Vol. 10 No. 1 (2019)

Journal of Nonlinear Analysis and Optimization: Theory & Applications

Editors-in-Chief:

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About the Journal

Journal of Nonlinear Analysis and Optimization: Theory & Applications is a peer-reviewed, open-access international journal, that devotes to the publication of original articles of current interest in every theoretical, computational, and applicational aspect of nonlinear analysis, convex analysis, fixed point theory, and optimization techniques and their applications to science and engineering. All manuscripts are refereed under the same standards as those used by the finest-quality printed mathematical journals. Accepted papers will be published in two issues annually in March and September, free of charge.

This journal was conceived as the main scientific publication of the Center of Excellence in Nonlinear Analysis and Optimization, Naresuan University, Thailand.

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Official Website: https://ph03.tci-thaijo.org/index.php/jnao

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Journal of Nonlinear Analysis and Optimization
Volume 10(1) (2019)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

BALL CONVERGENCE OF AN EIGHTH ORDER- ITERATIVE SCHEME WITH HIGH EFFICIENCY ORDER IN BANACH SPACE

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ABSTRACT. We present a local convergence analysis of an eighth order- iterative method in order to approximate a locally unique solution of an equation in Banach space setting. Earlier studies such as [13, 18] have used hypotheses up to the fourth derivative although only the first derivative appears in the definition of these methods. In this study, we only use the hypothesis of the first derivative. This way we expand the applicability of these methods. Moreover, we provide a radius of convergence, a uniqueness ball and computable error bounds based on Lipschitz constants. Numerical examples computing the radii of the convergence balls as well as examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

KEYWORDS: Banach space; eighth-order of convergence; local convergence; efficiency index.
AMS Subject Classification: Primary 65D10; Secondary 65D99, 65E99.

1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Using mathematical modeling, many problems in computational sciences and other disciplines can be expressed as a nonlinear equation (1.1) [1–30]. Closed form solutions of these nonlinear equations exist only for few special cases which may not be of much practical value. Therefore solutions of these nonlinear equations (1.1) are approximated by iterative methods.

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Article history : Received 4 July 2018; Accepted 3 November 2018.

In particular, the practice of Numerical Functional Analysis for approximating solutions iteratively is essentially connected to Newton-like methods [1–30]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1–30].

Newton's method is undoubtedly the most popular method for approximating a locally unique solution x^* provided that the initial point is close enough to the solution. In order to obtain a higher order of convergence Newton-like methods have been studied such as Potra-Ptak, Chebyshev, Cauchy Halley and Ostrowski method [3, 6, 23, 26]. The number of function evaluations per step increases with the order of convergence. In the scalar case the efficiency index [3, 6, 21] $EI = p^{\frac{1}{m}}$ provides a measure of balance where p is the order of the method and m is the number of function evaluations.

It is well known that according to the Kung-Traub conjuncture the convergence of any multi-point method without memory cannot exceed the upper bound 2^{m-1} [21] (called the optimal order). Hence the optimal order for a method with three function evaluations per step is 4. The corresponding efficiency index is $EI = 4^{\frac{1}{3}} = 1.58740...$ which is better than Newtons method which is $EI = 2^{\frac{1}{2}} = 1.414....$ Therefore, the study of new optimal methods of order four is important.

We present the local convergence analysis of the eighth-order method defined for each $n = 0, 1, 2...$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ w_n &= \frac{1}{2}(y_n + x_n) \\ z_n &= \frac{1}{3}(4w_n - x_n) \\ u_n &= w_n + (F'(x_n) - 3F'(z_n))^{-1}F(x_n) \\ v_n &= u_n + 2(F'(x_n) - 3F'(z_n))^{-1}F(u_n) \\ x_{n+1} &= v_n + (F'(x_n) - 3F'(z_n))^{-1}F(v_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point. The local convergence analysis of method (1.2) was given in [13] in the special case when $X = Y = \mathbb{R}^m$. The semi-local convergence analysis of method (1.2) in a Banach space was given in [18]. The computational efficiency of method (1.2) was also given in [18]. However, the convergence hypotheses for method (1.2) in these references require hypotheses up to the fourth derivative of operator F. These hypothesis limit the applicability of method (1.2) and the other comparable methods given in [13, 18]. As a motivational example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \; F'(1) = 3, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously function F does not have bounded third derivative in X. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on $\mathbb{R}$ [1–30]. These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) in Section 2. The same technique can be used to other methods.

In the present study we extend the applicability of the method (1.2) by using hypotheses up to the first derivative of function F and contractions on a Banach space setting. Moreover we avoid Taylor expansions and use instead Lipschitz parameters. Moreover, we do not have to use higher order derivatives to show the convergence of method (1.2). This way we expand the applicability of method (1.2).

The paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies using Taylor expansions. Special cases and numerical examples are presented in the concluding Section 3.

2. Local Convergence Analysis

We present the local convergence analysis of the method (1.2) in this section. Let $L_0 > 0$, $L > 0$ and $M \geq 1$ be parameters. It is convenient for the local convergence analysis of method (1.2) that follows to introduce some scalar functions and parameters. Define functions g_1, g_2 on the interval $[0, 1/L_0)$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)},$$

$$g_2(t) = \frac{1}{2}(1 + g_1(t))$$

and parameters r_A, r_0 by

$$r_A = \frac{2}{2L_0 + L}, \ r_0 = \frac{1}{3L_0}.$$

Moreover, define functions g_3, h_3, g_4, h_4, g_5 and h_5 on the interval $[0, r_0)$ by

$$g_3(t) = \frac{1}{2(1 - L_0 t)}(L + \frac{8ML_0}{1 - 3L_0 t})t, \ h_3(t) = g_3(t) - 1,$$

$$g_4(t) = (1 + \frac{2M}{1 - 3L_0 t})g_3(t), \ h_4(t) = g_4(t) - 1,$$

$$g_5(t) = (1 + \frac{2M}{1 - 3L_0 t})g_4(t)$$

and

$$h_5(t) = g_5(t) - 1.$$

We have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to r_0^-$. It then follows from the intermediate value theorem that function h_3 has zeros in the interval $(0, r_0)$. Denote by r_3 the smallest such zero. We also have that $h_4(0) = -1 < 0$ and $h_4(r_3) = \frac{2M}{1 - 3L_0 r_3} > 0$, since $g_3(r_3) = 1$ and $1 - 3L_0 r_3 > 0$. Denote by r_4 the smallest zero of function h_4 in the interval $(0, r_3)$. Finally, we have $h_5(0) = -1 < 0$ and $h_5(r_4) = \frac{2M}{1 - 3L_0 r_4} > 0$. Denote by r_5 the smallest zero of function h_5 in the interval $(0, r_4)$. Set

$$r = \min\{r_A, r_5\}. \tag{2.1}$$

Then, we have that

$$0 < r < r_A \tag{2.2}$$

and for each $t \in [0, r)$

$$0 \le g_1(t) < 1 \tag{2.3}$$

$$0 \le g_2(t) < 1 \tag{2.4}$$

$$0 \le g_3(t) < 1 \tag{2.5}$$

$$0 \le g_4(t) < 1 \tag{2.6}$$

and

$$0 \le g_5(t) < 1. \tag{2.7}$$

Let $U(\gamma, \rho)$, $\bar{U}(\gamma, \rho)$, respectively the open and closed balls in X with center $r \in X$ and of radius $r \in X$ and of $\rho > 0$. Next, we present the local convergence analysis of the method (1.2), using the preceding notation.

Theorem 2.1. *Let $F : D \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$ and $M \ge 1$ such that for each $x, y \in D$*

$$F(x^*) = 0, F'(x^*)^{-1} \in L(Y, X), \tag{2.8}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0\|x - x^*\|, \tag{2.9}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L\|x - x^*\|, \tag{2.10}$$

$$\|F'(x^*)^{-1}F'(x)\| \le M \tag{2.11}$$

and

$$\bar{U}(x^*, \frac{5}{3}r) \subseteq D, \tag{2.12}$$

where the radius r is given by (2.1). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined , remains in $U(x^*, r)$ for each $n = 0, 1, 2.....$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.13}$$

$$\|w_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.14}$$

$$\|z_n - x^*\| \le \frac{1}{3}(4\|w_n - x^*\| + \|x_n - x^*\|) < \frac{5}{3}\|x_n - x^*\|, \tag{2.15}$$

$$\|u_n - x^*\| \le g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.16}$$

$$\|v_n - x^*\| \le g_4(\|x_n - x^*)\|x_n - x^*\| < \|x_n - x^*\| \tag{2.17}$$

and

$$\|x_{n+1} - x^*\| \le g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*, \| \tag{2.18}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, if there exist $T \in [r, \frac{2}{L_0})$ and $\bar{U}(x^*, T) \in D$, then the limit point x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.

Proof: We shall show estimates (2.13)-(2.18) using mathematical induction. By (2.1), (2.9) and hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.19}$$

It follows from (2.19) and Banach Lemma on invertible operators [3, 6, 19, 23, 24, 28] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r}. \tag{2.20}$$

Hence, y_0, w_0 and z_0 are well defined. Using the first sub-step of method (1.2) for $n = 0$, (2.1), (2.2), (2.8), (2.10) and (2.20), we get in turn that

$$\begin{aligned}\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\ &\leq \|F'(x_0)^{-1}F'(x^*)\| \| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) \\ &- F'(x_0))(x_0 - x^*)d\theta\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.21}$$

which shows (2.13) for $n = 0$ and $y_0 \in U(x^*, r)$. Then, by the second sub-step of method (1.2) for $n = 0$, (2.1), (2.3) and (2.21), we obtain that

$$\begin{aligned}\|w_0 - x^*\| &\leq \frac{1}{2}(\|y_0 - x^*\| + \|x_0 - x^*\|) \\ &\leq \frac{1}{2}(1 + g_1(\|x_0 - x^*\|))\|x_0 - x^*\| \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.22}$$

which shows (2.14) and $w_0 \in U(x^*, r)$. In view of third sub-step of method (1.2) for $n = 0$, (2.1) and (2.22), we get that

$$\begin{aligned}\|z_0 - x^*\| & \| \frac{1}{3}(4(w_0 - x^*) - (x_0 - x^*)\| \\ &\leq \frac{1}{3}(4\|w_0 - x^*\| + \|x_0 - x^*\|) \\ &\leq \frac{1}{3}(4\|x_0 - x^*\| + \|x_0 - x^*\|) \\ &= \frac{5}{3}\|x_0 - x^*\| < \frac{5}{3}r, \end{aligned} \tag{2.23}$$

which shows (2.15) for $n = 0$ and $z_0 \in U(x^*, \frac{5}{3}r) \subset D$ (by (2.12)). Next, we shall show that $(F'(x_0) - 3F'(z_0))^{-1} \in L(Y, X)$. Using (2.1), (2.9) and (2.23), we get that

$$\begin{aligned}&\|(-2F'(x^*))^{-1}[F(x_0) - 3F'(z_0) - F'(x^*) + 3F'(x^*)]\| \\ &\leq \frac{1}{2}[\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\ &+ 3\|F'(x^*)^{-1}(F'(z_0) - F'(x^*))\|] \\ &\leq \frac{L_0}{2}[\|x_0 - x^*\| + 3\|z_0 - x^*\|] \\ &< \frac{L_0}{3}(\|x_0 - x^*\| + 3(\frac{5}{3})\|x_0 - x^*\|) \\ &= 3L_0\|x_0 - x^*\| < 3L_0 r < 1. \end{aligned} \tag{2.24}$$

Hence, we get that u_0 is well defined by the fourth sub-step of method (1.2) for $n = 0$ and

$$\begin{aligned}\|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\| &\leq \frac{1}{2(1 - \frac{L_0}{2}(\|x_0 - x^*\| + 3\|z_0 - x^*\|))} \\ &\leq \frac{1}{2(1 - 3L_0\|x_0 - x^*\|)}. \end{aligned} \tag{2.25}$$

Hence, u_0, v_0 and x_1 are well defined. We can write by (2.8) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.26}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$. That is $x^* + \theta(x_0 - x^*) \in U(x^*, r)$. Using (2.11) and (2.26), we obtain that

$$\begin{aligned} \|F'(x^*)^{-1}F(x_0)\| &= \|\int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta\| \\ &\leq M\|x_0 - x^*\|. \end{aligned} \tag{2.27}$$

We can write in turn by the first, second and fourth sub-step of method (1.2) for $n = 0$ that

$$\begin{aligned} u_0 - x^* &= \frac{1}{2}(y_0 - x^*) + \frac{1}{2}(x_0 - x^*) + (F'(x_0) - 3F'(z_0))^{-1}F(x_0) \qquad (2.28) \\ &= \frac{1}{2}(y_0 - x^*) + \frac{1}{2}(x_0 - x^* - F'(x_0)^{-1}F(x_0)) \\ &\quad + \frac{1}{2}F'(x_0)^{-1}F(x_0) + (F'(x_0) - 3F'(z_0))^{-1}F(x_0) \\ &= y_0 - x^* \\ &\quad + \frac{1}{2}F'(x_0)^{-1}[F'(x_0) - 3F'(z_0) + 2F'(x_0)](F'(x_0) - 3F'(z_0))^{-1}F(x_0) \\ &= y_0 - x^* \\ &\quad + \frac{3}{2}F'(x_0)^{-1}(F'(x_0) - F'(z_0))(F'(x_0) - 3F'(z_0))^{-1}F(x_0). \qquad (2.29) \end{aligned}$$

Using (2.1), (2.5), (2.20), (2.21), (2.25), (2.27) and (2.29), we obtain in turn that

$$\begin{aligned} \|u_0 - x^*\| &\leq \|y_0 - x^*\| + \frac{3}{2}\|F'(x_0)^{-1}F(x_0)\| \\ &\quad \times [\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(z_0) - F'(x^*))\|] \\ &\quad \times \|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} + \frac{3ML_0(\|x_0 - x^*\| + \|z_0 - x^*\|)\|x_0 - x^*\|}{2(1 - L_0\|x_0 - x^*\|)(1 - 3L_0\|x_0 - x^*\|)} \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.30}$$

which shows (2.16) for $n = 0$ and $u_0 \in U(x^*, r)$ (where, we also used the estimate $\|x_0 - x^*\| + \|z_0 - x^*\| \leq \|x_0 - x^*\| + \frac{5}{3}\|x_0 - x^*\| = \frac{8}{3}\|x_0 - x^*\| < \frac{8}{3}r$). Then, as in (2.27) for $x_0 = w_0$, we obtain that

$$\|F'(x^*)^{-1}F(w_0)\| \leq M\|w_0 - x^*\|. \tag{2.31}$$

Using the fifth sub-step of method (1.2) for $n = 0$, (2.1), (2.6), (2.25), (2.30) and (2.31), we have that

$$\begin{aligned} \|v_0 - x^*\| &\leq \|u_0 - x^*\| + 2\|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(u_0)\| \\ &\leq \|u_0 - x^*\| + \frac{2M\|u_0 - x^*\|}{1 - 3L_0\|x_0 - x^*\|} \\ &= (1 + \frac{2M}{1 - 3L_0\|x_0 - x^*\|})\|u_0 - x^*\| \\ &\leq g_4(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.32}$$

which shows (2.17) for $n = 0$ and $v_0 \in U(x^*, r)$.

Then, again as in (2.27) for $x_0 = v_0$, we get that

$$\|F'(x^*)^{-1}F(v_0)\| \leq M\|v_0 - x^*\|. \tag{2.33}$$

Using the sixth sub-step of method (1.2) for $n = 0$, (2.1), (2.7), (2.25), (2.32) and (2.33), we have that

$$\begin{aligned}\|x_1 - x^*\| &\leq \|v_0 - x^*\| + 2\|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(v_0)\| \\ &= (1 + \frac{2M}{1 - 3L_0\|x_0 - x^*\|})\|v_0 - x^*\| \\ &= g_5(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.34}$$

which shows (2.18) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing $x_0, y_0, w_0, z_0, v_0, x_1$ by $x_k, y_k, w_k, z_k, v_k, x_{k+1}$ in the preceding estimates we arrive at estimates (2.13) – (2.18). Then, from the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.9) we get that

$$\begin{aligned}|F'(x^*)^{-1}(Q - F'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\ &\leq \int_0^1 L_0(1-\theta)|x^* - y^*|d\theta \leq \frac{L_0}{2}T < 1. \end{aligned} \tag{2.35}$$

It follows from (2.35) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$.

□

Remark 2.1.

1. In view of (2.9) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\|\end{aligned}$$

   condition (2.11) can be dropped and be replaced by

$$M(t) = 1 + L_0 t,$$

   or

$$M = M(t) = 2,$$

   since $t \in [0, \frac{1}{L_0})$.
2. The results obtained here can be used for operators F satisfying autonomous differential equations [3,6,17] of the form

$$F'(x) = G(F(x))$$

   where T is a continuous operator. Then, since $F'(x^*) = G(F(x^*)) = G(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $G(x) = x + 1$.
3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3–7].

4. The parameter $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's method [3, 6]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.36}$$

under the conditions (2.8)–(2.10). It follows from the definitions of radii r that the convergence radius r of these preceding methods cannot be larger than the convergence radius r_A of the second order Newton's method (2.26). As already noted in [3, 6] r_A is at least as large as the convergence ball given by Rheinboldt [26]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ as \ \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [28].

5. It is worth noticing that the studied methods are not changing when we use the conditions of the preceding Theorems instead of the stronger conditions used in [13, 18]. Moreover, the preceding Theorems we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence.

3. Numerical Examples

The numerical examples are presented in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.9) conditions, we get $L_0 = e - 1, L = e, M = 2$. The parameters are

$$r_A = 0.3249, r_0 = 0.3880, r_3 = 0.0471, r_4 = 0.0117, r_5 = 0.0026 = r.$$

Example 3.2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$, $M = 2$. The parameters for method are

$$r_A = 0.0667, r_0 = 0.0889, r_3 = 0.0106, r_4 = 0.0026, r_5 = 0.0006 = r.$$

Example 3.3. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073$, $M = 2$. The parameters are

$$r_A = 0.0045 = r_0, r_3 = 0.0006, r_4 = 0.0001 = r, r_5 = 0.0091.$$

References

1. M. F. Abad, A. Cordero, J. R. Torregrosa, Fourth and Fifth-order methods for solving nonlinear systems of equations: An application to the global positioning system Abstract and Applied Analysis, (2013),
2. S. Amat, M.A. Hernández, N. Romero, Semilocal convergence of a sixth order iterative method for quadratic equations, Applied Numerical Mathematics, **62** (2012), 833-841.
3. I.K. Argyros, Computational theory of iterative methods. Series: Studies in Computational Mathematics, 15, Editors: C.K.Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.
4. Argyros, I.K., A semilocal convergence analysis for directional Newton methods. Math. Comput. **80** (2011), 327–343.
5. I.K. Argyros, S. Hilout, Weaker conditions for the convergence of Newton's method. J. Complexity. **28** (2012) 364–387.
6. I. K. Argyros and Said Hilout, Computational methods in nonlinear analysis. Efficient algorithms, fixed point theory and applications, World Scientific, 2013.
7. I.K. Aryros and H. Ren, Improved local analysis for certain class of iterative methods with cubic convergence, Numerical Algorithms, **59**(2012), 505-521.
8. V. Candela, A. Marquina, Recurrence relations for rational cubic methods I: The Halley method, Computing, **44**(1990), 169-184.
9. V. Candela, A. Marquina, Recurrence relations for rational cubic methods II: The Chebyshev method, Computing, **45(4)**(1990), 355-367.
10. C. Chun, P. Stănică and B. Neta, Third-order family of methods in Banach spaces, Computers & Mathematics with Applications, **61** (2011), 1665-1675.
11. A. Cordero, F. Martinez, J. R. Torregrosa, Iterative methods of order four and five for systems of nonlinear equations, J. Comput. Appl. Math. **231**(2009), 541-551.
12. A. Cordero, J. Hueso, E. Martinez, J. R. Torregrosa, A modified Newton-Tarratt's composition, Numer. Algor. **55** (2010), 87–99.
13. A. Cordero, J. R. Torregrosa, M. P. Vasileva, Increasing the order of convergence of iterative schemes for solving nonlinear systems, J. Comput. Appl. MAth. **252**, (2013), 86–94.
14. R. Ezzati, E. Azandegan, A simple iterative method with fifth order convergence by using Potra and Ptak's method, Mathematical Sciences, **3, 2**(2009), 191-200.
15. J. M. Gutiérrez, A.A. Magreñán and N. Romero, On the semi-local convergence of Newton-Kantorovich method under center-Lipschitz conditions, Applied Mathematics and Computation, **221** (2013), 79-88.
16. V. I. Hasanov, I. G. Ivanov and F. Nebzhibov, A new modification of Newton's method, Appl. Math. Eng. **27**(2002), 278-286.
17. M.A. Hernández, M.A. Salanova, Modification of the Kantorovich assumptions for semi-local convergence of the Chebyshev method, Journal of Computational and Applied Mathematics, **126** (2000), 131-143.

18. J. P. Jaiswal, Semilocal convergence of an eighth-order method in Banach spaces and its computational efficiency, Numer. Algor. **71, 4** (2016), 933–951.
19. L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.
20. J.S. Kou, Y. T. Li and X.H. Wang, A modification of Newton method with third-order convergence, Appl. Math. Comput. **181** (2006), 1106-1111.
21. A. A. Magrenan, Different anomalies in a Jarratt family of iterative root finding methods, Appl. Math. Comput. **233** (2014), 29-38.
22. A. A. Magrenan, A new tool to study real dynamics: The convergence plane, Appl. Math. Comput. **248** (2014), 29-38.
23. M. S. Petkovic, B. Neta, L. Petkovic, J. Džunič, Multipoint methods for solving nonlinear equations, Elsevier, 2013.
24. F. A. Potra and V. Pták, Nondiscrete Induction and Iterative Processes, in: Research Notes in Mathematics, Vol. 103, Pitman, Boston, 1984.
25. H. Ren and Q. Wu, Convergence ball and error analysis of a family of iterative methods with cubic convergence, Appl. Math. Comput., **209** (2009), 369-378.
26. W.C. Rheinboldt,An adaptive continuation process for solving systems of nonlinear equations, In: Mathematical models and numerical methods (A.N.Tikhonov et al. eds.) pub.3, (1977), 129-142 Banach Center, Warsaw Poland.
27. J.R. Sharma, P.K. Guha and R. Sharma, An efficient fourth order weighted-Newton method for systems of nonlinear equations, Numerical Algorithms, 62, **2**, (2013), 307-323.
28. J.F.Traub, Iterative methods for the solution of equations, AMS Chelsea Publishing, 1982.
29. S. Weerakoon, T.G.I. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett. **13**, (2000), 87-93.
30. T. Zhanlar, O. Chulunbaatar, G. Ankhbayar, On Newton-type methods with fourth and fifth-order convergence, Bulletin of PFUR Series Mathematics. Information Sciences, Physics, 2, **2**(2010), 29-34.

P. N. dutta et al., J. Nonlinear Anal. Optim. Vol. 10(1) (2019), 11-20

Journal of Nonlinear Analysis and Optimization
Volume 10(1) (2019)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

FIXED POINT FOR MAPPINGS SATISFYING KANNAN TYPE INEQUALITY IN FUZZY METRIC SPACES INVOLVING T-NORMS WITH EQUI-CONTINUOUS ITERATES

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ABSTRACT. In this paper, we define coupled weak compatible condition and use it to derive certain coupled coincidence point theorems for four mappings in fuzzy metric spaces. We use here a t-norm which has equicontinuous iterates at 1. Some coupled fixed point results in metric spaces are obtained by applications of the results. Our results are obtained without any assumption of continuity on the mappings. Our main result is supported by an illustrative example. Some corollaries are also obtained.

KEYWORDS: Fuzzy metric space, Hadzic type t-norm, Ψ-function, Cauchy sequence, Weak compatible mappings, Coupled fixed point, Coupled coincidence point.

AMS Subject Classification: 47H10; 54H25.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. Afterwards fuzzy concepts made quick headways in almost all branches of mathematics. In particular, fuzzy metric space was introduced by Kramosil and Michalek [2]. George and Veeramani modified the definition of Kramosil and Michalek in [3] for topological reasons. The topology in the space introduced by George and Veeramani is a Hausdroff topology. There are several fixed point results for mappings defined on fuzzy metric spaces in the sense of George and Veeramani. We have

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Article history : Received 30 June 2018; Accepted 2 November 2018.

noted some of these works in [4, 5, 6, 7, 8, 9, 10] and [11].

Coupled fixed point results attracted renewed interest after the publication of a coupled contraction mapping theorem in partially ordered metric spaces by Gnana-Bhasker and Lashmikanthm [12]. An interesting application of this result was also given in the same paper. The result in [12] was extended to coincidence point results in [13] and [14] under separate sets of sufficient conditions. Several other works on coupled fixed points have appeared in recent times. Some other works in this line of research are noted in [15, 16, 17]. Coupled fixed point problems have also been studied in probabilistic metric spaces [18], in cone metric spaces [19, 20] and in G- metric spaces [21]. We establish here coupled coincidence and fixed point results for the cases of coupled Kannan type mappings. Kannan type of mappings are considered to be important in metric fixed point theory for several reasons. We mention two of these in the following.

Banach contraction is continuous. A natural question is that whether there exists a class of mappings satisfying some contractive inequality which necessary have fixed points in complete metric spaces but need not necessarily be continuous. Kannan type mappings are such mappings [22, 23]. Another reason is its connection with metric completeness. A Banach contraction mapping may have a fixed point in a metric space which is not complete. In fact, Connell in [24] has given an example of a metric space which is not complete but every Banach contraction defined on which has a fixed point. It has been established in [25] that the metric completeness is implied by the necessary existence of fixed points of the class of Kannan type mappings. Some of the recent works on Kannan type mappings are noted in [26, 27, 28]. It may be noted that fuzzy functional analysis is a vast area of study of which some instances are [29, 30, 31, 32, 33, 34, 35].

In this paper we establish a common fixed point and coupled fixed point result for four mappings. The name 'Kannan type' is suggested by the form of the inequality we use here. We apply our result to obtain a new coupled Kannan type common fixed point result in metric spaces. An example illustrates our result in fuzzy metric spaces. In this paper we use Hadzic type t-norm which is a t-norm for which the iterates are equicontinuous at 1.

2. Preliminaries

Definition 2.1[36] A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a t-norm if the following properties are satisfied:
(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for all $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.
Generic examples of $t-$norms are $a *_1 b = \min\{a, b\}$, $a *_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for $0 < \lambda < 1$, $a *_3 b = ab$ and $a *_4 b = \max\{a + b - 1, 0\}$.

Kramosil and Michalek defined fuzzy metric space by extending probabilistic metric spaces.

Definition 2.2[2] The 3-tuple $(X, M, *)$ is called a fuzzy metric space in the sense of Kramosil and Michalek if X is a non-empty set, $*$ is a t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:
(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and

(v) $M(x,y,.) : (0,\infty) \longrightarrow [0,1]$ is left-continuous, where $t,s > 0$ and $x,y,z \in X$.
George and Veeramani in their paper [3] introduced a modification of the above definition. The motivation was to make the corresponding induced topology necessarily into a Hausdroff topology.
Definition 2.3[3] The 3-tuple $(X, M, *)$ is called a fuzzy metric space in the sense of George and Veeramani if X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for each $x,y,z \in X$ and $t,s > 0$:
(i) $M(x,y,t) > 0$,
(ii) $M(x,y,t) = 1$ if and only if $x = y$,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ and
(v) $M(x,y,.) : (0,\infty) \longrightarrow [0,1]$ is continuous.
The following details of this space are described in the introductory paper [3].
Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x,t,r)$ with center $x \in X$ is defined by

$$B(x,t,r) = \{y \in X : M(x,y,t) > 1 - r\}.$$

A subset $A \subset X$ is open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x,t,r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

In the present work we will only consider the space as described in definition 2.3 and will refer this space simply as fuzzy metric space.
There are several examples of the fuzzy metric space for which we refer to [3].
Lemma 2.4[37] Let $(X, M, *)$ be a fuzzy metric space. Then $M(x,y,.)$ is nondecreasing for all $x,y \in X$.
Definition 2.5[2] Let $(X, M, *)$ be a fuzzy metric space.
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
Lemma 2.6[38] M is a continuous function on $X^2 \times (0,\infty)$.
Definition 2.7[12] Let X be a nonempty set. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x,y) = x, \; F(y,x) = y.$$

Further Lakshmikantham and Ćirić have introduced the concept of coupled coincidence point.
Definition 2.8[14] Let X be a nonempty set. An element $(x,y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $h : X \to X$ if

$$F(x,y) = hx, \; F(y,x) = hy.$$

Definition 2.9[14] Let X be a nonempty set and the mappings $F : X \times X \to X$ and $h : X \to X$ are commuting if for all $x,y \in X$

$$hF(x,y) = F(hx,hy).$$

Definition 2.10[39] A t-norm $*$ is said to be Hadzic type t-norm if the family $\{*^p\}_{p\geq 0}$ of its iterates defined for each $s \in [0,1]$ by

$*^0(s) = 1$, $*^{p+1}(s) = *(*^p(s), s)$ for all $p \geq 0$, is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exists $\eta(\lambda) \in (0, 1)$ such that

$1 \geq s > \eta(\lambda) \Rightarrow *^{(p)}(s) > 1 - \lambda$ for all $p \geq 0$.

We will require the result of the following recently established lemma to prove our results.

Lemma 2.11 [40] Let $(X, M, *)$ be a fuzzy metric space with a Hadzic type t-norm $*$ such that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. If the sequences $\{x_n\}$ and $\{y_n\}$ in X are such that, for all $n \geq 1$, $t > 0$,

$$M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t) \geq M(x_{n-1}, x_n, \tfrac{t}{k}) * M(y_{n-1}, y_n, \tfrac{t}{k})$$

where $0 < k < 1$, then the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

We will use the following class of real mappings.

Definition 2.12 (Ψ-function) A function $\psi : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a ψ-function if

(i) ψ continuous and monotone increasing in both the variables,

(ii) $\psi(t, t) \geq t$ for all $0 \leq t \leq 1$.

3. Main Results

We next give the following definition.

Definition 3.1. Two maps $F : X \times X \to X$ and $h : X \to X$, where X is a nonempty set, are weakly compatible pair if they commute at their coincidence point, that is, for any $x, y \in X$, $hx = F(x, y)$ and $hy = F(y, x)$ implies that $h(F(x, y)) = F(hx, hy)$ and $h(F(y, x)) = F(hy, hx)$.

Theorem 3.2. *Let $(X, M, *)$ be a complete fuzzy metric space with a Hadzic type t-norm where $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let $F : X \times X \to X$, $G : X \times X \to X$, $h : X \to X$ and $g : X \to X$ be four mappings satisfying the following conditions:*

(i) $F(X \times X) \subseteq g(X), G(X \times X) \subseteq h(X)$ and $h(X), g(X)$ are two closed subsets of X,

(ii) (F, h) and (G, g) are weakly compatible pairs,

(iii) $$M(F(x, y), G(u, v), kt) \geq \psi(M(hx, F(x, y), t), M(gu, G(u, v), t)), \quad (3.1)$$

where $x, y, u, v \in X$, $t > 0$, $0 < k < 1$ and ψ is Ψ-function. Then there exist $x, y \in X$ such that $x = hx = gx = F(x, y) = G(x, y)$ and $y = hy = gy = F(y, x) = G(y, x)$, that is, there exist $x, y \in X$ such that x and y are common fixed points of h and g, and that (x, y) is a unique coupled fixed point of F and G.

Proof. Let x_0, y_0 be two points in X. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows, for all $n \geq 0$,

$$p_{2n} = gx_{2n+1} = F(x_{2n}, y_{2n}) \text{ and } q_{2n} = gy_{2n+1} = F(y_{2n}, x_{2n}). \quad (3.2)$$

$$p_{2n+1} = hx_{2n+2} = G(x_{2n+1}, y_{2n+1}) \text{ and } q_{2n+1} = hy_{2n+1} = G(y_{2n+1}, x_{2n+1}). \quad (3.3)$$

This construction is possible by the condition $F(X \times X) \subseteq g(X)$, $G(X \times X) \subseteq h(X)$. Now, for all $t > 0$, $n \geq 1$, we have

$$\begin{aligned} M(p_{2n}, p_{2n+1}, kt) &= M(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1}), kt) \text{ (by(3.2) and (3.3))} \\ &\geq \psi(M(hx_{2n}, F(x_{2n}, y_{2n}), t), M(gx_{2n+1}, G(x_{2n+1}, y_{2n+1}), t)) \text{ (by(3.1))} \\ &\geq \psi(M(p_{2n-1}, p_{2n}, t), M(p_{2n}, p_{2n+1}, t)). \end{aligned}$$

If, for some $s > 0$, and for some n, $M(p_{2n-1}, p_{2n}, s) \geq M(p_{2n}, p_{2n+1}, s)$, then, from the above inequality, and using the properties of ψ, we obtain

$$\begin{aligned} M(p_{2n+1}, p_{2n}, ks) &\geq \psi(M(p_{2n}, p_{2n+1}, s), M(p_{2n+1}, p_{2n}, s)) \\ &\geq M(p_{2n}, p_{2n+1}, s). \end{aligned}$$

But this contradicts our assumption that M is strictly increasing in the third variable. Hence we have

$$M(p_{2n}, p_{2n+1}, ks) > M(p_{2n-1}, p_{2n}, s) \text{ for all } n > 0.$$

Thus, for all $n > 0$ and $t > 0$, we have

$$M(p_{2n}, p_{2n+1}, kt) \geq \psi(M(p_{2n-1}, p_{2n}, t), M(p_{2n-1}, p_{2n}, t)),$$

that is, for all $n > 0$, $t > 0$, we have

$$M(p_{2n}, p_{2n+1}, kt) \geq M(p_{2n-1}, p_{2n}, t), \text{ (using the properties of } \psi\text{- function).} \quad (3.4)$$

Again, for all $t > 0$, $n \geq 0$, we have

$$\begin{aligned} M(p_{2n+1}, p_{2n+2}, kt) &= M(F(x_{2n+2}, y_{2n+2}), G(x_{2n+1}, y_{2n+1}), kt) \text{ (by(3.2) and (3.3))} \\ &\geq \psi(M(hx_{2n+2}, F(x_{2n+2}, y_{2n+2}), t), M(gx_{2n+1}, G(x_{2n+1}, y_{2n+1}), t)) \\ &\qquad \text{(by(3.1))} \\ &\geq \psi(M(p_{2n+1}, p_{2n+2}, t), M(p_{2n}, p_{2n+1}, t)). \end{aligned}$$

If, for some $s > 0$, and for some n, $M(p_{2n}, p_{2n+1}, s) \geq M(p_{2n+1}, p_{2n+2}, s)$, then, from the above inequality, and using the properties of ψ, we obtain

$$\begin{aligned} M(p_{2n+1}, p_{2n+2}, ks) &\geq \psi(M(p_{2n+1}, p_{2n+2}, s), M(p_{2n+1}, p_{2n+2}, s)) \\ &\geq M(p_{2n+1}, p_{2n+2}, s). \end{aligned}$$

But this contradicts our assumption that M is strictly increasing in the third variable. Hence we have

$$M(p_{2n+1}, p_{2n+2}, ks) > M(p_{2n}, p_{2n+1}, s) \text{ for all } n > 0.$$

Thus, for all $n > 0$ and $t > 0$, we have

$$M(p_{2n+1}, p_{2n+2}, kt) \geq \psi(M(p_{2n}, p_{2n+1}, t), M(p_{2n}, p_{2n+1}, t)),$$

that is, for all $n > 0$, $t > 0$, we have

$$M(p_{2n+1}, p_{2n+2}, kt) \geq M(p_{2n}, p_{2n+1}, t), \text{ (using the properties of } \psi\text{- function).} \quad (3.5)$$

From (3.4) and (3.5), for all $t > 0$, $n \geq 1$, we have

$$M(p_n, p_{n+1}, kt) \geq M(p_{n-1}, p_n, t), \text{ (using the properties of } \psi\text{- function).} \quad (3.6)$$

Now, for all $t > 0$, $n \geq 1$, we have

$$\begin{aligned} M(q_{2n}, q_{2n+1}, kt) &= M(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1}), kt) \text{ (by(3.2) and (3.3))} \\ &\geq \psi(M(hy_{2n}, F(y_{2n}, x_{2n}), t), M(gy_{2n+1}, G(y_{2n+1}, x_{2n+1}), t)) \\ &\qquad \text{(by(3.1))} \\ &\geq \psi(M(q_{2n-1}, q_{2n}, t), M(q_{2n}, q_{2n+1}, t)). \end{aligned}$$

If, for some $s > 0$, and for some n, $M(q_{2n-1}, q_{2n}, s) \geq M(q_{2n}, q_{2n+1}, s)$, then, from the above inequality, and using the properties of ψ, we obtain

$$\begin{aligned} M(q_{2n+1}, q_{2n}, ks) &\geq \psi(M(q_{2n}, q_{2n+1}, s), M(q_{2n+1}, q_{2n}, s)) \\ &\geq M(q_{2n}, q_{2n+1}, s). \end{aligned}$$

But this contradicts our assumption that M is strictly increasing in the third variable. Hence we have

$$M(q_{2n}, q_{2n+1}, ks) > M(q_{2n-1}, q_{2n}, s) \text{ for all } n > 0.$$

Thus, for all $n > 0$ and $t > 0$, we have

$$M(q_{2n}, q_{2n+1}, kt) \geq \psi(M(q_{2n-1}, q_{2n}, t), M(q_{2n-1}, q_{2n}, t)),$$

that is, for all $n > 0$, $t > 0$, we have

$$M(q_{2n}, q_{2n+1}, kt) \geq M(q_{2n-1}, q_{2n}, t), \text{ (using the properties of } \psi\text{- function).} \quad (3.7)$$

Again, for all $t > 0$, $n \geq 1$, we have

$$M(q_{2n+1}, q_{2n+2}, kt) = M(F(y_{2n+2}, x_{2n+2}), G(y_{2n+1}, x_{2n+1}), kt) \text{ (by(3.2) and (3.3))}$$

$$\geq \psi(M(hy_{2n+2}, F(y_{2n+2}, x_{2n+2}), t), M(gy_{2n+1}, G(y_{2n+1}, x_{2n+1}), t)) \quad \text{(by(3.1))}$$
$$\geq \psi(M(q_{2n+1}, q_{2n+2}, t), M(q_{2n}, q_{2n+1}, t)).$$

If, for some $s > 0$, and for some n, $M(q_{2n}, q_{2n+1}, s) \geq M(q_{2n+1}, q_{2n+2}, s)$, then, from the above inequality, and using the properties of ψ, we obtain

$$\begin{aligned} M(q_{2n+1}, q_{2n+2}, ks) &\geq \psi(M(q_{2n+1}, q_{2n+2}, s), M(q_{2n+1}, q_{2n+2}, s)) \\ &\geq M(q_{2n+1}, q_{2n+2}, s). \end{aligned}$$

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$$M(q_{2n+1}, q_{2n+2}, ks) > M(q_{2n}, q_{2n+1}, s) \text{ for all } n \geq 0.$$

Thus, for all $n \geq 0$ and $t > 0$, we have

$$M(q_{2n+1}, q_{2n+2}, kt) \geq \psi(M(q_{2n}, q_{2n+1}, t), M(q_{2n}, q_{2n+1}, t)),$$

that is, for all $n \geq 0$, $t > 0$, we have

$$M(q_{2n+1}, q_{2n+2}, kt) \geq M(q_{2n}, q_{2n+1}, t), \text{ (using the properties of } \psi\text{- function).} \quad (3.8)$$

From (3.7) and (3.8), for all $t > 0$, $n \geq 1$, we have

$$M(q_n, q_{n+1}, kt) \geq M(q_{n-1}, q_n, t), \text{ (using the properties of } \psi\text{- function).} \quad (3.9)$$

By (3.6) and (3.9), for all $n > 0$, $t > 0$, we have

$$M(p_n, p_{n+1}, kt) * M(q_n, q_{n+1}, kt) \geq M(p_{n-1}, p_n, t) * M(q_{n-1}, q_n, t). \quad (3.10)$$

In view of (3.10), by Lemma 2.11, we conclude that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$p_{2n} = gx_{2n+1} = F(x_{2n}, y_{2n}) = p_{2n+1} = hx_{2n+2} = G(x_{2n+1}, y_{2n+1}) \to x \text{ as } n \to \infty$$

and

$$q_{2n} = gy_{2n+1} = F(y_{2n}, x_{2n}) = q_{2n+1} = hy_{2n+1} = G(y_{2n+1}, x_{2n+1}) \to y \text{ as } n \to \infty.$$

Also $x, y \in h(X) \bigcap g(X)$.

Since, $G(X \times X) \subseteq h(X)$, there exists $u \in X$ such that $hu = x$ and also there exists $v \in X$ such that $hv = y$.

Now for all $t > 0$, we have

$$M(F(u, v), G(x_{2n+1}, y_{2n+1}), kt) \geq \psi(M(hu, F(u, v), t), M(gx_{2n+1}, G(x_{2n+1}, y_{2n+1}), t)).$$

Taking $n \to \infty$ on the both sides, for all $t > 0$, we have

$$M(F(u, v), x, kt) \geq \psi(M(x, F(u, v), t), M(x, F(u, v), t))$$
$$M(F(u, v), x, kt) \geq M(x, F(u, v), t),$$

which implies that $F(u, v) = x$.

Therefore, $F(u, v) = hu = x$.

Similarly, we can prove $F(v, u) = hv = y$.

Since, $F(X \times X) \subseteq g(X)$, there exists $r \in X$ such that $gr = x$ and also there exists $z \in X$ such that $gz = y$.

Now for all $t > 0$, we have

$$\begin{aligned} M(x, G(r, z), kt) &= M(F(x_{2n}, y_{2n}), G(r, z), kt) \\ &\geq \psi(M(hx_{2n}, F(x_{2n}, y_{2n}), t), M(gr, G(r, z), t)) \text{ (by (3.1))} \end{aligned}$$

Taking $n \to \infty$ on the both sides, for all $t > 0$, we have

$$M(x, G(r, z), kt) \geq \psi(M(x, G(r, z), t), M(x, G(r, z), t))$$
$$M(x, G(r, z), kt) \geq M(x, G(r, z), t),$$

which implies that $G(r, z) = x$.

Therefore, $G(r, z) = gr = x$.

Similarly, we can prove $G(z, r) = gz = y$.

Therefore, $F(u, v) = hu = G(r, z) = gr = x$ and $F(v, u) = hv = G(z, r) = gz = y$.

Since, (F, h) is weakly compatible,

therefore $hF(u, v) = F(hu, hv)$ and $hF(v, u) = F(hv, hu)$,

which implies $hx = F(x, y)$ and $hy = F(y, x)$.
Since, (G, g) is weakly compatible,
therefore $gG(r, z) = G(gr, gz)$ and $gG(z, r) = G(gz, gr)$,
which implies $gx = G(x, y)$ and $gy = G(y, x)$.
Now we will prove $x = hx = F(x, y)$.
For $t > 0$, we have

$$\begin{aligned} M(x, F(x, y), kt) &= M(F(x, y), G(r, z), kt) \\ &\geq \psi(M(hx, F(x, y), t), M(gr, G(r, z), t)) \text{ (by (3.1))} \\ &\geq \psi(1, 1) \\ &\geq 1. \end{aligned}$$

Therefore, $F(x, y) = x$. So $F(x, y) = hx = x$.
Similarly, $F(y, x) = hy = y$.
Again we will prove $x = gx = G(x, y)$.
For $t > 0$, we have

$$\begin{aligned} M(x, G(x, y), kt) &= M(F(u, v), G(x, y), kt) \\ &\geq \psi(M(hu, F(u, v), t), M(gx, G(x, y), t)) \text{ (by (3.1))} \\ &\geq \psi(1, 1) \\ &\geq 1. \end{aligned}$$

Therefore, $G(x, y) = x$. So $G(x, y) = gx = x$.
Similarly, $G(y, x) = gy = y$.
Therefore, $F(x, y) = G(x, y) = hx = gx = x$ and $F(y, x) = G(y, x) = hy = gy = y$.
So, (x, y) is the coupled common fixed point of F and G.
To show uniqueness, let (e_1, e_2) be another coupled common fixed point of F and G.
Therefore, $F(e_1, e_2) = G(e_1, e_2) = he_1 = ge_1 = e_1$
and

$$F(e_2, e_1) = G(e_2, e_1) = he_2 = ge_2 = e_2.$$

$$\begin{aligned} M(x, e_1, kt) &= M(F(x, y), G(e_1, e_2), kt) \\ &\geq \psi(M(hx, F(x, y), t), M(ge_1, G(e_1, e_2), t)) \text{ (by (3.1))} \\ &\geq \psi(1, 1) \\ &\geq 1. \end{aligned}$$

Therefore $e_1 = x$.

$$\begin{aligned} M(y, e_2, kt) &= M(F(y, x), G(e_2, e_1), kt) \\ &\geq \psi(M(hy, F(y, x), t), M(ge_2, G(e_2, e_1), t)) \text{ (by (3.1))} \\ &\geq \psi(1, 1) \\ &\geq 1. \end{aligned}$$

Therefore $e_2 = y$.
Therefore (x, y) is the unique coupled common fixed point of F and G.
Thus completes the proof. □

Corollary 3.3 Let $(X, M, *)$ be a complete fuzzy metric space with a Hadzic type t-norm where $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let $F : X \times X \to X$, $h : X \to X$ be two mappings satisfying the following conditions:
(i) $M(F(x, y), F(u, v), kt) \geq \psi(M(hx, F(x, y), t), M(hu, F(u, v), t))$, (3.11)
(ii) $F(X \times X) \subseteq h(X)$ and $h(X)$ are two closed subsets of X,
(iii) (F, h) is weakly compatible pair,
for every $x, y, u, v \in X$, for $t > 0$, $0 < k < 1$ and ψ is Ψ-function. Then there exist $x, y \in X$ such that $x = hx = F(x, y)$ and $y = hy = F(y, x)$, that is, there exist $x, y \in X$ such that x and y are fixed points of h, and that (x, y) is a unique coupled fixed point of F.

Proof. The proof follows by putting $F = G$, $h = g$ in Theorem 3.2.

Corollary 3.4 Let $(X, M, *)$ be a complete fuzzy metric space with a Hadzic type t-norm where $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let $F : X \times X \to X$ be a mapping satisfies the following condition:

$$M(F(x, y), F(u, v), kt) \geq \psi(M(x, F(x, y), t), M(u, F(u, v), t)),$$

for every $x, y, u, v \in X$, $0 < k < 1$ and ψ is Ψ-function. Then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$, that is, F has unique coupled fixed point in X.

Proof. The proof follows by putting $F = G$, $h = g = I$, the identity function, in Theorem 3.2.

Example 3.5 Let $X = [0, 1]$. Let $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ and $a * b = \min\{a, b\}$. Then $(X, M, *)$ is a complete fuzzy metric space with the property that M is strictly increasing in t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let the mappings $F : X \times X \to X$ and $G : X \times X \to X$ be defined as follows:

$$F(x, y) = G(x, y) = \begin{cases} 1, & \text{if } x > 1, \\ 0, & \text{if } otherwise, \end{cases}$$

and the mappings $h : X \to X$ and $g : X \to X$ be defined as follows:

$$hx = gx = \begin{cases} \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ 200, & \text{if } x > 1, \end{cases}$$

Let, for all $x, y \in X$, $\psi(x, y) = \min\{x, y\}$. Then all the conditions of Theorem 3.2 are satisfied. Here $(0, 0)$ is unique coupled common fixed point F and G, and 0 is a fixed point of h and g.

4. Applications to result in metric spaces

In this section we present a coupled coincidence point result in metric spaces. This is obtained by an application of the theorem established in the previous section.

Theorem 4.1 Let (X, d) be a complete metric space. Let $F : X \times X \to X$ and $h : X \to X$ be two mappings satisfying the following conditions:

(i) $F(X \times X) \subseteq h(X)$ and $h(X)$ is closed subsets of X,

(ii) (h, F) is weakly compatible pair,

(iii) $d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(hx, F(x, y)) + d(hu, F(u, v))]$, (4.1)

for all $x, y, u, v \in X$ and $0 < k < 1$. Then there exist $x, y \in X$ such that $x = hx = F(x, y)$ and $y = hy = F(y, x)$, that is, F and h have unique coupled common fixed point in X.

Proof. For all $x, y \in X$ and $t > 0$, we define

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

and $a * b = \min\{a, b\}$. Then, as noted earlier, $(X, M, *)$ is a fuzzy metric space.
Further, from the above definition of M, $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$.
Next we show that the inequality (4.1) implies (3.11) with $\psi(x, y) = \min\{x, y\}$. If otherwise, from (3.11), for some $t > 0$, $x, y, u, v \in X$ we have

$$\frac{t}{t + \frac{1}{k} d(F(x, y), F(u, v))} < min\{\frac{t}{t + d(hx, F(x, y))}, \frac{t}{t + d(hu, F(u, v))}\},$$

that is, $t + \frac{1}{k} d(F(x, y), F(u, v)) > t + d(hx, F(x, y))$ and

$$t + \frac{1}{k} d(F(x, y), F(u, v)) > t + d(hu, F(u, v)).$$

Combining the above two inequalities, we have that

$$d(F(x,y),F(u,v)) > \frac{k}{2}[d(hx,F(x,y))+d(hu,F(u,v))],$$

which is a contradiction with (4.1).
The proof is then completed by an application of corollary 3.2. □

5. Conclusion

We use Hadzic type t-norm in our main result. It has an advantage for use due to the fact that the iterates are equicontinuous at $s = 1$. The fuzzy metric space theory depends strongly on the type of t-norm used in its description. Our main theorem depends on a lemma which in turn depends on the aforesaid equicontinuous of the t-norm. Also the coupled fixed point for two maps are obtained under the assumption of weak compatibility between ordinary maps and coupled maps which is a concept defined in this paper. It may be used elsewhere under different conditions to obtain other fixed point results. Since this definition does not depend on the structure associated with the set on which the mappings are defined, it is possible that such definitions are used in some other spaces as well.

6. Acknowledgements

The suggestions of the learned referee have been acknowledged.

References

1. L. A. Zadeh, Fuzzy sets, Inform and Control., 8 (1965), 338-353.
2. I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica., 11 (1975), 326-334.
3. A. George and P. Veeramani, On some result in fuzzy metric space, Fuzzy Sets and Systems, 64 (1994), 395-399.
4. B.S. Choudhury and P.N. Dutta, Fixed point result for a sequence of mutually contractive self-mappings on fuzzy metric spaces, The Journal of Fuzzy Math., 13 (2005), 723-730.
5. L. Ćirić, Some new results for Banach contractions and Edelstein contractive mappings on fuzzy metric spaces, Chaos, Solitons and Fractals, 42 (2009), 146-154.
6. D. Mihet, On fuzzy contractive mappings in fuzzy metric spaces, Fuzzy Sets and Systems, 158 (2007), 915-921.
7. D. Mihet, On fuzzy ϵ-contractive mappings in fuzzy metric space, Fixed Point Theory. Appl., Article ID 87471 (2007).
8. A. Razani, A contraction theorem in fuzzy metric space, Fixed Point Theory. Appl., 2005, 257-265 (2005).
9. S. Sedghi, B. S. Choudhury and N. Shobe, Unique Common Fixed Point Theorem for Four weakly Compatible Mappings in Complete Fuzzy Metric Spaces, The Journal of Fuzzy Math., 18 (2010), 161-170.
10. B.S. Choudhury, K.P. Das and P. Das, Extensions of Banach's and Kannan's results in fuzzy metric spaces, Commun. Korean Math. Soc., 27 (2012), 265-277.
11. T. Zikic-Dosenovic, A multivalued generalization of Hick's C-contraction, Fuzzy Sets and Systems, 151 (2005), 549-562.
12. T. Gnana-Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., 65 (2006), 1379–1393.
13. B.S. Choudhury and A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal., 73 (2010), 2524-2531.
14. V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., 70 (2009), 4341-4349.
15. L. Ćirić, V. Lakshmikantham, Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, Stoch. Anal. Appl., 27 (2009), 1246-1259.
16. B. Samet, Coupled fixed point theorems for a generalised Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. TAM, 72 (2010), 4508-4517.

17. N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal., 74 (2011), 983-992.
18. J. Z. Xiao, X. H. Zhu, Y. F. Cao, Common coupled fixed point results for probabilistic φ-contractions in Menger spaces, Nonlinear Anal., 74 (2011), 4589-4600.
19. E. Karapinar, Coupled fixed point theorems for nonlinear contractions in cone metric spaces, Comput. Math. Appl., 59 (2010), 3656-3668.
20. F. Sabetghadam, H.P. Masiha, A.H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory Appl., 2009 (2009), 8. doi:10.1155/2009/125426. Art. ID 125426.
21. B.S. Choudhury and P. Maity, Coupled fixed point results in Generalised metric spaces, Mathematical and Computer Modeling, 54 (2011), 73-79.
22. R. Kannan, Some results on fixed point, Bull. Cal. Math. Soc., 60 (1968), 71-76.
23. R. Kannan, Some results on fixed points, Amer. Math. Monthly, 76 (1969), 405-408.
24. E.H. Connell, Properties of fixed point spaces, Proc. Amer. Math. Soc., 10 (1959), 974-979.
25. P.V. Subrahmanyam, Completeness and fixed points, Monatsh. Math., 80 (1975), 325-330.
26. B.S. Choudhury and K. Das, Fixed points of generalised Kannan type mappings in generalised Menger spaces, Commun. Korean Math. Soc., 24 (2009), 529-537.
27. Y. Enjouji, M. Nakanishi and T. Suzuki, A Generalization of Kannan's Fixed Point Theorem, Fixed Point Theory. Appl., Article ID 192872 (2009).
28. N. Shioji, T. Suzuki and W. Takahashi, Contractive mappings, Kannan mappings and metric completeness, Proc. Amer. Math. Soc. 126 (1998), 3117-3124.
29. P. Debnath, Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces, Computers and Mathematics with Applications, 63 (2012), 708-715.
30. P. Debnath and M. Sen, Some completeness results in terms of infinite series and quotient spaces in intuitionistic fuzzy n-normed linear spaces, Journal of Intelligent and Fuzzy Systems, 26 (2014), 975-982.
31. P. Debnath and M. Sen, Some results of calculus for functions having values in an intuitionistic fuzzy n-normed linear space, Journal of Intelligent and Fuzzy Systems, 26 (2014), 2983-2991.
32. P. Debnath, Results on lacunary difference ideal convergence in intuitionistic fuzzy normed linear spaces, Journal of Intelligent and Fuzzy Systems, 28 (2015), 1299-1306.
33. M. Sen and P. Debnath, Lacunary statistical convergence in intuitionistic fuzzy n-normed linear spaces, Mathematical and Computer Modelling, 54 (2011), 2978-2985.
34. M. Sen and P. Debnath, Statistical convergence in intuitionistic fuzzy n-normed linear spaces, Fuzzy Information and Engineering, 3 (2011), 259-273.
35. M. Sen and P. Debnath, Best approximation in intuitionistic fuzzy n-normed linear spaces, Journal of Nonlinear Analysis and Optimization: Theory and Applications, 3 (2012), 45-53.
36. B. Schweizer and A. Sklar, Statistical metric spaces. Pacific J. Math., 10 (1960), 313-334.
37. M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems, 27 (1988), 385-389.
38. J. Rodríguez López and S. Ramaguera, The Hausdorff fuzzy metric on compact sets, Fuzzy and Sets System, 147 (2004), 273-283.
39. O. *Hadžić* and E. Pap, Fixed Point Theory in Probabilistic Metric Space, Kluwer Academic Publishers, Dordrecht, (2001).
40. B.S. Choudhury, K.P. Das and P. Das, Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces, Fuzzy Sets and Systems, 222 (2013), 84-97.

D. Chumpungam & A. Kettapun, J. Nonlinear Anal. Optim. Vol. 10(1) (2019), 21-34

Journal of Nonlinear Analysis and Optimization
Volume 10(1) (2019)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

STABILITY AND RATE OF CONVERGENCE OF SOME ITERATION METHODS FOR BERINDE CONTRACTIONS

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ABSTRACT. In this paper, we first prove strong convergence theorems of a new iteration method finding a common fixed point of three Berinde nonexpansive mappings and introduce a new iteration method and study stability of the proposed method and Noor iteration for a class of Berinde contraction mappings in complete metric space. We also compare the rate of convergence between our iteration method and Noor iteration under some suitable control conditions.

KEYWORDS: stability, rate of convergence, Noor iteration, Berinde contractions.
AMS Subject Classification: 47H09, 47H10.

1. Introduction and preliminaries

Let C be a nonempty convex subset of a Banach space X, and $T : C \to C$ be a mapping. A point $x \in C$ is a fixed point of T if $Tx = x$. We denote $F(T)$ the set of all fixed points of T. There are two important problems in fixed point theory. The first one is the existence problem. Many mathematicians are interested in finding sufficient conditions to guarantee the existence of fixed point and common fixed point of mappings. The second problem is to study how to approximate a fixed point and a common fixed point of mappings. Many iteration methods were introduced and studied. Some conditions for convergence of those methods were given, see for instance [11, 9, 13, 18, 8].

In 2003, Berinde [3, 4] introduced and studied a weak contraction mapping. It is very interesting to study this mapping because it generalizes many well-known mappings such as contraction and Zamfirescu mappings.

In 2004, Berinde [5] provided the concept of how to compare the rate of convergence of the iterative methods and proved that Picard iteration converges faster than Mann iteration for a class of Zamfirescu operators and a class of quasi-contractive

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Article history : Received 4 July 2018; Accepted 3 November 2018.

operator in arbitrary Banach spaces. After that there are many works concerning comparision of the rate of convergence, see [2, 14, 19, 16, 13] for examples.

A mapping $T : X \to X$ is said to be

(1) a *contraction* if there exists $k \in [0, 1)$ such that for $x, y \in X$,

$$\|Tx - Ty\| \ \leq \ k\|x - y\|, \tag{1.1}$$

(2) *nonexpansive* if for $x, y \in X$,

$$\|Tx - Ty\| \ \leq \ \|x - y\|, \tag{1.2}$$

In 1968, Kannan extended a contraction mapping to mapping that need not be continuous. A mapping $T : X \to X$ is called a *Kannan mapping* if for $x, y \in X$, there is a constant $0 < b < \dfrac{1}{2}$ such that

$$\|Tx - Ty\| \ \leq \ b[\ \|x - Tx\| + \|y - Ty\|\ ]. \tag{1.3}$$

In 1972, Chatterjea introduced a mapping that is dual of Kannan mapping.

A mapping $T : X \to X$ is called a *Chatterjea mapping* if for $x, y \in X$, there exists $0 < c < \dfrac{1}{2}$ such that

$$\|Tx - Ty\| \ \leq \ c[\ \|x - Ty\| + \|y - Tx\|\ ]. \tag{1.4}$$

In 1972, Zamfirescu obtained a very interesting nonlinear mapping by combining (1.1), (1.3) and (1.4). A mapping $T : X \to X$ is said to be a *Zamfirescu operator* if there exist $a \in [0, 1), b, c \in (0, \dfrac{1}{2})$ such that for $x, y \in X$, satisfy at least one of the following :

(a) $\|Tx - Ty\| \ \leq \ a\|x - y\|$;
(b) $\|Tx - Ty\| \ \leq \ b[\ \|x - Tx\| + \|y - Ty\|\ ]$;
(c) $\|Tx - Ty\| \ \leq \ c[\ \|x - Ty\| + \|y - Tx\|\ ]$.

In 1974, Ciric introduced a mapping that is one of the most general contraction condition. A mapping $T : X \to X$ is called a *quasi-contraction mapping* if for $x, y \in X$, there exists $0 < h < 1$ such that

$$\|Tx - Ty\| \ \leq \ h \max\{\ \|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\ \}. \tag{1.5}$$

It is obvious any mapping that satisfies (1.1), (1.3), (1.4) and Zamfirescu mapping is a quasi-contraction mapping.

Definition 1.1. (condition $(*)$) Let X be a Banach space. A mapping $T : X \to X$ is said to satisfy condition $(*)$ if there exists a constant $\delta' \in (0, 1)$ and $L' \geq 0$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \ \leq \ \delta'\|x - y\| + L'\|y - Ty\|. \tag{1.6}$$

Definition 1.2. Let X be a Banach space. A mapping $T : X \to X$ is called a F-*contraction* if $F(T) \neq \emptyset$ and there exists $0 \leq k < 1$ such that for $x \in X,\ p \in F(T)$,

$$\|Tx - p\| \ \leq \ k\|x - p\|. \tag{1.7}$$

We can show that a F-contraction mapping with $F(T) \neq \emptyset$ has a unique fixed point and it easy to see that any mapping which satisfies condition $(*)$ (1.6) with $F(T) \neq \emptyset$ is F-contraction.

Definition 1.3. Let X be a Banach space. A mapping $T : X \to X$ with $F(T) \neq \emptyset$ is called a *quasi-nonexpansive* if for $x \in X,\ p \in F(T)$,

$$\|Tx - p\| \ \leq \ \|x - p\|. \tag{1.8}$$

It is clearly that any F-contraction mapping is quasi-nonexpansive.

Example 1.4. [7] Let $X = l^\infty$ and $C = \{\{x_n\} : -1 \leq x_1 \leq 3,\ -1 \leq x_2 \leq 1,\ x_n = 0,\ \forall n \geq 3\}$. Define $T : C \to C$ by

$$T(x_1, x_2, 0, ...) = (x_1, -x_2, 0, ...), \ \ \forall x_2 \neq 0,$$

$$T(x_1, 0, ...) = \begin{cases} (x_1, |x_1|, 0, ...), & \text{if } -1 \leq x_1 \leq 1, \\ (x_1, |x_1 - 2|, 0, ...), & \text{if } 1 \leq x_1 \leq 3. \end{cases}$$

Then T is a quasi-nonexpansive mapping with $F(T) = \{(0, 0, 0, ...), (2, 0, 0, ...)\}$.

In 2003, Berinde [3] introduced a weak contraction mapping and proved the following existence fixed point theorem in Banach spaces.

Definition 1.5. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called *weak contraction or Berinde contraction* if there exists a constant $\delta \in (0, 1)$ and $L \geq 0$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \ \leq \ \delta\|x - y\| + L\|y - Tx\|. \tag{1.9}$$

The class of Berinde contraction mappings includes classes of contraction, Kannan, Zamfirescu, Chatterjea and quasi-contraction mappings.

Proposition 1.6. [3] Let C be a nonempty closed subset of a Banach space X and $T : C \to C$ be a weak contraction, Then $F(T) \neq \emptyset$. Moreover, the Picard iteration $\{x_n\}$ defined by $x_1 \in C$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, converges to a fixed point of T.

Let C be a nonempty convex subset of a Banach space X, and $T : C \to C$ be a mapping. The *Mann iteration* is defined by $s_0 \in C$,

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n Ts_n, \ \text{for all} \ \ n \geq 0, \tag{1.10}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.
For $\alpha_n = \lambda$ (constant), the iteration (1.10) reduces to the so-called Krasnoselskij iteration. For $\alpha_n = 1$, we obtain the Picard iteration.
The *Ishikawa iteration* is defined by $s_0 \in C$,

$$\begin{aligned} w_n &= (1 - \beta_n)s_n + \beta_n Ts_n, \\ s_{n+1} &= (1 - \alpha_n)s_n + \alpha_n Tw_n, \text{ for all } n \geq 0, \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.
The *Noor iteration* is defined by $s_0 \in C$,

$$\begin{aligned} h_n &= (1 - \gamma_n)s_n + \gamma_n Ts_n, \\ w_n &= (1 - \beta_n)s_n + \beta_n Th_n, \\ s_{n+1} &= (1 - \alpha_n)s_n + \alpha_n Tw_n, \ \text{for all} \ \ n \geq 0, \end{aligned} \tag{1.12}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.
It is easy to see that Mann and Ishikawa iterations are special cases of Noor iterations.

In this paper, we introduce an iterative method as follows.
Let $\{x_n\}$ be a sequence defined by $x_0 \in C$,

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n, \\ y_n &= (1 - \beta_n)z_n + \beta_n Tz_n, \\ x_{n+1} &= (1 - \alpha_n)Tz_n + \alpha_n Ty_n, \ \text{for all} \ \ n \geq 0, \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. The main purpose of the paper is to study stability of the proposed method and Noor iteration for a class of Berinde contraction mappings in a complete metric space. We also compare rate of convergence between our iteration method and Noor iteration under some suitable control conditions.

2. Convergence Theorem and Stability

We first recall the definition of Berinde nonexpansive mappings introduced by Kosol [10] as follow:

Definition 2.1. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called *Berinde nonexpansive* if there exists a constant $L \geq 0$ such that for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\| + L\|y - Tx\|. \tag{2.1}$$

It is easy to see that all nonexpansive mappings and weak contraction mappings are Berinde nonexpansive.

Example 2.2. Let $X = \mathbb{R}$ and $C = [0,1]$. Define $T : C \to C$ by

$$T(x) = \begin{cases} x^2, & \text{if } x \in [0, \frac{1}{2}), \\ 1, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then T is a Berinde nonexpansive with $L = 4$, but is not a nonexpansive mapping.

Proof. (i). If $x, y \in [0, \frac{1}{2})$, we have

$$\begin{aligned} |Tx - Ty| = |x^2 - y^2| &= |(x+y)(x-y)| \\ &= |x+y||x-y| \\ &\leq |x-y|. \end{aligned}$$

(ii). If $x \in [0, \frac{1}{2}), y \in [\frac{1}{2}, 1]$,

$$\begin{aligned} |Tx - Ty| = |x^2 - 1| &= 1 - x^2 \\ &\leq 4 \cdot \frac{1}{4} \\ &\leq 4|y - Tx|, \end{aligned}$$

and $|Tx - Ty| \leq |x - y| + 4|x - Ty|$.

(iii). If $x, y \in [\frac{1}{2}, 1]$, then $|Tx - Ty| = 0 \leq |x - y|$.

So, we have $|Tx-Ty| \leq |x-y|+4|x-Ty|$ and $|Tx-Tx| \leq |x-y|+4|y-Tx|$, for all $x, y \in [0,1]$. So T is a Berinde nonexpansive mapping, but it is not a nonexpansive mapping because $|T(\frac{1}{3}) - T(1)| \geq |\frac{1}{3} - 1|$. □

Let $T_i : C \longrightarrow C, i = 1,2,3$ be mappings. In order to approximate a common fixed point of Berinde nonexpansive mappings, we introduce the following iterative method. Let $\{x_n\}$ be a sequence defined by $x_0 \in C$,

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T_1 x_n, \\ y_n &= (1-\beta_n)z_n + \beta_n T_2 z_n, \\ x_{n+1} &= (1-\alpha_n)T_3 z_n + \alpha_n T_3 y_n, \text{ for all } n \geq 0, \end{aligned} \tag{2.2}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.
The Noor iteration is defined by $s_0 \in C$ and

$$\begin{aligned} h_n &= (1-\gamma_n)s_n + \gamma_n T_1 s_n, \\ w_n &= (1-\beta_n)s_n + \beta_n T_2 h_n, \\ s_{n+1} &= (1-\alpha_n)s_n + \alpha_n T_3 w_n, \text{ for all } n \in \mathbb{N}, \end{aligned} \tag{2.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

First, we will prove a convergence theorem of the proposed iteration method (2.2) for finding a common fixed point of Berinde nonexpansive mappings.

Let $T_i : C \to C,\ i = 1,2,3,$ be Berinde nonexpansive mappings. Through out this thesis, we let $L_i, i = 1,2,3,$ be nonnegative real numbers such that for $x, y \in C$,

$$\|T_i x - T_i y\| \le \|x-y\| + L_i\|y - T_i x\|.$$

Lemma 2.3. [6] Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\},\ r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty),\ g(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z\|^2 \le \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta \cdot g(\|x-y\|),$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 2.4. Let X be a Banach space and C be a nonempty closed convex subset of X. For each $i = 1,2,3.$, let $T_i : C \to C$ be a quasi-nonexpansive mapping. Assume that $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and $\{x_n\}$ is a sequence generated by (2.2) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$. Then,

(i) $\|x_{n+1} - p\| \le \|x_n - p\|, \quad \forall n \in \mathbb{N}$ and $\forall p \in \bigcap_{i=1}^{3} F(T_i)$.

(ii) $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Let $p \in \bigcap_{i=1}^{3} F(T_i)$. By using (2.2), we have

$$\begin{aligned} \|z_n - p\| &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|T_1 x_n - p\| \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\|, \end{aligned}$$

$$\begin{aligned} \|y_n - p\| &\le (1-\beta_n)\|z_n - p\| + \beta_n\|T_2 z_n - p\| \\ &\le (1-\beta_n)\|z_n - p\| + \beta_n\|z_n - p\| \\ &\le \|x_n - p\|. \end{aligned}$$

From above inequalities, we get

$$\begin{aligned} \|x_{n+1} - p\| &\le (1-\alpha_n)\|T_3 z_n - p\| + \alpha_n\|T_3 y_n - p\| \\ &\le (1-\alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\| \\ &\le \|x_n - p\|, \end{aligned}$$

so $\|x_{n+1} - p\| \le \|x_n - p\|$. Since $\{\|x_n - p\|\}$ is a non-increasing sequence and bounded below by 0, $\lim_{n\to\infty} \|x_n - p\|$ exists. ∎

Theorem 2.5. Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. For each $i = 1, 2, 3$, let $T_i : C \to C$ be a Berinde nonexpansive and quasi-nonexpansive mapping. Assume that $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and T_i is demicompact, for some $i \in \{1, 2\}$. Let $\{x_n\}$ be a sequence generated by (2.2) where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ which satisfy the following conditions :

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{3}$.

Proof. Let $p \in \bigcap_{i=1}^{3} F(T_i)$. From Lemma 2.4, $\lim_{n\to\infty} \|x_n - p\|$ exists, then we have $\{\|x_n - p\|\}$ is bounded, that is, $\exists M > 0$ such that for each $n \in \mathbb{N}$, $\|x_n - p\| \leq M$. By quasi-nonexpansiveness of T_i, $\{x_n - p\}, \{T_1 x_n - p\}, \{z_n - p\}, \{T_2 z_n - p\}, \{T_3 z_n - p\}, \{T_3 y_n - p\} \subset B_M$. By Lemma 2.3, there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, with $g(0) = 0$ such that

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1-\gamma_n)(x_n - p) + \gamma_n(T_1 x_n - p)\|^2 \\
&\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|T_1 x_n - p\|^2 - (1-\gamma_n)\gamma_n g(\|x_n - T_1 x_n\|) \\
&\leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 - (1-\gamma_n)\gamma_n g(\|x_n - T_1 x_n\|) \\
&= \|x_n - p\|^2 - (1-\gamma_n)\gamma_n g(\|x_n - T_1 x_n\|),
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1-\beta_n)(z_n - p) + \beta_n(T_2 z_n - p)\|^2 \\
&\leq (1-\beta_n)\|z_n - p\|^2 + \beta_n\|T_2 z_n - p\|^2 - (1-\beta_n)\beta_n g(\|z_n - T_2 z_n\|) \\
&\leq (1-\beta_n)\|z_n - p\|^2 + \beta_n\|z_n - p\|^2 - (1-\beta_n)\beta_n g(\|z_n - T_2 z_n\|) \\
&= \|z_n - p\|^2 - (1-\beta_n)\beta_n g(\|z_n - T_2 z_n\|).
\end{aligned}$$

From above inequalities, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(T_3 z_n - p) + \alpha_n(T_3 y_n - p)\|^2 \\
&\leq (1-\alpha_n)\|z_n - p\|^2 + \alpha_n\|y_n - p\|^2 - (1-\alpha_n)\alpha_n g(\|T_3 z_n - T_3 y_n\|) \\
&\leq (1-\alpha_n)\|z_n - p\|^2 + \alpha_n\|z_n - p\|^2 - (1-\beta_n)\beta_n\alpha_n g(\|z_n - T_2 z_n\|) \\
&\quad - (1-\alpha_n)\alpha_n g(\|T_3 z_n - T_3 y_n\|) \\
&= \|z_n - p\|^2 - (1-\beta_n)\beta_n\alpha_n g(\|z_n - T_2 z_n\|) \\
&\quad - (1-\alpha_n)\alpha_n g(\|T_3 z_n - T_3 y_n\|) \\
&\leq \|x_n - p\|^2 - (1-\gamma_n)\gamma_n g(\|x_n - T_1 x_n\|) \\
&\quad - (1-\beta_n)\beta_n\alpha_n g(\|z_n - T_2 z_n\|) - (1-\alpha_n)\alpha_n g(\|T_3 z_n - T_3 y_n\|).
\end{aligned}$$

By assumptions on the control sequences, there exist $n_0 \in \mathbb{N}$ and $\eta_1, \eta_2 \in (0, 1)$ such that $0 < \eta_1 < \min\{\alpha_n, \beta_n, \gamma_n\}$ and $\max\{\alpha_n, \beta_n, \gamma_n\} < \eta_2 < 1$, for all $n \geq n_0$. Then,

$$\begin{aligned}
\eta_1(1-\eta_2) g(\|x_n - T_1 x_n\|) &\leq (1-\gamma_n)\gamma_n g(\|x_n - T_1 x_n\|) \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2,
\end{aligned}$$

$$\begin{aligned}
\eta_1^2(1-\eta_2)g(\|z_n - T_2z_n\|) &\leq (1-\beta_n)\beta_n\alpha_n g(\|z_n - T_2z_n\|)\\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2,\\
\eta_1(1-\eta_2)g(\|T_3z_n - T_3y_n\|) &\leq (1-\alpha_n)\alpha_n g(\|T_3z_n - T_3y_n\|)\\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists, it implies that $\lim_{n\to\infty} g(\|x_n - T_1x_n\|) = \lim_{n\to\infty} g(\|z_n - T_2z_n\|) = \lim_{n\to\infty} g(\|T_3z_n - T_3y_n\|) = 0$. Since g is continuous and $g(0) = 0$, we have $\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|z_n - T_2z_n\| = \lim_{n\to\infty}\|T_3z_n - T_3y_n\| = 0$. It follows that

$$\begin{aligned}
\|z_n - x_n\| &\leq \gamma_n\|x_n - T_1x_n\|\\
&\leq \|x_n - T_1x_n\| \quad \to 0,\\
\|y_n - z_n\| &\leq \beta_n\|z_n - T_2z_n\|\\
&\leq \|z_n - T_2z_n\| \quad \to 0,\\
\|x_{n+1} - T_3z_n\| &\leq \alpha_n\|T_3z_n - T_3y_n\|\\
&\leq \|T_3z_n - T_3y_n\| \quad \to 0.
\end{aligned}$$

By Berinde nonexpansiveness of T_2, we have

$$\begin{aligned}
\|x_n - T_2x_n\| &\leq \|x_n - z_n\| + \|z_n - T_2z_n\| + \|T_2z_n - T_2x_n\|\\
&\leq \|x_n - z_n\| + \|z_n - T_2z_n\| + (\ \|z_n - x_n\| + L_2\|x_n - T_2z_n\|\ )\\
&\leq \|x_n - z_n\| + \|z_n - T_2z_n\| + \|z_n - x_n\|\\
&\qquad + L_2(\ \|x_n - z_n\| + \|z_n - T_2z_n\|\ ) \to 0.
\end{aligned}$$

It implies that $\lim_{n\to\infty}\|x_n - T_2x_n\| = 0$. Now, suppose that T_{i_0} is demicompact, for some $i_0 \in \{1, 2\}$. Then $\exists\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to q$, $\exists q$. From above inequalities, we have

$$\begin{aligned}
\|T_1x_{n_k} - T_1q\| &\leq \|x_{n_k} - q\| + L_1\|q - T_1x_{n_k}\|\\
&\leq \|x_{n_k} - q\| + L_1(\ \|q - x_{n_k}\| + \|x_{n_k} - T_1x_{n_k}\|\ ) \to 0,
\end{aligned}$$

$$\begin{aligned}
\|T_2z_{n_k} - T_2q\| &\leq \|z_{n_k} - q\| + L_2\|q - T_2z_{n_k}\|\\
&\leq \|z_{n_k} - x_{n_k}\| + \|x_{n_k} - q\|\\
&\qquad + L_2(\ \|q - x_{n_k}\| + \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - T_2z_{n_k}\|\ ) \to 0,
\end{aligned}$$

and

$$\begin{aligned}
\|T_3y_{n_k} - T_3q\| &\leq \|y_{n_k} - q\| + L_3\|q - T_3y_{n_k}\|\\
&\leq \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| + \|x_{n_k} - q\|\\
&\qquad + L_3(\ \|q - x_{n_k+1}\| + \|x_{n_k+1} - T_3z_{n_k}\| + \|T_3z_{n_k} - T_3y_{n_k}\|\ ) \to 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|q - T_1q\| &\leq \|q - x_{n_k}\| + \|x_{n_k} - T_1x_{n_k}\| + \|T_1x_{n_k} - T_1q\| \to 0,\\
\|q - T_2q\| &\leq \|q - x_{n_k}\| + \|x_{n_k} - z_{n_k}\| + \|z_{n_k} - T_2z_{n_k}\|\\
&\qquad + \|T_2z_{n_k} - T_2q\| \to 0,\\
\|q - T_3q\| &\leq \|q - x_{n_k+1}\| + \|x_{n_k+1} - T_3z_{n_k}\| + \|T_3z_{n_k} - T_3y_{n_k}\|\\
&\qquad + \|T_3y_{n_k} - T_3q\| \to 0.
\end{aligned}$$

So $q \in \bigcap_{i=1}^{3} F(T_i)$. By Theorem 2.4, $\lim_{n\to\infty} \|x_n - q\|$ exists. Since $\|x_{n_k} - q\| \to 0$, it implies that $\lim_{n\to\infty} x_n = q$. □

Theorem 2.6. [20] Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. For each $i = 1, 2, 3.$, let $T_i : C \to C$ be a Berinde nonexpansive and quasi-nonexpansive mapping. Assume that $\bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and T_i is a demicompact, for some $i \in \{1, 2, 3\}$. Suppose $\{s_n\}$ is a sequence generated by Noor iteration (2.3) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ which satisfy the following conditions :

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then $\lim_{n\to\infty} \|s_n - T_i s_n\| = 0$, for all $i = 1, 2, 3$ and $\{s_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{3}$.

In concrete applications, when calculating $\{x_n\}$, we usually follow the step :

(i) We choose the initail approximation $x_0 \in X$.
(ii) We compute $x_1 = f(T, x_0)$. Because of the various error, we do not get the exact value of x_1, but a different one, say y_1, which is however closed enough to x_1, i.e., $y_1 \approx x_1$.
(iii) Consequently, when computing $x_2 = f(T, x_1)$ we will actually compute x_2 as $x_2 = f(T, y_1)$, and error again from the computations, we will obtain in fact another valued, say y_2, closed enough to x_2, i.e., $y_2 \approx x_2$, and so on.

In this way, instead of the theoretical sequence $\{x_n\}$ defined by the given iterative method, we will practically obtain an *approximate sequence* $\{y_n\}$. We shall consider the given fixed point iteration method to be numerically **stable** if and only if for $\{y_n\}$ closed enough to $\{x_n\}$ at each stage, the approximate sequence $\{y_n\}$ still converges to the fixed point of T.

Definition 2.7. Let $\{x_n\}$ be a sequence in above procedure and converge to a fixed point p of T. Let $\{y_n\}$ be an arbitary sequence in X and set

$$\varepsilon_n = \|y_{n+1} - f(T, y_n)\|.$$

We shall say that the fixed point iteration procedure $\{x_n\}$ is T-stable or stable with respect to T if

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} y_n = p.$$

Theorem 2.8. Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $T : C \to C$ be a weak contraction and F-contraction mapping. Suppose $\{x_n\}$ is a sequence generated by (1.13) and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ which satisfy the following conditions :

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Let $\{y_n\}$ be an arbitary sequence in C and define

$$s_n = (1-\gamma_n)y_n + \gamma_n T y_n,$$
$$h_n = (1-\beta_n)s_n + \beta_n T s_n,$$
$$\epsilon_n = \|y_{n+1} - ((1-\alpha_n)Ts_n + \alpha_n Th_n)\|.$$

Then $\{x_n\}$ is T-stable.

Proof. By Proposition 1.6 and Theorem 2.5, $\{x_n\}$ converges strongly to a unique fixed point of T, say x^*. Since T is a weak contraction and F-contraction, we have

$$\begin{aligned}
\|(1-\alpha_n)Ts_n + \alpha_n Th_n - x^*\| &\leq (1-\alpha_n)\|Ts_n - x^*\| + \alpha_n\|Th_n - x^*\| \\
&\leq (1-\alpha_n)\delta\|s_n - x^*\| + \alpha_n\delta\|h_n - x^*\| \\
&\leq (1-\alpha_n)\delta\|s_n - x^*\| \\
&\qquad + \alpha_n\delta\,[\,(1-\beta_n)\|s_n - x^*\| + \beta_n\delta\|s_n - x^*\|\,] \\
&= [\,\delta(1-\alpha_n + \alpha_n(1-\beta_n) + \alpha_n\beta_n\delta)\,]\|s_n - x^*\| \\
&= \delta(1-\alpha_n\beta_n(1-\delta))\|s_n - x^*\| \\
&\leq \delta(1-\alpha_n\beta_n(1-\delta))\,[\,(1-\gamma_n)\|y_n - x^*\| + \gamma_n\delta\|y_n - x^*\|\,] \\
&= \delta(1-\alpha_n\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|y_n - x^*\| \\
&\leq (1-\alpha_n\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|y_n - x^*\|.
\end{aligned}$$

Next, assume that $\lim_{n\to\infty}\varepsilon_n = 0$. By above inequality, we have

$$\begin{aligned}
\|y_{n+1} - x^*\| &\leq \|y_{n+1} - [(1-\alpha_n)Ts_n + \alpha_n Th_n]\| + \|[(1-\alpha_n)Ts_n + \alpha_n Th_n] - x^*\| \\
&\leq \varepsilon_n + (1-\alpha_n\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|y_n - x^*\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\varepsilon_n = 0$, by assumption of control sequences and Lemma 2.4, we conclude that $\lim_{n\to\infty} y_n = x^*$. Conversely, suppose that $\lim_{n\to\infty} y_n = x^*$, then

$$\begin{aligned}
\varepsilon_n &= \|y_{n+1} - ((1-\alpha_n)Ts_n + \alpha_n Th_n)\| \\
&\leq \|y_{n+1} - x^*\| + \|x^* - [(1-\alpha_n)Ts_n + \alpha_n Th_n]\| \\
&\leq \|y_{n+1} - x^*\| + (1-\alpha_n\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|y_n - x^*\| \\
&\leq \|y_{n+1} - x^*\| + \|y_n - x^*\|.
\end{aligned}$$

Since $y_n \to x^*$, we obtain that $\lim_{n\to\infty}\varepsilon_n = 0$.
Hence $\{x_n\}$ is T-stable. □

Theorem 2.9. [20] Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ be a weak contraction and F-contraction mapping. Suppose that $\{s_n\}$ is a sequence generated by Noor iteration (1.12) where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ which satisfy the following conditions :

(i) $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < 1$.

Let $\{y_n\}$ be an arbitary sequence in C and define

$$u_n = (1-\gamma_n)y_n + \gamma_n T y_n,$$
$$z_n = (1-\beta_n)y_n + \beta_n T u_n,$$
$$\epsilon_n = \|y_{n+1} - [\,(1-\alpha_n)y_n + \alpha_n T z_n\,]\|.$$

Then $\{s_n\}$ is T-stable.

3. The Rate of Convergence Theorem

There are a few papers concerning comparison of the rate of convergence of iteration methods. In 1976, Rhoades [15] introduced the concept to compare the rate of convergence of iterative methods as follows :

Definition 3.1. Let $\{x_n\}$ and $\{z_n\}$ be two iteration methods which converge to the same fixed point p, we shall say that $\{x_n\}$ *converges faster* than $\{z_n\}$ if

$$\|x_n - p\| \ \leq \ \|z_n - p\|, \quad \text{for all } n \in \mathbb{N}.$$

In 2004, Berinde [5] provided the following concept to compare the rate of convergence of the iterative methods.

Definition 3.2. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l \ = \ \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

(i) If $l = 0$, then it can be said that $\{a_n\}$ converges *faster* to a than $\{b_n\}$ to b.
(ii) If $0 < l < \infty$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Remark 3.3. (i) In the case 1. we use the notation $a_n - a = o(b_n - b)$.
(ii) If $l = \infty$, then the sequence $\{b_n\}$ converges faster than $\{a_n\}$, that is $b_n - b = o(a_n - a)$.

Suppose that for two fixed point iteration methods $\{x_n\}$ and $\{y_n\}$, both converging to the same fixed point p, the error estimates

$$\|x_n - p\| \ \leq \ a_n, \quad n = 0, 1, 2, 3, ...$$
$$\|y_n - p\| \ \leq \ b_n, \quad n = 0, 1, 2, 3, ...$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero). Then, in view of Definition 3.2, the following concept appears to be very natural.

Definition 3.4. Let $\{x_n\}$ and $\{y_n\}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy above inequalities. If $\{a_n\}$ converges faster than $\{b_n\}$, then it can be said that $\{x_n\}$ *converges faster* than $\{y_n\}$ to p.

To comparision the rate of convergence in above definition depends on the error estimate sequences. So, in 2013, Phuengrattana and Suantai [13] modified above definition to compare the rate of convergence as follows :

Definition 3.5. Let $\{x_n\}$ and $\{y_n\}$ be two iterative methods converging to the same fixed point z of a mapping T. We say that $\{x_n\}$ converges faster than $\{y_n\}$ to z if

$$\lim_{n\to\infty} \frac{\|x_n - z\|}{\|y_n - z\|} \ = \ 0.$$

Theorem 3.6. Let C be a nonempty closed convex subset of Banach space X and let $T : C \to C$ be a weak contraction and F-contraction mapping. Suppose $\{x_n\}, \{p_n\}$ and $\{s_n\}$ are sequences generated by (1.13) and Noor iteration (1.12), respectively, which converge to a fixed point of T where $x_0 = s_0$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$,

are sequences in $[0,1]$. Then,
if $0 < \alpha_n < \frac{1}{1+\delta}$, $\frac{\alpha_n(1+\delta)}{(1-\delta)} \leq \gamma_n < 1$ and $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$,
then $\{x_n\}$ converges faster than $\{s_n\}$.

Proof. By Proposition 1.6, $F(T)$ is nonempty. Since T is a F-contraction mapping, we obtain that a fixed point of map T is unique, say p. and by assumption, $\{x_n\}$ and $\{s_n\}$ converge to p.
First, from iteration (1.13), we have

$$\begin{aligned}
\|[x_{n+1} - p\| &\leq (1-\alpha_n)\|Tz_n - p\| + \alpha_n\|Ty_n - p\| \\
&\leq (1-\alpha_n)\delta\|z_n - p\| + \alpha_n\delta\|y_n - p\| \\
&\leq (1-\alpha_n)\delta\|z_n - p\| + \alpha_n\delta \left[\ (1-\beta_n)\|z_n - p\| + \beta_n\delta\|z_n - p\|\ \right] \\
&= \left[\ \delta(1-\alpha_n + \alpha_n(1-\beta_n) + \alpha_n\beta_n\delta)\ \right]\|z_n - p\| \\
&= \delta(1-\alpha_n\beta_n(1-\delta))\|z_n - p\| \\
&\leq \delta(1-\alpha_n\beta_n(1-\delta))\left[\ (1-\gamma_n)\|x_n - p\| + \gamma_n\delta\|x_n - p\|\ \right] \\
&= \delta(1-\alpha_n\beta_n(1-\delta))(1-\gamma_n(1-\delta))\|x_n - p\| \\
&\ \vdots \\
&\leq \delta^{n+1}\prod_{k=1}^{n+1}(1-\alpha_k\beta_k(1-\delta))(1-\gamma_k(1-\delta))\|x_0 - p\|. \qquad (3.1)
\end{aligned}$$

Next, by iteration (1.12), we have

$$\begin{aligned}
\|s_{n+1} - p\| &= \|(1-\alpha_n)(s_n - p) + \alpha_n(Tw_n - p)\| \\
&\geq (1-\alpha_n)\|s_n - p\| - \alpha_n\|Tw_n - p\| \\
&\geq (1-\alpha_n)\|s_n - p\| - \alpha_n\delta\|w_n - p\| \\
&\geq (1-\alpha_n)\|s_n - p\| - \alpha_n\delta(\ (1-\beta_n)\|s_n - p\| + \beta_n\|Th_n - p\|\ ) \\
&\geq (1-\alpha_n)\|s_n - p\| - \alpha_n\delta(\ (1-\beta_n)\|s_n - p\| + \beta_n\delta\|h_n - p\|\ ) \\
&\geq (1-\alpha_n - \alpha_n\delta(1-\beta_n))\|s_n - p\| \\
&\quad - \alpha_n\beta_n\delta^2(1-\gamma_n(1-\delta))\|s_n - p\| \\
&= (1-\alpha_n(1+\delta(1-\beta_n(1-\delta(1-\gamma_n(1-\delta))))))\|s_n - p\| \\
&\geq (1-\alpha_n(1+\delta))\|s_n - p\| \\
&\ \vdots \\
&\geq \prod_{k=1}^{n+1}(1-\alpha_k(1+\delta))\|s_0 - p\|.
\end{aligned}$$

Then

$$\frac{1}{\|s_{n+1} - p\|} \leq \frac{1}{\prod_{k=1}^{n+1}(1-\alpha_k(1+\delta))\|s_0 - p\|}. \qquad (3.2)$$

It follows by (3.1) and (3.2) that

$$\frac{\|x_{n+1} - p\|}{\|s_{n+1} - p\|} \leq \frac{\prod_{k=1}^{n+1}(1-\alpha_k\beta_k(1-\delta))(1-\gamma_k(1-\delta))}{\prod_{k=1}^{n+1}(1-\alpha_k(1+\delta))}$$

$$\leq \prod_{k=1}^{n+1}(1-\alpha_k\beta_k(1-\delta)) \quad \to 0.$$

Then $\{x_n\}$ converges faster than $\{s_n\}$. □

Example 3.7. [13] Let $T : [0,1] \to [0,1]$ be defined by

$$Tx = \begin{cases} \frac{x}{3}, & \text{if } x \in [0, \frac{2}{5}), \\ \frac{2x}{5}, & \text{if } x \in [\frac{2}{5}, 1]. \end{cases}$$

Then T is a weak contraction and F-contraction mapping.

Proof. Let $x, y \in [0,1]$.
If $x, y \in [0, \frac{2}{5})$,

$$|Tx - Ty| = \left|\frac{x}{3} - \frac{y}{3}\right| = \frac{1}{3}|x-y|.$$

If $x, y \in [\frac{2}{5}, 1]$,

$$|Tx - Ty| = \left|\frac{2x}{5} - \frac{2y}{5}\right| = \frac{2}{5}|x-y|.$$

If $x \in [0, \frac{2}{5})$ and $y \in [\frac{2}{5}, 1]$,

$$\begin{aligned} |Tx - Ty| &= \left|\frac{x}{3} - \frac{2y}{5}\right| \leq \frac{1}{3}|x-y| + \left|\frac{y}{3} - \frac{2y}{5}\right| \\ &\leq \frac{1}{3}|x-y| + \frac{1}{15}. \\ &\leq \frac{1}{3}|x-y| + |Tx-y|. \end{aligned}$$

Choose $\delta = \frac{2}{5}$ and $L = 1$, so T is a weak contraction. With the same argument as above, we can show that T satisfies condition (1.1) with $\delta' = \frac{2}{5}$, $L' = \frac{1}{4}$. So T is a F-contraction. □

Let $\{x_n\}$ and $\{s_n\}$ be sequences generated by iteration (1.13) and Noor iteration (1.12), respectively. The comparison of the convergence, we assume that the initial point $x_0 = s_0 = 1$ and the control conditions $\alpha_n = \beta_n = \lambda_n = \frac{1}{3(n^{0.2}+1)}$ and $\gamma_n = \frac{1}{n^{0.2}+1}$. Then these control conditions satisfy Theorem 3.6.

Proof. We know that $\{n^{0.2}+1\}$ is a strictly increasing sequence in $[2,\infty)$ and by above example, we have $1+\delta = \frac{7}{5}$ and $1-\delta = \frac{3}{5}$. Then

$$\frac{1}{3(n^{0.2}+1)} \leq \frac{1}{n^{0.2}+1} < \frac{5}{7} = \frac{1}{1+\delta},$$

that is $\alpha_n + \beta_n + \lambda_n < \frac{1}{1+\delta}$. It is clearly that $\lim_{n\to\infty} \frac{1}{n^{0.2}+1} = 0$. So we obtain that $\alpha_n + \beta_n + \lambda_n \to 0$. Next,

$$\alpha_n(1+\delta) = \frac{7}{5}\alpha_n \leq \frac{9}{5}\alpha_n = \gamma_n(1-\delta).$$

Then $\dfrac{\alpha_n(1+\delta)}{(1-\delta)} \leq \gamma_n < 1$. Next, we will show that $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.
Since $9(n^{0.2}+1)^2 \leq 9(n^{0.2}+n^{0.2})^2 = 9(2n^{0.2})^2 = 36n^{0.4}$, then we get $\dfrac{1}{36n^{0.4}} \leq \dfrac{1}{9(n^{0.2}+1)^2}$. By the p-series ($p \leq 1$), implies that $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$.
Moreover, by the same argument as above we can show that $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \gamma_n = \infty$.
It make all sequences, $\{x_n\}$ and $\{s_n\}$, converge to unique fixed point of T, that is 0. □

n	Iteration (1.13) $\{x_n\}$	Noor iteration $\{s_n\}$
1	0.2753333333333333	0.892
2	0.0622866840398323	0.8028618015300142
3	0.0143843416773601	0.7263593546244763
4	0.0033696365323228	0.6595078099795465
5	7.9797524777686E-4	0.6004531818917709
⋮	⋮	⋮
22	3.9117640537377E-14	0.1322059479152645
23	9.9227426469139E-15	0.1215354334180748
24	2.5215123540173E-15	0.1117829732321454
25	6.4183893837157E-16	0.1028630625344521
26	1.6364198777878E-16	0.0946989830319073
27	4.1786674254824E-17	0.0872217855864026
28	1.0686318847965E-17	0.0803694081047037
29	2.7367843754926E-18	0.0740859078126916
30	7.0186215448254E-19	0.0683207907741172

Table 1. Comparison of the rate of convergence of the iterative methods (1.13) and Noor iterations for the mapping given in Example 3.7

4. Acknowledgements

This research was supported by the Research fund for DPST graduate with first placement and Chiang Mai University. The authors would like to thank Professor Dr. Suthep Suantai for valuable comments and suggestions for improving this work.

References

1. M. Arshad, A. Azom, M. Abbas and A. Shoaib, Fixed points results of dominated mappings on a closed ball in ordered partial metric spaces without continuity, U.P.B. Sci. Bull., Series A 76 (2014), 123-134.
2. G. V. Babu and K. N. Prasad, Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators, Fixed Point Theory Appl. 2006 (2006), Article ID49615.
3. V. Berinde, Approximation fixed points of weak φ-contractions using the Picard iteration, Fixed Point Theory 4 (2003), 131-142.

4. V. Berinde, Approximation fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum 9 (2004), 43-53.
5. V. Berinde, Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, Fixed Point Theory Appl. 2 (2004), 97-105.
6. Y. J. Cho, H. Y. Zhou and G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Appl. Math. 47 (2004), 707-717.
7. W. G. Doston, Fixed points of quasi-nonexpansive mappings, J. Aust. Math. Soc. 13 (1972), 167-170.
8. F. Gürsoy, A Picard-S Iterative scheme for approximating fixed point of weak contraction mappings, arXiv:1404.0241 (2014).
9. S. Isikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974), 147-150.
10. S. Kosol, Weak and strong convergence theorems of some iterative methods for common fixed point of Berinde nonexpansive mappings, Thai J. Math., Series A 15(3) (2017), 629-639.
11. W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953), 506-510.
12. M. O. Osilike, Stability results for the Ishikawa fixed point procedures, Indian J. Pure Appl. Math. 26 (1995), 937-945.
13. W. Phuengrattana and S. Suantai , Comparison of the rate of convergence of various iterative methods for the class of weak contractions in Banach spaces, Thai J. Math. 11 (2013), 217-226.
14. Y. Qing and B. E. Rhoades, Comments on the rate of convergence between Mann and Ishikawa iterations applied to Zamfirescu operators, Fixed Point Theory Appl. 2008 (2008), Article ID 387504.
15. B. E. Rhoades, Comments on two fixed point iteration methods, Math. Anal. Appl. 56 (1976), 741-750.
16. B. E. Rhoades and Z. Xue, Comparison of the rate of convergence among Picard, Mann, Ishikawa, and Noor iterations applied to quasicontractive maps, Fixed Point Theory Appl. 2010 (2010), Article ID 169062.
17. B. E. Rhoades, Fixed point theorems and stability results for fixed iteration procedures, Indian J. Pure Appl. Math. 21 (1990), 1-9.
18. G. Sherly and P. Shaini, Convergence theorems for the class of Zamfiresu operators, International Mathematical Forum 7 (2012), 1785-1792.
19. Z. Xue, The comparison of the convergence speed between Picard, Mann, Krasnoselskij and Ishikawa iterations in Banach spaces, Fixed Point Theory Appl. 2008 (2008), Article ID 387056.
20. D. Chumpungam and S. Suantai, Strong convergence theorems of Noor iterative method for common fixed points of some generalized nonexpansive in Banach spaces, In Proceedings of the ”21st Annual Meeting in Mathematics and Annual Pure and Applied Mathematics Conference”, Chulalongkorn University (2016) , 179-186

S. Sharma & M. Chaudhary, J. Nonlinear Anal. Optim. Vol. 10(1) (2019), 35-47

Journal of Nonlinear Analysis and Optimization
Volume 10(1) (2019)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

OPTIMALITY AND DUALITY FOR SET-VALUED FRACTIONAL PROGRAMMING INVOLVING GENERALIZED CONE INVEXITY

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ABSTRACT. In this paper, a new class of generalized preinvex set-valued maps is introduced and its characterization in terms of their contingent epi-derivatives is obtained. Then we derive necessary and sufficient optimality conditions for a set-valued fractional programming problem using generalized cone invexity. Wolfe and Mond Weir type duals are formulated and various duality results are established.

KEYWORDS: Convex cone; set-valued map; optimality conditions; contingent epiderivative, duality.
AMS Subject Classification: Primary 90C46; Secondary 45N15.

1. INTRODUCTION

In the last decade, there has been increasing interest in the extension of vector optimization to set-valued optimization. The theory of set-valued optimization problems has wide applications in differential inclusion, variational inequality, optimal control, game theory, economic equilibrium problem, viability theory etc. Realizing the importance of the application of the set-valued maps, it becomes essential to study the notion of derivative for a set-valued map as it is most important for the formulation of optimality conditions. Aubin [1] introduced the notion of contingent derivative of a set-valued map. Later it was observed by Corley [4] that in case of contingent derivative, necessary and sufficient optimality conditions do not coincide under standard assumptions. Therefore, while characterizing optimality conditions, derivatives involving epigraph of set-valued maps were considered rather than their graph [7, 9]. These derivatives were termed as epiderivatives of different types. These epiderivatives differed either on the basis of their tangent cones or on the nature of the minimizers. Working in this direction, Jahn and Rauh [7] introduced contingent epiderivative in set-valued analysis.

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Article history : Received 1 July 2018 Accepted 3 November 2018.

Luc and Malivert [8] extended the study of invexity to set-valued maps and vector optimization problems with set-valued data. Sach and Craven [9] proved Wolfe type (WD) and Mond Weir type (MWD) duality theorems for set-valued optimization problems under invexity assumptions. Later on Bhatia and Mehra [3] introduced preinvex set-valued map as an extension of the notion of convex set-valued map. Bao-huai and San-Yang [2] investigated the KKT optimality conditions for preinvex set-valued optimization problems with the help of the generalized contingent epiderivative. Recently Das and Nahak [5, 6] and Yu and Kong [10] studied various types of generalized convexity notions for studying set-valued optimization problem via contingent epiderivatives.

In this paper, we study set-valued fractional programming problem (FP) and its associated parametric problem $(\text{FP})_{\lambda^*}$. It is structured as follows: In Section 2 we recall some well known definitions and results. We also introduce a new class of generalized setvalued cone preinvex maps. Then we give characterization of these maps in terms of contingent epiderivatives. In section 3 necessary and sufficient optimality conditions are obtained for a weak minimizer of the problems (FP) and $(\text{FP})_{\lambda^*}$. In section 4, we formulate Parametric type dual, Wolfe type dual and Mond Weir type dual of (FP) and establish weak duality, strong duality and converse duality results for the same.

2. Definitions and Preliminaries

Throughout this paper, let X and Y be real normed spaces. Let $K \subseteq Y$ be a closed, pointed convex cone with non-empty interior. Then its positive dual cone K^+ is defined as follows:

$$K^+ = \{y^* \in Y : \langle y^*, y\rangle \geq 0 \text{ for all } y \in K\}$$

Several kinds of tangent cones have been studied in literature. We now give the definition of tangent cone namely, the contingent (or Bouligand tangent) cone.

Definition 2.1. Let B be a non-empty subset of Y. Then the contingent (or Bouligand tangent) cone to B at $y^* \in B$ is denoted by $T(B, y^*)$ and is defined as

$$T(B, y^*) = \{y \in Y : \exists\, y_n \longrightarrow y^*, y_n \in B, t_n > 0, n \longrightarrow \infty \text{ such that } t_n(y_n - y^*) \longrightarrow y\}$$

or

$$T(B, y^*) = \{y \in Y : \exists\ y_n \longrightarrow y^*, y_n \in B, t_n \downarrow 0 \text{ such that } y^* + t_n y_n \in B\}$$

Let $F : X \longrightarrow 2^Y$ be a set-valued map where X and Y are real normed spaces. Let the space Y be partially ordered by a closed convex pointed cone $K \subseteq Y$ with nonempty interior. The domain, graph and epigraph of F are defined as

$$\begin{aligned} &\operatorname{dom} F = \{x \in X : F(x) \neq \phi\};\\ &\operatorname{gr} F = \{(x, y) : x \in X, y \in F(x)\};\\ &\operatorname{epi} F = \{(x, y) : x \in X, y \in F(x) + K\}. \end{aligned}$$

Jahn and Rauh [7] gave the following notion of contingent epiderivative relating epigraph of the derivative with the contingent cone.

Definition 2.2. A single valued map $DF(x^*, y^*) : X \longrightarrow Y$ whose epigraph is the contingent cone to $\operatorname{epi} F$ at $(x^*, y^*) \in \operatorname{gr} F$, that is,

$$\operatorname{epi} DF(x^*, y^*) = T(\operatorname{epi} F, (x^*, y^*))$$

is called the contingent epiderivative of F at (x^*, y^*).

In the present paper we assume condition C on η defined as follows:

Condition C ([2]). Let $\eta : X \times X \longrightarrow X$ be a map. Then η is said to be satisfy Condition C if for any $x, y \in X$.

(C1) $\eta(x,x) = 0$;

(C2) $\bigcup\limits_{x\in X} \eta(x,y) = X,\ \forall\ y \in X$;

(C3) $\eta(\lambda x, \lambda y) = \lambda\eta(x,y)$, $\eta(x - x_0, y - x_0) = \eta(x,y)$, for all $x, x_0, y \in X$

Consider the following set-valued fractional programming problem

$$\text{(FP)} \quad K\text{-minimize } \frac{F(x)}{G(x)} = \left(\frac{F_1(x)}{G(x)}, \frac{F_2(x)}{G(x)}, \ldots, \frac{F_m(x)}{G(x)}\right)$$
$$\text{subject to } H(x) \cap (-Q) \neq \phi,$$

where X is a real normed space and S is a non-empty subset of X, $F : S \longrightarrow 2^{R^m}$, $G : S \longrightarrow 2^{R^+}$ and $H : S \longrightarrow 2^{R^k}$ are set-valued maps.

K and Q are closed convex pointed cones in R^m and R^k respectively with non-empty interiors. The feasible set of the problem (FP) is

$$X^0 = \{x \in S : H(x) \cap (-Q) \neq \phi\}$$

Throughout the paper, we denote

$$0_{R^m} = (0, 0, \ldots, 0) \in R^m$$

Definition 2.3. A point $\left(x^*, \frac{y^*}{z^*}\right) \in X \times R^m$, with $x^* \in X^0$, $y^* \in F(x^*)$ and $z^* \in G(x^*)$ is called a minimizer of the problem (FP) if there exist no $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} - \frac{y^*}{z^*} \in -K \setminus \{0_{R^m}\}\,.$$

Definition 2.4. A point $\left(x^*, \frac{y^*}{z^*}\right) \in X \times R^m$, with $x^* \in X^0$, $y^* \in F(x^*)$ and $z^* \in G(x^*)$ is called a weak minimizer of the problem (FP) if there exist no $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} - \frac{y^*}{z^*} \in -\operatorname{int} K\,.$$

Consider the parametric problem $(\text{FP})_{\lambda^*}$ associated with the set-valued fractional programming problem (FP):

$$(\text{FP})_{\lambda^*} \quad K\text{-}\underset{x\in S}{\text{minimize}}\ F(x) - \lambda^* G(x)$$
$$\text{subject to } H(x) \cap (-Q) \neq \phi.$$

Definition 2.5. A point $(x^*, y^* - \lambda^* z^*) \in X \times R^m$, with $\lambda^* = \frac{y^*}{z^*}$, $x^* \in X^0$, $y^* \in F(x^*)$ and $z^* \in G(x^*)$ is called a minimizer of the problem $(\text{FP})_{\lambda^*}$ if there exist no $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$(y - \lambda^* z) - (y^* - \lambda^* z^*) \in -K \setminus \{0_{R^m}\}\,.$$

Definition 2.6. A point $(x^*, y^* - \lambda^* z^*) \in X \times R^m$, with $\lambda^* = \frac{y^*}{z^*}$, $x^* \in X^0$, $y^* \in F(x^*)$ and $z^* \in G(x^*)$ is called a weak minimizer of the problem $(\text{FP})_{\lambda^*}$ if there exist no $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$(y - \lambda^* z) - (y^* - \lambda^* z^*) \in -\operatorname{int} K\,.$$

Lemma 2.1 (3). *A point $\left(x^*, \frac{y^*}{z^*}\right) \in X \times R^m$ is a weak minimizer of the problem (FP) if and only if $(x^*, 0_{R^m})$ is a weak minimizer of the problem $(FP)_{\lambda^*}$, where $\lambda^* = \frac{y^*}{z^*}$.*

Let X, Y be real normed spaces. Let $\eta : S \times S \longrightarrow X$ be a vector valued function. Let $S \subseteq X$ be a non-empty set.

Definition 2.7. A subset $S \subseteq X$ is said to be an η-invex set if for every $x, x^* \in S$ there exists a map $\eta : S \times S \longrightarrow X$ such that

$$x^* + \lambda\eta(x, x^*) \in S, \text{ for all } \lambda \in [0, 1].$$

Now, we introduce the notion of ρ-cone preinvexity of set-valued maps.

Definition 2.8. Let $S \subseteq X$ be an η-invex set. Let $e \in \operatorname{int} K$ and $F : S \longrightarrow 2^Y$ be a set-valued map. Then F is called $\rho - K - \eta$-preinvex at $x^* \in S$ with respect to e on S if there exists $\rho \in R$ such that

$$(1-\lambda)F(x^*) + \lambda F(x) \subseteq F(x^* + \lambda\eta(x, x^*)) + \lambda(1-\lambda)\rho\|\eta(x, x^*)\|^2 e + K,$$
$$\text{for all } x \in S \text{ and } \lambda \in [0, 1].$$

F is $\rho - K - \eta$-preinvex with respect to e on S if F is $\rho - K - \eta$-preinvex with respect to e for all $x^* \in S$.

Remark 2.1. (i) If $\rho = 0$, then the definition of $\rho - K - \eta$ preinvex reduces to the usual notion of cone $K - \eta$ preinvexity of set-valued maps defined by Bhatia and Mehra [3].
(ii) If $\eta(x, x^*) = x - x^*$, then $\rho - K - \eta$ preinvex functions reduce to $\rho - K$-convex functions defined by Das and Nahak in [6].

Now we give a characterization of ρ-cone preinvexity of set-valued maps in terms of their contingent epiderivatives.

Theorem 2.1. *Let $S \subseteq X$ be an η-invex set, $e \in \operatorname{int} K$ and $F : S \longrightarrow 2^Y$ be $\rho - K - \eta$-preinvex with respect to e on S. Let $x^* \in S$ and $y^* \in F(x^*)$. Suppose that F is contingent epiderivable at (x^*, y^*). Then*

$$F(x) - y^* \subseteq DF(x^*, y^*)\eta(x, x^*) + \rho\|\eta(x, x^*)\|^2 e + K, \textit{ for all } x \in S.$$

Proof. Let $x \in S$ and $y \in F(x)$. As F is $\rho - K - \eta$ preinvex with respect to e on S, therefore

$$(1-\lambda)F(x^*) + \lambda F(x) \subseteq F(x^* + \lambda\eta(x, x^*)) + \rho\lambda(1-\lambda)\|\eta(x, x^*)\|^2 e + K,$$
$$\text{for all } x \in S \text{ and } \lambda \in [0, 1].$$

Define a sequence $\{(x_n, y_n)\}_{n \in N}$ with

$$x_n = x^* + \frac{1}{n}\eta(x, x^*)$$

and

$$y_n = \frac{1}{n}y + \left(1 - \frac{1}{n}\right)y^* - \rho\frac{1}{n}\left(1 - \frac{1}{n}\right)\|\eta(x, x^*)\|^2 e, \text{ for all } n \in N.$$

Therefore

$$y_n \in F\left(x^* + \frac{1}{n}\eta(x, x^*)\right) + K, \quad \text{for all } x \in S.$$

Thus

$$y_n \in F(x_n) + K, \quad \text{for all } x \in S.$$

It is clear that

$$x_n \longrightarrow x^*,\ y_n \longrightarrow y^*,\ n(x_n - x^*) \longrightarrow \eta(x, x^*), \text{when } n \longrightarrow \infty \text{ and}$$

$$n(y_n - y^*) = y - y^* - \rho\left(1 - \frac{1}{n}\right)\|\eta(x, x^*)\|^2 e$$

$$\longrightarrow y - y^* - \rho\|\eta(x,x^*)\|^2 e, \text{ when } \eta \longrightarrow \infty.$$

Therefore

$$(\eta(x,x^*), y - y^* - \rho\|\eta(x,x^*)\|^2 e) \in T(\text{epi}(F), (x,x^*)) = \text{epi}(DF(x^*,y^*)).$$

Consequently

$$y - y^* - \rho\|\eta(x,x^*)\|^2 e \in DF(x^*,y^*)\eta(x,x^*) + K,$$

which is true for all $y \in F(x)$.

Therefore

$$F(x) - y^* \subseteq DF(x^*,y^*)\eta(x,x^*) + \rho\|\eta(x,x^*)\|^2 e + K, \text{ for all } x \in S.$$

□

Remark 2.2. If F satisfies the above condition then it is said to be $\rho - K - \eta$ invex function.

3. Optimality Conditions

We shall use the following Slater type constraint qualification to prove the necessary optimality Kuhn Tucker conditions for $(\text{FP})_{\lambda^*}$.

Definition 3.1. A set-valued map $H : S \longrightarrow 2^{R^k}$ is said to satisfy the generalized Slater's constraint qualification if there exists an element $\hat{x} \in S$ such that $H(\hat{x}) \cap -\operatorname{int} Q \neq \phi$.

Bao-Huai and San Yang [2] investigated KKT necessary optimality conditions for a set-valued vector optimization problem in terms of alpha order contingent epiderivatives by assuming the objective and the constraint function to be alpha order preinvex.

If we take $\alpha = 1$, then we can get the following necessary optimality conditions for $(\text{FP})_{\lambda^*}$.

Theorem 3.1 (Karush-Kuhn-Tucker Necessary Optimality Conditions)**.** *Let $S \subseteq X$ be a η-invex set satisfying Condition C. Let $(x^*, y^* - \lambda^* z^*)$ be a weak minimizer of (FP)$_{\lambda^*}$. Let $F : S \longrightarrow 2^{R^m}$ be $K - \eta$ preinvex set-valued map, $-\lambda^* G : S \longrightarrow 2^{R^+}$ be $K - \eta$ preinvex set-valued map and $H : S \longrightarrow 2^{R^k}$ be $Q - \eta$ preinvex set-valued map. If H satisfies generalized Slater's constraint qualification and F is contingent epiderivable at (x^*, y^*), $-\lambda^* G$ is contingent epiderivable at $(x^*, -\lambda^* z^*)$ and H is contingent epiderivable at (x^*, w^*), where $w^* \in H(x^*) \cap (-Q)$, then there exists $(\tau^*, \mu^*) \in K^+ \times Q^+$, with $\tau^* \neq 0_{R^m}$ such that*

$$\langle \tau^*, DF(x^*,y^*)\eta(x,x^*) + D(-\lambda^* G)(x^*, -\lambda^* z^*)\eta(x,x^*)\rangle$$
$$\langle \mu^*, DH(x^*,w^*)\eta(x,x^*)\rangle \geq 0, \text{ for all } x \in S. \tag{3.1}$$
$$\langle \mu^*, w^* \rangle = 0 \tag{3.2}$$

In the light of Lemma 2.1, we have the following necessary optimality theorem for (FP).

Theorem 3.2 (Karush-Kuhn-Tucker Necessary Optimality Conditions)**.** *Let $S \subseteq X$ be an η-invex set satisfying Condition C. Let $\left(x^*, \dfrac{y^*}{z^*}\right)$ be a weak minimizer of (FP). Let $z^* F : S \longrightarrow 2^{R^m}$ be $K - \eta$ preinvex set-valued map, $-y^* G : S \longrightarrow 2^{R^+}$ be $K - \eta$ preinvex set-valued map and $H : S \longrightarrow 2^{R^k}$ be $Q - \eta$ preinvex set-valued map. If H satisfies generalized Slater's constraint qualification and $z^* F$ is contingent* *epiderivable at* (x^*, y^*z^*), $-y^*G$ *is contingent epiderivable at* $(x^*, -y^*z^*)$ *and* H *is contingent epiderivable at* (x^*, w^*), *where* $w^* \in H(x^*) \cap (-Q)$, *then there exists* $(\tau^*, \mu^*) \in K^+ \times Q^+$, *with* $\tau^* \neq 0_{R^m}$ *such that*

$$\langle \tau^*, D(z^*F)(x^*, y^*) + D(-y^*G)(x^*, -y^*z^*)\eta(x, x^*)\rangle + \langle \mu^*, DH(x^*, w^*)\eta(x, x^*)\rangle \geq 0, \text{ for all } x \in S \tag{3.3}$$

and condition (3.2) *hold.*

Now we establish sufficient optimality conditions for the problems (FP) and $(\mathrm{FP})_{\lambda^*}$ by assuming that the objective and constraint set-valued maps are ρ-cone invex as well as contingent epiderivable.

Theorem 3.3 (Sufficiency)**.** *Let* $S \subseteq X$ *be an* η*-invex set,* $x^* \in X^0$, $y^* \in F(x^*)$, $z^* \in G(x^*)$, $\lambda^* = \frac{y^*}{z^*}$ *and* $w^* \in H(x^*) \cap (-Q)$. *Assume that* F *is* $\rho_1 - K - \eta$ *invex with respect to* e, $-\lambda^*G$ *is* $\rho_2 - K - \eta$ *invex with respect to* e *and* H *is* $\rho_3 - Q - \eta$ *invex with respect to* e *on* S. *Let* F *be contingent epiderivable at* (x^*, y^*), $-\lambda^*G$ *be contingent epiderivable at* $(x^*, -\lambda^*z^*)$ *and* H *be contingent epiderivable at* (x^*, w^*).

Suppose there exist $0 \neq \tau^* \in K^+$ *and* $\mu^* \in Q^+$ *satisfying the conditions* (3.1) *and* (3.2), *then* $(x^*, y^* - \lambda^*z^*)$ *is a weak minimizer of the problem* $(FP)_{\lambda^*}$ *provided*

$$(\rho_1 + \rho_2)\langle \tau^*, e\rangle + \rho_3\langle \mu^*, e\rangle \geq 0 \tag{3.4}$$

Proof. Let if possible $(x^*, y^* - \lambda^*z^*)$ be not a weak minimizer of the problem $(\mathrm{FP})_{\lambda^*}$. Then there exist $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$(y - \lambda^*z) - (y^* - \lambda^*z^*) \in -\operatorname{int} K.$$

As $y^* - \lambda^*z^* = 0$, so we have

$$y - \lambda^*z \in -\operatorname{int} K\,.$$

Hence $\langle \tau^*, y - \lambda^*z\rangle < 0$.

Therefore we have $\langle \tau^*, y - \lambda^*z - (y^* - \lambda^*z^*)\rangle < 0$.

Since $x_0 \in X$, there exists an element $w \in H(x) \cap (-Q)$.

Therefore $\langle \mu^*, w\rangle \leq 0$.

So, $\langle \mu^*, w - w^*\rangle \leq 0$.

Hence

$$\langle \tau^*, y - \lambda^*z - (y^* - \lambda^*z^*)\rangle + \langle \mu^*, w - w^*\rangle < 0. \tag{3.5}$$

As F is $\rho_1 - K - \eta$ invex with respect to e, $-\lambda^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. We have

$$F(x) - y^* \subseteq DF(x^*, y^*)\eta(x, x^*) + \rho_1\|\eta(x, x^*)\|^2 e + K,$$
$$-\lambda^*G(x) + \lambda^*z^* \subseteq D(-\lambda^*G)(x^*, -\lambda^*z^*)\eta(x, x^*) + \rho_2\|\eta(x, x^*)\|^2 e + K$$

and

$$H(x) - w^* \subseteq DH(x^*, w^*)\eta(x, x^*) + \rho_3\|\eta(x, x^*)\|^2 e + Q.$$

Hence

$$y - y^* \in DF(x^*, y^*)\eta(x, x^*) + \rho_1\|\eta(x, x^*)\|^2 e + K,$$
$$-\lambda^*z + \lambda^*z^* \in D(-\lambda^*G)(x^*, -\lambda^*z^*)\eta(x, x^*) + \rho_2\|\eta(x, x^*)\|^2 e + K$$

and

$$w - w^* \in DH(x^*, w^*)\eta(x, x^*) + \rho_3\|\eta(x, x^*)\|^2 e + Q.$$

This gives

$$y - y^* - DF(x^*, y^*)\eta(x, x^*) - \rho_1\|\eta(x, x^*)\|^2 e \in K,$$

$$-\lambda^*z+\lambda^*z^*-D(-\lambda^*G)(x^*,-\lambda^*z^*)\eta(x,x^*)-\rho_2\|\eta(x,x^*)\|^2e\in K$$

and

$$w-w^*-DH(x^*,w^*)\eta(x,x^*)-\rho_3\|\eta(x,x^*)\|^2e\in Q.$$

This further gives

$$\langle\tau^*,y-y^*-\lambda^*z+\lambda^*z^*\rangle-\langle\tau^*,DF(x^*,y^*)\eta(x,x^*)+D(-\lambda^*G)(x^*,-\lambda^*z^*)\eta(x,x^*)\rangle$$
$$-(\rho_1+\rho_2)\langle\tau^*,e\rangle\|\eta(x,x^*)\|^2+\langle\mu^*,w-w^*\rangle-\langle\mu^*,DH(x^*,w^*)\eta(x,x^*)\rangle$$
$$-\rho_3\langle\mu^*,e\rangle\|\eta(x,x^*)\|^2\geq 0$$

By condition (3.4), this implies

$$\langle\tau^*,y-y^*-\lambda^*z+\lambda^*z^*\rangle-\langle\tau^*,DF(x^*,y^*)\eta(x,x^*)+D(-\lambda^*G)(x^*,-\lambda^*z^*)\eta(x,x^*)\rangle$$
$$+\langle\mu^*,w-w^*\rangle-\langle\mu^*-DH(x^*,w^*)\eta(x,x^*)\rangle\geq 0$$

By condition (3.1), this implies

$$\langle\tau^*,y-y^*-\lambda^*z+\lambda^*z^*\rangle+\langle\mu^*,w-w^*\rangle\geq 0$$

which contradicts (3.5).

Therefore $(x^*,y^*-\lambda^*z^*)$ is a weak minimizers of $(\text{FP})_{\lambda^*}$. □

Theorem 3.4 (Sufficiency)**.** *Let $S\subseteq X$ be an η-invex set, $x^*\in X^0$, $y^*\in F(x^*)$, $z^*\in G(x^*)$ and $w^*\in H(x^*)\cap(-Q)$. Assume that z^*F is $\rho_1-K-\eta$ invex with respect to e, $-y^*G$ is $\rho_2-K-\eta$ invex with respect to e and H is $\rho_3-Q-\eta$ invex with respect to e on S. Assume that z^*F is a contingent epiderivable at (x^*,y^*z^*), $-y^*G$ is contingent epiderivable at $(x^*,-y^*z^*)$ and H is contingent epiderivable at (x^*,w^*). Further suppose that there exist $(\tau^*,\mu^*)\times K^+\times Q^+$ with $\tau^*\neq 0_{R^m}$ such that condition*

$$\langle\tau^*,D(z^*F)(x^*,y^*z^*)\eta(x,x^*)+D(-y^*G)(x^*,-y^*z^*)\eta(x,x^*)\rangle$$
$$+\langle\mu^*,DH(x^*,w^*)\eta(x,x^*)\rangle\geq 0,\ \textit{for all } x\in S$$

and condition (3.2) are satisfied. Then $\left(x^*,\frac{y^*}{z^*}\right)$ is a weak minimizer of the problem (FP) provided condition (3.4) holds.

4. Duality

We now formulate parametric, Mond-Weir and Wolfe type duals for the problem (FP) and study duality theorems for the same.

Parametric type dual. We associate the following parametric type dual (PD) with the primal problem (FP).

$$\begin{aligned}
\text{(PD)}\quad & \text{maximize}\,\lambda\\
& \text{subject to}\\
& \langle\tau,DF(u,v)\eta(x,u)+D(-\lambda G)(u,-\lambda l)\eta(x,u)\rangle\\
& \quad+\langle\mu,DH(u,q)\eta(x,u)\rangle\geq 0,\ \text{for all } x_0\in X,\\
& \langle\mu,q\rangle\geq 0,\\
& u\in S,\ v\in F(u),\ l\in G(u),\ \lambda=\frac{v}{l},\ q\in H(u).\\
& 0\neq\tau\in K^+,\ \mu\in Q^+ \text{ and } \langle\tau,e\rangle=1.
\end{aligned}$$

A point $(u^*,v^*,l^*,\lambda^*,q^*,\tau^*,\mu^*)$ satisfying all the constraints of the problem (PD) is called a feasible point of (PD).

Definition 4.1. A feasible point $(u^*, v^*, l^*, \lambda^*, q^*, \tau^*, \mu^*)$ of the problem (PD) is called a weak maximizer of (PD) if there exists no feasible point $(u, v, l, \lambda, q, \tau, \mu)$ of (PD) such that

$$\lambda - \lambda^* \in \operatorname{int} K.$$

Theorem 4.1 (Weak Duality)**.** *Let $S \subseteq X$ be an η-invex set, $\bar{x} \in X$ and $(u^*, v^*, l^*, \lambda^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (PD). Suppose that F is $\rho_1 - K - \eta$ invex with respect to e, $-\lambda^* G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Further assume that F is contingent epiderivable at (u^*, v^*), $-\lambda^* G$ is contingent epiderivable at $(u^*, -\lambda^* l)$ and H is contingent epiderivable at (u^*, q^*). Then*

$$\frac{F(\bar{x})}{G(\bar{x})} - \lambda^* \subseteq R^m \setminus - \operatorname{int} K,$$

provided condition (3.4) holds.

Proof. Let if possible for some $\dot{v} \in F(\bar{x})$ and $\dot{l} \in G(\bar{x})$,

$$\frac{\dot{v}}{\dot{l}} - \lambda^* \in \operatorname{int} K.$$

Thus $\dot{v} - \lambda^* \dot{l} \in - \operatorname{int} K$.

Hence $\langle \tau^*, \dot{v} - \lambda^* \dot{l} \rangle < 0$.

Therefore, $\langle \tau^*, \dot{v} - \lambda^* \dot{l} - (v^* - \lambda^* l^*) \rangle < 0$.

Now as $\bar{x} \in X^0$, we have

$$H(\bar{x}) \cap (-Q) \neq \phi.$$

Let $\bar{q} \in H(\bar{x}) \cap (-Q)$. Then

$$\langle \mu^*, \bar{q} \rangle \leq 0.$$

Again, from the constraints of (PD), we have

$$\langle \mu^*, q^* \rangle \geq 0.$$

Hence $\langle \mu^*, \bar{q} - q^* \rangle \leq 0$.

Therefore

$$\langle \tau^*, \dot{v} - \lambda^* \dot{l} - (v^* - \lambda^* l^*) + \langle \mu^*, \bar{q} - q^* \rangle < 0 \,. \tag{4.1}$$

As F is $\rho_1 - K - \eta$ invex with respect to e, $-\lambda^* G$ is $\rho_2 - K - \eta$ with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S, we have

$$F(\bar{x}) - v^* \subseteq DF(u^*, v^*)\eta(\bar{x}, u^*) + \rho_1 \|\eta(\bar{x}, u^*)\|^2 e + K,$$
$$(-\lambda^* G)(\bar{x}) + \lambda^* l^* \subseteq D(-\lambda^* G)(u^*, -\lambda^* l^*) + \eta(\bar{x}, u^*) + \rho_2 \|\eta(\bar{x}, u^*)\|^2 e + K$$

and

$$H(\bar{x}) - q^* \subseteq DH(u^*, q^*)\eta(\bar{x}, u^*) + \rho_3 \|\eta(\bar{x}, u^*)\|^2 e \in Q.$$

Hence

$$\dot{v} - v^* \in DF(u^*, v^*)\eta(\bar{x}, u^*) + \rho_1 \|\eta(\bar{x}, u^*)\|^2 e + K,$$
$$- \lambda^* \dot{l} + \lambda^* l^* \in D(-\lambda^* G)(u^*, -\lambda^* l^*)\eta(\bar{x}, u^*) + \rho_2 \|\eta(\bar{x}, u^*)\|^2 e + K$$

and

$$\bar{q} - q^* \in DH(u^*, q^*)\eta(\bar{x}, u^*) + \rho_3 \|\eta(\bar{x}, u^*)\|^2 e + Q.$$

Hence, from the constraints of (PD) and condition (3.4), we have

$$\langle \tau^*, \dot{v} - \lambda^* \dot{l} - (v^* - \lambda^* l^*) \rangle + \langle u^*, \bar{q} - q^* \rangle \geq 0,$$

which contradicts equation (4.1).

Thus

$$\frac{\dot{v}}{\dot{l}} - \lambda^* \notin \operatorname{int} K.$$

Since $\dot{v} \in F(\bar{x})$ and $\dot{l} \in G(\bar{x})$ are arbitrary, therefore

$$\frac{F(\bar{x})}{G(\bar{x})} - \lambda^* \subseteq R^m \setminus -\operatorname{int} K.$$

By the Theorems 3.1 and 4.1, we get the following result. □

Theorem 4.2 (Strong Duality). *Let $S \subseteq X$ be an η-invex set satisfying Condition C. Let $(x^*, y^* - \lambda^* z^*)$ be a weak minimizer of (FP)$_{\lambda^*}$. Let $F : S \longrightarrow 2^{R^m}$ be $K - \eta$ preinvex set-valued map, $-\lambda^* G : S \longrightarrow 2^{R^+}$ be $K - \eta$ preinvex set-valued map and $H : S \longrightarrow 2^{R^k}$ be $Q - \eta$ preinvex set-valued map. Further assume that H satisfies generalized Slater's constraint qualification and F is contingent epiderivable at (x^*, y^*), $-\lambda^* G$ is contingent epiderivable at $(x^*, -\lambda^* z^*)$ and H is contingent epiderivable at (x^*, w^*), $w^* \in H(x^*) \cap -(Q)$. Then there exist $0 \neq \tau^* \in K^+$, $\mu^* \in Q^+$ such $(x^*, y^*, z^*, \lambda^*, w^*, \tau^*, \mu^*)$ is feasible for (PD). Moreover, if for each feasible point of (PD), hypothesis of Weak Duality Theorem 4.1 holds, then $(x^*, y^*, z^*, \lambda^*, w^*, \tau^*, \mu^*)$ is a weak maximizer of (PD).*

Theorem 4.3 (Converse Duality). *Let $S \subseteq X$ be an η-invex set and $(u^*, v^*, l^*, \lambda^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (PD), where $\lambda^* = \frac{v^*}{l^*}$. Suppose that F is $\rho_1 - K - \eta$ invex with respect to e, $-\lambda^* G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Also let F be contingent epiderivable at (u^*, v^*), $-\lambda^* G$ be contingent epiderivable at $(u^*, -\lambda^* l^*)$ and H be contingent epiderivable at (u^*, q^*). If u^* is a feasible point of the problem (FP)$_{\lambda^*}$, then $(u^*, v^* - \lambda^* l^*)$ is a weak minimizer of the problem (FP)$_{\lambda^*}$ provided condition (3.4) holds.*

Proof. Let if possible $(u^*, v^* - \lambda^* l^*)$ be not a weak minimizer of the problem (FP)$_{\lambda^*}$.

Then there exist $x \in X^0$, $v \in F(x)$ and $l \in G(x)$ such that

$$(v - \lambda^* l) - (v^* - \lambda^* l^*) \in -\operatorname{int} K.$$

This gives

$$v - \lambda^* l \in -\operatorname{int} K$$

Since $0_{R^m} \neq \tau^* \in K^+$, so this further gives

$$\langle \tau^*, v - \lambda^* l \rangle < 0$$

Therefore

$$\langle \tau^*, v - \lambda^* l - (v^* - \lambda^* l^*) \rangle < 0.$$

Proceeding on the same lines as in the proof of Theorem 4.1, we will get the result. □

Mond-Weir type dual. We now associate the following Mond-Weir type dual with the primal problem (FP).

$$\text{(MWD)} \quad \text{maximize } \frac{v}{l}$$

subject to

$$\langle \tau, D(lF)(u, vl)\eta(x, u) + D(-vG)(u, -vl)\eta(x, u) \rangle$$
$$+ \langle \mu, DH(u, q)\eta(x, u) \rangle \geq 0, \text{ for all } x \in X^0,$$
$$\langle \mu, q \rangle \geq 0,$$

$$u \in S,\ v \in F(u), l \in G(u),\ q \in H(u),\ 0 \neq \tau \in K^+,\ \mu \in Q^+$$

and $\langle \tau, e\rangle = 1$.

A point $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ which satisfies all the constraints of the dual problem (MWD) is a feasible point of (MWD).

Definition 4.2. A feasible point $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ of the problem (MWD) is called a weak maximizer of (MWD) if there exists no feasible point (u, v, l, q, τ, μ) of (MWD) such that

$$\frac{v}{l} - \frac{v^*}{l^*} \in \operatorname{int} K$$

Theorem 4.4 (Weak Duality)**.** *Let $S \subseteq X$ be an η-invex set, $\bar{x} \in X^0$ and $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (MWD). Suppose that l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Also let l^*F be contingent epiderivable at (u^*, v^*l^*), $-v^*G$ be contingent epiderivable at $(u^*, -v^*l^*)$ and H be contingent epiderivable at (u^*, q^*), then*

$$\frac{F(\bar{x})}{G(\bar{x})} - \frac{v^*}{l^*} \subseteq R^m \setminus -\operatorname{int} K,$$

provided condition (3.4) *holds.*

Proof. Let if possible for some $\dot{v} \in F(\bar{x})$ and $\dot{l} \in G(\bar{x})$,

$$\frac{\dot{v}}{\dot{l}} - \frac{v^*}{l^*} \in -\operatorname{int} K.$$

As $\dot{l}l^* \in R^+$, so this implies

$$\dot{v}l^* - v^*\dot{l} \in -\operatorname{int} K\,.$$

Thus $\langle \tau^*, \dot{v}l^* - v^*\dot{l}\rangle < 0$.

Now $\bar{x} \in X^0$, so there exists an element $\bar{q} \in H(\bar{x}) \cap (-Q)$.

Therefore

$$\langle \mu^*, \bar{q}\rangle \leq 0$$

Again, from the constraints of (MWD), we have

$$\langle \mu^*, q^*\rangle \geq 0.$$

So

$$\langle \mu^*, \bar{q} - q^*\rangle \leq 0.$$

Hence,

$$\langle \tau^*, \dot{v}l^* - v^*\dot{l}\rangle + \langle \mu^*, \bar{q} - q^*\rangle < 0. \tag{4.2}$$

Since l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S, we have

$$(l^*F)(\bar{x}) - v^*l^* \subseteq D(l^*F)(u^*, v^*l^*)\eta(\bar{x}, u^*) + \rho_1\|\eta(\bar{x}, u^*)\|^2 e + K,$$
$$(-v^*G)(\bar{x}) - v^*l^* \subseteq D(v^*G)(u^*, v^*l^*)\eta(\bar{x}, u^*) + \rho_2\|\eta(\bar{x}, u^*)\|^2 e + K$$

and

$$H(\bar{x}) - q^* \subseteq DH(u^*, q^*)\eta(\bar{x}, u^*) + \rho_3\|\eta(\bar{x}, u^*)\|^2 e + Q\,.$$

This gives

$$\dot{v}l^* - v^*l^* \in D(l^*F)(u^*, v^*l^*)\eta(\bar{x}, u^*) + \rho_1\|\eta(\bar{x}, u^*)\|^2 e + K,$$
$$-v^*\dot{l} + v^*l^* \in D(-v^*G)(u^*, -v^*l^*)\eta(\bar{x}, u^*) + \rho_2\|\eta(\bar{x}, u^*)\|^2 e + K$$

and

$$\bar{q} - q^* \in DH(u^*, q^*)\eta(\bar{x}, u^*) + \rho_3 \|\eta(\bar{x}, u^*)\|^2 e + Q.$$

Thus, from the constraints of (MWD) and condition (3.4), we have

$$\langle \tau^*, \dot{v}l^* - v^*\dot{l} \rangle + \langle \mu^*, \bar{q} - q^* \rangle \geq 0$$

which contradicts equation (4.2).

Hence

$$\frac{\dot{v}}{\dot{l}} - \frac{v^*}{l^*} \notin -\operatorname{int} K.$$

Since $\frac{\dot{v}}{\dot{l}} \in \frac{F(\bar{x})}{G(\bar{x})}$ is arbitrary, so

$$\frac{F(\bar{x})}{G(\bar{x})} - \frac{v^*}{l^*} \subseteq R^m \setminus -\operatorname{int} K. \qquad \square$$

By the Theorems 3.2 and 4.4, we will get the following result.

Theorem 4.5 (Strong Duality)**.** *Let $S \subseteq X$ be an η-invex set satisfying Condition C. Let $\left(x^*, \frac{y^*}{z^*}\right)$ be an weak minimizer of (FP). Let $z^*F : S \longrightarrow 2^{R^m}$ be $K - \eta$ preinvex set-valued map $-y^*G : S \longrightarrow 2^{R^+}$ be $K - \eta$ preinvex set-valued map and $H : S \longrightarrow 2^{R^k}$ be $Q - \eta$ preinvex set-valued map. If H satisfies generalized Slater's constraint qualification and z^*F is contingent epiderivable at (x^*, y^*z^*), $-y^*G$ is contingent epiderivable at $(x^*, -y^*z^*)$ and H is contingent epiderivable at (x^*, w^*), where $w^* \in H(x^*) \cap (-Q)$. Then there exists $0_{R^m} \neq \tau^* \in K^+$, $\mu^* \in Q^+$, such that $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$ is feasible for (MWD). Moreover, if for each feasible point of (MWD), hypothesis of Weak Duality Theorem 4.4 holds, then $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$ is a weak maximizer of (MWD).*

Theorem 4.6 (Converse Duality)**.** *Let $S \subseteq X$ be an η-invex set and $(u^*, v^*, l^*, q^*, \eta^*, \mu^*)$ be a feasible point of the problem (MWD). Suppose that l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Also let l^*F be contingent epiderivable at (u^*, v^*l^*), $-v^*G$ be contingent epiderivable at $(u^*, -v^*l^*)$ and H be contingent epiderivable at (u^*, q^*). If u^* is a feasible point of the problem (FP), then $\left(v^*, \frac{v^*}{l^*}\right)$ is a weak minimizer of the problem (FP) provided condition (3.4) holds.*

Proof. Let if possible $\left(v^*, \frac{v^*}{l^*}\right)$ be not a weak minimizer of the problem (FP). Then there exist $x \in X^0$, $v \in F(x)$ and $l \in G(x)$ such that

$$\frac{v}{l} - \frac{v^*}{l^*} \in -\operatorname{int} K.$$

This gives

$$vl^* - v^*l \in -\operatorname{int} K.$$

Thus $\langle \tau^*, vl^* - v^*l \rangle < 0$.

Proceeding on the same lines as in the proof of Theorem 4.4, we will get a contradiction which proves that $\left(u^*, \frac{v^*}{l^*}\right)$ must be a weak minimizer of (FP). $\square$

Wolfe Type Dual. We now associate the following Wolfe-type dual with the primal problem (FP).

$$\text{(WD)} \quad \text{maximize } \frac{v + \langle \mu, q\rangle e}{l}$$

subject to

$$\langle \tau, D(lF)(u, vl)\eta(x, u) + D(-vG)(u, -vl)\eta(x, u) \geq 0, \text{ for all } x \in X^0. \tag{4.3}$$

$$\langle \mu, DH(u, q)\eta(x, u)\rangle \geq 0, \text{ for all } x \in X^0, \tag{4.4}$$

$$u \in S,\ v \in F(u),\ l \in G(u),\ 0_{R^m} \neq \tau \in K^+,\ \mu \in Q^+$$

and $\langle \tau, e\rangle = 1$.

Definition 4.3. A feasible point $\langle u^*, v^*, l^*, q^*, \tau^*, \mu^*\rangle$ of the problem (WD) is called a weak maximizer of (WD) if there exists no feasible point (u, v, lq, τ, μ) of (WD) such that

$$\frac{v + \langle \mu, q\rangle}{l} - \frac{v + \langle \mu^*, q^*\rangle}{l^*} \in \operatorname{int} K.$$

The following results can easily be established for the Wolfe type dual.

Theorem 4.7 (Weak Duality)**.** *Let $S \subseteq X$ be an η-invex set, $\bar{x} \in X^0$ and $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (WD). Suppose l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e, on S. Also l^*F is contingent epiderivable at (u^*, v^*l^*), $-v^*G$ is contingent epiderivable at $(u^*, -v^*l^*)$ and H is contingent epiderivable at (u^*, q^*). Then*

$$\frac{F(\bar{x})}{G(\bar{x})} - \frac{v^* + \langle \mu^*, q^*\rangle e}{l^*} \subseteq R^m \setminus -\operatorname{int} K,$$

provided

$$(\rho_1 + \rho_2)\langle \tau^*, e\rangle \geq 0 \text{ and } \rho_3\langle \mu^*, e\rangle \geq 0 \tag{4.5}$$

Theorem 4.8 (Strong Duality)**.** *Let $\left(x^*, \frac{y^*}{z^*}\right)$ be a weak minimizer of the problem (FP) and $w^* \in H(x^*) \cap (-Q)$. Suppose that there exists $\tau^* \in K^+$, $\mu^* \in Q^+$ with $\langle \tau^*, e\rangle \geq 1$ such that conditions (4.3) and (4.4) are satisfied at $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$. Then $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$ is a feasible solution of the problem (WD). Furthermore, if for each feasible point of (WD) the conditions of Weak Duality Theorem 4.7 hold, then $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$ is a weak maximizer of (WD).*

Theorem 4.9 (Converse Duality)**.** *Let $S \subseteq X$ be an η-invex set and $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (WD) and $\langle \mu^*, q^*\rangle = 0$. Suppose that l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e, on S. Also let l^*F be contingent epiderivable at (u^*, v^*l^*), $-v^*G$ be contingent epiderivable at $(u^*, -v^*l^*)$ and H be contingent epiderivable at (u^*, q^*). If u^* is a feasible point of the problem (FP), then $\left(u^*, \frac{v^*}{l^*}\right)$ is a weak minimizer of the problem (FP) provided condition (4.5) hold.*

Acknowledgements

We express our sincere thanks to Dr. Surjeet K. Suneja for her valuable comments and suggestions that helped in improving the quality of our paper.

References

1. J. P. Aubin, Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, In: L. Nachbin (ed.), Mathematical Analysis and Applications, Part A pp. 160 – 229, Academic Press, New York (1981).
2. S. Bao-Huai and I. San-Yang, Kuhn-Tucker Condition and Wolfe-duality of preinvex set-valued optimization, Appl. Math. Mech. Eng. 27 (12) (2006), 1655 – 1644.
3. D. Bhatia and A. Mehra, Langrangian duality for preinvex set-valued function, J. Math. Anal. Appl. 214 (1997), 599 – 612.
4. H. W. Corley, Existence and Lagrangian duality for maximization of set-valued functions, J. Optim. Theory Appl. 54 (1987), 489 – 501.
5. K. Das and C. Nahak, Sufficient Optimality Conditions and duality theorems for set-valued optimization problem under generalized cone convexity, Rend. Circ. Mat. Palermo 63 (2014), 329 – 345.
6. K. Das and C. Nahak, Set-Valued fractional programming problems under generalized cone convexity, Opsearch 53(1) (2016), 157 – 177.
7. J. Jahn and R. Rauh, Contingent epiderivative and set-valued optimization, Math. Methods Oper. Res. 46 (1997), 193 – 211.
8. D. T. Luc and C. Malivert, Invex optimization problems, Bull. Aust. Math. Soc. 49 (1992), 47 – 66.
9. P. H. Sach and B. D. Craven, Invex multifunction and duality, Numer. Funct. Anal. Opt. 12 (1991), 575 – 591.
10. G. Yu and X. Kong, Approximate solutions for nonconvex set-valued optimization and vector variational inequality, Journal of Inequalities and Applications 2015 (2015), 324.