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COINCIDENCE POINT AND COMMON FIXED POINT THEOREMS IN CONE b-METRIC SPACES

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ABSTRACT. The main purpose of this paper is to obtain sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings in cone b-metric spaces. Our results extend and generalize several well known comparable results in the existing literature. Finally, some examples are provided to illustrate our results.

KEYWORDS: Cone b-metric space, point of coincidence, weakly compatible mappings, common fixed point.
AMS Subject Classification: 54H25, 47H10.

1. Introduction

Over the past two decades a considerable amount of research work for the development of fixed point theory have executed by several mathematicians. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a b-metric space initiated by Bakhtin[2]. In[6], Huang and Zhang introduced the concept of cone metric spaces as a generalization of metric spaces and proved some important fixed point theorems in such spaces. After that a series of articles have been dedicated to the improvement of fixed point theory. In most of those articles, the authors used normality property of cones in their results. Recently, Hussain and Shah[8] introduced the concept of cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. They studied some topological properties and improved some recent results about KKM mappings in the setting of a cone b-metric space. In this work, we shall establish sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings without the assumption of normality in cone b-metric spaces. The results generalize and improve some recent results in the literature. Furthermore, we support our results by examples.

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2. Preliminaries

In this section we need to recall some basic notations, definitions, and necessary results from existing literature. Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

$$\begin{aligned} &(i) && P \textit{ is closed, nonempty and } P \neq \{\theta\}; \\ &(ii) && a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \ \Rightarrow \ ax + by \in P; \\ &(iii) && P \cap (-P) = \{\theta\}. \end{aligned}$$

For any cone $P \subseteq E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$(equivalently, $y \succeq x$) if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \textit{ implies } \|x\| \leq k \|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. Throughout this paper, we suppose that E is a real Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\preceq$ is a partial ordering on E with respect to P.

Definition 2.1. [6] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.2. [8] Let X be a nonempty set and E a real Banach space with cone P. A vector valued function $d : X \times X \to E$ is said to be a cone b-metric function on X with the coefficient $s \geq 1$ if the following conditions are satisfied:

(i) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \preceq s\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

The pair (X, d) is called a cone b-metric space.

Observe that if $s = 1$, then the ordinary triangle inequality in a cone metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of cone b-metric spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b-metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example 2.3. [8] Let $X = \{-1, 0, 1\}$, $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\}$. Define $d : X \times X \to P$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = \theta$, $x \in X$ and $d(-1, 0) = (3, 3)$, $d(-1, 1) = d(0, 1) = (1, 1)$. Then (X, d) is a cone b-metric space, but not a cone metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = (1, 1) + (1, 1) = (2, 2) \prec (3, 3) = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.4. [9] Let $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\} \subseteq E$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (\mid x - y \mid^p, \alpha \mid x - y \mid^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space with $s = 2^{p-1}$, but not a cone metric space.

Definition 2.5. [8] Let (X, d) be a cone b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

- **(i):** (x_n) converges to x whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$;
- **(ii):** (x_n) is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$;
- **(iii):** (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Remark 2.6. [8] Let (X, d) be a cone b-metric space over the ordered real Banach space E with a cone P. Then the following properties are often used:

- **(i):** If $a \preceq b$ and $b \ll c$, then $a \ll c$.
- **(ii):** If $a \ll b$ and $b \ll c$, then $a \ll c$.
- **(iii):** If $\theta \preceq u \ll c$ for each $c \in int(P)$, then $u = \theta$.
- **(iv):** If $c \in int(P)$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.
- **(v):** Let $\theta \ll c$. If $\theta \preceq d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then eventually $d(x_n, x) \ll c$, where (x_n), x are a sequence and a given point in X.
- **(vi):** If $\theta \preceq a_n \preceq b_n$ and $a_n \to a$, $b_n \to b$, then $a \preceq b$, for each cone P.
- **(vii):** If E is a real Banach space with cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
- **(viii):** $\alpha\, int(P) \subseteq int(P)$ for $\alpha > 0$.
- **(ix):** For each $\delta > 0$ and $x \in int(P)$ there is $0 < \gamma < 1$ such that $\parallel \gamma x \parallel < \delta$.
- **(x):** For each $\theta \ll c_1$ and $c_2 \in P$, there is an element $\theta \ll d$ such that $c_1 \ll d$, $c_2 \ll d$.
- **(xi):** For each $\theta \ll c_1$ and $\theta \ll c_2$, there is an element $\theta \ll e$ such that $e \ll c_1$, $e \ll c_2$.

Definition 2.7. [1] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.8. [11] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \textit{ whenever } Sx = Tx.$$

Proposition 2.9. [1] *Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

Definition 2.10. Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. A mapping $T : X \to X$ is called expansive if there exists a real constant $k > s$ such that

$$d(Tx, Ty) \succeq k\, d(x, y)$$

for all $x, y \in X$.

3. TOPOLOGY IN CONE b-METRIC SPACES

In this section our concern is to introduce some topological aspects in cone b-metric spaces. This will facilate the initiation of open and closed sets, limit points of sets and other allied notions in the setting of cone b-metric spaces.

Definition 3.1. [8] Let (X, d) be a cone b-metric space and $B \subseteq X$.

- **(i):** $b \in B$ is called an interior point of B whenever there is $\theta \ll p$ such that $B_0(b, p) \subseteq B$, where $B_0(b, p) := \{y \in X : d(y, b) \ll p\}$.
- **(ii):** An element $x \in X$ is called a limit point of B whenever for every $\theta \ll e$, $B_0(x, e) \setminus (B \setminus \{x\}) \neq \emptyset$. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.
- **(iii):** A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A, that is, for any $a \in A$, there exists $c \in int\, P$ such that the open ball $B_0(a, c) \subseteq A$.
- **(iv):** A subset $B \subseteq X$ is called bounded whenever there exist $\theta \ll c$ and $x_0 \in X$ such that $d(b, x_0) \ll c$ for all $b \in B$.
- **(v):** A subset $B \subseteq X$ is called compact whenever every open cover of B has a finite subcover.

Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. Then the family of sets $\{B(x, c) : x \in X,\ \theta \ll c\}$ where $B(x, c) = \{y \in X : d(y, x) \ll c\}$ is a sub basis for a topology on X. This topology is denoted by τ_{cb}. It is to be noted that τ_{cb} is a Hausdorff topology. Suppose for each c with $\theta \ll c$, we have $B(x, c) \cap B(y, c) \neq \emptyset$. So, there exists $z \in X$ such that $d(z, x) \ll \frac{c}{2s}$ and $d(z, y) \ll \frac{c}{2s}$. Hence,

$$d(x, y) \preceq s(d(x, z) + d(z, y)) \ll c.$$

This implies that $d(x, y) = \theta$, that is, $x = y$.

Proposition 3.2. [8] *Let (X, d) be a cone b-metric space and τ_{cb} be the topology defined above. Then for any nonempty subset $A \subseteq X$ we have*
(i) A is closed if and only if for any sequence (x_n) in A which converges to x, we have $x \in A$;
(ii) If we define $\bar{A}$ to be the intersection of all closed subsets of X which contains A, then for any $x \in \bar{A}$ and for any $c \in int\, P$, we have $B_0(x, c) \cap A \neq \emptyset$.

Theorem 3.3. [8] *Let (X, d) be a cone b-metric space and τ_{cb} be the topology defined above. Then for any nonempty subset $A \subseteq X$, the following properties are equivalent:*
(i) A is compact.
(ii) For any sequence (x_n) in A, there exists a subsequence (x_{n_k}) of (x_n) which converges, and $\lim_{n\to\infty} x_n \in A$.

4. Main Results

In this section, we prove some point of coincidence and common fixed point results in cone b-metric spaces.

Theorem 4.1. *Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f, g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \preceq \lambda\, d(gx, gy) \tag{4.1}$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s})$ is a constant. If $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X.

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and choose $x_1 \in X$ such that $fx_0 = gx_1$. This is possible since $f(X) \subseteq g(X)$. Continuing this process, we can construct a sequence (x_n) in X such that $fx_n = gx_{n+1},\ n = 0, 1, 2, \cdots$.
By using (4.1), we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &\preceq \lambda d(gx_{n+1}, gx_n) \\ &= \lambda d(fx_n, fx_{n-1}) \\ &\preceq \lambda^2 d(gx_n, gx_{n-1}) \\ &= \lambda^2 d(fx_{n-1}, fx_{n-2}) \\ &\cdot \\ &\cdot \\ &\cdot \\ &\preceq \lambda^n d(fx_1, fx_0). \end{aligned} \tag{4.2}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (4.2) that

$$\begin{aligned} d(fx_n, fx_m) &\preceq s\left[d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_m)\right] \\ &\preceq sd(fx_n, fx_{n+1}) + s^2 d(fx_{n+1}, fx_{n+2}) + \cdots \\ &\quad + s^{m-n-1}\left[d(fx_{m-2}, fx_{m-1}) + d(fx_{m-1}, fx_m)\right] \\ &\preceq \left[s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n-1}\lambda^{m-1}\right] d(fx_0, fx_1) \\ &\preceq \left[s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}\right] d(fx_0, fx_1) \\ &= s\lambda^n\left[1 + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-2} + (s\lambda)^{m-n-1}\right] d(fx_0, fx_1) \\ &\preceq \frac{s\lambda^n}{1 - s\lambda}\, d(fx_0, fx_1). \end{aligned} \tag{4.3}$$

It is to be noted that $\frac{s\lambda^n}{1-s\lambda}\, d(fx_0, fx_1) \to \theta$ as $n \to \infty$. Let $\theta \ll c$ be given. Then we can find $m_0 \in \mathbb{N}$ such that

$$\frac{s\lambda^n}{1 - s\lambda}\, d(fx_0, fx_1) \ll c,$$

for each $n > m_0$.

Therefore, it follows from (4.3) that

$$d(fx_n, fx_m) \preceq \frac{s\lambda^n}{1 - s\lambda}\, d(fx_0, fx_1) \ll c$$

for all $m > n > m_0$.

So (fx_n) is a Cauchy sequence in $f(X)$. Suppose that $f(X)$ is a complete subspace of X. Then there exists $y \in f(X) \subseteq g(X)$ such that $fx_n \to y$ and also $gx_n \to y$. In case, $g(X)$ is complete, this holds also with $y \in g(X)$. Let $u \in X$ be such that $gu = y$. For $\theta \ll c$, one can choose a natural number $n_0 \in \mathbb{N}$ such that $d(y, fx_n) \ll \frac{c}{s(\lambda+1)}$ and $d(gx_n, gu) \ll \frac{c}{s(\lambda+1)}$ for all $n > n_0$.

Now,

$$\begin{aligned} d(y, fu) &\preceq s[d(y, fx_n) + d(fx_n, fu)] \\ &\preceq s[d(y, fx_n) + \lambda\, d(gx_n, gu)] \\ &\ll c,\ \textit{for all } n > n_0, \end{aligned}$$

which gives that $d(y, fu) = \theta$, i.e., $fu = y$ and hence $fu = gu = y$. Therefore, y is a point of coincidence of f and g.
For uniqueness, let v be another point of coincidence of f and g. So $fx = gx = v$ for some $x \in X$. Then

$$d(v, y) = d(fx, fu) \preceq \lambda\, d(gx, gu) = \lambda\, d(v, y).$$

By Remark 2.6(vii), we have $d(v, y) = \theta$ i.e., $v = y$.
Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.9, f and g have a unique common fixed point in X. □

The following Corollary is the Theorem 2.1[9].

Corollary 4.2. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \preceq \lambda\, d(x, y)$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s})$ is a constant. Then f has a unique fixed point in X. Furthermore, the iterative sequence $(f^n x)$ converges to the fixed point.

Proof. The proof follows from Theorem 4.1 by taking $g = I$, the identity mapping on X.

□

Corollary 4.3. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $g : X \to X$ is onto and satisfies*

$$d(gx, gy) \succeq k\, d(x, y)$$

for all $x, y \in X$, where $k > s$ is a constant. Then g has a unique fixed point in X.

Proof. Taking $f = I$ in Theorem 4.1, we obtain the desired result.

□

Remark 4.4. Corollary 4.3 gives a sufficient condition for the existence of unique fixed point of an expansive mapping in cone b-metric spaces.

Theorem 4.5. *Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f,\ g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \preceq a\, d(fx, gx) + b\, d(fy, gy) \tag{4.4}$$

for all $x, y \in X$, where $a,\ b \geq 0$ with $a + sb < 1$. If $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. As in Theorem 4.1, we can construct a sequence (x_n) in X such that $fx_n = gx_{n+1},\ n = 0,\ 1,\ 2,\ \cdots$.
Using (4.4), we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &\preceq a\, d(fx_{n+1}, gx_{n+1}) + b\, d(fx_n, gx_n) \\ &= a\, d(fx_{n+1}, fx_n) + b\, d(fx_n, fx_{n-1}) \end{aligned}$$

which implies that

$$d(fx_{n+1}, fx_n) \preceq k\, d(fx_n, fx_{n-1}) \tag{4.5}$$

where $k = \frac{b}{1-a}$. It is easy to see that $0 \leq k < \frac{1}{s}$.
By repeated application of (4.5), we obtain

$$d(fx_{n+1}, fx_n) \preceq kd(fx_n, fx_{n-1}) \preceq k^2 d(fx_{n-1}, fx_{n-2}) \preceq \cdots \preceq k^n d(fx_1, fx_0).$$

By an argument similar to that used in Theorem 4.1, it follows that (fx_n) is a Cauchy sequence in $f(X)$. If $f(X)$ is a complete subspace of X, then there exists $y \in f(X) \subseteq g(X)$ such that $fx_n \to y$ and also $gx_n \to y$. In case, $g(X)$ is complete, this holds also with $y \in g(X)$. Let $u \in X$ be such that $gu = y$. For $\theta \ll c$, one can choose a natural number $n_0 \in \mathbb{N}$ such that $d(y, fx_n) \ll \frac{1-bs}{2(s+as^2)}c$ and $d(gx_n, gu) \ll \frac{1-bs}{2as^2}c$ for all $n > n_0$.

Now,

$$\begin{aligned} d(y, fu) &\preceq s[d(y, fx_n) + d(fx_n, fu)] \\ &\preceq s[d(y, fx_n) + a\, d(fx_n, gx_n) + b\, d(fu, gu)] \\ &\preceq s[d(y, fx_n) + as\, d(fx_n, y) + as\, d(y, gx_n) + b\, d(fu, y)]. \end{aligned}$$

So it must be the case that

$$(1 - bs)d(y, fu) \preceq (s + as^2)d(y, fx_n) + as^2 d(y, gx_n). \tag{4.6}$$

Therefore, we obtain from condition (4.6) that

$$\begin{aligned} d(y, fu) &\preceq \frac{s + as^2}{1 - bs} d(y, fx_n) + \frac{as^2}{1 - bs} d(gu, gx_n) \\ &\ll c \text{ for all } n > n_0. \end{aligned} \tag{4.7}$$

This implies that $d(y, fu) = \theta$, i.e., $fu = y$ and hence $fu = gu = y$. Therefore, y is a point of coincidence of f and g.

For uniqueness, let v be another point of coincidence of f and g. So $fx = gx = v$ for some $x \in X$. Then

$$d(v, y) = d(fx, fu) \preceq a\, d(fx, gx) + b\, d(fu, gu) = \theta.$$

By Remark 2.6(vii), we have $d(v, y) = \theta$ i.e., $v = y$.
Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.9, f and g have a unique common fixed point in X. □

Corollary 4.6. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \preceq a\, d(fx, x) + b\, d(fy, y)$$

for all $x, y \in X$, where $a,\ b \geq 0$ with $a + sb < 1$. Then f has a unique fixed point in X.

Proof. Proof follows from Theorem 4.5 by taking $g = I$.

□

Theorem 4.7. *Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f,\ g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \preceq a\, d(fx, gy) + b\, d(fy, gx) \tag{4.8}$$

for all $x, y \in X$, where $a,\ b \geq 0$ with $max\{a,\ b\} < \frac{1}{s^2+s}$. If $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of *coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary. As in Theorem 4.1, we can construct a sequence (x_n) in X such that $fx_n = gx_{n+1},\ n = 0, 1, 2, \cdots$.
Using (4.8), we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &\preceq a\, d(fx_{n+1}, gx_n) + b\, d(fx_n, gx_{n+1}) \\ &= a\, d(fx_{n+1}, fx_{n-1}) \\ &\preceq as[d(fx_{n+1}, fx_n) + d(fx_n, fx_{n-1})]. \end{aligned}$$

This implies that

$$d(fx_{n+1}, fx_n) \preceq \frac{as}{1-as}\, d(fx_n, fx_{n-1}). \tag{4.9}$$

Therefore, we obtain from condition (4.9) that

$$d(fx_{n+1}, fx_n) \preceq k\, d(fx_n, fx_{n-1}) \tag{4.10}$$

where $k = \frac{as}{1-as}$. It is easy to see that $0 \leq k < \frac{1}{s}$.
By repeated application of (4.10), we obtain

$$d(fx_{n+1}, fx_n) \preceq kd(fx_n, fx_{n-1}) \preceq k^2 d(fx_{n-1}, fx_{n-2}) \preceq \cdots \preceq k^n d(fx_1, fx_0).$$

By an argument similar to that used in Theorem 4.1, it follows that (fx_n) is a Cauchy sequence in $f(X)$. If $f(X)$ is a complete subspace of X, then there exists $y \in f(X) \subseteq g(X)$ such that $fx_n \to y$ and also $gx_n \to y$. In case, $g(X)$ is complete, this holds also with $y \in g(X)$. Let $u \in X$ be such that $gu = y$. For $\theta \ll c$, one can choose a natural number $n_0 \in \mathbb{N}$ such that $d(y, fx_n) \ll \frac{1-bs^2}{2(s+as)}c$ and $d(gx_n, gu) \ll \frac{1-bs^2}{2bs^2}c$ for all $n > n_0$.

Now,

$$\begin{aligned} d(y, fu) &\preceq s[d(y, fx_n) + d(fx_n, fu)] \\ &\preceq s[d(y, fx_n) + a\, d(fx_n, gu) + b\, d(fu, gx_n)] \\ &\preceq s[d(y, fx_n) + a\, d(fx_n, y) + bs\, d(y, gx_n) + bs\, d(fu, y)]. \end{aligned}$$

So it must be the case that

$$(1 - bs^2)d(y, fu) \preceq (s + as)d(y, fx_n) + bs^2 d(y, gx_n). \tag{4.11}$$

Therefore, we obtain from condition (4.11) that

$$\begin{aligned} d(y, fu) &\preceq \frac{s+as}{1-bs^2} d(y, fx_n) + \frac{bs^2}{1-bs^2} d(gu, gx_n) \\ &\ll c \text{ for all } n > n_0. \end{aligned}$$

This implies that $d(y, fu) = \theta$, i.e., $fu = y$ and hence $fu = gu = y$. Therefore, y is a point of coincidence of f and g.

For uniqueness, let v be another point of coincidence of f and g. So $fx = gx = v$ for some $x \in X$. Then

$$\begin{aligned} d(v, y) = d(fx, fu) &\preceq a\, d(fx, gu) + b\, d(fu, gx) \\ &= a\, d(v, y) + b\, d(y, v) \\ &= (a + b)\, d(v, y). \end{aligned}$$

Since $a+b<1$, by Remark 2.6(vii), we have $d(v,y)=\theta$ i.e., $v=y$.
Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.9, f and g have a unique common fixed point in X. □

Corollary 4.8. *Let (X,d) be a complete cone b-metric space with the coefficient $s\geq 1$. Suppose the mapping $f:X\to X$ satisfies the contractive condition*

$$d(fx,fy)\preceq a\,d(fx,y)+b\,d(fy,x)$$

for all $x,y\in X$, where $a,\ b\geq 0$ with $max\{a,\ b\}<\frac{1}{s^2+s}$. Then f has a unique fixed point in X.

Proof. Proof follows from Theorem 4.7 by taking $g=I$. □

We conclude with some examples.

Example 4.9. Let $E=\mathbb{R}^2$, the Euclidean plane and $P=\{(x,y)\in\mathbb{R}^2 : x,y\geq 0\}$ a cone in E. Let $X=[0,1]$ and $p>1$ be a constant. We define $d:X\times X\to E$ as

$$d(x,y)=(\mid x-y\mid^p,\mid x-y\mid^p)$$

for all $x,y\in X$. Then (X,d) is a cone b-metric space with the coefficient $s=2^{p-1}$. Let us define $f,g:X\to X$ as

$$fx=\frac{x}{4}-\frac{x^2}{8},\ for\ all\ x\in X$$

and

$$gx=\frac{x}{2},\ for\ all\ x\in X.$$

Then, for every $x,y\in X$ one has

$$\begin{aligned}
d(fx,fy) &= (\mid fx-fy\mid^p,\mid fx-fy\mid^p)\\
&= \left(\mid \frac{1}{4}(x-y)-\frac{1}{8}(x-y)(x+y)\mid^p,\mid \frac{1}{4}(x-y)-\frac{1}{8}(x-y)(x+y)\mid^p\right)\\
&= \left(\mid \frac{x}{2}-\frac{y}{2}\mid^p.\mid \frac{1}{2}-\frac{1}{4}(x+y)\mid^p,\mid \frac{x}{2}-\frac{y}{2}\mid^p.\mid \frac{1}{2}-\frac{1}{4}(x+y)\mid^p\right)\\
&\preceq \frac{1}{2^p}\left(\mid \frac{x}{2}-\frac{y}{2}\mid^p,\mid \frac{x}{2}-\frac{y}{2}\mid^p\right)\\
&= \frac{1}{2^p}\,d(gx,gy).
\end{aligned}$$

Thus, we have all the conditions of Theorem 4.1 and $0\in X$ is the unique common fixed point of f and g.

Example 4.10. Let $E=\mathbb{R}^2$, the Euclidean plane and $P=\{(x,y)\in\mathbb{R}^2 : x,y\geq 0\}$ a cone in E. Let $X=[0,1]$ and $p>1$ be a constant. We define $d:X\times X\to E$ as

$$d(x,y)=(\mid x-y\mid^p,\mid x-y\mid^p)$$

for all $x,y\in X$. Then (X,d) is a cone b-metric space with the coefficient $s=2^{p-1}$. Let us define $f,g:X\to X$ as

$$\begin{aligned}
fx &= \frac{x}{16},\ for\ all\ x\in[0,\frac{1}{2})\\
&= \frac{x}{12},\ for\ all\ x\in[\frac{1}{2},1]
\end{aligned}$$

and

$$gx = \frac{x}{2},\ for\ all\ x \in X.$$

Now we verify that for every $x, y \in X$ one has

$$d(fx, fy) \preceq a\, d(fx, gx) + b\, d(fy, gy)$$

where $a,\ b \geq 0$ with $a + sb < 1$.

Case-I If $x, y \in [0, \frac{1}{2})$, then

$$\begin{aligned} d(fx, fy) &= (\mid fx - fy \mid^p, \mid fx - fy \mid^p) \\ &= \frac{1}{16^p}\, (\mid x - y \mid^p, \mid x - y \mid^p) \\ &\preceq \frac{1}{16^p}\, ((\mid x \mid + \mid y \mid)^p, (\mid x \mid + \mid y \mid)^p) \\ &\preceq \frac{2^p}{16^p}\, (x^p + y^p, x^p + y^p) \\ &= \frac{2^p}{7^p} . \frac{7^p}{16^p}\, (x^p + y^p, x^p + y^p)\, . \end{aligned}$$

Also,

$$\begin{aligned} d(fx, gx) + d(fy, gy) &= (\mid fx - gx \mid^p, \mid fx - gx \mid^p) + (\mid fy - gy \mid^p, \mid fy - gy \mid^p) \\ &= \left(\mid \frac{x}{16} - \frac{x}{2} \mid^p, \mid \frac{x}{16} - \frac{x}{2} \mid^p\right) + \left(\mid \frac{y}{16} - \frac{y}{2} \mid^p, \mid \frac{y}{16} - \frac{y}{2} \mid^p\right) \\ &= \frac{7^p}{16^p}\, (x^p + y^p, x^p + y^p)\, . \end{aligned}$$

Therefore,

$$d(fx, fy) \preceq \frac{2^p}{7^p}\, [d(fx, gx) + d(fy, gy)] \preceq \frac{2^p}{5^p}\, [d(fx, gx) + d(fy, gy)].$$

Case-II If $x, y \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} d(fx, fy) &\preceq \frac{2^p}{12^p}\, (x^p + y^p, x^p + y^p) \\ &= \frac{2^p}{5^p} . \frac{5^p}{12^p}\, (x^p + y^p, x^p + y^p)\, . \end{aligned}$$

and,

$$d(fx, gx) + d(fy, gy) = \frac{5^p}{12^p}\, (x^p + y^p, x^p + y^p)\, .$$

Therefore,

$$d(fx, fy) \preceq \frac{2^p}{5^p}\, [d(fx, gx) + d(fy, gy)].$$

Case-III If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} d(fx, fy) &= (\mid fx - fy \mid^p, \mid fx - fy \mid^p) \\ &= \left(\mid \frac{x}{16} - \frac{y}{12} \mid^p, \mid \frac{x}{16} - \frac{y}{12} \mid^p\right) \\ &\preceq 2^p \left((\frac{x}{16})^p + (\frac{y}{12})^p, (\frac{x}{16})^p + (\frac{y}{12})^p\right) \\ &= \frac{2^p}{5^p} \left((\frac{5x}{16})^p + (\frac{5y}{12})^p, (\frac{5x}{16})^p + (\frac{5y}{12})^p\right) \\ &\preceq \frac{2^p}{5^p} \left((\frac{7x}{16})^p + (\frac{5y}{12})^p, (\frac{7x}{16})^p + (\frac{5y}{12})^p\right). \end{aligned}$$

Also,

$$\begin{aligned} d(fx,gx)+d(fy,gy) &= (\mid fx-gx\mid^p, \mid fx-gx\mid^p)+(\mid fy-gy\mid^p, \mid fy-gy\mid^p) \\ &= \Big(\mid \frac{x}{16}-\frac{x}{2}\mid^p, \mid \frac{x}{16}-\frac{x}{2}\mid^p\Big)+\Big(\mid \frac{y}{12}-\frac{y}{2}\mid^p, \mid \frac{y}{12}-\frac{y}{2}\mid^p\Big) \\ &= \Bigg((\frac{7x}{16})^p+(\frac{5y}{12})^p, (\frac{7x}{16})^p+(\frac{5y}{12})^p\Bigg). \end{aligned}$$

Therefore,

$$d(fx,fy)\preceq \frac{2^p}{5^p}\,[d(fx,gx)+d(fy,gy)].$$

Thus, we have

$$d(fx,fy)\preceq \frac{2^p}{5^p}\,[d(fx,gx)+d(fy,gy)]$$

for all $x,y\in X$, where $a+sb=(1+s)\frac{2^p}{5^p}\le 2s.\frac{2^p}{5^p}=2^p.\frac{2^p}{5^p}=\frac{4^p}{5^p}<1$ since $s=2^{p-1}$. We see that $f(X)\subseteq g(X)$, $g(X)$ is complete, f and g are weakly compatible. Therefore, all the conditions of Theorem 4.5 are satisfied and $0\in X$ is the unique common fixed point of f and g.

Example 4.11. Let $X=\{1,2,3\}$, $E=\mathbb{R}^2$, $P=\{(x,y): x\ge 0, y\ge 0\}$. Define $d: X\times X\to P$ by $d(x,y)=d(y,x)$ for all $x,y\in X$, $d(x,x)=\theta$, $x\in X$ and $d(1,2)=(8,8)$, $d(2,3)=d(1,3)=(2,2)$. We observe that

$$d(1,2)=(8,8)\not\preceq d(1,3)+d(3,2)=(2,2)+(2,2)=(4,4).$$

This shows that the triangle inequality does not hold true and so (X,d) is not a cone metric space. It is easy to verify that (X,d) is a cone b-metric space with the coefficient $s=2$. Let us define $f,g: X\to X$ as

$$fx=3,\ for\ all\ x\in X$$

and

$$\begin{aligned} gx &= 3,\ for\ x\in\{1,3\} \\ &= 1,\ for\ x=2. \end{aligned}$$

Then for every $x,y\in X$ one has

$$d(fx,fy)\preceq a\,d(fx,gy)+b\,d(fy,gx)$$

for all $a,\ b\ge 0$.

We see that $f(X)\subseteq g(X)$, $f(X)$ is complete, f and g are weakly compatible. Therefore, all the conditions of Theorem 4.7 are satisfied and $3\in X$ is the unique common fixed point of f and g.

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References

1. M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008), 416 – 420.
2. I.A.Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal.,Gos. Ped. Inst. Unianowsk 30 (1989), 26 – 37.
3. I. Beg and M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, Fixed Point Theory and Applications 2006 (2006), Article ID 74503, 7 pages.

4. S.Czerwik, Nonlinear set valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fiz. Univ. Modena 46 (1998), 263 – 276.
5. Z.M.Fadail and A.G.B.Ahmad, Coupled coincidence point and common coupled fixed point results in cone b-metric spaces, Fixed Point Theory and Applications 2013 (2013), 2013:177.
6. L.-G.Huang, X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468 – 1476.
7. O.Hadzic, Common fixed point theorem for family of mappings in complete metric spaces, Math. Japon 29 (1984), 127 – 134.
8. N.Hussain and M.H.Shah, KKM mappings in cone b-metric spaces, comput. Math. Appl. 62 (2011), 1677 – 1684.
9. H.Huang and S.Xu, Fixed point theorems of contractive mappings in cone b-metric spaces and applications, Fixed Point Theory and Applications 2013 (2013), 2013:112.
10. D.Ilić, V.Rakočević, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. 341 (2008), 876 – 882.
11. G. Jungck, Common fixed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci. 4 (1996), 199 – 215.
12. J. O. Olaleru, Some generalizations of fixed point theorems in cone metric spaces, Fixed Point Theory and Applications 2009 (2009), Article ID 657914.
13. H.K.Pathak, Weak commuting mappings and fixed points, Ind. J. Pure Appl. Math. 17 (1986), 201 – 211.
14. B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Analysis: Theory, Methods and Applications 47 (2001), 2683 – 2693.
15. S.Rezapour, R.Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", J. Math. Anal. Appl. 345 (2008), 719 – 724.
16. S.Rezapour, M.Derafshpour and R.Hamlbarani, A review on topological properties of cone metric spaces, in Proceedings of the Conference on Analysis, Topology and Applications (ATA'08), Vrnjacka Banja, Serbia, May-June 2008.
17. S.P.Singh and B.A.Meade, On common fixed point theorem, Bull. Aus. Math. Soc. 16 (1977), 49 – 53.
18. P. Vetro, Common fixed points in cone metric spaces, Rend. Circ. Mat. Palermo 56 (2007), 464 – 468.

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ON THE SOLVABILITY OF GENERALIZED SET-VALUED EQUILIBRIUM PROBLEMS

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ABSTRACT. In this present article, we consider the generalized equilibria accompanied with certain multi-objective multifunctions. This class unifies numbers of problems surfaced in the area of optimization including the well-known mixed equilibrium problems. Some theorems are adopted, under the conventional assumptions of continuity, convexity, and coercivity on the objective functions, guaranteeing that these functions enjoy the existence of such equilibria. The consequences of this generalized class are of course studied and presented as well.

KEYWORDS : Generalized equilibrium, multiobjective optimization, set-valued optimization
AMS Subject Classification:

1. INTRODUCTION

The theory of optimization has always been an important subject as it evolved through its history. Amongst the developments in this area, the formulation of an equilibrium problem is one of the most prominent and promising advance which provides a unification to the classical approaches of variational inequalities, fixed point theory, saddle point theory, and several more optimization problems. Moreover, it has been a wealthy source for solving the problems frequently appeared in economics, management science, engineering, etc. The term equilibrium problem was coined by Blum and Oettli [1]. However, it was first discussed decades before by Fan [2, 3], under the influences of the minimax problems in economics. The study of equilibrium theory has then been a heavily investigated area, and it has been extended in various aspects and directions (see e.g., [4–9]).

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After the blossom of the scalar optimization theory, the vector cases were subsequently introduced. These extensions are admired for their capabilities implanted in multi-objective optimization programming. The initiation of the vector equilibria is motivated by the foundation of the vector variational inequalities introduced by Giannessi [10] in finite-dimensional Euclidean spaces. The enhanced versions where the objective functions take their values in arbitrary topological vector spaces were also inaugurated in [11]. As for the equilibrium problems, the vectorial variants were also conprehensively studied in [12–17].

On the other hand, the combinations between two kinds of optimization problems were introduced and studied. Typical important problems in this area are the mixed variational inequalities, which is notable for their distinguishing applications in engineering (see e.g., [18–26]). Due to the impact and utilities of mixed variational inequalities, several notions of mixed equilibria are then adopted in [27–29]. Although lots of numerical and computational techniques have been invented to approximate the solutions of mixed equilibrium problems (for examples, [30–37]), only a relatively small amount of qualitative results are known.

To enrich the theory of mixed equilibrium problems, we contemplate a class of generalized set-valued equilibrium problems, where various classes of the mixed equilibrium problems are determined to be embedded. This new class examines the situation where the objective functions are vectorial and multivalued in fashion. A qualitative study providing some sufficient conditions which guarantee the solvability of this class, including its significant consequences, is also conducted and employed.

The paper is organized in the following way: In section 2, we give a recollection of some background definitions and properties which are useful in our main results, and also introduce the class of generalized equilibrium problem we are interested to study. In section 3, we state and prove our main results providing the validity conditions for the problems suggested in preceding section. The consequential remarks are afterward given and studied.

2. Preliminaries

Suppose that E is a topological vector space and $Y \subset E$ being nonempty. A problem of finding a point $\widehat{x} \in Y$ with

$$f(\widehat{x}, y) \geq 0, \quad \forall y \in Y,$$

where $f : Y \times Y \to \mathbb{R}$, seems to be a fertile area for mathematicians over the past years. This problem is known today as the equilibrium problem and the point $\widehat{x}$ is referred to as an equilibrium. The very first existence theorems for solutions to this problem were introduced in [1, 3, 38] through the usages of continuity, convexity, monotonicity and compactness.

To realize the vectorial formulation, recall that a nonempty subset $C \subset E$ is said to be a cone if $\lambda C \subset C$ for $\lambda \geq 0$. If $C + C \subset C$ holds, then the cone C is called convex. By C°, we means the interiors of C. There should be no ambiguity to denote the zero element of any involved vector space by θ. A cone C is said to be pointed if $C \cap -C = \{\theta\}$ and is said to be solid if $C^\circ \neq \emptyset$.

With the perception of cones, we can define two partial ordering $\preceq$ and $\ll$ on E by

$$\begin{cases} x \preceq y \iff y - x \in C, \\ x \ll y \iff y - x \in C^\circ. \end{cases}$$

For instance, let $f : Y \times Y \to L$ a function valued in another topological vector space L with a cone $C \subset L$. A point $\widehat{x} \in Y$ is said to be a weak equilibrium (equilibrium) of f if

$$f(\widehat{x}, y) \notin -C^\circ \ (-C \setminus \{\theta\}), \quad \forall y \in Y.$$

Under many circumstances, a single action might bring more than one feasible outcomes at a time. This is where the powerful concept of multifunctions gets in. By the term multifunction, we shall refer to the function $F : A \to 2^B$ with nonempty values (it is to be understood that A, B are any nonempty sets). In this context, we shall write $F : A \rightrightarrows B$ instead. We are now consider the multi-objective multifunctions, which will be used mainly in this work: Suppose now that $F, G : Y \times Y \rightrightarrows L$. The point $\widehat{x} \in Y$ such that

$$F(y, \widehat{x}) - G(\widehat{x}, y) \not\subset C^\circ \ (C \setminus \{\theta\}), \quad \forall y \in Y \qquad \text{(GEP)}$$

is called a generalized weak equilibrium (generalized equilibrum) for F and G. This class of problem contains many important special cases as one shall see as we proceed further.

On the contrary, the KKM theory has been an astonishing area as it provides a key tool in nonlinear analysis and optimization (see e.g. [7, 15, 39–41]). Recall that a multifunction $T : Y \rightrightarrows L$ is said to be a KKM if

$$\operatorname{co}(A) \subset \bigcup_{y \in A} T(y),$$

for all $A \in \langle Y \rangle$, where $\operatorname{co}(A)$ denotes the convex hull of A and $\langle A \rangle$ denotes the family of all finite subsets of Y. The celebrated lemma of Fan [2] asserts that if a KKM multifunction possesses a compact value at some point $x_0 \in Y$, then $\bigcap_{x \in Y} T(x) \neq \emptyset$.

A replacement for the above compactness of $T(x_0)$ is a favor for many nonlinear analysts. In [42], the following coercivity conditions were introduced:

Definition 2.1 ([42]). Suppose that $T : Y \rightrightarrows L$ a multifunction. A family $\{(C_i, K_i)\}_{i \in I}$ is said to be coercing for T if the following properties are satisfied:

(C1) for each $i \in I$, $C_i \subset K$ for some compact convex set $K \subset Y$ and $K_i \subset L$ is compact;

(C2) for each $i, j \in I$, there exists $k \in I$ such that

$$C_i \cup C_j \subset C_k;$$

(C3) for each $i \in I$, there exists $k \in I$ with

$$\bigcap_{x \in C_k} T(x) \subset K_i.$$

The following KKM principle was subsequently proposed, which shows the above coercing conditions successfully overcome the necessity of the compactness of $T(x_0)$.

Lemma 2.2 ([42]). Let E be a topological vector space, $K \subset E$ be nonempty and convex, and $X \subset K$ be nonempty. Suppose that $T : X \rightrightarrows K$ is a KKM multifunction with compactly closed values (w.r.t. Y) at each $x \in X$. If T admits a coercing family, then $\bigcap_{x \in X} T(x) \neq \emptyset$.

Most existence results for any general class of an equilibrium imposed some kinds of continuity, monotonicity, convexity, and coercivity upon the objective (multi)functions. The rest of this section is thus devoted to recall some background definitions and properties which will be applied to the multifunctions F and G in (GEP).

Definition 2.3. A multifunction $T : Y \rightrightarrows L$ is said to be C-lower semi-continuous if for any point $y \in Y$ and any neighbourhood V of $T(y)$, there exists a neighbourhood $U \subset Y$ of y such that $T(U) \subset V + C$. Moreover, T is said to be C-upper semi-continuous if $-T$ is C-lower semi-continuous.

It is proved in [43] that the following conditions characterize each others:

(i) T is C-lower semi-continuous;

(ii) for each $e \in L$, the set
$$T^{-}(e + C^{\circ}) \overset{\textbf{def}}{=} \{x \in E\,;\, T(x) \cap (e + C^{\circ}) \neq \emptyset\}$$
is open;

(iii) for each $x \in E$ and each $e \in C^{\circ}$, there exists a neighbourhood U of x such that
$$T(U) \subset T(x) - e + C^{\circ}.$$

Definition 2.4. A multifunction $F : Y \times Y \rightrightarrows E$ is said to be C-monotone if
$$F(x, y) + F(y, x) \subset -C, \quad \text{for all } x, y \in Y.$$

Definition 2.5. Suppose that K is nonempty and convex, a multifunction $F : K \rightrightarrows L$ is said to be C-convex if
$$F\left(\sum_{i=1}^{n} \lambda_i x_i\right) \subset \sum_{i=1}^{n} \lambda_i F(x_i) - C.$$
where for each $i \in \{1, 2, \cdots, n\}$, $x_i \in Y$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. In addition, F is C-concave if $-F$ is C-convex.

3. Main results

Here, we shall consider the conditions under which the solvability is possible for the problem (GEP). The section is organized into the sequence of problems known throughout the literature in terms the generalized vector equilibria. Note that some problems can be seen explicitly, while some are not.

3.1. Generalized equilibria.

Theorem 3.1. Let E, L be two topological vector spaces, $K \subset E$ a nonempty closed convex set and $C \subset L$ a pointed closed convex cone. Suppose that $F, G : K \times K \rightrightarrows L$ are two multifunctions with the following properties:

(H1) F is C-monotone;

(H2) $\theta \in F(x, x) \cap G(x, x)$ for all $x \in K$;

(H3) $F(x, \cdot)$ is C-lower semi-continuous and $F(\cdot, y)$ is C-concave;

(H4) $G(\cdot, y)$ is C-upper semi-continuous and $G(x, \cdot)$ is C-convex;

(H5) there exists a collection $\{(C_i, K_i)\}_{i \in I}$ satisfying (C1), (C2) and for each $i \in I$, there exists $k \in I$ with $\{x \in K\,;\, F(y, x) - G(x, y) \not\subset C^{\circ}, \forall y \in C_k\} \subset K_i$.

Then, F and G possess at least one generalized weak equilibrium.

As for the proof, we shall consider this theorem through the following sequence of lemmas.

Lemma 3.2. The multifunction
$$H(y) \overset{\textbf{def}}{=} \{x \in K\,;\, F(y, x) - G(x, y) \not\subset C^{\circ}\}$$
has closed values at each $y \in K$.

Proof. Suppose that $y \in K$ and $(x_n) \subset H(y)$ with $x_n \longrightarrow x$. Assume that $x \notin H(y)$. Thus, it follows from (H3) and (H4) that

$$\begin{aligned} F(y, x_n) - G(x_n, y) &\subset F(y, x) - G(x, y) - 2d + C^\circ \\ &\subset -2d + C^\circ, \end{aligned}$$

for all $d \in C^\circ$. For each $m \in \mathbb{N}$, $\frac{1}{m}d \in C^\circ$. Hence, $F(y, x_n) - G(x_n, y) \subset \bigcap_{m\in\mathbb{N}}(-\frac{2}{m}d + C^\circ) = C^\circ$. This yields a contradiction. □ □

Lemma 3.3. *H is a KKM multifunction.*

Proof. Let $A \overset{\textbf{def}}{=} \{y_j\,;\, j \in J\} \in \langle K\rangle$ and $z \in \mathrm{co}(A)$. Thus, z can be expressed by $z = \sum_{j\in J} \lambda_j y_j$ with $\lambda_j \geq 0$ and $\sum_{j\in J} \lambda_j = 1$. Assume that H is not KKM so that $z \notin \bigcup_{j\in J} H(y_j)$. It means that

$$\bigcap_{j\in J} (F(y_j, z) - G(z, y_j)) \subset C^\circ.$$

From (H3) and (H4), we may deduce that

$$\begin{cases} \theta \in F(z, z) \subset \sum_{k\in J} \lambda_k F(y_k, z) + C, \\ \theta \in G(z, z) \subset \sum_{k\in J} \lambda_k G(z, y_k) - C. \end{cases}$$

We subsequently have

$$\theta \in F(z, z) - G(z, z) \subset \sum_{k\in J} [F(y_k, z) - G(z, y_k)] + C \subset C^\circ + C \subset C^\circ,$$

which leads to a contradiction (otherwise the cone cannot be pointed). □ □

Lemma 3.4. *For each $i \in I$, we can find $k \in I$ with*

$$\bigcap_{y\in C_k} H(y) \subset K_i.$$

Proof. The desired result follows immediately from (H5). □ □

With the lemmas above, we may obtain a simple proof of Theorem 3.1.

of Theorem 3.1. Since E is Hausdorff, Lemma 3.2 implies that H has compactly closed values for each $y \in K$. Now, from Lemmas 3.3, 3.4 and 2.2, resp., we have

$$\bigcap_{y\in K} H(y) \neq \emptyset.$$

Take any $\widehat{x} \in \bigcap_{y\in K} H(y)$, it follows directly that $\widehat{x}$ is a generalized weak equilibrium for F and G. □ □

As initial consequences, we might consider the following corollaries.

Corollary 3.5. *Let E, L be two topological vector spaces, $K \subset E$ a nonempty closed convex set and $C \subset L$ a pointed closed convex cone. Suppose that $G : K \times K \rightrightarrows L$ is a multifunction with the following properties:*

1. *$\theta \in G(x, x)$ for all $x \in K$;*
2. *$G(\cdot, y)$ is C-upper semi-continuous and $G(x, \cdot)$ is C-convex;*
3. *there exists a collection $\{(C_i, K_i)\}_{i\in I}$ satisfying (C1), (C2) and for each $i \in I$, there exists $k \in I$ with*

$$\{x \in K\,;\, G(x, y) \not\subset -C^\circ,\, \forall y \in C_k\} \subset K_i.$$

Then, there exists a point $\widehat{x} \in K$ with

$$G(\widehat{x}, y) \not\subset -C^\circ, \quad \text{for all } y \in Y.$$

Proof. Consider Theorem 3.1 as $F = \theta$. □ □

Corollary 3.6. Let E, L be two topological vector spaces, $K \subset E$ a nonempty closed convex set and $C \subset L$ a pointed closed convex cone. Suppose that the multifunction $H : K \times K \rightrightarrows L$ possesses the following properties:

(1) $H(x, \cdot)$ is C-upper semi-continuous and C-concave;
(2) and $H(\cdot, y)$ is C-lower semi-continuous and C-convex;
(3) there exists a collection $\{(C_i, K_i)\}_{i \in I}$ satisfying (C1), (C2) and for each $i \in I$, there exists $k \in I$ with

$$\{x \in K\,;\, H(y, x) - H(x, y) \not\subset -C^\circ,\, \forall y \in C_k\} \subset K_i.$$

Then, there exists a point $\widehat{x} \in K$ with

$$H(y, \widehat{x}) - H(\widehat{x}, y) \not\subset -C^\circ, \quad \text{for all } y \in Y.$$

Proof. Define a multifunction

$$G(x, y) \overset{\textbf{def}}{=} H(y, x) - H(x, y).$$

Then, $\theta \in G(x, x)$ for all $x \in K$, $G(\cdot, y)$ is C-upper semi-continuous and $G(x, \cdot)$ is C-convex. Apply Corollary 3.5 to complete the proof. □ □

3.2. Strong solutions. According to Theorem 3.1, we shall give a supplementary assumption to assure the existence of a generalized weak equilibrium.

Corollary 3.7. In addition to Theorem 3.1, if there exists a pointed closed convex cone $\widetilde{C} \subset L$ with $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, then F and G possess at least one 2-equilibrium.

Proof. With this assumption, we can replace the cone C in Theorem 3.1 with $\widetilde{C}$ and still obtain the result that

$$F(y, x) - G(x, y) \not\subset \widetilde{C}^\circ.$$

Since $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, we have

$$F(y, x) - G(x, y) \not\subset C \setminus \{\theta\}. \qquad \square$$

□

Corollary 3.8. In addition to Theorem 3.5, if there exists a pointed closed convex cone $\widetilde{C} \subset L$ with $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, then F and G possesses at least one generalized equilibrium.

Proof. As in the previous corollary, set $F = \theta$. □ □

3.3. Saddle points. We have mentioned in the earlier section the problem of finding a saddle point. For instance, let E_1, E_2, L be three topological vector spaces and K_1, K_2 be two nonempty closed convex subsets of E_1 and E_2, respectively. Suppose that $C \subset L$ is a pointed closed convex cone. Now, consider the multifunction $F : K_1 \times K_2 \rightrightarrows L$. A point $(\overline{x}_1, \overline{x}_2) \in K_1 \times K_2$ is called a weak saddle point (strong saddle point) if

$$F(y_1, \overline{x}_2) - F(\overline{x}_1, y_2) \not\subset -C^\circ(-C \setminus \{\theta\}), \quad \text{for all } (y_1, y_2) \in K_1 \times K_2.$$

Theorem 3.9. Let E_1, E_2, L be three topological vector spaces, K_1, K_2 two nonempty closed convex subsets of E_1, E_2, resp., $C \subset L$ a pointed closed convex cone. Suppose that $F : K_1 \times K_2 \rightrightarrows L$ is a multifunction with the following properties:

(i) $F(x, \cdot)$ is C-upper semi-continuous and C-concave;
(ii) $F(\cdot, y)$ is C-lower semi-continuous and C-convex;
(iii) there exists a collection $\{(C_i, K_i)\}_{i\in I}$ satisfying (C1), (C2) and for each $i \in I$, there exists $k \in I$ with

$$\{x \in K\, ;\, F(y_1, x_2) - F(x_1, y_2) \not\subset -C^\circ,\, \forall (y_1, y_2) \in C_k\} \subset K_i.$$

Then, F possesses at least one (K_1, K_2)-weak saddle point.

Proof. Set $K \overset{\textbf{def}}{=} K_1 \times K_2$. We may see that K is closed and convex. Consider the multifunction

$$G(x, y) \overset{\textbf{def}}{=} F(y_1, x_2) - F(x_1, y_2).$$

It is easy to verify that $\theta \in G(x, x)$ for all $x \in K_1 \times K_2$, $G(x, \cdot)$ is C-convex and $G(\cdot, y)$ is C-upper semi-continuous. By Corollary 3.5, G has a weak equilibrium $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \in K$ which is in turn a weak equilibrium of G and is in turn a (K_1, K_2)-weak saddle point of F. □ □

Corollary 3.10. In addition to Theorem 3.9, if there exists a pointed closed convex cone $\widetilde{C} \subset L$ with $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, then F possesses at least one saddle point.

Proof. Combine the proofs of Corollary 3.8 and Theorem 3.9. □ □

3.4. Non-cooperative game equilibrium. Let $I = \{1, 2, \cdots, n\}$ $(n \in \mathbb{N})$ denotes the set of players. To each player $i \in I$, we assign a set K_i of strategies with K_i being nonempty, closed and convex in some topological vector space E_i. Suppose that L is a topological vector space with a pointed closed convex cone C. A loss multifunction for each player i is the function $F_i : K = \prod_{i\in I} K_i \rightrightarrows L$. For $x = (x_i)_{i\in I} \in K$, we write $x^{-i}|y_i \overset{\textbf{def}}{=} (x_1, x_2, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)$, where $y_i \in K_i$.

A point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in K$ is said to be a weak non-cooperative game equilibrium (strong non-cooperative game equilibrium) if for each $i \in I$,

$$F_i(\widehat{x}^{-i}|y_i) - F_i(\widehat{x}) \not\subset -C^\circ, \quad \text{for all } y = (y_i)_{i\in I} \in K.$$

Theorem 3.11. Let $(E_i)_{i\in I}$ be a sequence of topological vector spaces, (K_i) a sequence of nonempty closed convex sets with $K_i \subset E_i$. Suppose that for each $i \in I$, $F_i : K = \prod_{i\in I} K_i \rightrightarrows L$, where L is a topological vector space with a pointed closed convex cone C. Also assume that the following properties hold for each $i \in I$:

(N1) F_i is C-continuous, i.e., F_i is both C-upper and C-lower semi-continuous;
(N2) at each $i \in I$, $F_i(x^{-i}|\cdot)$ is C-convex;
(N3) there exists a collection $\{(C_j, Q_j)\}_{j\in J}$, where for each $j \in J$, $C_j, Q_j \subset E \overset{\textbf{def}}{=} \prod_{i\in I} E_i$, satisfying (C1), (C2) and for each $j \in J$, there exists $k \in J$ with

$$\{(x_i)_{i\in I} \in K\, ;\, \sum_{i\in I}[F_i(x^{-i}|y_i) - F_i(x)] \not\subset -C^\circ,\, \forall (y_i)_{i\in I} \in C_k\} \subset Q_j.$$

Then, the sequence $(F_i)_{i\in I}$ has at least one weak non-cooperative game equilibrium.

Proof. Define a multifunction $G : K \times K \rightrightarrows L$ by

$$G(x, y) \overset{\textbf{def}}{=} \sum_{i\in I}[F_i(x^{-i}|y_i) - F_i(x)].$$

It is clear that $\theta \in G(x, x)$ for all $x \in K$, $G(x, \cdot)$ is C-convex and $G(\cdot, y)$ is C-lower semi-continuous. Applying Corollary 3.5, we obtain the existence of a weak equilibrium of H, which is in turn a weak equilibrium of G. That is, there exists a point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in K$ with

$$\sum_{i\in I}[F_i(\widehat{x}^{-i}|y_i) - F_i(\widehat{x})] \not\subset -C^\circ, \quad \forall y = (y_i)_{i\in I} \in K.$$

For each $\ell \in I$, we may take $y \in K$ such that $y_i = \widehat{x}_i$ for all $i \in I \setminus \{\ell\}$ into account and conclude that

$$F_\ell(\widehat{x}^{-\ell}|y_\ell) - F_\ell(\widehat{x}) \not\subset -C^\circ, \quad \forall y_\ell \in K_\ell. \qquad \square$$

$\square$

Corollary 3.12. In addition to Theorem 3.11, if there exists a pointed closed convex cone $\widetilde{C} \subset L$ with $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, then the sequence $(F_i)_{i\in I}$ has at least one non-cooperative game equilibrium.

Proof. Combine the proof of Corollary 3.8 and Theorem 3.11. $\square$ $\square$

3.5. Mixed equilibrium problems. There are several classes of different mixed equilibrium problems and generalized mixed equilibrium problems. However, in this paper, we shall consider only on the major ones.

According to the problem (GEP), we may consider this as a mixture of two multifunctions F and G, with $F(x, y) = -\widetilde{F}(y, x)$ for some $\widetilde{F} : Y \times Y \rightrightarrows L$ such that $\theta \in \widetilde{F}(x, x)$ for all $x \in Y$. Then, (GEP) can be rewritten as the problem of finding $\widehat{x} \in Y$ such that

$$F(\widehat{x}, y) + G(\widehat{x}, y) \not\subset -C^\circ, \quad \forall y \in Y.$$

In other words, the problem (GEP) conveys the mixed equilibrium problemsas it includes the two set-valued equilibrium problems corresponded to $\widetilde{F}$ and G, respectively. Thus, if $\widetilde{F}$ is $(-C)$-monotone, $\widetilde{F}(\cdot, y)$ is C-upper semi-continuous, and $\widetilde{F}(x, \cdot)$ is C-convex, then it follows from Theorem 3 that this mixed equilibrium problem has a solution.

The next theorem overcome the situation when the function $\widetilde{F}$ is not valid.

Theorem 3.13. In addition to Theorem 3.1, assume further that

(M1) $F(x, \cdot)$ is C-convex;

(M2) for each fixed $x, y \in K$, the multifunction
$$t \in [0, 1] \mapsto F(ty + (1 - t)x, y)$$
is C-upper semi-continuous at $t = 0$;

(M3) there is a solution $\widehat{x} \in K$ of the problem (GEP) that admits an absolute exclusion, i.e.,
$$F(y, \widehat{x}) - G(\widehat{x}, y) \subset L \setminus C^\circ, \quad \forall y \in K.$$

Then, $\widehat{x}$ also solves the following problem:

$$F(\widehat{x}, y) + G(\widehat{x}, y) \not\subset -C^\circ, \quad \text{for all } y \in K.$$

Proof. We first define two multifunctions $F', G' : K \times K \rightrightarrows L$ such that

$$F'(x, y) \overset{\textbf{def}}{=} \begin{cases} F(x, y), & \text{if } x \neq y, \\ \{\theta\}, & \text{otherwise,} \end{cases} \quad \text{and} \quad G'(x, y) \overset{\textbf{def}}{=} \begin{cases} G(x, y), & \text{if } x \neq y, \\ \{\theta\}, & \text{otherwise.} \end{cases}$$

It is clear that F' and G' preserves the C-convexity, C-semi continuity, and C-monotonicity of F and G, respectively. For $t \in [0, 1]$ and $y \in K$, we write

$$x_t \overset{\textbf{def}}{=} ty + (1 - t)\widehat{x}.$$

From the C-convexity of $F'(x,\cdot)$ and $G'(x,\cdot)$, we have

$$\begin{cases} \theta \in tF'(x_t,y) + (1-t)F'(x_t,\widehat{x}) - C, \\ \theta \in (1-t)tG'(\widehat{x},y) - (1-t)G'(\widehat{x},x_t) - C. \end{cases}$$

Consequently, we have

$$\theta \in tF'(x_t,y) + (1-t)[F'(x_t,\widehat{x}) - G'(\widehat{x},x_t)] + (1-t)tG'(\widehat{x},y) - C.$$

We now claim that $tF'(x_t,y) + (1-t)tG'(\widehat{x},y) \not\subset -C^\circ$. Otherwise, if we suppose to the contrary, it follows that for some $\xi \in tF'(x_t,y) + (1-t)tG'(\widehat{x},y)$ and $\xi' \in (1-t)[F'(x_t,\widehat{x}) - G'(\widehat{x},x_t)]$, we have

$$\theta \in \xi + \xi' - C \subset \xi' - C^\circ - C \subset \xi' - C^\circ.$$

Hence, it is the case that $\xi' \in C^\circ$, which contradicts (M3). So we have proved our claim. For $t \neq 0$, we further obtain that

$$tF'(x_t,y) + (1-t)tG'(\widehat{x},y) \not\subset -C^\circ. \tag{3.1}$$

Define a multifunction $H : [0,1] \rightrightarrows L$ by

$$H(t) \overset{\mathbf{def}}{=} F'(x_t,y) + (1-t)G'(\widehat{x},y), \quad \forall t \in [0,1].$$

We may see from (M2) that H is C-upper semi-continuous at $t = 0$.

Let us now verify that $H(0) \not\subset -C^\circ$, since if it is so, the combination with the fact that $F'(\widehat{x},y) + G'(\widehat{x},y) \subset F(\widehat{x},y) + G(\widehat{x},y)$ will eventually implies our desired result. Assume to the contrary that $H(0) \subset -C^\circ$. Thus, we may find an open set N with $H(0) \subset N \subset -C^\circ$, which immediately give

$$N - C \subset -C^\circ - C \subset -C^\circ.$$

Since H is C-upper semi-continuous, we can find an open set $P \subset \mathbb{R}$ such that

$$H(P \cap [0,1]) \subset N - C \subset -C^\circ.$$

Taking any $t \in P \cap (0,1]$, we have from the above inclusion that $H(t) \subset -C^\circ$. This contradicts with (3.1), and so this proves the theorem. □ □

Conclusion

We close this paper with recalling that a new class of generalized equilibrium problems is formulated. It turns out that many esteemed problems in optimization are included and unified. Explicit and implicit consequences are also deduced and studied. Ultimately and most importantly, our results enlarge the validity support of the approximation models for several generalized equilibrium problems.

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References

[1] E. Blum and W. Oettli. From optimization and variational inequalities to equilibrium problems. The Mathematics Student, 63(1-4):123–145, 1994.

[2] Ky Fan. A generalization of Tychonoff's fixed point theorem. Math. Ann., 142:305–310, 1961.

[3] K. Fan. A minimax inequality and applications. Inequalities, III (Proceedings of the Third Symposium, University of California, Los Angeles, CA, 1969; dedicated to the memory of Theodore S. Motzkin), pages 103–113, 1972.

[4] Q.H. Ansari, S. Schaible, and J.C. Yao. System of vector equilibrium problems and its applications. Journal of Optimization Theory and Applications, 107:547–557, 2000.

[5] I.V. Konnov and J.C. Yao. Existence of solutions for generalized vector equilibrium problems. Journal of Mathematical Analysis and Applications, 233(1):328 – 335, 1999.

[6] Werner Oettli and Dirk Schläger. Existence of equilibria for monotone multivalued mappings. Mathematical Methods of Operations Research, 48:219–228, 1998.

[7] Q.H. Ansari and J.-C. Yao. An existence result for the generalized vector equilibrium problem. Applied Mathematics Letters, 12(8):53 – 56, 1999.

[8] S. Park. Generalized equilibrium problems and generalized complementarity problems. Journal of Optimization Theory and Applications, 95:409–417, 1997.

[9] Marc Lassonde. On the use of kkm multifunctions in fixed point theory and related topics. Journal of Mathematical Analysis and Applications, 97(1):151 – 201, 1983.

[10] F. Giannessi. Theorems of alternative, quadratic programs and complementarity problems. , 1980.

[11] G.-Y. Chen and B.D. Craven. A vector variational inequality and optimization over an efficient set. Zeitschrift für Operations Research, 34(1):1–12, 1990.

[12] N. Hadjisavvas and S. Schaible. From scalar to vector equilibrium problems in the quasimonotone case. Journal of Optimization Theory and Applications, 96(2):297–309, 1998.

[13] A. P. Farajzadeh and B. S. Lee. On dual vector equilibrium problems. Appl. Math. Lett., 25(6):974–979, 2012.

[14] M. Bianchi, N. Hadjisavvas, and S. Schaible. Vector equilibrium problems with generalized monotone bifunctions. Journal of Optimization Theory and Applications, 92(3):527–542, 1997.

[15] Lai-Jiu Lin and Hsiu-Li Chen. The study of kkm theorems with applications to vector equilibrium problems and implict vector variational inequalities problems. Journal of Global Optimization, 32:135–157, 2005.

[16] Q.H. Ansari, I.V. Konnov, and J.C. Yao. Characterizations of solutions for vector equilibrium problems. Journal of Optimization Theory and Applications, 113(3):435–447, 2002.

[17] Ya-Ping Fang, Nan-Jing Huang, and JongKyu Kim. Existence results for systems of vector equilibrium problems. Journal of Global Optimization, 35(1):71–83, 2006.

[18] F.E. Browder. Existence and approximation of solutions of nonlinear variational inequalities. Proc. Natl. Acad. Sci. USA, 56:1080–1086, 1966.

[19] Xiao-bo Li and Fu-quan Xia. Hadamard well-posedness of a general mixed variational inequality in banach space. Journal of Global Optimization, pages 1–13, 2012.

[20] MuhammadAslam Noor. Some iterative schemes for general mixed variational inequalities. Journal of Applied Mathematics and Computing, 34(1-2):57–70, 2010.

[21] MuhammadAslam Noor. On a system of general mixed variational inequalities. Optimization Letters, 3(3):437–451, 2009.

[22] D. Goeleven. Existence and uniqueness for a linear mixed variational inequality arising in electrical circuits with transistors. Journal of Optimization Theory and Applications, 138(3):397–406, 2008.

[23] JuanCarlos Reyes. Optimization of mixed variational inequalities arising in flow of viscoplastic materials. Computational Optimization and Applications, 52(3):757–784, 2012.

[24] MuhammadAslam Noor. Resolvent dynamical systems for mixed variational inequalities. Korean Journal of Computational & Applied Mathematics, 9(1):15–26, 2002.

[25] Ding Fangyun, Zhang Xin, and Ding Rui. Boundary mixed variational inequality in friction problem. Applied Mathematics and Mechanics, 20(2):213–224, 1999.

[26] Ren-you Zhong and Nan-jing Huang. Strict feasibility for generalized mixed variational inequality in reflexive banach spaces. Journal of Optimization Theory and Applications, 152(3):696–709, 2012.

[27] Yonghong Yao, M. Aslam Noor, S. Zainab, and Yeong-Cheng Liou. Mixed equilibrium problems and optimization problems. Journal of Mathematical Analysis and Applications, 354(1):319 − 329, 2009.

[28] Shi-sheng Zhang. Generalized mixed equilibrium problem in banach spaces. Applied Mathematics and Mechanics, 30(9):1105–1112, 2009.

[29] I.V. Konnov, S. Schaible, and J.C. Yao. Combined relaxation method for mixed equilibrium problems. Journal of Optimization Theory and Applications, 126(2):309–322, 2005.

[30] Phayap Katchang and Poom Kumam. A new iterative algorithm of solution for equilibrium problems, variational inequalities and fixed point problems in a hilbert space. Journal of Applied Mathematics and Computing, 32(1):19–38, 2010.

[31] J.W. Peng. Iterative algorithms for mixed equilibrium problems, strict pseudocontractions and monotone mappings. Journal of Optimization Theory and Applications, 144(1):107–119, 2010.

[32] Yisheng Song and Qingnian Zhang. Proximal algorithms for a class of mixed equilibrium problems. Fixed Point Theory and Applications, 2012(1):1–15, 2012.

[33] JongSoo Jung. Iterative methods for mixed equilibrium problems and strictly pseudocontractive mappings. Fixed Point Theory and Applications, 2012(1):1–19, 2012.

[34] Lu-Chuan Ceng and Jen-Chih Yao. A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. Journal of Computational and Applied Mathematics, 214(1):186 − 201, 2008.

[35] X.P. Ding, Y.C. Liou, and J.C. Yao. Existence and algorithms for bilevel generalized mixed equilibrium problems in banach spaces. Journal of Global Optimization, 53(2):331–346, 2012.

[36] Zi-Ming Wang, Xiaolong Qin, Zhentao Gu, and Yongfu Su. Strong convergence theorems for a family non-expansive mappings and application to generalized mixed equilibrium problems systems. Journal of Applied Mathematics and

Computing, 36(1-2):507–519, 2011.

[37] A. Moudafi. Mixed equilibrium problems: Sensitivity analysis and algorithmic aspect. Computers & Mathematics with Applications, 44(8–9):1099 – 1108, 2002.

[38] George Allen. Variational inequalities, complementarity problems, and duality theorems. Journal of Mathematical Analysis and Applications, 58(1):1 – 10, 1977.

[39] Q.H. Ansari, I.V. Konnov, and J.C. Yao. On generalized vector equilibrium problems. Nonlinear Analysis: Theory, Methods & Applications, 47(1):543 – 554, 2001. Proceedings of the Third World Congress of Nonlinear Analysts.

[40] Qamrul Hasan Ansari, Abul Hasan Siddiqi, and Soon Yi Wu. Existence and duality of generalized vector equilibrium problems. Journal of Mathematical Analysis and Applications, 259(1):115 – 126, 2001.

[41] M. Fakhar and J. Zafarani. Generalized vector equilibrium problems for pseudomonotone multivalued bifunctions 1. Journal of Optimization Theory and Applications, 126:109–124, 2005.

[42] Hichem Ben-El-Mechaiekh, Souhail Chebbi, and Monique Florenzano. A generalized kkmf principle. Journal of Mathematical Analysis and Applications, 309(2):583 – 590, 2005.

[43] Tamaki Tanaka. Cone-quasiconvexity of vector-valued functions. Sci. Rep. Hirosaki Univ., 42(1):157–163, 1995.

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ON COUPLED FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

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ABSTRACT. The study of coupled fixed points of nonlinear operators, which was introduced about three decades ago, got a boost in 2006 when Bhaskar and Lakshmikantham (2006) studied the coupled fixed points of some contractive maps in partially ordered metric spaces and applied it to solve some first order ordinary differential equations with periodic boundary problems. Since then, coupled fixed points theorems have been proved by several authors for certain contractive maps in both partially ordered and cone metric spaces. The study of coupled fixed point, previously limited to quasi-contractive maps, was recently extended to asymptotically nonexpansive mappings in uniformly convex Banach spaces by Olaoluwa, Olaleru and Chang (2013). In this paper, their results (demiclosed principle and existence result) are extended to asymptotically nonexpansive maps in the intermediate sense in a wider class of spaces. The study naturally opens up new areas of research on the study of coupled fixed points of different classes of pseudocontractive maps.

KEYWORDS : Coupled fixed point; Asymptotically nonexpansive; Uniformly convex Banach spaces.
AMS Subject Classification: 47H10, 47H09

1. INTRODUCTION

The notion of coupled fixed point was introduced by Guo and Lakshmikantham [13] in 1987. Of recent, Gnana-Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone property for contractive operators of the form $F : X \times X \longrightarrow X$ satisfying

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)],\ k<1,$$

where (X,d) is a partially ordered metric space. Their results encompassed some coupled fixed point theorems and their applications to proving the existence and

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uniqueness of the solution for a periodic boundary value problem. Ever since, many authors have established many results on coupled fixed points of quasi-contractive maps in different contexts and spaces (see e.g. [9], [19], [1], [16]).
Results on fixed points of nonexpansive mappings and pseudocontractive mappings abound in literature. The mean ergodic theorem for contractions in uniformly convex Banach spaces was proved in [3] while the authors in [4] introduced the convex approximation property of a space, proved that contractions satisfy an inequality analogue to the Zarantonello inequality (see [22]) and then studied the asymptotic behavior of contractions.
Given a nonempty subset K of a real linear normed space X, a self-mapping $T : K \longrightarrow K$ is said to be nonexpansive if the inequality $\|Tx - Ty\| \leq \|x - y\|$ holds for all $x, y \in K$. Many more general classes of mappings have been considered, including the class of asymptotically nonexpansive mappings introduced by Goebel and Kirk [12], defined by the relation $\|T^n x - T^n y\| \leq k_n \|x - y\| \; \forall n \geq 1 \; \forall x, y \in K$, where the sequence $\{k_n\} \subset [1, \infty)$ converges to 1 as $n \longrightarrow \infty$. Bruck, Kuczumow and Reich [5] introduced the definition of an asymptotically nonexpansive mapping in the intermediate sense (which is more general than an asymptotically nonexpansive map) as a continuous mapping $T : K \longrightarrow K$ such that

$$\limsup_{n \longrightarrow \infty} \sup_{x,y \in K} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \tag{1.1}$$

for any bounded subset $K \in C$. It has been proved by Kirk [15] that asymptotically nonexpansive mappings in the intermediate sense in a nonempty closed convex bounded subset of a space with characteristic of convexity $\epsilon_0(X)$ less than one, have a fixed point.

Recall that the modulus of convexity of X is the function $\delta : [0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{1}{2}\|x + y\| : x, y \in X, \; \|x\|, \|y\| \leq 1, \; \|x - y\| \geq \epsilon \right\},$$

and the number $\epsilon_0(X) = \sup\{\epsilon : \delta(\epsilon) = 0\}$ is called the characteristic of convexity of X [11]. Spaces with characteristic of convexity less than one ($\epsilon_0(X) < 1$) are known to be uniformly non-square (see [11]) hence reflexive [14]. Also if X is uniformly convex [8] if $\delta(\epsilon) > 0$ whenever $\epsilon > 0$; hence $\epsilon_0(X) = 0$. Thus spaces with characteristic of convexity less than one, are a super-class of uniformly convex spaces.

Yang et al. [21] proved the demiclosedness principle for the same class of asymptotically nonexpansive mappings in the intermediate sense using Lemma 2.2 given in [16].

Recently, Olaoluwa et al. [18] extended –for the first time– the theory of coupled fixed points to pseudo-contractive-type mappings defined on a product space (algebraic product) by defining asymptotically nonexpansive maps in the context, and studying their asymptotic behaviour, the demiclosedness property and the conditions of existence of their coupled fixed points. Our interest and main purpose is to extend their results to asymptotically nonexpansive mappings in the intermediate sense defined in a product space.

We now recall the definitions, in product spaces, of nonexpansive maps and asymptotically nonexpansive maps as introduced by Olaoluwa et al. [18] and introduce in the same context, asymptotically nonexpansive mappings in the intermediate sense.

Let K be a nonempty bounded subet of a real normed linear space X.

Definition 1.1. [18] A mapping $T : K \times K \longrightarrow K$ is said to be nonexpansive if

$$\|T(x, y) - T(u, v)\| \leq \frac{1}{2}[\|x - u\| + \|y - v\|] \ \forall x, y, u, v \in X. \tag{1.2}$$

Definition 1.2. [18] A mapping $T : K \times K \longrightarrow K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} k_n = 1$ such that

$$\|T^n(x, y) - T^n(u, v)\| \leq \frac{k_n}{2}[\|x - u\| + \|y - v\|] \ \forall n \geq 1 \ \forall x, y, u, v \in X, \tag{1.3}$$

where the sequence $\{T^n\}$ is defined as follows:

$$\begin{cases} T^0(x, y) = x \\ T^{n+1}(x, y) = T(T^n(x, y), T^n(y, x)) \ n \geq 0. \end{cases} \tag{1.4}$$

The following definition is introduced as an extension of asymptotically nonexpansive mappings in the intermediate sense in product spaces:

Definition 1.3. $T : K \times K \longrightarrow K$ is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in K} (\|T^n(x, y) - T^n(u, v)\| - \|x - u\| - \|y - v\|) \leq 0 \tag{1.5}$$

Remark 1.4. The sequence $\{T^n(x, y)\}$ can be written as the sequence $\{x_n\}$ defined (see [1]) as follows:

$$\begin{cases} x_0 = x; \ y_0 = y \\ x_{n+1} = T(x_n, y_n), \ n \geq 0 \\ y_{n+1} = T(y_n, x_n), \ n \geq 0. \end{cases} \tag{1.6}$$

2. Demiclosedness principle

In [7], Chang et al. recalled the definition the definition of demi-closed maps at the origin as follows:

Definition 2.1. [7] Let X be a real Banach space and K be a closed subset of X. A mapping $T : K \longrightarrow K$ is said to be demi-closed at the origin if, for any sequence $\{x_n\}$ in K, the conditions $x_n \longrightarrow q$ weakly and $Tx_n \longrightarrow 0$ strongly, imply $Tq = 0$.

The definition of demi-closed mappings in product spaces can be proposed from the previous definition as follows:

Definition 2.2. [18] Let X be a real Banach space and K be a closed subset of K. A mapping $T : K \times K \longrightarrow K$ is said to be demi-closed at the origin if, for any sequence $\{(x_n, y_n)\}$ in $K \times K$, the conditions $x_n \longrightarrow q_1$, $y_n \longrightarrow q_2$ weakly and $F(x_n, y_n) \longrightarrow 0$, $F(y_n, x_n) \longrightarrow 0$ strongly imply $F(q_1, q_2) = F(q_2, q_1) = 0$.

In order to establish the demiclosedness principle for asymptotically nonexpansive mappings in the intermediate sense defined in a product space, it is important to estimate the difference between $T^k\left(\sum_{i=1}^n \lambda_i(x_i, y_i)\right)$ and $\sum_{i=1}^n \lambda_i T^k(x_i, y_i)$ for $\lambda \in \Delta^{n-1}$, $(x_1, y_1), \ldots, (x_n, y_n) \in K \times K$ and $k \geq 1$.
Here $\Delta^{n-1} = \{\lambda = (\lambda_1, \cdots, \lambda_n) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^n \lambda_i = 1\}$. The following lemmas are useful.

Lemma 2.3. Let E be a uniformly convex Banach space and K be a nonempty closed bounded convex subset of E. For $\epsilon > 0$, there exists an integer $N_\epsilon \geq 1$ and $\delta_{2,\epsilon} > 0$ such that if $k \geq N_\epsilon$, $x_1, x_2, y_1, y_2 \in K$ and

$$\|x_1 - x_2\| + \|y_1 - y_2\| - 2\|T^k(x_1, y_1) - T^k(x_2, y_2)\| \leq \delta_{2,\epsilon},$$

then

$$\|T^k(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)) - \lambda_1 T^k(x_1, y_1) - \lambda_2 T^k(x_2, y_2)\| < \epsilon$$

for all $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$.

Proof. Let δ be the modulus of uniform convexity of X and define $d : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by

$$d(t) = \begin{cases} \frac{1}{2}\int_0^t \delta(s)ds, & 0 \leq t \leq 2 \\ d(2) + \frac{1}{2}\delta(2)(t-2), & t > 2. \end{cases}$$

It is well known (e.g. see [3],[16]) that d is strictly increasing, continuous, convex, satisfying $d(0) = 0$ and

$$2\lambda_1\lambda_2 d(\|u - v\|) \leq 1 - \|\lambda_1 u + \lambda_2 v\| \tag{2.1}$$

for all $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$, and $u, v \in X$ such that $\|u\| \leq 1$ and $\|v\| \leq 1$.
For $\epsilon > 0$, choose $\eta_\epsilon > 0$ and $\frac{D_K}{2} d^{-1}\left(\frac{\eta_\epsilon}{D_K}\right) < \epsilon$ and put $\delta_{2,\epsilon} = \min\{\eta_\epsilon, D_K\}$. By (1.5), there exists an integer $N_\epsilon \geq 1$ (depending on K) such that if $k \geq N_\epsilon$,

$$2\|T^k(x, y) - T^k(u, v)\| - \|x - u\| - \|y - v\| < \delta_{2,\epsilon} \text{ for all } x, y, u, v \in K.$$

Let $k \geq N_\epsilon$ and let $(x_1, y_1), (x_2, y_2) \in K \times K$ with

$$\|x_1 - x_2\| + \|y_1 - y_2\| - 2\|T^k(x_1, y_1) - T^k(x_2, y_2)\| \leq \delta_{2,\epsilon}.$$

It suffices to show Lemma 2.3 in the case of $0 < \lambda_1, \lambda_2 < 1$. Put

$$u = 2\left[\frac{T^k(x_2, y_2) - T^k(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2))}{\lambda_1(\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon})}\right]$$

and

$$v = 2\left[\frac{T^k(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)) - T^k(x_1, y_1)}{\lambda_2(\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon})}\right].$$

We have $\|u\| \leq 1$, $\|v\| \leq 1$ and

$$\lambda_1 u + \lambda_2 v = 2\left[\frac{T^k(x_2, y_2) - T^k(x_1, y_1)}{\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon}}\right]. \tag{2.2}$$

Since $u - v = 2\left[\frac{\lambda_1 T^k(x_1, y_1) + \lambda_2 T^k(x_2, y_2) - T^k(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2))}{\lambda_1\lambda_2(\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon})}\right]$ and $\frac{1}{D_K}\lambda_1\lambda_2(\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon}) \leq \frac{1}{D_K} \cdot \frac{1}{4}(2D_K + D_K) < 1$, we have by (2.1) and (2.2) that

$$
\begin{aligned}
&d\left(\frac{2}{D_K}\left\|\lambda_1 T^k(x_1,y_1)+\lambda_2 T^k(x_2,y_2)-T^k(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))\right\|\right)\\
&\quad\le \frac{1}{D_K}\lambda_1\lambda_2\left(\|x_1-x_2\|+\|y_1-y_2\|+\delta_{2,\epsilon}\right)d(\|u-v\|)\\
&\quad\le \tfrac{1}{D_K}\lambda_1\lambda_2(\|x_1-x_2\|+\|y_1-y_2\|+\delta_{2,\epsilon})\cdot\tfrac{1}{2\lambda_1\lambda_2}\left\{1-2\tfrac{\|T^k(x_2,y_2)-T^k(x_1,y_1)\|}{\|x_1-x_2\|+\|y_1-y_2\|+\delta_{2,\epsilon}}\right\}\\
&\quad=\frac{1}{2D_K}(\|x_1-x_2\|+\|y_1-y_2\|-2\|T^k(x_2,y_2)-T^k(x_1,y_1)\|+\delta_{2,\epsilon})\\
&\quad\le \frac{2\delta_{2,\epsilon}}{2D_K}=\frac{\delta_{2,\epsilon}}{D_K}\le\frac{\eta_\epsilon}{D_K}.
\end{aligned}
$$

Here we have used the fact that $t\mapsto\frac{d(t)}{t}$ is strictly increasing; $t_1\le t_2 \implies \frac{d(t_1)}{t_1}\le \frac{d(t_2)}{t_2}$, with $t_1=\frac{2}{D_K}\|\lambda_1 T^k(x_1,y_1)+\lambda_2 T^k(x_2,y_2)-T^k(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))\|$ and $t_2=\|u-v\|$.
Consequently, from the choice of η_ϵ, we obtain
$\|T^k(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))-\lambda_1T^k(x_1,y_1)-\lambda_2T^k(x_2,y_2)\|\le\dfrac{D_K}{2}d^{-1}\left(\frac{\eta_\epsilon}{D_K}\right)<\epsilon.$ □

Lemma 2.4. Let E be a uniformly convex Banach space and K be a nonempty closed bounded convex subset of E. For each $\epsilon>0$ and each integer $n\ge 2$, there exists an integer $N_\epsilon\ge 1$ and $\delta_{n,\epsilon}>0$ (where N_ϵ is independent of n) such that if $k\ge N_\epsilon$, $(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\in K\times K$ and if

$$\|x_i-x_j\|+\|y_i-y_j\|-2\|T^k(x_i,y_i)-T^k(x_j,y_j)\|<\delta_{n,\epsilon}$$

for $1\le i,j\le n$, then

$$\left\|T^k\left(\sum_{i=1}^n\lambda_i(x_i,y_i)\right)-\sum_{i=1}^n\lambda_iT^k(x_i,y_i)\right\|<\epsilon$$

for all $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\Delta^{n-1}$.

Proof. Let $\epsilon>0$ and let $n\ge 2$ be an arbitrary integer. Choose an integer $N_\epsilon\ge 1$ in Lemma 2.3. We shall construct $\delta_{n,\epsilon}$ $(n=2,3,\dots)$ inductively. Let $\delta_{2,\epsilon}$ be as in Lemma 2.3. Suppose that all $\delta_{q,\epsilon}$ are constructed for $q=2,3,\dots,p$. Let $\epsilon'=\min\{\frac{1}{10}\delta_{p,\frac{\epsilon}{2}},\frac{\epsilon}{2}\}$ and put $\delta_{p+1,\epsilon}=\min\{\delta_{2,\epsilon'},\epsilon'\}$.
Let $\lambda\in\Delta^p$, $(x_1,y_1),\dots,(x_{p+1},y_{p+1})\in K\times K$, $k\ge N_\epsilon$ and

$$\|x_i-x_j\|+\|y_i-y_j\|-2\|T^k(x_i,y_i)-T^k(x_j,y_j)\|<\delta_{p+1,\epsilon}$$

for $1\le i,j\le p+1$.
The case $\lambda_{p+1}=1$ is trivial and so we assume $\lambda_{p+1}\ne 1$. Put for $j=1,2,\dots,p$ and $i=1,2,\dots,p+1$,

$$\begin{pmatrix}u_j\\ v_j\end{pmatrix}=(1-\lambda_{p+1})\begin{pmatrix}x_j\\ y_j\end{pmatrix}+\lambda_{p+1}\begin{pmatrix}x_{p+1}\\ y_{p+1}\end{pmatrix};\ \mu_j=\frac{\lambda_j}{1-\lambda_{p+1}},$$

$$\begin{pmatrix}u'_j\\ v'_j\end{pmatrix}=(1-\lambda_{p+1})\begin{pmatrix}x'_j\\ y'_j\end{pmatrix}+\lambda_{p+1}\begin{pmatrix}x'_{p+1}\\ y'_{p+1}\end{pmatrix},\ \text{with}\ \begin{pmatrix}x'_i\\ y'_i\end{pmatrix}=\begin{pmatrix}T(x_i,y_i)\\ T(y_i,x_i)\end{pmatrix}.$$

We have:

$$\sum_{i=1}^{p+1}\lambda_i\begin{pmatrix}x_i\\ y_i\end{pmatrix}=\sum_{j=1}^{p}\mu_j\begin{pmatrix}u_j\\ v_j\end{pmatrix};\ \sum_{i=1}^{p+1}\lambda_i\begin{pmatrix}x'_i\\ y'_i\end{pmatrix}=\sum_{j=1}^{p}\mu_j\begin{pmatrix}u'_j\\ v'_j\end{pmatrix}.$$

Therefore:

$$\begin{aligned}
&\left\|T^k\left(\sum_{i=1}^{p+1}\lambda_i(x_i,y_i)\right)-\sum_{i=1}^{p+1}\lambda_iT^k(x_i,y_i)\right\| \\
&\qquad=\left\|T^k\left(\sum_{j=1}^{p}\mu_j(u_j,v_j)\right)-\sum_{j=1}^{p}u'_j\right\| \\
&\qquad\le\left\|T^k\left(\sum_{j=1}^{p}\mu_j(u_j,v_j)\right)-\sum_{j=1}^{p}\mu_jT^k(u_j,v_j)\right\|+\sum_{j=1}^{p}\mu_j\|T^k(u_j,v_j)-u'_j\|.
\end{aligned} \tag{2.3}$$

Since $\|x_j-x_{p+1}\|+\|y_j-y_{p+1}\|-2\|T^k(x_j,y_j)-T^k(x_{p+1},y_{p+1})\|<\delta_{p+1,\epsilon}\le\delta_{2.\epsilon'}$, we have by Lemma 2.3:

$$\begin{aligned}
\|u'_j-T^k(u_j,v_j)\| &= \|(1-\lambda_{p+1})T^k(u_j,v_j)+\lambda_{p+1}T^k(u_j,v_j) \\
&\quad -T^k((1-\lambda_{p+1})(x_j,y_j)+\lambda_{p+1}(x_{p+1},y_{p+1}))\| \\
&< \epsilon',
\end{aligned}$$

$$\begin{aligned}
\|v'_j-T^k(v_j,u_j)\| &= \|(1-\lambda_{p+1})T^k(v_j,u_j)+\lambda_{p+1}T^k(v_j,u_j) \\
&\quad -T^k((1-\lambda_{p+1})(y_j,x_j)+\lambda_{p+1}(y_{p+1},x_{p+1}))\| \\
&< \epsilon',
\end{aligned}$$

$$\begin{aligned}
&\|u_j-u_l\|+\|v_j-v_l\|-\|u'_j-u'_l\|-\|v'_j-v'_l\| \\
&\qquad=(1-\lambda_{p+1})\left\{\|x_j-x_l\|+\|y_j-y_l\|-\|T^k(x_j,y_j)-T^k(x_l,y_l)\|\right. \\
&\qquad\qquad\left.-\|T^k(y_j,x_j)-T^k(y_l,x_l)\|\right\} \\
&\qquad=(1-\lambda_{p+1})\left\{\frac{\|x_j-x_l\|+\|y_j-y_l\|-2\|T^k(x_j,y_j)-T^k(x_l,y_l)\|}{2}\right. \\
&\qquad\qquad\left.+\frac{\|y_j-y_l\|+\|x_j-x_l\|-2\|T^k(y_j,x_j)-T^k(y_l,x_l)\|}{2}\right\} \\
&\qquad\le(1-\lambda_{p+1})\left\{\frac{\delta_{p+1,\epsilon}}{2}+\frac{\delta_{p+1,\epsilon}}{2}\right\}=(1-\lambda_{p+1})\delta_{p+1,\epsilon}\le\epsilon'
\end{aligned}$$

for all $1\le j,l\le p$.
Therefore we obtain by the triangle inequality:

$$\begin{aligned}
&\|u_j-u_l\|+\|v_j-v_l\|-\|T^k(u_j,v_j)-T^k(u_l,v_l)\|-\|T^k(v_j,u_j)-T^k(v_l,u_l)\| \\
&\qquad\le\ \|u_j-u_l\|+\|v_j-v_l\|-\|u'_j-u'_l\|-\|v'_j-v'_l\| \\
&\qquad\quad+\|u'_j-T^k(u_j,v_j)\|+\|u'_l-T^k(u_l,v_l)\| \\
&\qquad\quad+\|v'_j-T^k(v_j,u_j)\|+\|v'_l-T^k(v_l,u_l)\| \\
&\qquad\le\ 5\epsilon'\le\tfrac{1}{2}\delta_{p,\frac{\epsilon}{2}}
\end{aligned}$$

for $1 \le j, l \le p$. Since

$$\begin{aligned} & \|u_j - u_l\| + \|v_j - v_l\| - \|T^k(u_j, v_j) - T^k(u_l, v_l)\| - \|T^k(v_j, u_j) - T^k(v_l, u_l)\| \\ & = \tfrac{1}{2}\left\{\|u_j - u_l\| + \|v_j - v_l\| - 2\|T^k(u_j, v_j) - T^k(u_l, v_l)\|\right\} \\ & \quad + \tfrac{1}{2}\left\{\|v_j - v_l\| + \|u_j - u_l\| - 2\|T^k(v_j, u_j) - T^k(v_l, u_l)\|\right\} \\ & \le \tfrac{1}{2}\delta_{p, \frac{\epsilon}{2}}, \end{aligned}$$

then $\|u_j - u_l\| + \|v_j - v_l\| - 2\|T^k(u_j, v_j) - T^k(u_l, v_l)\| \le \delta_{p, \frac{\epsilon}{2}}$. Thus by inductive assumption and (2.3), the desired conclusion holds. $\square$

The following Lemma shows that the positive number $\delta_{n,\epsilon}$ in Lemma 2.4 can be chosen independently of n.

Lemma 2.5. Let E be a uniformly convex Banach space, K be a nonempty closed bounded convex subset of E. For every $\epsilon > 0$ and every integer $n \ge 2$, there exist an integer $N_\epsilon \ge 1$ and $\delta_\epsilon > 0$ (where both N_ϵ and δ_ϵ are independent of n) such that if $k \ge N_\epsilon$, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in K \times K$ and if

$$\|x_i - x_j\| + \|y_i - y_j\| - 2\|T^k(x_i, y_i) - T^k(x_j, y_j)\| \le \delta_\epsilon$$

for $1 \le i, j \le n$, then

$$\left\| T^k\left(\sum_{i=1}^n \lambda_i (x_i, y_i)\right) - \sum_{i=1}^n T^k(x_i, y_i) \right\| < \epsilon$$

for all $\lambda \in \Delta^{n-1}$.

Proof. Fix $\epsilon > 0$ and an integer $n \ge 2$ arbitrarily. Denote by $N_{1,\epsilon}$ the integer $N_{\epsilon/4}$ in Lemma 2.4. By (1.5) there is an integer $N_{2,\epsilon} \ge 1$ such that if $k \ge N_{2,\epsilon}$, then we have

$$2\|T^k(x, y) - T^k(u, v)\| - \|x - u\| - \|y - v\| < \frac{\epsilon}{4} \text{ for all } x, y, u, v \in K \tag{2.4}$$

Put $N_\epsilon = \max\{N_{1,\epsilon}, N_{2,\epsilon}\}$. Let $\delta_{n,\epsilon}$ $(n = 2, 3, \ldots)$ be positive numbers determined in Lemma 2.4. Since X is uniformly convex, X is B-convex (see [4]) and since the product of B-convex spaces is also B-convex (see [10]), X^3 is B-convex, hence has the convex approximation property (C.A.P.) (see [4]) so we can choose an integer $p = p(\epsilon) \ge 1$ (independent of n) such that $coM \subset co_pM + B_{\epsilon/4} \times B_{\epsilon/4} \times B_{\epsilon/4}$ for all subsets $M \subset X^3$ whose diameters are uniformly bounded, where B_r is the open sphere centered at the origin and with r as radius, coM is the convex hall of M and

$$co_pM = \left\{ \sum_{i=1}^p t_i X_i;\ t \in \Delta^{p-1}; X_i \in M \text{ for all } i \in \{1, \ldots, p\},\ p \text{ fixed} \right\}.$$

Put $\delta_\epsilon = \delta_{p, \frac{\epsilon}{4}}$. Let $k \ge N_\epsilon$, $(x_1, y_1), \ldots, (x_n, y_n) \in K \times K$ and

$$\|x_i - x_j\| + \|y_i - y_j\| - 2\|T^k(x_i, y_i) - T^k(x_l, y_l)\| \le \delta_\epsilon \ (1 \le i, j \le n).$$

Consider $M = \{[x_i, y_i, T^k(x_i, y_i)] \in X^3 : i = 1, 2, \cdots, n\}$. Note that there exists $r > 0$ (independent from k and n) such that $\sup_{(x,y,z)\in M} \|(x, y, z)\|_{X^3} \le r$.
Then for each $\lambda \in \Delta^{n-1}$, there exist $\mu \in \Delta^{p-1}$ and $i_1, \cdots, i_p \in \{1, \cdots, n\}$ such that

$$\left\| \sum_{i=1}^n \lambda_i x_i - \sum_{j=1}^p \mu_j x_{i_j} \right\| < \frac{\epsilon}{4}, \ \left\| \sum_{i=1}^n \lambda_i y_i - \sum_{j=1}^p \mu_j y_{i_j} \right\| < \frac{\epsilon}{4}, \text{ and}$$

$$\left\|\sum_{i=1}^{n}\lambda_i T^k(x_i,y_i)-\sum_{j=1}^{p}\mu_j T^k(x_{i_j},y_{i_j})\right\|<\frac{\epsilon}{4}.$$

By (2.4) and the choice of δ_ϵ we have

$$\begin{aligned}2&\left\|T^k\left(\sum_{i=1}^{n}\lambda_i(x_i,y_i)\right)-T^k\left(\sum_{j=1}^{p}\mu_j(x_{i_j},y_{i_j})\right)\right\|\\ &\leq\left\|\sum_{i=1}^{n}\lambda_i x_i-\sum_{j=1}^{p}\mu_j x_{i_j}\right\|+\left\|\sum_{i=1}^{n}\lambda_i y_i-\sum_{j=1}^{p}\mu_j y_{i_j}\right\|+\frac{\epsilon}{4}\leq\frac{3\epsilon}{4}<\epsilon\end{aligned}$$

and

$$\left\|T^k\left(\sum_{j=1}^{p}\mu_j(x_{i_j},y_{i_j})\right)-\sum_{j=1}^{p}\mu_j T^k(x_{i_j},y_{i_j})\right\|<\frac{\epsilon}{4}.$$

Therefore

$$\begin{aligned}&\left\|T^k\left(\sum_{i=1}^{n}\lambda_i(x_i,y_i)\right)-\sum_{i=1}^{n}T^k(x_i,y_i)\right\|\\ &\quad\leq\left\|T^k\left(\sum_{i=1}^{n}\lambda_i(x_i,y_i)\right)-T^k\left(\sum_{j=1}^{p}\mu_j(x_{i_j},y_{i_j})\right)\right\|\\ &\quad\quad+\left\|T^k\left(\sum_{j=1}^{p}\mu_j(x_{i_j},y_{i_j})\right)-\sum_{j=1}^{p}\mu_j T^k(x_{i_j},y_{i_j})\right\|\\ &\quad\quad\left\|\sum_{j=1}^{p}\mu_j T^k(x_{i_j},y_{i_j})-\sum_{j=1}^{n}\lambda_i T^k(x_i,y_i)\right\|\\ &\quad<\epsilon.\end{aligned}$$

□

Lemma 2.5 is an extension of Lemma 1.5 of Yang et al. [21] to asymptotically nonexpansive maps in the intermediate sense defined on product spaces. From Lemma 2.5, we can now state the following theorem which is likewise an extension of their Lemma 1.6:

Theorem 2.1. (Demiclosedness Principle): Let X be a real uniformly convex Banach space and K a nonempty bounded closed convex subset of X. Let $T: K\times K\longrightarrow K$ be a mapping which is asymptotically nonexpansive in the intermediate sense. If $\{x_n\}$ and $\{y_n\}$ are sequences in K converging weakly to x^* and y^* and if

$$\begin{cases}\lim\limits_{k\longrightarrow\infty}(\limsup\limits_{n}\|x_n-T^k(x_n,y_n)\|)=0\\ \lim\limits_{k\longrightarrow\infty}(\limsup\limits_{n}\|y_n-T^k(y_n,x_n)\|)=0\end{cases}$$

then p_1-T is demiclosed at zero, i.e., for each sequences $\{x_n\},\{y_n\}\in K$, if they converge weakly to $x^*\in K$ and $y^*\in K$ respectively and $\{x_n-T(x_n,y_n)\}$ and $\{y_n-T(y_n,x_n)\}$ converge strongly to 0, then $x^*=T(x^*,y^*)$ and $y^*=T(y^*,x^*)$.

Proof. The sequences $\{x_n\}$ and $\{y_n\}$ are bounded so there exists $r>0$ such that $\{x_n\},\{y_n\}\subset C:=K\cap B_r$, where B_r is the closed ball in X with center 0 and radius r. So C is a nonempty bounded closed convex subset in K. Let us prove that $T^k(x^*,y^*)\longrightarrow x^*$ and $T^k(y^*,x^*)\longrightarrow y^*$.
For $\epsilon>0$, choose an integer $N_1(\epsilon)$ such that if $k\geq N_1(\epsilon)$, then $2\|T^k(x,y)-T^k(u,v)\|-\|x-u\|-\|y-v\|<\frac{\epsilon}{5}$ for $(x,y),(u,v)\in C\times C$ and $\limsup_n\|x_n-T^k(x_n,y_n)\|+\limsup_n\|y_n-T^k(y_n,x_n)\|<\frac{1}{4}\delta_{\frac{\epsilon}{5}}$.
Thus there exists $n_{\epsilon,k}$ such that $\|x_n-T^k(x_n,y_n)\|+\|y_n-T^k(y_n,x_n)\|<\frac{1}{4}\delta_{\epsilon/5}$ for $n\geq n_{\epsilon,k}$.
Set $\epsilon'=\min\{\frac{1}{4}\delta_{\frac{\epsilon}{5}},\frac{\epsilon}{5}\}$. Then we have $N_1(\epsilon')\geq 1$. Let $N_2(\epsilon)=\max\{N_{\frac{\epsilon}{5}},N_1(\epsilon),N_1(\epsilon')\}$ and let $j\geq N_2(\epsilon)$. Since $\{x_n\}$ and $\{y_n\}$ converge weakly to x^* and y^*, by Mazur's theorem, for each positive integer $n\geq 1$, there exist convex combinations $A_n=\sum_{i=1}^{m(n)}\lambda_i^{(n)}x_{i+n}$ and $B_n=\sum_{i=1}^{m(n)}\lambda_i^{(n)}y_{i+n}$ with $\lambda_i^{(n)}\geq 0$ and $\sum_{i=1}^{m(n)}\lambda_i^{(n)}=1$

such that $\|A_n - x^*\| \longrightarrow 0$ and $\|B_n - y^*\| \longrightarrow 0$ as $n \longrightarrow \infty$.
Since

$$\left\{\begin{array}{l} \|x_{i+n} - x_{j+n}\| + \|y_{i+n} - y_{j+n}\| \\ -\|T^k(x_{i+n}, y_{i+n}) - T^k(x_{j+n}, y_{j+n})\| \\ -\|T^k(y_{i+n}, x_{i+n}) - T^k(y_{j+n}, x_{j+n})\| \end{array}\right\} \leq \left\{\begin{array}{l} \|x_{i+n} - T^k(x_{i+n}, y_{i+n})\| \\ +\|y_{i+n} - T^k(x_{i+n}, y_{i+n})\| \\ +\|x_{j+n} - T^k(x_{j+n}, y_{j+n})\| \\ +\|y_{j+n} - T^k(y_{j+n}.x_{j+n})\| \end{array}\right\} \leq \frac{1}{2}\delta_{\frac{\epsilon}{5}}$$

and

$$\left\{\begin{array}{l} \|x_{i+n} - x_{j+n}\| + \|y_{i+n} - y_{j+n}\| \\ -\|T^k(x_{i+n}, y_{i+n}) - T^k(x_{j+n}, y_{j+n})\| \\ -\|T^k(y_{i+n}, x_{i+n}) - T^k(y_{j+n}, x_{j+n})\| \end{array}\right\} = \left\{\begin{array}{l} \frac{1}{2}\left[\|x_{i+n} - x_{j+n}\| + \|y_{i+n} - y_{j+n}\|\right] \\ -\|T^k(x_{i+n}, y_{i+n}) - T^k(x_{j+n}, y_{j+n})\| \\ +\frac{1}{2}\left[\|y_{i+n} - y_{j+n}\| + \|x_{i+n} - x_{j+n}\|\right] \\ -\|T^k(y_{i+n}, x_{i+n}) - T^k(y_{j+n}, x_{j+n})\| \end{array}\right\} \leq \frac{1}{2}\delta_{\epsilon/5},$$

we therefore have

$$\|x_{i+n} - x_{j+n}\| + \|y_{i+n} - y_{j+n}\| - 2\|T^k(x_{i+n}, y_{i+n}) - T^k(x_{j+n}, y_{j+n})\| \leq \delta_{\epsilon/5}$$

for $1 \leq i, j \leq m(n)$; by Lemma 2.5, we have

$$\left\| T^k(A_n, B_n) - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k(x_{i+n}, y_{i+n}) \right\| < \frac{\epsilon}{5},$$

$$\left\| T^k(B_n, A_n) - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k(y_{i+n}, x_{i+n}) \right\| < \frac{\epsilon}{5}.$$

There is $L_{k,\epsilon} \geq 1$ such that $\|A_n - x^*\| + \|B_n - y^*\| < \frac{\epsilon}{5}$ for all $n \geq L_{k,\epsilon}$.
Since $x^*, y^* \in K$,

$$\begin{aligned} \|T^k(x^*, y^*) - x^*\| &\leq \|T^k(x^*, y^*) - T^k(A_n, B_n)\| \\ &\quad + \left\| T^k(A_n, B_n) - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k(x_{i+n}, y_{i_n}) \right\| \\ &\quad + \left\| \sum_{i=1}^{m(n)} \lambda_i^{(n)} (T^k(x_{i+n}, y_{i+n}) - x_{i+n}) \right\| + \|A_n - x^*\| \\ &< \epsilon \end{aligned}$$

for $n \geq L_{k,\epsilon}$ and $k \geq N_2(\epsilon)$. Thus $\|T^k(x^*, y^*) - x^*\| < \epsilon$ for $k \geq N_2(\epsilon)$ and so $\|T^k(x^*, y^*) - x^*\| \longrightarrow 0$ as $k \longrightarrow \infty$. Similarly, $\|T^k(y^*, x^*) - y^*\| \longrightarrow 0$ as $k \longrightarrow \infty$. By the continuity of T, we have

$$\left\{\begin{array}{l} x^* = \lim\limits_{k \longrightarrow \infty} T^{k+1}(x^*, y^*) = \lim\limits_{k \longrightarrow \infty} T(T^k(x^*, y^*), T^k(y^*, x^*)) = T(x^*, y^*) \\ y^* = \lim\limits_{k \longrightarrow \infty} T^{k+1}(y^*, x^*) = \lim\limits_{k \longrightarrow \infty} T(T^k(y^*, x^*), T^k(x^*, y^*)) = T(y^*, x^*) \end{array}\right.$$

This completes the proof. □

Theorem 2.1 extends Theorem 2.1 of Olaoluwa et al. [18] to asymptotically nonexpansive maps in the intermediate sense defined on a product space.

3. Existence of coupled fixed points

The following theorem relative to the existence of coupled fixed points of asymptotically nonexpansive maps in the intermediate sense extends the results of Kirk [15] to product spaces. The spaces considered have a characteristic of convexity less than one. Thus the result remain valid for uniformly convex Banach spaces and consequently generalize Theorem 3.1 of Olaoluwa et al. [18] on existence of coupled fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces.

Theorem 3.1. Let X be a Banach space for which $\epsilon_0 = \epsilon_0(X) < 1$ and let $K \subset X$ be nonempty, bounded, closed and convex. Suppose $T : K \times K \longrightarrow K$ is asymptotically nonexpansive in the intermediate sense. Then T has a fixed point in $K \times K$.

Proof. Let $(x, y) \in K \times K$ be fixed. Define the set $R(x, y)$ as follows:

$$R(x,y) = \left\{ \rho \in \mathbb{R} \ / \ \exists k_\rho \in \mathbb{N} \ : \ (K \times K) \bigcap \left(\bigcap_{i=k_\rho}^{\infty} B(T^i(x,y), \rho) \times B(T^i(y,x), \rho) \right) \neq \emptyset \right\}.$$

where $B(x, r)$ is the open sphere in X, of center x and radius r. K is bounded, so, if $D_K := diamK$ (diameter of K), $D_K \in R(x, y)$, hence $R(x, y) \neq \emptyset$. Let ρ^* be the g.l.b. of $R(x, y)$.
For any $\epsilon > 0$, define the sets $C_\epsilon = \bigcup_{k=1}^{\infty} \left(\bigcap_{i=k}^{\infty} B\left(T^i(x,y), \rho^* + \epsilon\right)\right)$ and $D_\epsilon = \bigcup_{k=1}^{\infty} \left(\bigcap_{i=k}^{\infty} B\left(T^i(y,x), \rho^* + \epsilon\right)\right)$. The sets C_ϵ and D_ϵ are nonempty, bounded and convex hence by the reflexivity of X the closures $\bar{C}_\epsilon$ and $\bar{D}_\epsilon$ are weakly compact and $C = \bigcap_{\epsilon>0}(\bar{C}_\epsilon \cap K) \neq \emptyset$ and $D = \bigcap_{\epsilon>0}(\bar{D}_\epsilon \cap K) \neq \emptyset$.
Let $(u, v) \in C \times D$ and let $d(u, v) = \limsup_{i \longrightarrow \infty} \|u - T^i(u, v)\| + \|v - T^i(v, u)\|$.
Suppose $\rho^*(x, y) = 0$. Then $T^n(x, y) \longrightarrow u$ and $T^n(y, x) \longrightarrow v$ as $n \longrightarrow \infty$. Let $\eta > 0$ and using (1.5), choose L such that $i \geq L$ implies

$$\sup_{(u,v),(z,t) \in K \times K} [2\|T^i(u,v) - T^i(z,t)\| - \|u - z\| - \|v - t\|] \leq \frac{1}{3}\eta.$$

Given $i \geq L$, since $T^n(x, y) \longrightarrow u$ and $T^n(y, x) \longrightarrow v$, there exists $l > i$ such that $\|T^l(x, y) - u\| + \|T^l(y, x) - v\| \leq \frac{1}{3}\eta$ and $\|T^{l-i}(x, y) - u\| + \|T^{l-i}(y, x) - v\| \leq \frac{1}{3}\eta$.

Thus if $i \geq L$,

$$\|u - T^i(u,v)\| + \|v - T^i(v,u)\|$$

$$\leq \|u - T^l(x,y)\| + \|T^l(x,y) - T^i(u,v)\| + \|v - T^l(x,y)\| + \|T^l(x,y) - T^i(u,v)\|$$

$$\begin{aligned}\leq &\|u - T^l(x,y)\| + \|T^i(u,v) - T^i(T^{l-i}(x,y), T^{l-i}(y,x))\| - \|u - T^{l-i}(x,y)\| \\ &+\|u - T^{l-i}(x,y)\| + \|v - T^l(y,x)\| + \|T^i(v,u) - T^i(T^{l-i}(y,x), T^{l-i}(x,y))\| \\ &-\|v - T^{l-i}(y,x)\| + \|v - T^{l-i}(y,x)\|\end{aligned}$$

$$\begin{aligned}\leq &\ \frac{2}{3}\eta \\ &+\frac{1}{2}\left[2\|T^i(T^{l-i}(x,y), T^{l-i}(y,x)) - T^i(u,v)\| - \|u - T^{l-i}(x,y)\| - \|v - T^{l-i}(y,x)\|\right] \\ &+\frac{1}{2}\left[2\|T^i(T^{l-i}(y,x), T^{l-i}(x,y)) - T^i(v,u)\| - \|u - T^{l-i}(x,y)\| - \|v - T^{l-i}(y,x)\|\right]\end{aligned}$$

$$\leq \frac{2}{3}\eta + \sup\left[2\|T^i(u,v) - T^i(z,t)\| - \|u - z\| - \|v - t\|\right]$$

$$\leq \eta.$$

This proves that $T^n(u,v) \longrightarrow u$ and $T^n(v,u) \longrightarrow v$ as $n \longrightarrow \infty$, that is, $d(u,v) = 0$. But $d(u,v) = 0$ implies $T^{N_i}(u,v) \longrightarrow u$ and $T^{N_i}(v,u) \longrightarrow v$ as $i \longrightarrow \infty$ and with the continuity of T^N this yields $T^N(u,v) = u$ and $T^N(v,u) = v$. Thus, as $i \longrightarrow \infty$,

$$\begin{cases} T(u,v) = T\left(T^{N_i}(u,v), T^{N_i}(v,u)\right) = T^{N_i+1}(u,v) \longrightarrow u \\ \\ T(v,u) = T\left(T^{N_i}(v,u), T^{N_i}(u,v)\right) = T^{N_i+1}(v,u) \longrightarrow v \end{cases}$$

so $T(u,v) = u$ and $T(v,u) = v$.

Now we assume that $\rho^*(x,y) > 0$ and $d(u,v) > 0$. In fact, we may assume this for any $x, y, u, v \in K$.

Let $\epsilon > 0$, $\epsilon \leq d(u,v)$. By definition of ρ^* there exists an integer N^* such that if $i \geq N^*$ then

$$\|u - T^i(x,y)\| + \|v - T^i(y,x)\| \leq \rho^* + \epsilon, \tag{3.1}$$

and by (1.5) there exists N^{**} such that if $i \geq N^{**}$ then

$$\sup\left[2\|T^i(u,v) - T^i(z,t)\| - \|u - z\| - \|v - t\|\right] \leq \epsilon.$$

Select j so that $j \geq N^{**}$ and so that

$$\|u - T^j(u,v) + v - T^j(v,u)\| \geq d(u,v) - \epsilon. \tag{3.2}$$

Thus if $i - j \geq N^{**}$,

$$\|T^j(u,v) - T^i(x,y)\| + \|T^j(v,u) - T^i(y,x)\|$$

$$\begin{aligned}= &\|T^j(u,v) - T^j(T^{i-j}(x,y), T^{i-j}(y,x))\| - \|u - T^{i-j}(x,y)\| + \|u - T^{i-j}(x,y)\| \\ &+\|T^j(v,u) - T^j(T^{i-j}(y,x), T^{i-j}(x,y))\| - \|v - T^{i-j}(y,x)\| + \|v - T^{i-j}(y,x)\|\end{aligned}$$

$$\begin{aligned}= &\tfrac{1}{2}\left\{2\|T^j(u,v) - T^j(T^{i-j}(x,y), T^{i-j}(y,x))\| - \|u - T^{i-j}(x,y)\| - \|v - T^{i-j}(y,x)\|\right\} \\ &+\tfrac{1}{2}\left\{2\|T^j(v,u) - T^j(T^{i-j}(y,x), T^{i-j}(x,y))\| - \|u - T^{i-j}(x,y)\| - \|v - T^{i-j}(y,x)\|\right\} \\ &+\|u - T^{i-j}(x,y)\| + \|v - T^{i-j}(y,x)\|\end{aligned}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} + (\rho^* + \epsilon) = 2\epsilon + \rho^*. \tag{3.3}$$

Letting $m = \dfrac{1}{2}[u + T^j(u,v) + v + T^j(v,u)]$, by (3.3) we have:

$$\|m - T^i(x,y) - T^i(y,x)\| \leq \left(1 - \delta\left(\frac{d(u,v)-\epsilon}{\rho^* + 2\epsilon}\right)\right)(\rho^* + 2\epsilon),\ i \geq N^* + j.$$

By the minimality of ρ^* this implies that

$$\rho^* \leq \left(1 - \delta\left(\frac{d(u,v)-\epsilon}{\rho + 2\epsilon}\right)\right)(\rho^* + 2\epsilon).$$

Letting $\epsilon \longrightarrow 0$, $\rho^* \leq \left(1 - \delta\left(\frac{d(u,v)}{\rho^*}\right)\right)\rho^*$. This implies that $1 - \delta\left(\frac{d(u,v)}{\rho^*}\right) \geq 1$ and hence $\delta\left(\frac{d(u,v)}{\rho^*}\right) = 0$. It follows from the definition of ϵ_0 that $\frac{d(u,v)}{\rho^*} \leq \epsilon_0$. Hence $d(u,v) \leq \epsilon_0 \rho^*(x,y)$ and letting $d(x,y) = \limsup_{i\longrightarrow\infty} \|x - T^i(x,y)\| + \|y - T^i(y,x)\|$ we have $\rho_0(x) \leq d(x,y)$ so

$$d(u,v) \leq \epsilon_0 d(x,y) \tag{3.4}$$

Also notice that $\|u - x\| + \|v - y\| \leq d(x,y) + \rho_0(x,y) \leq 2d(x,y)$.
Fix $(x_0,y_0) \in K \times K$ and define the sequence $\{(x_n,y_n)\}$ for all $n \in \mathbb{N}$ by

$$\begin{cases} x_{n+1} = u(x_n,y_n) \\ y_{n+1} = v(y_n,x_n), \end{cases}$$

where $u(x_n,y_n)$ is obtained from x_n and y_n in the same manner as $u(x,y)$ from x and y.
If for any n we have $\rho(x_n,y_n) = 0$ and $\rho(y_n,x_n) = 0$ then, as seen above, $T(x_{n+1},y_{n+1}) = x_{n+1}$ and $T(y_{n+1},x_{n+1}) = y_{n+1}$. Otherwise, by 3.4 we have $\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq 2d(x_n,y_n) \leq 2\epsilon^n d(x_0,y_0)$ and since $\epsilon_0 < 1$, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Therefore there exists $(x,y) \in K \times K$ such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ as $n \longrightarrow \infty$. Also:

$$\begin{aligned}
&\|x - T^i(x,y)\| + \|y - T^i(y,x)\| \\
&\quad \leq \|x - x_n\| + \|x_n - T^i(x_n,y_n)\| + \|T^i(x_n,y_n) - T^i(x,y)\| \\
&\qquad + \|y - y_n\| + \|y_n - T^i(y_n,x_n)\| + \|T^i(y_n,x_n) - T^i(y,x)\| \\
&\quad \leq \|x - x_n\| + \|y - y_n\| + \|x_n - T^i(x_n,y_n)\| \\
&\qquad + \|y_n - T^i(y_n,x_n)\| + \|x_n - x\| + \|y_n - y\| \\
&\qquad\quad + \tfrac{1}{2}\left[2\|T^i(x_n,y_n) - T^i(x,y)\| - \|x_n - x\| - \|y_n - y\|\right] \\
&\qquad\qquad + \tfrac{1}{2}\left[2\|T^i(y_n,x_n) - T^i(y,x)\| - \|x_n - x\| - \|y_n - y\|\right]
\end{aligned}$$

Thus

$$\begin{aligned}
d(x,y) &= \limsup_{i\longrightarrow\infty} \|x - T^i(x,y)\| + \|y - T^i(y,x)\| \\
&\leq \limsup_{i\longrightarrow\infty} 2[\|x - x_n\| + \|y - y_n\|] \\
&\quad + \limsup_{i\longrightarrow\infty} [\|x_n - T^i(x_n,y_n)\| + \|y_n - T^i(y_n,x_n)\|] \\
&\quad + \limsup_{i\longrightarrow\infty} [\sup_{x,y} 2\|T^i(x,y) - T^i(u,v)\| - \|x - u\| - \|y - v\|] \\
&\leq d(x_n,y_n) + 2[\|x - x_n\| + \|y - y_n\|]
\end{aligned}$$

Since $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $d(x_n,y_n) \longrightarrow 0$ as $n \longrightarrow \infty$, this implies that $d(x,y) = 0$. But as seen before, it implies that $T(x,y) = x$ and $T(y,x) = y$. $\square$

References

1. M. Abbas, M. Ali Khan, S. Radenovic, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, Appl. Math. Comput. 217(2010), 195–202.
2. T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65(2006), 1379–1393.

3. R. E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, Israel J. Math. 32(1979), 107–116.
4. R. E. Bruck, On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, Israel J. Math. 38(4) (1981), 304 –314.
5. R.E. Bruck, T. Kuczumow, S. Reich, Convergence of iterates of asymptotically non- expansive mappings in Banach spaces with the uniform Opial property, Colloq. Math. LXV(1993), 169–179. MR 94h:47106.
6. S. S. Chang, Y. J. Cho, N. J. Huan, Coupled fixed point theorems with applications, J. Korean Math. Soc. 33(3) (1996), 575–585.
7. S. S. Chang, Y. J. Cho, and H. Zhao, Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings, J. Korean Math. Soc. 38(6) (2001), 1245–1260.
8. J.A. Clarkson, Uniformly convex spaces, Trans. Amer. Math. Soc. 40(1936), 396–414.
9. J.X. Fang, Common fixed point theorems of compatible and weakly compatible maps in Menger spaces, Nonlinear Anal. 71(5-6) (2009), 1833 – 1843.
10. D.P. Giesy, On a convexity condition in normed linear spaces, Trans. Amer. Math. Soc. 125(1966), 114–146.
11. K. Goebel, Convexity of balls and fixed-point theorems for mappings with nonexpansive square, Compositio Math. 22(1970), 269–274.
12. K. Goebel; W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35(1972), 171–174.
13. D. Guo and V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal. TMA. 11(1987), 623–632.
14. R.C. James, Uniformly non-square Banach spaces, Ann. of Math. 80(1964), 542 – 550.
15. W.A. Kirk, Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type, Israel J. Math. 17(1974), 339–346. MR 49:11330.
16. H. Oka, An ergodic theorem for assymptotically nonexpansive mappings in the intermediate sense, Proc. Amer. Math. Soc. 125(1997), 1693–1703.
17. J.O. Olaleru and G.A. Okeke, Strong convergence theorems for asymptotically pseudocontractive mappings in the intermediate sense, British Journal of Mathematics and Computer Science 2(3) (2012), 151–162.
18. H. Olaoluwa, J.O. Olaleru and S.S. Chang, Coupled fixed point theorems for asymptotically nonexpansive mappings, Fixed Point Theory Appl. 2013, 2013:68.
19. S. Shakeri, L.J.B. Ciric, R. Saadati, Common Fixed Point Theorem in Partially Ordered L-Fuzzy Metric Spaces, Fixed Point Theory Appl. 2010, Article ID 125082.
20. P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge Univ. Press, 1991.
21. L.P. Yang, X. Xie, S. Peng, G. Hu, Demiclosed principle and convergence for modified three step iterative process with errors of non-Lipschitzian mappings, J. Comput. Appl. Math. 234(2010), 972–984.
22. E.H. Zarantonello, Protections on convex sets in Hilbert space and spectral theory, Contributions to Nonlinear Functional Analysis, Academic Press, New York 1971, 237–244.

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GENERALIZED CONTRACTIVE MAPPINGS IN b-METRIC SPACES

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ABSTRACT. We will apply a new method to generalize some fixed point theorems for a special class of contractive type mappings in complete b-metric spaces. We also discuss about the existence of a unique fixed point for a family of self mappings. Our results enable us to generalize some known fixed point theorems in the literature. Finally, we present some applications of our results.

KEYWORDS: Fixed points, b-meric spaces, contraction-type mappings.
AMS Subject Classification: Primary 47H10, 54E40; Secondary 47H9.

1. Introduction and preliminaries

Let (X, d) be a metric space and T be a self mapping on X. The function T is said to be a contraction on X if there is some $0 \leq r < 1$ such that

$$d\big(T(x), T(y)\big) \leq rd(x, y) \quad (x, y \in X). \tag{1.1}$$

A celebrated result due to Banach [3] states that every contraction function on a complete metric space has a unique fixed point. The Banach contraction mapping principle is considered to be the core of many extended fixed point theorems (see. e. g. [6, 8, 10, 14, 18, 20, 21, 22]).

Despite these important features, Banach fixed point theorem suffers from one serious drawback - the contractive condition (1.1) forces T to be continuous on the entire space X. It was then naturally to ask if there exist contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [13] in 1968, who proved a fixed point theorem which extends Banach's theorem to mappings that need not be continuous, by considering instead of (1.1) the next condition:

$$d\big(T(x), T(y)\big) \leq r\big\{d\big(x, T(x)\big) + d\big(y, T(y)\big)\big\} \text{ for some } 0 \leq r < \frac{1}{2} \text{ and all } x, y \in X. \tag{1.2}$$

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Kannan's theorem has been generalized by some authors [16, 17]. In particular, G. Hardy and T. Rogers [12] proved the following results:

THEOREM 1.1. [12, Theorem 1] *Let X be a complete metric space with metric d, and let $T : X \longrightarrow X$ be a function with the following property:*

$$d\big(T(x),T(y)\big) \leq a\ d\big(x,T(x)\big)+b\ d\big(y,T(y)\big)+c\ d\big(x,T(y)\big)+e\ d\big(y,T(x)\big)+f\ d(x,y), \tag{1.3}$$

where $0 \leq a, b, c, e, f < 1$ and $a+b+c+e+f < 1$. Then T has a unique fixed point.

THEOREM 1.2. [12, Theorem 2] *Let (X, d) be a complete metric space, a, b, c, e, f be monotonically decreasing functions from $[0,\infty)$ to $[0, 1)$, and let the sum of these five functions be less than 1. Suppose $T : X \longrightarrow X$ satisfies condition (1.3) with $a = a(d(x, y)), \cdots, f = f(d(x, y))$ for all $x, y \in X$. Then T has a unique fixed point.*

The above results were extended for some metric-like spaces (see e.g. [1, 15, 19]). The following generalization of a metric is due to S. Czerwik [7]:

DEFINITION 1.3. Let X be a space and $d : X \times X \longrightarrow [0,\infty)$ be a function such that for each x, y and z in X,

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) for each $\varepsilon > 0$, if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$.

Then d is called a b-metric on X. It is easy to verify that the condition (iii) is equivalent to the following (see e. g. [7, Lemma 1]).

(iv) For each $\varepsilon > 0$ and $x, y, z \in X$, if $d(x, y) \leq \varepsilon$ and $d(y, z) \leq \varepsilon$, then $d(x, z) \leq 2\varepsilon$.

Clearly, every metric space is a b-metric space. However, the converse is not true in general.

EXAMPLE 1.4. Let $X = \{1, 2, 3\}$. Define a symmetric function $d : X \times X \longrightarrow [0,\infty)$ by

$$d(1, 1) = d(2, 2) = d(3, 3) = 0,\ d(1, 2) = 3,\ d(2, 3) = 2,\ d(1, 3) = 6.$$

It is easy to verify that (iv) holds, hence (X, d) is a b-metric space but it is not a metric space, since $d(1, 3) \nleq d(1, 2) + d(2, 3)$.

In order to prove some fixed point theorems of Banach type for b-metric spaces, S. Czerwik [7] replaced (iv) by the following weaker condition:

$$d(x, y) \leq 2d(x, z) + 2d(z, y) \quad (x, y, z \in X). \tag{1.4}$$

The following example shows that (1.4) is strictly weaker than (iv).

EXAMPLE 1.5. Let $X = \mathbb{R}$ and define $d : X \times X \longrightarrow [0,\infty)$ by

$$d(x, y) = |x - y|^2 \quad (x, y \in \mathbb{R}).$$

Then for $\varepsilon = 1$, we have $d(1, 0) = \varepsilon,\ d(0,-1) = \varepsilon$ but $d(1,-1) = 4 \nleq 2\varepsilon = 2$. Hence (iv) is not true. However, it is easy to verify that (1.4) holds.

Following [7], a few mathematicians investigated the existence of fixed points for self-mappings on special kind of b-metrics X; i. e. the triangle inequality is replaced by

$$d(x, y) \leq k\big(d(x, z) + d(z, y)\big) \quad (x, y, z \in X),$$

for some $k \geq 1$ (see e. g. [2, 4, 5, 22]).

In this paper, we will assume that our b-metric space satisfies (iv). We will show that Frink's lemma and the inequality (iv) enable us to improve some known fixed point theorems. More precisely, we will establish Theorems 1.1 and 1.2 for b-metric spaces, which improves some results in [7] and [16]. We also discuss about the existence of a unique fixed point for a family of self mappings on b-metric spaces.

2. Results

In order to state main results of this paper, we need to the following result.

LEMMA 2.1. *(see [9] or [11]). Suppose $d : X \times X \longrightarrow [0, \infty)$ satisfies the following condition:*
For any $\varepsilon > 0$ and $x, y, z \in X$, if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$.
Then the function $\rho : X \times X \longrightarrow [0, \infty)$, defined by

$$\rho(x, y) = inf\Big\{ \sum_{i=1}^{n} d(x_{i-1}, x_i);\ where\ n \in \mathbb{N}, x_0 = x\ and\ x_n = y\Big\}, \quad \big((x, y) \in X\times X\big), \tag{2.1}$$

has the following properties:

(i) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, *for all $x, y, z \in X$.*
(ii) $\frac{d(x,y)}{4} \leq \rho(x, y) \leq d(x, y)$ *for all $x, y \in X$. Further, ρ is symmetric (i.e. $\rho(x, y) = \rho(y, x)$) if d is.*

Now, we are ready to state one of the main results of this paper.

THEOREM 2.2. *Let (X, d) be a b-metric space and T be a self-mapping on X such that for all $x, y \in X$*

$$d\big(T(x), T(y)\big) \leq a\ d\big(x, T(x)\big) + b\ d\big(y, T(y)\big) + c\ d\big(x, T(y)\big) + e\ d\big(y, T(x)\big) + f\ d(x, y), \tag{2.2}$$

where a, b, c, e, f are nonnegative, $a + b + 2(c + e) + f < 1$. Then T has a unique fixed point.

Proof. We first show that T has at most one fixed point. Let x^*, y^* be fixed points of T. We have

$$\begin{aligned} d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\leq a\ d\big(x^*, T(x^*)\big) + b\ d\big(y^*, T(y^*)\big) + c\ d(y^*, T(x^*)) + e\ d\big(x^*, T(y^*)\big) + f\ d(x^*, y^*) \\ &= (c + e + f)\ d(x^*, y^*). \end{aligned}$$

Hence $x^* = y^*$. By symmetry, it follows from (2.2) that

$$d\big(T(x), T(y)\big) \leq a\ d\big(y, T(y)\big) + b\ d\big(x, T(x)\big) + c\ d\big(y, T(x)\big) + e\ d\big(x, T(y)\big) + f\ d(x, y), \tag{2.3}$$

for all $x, y \in X$. By (2.2) and (2.3), we have

$$d\big(T(x), T(y)\big) \leq \frac{a + b}{2}\left[d\big(x, T(x)\big) + d\big(y, T(y)\big)\right] + \frac{c + e}{2}\left[d\big(x, T(y)\big) + d\big(y, T(x)\big)\right] + f\ d(x, y)$$

for all $x, y \in X$. Let $\alpha = \frac{a+b}{2}$, $\beta = \frac{c+e}{2}$ and put $y = T(x)$, then we have

$$\begin{aligned} d\big(T(x), T^2(x)\big) &\leq \alpha[d\big(x, T(x)\big) + d\big(T(x), T^2(x)\big)] + \beta[d\big(x, T^2(x)\big) + d\big(T(x), T(x)\big)] \\ &+ f d\big(x, T(x)\big) \\ &= (\alpha + f)\ d\big(x, T(x)\big) + \alpha\ d\big(T(x), T^2(x)\big) + \beta\ d\big(x, T^2(x)\big) \\ &\leq (\alpha + 2\beta + f)\ d\big(x, T(x)\big) + (\alpha + 2\beta)\ d\big(T(x), T^2(x)\big) \quad (x \in X). \end{aligned}$$

Let $r = \frac{\alpha+2\beta+f}{1-\alpha-2\beta}$. Then $0 \le r < 1$, since $a+b+2(c+e)+f < 1$. By the above inequality

$$d\big(T(x), T^2(x)\big) \le r d\big(x, T(x)\big) \quad (x \in X).$$

By induction, it follows that for all $n \in \mathbb{N}$,

$$d\big(T^n(x), T^{n+1}(x)\big) \le r^n d\big(x, T(x)\big) \quad (x \in X). \tag{2.4}$$

Take some arbitrary point $x_0 \in X$ and define

$$x_1 = T(x_0), x_2 = T(x_1), \ldots, x_n = T(x_{n-1}), \ldots.$$

By (2.4), we have

$$d(x_n, x_{n+1}) \le r^n d(x_0, x_1).$$

According to Lemma 2.1, there is a metric ρ on X such that $\frac{d(x,y)}{4} \le \rho(x,y) \le d(x,y)$ for all $x, y \in X$. For each $m > n$, we have

$$\begin{aligned} d(x_n, x_m) \le 4\rho(x_n, x_m) &\le 4\sum_{i=n}^{m-1} \rho(x_{i+1}, x_i) \\ &\le 4\sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\le 4d(x_1, x_0)\sum_{i=n}^{m-1} r^i \le \frac{4r^n d(x_0, x_1)}{1-r}, \end{aligned}$$

which tends to zero as n tends to infinity. Hence $\{x_n\}$ is a Cauchy sequence in complete b-metric space (X, d). Therefore, $\{x_n\}$ is convergent to some $x^* \in X$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, T(x^*)) &\le 2d\big(T^{n+1}(x_0), T(x^*)\big) + 2d\big(T^{n+1}(x_0), x^*\big) \\ &\le 2a\ d(T^n(x_0), T^{n+1}(x_0)\big) + 2b\ d\big(x^*, T(x^*)\big) + 2c\ d\big(T^n(x_0), T(x^*)\big) \\ &\quad +(2e+2)\ d\big(T^{n+1}(x_0), x^*\big) + 2f\ d\big(T^n(x_0), x^*\big) \\ &\le 2a\ d(T^n(x_0), T^{n+1}(x_0)\big) + 2b\ d\big(x^*, T(x^*)\big) + 4c\ d\big(T^n(x_0), x^*\big) \\ &\quad +4c\ d\big(T(x^*), x^*\big) + (2e+2)\ d\big(T^{n+1}(x_0), x^*\big) + 2f\ d\big(T^n(x_0), x^*\big). \end{aligned}$$

By taking limit as n tends to infinity, it follows that

$$d(x^*, T(x^*)) \le (2b+4c)\ d(x^*, T(x^*)). \tag{2.5}$$

By symmetry,

$$d(x^*, T(x^*)) \le (2a+4e)\ d(x^*, T(x^*)). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned} d(x^*, T(x^*)) &\le \frac{1}{2}\big[(2b+4c)\ d(x^*, T(x^*)) + (2a+4e)\ d(x^*, T(x^*))\big] \quad (2.7) \\ &= \big(a+b+2(c+e)\big)\ d(x^*, T(x^*)). \end{aligned}$$

In view of (2.7) and the fact that $a+b+2(c+e) < 1-f$, we have $T(x^*) = x^*$. $\square$

In the following result, we will show that under certain condition, if $T : X \longrightarrow X$ satisfies (2.2) on a complete subset, not necessarily on the entire space, then T has a unique fixed point.

COROLLARY 2.3. *Let T be a self mapping on a b-metric space (X, d). Let Y be a complete subset of X such that $T(Y) \subseteq Y$ and for each $x, y \in Y$,*

$$d\big(T(x), T(y)\big) \le a\ d\big(x, T(x)\big) + b\ d\big(y, T(y)\big) + c\ d\big(x, T(y)\big) + e\ d\big(y, T(x)\big) + f\ d(x, y), \tag{2.8}$$

where a, b, c, e, f are nonnegative and $a + b + 2(c + e) + f < 1$. If for each $z, w \in X$ with $z \neq w$,

$$d\big(T(z), T(w)\big) < d\big(T(z), z\big) + d\big(T(w), w\big) + d(z, w),$$

then T has a unique fixed point on X.

Proof. By Theorem 2.2, $T|_Y : Y \longrightarrow Y$ has a fixed point $y^* \in Y$. Let $x^* \in X$ be another fixed point of T on X. Then we have

$$d(x^*, y^*) = d\big(T(x^*), T(y^*)\big) < d\big(T(x^*), x^*\big) + d\big(T(y^*), y^*\big) + d(x^*, y^*) = d(x^*, y^*).$$

This contradiction shows that $T : X \longrightarrow X$ has a unique fixed point. □

Reich's theorem [16] for b-metric spaces follows immediately from Theorem 2.2:

COROLLARY 2.4. *Let (X, d) be a complete b-metric space and $T : X \longrightarrow X$ satisfy*

$$d\big(T(x), T(y)\big) \le a\ d(x, T(x)) + b\ d(y, T(y)) + f\ d(x, y) \quad (x, y \in X),$$

where a, b and f are nonnegative and $a + b + f < 1$. Then T has a unique fixed point.

In the following, we use our main result to show that under certain circumstances a family of self-mappings has a unique common fixed point.

COROLLARY 2.5. *Let (X, d) be a complete b-metric space and $\mathcal{F}$ be a family of self mappings on X such that for each $T, S \in \mathcal{F}$ and $x, y \in X$,*

$$d\big(T(x), S(y)\big) \le a\ d\big(x, T(x)\big) + b\ d\big(y, S(y)\big) + c\ d\big(x, S(y)\big) + e\ d\big(y, T(x)\big) + f\ d(x, y), \tag{2.9}$$

where a, b, c, e, f are nonnegative and $a + b + 2(c + e) + f < 1$. Then $\mathcal{F}$ has a unique common fixed point.

Proof. Assume that $T \in \mathcal{F}$. By putting $S = T$ in (2.9) and applying Theorem 2.2, we conclude T has a unique fixed point. So every element of $\mathcal{F}$ has a unique fixed point. Now we show that all elements of $\mathcal{F}$ have a common unique fixed point. Let $S, T \in \mathcal{F}$ and let x^* and y^* be fixed points of T and S respectively. Then

$$\begin{array}{rcl} d(x^*, y^*) & = & d\big(T(x^*), S(y^*)\big) \\ & \le & a\ d\big(x^*, T(x^*)\big) + b\ d\big(y^*, S(y^*)\big) + c\ d\big(x^*, S(y^*)\big) + e\ d\big(y^*, T(x^*)\big) \\ +f\ d(x^*, y^*) & = & (c + e + f) d(x^*, y^*). \end{array}$$

Since $c + e + f < 1$, we have $x^* = y^*$. □

In 1995, S. Czerwick proved the following result which is a generalization one of the main results in [13, 23]:

THEOREM 2.6. [7, Theorem 3] *Let $a : (0, \infty) \longrightarrow (0, \frac{1}{2})$ be a decreasing function. Let (X, d) be a complete b-metric space and let $T : X \longrightarrow X$ satisfy*

$$d\big(T(x), T(y)\big) \le a\big(d(x, y)\big)\big[d\big(x, T(x)\big) + d\big(y, T(y)\big)\big] \quad (x \neq y; x, y \in X).$$

Then T has a unique fixed point x^* and $\lim_{n \longrightarrow \infty} d\big(T^n(x), x^*\big) = 0$ for each $x \in X$.

Here we will prove Theorem 1.2 for b-metric spaces under some extra conditions. This result also generalizes Theorem 2.6.

THEOREM 2.7. *Let (X,d) be a complete b-metric space, a,b,c,e,f be monotonically decreasing functions from $(0,\infty)$ to $[0,1)$ such that $a(t)+b(t)+2(c(t)+e(t))+f(t)<1$, $2(c(t)+e(t))+4f(t)<1$ for each $t\in(0,\infty)$ and $f(t_0)>0$ for some $t_0\in(0,\infty)$. Suppose that $T:X\longrightarrow X$ satisfies*

$$d\big(T(x),T(y)\big)\le a(t)\ d\big(x,T(x)\big)+b(t)\ d\big(y,T(y)\big)+c(t)\ d\big(x,T(y)\big)\\ +e(t)\ d\big(y,T(x)\big)+f(t)\ d(x,y), \tag{2.10}$$

where $t=d(x,y)$ and $x,y\in X$ with $x\neq y$. Then T has a unique fixed point.

Proof. The proof is divided into several steps:

Step 1. *T has at most one fixed point.*
Let x^*,y^* be fixed points of T. If $x^*\neq y^*$, then

$$\begin{aligned} d(x^*,y^*) &= d(T(x^*),T(y^*))\\ &\le a\ d\big(x^*,T(x^*)\big)+b\ d\big(y^*,T(y^*)\big)+c\ d(y^*,T(x^*))+e\ d\big(x^*,T(y^*)\big)+f\ d(x^*,y^*)\\ &= (c+e+f)\ d(x^*,y^*)<d(x^*,y^*), \end{aligned}$$

where $a=a(d(x^*,y^*)),\ldots,f=f(d(x^*,y^*))$. This is a contradiction. Hence $x^*=y^*$.

Step 2. *There exists a monotone decreasing function $r:(0,\infty)\longrightarrow[0,1)$ such that for $t=d\big(x,T(x)\big)\neq 0$,*

$$d\big(T^2(x),T(x)\big)\le r(t)d\big(x,T(x)\big)\quad (x\in X). \tag{2.11}$$

By symmetry, it follows from (2.10) that for $a=a(d(x,y))$, $b=b(d(x,y))$, $c=c(d(x,y))$, $e=e(d(x,y))$ and $f=f(d(x,y))$, we have

$$d\big(T(x),T(y)\big)\le a\ d\big(y,T(y)\big)+b\ d\big(x,T(x)\big)+c\ d\big(y,T(x)\big)+e\ d\big(x,T(y)\big)+f\ d(x,y), \tag{2.12}$$

where $x,y\in X$ and $x\neq y$. By (2.10) and (2.3), for the same a,b,c,e and f, we have

$$d\big(T(x),T(y)\big)\le\frac{a+b}{2}\,[d\big(x,T(x)\big)+d\big(y,T(y)\big)]+\frac{c+e}{2}\,[d\big(x,T(y)\big)+d\big(y,T(x)\big)]+f\ d(x,y) \tag{2.13}$$

for all $x,y\in X$ with $x\neq y$. Fix some $x\in X$, then for $t=d\big(x,T(x)\big)\neq 0$, $\alpha(t)=\frac{a(t)+b(t)}{2}$, $\beta(t)=\frac{c(t)+e(t)}{2}$, we have

$$\begin{aligned} d\big(T(x),T^2(x)\big) &\le \alpha(t)[d\big(x,T(x)\big)+d\big(T(x),T^2(x)\big)]+\beta(t)[d\big(x,T^2(x)\big)+d\big(T(x),T(x)\big)]\\ &+ f(t)d\big(x,T(x)\big)\\ &= (\alpha(t)+f(t))\ d\big(x,T(x)\big)+\alpha(t)\ d\big(T(x),T^2(x)\big)+\beta(t)\ d\big(x,T^2(x)\big)\\ &\le (\alpha(t)+2\beta(t)+f(t))\ d\big(x,T(x)\big)+(\alpha(t)+2\beta(t))\ d\big(T(x),T^2(x)\big)\quad (x\in X). \end{aligned}$$

Let $r(t)=\frac{\alpha(t)+2\beta(t)+f(t)}{1-\alpha(t)-2\beta(t)}$. Then r is monotonically decreasing and (2.11) holds.

Step 3. *For each $x\in X$, $\lim_{n\longrightarrow\infty}d\big(T^n(x),T^{n+1}(x)\big)=0$.*
Let $x\in X$, $n\in\mathbb{N}$ and $t=d\big(T^{n-1}(x),T^n(x)\big)$. If $T^{n-1}(x)=T^n(x)$, then the claim is proved. So we may assume that $t\neq 0$. By step 2, we have

$$d\big(T^n(x),T^{n+1}(x)\big)\le r(t)d\big(T^{n-1}(x),T^n(x)\big)<d\big(T^{n-1}(x),T^n(x)\big).$$

Therefore $\{d\big(T^{n+1}(x),T^n(x)\big)\}$ is a decreasing sequence. Let $\lim_{n\longrightarrow\infty} d\big(T^{n+1}(x),T^n(x)\big) = p$. If $p>0$, then for each $n\in\mathbb{N}$, we have $d\big(T^{n+1}(x),T^n(x)\big)\geq p$. Since r is monotone deceasing,

$$d\big(T^{n+1}(x),T^n(x)\big) < r(p)d\big(T^n(x),T^{n-1}(x)\big) \leq \cdots \leq r^n(p)d\big(x,T(x)\big).$$

Since the right hand side of the above inequality tends to zero as $n\longrightarrow\infty$, we have

$$\lim_{n\longrightarrow\infty} d\big(T^{n+1}(x),T^n(x)\big) = 0.$$

This contradiction shows that $p=0$.

Step 4. *For each $x\in X$, $\{T^n(x)\}$ converges and $x^*=\lim_{n\longrightarrow\infty}T^n(x)$ is the fixed point of T.*

Let $\alpha(t)=\frac{a(t)+b(t)}{2}$, $\beta(t)=\frac{c(t)+e(t)}{2}$, $t\in(0,\infty)$. By our assumption,

$$4\beta(t)+4f(t) = 2(c(t)+e(t))+4f(t)<1 \quad (t\in(0,\infty)).$$

Therefore we can define

$$\gamma_1(t)=\tfrac{\alpha(t)+2\beta(t)+2f(t)}{1-4\beta(t)-4f(t)} \text{ and } \gamma_2(t)=\tfrac{\alpha(t)+2\beta(t)+4f(t)}{1-4\beta(t)-4f(t)} \text{ for } t\in(0,\infty).$$

Suppose that $\varepsilon>0$ and $x\in X$. Choose $n_0\in\mathbb{N}$ so that for each $n,m\geq n_0$,

$$d\big(T^n(x),T^{n-1}(x)\big) < \min\left\{\frac{\varepsilon}{8}, \frac{\varepsilon}{2\gamma_2(\frac{\varepsilon}{8})}\right\}.$$

Let $n,m>n_0$ and $x\in X$ and $t=d\big(T^{n-1}(x),T^{m-1}(x)\big)$. If $T^{n-1}(x)=T^{m-1}(x)$, then $d\big(T^n(x),T^m(x)\big)=0$. So that we may assume that $t\neq 0$. By (1.4) and (2.13), we have

$$\begin{aligned}
&d\big(T^n(x),T^m(x)\big)\\
\leq\ & \alpha(t)\big[d\big(T^{n-1}(x),T^n(x)\big)+d\big(T^{m-1}(x),T^m(x)\big)\big]\\
&+\beta(t)\big[d\big(T^{n-1}(x),T^m(x)\big)+d\big(T^{m-1}(x),T^n(x)\big)\big]+f(t)\ d\big(T^{n-1}(x),T^{m-1}(x)\big)\\
\leq\ & \alpha(t)\big[d\big(T^{n-1}(x),T^n(x)\big)+d\big(T^{m-1}(x),T^m(x)\big)\big]\\
&+\beta(t)\big[2d\big(T^{n-1}(x),T^n(x)\big)+4d\big(T^n(x),T^m(x)\big)+2d\big(T^{m-1}(x),T^m(x)\big)\big]\\
&+f(t)\ \Big[2d\big(T^{n-1}(x),T^n(x)\big)+4\ d\big(T^n(x),T^m(x)\big)+4d\big(T^m(x),T^{m-1}(x)\big)\Big].
\end{aligned}$$

It follows that

$$d\big(T^n(x),T^m(x)\big)\leq\gamma_1(t)d\big(T^n(x),T^{n-1}(x)\big)+\gamma_2(t)d\big(T^m(x),T^{m-1}(x)\big).$$

If $d\big(T^{n-1}(x),T^{m-1}(x)\big)\geq\frac{\varepsilon}{8}$, then

$$\begin{aligned}
d\big(T^n(x),T^m(x)\big)\ &\leq\ \gamma_1(\frac{\varepsilon}{8})d\big(T^n(x),T^{n-1}(x)\big)+\gamma_2(\frac{\varepsilon}{8})d\big(T^m(x),T^{m-1}(x)\big)\\
&<\ \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \qquad (2.14)
\end{aligned}$$

Let $d\big(T^{n-1}(x),T^{m-1}(x)\big)<\frac{\varepsilon}{8}$, then

$$\begin{aligned}
d\big(T^n(x),T^m(x)\big)\ &\leq\ 2d\big(T^n(x),T^{n-1}(x)\big)+4d\big(T^{n-1}(x),T^{m-1}(x)\big)+2d\big(T^m(x),T^{m-1}(x)\big)\\
&<\ 2d\big(T^n(x),T^{n-1}(x)\big)+4\frac{\varepsilon}{8}+2d\big(T^m(x),T^{m-1}(x)\big) \qquad (2.15)\\
&<\ \frac{\varepsilon}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon}{4}=\varepsilon.
\end{aligned}$$

It follows from (2.14) and (2.15) that $\{T^n(x)\}$ is a Cauchy sequence in (X,d). Hence $x^*=\lim_{n\longrightarrow\infty}T^n(x)$ exists. So that for some $n_1\in\mathbb{N}$, $d\big(T^{n-1}(x),x^*\big)<t_0$ provided

that $n \geq n_1$. If $T^{n-1}(x) = x^*$ for infinity many n, then $T^n(x^*) = Tx^*$ for infinity many n. It follows that $T(x^*) = x^*$. Therefore, we can assume that $T^{n-1}(x) \neq x^*$ for all $n \in \mathbb{N}$. Let $t_n = d\big(T^{n-1}(x), x^*\big)$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} d\big(x^*, T(x^*)\big) &\leq 2d\big(x^*, T^n(x)\big) + 2d\big(T^n(x), T(x^*)\big) \\ &\leq 2d\big(x^*, T^n(x)\big) + 2\alpha(t_n)[d\big(x^*, T(x^*)\big) + d\big(T^n(x), T^{n-1}(x)\big)] \\ &\quad +2\beta(t_n)[2d\big(T^{n-1}(x), x^*\big) + 2d\big(x^*, T(x^*)\big) + d\big(x^*, T^n(x)\big)] \\ &\quad +2f(t_n)d\big(T^{n-1}(x), x^*\big) \quad (n \in \mathbb{N}). \end{aligned}$$

For each $n > n_1$, we have

$$2\alpha(t_n) + 4\beta(t_n) = a(t_n) + b(t_n) + 2(c(t_n) + e(t_n)) < 1 - f(t_n) \leq 1 - f(t_0).$$

Therefore

$$\begin{aligned} d\big(x^*, T(x^*)\big) &\leq \big(2\alpha(t_n) + 4\beta(t_n)\big)d\big(x^*, T(x^*)\big) + 4d\big(x^*, T^n(x)\big) \\ &\quad +2d\big(T^n(x), T^{n-1}(x)\big) + 6d\big(T^{n-1}(x), x^*\big) \\ &\leq \big[1 - f(t_0)\big]d\big(x^*, T(x^*)\big) + 4d\big(x^*, T^n(x)\big) \\ &\quad +2d\big(T^n(x), T^{n-1}(x)\big) + 6d\big(T^{n-1}(x), x^*\big). \end{aligned}$$

By taking limit as $n \longrightarrow \infty$, we see that

$$d\big(x^*, T(x^*)\big) \leq \big[1 - f(t_0)\big]d\big(x^*, T(x^*)\big).$$

Since $1 - f(t_0) < 1$, we have $T(x^*) = x^*$. □

The following result is a special case of Theorem 2.7, which is also a generalization of Theorem 2.6.

COROLLARY 2.8. *Let $\alpha, \gamma : (0, \infty) \longrightarrow [0, 1)$ be two decreasing function with $2\alpha(t) + \gamma(t) < 1$ and $\gamma(t) < \frac{1}{4}$ for all $t > 0$. Let (X, d) be a complete b-metric space and let $T : X \longrightarrow X$ satisfy*

$$d\big(T(x), T(y)\big) \leq \alpha\big(d(x, y)\big)\big[d\big(x, T(x)\big) + d\big(y, T(y)\big)\big] + \gamma(d(x, y))d(x, y) \quad (x \neq y; x, y \in X),$$

where γ is non-zero function. Then T has a unique fixed point x^* and $\lim_{n \longrightarrow \infty} d\big(T^n(x), x^*\big) = 0$ for each $x \in X$.

Proof. Put $\alpha(t) = a(t) = b(t)$, $c(t) = e(t) = 0$ and $\gamma(t) = f(t)$ for all $t > 0$ in Theorem 2.7. □

The following example shows that the above result is a genuine extension of Theorem 2.6.

EXAMPLE 2.9. Let $X = \{t_1, t_2, t_3\}$ and $d : X \times X \longrightarrow [0, \infty)$ be a symmetric function with $d(x, x) = 0$ for all $x \in X$,

$$d(t_1, t_2) = \frac{7}{4}, \; d(t_2, t_3) = 1 \text{ and } d(t_3, t_1) = 3.$$

It is easy to see that (X, d) is a complete b-metric space. Since $d(t_1, t_3) \nleq d(t_1, t_2) + d(t_2, t_3)$, d is not a metric. Define $T : X \longrightarrow X$ by

$$T(t_1) = t_2, \; T(t_2) = t_3, \; T(t_3) = t_3.$$

Let $\alpha = \frac{3}{10}$ and $\gamma = \frac{2}{10}$. Then $2\alpha + \gamma < 1$,

$$d(t_1, T(t_1)) = d(t_1, t_2) = \frac{7}{4}, \; d(t_2, T(t_2)) = d(t_2, t_3) = 1 \text{ and } d(t_3, T(t_3)) = 0.$$

Therefore

$$d(T(x),T(y)) \leq \quad \alpha\left[d(x,T(x))+d(y,T(y))\right]+\gamma d(x,y), \quad (x\neq y; x,y\in X).$$

It follows from Corollary 2.8 that T has a unique fixed point. However, Theorem 2.6 can't be used, since

$$d(T(t_1),T(t_3)) > \frac{1}{2}\left[d(t_1,T(t_1))+d(t_3,T(t_3))\right].$$

Conclusion. In this article, we establish some generalizations of Hardy-Rogers's theorem in complete b-metric space. The above consequences improve some known fixed point theorems in complete b-metric space and enable us to obtain new outcome . By our results, we can also conclude the existence of unique fixed point in some spaces which are not metric

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References

1. M. Abbas, H. Aydi and S. Radenovic, *Fixed point of T-Hardy-Rogers contractive mappings in partially ordered partial metric spaces*, Internat. J. Math., Article ID 313675 (2012), 11 pages.
2. H. Aydi, M. Bota, E. Karapinar and S. Mitrovic, *A fixed Point for set-valued quasi-contractions in b-metric spaces*, Fixed point theory Appl. (2012), **2012**:88.
3. S. Banach, *Sur les opérations dans les ensemble abstraits et leur application aux équations intégrales,* Fund. Mat. **3** (1922), 133–181.
4. M. Boriceanu, *Strict fixed point theorems for multivalued operators in b-metric spaces*, Int. J. Modern Math. **4**(2), (2009), 285Â–-301.
5. M. Bota, A. Molnar and C. Varga, *On EkelandÂ's variational principle in b-metric spaces*, Fixed Point Theory. **12**(2), (2011), 21-Â–28.
6. J. Caristi, *Fixed point theorems for mppings satisfying inwardness condition,* Trans. Amer. Math. Soc. **215** (1976), 241–251.
7. S. Czerwik, *Contraction mappings in b-metric spaces,* Acta Math. Inform. Univ. Ostraviensis **1**, (1995), 5–11.
8. M. Edelstein, *An extension of Banach contraction principle*, Proc. Amer. Math. Soc. 12(1961), 7–10.
9. A. H. Frink, *Generalization of the G_δ-property of complete metric spaces,* Bull. Amer. Math. Soc. **43**, (1937), 133–142.
10. A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, (2003).
11. G. Gruenhage, *Generalized Metric Spaces*, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, (1984), 423–501.
12. G. Hardy and T. Rogers, *A generalization of fixed point theorem of Reich,* Canad. Math. Bull. **16**(2) (1973), 201–206.
13. R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc. **60**(1968), 71–76.
14. W. A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, , Kluwer Academic, London (2001).
15. H. K. Nashine, Z. Kadelburg, S. N. Radenović and J. K. Kim, *Fixed point theorems under Hardy-Rogers contractive conditions on* 0*-complete ordered partial metric spaces*, Fixed Point Theory Appl. (2012), **2012**:180.
16. S. Reich, *Kannan's fixed point theorem,* Bull. Univ. Italiana, **4**(4) (1971), 1–11.
17. S. Reich, *Some remarks concerning contraction mappings,* Canad. Math. Bull. Vol. **14** (1), (1971), 121–124.
18. B. E. Rhoades, *A comparison of various definitions of contractive mappings,* Trans. Amer. Math. Soc. **236**, (1997), 257–290.
19. B. E. Rhoades, S. Sessa, M. S. Khan and M. D. Khan, *Some fixed point theorems for Hardy-Rogers mappings,* Internat. J. Math. **7**(1), (1984), 75–87.

20. A. Meir and E. Keeler, *A theorem on contraction mappings,* J. Math. Anal. Appl. **28** (1969), 326–329.
21. T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness,* Proc. Amer. Math. Soc. **136**(5), (2008), 1861–1869.
22. H. Yingtaweesittikul, *Suzuki type fixed point theorems for generalized muli-valued mappins in b-metric spaces,* Fixed Point Theory Appl. (2013), **2013**:215.
23. T. Zamfirescu, *Fixed point theorems in metric spaces,* Arch. Math. **23**(1), (1972), 292–298.

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EXISTENCE AND STABILITY OF A DAMPED WAVE EQUATION WITH TWO DELAYED TERMS IN BOUNDARY

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ABSTRACT. This paper considers a linear damped wave equation with dynamic bouandary conditions where two feedback terms have a delay. In bounded domain, we first establish the question of well-posedness and uniqueness of the solution for the initial-boundary value problem, using semigroup arguments in [13, 14, 29]. Next, by introducing suitable Lyapunov functionals, exponential stability estimates are obtained under conditions on the delay terms.

KEYWORDS: Damped wave equation; delay feedback; stabilization; semigroup formulation.
AMS Subject Classification: Primary 35B40, 35L05; Secondary 93D15.

1. INTRODUCTION

It is well known that the PDEs with time delay have been much studied during the last years and their results is by now rather developed. See [5, 7, 1, 26, 24, 32, 31]

In the classical theory of delayed wave equations, several main parts are joined in a fruitful way, it is very remarkable that the damped wave equation with two delays occupies a similar position and arise in many applied problems, when it comes to boundary conditions.

Dynamic boundary conditions arise in many physical applications, in particular they occur in elastic models. These conditions appear in modelling dynamic vibrations of linear viscoelastic rods and beams which have attached tip masses at their free ends. See [2, 4, 6, 22, 10].

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In this paper, we consider n-dimensional wave equation with strong damping and boundary conditions when two terms acting on the boundary are delayed in the following problem

$$u'' - \Delta u - a\Delta u' = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$
$$u = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \tag{1.2}$$
$$\mu u'' + \tfrac{\partial(u+au')}{\partial \nu} = -k_1 u'(x, t-\tau_1) - k_2 u'(x, t-\tau_2) \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{1.3}$$
$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.4}$$
$$u' = f_0 \quad \text{in } \Gamma_1 \times \left(-\max\left(\tau_1(0), \tau_2(0)\right), 0\right), \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set with boundary Γ of class C^2. We assume that Γ is divided into two parts Γ_0 and Γ_1; i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and $meas\Gamma_0 \neq 0$.

The vector $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, $\tau_i = \tau_i(t)$, $i = 1, 2$ is the time delay, μ, a, k are real numbers, with $\mu \geq 0, a > 0$, and the initial datum (u_0, u_1, f_0) belongs to a suitable space.

We define the energy of system (1.1)–(1.5) as

$$\begin{aligned} E(t) := \frac{1}{2}\int_\Omega \{u'^2 + |\nabla u|^2\} dx + \frac{\xi_1}{2}\int_{t-\tau_1}^t \int_{\Gamma_1} e^{\lambda(s-t)} u'^2(x,s) d\Gamma ds \\ + \frac{\xi_2}{2}\int_{t-\tau_2}^t \int_{\Gamma_1} e^{\lambda(s-t)} u'^2(x,s) d\Gamma ds + \frac{\mu}{2}\int_{\Gamma_1} u'^2 d\Gamma, \end{aligned} \tag{1.6}$$

where ξ_i, λ are suitable positive constants.

To motivate our work, let us mention the major work [25], when the authors studied well-posedness and exponential stability of the problem (1.1)–(1.5) with structural damping and boundary delay in both cases $\mu > 0$ and $\mu = 0$ in a bounded and smooth domain, where $k_2 = 0$. The analogous problem with boundary feedback has been introduced and studied by Xu, Yung, Li [31] in one-space dimension using a fine spectral analysis and in higher space dimension by the authors [26]. The case of time-varying delay has been already studied in [28] in one space dimension and in general dimension, with a possibly degenerate delay, in [27]. Both these papers deal with boundary feedback.

When $\tau_1(t) \equiv \tau_2(t) \equiv 0$ (in absence of delays), it is well-known that the above problem is exponentially stable. See in this direction [3, 19, 18, 20, 15, 17, 16, 33, 12, 23, 8, 30, 10]. When $\mu = 0, k_2 = 0$, in presence of a constant delay, and the condition (1.3) is substituted by

$$\frac{\partial u}{\partial \nu} = -ku_t(x, t-\tau), \quad \Gamma_1 \times (0, +\infty),$$

the system becomes unstable for arbitrarily small delays (see [6]).

The above model without delay (e.g. $\tau = 0$) has been proposed in one dimension by Pellicer and Sòla-Morales [30] as an alternative model for the classical spring-mass damper system. In both cases, no rates of convergence are proved. In dimension higher than 1, we refer to Gerbi and Said-Houari [10] where a nonlinear boundary feedback is even considered and the exponential growth of the energy is proved if the initial data are large enough. A different problem with a dynamic boundary condition (without delay), motivated by the study of flows of gas in a channel with porous walls, is analyzed in [8] where exponential decay is proved.

2. Assumptions

We assume, on the time-delay functions, that there exist positive constants $\overline{\tau}_0, \widetilde{\tau}_0, \overline{\tau}, \widetilde{\tau}$ such that

$$0 \quad < \quad \overline{\tau}_0 \le \tau_1 \le \overline{\tau}, \quad \forall t > 0, \tag{2.1}$$

$$0 \quad < \quad \widetilde{\tau}_0 \le \tau_2 \le \widetilde{\tau}, \quad \forall t > 0. \tag{2.2}$$

Moreover, we assume

$$\tau_i \in W^{2,\infty}([0,T]), \quad \forall T > 0, i = 1, 2, \tag{2.3}$$

$$\max\{\tau_1', \tau_2'\} \le d < 1, \quad \forall t > 0, \tag{2.4}$$

where d is the positive constant.

Under (2.3)-(2.4) we will prove that an exponential stability result holds under a suitable assumption between the coefficients a and k_1, k_2.

Let C^* be a Poincaré's type constant defined as the smallest positive constant such that

$$\int_{\Gamma_1} |v|^2 d\Gamma \le C^* \int_\Omega |\nabla v|^2 dx, \quad \forall v \in H^1_{\Gamma_0}(\Omega), \tag{2.5}$$

where, as usual,

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u = 0 \quad \text{on } \Gamma_0\}.$$

We will give a well-posedness result under the assumption

$$\frac{|k| C^*}{\sqrt{1-d}} \le \frac{a}{2}, \quad k = \max\{k_1, k_2\}. \tag{2.6}$$

We omit the space variable x of $u(x,t)$, $u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.

3. EXISTENCE OF SOLUTION

First as in [26] we introduce the new variables

$$z(x,\rho,t) \quad = \quad u'(x, t - \tau_1 \rho) \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty), \tag{3.1}$$

$$w(x,\rho,t) \quad = \quad u'(x, t - \tau_2 \rho) \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty). \tag{3.2}$$

Then we have

$$\tau_1 z'(x,\rho,t) + z_\rho(x,\rho,t) \quad = \quad 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \tag{3.3}$$

$$\tau_2 w'(x,\rho,t) + w_\rho(x,\rho,t) \quad = \quad 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{3.4}$$

Therefore problem (1.1)–(1.5) is equivalent to

$$u'' - \Delta u - a\Delta u' = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{3.5}$$

$$\tau_1 z'(x,\rho,t) + (1 - \tau_1'\rho) z_\rho(x,\rho,t) = 0 \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty), \tag{3.6}$$

$$\tau_2 w'(x,\rho,t) + (1 - \tau_2'\rho) z_\rho(x,\rho,t) = 0 \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty), \tag{3.7}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0,+\infty), \tag{3.8}$$

$$\mu u'' = -\tfrac{\partial(u + au')}{\partial \nu} - k_1 z(x,1,t) - k_2 w(x,1,t) \quad \text{on } \Gamma_1 \times (0,+\infty), \tag{3.9}$$

$$z(x,0,t) = w(x,0,t) = u' \quad \text{on } \Gamma_1 \times (0,\infty), \tag{3.10}$$

$$u(x,0) = u_0(x) \text{ and } u'(x,0) = u_1(x) \quad \text{in } \Omega, \tag{3.11}$$

$$z(x,\rho,0) = f_0(x, -\rho\tau_1(0)) \quad \text{in } \Gamma_1 \times (0,1), \tag{3.12}$$

$$w(x,\rho,0)=f_0(x,-\rho\tau_2(0)) \quad \text{in } \Gamma_1\times(0,1). \tag{3.13}$$

Let us denote

$$U=(u,u',\gamma u',z,w)^T,$$

where γ is the trace operator on Γ_1. Then problem (1.1)–(1.5) equivalent to

$$\begin{aligned} U' &= (u',u'',\gamma_1 u'',z',w')^T \\ &= (u',\Delta u+a\Delta u',-\mu^{-1}(\tfrac{\partial(u+au')}{\partial\nu}+k_1 z(\cdot,1,\cdot)+k_2 w(\cdot,1,\cdot)),\tfrac{\tau_1'(t)\rho-1}{\tau_1}z_\rho,\tfrac{\tau_2'(t)\rho-1}{\tau_2}w_\rho)^T. \end{aligned}$$

Therefore, problem (1.1)–(1.5) can be rewritten as

$$\begin{aligned} U' &= A(t)U, \\ U(0) &= (u_0,u_1,\gamma_1 u_1,f_0(\cdot,-\cdot\,\tau_1),f_0(\cdot,-\cdot\,\tau_2))^T, \end{aligned} \tag{3.14}$$

where $A(t)$ is defined by

$$\begin{aligned} &A(t)(u,v,v_1,z,w)^T \\ &= \left(v,\Delta(u+av),-\mu^{-1}\left(\tfrac{\partial(u+av)}{\partial\nu}+k_1 z(\cdot,1)+k_2 w(\cdot,1)\right),\tfrac{\tau_1'(t)\rho-1}{\tau_1(t)}z_\rho,\tfrac{\tau_2'(t)\rho-1}{\tau_2(t)}w_\rho\right)^T, \end{aligned}$$

with domain of $A(t)$ given by

$$\begin{aligned} &D(A(t)) \\ &= \Big\{(u,v,v_1,z,w)^T\in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1) \\ &\qquad\qquad \times (L^2(\Gamma_1\times(0,1)))^2\times L^2(\Gamma_1\times(0,1)), \\ &u+av\in E\left(\Delta,L^2(\Omega)\right),\tfrac{\partial(u+av)}{\partial\nu}\in L^2(\Gamma_1),v=v_1=z(\cdot,0)=w(\cdot,0)\quad\text{on }\Gamma_1\Big\} \end{aligned} \tag{3.15}$$

is independent of the time t, i.e.,

$$D(A(t))=D(A(0)),\quad t>0. \tag{3.16}$$

where

$$E(\Delta,L^2(\Omega))=\{u\in H^1(\Omega):\Delta u\in L^2(\Omega)\}.$$

For a function $u\in E(\Delta,L^2(\Omega))$, $\frac{\partial u}{\partial\nu}$ belongs to $H^{-1/2}(\Gamma_1)$ and the next Green formula

$$\int_\Omega \nabla u\nabla q dx=-\int_\Omega \Delta u q dx+\langle\frac{\partial u}{\partial\nu};q\rangle_{\Gamma_1},\ \forall q\in H^1_{\Gamma_0}(\Omega), \tag{3.17}$$

is valid (see [11]), where $\langle\cdot;\cdot\rangle_{\Gamma_1}$ means the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$.

Let us introduce a Hilbert space $\tilde H$ defined by

$$\tilde H=H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1)\times(L^2(\Gamma_1\times(0,1)))^2,$$

equipped with the standard inner product

$$\begin{aligned} &\left\langle(u,v,v_1,z,w)^T,(\tilde u,\tilde v,\tilde v_1,\tilde z,\widetilde w)^T\right\rangle_{\tilde H} \\ &=\int_\Omega\{\nabla u(x)\nabla\tilde u(x)+v(x)\tilde v(x)\}dx \\ &+\mu\int_{\Gamma_1}v_1(x)\tilde v_1(x)d\Gamma+\xi_1\tau_1(t)\int_{\Gamma_1}\int_0^1 z(x,\rho)\tilde z(x,\rho)d\rho d\Gamma \\ &+\xi_2\tau_2(t)\int_{\Gamma_1}\int_0^1 w(x,\rho)\,\widetilde w(x,\rho)\,d\rho d\Gamma. \end{aligned} \tag{3.18}$$

Remark 3.1. The time varying operator $A(t)$ is an unbounded in $\tilde H$.

The next theorem is our main tool to prove well-posedness results, its proof is similar in [13].

Theorem 3.1. *Assume that*

(i) $D(A(0))$ *is a dense subset of* $\tilde{H}$,
(ii) $D(A(t)) = D(A(0))$ *for all* $t > 0$,
(iii) *for all* $t \in [0, T]$, $A(t)$ *generates a strongly continuous semigroup on* $\tilde{H}$ *and the family* $A = \{A(t) : t \in [0, T]\}$ *is stable with stability constants* C *and* m *independent of* t *(i.e. the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $A(t)$ *satisfies* $\|S_t(s)u\|_{\tilde{H}} \leq Ce^{ms}\|u\|_{\tilde{H}}$, *for all* $u \in \tilde{H}$ *and* $s \geq 0$*),*
(iv) $\partial_t A$ *belongs to* $L^\infty_*([0, T], B(D(A(0)), \tilde{H}))$, *the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0, T]$ *into the set* $B(D(A(0)), \tilde{H})$ *of bounded operators from* $D(A(0))$ *into* $\tilde{H}$.

Then, problem (3.14) has a unique solution $U \in C([0, T], D(A(0))) \cap C^1([0, T], \tilde{H})$ *for any initial datum in* $D(A(0))$.

Let ξ_1 and ξ_2 are a positive constants such that

$$\frac{|k|}{\sqrt{1-d}} \leq \xi_i \leq \frac{a}{C^*} - \frac{|k|}{\sqrt{1-d}}, \quad \text{for } i = 1, 2. \tag{3.19}$$

In order to deduce a well-posedness result, we define on $\tilde{H}$ the time dependent inner product

$$\begin{aligned}
&\left\langle (u, v, v_1, z, w)^T, (\tilde{u}, \tilde{v}, \tilde{v}_1, \tilde{z}, \widetilde{w})^T \right\rangle_t \\
&= \int_\Omega \{\nabla u(x)\nabla\tilde{u}(x) + v(x)\tilde{v}(x)\}dx \\
&+ \mu \int_{\Gamma_1} v_1(x)\tilde{v}_1(x)d\Gamma \\
&+ \xi_1\tau_1(t)\int_{\Gamma_1}\int_0^1 z(x,\rho)\tilde{z}(x,\rho)d\rho\, d\Gamma \\
&+ \xi_2\tau_2(t)\int_{\Gamma_1}\int_0^1 w(x,\rho)\,\widetilde{w}(x,\rho)\,d\rho d\Gamma.
\end{aligned} \tag{3.20}$$

and using Theorem 3.1.

Theorem 3.2. *Assume that (2.1)–(2.4) and (2.6) hold. Then for any initial datum* $U_0 \in \tilde{H}$ *there exists a unique solution* $U \in C([0, +\infty), \tilde{H})$ *of problem (3.14). Moreover, if* $U_0 \in D(A(0))$, *then*

$$U \in C([0, +\infty), D(A(0))) \cap C^1([0, +\infty), \tilde{H}).$$

We need to check assumptions of Theorem 3.1 for problem (3.14).

Lemma 3.2. $D(A(0))$ *is dense in* $\tilde{H}$.

Proof. Let $(f, g, g_1, h_1, h_2)^T \in \tilde{H}$ be orthogonal to all elements of $D(A(0))$, that is,

$$\begin{aligned}
0 &= \left\langle (u, v, v_1, z, w)^T, (f, g, g_1, h_1, h_2)^T \right\rangle_{\tilde{H}} \\
&= \int_\Omega \{\nabla u(x)\nabla f(x) + v(x)g(x)\}dx \\
&+ \int_{\Gamma_1} v_1 g_1 d\Gamma + \int_{\Gamma_1}\int_0^1 z(x,\rho)h_1(x,\rho)d\rho d\Gamma
\end{aligned}$$

$$+\int_{\Gamma_1}\int_0^1 w(x,\rho)h_2(x,\rho)d\rho d\Gamma$$

$\forall (u,v,v_1,z,w)^T \in D(A(0))$.

Taking $u=v=0$ (then $v_1=0$) , $z=0$ and $w\in D(\Gamma_1\times(0,1))$. As $(0,0,0,0,w)^T \in D(A(0))$, we obtain

$$\int_{\Gamma_1}\int_0^1 w(x,\rho)h_2(x,\rho)d\rho d\Gamma = 0.$$

Since $D(\Gamma_1\times(0,1))$ is dense in $L^2(\Gamma_1\times(0,1))$, we deduce that $h_2=0$.

In the same way, by taking $u=v=0$ (then $v_1=0$), $w=0$ and $z\in D(\Gamma_1\times(0,1))$. As $(0,0,0,z,0)^T \in D(A(0))$, we obtain

$$\int_{\Gamma_1}\int_0^1 z(x,\rho)h_1(x,\rho)d\rho d\Gamma = 0.$$

Since $D(\Gamma_1\times(0,1))$ is dense in $L^2(\Gamma_1\times(0,1))$, we deduce that $h_1=0$. Also for $u=z=w=0$ and $v\in D(\Omega)$ (then $v_1=0$) we see that $g=0$. Therefore, for $u=0$, $z=0$ and $w=0$, we deduce also

$$\int_{\Gamma_1} g_1 v_1 d\Gamma = 0,\ \forall v_1\in D(\Gamma_1),$$

and so $g_1=0$.

The above orthogonality condition is then reduced to

$$0=\int_\Omega \nabla u\nabla f dx,\ \forall (u,v,v_1,z,w)^T\in D(A(0)).$$

By restricting ourselves to $v=z=w=0$, we obtain

$$\int_\Omega \nabla u(x)\nabla f(x)dx = 0,\ \forall (u,0,0,0,0)^T\in D(A(0)).$$

But we easily see that $(u,0,0,0,0)^T\in D(A(0))$ if and only if $u\in E(\Delta,L^2(\Omega))\cap H^1_{\Gamma_0}(\Omega)$. This set is dense in $H^1_{\Gamma_0}(\Omega)$ (equipped with the inner product $\langle .,.\rangle_{H^1_{\Gamma_0}(\Omega)}$), thus we conclude that $f=0$. $\square$

Lemma 3.3. *Let $\Phi=(u,v,v_1,z,w)^T$, then*

$$\|\Phi\|_t \le \|\Phi\|_s e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|},\ \forall t,s\in[0,T], \tag{3.21}$$

where d is a positive constant.

Proof. For all $s,t\in[0,T]$, we have

$$\begin{aligned}
&\|\Phi\|_t^2-\|\Phi\|_s^2 e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}\\
&=\left(1-e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}\right)\left(\int_\Omega(|\nabla u(x)|^2+v^2)dx+\mu\int_{\Gamma_1}v_1^2 d\Gamma\right)\\
&+\xi_1\left(\tau_1(t)-\tau_1(s)e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}\right)\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma\\
&+\xi_2\left(\tau_2(t)-\tau_2(s)e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}\right)\int_{\Gamma_1}\int_0^1 w^2(x,\rho)d\rho d\Gamma.
\end{aligned}$$

We notice that

$$e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}\ge 1.$$

Moreover
$$\tau_1(t)-\tau_1(s)e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}\leq 0,$$
and
$$\tau_2(t)-\tau_2(s)e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}\leq 0,$$
for some $d>0$.

Indeed,
$$\tau_1(t)=\tau_1(s)+\tau_1'(a)(t-s),$$
and
$$\tau_2(t)=\tau_2(s)+\tau_2'(b)(t-s),$$
where $a,b\in(s,t)$, and thus,
$$\begin{aligned}\frac{\tau_1(t)}{\tau_1(s)} &= 1+\frac{|\tau_1'(a)|}{\tau_1(s)}|t-s|,\\ \frac{\tau_2(t)}{\tau_2(s)} &= 1+\frac{|\tau_2'(b)|}{\tau_2(s)}|t-s|.\end{aligned}$$
By (2.3), τ_1' and τ_2' are bounded on $[0,T]$ and therefore, recalling also (2.1), (2.2),
$$\frac{\tau_1(t)}{\tau_1(s)}\leq 1+\frac{d}{\overline{\tau}_0}|t-s|\leq e^{\frac{d}{\overline{\tau}_0}|t-s|},$$
$$\frac{\tau_2(t)}{\tau_2(s)}\leq 1+\frac{d}{\tilde{\tau}_0}|t-s|\leq e^{\frac{d}{\tilde{\tau}_0}|t-s|},$$
thus
$$\frac{\tau_1(t)}{\tau_1(s)}\leq e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|},$$
and
$$\frac{\tau_2(t)}{\tau_2(s)}\leq e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}.$$
This complete the proof. □

Lemma 3.4. *Under condition (3.19), the operator $\tilde{A}(t)=A(t)-\kappa(t)I$ is dissipative, and*
$$\frac{d}{dt}\tilde{A}(t)\in L_*^\infty([0,T],B(D(A(0)),\tilde{H})),$$
where
$$\kappa(t)=\frac{\sqrt{\tau_1'^2(t)+1}}{2\tau_1(t)}+\frac{\sqrt{\tau_2'^2(t)+1}}{2\tau_2(t)}. \tag{3.22}$$

Proof. Taking $U=(u,v,v_1,z,w)^T\in D(A(t))$. Then, for a fixed t,
$$\begin{aligned}\langle A(t)U,U\rangle_t &= \int_\Omega\{\nabla v(x)\nabla u(x)+v(x)\Delta(u(x)+av(x))\}dx\\ &-\xi_1\int_{\Gamma_1}\int_0^1(1-\tau_1'(t)\rho)z_\rho(x,\rho)z(x,\rho)d\rho d\Gamma\\ &-\xi_2\int_{\Gamma_1}\int_0^1(1-\tau_2'(t)\rho)w_\rho(x,\rho)w(x,\rho)d\rho d\Gamma\\ &-\int_{\Gamma_1}\left(\frac{\partial(u+av)}{\partial\nu}(x)+k_1z(x,1)+k_2w(x,1)\right)v(x)d\Gamma.\end{aligned}$$

By Green's formula,

$$\begin{aligned}\langle A(t)U,U\rangle_t &= -k\int_{\Gamma_1} z(x,1)v(x)d\Gamma - a\int_\Omega |\nabla v(x)|^2dx\\ &- \xi_1\int_{\Gamma_1}\int_0^1 (1-\tau_1'(t)\rho)z_\rho(x,\rho)z(x,\rho)d\rho d\Gamma\\ &- \xi_2\int_\Gamma\int_0^1 (1-\tau'(t)\rho)\,w_\rho(x,\rho)\,w(x,\rho)\,d\rho d\Gamma. \end{aligned}\tag{3.23}$$

Integrating by parts in ρ and ρ we obtain

$$\begin{aligned}&\int_{\Gamma_1}\int_0^1 z_\rho(x,\rho)z(x,\rho)(1-\tau_1'(t)\rho)d\rho d\Gamma\\ &=\int_{\Gamma_1}\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}z^2(x,\rho)(1-\tau_1'(t)\rho)d\rho d\Gamma\\ &=\frac{\tau_1'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma+\frac{1}{2}\int_{\Gamma_1}\{z^2(x,1)(1-\tau_1'(t))-z^2(x,0)\}d\Gamma, \end{aligned}\tag{3.24}$$

and

$$\begin{aligned}&\int_{\Gamma_1}\int_0^1 w_\rho(x,\rho)\,w(x,\rho)\,(1-\tau_2'(t))\,d\rho d\Gamma\\ &=\int_{\Gamma_1}\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}w^2(x,\rho)\,(1-\tau_2'(t))\,d\rho d\Gamma\\ &=\frac{\tau_2'(t)}{2}\int_{\Gamma_1}\int_0^1 w^2(x,\rho)\,d\rho d\Gamma+\int_{\Gamma_1}\left\{w^2(x,1)\,(1-\tau_2'(t))-w^2(x,0)\right\}d\Gamma. \end{aligned}\tag{3.25}$$

Therefore, from (3.23), (3.24) and (3.25),

$$\begin{aligned}&\langle A(t)U,U\rangle_t\\ &=-k_1\int_{\Gamma_1} z(x,1)v(x)d\Gamma-k_2\int_{\Gamma_1} w(x,1)v(x)d\Gamma-a\int_\Omega|\nabla v(x)|^2dx\\ &-\frac{\xi_1}{2}\int_{\Gamma_1}\{z^2(x,1)(1-\tau_1'(t))-z^2(x,0)\}d\Gamma-\frac{\xi_1\tau_1'(t)}{2}\int_{\Gamma_1}\int_0^1 Z^2(x,\rho)d\rho d\Gamma\\ &-\frac{\xi_2}{2}\int_{\Gamma_1}\{w^2(x,1)(1-\tau_2'(t))-w^2(x,0)\}d\Gamma-\frac{\xi_2\tau_2'(t)}{2}\int_{\Gamma_1}\int_0^1 w^2(x,\rho)d\rho d\Gamma\\ &=-k_1\int_{\Gamma_1} z(x,1)v(x)d\Gamma-k_2\int_{\Gamma_1} w(x,1)v(x)d\Gamma\\ &-a\int_\Omega|\nabla v(x)|^2dx-\frac{\xi_1}{2}\int_{\Gamma_1} z^2(x,1)(1-\tau_1'(t))d\Gamma\\ &-\frac{\xi_2}{2}\int_{\Gamma_1} w^2(x,1)(1-\tau_2'(t))d\Gamma+\frac{\xi_1}{2}\int_{\Gamma_1} v^2(x)d\Gamma-\frac{\xi_1\tau_1'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma\\ &+\frac{\xi_2}{2}\int_{\Gamma_1} v^2(x)d\Gamma-\frac{\xi_2\tau_2'(t)}{2}\int_{\Gamma_1}\int_0^1 w^2(x,\rho)d\rho d\Gamma. \end{aligned}$$

Using Cauchy-Schwarz's and Poincaré's inequalities, a trace estimate, it follows that

$$\begin{aligned}&\langle A(t)U,U\rangle_t\\ &\leq-\left[\left(\frac{a}{2}-\frac{|k|C^*}{2\sqrt{1-d}}-\frac{\xi_1}{2}C^*\right)+\left(\frac{a}{2}-\frac{|k|\,C^*}{2\sqrt{1-d}}-\frac{\xi_2}{2}C^*\right)\right]\int_\Omega|\nabla v(x)|^2dx \end{aligned}$$

$$- \left(\frac{\xi_1}{2}(1-d) - \frac{|k|}{2}\sqrt{1-d} \right) \int_{\Gamma_1} z^2(x,1) d\Gamma$$
$$- \left(\frac{\xi_2}{2}(1-d) - \frac{|k|}{2}\sqrt{1-d} \right) \int_{\Gamma_1} w^2(x,1) d\Gamma + \kappa(t)\langle U, U \rangle_t, \tag{3.26}$$

where

$$\kappa(t) = \frac{(\tau_1'^2(t)+1)^{\frac{1}{2}}}{2\tau_1(t)} + \frac{\left(\tau_2'^2(t)+1\right)^{\frac{1}{2}}}{2\tau_2(t)}.$$

then, from (3.19),

$$\langle A(t)U, U\rangle_t \le \kappa(t)\langle U, U\rangle_t. \tag{3.27}$$

Moreover,

$$\kappa'(t) = \frac{\tau_1''(t)\tau_1'(t)}{2\tau_1(t)(\tau_1'^2+1)^{\frac{1}{2}}} - \frac{\tau_1'(t)(\tau_1'^2(t)+1)^{\frac{1}{2}}}{2\tau_1(t)^2}$$
$$+ \frac{\tau_2''(t)\tau_2'(t)}{2\tau_2(t)(\tau_2'^2(t)+1)^{\frac{1}{2}}} - \frac{\tau_2'(t)(\tau_2'^2(t)+1)^{\frac{1}{2}}}{2\tau_2(t)^2},$$

is bounded on $[0,T]$ for all $T>0$ (by (2.1) and (2.3) and we have

$$\frac{d}{dt}A(t)U$$
$$= (0,0,0, \frac{\tau_1''(t)\tau_1(t)\rho - \tau_1'(t)(\tau_1'(t)\rho - 1)}{\tau_1(t)^2} z_\rho, \frac{\tau_2''(t)\tau_2(t)\rho - \tau_2'(t)\left(\tau_2'(t)\rho - 1\right)}{\tau_2(t)^2} w_\rho)^T \tag{3.28}$$

with $\frac{\tau_1''(t)\tau_1(t)\rho - \tau_1'(t)(\tau_1'(t)\rho-1)}{\tau_1(t)^2}$ and $\frac{\tau_2''(t)\tau_2(t)\rho - \tau_2'(t)\left(\tau_2'(t)\rho-1\right)}{\tau_2(t)^2}$ are bounded on $[0,T]$. Thus

$$\frac{d}{dt}\tilde{A}(t) \in L_*^\infty([0,T], B(D(A(0)), \tilde{H})), \tag{3.29}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(D(A(0)), \tilde{H})$. $\square$

Lemma 3.5. *For fixed $t>0$ and $\lambda>0$, the operator $\lambda I - A(t)$ is surjective.*

Proof. Let $(f, g, g_1, h_1, h_2)^T \in \tilde{H}$, we seek $U = (u, v, v_1, z, w)^T \in D(A(t))$ solution of

$$(\lambda I - A(t))(u, v, v_1, z, w)^T = (f, g, g_1, h_1, h_2)^T,$$

that is verifying

$$\begin{gathered} \lambda u - v = f, \\ \lambda v - \Delta(u + av) = g, \\ \lambda v_1 + \mu^{-1}\left(\frac{\partial(u+av)}{\partial\nu}(x) + k_1 z(x,1) + k_2 w(x,1)\right) = g_1, \\ \lambda z + \frac{1-\tau_1'(t)\rho}{\tau_1(t)} z_\rho = h_1, \\ \lambda w + \frac{1-\tau_2'(t)\rho}{\tau_2(t)} w_\rho = h_2. \end{gathered} \tag{3.30}$$

Suppose that we have found u with the appropriate regularity. Then

$$v = \lambda u - f, \tag{3.31}$$

and we can determine z, w. Indeed, by (3.15),

$$z(x,0) = v(x), \quad \text{for } x \in \Gamma_1, \tag{3.32}$$

and, from (3.30),

$$\lambda z(x,\rho) + \frac{1-\tau_1'(t)\rho}{\tau_1(t)} z_\rho(x,\rho) = h_1(x,\rho), \quad \text{for } x \in \Gamma_1,\ \rho \in (0,1). \tag{3.33}$$

Then, by (3.32) and (3.33), we obtain

$$z(x,\rho) = v(x)e^{-\lambda\rho\tau_1(t)} + \tau_1(t)e^{-\lambda\rho\tau_1(t)} \int_0^\rho h_1(x,\sigma)e^{\lambda\sigma\tau_1(t)}d\sigma,$$

if $\tau_1'(t) = 0$, and

$$\begin{aligned} z(x,\rho) &= v(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} \\ &\quad + e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} \int_0^\rho \frac{h_1(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\sigma)}d\sigma, \end{aligned}$$

otherwise. From (3.31),

$$\begin{aligned} z(x,\rho) &= \lambda u(x)e^{-\lambda\rho\tau_1(t)} - f(x)e^{-\lambda\rho\tau_1(t)} \\ &\quad + \tau_1(t)e^{-\lambda\rho\tau_1(t)} \int_0^\rho h_1(x,\sigma)e^{\lambda\sigma\tau_1(t)}d\sigma, \end{aligned} \tag{3.34}$$

on $\Gamma_1 \times (0,1)$.

If $\tau_1'(t) = 0$, and

$$\begin{aligned} z(x,\rho) &= \lambda u(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} - f(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} \\ &\quad + e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} \int_0^\rho \frac{h_1(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\sigma)}d\sigma, \end{aligned} \tag{3.35}$$

on $\Gamma_1 \times (0,1)$ otherwise.

In particular, if $\tau_1'(t) = 0$,

$$z(x,1) = \lambda u(x)e^{-\lambda\tau_1(t)} + z_0(x), \quad x \in \Gamma_1, \tag{3.36}$$

with $z_0 \in L^2(\Gamma_1)$ defined by

$$z_0(x) = -f(x)e^{-\lambda\tau_1(t)} + \tau_1(t)e^{-\lambda\tau_1(t)} \int_0^1 h_1(x,\sigma)e^{\lambda\sigma\tau_1(t)}d\sigma, \quad x \in \Gamma_1, \tag{3.37}$$

and, if $\tau_1'(t) \neq 0$,

$$z(x,1) = \lambda u(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} + z_0(x), \quad x \in \Gamma_1, \tag{3.38}$$

with $z_0 \in L^2(\Gamma_1)$ defined by

$$\begin{aligned} z_0(x) &= -f(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} \\ &\quad + e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} \int_0^1 \frac{h_1(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\sigma)}d\sigma. \end{aligned} \tag{3.39}$$

Now we will determine w, again by (3.15),

$$\lambda w(x,\rho) + \frac{1-\tau_2'(t)\rho}{\tau_2(t)} w_\rho(x,\rho) = h_2(x,\rho),$$

then

$$w(x,\rho) = v(x)e^{-\lambda\rho\tau_2(t)} + \tau_2(t)e^{-\lambda\rho\tau_2(t)} \int_0^\rho h_2(x,\sigma)e^{\lambda\sigma\tau_2(t)}d\sigma,$$

if $\tau_2'(t) = 0$, and

$$w(x,\rho) = v(x)e^{\lambda \frac{\tau_2(t)}{\tau_2'} \ln(1-\tau_2'(t)\rho)} + e^{\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t)\rho)} \int_0^\rho \frac{h_2(x,\sigma)\tau_2(t)}{1-\tau_2'(t)\sigma} e^{-\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t)\sigma)} d\sigma,$$

otherwise. From (3.31),

$$w(x,\rho) = \lambda u(x)e^{-\lambda\rho\tau_2(t)} - f(x)e^{-\lambda\rho\tau_2(t)} + \tau_2(t)e^{-\lambda\rho\tau_2(t)} \int_0^\rho h_2(x,\sigma)e^{\lambda\sigma\tau_2(t)} d\sigma, \tag{3.40}$$

if $\tau_2'(t) = 0$, and

$$w(x,\rho) = \lambda u(x)e^{\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t)\rho)} - f(x)e^{\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t)\rho)} + e^{\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t)\rho)} \int_0^\rho \frac{h_2(x,\sigma)\tau_2(t)}{1-\tau_2'(t)\sigma} e^{-\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t)\sigma)} d\sigma, \tag{3.41}$$

on $\Gamma_1 \times (0,1)$ otherwise.

In particular, if $\tau_2'(t) = 0$,

$$w(x,1) = \lambda u(x)e^{-\lambda\tau_2(t)} + w_0(x), \quad x \in \Gamma_1, \tag{3.42}$$

with $w_0 \in L^2(\Gamma_1)$ defined by

$$w_0(x) = -f(x)e^{-\lambda\tau_2(t)} + \tau_2(t)e^{-\lambda\tau_2(t)} \int_0^1 h_2(x,\sigma)e^{\lambda\sigma\tau_2(t)} d\sigma, \quad x \in \Gamma_1, \tag{3.43}$$

and, if $\tau_2'(t) \neq 0$,

$$w(x,1) = \lambda u(x)e^{\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t))} + w_0(x), \quad x \in \Gamma_1, \tag{3.44}$$

with $w_0 \in L^2(\Gamma_1)$ defined by

$$w_0(x) = -f(x)e^{\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t))} + e^{\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t))} \int_0^1 \frac{h_2(x,\sigma)\tau_2(t)}{1-\tau_2'(t)\sigma} e^{-\lambda \frac{\tau_2(t)}{\tau_2'(t)} \ln(1-\tau_2'(t)\sigma)} d\sigma, \tag{3.45}$$

for $x \in \Gamma_1$. Then, we have to find u. In view of the equation

$$\lambda v - \Delta(u + av) = g,$$

we set $s = u + av$ and look at s. Now according to (3.31), we may write

$$v = \lambda u - f = \lambda s - f - \lambda av,$$

or equivalently

$$v = \frac{\lambda}{1+\lambda a}s - \frac{1}{1+\lambda a}f. \tag{3.46}$$

Hence once s will be found, we will get v by (3.46) and then u by $u = s - av$, or equivalently

$$u = \frac{1}{1+\lambda a}s + \frac{a}{1+\lambda a}f. \tag{3.47}$$

By (3.46) and (3.30), the function s satisfies

$$\frac{\lambda^2}{1+\lambda a}s - \Delta s = g + \frac{\lambda}{1+\lambda a}f \quad \text{in } \Omega, \tag{3.48}$$

with the boundary conditions

$$s = 0 \quad \text{on } \Gamma_0, \tag{3.49}$$

as well as (at least formally)

$$\frac{\partial s}{\partial \nu} = \mu g_1 - \mu\lambda v_1 - k_1 z(\cdot,1) - k_2 w(\cdot,1) \quad \text{on } \Gamma_1,$$

which becomes due to (3.46), (3.47), (3.36), (3.38), (3.40) and the requirement that $v_1 = \gamma_1 v$ on Γ_1:

$$\frac{\partial s}{\partial \nu} = -\frac{\lambda(k_1 e^{-\lambda\tau_1(t)} + k_2 e^{-\lambda\tau_2(t)} + \mu\lambda)}{1+\lambda a} s + l \quad \text{on } \Gamma_1, \tag{3.50}$$

where

$$l = \mu g_1 + \frac{\lambda(\mu - k_1 a e^{-\lambda\tau_1(t)} - k_2 a e^{-\lambda\tau_2(t)})}{1+\lambda a} f - k_1 z_0 - k_2 w_0 \quad \text{on } \Gamma_1,$$

if $\tau_1'(t) = \tau_2'(t) = 0$, before $\tau_1'(t) \neq 0$ and $\tau_2'(t) \neq 0$ we obtain

$$\frac{\partial s}{\partial \nu} = -\frac{\lambda(k_1 e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} + k_2 e^{-\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t))} + \mu\lambda)}{1+\lambda a} s + \tilde{l} \quad \text{on } \Gamma_1, \tag{3.51}$$

where

$$\tilde{l} = \mu g_1 + \frac{\lambda(\mu - k_1 a e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} - k_2 a e^{-\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t))})}{1+\lambda a} f - k_1 z_0 - k_2 w_0 \quad \text{on } \Gamma_1.$$

From (3.48), integrating by parts, and using (3.49), (3.50), (3.51) we find the variational problem

$$\begin{aligned} &\int_\Omega (\frac{\lambda^2}{1+\lambda a} sq + \nabla s \cdot \nabla q) dx + \int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\tau_1} + k_2 e^{-\lambda\tau_2} + \mu\lambda)}{1+\lambda a} sq d\Gamma \\ &= \int_\Omega (g + \frac{\lambda}{1+\lambda a} f) q dx + \int_{\Gamma_1} l q d\Gamma \quad \forall q \in H^1_{\Gamma_0}(\Omega), \end{aligned} \tag{3.52}$$

if $\tau_1'(t) = \tau_2'(t) = 0$, before $\tau_1'(t) \neq 0$ and $\tau_2'(t) \neq 0$ we obtain

$$\begin{aligned} &\int_\Omega (\frac{\lambda^2}{1+\lambda a} sq + \nabla s \cdot \nabla q) dx + \int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\frac{\tau_1}{\tau_1'}\ln(1-\tau_1')} + k_2 e^{-\lambda\frac{\tau_2}{\tau_2'}\ln\left(1-\tau_2'\right)} + \mu\lambda)}{1+\lambda a} sq d\Gamma \\ &= \int_\Omega (g + \frac{\lambda}{1+\lambda a} f) q dx + \int_{\Gamma_1} \tilde{l} q d\Gamma \quad \forall q \in H^1_{\Gamma_0}(\Omega). \end{aligned} \tag{3.53}$$

If $\tau_1'(t) = 0$ and $\tau_2'(t) \neq 0$ we have

$$\begin{aligned} &\int_\Omega (\frac{\lambda^2}{1+\lambda a} sq + \nabla s \cdot \nabla q) dx + \int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\tau_1} + k_2 e^{-\lambda\frac{\tau_2}{\tau_2'}\ln\left(1-\tau_2'\right)} + \mu\lambda)}{1+\lambda a} sq d\Gamma \\ &= \int_\Omega (g + \frac{\lambda}{1+\lambda a} f) q dx + \int_{\Gamma_1} \tilde{l} q d\Gamma \quad \forall q \in H^1_{\Gamma_0}(\Omega). \end{aligned} \tag{3.54}$$

Otherwise, we get

$$\begin{aligned} &\int_\Omega (\frac{\lambda^2}{1+\lambda a} sq + \nabla s \cdot \nabla q) dx + \int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\frac{\tau_1}{\tau_1'}\ln(1-\tau_1')} + k_2 e^{-\lambda\tau_2} + \mu\lambda)}{1+\lambda a} sq d\Gamma \\ &= \int_\Omega (g + \frac{\lambda}{1+\lambda a} f) q dx + \int_{\Gamma_1} \tilde{l} q d\Gamma \quad \forall q \in H^1_{\Gamma_0}(\Omega), \end{aligned} \tag{3.55}$$

As the left-hand side of (3.52), (3.53), (3.54), (3.55) is coercive on $H^1_{\Gamma_0}(\Omega)$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $s \in H^1_{\Gamma_0}(\Omega)$ of (3.52), (3.53), (3.54), (3.55).

If we consider $q \in D(\Omega)$ in (3.52), (3.53), we have that s solves (3.48) in $D'(\Omega)$ and thus $s = u + av \in E(\Delta, L^2(\Omega))$.

Using Green's formula (3.17) in (3.52) and using (3.48), we obtain

$$\int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\tau_1} + k_2 e^{-\lambda\tau_2} + \mu\lambda)}{1+\lambda a} sq d\Gamma + \langle \frac{\partial s}{\partial \nu}; q\rangle_{\Gamma_1} = \int_{\Gamma_1} lq\, d\Gamma,$$

leading to (3.50) and then to the third equation of (3.30) due to the definition of l and the relations between u, v and s. We find the same result if $\tau_i'(t) \neq 0$, $i = 1, 2$.

In conclusion, we have found $(u, v, v_1, z, w)^T \in D(A)$, which verifies (3.30), and thus $\lambda I - A(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this proves that

$$\lambda I - \tilde{A}(t) = (\lambda + \kappa(t))I - A(t) \text{ is surjective,} \tag{3.56}$$

for any $\lambda > 0$ and $t > 0$. □

Proof. (of Theorem 3.2) Then, (3.21), (3.27) and (3.56) imply that the family $\tilde{A} = \{\tilde{A}(t) : t \in [0, T]\}$ is a stable family of generators in $\tilde{H}$ with stability constants independent of t. Therefore, all assumptions of Theorem 3.1 are satisfied by (3.16), Lemma3.2– Lemma3.5, and thus, the problem

$$\begin{aligned} \tilde{U}' &= \tilde{A}(t)\tilde{U}, \\ \tilde{U}(0) &= U_0, \end{aligned}$$

has a unique solution $\tilde{U} \in C([0, +\infty), D(A(0))) \cap C^1([0, +\infty), \tilde{H})$ for $U_0 \in D(A(0))$. The requested solution is then given by

$$U(t) = e^{\int_0^t \kappa(s)ds}\tilde{U}(t).$$

This concludes the proof. □

4. STABILITY RESULT

Now, we show that problem (1.1)–(1.5) is uniformly exponentially stable under the assumption

$$\frac{a}{2} > \frac{C^*|k|}{\sqrt{1-d}}. \tag{4.1}$$

We fix ξ_i, (given in (1.6)) such that

$$\frac{|k|}{\sqrt{1-d}} < \xi_i < \frac{a}{C^*} - \frac{|k|}{\sqrt{1-d}}, \ i = 1, 2. \tag{4.2}$$

Moreover, the parameter λ (given in (1.6)) is fixed to satisfy

$$\lambda < \min\left\{\frac{1}{\overline{\tau}}\left|\log \frac{|k|}{\xi_1\sqrt{1-d}}\right|, \frac{1}{\overline{\tau}}\left|\log \frac{|k|}{\xi_2\sqrt{1-d}}\right|\right\}. \tag{4.3}$$

We start with giving an explicit formula for the derivative of the energy.

Lemma 4.1. *Assume (2.1)–(2.4) and (4.1). Then, for any regular solution of problem (1.1)–(1.5) the energy is decreasing and, for a suitable positive constant C, we have*

$$\begin{aligned} E'(t) \leq -C&\left\{\int_\Omega |\nabla u'|^2 dx + \int_{\Gamma_1} u'^2(x, t-\tau_1(t))d\Gamma + \int_{\Gamma_1} u'^2(x, t-\tau_2(t))\, d\Gamma\right\} \\ &- C\left\{\int_{t-\tau_1(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)}u'^2(x,s)d\Gamma ds + \int_{t-\tau_2(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)}u'^2(x,s)\, d\Gamma ds\right\}. \end{aligned} \tag{4.4}$$

Proof. Differentiating (1.6), we obtain

$$
\begin{aligned}
E'(t) &= \int_\Omega \{u'u'' + \nabla u \nabla u'\}dx + \frac{\xi_1+\xi_2}{2}\int_{\Gamma_1} u'^2 d\Gamma + \mu\int_{\Gamma_1} u'u''\,d\Gamma \\
&- \frac{\xi_1(1-\tau_1')}{2}\int_{\Gamma_1} e^{-\lambda\tau_1}u'^2(x,t-\tau_1)d\Gamma - \frac{\xi_2(1-\tau_2')}{2}\int_{\Gamma_1} e^{-\lambda\tau_2}u'^2(x,t-\tau_2)d\Gamma \\
&- \lambda\frac{\xi_1}{2}\int_{t-\tau_1}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds - \lambda\frac{\xi_2}{2}\int_{t-\tau_2}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds,
\end{aligned}
$$

and then, applying Green's formula,

$$
\begin{aligned}
E'(t) &= \int_\Omega au'\Delta u' dx + \int_{\Gamma_1} u'\frac{\partial u}{\partial \nu}d\Gamma \\
&+ \frac{\xi_1+\xi_2}{2}\int_{\Gamma_1} u'^2\,(x,t)\,d\Gamma - \frac{\xi_1\,(1-\tau_1'\,(t))}{2}\int_{\Gamma_1} e^{-\lambda\tau_1(t)}u'^2(x,t-\tau_1\,(t))d\Gamma \\
&- \frac{\xi_2\,(1-\tau_2'\,(t))}{2}\int_{\Gamma_1} e^{-\lambda\tau_2(t)}u'^2\,(x,t-\tau_2\,(t))\,d\Gamma \\
&- \lambda\frac{\xi_1}{2}\int_{t-\tau_1(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds \\
&- \lambda\frac{\xi_2}{2}\int_{t-\tau_2(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2\,(x,s)\,d\Gamma ds + \mu\int_{\Gamma_1} u'u''d\Gamma. \qquad (4.5)
\end{aligned}
$$

Integrating once more by parts and using the boundary conditions we obtain

$$
\begin{aligned}
F'(t) &= -a\int_\Omega \left|\nabla u'\right|^2 dx - k_1\int_{\Gamma_1} u'u'(x,t-\tau_1\,(t))d\Gamma \\
&- k_2\int_{\Gamma_1} u'\,(x,t)\,u'\,(x,t-\tau_2\,(t))\,d\Gamma \\
&- \frac{\xi_1\,(1-\tau_1'\,(t))}{2}\int_{\Gamma_1} e^{-\lambda\tau_1}u'^2(x,t-\tau_1\,(t))d\Gamma \\
&- \frac{\xi_2\,(1-\tau_2'\,(t))}{2}\int_{\Gamma_1} e^{-\lambda\tau_2(t)}u'^2\,(x,t-\tau_2\,(t))\,d\Gamma \\
&+ \frac{\xi_1+\xi_2}{2}\int_{\Gamma_1} u'^2 d\Gamma - \lambda\frac{\xi_1}{2}\int_{t-\tau_1(t)}^t\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds \\
&- \lambda\frac{\xi_2}{2}\int_{t-\tau_2(t)}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2\,(x,s)\,d\Gamma ds. \qquad (4.6)
\end{aligned}
$$

Applying Cauchy-Schwarz's and Poincaré's inequalities, a trace estimate and recalling the assumptions (2.1)–(2.4), we obtain

$$
\begin{aligned}
E'(t) &\le -a\int_\Omega |\nabla u'|^2 dx + \frac{\xi_1+\xi_2}{2}\int_{\Gamma_1} u'^2 d\Gamma \\
&+ \frac{|k|}{2\sqrt{1-d}}\int_{\Gamma_1} u'^2 d\Gamma + \frac{|k|}{2\sqrt{1-d}}\int_{\Gamma_1} u'^2 d\Gamma \\
&+ \frac{|k|}{2}\sqrt{1-d}\int_{\Gamma_1} u'^2(t-\tau_1)d\Gamma + \frac{|k|}{2}\sqrt{1-d}\int_{\Gamma_1} u'^2(t-\tau_2)d\Gamma \\
&- \frac{\xi_1}{2}(1-d_1)e^{-\lambda\overline{\tau}}\int_{\Gamma_1} u'^2(x,t-\tau_1)d\Gamma - \frac{\xi_2}{2}(1-d_2)e^{-\lambda\widetilde{\tau}}\int_{\Gamma_1} u'^2(x,t-\tau_2)d\Gamma
\end{aligned}
$$

$$
\begin{aligned}
& -\lambda\frac{\xi_1}{2}\int_{t-\tau_1}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds - \lambda\frac{\xi_2}{2}\int_{t-\tau_2}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds \\
\leq\ & -\left(\frac{a}{2}-\frac{|k|C^*}{2\sqrt{1-d}}-\frac{\xi_1}{2}C^*\right)\int_\Omega |\nabla u'|^2\,dx \\
& -\left(\frac{a}{2}-\frac{|k|C^*}{2\sqrt{1-d}}-\frac{\xi_2}{2}C^*\right)\int_\Omega |\nabla u'|^2\,dx \\
& -\left(e^{-\lambda\overline{\tau}}\frac{\xi_1}{2}(1-d)-\frac{|k|}{2}\sqrt{1-d}\right)\int_{\Gamma_1} u'^2(x,t-\tau_1)d\Gamma \\
& -\left(e^{-\lambda\overline{\tau}}\frac{\xi_2}{2}(1-d)-\frac{|k|}{2}\sqrt{1-d}\right)\int_{\Gamma_1} u'^2(x,t-\tau_2 d\Gamma \\
& -\lambda\frac{\xi_1}{2}\int_{t-\tau_1}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds - \lambda\frac{\xi_2}{2}\int_{t-\tau_2}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds.
\end{aligned}
$$

Therefore, (4.4) immediately follows recalling (4.2) and (4.3). □

We will use an appropriate Lyapunov functional. For this purpose, let us define the Lyapunov functional

$$F(t) = E(t) + \varepsilon\left[\int_\Omega uu'dx + \mu\int_{\Gamma_1} uu'd\Gamma\right], \tag{4.7}$$

where ε is a positive small constant that we will choose later on.

Remark 4.2. From Poincaré's inequality, it is easy to verify that the functional F is equivalent to the energy E, that is, for ε small enough, there exist two positive constant $\varepsilon_1, \varepsilon_2$ such that

$$\varepsilon_1 F(t) \leq E(t) \leq \varepsilon_2 F(t), \quad \forall t \geq 0. \tag{4.8}$$

Lemma 4.3. *For any regular solution (u, z, w) of problem (1.1)–(1.5), we have*

$$
\begin{aligned}
& \frac{d}{dt}\left\{\int_\Omega uu'dxdt + \mu\int_{\Gamma_1} uu'd\Gamma\right\} \\
& \leq C\left\{\int_\Omega |\nabla u'|^2\,dx + \int_{\Gamma_1} u'^2(x,t-\tau_1(t))d\Gamma + \int_{\Gamma_1} u'^2\left(x,t-\tau_2\left(t\right)\right)d\Gamma\right\} \\
& -\frac{1}{2}\int_\Omega |\nabla u|^2 dx,
\end{aligned} \tag{4.9}
$$

for a suitable positive constant C.

Proof. Differentiating and integrating by parts we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega uu'dx &= \int_\Omega u'^2dx + \int_\Omega u(\Delta u + a\Delta u')dx \\
&= \int_\Omega u'^2dx - \int_\Omega |\nabla u|^2dx - a\int_\Omega \nabla u\cdot\nabla u'dx \\
&+ \int_{\Gamma_1} u(t)\frac{\partial(u+au')}{\partial\nu}(t)d\Gamma.
\end{aligned} \tag{4.10}
$$

From (4.10), using the boundary condition on Γ_1, we obtain

$$
\begin{aligned}
& \frac{d}{dt}\left\{\int_\Omega uu'dx + \mu\int_{\Gamma_1} uu'd\Gamma\right\} \\
&= \int_\Omega u'^2dx + \int_\Omega u(\Delta u + a\Delta u')dx + \mu\int_{\Gamma_1} u'^2d\Gamma + \mu\int_{\Gamma_1} uu''d\Gamma
\end{aligned}
$$

$$= \int_\Omega u'^2 dx - \int_\Omega |\nabla u|^2 dx - a \int_\Omega \nabla u \cdot \nabla u' dx$$
$$- k_1 \int_{\Gamma_1} u(t)u'(t-\tau_1(t))d\Gamma - k_2 \int_{\Gamma_1} u\,(t)\,u'\,(t-\tau_2\,(t))\,d\Gamma + \mu \int_{\Gamma_1} u'^2 d\Gamma. \quad (4.11)$$

We can conclude by using Young's, Poincaré's inequalities and a trace estimate. □

Now we can deduce our last result.

Theorem 4.1. *Assume (2.1)–(2.4) and (4.1). Then there exist positive constants C_1, C_2 such that for any solution of problem (1.1)-(1.5),*

$$F(t) \le C_1 F(0) e^{-C_2 t}, \quad \forall t \ge 0. \quad (4.12)$$

Proof. From Lemma 4.3, taking ε sufficiently small in the definition of the Lyapunov functional F, we have

$$\frac{d}{dt}F(t) \le -C\left\{\int_\Omega |\nabla u'|^2\,dx + \int_{\Gamma_1} u'^2(x,t-\tau_1(t))dx + \int_{\Gamma_1} u'^2\,(x;t-\tau_2\,(t))\,dx\right\}$$
$$- C\int_{t-\tau_1(t)}^t e^{-\lambda(t-s)}\int_{\Gamma_1} u'^2(x,s)d\Gamma ds - C\int_{t-\tau_2(t)}^t e^{-\lambda(t-s)}\int_{\Gamma_1} u'^2(x,s)d\Gamma ds$$
$$- \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx, \quad (4.13)$$

for a suitable positive constant C. Poincaré's inequality implying

$$\int_\Omega |u'|^2\,dx + \int_{\Gamma_1} |u'|^2\,ds \le C_1^* \int_\Omega |\nabla u'|^2\,dx,$$

for some $C_1^* > 0$, we obtain

$$\frac{d}{dt}F(t) \le -C' E(t), \quad (4.14)$$

for a suitable positive constant C'. (4.8) permits us to conclude our result. □

References

1. C. Abdallah, P. Dorato, J. Benitez-Read and R. Byrne, Delayed positive feedback can stabilize oscillatory system, Proceedings of the 1993 American Control Conference, pp. 3106–3107, San Francisco, CA, USA, 1993.
2. K. T. Andrews, K. L. Kuttler and M. Shillor, Second order evolution equations with dynamic boundary conditions, J. Math. Anal. Appl., **197(3)** (1996), 781–795.
3. G. Chen, Control and stabilization for the wave equation in a bounded domain I-II, SIAM J. Control Optim., **19** (1981), 114–122.
4. F. Conrad and Ö. Morgül, On the stabilization of a flexible beam with a tip mass, SIAM J. Control Optim., **36(6)** (1998), 1962–1986.
5. R. Datko, Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, SIAM J. Control Optim., **26** (1988), 697–713.
6. R. Datko, Two questions concerning the boundary control of certain elastic systems, J. Differential Equations, **92** (1991), 27–44.
7. R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, SIAM J. Control Optim., **24** (1986), 152–156.
8. G. G. Doronin and N. A. Larkin, Global solvability for the quasilinear damped wave equation with nonlinear second-order boundary conditions, Nonlinear Anal., **50** (2002), 1119–1134.
9. E. Fridman, S. Nicaise and J. Valein, Stabilization of second order evolution equations with unbounded feedback with time-dependent delay, SIAM J. Control and Optim., **48** (2010), 5028–5052.
10. S. Gerbi and B. Said-Houari, Local existence and exponential growth for a semilinear damped wave equation with dynamic boundary conditions, Adv. Differential Equations, **13** (2008), 1051–1074.

11. P. Grisvard, Elliptic problems in nonsmooth domains, volume 21 of Monographs and Studies in Mathematics,Pitman, Boston-London-Melbourne, 1985.
12. M. Grobbelaar-Van Dalsen, On fractional powers of a closed pair of operators and a damped wave equation with dynamic boundary conditions, Appl. Anal., **53** (1994), 41–54.
13. T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc. Japan, **19** (1967), 508–520.
14. T. Kato, Linear and quasilinear equations of evolution of hyperbolic type, C.I.M.E., **II** ciclo (1976), 125–191.
15. V. Komornik, Rapid boundary stabilization of the wave equation, SIAM J. Control Optim., **29** (1991), 197–208.
16. V. Komornik, Exact controllability and stabilization, the multiplier method, volume 36 of RMA, Masson, Paris, 1994.
17. V. Komornik and E. Zuazua, A direct method for the boundary stabilization of the wave equation, J. Math. Pures Appl., **69** (1990), 33–54.
18. J. Lagnese, Decay of solutions of the wave equation in a bounded region with boundary dissipation, J. Differential Equations, **50** (1983), 163–182.
19. J. Lagnese, Note on boundary stabilization of wave equations, SIAM J. Control and Optim., **26** (1988), 1250–1256.
20. I. Lasiecka and R. Triggiani, Uniform exponential decay in a bounded region with $L_2(0,T;L_2(\Sigma))$-feedback control in the Dirichlet boundary conditions, J. Differential Equations, **66** (1987), 340–390.
21. I. Lasiecka, R. Triggiani and P. F. Yao, Inverse observability estimates for second-order hyperbolic equations with variable coefficients, J. Math. Anal. Appl., **235** (1999), 13–57.
22. W. Littman and L. Markus, Stabilization of a hybrid system of elasticity by feedback boundary damping, Ann. Mat. Pura Appl., **152:4** (1988), 281–330.
23. Ö. Mörgul, On the stabilization and stability robustness against small delays of some damped wave equations, IEEE Trans. Automat. Control., 40:1626–1630, 1995.
24. S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, Differential Integral Equations, **21:9-10** (2008), 935–958.
25. S. Nicaise and C. Pignotti, Interior feedback stabilisations of wave equations with time dependent delay, Elect. j. Deff. Eq., **2011:41** (2011), 01–20.
26. S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedback, SIAM J. Control and Optim., **45** (2006), 1561–1585.
27. S. Nicaise, C. Pignotti and J. Valein, Exponential stability of the wave equation with boundary time-varying delay, Discrete Contin. Dyn. Syst. Ser. S, **4** (2011), 693 – 722.
28. S. Nicaise, J. Valein and E. Fridman, Stability of the heat and of the wave equations with boundary time-varying delays, Discrete Contin. Dyn. Syst. Ser. S, **2** (2009), 559–581.
29. A. Pazy. Semigroups of linear operators and applications to partial differential equations, volume 44 of Applied Math. Sciences, Springer-Verlag, New York, 1983.
30. M. Pellicer and J. Sòla-Morales, Analysis of a viscoelastic spring-mass model, J. Math. Anal. Appl., **294** (2004), 687–698.
31. G. Q. Xu, S. P. Yung and L. K. Li, Stabilization of wave systems with input delay in the boundary control, ESAIM Control Optim. Calc. Var., **12(4)** (2006), 770–785.
32. I. H. Suh and Z. Bien, Use of time delay action in the controller design, IEEE Trans. Autom. Control, **25:3** (1980), 600–603.
33. E. Zuazua, Exponential decay for the semi-linear wave equation with locally distributed damping, Comm. Partial Differential Equations, **15** (1990), 205–235.

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CONTROLLABILITY RESULTS FOR A NONLOCAL IMPULSIVE NEUTRAL STOCHASTIC FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY AND POISSON JUMPS

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ABSTRACT. The current paper is concerned with the controllability of impulsive neutral stochastic delay partial functional integro-differential equations with Poisson jumps in Hilbert spaces. Sufficient conditions are established using the theory of resolvent operators developed by Grimmer [Resolvent operators for integral equations in Banach spaces, Trans. Amer. Math. Soc., 273(1982):333–349] combined with a fixed point approach for achieving the required result. An example is presented to illustrate the application of the obtained results.

KEYWORDS: Controllability, Resolvent operators, C_0-semigroup, impulsive integrodifferential equations, fixed point theory.
AMS Subject Classification: 34A37; 93B05; 93E03; 60H20; 34K50

1. Introduction

The theory of semigroups of bounded linear operators is closely related to solving differential and integrodifferential equations in Banach spaces. In recent years, this theory has been applied to a large class of nonlinear differential equations in Banach spaces. Using the method of semigroups, various types of solutions to semilinear evolution equations have been discussed by Pazy [22]. Various evolutionary processes from fields as diverse as physics, population dynamics, aeronautics, economics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. Because the duration of these changes are often negligible compared to the

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total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. These processes can be more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state.
The development of the theory of functional differential equations with infinite delay heavily depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space requires a separate development of the theory [13]. The common space is the phase space $\mathcal{B}$ proposed by Hale and Kato in [12], which is widely applied in functional differential equations with infinite delay. However, this phase space is not correct for the impulsive case. Generally, the theory of impulsive functional differential equations or inclusions is based on the phase space defined later (see [15]).

In many cases, deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. Taking the disturbances into account, the theory of differential inclusions has been generalized to stochastic functional differential inclusions (see [7, 6] and the references therein). The existence, uniqueness, stability, controllability and other quantitative and qualitative properties of solutions of stochastic evolution equations or inclusions have recently received a lot of attention (see [14] and the references therein). As one of the fundamental concepts in mathematical control theory, controllability plays an important role both in deterministic and stochastic control theory. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. The controllability of nonlinear stochastic systems in infinite dimensional spaces has been extensively studied by several authors, see [21] and the references therein. Recently, Park, Balachandran, and Arthi [19] investigated the controllability of impulsive neutral integro-differential systems with infinite delay in Banach spaces using Schauder-type fixed point theorem. Arthi and Balachandran [2] established the controllability of damped second-order impulsive neutral functional differential systems with infinite delay by means of the Sadovskii fixed point theorem combined with a noncompact condition on the cosine family of operators. Very recently, also using Sadovski's fixed point theorem, Muthukumar and Rajivganthi [18] proved sufficient conditions for the approximate controllability of fractional order neutral stochastic integro-differential systems with nonlocal conditions and infinite delay.

Motivated by the previously mentioned works, in this paper, we will extend some such results of mild solution for the following neutral stochastic partial functional integrodifferential equations with infinite delay and Poisson jumps.

$$
\begin{cases}
d\left[x(t) - g\left(t, x_t, \int_0^t \sigma_1(t,s,x_s)ds\right)\right] = \left[A\left[x(t) - g\left(t, x_t, \int_0^t \sigma_1(t,s,x_s)ds\right)\right]\right. \\
\left. +f\left(t, x_t, \int_0^t \sigma_2(t,s,x_s)ds\right)\right]dt + \left[\int_0^t B(t-s)[x(s) - g\left(s, x_s, \int_0^s \sigma_1(s,\tau,x_\tau)d\tau\right)]ds\right]dt \\
+Cu(t)dt + \int_{-\infty}^t \sigma(t,s,x_s)dw(s) + \int_{\mathfrak{U}} \gamma(t, x(t-), v)d\tilde{N}(dt,dv), \quad t_k \neq t \in J := [0,T], \\
\Delta x(t_k) = I_k(x_{t_k}), \quad k = \{1,\cdots,m\} =: \overline{1,m}, \\
x(s) - q(x_{t_1}, x_{t_2}, \cdots, x_{t_n})(s) = \varphi(s) \in \mathcal{L}_2(\Omega, \mathcal{B}), \text{ for a.e } \ s \in J_0 := (-\infty, 0],
\end{cases}
\tag{1.1}
$$

where $0 < t_1 < t_2 < \cdots < t_n < T, n \in \mathbb{N}; x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space $\mathbb{H}$, $A: \ D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{H}, (B(t))_{t\geq 0}$ is a family of closed linear operator on $\mathbb{H}$ having the same domain $D(B)$ which contains the domain of A. The history $x_t : J_0 \to \mathbb{H}, x_t(\theta) = x(t+\theta)$ for $t \geq 0$, belongs to the phase space $\mathcal{B}$, which will be described in Section 2. Assume that the mappings $f, g : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}, \sigma : J \times J \times \mathcal{B} \to \mathcal{L}_2^0, \sigma_i : J \times J \times \mathcal{B} \to \mathbb{H}$, i =1, 2, $I_k : \mathcal{B} \to \mathbb{H}, k = \overline{1,m}, q : \mathcal{B}^n \to \mathcal{B}$, and $\gamma : J \times \mathbb{H} \times \mathfrak{U} \to \mathbb{H}$ are appropriate functions to be specified later. The control function $u(\cdot)$ takes values in $L^2(J,U)$ of admissible control functions for a separable Hilbert space U and C is a bounded linear operator from U into $\mathbb{H}$. Furthermore, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ be prefixed points, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump of the function x at time t_k with I_k, determining the size of the jump, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The initial data $\varphi(t) = \{\varphi(t) \ : \ -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-measurable $\mathcal{B}$-valued random variables independent of the Wiener process $\{w(t)\}$ and the Poisson point process $p(\cdot)$ with a finite second moment.

The aim of our paper is to study the controllability of nonlocal impulsive neutral stochastic functional integrodifferential equations with infinite delay and Poisson jumps in Hilbert spaces. The main techniques used here include the Banach contraction principle and techniques based on the use of a strongly continuous family of operators $R(t); t \geq 0$ defined on the Hilbert space $\mathbb{H}$ and called their resolvent (the precise definition will be given below). The resolvent operator is similar to semigroup operator for abstract diferential equations in Banach spaces.

The structure of this paper is as follows: in Section 2, we briefly present some basic notations, preliminaries, and assumptions. The main results in Section 3 are devoted to study the controllability for the system (1.1) with their proofs. An example is given in Section 4 to illustrate the theory.

2. Preliminaries

In this section, we briefly recall some basic definitions and results for stochastic equations in infinite dimensions. For more details on this section, we refer the reader to Da Prato and Zabczyk (1992)[24] and Protter (2004)[25]. Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ denote two real separable Hilbert spaces, with their vectors, norms, and their inner products, respectively. We denote by $\mathcal{L}(\mathbb{K};\mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, which is equipped with the usual operator norm $\|\cdot\|$. In this paper, we use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises.

2.1. **Basic preliminaries on the stochastic integration and the abstract phase space.** Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space satisfying the usual condition (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w = (w(t))_{t\geq 0}$ be a Q-Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the covariance operator Q such that $Tr(Q) < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \cdots$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$\langle w(t), e\rangle_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k, e\rangle_{\mathbb{K}}\beta_k(t), \quad e \in \mathbb{K}, t \geq 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{1/2}\mathbb{K}; \mathbb{H})$ be the space of all Hilbert–Schmidt operators from $Q^{1/2}\mathbb{K}$ into $\mathbb{H}$ with the inner product $\langle \Psi, \phi\rangle_{\mathcal{L}_2^0} = Tr[\Psi Q\phi^*]$, where ϕ^* is the adjoint of the operator ϕ. Let $p = p(t), t \in D_p$ (the domain of p(t)) be a stationary $\mathcal{F}_t$-Poisson point process taking its value in a measurable space $(\mathfrak{U}, \mathcal{B}(\mathfrak{U}))$ with a σ-finite intensity measure $\lambda(dv)$ by $N(dt, dv)$ the Poisson counting measure associated with p, that is,

$$N(t, \mathfrak{U}) = \sum_{s\in D_p, s\leq t} \mathbb{I}_{\mathfrak{U}}(p(s));$$

for any measurable set $\mathfrak{U} \in \mathcal{B}(\mathbb{K} - \{0\})$, which denotes the Borel σ-field of $(\mathbb{K} - \{0\})$. Let

$$\tilde{N}(dt, dv) = N(dt, dv) - \lambda(dv)dt,$$

be the compensated Poisson measure that is independent of $w(t)$. Denote by $\mathcal{P}^2(J \times \mathfrak{U}; \mathbb{H})$ the space of all predictable mappings $\gamma : J \times \mathfrak{U} \to \mathbb{H}$ for which

$$\int_0^t \int_{\mathfrak{U}} \mathbb{E}\|\gamma(t, v)\|_{\mathbb{H}}^2 \lambda(dv)dt < \infty.$$

We may then define the $\mathbb{H}$-valued stochastic integral $\int_0^t \int_{\mathfrak{U}} \gamma(t, v)\tilde{N}(dt, dv)$, which is a centered square integrable martingale. For the construction of this kind of integral, we can refer to Protter [25].

The collection of all strongly measurable, square-integrable $\mathbb{H}$-valued random variables, denoted by $\mathcal{L}_2(\Omega, \mathbb{H})$, is a Banach space equipped with norm $\|x\|_{\mathcal{L}_2} = (\mathbb{E}\|x\|^2)^{1/2}$. Let $\mathcal{C}(J, \mathcal{L}_2(\Omega, \mathbb{H}))$ be the Banach space of all continuous maps from J to $\mathcal{L}_2(\Omega, \mathbb{H})$, satisfying the condition $\sup_{t\in J} \mathbb{E}\|x(t)\|^2 < \infty$. An important subspace is given by $\mathcal{L}_2^0(\Omega, \mathbb{H}) = \{f \in \mathcal{L}_2(\Omega, \mathbb{H}) : \text{f is } \mathcal{F}_0 - \text{measurable}\}$. Further, let

$$\mathcal{L}_2^{\mathbb{F}}(0, T; \mathbb{H}) = \{g : J \times \Omega \to \mathbb{H} : g \,\text{is}\ \mathbb{F} - \text{progressively measurable and}$$
$$\mathbb{E}\left(\int_J \|g(t)\|_{\mathbb{H}}^2 dt\right) < \infty\bigg\}.$$

Since the system (1.1) has impulsive effects, the phase space used in Balasubramaniam and Ntouyas [5] and Park et al. [21] cannot be applied to these systems. So, we need to introduce an abstract phase space $\mathcal{B}$, as follows:

Assume that $l : J_0 \to (0, +\infty)$ is a continuous function with $l_0 = \int_{J_0} l(t)dt < \infty$. For any $a > 0$, we define

$$\mathcal{B} := \bigg\{\psi : J_0 \to \mathbb{H} : (\mathbb{E}\|\psi(\theta)\|^2)^{1/2} \text{ is a bounded and measurable function on } [-a, 0]$$
$$\text{and} \int_{J_0} l(s) \sup_{\theta\in[s,0]} (\mathbb{E}\|\psi(\theta)\|^2)^{1/2} ds < \infty\bigg\}.$$

If $\mathcal{B}$ is endowed with the norm

$$\|\psi\|_{\mathcal{B}} = \int_{J_0} \sup_{\theta \in [s,0]} (\mathbb{E}\|\psi(\theta)\|^2)^{1/2} ds, \quad \forall \psi \in \mathcal{B},$$

then, it is clear that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space (Hino, Murakami, & Naito, 1991).

Let $J_T = (-\infty, T]$. We consider the space

$$\mathcal{B}_T := \{x : J_T \to \mathbb{H} \text{ such that } x_k \in \mathcal{C}(J_k, \mathbb{H}) \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+) \text{ with } x(t_k^-) = x(t_k^+), x(0) - q(x_{t_1}, x_{t_2}, \cdots, x_{t_n}) = \varphi \in \mathcal{B}, k = \overline{1,m}\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}], k = \overline{1,m}$. Set $\|\cdot\|_T$ be a seminorm in $\mathcal{B}_T$ defined by

$$\|x\|_T = \|\varphi\|_{\mathcal{B}} + \sup_{s \in J} (\mathbb{E}\|x(s)\|^2)^{1/2}, \quad x \in \mathcal{B}_T.$$

Now, recall the following useful lemma that appeared in Chang [9]

Lemma 2.1. *(Chang, [9]) Assume that $x \in \mathcal{B}_T$, then for $t \in J, x_t \in \mathcal{B}$. Moreover,*

$$l_0(\mathbb{E}\|x(t)\|^2)^{1/2} \leq \|x_t\|_{\mathcal{B}} \leq \|x_0\|_{\mathcal{B}} + l_0 \sup_{s \in [0,t]} (\mathbb{E}\|x(s)\|^2)^{1/2}.$$

2.2. **Partial integrodifferential equation in Banach space.** In the present section we recall some definitions, notations and properties needed in what follows. Let Z_1 and Z_2 be Banach spaces. We denote by $\mathcal{L}(Z_1, Z_2)$ the Banch space of bounded linear operators from Z_1 into Z_2 endowed with the operator norm and we abbreviate this notation to $\mathcal{L}(Z_1)$ when $Z_1 = Z_2$.

In what follows, $\mathbb{H}$ will denote a Banach space, A and B(t) are closed linear operators on $\mathbb{H}$. Y represents the Banach space $\mathcal{D}(A)$, the domain of operator A, equiped with the graph norm

$$\|y\|_Y := \|Ay\| + \|y\| \text{ for } \quad y \in Y.$$

The notation $C([0, +\infty); Y)$ stands for the space of all continuous function from $[0, +\infty)$ into Y. We then consider the following Cauchy problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds \text{ for } \quad t \geq 0, \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \tag{2.1}$$

Definition 2.2. ([11]) A resolvent operator of Eq.(2.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{H})$ for $t \geq 0$, satisfying the following propreties:

(i) R(0)=I and $\|R(t)\| \leq Ne^{\beta t}$ for some constant N and β.
(ii) For each $x \in \mathbb{H}$, R(t)x is strongly continuous for $t \geq 0$.
(iii) For $x \in Y$, $R(.)x \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)xds \\ &= R(t)Ax + \int_0^t R(t-s)xds \text{ for } \quad t \geq 0. \end{aligned}$$

For additional details on resolvent operators, we refer the reader to [11, 10]. The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants for non-linear systems. For this reason, we need to know when the linear system(2.1) possesses a resolvent operator. Theorem 2.1 below provides a satisfactory answer to this problem.

In what follows we suppose the following assumptions:

(H1) A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t \geq 0}$ on $\mathbb{H}$

(H2) For all $t \geq 0$, B(t) is a closed linear operator from $\mathcal{D}(A)$ to $\mathbb{H}$, and $B(t) \in \mathcal{L}(Y, \mathbb{H})$. For any $y \in Y$, the map $t \to B(t)y$ is bounded, differentiable and the derivative $t \to B^{'}(t)y$ is bounded uniformly continuous on $\mathbb{R}^{+}$.

Theorem 2.1. *([11]) Assume that hypotheses **(H1)** and **(H2)** hold. Then the Eq. (2.1) admits a resolvent operator* $(R(t))_{t\geq 0}$.

In the sequel, we recall some results on the existence of solutions for the following integro-differentiel equation

$$\begin{cases} v^{'}(t) = Av(t) + \int_0^t B(t-s)v(s)ds + q(t) \ \text{ for } \ \ t \geq 0, \\ v(0) = v_0 \in \mathbb{H}. \end{cases} \tag{2.2}$$

where $q \ : \ [0, +\infty[\to \mathbb{H}$ is continuous function.

Definition 2.3. A continuous function $v : [0, +\infty) \to \mathbb{H}$ is said to be a strict solution of the Eq.(2.2) if

(i) $v \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); Y)$,
(ii) v satisfies Eq.(2.2) for $t \geq 0$.

Remark 2.4. From this definition we deduce that $v(t) \in \mathcal{D}(A)$, and the function $B(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0, +\infty)$.

Theorem 2.2. *([11]) Assume that hypotheses **(H1)** and **(H2)** hold. If v is a strict solution of the Eq.(2.2), then the following variation of constant formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \ \ \textit{for} \ \ \ t \geq 0. \tag{2.3}$$

Accordingly, we can establish the following definiton.

Definition 2.5. A function $v : [0, +\infty) \to \mathbb{H}$ is called mild solution of the Eq.(2.2), for $v_0 \in \mathbb{H}$, if v satisfies the variation of constants formula (2.3).

The next theorem provides sufficient conditions ensuring the regularity of solutions of the Eq.(2.2).

Theorem 2.3. *Let $q \in C^1([0, +\infty); \mathbb{H})$ and v be defined by (2.3). If $v_0 \in \mathcal{D}(A)$, then v is a strict solution of the Eq.(2.2).*

Now, we give the definition of mild solution for (1.1).

Definition 2.6. An $\mathcal{F}_t$-adapted càdlàg stochastic process $x : J_T \to \mathbb{H}$ is called a mild solution of (1.1) on J_T if $x(0) - q(x_{t_1}, x_{t_2}, \cdots, x_{t_n})(0) = x_0 = \varphi \in \mathcal{B}$, satisfying $\varphi, q \in \mathcal{L}_2^0(\Omega, \mathbb{H})$; is such that the following conditions hold:

(i) $\{x_t : t \in J\}$ is a $\mathcal{B}$-valued stochastic process;
(ii) For arbitrary $t \in J, x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) \quad &= \quad R(t)[x_0 + q(x_{t_1}, x_{t_2}, \cdots, x_{t_n})(0) - g(0, x_0, 0)] + g\left(t, x_t, \int_0^t \sigma_1(t, s, x_s)ds\right) \\ &+ \quad \int_0^t R(t-s)Cu(s)ds + \int_0^t R(t-s)f\left(s, x_s, \int_0^s \sigma_2(s, \xi, x_\tau)d\tau\right)ds \\ &+ \quad \int_0^t R(t-s)\int_{-\infty}^s \sigma(s, \tau, x_\tau)dw(\tau)ds + \int_0^t R(t-s)\int_{\mathfrak{U}} \gamma(t, x(t-), v)\tilde{N}(dt, dv) \\ &+ \quad \sum_{0<t_k<t} R(t-s)I_k(x_{t_k}), \quad \text{and} \end{aligned} \tag{2.4}$$

(iii) $\Delta x(t_k) = I_k(x_{t_k}), k = \overline{1, m}$.

Definition 2.7. The system (1.1) is said to be controllable on the interval J_T, if for every initial stochastic process $\varphi \in \mathcal{B}$ defined on J_0 and $y_1 \in \mathbb{H}$; there exists a stochastic control $u \in L^2(J, U)$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t\in J}$ such that the solution $x(\cdot)$ of the system (1.1) satisfies $x(T) = y_1$, where y_1 and T are the preassigned terminal state and time, respectively.

To prove our main results, we shall impose the following assumptions.

(H3) There exists positive constants M_R and M_{σ_1} such that for all $t, s \in J, x, y \in \mathcal{B}$

$$\|R(t)\|^2 \leq M_R;$$
$$\mathbb{E}\left\|\int_0^t [\sigma_1(t, s, x) - \sigma_1(t, s, y)]ds\right\|^2 \leq M_{\sigma_1}\|x - y\|^2_{\mathcal{B}}.$$

(H4) The function $g : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}$ is continuous and there exists a positive constant M_g such that for all $t \in J, x_1, x_2 \in \mathcal{B}, y_1, y_2 \in \mathcal{L}_2(\Omega, \mathbb{H})$

$$\mathbb{E}\|g(t, x_1, y_1) - g(t, x_2, y_2)\|^2 \leq M_g(\|x_1 - x_2\|^2_{\mathcal{B}} + \mathbb{E}\|y_1 - y_2\|^2).$$

(H5) For each $(t, s) \in J \times J$, the function $\sigma_2 : J \times J \times \mathcal{B} \to \mathbb{H}$ is continuous and there exists a positive constant M_{σ_2} such that for all $t, s \in J, x, y \in \mathcal{B}$

$$\mathbb{E}\left\|\int_0^t [\sigma_2(t, s, x) - \sigma_2(t, s, y)]ds\right\|^2 \leq M_{\sigma_2}\|x - y\|^2_{\mathcal{B}}.$$

(H6) The function $f : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}$ is continuous and there exists a positive constant M_f such that for all $t \in J, x_1, x_2 \in \mathcal{B}, y_1, y_2 \in \mathcal{L}_2(\Omega, \mathbb{H})$

$$\mathbb{E}\|f(t, x_1, y_1) - f(t, x_2, y_2)\|^2 \leq M_f(\|x_1 - x_2\|^2_{\mathcal{B}} + \mathbb{E}\|y_1 - y_2\|^2).$$

(H7) The functions $I_k, k \in \mathcal{C}(\mathcal{B}, \mathbb{H}), k = \overline{1, m}$ and there exist positive constants M_{I_k} and $\overline{M}_{I_k}$ such that for all $x, y \in \mathcal{B}$

$$\mathbb{E}\|I_k(x)\|^2 \leq M_{I_k};$$
$$\mathbb{E}\|I_k(x) - I_k(y)\|^2 \leq \overline{M}_{I_k}\|x - y\|^2_{\mathcal{B}}.$$

(H8) For each $\varphi \in \mathcal{B}, h(t) = \lim_{c\to\infty} \int_{-c}^0 \sigma(t, s, \varphi)dw(s)$ exists and continuous. Further, there exists a positive constant M_h such that

$$\mathbb{E}\|h(t)\|^2 \leq M_h.$$

(H9) The function $\sigma : J \times J \times \mathcal{B} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ is continuous and there exists positive constants $M_\sigma, \overline{M}_\sigma$ such that for all $s, t \in J$ and $x, y \in \mathcal{B}$

$$\mathbb{E}\|\sigma(t, s, x)\|^2_{\mathcal{L}_2^0} \leq M_\sigma;$$
$$\mathbb{E}\|\sigma(t, s, x) - \sigma(t, s, y)\|^2_{\mathcal{L}_2^0} \leq \overline{M}_\sigma\|x - y\|^2_{\mathcal{B}}.$$

(H10) The function $q : \mathcal{B}^n \to \mathcal{B}$ is continuous and there exist positive constants $M_q, \overline{M_q}$ such that for all $x, y \in \mathcal{B}, t \in J_0$

$$\mathbb{E}\|q(x_{t_1}, x_{t_2}, ..., x_{t_n})(t)\|^2 \leq M_q;$$
$$\mathbb{E}\|q(x_{t_1}, x_{t_2}, ..., x_{t_n})(t) - q(y_{t_1}, y_{t_2}, ..., y_{t_n})(t)\|^2 \leq \overline{M_q}\|x - y\|^2_{\mathcal{B}}.$$

(H11) The linear operator $W : L^2(J,U) \to L^2(\Omega,\mathbb{H})$ defined by

$$Wu = \int_J R(T-s)Cu(s)ds$$

has an induced inverse W^{-1} which takes values in $L^2(J,U)/Ker\,W$ (see Carimichel & Quinn, 1984) and there exist two positive constants M_C and M_W such that

$$\|C\|^2 \le M_C \quad \text{and} \quad \|W^{-1}\|^2 \le M_W.$$

(H12) The function $\gamma : J\times\mathbb{H}\times\mathfrak{U} \to \mathbb{H}$ is a Borel measurable function and satisfies the Lipschitz continuity condition, the linear growth condition, and there exists positive constants $M_\gamma, \overline{M}_\gamma$ such that for any $x, y \in \mathcal{L}_2^{\mathbb{F}}(0,T;\mathbb{H}), t \in J$

$$\mathbb{E}\left(\int_0^t\int_{\mathfrak{U}}\|\gamma(t,x(s-),v)\|^2_{\mathbb{H}}\lambda(dv)ds\right) \vee \mathbb{E}\left(\int_0^t\int_{\mathfrak{U}}\|\gamma(t,x(s-),v)\|^4_{\mathbb{H}}\lambda(dv)ds\right)^{1/2}$$
$$\le M_\gamma\mathbb{E}\int_0^t(1+\|x(s)\|^2_{\mathbb{H}})ds;$$
$$\mathbb{E}\left(\int_0^t\int_{\mathfrak{U}}\|\gamma(t,x(s-),v)-\gamma(t,y(s-),v)\|^2_{\mathbb{H}}\lambda(dv)ds\right)$$
$$\vee\,\mathbb{E}\left(\int_0^t\int_{\mathfrak{U}}\|\gamma(t,x(s-),v)-\gamma(t,y(s-),v)\|^4_{\mathbb{H}}\lambda(dv)ds\right)^{1/2} \le \overline{M}_\gamma\mathbb{E}\int_0^t\|x(s)-y(s)\|^2_{\mathbb{H}}ds$$

3. Main Results

In this section, we shall investigate the controllability of nonlocal impulsive neutral stochastic functional integro-differential equations with infinite delay and Poisson jumps in Hilbert spaces.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that the assumptions **(H1)**–**(H12)** hold. If* $\Xi < 1$ *and* $\Theta < 1$*, then the system (1.1) is controllable on* J_T*, where*

$$\Xi := 28(1+7T^2M_WM_C)\left[2l_0^2\left(M_g(1+2M_{\sigma_1})+T^2M_RM_f(1+2M_{\sigma_2})\right)+T^2M_\gamma\tilde{C}\right]+7M_g$$
$$\Theta := \{84l_0^2T^2M_CM_WM_R^2\bar{M}_q+12l_0^2(1+7T^2M_CM_RM_W)\,[M_g(1+M_{\sigma_1})$$
$$+\;T^2M_RM_f(1+M_{\sigma_2})+T^3M_R\bar{M}_\sigma Tr(Q)+\frac{T\bar{M}_\gamma\tilde{C}}{2l_0^2}mM_R\sum_{k=1}^m\bar{M}_{I_k}\Big]\Big]\Big\}.$$

Proof. Using the assumption **(H9)**, for an arbitrary function $x(\cdot)$, we define the control process

$$u_x^T(t) = W^{-1}\Bigg\{x_1 - R(T)[x_0+q(x_{t_1},\cdots,x_{t_n})-g(0,x_0,0)] - g(T,x_T,\int_0^T\sigma_1(T,s,x_s)ds)$$
$$-\int_0^T R(T-s)f(s,x_s,\int_0^s\sigma_2(s,\tau,x_\tau)d\tau)ds - \int_0^T R(T-s)[h(s)+\int_0^s\sigma(s,\tau,x_\tau)dw(\tau)]ds$$
$$-\int_0^T R(T-s)\int_{\mathfrak{U}}\gamma(t,x(t-),v)\tilde{N}(dt,dv) - \sum_{0<t_k<T}R(T-t_k)I_k(x_{t_k})\Bigg\}(t) \tag{3.1}$$

Let's put (1.1) into a fixed point problem. Consider the operator $\Pi : \mathcal{B}_T \to \mathcal{B}_T$ defined by

$$\Pi x(t) = x_0 + q(x_{t_1},\cdots,x_{t_n})(t), \quad t \in J_0;$$

$$\begin{aligned}\Pi x(t) &= R(t)[x_0 + q(x_{t_1}, \cdots, x_{t_n})(0) - g(0, x_0, 0)] + g\left(t, x_t, \int_0^t \sigma_1(t, s, x_s) ds\right) \\ &+ \int_0^t R(t-s) C u_x^T(s) ds + \int_0^t R(t-s) f\left(s, x_s, \int_0^s \sigma_2(s, \tau, x_\tau) d\tau\right) ds \\ &+ \int_0^t R(t-s)\left[h(s) + \int_0^s \sigma(s, \tau, x_\tau) dw(\tau)\right] ds + \int_0^t R(t-s) \int_{\mathfrak{U}} \gamma(t, x(t-), v) \tilde{N}(dt, dv) \\ &+ \sum_{0<t_k<t} R(t-t_k) I_k(x_{t_k}), \quad \text{for a.e } t \in J.\end{aligned}$$

In what follows, we shall show that using the control $u_x^T(\cdot)$, the operator Π has a fixed point, which is then a mild solution for system (1.1).

Clearly, $\Pi x(T) = y_1$.
For $\varphi \in \mathcal{B}$, we define $\tilde{\varphi}$ by

$$\tilde{\varphi}(t) = \begin{cases} x_0 + q(x_{t_1}, x_{t_2}, \cdots, x_{t_n})(t) & \text{if } t \in J_0, \\ R(t)[x_0 + q(x_{t_1}, x_{t_2}, \cdots, x_{t_n})(0)] & \text{if } t \in J. \end{cases}$$

then $\tilde{\varphi}(t) \in \mathcal{B}_T$. Set $x(t) = z(t) + \tilde{\varphi}(t), t \in J_T$.
It is easy to see that x satisfies (2.4) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned}z(t) &= -R(t)g(0, x_0, 0) + g(t, z_t + \tilde{\varphi}_t, \int_0^t \sigma_1(t, s, z_s + \tilde{\varphi}_s) ds) + \int_0^t R(t-s) C u_{z+\tilde{\varphi}}^T(s) ds \\ &+ \int_0^t R(t-s) f(s, z_s + \tilde{\varphi}_s, \int_0^s \sigma_2(s, \tau, z_\tau + \tilde{\varphi}_\tau) d\tau) ds \\ &+ \int_0^t R(t-s)[h(s) + \int_0^s \sigma(s, \tau, z_\tau + \tilde{\varphi}_\tau) dw(\tau)] ds + \sum_{0<t_k<t} R(t-t_k) I_k(z_{t_k} + \tilde{\varphi}_{t_k}) \\ &+ \int_0^t R(t-s) \int_{\mathfrak{U}} \gamma(t, z(t-) + \tilde{\varphi}(t-), v) \tilde{N}(dt, dv), \quad t \in J,\end{aligned}$$

where $u_{z+\tilde{\varphi}}^T(t)$ is obtained from (3.1) by replacing $x_t = z_t + \tilde{\varphi}_t$

Let $\mathcal{B}_T^0 = \{y \in \mathcal{B}_T : y_0 = 0 \in \mathcal{B}\}$. For any $y \in \mathcal{B}_T^0$, we have

$$\|y\|_T = \|y_0\|_{\mathcal{B}} + \sup_{s \in J} (\mathbb{E}\|y(s)\|^2)^{1/2} = \sup_{s \in J} (\mathbb{E}\|y(s)\|^2)^{1/2},$$

and thus $(\mathcal{B}_T^0, \|\cdot\|_T)$ is a Banach space. Set $B_r = \{y \in \mathcal{B}_T^0 : \|y\|_T^2 \leq r\}$ for some $r \geq 0$, then $B_r \subseteq \mathcal{B}_T^0$ is uniformly bounded, and for $u \in B_r$, by Lemma 2.1, we have

$$\begin{aligned}\|z_t + \tilde{\varphi}_t\|_{\mathcal{B}} &\leq 2(\|z_t\|_{\mathcal{B}}^2 + \|\tilde{\varphi}_t\|_{\mathcal{B}}^2) \\ &\leq 4(l_0^2 \sup_{s \in [0,t]} (\mathbb{E}\|z(s)\|^2 + \|z_0\|_{\mathcal{B}}^2 + l_0^2 \sup_{s \in [0,t]} (\mathbb{E}\|\tilde{\varphi}(s)\|^2 + \|\tilde{\varphi}_0\|_{\mathcal{B}}^2) \\ &\leq 4l_0^2(r + 2M_R[\mathbb{E}\|\varphi(0)\|^2 + M_q) + 4\|\tilde{\varphi}\|_{\mathcal{B}}^2 \qquad (3.2) \\ &:= r^*.\end{aligned}$$

Consider the map $\bar{\Pi} : \mathcal{B}_T^0 \to \mathcal{B}_T^0$ defined by $\bar{\Pi} z(t) = 0$, for $t \in J_0$ and

$$\begin{aligned}\bar{\Pi} z(t) &= -R(t)g(0, x_0, 0) + g(t, z_t + \tilde{\varphi}_t, \int_0^t \sigma_1(t, s, z_s + \tilde{\varphi}_s) ds) + \int_0^t R(t-s) C u_{z+\tilde{\varphi}}^T(s) ds \\ &+ \int_0^t R(t-s) f(s, z_s + \tilde{\varphi}_s, \int_0^s \sigma_2(s, \tau, z_\tau + \tilde{\varphi}_\tau) d\tau) ds \\ &+ \int_0^t R(t-s)[h(s) + \int_0^s \sigma(s, \tau, z_\tau + \tilde{\varphi}_\tau) dw(\tau)] ds + \sum_{0<t_k<t} R(t-t_k) I_k(z_{t_k} + \tilde{\varphi}_{t_k})\end{aligned}$$

$$+\int_0^t R(t-s)\int_{\mathfrak{U}}\gamma(t,z(t-)+\tilde{\varphi}(t-),v)\tilde{N}(dt,dv),\quad t\in J.$$

Obviously, the operator Π has a fixed point which is equivalent to prove that $\bar{\Pi}$ has a fixed point. Note that, by our assumptions, we infer that all the functions involved in the operator are continuous, therefore $\bar{\Pi}$ is continuous.

Let $z,\bar{z}\in\mathcal{B}_T^0$. From (3.1), by our assumptions, Hölder's inequality, the Doob martingale inequality, and the Burkholder-Davis-Gundy inequality for pure jump stochastic integral in Hilbert space (see Luo & Liu, 2008, [17]), Lemma 2.1, and in view of (3.2), for $t\in J$, we obtain the following estimates.

$$\begin{aligned}
&\mathbb{E}\|u^T_{z+\tilde{\varphi}}\|^2\\
&\le 7M_W\left\{\mathbb{E}\|x_1\|^2+3M_R[\|x_0\|^2+M_q+C_2]+2M_g([2+2M_{\sigma_1}]r^*+2C_1)+C_2\right.\\
&+2TM_R(M_f([1+2M_{\sigma_2}]r^*+2C_3)+C_4)+2M_R(M_h+tTr(Q)M_\sigma)+T^2M_\gamma\tilde{C}(1+\frac{r^*}{l_0^2})\\
&\left.+mM_R\sum_{k=1}^m M_{I_k}\right\}:=\mathfrak{L},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\|u^T_{z+\tilde{\varphi}}(t)-u^T_{\bar{z}+\tilde{\varphi}}(t)\|^2\\
&\le 12l_0^2M_W\left\{2M_R\bar{M}_q+M_g(1+M_{\sigma_1})+T^2M_RM_f(1+M_{\sigma_2})+T^3M_R\bar{M}_\sigma Tr(Q)\right.\\
&\qquad\left.+\frac{T\bar{M}_\gamma\tilde{C}}{2l_0^2}+mM_R\sum_{k=1}^m\bar{M}_{I_k}\right\}\sup_{s\in J}\mathbb{E}\|z(t)-\bar{z}(t)\|^2
\end{aligned}$$

where $\tilde{C}>0$ is a positive constant and

$$C_1:=T\sup_{(r,s)\in J\times J}\sigma_1^2(r,s,0),\quad C_2:=\sup_{(t,s)\in J\times\mathcal{B}}\|g(t,s,0)\|^2,$$
$$C_3:=T\sup_{(r,s)\in J\times J}\sigma_2^2(r,s,0),\quad C_4:=\sup_{t\in J}\|f(t,0,0)\|^2$$

Lemma 3.2. *Under the assumptions of Theorem 3.1, there exists $r>0$ such that $\bar{\Pi}(B_r)\subseteq B_r$.*

Proof. If this property is false, then for each $r>0$, there exists a function $z^r(\cdot)\in B_r$, but $\bar{\Pi}(z^r)\notin B_r$, i.e. $\|\bar{\Pi}(z^r)(t)\|^2>r$ for some $t\in J$. However, by our assumptions, Hölder's inequality and the Burkholder–Davis–Gundy inequality, we have

$$\begin{aligned}
&r<\mathbb{E}\|\bar{\Pi}(z^r)(t)\|^2\\
&\le 7\left[2M_g([1+2M_{\sigma_1}]r^*+2C_1)+C_2+M_RC_2+2T^2M_R[(M_f[1+2M_{\sigma_2}]r^*+2C_3)+C_4]\right.\\
&\left.+T^2M_RM_C\mathfrak{L}+2T^2M_R(M_h+TTr(Q)M_\sigma)+TM_\gamma\tilde{C}(1+\frac{r^*}{l_0^2})+mM_R\sum_{k=1}^m M_{I_k}\right]\\
&\le M^{**}+7(1+7T^2M_WM_C)\left[2\left(M_g(1+2M_{\sigma_1})+T^2M_RM_f(1+2M_{\sigma_2})\right)+\frac{T^2M_\gamma\tilde{C}}{l_0^2}\right]r^*\\
&+7M_gr^* \qquad (3.3)
\end{aligned}$$

where

$$M^{**}:=7\left(2C_1+C_2(1+M_R)+2T^2M_R[2C_3+C_4]+mM_R\sum_{k=1}^m M_{I_k}\right)$$

$$+ 49T^2 M_W M_R M_C \left(\|x_1\|^2 + 3M_r[\|x_0\|^2 + M_q + C_2] + 2M_g C_1 + C_2 + 2TM_R[2C_3 + C_4] \right.$$
$$\left. +2M_R(M_h + TM_\sigma Tr(Q)) + T^2 M_\gamma \tilde{C} + mM_R \sum_{k=1}^{m} M_{I_k} \right) \tag{3.4}$$

Dividing both sides of (3.3) by r and noting that

$$r^* = 4l_0^2(r + 2M_R\mathbb{E}\|\varphi(0)\|^2 + M_q) + 4\|\tilde{\varphi}\|^2_{\mathcal{B}}$$
$$\longrightarrow \infty, \quad r \to \infty$$

and taking the limit as $r \to \infty$, we obtain $1 \le \Xi$ which contradicts our assumption. Thus, for some positive number $r, \bar{\Pi}(B_r) \subseteq B_r$. This completes the proof of Lemma 3.2. □

Lemma 3.3. *Under the assumptions of Theorem 3.1, $\bar{\Pi} : \mathcal{B}^0_T \to \mathcal{B}^0_T$ is a contraction mapping.*

Proof. Let $z, \bar{z} \in \mathcal{B}^0_T$. Then, by our assumptions, Hölder's inequality, Burkholder-Davis-Gundy's inequality, Lemma 2.1, and since $\|z_0\|^2_{\mathcal{B}} = 0$ and $\|\bar{z}_0\|^2_{\mathcal{B}} = 0$, for each $t \in J$, we see that

$$\mathbb{E}\|(\bar{\Pi}z)(t) - (\bar{\Pi}\bar{z})(t)\|^2$$
$$\le 14 \left\{ M_g(1 + M_{\sigma_1}) + T^2 M_R M_f(1 + M_{\sigma_2}) + T^3 M_R \bar{M}_\sigma Tr(Q) + \frac{T\bar{M}_\gamma \tilde{C}}{2l_0^2} \right.$$
$$\left. +mM_R \sum_{k=1}^{m} \bar{M}_{I_k} \right\} \sup_{s \in J} \mathbb{E}\|z(t) - \bar{z}(t)\|^2 + 7T^2 M_R M_C \mathbb{E}\|u^T_{z+\tilde{\varphi}}(t) - u^T_{\bar{z}+\tilde{\varphi}}(t)\|^2$$
$$\le \left\{ 84l_0^2 T^2 M_C M_W M_R^2 \bar{M}_q + 14l_0^2(1 + 7T^2 M_C M_R M_W) \right.$$
$$\times \left[M_g(1 + M_{\sigma_1}) + T^2 M_R M_f(1 + M_{\sigma_2}) T^3 M_R \bar{M}_\sigma Tr(Q) + \frac{T\bar{M}_\gamma \tilde{C}}{2l_0^2} \right.$$
$$\left.\left. mM_R \sum_{k=1}^{m} \bar{M}_{I_k} \right] \right\} \sup_{s \in J} \mathbb{E}\|z(t) - \bar{z}(t)\|^2$$

Taking the supremum over t, we obtain

$$\|(\bar{\Pi}z)(t) - (\bar{\Pi}\bar{z})(t)\|^2_T \le \Theta\|z - \bar{z}\|^2_T.$$

By our assumption, we conclude that $\bar{\Pi}$ is a contraction on $\mathcal{B}^0_T$. Thus, we have completed the proof of Lemma 3.3. □

On the other hand, by Banach fixed point theorem, there exists a unique fixed point $x(\cdot) \in \mathcal{B}^0_T$ such that $(\Pi x)(t) = x(t)$. This fixed point is then the mild solution of the system (1.1). Clearly, $x(T) = (\Pi x)(T) = y_1$. Thus, the system (1.1) is controllable on J_T. The proof for Theorem 3.1 is thus complete. □

Now, let us consider a special case for the system (1.1).

If $\gamma(t, x(t-), v) \equiv 0$, the system (1.1) becomes the following nonlocal impulsive neutral stochastic functional integrodifferential equations with infinite delay without

Poisson jumps

$$\begin{cases} d\left[x(t) - g\left(t, x_t, \int_0^t \sigma_1(t,s,x_s)ds\right)\right] = \left[A\left[x(t) - g\left(t, x_t, \int_0^t \sigma_1(t,s,x_s)ds\right)\right]\right. \\ + f\left(t, x_t, \int_0^t \sigma_2(t,s,x_s)ds\right)\Big] dt + \left[\int_0^t B(t-s)[x(s) - g\left(s, x_s, \int_0^s \sigma_1(s,\tau,x_\tau)d\tau\right)]ds\right] dt \\ +Cu(t)dt + \int_{-\infty}^t \sigma(t,s,x_s)dw(s) \ t_k \neq t \in J := [0,T], \\ \Delta x(t_k) = I_k(x_{t_k}), \quad k = \{1, \cdots, m\} =: \overline{1,m}, \\ x(0) - q(x_{t_1}, x_{t_2}, \cdots, x_{t_n}) = x_0 = \varphi \in \mathcal{B}, \text{ for a.e } \ s \in J_0 := (-\infty, 0], \end{cases} \tag{3.5}$$

Corollary 3.4. *Assume that all assumptions of Theorem 3.1 hold except that **(H12)** and Ξ, Θ replaced by $\hat{\Xi}, \hat{\Theta}$ such that*

$$\hat{\Xi} := 24l_0^2(1 + 6T^2 M_B M_R M_W)[M_g(1 + 2M_{\sigma_1}) + T^2 M_R M_f(1 + M_{\sigma_1})] + 6M_g$$

and

$$\hat{\Theta} := \Big\{72l_0^2 T^2 M_C M_W M_R \bar{M}_q + 12l_0^2(1 + 6T^2 M_C M_R M_W) \\ \times \left[M_g(1 + M_{\sigma_1}) + T^2 M_R M_f(1 + M_{\sigma_2}) + T^3 M_R \bar{M}_\sigma Tr(Q) + mM_R \sum_{k=1}^m \bar{M}_{I_k}\right]\Big\}$$

If $\hat{\Xi} < 1$ and $\hat{\Theta} < 1$, then the system (3.5) is controllable on J_T.

4. Application

In this section, the established previous results are applied to study the controllability of the stochastic nonlinear wave equation with infinite delay and Poisson jumps. Specifically, we consider the following controllability of nonlocal impulsive neutral stochastic functional integrodifferential equations with infinite delay and Poisson jumps of the form:

$$\begin{cases} \dfrac{\partial}{\partial t}\left[y(t,\xi) - G_1\left(t, y(t-\tau,\xi), \displaystyle\int_0^t g(t,s,y(s-\tau,\xi))ds\right)\right] \\ = \dfrac{\partial^2}{\partial \xi^2}\left[y(t,\xi) - G_1\left(t, y(t-\tau,\xi), \displaystyle\int_0^t g(t,s,y(s-\tau,\xi))ds\right)\right] dt \\ + \left[\displaystyle\int_0^t \Gamma(t-s)\left[y(s,\xi) - G_1\left(t, y(t-\tau,\xi), \int_0^t g(t,s,y(s-\tau,\xi))ds\right)\right] + b(\xi)u(t)\right] dt \\ \qquad + \left[g_1\left(t, y(t-\tau,\xi), \displaystyle\int_0^t g_2(t,s,y(s-\tau,\xi))ds\right)\right] dt \\ \qquad + \displaystyle\int_{-\infty}^t \delta(s-t)y(t,\xi)d\beta(s) + \int_{\mathfrak{U}} y(t-\xi)v\tilde{N}(dt,dv) \text{ for } t_k \neq t \in J, \xi \in [0,\pi], \\ \Delta y(t_k)(\xi) = \displaystyle\int_{-\infty}^{t_k} \eta_k(t_k - s)y(s,\xi)ds, \quad k = \{1,\cdots,m\} =: \overline{1,m}, \ \ \xi \in [0,\pi], \\ y(0,\xi) = y(t,\pi) = 0, \ \ t \in J, \\ y(t,\xi) - \displaystyle\sum_{i=1}^n \int_0^\pi p_i(\xi,\zeta)y(t,\zeta)d\zeta = \varphi(t,\xi), \quad t \in J_0, \qquad \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where, $\gamma : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function and $\beta(t)$ is a standard one-dimensional Wiener process in $\mathbb{H}$, defined on a stochastic basis $(\Omega, \mathcal{F}, P)$; $\mathfrak{U} = \{v \in \mathbb{R} : 0 < \|v\|_{\mathbb{R}} \le a, a > 0\}$; $0 < t_1 < t_2 < \cdots < t_n < T, n \in \mathbb{N}$; $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < T$ are prefixed numbers, and $\varphi \in \mathcal{B}$. Let $p = p(t), t \in D_p$ be a $\mathbb{K}$-valued σ-finite stationary Poisson point process (independent of $\beta(t)$) on a complete probability space with the usual condition $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$. Let $\tilde{N}(ds,dv) := N(ds,dv) - \lambda(dv)ds$, with the characteristic measure $\lambda(dv)$ on $\mathfrak{U} \in \mathfrak{B}(\mathbb{K} - \{0\})$. Assume that

$$\int_{\mathfrak{U}} v^2 \lambda(dv) < \infty.$$

To rewrite (4.1) into the abstract from of (1.1), we consider the space $\mathbb{H} = L^2([0,\pi])$ with the norm $\|\cdot\|$. Let $e_n(\xi) := \sqrt{\frac{2}{\pi}} \sin n\xi, n = 1,2,3,...$ denote the completed orthogonal basics in $\mathbb{H}$ and $\beta(t) = \sum_{n=1}^\infty \sqrt{\lambda_n}\beta_n(t)e_n, t \ge 0, \lambda > 0$, where $\{\beta_n(t)\}_{n\ge 0}$ are one-dimensional standard Brownian motions mutuallyi ndependent on a usual complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$.

Defined $A : \mathbb{H} \to \mathbb{H}$ by $A = \frac{\partial^2}{\partial \xi^2}$, with domain $D(A) = \mathbb{H}^2([0,\pi]) \cap \mathbb{H}^1_0([0,\pi])$, where
$\mathbb{H}^1_0([0,\pi]) = \{w \in L^2([0,\pi]), w(0) = w(\pi) = 0\}$ and $\mathbb{H}^2([0,\pi]) = \{w \in L^2([0,\pi]) : \frac{\partial w}{\partial z}, \frac{\partial^2 w}{\partial z^2} \in L^2([0,\pi])\}$.
Then,

$$Ax = \sum_{n=1}^\infty n^2 \langle x, e_n\rangle e_n, \quad x \in D(A), \tag{4.2}$$

It is well known that A is the infinitesimal generator of a strongly continuous semigroup on $\mathbb{H}$; thus, **(H1)** is true.
Let $\Gamma : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}$ be the operator defined by $\Gamma(t)(z) = B(t)Az$ for $t \geq 0$ and $z \in \mathcal{D}(A)$.

Now, we give a special $\mathcal{B}$-space. Let $l(s) = e^{2s}, s \leq 0$, then $l_0 = \int_{J_0} l(s)ds = \frac{1}{2}$ and define

$$\|\psi\|_{\mathcal{B}} = \int_{J_0} e^{2s} \sup_{\theta\in[s,0]} (\mathbb{E}\|\psi(\theta)\|^2)^{\frac{1}{2}} ds, \quad \forall \psi \in \mathcal{B}.$$

It follows from Hino et al. [13] that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space. Additionally, we assume that the following conditions hold:

(1) Let $C \in \mathcal{L}(\mathbb{R}, \mathbb{H})$ be defined as

$$Cu(\xi) = b(\xi)u, \quad 0 \leq \xi \leq \pi, \quad u \in \mathbb{R}, \quad b(\xi) \in L^2([0, \pi])$$

(2) The linear operator $W : L^2(J, U) \to \mathbb{H}$ defined by

$$Wu = \int_J R(T - s)b(\xi)u(s)ds$$

is a bounded linear operator but not necessarily one-to-one. Let $KerW = \{u \in L^2(J, U) : Wu = 0\}$ be null space of W and $[KerW]^{\perp}$ be its orthogonal complement in $L^2(J, U)$. Let $W^* : [KerW]^{\perp} \to \text{Range}(W)$ be the restriction of W to $[KerW]^{\perp}, W^*$ is necessarily one-to-one operator. The inverse mapping theorem says that $(W^*)^{-1}$ is bounded since $[KerW]^{\perp}$ and Range(W)are Banach spaces. Since the inverse operator W^{-1} is bounded and takes values in $L^2(J, U)/KerW$, the assumption **(H11)** is satisfied.

(3) The functions $p_i : [0, \pi] \times [0, \pi] \to \mathbb{H}$ are $\mathcal{C}^2$-functions, for each $i = \overline{1, n}$.

(4) The function $\nu_1(\theta) \geq 0$ is continuous in $]-\infty, 0]$ satisfying

$$\int_{-\infty}^0 \nu_1^2(\theta)d\theta < \infty, \qquad \left(\int_{-\infty}^0 \frac{(\nu_1(s))^2}{l(s)}ds\right)^{\frac{1}{2}} < \infty.$$

(5) $b_2, b_3 : \mathbb{R} \to \mathbb{R}$ are continuous, and

$$\left(\int_{-\infty}^0 \frac{(b_3(s))^2}{l(s)}ds\right)^{\frac{1}{2}} < \infty.$$

(6) The functions $\tilde{b}_2 : \mathbb{R} \to \mathbb{R}$ is continuous and $\tilde{b}_i : \mathbb{R} \to \mathbb{R}, \ i = 1, 3$ are continuous and there exist continuous functions $r_j : \mathbb{R} \to \mathbb{R}, j = 1, 2, 3, 4$ such that

$$\begin{aligned} |\tilde{b}_1(t, s, x, y)| &\leq r_1(t)r_2(s)|y|; \quad (t, s, x, y) \in \mathbb{R}^4, \\ |\tilde{b}_3(t, s, x, y)| &\leq r_3(t)r_4(s)|y|; \quad (t, s, x, y) \in \mathbb{R}^4, \end{aligned}$$

with $\tilde{L}_1^b = \left(\int_{-\infty}^0 \frac{(r_2(s))^2}{l(s)}ds\right)^{\frac{1}{2}} < \infty, \quad \tilde{L}_2^b = \left(\int_{-\infty}^0 \frac{(r_4(s))^2}{l(s)}ds\right)^{\frac{1}{2}} < \infty.$

(7) The functions $d_i \in C(\mathbb{R}, \mathbb{R})$, and $L_{I_i} = \left(\int_{-\infty}^0 d_i^2(s)ds\right)^{\frac{1}{2}} < \infty$, where $i = 1, \cdots, m$ are finite.

Let $\phi(\theta)(\xi) = \phi(\theta, \xi), (\theta, \xi) \in J \times \mathcal{B}$, with $\phi(\theta)x = \phi(\theta, x), (\theta, x) \in J_0 \times [0, \pi]$. Let $y(t)(\xi) = y(t, \xi).g,\ f : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}, \sigma : J \times J \times \mathcal{B} \to \mathcal{L}_2^0, \gamma : J \times \mathbb{H} \times \mathfrak{U} \to \mathbb{H}$, and $I_k : \mathcal{B} \to \mathbb{H}, k = \overline{1, m}$ be the operators defined by

$$\begin{aligned}
\sigma_1(t, s, \phi)(\tau) &= g(t, s, \phi(\theta, \tau)),\\
g(t, \phi, \int_0^t \sigma_1(t, s, \phi)ds)(\tau) &= G_1(t, \phi(\theta, \tau), \int_0^t g(t, s, \phi(\theta, \xi))ds),\\
&= \int_{-\infty}^0 \nu_1 \phi(\theta)(\xi)d\theta + \int_0^t \int_{-\infty}^0 b_2(t)b_3(l)\phi(l, \xi)dlds,\\
\sigma_2(t, s, \phi)(\tau) &= g_1(t, s, \phi(\theta, \xi)),\\
f(t, \phi, \int_0^t \sigma_2(t, s, \phi)ds)(\tau) &= g_1(t, \phi(\theta, \xi), \int_0^t g_2(t, s, \phi(\theta, \xi))ds),\\
&= \int_{-\infty}^0 \tilde{b}_1(t, s, \xi, \phi(s, \xi))ds\\
&\quad + \int_0^t \int_{-\infty}^0 \tilde{b}_2(t)\tilde{b}_3(s, l, \xi, \phi(l, \xi))dlds,\\
\sigma(t, s, \phi)(\tau) &= \delta(s - t)\phi(s)(\tau),\\
\gamma(t, \phi(\tau), v) &= \phi(\tau)v,\\
I_i(t, \phi)(\xi) &= \int_{-\infty}^0 d_i(t - s)\phi(\theta)\xi ds,
\end{aligned}$$

Moreover, if Γ is bounded and C^1 function such that Γ' is bounded and uniformly continuous, then **(H2)** is satisfied and hence, by Theorem 2.1, Eq. (1.1) has a resolvent operator $(R(t))_{t\geq 0}$ on $\mathbb{H}$.
Thus it is easy to show that conditions (4) and (5) implies that g satisfies conditions in **(H4)**, in fact for any $\phi_1, \phi_2 \in \mathcal{B}, y_1, y_2 \in \mathcal{L}_2(\Omega, L^2([0, \pi]))$, we have

$$\mathbb{E}\|g(t, \phi_1, y_1) - g(t, \phi_2, y_2)\|^2 \leq M_g(\|x_1 - x_2\|_{\mathcal{B}}^2 + \mathbb{E}\|y_1 - y_2\|^2),$$

where

$$M_g = [\gamma_g^1 + T\|b_2\|_\infty \gamma_G^2]^2.$$

Similary we can verify that other assumptions are satisfied and therefore, by Theorem 3.1, we can conclude that the system (4.1) is controllable on J_T.

5. Conclusion

This paper has studied the controllability of a new class of impulsive nonlocal delayed stochastic functional integrodifferential equations of neutral type, driven by a Wiener process and Poisson jumps. Clearly, using the stocastic analysis theory, the resolvent operator theory in the sense of Grimmer, combinated with the Banach fixed point theory, we established the conditions for the controllability of the aforementioned system. Finally, we give an application to illustrate the obtained results.

There are two direct issues which require further study. We will study the conditions for the approximate controllability of the system (1.1). Also, we will investigate the optimal controls problems for the nonlocal integro-differential system.

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References

1. N.U. Ahmed, Semigroup theory with Applications to Systems and Control, of Pitman Research Notes in Mathematics Series, Longman Scientific and technical, Harlow, vol.246 UK, 1991.
2. G. Arthi, K. Balachandran, Controllability of damped second-order impulsive neutral functional differential systems with infinite delay, Journal of Optimization Theory and Applications, 152 (2012), 799–813.
3. P. Balasubramaniam, J. P. Dauer, Controllability of semilinear stochastic delay evolution equations in Hilbert spaces. International Journal of Mathematics and Mathematical Sciences, 31 (2002), 157–166.
4. P. Balasubramaniam, P. Muthukumar, Approximate controllability of second-order stochastic distributed implicit functional differential systems with infinite delay. Journal of Optimization Theory and Applications, 143 (2009), 225–244.
5. P. Balasubramaniam, Ntouyas, Controllability for neutral stochastic functional differential inclusions with infini S. K.te delay in abstract space. Journal of Mathematical Analysis, 324 (2006), 161–176.
6. P. Balasubramaniam, D. Vinayagam, Existence of solutions of nonlinear neutral stochastic differential inclusions in a Hilbert space, Stochastic Anal. Appl. 23 (2005), 137–151.
7. T. Caraballo, M. A. Diop, A. Mané, Controllability for neutral stochastic functional integrodifferential equations with infinite delay, Applied Mathematics and Nonlinear Sciences, 1(2) (2016), 493–506.
8. N. Carimichel, M. D. Quinn, Fixed point methods in nonlinear control, Lecture Notes in Control and Information Society, Vol. 75. Berlin: Springer, 1984.
9. Y. K. Chang, Controllability of impulsive functional differential systems with infinite delay in Banach spaces. Chaos, Solitons & Fractals, 33 (2007), 1601–1609.
10. R. C. Grimmer, A. J. Pritchard, Analytic resolvent operators for integral equations in Banach space, Journal of Differential Equations, 50(2) (1983), 234–259.
11. R. C. Grimmer, Resolvent operators for integral equations in a Banach space, Transactions of the American Mathematical Society, 273(1) (1982),333–349.
12. J. Hale, J. Kato, Phase spaces for retarded equations with infinite delay, Funkcial Ekvac. 21 (1978), 11–41.
13. Y. Hino, S. Murakami, T. Naito, Functional-Differential Equations with Infinite Delay, in: Lecture Notes in Mathematics, Springer-Verlag, Berlin vol. 1473, 1991.
14. L. Hu, Y. Ren, Existence results for impulsive neutral stochastic functional integro-differential equations with infinite delays, Acta Appl. Math. 111 (2010), 303–317.
15. D. D. Huan, H. Gao, Controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with delay and Poisson jumps, Cogent Engineering, 2 (2015), 1–16.
16. S. Karthikeyan, K. Balachandran, On controllability for a class of stochastic impulsive systems with delays in control. International Journal of Systems Science, 44 (2013), 67–76.
17. J. Luo, K. Liu, Stability of infinite dimensional stochastic evolution equations with memory and Markovian jumps, Stochastic Process. Appl., 118 (2008), 864–895.
18. P. Muthukumar, C. Rajivganthi, Approximate controllability of fractional order neutral stochastic integro-differential systems with nonlocal conditions and infinite delay. Taiwanese Journal of Mathematics, 17 (2013), 1693–1713.
19. J. Y. Park, K. Balachandran, G. Arthi, Controllability of impulsive neutral integrodifferential systems with infinite delay in Banach spaces. Nonlinear Analysis: Hybrid Systems, 3 (2009), 184–194.
20. J. Y. Park, K. Balachandran, N. Annapoorani, Existence results for impulsive neutral functional integrodifferential equations with infinite delay. Nonlinear Analysis, 71 (2009), 3152–3162.
21. J. Y. Park, P. Balasubramaniam, N. Kumaresan, Controllability for neutral stochastic functional integrodifferential infinite delay systems in abstract space. Numerical Functional Analysis and Optimization, 28 (2007), 1369–1386.
22. A. Pazy, Semigroups of Linear Operators and Applications to Partial Diferential Equations, Springer-Verlag, New York, 1983.

23. S. Peszat, J. Zabczyk, Stochastic partial differential equations with Levy noise, Cambridge, 2007.
24. G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.
25. P. E. Protter, Stochastic integration and differential equations, New York, NY: Springer, 2004.
26. T. Yang, Impulsive systems and control: Theory and applications. Berlin: Springer, 2001.
27. J. Zabczyk, Controllability of stochastic linear systems. Systems & Control Letters, 1 (1991), 25–31.