational inequality and related optimization problems. Hierarchical optimization was first defined by Bracken and McGill [2, 3] as a generalization of mathematical programming. In this context, the constraint region is implicitly determined by a series of optimization problems which must be solved in a predetermined sequence.

Inspired and motivated by the recent works [1, 4, 5, 6, 9, 14, 16, 17], we introduced the system of generalized hierarchical variational inequality problems and

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investigate a more general form of the schemes to solve the system of generalized hierarchical variational inequality problems.

2. Preliminaries

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. Let C be a nonempty closed convex subsets of H. $F(T)$ denotes the set of fixed points of $T : C \longrightarrow C$, *that is,* $F(T) = \{x \in C : Tx = x\}$ and a variational inequality problems [8] is the problem of finding a point $x \in C$ such that

$$\langle Qx, y - x \rangle \geq 0, \forall y \in C, \tag{2.1}$$

where $Q : C \longrightarrow C$ is a nonlinear mapping and solution set of (2.1) is denoted by $\Upsilon(Q, C)$.
The hierarchical fixed point problems [11, 12, 13, 18, 19] is the problem of finding a point $x^* \in F(T)$ such that

$$\langle Qx^*, x - x^* \rangle \geq 0, \ \forall x \in F(T). \tag{2.2}$$

When the set $F(T)$ is replaced by the solution set of variational inequality (2.1), then (2.2) is known as hierarchical variational inequality problems.

In this paper, we define the system of generalized hierarchical variational inequality problems for finding $x_i^* \in \Upsilon(Q_i, C)$ such that for given positive real number $\eta_i, \ (i = 1, 2, \cdots, N)$ the following inequalities are hold:

$$\begin{aligned}
\langle \eta_1 F(x_2^*) + x_1^* - x_2^*, \ x_1 - x_1^* \rangle &\geq 0, \ \forall \ x_1 \in \Upsilon(Q_1, C), \\
\langle \eta_2 F(x_3^*) + x_2^* - x_3^*, \ x_2 - x_2^* \rangle &\geq 0, \ \forall \ x_2 \in \Upsilon(Q_2, C), \\
&\vdots \\
\langle \eta_{N-1} F(x_N^*) + x_{N-1}^* - x_N^*, \ x_{N-1} - x_{N-1}^* \rangle &\geq 0, \ \forall \ x_{N-1} \in \Upsilon(Q_{N-1}, C), \\
\langle \eta_N F(x_1^*) + x_N^* - x_1^*, \ x_N - x_N^* \rangle &\geq 0, \ \forall \ x_N \in \Upsilon(Q_N, C),
\end{aligned} \tag{2.3}$$

where $F, Q_i : H \longrightarrow H \ (i = 1, 2, \cdots, N)$ are mappings.

Definition 2.1. Let $T, F : H \longrightarrow H$ be the single valued mappings. Then

(i) T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in H;$$

(ii) T is said to be quasi nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \ \forall x \in H, p \in F(T);$$

(iii) T is quasi nonexpansive if and only if for all $x \in H, p \in F(T)$

$$\langle x - Tx, \ x - p \rangle \geq \frac{1}{2}\|x - Tx\|^2;$$

(iv) T is said to be strongly quasi nonexpansive if T is quasi nonexpansive and

$$x_n - Tx_n \longrightarrow 0$$

whenever $\{x_n\}$ is a bounded sequence in H and

$$\|x_n - p\| - \|Tx_n - p\| \longrightarrow 0, \ \text{for some } p \in F(T);$$

(v) F is said to be μ-Lipschitzian if there exists $\mu > 0$ such that

$$\|F(x) - F(y)\| \leq \mu\|x - y\|, \ \forall x, y \in H;$$

(vi) F is said to be r-strongly monotone if there exists $r > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq r\|x - y\|^2, \ \forall x, y \in H;$$

(vii) If F is a μ-Lipschitzian and r-strongly monotone mapping and $\rho \in (0, \frac{2r}{\mu^2})$, then

$$I - \rho F$$

is a contraction mapping;

(viii) F is said to be α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \alpha\|F(x) - F(y)\|^2, \ \forall x, y \in H.$$

Lemma 2.2. [20] *Let $Q : H \longrightarrow H$ be an α-inverse strongly monotone mapping. Then*

(i) *Q is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping;*
(ii) $\|(I - \lambda Q)x - (I - \lambda Q)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Qx - Qy\|^2,$ *for* $\lambda > 0$;
(iii) *if $\lambda \in (0, 2\alpha]$, then $I - \lambda Q$ is a nonexpansive mapping where I is an identity mapping on H.*

Lemma 2.3. *Let $x \in H$ and $z \in C$ be any points. Then the following statements are hold:*

(i) $z = P_C(x) \iff \langle x - z, \ y - z\rangle \geq 0, \ \forall y \in C.$
(ii) $z = P_C(x) \Rightarrow \ \|x - z\|^2 \geq \|x - y\|^2 - \|y - z\|^2, \ \forall y \in C.$
(iii) $\langle P_C(x) - P_C(y), \ x - y\rangle \geq \|P_C(x) - P_C(y)\|^2, \ \forall x, y \in H.$
(iv) $u \in \Upsilon(Q, C) \Leftrightarrow u \in F(P_C(I - \lambda Q)), \ \forall \lambda > 0.$

Lemma 2.4. [15] *For $x, y \in H$ and $\omega \in (0, 1)$ the following statements are hold:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle;$
(ii) $\|(1 - \omega)x + \omega y\|^2 = (1 - \omega)\|x\|^2 + \omega\|y\|^2 - \omega(1 - \omega)\|x - y\|^2.$

Lemma 2.5. [10] *Let $\{a_n\}$ be a sequence of real numbers and there exists a subsequence $\{a_{m_j}\}$ of $\{a_n\}$ such that $a_{m_j} < a_{m_j+1}$ for all $j \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers. Then there exists a non decreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\longrightarrow\infty} n_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$*

$$a_{n_k} \leq a_{n_k+1}, \ a_k \leq a_{n_k+1}.$$

In fact, n_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $a_n < a_{n+1}$ hold.

Lemma 2.6. [7] *Let $\{a_n\} \subset [0, \infty), \ \{\alpha_n\} \subset [0, 1), \{b_n\} \subset (-\infty, +\infty)$ and $\hat{\alpha} \in [0, 1]$ be such that*

(i) *$\{a_n\}$ is a bounded sequence;*
(ii) $a_{n+1} \leq (1 - \alpha_n)^2 a_n \hat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n, \ \forall n \geq 1;$
(iii) *whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying*

$$\lim_{k\longrightarrow\infty} \inf(a_{n_{k+1}} - a_{n_k}) \geq 0$$

it follows that

$$\lim_{k\longrightarrow\infty} \sup b_{n_k} \leq 0;$$

(iv) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty.$

Then $\lim_{n\longrightarrow\infty} a_n = 0.$

3. Main Results

First, we prove the following lemma.

Lemma 3.1. *Let $Q : H \longrightarrow H$ be an α-inverse strongly monotone mapping. Let $\Upsilon(Q, C) \neq \emptyset$ be the solution set of (2.1). Then the following are hold:*

1. *the mapping $\Omega : H \longrightarrow C$ is defined by*
$$\Omega = P_C(I - \lambda Q), \text{ for } \lambda \in (0, 2\alpha],$$
is quasi nonexpansive, where I is an identity mapping;
2. *the mapping $I-\Omega : H \longrightarrow H$ is demiclosed at zero, that is, for any sequence $\{x_n\} \subset H$ if $x_n \rightharpoonup x$ and $(I - \Omega)x_n \longrightarrow 0$, then $x = \Omega x$;*
3. *the mapping Ω_β defined by*
$$\Omega_\beta = (I - \beta)I + \beta\Omega, \text{ for } \beta \in (0, 1) \tag{3.1}$$
is strongly quasi nonexpansive mapping and $F(\Omega_\beta) = F(\Omega)$.
4. *$I - \Omega_\beta,\ \beta(0, 1)$ is demiclosed at zero.*

Proof. (i) From Lemma 2.2(iii) and Lemma 2.3(iv), the mapping Ω is nonexpansive and $\Upsilon(Q, C) = F(\Omega) \neq \emptyset$. then this show that Ω is quasi nonexpansive.

(ii) Since Ω is a nonexpansive mapping on C, $I - \Omega$ is demiclosed at zero.

(iii) It is obvious that $F(\Omega_\beta) = F(\Omega)$.
Next, we prove that $\Omega_\beta, \beta \in (0, 1)$ is a strongly quasi nonexpansive mapping.
Let $\{x_n\}$ be any bounded sequence in H and $p \in \Omega_\beta$ be a given point such that
$$\|x_n - p\| - \|\Omega_\beta x_n - p\| \longrightarrow 0. \tag{3.2}$$
First, we prove that $\Omega_\beta, \beta \in (0, 1)$ is a quasi nonexpansive mapping.
From (3.1) and the fact that Ω is quasi nonexpansive, we have
$$\begin{aligned}\|\Omega_\beta x - p\| &= \|(1 - \beta)[x - p] + \beta(\Omega x - p)\|\\ &\leq (1 - \beta)\|x - p\| + \beta\|\Omega x - p\|\\ &\leq \|x - p\|, \ \forall x \in C.\end{aligned}$$
Therefore, Ω_β is a quasi nonexpansive mapping.
Next, we prove that
$$\|\Omega_\beta x_n - x_n\| \longrightarrow 0.$$
In fact, it follows from (3.1) that
$$\begin{aligned}\|\Omega_\beta x_n - p\|^2 &= \|x_n - p - \beta(x_n - \Omega x_n)\|^2\\ &= \|x_n - p\|^2 - 2\beta\langle x_n - p, x_n - \Omega x_n\rangle + \beta^2\|x_n - \Omega x_n\|^2\\ &\leq \|x_n - p\|^2 - \beta(1 - \beta)\|x_n - \Omega x_n\|^2.\end{aligned}$$
From (3.2), we have
$$\beta(1 - \beta)\|x_n - \Omega x_n\|^2 \leq \|x_n - p\|^2 - \|\Omega_\beta x_n - p\|^2 \longrightarrow 0.$$
Since $\beta(1 - \beta) > 0$, then
$$\|x_n - \Omega x_n\| \longrightarrow 0.$$
Hence
$$\|x_n - \Omega_\beta x_n\| = \beta\|x_n - \Omega x_n\| \longrightarrow 0.$$

(iv) Since $I - \Omega_\beta = \beta(I - \Omega)$ and $I - \Omega$ is demiclosed at zero, hence $I - \Omega_\beta$ is demiclosed at zero. This completes the proof.

□

Throughout this section, we always assume that the following conditions are satisfied:

(C1) $Q_i : H \longrightarrow H$ is an α_i-inverse strongly monotone mapping and $\Upsilon(Q_i, C) \neq \emptyset$ is the solution set of (2.1) with $Q = Q_i$ $(i = 1, 2, \cdots, N)$.

(C2) Ω_i and $\Omega_{i,\beta}, \beta \in (0, 1)$ are the mappings defined by

$$\begin{aligned} \Omega_i &= P_{C_i}(I - \lambda Q_i), \ \lambda \in (0, 2\alpha_i]; \\ \Omega_{i,\beta} &= (1 - \beta)I + \beta\Omega_i, \ \beta \in (0, 1), (i = 1, 2, \cdots, N) \textit{ respectively.} \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let Q_i and $\Upsilon(Q_i, C)$ satisfying the conditions $(C1)$ and $f_i : H \longrightarrow H$ be contraction with a contractive constant $\vartheta_i \in (0, 1)$, $(i = 1, 2, \cdots, N)$. Then there exists a unique elements $x_i^* \in \Upsilon(Q_i, C)$ such that the following are hold:*

$$\begin{aligned} \langle x_1^* - f_1(x_2^*),\ x_1 - x_1^*\rangle &\geq 0,\ \forall x_1 \in \Upsilon(Q_1, C), \\ \langle x_2^* - f_2(x_3^*),\ x_2 - x_2^*\rangle &\geq 0,\ \forall x_2 \in \Upsilon(Q_2, C), \\ &\vdots \\ \langle x_{N-1}^* - f_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^*\rangle &\geq 0,\ \forall x_{N-1} \in \Upsilon(Q_{N-1}, C), \\ \langle x_N^* - f_N(x_1^*),\ x_N - x_N^*\rangle &\geq 0,\ \forall x_N \in \Upsilon(Q_N, C)(i = 1, 2, \cdots, N). \end{aligned} \tag{3.4}$$

Proof. The proof is a consequence of Banach's contraction principle but it is given here for the sake of completeness. From Lemma 2.2(iii) and Lemma 2.3(iv), $\Upsilon(Q_i, C)$ $(i = 1, 2, \cdots, N)$ are nonempty closed convex. Therefore the metric projection $P_{\Upsilon(Q_i,C)}$ is well defined for each $i = 1, 2, \cdots, N$. Since $f_i(i = 1, 2, \cdots, N)$ is a contraction mapping. Then

$$P_{\Upsilon(Q_i,C)} f_i$$

and

$$P_{\Upsilon(Q_1,C)} f_1 \circ P_{\Upsilon(Q_2,C)} f_2 \circ \cdots \circ P_{\Upsilon(Q_N,C)} f_N \tag{3.5}$$

are contraction mappings. Hence there exists a unique element $x^* \in H$ such that

$$x^* = (P_{\Upsilon(Q_1,C)} f_1 \circ P_{\Upsilon(Q_2,C)} f_2 \circ \cdots \circ P_{\Upsilon(Q_N,C)} f_N) x^*. \tag{3.6}$$

Putting $x_N^* = P_{\Upsilon(Q_N,C)} f_N(x_1^*), \cdots, x_2^* = P_{\Upsilon(Q_2,C)} f_2(x_3^*), x_1^* = P_{\Upsilon(Q_1,C)} f_1(x_2^*)$ and $x_N^* \in \Upsilon(Q_N, C), \cdots, x_1^* \in \Upsilon(Q_1, C)$.

Suppose that $(\bar{x}_1, \cdots, \bar{x}_N) \in \Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$ such that the following are satisfied:

$$\begin{aligned} \langle \bar{x}_1 - f_1(\bar{x}_2),\ x_1 - \bar{x}_1\rangle &\geq 0,\ \forall x_1 \in \Upsilon(Q_1, C), \\ \langle \bar{x}_2 - f_2(\bar{x}_3),\ x_2 - \bar{x}_2\rangle &\geq 0,\ \forall x_2 \in \Upsilon(Q_2, C), \\ &\vdots \\ \langle \bar{x}_{N-1} - f_{N-1}(\bar{x}_N),\ x_{N-1} - \bar{x}_{N-1}\rangle &\geq 0,\ \forall x_{N-1} \in \Upsilon(Q_{N-1}, C), \\ \langle \bar{x}_N - f_N(\bar{x}_1),\ x_N - \bar{x}_N\rangle &\geq 0,\ \forall x_N \in \Upsilon(Q_N, C). \end{aligned} \tag{3.7}$$

Then

$$\begin{aligned} \bar{x}_1 &= P_{\Upsilon(Q_1,C)} f_1(\bar{x}_2), \\ \bar{x}_2 &= P_{\Upsilon(Q_2,C)} f_2(\bar{x}_3), \end{aligned}$$

$$\vdots$$
$$\bar{x}_N = P_{\Upsilon(Q_N,C)} f_N(\bar{x}_1). \tag{3.8}$$

Therefore
$$\bar{x}_1 = (P_{\Upsilon(Q_1,C)} f_1 \circ P_{\Upsilon(Q_2,C)} f_2 \circ \cdots \circ P_{\Upsilon(Q_N,C)} f_N)\bar{x}_1. \tag{3.9}$$

This implies that $\bar{x}_1 = x_1^*, \bar{x}_2 = x_2^*, \cdots, \bar{x}_N = x_N^*$, the proof is completed. □

Theorem 3.2. *Let $Q_i, \Upsilon(Q_i, C), \Omega_i$ and $\Omega_{i,\beta}$ satisfying the conditions $(C1 - C2)$, and $f_i : H \longrightarrow H$ be the contraction with a contractive constant $\vartheta_i \in (0, 1)(i = 1, 2, \cdots, N)$. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned}
x_i^0 &\in H, \qquad i = 1, 2, \cdots, N\\
x_1^{n+1} &= (1 - \alpha_n)\Omega_{1,\beta} x_1^n + \alpha_n f_1(\Omega_{2,\beta} x_2^n),\\
x_2^{n+1} &= (1 - \alpha_n)\Omega_{2,\beta} x_2^n + \alpha_n f_2(\Omega_{3,\beta} x_3^n),\\
&\vdots\\
x_N^{n+1} &= (1 - \alpha_n)\Omega_{N,\beta} x_N^n + \alpha_n f_N(\Omega_{1,\beta} x_1^n),
\end{aligned} \tag{3.10}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_i^n\}$ $(i = 1, 2, \cdots, N)$ defined by (3.10) converge to x_i^*, where $(x_1^*, \cdots, x_N^*)$ is the unique elements in $\Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$, verifying (3.4).

Proof. **(i)** We first prove that the sequence $\{x_1^n\}, \cdots, \{x_N^n\}$ are bounded. From Lemma 3.1, it follows that $\Omega_{i,\beta}$ is strongly quasi nonexpansive and $F(\Omega_{i,\beta}) = F(\Omega_i) = \Upsilon(Q_i, C)$ $(i = 1, \cdots, N)$. Since f_i is contraction with constant ϑ_i $(i = 1, \cdots, N)$ and $x_1^* \in F(\Omega_{1,\beta}), x_2^* \in F(\Omega_{2,\beta}), \cdots, x_N^* \in F(\Omega_{N,\beta})$, we have

$$\begin{aligned}
\|x_1^{n+1} - x_1^*\| &\le (1 - \alpha_n)\|\Omega_{1,\beta} x_1^n - x_1^*\| + \alpha_n\|f_1(\Omega_{2,\beta} x_2^n) - x_1^*\|\\
&\le (1 - \alpha_n)\|x_1^n - x_1^*\| + \alpha_n\|f_1(\Omega_{2,\beta} x_2^n) - f_1(x_2^*)\| + \alpha_n\|f_1(x_2^*) - x_1^*\|\\
&\le (1 - \alpha_n)\|x_1^n - x_1^*\| + \alpha_n\vartheta_1\|\Omega_{2,\beta} x_2^n - x_2^*\| + \alpha_n\|f_1(x_2^*) - x_1^*\|\\
&\le (1 - \alpha_n)\|x_1^n - x_1^*\| + \alpha_n\vartheta_1\|x_2^n - x_2^*\| + \alpha_n\|f_1(x_2^*) - x_1^*\|\\
&\le (1 - \alpha_n)\|x_1^n - x_1^*\| + \alpha_n\vartheta_1\|x_2^n - x_2^*\| + \alpha_n\|f_1(x_2^*) - x_1^*\|.
\end{aligned} \tag{3.11}$$

Similarly, we can also compute that

$$\begin{aligned}
\|x_2^{n+1} - x_2^*\| &\le (1 - \alpha_n)\|x_2^n - x_2^*\| + \alpha_n\vartheta_2\|x_3^n - x_3^*\| + \alpha_n\|f_2(x_3^*) - x_2^*\|,\\
&\vdots\\
\|x_N^{n+1} - x_N^*\| &\le (1 - \alpha_n)\|x_N^n - x_N^*\| + \alpha_n\vartheta_N\|x_1^n - x_1^*\| + \alpha_n\|f_N(x_1^*) - x_N^*\|.
\end{aligned} \tag{3.12}$$

This implies that

$$\begin{aligned}
&\|x_1^{n+1} - x_1^*\| + \|x_2^{n+1} - x_2^*\| + \cdots + \|x_N^{n+1} - x_N^*\| \le (1 - \alpha_n)[\|x_1^n - x_1^*\|\\
&+ \cdots + \|x_N^n - x_N^*\|] + \alpha_n[\vartheta_N\|x_1^n - x_1^*\| + \vartheta_1\|x_2^n - x_2^*\| + \cdots + \vartheta_{N-1}\|x_N^n - x_N^*\|]\\
&+\alpha_n[\|f_1(x_2^*) - x_1^*\| + \cdots + \|f_N(x_1^*) - x_N^*\|]\\
&\le (1 - \alpha_n)[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|] + \alpha_n\vartheta[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|]\\
&+\alpha_n[\|f_1(x_2^*) - x_1^*\| + \cdots + \|f_N(x_1^*) - x_N^*\|]\\
&\le (1 - \alpha_n(1 - \vartheta))[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|]\\
&+\alpha_n(1 - \vartheta)\frac{\|f_1(x_2^*) - x_1^*\| + \|f_2(x_3^*) - x_2^*\| + \cdots + \|f_N(x_1^*) - x_N^*\|}{1 - \vartheta}
\end{aligned}$$

$$\leq \max\{\|x_1^n-x_1^*\|+\cdots+\|x_N^n-x_N^*\|, \frac{\|f_1(x_2^*)-x_1^*\|+\cdots+\|f_N(x_1^*)-x_N^*\|}{1-\vartheta}\}, \quad (3.13)$$

where $\vartheta = \max\{\vartheta_1, \vartheta_2, \cdots, \vartheta_N\}$.
By induction, we have

$$\|x_1^{n+1}-x_1^*\|+\|x_2^{n+1}-x_2^*\|+\cdots+\|x_N^{n+1}-x_N^*\|$$
$$\leq \max\{\|x_1^0-x_1^*\|+\cdots+\|x_N^0-x_N^*\|, \frac{\|f_1(x_2^*)-x_1^*\|+\cdots+\|f_N(x_1^*)-x_N^*\|}{1-\vartheta}\}, \forall n \geq 1. \quad (3.14)$$

Hence $\{x_1^n\}, \cdots, \{x_N^n\}$ are bounded, consequently $\{\Omega_{1,\beta}x_1^*\}, \cdots, \{\Omega_{N,\beta}x_N^*\}$ are bounded.

(ii) Next, we prove that for each $n \geq 1$ the following inequalities are hold:

$$\begin{aligned}\|x_1^{n+1}-x_1^*\|^2+\|x_2^{n+1}-x_2^*\|^2+\cdots+\|x_N^{n+1}-x_N^*\|^2 &\leq (1-\alpha_n)^2(\|x_1^n-x_1^*\|^2 \\ +\|x_2^n-x_2^*\|^2+\cdots+\|x_N^n-x_N^*\|^2)+2\alpha_n\vartheta(\|x_1^{n+1}-x_1^*\|\|x_2^n-x_2^*\| \\ +\|x_2^{n+1}-x_2^*\|\|x_3^n-x_3^*\|+\cdots+\|x_N^{n+1}-x_N^*\|\|x_1^n-x_1^*\|) \\ +2\alpha_n(\langle f_1(x_2^*)-x_1^*,\ x_1^{n+1}-x_1^*\rangle+\langle f_2(x_3^*)-x_2^*,\ x_2^{n+1}-x_2^*\rangle \\ +\cdots+\langle f_N(x_1^*)-x_N^*,\ x_N^{n+1}-x_N^*\rangle).\end{aligned} \quad (3.15)$$

From (3.10) and Lemma 2.4, we have

$$\begin{aligned}\|x_1^{n+1}-x_1^*\|^2 &= \|(1-\alpha_n)(\Omega_{1,\beta}(x_1^n)-x_1^*)+\alpha_n(f_1(\Omega_{2,\beta}(x_2^n))-x_1^*)\|^2 \\ &\leq \|(1-\alpha_n)(\Omega_{1,\beta}(x_1^n)-x_1^*)\|^2+2\alpha_n\langle f_1(\Omega_{2,\beta}(x_2^n))-x_1^*,\ x_1^{n+1}-x_1^*\rangle \\ &\leq (1-\alpha_n)^2\|\Omega_{1,\beta}(x_1^n)-x_1^*\|^2+2\alpha_n\langle f_1(\Omega_{2,\beta}(x_2^n))-f_1(x_2^*),\ x_1^{n+1}-x_1^*\rangle \\ &\quad +2\alpha_n\langle f_1(x_2^n)-x_1^*,\ x_1^{n+1}-x_1^*\rangle \\ &\leq (1-\alpha_n)^2\|x_1^n-x_1^*\|^2+2\alpha_n\|f_1(\Omega_{2,\beta}(x_2^n))-f_1(x_2^*)\|\|x_1^{n+1}-x_1^*\| \\ &\quad +2\alpha_n\langle f_1(x_2^n)-x_1^*,\ x_1^{n+1}-x_1^*\rangle \\ &\leq (1-\alpha_n)^2\|x_1^n-x_1^*\|^2+2\alpha_n\vartheta_1\|\Omega_{2,\beta}(x_2^n)-x_2^*\|\|x_1^{n+1}-x_1^*\| \\ &\quad +2\alpha_n\langle f_1(x_2^*)-x_1^*,\ x_1^{n+1}-x_1^*\rangle \\ &\leq (1-\alpha_n)^2\|x_1^n-x_1^*\|^2+2\alpha_n\vartheta_1\|x_2^n-x_2^*\|\|x_1^{n+1}-x_1^*\| \\ &\quad +2\alpha_n\langle f_1(x_2^*)-x_1^*,\ x_1^{n+1}-x_1^*\rangle.\end{aligned} \quad (3.16)$$

Similarly, we can also prove that

$$\begin{aligned}\|x_2^{n+1}-x_2^*\|^2 &\leq (1-\alpha_n)^2\|x_2^n-x_2^*\|^2+2\alpha_n\vartheta_2\|x_3^n-x_3^*\|\|x_2^{n+1}-x_2^*\| \\ &\quad +2\alpha_n\langle f_2(x_3^*)-x_2^*,\ x_2^{n+1}-x_2^*\rangle, \\ &\vdots \\ \|x_N^{n+1}-x_N^*\|^2 &\leq (1-\alpha_n)^2\|x_N^n-x_N^*\|^2+2\alpha_n\vartheta_N\|x_1^n-x_1^*\|\|x_N^{n+1}-x_N^*\| \\ &\quad +2\alpha_n\langle f_N(x_1^*)-x_N^*,\ x_N^{n+1}-x_N^*\rangle.\end{aligned} \quad (3.17)$$

Adding (3.16) and (3.17), and assume that $\vartheta = \max\{\vartheta_1, \cdots, \vartheta_N\}$, inequalities (3.15) is proved.

(iii) Next, we prove that if there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$\lim_{k\longrightarrow\infty} \inf\{(\|x_1^{n_k+1}-x_1^*\|^2+\cdots+\|x_N^{n_k+1}-x_N^*\|^2)-(\|x_1^{n_k}-x_1^*\|^2 \\ +\cdots+\|x_N^{n_k}-x_N^*\|^2)\} \geq 0. \quad (3.18)$$

Then

$$\lim_{k\longrightarrow\infty}\sup\{\langle f_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle+\langle f_2(x_3^*)-x_2^*,\ x_2^{n_k+1}-x_2^*\rangle \\ +\cdots+\langle f_N(x_1^*)-x_N^*,\ x_N^{n_k+1}-x_N^*\rangle\}\leq 0. \tag{3.19}$$

Since the norm $\|\cdot\|^2$ is convex and $\lim_{n\longrightarrow\infty}\alpha_n=0$, by (3.10) we have

$$\begin{aligned} 0 &\leq \lim_{k\longrightarrow\infty}\inf\{(\|x_1^{n_k+1}-x_1^*\|^2+\cdots+\|x_N^{n_k+1}-x_N^*\|^2)-(\|x_1^{n_k}-x_1^*\|^2 \\ &\quad +\cdots+\|x_N^{n_k}-x_N^*\|^2)\} \\ &\leq \lim_{k\longrightarrow\infty}\inf\{(1-\alpha_{n_k})\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|^2+\alpha_{n_k}\|f_1(\Omega_{2,\beta}(x_2^{n_k}))-x_1^*\|^2 \\ &\quad +(1-\alpha_{n_k})\|\Omega_{2,\beta}x_2^{n_k}-x_2^*\|^2+\alpha_{n_k}\|f_2(\Omega_{3,\beta}(x_3^{n_k}))-x_2^*\|^2 \\ &\quad +\cdots+(1-\alpha_{n_k})\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|^2+\alpha_{n_k}\|f_N(\Omega_{1,\beta}(x_1^{n_k}))-x_N^*\|^2 \\ &\quad -(\|x_1^{n_k}-x_1^*\|^2+\cdots+\|x_N^{n_k}-x_N^*\|^2)\} \\ &\leq \lim_{k\longrightarrow\infty}\inf\{(\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2)+(\|\Omega_{2,\beta}(x_2^{n_k})-x_2^*\|^2 \\ &\quad -\|x_2^{n_k}-x_2^*\|^2)+\cdots+(\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2)\} \\ &\leq \lim_{k\longrightarrow\infty}\sup\{(\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2)+(\|\Omega_{2,\beta}(x_2^{n_k})-x_2^*\|^2 \\ &\quad -\|x_2^{n_k}-x_2^*\|^2)+\cdots+(\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2)\}\leq 0. \end{aligned} \tag{3.20}$$

This implies that

$$\begin{aligned} &\lim_{k\longrightarrow\infty}(\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2) \\ &=\lim_{k\longrightarrow\infty}(\|\Omega_{2,\beta}x_2^{n_k}-x_2^*\|^2-\|x_2^{n_k}-x_2^*\|^2) \\ &=\cdots=\lim_{k\longrightarrow\infty}(\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2)=0. \end{aligned} \tag{3.21}$$

Since the sequences $\{\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|+\|x_1^{n_k}-x_1^*\|\},\{\|\Omega_{2,\beta}x_2^{n_k}-x_2^*\|+\|x_2^{n_k}-x_2^*\|\},\cdots,\{\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|+\|x_N^{n_k}-x_N^*\|\}$ are bounded. Therefore, we have

$$\begin{aligned} &\lim_{k\longrightarrow\infty}(\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|-\|x_1^{n_k}-x_1^*\|) \\ &=\lim_{k\longrightarrow\infty}(\|\Omega_{2,\beta}x_2^{n_k}-x_2^*\|-\|x_2^{n_k}-x_2^*\|) \\ &=\cdots=\lim_{k\longrightarrow\infty}(\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|-\|x_N^{n_k}-x_N^*\|)=0. \end{aligned} \tag{3.22}$$

By Lemma 3.1, $\Omega_{1,\beta},\cdots,\Omega_{N,\beta}$ are strongly quasi nonexpansive, then

$$\Omega_{1,\beta}x_1^{n_k}-x_1^{n_k}\longrightarrow 0,\ \Omega_{2,\beta}x_2^{n_k}-x_2^{n_k}\longrightarrow 0,\ \cdots,\ \Omega_{N,\beta}x_N^{n_k}-x_N^{n_k}\longrightarrow 0. \tag{3.23}$$

Consequently, we obtain that

$$x_1^{n_k}-x_1^{n_k+1}\longrightarrow 0,\ x_2^{n_k}-x_2^{n_k+1}\longrightarrow 0,\ \cdots,\ x_N^{n_k}-x_N^{n_k+1}\longrightarrow 0. \tag{3.24}$$

It follows from the boundedness of $\{x_1^{n_k}\}$ that there exists a subsequence $\{x_1^{n_{k_\ell}}\}$ of $\{x_1^{n_k}\}$ such that $x_1^{n_{k_\ell}}\rightharpoonup p$ and

$$\begin{aligned} \lim_{\ell\longrightarrow\infty}\langle f_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle &=\lim_{k\longrightarrow\infty}\sup\langle f_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle \\ &=\lim_{k\longrightarrow\infty}\sup\langle f_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle. \end{aligned} \tag{3.25}$$

By Lemma 3.1, $I-\Omega_{1,\beta}$ is demiclosed at zero and $p\in Fix(\Omega_{1,\beta})=\Upsilon(Q_1,C)$. Hence from (3.4) we have

$$\lim_{\ell\longrightarrow\infty}\langle f_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle=\langle f_1(x_2^*)-x_1^*,\ p-x_1^*\rangle\leq 0. \tag{3.26}$$

Therefore

$$\lim_{k\longrightarrow\infty}\sup\langle f_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle=\lim_{\ell\longrightarrow\infty}\langle f_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle\leq 0. \quad (3.27)$$

Similarly, we can also prove that

$$\begin{aligned}\lim_{k\longrightarrow\infty}\sup\langle f_2(x_3^*)-x_2^*,\ x_2^{n_k+1}-x_2^*\rangle\leq 0,\\ \vdots\\ \lim_{k\longrightarrow\infty}\sup\langle f_N(x_1^*)-x_N^*,\ x_N^{n_k+1}-x_N^*\rangle\leq 0.\end{aligned} \quad (3.28)$$

Hence, we have the desired inequalities.

(iv) Finally, we prove that the sequences $\{x_1^n\},\cdots,\{x_N^n\}$ generated by (3.10) converge to $x_1^*,\cdots,x_N^*$, *respectively.* It is clear that

$$\begin{aligned}&\|x_1^{n+1}-x_1^*\|\|x_2^n-x_2^*\|+\|x_2^{n+1}-x_2^*\|\|x_3^n-x_3^*\|+\cdots\\&+\|x_N^{n+1}-x_N^*\|\|x_1^n-x_1^*\|\leq\sqrt{\|x_1^n-x_1^*\|^2+\cdots+\|x_N^n-x_N^*\|^2}\times\\&\sqrt{\|x_1^{n+1}-x_1^*\|^2+\cdots+\|x_N^{n+1}-x_N^*\|^2}.\end{aligned} \quad (3.29)$$

Substituting (3.29) into (3.15) we have

$$\begin{aligned}&\|x_1^{n+1}-x_1^*\|^2+\|x_2^{n+1}-x_2^*\|^2+\cdots+\|x_N^{n+1}-x_N^*\|^2\leq(1-\alpha_n)^2(\|x_1^n-x_1^*\|^2\\&+\cdots+\|x_N^n-x_N^*\|^2)+2\alpha_n\vartheta\{\sqrt{\|x_1^n-x_1^*\|^2+\cdots+\|x_N^n-x_N^*\|^2}\times\\&\sqrt{\|x_1^{n+1}-x_1^*\|^2+\cdots+\|x_N^{n+1}-x_N^*\|^2}\}+2\alpha_n(\langle f_1(x_2^*)-x_1^*,\ x_1^{n+1}-x_1^*\rangle\\&+\langle f_2(x_3^*)-x_2^*,\ x_2^{n+1}-x_2^*\rangle+\cdots+\langle f_N(x_1^*)-x_N^*,\ x_N^{n+1}-x_N^*\rangle).\end{aligned} \quad (3.30)$$

Set

$$\begin{aligned}a_n&=\|x_1^n-x_1^*\|^2+\|x_2^n-x_2^*\|^2+\cdots+\|x_N^n-x_N^*\|^2,\\ b_n&=2(\langle f_1(x_2^*)-x_1^*,\ x_1^{n+1}-x_1^*\rangle+\cdots+\langle f_N(x_1^*)-x_N^*,\ x_N^{n+1}-x_N^*\rangle).\end{aligned} \quad (3.31)$$

Then, we have the following statements:

(i) From (i), $\{a_n\}$ is bounded sequence.
(ii) From (3.30) $a_{n+1}\leq(1-\alpha_n)^2a_n\vartheta\sqrt{a_n}\sqrt{a_{n+1}}+\alpha_nb_n,\ \forall n\geq 1.$
(iii) From (iii) whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying

$$\lim_{k\longrightarrow\infty}\inf(a_{n_k+1}-a_{n_k})\geq 0, \quad (3.32)$$

it follows that

$$\lim_{k\longrightarrow\infty}\sup b_{n_k}\leq 0.$$

By Lemma 2.6, we have

$$\lim_{n\longrightarrow\infty}(\|x_1^n-x_1^*\|^2+\cdots+\|x_N^n-x_N^*\|^2)=0. \quad (3.33)$$

Hence, we obtain that

$$\lim_{n\longrightarrow\infty}\|x_1^n-x_1^*\|=\lim_{n\longrightarrow\infty}\|x_2^n-x_2^*\|=\cdots=\lim_{n\longrightarrow\infty}\|x_N^n-x_N^*\|=0. \quad (3.34)$$

The proof is completed. □

Theorem 3.3. *Let $Q_i, \Upsilon(Q_i, C), \Omega_i$ and $\Omega_{i,\beta}$ $(i = 1, 2, \cdots, N)$ satisfying the conditions $(C1) - (C2)$ and $F : H \longrightarrow H$ be μ-Lipschitz continuous and r-strongly monotone mapping. Let $\{x_1^n\}, \cdots, \{x_N^n\}$ be the sequences defined by*

$$\begin{aligned} x_1^0, \cdots, x_N^0 &\in H \\ x_1^{n+1} &= (1-\alpha_n)\Omega_{1,\beta}x_1^n + \alpha_n f_1(\Omega_{2,\beta}(x_2^n)), \\ x_2^{n+1} &= (1-\alpha_n)\Omega_{2,\beta}x_2^n + \alpha_n f_2(\Omega_{3,\beta}(x_3^n)), \\ &\vdots \\ x_N^{n+1} &= (1-\alpha_n)\Omega_{N,\beta}x_N^n + \alpha_n f_N(\Omega_{1,\beta}(x_1^n)), \text{ for } n = 0, 1, 2, \cdots \end{aligned} \tag{3.35}$$

where $f_i = I - \eta_i F$ with $\eta_i \in (0, \frac{2r}{\mu})(i = 1, 2, \cdots, N)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_1^n\}, \{x_2^n\} \cdots, \{x_N^n\}$ converges to $x_1^*, x_2^*, \cdots, x_N^*$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $\Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$ such that (2.3) is satisfied.

Proof. It is easy to see that $f_i(i = 1, 2, \cdots, N)$ are contraction mappings and all the conditions in Theorem 3.2 are satisfied. By Theorem 3.2, we have the sequences $\{x_1^n\}, \cdots, \{x_N^n\}$ which converges to $(x_1^*, \cdots, x_N^*) \in \Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$ such that the following are satisfied.

$$\begin{aligned} \langle x_1^* - f_1(x_2^*),\ x_1 - x_1^*\rangle &\geq 0,\ \forall x_1 \in \Upsilon(Q_1, C), \\ \langle x_2^* - f_2(x_3^*),\ x_2 - x_2^*\rangle &\geq 0,\ \forall x_2 \in \Upsilon(Q_2, C), \\ &\vdots \\ \langle x_{N-1}^* - f_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^*\rangle &\geq 0,\ \forall x_{N-1} \in \Upsilon(Q_{N-1}, C), \\ \langle x_N^* - f_N(x_1^*),\ x_N - x_N^*\rangle &\geq 0,\ \forall x_N \in \Upsilon(Q_N, C). \end{aligned} \tag{3.36}$$

Substituting $f_1 = I - \eta_1 F,\ f_2 = I - \eta_2 F, \cdots, f_N = I - \eta_N F$ in (3.36), we obtain that the sequences $\{x_1^n\}, \cdots, \{x_N^n\}$ converges to $(x_1^*, \cdots, x_N^*) \in \Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$ such that (2.3) are hold and proof is completed. □

If setting $Q_i = I - T_i$, where $T_i : H \longrightarrow H$ is a nonexpansive mapping in Theorem 3.2 and Theorem 3.3, Then, Q_i is $\frac{1}{2}$-inverse strongly monotone and $\Upsilon(Q_i, C) = F(T_i)$ $(i = 1, 2, \cdots, N)$. Hence, we obtain the following corollary.

Corollary 3.2. *Let $T_i : H \longrightarrow H$ be a nonexpansive mapping and $Q_i = I - T_i, \Upsilon(Q_i, C), \Omega_i$ and $\Omega_{i,\beta}$ satisfying the conditions $(C1) - (C2)$ $(i = 1, 2, \cdots, N)$. Let $f_i : H \longrightarrow H$ be contraction with a contractive constant $\vartheta_i \in (0, 1)$ for $i = 1, 2, \cdots, N$. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned} x_i^0 \in H, \quad & i = 1, 2, \cdots, N \\ x_1^{n+1} &= (1-\alpha_n)\Omega_{1,\beta}x_1^n + \alpha_n f_1(\Omega_{2,\beta}x_2^n), \\ x_2^{n+1} &= (1-\alpha_n)\Omega_{2,\beta}x_2^n + \alpha_n f_2(\Omega_{3,\beta}x_3^n), \\ &\vdots \\ x_N^{n+1} &= (1-\alpha_n)\Omega_{N,\beta}x_N^n + \alpha_n f_N(\Omega_{1,\beta}x_1^n), \end{aligned} \tag{3.37}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the sequences $\{x_i^n\}$ converge to x_i^* $(i = 1, 2, \cdots, N)$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $F(T_1) \times F(T_2) \times \cdots \times F(T_N)$ such that following are satisfied:

$$\begin{aligned} \langle x_1^* - f_1(x_2^*),\ x_1 - x_1^*\rangle &\geq 0,\ \forall x_1 \in F(T_1), \\ \langle x_2^* - f_2(x_3^*),\ x_2 - x_2^*\rangle &\geq 0,\ \forall x_2 \in F(T_2), \end{aligned}$$

$$\begin{aligned}
&\vdots\\
\langle x_{N-1}^* - f_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^*\rangle &\geq 0,\ \forall x_{N-1} \in F(T_{N-1}),\\
\langle x_N^* - f_N(x_1^*),\ x_N - x_N^*\rangle &\geq 0,\ \forall x_N \in F(T_N). \qquad (3.38)
\end{aligned}$$

Corollary 3.3. *Let $T_i : H \longrightarrow H$ be a nonexpansive mapping and $Q_i = I - T_i$, $\Upsilon(Q_i, C)$, Ω_i and $\Omega_{i,\beta}$ satisfying the conditions $(C1) - (C2)$ $(i = 1, 2, \cdots, N)$. Let $F : H \longrightarrow H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned}
x_i^0 \in H, \qquad & i = 1, 2, \cdots, N\\
x_1^{n+1} &= (1-\alpha_n)\Omega_{1,\beta}x_1^n + \alpha_n f_1(\Omega_{2,\beta}x_2^n),\\
x_2^{n+1} &= (1-\alpha_n)\Omega_{2,\beta}x_2^n + \alpha_n f_2(\Omega_{3,\beta}x_3^n),\\
&\vdots\\
x_N^{n+1} &= (1-\alpha_n)\Omega_{N,\beta}x_N^n + \alpha_n f_N(\Omega_{1,\beta}x_1^n), \qquad (3.39)
\end{aligned}$$

where $f_i = I - \eta_i F$ with $\eta_i \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the sequences $\{x_i^n\}$ converge to x_i^* $(i = 1, 2, \cdots, N)$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $F(T_1) \times F(T_2) \times \cdots \times F(T_N)$ such that the following are satisfied:

$$\begin{aligned}
\langle \eta_1 F(x_2^*) + x_1^* - x_2^*,\ x_1 - x_1^*\rangle &\geq 0,\ \forall\ x_1 \in F(T_1),\\
\langle \eta_2 F(x_3^*) + x_2^* - x_3^*,\ x_2 - x_2^*\rangle &\geq 0,\ \forall\ x_2 \in F(T_2),\\
&\vdots\\
\langle \eta_{N-1} F(x_N^*) + x_{N-1}^* - x_N^*,\ x_{N-1} - x_{N-1}^*\rangle &\geq 0,\ \forall\ x_{N-1} \in F(T_{N-1}),\\
\langle \eta_N F(x_1^*) + x_N^* - x_1^*,\ x_N - x_N^*\rangle &\geq 0,\ \forall\ x_N \in F(T_N). \qquad (3.40)
\end{aligned}$$

Corollary 3.4. *Let C_i be a nonempty closed convex subset of H and $Q_i = I - P_{C_i}$, $\Upsilon(Q_i, C)$, Ω_i and $\Omega_{i,\beta}$ satisfying the conditions $(C1) - (C2)$ $(i = 1, 2, \cdots, N)$. Let $f_i : H \longrightarrow H$ be contraction with a contractive constant $\vartheta_i \in (0,1)$ $(i = 1, 2, \cdots, N)$. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned}
x_i^0 \in H, \qquad & i = 1, 2, \cdots, N\\
x_1^{n+1} &= (1-\alpha_n)\Omega_{1,\beta}x_1^n + \alpha_n f_1(\Omega_{2,\beta}x_2^n),\\
x_2^{n+1} &= (1-\alpha_n)\Omega_{2,\beta}x_2^n + \alpha_n f_2(\Omega_{3,\beta}x_3^n),\\
&\vdots\\
x_N^{n+1} &= (1-\alpha_n)\Omega_{N,\beta}x_N^n + \alpha_n f_N(\Omega_{1,\beta}x_1^n), \qquad (3.41)
\end{aligned}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_i^n\}$ converge to x_i^* $(i = 1, 2, \cdots, N)$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $C_1 \times C_2 \times \cdots \times C_N$ such that the following are satisfied:

$$\begin{aligned}
\langle x_1^* - f_1(x_2^*),\ x_1 - x_1^*\rangle &\geq 0,\ \forall x_1 \in C_1,\\
\langle x_2^* - f_2(x_3^*),\ x_2 - x_2^*\rangle &\geq 0,\ \forall x_2 \in C_2,\\
&\vdots\\
\langle x_{N-1}^* - f_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^*\rangle &\geq 0,\ \forall x_{N-1} \in C_{N-1},\\
\langle x_N^* - f_N(x_1^*),\ x_N - x_N^*\rangle &\geq 0,\ \forall x_N \in C_N. \qquad (3.42)
\end{aligned}$$

Corollary 3.5. *Let C_i be a nonempty closed convex subset of H and $Q_i = I - P_{C_i}$, $\Upsilon(Q_i, C)$, Ω_i and $\Omega_{i,\beta}$ satisfying the conditions $(C1) - (C2)$ $(i = 1, 2, \cdots, N)$. Let $F_i : H \longrightarrow H$ $(i = 1, 2, \cdots, N)$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned} x_i^0 &\in H, \qquad i = 1, 2, \cdots, N \\ x_1^{n+1} &= (1-\alpha_n)\Omega_{1,\beta} x_1^n + \alpha_n f_1(\Omega_{2,\beta} x_2^n), \\ x_2^{n+1} &= (1-\alpha_n)\Omega_{2,\beta} x_2^n + \alpha_n f_2(\Omega_{3,\beta} x_3^n), \\ &\vdots \\ x_N^{n+1} &= (1-\alpha_n)\Omega_{N,\beta} x_N^n + \alpha_n f_N(\Omega_{1,\beta} x_1^n), \end{aligned} \tag{3.43}$$

where $f_i = I - \eta_i F$ with $\eta_i \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the sequences $\{x_i^n\}$ converge to x_i^* $(i = 1, 2, \cdots, N)$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $C_1 \times C_2 \times \cdots \times C_N$ such that the following are satisfied:

$$\begin{aligned} \langle \eta_1 F(x_2^*) + x_1^* - x_2^*,\ x_1 - x_1^* \rangle &\geq 0,\ \forall\ x_1 \in C_1, \\ \langle \eta_2 F(x_3^*) + x_2^* - x_3^*,\ x_2 - x_2^* \rangle &\geq 0,\ \forall\ x_2 \in C_2, \\ &\vdots \\ \langle \eta_{N-1} F(x_N^*) + x_{N-1}^* - x_N^*,\ x_{N-1} - x_{N-1}^* \rangle &\geq 0,\ \forall\ x_{N-1} \in C_{N-1}, \\ \langle \eta_N F(x_1^*) + x_N^* - x_1^*,\ x_N - x_N^* \rangle &\geq 0,\ \forall\ x_N \in C_N. \end{aligned} \tag{3.44}$$

4. Applications

Let $Q_1 = Q_2 = \cdots = Q_N$, $f_1 = \cdots = f_N$ and $x_1^0 = \cdots = x_N^0$ in Theorem 3.2, then we have the following:

Theorem 4.1. *Let Q, $\Upsilon(Q, C)$, Ω and Ω_β satisfy the conditions $(C1) - (C2)$ and $f : H \longrightarrow H$ be a contraction with a contractive constant $\vartheta \in (0, 1)$. Let $\{x_n\}$ be a sequence suggested by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)\Omega_\beta x_n + \alpha_n f(\Omega_\beta x_n),\ n = 0, 1, 2, \cdots, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then a sequence $\{x_n\}$ converges to $x^* \in \Upsilon(Q, C)$ such that the following inequality is satisfied:

$$\langle x^* - f(x^*), x - x^* \rangle \geq 0,\ \forall x \in \Upsilon(Q, C).$$

Theorem 4.2. *Let Q, $\Upsilon(Q, C)$, Ω and Ω_β satisfy the conditions $(C1) - (C2)$ and $F : H \longrightarrow H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}$ be a sequence suggested by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)\Omega_\beta x_n + \alpha_n (I - \eta F)(\Omega_\beta x_n),\ n = 0, 1, 2, \cdots, \end{cases} \tag{4.2}$$

where $\eta \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then a sequence $\{x_n\}$ converges to $x^* \in \Upsilon(Q, C)$ such that the following are satisfied:

$$\langle F(x^*), x - x^* \rangle \geq 0,\ \forall x \in \Upsilon(Q, C).$$

Corollary 4.1. *Let $T : H \longrightarrow H$ be a nonexpansive mapping and $Q = I - T$, $\Upsilon(Q, C)$, Ω and Ω_β satisfy the conditions $(C1) - (C2)$ and $f : H \longrightarrow H$ be contraction with a contractive constant $\vartheta \in (0, 1)$. Let $\{x_n\}$ be a sequence suggested by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)\Omega_\beta x_n + \alpha_n f(\Omega_\beta x_n), \ n = 0, 1, 2, \cdots, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges to $x^* \in F(T)$ such that the following inequality is satisfied:

$$\langle x^* - f(x^*), x - x^* \rangle \geq 0, \ \forall x \in F(T).$$

Corollary 4.2. *Let $T : H \longrightarrow H$ be a nonexpansive mapping and $Q = I - T$, $\Upsilon(Q, C)$, Ω and Ω_β satisfy the conditions $(C1) - (C2)$ and $F : H \longrightarrow H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}$ be a sequence suggested by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)\Omega_\beta x_n + \alpha_n (I - \eta F)(\Omega_\beta x_n), \ n = 0, 1, 2, \cdots, \end{cases} \tag{4.4}$$

where $\eta \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then a sequence $\{x_n\}$ converges to $x^* \in F(T)$ such that the following inequality is satisfied:

$$\langle F(x^*), x - x^* \rangle \geq 0, \ \forall x \in F(T).$$

5. Conclusion

We propose a new class of system of generalized hierarchical variational inequality problems in Hilbert spaces, that seems to be a useful extension of the class of hierarchical variational inequality. Further, we established some fundamental properties belonging to this class. Based on these properties and well-known result due to concepts of Mainge′s, we obtained some existence of the solutions of system of generalized hierarchical variational inequality problems. Also, we established a result, that may be viewed as an applications for system of generalized hierarchical variational inequality problems.

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SHRINKING PROJECTION METHOD WITH ALLOWABLE RANGES

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ABSTRACT. In this note, we introduce an iterative method with allowable ranges which is a revised version of the shrinking projection method. Allowable ranges associated with the method are related to errors caused by a corresponding numerical calculation method.

KEYWORDS: Shrinking projection method, allowable ranges, errors.
AMS Subject Classification: : Primary 47H05, 47H09; Secondary 47J25.

1. Introduction

In 2008, Takahashi, Takeuchi and Kubota [10] introduced an iterative method finding a common fixed point of some families of nonlinear mappings in Hilbert spaces. In 2009, Kimura and Takahashi [5] improved the method. Before them, there were some results in Banach spaces; see [5] and its references. Nevertheless, by reviewing the structure, they improved the method itself. In their direction, we can deal with wider classes of mappings in wider spaces. Typically, we can apply the method to find a common fixed point of a family of mappings of Type P in suitable Banach spaces. This iterative method is called shrinking projection method. Also, in 2014, Kimura [4] considered the method with non–summable errors.

Let C, Q be closed convex subsets of a Banach space with $Q \subset C$. For simplicity, consider a method which generate $v = n_{xt}(w) \in C$ from $w \in C$ theoretically. So, for $x_1 \in C$, we can generate $\{x_n\}$ in theory, where $x_{n+1} = n_{xt}(x_n)$. Then, $\{x_n\}$ is required to converge strongly to a point of Q. Also, consider a corresponding numerical calculation procedure. Let $x_1 = z_1 = y_1$ and generate $z_2 = n_{xt}(y_1) \in C$. By actual restrictions, usually we can only have $y_2 \in C$ which is slightly different from z_2. Generate $z_3 = n_{xt}(y_2)$. In this way, practically, we can only have $\{y_n\}$; $\{z_n\}$ is also in theory. For step n, we call $\|z_n - y_n\|$ error. We may consider that there are $\{b_n\} \subset (0, \infty)$ and $M \in (0, \infty)$ satisfying $\|z_n - y_n\| \le b_n \le M$ for $n \in N$.

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As a matter of course we face a difficulty: We do not know the size of $\|x_n - y_n\|$, that is, we do not know whether $\{y_n\}$ converges. Some researchers studied the cases as below: Assuming $\sum_{n\in N} b_n < \infty$, "Strong convergence of $\{y_n\}$ to a point of Q" is guaranteed. However, errors can satisfy neither $\sum_{n\in N} b_n < \infty$ nor $\lim_n b_n = 0$. Recently, avoiding the conditions, some replacements of "Strong convergence of $\{y_n\}$ to a point of Q" are studied. So, maybe handling of errors is unsatisfactory.

On the other hand, an allowable range A_n for step n is a subset of C associated with such a method. Then, $\{y_n\}$ is required to converge strongly to some $u \in Q$ theoretically if $y_n \in A_n$ for $n \in N$. Suppose, by actual restrictions, we cannot get a point $y_{n_0+1} \in A_{n_0+1}$. Then, our procedure has to be stopped. Nevertheless, for the method, maybe y_{n_0} is a best approximate point of u even if $\|y_{n_0} - u\|$ is unknown.

In this note, motivated by the works as above, we present a shrinking projection method which has an allowable range for each step. In a sense, we give another interpretation of Kimura's idea [4]. To clarify basic structures of our method, we only deal with mappings related to Type P, and present only typical and basic applications. The concept of allowable ranges is not bounded by an iterative method.

2. Preliminaries

For details of this section, consult Takahashi [9] and Aoyama and co–authors [2]. In the sequel, without notice, sometimes we use the facts and symbols below.

N and R denote sets of positive integers and real numbers, respectively. For $k \in N$, N_k denotes $\{j \in N : 1 \le j \le k\}$. E denotes a real Banach space with norm $\|\cdot\|$, and E^* denotes the dual of E. C always denotes a non–empty set; in this note, normally "non–empty" is omitted.

Let E be a Banach space. The normalized duality mapping J is the set valued mapping from E into E^* as below:

$$Jx = \{x^* \in E : \langle x, x^* \rangle = \|x\|\|x^*\|,\ \|x^*\| = \|x\|\} \quad \text{for} \quad x \in E.$$

Let C be a subset of E. Then, C is weakly closed if C is closed and convex. Let T be a mapping from C into E. $F(T)$ denotes $\{x \in C : x = Tx\}$, that is, $F(T)$ is the fixed point set of T. T is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for $x, y \in C$.

In the canonical way, E is embedded in E^{**}; we may consider E as a subset of E^{**}. E is called reflexive if the embedding of E is E^{**}. In this case, we may consider $E = E^{**}$. So, weak topology and weak* topology of E^* are coincide; we only use "weak topology". E is called strictly convex if $\|\cdot\|^2$ is strictly convex, that is, for $x, y \in E$ with $x \ne y$ and $a \in (0,1)$, $\|(1-a)x + ay\|^2 < (1-a)\|x\|^2 + a\|y\|^2$. E is called smooth if $\lim_{t\longrightarrow 0}(\|x+ty\| - \|x\|)/t$ exists for $x, y \in E$ with $\|x\| = \|y\| = 1$. E is said to have the Kadec–Klee property if a sequence $\{x_n\}$ in E converges strongly to $x \in E$ whenever $\{x_n\}$ converges weakly to x and $\{\|x_n\|\}$ converges to $\|x\|$.

Let E be reflexive. Then, any bounded sequence $\{x_n\}$ in E has a weakly convergent subsequence. A sequence $\{x_n\}$ in E converges weakly to $z \in E$ if every weak cluster point of $\{x_n\}$ and z are the same. Let E be a strictly convex reflexive Banach space and let C be a closed convex subset of E. Then, for $x \in E$, there is a unique $z_x \in C$ satisfying $\|x - z_x\| = \inf_{z\in C} \|x - z\|$. Define a mapping P_C from E onto C by $P_C x = z_x$ for $x \in E$. P_C is called the metric projection from E onto C. We know the following: $z = P_C x$ if and only if $z \in C$ and $\inf_{y\in C}\langle y - z, J(z-x)\rangle \ge 0$. So, $\inf_{y\in C}\langle y - P_C x, J(P_C x - x)\rangle \ge 0$ holds.

In this note, we mainly deal with smooth strictly convex reflexive Banach spaces. Let E be such a Banach space. Then, we refer to some basic concepts and facts needed in the sequel; of course, some assertions hold under more weak conditions.

In the setting, Jx is singleton for $x \in E$. So, we can regard J as a mapping from E into E^*. Of course, for $x \in E$, $y^* \in E^*$ and $z^{**} \in E^{**}$, $\langle x, y^* \rangle$ and $\langle x^{**}, y^* \rangle$ denote $y^*(x)$ and $z^{**}(y^*)$. Define a mapping ϕ by $\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for $x, y \in E$. ϕ is called Alber's bi–function [1] from $E \times E$ into R. We denote by ϕ^* Alber's bi–function from $E^* \times E^*$ into R. Let A be a mapping from a subset C of E into E^*. A is called monotone if $\langle x - y, Ax - Ay \rangle \geq 0$ for $x, y \in C$. A is called strictly monotone if A is monotone and $\langle x - y, Ax - Ay \rangle = 0$ implies $x = y$.

In the setting, the following hold:

(1) E^* is smooth, strictly convex and reflexive.
(2) J is a bijection from E onto E^*.
(3) J is norm to weak continuous.
(4) The normalized duality mapping J^* from E^* onto E and J^{-1} are coincide.
(5) For $y \in E$, $\langle \cdot, Jy \rangle$ is continuous and linear.
(6) For $x, y \in E$, the following hold: $\phi(x,y) = \phi^*(Jy, Jx) \geq (\|x\| - \|y\|)^2 \geq 0$, $\langle x - y, Jy \rangle \leq \frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2 \leq \langle x - y, Jx \rangle$, $\langle x - y, Jx - Jy \rangle = \frac{1}{2}\phi(y,x) + \frac{1}{2}\phi(x,y) \geq (\|x\| - \|y\|)^2 \geq 0$.
(7) For $y \in E$, $\phi(\cdot, y)$ is weakly lower semi–continuous and strictly convex.
(8) For $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$.
(9) Suppose a sequence $\{x_n\}$ in E satisfies $\lim_n \langle x_n - y, Jx_n - Jy \rangle = 0$. Then, $\lim_n \|x_n\| = \|y\| = \lim_n \|Jx_n\| = \|Jy\|$, $\lim_n \phi(y, x_n) = \lim_n \phi(x_n, y) = \lim_n \phi^*(Jy, Jx_n) = \lim_n \phi^*(Jx_n, Jy) = 0$.
(10) J is strictly monotone.

We give short explanations of (6)–(10). Fix any $x, y \in E$. By the definitions of J, ϕ and ϕ^*, obviously $\|x\| = \|Jx\|$ and $\phi(x,y) = \phi^*(Jy, Jx)$ hold. Also, we see $\phi(x,y) \geq (\|x\| - \|y\|)^2 \geq 0$ by $-\langle x, Jy \rangle \geq -\|x\|\|y\|$. Then,

$$\langle x - y, Jx \rangle - \tfrac{1}{2}(\|x\|^2 - \|y\|^2) = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \langle y, Jx \rangle = \tfrac{1}{2}\phi(y,x).$$

From these, the following immediately follow: For $x, y \in E$,

$$\langle x-y, Jy \rangle \leq \tfrac{1}{2}\|x\|^2 - \tfrac{1}{2}\|y\|^2 \leq \langle x - y, Jx \rangle,$$
$$\langle x-y, Jx - Jy \rangle = \tfrac{1}{2}\phi(y,x) + \tfrac{1}{2}\phi(x,y) \geq (\|x\| - \|y\|)^2 \geq 0.$$

Since $\|\cdot\|^2$ is weakly lower semi–continuous and strictly convex, by (5), so is $\phi(\cdot, y)$. Suppose $\phi(x,y) = 0$ and $x \neq y$. Then, $0 \leq \phi(\frac{1}{2}(x+y), y) < \frac{1}{2}\phi(x,y) + \frac{1}{2}\phi(y,y) = 0$. So, $\phi(x,y) = 0$ implies $x = y$. We confirmed that (6)–(8) hold. By (8) and the last inequality in (6), we immediately see that (9) and (10) hold.

Some classes of mappings.

Let C be a subset of a smooth strictly convex reflexive Banach space E. We denote by $\mathcal{F}^C_{E^*}$ the class of all mappings from C into E^*, by $\mathcal{F}^C$ the class of all mappings from C into E. Consider the following:

$$\mathcal{M}^C_{E^*} = \{A \in \mathcal{F}^C_{E^*} : A \text{ is norm to weak continuous and monotone}\},$$
$$\mathcal{M}^C_P = \{S \in \mathcal{F}^C : J(I - S) \text{ is norm to weak continuous and monotone}\},$$
$$\mathcal{M}^C_R = \{U \in \mathcal{F}^C : JU \text{ is norm to weak continuous and monotone}\},$$
$$\mathcal{T}^C_P = \{S \in \mathcal{F}^C : \langle Sx - Sy, J(x - Sx) - J(y - Sy) \rangle \geq 0 \quad \text{for} \quad x, y \in C\},$$
$$\mathcal{T}^C_R = \{U \in \mathcal{F}^C : \langle (x - Ux) - (y - Uy), JUx - JUy \rangle \geq 0 \quad \text{for} \quad x, y \in C\}.$$

$\mathcal{T}^C_P$ and $\mathcal{T}^C_R$ are called Type P and Type R, respectively. For details of Type P, Type Q, and Type R, see Aoyama and co–authors [2]. In a Hilbert space, T is called firmly nonexpansive if $\langle (x-y)-(Tx-Ty), Tx-Ty\rangle \geq 0$ for $x, y \in C$. In this setting, these four classes are coincide. However, in our setting, the difference in mathematical properties between Type P and Type Q is not small. In a sense, Type P and Type R are dual each other; see (11). Then, we do not deal with Type Q.

Following [2], by considering (1)–(10), we can confirm (11)–(14) below.

(11) $S \in \mathcal{T}_P^C$ if and only if $U = I - S \in \mathcal{T}_R^C$.
$S \in \mathcal{M}_P^C$ if and only if $U = I - S \in \mathcal{M}_R^C$.

(12) $\mathcal{T}_R^C \subset \mathcal{M}_R^C$; JU is norm to weak continuous and monotone for $U \in \mathcal{T}_R^C$.
$\mathcal{T}_P^C \subset \mathcal{M}_P^C$; $J(I-S)$ is norm to weak continuous and monotone for $S \in \mathcal{T}_P^C$.

Suppose further that E has the Kadec–Klee property. Then, the following holds:

(13) U is continuous if $U \in \mathcal{T}_R^C$, and S is continuous if $S \in \mathcal{T}_P^C$.

Suppose E^* has the Kadec–Klee property. Then, the following holds:

(14) JU is norm to norm continuous if $U \in \mathcal{T}_R^C$.
$J(I-S)$ is norm to norm continuous if $S \in \mathcal{T}_P^C$.

For (11), we only show the following: Let $x, y \in C$, $S \in \mathcal{F}^C$ and $U = I - S$. Then,

$$\langle Sx - Sy, J(I-S)x - J(I-S)y\rangle = \langle (I-U)x - (I-U)y, JUx - JUy\rangle.$$

To reduce the burden of readers, we confirm that (12)–(14) hold.

Let $U \in \mathcal{T}_R^C$. Fix any $x, y \in C$. Since J is monotone, by (6), we see

$$\begin{aligned} &\langle x-y, JUx-JUy\rangle - \langle (x-Ux)-(y-Uy), JUx-JUy\rangle \\ &\quad = \langle Ux-Uy, JUx-JUy\rangle \geq 0. \end{aligned} \tag{2.1}$$

Then, by $U \in \mathcal{T}_R^C$, we see $\langle x-y, JUx-JUy\rangle \geq 0$. So, JU is monotone.

Let $\{x_n\}$ be a sequence in C converging strongly to $u \in C$. For $n \in N$, set $a_n = \|Ux_n\| + \|Uu\| = \|JUx_n\| + \|JUu\|$. By $U \in \mathcal{T}_R^C$, (6) and (2.1), we see

$$\begin{aligned} \|x_n-u\|(\|Ux_n\|+\|Uu\|) &\geq \|x_n-u\|\|JUx_n-JUu\| \geq \langle x_n-u, JUx_n-JUu\rangle \\ &\geq \langle Ux_n-Uu, JUx_n-JUu\rangle \geq (\|Ux_n\|-\|Uu\|)^2 \geq 0. \end{aligned}$$

In the case of $Uu = 0$, we immediately see $\|Ux_n\|^2 \leq \|x_n-u\|\|Ux_n\|$. Then $\{Ux_n\}$ converges strongly to $0 = Uu$ and $\{JUx_n\}$ converges strongly to $0 = JUu$.

In the case of $Uu \neq 0$, by $a_n \geq \|Uu\| > 0$ and the inequality as above, we see

$$\begin{aligned} \|x_n-u\| &\geq \tfrac{1}{a_n}\langle Ux_n-Uu, JUx_n-JUu\rangle \\ &\geq \tfrac{1}{a_n}(\|Ux_n\|-\|Uu\|)^2 = \tfrac{1}{a_n}(\|Ux_n\|+\|Uu\|-2\|Uu\|)^2 \\ &= a_n\left(1-\tfrac{2\|Uu\|}{a_n}\right)^2 \geq \|Uu\|\left(1-\tfrac{2\|Uu\|}{a_n}\right)^2 \geq 0. \end{aligned}$$

So, $\{a_n\}$ must converge to $2\|Uu\| > 0$, and $\lim_n\langle Ux_n-Uu, JUx_n-JUu\rangle = 0$. Then, by (9), we also see $\lim_n\|Ux_n\| = \|Uu\| = \lim_n\|JUx_n\| = \|JUu\|$, and

$$\lim_n \phi(Ux_n, Uu) = \lim_n \phi^*(JUx_n, JUu) = 0.$$

Since $\{Ux_n\}$ is bounded, $\{Ux_n\}$ has a weakly convergent subsequence. Let $\{Ux_{n_j}\}$ be a subsequence of $\{Ux_n\}$ which converges weakly to some $v \in E$. Then, since $\phi(\,\cdot, Uu)$ is weakly lower semi–continuous, we have

$$0 = \lim_j \phi(Ux_{n_j}, Uu) = \liminf_j \phi(Ux_{n_j}, Uu) \geq \phi(v, Uu).$$

Thus $\phi(v, Uu) = 0$, that is, $v = Uu$. From these, any weakly convergent subsequence of $\{Ux_n\}$ converges weakly to Uu. Then, $\{Ux_n\}$ itself converges weakly to Uu. We confirmed that U is norm to weak continuous. By replacing $\{Ux_n\}$, Uu and ϕ by $\{JUx_n\}$, JUu and ϕ^*, we also see that JU is norm to weak continuous.

Suppose further that E has the Kadec–Klee property. By the argument as above, it is immediate that $\{Ux_n\}$ converges strongly to Uu. Suppose E^* has the Kadec–Klee property. Similarly, we see that $\{JUx_n\}$ converges strongly to JUu. Finally, from the argument so far, by (11), we see that (12)–(14) hold.

Let C be a subset of a smooth strictly convex reflexive Banach space E. For $V \in \mathcal{F}^C$, define A^V and D_x^V as below:

$$A^V = J(I - V), \quad D_x^V = \{y \in C : \langle Vx - y, A^V x\rangle \geq 0\} \text{ for } x \in C.$$

For simplicity, we use A and D_x instead of A^V and D_x^V if it causes no confusion.

Let $S \in \mathcal{T}_P^C$. By the definition of $\mathcal{T}_P^C$, for $x \in C$ and $u \in F(S)$, we easily see $0 \leq \langle Sx - Su, J(x - Sx) - J(u - Su)\rangle = \langle Sx - u, J(x - Sx)\rangle$. Then, $F(S) \subset \cap_{x\in C} D_x$. Suppose $z \in \cap_{x\in C} D_x$. Then, $z \in D_z$ and $-\|z - Sz\|^2 = \langle Sz - z, J(z - Sz)\rangle \geq 0$. So, we see $z \in F(S)$, that is, $\cap_{x\in C} D_x \subset F(S)$ holds.

(15) $F(S) = \cap_{x\in C} D_x$ for $S \in \mathcal{T}_P^C$.

In the case that C is closed and convex, so is D_x. By (15), the following follows:

(16) $F(S)$ is closed and convex for $S \in \mathcal{T}_P^C$.

We use the expression $B \in \mathcal{C}^C$ if $B \in \mathcal{F}^C$ is continuous and $\cap_{x\in C} D_x \neq \emptyset$. Let $B \in \mathcal{C}^C$. Then, since J is norm to weak continuous, $A = J(I - B)$ is also norm to weak continuous. Obviously, $v \in \cap_{x\in C} D_x$ implies $v \in F(B)$. However, in general, $u \in F(B)$ does not imply $u \in \cap_{x\in C} D_x$. Suppose further that E has the Kadec–Klee property. In this case, by (13) and (15), $S \in \mathcal{T}_P^C$ and $F(S) \neq \emptyset$ imply $S \in \mathcal{C}^C$. On the other hand, it is easy to find C and $B \in \mathcal{C}^C$ satisfying $B \notin \mathcal{T}_P^C$.

Let $C = [0, 1]$. Consider B such that $Bx = x^2$ for $x \in C$. Then, B is continuous and $F(B) = \{0, 1\}$. We know $0 \in \cap_{x\in C} D_x$; $\langle Bx - 0, Ax\rangle = (x^2 - 0)(x - x^2) \geq 0$ for $x \in C$. So, $B \in \mathcal{C}^C$. However, $\langle By - Bz, Ay - Az\rangle = (\frac{1}{4} - 1)((\frac{1}{2} - \frac{1}{4}) - (1 - 1)) < 0$, where $y = 1/2$ and $z = 1$. We see $B \notin \mathcal{T}_P^C$. Similarly, $B \notin \mathcal{M}_P^C$. Confirm that $F(B)$ is not convex, $F(B) \not\subset \cap_{x\in C} D_x$ ($1 \notin \cap_{x\in C} D_x$), and $A = J(I - B)$ is not monotone.

In later sections, we deal with $B \in \mathcal{C}^C$. Then, we refer to the following: Let $B \in \mathcal{C}^C$. Even if $I - B$ is demiclosed at 0, maybe it plays no important roll to find a fixed point of B. Let $a \in [0, 1)$ and $T = aI + (1 - a)B$. Then,

(17) $\langle Tx - u, J(x - Bx)\rangle \geq 0$ for $x \in C$ and $u \in \cap_{x\in C} D_x \subset F(B)$.

It follows from $\langle x - u, J(x - Bx)\rangle - \langle Bx - u, J(x - Bx)\rangle = \|x - Bx\|^2 \geq 0$.

3. Basic structures

The following lemma is the origin of shrinking projection method; see section 5.

Lemma 3.1. *Let E be a strictly convex reflexive Banach space. Let $x_0 \in E$ and let D be a non–empty closed convex subset of E. Let $\{x_n\}$ be a sequence in E satisfying*

$$\limsup_n \|x_0 - x_n\| \leq \|x_0 - P_D x_0\|. \tag{3.1}$$

Then the following hold:

(1) *Suppose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to $u \in D$.*
Then, $u = P_D x_0$; $\{x_{n_j}\}$ converges weakly to $P_D x_0$.
When E has the Kadec–Klee property, $\{x_{n_j}\}$ converges strongly to $P_D x_0$.

(2) *Suppose every weak cluster point of $\{x_n\}$ is a point of D.*
Then, $\{x_n\}$ converges weakly to $P_D x_0$.
When E has the Kadec–Klee property, $\{x_n\}$ converges strongly to $P_D x_0$.

Remark. *Of course, we can replace (3.1) by the following:*

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \leq \|x_0 - P_D x_0\| \quad \text{for } n \in N. \tag{3.2}$$

Proof. By (3.1), $\{x_n\}$ is bounded, that is, $\{x_n\}$ has a weakly convergent subsequence. Obviously, every subsequence of $\{x_n\}$ satisfies (3.1).

We show (1). Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $u \in D$, that is, $\{x_0 - x_{n_j}\}$ converges weakly to $x_0 - u$. Since $\{x_{n_j}\}$ satisfies (3.1) and $\|\cdot\|$ is weakly lower semi–continuous, by $u \in D$ and (3.1),

$$\|x_0 - u\| \leq \liminf_j \|x_0 - x_{n_j}\| \leq \limsup_j \|x_0 - x_{n_j}\| \leq \|x_0 - P_D x_0\| \leq \|x_0 - u\|.$$

Then, $\|x_0 - u\| = \|x_0 - P_D x_0\| = \lim_j \|x_0 - x_{n_j}\|$. Since $P_D x_0$ is unique, we see $u = P_D x_0$. So, $\{x_{n_j}\}$ converges weakly to $P_D x_0$. Suppose E has the Kadec–Klee property. Then, by the argument as above, we immediately see that $\{x_0 - x_{n_j}\}$ converges strongly to $x_0 - u$. Thus, $\{x_{n_j}\}$ converges strongly to $P_D x_0$.

We show (2). Suppose every weak cluster point of $\{x_n\}$ is a point of D, that is, every weakly convergent subsequence of $\{x_n\}$ converges weakly to a point of D. By (1), every weakly convergent subsequence of $\{x_n\}$ converges weakly to $P_D x_0$. Then, $\{x_n\}$ itself converges weakly to $P_D x_0$. Thus, by (1), we see that (2) holds. □

The following lemma expresses a basic structure of our method; it follows from Lemma 3.1. This lemma is closely connected with Tsukada's lemma [11].

Lemma 3.2. *Let E be a strictly convex reflexive Banach space. Let $x_0 \in E$ and let $\{D_n\}$ be a sequence of closed convex subsets of E satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $D = \cap_n D_n \neq \emptyset$. Let $x_1 = P_{D_1} x_0$. For $n \in N$, define x_{n+1}, K_n and z_n by*

$$x_{n+1} = P_{D_{n+1}} x_0, \quad K_n = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|\}, \quad z_n \in K_n.$$

Then $\{x_n\}$ and $\{z_n\}$ converge weakly to $P_D x_0$. Furthermore, when E has the Kadec–Klee property, $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_D x_0$.

Proof. Since $\emptyset \neq D \subset D_{n+1} \subset D_n$ for $n \in N$ and properties of metric projection, we know that $\{x_n\}$ satisfies (3.2). Then, $\{x_n\}$ has a weakly convergent subsequence. Also, every subsequence of $\{x_n\}$ satisfies (3.2). Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $u \in E$. For $m \in N$, since $\{x_{n_j}\}_{n_j \geq m} \subset D_m$ and D_m is weakly closed, $u \in D_m$ is immediate. So, $u \in D$. We confirmed that every weakly convergent subsequence of $\{x_n\}$ converges weakly to a point of D.

By $K_n \neq \emptyset\, (x_n, x_{n+1} \in K_n)$, such $\{z_n\}$ exists. We easily see that, for $n \in N$,

$$\|x_0 - z_n\| \leq \|x_0 - x_{n+1}\| \leq \|x_0 - z_{n+1}\| \leq \|x_0 - x_{n+2}\| \leq \|x_0 - P_D x_0\|.$$

Then, $\{z_n\}$ also satisfies (3.2). Note $z_k \in K_k \subset D_k \subset D_m$ for $k \geq m$. So, every weakly convergent subsequence of $\{z_n\}$ converges weakly to a point of D.

From these, by Lemma 3.1 (2), we immediately have the desired results. □

The following lemma expresses another basic structure of our method. Recall, in the setting, $A = J(I - V)$, $D_x = \{y \in C : \langle Vx - y, Ax \rangle \geq 0\}$ for $V \in \mathcal{F}^C$, $x \in C$.

Lemma 3.3. *Let E be a smooth strictly convex reflexive Banach space. Let C be a closed convex subset of E and let $B \in \mathcal{C}^C$. Let $\{y_n\}$ be a sequence in C. Set $D_n = \cap_{j \in N_n} D_{y_j}$ for $n \in N$ and $D = \cap_{n \in N} D_n$. Then, the following hold:*

(1) *For $x \in C$, D_x is non–empty, closed and convex.*
(2) *Each D_n and D are non–empty, closed and convex.*
(3) *Suppose $\{y_n\}$ converges strongly to some $v \in D$. Then, $v \in F(B)$.*

Proof. By $B \in \mathcal{C}^C$, we know $\emptyset \neq \cap_{y \in C} D_y \subset D_x$ for $x \in C$. Since C is closed and convex, by the definition of D_x and properties of dual pair, D_x is closed and convex. From these, (1) and (2) are immediate. Note $D = \cap_{n \in N} D_n = \cap_{n \in N} D_{y_n}$.

We show (3). In this setting, J is norm to weak continuous. Then, since B is continuous, $A = J(I - B)$ is also norm to weak continuous. Since $\{y_n\}$ converges strongly to $v \in D$, we see that $\{Ay_n\}$ converges weakly to Av and $\{Ay_n\}$ is bounded. For $n \in N$, by $v \in D = \cap_{n \in N} D_{y_n} \subset D_{y_n}$, we see $0 \le \langle By_n - v, Ay_n \rangle$,

$$\langle By_n - v, Ay_n \rangle = \langle By_n - v, Ay_n - Av \rangle + \langle By_n - v, Av \rangle,$$
$$\langle By_n - v, Ay_n - Av \rangle \le \|By_n - Bv\| \|Ay_n - Av\| + \langle Bv - v, Ay_n - Av \rangle.$$

From these, since $\{y_n\}$ converges strongly to $v \in D$, the following hold:

$$\lim_n \langle By_n - v, Ay_n - Av \rangle = 0, \quad \lim_n \langle By_n - v, Av \rangle = \langle Bv - v, Av \rangle,$$
$$0 \le \lim_n \langle By_n - v, Ay_n \rangle = \langle Bv - v, Av \rangle.$$

By $A = J(I - B)$, we see $0 \le \langle Bv - v, J(v - Bv) \rangle = -\|v - Bv\|^2$. Thus $v \in F(B)$. $\square$

Remark 3.4. In Lemma 3.2, for $\{D_n\}$, we only require that $\{D_n\}$ is a sequence of closed convex subsets of E satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $\emptyset \ne D = \cap_n D_n$. Then, the lemma has no relation with method of generating $\{D_n\}$. Also, any subsequence $\{D_{n_k}\}$ of $\{D_n\}$ satisfies $D_{n_{k+1}} \subset D_{n_k}$ for $k \in N$ and $D = \cap_{k \in N} D_{n_k}$. In Lemma 3.3 (3), for $\{y_n\}$, we only require that $\{y_n\}$ converges strongly to some $v \in D$. So, (3) has no relation with method of generating $\{y_n\}$. The importance of these facts were suggested in Kimura and Takahashi [5]. For example, from these properties of $\{D_n\}$ and $\{y_n\}$, we can present Theorem 4.4. Also, for our method, we can confirm that the difference between to find a fixed point of a mapping and to find a common fixed point of a family of such mappings is so slight.

4. Applications

In this section, we present some strong convergence theorems as typical and basic applications of shrinking projection method with allowable ranges.

Theorem 4.1. *Let E be a smooth strictly convex reflexive Banach space which has the Kadec–Klee property. Let C be a closed convex subset of E and let $B \in \mathcal{C}^C$. Consider an iterative procedure as below: Let $x_0 \in E$, $w_1 \in C$, $D_1 = D_{w_1}$ and $x_1 = P_{D_1} x_0$. Let $A_1 = C \backslash (D_1 \cup \{w_1\})$ and let $y_1 \in A_1$. For $n \in N$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and y_{n+1} by*

$$D_{n+1} = D_n \cap D_{y_n}, \quad x_{n+1} = P_{D_{n+1}} x_0,$$
$$A_{n+1} = \{y \in D_n : \|x_0 - y\| \le \|x_0 - x_{n+1}\|, y \ne y_n\}, \quad y_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. *$A_n \ne \emptyset$ for $n \in N$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0 \in F(B)$, where $D = \cap_{n \in N} D_n$.*
2. *$A_k = \emptyset$ for some $k \in N$; the procedure is stopped. In this case, either $y_{k-1} \in F(B)$ or $w_1 \in F(B)$ holds.*

Proof. Recall Lemma 3.3 (1)–(2). For $x \in C$, by $B \in \mathcal{C}^C$, $\emptyset \ne \cap_{y \in C} D_y \subset D_x$ and D_x is closed and convex. So, C, $\{w_1\}$ and $D_1 = D_{w_1}$ are non–empty closed and convex. Then $x_1 = P_{D_1} x_0$ exists. A_1 may be empty. In the case of $A_1 \ne \emptyset$, we can find $y_1 \in A_1$ and generate D_2, x_2 and A_2; D_2 is nonempty closed and convex. A_2 may be empty. In the case of $A_2 \ne \emptyset$. we can find $y_2 \in A_2$ and continue this process. So, the procedure is stopped when we meet $k \in N$ satisfying $A_k = \emptyset$.

In the case of (1), we can generate sequences $\{D_n\}$, $\{x_n\}$, $\{A_n\}$ and $\{y_n\}$ inductively. By our generating method, $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $D = \cap_n D_n \ne \emptyset$. For $n \in N$, let

K_n be as in Lemma 3.2, that is, $K_n = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|\}$. Then, $y_{n+1} \in A_{n+1} \subset K_n$ for $n \in N$. By Lemma 3.2, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0$. Since $\{y_n\}$ converges strongly to $P_D x_0$, by Lemma 3.3 (3), we see $P_D x_0 \in F(B)$. These complete the proof of (1).

We show (2). Assume $A_1 = C \backslash (D_1 \cup \{w_1\}) = \emptyset$. We know that $\{w_1\}$ and D_1 are non–empty closed and convex. Then, since C is connected, by $C = D_1 \cup \{w_1\}$, we see $w_1 \in D_{w_1}$; $-\|Bw_1 - w_1\|^2 = \langle Bw_1 - w_1, Aw_1 \rangle \geq 0$. Thus, $w_1 \in F(B)$. Suppose we generated D_{k+1}, x_{k+1} and $A_{k+1} = \emptyset$ for some $k \in N$. Then, D_k and D_{k+1} are non–empty closed and convex. By $K_k = \{y \in D_k : \|x_0 - y\| \leq \|x_0 - x_{k+1}\|\}$, we see $x_k, x_{k+1} \in K_k$ and $K_k \neq \emptyset$. By $\emptyset = A_{k+1} = K_k \backslash \{y_k\}$, we see that $y_k \in K_k$ and K_k is singleton. From these, $y_k = x_k = x_{k+1}$ holds. So, by $y_k = x_{k+1} \in D_{k+1} \subset D_{y_k}$, we see $-\|By_k - y_k\|^2 = \langle By_k - y_k, Ay_k \rangle \geq 0$. Thus, $y_k \in F(B)$. □

Remark 4.2. In the setting of Theorem 4.1, neither $F(B) \subset D$ nor the convexity of $F(B)$ are guaranteed. Suppose $B \in \mathcal{T}_P^C$ and $F(B) \neq \emptyset$. Then, by section 2 (15)–(16), $F(B) = \cap_{x \in C} D_x \subset D$, and $F(B)$ is closed and convex. In this case, we can see that $P_D x_0 \in F(B)$ implies $P_D x_0 = P_{F(B)} x_0$. Refer to section 2 (17) and Remark 3.4. So, we can apply the method to have a conventional expression of Theorem 4.1 and to find a common fixed point of some families of mappings.

Consider a corresponding numerical calculation procedure. By considering actual restrictions, we should think that the boundary of D_n and the position of $P_{D_n} x_0$ are obscure. So, $P_{D_n} x_0$ exists only in theory; we cannot generate D_{n+1} from $P_{D_n} x_0$ practically. Then, we considered an allowable range A_n for each $n \in N$. The boundary of A_n is also obscure. However, we may get a practical $y_n \in A_n$ if A_n has a certain size; we can generate D_{n+1} from y_n even if its boundary is obscure.

Recall section 1. We repeat the following: Consider a method finding a point of $Q \subset C$. Let $b_n \in (0, \infty)$ for $n \in N$. It is strange to assume either $\sum_{n \in N} b_n < \infty$ or $\lim_n b_n = 0$ if we regard b_n as an upper bound of error for $n \in N$. Because errors cannot satisfy such conditions. For example, suppose we can consider an allowable range A_n for $n \in N$ such that $\|y - z_n\| \leq b_n$ for $y \in A_n$, where z_n is our target. In this case, $\{b_n\}$ may satisfy either $\sum_{n \in N} b_n < \infty$ or $\lim_n b_n = 0$. However, maybe the condition of $\{b_n\}$ is related to effectiveness of the method. The procedure has to be stopped when we cannot have $y_{n_0+1} \in A_{n_0+1}$. Even if $\{y_n\}$ converges strongly to $u \in Q$ theoretically, by actual restrictions, it may be stopped at some step.

The following is a variant of Ibaraki and Kimura's theorem [3]. They considered b_n as an upper bound of error; they mainly studied the case of $\limsup_n b_n < \infty$. With respect to their method, we select the allowable range A_n for $n \in N$.

Theorem 4.3. *Let E be a smooth strictly convex reflexive Banach space which has the Kadec–Klee property. Let C be a closed convex subset of E and let $B \in \mathcal{C}^C$. Let $\{b_n\}$ be a sequence in $(0, \infty)$ satisfying $\lim_n b_n = 0$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = C$ and $x_1 = P_{D_1} x_0$. Let $A_1 = C$ and let $y_1 \in A_1$. For $n \in N$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and y_{n+1} by*

$$D_{n+1} = D_n \cap D_{y_n}, \quad x_{n+1} = P_{D_{n+1}} x_0,$$
$$A_{n+1} = \{y \in D_{n+1} : \|x_{n+1} - y\| \leq b_n\}, \quad y_{n+1} \in A_{n+1}.$$

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0 \in F(B)$, where $D = \cap_{n \in N} D_n$.

Proof. Recall Lemma 3.3 (1)–(2). By $B \in \mathcal{C}^C$, inductively, we can generate sequences $\{D_n\}$, $\{x_n\}$, $\{A_n\}$ and $\{y_n\}$. By our generating method, $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $D = \cap_n D_n \neq \emptyset$.

Note $\lim_n \|x_{n+1} - y_{n+1}\| \le \lim_n b_n = 0$. Then, by Lemma 3.2, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0$. Since $\{y_n\}$ converges strongly to $P_D x_0 \in D$, by Lemma 3.3 (3), we see $P_D x_0 \in F(B)$. This completes the proof. □

By taking notice of Remarks 3.4, we can combine Theorems 4.1 and 4.3.

Theorem 4.4. *Let E be a smooth strictly convex reflexive Banach space which has the Kadec–Klee property. Let C be a closed convex subset of E and let $B \in \mathcal{C}^C$. Let $\{b_n\}$ be a sequence in $(0,\infty)$ satisfying $\lim_n b_n = 0$. Consider an iterative procedure as below: Let $x_0 \in E$, $w_1 \in C$, $D_1 = D_{w_1}$ and $x_1 = P_{D_1} x_0$. Let $A_1 = C \backslash (D_1 \cup \{w_1\})$ and let $y_1 \in A_1$. For $n \in N$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and y_{n+1} by*

$$\begin{aligned}
&D_{n+1} = D_n \cap D_{y_n}, \quad x_{n+1} = P_{D_{n+1}} x_0,\\
&A^1_{n+1} = \{y \in D_{n+1} : \|x_{n+1} - y\| \le b_n\},\\
&A^2_{n+1} = \{y \in D_n : \|x_0 - y\| \le \|x_0 - x_{n+1}\|, y \ne y_n\},\\
&A_{n+1} = A^1_{n+1} \cup A^2_{n+1}, \quad y_{n+1} \in A_{n+1}.
\end{aligned}$$

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0 \in F(B)$, where $D = \cap_{n\in N} D_n$.
Remark. *We may ignore the case of $A_1 = \emptyset$ because $w_1 \in F(B)$ if $A_1 = \emptyset$.*

Proof. Recall Lemma 3.3 (1)–(2). By $B \in \mathcal{C}^C$, inductively, we can generate sequences $\{D_n\}$, $\{x_n\}$, $\{A_n\}$ and $\{y_n\}$. By our generating method, $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $D = \cap_n D_n \ne \emptyset$.

By Lemma 3.2, $\{x_n\}$ converges strongly to $P_D x_0$. By collecting $n \in N$ satisfying $y_{n+1} \in A^1_{n+1}$, we have a subsequence $\{y_{n_i+1}\}$ of $\{y_n\}$. By collecting $n \in N$ satisfying $y_{n+1} \in A^2_{n+1} \backslash A^1_{n+1}$, we have another subsequence $\{y_{n_j+1}\}$ of $\{y_n\}$. Then we easily see that $N = \{n_i\} \cup \{n_j\}$, and either $\{n_i\}$ or $\{n_j\}$ has infinite terms.

Suppose $\{n_i\}$ has infinite terms. Then, since $\{x_{n_i+1}\}$ converges strongly to $P_D x_0$, by $\lim_i b_{n_i} = 0$, $\{y_{n_i+1}\}$ also converges strongly to $P_D x_0$.

Suppose $\{n_j\}$ has infinite terms. Then, $\{D_{n_j}\}$ is a sequence of closed convex subsets of C satisfying $D_{n_{j+1}} \subset D_{n_j}$ for $j \in N$ and $D = \cap_{j\in N} D_{n_j} \ne \emptyset$. For $j \in N$, we know $D_{n_{j+1}} \subset D_{n_j+1} \subset D_{n_j}$, and $y_{n_j+1} \in A^2_{n_j+1}$; $y_{n_j+1} \in D_{n_j}$. Then, we see $y_{n_j+1} \in K'_{n_j} = \{y \in D_{n_j} : \|x_0 - y\| \le \|x_0 - x_{n_{j+1}}\|\}$ because

$$\|x_0 - y_{n_j+1}\| \le \|x_0 - x_{n_j+1}\| \le \|x_0 - x_{n_{j+1}}\| \quad \text{for} \quad j \in N.$$

By Lemma 3.2, $\{y_{n_j+1}\}$ converges strongly to $P_D x_0$.

From these, we see that $\{y_n\}$ converges strongly to $P_D x_0 \in D$. By considering Lemma 3.3 (3), this implies that $\{y_n\}$ converges strongly to $P_D x_0 \in F(B)$. □

5. Additional explanations

In this section, by taking account of historical viewpoints, we give a summary of shrinking projection method to support the main issue. For simplicity, let C be a closed convex subset of a real Hilbert space H. Of course, H is a smooth strictly convex reflexive Banach space having the Kadec–Klee property. In this setting, $S \in \mathcal{T}^C_P$ is nonexpansive, however, the reverse is not always true. Nevertheless, for a nonexpansive mapping $S' \in \mathcal{F}^C$, $S = \frac{1}{2}(I + S') \in \mathcal{T}^C_P$ and $F(S) = F(S')$ hold.

Let S be a mapping from C into H such that $F(S)$ is non–empty closed and convex. Let $x_0 \in H$, $w = P_{F(S)} x_0$ and $\{u_n\}$ be a sequence in C. Here we present a sufficient condition to guarantee that $\{u_n\}$ converges strongly to $w = P_{F(S)} x_0$.

(∗) Suppose $\limsup_n \|x_0 - u_n\| \le \|x_0 - w\|$ and $\lim_n \|Su_n - u_n\| = 0$.
Suppose further that $I - S$ is demiclosed at 0.
Then, $\{u_n\}$ converges strongly to $w = P_{F(S)} x_0$.

Takahashi, Takeuchi and Kubota focused on (∗). Correctly, they studied properties of Browder's sequence and reach (∗). In a sense, Browder's fixed point theorem and the prototype of shrinking projection method are related through the fact that $I - S$ is demiclosed at 0. Even if $\limsup_n \|x_0 - u_n\| \leq \|x_0 - u\|$, $\{\|x_0 - u_n\|\}$ need not be non–decreasing. Nevertheless, to simplify their assignment, they placed importance on the case that $\{\|x_0 - u_n\|\}$ is non–decreasing.

From now on, C denotes a closed convex subset of H and S denotes a nonexpansive mapping from C into H which satisfies $F(S) \neq \emptyset$. Also $\{a_n\}$ denotes a sequence in $(0, 1)$ satisfying $\lim_n a_n = 0$. Then, $F(S)$ is closed and convex. Also $I - S$ is demiclosed at 0: $u \in F(S)$ holds if there is a sequence $\{u_n\}$ in C such that $\{u_n\}$ converges weakly to $u \in C$ and $\{\|Su_n - u_n\|\}$ converges to 0.

For reference, we show some typical properties of a Browder's sequence. For our purpose, we have to assume that $x_0 \in C$ and S is a self–mapping on C. In advance, we confirm the following: Suppose $x_0 \neq w = P_{F(S)}x_0$ and there is $k \in N$ satisfying $a_k < a_{k+1}$. For $n \in N$, let $u_n = a_n x_0 + (1 - a_n)w \in C$. Then, for $n \in N$,

$$\|x_0 - u_n\| = \|x_0 - (a_n x_0 + (1 - a_n)w)\| = (1 - a_n)\|x_0 - w\| < \|x_0 - w\|.$$

So, for $\{u_n\}$, we see $\limsup_n \|x_0 - u_n\| \leq \|x_0 - w\|$ and $\|x_0 - u_{k+1}\| < \|x_0 - u_k\|$.

For $n \in N$, let S_n be the contraction on C defined by $S_n x = a_n x_0 + (1 - a_n)Sx$ for $x \in C$ and let $x_n \in C$ be the unique fixed point of S_n. So, we call $\{x_n\}$ a Browder's sequence. Fix any $u \in F(S)$ and $n \in N$. Then,

$$\begin{aligned}\|x_n - u\|^2 &= \langle a_n(x_0 - u) + (1 - a_n)(Sx_n - Su), x_n - u\rangle \\ &= a_n\langle x_0 - u, x_n - u\rangle + (1 - a_n)\langle Sx_n - Su, x_n - u\rangle \\ &\leq a_n\langle x_0 - u, x_n - u\rangle + (1 - a_n)\|x_n - u\|^2.\end{aligned}$$

By $a_n > 0$, we see $\langle x_0 - u, x_n - u\rangle \geq \|x_n - u\|^2 \geq 0$. Also, we see

$$a_n\langle x_0 - u, x_n - u\rangle + (1 - a_n)\|x_n - u\|^2 = a_n\langle x_0 - x_n, x_n - u\rangle + \|x_n - u\|^2.$$

So, very interesting inequalities $\langle x_0 - x_n, x_n - u\rangle \geq 0$ and $\langle x_0 - u, x_n - u\rangle \geq 0$ hold.

Set $C_{x_n} = \{y \in C : \langle x_0 - x_n, x_n - y\rangle \geq 0\}$. By $\langle x_0 - x_n, x_n - u\rangle \geq 0$, we see $u \in C_{x_n}$. Then C_{x_n} is closed and convex, and $P_{C_{x_n}} x_0 = x_n \in C_{x_n}$. Set $D = \cap_n C_{x_n}$. So, we confirmed $F(S) \subset D \subset C_{x_n}$. For $y \in C_{x_n}$, by $P_{C_{x_n}} x_0 = x_n$, we know

$$\|x_n - y\|^2 \leq \|x_0 - y\|^2 - \|x_0 - x_n\|^2, \quad \text{that is,} \quad \|x_0 - x_n\| \leq \|x_0 - y\|.$$

Also, set $v = P_D x_0$ and $w = P_{F(S)}x_0$. From these, $\{x_n\}$ satisfies the following:

$$F(S) \subset D \subset C_{x_n}, \quad \|x_0 - x_n\| \leq \|x_0 - v\| \leq \|x_0 - w\| \quad \text{for} \quad n \in N. \tag{5.1}$$

Of course, $\limsup_n \|x_0 - x_n\| \leq \|x_0 - w\|$ follows. At present, we do not know whether there is $k > m$ satisfying $x_k \in C_{x_m}$ for $m \in N$. Also, we do not know whether $\{\|x_0 - x_n\|\}$ has a non–decreasing subsequence. Nevertheless, by (5.1), we can have a result which contains the assertion of Browder's fixed point theorem.

Since S is nonexpansive and $F(S) \neq \emptyset$, by (5.1), $\{x_n\}$ and $\{Sx_n\}$ are bounded. Then, $\{x_n\}$ has a weakly convergent subsequence. By $\lim_n a_n = 0$ and $x_n = a_n x_0 + (1 - a_n)Sx_n$, we see $\lim_n \|x_n - Sx_n\| = \lim_n a_n\|x_0 - Sx_n\| = 0$.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $z \in C$. By $\lim_n \|x_n - Sx_n\| = 0$, we see $z \in F(S)$. Obviously, $\{x_0 - x_{n_j}\}$ converges weakly to $x_0 - z$. By $w = P_{F(S)}x_0$, $z \in F(S)$ and (5.1), we have

$$\begin{aligned}\|x_0 - z\| &\leq \liminf_j \|x_0 - x_{n_j}\| \\ &\leq \limsup_j \|x_0 - x_{n_j}\| \leq \|x_0 - v\| \leq \|x_0 - w\| \leq \|x_0 - z\|.\end{aligned}$$

Then, $\lim_j \|x_0 - x_{n_j}\| = \|x_0 - z\| = \|x_0 - v\| = \|x_0 - w\|$.

Since H has the Kadec–Klee property, $\{x_0 - x_{n_j}\}$ converges strongly to $x_0 - z$, that is, $\{x_{n_j}\}$ converges strongly to z. Since $v = P_D x_0$ is unique, by $z, w \in D$, $z = w = v$ holds. So, $\{x_{n_j}\}$ converges strongly to $P_{F(S)} x_0 = P_D x_0$. Since $\{x_{n_j}\}$ is arbitrary, $\{x_n\}$ converges strongly to $P_{F(S)} x_0 = P_D x_0$.

Takahashi, Takeuchi and Kubota considered as below: For such S, maybe we can choose a non–empty closed convex subset D of C with $F(S) \subset D$ and a sequence $\{u_n\}$ which satisfy $\|x_0 - u_n\| \le \|x_0 - u_{n+1}\| \le \|x_0 - P_D x_0\|$ for $n \in N$. Then, maybe $\{u_n\}$ converges strongly to $P_{F(S)} x_0$ if we choose suitable D and $\{u_n\}$.

Details of their way of thinking is presented below. For $\{x_n\}$ as above, $\{C_{x_n}\}$ is a sequence of non–empty closed convex subsets of C satisfying $F(S) \subset D = \cap_{n\in N} C_{x_n}$. By setting $D_n = \cap_{i=1}^n C_{x_i}$ and $u_n = P_{D_n} x_0$, we have $\{u_n\}$ satisfying

$$\|x_0 - u_n\| \le \|x_0 - u_{n+1}\| \le \|x_0 - P_D x_0\| \le \|x_0 - P_{F(S)} x_0\| \quad \text{for} \quad n \in N. \tag{5.2}$$

By $u_n \in D_n$, (5.2) and uniqueness of $P_D x_0$, $\{u_n\}$ converges strongly to $P_D x_0$. By $P_{F(S)} x_0 = P_D x_0$, $\{u_n\}$ has to converge strongly to $P_{F(S)} x_0 = P_D x_0$.

This is the prototype of shrinking projection method. Also, as an abstract of the argument, they considered Lemma 3.1 which is the origin of their method. Here, we must refer to a lemma in Martinez–Yanes and Xu [6]. In a Hilbert space, they are almost the same. The author did not know about it until he reads Saejung [8].

For the prototype, $\{D_n\}$ is made from $\{C_{x_n}\}$. Maybe there are some generating methods of $\{D_n\}$ such that $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$, and $F(S) \subset D = \cap_n D_n$. For such $\{D_n\}$, $\{u_n\} = \{P_{D_n} x_0\}$ converges strongly to $P_D x_0$. Then, there remain to find a suitable $\{D_n\}$ and to confirm $P_D x_0 \in F(S)$. From this viewpoint, their method appeared.

So far, to prove $P_D x_0 = P_{F(S)} x_0 \in F(S)$, we used the facts that $I - S$ is demiclosed at 0 and $F(S)$ is closed and convex. However, observing proofs in Kimura and Takahashi [5] and Ibaraki and Kimura [3], we notice the following: To prove $P_D x_0 \in F(S)$, we need not know whether these hold if we can choose suitable $\{D_n\}$. So, we use neither of them to show $P_D x_0 \in F(S)$ in the argument below.

Let C be a closed convex subset of H and let S be a nonexpansive mapping from C into H with $F(S) \neq \emptyset$. Let $x_0 \in H$. Following Nakajo and Takahashi [7], they took notice of the following equality: For $x, y, z \in H$,

$$\|y - z\|^2 + 2\langle y - x, z - y\rangle = \|z - x\|^2 - \|y - x\|^2.$$

Let $x, y, z \in H$. By $y - z = y - x + x - z$, we easily have the equality by

$$\begin{aligned}\|y - z\|^2 &= \|y - x\|^2 + \|x - z\|^2 + 2\langle y - x, (x - y) + (y - z)\rangle \\ &= \|z - x\|^2 - \|y - x\|^2 - 2\langle y - x, z - y\rangle.\end{aligned}$$

For $y, z \in C$, define a function $f_{y,z}$ from C into R by

$$f_{y,z}(x) = \|y - z\|^2 + 2\langle y - x, z - y\rangle = \|z - x\|^2 - \|y - x\|^2 \quad \text{for} \quad x \in C.$$

It follows from only properties of inner product that $f_{y,Sy}$ is continuous and convex. Set $L_y = \{x \in C : f_{y,Sy}(x) \le 0\}$. Then, L_y is a closed convex subset of C.

For $y \in C$ and $u \in F(S)$, we see the following:

$$f_{y,Sy}(u) = \|Sy - u\|^2 - \|y - u\|^2 \le 0, \quad \text{that is,} \quad u \in L_y.$$

According to [10], we may replace $f_{y,Sy}$ by $f_{y,ay+(1-a)Sy}$, where $a \in [0, 1)$.

Let $D_1 = C$ and $u_1 = P_{D_1} x_0$. Inductively, generate D_{n+1} and u_{n+1} as below: $D_{n+1} = D_n \cap L_{u_n}$ and $u_{n+1} = P_{D_{n+1}} x_0$ for $n \in N$. Set $D = \cap_{n\in N} D_n$. Then, $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$, and $\emptyset \neq F(S) \subset D = \cap_n D_n$. So, $\{u_n\} = \{P_{D_n} x_0\}$ converges strongly to $P_D x_0$.

At present, we know that $\{u_n\}$ converges strongly to $v = P_D x_0 \in D$ and S is continuous. Then, by $u_{n+1} \in D_{n+1} = D_n \cap L_{u_n} \subset L_{u_n}$ for $n \in N$, we see

$$\|Sv - v\|^2 = \lim_n \|Su_n - u_{n+1}\|^2 \leq \lim_n \|u_n - u_{n+1}\|^2 = 0, \text{ that is, } v \in F(S).$$

In this argument, the continuity of S and the fact $D = \cap_{n\in N} D_n \neq \emptyset$ play important rolls. Referring $f_{y,z}(x) = \|y - z\|^2 + 2\langle y - x, z - y\rangle$, consider $h_{y,z}$ such that

$$h_{y,z}(x) = \|y - z\|^2 + \langle y - x, z - y\rangle = \langle x - z, y - z\rangle \quad \text{for} \quad x \in C.$$

Let B be a continuous mapping from C into R. For $y \in C$ and $z \in H$, let $h_{y,z}(x) = \langle x - z, y - z\rangle$ for $x \in C$ and let $D_y = \{x \in C : h_{y,By}(x) \leq 0\}$. Then, D_y is closed and convex. Assume $\cap_{y\in C} D_y \neq \emptyset$. Let $D_1 = C$ and $u_1 = P_{D_1} x_0$. Inductively, generate D_{n+1} and u_{n+1} by $D_{n+1} = D_n \cap D_{u_n} \supset \cap_{y\in C} D_y \neq \emptyset$ and $u_{n+1} = P_{D_{n+1}} x_0$ for $n \in N$. Set $D = \cap_{n\in N} D_n \supset \cap_{y\in C} D_y \neq \emptyset$. In this setting, we know that $\{u_n\}$ converges strongly to $v = P_D x_0$. For $n \in N$, by $v = P_D x_0 \in D \subset D_{u_n}$, we see

$$\begin{aligned} 0 &\geq \langle v - Bu_n, u_n - Bu_n\rangle = \langle v - Bv + Bv - Bu_n, u_n - Bu_n\rangle \\ &= \langle v - Bv, u_n - Bu_n\rangle - \|Bv - Bu_n\|\|u_n - Bu_n\|. \end{aligned}$$

So, since $\{u_n\}$ converges strongly to $v = P_D x_0$ and B is continuous, we easily see $0 \geq \lim_n \langle v - Bv, u_n - Bu_n\rangle = \|v - Bv\|^2$, that is, $v \in F(B)$.

Essentially, this is a revised version of shrinking projection method which is used in this note. For a nonexpansive mapping $S \in \mathcal{F}^C$ with $F(S) \neq \emptyset$, set $B = \frac{1}{2}(I + S)$. Then, by $B \in \mathcal{T}_P^C$, B is continuous and $\emptyset \neq F(S) = F(B) = \cap_{y\in C} D_y$.

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A VISCOSITY NONLINEAR MIDPOINT ALGORITHM FOR NONEXPANSIVE SEMIGROUP

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ABSTRACT. In this paper, we propose a viscosity nonlinear midpoint algorithm (VNMA) for finding a solution of fixed point problem for a nonexpansive semigroup in real Hilbert spaces. Under certain conditions control on parameters, the iteration sequences generated by the proposed algorithm are proved to be strongly convergent to a solution of fixed point problem for a nonexpansive semigroup. Some numerical examples are presented to illustrate the convergence result. Our results improve and extend the corresponding results in the literature.

KEYWORDS:Nonexpansive semigroup, Equilibrium problem, Midpoint method, Strongly positive linear bounded operator, Fixed point, Hilbert space.
AMS Subject Classification: Primary: 47H09, 47H10; Secondary: 47J20.

1. Introduction

The explicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, we refer to [2, 3, 4, 7, 8, 14, 17, 16] and the references cited therein. For instance, consider the initial value problem for the differential equation $y^{'}(t) = f(y(t))$ with the initial condition $y(0) = y_0$, where f is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}^d$. The explicit midpoint rule which generates a sequence $\{y_n\}$ by following the recurrence relation

$$\frac{1}{h}(y_{n+1} - y_n) = f(\frac{y_{n+1} - y_n}{2}).$$

In 2015, Xu et al. [19] extended and generalized the results of Alghamdi et al. [1] and applied the viscosity method on the midpoint rule for nonexpansive mappings and they give the generalized viscosity explicit method:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}).$$

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In 2016, Rizvi [13] introduced the following iterative method for the explicit midpoint rule of nonexpansive mappings:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B)T(\frac{x_n + x_{n+1}}{2}).$$

Motivated and inspired by the results mentioned and related literature in [1, 13, 19], we propose an iterative midpoint algorithm based on the viscosity method for finding a common element of the set of solutions of nonexpansive semigroup in Hilbert spaces. Then we prove strong convergence theorems that extend and improve the corresponding results of Rizvi [13], Xu [19], and others. Finally, we give examples and numerical result to illustrate our main result.

The rest of paper is organized as follows. The next section presents some preliminary results. Section 3 is devoted to introduce midpoint algorithm for solving it. The last section presents a numerical example to demonstrate the proposed algorithms.

2. Preliminaries

Let $\mathbb{R}$ denote the set of all real numbers, H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and C be a nonempty closed convex subset of H.
A mapping $T : C \to C$ is said to be a contraction if there exists a constant $\alpha \in (0, 1)$ such that $\|T(x) - T(y)\| \leq \alpha \|x - y\|$, for all $x, y \in C$. If $\alpha = 1$ then T is called nonexpansive on C.
The fixed point problem (FPP) for a nonexpansive mapping T is: To find $x \in C$ such that $x \in \text{Fix}(T)$, where $\text{Fix}(T)$ is the fixed point set of the nonexpansive mapping T.
In 2006, Marino and Xu [11] considered the following iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)T x_n$$

with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and prove that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality $\langle (B - \gamma f)z, x - z \rangle \geq 0$, $\forall x \in \text{Fix}(T)$ which is the optimality condition for minimization problem

$$\min_{x \in \text{Fix}(T)} \frac{1}{2} \langle Bx, x \rangle - h(x)$$

where h is the potential function for γf and $B : H \to H$ is a strongly positive linear bounded operator, i.e., if there exists a constant $\bar{\gamma} > 0$ such that $\langle Bx, x \rangle \geq \bar{\gamma} \|x\|^2$, $\forall x \in \text{Fix}(T)$.
A family $S := \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$
(iv) For all $x \in C, s \to T(s)x$ is continuous.

Chen and Song [6] introduced and studied the following iterative method to prove a strong convergence theorem for FPP in a real Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) \tfrac{1}{s_n} \int_0^{s_n} T(s) x_n ds.$$

where f is a contraction mapping. For each point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C

is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C y \rangle \leq 0 \tag{2.1}$$

Further, it is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leq (\frac{1}{2}) \|(T(x) - x) - (T(y) - y)\|^2. \tag{2.2}$$

It is also known that H satisfies Opial's condition [12], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.3}$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. [5] *The following inequality holds in real space H:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \qquad \forall x, y \in H.$$

Definition 2.2. A mapping $M : C \to H$ is said to be monotone, if

$$\langle Mx - My, x - y \rangle \geq 0, \qquad \forall x, y \in C.$$

M is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Mx - My, x - y \rangle \geq \alpha \|Mx - My\|^2, \qquad \forall x, y \in C.$$

Lemma 2.3. [11] *Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.4. [15] *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $S := \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, for each $x \in C$ and $t > 0$. Then, for any $0 \leq h < \infty$,*

$$\lim_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)x ds - T(h)(\frac{1}{t}\int_0^t T(s)x ds)\| = 0.$$

Lemma 2.5. [18] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n$, $n \geq 0$ where α_n is a sequence in $(0, 1)$ and δ_n is a sequence in $\mathbb{R}$ such that*

(i) $\Sigma_{n=1}^{\infty} \alpha_n = \infty$; *(ii)* $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\Sigma_{n=1}^{\infty} \delta_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. **Viscosity Nonlinear Midpoint Algorithm**

In this section, we prove a strong convergence theorem based on the explicit iterative for fixed point of nonexpansive semigroup. We firstly present the following unified algorithm.
Let C be a nonempty closed convex subset of real Hilbert space H. Let $S = \{T(s) : s \in [0, +\infty)\}$ be a nonexpansive semigroup on C such that $\mathrm{Fix}(S) \neq \emptyset$. Also $f : C \to H$ be a α-contraction mapping and A be a strongly positive bounded linear self adjoint operator on H with coefficient $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$ and $\bar{\gamma} \leq \|A\| \leq 1$.

Algorithm 3.1. For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds. \quad (3.1)$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{s_n\} \subset [s,\infty)$ with $s > 0$ satisfying conditions:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0,\ \Sigma_{n=1}^{\infty}\alpha_n = \infty;$

$(C2)$ $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty \quad or \quad \lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1;$

$(C3)$ $\lim_{n\to\infty} s_n = \infty,\ \sup_{n\in\mathbb{N}} |s_{n+1} - s_n|$ *is bounded.*

In the next remark, we observe that the iterative Algorithm 3.1 is well defined for all n.

Remark 3.2. For all $t \in (0, \|A\|^{-1})$ and $u \in C$ fixed, the mapping

$$x \mapsto V_t x := t\gamma f(u) + (1 - tA)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{u + x}{2})ds$$

is a contraction with coefficient $\frac{1}{2}(1 - t\overline{\gamma}) \in (0,1)$. This is immediately clear, due to the nonexpansivity semigroup of $S = \{T(s) : s \in [0,+\infty)\}$ and the inequality (2.3). In fact, we have, for all $x, y \in H$,

$$\begin{aligned}
\|V_t x - V_t y\| &= \|t\gamma f(u) + (1 - tA)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\tfrac{u+x}{2})ds - t\gamma f(u) - (1 - tA)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{u+y}{2})ds\| \\
&\le (1 - t\overline{\gamma})\tfrac{1}{s_n}\textstyle\int_0^{s_n} \|T(s)(\tfrac{u+x}{2})ds - T(s)(\tfrac{u+y}{2})ds\| \\
&\le \tfrac{1}{2}(1 - t\overline{\gamma})\|x - y\|.
\end{aligned}$$

Hence the Algorithm 3.1 is well defined. Moreover, V_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = t\gamma f(u) + (1 - tA)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{u + x_t}{2})ds. \quad (3.2)$$

Lemma 3.3. *Let $p \in Fix(S)$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. Let $p \in \mathrm{Fix}(S)$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (1 - \alpha_n A)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\tfrac{x_n + x_{n+1}}{2})ds - p\| \\
&\le \alpha_n\|\gamma f(x_n) - Ap\| + (1 - \alpha_n\bar{\gamma})\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\tfrac{x_n + x_{n+1}}{2}) - T(s)p\|ds \\
&\le \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Ap\|) + (1 - \alpha_n\bar{\gamma})\|\tfrac{x_n + x_{n+1}}{2} - p\| \\
&\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \tfrac{(1-\alpha_n\bar{\gamma})}{2}(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

which implies that

$$\tfrac{1+\alpha_n\bar{\gamma}}{2}\|x_{n+1} - p\| \le (\alpha_n\gamma\alpha + \tfrac{1-\alpha_n\bar{\gamma}}{2})\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|.$$

Then

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1+\alpha_n\bar{\gamma}})\|x_n-p\|+\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1+\alpha_n\bar{\gamma}}\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\\ &\leq \max\{\|x_n-p\|,\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\}\\ &\vdots\\ &\leq \max\{\|x_0-p\|,\frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\}.\end{aligned} \tag{3.3}$$

Hence $\{x_n\}$ is bounded. □

Now, set $t_n := \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds$. Then $\{t_n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 3.4. *The following properties are satisfying for the Algorithm 3.1*

$P1.$ $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$

$P2.$ $\lim_{n\to\infty}\|x_n-t_n\|=0.$

$P3.$ $\lim_{n\to\infty}\|T(s)t_n-t_n\|=0.$

Proof. P1: Let $p\in Fix(S)$, we have,

$$\begin{aligned}&\|t_{n+1}-t_n\|\\ &=\|\frac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)(\frac{x_{n+1}+x_{n+2}}{2})ds-\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\|\\ &=\|\frac{1}{s_{n+1}}\int_0^{s_{n+1}}(T(s)(\frac{x_{n+1}+x_{n+2}}{2})-T(s)(\frac{x_n+x_{n+1}}{2}))ds+(\frac{1}{s_{n+1}}-\frac{1}{s_n})\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\\ &\quad+\frac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}} T(s)(\frac{x_n+x_{n+1}}{2})ds\|\\ &=\|\frac{1}{s_{n+1}}\int_0^{s_{n+1}}(T(s)(\frac{x_{n+1}+x_{n+2}}{2})-T(s)(\frac{x_n+x_{n+1}}{2}))ds\\ &\quad+(\frac{1}{s_{n+1}}-\frac{1}{s_n})\int_0^{s_n}(T(s)(\frac{x_n+x_{n+1}}{2})-T(s)p)ds+\frac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}(T(s)(\frac{x_n+x_{n+1}}{2})-T(s)p)ds\|\\ &\leq\|\frac{x_{n+1}+x_{n+2}}{2}-\frac{x_n+x_{n+1}}{2}\|+\frac{|s_{n+1}-s_n|s_n}{s_{n+1}s_n}\|\frac{x_n+x_{n+1}}{2}-p\|+\frac{|s_{n+1}-s_n|}{s_{n+1}}\|\frac{x_n+x_{n+1}}{2}-p\|\\ &\leq\frac{1}{2}(\|x_{n+1}-x_n\|+\|x_{n+2}-x_{n+1}\|)+\frac{|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\|+\|x_{n+1}-p\|).\end{aligned} \tag{3.4}$$

Next, we show that the sequence $\{x_n\}$ is asymptotically regular, i.e.,
$\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\|=0.$
By (3.4) we estimate that

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&= \|(\alpha_{n+1}\gamma f(x_{n+1}) + (1-\alpha_{n+1}A)\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)(\tfrac{x_{n+1}+x_{n+2}}{2})ds) \\
&\quad -(\alpha_n\gamma f(x_n) + (1-\alpha_n A)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds)\| \\
&= \|(1-\alpha_{n+1}A)(\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)(\tfrac{x_{n+1}+x_{n+2}}{2})ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds) \\
&\quad +(\alpha_n A-\alpha_{n+1}A)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds + (\alpha_{n+1}-\alpha_n)\gamma f(x_n) \\
&\quad +\alpha_{n+1}(\gamma f(x_{n+1})-\gamma f(x_n))\| \\
&\le (1-\alpha_{n+1}\bar{\gamma})\|t_{n+1}-t_n\| + M|\alpha_n-\alpha_{n+1}| + \alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\| \\
&\le (1-\alpha_{n+1}\bar{\gamma})\|t_{n+1}-t_n\| + M|\alpha_n-\alpha_{n+1}| + \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\| \\
&\le \tfrac{1-\alpha_{n+1}\bar{\gamma}}{2}(\|x_{n+1}-x_n\| + \|x_{n+2}-x_{n+1}\|) + (1-\alpha_{n+1}\bar{\gamma})\tfrac{|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\| \\
&\quad +\|x_{n+1}-p\|) + M|\alpha_n-\alpha_{n+1}| + \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|,
\end{aligned}$$

where $M := \sup\{\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds + \gamma\|f(x_n)\|\}$.

Then

$$\begin{aligned}
(1+\alpha_{n+1}\bar{\gamma})\|x_{n+2}-x_{n+1}\| &\le (1+(2\alpha\gamma-\bar{\gamma})\alpha_{n+1})\|x_{n+1}-x_n\| \\
&\quad +(1-\alpha_{n+1}\bar{\gamma})\tfrac{2|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\| + \|x_{n+1}-p\|) \\
&\quad +2M|\alpha_n-\alpha_{n+1}|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\le (1-\tfrac{2(\bar{\gamma}-\alpha\gamma)\alpha_{n+1}}{1+\alpha_{n+1}\bar{\gamma}})\|x_{n+1}-x_n\| + (\tfrac{1-\alpha_{n+1}\bar{\gamma}}{1+\alpha_{n+1}\bar{\gamma}})(\tfrac{2|s_{n+1}-s_n|}{s_{n+1}})(\|x_n-p\| \\
&\quad +\|x_{n+1}-p\|) + \tfrac{2M}{1+\alpha_{n+1}\bar{\gamma}}|\alpha_n-\alpha_{n+1}|.
\end{aligned}$$

Hence, it follows by Lemma 2.5 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.5}$$

And similarly, we have

$$\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\| = 0. \tag{3.6}$$

Also by (3.4), (3.5),(3.6) and (C3) we have $\lim_{n\to\infty}\|t_{n+1}-t_n\| = 0$.

P2: We can write

$$\begin{aligned}
\|x_n-t_n\| &\le \|x_{n+1}-x_n\| + \|\alpha_n\gamma f(x_n) + (1-\alpha_n A)t_n - t_n\| \\
&\le \|x_n-x_{n+1}\| + \alpha_n\|\gamma f(x_n) - At_n\|.
\end{aligned}$$

By (C1) and (3.5), we obtain

$$\lim_{n\to\infty}\|x_n-t_n\| = 0. \tag{3.7}$$

P3: Let $K := \{w \in C : \|w - p\| \leq \|x_0 - p\|, \frac{1}{\bar{\gamma}-\gamma\alpha}\|\gamma f(p) - Bp\|\}$. Then K is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $s \in [0, +\infty)$ and contains $\{x_n\}$. So, without loss of generality, we may assume that $S := \{T(s) : s \in [0, +\infty)\}$ is a nonexpansive semigroup on K.

$$\begin{aligned}
\|T(s)x_n - x_n\| &= \|T(s)x_n - T(s)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds + T(s)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds \\
&\quad -\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds + \tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\leq \|T(s)x_n - T(s)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\leq \|x_n - \tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds - x_n\| \\
&= 2\|\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds\|
\end{aligned}$$

Since $\frac{x_n+x_{n+1}}{2} \in C$, from (3.7) and Lemma 2.4, we obtain $\lim_{n\to\infty}\|T(s)x_n - x_n\| = 0$.
Therefore

$$\begin{aligned}
\|T(s)t_n - t_n\| &\leq \|T(s)t_n - T(s)x_n\| + \|T(s)x_n - x_n\| + \|x_n - t_n\| \\
&\leq \|t_n - x_n\| + \|T(s)x_n - x_n\| + \|x_n - t_n\|.
\end{aligned}$$

Then we have $\lim_{n\to\infty}\|T(s)t_n - t_n\| = 0$. □

4. **Convergence Algorithm**

Theorem 4.1. *The Algorithm defined by (3.1) convergence strongly to $z \in Fix(S)$, which is a unique solution in of the variational inequality $\langle(\gamma f - A)z, y - z\rangle \leq 0, \quad \forall y \in Fix(S)$.*

Proof. Let $s = P_{\mathrm{Fix}(S)}$. We get

$$\begin{aligned}
\|s(I - A + \gamma f)(x) - s(I - A + \gamma f)(y)\| &\leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\
&\leq \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\
&\leq (1 - \bar{\gamma})\|x - y\| + \gamma\alpha\|x - y\| \\
&= (1 - (\bar{\gamma} - \gamma\alpha))\|x - y\|.
\end{aligned}$$

Then $s(I - A + \gamma f)$ is a contraction mapping from H into itself. Therefore by Banach contraction principle, there exists $z \in H$ such that $z = s(I - A + \gamma f)z =$

$P_{\mathrm{Fix}(S)}(I - A + \gamma f)z$.
We show that $\langle(\gamma f - A)z, x_n - z\rangle \leq 0$. To show this inequality, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)z, t_n - z\rangle = \lim_{i\to\infty}\langle(\gamma f - A)z, t_{n_i} - z\rangle. \tag{4.1}$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\} \subseteq K$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $t_{n_i} \rightharpoonup w$. Now, we prove that $w \in \mathrm{Fix}(S)$. Assume that $w \notin \mathrm{Fix}(S)$. Since $t_{n_i} \rightharpoonup w$ and $T(s)w \neq w$, from Opial's conditions (2.3) and Lemma 3.4 (P3), we have

$$\begin{aligned}\liminf_{i\to\infty}\|t_{n_i} - w\| &< \liminf_{i\to\infty}\|t_{n_i} - T(s)w\| \\ &\leq \liminf_{i\to\infty}(\|t_{n_i} - T(s)t_{n_i}\| + \|T(s)t_{n_i} - T(s)w\|) \\ &\leq \liminf_{i\to\infty}\|t_{n_i} - w\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in \mathrm{Fix}(S)$. Now from (2.1), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle &= \limsup_{n\to\infty}\langle(\gamma f - A)z, t_n - z\rangle \\ &\leq \limsup_{i\to\infty}\langle(\gamma f - A)z, t_{n_i} - z\rangle \\ &= \langle(\gamma f - A)z, w - z\rangle \\ &\leq 0.\end{aligned} \tag{4.2}$$

Now we prove that x_n is strongly convergence to z.

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \alpha_n\langle\gamma f(x_n) - Az, x_{n+1} - z\rangle + \langle(1 - \alpha_n A)(t_n - z), x_{n+1} - z\rangle \\ &\leq \alpha_n(\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle + \langle\gamma f(z) - Az, x_{n+1} - z\rangle) \\ &\quad + \|1 - \alpha_n A\|\|t_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\alpha\gamma\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\ &\quad + (1 - \alpha_n\bar{\gamma})\|\tfrac{x_n + x_{n+1}}{2} - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\alpha\gamma\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\ &\quad + \tfrac{1 - \alpha_n\bar{\gamma}}{2}(\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - z\| \\ &= \tfrac{1 - \alpha_n\bar{\gamma} + 2\alpha_n\alpha\gamma}{2}\|x_n - z\|\|x_{n+1} - z\| + \tfrac{1 - \alpha_n\bar{\gamma}}{2}\|x_{n+1} - z\|^2 \\ &\quad + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\ &\leq \tfrac{1 - \alpha_n(\bar{\gamma} - 2\alpha\gamma)}{4}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \tfrac{1 - \alpha_n\bar{\gamma}}{2}\|x_{n+1} - z\|^2 \\ &\quad + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\ &\leq \tfrac{1 - \alpha_n(\bar{\gamma} - 2\alpha\gamma)}{4}\|x_n - z\|^2 + \tfrac{3}{4}\|x_{n+1} - z\|^2 + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle.\end{aligned}$$

This implies that

$$4\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(\bar{\gamma} - 2\alpha\gamma))\|x_n - z\|^2 + 3\|x_{n+1} - z\|^2 + 4\alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle.$$

Then

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1-\alpha_n(\bar{\gamma}-2\alpha\gamma))\|x_n-z\|^2+4\alpha_n\langle \gamma f(z)-Az, x_{n+1}-z\rangle \\ &= (1-l_n)\|x_n-z\|^2+4\alpha_n\langle \gamma f(z)-Az, x_{n+1}-z\rangle,\end{aligned} \tag{4.3}$$

where $l_n = \alpha_n(\bar{\gamma}-2\alpha\gamma)$.
Since $\lim_{n\to\infty}\alpha_n = 0$ and $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, it is easy to see that $\lim_{n\to\infty} l_n = 0$, $\Sigma_{n=0}^{\infty} l_n = \infty$. Hence, from (4.2) and (4.3) and Lemma 2.5, we deduce that $x_n \to z$, where $z = P_{\Theta}(I-A+\gamma f)z$. □

5. Numerical Examples

In this section, we give some examples and numerical results for supporting our main theorem. All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory

Example 5.1. *Consider a Fredholm integral equation of the following form*

$$x(t) = g(t) + \int_0^t F(t,k,x(k))dk, t \in [0,1], \tag{5.1}$$

where g is a continuous function on $[0,1]$ and $F : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following condition

$$|F(t,k,x)-F(t,k,y)| \leq |x-y|, \qquad \forall t,s \in [0,1], \quad x,y \in \mathbb{R},$$

then equation (5.1) has at least one solution in $L^2[0,1]$ (see [9]).
Define a mapping $T(s) : L^2[0,1] \to L^2[0,1]$ by

$$(T(s)x)(t) = e^{-2s}(g(t) + \int_0^t F(t,k,x(k))\ dk, \qquad t \in [0,1].$$

It is easy to observe that $S = \{T(s) : s \in [0,+\infty)\}$ is nonexpansive semigroup. In fact, we have, for $x, y \in L^2[0,1]$,

$$\begin{aligned}\|T(s)x-T(s)y\|^2 &= \int_0^1 |(T(s)x)(t)-(T(s)y)(t)|^2\ dt \\ &= \int_0^1 |e^{-2s}\int_0^1(F(t,k,x(k))-F(t,k,y(k)))\ dk|^2\ dt \\ &\leq \int_0^1(\int_0^1 |x(k)-y(k)|^2\ dk)\ dt \\ &= \int_0^1 |x(k)-y(k)|^2\ dk \\ &= \|x-y\|^2.\end{aligned}$$

This means that to find the solution of integral equation (5.1) is reduced to find a fixed point of the nonexpansive semigroup S in $L^2[0,1]$.
For any given function $x_0 \in L^2[0,1]$, define a sequence of functions x_n in $L^2[0,1]$ by

$$x_{n+1} = \alpha_n\gamma f(x_n) + (1-\alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds$$

satisfying the conditions of Algorithm 3.1. Then the sequence $\{x_n\}$ converges strongly in $L^2[0,1]$ to the solution of integral equation (5.1) which is also a solution of the following variational inequality

$$\langle(\gamma f-A)z, y-z\rangle \leq 0, \quad \forall y \in Fix(S).$$

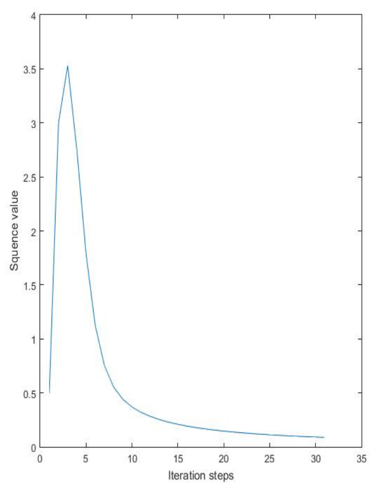
Example 5.2. *Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $\forall x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [-3, 0]$; Let $f(x) = \frac{1}{5}(x+2)$, $A(x) = \frac{1}{3}x$ and let, for each $x \in C$, $T(s)x = \frac{1}{1+3s}x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ generated by the iterative scheme*

$$x_{n+1} = \frac{6}{5}(x_n + 2) + (1 - \frac{3}{n}A)\frac{1}{s_n}\int_0^{s_n} \frac{1}{1+3s}(\frac{x_n + x_{n+1}}{2})ds \tag{5.2}$$

where $\alpha_n = \frac{3}{n}$ and $s_n = n$. Then $\{x_n\}$ converges to $\{0\} \in Fix(S)$. f is contraction mapping with constant $\alpha = \frac{1}{3}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma} = 1$ on C. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $Fix(S) = \{0\} \neq \emptyset$. After simplification, scheme (5.2) reduce to

$$x_{n+1} = \frac{\frac{12}{5n} + \frac{1}{n}(\frac{6}{5} + \frac{1}{6}(1-\frac{1}{n})\ln(1+3n))x_n}{1 - \frac{1}{6n}(1-\frac{1}{n})\ln(1+3n)}.$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\}$ converges strongly to $w = \{0\} \in Fix(S)$.



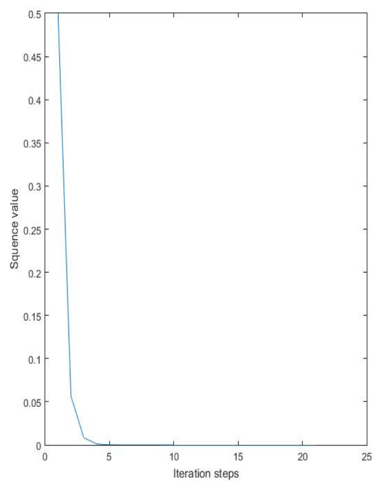
Example 5.3. *Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $\forall x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [0, 2]$; Let $f(x) = \frac{1}{8}x$, $A(x) = 2x$ and let, for each $x \in C$, $T(s)x = e^{-2s}x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ generated by the iterative scheme*

$$x_{n+1} = \frac{1}{4\sqrt{n}}x_n + (1 - \frac{1}{\sqrt{n}}A)\frac{1}{s_n}\int_0^{s_n} e^{-2s}(\frac{x_n + x_{n+1}}{2})ds \tag{5.3}$$

where $\alpha_n = \frac{1}{\sqrt{n}}$ and $s_n = 2n$. Then $\{x_n\}$ converges to $\{0\} \in Fix(S)$. f is contraction mapping with constant $\alpha = \frac{1}{5}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma} = 1$ on C. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $Fix(S) = \{0\} \neq \emptyset$. After simplification, scheme (5.3) reduce to

$$x_{n+1} = \frac{\frac{1}{4\sqrt{n}} - \frac{1}{8n}(1-\frac{2}{\sqrt{n}})(e^{-4n}-1)}{1 + \frac{1}{8n}(1-\frac{2}{\sqrt{n}})(e^{-4n}-1)}x_n.$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\}$ *converges strongly to* $w = \{0\} \in Fix(S)$.



6. Conculsion

We have proposed a viscosity nonlinear midpoint algorithm (VNMA) in real Hilbert spaces. The strong convergence of iteration sequence generated by the algorithm to a solution of VNMA is obtained. Some numerical examples are also provided to illustrate the convergence of proposed algorithm.

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ON LOWER SEMICONTINUITY OF THE SOLUTION MAPPINGS OF THE VECTOR EQUILIBRIUM PROBLEMS

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ABSTRACT. By using a new assumption, the lower semicontinuity of the solution mapping of the vector equi- librium mappings in the setting of topological vector spaces without using a continuity assumption are established. Some examples in order to illustrate the main results are given. The results of the article can be viewed as improvements to the results published in this area.

KEYWORDS:Lower semicontinuity, Efficient solutions,vector equilibrium.
AMS Subject Classification: 49J35.

1. Introduction

complementarity problem, the vector optimization problem and the vector saddle point problem, the vector equilibrium problem has been intensively studied in the literature. The stability analysis of the solution mappings for vector equilibrium problems is an important topic in optimization theory. Recently, the semicontinuity, especially the lower semicontinuity, of the solution mappings for parametric vector equilibrium problems has been of considerable interest.

Inspired by the pioneer work of Giannessi [14], the theory of vector equilibrium problems was started during the last decade of last century. The vector equilibrium problems (for short, VEP) are among the most interesting and intensively studied classes of nonlinear problems. They include fundamental mathematical problems, namely, vector optimization problems, vector variational inequality problems, Nash equilibrium problem for vector-valued mappings and fixed point problems as special cases. A large number of research papers have been published on different aspects of vector equilibrium problems, see, for example [1, 4, 5, 8, 9, 10, 11, 12, 13, 18, 19, 23, 26, 27, 28] and the references therein.

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There are several possible ways to generalize vector equilibrium problems for set-valued mappings, see, for example, [23, 28] and the references therein. Such generalizations are based on the concepts, namely, weak efficient solution, efficient solutions, strong efficient solutions, etc., of vector optimization problems.

In [2], Anh and Khanh studied the semicontinuity of the solution mappings of parametric multivalued vector quasi-equilibrium problems under the assumption C-inclusion property. In [17], Huang et. al. ,using local existence results for the models, considered the lower semicontinuity of solution mappings for parametric implicit vector equilibrium problems. Anh and Khanh [3] and Kimura and Yao [21] discussed the semicontinuity of solution mappings of parametric vector quasi-equilibrium problems by virtue of the closedness or openness assumptions for some certain sets, respectively. By using the ideas of Cheng and Zhu [7], Gong [15] discussed the continuity of the solution mappings for parametric weak vector equilibrium problems in topological vector spaces. In [6], by using a new proof which is different from the ones of [16, 7], Chen et al. established the lower semicontinuity and continuity of the solution mappings to a parametric generalized vector equilibrium problem involving set-valued mappings. In [22], Li and Fang investigated the lower semicontinuity of the solutions mapping to parametric weak vector equilibrium problems, called weak vector solutions to a generalized Ky Fan inequality, under a weaker assumption than C-strict monotonicity.

2. Preliminaries

Throughout this paper, let X, Y and Z be topological vector spaces and C be a pointed closed and convex cone in Y with nonempty interior $(, intC \neq \emptyset)$. Let B be a nonempty subset of X and $F : B \times B \longrightarrow Y$ be a vector valued mapping. The vector equilibrium problem (VEP) consists of finding $x \in B$ such that

$$f(x, y) \notin -C\backslash\{0\}, \ \forall y \in B.$$

When the subset B and the function f are perturbed by parameter μ which varies over a subset Λ of Z, we consider the following parametric vector equilibrium problem (PVEP)

$$\text{Finding } x \in A(\mu) \text{ such that } f(x, y, \mu) \notin -C\backslash\{0\}, \ \forall y \in A(\mu),$$

where $A : \Lambda \longrightarrow 2^B \setminus \{\emptyset\}$ is a set valued mapping and $f : B \times B \times \Lambda \longrightarrow Y$is a vector valued mapping. The solution set of PVEP is denoted by $S(\mu)$, i.e,

$$S(\mu) = \{x \in A(\mu) | f(x, y, \mu) \notin -C \setminus \{0\}, \ \forall y \in A(\mu)\} \tag{2.1}$$

Throughout this article we always assume $S(\mu) \neq \emptyset$ for all $\mu \in \Lambda$. The main of this note is to investigate the continuity of the solution set map $S(\mu)$ as a set valued mapping from the set Λ into X.

The following definitions and results are needed in the next section.

Definition 2.1. Let X and Y be topological spaces and $F : X \longrightarrow 2^Y$ be a set valued mapping. The set valued mapping F is called:

- **lower semicontinuous** (lsc) at x if for any open set V satisfying $V \cap F(x) \neq \emptyset$, there exists open set U of X such that

$$V \cap F(y) \neq \emptyset, \ \ \forall y \in U.$$

- **upper semicontinuous** (usc) at x if for any open set V satisfying $F(x) \subset V$, there exists open set U of X such that
$$F(x) \subset V, \ \ \forall y \in U.$$
- **continuous** at x if it is both l.s.c and u.s.c at x.

Note F is called respectively lsc, usc and continuous on $A \subset X$ if it is respectively, lsc, usc and continuous at each $x \in A$.

Proposition 2.2. *([20]) Let X and Y be topological spaces and $F : X \longrightarrow 2^Y$ be a set valued mapping. Then*

- *F is lsc at $x \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \longrightarrow x$ and any $z \in F(x))$ there exists $z\alpha \in f(x_\alpha)$ such that $z_\alpha \longrightarrow x$.*
- *If F has compact values (i.e, $F(x)$ is compact set for each $x \in X$), then F is usc at x if and only if for any net $x_\alpha \subset X$ with $x_\alpha \longrightarrow x$and for any $z_\alpha \in F(x_\alpha)$, there exists $z \in F(x)$ and a subnet z_β of z_α such that $z_\beta \longrightarrow z$.*

We are going to recall the linear scalarization method. Let Y be topological vector space. The topological dual of Y is denoted by Y^* and it is consists of all continuous linear mappings from Y into the real line ($\mathbb{R}$). Let C be a subset of Y, The (positive) polar cone of C is defined by
$$C^* := \{c^* \in Y^* : \langle c^*, c\rangle \geq 0, \ \forall c \in C\},$$
and quasi interior of C^* is defined by
$$C^*_+ := \{c^* \in C^* : \langle c^*, c\rangle > 0, \ \forall x \in C\backslash\{0\}\}.$$

It follows from the bipolar theorem (see [4]) that if Y is a locally convex space and C is a closed convex cone with nonempty interior then the following assertions hold:
$$y \in C \Longleftrightarrow [\langle y^*, y\rangle \geq 0, \ \forall y^* \in C^*],$$
$$y \in intC \Longleftrightarrow [\langle y^*, y\rangle > 0, \forall y^* \in C^*_+].$$

Definition 2.3. Let X be a topological space and C be a convex cone with nonempty interior of the topological vector space Y. The vector valued mapping $g : X \longrightarrow Y$ is called:

- $C-$lower semicontinuous on X if for each fixed $x \in X$ and for any $y \in intC$, there exists a neighborhood $U(x)$ such that
$g(x) \in g(u) + y - intC, \ \forall u \in U(x)$.
- $C-$upper semicontinuous on X if for each fixed $x \in X$ and for any $y \in intC$, there exists a neighborhood $U(x)$ such that
$g(u) \in g(x) + y - intC, \ \forall u \in U(x)$.

Proposition 2.4. *If $g : K \longrightarrow Y$ is $C-$lower semicontinuous then the set $A := \{x \in K; g(x) \notin intC\}$ is closed in K.*

Proof. Suppose that $x \notin A$ then $g(x) \in intC$. Hence by Definition **??** (a) there exists $U(x)$ such that
$$g(x) \in g(u) + g(x) - intC, \ \forall u \in U(x),$$
which implies that $g(u) \in intC$. Therefore $U(x) \subset K \setminus A$. This shows that A is closed in K. □

Proposition 2.5. *([4]) If $g : X \longrightarrow Y$ is a vector valued mapping, $C-$lower semicontinuous ,and $c^* \in C^*$ then the mapping $\langle c^*, g(.)\rangle$ is lower semicontinuous.*

3. Main Results

In this section we present sufficient conditions in order to guarantee the lower semicontinuity of the solution set mapping PVEP.

Theorem 3.1. *Suppose that the following are satisfied:*
(i) $A(.)$ *is continuous with compact values on* Λ,
(ii) $f(.,.,.)$ *is* $C-$*lowe semicontinuous on* $B \times B \times \Lambda$,
(iii) For each $\mu \in \Lambda,\ x \in A(\mu) \setminus S(\mu)$, *there exists* $y \in S(\mu)$ *such that*

$$f(x, y, \mu) \in -C \setminus \{0\}.$$

Then $S(.)$ *is lower semicontinuous on* Λ.

Proof. Suppose to the contrary that there exists μ_0 such that $S(.)$ is not lower semicontinuous at μ_0. Then there exist a net $\{\mu_\alpha\}$ with $\mu_\alpha \longrightarrow \mu_0$ and $x_0 \in S(\mu_0)$, such that for any $x_\alpha \in S(\mu_\alpha),\ x_\alpha \nrightarrow x_0$. From $x_0 \in S(\mu_0)$ we have $x_0 \in A(\mu_0)$ and

$$f(x_0, y, \mu_0) \notin -C \setminus \{0\}\ \forall y \in A(\mu_0). \tag{3.1}$$

Since $A(.)$ is l.s.c at μ_0 there exists a net $\{\bar{x}_\alpha\} \subset A(\mu_\alpha)$ such that $\bar{x}_\alpha \longrightarrow x_0$. Obviously, $\bar{x}_\alpha \in A(\mu_\alpha) \setminus S(\mu_\alpha)$. By (iii) there exists $y_\alpha \in S(\mu_\alpha)$ such that
$f(x_\alpha, y_\alpha, \mu_\alpha) \in -C$. Since $y_\alpha \in A(\mu_\alpha)$, it follows from the upper continuity and compactness of $A(.)$ at μ_0 that there exist $y_0 \in A(\mu_0)$ and subnet y_{α_β} of y_α such that $y_{\alpha_\beta} \longrightarrow y_0$, Suppose that $c^* \in C^*$ be arbitrary, since $f(x_{\alpha_\beta}, y_{\alpha_\beta}, \mu_{\alpha_\beta}) \in -C$ we have $\langle c^*, f(x_{\alpha_\beta}, y_{\alpha_\beta}, \mu_{\alpha_\beta})\rangle \leq 0$ then by Proposition 2.5 we can obtain

$$\langle c^*, f(x_0, y_0, \mu_0)\rangle \leq \liminf_{\beta} \langle c^*, f(x_{\alpha_\beta}, y_{\alpha_\beta}, \mu_{\alpha_\beta})\rangle \leq 0.$$

Hence $f(x_0, y_0, \mu_0) \in -C$ and $y_0 \in A(\mu_0)$ which is contradicted by (3.1). This completes the proof. □

The following example indicates that assumption (iii) in Theorem (3.1) is essential.

Example 3.2. Let $X = Y = Z = \mathbb{R}$ and $C = \mathbb{R}_+, \Lambda = [0, 1], A(\mu) = B = [0, 1]$ and $f(x, y, \mu) = 2x - y + \mu$. It is easy to see that the assumptions (i) and (ii) of Theorem 3.1 are satisfied. It follows from a direct computation by Definition2.1 that

$$S(\mu) = \begin{cases} [\frac{1}{2}, 1],\ \mu = 0; \\ [\frac{1+\mu}{2}, 1],\ \mu \neq 0, \end{cases}$$

which is not lower semicotinuous at $\mu = 0$ and hence the condition (iii) of Theorem (3.1) is dropped.

Remark 3.3. Theorem 3.1 improves Theorem 3.1 of [29] by relaxing the continuity of the mapping f and mertizability of the topological vector space. Further, our approach can be also applied to study the lower semicontinuity of the following problem which is called weakly parametric vector equilibrium problem (WPVEP). Also one can consider Theorem 3.1 is an improvement of Theorem 3.6 of [24] for single valued mappings.

A vector $x \in A(\mu)$ is called a solution of WPVEP if,
$f(x, y, \mu) \notin -intC,\ \forall y \in A(\mu)$.
The set of WPVEP solutions is denoted by

$$S_1(\mu) = \{x \in A(\mu) | f(x, y, \mu) \notin -intC,\ \forall y \in A(\mu)\}.$$

By using a similar proof as given for Theorem 3.1 we can establish the following result about the semicontinuity of the solution mapping of WPVEP.

Theorem 3.4. *Suppose that the following condition are satisfied:*
(i) A(.) is continuous with compact values on Λ
(ii) f(.,.,.) is $C-l.s.c$ *on* $B \times B \times \Lambda$
(iii) $\mu \in \Lambda,\ x \in A(\mu) \setminus S_1(\mu)$, *there exists* $y \in S_1(\mu)$ *such that* $f(x, y, \mu) \in -intC \setminus \{0\}$.

Then $S_1(.)$ *is lower semicontinuous on* Λ

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