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Contact

Narin Petrot (narinp@nu.ac.th)
Center of Excellence in Nonlinear Analysis and Optimization,
Department of Mathematics, Faculty of Science,
Naresuan University, Phitsanulok, 65000, Thailand.

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SOME INTEGRAL INEQUALITIES OF THE HADAMARD AND THE FEJÉR-HADAMARD TYPE VIA GENERALIZED FRACTIONAL INTEGRAL OPERATOR

GHULAM ABBAS[1] AND GHULAM FARID*[2]

[1] Department of Mathematics, Government College Bhalwal, Sargodha, Pakistan
[2] Department of Mathematics, COMSATS University Islamabad, Attock, Pakistan

ABSTRACT.

In this paper we give the Hadamard and the Fejér-Hadamard type integral inequalities for convex and relative convex functions by involving a generalization of the Riemann-Liouville fractional integral. Also some connections with known results have been obtained.
KEYWORDS: Convex function; Hadamad inequality; Fejér-Hadamard inequality; Fractional integral operators.
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1. Preliminaries

Convex functions are very useful for diverse fields of Mathematics, a rich literature has been built since their discovery [15].

Definition 1.1. Let I be an interval of real numbers. Then a function $f : I \to \mathbb{R}$ is said to be convex function if for all $x, y \in I$ and $0 \leq \lambda \leq 1$ the following inequality holds

$$f(x\lambda + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Convex functions are naturally obey the following inequality which is well known as the Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

where $f : I \to \mathbb{R}$ is a convex function on I and $a, b \in I, a < b$.
Following definitions are given in [14].

* *Corresponding author.*
Email address : prof.abbas6581@gmail.com (Ghulam Abbas), faridphdsms@hotmail.com, ghlmfarid@cuiatk.edu.pk (Ghulam Farid).

Definition 1.2. Let T_g be a set of real numbers. This set T_g is said to be relative convex with respect to an arbitrary function $g:\mathbb{R}\to\mathbb{R}$ if

$$(1-t)x+tg(y)\in T_g$$

where $x,y\in\mathbb{R}$ such that $x,g(y)\in T_g$, $0\le t\le 1$.

Note that every convex set is relative convex, but the converse is not true. For example $T_g=[-1,\frac{-1}{2}]\bigcup[0,1]$ and $g(x)=x^2$, for all $x\in\mathbb{R}$. This set is relative convex but not convex set. Another possibility may be occure that a realtive covex set is convex set for example if $T_g=[-1,1]$ and $g(x)=(|x|)^{\frac{1}{4}}$ for all $x\in\mathbb{R}$ (see[9]). If $g=I$ the identity function, then the definition of relative convex set recaptures the definition of classical convex set.

Definition 1.3. A function $f:T_g\to\mathbb{R}$ is said to be relative convex, if there exists an arbitrary function $g:\mathbb{R}\to\mathbb{R}$ such that

$$f((1-t)x+tg(y))\le(1-t)f(x)+tf(g(y)),$$

holds, where $x,y\in\mathbb{R}$ such that $x,g(y)\in T_g$, $0\le t\le 1$.

Noor et al proved the following Hadamard type integral inequality in [14] for relative convex functions via Riemann-Liouville fractional integral operators.

Theorem 1.4. *Let f be a positive relative convex function and integrable on $[a,g(b)]$. Then the following inequality holds*

$$f\left(\frac{a+g(b)}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(g(b)-a)^{\alpha}}[I^{\alpha}_{a^+}fg(b)+I^{\alpha}_{b_-}f(a)]\le\frac{f(a)+f(g(b))}{2}$$

$\alpha>0$.

In the following we give some definitions and known facts about fractional integral operators [17].

Definition 1.5. Let $\omega\in\mathbb{R}$ and α,β,k,l,γ be positive real numbers. The generalized fractional integral operators $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ and $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}$ for a real valued continuous function f are defined as follows

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(x)=\int_a^x(x-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha})f(t)dt, \tag{1.1}$$

and

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}f\right)(x)=\int_x^b(t-x)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(t-x)^{\alpha})f(t)dt,$$

where the function $E^{\gamma,\delta,k}_{\alpha,\beta,l}$ is the generalized Mittag-Leffler function defined as

$$E^{\gamma,\delta,k}_{\alpha,\beta,l}(t)=\sum_{n=0}^{\infty}\frac{(\gamma)_{kn}t^n}{\Gamma(\alpha n+\beta)(\delta)_{ln}}, \tag{1.2}$$

the Pochhammer symbol $(a)_n$ is defined by $(a)_n= a(a+1)(a+2)...(a+n-1)$, $(a)_0$=1.
For $\omega=0$, (1.1) produces the definition of Riemann-Liouville fractional integral operators [17]

$$I^{\beta}_{a^+}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}f(t)dt,\ x>a$$

and

$$I^{\beta}_{b_-}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b(t-x)^{\beta-1}f(t)dt,\ x<b.$$

In [17] properties of the generalized Mittag-Leffler function are discussed and it is given that $E^{\gamma,\delta,k}_{\alpha,\beta,l}(t)$ is absolutely convergent for $k < l + \alpha$. Let S be the sum of series of absolute terms of the Mittag-Leffler function $E^{\gamma,\delta,k}_{\alpha,\beta,l}(t)$, then we have $\left|E^{\gamma,\delta,k}_{\alpha,\beta,l}(t)\right| \leq S$. We use this property of Mittag-Leffler function in our results where we need.

In [10] the following Hadamard and the Fejér-Hadamard inequalities for convex functions via generalized fractional integral operator containing the Mittag-Leffler function have been proved.

Theorem 1.6. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is convex on $[a, b]$, then the following inequality for generalized fractional integrals holds*

$$f\left(\frac{a+b}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}1)(b) \leq \frac{(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f)(b) + (\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}f)(a)}{2} \leq \frac{f(a)+f(b)}{2}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}1)(a), \tag{1.3}$$

where $\omega' = \frac{w}{(b-a)^\alpha}$.

Theorem 1.7. *Let $f : [a, b] \to \mathbb{R}$ be a convex function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also, let $g : [a, b] \to \mathbb{R}$ be a function which is non-negative, integrable and symmetric about $\frac{a+b}{2}$. Then the following inequality for generalized fractional integrals holds*

$$f\left(\frac{a+b}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}g)(b) \leq \frac{(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}fg)(b) + (\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}fg)(a)}{2} \leq \frac{f(a)+f(b)}{2}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}g)(a), \tag{1.4}$$

where $\omega' = \frac{w}{(b-a)^\alpha}$.

In [12, 14] the Hadamard and the Fejér-Hadamard type inequalities for convex and relative convex functions via Riemann-Liouville fractional integral operators have been proved. In this paper we give fractional integral inequalities of the Hadamard and the Fejér-Hadamard type for convex and relative convex functions by using the fractional integral operators involving the generalized Mittag-Leffler function. We also produce the results which are given in [12, 14] by setting particular values of parameters.

2. MAIN RESULTS

Following lemmas are useful to establish new results.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be an integrable and symmetric function about $\frac{a+b}{2}$. Then the following equality holds*

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b) = \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}f\right)(a) = \frac{\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}f\right)(a)}{2}. \tag{2.1}$$

Proof. As f is symmetric about $\frac{a+b}{2}$, therefore $f(a + b - t) = f(t)$. By definition we have

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b) = \int_a^b (b-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-t)^\alpha)f(t)dt, \tag{2.2}$$

replacing t by $a+b-t$ in equation (2.2) we have

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b)=\int_a^b (t-a)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(t-a)^{\alpha})f(t)dt.$$

This implies

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b)=\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}f\right)(a). \tag{2.3}$$

Therefore we get (2.1).

□

Lemma 2.2. *Let $f:[a.b]\to\mathbb{R}$ be a differentiable function on (a,b) and $f'\in L[a,b]$. If $g:[a.b]\to\mathbb{R}$ is integrable and symmetric about $\frac{a+b}{2}$, then we have the following equality*

$$\begin{aligned}&\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}g\right)(a)\right]\\&-\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}gf\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}gf\right)(a)\right]\\&=\int_a^b\left[\int_a^t (b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds\right.\\&\left.-\int_t^b (s-a)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^{\alpha})g(s)ds\right]f'(t)dt.\end{aligned}$$

Proof. To prove this lemma we take terms of the right hand side, on integrating by parts and after simplification we have

$$\begin{aligned}&\int_a^b\left[\int_a^t (b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds\right]f'(t)dt\\&=f(b)\int_a^b (b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds-\int_a^b (b-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-t)^{\alpha})gf(t)dt\\&=f(b)\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)-\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}gf\right)(b).\end{aligned}$$

By using Lemma 2.1 we have

$$\begin{aligned}&\int_a^b\left[\int_a^t (b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds\right]f'(t)dt \\&=\frac{f(b)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}g\right)(a)\right]-\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}gf\right)(b).\end{aligned} \tag{2.4}$$

Similarly

$$\begin{aligned}&-\int_t^b\left[(s-a)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^{\alpha})g(s)ds\right]f'(t)dt \\&=\frac{f(a)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}g\right)(a)\right]-\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}gf\right)(a).\end{aligned} \tag{2.5}$$

Adding (2.4) and (2.5) we get the left hand side. □

In the following we give our first integral inequality of the Hadamard type.

Theorem 2.3. *Let $f:I\to\mathbb{R}$ be a differentiable mapping in the interior of I with $f'\in L[a,b]$, $a<b$. If $|f'|$ is convex on $[a,b]$ and $g:I\to\mathbb{R}$ is continuous and symmetric function about $\frac{a+b}{2}$, then we have the following inequality*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}g\right)(a)\right]\right.$$

$$- \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \Big|$$
$$\leq \frac{\| g \|_\infty S (b-a)^{\beta+1}}{\beta(\beta+1)} \left(1 - \frac{1}{2^\beta} \right) [|f'(a) + f'(b)|],$$

for $k < l + \alpha$ *and* $\| g \|_\infty = \sup_{t \in [a,b]} |g(t)|$.

Proof. By using Lemma 2.2 we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g \right)(a) \right] \right. \tag{2.6}$$
$$- \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \Big|$$
$$\leq \int_a^b \left| \left[\int_a^t (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds \right. \right.$$
$$\left. \left. - \int_t^b (s-a)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^\alpha) g(s) ds \right] \right| |f'(t)| dt.$$

Using the convexity of $|f'|$ we have

$$|f'(t)| \leq \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)|;\ t \in [a,b]. \tag{2.7}$$

By using symmetry of function g we have

$$\int_t^b (s-a)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^\alpha) g(s) ds$$
$$= \int_a^{a+b-t} (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(a+b-s) ds$$
$$= \int_a^{a+b-t} (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds.$$

This implies

$$\left| \int_a^t (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds - \int_t^b (s-a)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^\alpha) g(s) ds \right| \tag{2.8}$$
$$= \left| \int_t^{a+b-t} (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds \right|$$
$$\leq \begin{cases} \int_t^{a+b-t} |(b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s)| ds, t \in [a, \frac{a+b}{2}] \\ \int_{a+b-t}^t |(b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s)| ds, t \in [\frac{a+b}{2}, b]. \end{cases}$$

By (2.6), (2.7), (2.8) and absolute convergence of Mittag-Leffler function, we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g \right)(a) \right] \right. \tag{2.9}$$
$$- \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \Big|$$
$$\leq \int_a^{\frac{a+b}{2}} \left(\int_a^{a+b-t} |(b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s)| ds \right) \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt$$
$$+ \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t |(b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s)| ds \right) \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt.$$

$$\leq \frac{\parallel g \parallel_\infty S}{\beta(b-a)}\left[\int_a^{\frac{a+b}{2}} ((b-t)^\beta - (t-a)^\beta (b-t)|f'(a)|)dt + \int_a^{\frac{a+b}{2}} ((b-t)^\beta - (t-a)^\beta (t-a)|f'(b)|)dt\right.$$
$$\left. + \int_{\frac{a+b}{2}}^b ((t-a)^\beta - (b-t)^\beta (b-t)|f'(a)|)dt + \int_{\frac{a+b}{2}}^b ((t-a)^\beta - (b-t)^\beta (t-a)|f'(b)|)dt\right].$$

Since we have

$$\int_a^{\frac{a+b}{2}} \left((b-t)^\beta - (t-a)^\beta\right)(b-t)dt = \frac{(b-a)^{\beta+2}}{\beta+1}\left(\frac{\beta+1}{\beta+2} - \frac{1}{2^{\beta+1}}\right)$$

and

$$\int_a^{\frac{a+b}{2}} \left((b-t)^\beta - (t-a)^\beta\right)(t-a)dt = \frac{(b-a)^{\beta+2}}{\beta+1}\left(\frac{1}{\beta+2} - \frac{1}{2^{\beta+1}}\right).$$

Using the above calculations in (2.9) we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a)\right]\right.$$
$$\left. - \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf\right)(a)\right]\right|$$
$$\leq \frac{\parallel g \parallel_\infty S}{\beta(b-a)}\frac{(b-a)^{\beta+2}}{\beta+1}\left[\left(\frac{\beta+1}{\beta+2} - \frac{1}{2^{\beta+1}}\right) + \left(\frac{1}{\beta+2} - \frac{1}{2^{\beta+1}}\right)\right][|f'(a)| + |f'(b)|]$$
$$= \frac{\parallel g \parallel_\infty S}{\beta(\beta+1)}(b-a)^{\beta+1}\left(1 - \frac{1}{2^\beta}\right)[|f'(a)| + |f'(b)|].$$

□

A special case is stated in the following, which is inequality of the Hadamard type for Riemann-Liouville fractional integrals.

Corollary 2.4. *Setting $\omega = 0$ in Theorem 2.3 we have the following inequality for Riemann-Liouville fractional integral operators*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[I^\beta_{a^+} g(b) + I^\beta_{b_-} g(a)\right] - \left[I^\beta_{a^+} fg(b) + I^\beta_{b_-} fg(a)\right]\right| \tag{2.10}$$
$$\leq \frac{\parallel g \parallel_\infty (b-a)^{\beta+1}}{\Gamma(\beta+2)}\left(1 - \frac{1}{2^\beta}\right)[|f'(a)| + |f'(b)|].$$

Remark 2.5. The above inequality (2.10) is proved in [12].

Theorem 2.6. *Let $f : I \to \mathbb{R}$ be a differentiable function in the interior of I, also let $f' \in L[a,b]$, $a < b$. If $|f'|^q$, $q > 0$ is convex on $[a,b]$ and $g : I \to \mathbb{R}$ is continuous and symmetric function about $\frac{a+b}{2}$, then we have the following inequality*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a)\right]\right. \tag{2.11}$$
$$\left. - \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf\right)(a)\right]\right|$$
$$\leq \frac{2 \parallel g \parallel_\infty S(b-a)^{\beta+\frac{1}{p}}}{\beta(\beta+1)}\left(1 - \frac{1}{2^\beta}\right)\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}},$$

for $k < l + \alpha$ and $\parallel g \parallel_\infty = \sup_{t\in[a,b]} |g(t)|$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.2, Hölder inequality, inequality (2.8) one can has

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a)\right]\right. \tag{2.12}$$

$$
\begin{aligned}
&-\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf\right)(a)\right]\Bigg|\\
&\leq \left[\int_a^b \left|\int_t^{a+b-t}(b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds\right| dt\right]^{1-\frac{1}{q}}\\
&\left[\int_a^b \left|\int_t^{a+b-t}(b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds\right| |f'(t)|^q dt\right]^{\frac{1}{q}}.
\end{aligned}
$$

Using absolute convergence of Mittag-Leffler function and $\| g \|_\infty = \sup_{t\in[a,b]} |g(t)|$ we have

$$
\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a)\right]\right.\\
&\left.-\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf\right)(a)\right]\right|\\
&\leq \| g \|_\infty^{1-\frac{1}{q}} S^{1-\frac{1}{q}}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}(b-s)^{\beta-1}ds\right)dt+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^{t}(b-s)^{\beta-1}ds\right)dt\right]^{1-\frac{1}{q}}\\
&\times \| g \|_\infty^{\frac{1}{q}} S^{\frac{1}{q}}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}(b-s)^{\beta-1}ds\right)|f'(t)|^q dt\right.\\
&\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^{t}(b-s)^{\beta-1}ds\right)|f'(t)|^q dt\right]^{\frac{1}{q}}.
\end{aligned}
$$

By some calculation we have

$$
\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a)\right]\right.\\
&\left.-\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf\right)(a)\right]\right|\\
&\leq \| g \|_\infty S\left[\frac{(b-a)^{\beta+1}}{\beta+1}(1-\frac{1}{2^\beta})+\frac{(b-a)^{\beta+1}}{\beta+1}(1-\frac{1}{2^\beta})\right]^{1-\frac{1}{q}}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left((b-t)^\beta-(t-a)^\beta\right)|f'(t)|^q dt+\int_{\frac{a+b}{2}}^b\left((b-t)^\beta-(t-a)^\beta.\right)|f'(t)|^q dt\right]^{\frac{1}{q}}.
\end{aligned}
$$

Since $|f'|^q$ is convex on $[a,b]$, therefore we have

$$
|f'(t)|^q \leq \frac{b-t}{b-a}|f'(a)|^q+\frac{t-a}{b-a}|f'(b)|^q. \tag{2.13}
$$

Hence

$$
\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a)\right]\right.\\
&\left.-\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf\right)(a)\right]\right|\\
&\leq \| g \|_\infty S\left[2\frac{(b-a)^{\beta+1}}{\beta+1}(1-\frac{1}{2^\beta})\right]^{1-\frac{1}{q}}\\
&\times\left[\int_a^{\frac{a+b}{2}}\left((b-t)^\beta-(t-a)^\beta\right)\left(\frac{b-t}{b-a}|f'(a)|^q+\frac{t-a}{b-a}|f'(b)|^q\right)dt\right.
\end{aligned}
$$

$$+\int_{\frac{a+b}{2}}^{b}\left((b-t)^{\beta}-(t-a)^{\beta}\right)\left(\frac{b-t}{b-a}|f'(a)|^{q}+\frac{t-a}{b-a}|f'(b)|^{q}\right)dt\Bigg]^{\frac{1}{q}}.$$

From which one can have (2.11). □

Corollary 2.7. *Setting $\omega = 0$ in Theorem 2.6 we have the following result for Riemann-Liouville fractional integral operators*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)[I_{a^+}^{\beta}g(b)+I_{b_-}^{\beta}g(a)]-[I_{a^+}^{\beta}fg(b)+I_{b_-}^{\beta}fg(a)]\right|$$
$$\leq \frac{2\parallel g\parallel_{\infty}(b-a)^{\beta+1-\frac{1}{q}}}{\Gamma(\beta+2)}\left(1-\frac{1}{2^{\beta}}\right)\left(|f'(a)|^{q}+|f'(b)|^{q}\right)^{\frac{1}{q}},$$

$\beta > 0$.

In the following we give the Hadamard inequality for relative convex functions via generalized fractional integral operators.

Theorem 2.8. *Let $f : [a, g(b)] \to \mathbb{R}$ be a positive relative convex function and $f \in L[a, g(b)]$. Then the following inequalities for generalized fractional integral operators hold*

$$f\left(\frac{a+g(b)}{2}\right)\left(\epsilon_{\alpha,\beta,l,\omega',a^+}^{\gamma,\delta,k}1\right)(g(b)) \leq \frac{1}{2}\left[\left(\epsilon_{\alpha,\beta,l,\omega',a^+}^{\gamma,\delta,k}f\right)(g(b))+\left(\epsilon_{\alpha,\beta,l,\omega',g(b)_-}^{\gamma,\delta,k}f\right)(a)\right]$$
$$\leq \frac{f(a)+f(g(b))}{2}\left(\epsilon_{\alpha,\beta,l,\omega',g(b)_-}^{\gamma,\delta,k}1\right)(a),$$

where $\omega' = \frac{\omega}{(g(b)-a)^{\alpha}}$.

Proof. Since f is relative convex on $[a, g(b)]$, we have

$$f\left(\frac{a+g(b)}{2}\right) = f\left[\left(\frac{1}{2}(ta+(1-t)g(b)\right)+\left(1-\frac{1}{2}\right)((1-t)a+tg(b))\right]$$
$$\leq \frac{1}{2}f\left(ta+(1-t)g(b)\right)+\frac{1}{2}f\left((1-t)a+tg(b)\right).$$

Multiplying both sides by $2t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^{\alpha})$ and integrating over $[0, 1]$ we have

$$2f\left(\frac{a+g(b)}{2}\right)\int_{0}^{1}t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^{\alpha})dt \leq \int_{0}^{1}t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^{\alpha})f\left(ta+(1-t)g(b)\right)dt$$
$$+\int_{0}^{1}t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^{\alpha})f\left((1-t)a+tg(b)\right)dt. \quad (2.14)$$

Setting $ta + (1-t)g(b) = x$ that is $t = \frac{g(b)-x}{g(b)-a}$ and $(1-t)a + tg(b) = y$ that is $t = \frac{y-a}{g(b)-a}$ we have

$$2f\left(\frac{a+g(b)}{2}\right)\int_{g(b)}^{a}\left(\frac{g(b)-x}{g(b)-a}\right)^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}\left(\omega\left(\frac{g(b)-x}{g(b)-a}\right)^{\alpha}\right)\left(\frac{-dx}{g(b)-a}\right) \quad (2.15)$$
$$\leq \int_{g(b)}^{a}\left(\frac{g(b)-x}{g(b)-a}\right)^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}\left(\omega\left(\frac{g(b)-x}{g(b)-a}\right)^{\alpha}\right)f(x)\left(\frac{-dx}{g(b)-a}\right)$$
$$+\int_{a}^{g(b)}\left(\frac{y-a}{g(b)-a}\right)^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}\left(\omega\left(\frac{y-a}{g(b)-a}\right)^{\alpha}\right)f(y)\left(\frac{dy}{g(b)-a}\right).$$

After simplification we get

$$2f\left(\frac{a+g(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}1\right)(g(b)) \leq\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f\right)(g(b))+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}f\right)(a)\right]. \tag{2.16}$$

By using the relative convexity of f on $[a, g(b)]$ one can has

$$f(ta+(1-t)g(b))+f((1-t)a+tg(b)) \leq tf(a)+(1-t)f(g(b))+(1-t)f(a)+tf(g(b)).$$

Multiplying $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})$ on both sides and integrating over $[0,1]$ we have

$$\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f(ta+(1-t)g(b))dt + \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f((1-t)a+tg(b))dt$$
$$\leq \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})(tf(a)+(1-t)f(g(b)))dt + \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})((1-t)f(a)+tf(g(b)))dt.$$

Setting $ta + (1-t)g(b) = x$ that is $t = \frac{g(b)-x}{g(b)-a}$ and $(1-t)a + tg(b) = y$ that is $t = \frac{y-a}{g(b)-a}$ and after simple calculation we have

$$\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f\right)(g(b)) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}f\right)(a)\right] \leq [f(a)+f(g(b))]\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}1\right)(a). \tag{2.17}$$

Combinig (2.16) and (2.17) we get the result. □

Remark 2.9. (i) If we put $\omega = 0$ and $k = 1$ in Theorem 2.8 we obtain Theorem 1.4.
(ii) If we put $\omega = 0$ and $\beta = \frac{\alpha}{k}$ in Theorem 2.8, then we get [11, Theorem 3].

In the upcoming theorem we give the generalization of previous result.

Theorem 2.10. *Let $f : [g(a), g(b)] \to \mathbb{R}$ be a positive relative convex function and $f \in L[g(a), g(b)]$. Then the following inequalities for generalized fractional integral operator holds*

$$f\left(\frac{g(a)+g(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(a)^+}1\right)(g(b))$$
$$\leq \frac{1}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(a)^+}f\right)(g(b)) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}f\right)(a)\right]$$
$$\leq \frac{f(g(a))+f(g(b))}{2}\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}1\right)(g(a)),$$

where $\omega' = \frac{\omega}{(g(b)-g(a))^{\alpha}}$.

Proof. Proof of this theorem is on the same lines of the proof of Theorem 2.8. □

Corollary 2.11. *For $\omega = 0$ we obtain the following inequality for Riemann-Liouville integral operator from Theorem 2.10*

$$f\left(\frac{g(a)+g(b)}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(g(b)-g(a))^{\beta}}[I^{\beta}_{g(a)^+}f(g(b)) + I^{\beta}_{g(b)_-}f(g(a))]$$
$$\leq \frac{f(g(a))+f(g(b))}{2},$$

with $\beta > 0$.

Remark 2.12. In Theorem 2.10 if we take $\omega = 0$, $\beta = \frac{\alpha}{k}$, then we get [11, Theorem 5].

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NEW CONVERGENCE THEOREMS FOR COMMON FIXED POINTS OF A WIDE RANGE OF NONLINEAR MAPPINGS

TAKANORI IBARAKI*[1] AND YUKIO TAKEUCHI[2]

[1] Department of Mathematics Education, Yokohama National University,
79-2 Tokiwadai, Hodogaya, Yokohama 240-8501, Japan
[2] Takahashi Institute for Nonlinear Analysis
1-11-11 Nakazato, Minami, Yokohama 232-0063, Japan

ABSTRACT. In this article, we present new convergence theorems for common fixed points of a wide range of nonlinear mappings in the Hilbert space setting.

KEYWORDS: Attractive point, common fixed point, convergence theorems.
AMS Subject Classification: Primary 47H09, 47H10; Secondary 41A65.

1. Introduction

In 1963, DeMarr [11] proved a common fixed point theorem for families of commuting nonexpansive mappings. After DeMarr, many researchers studied this subject and many results for families of nonexpansive mappings appeared; refer to Linhart [27], Ishikawa [15], Kuhfittig [25], Kitahara and Takahashi [16], Takahashi and Tamura [39], Suzuki [36, 35] and so on. For example, in the strictly convex Banach space setting, Linhart [27] presented an iteration scheme for common fixed points of infinite families of commuting nonexpansive self–mappings on a compact convex set. Motivated by Linhart's result, Suzuki [36] presented the following.

Theorem S. Let C be a compact convex subset of a strictly convex Banach space E. Let $\{T_n\}$ be a sequence of nonexpansive mappings on C with $\cap_{n=1}^{\infty} F(T_n) \neq$. Let $\{a_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n < 1$ and let $\{I_n\}$ be a sequence of subsets of N satisfying $I_n \subset I_{n+1}$ for $n \in N$ and $\cup_{n=1}^{\infty} I_n = N$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = (1 - \textstyle\sum_{i \in I_n} a_i) x_n + \sum_{i \in I_n} a_i T_i x_n \qquad \text{for} \quad n \in N.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

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* *Corresponding author.*
Email address : ibaraki@ynu.ac.jp, aho314159@yahoo.co.jp.

On the other hand, in 1975, Baillon [6] proved the first nonlinear ergodic theorem for a nonexpansive mapping in a Hilbert space. After Baillon, many mean convergence theorems appeared. Furthermore, Takahashi and Takeuchi [40] proved a mean convergence theorem for attractive points of generalized hybrid mappings with neither closeness nor convexity of the domain. Also, Aoyama [1] and Kohsaka [19] proved convergence theorems for quasi–nonexpansive type mappings.

In 1997, Shimizu and Takahashi [32] studied a common fixed point problem for finite families of commutative nonexpansive mappings. They introduced an iteration scheme combined Halpern type and Baillon type, and proved a strong convergence theorem in Hilbert spaces. In 1998, Atsushiba and Takahashi [4] introduced an iteration scheme combined Mann type and Baillon type, and proved a weak convergence theorem for commutative two nonexpansive mappings, in uniformly convex Banach spaces. Suzuki [34] and Takeuchi [42] studied this problem in general Banach spaces.

Very recently, in the Hilbert space setting, Kohsaka [20] replaced nonexpansive mappings by (λ)–hybrid mappings in the main theorems of [32, 4]. Kohsaka [20] also presented the following theorem; also see Ibaraki and Takeuchi [13].

Theorem K. Let C be a bounded closed and convex subset of a Hilbert space H. Let S and T be (λ)–hybrid self–mappings on C with λ and μ. Assume $ST = TS$. Set $F = F(S) \cap F(T)$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \tfrac{1}{(n+1)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} S^i T^j x_1 \qquad \text{for} \quad n \in N.$$

Then the following hold.

(1) $\{P_F S^i T^j x_1\}_{(i,j)\in N_0^2}$ converges strongly to $u \in F$ in the sense of net.
(2) $\{x_n\}$ converges weakly to $u \in F$.

Remark. Of course, we can replace the boundedness of C by $F = F(S) \cap F(T) \neq$.

Motivated by the works as above, we hope to add something new. Then, specifically, we prove some convergence theorems for common fixed points of a wide range of nonlinear self–mappings on a closed convex subset of a Hilbert space.

2. Preliminaries

In this article, N and N_0 denote the sets of positive integers and non–negative integers, respectively. $N(i,j)$ denotes the set $\{k \in N_0 : i \leq k \leq j\}$ for $i, j \in N_0$ with $i \leq j$. In the case of $j < i$, we define $N(i,j) =$ and $\sum_{k=i}^{j}(\cdot) = 0$.

H denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ derived from $\langle \cdot, \cdot \rangle$. C always denotes a non–empty subset of H unless otherwise noted. Then, normally, "non–empty" is omitted. The following are basic:

(1) A closed convex subset C of H is weakly closed. A bounded sequence in H has a weakly convergent subsequence.

(2) Let $\{u_n\}$ be a sequence in H. Then $\{u_n\}$ converges weakly to $z \in H$ if every weak cluster point of $\{u_n\}$ and z are the same.

(3) H has the Opial property [30], that is, if $\{u_n\}$ is a sequence in H which converges weakly to $u \in H$, then, for $v \in H$ with $v \neq u$,

$$\liminf_n \|u_n - u\| < \liminf_n \|u_n - v\|.$$

(4) Let C be a closed convex subset of H. For $x \in H$, there is the unique point z_x of C satisfying $\|x - z_x\| = \inf\{\|x - z\| : z \in C\}$. z_x is called the unique nearest

point of C to x. Define a mapping P_C by $P_C x = z_x$ for $x \in H$. P_C is called the metric projection from H onto C. P_C satisfies the following: For $x \in H$ and $y \in C$,

$$0 \leq \langle x - P_C x, P_C x - y \rangle \quad \text{and} \quad \|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2.$$

Let C be a subset of H and T be a mapping from C into H. I denotes the identity mapping on C. Sometimes we denote I by T^0. $F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. $A(T)$ denotes the set of attractive points of T, that is, $A(T) = \{x \in H : \|Ty - x\| \leq \|x - y\| \text{ for all } y \in C\}$; for the notion of attractive points, see Takahashi and Takeuchi [40]. $I - T$ is said to be demiclosed at 0 if $u \in F(T)$ holds whenever there is a sequence $\{x_n\}$ in C which converges weakly to $u \in C$ and satisfies $\lim_n \|Tx_n - x_n\| = 0$. In the case that C is compact and convex, $I - T$ is demiclosed at 0 if T is continuous on C.

T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in C$. Also T is called quasi–nonexpansive if $\neq F(T) \subset A(T)$. A nonexpansive mapping T with $F(T) \neq$ is quasi–nonexpansive.

T is said to satisfy condition (N_2) if there is $s \in [0, \infty)$ such that

$$\|x - Ty\| \leq \|x - y\| + s\|x - Tx\| \quad \text{for} \quad x, y \in C. \qquad (N_2)$$

A nonexpansive mapping satisfies (N_2) as $s = 1$. T satisfies $F(T) \subset A(T)$ if T satisfies (N_2). Then T is quasi–nonexpansive if T satisfies (N_2) and $F(T) \neq$.

Recently, some researchers study (N_2); see Suzuki [37], Falset and co–authors [12], Takahashi and Takeuchi [40], Kubota and Takeuchi [22], and Kubota and co–authors [21]. Also, some researchers study generalized hybrid mappings introduced by Kocourek and co–authors [18] or (λ)–hybrid mappings introduced by Aoyama and co–authors [2]. The class of generalized hybrid mappings is wider than the class of (λ)–hybrid mappings. Even so, the class of (λ)–hybrid mappings contains some important classes of nonlinear mappings.

In [2], they say as below: Let $\lambda \in R$. T is called λ–hybrid if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty \rangle \quad \text{for} \quad x, y \in C. \qquad (\lambda_h)$$

For example, the following expression appeared in Kohsaka [20]: Let S be a λ–hybrid self–mapping on C and T be a μ–hybrid self–mapping on C. To avoid confusion, we call T (λ)–hybrid if there is $\lambda \in R$ satisfying (λ_h). Then the expression becomes as below: Let S and T be (λ)–hybrid self–mappings on C with λ and μ.

A nonexpansive mapping is (λ)–hybrid as $\lambda = 1$. T satisfies $F(T) \subset A(T)$ if T is (λ)–hybrid. So a (λ)–hybrid mapping T is quasi–nonexpansive if $F(T) \neq$. Since the last term in (λ_h) is written by inner product, it is easy to deal with.

We had better give remarks for our way of thinking in this article.

Our way of thinking. Let C be a closed convex subset of a Hilbert space H. In later sections, we deal with a sequence $\{T_j\}$ of nonlinear self–mappings on C. To have a convergence theorem for common fixed points of $\{T_j\}$, maybe it is difficult to ignore the condition that $I - T_j$ is demiclosed at 0 for $j \in N$. Then we will assume the condition to express our assertions. Also we consider the following conditions:

$$\neq A = \cap_{j \in N} A(T_j), \qquad F = \cap_{j \in N} F(T_j) \subset A.$$

We give some notes for the conditions. For simplicity, we consider $\{S, T\}$ as $\{T_j\}$.

Let S and T be self–mappings on a closed convex subset C of a Hilbert space H. We denote by F the common fixed point set $F(S) \cap F(T)$ and by A the common attractive point set $A(S) \cap A(T)$. To have a convergence theorem finding a common fixed point of $\{S, T\}$, usually, we assume $F \neq$. In the case that both S and T are nonexpansive, $F \neq$ asserts $\neq F \subset A$ in cooperation with properties of nonexpansive mappings. However, we should be more careful about the fact that we do not make some beneficial results from $F \neq$ itself. In proofs of many such theorems, it seems that conditions corresponding to $A \neq$ and $F \subset A$ are essential.

We note the following:

(a) $F \subset A$ implies neither $F(S) \subset A(S)$ nor $F(T) \subset A(T)$.
(b) $F \neq$ does not imply $A \neq$ without the assumption $F \subset A$.
(c) $A \neq$ implies $F \neq$; see Lemma 3.7.
However, $A \neq$ does not imply $F \subset A$.
(d) Usually, it follows from the assumption $F \subset A$ that F is closed and convex.

In the case that $S = T$, $F \subset A$ and $F(S) \subset A(S)$ are equivalent.

Suppose both S and T are quasi–nonexpansive. In this case, $\{S, T\}$ has so good properties, that is, we know the following:

(e) $F = F(S) \cap F(T) \subset A(S) \cap A(T) = A$.
(f) $F(S)$, $F(T)$ and F are closed and convex.

It is important that, even if $\neq F \subset A$, neither S nor T need be quasi–nonexpansive. Furthermore, we easily find pairs of C and $\{S, T\}$ such that neither S nor T is quasi–nonexpansive, $\neq F \subset A$ and $ST \neq TS$. However, in general, we may need strict constraints on properties of $\{S, T\}$ to guaranty $A \neq$ in theory. Even so, to find a point of A is easier than to find directly a point of $F \cap A$.

Due to the reasons as above, to express our assertions connected with common fixed points of $\{T_j\}$, we assume the following:

(i) $I - T_j$ is demiclosed at 0 for $j \in N$.
(ii) $\neq A = \cap_{j \in N} A(T_j)$ and $F = \cap_{j \in N} F(T_j) \subset A$.

Here we present an example. For simplicity, we consider R^2 with the Euclidean norm. Maybe T_1 and T_2 in the example are closed to us and just ordinary mappings.

Example 2.1. Let $D = \{x = (s,t) \in R^2 : s \in [0,1], t \in [\frac{1}{2}s, 2s]\}$. Then D is compact and convex. For $x = (s,t) \in D$, set $u_x = (\frac{1}{2}t, t)$ and $z_x = (s, \frac{1}{2}s)$. Let T_1 and T_2 be self–mappings on D defined by

$$T_1 x = \tfrac{1}{2}(x + u_x) = \tfrac{1}{2}((s,t) + (\tfrac{1}{2}t, t)) = (\tfrac{1}{2}s + \tfrac{1}{4}t, t),$$
$$T_2 x = \tfrac{1}{2}(x + z_x) = \tfrac{1}{2}((s,t) + (s, \tfrac{1}{2}s)) = (s, \tfrac{1}{4}s + \tfrac{1}{2}t) \qquad \text{for } x = (s,t) \in D.$$

Then we can easily observe the following:

- (i) holds, that is, $I - T_j$ is demiclosed at 0 for $j = 1, 2$.
- (ii) holds, that is, $\neq \cap_{j=1}^2 F(T_j) \subset \cap_{j=1}^2 A(T_j)$.

Also, we can easily confirm the following:

- Neither T_1 nor T_2 is quasi–nonexpansive (hemi–contractive).
- T_1 and T_2 are not commutative.
- $B = \frac{1}{2}T_1 + \frac{1}{2}T_2$ is nonexpansive and $F(B) = \{(0,0)\} = \cap_{j=1}^2 F(T_j)$.

We had better note the following: A real linear space L may have more than one norms. Then it may depend on norm whether $\neq F \subset A$ holds or not. In some cases, nonexpansiveness of T and $A(T)$ depend on norm. Quasi–nonexpansiveness of T depends on norm and the domain of T. However, $F(T)$ has no connection with norms if the formula of T does not contain any norm on L. Especially, in finite dimensional linear spaces, we may choose a convenient norm to find a point of F.

3. Lemmas

Many researchers take the following assertion or a similar assertion in their articles; for example, see Weng [43], Xu [45], and Aoyama and co–authors [2].

Lemma 3.1. *Let $\{\alpha_n\}$ be a sequence in $[0,1]$. Let $\{a_n\}$ and $\{c_n\}$ be sequences of non–negative real numbers and let $\{b_n\}$ be a sequence of real numbers. Suppose $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_n b_n \le 0$, $\sum_{n=1}^{\infty} c_n < \infty$, and $a_{n+1} \le (1-\alpha_n)a_n + \alpha_n b_n + c_n$ for $n \in N$. Then $\lim_n a_n = 0$.*

In the Hilbert space setting, we present some lemmas needed in the sequel; some of them are obtained in suitable Banach spaces. The following is well–known.

Lemma 3.2. *Let $c \in [0,1]$ and $x, y \in H$. Then, the following holds:*

$$\|cx + (1-c)y\|^2 = c\|x\|^2 + (1-c)\|y\|^2 - c(1-c)\|x-y\|^2.$$

In the Hilbert space setting, the following lemma is an extension of Browder's demiclosed principle. This lemma was essentially proved in Suzuki [37].

Lemma 3.3. *Let C be a subset of H and let S be a mapping from C into H which satisfies (N_2). Suppose $\{x_n\}$ is a sequence in C which converges weakly to some $u \in C$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in F(S)$.*

Proof. We know that $\{x_n\}$ converges weakly to u, S satisfies condition (N_2) for some $s \in [0,\infty)$, and $\lim_n \|Sx_n - x_n\| = 0$. Then the following holds:

$$\begin{aligned} \liminf_n \|x_n - Su\| &\le \liminf_n (\|x_n - u\| + s\|x_n - Sx_n\|) \\ &= \liminf_n \|x_n - u\|. \end{aligned} \tag{3.1}$$

Arguing by contradiction, assume $u \neq Su$. Then, by the Opial property, we have $\liminf_n \|x_n - u\| < \liminf_n \|x_n - Su\|$. This contradicts to (3.1). □

The following is another extension due to Aoyama and co–authors [2].

Lemma 3.4. *Let C be a subset of H and let S be a (λ)–hybrid mapping with λ from C into H. Suppose $\{x_n\}$ is a sequence in C which converges weakly to some $u \in C$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in F(S)$.*

Proof. We know that $\{x_n\}$ converges weakly to u and S is (λ)–hybrid with λ. By $\lim_n \|Sx_n - x_n\| = 0$, the following hold:

$$\liminf_n \|x_n - Su\| \le \liminf_n (\|x_n - Sx_n\| + \|Sx_n - Su\|) = \liminf_n \|Sx_n - Su\|,$$

$$\begin{aligned} \liminf_n \|x_n - Su\|^2 &\le \liminf_n \|Sx_n - Su\|^2 \\ &\le \liminf_n (\|x_n - u\|^2 + 2|1-\lambda|\|Sx_n - x_n\|\|Su - u\|) = \liminf_n \|x_n - u\|^2. \end{aligned}$$

Then we have

$$\liminf_n \|x_n - Su\| \le \liminf_n \|x_n - u\|. \tag{3.2}$$

Arguing by contradiction, assume $u \neq Su$. Then, by the Opial property, we have $\liminf_n \|x_n - u\| < \liminf_n \|x_n - Su\|$. This contradicts to (3.2). □

The following lemma is useful when we consider weak convergence theorems in the Hilbert space setting; for example, see Atsushiba and co–authors [3].

Lemma 3.5. *Let D be a subset of H. Let $\{u_n\}$ be a sequence in H such that $\{\|u_n - w\|\}$ converges for each $w \in D$. Suppose $\{u_{n_i}\}$ and $\{u_{n_j}\}$ are subsequences of $\{u_n\}$ which converge weakly to $u, v \in D$, respectively. Then $u = v$.*

Proof. Let $w \in D$. Then, since $\{\|u_n - w\|\}$ converges, any subsequence of $\{\|u_n - w\|\}$ converges to the same real number. Arguing by contradiction, assume $u \neq v$. Then, by $u, v \in D$ and the Opial property, we have the following:

$$\liminf_i \|u_{n_i} - u\| < \liminf_i \|u_{n_i} - v\| = \liminf_j \|u_{n_j} - v\|,$$

$$\liminf_j \|u_{n_j} - v\| < \liminf_j \|u_{n_j} - u\| = \liminf_i \|u_{n_i} - u\|.$$

Thus we have $\liminf_i \|u_{n_i} - u\| < \liminf_i \|u_{n_i} - u\|$. This is a contradiction. □

The following two lemmas are due to Takahashi and Takeuchi [40].

Lemma 3.6. *Let C be a subset of H and let T be a mapping from C into H. Then, $A(T)$ is a closed convex subset of H.*

Lemma 3.7. *Let C be a subset of H and let T be a self–mapping on C. Suppose $x \in A(T)$ and z_x is the unique nearest point of C to x. Then $z_x \in F(T)$. In particular, $A(T) \cap C \subset F(T)$. Furthermore, $A(T) \cap C = F(T)$ holds if $F(T) \subset A(T)$.*

We need the following lemma in the sequel.

Lemma 3.8. *Let C be a subset of H and let T be a mapping from C into H. Let $a \in [0,1]$, $x \in C$ and $w = ax + (1-a)Tx$. Suppose $v \in A(T)$. Then,*

$$a(1-a)\|Tx - x\|^2 \le \|x - v\|^2 - \|w - v\|^2. \tag{1}$$

Suppose further that C is bounded. Let $r > \sup_{x \in C} \|x - v\|$. Then,

$$\tfrac{a(1-a)}{2r}\|Tx - x\|^2 \le \|x - v\| - \|w - v\|. \tag{2}$$

Proof. We show (1). By $v \in A(T)$ and Lemma 3.2, we have

$$\begin{aligned} \|w - v\|^2 &= \|a(x - v) + (1-a)(Tx - v)\|^2 \\ &= a\|x - v\|^2 + (1-a)\|Tx - v\|^2 - a(1-a)\|Tx - x\|^2 \\ &\le \|x - v\|^2 - a(1-a)\|Tx - x\|^2. \end{aligned}$$

Then we see $\|w - v\| \le \|x - v\|$ and $a(1-a)\|Tx - x\|^2 \le \|x - v\|^2 - \|w - v\|^2$.

There is $r \in (0, \infty)$ satisfying $r > \sup_{x \in C} \|x - v\|$ if C is bounded. We show (2). Set $s = \|x - v\|$ and $t = \|w - v\| \le \|x - v\|$. Then we know $0 \le s + t < 2r$ and

$$a(1-a)\|Tx - x\|^2 \le s^2 - t^2 = (s - t)(s + t).$$

In the case of $0 < s + t < 2r$, we immediately have

$$\tfrac{a(1-a)}{2r}\|Tx - x\|^2 \le \tfrac{a(1-a)}{s+t}\|Tx - x\|^2 \le \|x - v\| - \|w - v\|.$$

In the case of $s + t = 0$, it is trivial that $\frac{a(1-a)}{2r}\|Tx - x\|^2 \le \|x - v\| - \|w - v\|$. □

4. Convergence theorems

In this section, we present our main results. We begin our argument with considering the following sequences. Let $\{c_j\}$ be a sequence satisfying the following:

$$c_j \in (0,1) \quad \text{for} \quad j \in N, \qquad \textstyle\sum_{j=1}^{\infty} c_j = 1. \tag{s}$$

Let $\{c_{n,j}\}$ be the double sequence such that, for each $n \in N$,

$$c_{n,j} = c_j \quad \text{for} \quad j \in N(1, n-1), \qquad c_{n,n} = \textstyle\sum_{j=n}^{\infty} c_j = 1 - \sum_{j=1}^{n-1} c_j. \tag{ds}$$

Note $N(1,0) =$, $\sum_{j=1}^{0}(\cdot) = 0$ and $c_{1,1} = 1$. For $j \in N$, $c_{n,j} = c_j$ holds for $n > j$. Then the double sequence $\{c_{n,j}\}$ has the following properties:

$$\lim_n c_{n,j} = c_j \quad \text{for } j \in N, \qquad \textstyle\sum_{j=1}^{n} c_{n,j} = 1 \quad \text{for } n \in N.$$

For reference, we present a typical example of $\{c_j\}$ satisfying (s). Set $c_j = 1/2^j$ for $j \in N$. Then $\{c_j\}$ satisfies (s). For example, $\{c_{5,j}\}_{j \in N(1,5)} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\}$.

The following lemma is important to have our weak convergence theorems.

Lemma 4.1. *Let $a, b \in (0,1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a convex subset of H and let $\{T_j\}$ be a sequence of self–mappings on C. Assume $A = \cap_{j\in N} A(T_j) \neq$. Let $S_n = \sum_{j=1}^n c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1-a_n) S_n x_n \qquad \text{for } n \in N.$$

Then the following hold:

(1) *$\{\|x_n - u\|\}$ converges for $u \in A$.*
(2) *$\lim_n \|T_j x_n - x_n\| = 0$ for $j \in N$.*

Proof. Fix any $u \in A = \cap_{j\in N} A(T_j)$. We know that, for $n \in N$ and $x \in C$,

$$\|S_n x - u\| \le \sum_{j=1}^n c_{n,j} \|T_j x - u\| \le \|x - u\|.$$

So $u \in \cap_{n\in N} A(S_n)$. Then, $A \subset \cap_{n\in N} A(S_n)$. Set $D = \{x \in C : \|x-u\| \le \|x_1 - u\|\}$. Then D is bounded and convex. By the inequality as above, we easily see that each S_n is a self–mapping on D. Then $\{x_n\}$ is a sequence in D.

We show (1). By Lemma 3.8 (1), we see that, for $n \in N$,

$$0 \le a_n(1-a_n)\|S_n x_n - x_n\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Then $\{\|x_n - u\|\}$ is non–increasing and converges.

We show (2). Since D is bounded, let $r \in (0,\infty)$ satisfy $r > \sup_{x\in D} \|x-u\|$. Recall properties of $\{c_{n,j}\}$. By using Lemma 3.8 (2), we easily see that, for $n \in N$,

$$\begin{aligned}
\|x_{n+1} - u\| &= \|a_n x_n + (1-a_n) S_n x_n - u\| \\
&\le \sum_{j=1}^n c_{n,j} \|a_n x_n + (1-a_n) T_j x_n - u\| \\
&\le \sum_{j=1}^n c_{n,j} (\|x_n - u\| - \tfrac{a_n(1-a_n)}{2r} \|T_j x_n - x_n\|^2) \\
&\le \|x_n - u\| - \tfrac{a(1-b)}{2r} \sum_{j=1}^n c_{n,j} \|T_j x_n - x_n\|^2.
\end{aligned}$$

From this inequality, the following follows:

$$\tfrac{a(1-b)}{2r} \sum_{j=1}^n c_{n,j} \|T_j x_n - x_n\|^2 \le \|x_n - u\| - \|x_{n+1} - u\|.$$

Since $\{\|x_n - u\|\}$ converges and $\frac{a(1-b)}{2r} > 0$, we see that, for $j \in N$,

$$\limsup_n c_{n,j} \|T_j x_n - x_n\|^2 \le \limsup_n (\sum_{j=1}^n c_{n,j} \|T_j x_n - x_n\|^2) \le 0.$$

Then we have the following:

$$\lim_n \|T_j x_n - x_n\| = 0 \quad \text{for } \ j \in N.$$

□

4.1. **Weak convergence theorems.**

We present a weak convergence theorem which is one of our main results.

Theorem 4.2. *Let $a, b \in (0,1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N$. Set $F = \cap_{j\in N} F(T_j)$ and $A = \cap_{j\in N} A(T_j)$. Assume $\neq A$. Let $S_n = \sum_{j=1}^n c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1-a_n) S_n x_n \qquad \text{for } n \in N.$$

Then the following hold:

(1) *Every weak cluster point of* $\{x_n\}$ *is a point of* F.
(2) *In the case of* $F \subset A$, $\{x_n\}$ *converges weakly to some* $z \in F$.

Proof. We know that C is weakly closed and $\{x_n\}$ is a sequence in C. By Lemma 4.1, we also know that $\{\|x_n - u\|\}$ converges for $u \in A$ and

$$\lim_n \|T_j x_n - x_n\| = 0 \quad \text{for} \quad j \in N. \tag{4.1}$$

Since $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence.

We show (1). Let $\{x_{n_l}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $z \in C$. Since $I - T_j$ is demiclosed at 0 for $j \in N$, by (4.1), $z \in F = \cap_{j \in N} F(T_j)$. Thus every weak cluster point of $\{x_n\}$ is a point of F. We show (2). Suppose $F = \cap_j F(T_j) \subset A$. Then, $\{\|x_n - u\|\}$ converges for $u \in F \subset A$. Let z be a weak cluster point of $\{x_n\}$. Then, by Lemma 3.5 and (1), every weak cluster point of $\{x_n\}$ and $z \in F$ are the same. Thus $\{x_n\}$ converges weakly to $z \in F$. □

Remark 4.3. Let $m \in N_0$. By observing proofs of Lemma 4.1 and Theorems 4.2, it is obvious that we can replace S_n by S_{n+m} in the iteration scheme in Theorems 4.2.

Here we present some results derived from Theorems 4.2.

Theorem 4.4. *Let* $a, b \in (0, 1)$ *satisfy* $a \le b$ *and let* $\{a_n\}$ *be a sequence in* $[a, b]$. *Let* $k \in N$. *Let* C *be a closed convex subset of* H *and let* $\{T_j\}_{j \in N(1,k)}$ *be a finite sequence of self–mappings on* C *such that* $I - T_j$ *is demiclosed at* 0 *for* $j \in N(1, k)$. *Set* $F = \cap_{j \in N(1,k)} F(T_j)$ *and* $A = \cap_{j \in N(1,k)} A(T_j)$. *Assume* $\neq A$. *Let* $S = \frac{1}{k}\sum_{j=1}^{k} T_j$. *Define a sequence* $\{x_n\}$ *by* $x_1 \in C$ *and*

$$x_{n+1} = a_n x_n + (1 - a_n) S x_n \qquad \text{for } n \in N.$$

Then the following hold:

(1) *Every weak cluster point of* $\{x_n\}$ *is a point of* F.
(2) *In the case of* $F \subset A$, $\{x_n\}$ *converges weakly to some* $z \in F$.

Proof. Define $\{c_j\}$ and $\{U_j\}$ by

$$c_j = \tfrac{1}{k},\ U_j = T_j \quad \text{for} \quad j \in N(1, k-1), \quad c_j = \tfrac{1}{2^{j-k+1}} \times \tfrac{1}{k},\ U_j = T_k \quad \text{for} \quad j \ge k.$$

Then we know $\sum_{j=k}^{\infty} c_j = \frac{1}{k}$ and $\sum_{j=1}^{\infty} c_j = 1$. Also, we easily see that each $I - U_j$ is demiclosed at 0, $\cap_{j \in N} A(U_j) = \cap_{j \in N(1,k)} A(T_j)$ and $\cap_{j \in N} F(U_j) = \cap_{j \in N(1,k)} F(T_j)$. Let $\{c_{n,j}\}$ be the sequence satisfying (ds). Then $\{c_{n,j}\}$ and $\{U_j\}$ satisfy all assumptions in Theorem 4.2. Fix any $n \ge k$. We confirm that $S_n = \sum_{j=1}^{n} c_{n,j} U_j$ becomes S. By the definitions of $\{c_{n,j}\}$ and $\{U_j\}$, we have

$$\begin{aligned} \sum_{j=1}^{n} c_{n,j} U_j &= \sum_{j=1}^{k-1} \tfrac{1}{k} T_j + \sum_{j=k}^{n-1} \left(\tfrac{1}{2^{j-k+1}} \times \tfrac{1}{k}\right) T_k + \sum_{j=n}^{\infty} \left(\tfrac{1}{2^{j-k+1}} \times \tfrac{1}{k}\right) T_k \\ &= \tfrac{1}{k} \sum_{j=1}^{k} T_j. \end{aligned}$$

From these, by Theorems 4.2 and Remark 4.3, we have the results. □

Theorem 4.5. *Let* $a, b \in (0, 1)$ *satisfy* $a \le b$ *and let* $\{a_n\}$ *be a sequence in* $[a, b]$. *Let* $\{c_j\}$ *be a sequence satisfying* (s) *and let* $\{c_{n,j}\}$ *be the double sequence satisfying* (ds). *Let* C *be a closed convex subset of* H *and let* $\{T_j\}$ *be a sequence of self–mappings on* C *such that* T_j *satisfies* (N_2) *for* $j \in N$. *Assume* $F = \cap_{j \in N} F(T_j) \neq$. *Let* $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ *for* $n \in N$. *Define a sequence* $\{x_n\}$ *by* $x_1 \in C$ *and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ *converges weakly to some* $z \in F$.

Proof. Since each T_j satisfies (N_2), $\neq F \subset A = \cap_{j\in N} A(T_j)$ holds. By Lemma 3.3, $I - T_j$ is demiclosed at 0 for $j \in N$. By Theorem 4.2, we have the result. □

Theorem 4.6. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of (λ)–hybrid self–mappings on C. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F$.

Proof. Since each T_j is (λ)–hybrid, $\neq F \subset A = \cap_{j\in N} A(T_j)$ holds. By Lemma 3.4, $I - T_j$ is demiclosed at 0 for $j \in N$. By Theorem 4.2, we have the result. □

Theorem 4.7. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of quasi–nonexpansive self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N$. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F$.

Proof. Since each T_j is quasi–nonexpansive, $\neq F \subset A = \cap_{j\in N} A(T_j)$ holds. By Theorem 4.2, we have the result. □

Theorem 4.8. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let C be a closed convex subset of H and let T be a quasi–nonexpansive self–mapping on C such that $I - T$ is demiclosed at 0. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) T x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F(T)$.

The following is corresponding to Theorem S due to Suzuki.

Theorem 4.9. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of nonexpansive self–mappings on C. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F$.

We present some convergence theorems for sequences of non–self mappings which are derived from Theorem 4.2. In advance, we prepare a lemma.

Lemma 4.10. *Let C be a closed convex subset of H and let T be a quasi–nonexpansive mapping from C into H. Then $F(T) = F(P_C T)$.*

Proof. Note $F(T) \neq$. In general, $F(T) \subset F(P_C T)$ holds. We show the reverse.

Let $z \in F(P_C T)$ and $u \in F(T)$. Since T is quasi–nonexpansive, we have

$$\|Tz - z\|^2 + \|z - u\|^2 = \|Tz - P_C Tz\|^2 + \|P_C Tz - u\|^2 \leq \|Tz - u\|^2 \leq \|z - u\|^2.$$

This implies $Tz = z$. Thus we have $F(P_C T) \subset F(T)$. □

The following is a direct consequence of Theorem 4.2.

Theorem 4.11. *Let $a, b \in (0,1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of mappings from C into H such that $I - P_C T_j$ is demiclosed at 0 for $j \in N$. Set $F' = \cap_{j\in N} F(P_C T_j)$ and $A' = \cap_{j\in N} A(P_C T_j)$. Assume $A' \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} P_C T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then the followings hold.

(1) *Every weak cluster point of $\{x_n\}$ is a point of F'.*
(2) *In the case of $F' \subset A'$, $\{x_n\}$ converges weakly to some $z \in F'$.*

Remark 4.12. Here we give an additional explanation for Theorem 4.11.

Set $F = \cap_{j\in N} F(T_j)$ and $A = \cap_{j\in N} A(T_j)$. We consider the case of $F' = F$. Suppose $\{T_j\}$ is a sequence of quasi–nonexpansive mappings with $\neq F$. By Lemma 4.10, $F' = F$. Then, $\neq F' = F \subset A \cap A'$ holds because $F \subset A$ and

$$\|P_C T_j y - u\| = \|P_C T_j y - P_C u\| \le \|T_j y - u\| \le \|y - u\| \qquad \text{for } y \in C,\ u \in F.$$

We note the following: For $j \in N$, $P_C T_j$ $(T_j P_C)$ is nonexpansive if T_j is nonexpansive; $I - P_C T_j$ is demiclosed at 0. Furthermore, for example, we know the following: Let T be a k–strictly pseudo–contractive mapping from C into H, where $k \in [0,1)$. Then, we can easily find a nonexpansive mapping S satisfying $A(T) = A(S)$ and $F(T) = F(S)$; see Zhou [46], and Atsushiba and co–authors [3].

Suppose further that C is compact and every T_j is continuous. Then $P_C T_j$ is continuous; $I - P_C T_j$ is demiclosed at 0. Such pairs of C and $\{T_j\}$ are typical examples. So, $\neq F' = F \subset A \cap A'$ and assumptions in Theorem 4.11 are satisfied.

Theorem 4.13. *Let $a, b \in (0,1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of mappings from C into H satisfying the following:*

(a) *T_1 is a self–mappings on C, and $T_j T_1$ are self–mappings on C for $j \ge 2$.*
(b) *$I - T_j$ is demiclosed at 0 for $j \in N$.*

Let $V_1 = T_1$ and $V_j = T_j T_1$ for $j \ge 2$. Set $F = \cap_{j\in N} F(T_j)$ and $A' = \cap_{j\in N} A(V_j)$. Assume $A' \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} V_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then the following hold:

(1) *Every weak cluster point of $\{x_n\}$ is a point of F.*
(2) *In the case of $F \subset A'$, $\{x_n\}$ converges weakly to some $z \in F$.*

Proof. By Lemma 4.1, we know that $\{\|x_n - u\|\}$ converges for $u \in A'$. Also we know $\lim_n \|V_j x_n - x_n\| = 0$ for $j \in N$. Then we have

$$\text{(i) } \lim_n \|T_1 x_n - x_n\| = 0, \qquad \text{(ii) } \lim_n \|T_j T_1 x_n - x_n\| = 0 \ \text{ for } \ j \ge 2.$$

Since $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence.

We show (1). Let $\{x_{n_l}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $z \in C$. Since $I - T_1$ is demiclosed at 0, by (i), we see $z \in F(T_1)$. Also, by (i), $\{T_1 x_{n_l}\}$ converges weakly to z. Furthermore, by (i) and (ii), we see

$$\lim_l \|T_j T_1 x_{n_l} - T_1 x_{n_l}\| = 0 \ \text{ for } \ j \ge 2. \tag{4.2}$$

From these, since $I - T_j$ is demiclosed at 0 for $j \in N$, we have $z \in F = \cap_{j\in N} F(T_j)$. Thus every weak cluster point of $\{x_n\}$ is a point of F.

We show (2). Suppose $F \subset A'$. Then $\{\|x_n - u\|\}$ converges for $u \in F \subset A'$. Let z be a weak cluster point of $\{x_n\}$. By Lemma 3.5 and (1), every weak cluster point of $\{x_n\}$ and $z \in F$ are coincide. Thus $\{x_n\}$ converges weakly to $z \in F$. □

Remark 4.14. In Theorem 4.13, set $A = \cap_{j\in N} A(T_j)$. For reference, we show $A \subset A'$. Let $u \in A$. Then, since T_1 is a self–mapping on C, we see

$$\|V_j x - u\| = \|T_j T_1 x - u\| \leq \|T_1 x - u\| \leq \|x - u\| \quad \text{for} \quad x \in C.$$

Note that we do not claim $A \neq$. For the theorem, we only present the following typical example: Let $C = [-1, 1] \subset R$ and let T_1 and T_2 be mappings defined by $T_1 x = x/2$ and $T_2 x = 2x$ for $x \in [-1, 1]$. Then it is obvious that $A(T_1) = \{0\}$, $A(T_2) =$, $F(T_2) = F(T_1) = \{0\}$, $T_2T_1 = I$, $A(T_2T_1) = R$ and $F(T_2T_1) = C$. Furthermore, T_2 is not a self–mapping on C, $I - T_1$ and $I - T_2$ are demiclosed at 0, $A = A(T_1) \cap A(T_2) =$ and

$$\{0\} = F = F(T_1) \cap F(T_2) = F(T_1) \cap F(T_2T_1) = A(T_1) \cap A(T_2T_1) = A' = \{0\}.$$

4.2. **Strong convergence theorems.**

We present a strong convergence theorem which is our another main result. This theorem is connected with works of Aoyama [1], and Atsushiba and co–authors [3]; also see Maingé and Măruşter [28].

Theorem 4.15. *Let $b \in (0, 1)$ and let $\{a_n\}$ be a sequence in $(0, 1)$ satisfying*

$$\lim_n a_n = 0, \qquad \textstyle\sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N$. Set $F = \cap_{j\in N} F(T_j)$ and $A = \cap_{j\in N} A(T_j)$. Assume $\neq A$ and $F \subset A$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1 - b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_A q = P_F q \in F$.

Proof. Since C is convex, each S_n, each U_n and each $a_n q + (1 - a_n)U_n$ are self–mappings on C. Then $\{u_n\}$ is a sequence in C. By Lemma 3.6, A is closed and convex. Then we can consider the metric projection P_A. Set $v = P_A q \in A$ and $D = \{x \in C : \|x - v\| \leq \|u_1 - v\| + \|q - v\|\}$. Then D is bounded closed and convex. We know $q, u_1 \in D$ and $v \in A \subset \cap_{n\in N} A(S_n)$. Then, for $x \in D$ and $n \in N$, we have

$$\|U_n x - v\| \leq b\|x - v\| + (1 - b)\|S_n x - v\| \leq \|x - v\| \leq \|u_1 - v\| + \|q - v\|,$$

$$\begin{aligned}&\|a_n q + (1 - a_n)U_n x - v\| \\ &\qquad \leq a_n\|q - v\| + (1 - a_n)\|U_n x - v\| \leq \|u_1 - v\| + \|q - v\|.\end{aligned}$$

We confirmed that each U_n and each $a_n q + (1 - a_n)U_n$ are self–mappings on D, that is, we confirmed that $\{u_n\}$ and $\{U_n u_n\}$ are sequences in D.

We show that $\{u_n\}$ converges strongly to $v = P_A q$. We easily see that, for $n \in N$,

$$\|U_n u_n - u_n\| = \|(bu_n + (1 - b)S_n u_n) - u_n\| = (1 - b)\|S_n u_n - u_n\|. \tag{4.3}$$

By Lemma 3.8, we also see that, for $n \in N$,

$$\|U_n u_n - v\|^2 \leq \|u_n - v\|^2 - b(1 - b)\|S_n u_n - u_n\|^2. \tag{4.4}$$

Furthermore, it follows from (4.4) that, for $n \in N$,

$$\begin{aligned}
\|u_{n+1} - v\|^2 &= \|a_n q + (1-a_n)U_n u_n - v\|^2 \\
&= \|(1-a_n)(U_n u_n - v) + a_n(q-v)\|^2 \\
&\leq (1-a_n)\|U_n u_n - v\|^2 + a_n^2\|q-v\|^2 + 2a_n(1-a_n)\langle U_n u_n - v, q-v\rangle \\
&\leq (1-a_n)\left(\|u_n - v\|^2 - b(1-b)\|S_n u_n - u_n\|^2\right) \\
&\qquad + a_n^2\|q-v\|^2 + 2a_n(1-a_n)\langle U_n u_n - v, q-v\rangle \\
&= (1-a_n)\|u_n - v\|^2 + a_n K_n, \qquad (4.5)
\end{aligned}$$

$$\begin{aligned}
\text{where}\quad K_n &= a_n\|q-v\|^2 + 2(1-a_n)\langle U_n u_n - v, q-v\rangle \\
&\qquad - \tfrac{(1-a_n)}{a_n}b(1-b)\|S_n u_n - u_n\|^2. \qquad (4.6)
\end{aligned}$$

By $a_n, b \in (0,1)$ and (4.6), we easily see

$$\begin{aligned}
K_n &\leq a_n\|q-v\|^2 + 2(1-a_n)\langle U_n u_n - v, q-v\rangle \\
&\leq \|q-v\|^2 + 2\|U_n u_n - v\|\|q-v\|. \qquad (4.7)
\end{aligned}$$

Then, since D is bounded, we know $\limsup_n K_n < \infty$. We show $\limsup_n K_n \leq 0$.

Since D is weakly compact, there is a subsequence $\{n_l\}$ of $\{n\}$ such that $\{u_{n_l}\}$ converges weakly to some $u \in D$ and $\limsup_n K_n = \lim_l K_{n_l}$.

Consider the case of $\liminf_l \|S_{n_l}u_{n_l} - u_{n_l}\|^2 > 0$. Then there is $M > 0$ and a subsequence $\{n_{l_i}\}$ of $\{n_l\}$ satisfying $\|S_{n_{l_i}}u_{n_{l_i}} - u_{n_{l_i}}\|^2 > M > 0$. By $a_n, b \in (0,1)$, $\lim_n a_n = 0$ and (4.6), we know $K_{n_{l_i}} < 0$ for sufficiently large $i \in N$. Thus we have

$$\limsup_n K_n = \lim_l K_{n_l} = \lim_i K_{n_{l_i}} \leq 0.$$

In the case of $\liminf_l \|S_{n_l}u_{n_l} - u_{n_l}\|^2 = 0$, by passing to subsequences, we may consider that $\{u_{n_l}\}$ converges weakly to $u \in D$ and satisfies the following:

$$\limsup_n K_n = \lim_l K_{n_l}, \qquad \lim_l \|S_{n_l}u_{n_l} - u_{n_l}\|^2 = 0.$$

By (4.3), $\lim_l \|U_{n_l}u_{n_l} - u_{n_l}\|^2 = 0$, that is, $\{U_{n_l}u_{n_l}\}$ also converges weakly to u.

Since D is bounded, there is $r \in (0,\infty)$ satisfying $r > \sup_{x\in D}\|x - v\|$. Recall properties of $\{c_{n,j}\}$. Then, by Lemma 3.8 (2), we see that, for $l \in N$,

$$\begin{aligned}
\|U_{n_l}u_{n_l} - v\| &= \|bu_{n_l} + (1-b)S_{n_l}u_{n_l} - v\| \\
&\leq \textstyle\sum_{j=1}^{n_l} c_{n_l,j}\|bu_{n_l} + (1-b)T_j u_{n_l} - v\| \\
&\leq \textstyle\sum_{j=1}^{n_l} c_{n_l,j}(\|u_{n_l} - v\| - \frac{b(1-b)}{2r}\|T_j u_{n_l} - u_{n_l}\|^2) \\
&= \|u_{n_l} - v\| - \tfrac{b(1-b)}{2r}\textstyle\sum_{j=1}^{n_l} c_{n_l,j}\|T_j u_{n_l} - u_{n_l}\|^2.
\end{aligned}$$

From this inequality, the following follows:

$$\tfrac{b(1-b)}{2r}\textstyle\sum_{j=1}^{n_l} c_{n_l,j}\|T_j u_{n_l} - u_{n_l}\|^2 \leq \|u_{n_l} - v\| - \|U_{n_l}u_{n_l} - v\| \leq \|U_{n_l}u_{n_l} - u_{n_l}\|.$$

By $\lim_l \|U_{n_l}u_{n_l} - u_{n_l}\| = 0$ and $\frac{b(1-b)}{2r} > 0$, we see that, for $j \in N$,

$$\limsup_l c_{n_l,j}\|T_j u_{n_l} - u_{n_l}\|^2 \leq \limsup_l (\textstyle\sum_{j=1}^{n_l} c_{n_l,j}\|T_j u_{n_l} - u_{n_l}\|^2) \leq 0.$$

Then we have

$$\lim_l \|T_j u_{n_l} - u_{n_l}\| = 0 \quad \text{for} \quad j \in N.$$

Since $I - T_j$ is demiclosed at 0 for $j \in N$, we have $u \in F \subset A$.

Reconfirm that $\lim_l a_{n_l} = 0$, $v = P_A q$, and $\{U_{n_l} u_{n_l}\}$ converges weakly to $u \in A$. Then, by (4.7), we have the following:

$$\begin{aligned} \lim_l K_{n_l} &\le \lim_l \left(a_{n_l} \|v - q\|^2 + 2(1 - a_{n_l}) \langle U_{n_l} u_{n_l} - v, q - v \rangle \right) \\ &= 2\langle u - P_A q, q - P_A q \rangle \le 0. \end{aligned}$$

Thus we have $\limsup_n K_n = \lim_l K_{n_l} \le 0$.

We know that (4.5) holds. Then, by properties of $\{a_n\}$ and $\limsup_n K_n \le 0$, Lemma 3.1 asserts $\lim_n \|u_n - v\|^2 = 0$, that is, $\{u_n\}$ converges strongly to $v = P_A q$.

Finally, we show $v = P_A q = P_F q \in F$. Since D is closed and $\{u_n\} \subset D$, we know that $v = P_A q \in A \cap D$. By Lemma 3.7 and $F \subset A \cap C$, we can easily see that

$$A \cap C = (\cap_{j \in N} A(T_j)) \cap C = \cap_{j \in N} (A(T_j) \cap C) \subset \cap_{j \in N} F(T_j) = F \subset A \cap C.$$

Then $v = P_A q \in A \cap D \subset A \cap C = F$. We know that $F = A \cap C$ is closed and convex. Then we can consider the metric projection P_F. By $v \in F \subset A$, we know

$$\|q - v\| = \min_{y \in A} \|q - y\| \le \inf_{y \in F} \|q - y\| \le \|q - v\|.$$

This implies $\|q - v\| = \min_{y \in F} \|q - y\|$ and $v = P_F q$. Thus $v = P_A q = P_F q \in F$. □

We present some results follow from Theorem 4.15; refer to previous subsection.

Theorem 4.16. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $k \in N$. Let C be a closed convex subset of H and let $\{T_j\}_{j \in N(1,k)}$ be a finite sequence of self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N(1,k)$. Set $F = \cap_{j \in N(1,k)} F(T_j)$ and $A = \cap_{j \in N(1,k)} A(T_j)$. Assume $\neq F \subset A$. Let $S = \frac{1}{k} \sum_{j=1}^{k} T_j$ and $U = bI + (1-b)S$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1 - a_n) U u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_A q = P_F q \in F$.

Theorem 4.17. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of self–mappings on C such that T_j satisfies (N_2) for $j \in N$. Assume $F = \cap_{j \in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1-b) S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Theorem 4.18. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of (λ)–hybrid self–mappings on C. Assume $F = \cap_{j \in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1-b) S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Theorem 4.19. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \textstyle\sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) *and let $\{c_{n,j}\}$ be the double sequence satisfying* (ds)*. Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of quasi–nonexpansive self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N$. Assume $F = \cap_{j \in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and*

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Theorem 4.20. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \textstyle\sum_{n=1}^{\infty} a_n = \infty.$$

Let C be a closed convex subset of H and let T be a quasi–nonexpansive self–mapping on C such that $I - T$ is demiclosed at 0. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1 - a_n)(b u_n + (1-b) T u_n) \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_{F(T)} q \in F(T)$.

Theorem 4.21. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \textstyle\sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) *and let $\{c_{n,j}\}$ be the double sequence satisfying* (ds)*. Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of nonexpansive self–mappings on C. Assume $F = \cap_{j \in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and*

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Here we present strong convergence theorems for sequences of non–self mappings which are corresponding to Theorems 4.11 and 4.13; also see Remarks 4.12 and 4.14.

The following is a direct consequence of Theorem 4.15.

Theorem 4.22. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \textstyle\sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) *and let $\{c_{n,j}\}$ be the double sequence satisfying* (ds)*. Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of mappings from C into H such that $I - P_C T_j$ is demiclosed at 0 for $j \in N$. Set $F' = \cap_{j \in N} F(P_C T_j)$ and $A' = \cap_{j \in N} A(P_C T_j)$. Assume $\neq F' \subset A'$. Let $S_n = \sum_{j=1}^{n} c_{n,j} P_C T_j$ and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and*

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_{F'} q \in F'$.

Theorem 4.23. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \textstyle\sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) *and let $\{c_{n,j}\}$ be the double sequence satisfying* (ds)*. Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of mappings from C into H satisfying the following:*

(a) *T_1 is a self–mappings on C and T_jT_1 are self–mappings on C for $j \geq 2$.*
(b) *$I - T_j$ is demiclosed at 0 for $j \in N$.*

Let $V_1 = T_1$ and $V_j = T_jT_1$ for $j \geq 2$. Set $F = \cap_{j\in N}F(T_j)$ and $A' = \cap_{j\in N}A(V_j)$. Assume $\neq F \subset A'$. Let $S_n = \sum_{j=1}^{n} c_{n,j}V_j$ for $n \in N$and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n)U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Proof. Fix any $j \geq 2$. Then, $T_jT_1u = T_ju = u$ for $u \in F(T_1) \cap F(T_j)$ and $T_jw = T_jT_1w = w$ for $w \in F(T_1) \cap F(T_jT_1)$. So, $F(T_1) \cap F(T_jT_1) = F(T_1) \cap F(T_j)$ for $j \geq 2$. Set $F' = \cap_{j\in N}F(V_j)$. Then we have

$$\begin{aligned} F' &= \cap_{j\in N}F(V_j) = F(T_1) \cap (\cap_{j\geq 2}F(T_jT_1)) \\ &= \cap_{j\geq 2}(F(T_1) \cap F(T_jT_1)) = \cap_{j\geq 2}(F(T_1) \cap F(T_j)) = \cap_{j\in N}F(T_j) = F. \end{aligned}$$

By our assumption, we see $\neq F = F' = \cap_{j\in N}F(V_j) \subset A' = \cap_{j\in N}A(V_j)$.

Then, replace T_j by V_j in the proof of Theorem 4.15. So, the rest of our proof and the proof of Theorem 4.15 are the same without the part below.

Let $\{u_{n_l}\}$ be a sequence in C. Suppose $\{u_{n_l}\}$ converges weakly to $u \in C$ and $\lim_l \|V_ju_{n_l} - u_{n_l}\| = 0$ for $j \in N$. However, from this, $u \in F' = \cap_{j\in N}F(V_j)$ does not follows directly. Because we do not know whether $I - V_j$ is demiclosed at 0 for $j \geq 2$. Instead, we know that $I - T_j$ is demiclosed at 0 for $j \in N$.

We show $u \in F' = \cap_{j\in N}F(V_j)$. Since $\lim_l \|V_ju_{n_l} - u_{n_l}\| = 0$ for $j \in N$, we know

$$\text{(i)} \ \lim_l \|T_1u_{n_l} - u_{n_l}\| = 0, \qquad \text{(ii)} \ \lim_l \|T_jT_1u_{n_l} - u_{n_l}\| = 0 \quad \text{for} \quad j \geq 2.$$

Furthermore, by (i) and (ii), we see

$$\lim_l \|T_jT_1u_{n_l} - T_1u_{n_l}\| = 0 \quad \text{for} \quad j \geq 2. \tag{4.8}$$

Thus, by (i) and (4.8), we see $u \in F = \cap_{j\in N}F(T_j) = \cap_{j\in N}F(V_j) = F'$. □

5. Existence theorems and convergence theorems

The authors think that Theorems 4.2 and 4.15 are interesting. The theorems may have many useful applications because they are expressed in so wide setting. However, to guaranty $\neq A$ in theory, maybe $\{T_j\}$ need satisfy some strict constraints. Even so, we are interested in finding such $\{T_j\}$ and having related results.

We begin our argument with presenting two lemmas: for details, see Takahashi and Takeuchi [40], and Ibaraki and Takeuchi [13].

Lemma 5.1. *Let $x, v, w \in H$. Then the following equality holds:*

$$\langle (x-v) + (x-w), v-w \rangle = \|x-w\|^2 - \|x-v\|^2.$$

Remark 5.2. Let $v, w \in H$ and let $\{z_i\}$ be a sequence in H. Set $s_n = \frac{1}{n}\sum_{i=1}^{n} z_i$ for $n \in N$. Then, by Lemma 5.1, the following is immediate: For each $n \in N$,

$$\langle (s_n - v) + (s_n - w), v - w \rangle = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} \|z_i - w\|^2 - \tfrac{1}{n}\sum_{i=1}^{n} \|z_i - v\|^2.$$

Lemma 5.3. *Let C be a subset of H and let T be a mapping from C into H. Let $\{u_n\}$ be a sequence in H which satisfies*

$$\limsup_n \sup_{y\in C} \langle (u_n - y) + (u_n - Ty), y - Ty \rangle \leq 0.$$

Suppose $\{u_n\}$ converges weakly to $u \in H$. Then, $u \in A(T)$.

In the rest of this section, we deal with (λ)–hybrid mappings. We prepare the following lemma. For the lemma, there are previous studies; refer to Kohsaka [20], Brézis and Browder [7], Shimizu and Takahashi [32], and Takahashi and Takeuchi [40].

Lemma 5.4. *Let $k \in N$. Let C be a bounded subset of H. Set $L = \sup_{x,y\in C} \|x-y\|$. Let $\{T_j\}_{j\in N(1,k)}$ be a finite sequence of self–mappings on C. Assume that T_1 is (λ)–hybrid with λ. For $n \in N$, define a mapping S_n from C into H by*

$$S_n = \tfrac{1}{n^k} \textstyle\sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T_1^{i_1} \cdots T_k^{i_k}.$$

Then, for $n \in N$, the following holds:

$$\sup_{x,y\in C} \langle (S_n x - y) + (S_n x - T_1 y), y - T_1 y\rangle \le \tfrac{1+2|1-\lambda|}{n} L^2.$$

Remark. *Each S_n need not be a self–mapping on C.*

Proof. Fix any $x, y \in C$ and $n \in N$. We easily have

$$\begin{aligned} |\textstyle\sum_{i_1=1}^{n-1} \langle T_1^{i_1-1}x - T_1^{i_1}x, y - T_1 y\rangle| &= |\langle x - T_1^{n-1}x, y - T_1 y\rangle| \\ &\le \|x - T_1^{n-1}x\|\|y - T_1 y\| \le L^2. \end{aligned}$$

Then, since T_1 is (λ)–hybrid with λ, we have

$$\begin{aligned} \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} \|T_1^{i_1}x - T_1 y\|^2 &= \tfrac{1}{n}\|x - T_1 y\|^2 + \tfrac{1}{n} \textstyle\sum_{i_1=1}^{n-1} \|T_1^{i_1}x - T_1 y\|^2 \\ &\le \tfrac{1}{n} L^2 + \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-2} \|T_1^{i_1}x - y\|^2 \\ &\quad + \tfrac{2(1-\lambda)}{n} \textstyle\sum_{i_1=1}^{n-1} \langle T_1^{i_1-1}x - T_1^{i_1}x, y - T_1 y\rangle \\ &\le \tfrac{1}{n} L^2 + \tfrac{2|1-\lambda|}{n} \times L^2 + \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} \|T_1^{i_1}x - y\|^2. \end{aligned} \tag{5.1}$$

In Remark 5.2, set $z_i = T_1^{i_1-1}x \in C$, $w = T_1 y$ and $v = y$. Then, by (5.1), we have

$$\begin{aligned} &\langle (\tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} T_1^{i_1}x - y) + (\tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} T_1^{i_1}x - T_1 y), y - T_1 y\rangle \\ &= \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} \|T_1^{i_1}x - T_1 y\|^2 - \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} \|T_1^{i_1}x - y\|^2 \le \tfrac{1+2|1-\lambda|}{n} L^2. \end{aligned} \tag{5.2}$$

Fix any $i_2, \cdots, i_k \in N(0, n-1)$. By replacing x by $T_2^{i_2} \cdots T_k^{i_k}x$ in (5.2), we have

$$\begin{aligned} &\langle (\tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} T_1^{i_1}T_2^{i_2} \cdots T_k^{i_k}x - y) + (\tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} T_1^{i_1}T_2^{i_2} \cdots T_k^{i_k}x - T_1 y), y - T_1 y\rangle \\ &\le \tfrac{1+2|1-\lambda|}{n} L^2. \end{aligned} \tag{5.3}$$

Since $i_2, \cdots, i_k \in N(0, n-1)$ are arbitrary, the following holds:

$$\begin{aligned} &\tfrac{1}{n^{k-1}} \textstyle\sum_{i_2=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} (\tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} T_1^{i_1}T_2^{i_2} \cdots T_k^{i_k}x) \\ &= \tfrac{1}{n^k} \textstyle\sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T_1^{i_1}T_2^{i_2} \cdots T_k^{i_k}x = S_n x. \end{aligned} \tag{5.4}$$

By (5.3) and (5.4), we have

$$\langle (S_n x - y) + (S_n x - T_1 y), y - T_1 y\rangle \le \tfrac{1+2|1-\lambda|}{n} L^2.$$

Finally, since x, y, n are arbitrary, we see that, for $n \in N$,

$$\sup_{x,y\in C} \langle (S_n x - y) + (S_n x - Sy), y - Sy\rangle \le \tfrac{1+2|1-\lambda|}{n} L^2.$$

□

We denote by $\lambda(C)$ the set of all (λ)–hybrid self–mappings on a subset C of a Hilbert space H. Also, we denote by $\lambda_1(C)$ the subset of $\lambda(C)$ such that a (λ)–hybrid self–mapping on C with λ is an element of $\lambda_1(C)$ if and only if $|1-\lambda| \le 1$. Then $\lambda_1(C)$ is the principal part of $\lambda(C)$; refer to Aoyama and co–authors [2].

Remark 5.5. Let C be a bounded convex subset of H. Under this setting, consider (5.2) in the proof of Lemma 5.4. Fix any $S \in \lambda_1(C)$ and $x \in C$. Set $T_1 = S$ and $y = \frac{1}{n}\sum_{i=0}^{n-1} S^i x \in C$. Then, (5.2) becomes

$$\langle \tfrac{1}{n}\textstyle\sum_{i=0}^{n-1} S^i x - S(\tfrac{1}{n}\sum_{i=0}^{n-1} S^i x), \tfrac{1}{n}\sum_{i=0}^{n-1} S^i x - S(\tfrac{1}{n}\sum_{i=0}^{n-1} S^i x)\rangle \le \tfrac{3}{n} L^2.$$

Then we easily see that the following equality holds without closeness of C:

$$\lim_n \sup_{S\in\lambda_1(C),\, x\in C} \left\| S\left(\tfrac{1}{n}\textstyle\sum_{i=0}^{n-1} S^i x\right) - \tfrac{1}{n}\sum_{i=0}^{n-1} S^i x \right\| = 0.$$

We know that nonexpansive self–mappings on C are elements of $\lambda_1(C)$. So, in the Hilbert space setting, we obtained an extension of Bruck's well known lemma [10].

Lemma 5.6. *Let $k \in N$. Let C be a bounded subset of H. Set $L = \sup_{x,y\in C}\|x-y\|$. Let $\{T_j\}_{j\in N(1,k)}$ be a finite sequence of self–mappings on C. Assume that T_1 is (λ)–hybrid with λ. For $n \in N$, define a mapping S_n from C into H by*

$$S_n = \tfrac{1}{n^k}\textstyle\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} T_1^{i_1}\cdots T_k^{i_k}.$$

Then, the following hold:

1. $\limsup_n \sup_{x,y\in C}\langle (S_n x - y) + (S_n x - T_1 y), y - T_1 y\rangle \le 0$.
2. *For $x_1 \in C$, $\{S_n x_1\}$ is bounded.*
3. *For $x_1 \in C$, every weak cluster point of $\{S_n x_1\}$ is a point of $A(T_1)$.*
4. *$A(T_1)$ is non–empty closed and convex.*

Suppose further that C is closed and convex. Then the following hold:

5. *For $x_1 \in C$, every weak cluster point of $\{S_n x_1\}$ is a point of $F(T_1)$.*
6. *$F(T_1)$ is non–empty bounded closed and convex.*

Proof. By $\limsup_n \frac{1+2|1-\lambda|}{n} L^2 = 0$ and Lemma 5.4, we immediately see that (1) holds. We show (2)–(4). Fix any $x_1 \in C$ and consider $\{S_n x_1\}$.

Fix any $y \in C$. Then, by $T_1^{i_1}\cdots T_k^{i_k} x_1 \in C$ for $i_1, \cdots, i_k \in N_0$, we see that

$$\|S_n x_1 - y\| \le \tfrac{1}{n^k}\textstyle\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1}\|T_1^{i_1}\cdots T_k^{i_k} x_1 - y\| \le L.$$

Then $\{S_n x_1\}$ is bounded and has a weakly convergent subsequence. Let $\{S_{n_l} x_1\}$ be a subsequence of $\{S_n x_1\}$ which converges weakly to $u \in H$. By (1), we know

$$\limsup_l \sup_{y\in C}\langle (S_{n_l} x_1 - y) + (S_{n_l} x_1 - T_1 y), y - T_1 y\rangle \le 0.$$

Then, by Lemma 5.3, we know $u \in A(T_1)$. We confirmed that (3) holds. We also confirmed $A(T_1) \ne$. By Lemma 3.6, $A(T_1)$ is closed and convex. Then (4) holds.

Suppose further that C is closed and convex. We show (5) and (6). In the same way as in the proof of (3), we know $u \in A(T_1)$. Also $\{S_{n_l} x_1\}$ is in the weakly closed set C. Then, $u \in A(T_1) \cap C$. By Lemma 3.7, we see $u \in A(T_1)\cap C \subset F(T_1)$. So, we confirmed that (5) holds. Also we confirmed $F(T_1) \ne$. Since T_1 is (λ)–hybrid, by Lemma 3.7, we have $\ne F(T_1) = A(T_1) \cap C$. Then, (6) follows from (4). □

The following is a direct consequence of Lemma 5.6.

Lemma 5.7. *Let $k \in N$. Let C be a bounded subset of H and let $\{T_j\}_{j\in N(1,k)}$ be a finite family of commuting (λ)–hybrid self–mappings on C. Set $F = \cap_{j\in N(1,k)} F(T_j)$ and $A = \cap_{j\in N(1,k)} A(T_j)$. Then, A is non–empty closed and convex. Suppose further that C is closed and convex. Then F is non–empty bounded closed and convex.*

Proof. Since $\{T_j\}_{j\in N(1,k)}$ is commuting, for example, $T_1^{i_1}T_2^{i_2}\cdots T_k^{i_k} = T_2^{i_2}T_1^{i_1}\cdots T_k^{i_k}$. Since each T_j is (λ)–hybrid with λ_j, by Lemma 5.6 (4)–(6), the proof is trivial. □

In the Hilbert space setting, by using Lemma 5.7, we can have an extension of DeMarr's well–known common fixed point theorem; see DeMarr [11].

Theorem 5.8. *Let C be a bounded closed convex subset of H and let $\{T_j\}_{j\in J}$ be a family of commuting (λ)–hybrid self–mappings on C. Then $F = \cap_{j\in J} F(T_j)$ is non–empty bounded closed and convex.*

Proof. Since each T_j is (λ)–hybrid, we already know that $F(T_j)$ is closed and convex for $j \in J$. So $\{F(T_j)\}_{j\in J}$ consists of weakly closed subsets of C. By Lemma 5.7, $\{F(T_j)\}_{j\in J}$ has the finite intersection property. Thus, since C is weakly compact, we see $F = \cap_{j\in J} F(T_j) \neq$. It is obvious that F is bounded closed and convex. □

By Theorems 4.2 and 5.8, we have the following weak convergence theorem.

Theorem 5.9. *Let $a, b \in (0, 1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a, b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a bounded closed convex subset of H and let $\{T_j\}$ be a sequence of commuting (λ)–hybrid self–mappings on C. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F = \cap_{j\in N} F(T_j)$.

Proof. Set $A = \cap_{j\in N} A(T_j)$. By Lemma 3.4, $I - T_j$ is demiclosed at 0 for $j \in N$. Since T_j is (λ)–hybrid for $j \in N$, by Theorem 5.8, we know

$$\neq F = \cap_{j\in N} F(T_j) \subset \cap_{j\in N} A(T_j) = A.$$

Thus, by Theorems 4.2 (2), $\{x_n\}$ converges weakly to some $z \in F$. □

By Theorems 4.15 and 5.8, we have the following strong convergence theorem.

Theorem 5.10. *Let $b \in (0, 1)$ and let $\{a_n\}$ be a sequence such that $a_n \in (0, 1)$,*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a bounded closed convex subset of H and let $\{T_j\}$ be a sequence of commuting (λ)–hybrid self–mappings on C. Set $F = \cap_{j\in N} F(T_j)$ and $A = \cap_{j\in N} A(T_j)$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1 - b) S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_A q = P_F q \in F$.

Proof. By Lemma 3.4, for $j \in N$, $I - T_j$ is demiclosed at 0. Since T_j is (λ)–hybrid for $j \in N$, by Theorem 5.8, we know

$$\neq F = \cap_{j\in N} F(T_j) \subset \cap_{j\in N} A(T_j) = A.$$

Thus, by Theorem 4.15, $\{u_n\}$ converges strongly to $v = P_A q = P_F q \in F$. □

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BEST PROXIMITY POINTS INVOLVING SIMULATION FUNCTIONS WITH τ-DISTANCE

AREERAT ARUNCHAI[1*], SIWAKON SUPPALAP[2] AND WANCHAI TAPANYO[3]

[1,3] Department of Mathematics and Statistics, Nakhonsawan Rajabhat University, Nakhonsawan, Thailand
[2] Department of Mathematics, Naresuan University, Phisanulok, Thailand

ABSTRACT. In this paper, we illustrate the best proximity point theorems in complete metric spaces for $\mathscr{Z}$-p-proximal contractions of the first kind and of the second kind involving the simulation functions using τ-distance with lower semicontinuity in its first variable. Ous results extend generalize the results in literature.

KEYWORDS: Best proximity point, Simulation functions, τ-Distance.
AMS Subject Classification: Primary 47H10, 47H09; Secondary 54E50

1. Introduction

In 1922, Banach proved that if (X, d) is a complete metric space and the mapping T satisfies Banach contraction mapping principle, then T has a unique fixed point, that is $T(u) = u$ for some $u \in X$. Since the results of Banach, many authors have been studying fixed point and best proximity points of mappings in metric spaces. Their reseach are still being studied in many directions. In 1999, Suzuki [15] introduced the concept of τ-distance on a metric space, which is a generalized concept of w-distance. They also improve the generalizations of the Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle, including they discuss the relation between w-distance. In 2015, Khojasteh *et al.* [10] introduced the simulation function. Recently, simulation function have been used to study the best proximity points in metric spaces (see [12, 14, 2]).

In 2018, Kostić *et al.*[2] introduced a special type of w-distance, the w_0-distance, to extend best proximity results of Tchier *et al.* [14] involving simulation functions. In this paper, we generalize some best proximity points results in metric spaces involving simulation functions with τ-distance.

In this paper we prove the best proximity point results involving simulation

[*] *Corresponding author.*
Email address : areerat.a@nsru.ac.th.

functions with τ-distance, given by τ-distance is lower semicontinuous in its first variable.

2. Preliminaries

Here we recall some definition and some example of the simulation function ([1, 9, 10, 11, 12, 14]).

Definition 2.1. [1] Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping. Then ζ is called a *simulation function* if it satisfies the following conditions:
(ζ_1) $\zeta(t,s) < s - t$ for all $t,s > 0$
(ζ_2) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

In 2015, Khojasteh *et al.*[10] introduced the simulation function as a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying $\zeta(0,0) = 0$ alongside the conditions (ζ_1) and (ζ_2) of Definition 2.1. On the other hand, Argoubi *et al.* [1] slightly modified the definition of Khojasteh *et al.* [10] by removing the condition $\zeta(0,0) = 0$. In this paper, we use a modified definition of Argoubi *et al.* [1].

The set of all simulation functions will be denoted by $\mathscr{Z}$.

The following, we recall some examples of simulation functions.

Example 1.1 [9] Let $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}, i = 1, ..., 6$ be defined by

1. $\zeta_1(t,x) = \psi(s) - \phi(t)$ for all $t,s \in [0,\infty)$, where $\phi, \psi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\phi(t) = \psi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$.
2. $\zeta_2(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t,s \in [0,\infty)$, where $f, g : [0,\infty)^2 \to (0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t,s > 0$
3. $\zeta_3(t,s) = s - \varphi(s) - t$ for all $t,s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
4. If $\varphi : [0,\infty) \to [0,1)$ is a function such that $\limsup_{t\to r+} \varphi(t) < 1$ for all $r > 0$ we define
$$\zeta_4(t,s) = s\varphi(s) - t \text{ for all } t,s \in [0,\infty).$$
5. If $\eta : [0,\infty) \to [0,\infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, we define
$$\zeta_5(t,s) = \eta(s) - t \text{ for all } t,s \in [0,\infty).$$
6. If $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^\varepsilon \phi(u)du$ exists and $\int_0^\varepsilon \phi(u)du > \varepsilon$ for each $\varepsilon > 0$, we define
$$\zeta_6(t,s) = s - \int_0^t \phi(u)du \text{ for all } t,s \in [0,\infty).$$

It is easily verified that each function $\zeta_i (i = 1, ..., 6)$ is the simulation function.

Definition 2.2. [2] A nonself mapping $T : A \to B$ is said to be a $\mathscr{Z}$-p-proximal contraction of the first kind if there exists $\zeta \in \mathscr{Z}$ such that

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(u,Ty) = d(A,B) \end{aligned}\right\} \Rightarrow \zeta(\mu(u,v), \mu(x,y)) \geq 0$$

for all $u, v, x, y \in A$.

Definition 2.3. [2] A non-self-mapping $T : A \to B$ is said to be a $\mathscr{Z}$-p-proximal contraction of the second kind if there exists $\zeta \in \mathscr{Z}$ such that

$$\left.\begin{array}{r} d(u,Tx)=d(A,B) \\ d(u,Ty)=d(A,B) \end{array}\right\} \Rightarrow \zeta(\mu(Tu,Tv),\mu(Tx,Ty)) \geq 0$$

for all $u, v, x, y \in A$.

In the case $p = d$, the notaion of $\mathscr{Z}$-p-proximal contraction are reduced to $\mathscr{Z}$-proximal contraction of Tchier *et al.* [14].

In Definition 2.2, if the simulation function ζ is given by $\zeta(t,s) = \alpha s - t$ for some $\alpha \in [0,1)$, then the mapping T is called a *p-proximal contraction of the first kind.* Moreover if $p = d$, then T is a proximal contraction of the first kind.

We recall the following notation:

$\mathscr{G}_{A,p} = \{g : g$ is a continuous functions from (A,d) to (A,d) and $p(x,y) \leq p(gx,gy)$ for all $x,y \in A\}$

$\mathscr{T}_{g,p} = \{T : T$ is a function from A to B and $p(Tx,Ty) \leq p(Tgx,Tgy)$ for all $x,y \in A\}$.

In the case $p = d, \mathscr{G}_{A,p}$ is denoted by $\mathscr{G}_A$ and $\mathscr{T}_{g,p}$ by $\mathscr{T}_g$ (see [14]).

In 1999, Suzuki [15] introduced the concept of τ-distance on a metric space, which is a generalized concept of w-distance. They gave example of the τ-distance. Further They discuss the relation between w-distance. Kostić *et al.*[2] introduced the concept of w_0-distance, which is slightly different to the original w-distance of [8], in regard that the lower semicontinuity with respect to both variables is supposed.

Definition 2.4. [15] Let X be a metric space with metric d. Then a function $p : X \times X \to [0,\infty)$ is called the τ-distance on X if there exists a function η from $X \times [0,\infty) \to [0,\infty)$ and the following are satisfied:

- (τ_1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x,y,z \in X$;
- (τ_2) $\eta(x,0) = 0$ and $\eta(x,t) \geq t$ for all $x \in X$ and $t \in [0,\infty)$, and η is concave and continuous in its second variable;
- (τ_3) $\lim_n x_n = x$ and $\limsup_n \{\eta(z_n, p(z_n,x_m)) : m \geq n\} = 0$ imply $p(w,x) \leq \liminf_n p(w,x_n)$ for all $w \in X$;
- (τ_4) $\limsup_n \{p(x_n,y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n,t_n) = 0$ imply $\lim_n \eta(y_n,t_n) = 0$;
- (τ_5) $\lim_n \eta(z_n,p(z_n,x_n)) = 0$ and $\lim_n \eta(z_n,p(z_n,y_n)) = 0$ imply $\lim_n d(x_n,y_n) = 0$

We may replace (τ_2) by the following $(\tau_2)'$

- $(\tau_2)'$ $\inf\{\eta(x,t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in its second variable.

We recall some properties of τ-distance. Let X be a metric space with metric d and let p be a τ-distance on X. Then a sequence $\{x_n\}$ of X is called p-Cauchy if there exists a function η from $X \times [0,\infty) \to [0,\infty)$ satisfying $(\tau_2) - (\tau_5)$ and a sequence $\{z_n\}$ of X such that $\limsup_n \{\eta(z_n,p(z_n,x_m)) : m \geq n\} = 0$.

We recall the following lemma, which can be found in [15].

Lemma 2.5. [15] *Let X be a metric space with metric d and let p be a τ-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence. Moreover,* *if* $\{y_n\}$ *is a sequence satisfying* $\limsup_n \{p(x_n, y_m) : m \geq n\} = 0$, *then* $\{y_n\}$ *is also a p-Cauchy sequence and* $\lim_n d(x_n, y_n) = 0$.

Lemma 2.6. [15] *Let* X *be a metric space with metric* d *and* p *be a* τ*-distance on* X. *If a sequence* $\{x_n\}$ *of* X *satisfies* $\lim_n p(z, x_n) = 0$ *for some* $z \in X$, *then* $\{x_n\}$ *is a p-Cauchy sequence. Moreover, if a sequence* $\{y_n\}$ *of* X *also satisfies* $\lim_n p(z, y_n) = 0$, *then* $\lim_n d(x_n, y_n)$. *In particular for* $x, y, z \in X$, $p(z, x) = 0$ *and* $p(z, y) = 0$ *imply* $x = y$.

Lemma 2.7. [15] *Let* X *be metric space with metric* d *and let* p *be a* τ*-distance on* X. *If a sequence* $\{x_n\}$ *of* X *satisfies* $\limsup_n \{p(x_n, x_m) : m > n\} = 0$, *then* $\{x_n\}$ *is a p-Cauchy sequence. Moreover if a sequence* $\{y_n\}$ *of* X *satisfies* $\lim_n p(x_n, y_n) = 0$, *then* $\{y_n\}$ *is also a p-Cauchy sequence and* $\lim_n d(x_n, y_n) = 0$.

Let (X, d) be a metric space, A and B two nonempty subsets of X and $T : A \to B$ a non-self-mapping. The following notations will be used throughout the paper.

$$\begin{aligned} d(A, B) &= \inf\{d(x, y) : x \in A, y \in B\}; \\ d(y, A) &= \inf\{d(x, y) : x \in A\} = d(\{y\}, A); \\ A_0 &= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}; \\ B_0 &= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned}$$

Throughout this paper, the set of all best proximity points of a non-self mapping $T : A \to B$ will be denoted by

$$B_{est}(T) = \{x \in A : d(x, Tx) = d(A, B)\}.$$

If $g : A \to A$, then we have

$$B^g_{est}(T) = \{x \in A : d(gx, Tx) = d(A, B)\}.$$

3. Main Results

Let (X, d) be a metric space, $p : X \times X \to [0, \infty)$ a τ-distance on X, and let A and B be two nonempty subsets of X (which need not be equal). For every $x, y \in X$,

$$\mu(x, y) := \max\{p(x, y), p(y, x)\}.$$

It is easily checked that the function $\mu : X \times X \to [0, \infty)$ has the following properties, for all $x, y, z \in X$;

(1) $\mu(x, y) = 0 \Rightarrow x = y$;
(2) $\mu(x, y) = \mu(y, x)$, i.e. μ is symmetic;
(3) $\mu(x, y) \leq \mu(x, z) + \mu(z, y)$, i.e. μ satisfies the triangle inequality.

Lemma 3.1. *Suppose that* $\{x_n\}$ *is sequence such that* $\lim_n \mu(x_n, x_{n+1}) = 0$. *If* $\lim_{n,m} \mu(x_n, x_m) \neq 0$, *then there are* $\epsilon > 0$ *and two subsequence* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *of* $\{x_n\}$ *such that* $\lim_k \mu(x_{n_k}, x_{m_k}) = \lim_k \mu(x_{n_k+1}, x_{m_k+1}) = \epsilon$.

Next, we prove our main results.

Theorem 3.2. *Let* A *and* B *be two nonempty subsets of a complete metric space (X,d) with a* τ*-distance* p, *such that* A_0 *is nonempty and closed. Let* $p(\cdot, x) : X \to [0, \infty)$ *be lower semicontinuous for any* $x \in X$. *Suppose that the mappings* $g : A \to A$ *and* $T : A \to B$ *satisfy the folowing conditions:*

(a) *T is a $\mathscr{Z}$-p-proximal contraction of the first kind;*
(b) $g \in \mathscr{G}_{A,p}$;

(c) $A_0 \subseteq g(A_0)$;
(d) $T(A_0) \subseteq B_0$.

Then there exists a unique element $x \in A_0$ such that $d(gx, Tx) = d(A, B)$ and $p(x, x) = 0$. Moreover, for any initial $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B).$$

Again, since Tx_1 is an element of $T(A_0)$ which is contained in B_0, and A_0 is contained in $g(A_0)$, it follows that there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B).$$

This process can be continued, for any $x_n \in A_0$ it is possible to find $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Tx_n) = d(A, B).$$

If there exists $n_0 \in \mathbb{N}$ such that $\mu(x_{n_0}, x_{n_0-1}) = 0$, then $x_{n_0-1} = x_{n_0}$, which implies that $d(gx_{n_0-1}, Tx_{n_0-1}) = d(A, B)$. That is, x_{n_0-1} is a best proximity point of T under mapping g.

Assume that $\mu(x_n, x_{n-1}) > 0$ for all $n \in \mathbb{N}$. Since $g \in \mathscr{G}_{A,p}$, we have $\mu(gx_n, gx_{n-1}) > 0$ for all $n \in \mathbb{N}$. Since T is a $\mathscr{Z}$-p-proximal contraction of the first kind and $g \in \mathscr{G}_{A,p}$, we obtain

$$\begin{aligned} 0 \le \zeta(\mu(gx_{n+1}, gx_n), \mu(x_n, x_{n-1})) &< \mu(x_n, x_{n-1}) - \mu(gx_{n+1}, gx_n) \\ &\le \mu(x_n, x_{n-1}) - \mu(x_{n+1}, x_n). \end{aligned} \tag{3.1}$$

Thus

$$\mu(x_{n+1}, x_n) < \mu(x_n, x_{n-1}), \forall n \in \mathbb{N}. \tag{3.2}$$

This implies that the sequence $\{\mu(x_n, x_{n-1})\}$ is decreasing and so there exists

$$\lim_{n\to\infty} \mu(x_n, x_{n-1}) = r \ge 0. \tag{3.3}$$

Suppose that $r > 0$. From (3.1),

$$\mu(gx_{n+1}, gx_n) \le \mu(x_n, x_{n-1})$$

for every $n \in \mathbb{N}$. On the other hand, $g \in \mathscr{G}_{A,p}$ and hence

$$\mu(x_{n+1}, x_n) \le \mu(gx_{n+1}, gx_n) \le \mu(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$. Let $n \to \infty$,

$$\lim_{n\to\infty} \mu(gx_{n+1}, gx_n) = r. \tag{3.4}$$

Now, using the simulation function property (ζ_2) we obtain

$$0 \le \limsup_{n\to\infty} \zeta(\mu(gx_{n+1}, gx_n), \mu(x_n, x_{n-1})) < 0$$

which is a contradiction. Hence we have $r = 0$ which imples that

$$\lim_{n\to\infty} \mu(x_n, x_{n-1}) = 0. \tag{3.5}$$

Now, let us prove that

$$\lim_{m,n\to\infty} \mu(x_n, x_m) = 0. \tag{3.6}$$

If (3.6) is not true, then

$$\lim_{m,n\to\infty} \mu(x_n, x_m) \ne 0. \tag{3.7}$$

From Lemma 3.1, then there exists $\epsilon > 0$ and two subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \mu(x_{n_k}, x_{m_k}) = \varepsilon. \tag{3.8}$$

and

$$\lim_{k\to\infty} \mu(x_{n_k+1}, x_{m_k+1}) = \varepsilon. \tag{3.9}$$

We can assume that $\mu(x_{n_k+1}, x_{m_k+1}) > 0$ for all $k \in \mathbb{N}$. Again, T is a $\mathscr{Z}$-p-proximal contraction of the first kind and $d(gx_{n_k+1}, Tx_{n_k}) = d(A, B) = d(gx_{m_k+1}, Tx_{m_k})$. By the property (ζ_1), we obtain

$$\begin{aligned} 0 &\le \zeta(\mu(gx_{n_k+1}, gx_{m_k+1}), \mu(x_{n_k}, x_{m_k})) \\ &< \mu(x_{n_k}, x_{m_k}) - \mu(gx_{n_k+1}, gx_{m_k+1}) \\ &\le \mu(x_{n_k}, x_{m_k}) - \mu(x_{n_k+1}, x_{m_k+1}) \end{aligned}$$

for all $k \in \mathbb{N}$. Thus the previous inequality with (3.8) and (3.9) imply that

$$\lim_{k\to\infty} \mu(gx_{n_k+1}, gx_{m_k+1}) = \varepsilon. \tag{3.10}$$

From (3.8) and (3.10) we see that the sequance $t_k := \mu(gx_{n_k+1}, gx_{m_k+1})$ and $s_k := \mu(x_{n_k}, x_{m_k})$ have the same positive limit. By the property (ζ_2), we conclude that

$$0 \le \limsup_{k\to\infty} \zeta(t_k, s_k) < 0$$

which is a contradiction and hence (3.6) holds.
Since

$$\lim_{m,n\to\infty} \mu(x_n, x_m) = 0,$$

we have

$$\limsup_{m,n\to\infty} \{p(x_n, x_m) : m > n\} = 0.$$

By Lemma 2.7, we have $\{x_n\}$ is a p-Cauchy sequance in A_0. And by Lemma 2.5 we have $\{x_n\}$ is a Cauchy sequance in A_0. Since (X, d) is complete metric space and A_0 is a closed subset of X, there exists $\lim_{n\to\infty} x_n = x \in A_0$. Moreover, by the continuity of g we have $\lim_{n\to\infty} gx_n = gx$. Since $gx_n \in A_0$ for all $n \in \mathbb{N}$ and A_0 is closed, we also have $gx \in A_0$. On the other hand, since $x \in A_0$ and $T(A_0) \subseteq B_0$, there exists $z \in A_0$ such that $d(z, Tx) = d(A, B)$.

Let us prove that $z = gx$. If $z = gx_n$ for infinetely many $n \in \mathbb{N}$, then $z = gx$. Assume that $z \ne gx$, in which case there exists $n_0 \in \mathbb{N}$ such that $z \ne gx_n$ for all $n \ge n_0$. If $\mu(gx_n, z) = 0$ for some $n \ge n_0$, then $gx_n = z$. That is $\mu(gx_n, z) > 0$ for all $n \ge n_0$. Also there exists a subsequance $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \ne x$ for every $k \in \mathbb{N}$ (if that is not true, then $x_n = x$ for all $n \in \mathbb{N}$ and so $\mu(x_n, x_{n-1}) = 0$ for all $n \in \mathbb{N}$, which is contrary to (3.2)). Similary, we have $\mu(x_{n_k}, x) > 0$ for every $k \in \mathbb{N}$. Since T is a $\mathscr{Z}$-p-proximal contraction of the first kind and $g \in \mathscr{G}_{A,p}$, we obtain

$$\begin{aligned} 0 &\le \zeta(\mu(gx_{n_k+1}, z), \mu(x_{n_k}, x)) \\ &< \mu(x_{n_k}, x) - \mu(gx_{n_k+1}, z) \\ &\le \mu(gx_{n_k}, gx) - \mu(gx_{n_k+1}, z). \end{aligned}$$

This imples that

$$\mu(gx_{n_k+1}, z) < \mu(gx_{n_k}, gx) \tag{3.11}$$

for every $k \in \mathbb{N}$ such that $n_k \geq n_0$.

Similarly argument as before we can show that

$$\lim_{m,n\to\infty} \mu(gx_n, gx_m) = 0.$$

This means that for any $\epsilon > 0$ there exists a $N_\epsilon \in \mathbb{N}$ such that $\mu(gx_n, gx_m) < \epsilon$ for all $m > n \geq N_\epsilon$. For a fixed $n \in \mathbb{N}$ with $n \geq \max\{n_0, N_\epsilon\}$ and

$$\lim_{m,n\to\infty} \mu(gx_n, gx_m) = 0,$$

we have

$$\limsup_{m,n\to\infty} \{p(gx_n, gx_m) : m > n\} = 0.$$

By Lemma 2.7, we have $\{gx_n\}$ is a p-Cauchy sequance in A_0.
Since gx_n is a p-Cauchy sequance in A_0, there exists a function η from $A_0 \times [0,\infty) \to [0,\infty)$ satisfying $(\tau_2) - (\tau_5)$ and a sequence $\{z_n\}$ of A_0 such that

$$\limsup_{n\to\infty} \{\eta(z_n, p(z_n, gx_m)) : m \geq n\} = 0.$$

By (τ_3) and $p(\cdot, x) : X \to [0,\infty)$ is lower semicontinuous imply that

$$p(gx_n, gx) \leq \liminf_m p(gx_n, gx_m) < \epsilon.$$

and

$$p(gx, gx_n) \leq \liminf_m p(gx_m, gx_n) < \epsilon.$$

Therefore

$$\lim_{k\to\infty} p(gx_{n_k}, gx) = 0. \tag{3.12}$$

Similarly, $\lim_{k\to\infty} p(gx, gx_{n_k}) = 0$ which combind with (3.12) yields

$$\lim_{k\to\infty} \mu(gx_{n_k}, gx) = 0.$$

Then from (3.11) we have

$$\lim_{k\to\infty} \mu(gx_{n_k+1}, z) = 0. \tag{3.13}$$

Letting $k \to \infty$ in the following inequality and by (3.5), (3.13)

$$\mu(gx_{n_k}, z) \leq \mu(gx_{n_k}, gx_{n_k+1}) + \mu(gx_{n_k+1}, z),$$

we get $\lim_{k\to\infty} \mu(gx_{n_k}, z) = 0$. This implies

$$\lim_{k\to\infty} p(gx_{n_k}, z) = 0. \tag{3.14}$$

Since $\lim_{k\to\infty} gx_{n_k} = gx$, we obtain

$$p(gx, gx) = 0 \text{ and } p(gx, z) = 0.$$

By Lemma 2.6, imply that $z = gx$. Finally, from $d(z, Tx) = d(A, B)$, we get $d(gx, Tx) = d(A, B)$.

To prove the uniqueness, let y be in A_0 such that

$$d(gy, Ty) = d(A, B).$$

Assume that $\mu(gx, gy) \geq \mu(x, y) > 0$. Since $g \in \mathscr{G}_{A,p}$ and T is a $\mathscr{Z}$-p-proximal contraction of the first kind, we obtain

$$\begin{aligned} 0 &\leq \zeta(\mu(gx, gy), \mu(x, y)) \\ &< \mu(x, y) - \mu(gx, gy) \\ &\leq \mu(x, y) - \mu(x, y) = 0 \end{aligned}$$

which leads to a contradiction. Hence $\mu(x,y)=0$, which implies $x=y$.

By a similar argument we prove $p(x,x)=0$. Suppose to the contrary, that $\mu(x,x)=p(x,x)>0$. Then $\mu(gx,gx)>0$. Again, we have

$$\begin{aligned}0 &\leq \zeta(\mu(gx,gx),\mu(x,x))\\ &< \mu(x,x)-\mu(gx,gx)\\ &\leq \mu(x,x)-\mu(x,x)=0\end{aligned}$$

which is a contradiction. □

If g is the identity mapping on A, then the preceding theorem yields the following corollary.

Corollary 3.3. *Let A and B be two nonempty subset of a complete metric space (X,d) with a τ-distance p, such that A_0 is nonempty and closed. Let $p(\cdot,x):X\to[0,\infty)$ is lower semicontinuous for any $x\in X$. Suppose that the mappings $T:A\to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-p-proximal contraction of the first kind;*
(b) *$T(A_0)\subseteq B_0$.*

Then there exists a unique best proximity point $x\in A_0$ of the mapping T, such that $p(x,x)=0$. Moreover, for every $x_0\in A_0$ there exists a sequence $\{x_n\}\subseteq A_0$ converging to x, such that $d(gx_{n+1},Tx_n)=d(A,B)$ for all $n\in\mathbb{N}\cup\{0\}$.

From Theorem 3.2 we can also obtain an interesting g-best proximity point result for a p-proximal contraction of the first kind.

Corollary 3.4. *Let A and B be two nonempty subset of a complete metric space (X,d) with a τ-distance p, such that A_0 is nonempty and closed. Let $p(\cdot,x):X\to[0,\infty)$ is lower semicontinuous for any $x\in X$. Suppose that the mappings $g:A\to A$ and $T:A\to B$ satisfies the folowing conditions:*

(a) *T is a p-proximal contraction of the first kind with respect $\alpha\in[0,1)$;*
(b) *$g\in\mathscr{G}_{A,p}$;*
(c) *$A_0\subseteq g(A_0)$;*
(d) *$T(A_0)\subseteq B_0$.*

Then there exists a unique element $x\in A_0$ such that $d(gx,Tx)=d(A,B)$ and $p(x,x)=0$. Moreover, for every $x_0\in A_0$ there exists a sequence $\{x_n\}\subseteq A_0$ converging to x, such that $d(gx_{n+1},Tx_n)=d(A,B)$ for all $n\in\mathbb{N}\cup\{0\}$.

Proof. Note that a p-proximal contraction of the first kind with respect to $\alpha\in[0,1)$ is a $\mathscr{Z}$-p-proximal contraction of the first kind with respect to the simulation function $\zeta:[0,\infty)\times[0,\infty)\Rightarrow\mathbb{R}$ defined by $\zeta(t,s)=\alpha s-t$ for all $t,s\geq 0$. □

By taking $p=d$ in Theorem 3.2 the main result of [14] is obtained.

Corollary 3.5. *Let A and B be two nonempty subset of a complete metric space (X,d), such that A_0 is nonempty and closed. Suppose that the mappings $g:A\to A$ and $T:A\to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-proximal contraction of the first kind;*
(b) *$g\in\mathscr{G}_A$;*
(c) *$A_0\subseteq g(A_0)$;*
(d) *$T(A_0)\subseteq B_0$.*

Then there exists a unique element $x\in A_0$ such that $d(gx,Tx)=d(A,B)$. Moreover, for every $x_0\in A_0$ there exists a sequence $\{x_n\}\subseteq A_0$ such that $d(gx_{n+1},Tx_n)=d(A,B)$ for all $n\in\mathbb{N}\cup\{0\}$, and $\{x_n\}$ converging to x.

Theorem 3.6. *Let A and B be two nonempty subsets of a complete metric space (X,d) with a τ-distance p, such that $T(A_0)$ is nonempty and closed. Let $p(\cdot,x): X \to [0,\infty)$ is lower semicontinuous for any $x \in X$. Suppose that the mappings $g: A \to A$ and $T: A \to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-p-proximal contraction of the second kind;*
(b) *T is injective on A_0*
(c) *$T \in \mathscr{T}_{g,p}$;*
(d) *$A_0 \subseteq g(A_0)$;*
(e) *$T(A_0) \subseteq B_0$.*

Then there exists a unique element $x \in A_0$ such that $d(gx,Tx) = d(A,B)$ and $p(Tx,Tx) = 0$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1},Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Proceeding as in Theorem 3.2 we can construct a sequance $\{x_n\}$ such that $d(gx_{n+1},Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$. In the constructive process of $\{x_n\}$, if we have $Tx_n = Tx_m$ for some $m > n$, then we choose $x_{m+1} = x_{n+1}$.
Since T is a $\mathscr{Z}$-p-proximal constraction of the second kind, we have

$$\zeta(\mu(Tgx_n,Tgx_{n+1}),\mu(Tx_{n-1},Tx_n)) \geq 0$$

for every $n \in \mathbb{N}$. Since T is injective on A_0 and $T \in \mathscr{T}_{g,p}$, using the property (ζ_1) of a simulation function, we obtain that

$$\begin{aligned}0 &\leq \zeta(\mu(Tgx_n,Tgx_{n+1}),\mu(Tx_{n-1},Tx_n))\\ &< \mu(Tx_{n-1},Tx_n) - \mu(Tgx_n,Tgx_{n+1})\\ &\leq \mu(Tx_{n-1},Tx_n) - \mu(Tx_n,Tx_{n+1})\end{aligned} \tag{3.15}$$

for every $n \in \mathbb{N}$. Then we have

$$\mu(Tx_n,Tx_{n+1}) < \mu(Tx_{n-1},Tx_n),\ \forall n \in \mathbb{N} \tag{3.16}$$

which implies that the sequence $\{\mu(Tx_{n-1},Tx_n)\}$ is decreasing.

If there exists $n_0 \in \mathbb{N}$ such that $\mu(Tx_{n_0-1},Tx_{n_0}) = 0$, then $Tx_{n_0-1} = Tx_{n_0}$. By the injective of T on A_0 follows $x_{n_0-1} = x_{n_0}$. Then $d(gx_{n_0-1},Tx_{n_0}) = d(gx_{n_0},Tx_{n_0}) = d(A,B)$ and x_{n_0} is the best proximity point of T under mapping g. That is, $x_{n_0} \in B^g_{est}(T)$.
Now, let $\mu(Tx_{n-1},Tx_n) > 0$ for all $n \in \mathbb{N}$. Then there exists

$$\lim_{n\to\infty} \mu(Tx_{n-1},Tx_n) = r \geq 0.$$

Suppose $r > 0$. From (16) we can also deduce that

$$\mu(Tgx_n,Tgx_{n+1}) < \mu(Tx_{n-1},Tx_n).$$

On the other hand, $T \in \mathscr{T}_{g,p}$ and hence

$$\mu(Tx_n,Tx_{n+1}) \leq \mu(Tgx_n,Tgx_{n+1}) < \mu(Tx_{n-1},Tx_n)$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mu(Tgx_n,Tgx_{n+1}) = r.$$

Using the property (ζ_2) of a simulation function, we get

$$0 \leq \limsup_{n\to\infty} \zeta(\mu(Tgx_{n+1},Tgx_n),\mu(Tx_{n-1},Tx_n)) < 0$$

which is a contradiction, and hence $r = 0$.
Therefore

$$\lim_{n\to\infty} \mu(Tx_{n-1}, Tx_n) = 0. \tag{3.17}$$

Next, we claim that

$$\lim_{m,n\to\infty} \mu(Tx_n, Tx_m) = 0. \tag{3.18}$$

Assume that (3.18) is not true, that is

$$\lim_{m,n\to\infty} \mu(Tx_n, Tx_m) \neq 0.$$

From Lemma 3.1, then there exists $\epsilon > 0$ and two subsequence $\{Tx_{n_k}\}$ and $\{Tx_{m_k}\}$ of $\{Tx_n\}$ such that

$$\lim_{k\to\infty} \mu(Tx_{n_k}, Tx_{m_k}) = \lim_{k\to\infty} \mu(Tx_{n_{k+1}}, Tx_{m_{k+1}}) = \epsilon. \tag{3.19}$$

Then there exists a subsequence of $\{x_{n_k}\}$, which we assume it is the whole sequence $\{x_{n_k}\}$, such that $\mu(Tx_{n_k}, Tx_{m_k}) > 0$ for all $k \in \mathbb{N}$. Since T is a $\mathscr{Z}$-p-proximal contraction of the second kind and $d(gx_{n_k+1}, Tx_{n_k}) = d(A, B) = d(gx_{m_k+1}, Tx_{m_k})$, we have

$$\begin{aligned} 0 &\leq \zeta(\mu(Tgx_{n_k+1}, Tgx_{m_k+1}), \mu(Tx_{n_k}, Tx_{m_k})) \\ &< \mu(Tx_{n_k}, Tx_{m_k}) - \mu(Tgx_{n_k+1}, Tgx_{m_k+1}) \\ &\leq \mu(Tx_{n_k}, Tx_{m_k}) - \mu(Tx_{n_k+1}, Tx_{m_k+1}) \end{aligned}$$

for all $k \in \mathbb{N}$. From the above ineqauality and (3.19),

$$\lim_{k\to\infty} \mu(Tgx_{n_k+1}, Tgx_{m_k+1}) = \epsilon.$$

Using the property (ζ_2) of a simulation function with $t_k := \mu(Tgx_{n_k+1}, Tgx_{m_k+1})$ and $s_k := \mu(Tx_{n_k}, Tx_{m_k})$, we get

$$0 \leq \limsup_{k\to\infty} \zeta(t_k, s_k) < 0$$

which is a contradiction and hence (3.18) holds.
Since

$$\lim_{m,n\to\infty} \mu(Tx_n, Tx_m) = 0$$

we have

$$\limsup_{m,n\to\infty} \{p(Tx_n, Tx_m) : m > n\} = 0.$$

It follows from Lemma 2.7 that $\{Tx_n\}$ is a p-Cauchy sequance in B_0, And from Lemma 2.5, we have $\{Tx_n\}$ is a Cauchy sequance in B_0.

Since (X, d) is a complete metric space and $T(A_0)$ is a closed subset of X, there exists $\lim_{n\to\infty} Tx_n = Tu \in T(A_0) \subseteq B_0$. Moreover, there exists $z \in A_0$ such that

$$d(z, Tu) = d(A, B).$$

Since $A_0 \subseteq g(A_0)$, we obtain that $z = gx$ for some $x \in A_0$. Hence

$$d(gx, Tu) = d(A, B). \tag{3.20}$$

If $x_n = x$ holds for infinite values of $n \in \mathbb{N}$, then $Tx = Tu$. Assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \neq x$ for all $n \geq n_0$. Then there exists a sunsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \neq Tu$ for all $k \in \mathbb{N}$. Since T is a $\mathscr{Z}$-p-proximal contraction of the second kind, we get

$$0 \leq \zeta(\mu(Tgx_{n_k+1}Tgx), \mu(Tx_{n_k}, Tu)) < \mu(Tx_{n_k}, Tu) - \mu(Tgx_{n_k+1}, Tgx).$$

Hence

$$\mu(Tx_{n_k+1}, Tx) \leq \mu(Tgx_{n_k+1}, Tgx) < \mu(Tx_{n_k}, Tu) \tag{3.21}$$

for all $k \in \mathbb{N}$ such that $n_k \geq n_0$, because $T \in \mathscr{T}_{g,p}$.

It follows from (3.18) we obtain that for any $\epsilon > 0$ there exists a $N_\epsilon \in \mathbb{N}$ such that $\mu(Tx_n, Tx_m) < \epsilon$ for every $m > n \geq N_\epsilon$.

Since $\{Tx_n\}$ is a p-Cauchy sequance in B_0, there exists a function η from $B_0 \times [0,\infty) \to [0,\infty)$ satisfying $(\tau_2) - (\tau_5)$ and a sequence $\{z_n\}$ of B_0 such that

$$\limsup_{n\to\infty}\{\eta(z_n, p(z_n, Tx_m)) : m \geq n\} = 0.$$

By (τ_3) and $p(\cdot, x) : X \to [0,\infty)$ is lower semicontinuous, imply that

$$p(Tx_n, Tu) \leq \liminf_m p(Tx_n, Tx_m) < \epsilon$$

and

$$p(Tu, Tx_n) \leq \liminf_m p(Tx_m, Tx_n) < \epsilon$$

for any fixed $n \geq \max\{n_0, N_\epsilon\}$, which implies that

$$\lim_{k\to\infty} p(Tx_{n_k}, Tu) = 0. \tag{3.22}$$

Similarly $\lim_{k\to\infty} p(Tu, Tx_{n_k}) = 0$, and hence $\lim_{k\to\infty} \mu(Tx_{n_k}, Tu) = 0$. Combine this and (3.21) to get $\lim_{k\to\infty} \mu(Tx_{n_k+1}, Tx) = 0$. By triangle inequality of μ,

$$\mu(Tx_{n_k}, Tx) \leq \mu(Tx_{n_k}, Tx_{n_k+1}) + \mu(Tx_{n_k+1}, Tx).$$

From (3.17) and passing to limit as $k \to \infty$, we obtain $\lim_{k\to\infty} \mu(Tx_{n_k}, Tx) = 0$. This implies that

$$\lim_{k\to\infty} p(Tx_{n_k}, Tx) = 0 \tag{3.23}$$

Since $\lim_{k\to\infty} Tx_{n_k} = Tu$, we obtain

$$p(Tu, Tu) = 0 \text{ and } p(Tu, Tx) = 0.$$

Using (3.22), (3.23) and Lemma 2.6 imply that $Tx = Tu$. By substituting this in (3.20), we get $d(gx, Tx) = d(A, B)$.

We will show the uniqueness, let y be in A_0 such that

$$d(gy, Ty) = d(A, B),$$

i.e, $y \in B^g_{est}(T)$. Suppose that $\mu(Tgx, Tgy) \geq \mu(Tx, Ty) > 0$. Since $T \in \mathscr{T}_{g,p}$ is a $\mathscr{Z}$-p-proximal contraction of the second kind, we have

$$\begin{aligned} 0 &\leq \zeta(\mu(Tgx, Tgy), \mu(Tx, Ty)) \\ &< \mu(Tx, Ty) - \mu(Tgx, Tgy) \\ &\leq \mu(Tx, Ty) - \mu(Tx, Ty) = 0 \end{aligned}$$

which is a contraction. Hence $\mu(Tx, Ty) = 0$, which means that $Tx = Ty$. From T is injective on A_0, imply that $x = y$.

Finally, suppose that $\mu(Tx, Tx) = p(Tx, Tx) > 0$. Then $\mu(Tgx, Tgx) > 0$. Using a similar argument as above, we have

$$\begin{aligned} 0 &\leq \zeta(\mu(Tgx, Tgx), \mu(Tx, Tx)) \\ &< \mu(Tx, Tx) - \mu(Tgx, Tgx) \\ &\leq \mu(Tx, Tx) - \mu(Tx, Tx) = 0 \end{aligned}$$

which is a contraction. Therefore $p(Tx, Tx) = 0$. □

The following best proximity point result is a special case of Theorem 3.6 when g is an identity map on A.

Corollary 3.7. *Let A and B be two nonempty subsets of a complete metric space (X,d) with a τ-distance p, such that $T(A_0)$ is nonempty and closed. Let $p(\cdot,x) : X \to [0,\infty)$ is lower semi continuous for any $x \in X$, Suppose that the mappings $T : A \to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-p-proximal contraction of the second kind;*
(b) *T is injective on A_0;*
(c) *$T(A_0) \subseteq B_0$.*

Then there exists a unique best proximity point $x \in A_0$ of T with $p(Tx,Tx) = 0$, and for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1}, Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$.

By setting $p = d$ in Theorem 3.6 the main result of [14] is obtained.

Corollary 3.8. *Let A and B be two nonempty subsets of a complete metric space (X,d), such that $T(A_0)$ is nonempty and closed. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-proximal contraction of the second kind;*
(b) *T is injective on A_0*
(c) *$T \in \mathscr{T}_g$;*
(d) *$A_0 \subseteq g(A_0)$;*
(e) *$T(A_0) \subseteq B_0$.*

Then there exists a unique point $x \in A$ such that $d(gx,Tx) = d(A,B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} x_n = x$.

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BALL COMPARISON OF THREE METHODS OF CONVERGENCE ORDER SIX UNDER THE SAME SET OF CONDITIONS

IOANNIS K. ARGYROS[1] AND SANTHOSH GEORGE*[2]

[1] Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
[2] Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-575 025

ABSTRACT. The aim of this paper is to compare the convergence radii of three methods of convergence order six under the same conditions. Moreover, we expand the applicability of these methods using only the first derivative in contrast to earlier works using hypotheses on derivatives up to order seven although these derivatives do not appear in the methods. Numerical examples complete this study.

KEYWORDS: High order methods; Banach space; local convergence; $\omega-$conditions.
AMS Subject Classification: Primary 65D10; Secondary 65D99

1. Introduction

In this paper we compare the convergence radii of following three sixth order iterative methods defined for $n = 0, 1, 2, \ldots,$ by [12]:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - [-\frac{1}{2}I + \frac{9}{8}F'(y_n)^{-1}F'(x_n) \\ &\quad + \frac{3}{8}F'(x_n)^{-1}F'(y_n)]F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - \frac{9}{4}I + \frac{15}{8}F'(y_n)^{-1}F'(x_n)) \\ &\quad + \frac{11}{8}F'(x_n)^{-1}F'(y_n)]F'(y_n)^{-1}F(z_n), \end{aligned} \tag{1.1}$$

* *Corresponding author.*
Email address : iargyros@cameron.edu, sgeorge@nitk.edu.in.

[12]

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - [\frac{5}{8}I + \frac{3}{8}(F'(y_n)^{-1}F'(x_n))^2] \\ &\quad \times F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - [-\frac{9}{4}I + \frac{15}{8}F'(y_n)^{-1}F'(x_n)) \\ &\quad + \frac{11}{8}F'(x_n)^{-1}F'(y_n)]F'(y_n)^{-1}F(z_n), \end{aligned} \tag{1.2}$$

and [14]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - [\frac{23}{8}I - 3F'(x_n)^{-1}F'(y_n) + \frac{9}{8}(F'(x_n)^{-1}F'(y_n))^2] \\ &\quad \times F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - [\frac{5}{2}I - \frac{3}{2}F'(x_n)^{-1}F'(y_n)] \\ &\quad \times F'(x_n)^{-1}F(x_n) \end{aligned} \tag{1.3}$$

used for approximating a solution α of the equation

$$F(x) = 0. \tag{1.4}$$

Here: $F : \Omega \subset \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is a differentiable operator in the sense of Fréchet, $\mathcal{E}_1$ and $\mathcal{E}_2$ are Banach spaces and Ω is convex and open.

Earlier convergence analysis of these methods when $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^k$ used, assumptions of the Fréchet derivatives of F of order up to seven [1, 2, 14] although these derivatives do not appear in these methods, limiting the applicability.

Example 1.1. Let $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}$, $\Omega = [-\frac{5}{2}, \frac{3}{2}]$. Define F on Ω by

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

Then

$$\begin{aligned} F'(x) &= 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \log x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \log x^2 + 60x^2 = 24x + 22. \end{aligned}$$

Obviously $F'''(x)$ is not bounded on Ω. So, the convergence of methods (1.1), (1.2) and (1.3) is not guaranteed by the analysis in the earlier studies.

In this study, our analysis uses only the assumptions on the first Fréchet derivative of F. Thus, we extend the applicability of these methods and in the more general setting of Banach space valued operators. This technique can be used to extend the applicability of other iterative methods.

Notice that, solutions methods for equation (1.4) is an important area of research, since a plethora of problems from diverse disciplines such that Mathematics, Optimization, Mathematical Programming, Chemistry, Biology, Physics, Economics, Statistics, Engineering and other disciplines can be modeled into an equation of the form (1.4) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

The rest of the study is organized as follows. In Section 2 , the local convergence analysis is given and numerical examples are given in the last Section 4.

2. Local convergence

Let us introduce some real functions and parameters needed in the local convergence analysis. Consider a function $\omega_0 : S \longrightarrow S$ continuous and increasing with $w_0(0) = 0$, where $S = [0, \infty)$. Suppose that equation

$$\omega_0(t) = 1 \tag{2.1}$$

has at least one positive solution. We denote by ρ_0 the smallest such solution. Set $S_0 = [0, \rho_0)$. Let also $\omega : S_0 \longrightarrow S$ and $\omega_1 : S_0 \longrightarrow S$ be continuous and increasing functions with $\omega(0) = 0$. Define functions g_1 and $\bar{g}_1$ on the interval S_0 by

$$g_1(t) = \frac{\omega((1-\theta)t)d\theta + \frac{1}{3}\int_0^1 \omega_1(\theta t)d\theta}{1 - \omega_0(t)}$$

and

$$\bar{g}_1(t) = g_1(t) - 1.$$

Suppose that

$$\omega_1(0) < 3. \tag{2.2}$$

We obtain that $\bar{g}_1(0) = \frac{\omega_1(0)}{3} - 1 < 0$ and $\bar{g}_1(t) \longrightarrow \infty$ as $t \longrightarrow \rho_0^-$. The intermediate value theorem guarantees the existence of at least one solution of the equation $\bar{g}_1(t) = 0$ in $(0, \rho_0)$. Denote by R_1 the smallest such solution. Suppose that equation

$$\omega_0(g_1(t)t) = 1 \tag{2.3}$$

has at least one positive solution. Denote by ρ_1 the smallest such solution. Set $S_1 = [0, \rho_2)$, where $\rho_2 = \min\{\rho_0, \rho_1\}$. Define functions g_2 and $\bar{g}_2$ on S_1 by

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)} + \frac{3}{8}\left[3\frac{\omega_0(g_1(t)t) + \omega_0(t)}{1 - \omega_0(g_1(t)t)}\right. \\ &\quad \left. + \frac{w_0(g_1(t)t) + \omega_0(t)}{1 - \omega_0(t)}\right] \frac{\int_0^1 \omega_1(\theta t)d\theta}{1 - \omega_0(t)} \end{aligned}$$

and

$$\bar{g}_2(t) = g_2(t) - 1.$$

We also get $\bar{g}_2(0) = -1$ and $\bar{g}_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. Denote by R_2 the smallest solution of equation $\bar{g}_2(t) = 0$ in $(0, \rho_2)$. Suppose that

$$\omega_0(g_2(t)t) = 1 \tag{2.4}$$

has at least one positive solution. Denote by ρ_3 the smallest such solution. Set $S_2 = [0, \rho)$, where $\rho = \min\{\rho_2, \rho_3\}$. Define functions g_3 and $\bar{g}_3$ by

$$\begin{aligned} g_3(t) &= \left\{ \frac{\int_0^1 \omega((1-\theta)g_2(t)t)d\theta}{1 - \omega_0(g_2(t)t)} \right. \\ &\quad + \frac{(\omega_0(g_2(t)t) + \omega_0(g_1(t)t))\int_0^1 \omega_1(\theta g_2(t)t)d\theta}{(1 - \omega_0(g_2(t)t))(1 - \omega_0(g_1(t)t))} \\ &\quad \frac{1}{8}\left[\frac{15(\omega_0(g_1(t)t) + \omega_0(t))}{1 - \omega_0(g_1(t)t)}\right. \\ &\quad \left.\left. + \frac{11(w_0(g_1(t)t) + \omega_0(t))}{1 - \omega_0(t)}\right] \frac{\int_0^1 \omega_1(\theta g_2(t)t)d\theta}{1 - \omega_0(g_1(t)t)} \right\} \end{aligned}$$

and

$$\bar{g}_3(t) = g_3(t) - 1.$$

We have again $\bar{g}_3(0) = -1$ and $\bar{g}_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho^-$. Moreover, define a radius of convergence R by

$$R = \min\{R_i\}, \quad i = 1, 2, 3. \tag{2.5}$$

It follows that for each $t \in [0, R)$

$$0 \leq \omega_0(t) < 1,\ 0 \leq \omega_0(g_1(t)t) < 1,\ 0 \leq \omega_0(g_2(t)t) < 1, \tag{2.6}$$

and

$$0 \leq g_i(t) < 1,\ \ i = 1, 2, 3. \tag{2.7}$$

We base the local convergence analysis of method (1.1) on conditions (A):

(a1) $F : \Omega \longrightarrow \mathcal{E}_2$ is a continuously differentiable operator in the sense of Fréchet and there exists $\alpha \in \Omega$ such that $F(\alpha) = 0$ and $F'(\alpha)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$.

(a2) There exists function $\omega_0 : S \longrightarrow S$ continuous and increasing with $\omega_0(0) = 0$ and for each $x \in \Omega$

$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \leq \omega_0(\|x - \alpha\|).$$

Set $\Omega_0 = \Omega \cap U(\alpha, \rho_0)$, where ρ_0 is given in (2.1).

(a3) There exist functions $\omega : S_0 \longrightarrow S, \omega_1 : S_0 \longrightarrow S$ such that for each $x, y \in \Omega_0$

$$\|F'(\alpha)^{-1}(F'(y) - F'(x))\| \leq \omega(\|y - x\|)$$

and

$$\|F'(\alpha)^{-1}F'(x)\| \leq \omega_1(\|x - \alpha\|)$$

where S_0 and S are defined previously.

(a4) $\bar{U}(\alpha, R) \subset \Omega, \rho_0, \rho_1, \rho_2$ exist and are given by (2.1), (2.3) and (2.4), respectively, (2.2) holds and R is given by (2.5).

(a5) There exists $R_1 \geq R$ such that

$$\int_0^1 \omega_0(\theta R_1) d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{U}(\alpha, R_1)$.

Next, the local convergence analysis of method (1.1)is provided using the conditions (A) and the preceding notation.

Theorem 2.1. *Suppose that the conditions (A) hold. Then, sequence $\{x_n\}$ generated by (1.1), for $x_0 \in U(\alpha, R) - \{\alpha\}$ is well defined, remains in $U(\alpha, R)$ for each $n = 0, 1, 2, 3, \ldots$ and converges to α. Moreover, the following estimates hold*

$$\|y_n - \alpha\| \leq g_1(\|x - \alpha\|)\|x - \alpha\| \leq \|x - \alpha\| < R, \tag{2.8}$$

$$\|z_n - \alpha\| \leq g_2(\|x - \alpha\|)\|x - \alpha\| \leq \|x - \alpha\| \tag{2.9}$$

and

$$\|x_{n+1} - \alpha\| \leq g_3(\|x - \alpha\|)\|x - \alpha\| \leq \|x - \alpha\|, \tag{2.10}$$

where functions g_i are given previously and R is defined in (2.5). Furthermore, the limit point α is the only solution of equation $F(x) = 0$ in the set Ω_1.

Proof. We use mathematical induction to show (2.8) – (2.10). Let $x \in U(\alpha, R) - \{\alpha\}$. Using (2.5), (a1) and (a2), we get that

$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \leq \omega_0(\|x - \alpha\|) \leq \omega_0(R) < 1. \tag{2.11}$$

By the Banach perturbation lemma [6, 7, 10], $F'(x)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$,

$$\|F'(x)^{-1}F'(\alpha)\| \leq \frac{1}{1 - \omega(\|x - \alpha\|)} \tag{2.12}$$

and the iterate y_0 is well defined by the first substep of method (1.1) for $n=0$. We can write by (a1)that

$$y_0-\alpha = x_0-\alpha-F'(x_0)^{-1}F(x_0)+\frac{1}{3}F'(x_0)^{-1}F(x_0), \tag{2.13}$$

so by (2.5), (2.7) (for $i=1$), (2.12) (for $x=x_0$) and (2.13), we have in turn that

$$\begin{aligned}
\|y_0-\alpha\| &\le \|F'(x_0)^{-1}F'(\alpha)\| \\
&\quad \|\int_0^1 F'(\alpha)^{-1}(F'(\alpha+\theta(x_0-\alpha))-F'(x_0))d\theta(x-\alpha)\| \\
&\quad +\frac{1}{3}\|F'(x_0)^{-1}F'(\alpha)\| \\
&\quad \|\int_0^1 F'(\alpha)^{-1}(F'(\alpha+\theta(x_0-\alpha))-F'(x_0))d\theta(x-\alpha)\| \\
&\le \left[\frac{[\int_0^1\omega((1-\theta)\|x_0-\alpha\|)d\theta+\frac{1}{3}\int_0^1\omega_1(\theta\|x_0-\alpha\|)d\theta]}{1-\omega_0(\|x_0-\alpha\|)}\right] \\
&\quad \times\|x_0-\alpha\| \\
&= g_1(\|x_0-\alpha\|)\|x_0-\alpha\|\le\|x_0-\alpha\|<R,
\end{aligned} \tag{2.14}$$

so (2.8) holds for $n=0$ and $y_0\in U(\alpha,R)$. Moreover, z_0 exists by (2.12) (for $x=y_0$). We can write

$$\begin{aligned}
z_0-\alpha &= x_0-\alpha-F'(x_0)^{-1}F(x_0) \\
&\quad -[-\frac{3}{2}I+\frac{9}{8}F'(y_0)^{-1}F'(x_0)+\frac{3}{8}F'(x_0)^{-1}F'(y_0)]F'(x_0)^{-1}F(x_0) \\
&= x_0-\alpha-F'(x_0)^{-1}F(x_0)+\frac{3}{8}[3F'(y_0)^{-1}(F'(y_0)-F'(x_0)) \\
&\quad +F'(x_0)^{-1}(F'(x_0)-F'(y_0))]F'(x_0)^{-1}F(x_0),
\end{aligned} \tag{2.15}$$

where we used the estimations

$$\begin{aligned}
&\quad -\frac{12}{8}I+\frac{9}{8}F'(y_0)^{-1}F'(x_0)+\frac{3}{8}F'(x_0)^{-1}F'(y_0) \\
&= -\frac{9}{8}(I-F'(y_0)^{-1}F'(x_0))-\frac{3}{8}(I-F'(x_0)^{-1}F'(y_0)) \\
&= -\frac{3}{8}[3F'(y_0)^{-1}(F'(y_0)-F'(x_0))+F'(x_0)^{-1}(F'(x_0)-F'(y_0))].
\end{aligned}$$

Then, by (2.5), (2.7) (for $i=2$, (2.12) (for $x=y_0$), and (2.14), we have in turn that

$$\begin{aligned}
\|z_0-\alpha\| &\le \|x_0-\alpha-F'(x_0)^{-1}F(x_0)\|+\frac{3}{8}[3\|F'(y_0)^{-1}F'(\alpha)\| \\
&\quad (\|F'(\alpha)^{-1}(F(y_0)-F'(\alpha))\|+\|F'(\alpha)^{-1}(F'(x_0)-F'(\alpha))\|) \\
&\quad +\|F'(x_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}(\|F'(\alpha)^{-1}(F'(y_0)-F'(\alpha))\| \\
&\quad +\|F'(\alpha)^{-1}(F'(x_0)-F'(\alpha))\|)] \\
&\quad \|F'(x_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}F(x_0)\| \\
&\le \Bigg\{\frac{\int_0^1\omega((1-\theta)\|x_0-\alpha\|)d\theta}{1-\omega_0(\|y_0-\alpha\|)} \\
&\quad +\frac{3}{8}\Bigg[\frac{3(\omega_0(\|y_0-\alpha\|)+\omega_0(\|x_0-\alpha\|))}{1-\omega_0(\|y_0-\alpha\|)}
\end{aligned}$$

$$
\begin{aligned}
& + \left.\frac{\omega_0(\|x_0-\alpha\|)+\omega_0(\|y_0-\alpha\|)}{1-\omega_0(\|x_0-\alpha\|)}\right] \\
& \left.\frac{\int_0^1 \omega_1(\theta\|x_0-\alpha\|)d\theta}{1-\omega_0(\|x_0-\alpha\|)}\right\}\|x_0-\alpha\| \\
\leq\ & g_2(\|x_0-\alpha\|)\|x_0-\alpha\| \leq \|x_0-\alpha\|, \qquad (2.16)
\end{aligned}
$$

so (2.9) holds for $n=0$ and $z_0 \in U(\alpha, R)$. We also have by (2.12) (for $x=z_0$) that $F'(z_0)^{-1}$ exists. Then, we can write by the second substep of method (1.1) that

$$
\begin{aligned}
x_1-\alpha \;=\; & z_0-\alpha-F'(z_0)^{-1}F(z_0) \\
& +F'(z_0)^{-1}(F'(y_0)-F'(z_0))F'(y_0)^{-1}F(z_0) \\
& +\frac{1}{8}[15F'(y_0)^{-1}(F'(y_0)-F'(x_0))+11F'(x_0)^{-1}(F'(x_0)-F'(y_0))] \\
& F'(y_0)^{-1}F(z_0), \qquad (2.17)
\end{aligned}
$$

where we used estimations

$$
\begin{aligned}
& \frac{1}{8}[-26I+15F'(y_0)^{-1}F'(x_0)-11I+11F'(x_0)^{-1}F'(y_0)] \\
=\ & -\frac{1}{8}[15(I-F'(y_0)^{-1}F'(x_0))+11(I-F'(x_0)^{-1}F'(y_0))] \\
=\ & -\frac{1}{8}[15F'(y_0)^{-1}(F'(y_0)-F'(x_0))+11F'(x_0)^{-1}(F'(x_0)-F'(y_0))].
\end{aligned}
$$

Next, by (2.5), (2.7) (for $i=3$), (2.12) (for $x=x_0, z_0$), (2.16) and (2.17), we obtain in turn that

$$
\begin{aligned}
\|x_1-\alpha\| \;\leq\; & \|z_0-\alpha-F'(z_0)^{-1}F(z_0)\| \\
& +[\|F'(z_0)^{-1}F'(\alpha)\|(\|F'(\alpha)^{-1}(F'(y_0)-F'(x_0))\| \\
& +\|F'(\alpha)^{-1}(F'(z_0)-F'(\alpha))\|) \\
& \times\|F'(y_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}F(z_0)\| \\
& +\frac{1}{8}[15\|F'(y_0)^{-1}F'(\alpha)\|(\|F'(\alpha)^{-1}(F'(y_0)-F'(\alpha)\| \\
& +\|F'(\alpha)^{-1}(F'(x_0)-F'(\alpha))\|)] \\
& +11\|F'(x_0)^{-1}F'(\alpha)\|(\|F'(\alpha)^{-1}(F'(x_0)-F'(\alpha))\| \\
& +\|F'(\alpha)^{-1}(F'(y_0-F'(\alpha))\|)] \\
& \times\|F'(y_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}F(z_0)\| \\
\leq\; & \left\{\frac{\int_0^1 \omega((1-\theta)\|z_0-\alpha\|)d\theta}{1-\omega_0(\|z_0-\alpha\|)}\right. \\
& + \frac{(\omega_0(\|z_0-\alpha\|)+\omega_0(\|y_0-\alpha\|))\int_0^1 \omega_1(\theta\|z_0-\alpha\|)d\theta}{(1-\omega_0(\|z_0-\alpha\|))(1-\omega_0(\|y_0-\alpha\|))} \\
& +\frac{1}{8}\left[\frac{15(\omega_0(\|x_0-\alpha\|)+\omega_0(\|y_0-\alpha\|)}{1-\omega_0(\|y_0-\alpha\|)}\right. \\
& + \left.\frac{11(\omega_0(\|x_0-\alpha\|)+\omega_0(\|y_0-\alpha\|)}{1-\omega_0(\|x_0-\alpha\|)}\right] \\
& \times \left.\frac{\int_0^1 \omega_1(\theta\|z_0-\alpha\|)d\theta}{1-\omega_0(\|y_0-\alpha\|)}\right\}\|z_0-\alpha\| \\
\leq\; & g_3(\|x_0-\alpha\|)\|x_0-\alpha\| \leq \|x_0-\alpha\|, \qquad (2.18)
\end{aligned}
$$

so (2.10) holds for $n = 0$ and $x_1 \in U(\alpha, R)$. The induction for (2.8)– (2.10) is completed, if x_0, y_0, z_0, x_1 are replaced by x_j, y_j, z_j, x_{j+1} respectively, in the preceding estimations. It then follows from

$$\|x_{j+1} - \alpha\| \leq a\|x_j - \alpha\| < R,\ a = g_3(\|x_0 - \alpha\|) \in [0, 1) \tag{2.19}$$

that $\lim_{j \longrightarrow \infty} x_j =$ and $x_{j+1} \in U(\alpha, R)$. Finally, for the uniqueness part, let $\alpha_1 \in \Omega_1$ with $F(p_1) = 0$ with $F(\alpha_1) = 0$. Then, by (a2) and (a5), we get in turn that for $T = \int_0^1 F'(\alpha_1 + \theta(\alpha - \alpha_1))d\theta$ for

$$\begin{aligned} \|F'(\alpha)^{-1}(T - F'(\alpha))\| &\leq \int_0^1 \omega_0(\theta\|\alpha - \alpha_1\|)d\theta \\ &\leq \int_0^1 \omega_0(\theta R^*)d\theta < 1 \end{aligned} \tag{2.20}$$

leading to $T^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$. Then, by the identity

$$0 = F(\alpha) - F(\alpha_1) = T(\alpha - \alpha_1),$$

we deduce that $\alpha_1 = \alpha$.

□

Remark 2.1. The convergence order of method (1.1) can be determined using computing the computational order of convergence (COC) [7, 8, 11] given by

$$\xi = \frac{\ln(\frac{\|x_{n+2}-\alpha\|}{\|x_{n+1}-\alpha\|})}{\ln(\frac{\|x_{n+1}-\alpha\|}{\|x_n-\alpha\|})} \tag{2.21}$$

or the approximate computational order of convergence (ACOC) [7, 8, 11] given by

$$\xi^* = \frac{\ln(\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|})}{\ln(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|})}. \tag{2.22}$$

It turns out that the local convergence of method (1.2) (or method(1.3)) are given under the conditions (A) by modifying the definition of g_i functions to fit these methods as follows:

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)} + \frac{3}{8}\Bigg[\left(\frac{(\omega_0(g_1(t)t) + \omega_0(t))}{1-\omega_0(g_1(t)t)}\right)^2 \\ &+ \frac{2(w_0(g_1(t)t) + \omega_0(t))}{(1-\omega_0(g_1(t)t))^2}\Bigg] \frac{\int_0^1 \omega_1(\theta t)d\theta}{1-\omega_0(t)}, \\ \bar{g}_2(t) &= g_2(t) - 1, \end{aligned}$$

and g_3 and $\bar{g}_3$ as previously. The corresponding (2.15) Ostrowski-type representation in method (1.2) is:

$$\begin{aligned} z_n - \alpha &= x_n - \alpha - F'(x_n)^{-1}F(x_n) \\ &+ \frac{3}{8}[(F'(y_n)^{-1}(F'(y_n) - F'(x_n)))^2 \\ &+ 2F'(y_n)^{-1}(F'(y_n) - F'(x_n))F'(y_n)^{-1}F'(x_n)] \\ &\times F'(x_n)^{-1}F(x_n), \end{aligned} \tag{2.23}$$

where the representations for functions g_1 and g_3 are the same. Moreover, the corresponding to (2.15) and (2.17) representations for method (1.3) are:

$$z_n - \alpha = x_n - \alpha - F'(x_n)^{-1}F(x_n)$$

$$\begin{aligned} &+\frac{1}{8}[15F'(x_n)^{-1}(F'(x_n)-F'(y_n)) \\ &+9F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}(F'(y_n)-F'(x_n))] \\ &\times F'(x_n)^{-1}F(z_n). \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} x_{n+1}-\alpha &= z_n-\alpha-F'(z_n)^{-1}F(z_n) \\ &-\frac{3}{2}F'(x_n)^{-1}(F'(x_n)-F'(y_n)) \\ &\times F'(x_n)^{-1}F(z_n). \end{aligned} \tag{2.25}$$

The g functions are:

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)} + \frac{1}{8}\left[\frac{15(\omega_0(g_1(t)t)+\omega_0(t))}{1-\omega_0(t)}\right. \\ &+ \left.\frac{9w_1(g_1(t)t)(\omega_0(g_1(t)t)+\omega_0(t))\int_0^1 \omega_1(\theta t)d\theta}{(1-\omega_0(t))^3}\right] \end{aligned}$$

and

$$\begin{aligned} g_3(t) &= \left\{\frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)}\right. \\ &+ \left.\frac{3}{2}\frac{(\omega_0(t)+\omega_0(g_1(t)t))\int_0^1 \omega_1(\theta g_2(t)t)d\theta}{(1-\omega_0(t))^2}\right\} g_2(t). \end{aligned}$$

With the above changes and following the proof of Theorem 2.1, we arrive at the corresponding results for method (1.2) and method (1.3).

Theorem 2.2. *Suppose that the conditions (A) hold. Then, the conclusions of Theorem 2.1 hold for method (1.2) or method (1.3) with the above indicated changes.*

3. Numerical examples

Example 3.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, \Omega = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on Ω for $u = (x,y,z)^T$ by

$$F(u) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.8)-(2.12), conditions, we get $\omega_0(t) = (e-1)t, \omega(t) = e^{\frac{1}{e-1}}t, \omega_1(t) = e^{\frac{1}{e-1}}$.

Then using the definition of r, we have that
$R_1 = 0.154406951357154070825217218043 69$
$R_2 = 0.0837447893717740837749019533475 8 = R$
$R_3 = 0.113329320170320893557125430106 68.$

Example 3.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.$$

Then, we get that $x^* = 0$, $\omega_0(t) = 7.5t, \omega(t) = 15t, \omega_1(t) = 2$. This way, we have that
$R_1 = 0.0222222222222222222222222222222222$
$R_2 = 0.0159516980984292580658667759507781 = R$
$R_3 = 0.0219551063175956531758892253947\,12.$

Example 3.3. Let $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}$, $\Omega = [-\frac{5}{2}, \frac{1}{2}]$. Define F on Ω by

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

Then

$$F'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

Then, we get that $\varphi_0(t) = \varphi(t) = 147t, \psi(t) = 2$. So, we obtain
$R_1 = 0.00151171579743008314436885865457\,29$
$R_2 = 0.000881401706163512186492647870750\,26 = R$
$R_3 = 0.00122348030471346267550325492834\,42.$

Example 3.4. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1], \Omega = \bar{U}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1, 2, 3, 5, 11] defined by

$$x(s) = \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt,$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s \\ s(1-t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.4), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt.$$

Notice that

$$\|\int_0^1 G(s,t)dt\| \le \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)(\frac{3}{2}x(t)^{1/2} + x(t))dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \frac{1}{8}(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|).$$

Then, we get that $\omega_0(t) = \omega(t) = \frac{1}{8}(\frac{3}{2}t^{1/2} + t), \omega_1(t) = 1 + \omega_0(t)$. So, we obtain
$R_1 = 1.2$
$R_2 = 0.60784148620540678908952259007492 = R$
$R_3 = 0.77695598964350998105743428823189$.

4. Conclusion

A very important aspect in the study of iterative methods is the convergence region, since it determines the choices of the initial point. That is why we studied the convergence of three popular sixth order methods for solving nonlinear equations under the same set of conditions. The radii of convergence were evaluated on three test examples showing that in each example a different method has the largest radius of convergence.

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STRONG CONVERGENCE ALGORITHMS FOR EQUILIBRIUM PROBLEMS WITHOUT MONOTONICITY

BUI VAN DINH[1,*], NGUYEN THI THANH HA[1],
NGUYEN NGOC HAI[2], AND TRAN THI HUYEN THANH[1]

[1]Department of Mathematics, Faculty of Information Technology,
Le Quy Don Technical University, Hanoi, Vietnam
[2]Department of Scientific Fundamentals, Vietnam Trade Union University, Hanoi, Vietnam

ABSTRACT. In this paper, we introduce two new linesearch algorithms for solving a non-monotone equilibrium proplem in a real Hilbert space. Each method can be considered as a combination of the extragradient method with linesearch and shrinking projection methods. Then we show that the iterative sequence generated by each method converges strongly to a solution of the considered problem. A numerical example is also provided.

KEYWORDS: Non-monotonicity; equilibria; shrinking projection methods; strong convergence; Armijo linesearch; Hilbert space.
AMS Subject Classification: 90C25; 90C33; 65K10; 65K15

1. Introduction

Let $\mathbb{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. The strong convergence and the weak convergence in the Hilbert space $\mathbb{H}$ are denoted by '$\to$' and '$\rightharpoonup$', respectively.
Let Ω be an open convex subset in $\mathbb{H}$ containing a nonempty closed convex C, and $f : \Omega \times \Omega \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for every $x \in C$.

The equilibrium problem (shortly $\mathrm{EP}(C, f)$), in the sense of Blum, Muu and Oettli [4, 21] (see also [15]), consists of finding $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C,$$

and its associated equilibrium problem

$$\text{Find } y^* \in C \text{ such that } f(x, y^*) \leq 0, \ \forall x \in C. \tag{1}$$

* *Corresponding author.*
Email address: vandinhb@gmail.com, nttha711@gmail.com, hainn@dhcd.edu.vn, thanh0712@gmail.com.

Note that problem (1) is called as the Minty equilibrium problem (MEP(C, f) for short) due to M. Castellani and M. Giuli [6]. We denote the solution set of EP(C, f) and MEP(C, f) by S_E and S_M, respectively.

Although problem EP(C, f) has a simple formulation, it encompasses, among its particular cases, many important problems in applied mathematics: convex optimization problem, variational inequality problem, fixed point problem, saddle point problem, Nash equilibrium problem in noncooperative game, and others; see, for example, [3, 4, 21], and the references quoted therein.

Recall that a bifunction f is said to be monotone on C if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C,$$

and pseudo-monotone on C if

$$\forall x, y \in C, f(x, y) \geq 0 \implies f(y, x) \leq 0.$$

Solution methods for equilibrium problems with monotone or pseudo-monotone bifunctions [1, 9, 12, 13, 16, 17, 19, 22, 29] have been studied extensively by many researchers and they have been usually extended from those for variational inequality problems and other related problems [5, 14].

For obtaining a solution of a non Lipschitz type and pseudo-monotone equilibrium problem in Euclidean space, Tran *et al.* [27] proposed to combine extragradient algorithms [18] with Armijo linesearch rule [2] to get the following algorithm.

Algorithm 1.

Initialization. Pick $x^0 \in C$, $\eta, \mu \in (0, 1)$; $0 < \rho$;
$\gamma_k \in [\gamma, \bar{\gamma}] \subset (0, 2)$.
Iteration k (k = 0, 1, 2, ...). Having x^k do the following steps:
Step 1. Solve the strongly convex program

$$\min \left\{ f(x^k, y) + \frac{1}{2\rho}\|y - x^k\|^2 : \ y \in C \right\} \qquad CP(x^k)$$

to obtain its unique solutions y^k.
If $y^k = x^k$, then stop. Otherwise, go to Step 2.
Step 2. (Armijo linesearch rule) Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\mu}{2\rho}\|x^k - y^k\|^2. \end{cases}$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$.
Step 3. Select $w^k \in \partial_2 f(z^k, x^k)$, take $\sigma_k = \frac{f(z^k, x^k)}{\|w^k\|^2}$, and compute $x^{k+1} = P_C(x^k - \gamma_k.\sigma_k.w^k)$, and go to Step 1 with k is replaced by $k + 1$.

They showed that the sequence $\{x^k\}$ generated by the above algorithm converges to a solution of EP(C, f) provided that $S_E \neq \emptyset$.

In addition, to find a fixed point of a non-expansive self mapping T in real Hilbert spaces, i.e., $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. Takahashi *et al.* [26] introduced the following iterative method, known as the shrinking projection method, which is the following:

Algorithm 2

Initialization. Pick $x^0 = x^g \in C$, choose parameters $\alpha \in [0, 1)$, $\{\alpha_k\} \subset [0, \alpha]$ and set $C_0 = C$.

Iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:
Step 1. Compute

$$y^k = \alpha_k x^k + (1 - \alpha_k)Tx^k,$$
$$C_{k+1} = \{x \in C_k : \|x - u^k\| \leq \|x - x^k\|\}.$$

Step 2. Compute $x^{k+1} = P_{C_{k+1}}(x^g)$, and go to Step 1 with k is replaced by $k+1$.

They proved that $\{x^k\}$ generated by Algorithm 2 converges strongly to $x^* = P_{Fix(T)}(x^g)$. In spired by above algorithms and recent works [7, 10, 25, 31], in this paper, we introduce algorithms for solving an equilibrium problem in a real Hilbert space without pseudo-monotonicity assumption of the bifunctions by combining Algorithm 1 with Algorithm 2. Then, we proved that the sequences generated by proposed algorithms strongly converges to a solution of S_E.

The rest of paper is organized as follows. The next section contains some preliminaries on the metric projection and equilibrium problems. The third section is devoted to introduce two algorithms for EP(C, f) and their strong convergence. In the last section, we present an application of the proposed algorithm for Nash-Cournot equilibrium models of electricity markets and its implementation.

2. Preliminaries

In this paper, we denote the metric projection operator on C by P_C, that is

$$P_C(x) \in C : \ \|x - P_C(x)\| \leq \|y - x\|, \ \forall y \in C.$$

It is well known that the projection operator onto a closed convex has the following properties.

Lemma 2.1. *Suppose that C is a nonempty closed convex subset in $\mathbb{H}$. Then*

(a) *$P_C(x)$ is singleton and well defined for every x;*
(b) *$z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \forall y \in C$;*
(c) *$\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \ \forall x, y \in C$.*

Definition 2.1. *A bifunction $\varphi : C \times C \to \mathbb{R}$ is said to be jointly weakly continuous on $C \times C$ if for all $x, y \in C$ and $\{x^k\}$, $\{y^k\}$ are two sequences in C converging weakly to x and y respectively, then $\varphi(x^k, y^k)$ converges to $\varphi(x, y)$.*

In the sequel, we need the following blanket assumptions

$(\mathcal{A}_1)$ $f(x, .)$ is convex on Ω for every $x \in C$;
$(\mathcal{A}_2)$ f is jointly weakly continuous on $\Omega \times \Omega$.

For each $z, \ x \in C$, by $\partial_2 f(z, x)$ we denote the subdifferential of the convex function $f(z, .)$ at x, i.e.,

$$\partial_2 f(z, x) := \{w \in \mathbb{H} : f(z, y) \geq f(z, x) + \langle w, y - x\rangle, \ \forall y \in C\}.$$

In particular,

$$\partial_2 f(z, z) = \{w \in \mathbb{H} : f(z, y) \geq \langle w, y - z\rangle, \ \forall y \in C\}.$$

The next lemma can be considered as an infinite-dimensional version of Theorem 24.5 in [24]

Lemma 2.2. [28, Proposition 4.3] *Let $f : \Omega \times \Omega \to \mathbb{R}$ be a function satisfying conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Let $\bar{x}, \bar{y} \in \Omega$ and $\{x^k\}$, $\{y^k\}$ be two sequences in Ω converging weakly to $\bar{x}, \bar{y}$, respectively. Then, for any $\epsilon > 0$, there exist $\eta > 0$ and $k_\epsilon \in \mathbb{N}$ such that*

$$\partial_2 f(x^k, y^k) \subset \partial_2 f(\bar{x}, \bar{y}) + \frac{\epsilon}{\eta} B,$$

for every $k \geq k_\epsilon$, where B denotes the closed unit ball in $\mathbb{H}$.

Lemma 2.3. [20] *Under assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$, *a point* $x^* \in C$ *is a solution of* $EP(C, f)$ *if and only if it is a solution to the equilibrium problem:*

$$\text{Find } x^* \in C : f(x^*, y) + \frac{1}{2\rho}\|y - x^*\|^2 \geq 0, \ \forall y \in C. \qquad (AEP)$$

Lemma 2.4. [30] *Let* C *be a nonempty closed convex subset of* $\mathbb{H}$. *Let* $\{x^k\}$ *be a sequence in* $\mathbb{H}$ *and* $u \in \mathbb{H}$. *If any weak limit point of* $\{x^k\}$ *belongs to* C *and*

$$\|x^k - u\| \leq \|u - P_C(u)\|, \ \forall k.$$

Then $x^k \to P_C(u)$.

Lemma 2.5. [10] *Under assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$, *if* $\{z^k\} \subset C$ *is a sequence such that* $\{z^k\}$ *converges strongly to* $\bar{z}$ *and the sequence* $\{w^k\}$, *with* $w^k \in \partial_2 f(z^k, z^k)$, *converges weakly to* $\bar{w}$, *then* $\bar{w} \in \partial_2 f(\bar{z}, \bar{z})$.

Lemma 2.6. [11] *Let the equilibrium bifunction* f *satisfy the assumptions* $(\mathcal{A}_1)$ *on* Ω *and* $(\mathcal{A}_2)$ *on* C, *and* $\{x^k\} \subset C$, $0 < \underline{\rho} \leq \bar{\rho}$, $\{\rho_k\} \subset [\underline{\rho}, \ \bar{\rho}]$. *Consider the sequence* $\{y^k\}$ *defined as follows*

$$y^k = \arg\min\Big\{\varphi(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 : \ y \in C\Big\}.$$

Then, if $\{x^k\}$ *is bounded, then* $\{y^k\}$ *is also bounded.*

3. Main Results

Now we are in a position to present the first algorithm for solving a non-monotone equilibrium problem in a Hilbert space.

Algorithm 3.

Initialization. Pick $x^0 = x^g \in C$, choose parameters $\eta, \mu \in (0, 1)$, $0 < \rho \leq \bar{\rho}$, $\{\rho_k\} \subset [\rho, \ \bar{\rho}]$, $\gamma_k \in [\gamma, \bar{\gamma}] \subset (0, 2)$. and set $C_0 = C$.

At each iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min\Big\{f(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 : \ y \in C\Big\} \qquad CP(x^k)$$

to obtain its unique solution y^k. If $y^k = x^k$, then stop. Otherwise, do Step 2.

Step 2. (The first Armijo linesearch rule) Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\mu}{2\rho_k}\|x^k - y^k\|^2. \end{cases} \qquad (2)$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$.

Step 3. Select $w^k \in \partial_2 f(z^k, x^k)$, and compute $u^k = P_C(x^k - \gamma_k \sigma_k w^k)$, where $\sigma_k = \frac{f(z^k, x^k)}{\|w^k\|^2}$.

Step 4. Compute

$$x^{k+1} = P_{C_{k+1}}(x^g),$$

where $C_{k+1} = \{x \in C_k : \|x - u^k\| \leq \|x - x^k\|\}$, and go to Step 1 with k is replaced by $k + 1$.

Remark 3.1. *If* $y^k = x^k$ *then* x^k *is a solution to* $EP(C, f)$.

Before proving the convergence of Algorithm 1, let us recall the following lemma which was proved in [27].

Lemma 3.1. [27] *Suppose that the bifunction f satisfies assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, then we have:*

(a) *The linesearch is well-defined;*
(b) $f(z^k, x^k) > 0$*;*
(c) $0 \notin \partial_2 f(z^k, x^k)$*;*
(d) *In addition, if $S_M \neq \emptyset$, then*

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma_k(2-\gamma_k)(\sigma_k\|w^k\|)^2, \quad \textit{for all } x^* \in S_M. \tag{3}$$

Lemma 3.1 implies that the sequence $\{x^k\}$ generated by Algorithm 1 is well-defined. The following theorem establishes the strong convergence of $\{x^k\}$ to a solution of EP(C, f).

Theorem 3.2. *Suppose that bifunction f satisfies assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$. If the set S_M is nonempty, then the sequence $\{x^k\}$, $\{u^k\}$ generated by Algorithm 3 converge strongly to a solution x^* of EP(C, f).*

Proof. Take $\bar{x} \in S_M \subset C = C_0$. From Lemma 3.1, we have

$$\|u^k - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 - \gamma_k(2-\gamma_k)(\sigma_k\|w^k\|)^2. \tag{4}$$

Since $\gamma_k \in [\gamma, \bar{\gamma}] \subset (0, 2)$, we get

$$\|\bar{x} - u^k\| \leq \|\bar{x} - x^k\|. \tag{5}$$

By induction, we can conclude that $\bar{x} \in C_k$ for all k.

By Step 4, $x^k = P_{C_k}(x^g)$, we have

$$\|x^k - x^g\| \leq \|x - x^g\|, \ \forall x \in C_k, \tag{6}$$

so,

$$\|x^k - x^g\| \leq \|\bar{x} - x^g\|, \ \forall k. \tag{7}$$

Therefore, $\{x^k\}$ is bounded. Together with Lemma 2.2, $\{w^k\}$ is bounded. Combining with (5) we have $\{u^k\}$ is also bounded.
Since, $x^{k+1} \in C_k$ and (6), we have

$$\|x^k - x^g\| \leq \|x^{k+1} - x^g\|, \ \forall k. \tag{8}$$

Because $\{x^k\}$ is bounded, we get

$$\lim_{k\to\infty} \|x^k - x^g\| = \tau \geq 0. \tag{9}$$

In addition,

$$\begin{aligned}
\|x^{k+1} - x^k\|^2 &= \|x^{k+1} - x^g + x^g - x^k\|^2 \\
&= \|x^{k+1} - x^g\|^2 + \|x^g - x^k\|^2 + 2\langle x^{k+1} - x^g, x^g - x^k\rangle \\
&= \|x^{k+1} - x^g\|^2 + \|x^g - x^k\|^2 + 2\langle x^{k+1} - x^k, x^g - x^k\rangle - 2\|x^g - x^k\|^2 \\
&\leq \|x^{k+1} - x^g\|^2 - \|x^k - x^g\|^2,
\end{aligned}$$

where the last inequality follows from the fact that $x^k = P_{C_k}(x^g)$ and $x^{k+1} \in C_k$, then $\langle x^{k+1} - x^k, x^g - x^k\rangle \leq 0$.
From (9), we obtain

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{10}$$

Because $x^{k+1} \in C_{k+1}$, one has

$$\begin{aligned}\|x^k - u^k\| &\le \|x^k - x^{k+1}\| + \|x^{k+1} - u^k\| \\ &\le 2\|x^k - x^{k+1}\|\end{aligned}$$

Take into account with (10) we get

$$\lim_{k\to\infty} \|u^k - x^k\| = 0. \tag{11}$$

Next, we show that $\{x^k\}$, $\{u^k\}$ converge strongly to $x^* = P_{\cap_{k=0}^{\infty} C_k}(x^g)$.

It is clear that C_k is nonempty, closed and convex set, it is also weakly closed. Since $C_{k+1} \subset C_k, \forall k$ and $x^k \in C_k$, $x^k \in C_{k_0}$ for all $k \ge k_0$. Let $\hat{x}$ be any weak accumulation point of the sequence $\{x^k\}$, i.e., there exists $\{x^{k_j}\} \subset \{x^k\}$ such that $x^{k_j} \rightharpoonup \hat{x}$ as $j \to \infty$. Since $\{x^{k_j}\} \subset C_{k_i}, \forall j \ge i$ and the weak closedness of C_{k_i}, it implies that $\hat{x} \in C_{k_i}, \forall i$. Hence $\hat{x} \in C_k, \forall k$, or $\hat{x} \in \cap_{k=0}^{\infty} C_k$.

Set $x^* = P_{\cap_{k=0}^{\infty} C_k}(x^g)$. From (7) we have,

$$\|x^k - x^g\| \le \|x^* - x^g\|, \ \forall k. \tag{12}$$

We can conclude that x^k converges strongly to x^* by Lemma 2.4. Together with (11) we have u^k also converges strongly to x^*.

Next, we show that x^* solves EP(C, f).

In view of (4), it yields

$$\gamma_k(2-\gamma_k)(\sigma_k\|w^k\|)^2 \le \|x^k - u^k\|\big[\|x^k - \bar{x}\| + \|u^k - \bar{x}\|\big]. \tag{13}$$

Since $\gamma_k \in [\gamma, \bar{\gamma}] \subset (0, 2)$, and (11), we get from (13) that

$$\lim_{k\to\infty} \sigma_k\|w^k\| = 0. \tag{14}$$

Since $\{x^k\}$ is bounded and Lemma 2.6, $\{y^k\}$ is bounded. Consequently, $\{z^k\}$ is also bounded. Using Lemma 2.5, $\{w^k\}$ is bounded, In view of (14) yields

$$\lim_{k\to\infty} f(z^k, x^k) = \lim_{k\to\infty} [\sigma_k\|w^k\|]\|w^k\| = 0. \tag{15}$$

We have

$$\begin{aligned}0 = f(z^k, z^k) &= f(z^k, (1-\eta_k)x^k + \eta_k y^k) \\ &\le (1-\eta_k) f(z^k, x^k) + \eta_k f(z^k, y^k),\end{aligned}$$

so, we get from (2) that

$$\begin{aligned}f(z^k, x^k) &\ge \eta_k[f(z^k, x^k) - f(z^k, y^k)] \\ &\ge \frac{\mu}{2\rho_k}\eta_k\|x^k - y^k\|^2.\end{aligned}$$

Combining with (15) one has

$$\lim_{k\to\infty} \eta_k\|x^k - y^k\|^2 = 0. \tag{16}$$

We now consider two distinct cases:

Case 1. $\limsup_{k\to\infty} \eta_k > 0$.

Then there exists $\bar{\eta} > 0$ and a subsequence $\{\eta_{k_i}\} \subset \{\eta_k\}$ such that $\eta_{k_i} > \bar{\eta}$, $\forall i$, and from (16), one has

$$\lim_{i\to\infty} \|x^{k_i} - y^{k_i}\| = 0. \tag{17}$$

Remember that $x^k \to x^*$ and (17), it implies that $y^{k_i} \to x^*$ as $i \to \infty$.
By definition of y^{k_i} we have

$$f(x^{k_i}, y) + \frac{1}{2\rho_{k_i}}\|y - x^{k_i}\|^2 \geq f(x^{k_i}, y^{k_i}) + \frac{1}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2, \ \forall y \in C. \tag{18}$$

Without loss of generality, we assume that $\lim_{i\to\infty} \rho_{k_i} = \rho^*$. Letting $i \to \infty$, by jointly weak continuity of f and $x^{k_i} \to x^*$, $y^{k_i} \to x^*$, we obtain in the limit that

$$f(x^*, y) + \frac{1}{2\rho^*}\|y - x^*\|^2 \geq 0.$$

By Lemma 2.3, we conclude that

$$f(x^*, y) \geq 0, \ \forall y \in C.$$

Therefore, x^* is a solution of EP(C, f).

Case 2. $\lim_{k\to\infty} \eta_k = 0$.
Since $\{y^k\}$ is bounded, it implies that there exists $\{y^{k_i}\} \subset \{y^k\}$ such that $y^{k_i} \rightharpoonup \bar{y}$ as $i \to \infty$.

By the definition of y^{k_i}, we have

$$f(x^{k_i}, y^{k_i}) + \frac{1}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2 \leq 0. \tag{19}$$

In the other hand, by the Armijo linesearch rule (2), for $m_{k_i} - 1$, we have

$$f(z^{k_i, m_{k_i}-1}, x^{k_i}) - f(z^{k_i, m_{k_i}-1}, y^{k_i}) < \frac{\mu}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2. \tag{20}$$

Combining with (19) we get

$$f(x^{k_i}, y^{k_i}) \leq -\frac{1}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2 \leq \frac{1}{\mu}\big[f(z^{k_i, m_{k_i}-1}, y^{k_i}) - f(z^{k_i, m_{k_i}-1}, x^{k_i})\big]. \tag{21}$$

According to the linesearch rule, $z^{k_i, m_{k_i}-1} = (1-\eta^{m_{k_i}-1})x^{k_i} + \eta^{m_{k_i}-1}y^{k_i}$, $\eta^{m_{k_i}-1} \to 0$. Since x^{k_i} converges strongly to x^*, y^{k_i} converges weakly to $\bar{y}$, it implies that $z^{k_i, m_{k_i}-1}$ converges strongly to x^* as $i \to \infty$. In addition, $\{\frac{1}{\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2\}$ is bounded, without loss of generality, we may assume that $\lim_{i\to+\infty} \frac{1}{\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2$ exists. Hence, we get in the limit from (21) that

$$f(x^*, \bar{y}) \leq -\lim_{i\to+\infty} \frac{1}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2 \leq \frac{1}{\mu} f(x^*, \bar{y}).$$

Therefore, $f(x^*, \bar{y}) = 0$ and $\lim_{i\to+\infty} \|y^{k_i} - x^{k_i}\|^2 = 0$. By the Case 1, we get that x^* is a solution of EP(C, f).

□

Replacing the linesearch rule 2 by the other one, we get the following algorithm.

Algorithm 4.

Initialization. Pick $x^0 = x^g \in C$, choose parameters $\eta, \mu \in (0,1)$, $0 < \rho \leq \bar{\rho}$, $\{\rho_k\} \subset [\rho,\ \bar{\rho}]$, $\gamma_k \in [\gamma, \bar{\gamma}] \subset (0,2)$. and set $C_0 = C$.

At each iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min \left\{ f(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 :\ y \in C \right\} \qquad CP(x^k)$$

to obtain its unique solution y^k. If $y^k = x^k$, then stop. Otherwise, do Step 2.

Step 2. (The second Armijo linesearch rule) Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k \\ f(z^{k,m}, y^k) + \frac{\mu}{2\rho_k}\|x^k - y^k\|^2 \leq 0. \end{cases} \tag{22}$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$. If $0 \in \partial_2 f(z^k, z^k)$, then Stop. Otherwise, go to Step 3.

Step 3. Select $w^k \in \partial_2 f(z^k, z^k)$, and compute $u^k = P_C(x^k - \gamma_k \sigma_k w^k)$, where $\sigma_k = \frac{f(z^k, x^k)}{\|w^k\|^2}$.

Step 4. Compute

$$x^{k+1} = P_{C_{k+1}}(x^g),$$

where $C_{k+1} = \{x \in C_k : \|x - u^k\| \leq \|x - x^k\|\}$, and go to Step 1 with k is replaced by $k + 1$.

Remark 3.2.
- *If $y^k = x^k$ then x^k is a solution to EP(C, f);*
- *If $0 \in \partial_2 f(z^k, z^k)$, then z^k is a solution to EP(C, f).*

Lemma 3.3. [27] *Suppose that the bifunction f satisfies assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, then we have:*

(a) *The linesearch is well-defined;*
(b) $f(z^k, y^k) < 0$*;*
(c) *If $S_M \neq \emptyset$, then*

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma_k(2 - \gamma_k)(\sigma_k\|w^k\|)^2, \quad \textit{forall } x^* \in S_M. \tag{23}$$

Lemma 3.3 implies that the sequence $\{x^k\}$ generated by Algorithm 4 is well-defined.

The following theorem show us the convergence of Algorithm 4.

Theorem 3.4. *Suppose that bifunction f satisfies assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$. If the set S_M is nonempty, then the sequence $\{x^k\}$, $\{u^k\}$ generated by Algorithm 4 converge strongly to a solution x^* of EP(C, f).*

Proof. This theorem can be proved by the same arguments as in Theorem 3.2 so we obmit it.

4. Numerical examples

To illustrate the proposed algorithms, in this section, we consider an equilibrium problem arising in Nash-Cournot oligopolistic electricity market equilibrium model [8, 27]. In this model, there are n^c companies, each company i may possess I_i generating units. Let n^g be number of all generating units and x be the vector whose entry x_i stands for the power generating by unit i and $\sigma = \sum_{i=1}^{n^g} x_i$. We assume that the price p is a decreasing affine function of σ, that is

$$p(x) = 378.4 - 2\sum_{i=1}^{n^g} x_i = p(\sigma).$$

Then the profit made by company i is given by

$$f_i(x) = p(\sigma)\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j),$$

where $c_j(x_j)$ is the cost for generating x_j given by

$$c_j(x_j) := \max\{c_j^0(x_j), c_j^1(x_j)\}$$

with

$$c_j^0(x_j) := \frac{\alpha_j^0}{2}x_j^2 + \beta_j^0 x_j + \gamma_j^0,\ c_j^1(x_j) := \alpha_j^1 x_j + \frac{\beta_j^1}{\beta_j^1+1}\gamma_j^{-1/\beta_j^1}(x_j)^{(\beta_j^1+1)/\beta_j^1},$$

where $\alpha_j^k, \beta_j^k, \gamma_j^k$ $(k = 0, 1)$ are given parameters.

Denote $x_j^{\min}$ and $x_j^{\max}$ is the lower and upper bounds for the power generating by the unit j. Then the strategy set of the model takes the form

$$C := \{x = (x_1, ..., x^{n^g})^T :\ x_j^{\min} \le x_j \le x_j^{\max},\ \forall j\}.$$

By setting $q^i := (q_1^i, ..., q_{n^g}^i)^T$ with

$$q_j^i = \begin{cases} 1 & \text{if } j \in I_i \\ 0 & \text{if } j \notin I_i \end{cases},$$

and define

$$A := 2\sum_{i=1}^{n^c}(1 - q^i)(q^i)^T,\quad B := 2\sum_{i=1}^{n^c} q^i(q^i)^T, \tag{24}$$

$$a := -387.4\sum_{i=1}^{n^c} q^i, \text{and } c(x) := \sum_{j=1}^{n^g} c_j(x_j). \tag{25}$$

Then this oligopolistic equilibrium model can be written by the following equilibrium problem EP(C, f) (see [23, Page 155]):

Find $x^* \in C : f(x^*, y) = [(A+B)x^* + By + a]^T(y - x^*) + c(y) - c(x^*) \ge 0,\ \forall y \in C.$

It can be seen that, the matrix A is not positive semidefinite and $f(x, y) + f(y, x) = -(y - x)^T A(y - x)$, hence the bifunction f is nonmonotone and nonsmooth.

We test Algorithm 3 for this problem with corresponds to the first model in [8] where $n^c = 3$, and the parameters are given in the following tables:

We implement Algorithm 1 in Matlab R2014a running on a Laptop with Intel(R) Core(TM) i5-3230M CPU@2.60 GHz with 4 GB Ram. To terminate the Algorithm, we use the stopping criteria $\frac{\|x^{k+1}-x^k\|}{\max\{1,\|x^k\|\}} \le \epsilon$ with a tolerance $\epsilon = 10^{-3}$. The computation results are reported in Table 3 with some starting points and regularized parameters.

Com.	Gen.	$x^g_{\min}$	$x^g_{\max}$	$x^c_{\min}$	$x^c_{\max}$
1	1	0	80	0	80
2	2	0	80	0	130
2	3	0	50	0	130
3	4	0	55	0	125
3	5	0	30	0	125
3	6	0	40	0	125

TABLE 1. The lower and upper bounds of the power generation of the generating units and companies.

Gen.	α^0_j	β^0_j	γ^0_j	α^1_j	β^1_j	γ^1_j
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

TABLE 2. The parameters of the generating unit cost functions.

Iter(k)	ρ	x^k_1	x^k_2	x^k_3	x^k_4	x^k_5	x^k_6	Cpu(s)
0 691	0.1	0 46.6583	0 32.0728	0 15.0832	0 21.9862	0 12.3870	0 12.4071	136.0017
0 1166	0.5	0 46.6541	0 32.0750	0 15.0845	0 21.9224	0 12.4209	0 12.4389	151.3664
0 847	0.9	0 46.6440	0 31.9437	0 15.2014	0 21.6995	0 12.5953	0 12.4952	162.2410
0 629	0.1	30 46.6531	20 32.1041	10 15.0509	15 22.0089	10 12.4180	10 12.3606	122.1176
0 711 504	0.5	30 46.6416	20 31.9645	10 15.1811	15 21.6667	10 12.5630	10 12.5629	135.5798
0 711	0.9	30 46.6482	20 32.0263	10 15.1150	15 21.6827	10 12.5460	10 12.5657	147.0316

TABLE 3. Results computed with some starting points and regularized parameters.

5. Conclusion

. We have introduced two projection algorithms for finding a solution of a nonmonotone equilibrium problem in a real Hilbert space. The strong convergence of the proposed algorithms are obtained. We then have applied a proposed algorithm for a Nash-Cournot oligopolistic equilibrium model of electricity market. Some computation results are reported.

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