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HARDY-ROGERS TYPE MAPPINGS ON DISLOCATED QUASI METRIC SPACES

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ABSTRACT. In this paper, We prove some common fixed point results for two α-dominated mappings satisfying Hardy-Rogers Type on a closed ball of left (right) K-sequentially complete dislocated quasi-metric space and give some example for support our result.

KEYWORDS: Hardy-Rogers Type, Dislocated Quasi Metric Spaces, Quasi Metric Spaces.
AMS Subject Classification: 47H09, 47H10, 49M05

1. Introduction

The partial metric spaces have applications in theoretical computer science (see [14]). The notion of dislocated topologies has useful applications in the context of logic programming semantics (see [15]). Dislocated metric (metric-like) spaces (see [4, 16, 17, 18]) are generalizations of partial metric spaces. Furthermore, dislocated quasi metric spaces (quasi-metric-like spaces) generalize the idea of dislocated metric spaces and quasi-partial metric spaces.

Samet et al. [19] introduced the notion of α-admissible mappings. They weakened and generalized the contractive condition and several other known results.

In this paper, we proof common fixed point results for two α-dominated mappings in a closed ball in complete dislocated quasi metric space, under Hardy-Rogers Type.

2. Preliminaries

Definition 2.1. [10] Let X be a nonempty set. A quasi-partial metric on X is a function $q : X \times X \to \mathbb{R}^+$ satisfying, for all $x, y, z \in X$,

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(a) $0 \leq q(x,x) = q(x,y) = q(y,y)$ implies $x = y$ (equality),
(b) $q(x,x) \leq q(y,x)$ (small self-distances),
(c) $q(x,x) \leq q(x,y)$ (small self-distances),
(d) $q(x,y) + q(z,z) \leq q(x,z) + q(z,y)$ (triangleinequality).

The pair (X,q) is called a quasi-partial metric space.

Definition 2.2. [13] Let X be a nonempty set. A function $d_q : X \times X \to [0,\infty)$ is called a dislocated quasi metric (or simply d_q-metric) if the following conditions hold for any $x,y,z \in X$:

(a) If $d_q(x,y) = d_q(y,x) = 0$, then $x = y$,
(b) $d_q(x,y) \leq d_q(x,z) + d_q(z,y)$.

In this case, the pair (X,d_q) is called a dislocated quasi metric space.

It is clear that, if $d_q(x,y) = d_q(y,x) = 0$, then from (a) we have $x = y$. But, if $x = y$, then $d_q(x,y)$ may not be 0. It can be observed that, if $d_q(x,y) = d_q(y,x)$ for all $x,y \in X$, then (X,d_q) becomes a dislocated metric space (metric-like space)[1, 4, 5, 6, 9]. We will denote by (X,d_l) a dislocated metric space. For $x \in X$ and $\epsilon > 0, \overline{B_{d_q}(x,\epsilon)} = \{y \in X : d_q(x,y) \leq \epsilon\}$ is a closed ball in (X,d_q). Every quasi-partial metric space is a dislocated quasi metric space, but the converse is not true in general.

Example 2.3. If $X = \mathbb{R}^+ \cup \{0\}$, then $d_q(x,y) = x + \max\{x,y\}$ defines a dislocated quasi metric d_q on X. But, it is not a quasi-partial metric space. Indeed,

$$d_q(3,3) = 6 > d_q(2,3) = 5.$$

Reilly et al. [11] introduced the notion of left (right) K-Cauchy sequence and left (right) K-sequentially complete spaces.

Definition 2.4. Let (X,d_q) be a dislocated quasi metric space.

(a) A sequence $\{x_n\}$ in (X,d_q) is called left(right) K-Cauchy if $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n > m \leq n_0, d_q(x_m,x_n) < \epsilon$ (respectively $d_q(x_n,x_m) < \epsilon$).
(b) A sequence $\{x_n\}$ in (X,d_q) dislocated quasi-converges (for short d_q-converges) to x
if $\lim_{n\to\infty} d_q(x_n,x) = \lim_{n\to\infty} d_q(x,x_n) = 0$. In this case, the point x is called a d_q-limit of $\{x_n\}$.
(c) (X,d_q) is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence in $(X,d_q), d_q$ -converges to a point $x \in X$ such that $d_q(x,x) = 0$.

One can easily observe that every complete dislocated quasi metric space is also left K-sequentially complete dislocated quasi metric space, but the converse is not true in general.

Remark 2.5. [3] It is easy to see that, if $x_n \in B_{d_q}(x_0,r)$ for all $n \in \mathbb{N}$ and for some $x_0 \in X$, $r > 0$, and the sequence $\{x_n\}$, d_q-converges to a point $z \in X$, then $z \in B_{d_q}(x_0,r)$.

Definition 2.6. [12] Let (X,q) be a quasi-partial metric space.

(a) A sequence $\{x_n\}$ in (X,q) is called 0-Cauchy if $\lim_{n,m\to\infty} q(x_n,x_m) = 0$ or $\lim \lim_{n,m\to\infty} q(x_m,x_n) = 0$.
(b) The space (X,q) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $q(x,x) = 0$.

Remark 2.7. [3] By definitions, one can easily observe that if X is a 0-complete quasi-partial metric space then it is also a K-sequentially complete dislocated quasi metric space. But a K-sequentially complete dislocated quasi metric space may not be a 0-complete quasi-partial metric space. Therefore, the results in a K-sequentially complete dislocated quasi metric space are more general than those in a 0-complete quasi-partial metric space.

Let X be a non-empty set and $T, f : X \to X$ be two mappings. A point $y \in X$ is called a point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$, here x is called a coincidence point of T and f. The mappings T, f are said to be weakly compatible if they commute at their coincidence points i.e., $Tfx = fTx$ whenever $Tx = fx$.

Let Ψ denote the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t \geq 0$, where ψ^n is the n^{th} iterate of ψ. The following lemma is a consequence of definition of Ψ.

Lemma 2.8. *If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.*

Definition 2.9. [3] Let (X, d_q) be a dislocated quasi metric space, $A \subseteq X$, $T : X \to X$ be a selfmapping and $\alpha : X \times X \to [0, +\infty)$. Then:

(a) The mapping T is said to be α-dominated on A, if $\alpha(x, Tx) \geq 1$ for all $x \in A$.
(b) The function α is said to be a triangular function on A, if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies that $\alpha(x, z) \geq 1$ for all $x, y, z \in A$.
(b) (X, d_q) $is\alpha$-regularon A if for any sequence $\{x_n\}$ in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x \in A$ as $n \to \infty$ we have $\alpha(x_n, x) \geq 1$ for all $n \geq 0$.

It is clear that if T is an α-dominated mapping on X then T is α-dominated on each subset of X, but T can be α-dominated on some $A \subseteq X$, without being α-dominated mapping on X.

3. Main Results

Theorem 3.1. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $T, S : X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Suppose that $x_0 \in B_{d_q}(x_0, r)$ and there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + 2\delta \in (0, 1)$ and the following condition holds: if $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and $x, y \in \overline{B_{d_q}(x_0, r)}$, then*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)] \quad (3.1)$$

$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)] \quad (3.2)$$

and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r, \quad (3.3)$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X, d_q) is α-regular on $\overline{B_{d_q}(x_0, r)}$. Then there exists a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ of S and T. Moreover, $d_q(z, z) = 0$.

Proof. Let $x_0 \in X$, define $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence $\{x_n\}$ of points in X, such that

$$x_{2k+1} = Sx_{2k} \text{ and } x_{2k+2} = Tx_{2k+1}, \ \forall k = 0, 1, 2, \ldots.$$

By mathematical induction, we can show that

$$\begin{cases} x_{n+1} \in \overline{B_{d_q}(x_0, r)},\ \alpha(x_n, x_{n+1}) \geq 1, \\ d_q(x_n, x_{n+1}) \leq \lambda^n d_q(x_0, x_1), \quad \forall n \in \mathbb{N}. \end{cases} \tag{P_n}$$

By using (3.3) and $0 < \lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} < 1$, we obtain

$$d_q(x_0, x_1) = d_q(x_0, Sx_0) \leq (1-\lambda)r \leq r.$$

Hence, $x_1 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_0, x_1) \geq 1$. Therefore, using (3.1), we get that

$$\begin{aligned} d_q(x_1, x_2) &= d_q(Sx_0, Tx_1) \\ &\leq \beta d_q(x_0, x_1) + \gamma[d_q(x_0, Sx_0) + d_q(x_1, Tx_1)] \\ &\quad + \delta[d_q(x_1, Sx_0) + d_q(x_0, Tx_1)] \\ &= \beta d_q(x_0, x_1) + \gamma[d_q(x_0, x_1) + d_q(x_1, x_2)] \\ &\quad + \delta[d_q(x_1, x_1) + d_q(x_0, x_2)] \\ &\leq \beta d_q(x_0, x_1) + \gamma[d_q(x_0, x_1) + d_q(x_1, x_2)] \\ &\quad + \delta[d_q(x_0, x_1) + d_q(x_1, x_2)] \end{aligned}$$

Thus,

$$d_q(x_1, x_2) \leq \lambda d_q(x_0, x_1). \tag{3.4}$$

By using (3.4), we get that

$$\begin{aligned} d_q(x_0, x_2) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) \leq d_q(x_0, x_1) + \lambda d_q(x_0, x_1) \\ &= (1+\lambda) d_q(x_0, x_1) \leq (1+\lambda)(1-\lambda)r = (1-\lambda^2)r \leq r. \end{aligned}$$

Hence, $x_2 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_1, Tx_1) \geq 1$. Therefore, from (P_1) holds and using (3.2), we get that

$$\begin{aligned} d_q(x_2, x_3) &= d_q(Tx_1, Sx_2) \\ &\leq \beta d_q(x_1, x_2) + \gamma[d_q(x_1, Tx_1) + d_q(x_2, Sx_2)] \\ &\quad + \delta[d_q(x_2, Tx_1) + d_q(x_1, Sx_2)] \\ &= \beta d_q(x_1, x_2) + \gamma[d_q(x_1, x_2) + d_q(x_2, x_3)] \\ &\quad + \delta[d_q(x_2, x_2) + d_q(x_1, x_3)] \\ &\leq \beta d_q(x_1, x_2) + \gamma[d_q(x_1, x_2) + d_q(x_2, x_3)] \\ &\quad + \delta[d_q(x_1, x_2) + d_q(x_2, x_3)] \end{aligned}$$

By using (3.4), we get that

$$d_q(x_2, x_3) \leq \lambda d_q(x_1, x_2) \leq \lambda^2 d_q(x_0, x_1). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} d_q(x_0, x_3) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + d_q(x_2, x_3) \\ &\leq= d_q(x_0, x_1) + \lambda d_q(x_0, x_1) + \lambda^2 d_q(x_0, x_1) \\ &= (1+\lambda+\lambda^2) d_q(x_0, x_1) = \frac{1-\lambda^3}{1-\lambda} d_q(x_0, x_1) \\ &\leq \frac{1-\lambda^3}{1-\lambda}(1-\lambda)r = (1-\lambda^3)r \leq r. \end{aligned}$$

Hence, $x_3 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_1, Tx_1) \geq 1$. Therefore, from (P_2) holds. Suppose, $(P_1), (P_2), \ldots, (P_i)$

be the inductive hypothesis. We shall show that (P_{i+1}) holds. For this, we consider two possible cases. First, suppose that i is even. Then, since $\alpha(x_i, x_{i+1}) \geq 1$ and using (3.1), we get that

$$\begin{aligned} d_q(x_{i+1}, x_{i+2}) &= d_q(Sx_i, Tx_{i+1}) \\ &\leq \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, Sx_i) + d_q(x_{i+1}, Tx_{i+1})] \\ &\quad + \delta[d_q(x_{i+1}, Sx_i) + d_q(x_i, Tx_{i+1})] \\ &= \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, x_{i+1}) + d_q(x_i, x_{i+2})] \\ &\quad + \delta[d_q(x_{i+1}, x_{i+1}) + d_q(x_i, x_{i+2})] \\ &\leq \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, x_{i+1}) + d_q(x_{i+1}, x_{i+2})] \\ &\quad + \delta[d_q(x_i, x_{i+1}) + d_q(x_{i+1}, x_{i+2})] \end{aligned}$$

Since (P_i) holds, we get that

$$d_q(x_{i+1}, x_{i+2}) \leq \lambda d_q(x_i, x_{i+1}) \leq \lambda^{i+1} d_q(x_0, x_1).$$

Thus,

$$\begin{aligned} d_q(x_0, x_{i+2}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_{i+1}, x_{i+2}) \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^{i+2}) d_q(x_0, x_1) \\ &\leq \frac{1 - \lambda^{i+2}}{1 - \lambda}(1 - \lambda)r \leq (1 - \lambda^{i+2})r \leq r. \end{aligned}$$

Hence, $x_{i+2} \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$., we have $\alpha(x_{i+1}, Sx_{i+1}) = \alpha(x_{i+1}, x_{i+2}) \geq 1$. Therefore, (P_{i+1}) holds. Similarly, one can see that if i is odd, then (P_{i+1}) holds, which completes the inductive proof. Thus, we can write

$$d_q(x_n, x_{n+1}) \leq \lambda^n d_q(x_0, x_1), \ \ \forall n \in \mathbb{N}. \tag{3.6}$$

Next, we will show that the sequence $\{x_n\}$ is a left K-Cauchy sequence. Indeed, for $n, m \in \mathbb{N}$ with $m > n$ using (3.7) we have

$$\begin{aligned} d_q(x_n, x_m) &\leq d_q(x_n, x_{n+1}) + d_q(x_{n+1}, x_{n+2}) + \cdots + d_q(x_{m-1}, x_m) \\ &\leq \lambda^n d_q(x_0, x_1) + \lambda^{n+1} d_q(x_0, x_1) + \cdots + \lambda^{m-1} d_q(x_0, x_1). \end{aligned}$$

Thus,

$$d_q(x_n, x_m) \leq \frac{\lambda^n}{1 - \lambda} d_q(x_0, x_1), \ \forall n, m \in \mathbb{N}, \ m > n. \tag{3.7}$$

Since $0 < \lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} < 1$, for every $\epsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such tha $\lambda^n < \frac{1-\lambda}{d_q(x_0,x_1)}\epsilon$ for all $n > n_0$. Therefore, it follows from (3.7) that

$$d_q(x_n, x_m) < \epsilon, \ \ \forall m > n > n_0.$$

Therefore, the sequence $\{x_n\}$ is a left K-Cauchy sequence in X. By left K-sequential completeness of X, there exists $x \in X$ such that

$$\lim_{n\to\infty} d_q(x_n, z) = \lim_{n\to\infty} d_q(z, x_n) = 0. \tag{3.8}$$

We will show that z is a common fixed point of the mappings S and T. By Remark 2.5, we have $z \in B_{d_q}(x_0, r)$. Now, by the assumption we have for all $n \in \mathbb{N}$, therefore for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d_q(z, Sz) &\leq d_q(z, x_{2n+2}) + d_q(x_{2n+2}, Sz) \\ &\leq d_q(z, x_{2n+2}) + d_q(Tx_{2n+1}, Sz) \\ &\leq d_q(z, x_{2n+2}) + \beta d_q(x_{2n+1}, z) \\ &\quad + \gamma[d_q(x_{2n+1}, Sx_{n+1}) + d_q(z, Sz)] \\ &\quad + \delta[d_q(z, Tx_{2n+1}) + d_q(x_{2n+1}, Sz)] \\ &\leq d_q(z, x_{2n+2}) + \beta d_q(x_{2n+1}, z) \\ &\quad + \gamma[d_q(x_{2n+1}, Sx_{n+1}) + d_q(z, Sz)] \\ &\quad + \delta[d_q(z, x_{2n+2}) + d_q(x_{2n+1}, z) + d_q(z, Sz)]. \end{aligned}$$

By using (3.7) and (3.8), we obtain

$$(1 - \gamma + \delta) d_q(z, Sz) \leq 0 \tag{3.9}$$

which implies that $d_q(z, Sz) = 0$. Similarly, one can show that $d_q(Sz, z) = 0$. Thus, $d_q(z, Sz) = d_q(Sz, z) = 0$, i.e., $z = Sz$ Similarly, one can show that $z = Tz$.

Hence, S and T have a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$. As is an dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(z, Sz) = \alpha(z, z) \geq 1$. Therefore,

$$\begin{aligned} d_q(z, z) &\leq d_q(Sz, Tz) \\ &\leq \beta d_q(z, z) + \gamma[d_q(z, Sz) + d_q(z, Tz)] \\ &\quad + \delta[d_q(z, Sz) + d_q(z, Tz)] \\ &\leq (\beta + 2\gamma + 2\delta) d_q(z, z), \end{aligned}$$

and this implies that

$$d_q(z, z) = 0.$$

□

Example 3.2. Let $X = \mathbb{Q}^+ \cup \{0\}$ and let $d_q : X^2 \times X^2 \to X$ be defined by $d_q((x_1, y_1), (x_2, y_2)) = x_1 + 4y_1 + \frac{x_2}{4} + y_2$. Then it is easy to show that (X^2, d_q) is a left K-sequentially complete dislocated quasi metric space. If $(x_0, y_0) = (4, 1), r = 28$, then

$$\overline{B_{d_q}((4, 1), 28)} = \{(x, y) \in X : x + 4y \leq 42\}.$$

In particular, $(4, 1) \in \overline{B_{d_q}((4, 1), 28)}$.

Let $S, T : X^2 \to X^2$ be defined by

$$S(x, y) = \begin{cases} \left(\dfrac{x}{7}, \dfrac{y}{7}\right), & \text{if } x + 4y \leq 42 \\ \\ (2x^2 - 2, 4x + 5), & \text{if } x + 4y > 42 \end{cases}$$

and

$$T(x, y) = \begin{cases} \left(\dfrac{x}{6}, \dfrac{y}{9}\right), & \text{if } x + 4y \leq 42 \\ \\ (3x^2 - 3, y), & \text{if } x + 4y > 42. \end{cases}$$

Also, define $\alpha : X^2 \times X^2 \to [0,\infty)$ by

$$\alpha((x_1,y_1),(x_2,y_2)) = \begin{cases} 1, & \text{if } \frac{x_1}{4} + y_1 + x_2 + y_2 \leq 42 \\ 0, & \text{if } \frac{x_1}{4} + y_1 + x_2 + y_2 > 42. \end{cases}$$

Clearly, S and T are α-dominated mappings on $\overline{B_{d_q}((4,1),28)}$. Let $\beta = \frac{1}{7}$, $\gamma = \delta = \frac{1}{10}$, then $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} = \frac{3}{10} \in (0,1)$ and $(1-\lambda)r = 16$, $d_q((x_0,y_0), S(x_0,y_0) = d_q((4,1),S(4,1)) = \frac{104}{7} < 16 = (1-\lambda)r$. Observe that, for $(43,0) \notin \overline{B_{d_q}((4,1),28)}$, we have $d_q(S(43,0),T(43,0)) = d_q((3696,5),(5544,0)) = 5104$, $d_q((43,0),T(43,0)) + d_q((43,0),S(43,0)) = 2401$, and $d_q((43,0),(43,0)) = \frac{215}{4}$. Hence, there are no β,γ,δ such that $\beta+2\gamma+\delta \in (0,1)$ and (3.1) is satisfied. So the contractive condition does not hold on X^2. On the other hand, if $(x_1,y_1),(x_2,y_2) \in \overline{B_{d_q}((4,1),28)}$, then

$$\begin{aligned} d_q(S(x_1,y_1),T(x_2,y_2)) &= d_q\Big(\Big(\frac{x_1}{7},\frac{y_1}{7}\Big),\Big(\frac{x_2}{6},\frac{y_2}{9}\Big)\Big) \\ &= \frac{x_1}{7} + \frac{4y_1}{7} + \frac{x_2}{24} + \frac{y_2}{9} \\ &\leq \frac{1}{7} d_q((x_1,y_1),(x_2,y_2)) \\ &\quad + \frac{1}{10}[d_q((x_1,y_1),S(x_1,y_1)) + d_q((x_2,y_2),T(x_2,y_2))] \\ &\quad + \frac{1}{10}[d_q((x_2,y_2),S(x_1,y_1)) + d_q((x_1,y_1),T(x_2,y_2))]. \end{aligned}$$

Also,

$$\begin{aligned} d_q(T(x_1,y_1),S(x_2,y_2)) &= d_q\Big(\Big(\frac{x_1}{6},\frac{y_1}{9}\Big),\Big(\frac{x_2}{7},\frac{y_2}{7}\Big)\Big) \\ &= \frac{x_1}{6} + \frac{4y_1}{9} + \frac{x_2}{28} + \frac{y_2}{7} \\ &\leq \frac{1}{7} d_q((x_1,y_1),(x_2,y_2)) \\ &\quad + \frac{1}{10}[d_q((x_1,y_1),S(x_1,y_1)) + d_q((x_2,y_2),T(x_2,y_2))] \\ &\quad + \frac{1}{10}[d_q((x_2,y_2),S(x_1,y_1)) + d_q((x_1,y_1),T(x_2,y_2))]. \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are satisfied. Moreover, $(0,0)$ is the common fixed point of S and T.

Corollary 3.3. *Let (X,d_q) be a left K-sequentially complete dislocated quasi metric space and $S : X \to X$ be a mapping. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0,+\infty)$ such that S be an α-dominated mappings on $B_{d_q}(x_0,r)$. Suppose that $x_0 \in B_{d_q}(x_0,r)$ and there exist nonnegative real numbers β,γ,δ such that $\beta+2\gamma+\delta \in (0,1)$ and the following condition holds: if $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$ and $x,y \in B_{d_q}(x_0,r)$, then*

$$d_q(Sx,Sy) \leq \beta d_q(x,y) + \gamma[d_q(x,Sx) + d_q(y,Sy)] + \delta[d_q(y,Sx) + d_q(x,Sy)]$$

and

$$d_q(x_0,Sx_0) \leq (1-\lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X,d_q) is α-regular on $\overline{B_{d_q}(x_0,r)}$. Then there exists a point $z \in \overline{B_{d_q}(x_0,r)}$ such that $z = Sz$ and $d_q(z,z) = 0$.

Proof. Letting $T = S$ in Theorem 3.1, we obtain the following result. □

Corollary 3.4. *Let (X, d) be a complete dislocated metric space and $S, T : X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mappings on $\overline{B_d(x_0, r)}$. Suppose that $x_0 \in \overline{B_d(x_0, r)}$ and there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + \delta \in (0, 1)$ and the following condition holds: if $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and $x, y \in B_d(x_0, r)$, then*

$$d(Sx, Ty) \leq \beta d(x, y) + \gamma[d(x, Sx) + d(y, Ty)] + \delta[d(y, Sx) + d(x, Ty)]$$

and

$$d(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X, d) is α-regular on $\overline{B_d(x_0, r)}$. Then there exists a point $z \in \overline{B_d(x_0, r)}$ such that $z = Sz$ and $d(z, z) = 0$.

Proof. By Theorem 3.1, we obtain the following result. □

Theorem 3.5. *Suppose that all the conditions of Theorem 3.1 are satisfied. In addition suppose that:*

(a) *The function α is a triangular function on $\overline{B_{d_q}(x_0, r)}$.*

(b) *For $x, y \in \overline{B_{d_q}(x_0, r)}$ there exists $u_0 \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(x, u_0) \geq 1, \alpha(y, u_0) \geq 1$.*

(c) *For all $u \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(Sx_0, u) \geq 1$ the following condition holds*

$$d_q(x_0, Sx_0) + d_q(u, Tu) + d_q(u, Sx_0) + d_q(x_0, Tu) \leq d_q(x_0, u) + d_q(Sx_0, Tu).$$

Then S and T have a unique common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ and $d_q(z, z) = 0$.

Proof. Define the sequence fxng as in the proof Theorem 3.1. Then, $\{x_n\}, d_q$-converges to a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ of the mappings S and T such that $\alpha(x_n, z) \geq 1$ for all $n \geq 0, (P_n)$ holds and $d_q(z, z) = 0$. In order to prove uniqueness of z, suppose that z^* is another point in $\overline{B_{d_q}(x_0, r)}$ such that $z^* = Sz^* = Tz^*$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(z^*, Sz^*) = \alpha(z^*, z^*) \geq 1$. Therefore,

$$\begin{aligned} d_q(z^*, z^*) &\leq d_q(Sz^*, Tz^*) \\ &\leq \beta d_q(z^*, z^*) + \gamma[d_q(z^*, Sz^*) + d_q(z^*, Tz^*)] \\ &\quad + \delta[d_q(z^*, Sz^*) + d_q(z^*, Tz^*)] \\ &\leq (\beta + 2\gamma + 2\delta) d_q(z^*, z^*), \end{aligned}$$

and this implies that

$$d_q(z^*, z^*) = 0.$$

By assumption, there exists a point $u_0 \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(z, u0) \geq 1$ and $\alpha(z^*, u_0) \geq 1$. Define a sequence $\{u_n\}$ in X such that,

$$u_{2k+1} = Su_{2k} \text{ and } u_{2k+2} = Tu_{2k+1}, \ \ \forall k = 0, 1, 2, \ldots.$$

By mathematical induction, we can show that

$$\begin{cases} \alpha(u_n, u_{n+1}) \geq 1, \alpha(x_n, u_n) \geq 1, \ \ \forall n \in \mathbb{N}; \\ d_q(u_n, u_{n+1}) \leq \lambda^n d_q(u_0, u_1), \ \ \forall n \in \mathbb{N}; \\ d_q(x_n, z_n) \leq \lambda^n r, \, u_n \in \overline{B_{d_q}(x_0, r)}, \ \ \forall n \in \mathbb{N}. \end{cases} \tag{P'_n}$$

Since T is α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(u_0, Tu_0) = \alpha(u_0, u_1) \geq 1$. Since α is triangular function on $\overline{B_{d_q}(x_0, r)}$, and $\alpha(x_n, z) \geq 1, \alpha(z, u_0) \geq 1$, we have $\alpha(x_n, u_0) \geq 1$ for all $n \geq 0$. Therefore, using (c), we get that

$$\begin{aligned} d_q(x_1, u_1) &= d_q(Sx_0, Tu_0) \\ &\leq \beta d_q(x_0, u_0) + \gamma[d_q(x_0, Sx_0) + d_q(u_0, Tu_0)] \\ &\quad + \delta[d_q(u_0, Sx_0) + d_q(x_0, Tu_0)] \\ &\leq \beta d_q(x_0, u_0) + \gamma[d_q(x_0, u_0) + d_q(Sx_0, Tu_0)] \\ &\quad + \delta[d_q(x_0, u_0) + d_q(Sx_0, Tu_0)] \\ &\leq \beta d_q(x_0, u_0) + \gamma[d_q(x_0, u_0) + d_q(x_1, u_1)] \\ &\quad + \delta[d_q(u_0, x_0) + d_q(x_1, u_1)]. \end{aligned}$$

Thus,

$$d_q(x_1, u_1) \leq \frac{\beta + \gamma + \delta}{1 - \gamma - \delta} d_q(x_0, u_0) = \lambda d_q(x_0, u_0) \leq \lambda r. \tag{3.10}$$

Since $u_0 \in \overline{B_{d_q}(x_0, r)}$, using (3.10), we get

$$\begin{aligned} d_q(x_0, u_1) &\leq d_q(x_0, x_1) + d_q(x_1, u_1) \\ &\leq (1 - \lambda)r + \lambda d_q(x_0, u_0) \\ &\leq (1 - \lambda)r + \lambda r \leq r \end{aligned}$$

Hence, $u_1 \in \overline{B_{d_q}(x_0, r)}$. Since $\alpha(u_0, u_1) \geq 1$, by using (3.2), we get that

$$d_q(u_1, u_2) \leq \frac{\beta + \gamma + \delta}{1 - \gamma - \delta} d_q(u_0, u_1) = \lambda d_q(u_0, u_1).$$

Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(u_1, Su_1) = \alpha(u_1, u_1) \geq 1$. As, α is a triangular function on $\overline{B_{d_q}(x_0, r)}$, and $\alpha(x_1, u_0) \geq 1$, $\alpha(u_0, u_1) \geq 1$, we have $\alpha(x_1, u_1) \geq 1$. Therefore, from (P_1') holds. Since $\alpha(u_1, u_2) \geq 1$ and using (3.1), we get that

$$d_q(u_2, u_3) \leq \lambda d_q(u_1, u_2) \leq \lambda^2 d_q(u_0, u_1).$$

Since $\alpha(x_1, u_1) \geq 1$, using (3.2) that

$$\begin{aligned} d_q(x_2, u_2) &= d_q(Tx_1, Su_1) \\ &\leq \beta d_q(x_1, u_1) + \gamma[d_q(x_1, Tx_1) + d_q(u_1, Su_1)] \\ &\quad + \delta[d_q(u_1, Tx_1) + d_q(x_1, Su_1)] \\ &\leq \beta d_q(x_1, x_2) + \gamma\lambda[d_q(x_0, Sx_0) + d_q(u_0, Tu_0)] \\ &\quad + \delta\lambda[d_q(u_0, Sx_0) + d_q(x_0, Tu_0)] \end{aligned}$$

which gives wiyh (c)

$$\begin{aligned} d_q(x_2, u_2) &\leq \beta d_q(x_1, x_2) + \gamma\lambda[d_q(x_0, u_0) + d_q(Sx_0, Tu_0)] \\ &\quad + \delta\lambda[d_q(u_0, x_0) + d_q(Sx_0, Tu_0)] \\ &\leq (\beta + \lambda\gamma + \lambda\delta) d_q(x_1, u_1) + (\gamma\lambda + \delta\lambda) r. \end{aligned}$$

By using (3.10) and fact that $u_0 \in \overline{B_{d_q}(x_0, r)}$, in above inequality we obtain

$$\begin{aligned} d_q(x_2, u_2) &\leq (\beta + \lambda\gamma + \lambda\delta)\lambda r + (\gamma\lambda + \delta\lambda) r \\ &= (\beta + \lambda\gamma + \lambda\delta + \gamma + \delta)\lambda r = \lambda^2 r. \end{aligned}$$

Thus,

$$\begin{aligned} d_q(x_0,u_2) &\le d_q(x_0,x_1)+d_q(x_1,x_2)+d_q(x_2,u_2) \\ &\le d_q(x_0,x_1)+\lambda d_q(x_0,x_1)+\lambda^2 \le r. \end{aligned}$$

Hence, $u_2 \in \overline{B_{d_q}(x_0,r)}$. Since T is an α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$, we have $\alpha(x_0,Sx_0)=\alpha(u_2,u_3)\ge 1$. Therefore, from (P_2') holds. Suppose, $(P_1'),(P_2'),\dots,(P_i')$ be the inductive hypothesis. We shall show that (P_{i+1}') holds. For this, we consider two possible cases. First, suppose that i is even. Then, since $\alpha(u_i,u_{i+1})\ge 1$ and using (3.2), we get that

$$d_q(u_{i+1},u_{i+2}) \le \frac{\beta+\gamma+\delta}{1-\gamma-\delta} d_q(u_i,u_{i+1}) = \lambda^{i+1} d_q(u_0,u_1).$$

Since $\alpha(x_i,u_i)\ge 1$, using (3.1) that

$$\begin{aligned} d_q(x_{i+1},u_{i+1}) &= d_q(Sx_i,Tu_i) \\ &\le \beta d_q(x_i,u_i)+\gamma[d_q(x_i,Sx_i)+d_q(u_i,Tu_i)] \\ &\quad +\delta[d_q(u_i,Sx_i)+d_q(x_i,Tu_i)] \\ &\le \beta d_q(x_i,x_{i+1})+\gamma\lambda[d_q(x_0,Sx_0)+d_q(u_0,Tu_0)] \\ &\quad +\delta\lambda[d_q(u_0,Sx_0)+d_q(x_0,Tu_0)] \end{aligned}$$

which gives wiyh (c) and P_i'

$$\begin{aligned} d_q(x_{i+1},u_{i+1}) &\le \beta d_q(x_i,u_i)+\gamma\lambda^i[d_q(x_0,u_0)+d_q(Sx_0,Tu_0)] \\ &\quad +\delta\lambda^i[d_q(u_0,x_0)+d_q(Sx_0,Tu_0)] \\ &\le \beta\lambda^i r+\gamma\lambda^i[r+\lambda r]+\delta\lambda^i[r+\lambda r] \\ &= (\beta+\lambda\gamma+\lambda\delta+\gamma+\delta)\lambda^i r = \lambda^{i+1} r. \end{aligned}$$

Thus,

$$\begin{aligned} d_q(x_0,u_{i+1}) &\le d_q(x_0,x_1)+d_q(x_1,x_2)+\cdots+d_q(x_i,x_{i+1})+d_q(x_{i+1},u_{i+1}) \\ &\le (1+\lambda+\lambda^2+\cdots+\lambda^i)d_q(x_0,x_1)+\lambda^{i+1}r \\ &\le (1+\lambda+\lambda^2+\cdots+\lambda^i)(1-\lambda)r+\lambda^{i+1}r = r. \end{aligned}$$

Hence, $u_{i+1} \in \overline{B_{d_q}(x_0,r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$., we have $\alpha(u_{i+1},Su_{i+1})=\alpha(u_{i+1},u_{i+2})\ge 1$. Also, since $\alpha(x_{i+1},u_0)\ge 1$, $\alpha(u_n,u_{n+1})\ge 1$, $n=0,1,2,\dots,i+1$, by triangular nature of α, we have $\alpha(x_{i+1},u_{i+1})\ge 1$. Therefore, (P_{i+1}') holds. Similarly, one can see that if i is odd, then (P_{i+1}') holds, which completes the inductive proof. Thus, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d_q(z, u_{2n}) &= d_q(Tz, Su_{2n-1}) \\
&\leq \beta d_q(z, u_{2n-1}) + \gamma[d_q(z, Tz) + d_q(u_{2n-1}, Su_{2n-1})] \\
&\quad + \delta[d_q(u_{2n-1}, Tz) + d_q(z, Su_{2n-1})] \\
&\leq \beta d_q(z, u_{2n-1}) + \gamma d_q(u_{2n-1}, u_{2n}) \\
&\quad + \delta[d_q(u_{2n-1}, z) + d_q(z, u_{2n})] \\
&= (\beta + \delta) d_q(z, u_{2n-1}) + \gamma d_q(u_{2n-1}, u_{2n}) + \delta d_q(z, u_{2n}) \\
&\leq (\beta + 2\delta) d_q(z, u_{2n-1}) + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\leq (\beta + 2\delta) d_q(Tz, Su_{2n-2}) + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\leq (\beta + 2\delta)^2 d_q(z, u_{2n-2}) + (\beta + 2\delta)(\gamma + \delta) d_q(u_{2n-2}, u_{2n-1})] \\
&\quad + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\leq (\beta + 2\delta)^2 d_q(Tz, Su_{2n-3}) + (\beta + 2\delta)(\gamma + \delta) d_q(u_{2n-2}, u_{2n-1}) \\
&\quad + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\leq (\beta + 2\delta)^3 d_q(z, u_{2n-3}) + (\beta + 2\delta)^2 (\gamma + \delta) d_q(u_{2n-3}, u_{2n-2}) \\
&\quad + (\beta + 2\delta)(\gamma + \delta) d_q(u_{2n-2}, u_{2n-1}) + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\quad \vdots \\
&\leq (\beta + 2\delta)^{2n} d_q(z, u_0) + (\beta + 2\delta)^{2n-1}(\gamma + \delta) d_q(u_0, u_1) + \dots \\
&\quad + (\beta + 2\delta)(\gamma + \delta) d_q(u_{2n-2}, u_{2n-1}) + (\gamma + \delta) d_q(u_{2n-1}, u_{2n})
\end{aligned}$$

Since $\frac{\beta+2\delta}{\lambda} = \frac{(\beta+2\delta)(1-\gamma-\delta)}{\beta+\gamma+\delta} < 1$, using (P'_n) in the above inequality we obtain

$$\begin{aligned}
d_q(z, u_{2n}) &\leq (\beta + 2\delta)^{2n} d_q(z, u_0) + (\beta + 2\delta)^{2n-1}(\gamma + \delta) d_q(u_0, u_1) + \dots \\
&\quad + (\beta + 2\delta)(\gamma + \delta)\lambda^{2n-2} d_q(u_0, u_1) + (\gamma + \delta)\lambda^{2n-1} d_q(u_0, u_1) \\
&= (\beta + 2\delta)^{2n} d_q(z, u_0) \\
&\quad + (\gamma + \delta)\lambda^{2n-1} d_q(u_0, u_1)\Big[1 + \frac{\beta + 2\delta}{\lambda} + \dots + \Big(\frac{\beta + 2\delta}{\lambda}\Big)^{2n-1}\Big] \\
&\leq (\beta + 2\delta)^{2n} d_q(z, u_0) + \frac{(\gamma + \delta)\lambda^{2n-1} d_q(u_0, u_1)}{1 - \frac{\beta+2\delta}{\lambda}}
\end{aligned}$$

Since $\beta + 2\delta, \lambda \in [0, 1)$, it follows from the above inequality that

$$\lim_{n\to\infty} d_q(z, u_{2n}) = 0. \tag{3.11}$$

Similarly, we can show that

$$\lim_{n\to\infty} d_q(u_{2n}, z) = \lim_{n\to\infty} d_q(u_{2n}, z^*) = \lim_{n\to\infty} d_q(z^*, u_{2n}) = 0. \tag{3.12}$$

By using (3.11) and (3.12), we obtain

$$d_q(z, z^*) \leq d_q(z, u_{2n}) + d_q(u_{2n}, z^*) \to 0, \text{ as } n \to \infty.$$

$$d_q(z^*, z) \leq d_q(z^*, u_{2n}) + d_q(u_{2n}, z) \to 0, \text{ as } n \to \infty.$$

Hence, $d_q(z, z^*) = d_q(z^*, z) = 0$, i.e., $z = z^*$ □

Corollary 3.6. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $T, S : X \to X$ be two mappings. Let $x_0 \in X, r > 0, x_0 \in \overline{B_{d_q}(x_0, r)}$ and* *there exist nonnegative real numbers* β, γ, δ *such that* $\beta + 2\gamma + 2\delta \in (0, 1)$ *and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. *Then there exists a unique point* $z \in \overline{B_{d_q}(x_0, r)}$ *such that* $z = Sz = Tz$ *and* $d_q(z, z) = 0$. *Moreover,* S *and* T *have no fixed point in* $\overline{B_{d_q}(x_0, r)}$ *other than* z.

Proof. The proof follows by the previous results, taking $\alpha : X \times X \to [0, \infty)$ with $\alpha(x, y) = 1$ for all $x, y \in X$. □

Theorem 3.7. *Let* (X, d_q) *be a left* K*-sequentially complete dislocated quasi metric space. Suppose, there exist a function* $\alpha : X \times X \to [0, +\infty)$ *and nonnegative constants* β, γ, δ *such that* $\beta + 2\gamma + 2\delta \in (0, 1)$ *and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

for all $x, y \in X$ *such that* $\alpha(x, y) \geq 1$ *or* $\alpha(y, x) \geq 1$. *If* (X, d_q) *is* α*-regular,then there exists a point* z *in* X *such that* $z = Sz = Tz$ *and* $d_q(z, z) = 0$.

Proof. By Theorem 3.1, the condition (3.3) is imposed in order to restrict the contractive conditions (3.1) and (3.2) to $\overline{B_{d_q}(x_0, r)}$. However, the condition (3.3) can be relaxed by imposing the conditions (3.1) and (3.2) to all elements $x, y \in X$ such that $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$, we obtain the following result. □

Recall that, if $(X, \preceq)$ is a pre-ordered set and $T : X \to X$ is such that $Tx = x$ for all $x \in A \subseteq X$, then the mapping T is said to be dominated on A. Define the set ∇ by

$$\nabla = \{(x, y) \in X \times X : x \preceq y \,\text{or}\, y \preceq x\}.$$

Theorem 3.8. *Let* $(X, \preceq, d_q)$ *be a pre-ordered left* K*-sequentially complete dislocated quasi metric space,* $x_0 \in X$*,* $r > 0$ *and* $S, T : X \to X$ *be two dominated mappings on* $\overline{B_{d_q}(x_0, r)}$. *Suppose that there exist nonnegative real numbers* β, γ, δ *such that* $\beta + 2\gamma + \delta \in (0, 1)$ *and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

for all $(x, y) \in \overline{B_{d_q}(x_0, r)} \times \overline{B_{d_q}(x_0, r)} \cap \nabla$ *and*

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$ *If for any sequence* $\{x_n\} \in \overline{B_{d_q}(x_0, r)}$ *such that* $(x_n, x_{n+1}) \in \nabla$, $x_n \to w$ *as* $n \to \infty$ *implies that* $(w, x_n) \in \nabla$ *for all* $n \geq 0$, *then there exists a point* $z \in \overline{B_{d_q}(x_0, r)}$ *such that* $z = Sz = Tz$ *and* $d_q(z, z) = 0$. *In addition, suppose that:*

(a) $(x, y), (y, z) \in \nabla$ *implies* $(x, z) \in \nabla$.
(b) *For* x, $y \in \overline{B_{d_q}(x_0, r)}$ *there exists* $u_0 \in \overline{B_{d_q}(x_0, r)}$ *such that* (x, u_0), $(y, u_0) \in \nabla$.
(c) *For all* $u \in \overline{B_{d_q}(x_0, r)}$ *such that* $(u, Sx_0) \in \nabla$ *the following condition holds*

$$d_q(x_0, Sx_0) + d_q(u, Tu) + d_q(u, Sx_0) + d_q(x_0, Tu) \leq d_q(x_0, u) + d_q(Sx_0, Tu).$$

Then, z is the unique common fixed point of S and T in $\overline{B_{d_q}(x_0, r)}$.

Proof. This follows from Theorem 3.6 taking $\alpha : X \times X \to [0, +\infty)$ defined as

$$\alpha(x, y) = \begin{cases} 1, & \text{If } (x, y) \in \nabla, \\ 0, & \text{otherwise.} \end{cases}$$

□

4. Conclusions

We prov some common fixed point theorems for mappings under Hardy Rogers contractive conditions on a left K-sequentially complete dislocated quasi metric space.

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HARDY-ROGERS TYPE MAPPINGS ON DISLOCATED QUASI METRIC SPACES

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ABSTRACT. In this paper, We prove some common fixed point results for two α-dominated mappings satisfying Hardy-Rogers Type on a closed ball of left (right) K-sequentially complete dislocated quasi-metric space and give some example for support our result.

KEYWORDS: Hardy-Rogers Type, Dislocated Quasi Metric Spaces, Quasi Metric Spaces.
AMS Subject Classification: 47H09, 47H10, 49M05

1. Introduction

The partial metric spaces have applications in theoretical computer science (see [14]). The notion of dislocated topologies has useful applications in the context of logic programming semantics (see [15]). Dislocated metric (metric-like) spaces (see [4, 16, 17, 18]) are generalizations of partial metric spaces. Furthermore, dislocated quasi metric spaces (quasi-metric-like spaces) generalize the idea of dislocated metric spaces and quasi-partial metric spaces.

Samet et al. [19] introduced the notion of α-admissible mappings. They weakened and generalized the contractive condition and several other known results.

In this paper, we proof common fixed point results for two α-dominated mappings in a closed ball in complete dislocated quasi metric space, under Hardy-Rogers Type.

2. Preliminaries

Definition 2.1. [10] Let X be a nonempty set. A quasi-partial metric on X is a function $q : X \times X \to \mathbb{R}^+$ satisfying, for all $x, y, z \in X$,

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(a) $0 \leq q(x,x) = q(x,y) = q(y,y)$ implies $x = y$ (equality),
(b) $q(x,x) \leq q(y,x)$ (small self-distances),
(c) $q(x,x) \leq q(x,y)$ (small self-distances),
(d) $q(x,y) + q(z,z) \leq q(x,z) + q(z,y)$ (triangleinequality).

The pair (X,q) is called a quasi-partial metric space.

Definition 2.2. [13] Let X be a nonempty set. A function $d_q : X \times X \to [0,\infty)$ is called a dislocated quasi metric (or simply d_q-metric) if the following conditions hold for any $x,y,z \in X$:

(a) If $d_q(x,y) = d_q(y,x) = 0$, then $x = y$,
(b) $d_q(x,y) \leq d_q(x,z) + d_q(z,y)$.

In this case, the pair (X,d_q) is called a dislocated quasi metric space.

It is clear that, if $d_q(x,y) = d_q(y,x) = 0$, then from (a) we have $x = y$. But, if $x = y$, then $d_q(x,y)$ may not be 0. It can be observed that, if $d_q(x,y) = d_q(y,x)$ for all $x,y \in X$, then (X,d_q) becomes a dislocated metric space (metric-like space)[1, 4, 5, 6, 9]. We will denote by (X,d_l) a dislocated metric space. For $x \in X$ and $\epsilon > 0, \overline{B_{d_q}(x,\epsilon)} = \{y \in X : d_q(x,y) \leq \epsilon\}$ is a closed ball in (X,d_q). Every quasi-partial metric space is a dislocated quasi metric space, but the converse is not true in general.

Example 2.3. If $X = \mathbb{R}^+ \cup \{0\}$, then $d_q(x,y) = x + \max\{x,y\}$ defines a dislocated quasi metric d_q on X. But, it is not a quasi-partial metric space. Indeed,

$$d_q(3,3) = 6 > d_q(2,3) = 5.$$

Reilly et al. [11] introduced the notion of left (right) K-Cauchy sequence and left (right) K-sequentially complete spaces.

Definition 2.4. Let (X,d_q) be a dislocated quasi metric space.

(a) A sequence $\{x_n\}$ in (X,d_q) is called left(right) K-Cauchy if $\forall\epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n > m \leq n_0, d_q(x_m,x_n) < \epsilon$ (respectively $d_q(x_n,x_m) < \epsilon$).
(b) A sequence $\{x_n\}$ in (X,d_q) dislocated quasi-converges (for short d_q-converges) to x
if $\lim_{n\to\infty} d_q(x_n,x) = \lim_{n\to\infty} d_q(x,x_n) = 0$. In this case, the point x is called a d_q-limit of $\{x_n\}$.
(c) (X,d_q) is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence in $(X,d_q), d_q$ -converges to a point $x \in X$ such that $d_q(x,x) = 0$.

One can easily observe that every complete dislocated quasi metric space is also left K-sequentially complete dislocated quasi metric space, but the converse is not true in general.

Remark 2.5. [3] It is easy to see that, if $x_n \in B_{d_q}(x_0,r)$ for all $n \in \mathbb{N}$ and for some $x_0 \in X$, $r > 0$, and the sequence $\{x_n\}$, d_q-converges to a point $z \in X$, then $z \in B_{d_q}(x_0,r)$.

Definition 2.6. [12] Let (X,q) be a quasi-partial metric space.

(a) A sequence $\{x_n\}$ in (X,q) is called 0-Cauchy if $\lim_{n,m\to\infty} q(x_n,x_m) = 0$ or $\lim\lim_{n,m\to\infty} q(x_m,x_n) = 0$.
(b) The space (X,q) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $q(x,x) = 0$.

Remark 2.7. [3] By definitions, one can easily observe that if X is a 0-complete quasi-partial metric space then it is also a K-sequentially complete dislocated quasi metric space. But a K-sequentially complete dislocated quasi metric space may not be a 0-complete quasi-partial metric space. Therefore, the results in a K-sequentially complete dislocated quasi metric space are more general than those in a 0-complete quasi-partial metric space.

Let X be a non-empty set and $T, f : X \to X$ be two mappings. A point $y \in X$ is called a point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$, here x is called a coincidence point of T and f. The mappings T, f are said to be weakly compatible if they commute at their coincidence points i.e., $Tfx = fTx$ whenever $Tx = fx$.

Let Ψ denote the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t \geq 0$, where ψ^n is the n^{th} iterate of ψ. The following lemma is a consequence of definition of Ψ.

Lemma 2.8. *If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.*

Definition 2.9. [3] Let (X, d_q) be a dislocated quasi metric space, $A \subseteq X$, $T : X \to X$ be a selfmapping and $\alpha : X \times X \to [0, +\infty)$. Then:

(a) The mapping T is said to be α-dominated on A, if $\alpha(x, Tx) \geq 1$ for all $x \in A$.
(b) The function α is said to be a triangular function on A, if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies that $\alpha(x, z) \geq 1$ for all $x, y, z \in A$.
(b) (X, d_q) $is\alpha$-regularon A if for any sequence $\{x_n\}$ in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x \in A$ as $n \to \infty$ we have $\alpha(x_n, x) \geq 1$ for all $n \geq 0$.

It is clear that if T is an α-dominated mapping on X then T is α-dominated on each subset of X, but T can be α-dominated on some $A \subseteq X$, without being α-dominated mapping on X.

3. Main Results

Theorem 3.1. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $T, S : X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Suppose that $x_0 \in B_{d_q}(x_0, r)$ and there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + 2\delta \in (0, 1)$ and the following condition holds: if $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and $x, y \in \overline{B_{d_q}(x_0, r)}$, then*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)] \quad (3.1)$$

$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)] \quad (3.2)$$

and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r, \quad (3.3)$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X, d_q) is α-regular on $\overline{B_{d_q}(x_0, r)}$. Then there exists a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ of S and T. Moreover, $d_q(z, z) = 0$.

Proof. Let $x_0 \in X$, define $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence $\{x_n\}$ of points in X, such that

$$x_{2k+1} = Sx_{2k} \text{ and } x_{2k+2} = Tx_{2k+1}, \ \forall k = 0, 1, 2, \ldots.$$

By mathematical induction, we can show that

$$\begin{cases} x_{n+1} \in \overline{B_{d_q}(x_0, r)},\ \alpha(x_n, x_{n+1}) \geq 1, \\ d_q(x_n, x_{n+1}) \leq \lambda^n d_q(x_0, x_1), \quad \forall n \in \mathbb{N}. \end{cases} \tag{P_n}$$

By using (3.3) and $0 < \lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} < 1$, we obtain

$$d_q(x_0, x_1) = d_q(x_0, Sx_0) \leq (1-\lambda)r \leq r.$$

Hence, $x_1 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_0, x_1) \geq 1$. Therefore, using (3.1), we get that

$$\begin{aligned} d_q(x_1, x_2) &= d_q(Sx_0, Tx_1) \\ &\leq \beta d_q(x_0, x_1) + \gamma[d_q(x_0, Sx_0) + d_q(x_1, Tx_1)] \\ &\quad + \delta[d_q(x_1, Sx_0) + d_q(x_0, Tx_1)] \\ &= \beta d_q(x_0, x_1) + \gamma[d_q(x_0, x_1) + d_q(x_1, x_2)] \\ &\quad + \delta[d_q(x_1, x_1) + d_q(x_0, x_2)] \\ &\leq \beta d_q(x_0, x_1) + \gamma[d_q(x_0, x_1) + d_q(x_1, x_2)] \\ &\quad + \delta[d_q(x_0, x_1) + d_q(x_1, x_2)] \end{aligned}$$

Thus,

$$d_q(x_1, x_2) \leq \lambda d_q(x_0, x_1). \tag{3.4}$$

By using (3.4), we get that

$$\begin{aligned} d_q(x_0, x_2) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) \leq d_q(x_0, x_1) + \lambda d_q(x_0, x_1) \\ &= (1+\lambda) d_q(x_0, x_1) \leq (1+\lambda)(1-\lambda)r = (1-\lambda^2)r \leq r. \end{aligned}$$

Hence, $x_2 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_1, Tx_1) \geq 1$. Therefore, from (P_1) holds and using (3.2), we get that

$$\begin{aligned} d_q(x_2, x_3) &= d_q(Tx_1, Sx_2) \\ &\leq \beta d_q(x_1, x_2) + \gamma[d_q(x_1, Tx_1) + d_q(x_2, Sx_2)] \\ &\quad + \delta[d_q(x_2, Tx_1) + d_q(x_1, Sx_2)] \\ &= \beta d_q(x_1, x_2) + \gamma[d_q(x_1, x_2) + d_q(x_2, x_3)] \\ &\quad + \delta[d_q(x_2, x_2) + d_q(x_1, x_3)] \\ &\leq \beta d_q(x_1, x_2) + \gamma[d_q(x_1, x_2) + d_q(x_2, x_3)] \\ &\quad + \delta[d_q(x_1, x_2) + d_q(x_2, x_3)] \end{aligned}$$

By using (3.4), we get that

$$d_q(x_2, x_3) \leq \lambda d_q(x_1, x_2) \leq \lambda^2 d_q(x_0, x_1). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} d_q(x_0, x_3) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + d_q(x_2, x_3) \\ &\leq= d_q(x_0, x_1) + \lambda d_q(x_0, x_1) + \lambda^2 d_q(x_0, x_1) \\ &= (1+\lambda+\lambda^2) d_q(x_0, x_1) = \frac{1-\lambda^3}{1-\lambda} d_q(x_0, x_1) \\ &\leq \frac{1-\lambda^3}{1-\lambda}(1-\lambda)r = (1-\lambda^3)r \leq r. \end{aligned}$$

Hence, $x_3 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_1, Tx_1) \geq 1$. Therefore, from (P_2) holds. Suppose, $(P_1), (P_2), \ldots, (P_i)$

be the inductive hypothesis. We shall show that (P_{i+1}) holds. For this, we consider two possible cases. First, suppose that i is even. Then, since $\alpha(x_i, x_{i+1}) \geq 1$ and using (3.1), we get that

$$\begin{aligned} d_q(x_{i+1}, x_{i+2}) &= d_q(Sx_i, Tx_{i+1}) \\ &\leq \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, Sx_i) + d_q(x_{i+1}, Tx_{i+1})] \\ &\quad + \delta[d_q(x_{i+1}, Sx_i) + d_q(x_i, Tx_{i+1})] \\ &= \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, x_{i+1}) + d_q(x_i, x_{i+2})] \\ &\quad + \delta[d_q(x_{i+1}, x_{i+1}) + d_q(x_i, x_{i+2})] \\ &\leq \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, x_{i+1}) + d_q(x_{i+1}, x_{i+2})] \\ &\quad + \delta[d_q(x_i, x_{i+1}) + d_q(x_{i+1}, x_{i+2})] \end{aligned}$$

Since (P_i) holds, we get that

$$d_q(x_{i+1}, x_{i+2}) \leq \lambda d_q(x_i, x_{i+1}) \leq \lambda^{i+1} d_q(x_0, x_1).$$

Thus,

$$\begin{aligned} d_q(x_0, x_{i+2}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_{i+1}, x_{i+2}) \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^{i+2}) d_q(x_0, x_1) \\ &\leq \frac{1 - \lambda^{i+2}}{1 - \lambda}(1 - \lambda)r \leq (1 - \lambda^{i+2})r \leq r. \end{aligned}$$

Hence, $x_{i+2} \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$., we have $\alpha(x_{i+1}, Sx_{i+1}) = \alpha(x_{i+1}, x_{i+2}) \geq 1$. Therefore, (P_{i+1}) holds. Similarly, one can see that if i is odd, then (P_{i+1}) holds, which completes the inductive proof. Thus, we can write

$$d_q(x_n, x_{n+1}) \leq \lambda^n d_q(x_0, x_1), \ \ \forall n \in \mathbb{N}. \tag{3.6}$$

Next, we will show that the sequence $\{x_n\}$ is a left K-Cauchy sequence. Indeed, for $n, m \in \mathbb{N}$ with $m > n$ using (3.7) we have

$$\begin{aligned} d_q(x_n, x_m) &\leq d_q(x_n, x_{n+1}) + d_q(x_{n+1}, x_{n+2}) + \cdots + d_q(x_{m-1}, x_m) \\ &\leq \lambda^n d_q(x_0, x_1) + \lambda^{n+1} d_q(x_0, x_1) + \cdots + \lambda^{m-1} d_q(x_0, x_1). \end{aligned}$$

Thus,

$$d_q(x_n, x_m) \leq \frac{\lambda^n}{1 - \lambda} d_q(x_0, x_1), \ \forall n, m \in \mathbb{N}, \ m > n. \tag{3.7}$$

Since $0 < \lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} < 1$, for every $\epsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such tha $\lambda^n < \frac{1-\lambda}{d_q(x_0,x_1)}\epsilon$ for all $n > n_0$. Therefore, it follows from (3.7) that

$$d_q(x_n, x_m) < \epsilon, \ \ \forall m > n > n_0.$$

Therefore, the sequence $\{x_n\}$ is a left K-Cauchy sequence in X. By left K-sequential completeness of X, there exists $x \in X$ such that

$$\lim_{n\to\infty} d_q(x_n, z) = \lim_{n\to\infty} d_q(z, x_n) = 0. \tag{3.8}$$

We will show that z is a common fixed point of the mappings S and T. By Remark 2.5, we have $z \in B_{d_q}(x_0, r)$. Now, by the assumption we have for all $n \in \mathbb{N}$, therefore for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d_q(z, Sz) &\leq d_q(z, x_{2n+2}) + d_q(x_{2n+2}, Sz) \\ &\leq d_q(z, x_{2n+2}) + d_q(Tx_{2n+1}, Sz) \\ &\leq d_q(z, x_{2n+2}) + \beta d_q(x_{2n+1}, z) \\ &\quad + \gamma[d_q(x_{2n+1}, Sx_{n+1}) + d_q(z, Sz)] \\ &\quad + \delta[d_q(z, Tx_{2n+1}) + d_q(x_{2n+1}, Sz)] \\ &\leq d_q(z, x_{2n+2}) + \beta d_q(x_{2n+1}, z) \\ &\quad + \gamma[d_q(x_{2n+1}, Sx_{n+1}) + d_q(z, Sz)] \\ &\quad + \delta[d_q(z, x_{2n+2}) + d_q(x_{2n+1}, z) + d_q(z, Sz)]. \end{aligned}$$

By using (3.7) and (3.8), we obtain

$$(1 - \gamma + \delta) d_q(z, Sz) \leq 0 \tag{3.9}$$

which implies that $d_q(z, Sz) = 0$. Similarly, one can show that $d_q(Sz, z) = 0$. Thus, $d_q(z, Sz) = d_q(Sz, z) = 0$, i.e., $z = Sz$ Similarly, one can show that $z = Tz$.

Hence, S and T have a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$. As is an dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(z, Sz) = \alpha(z, z) \geq 1$. Therefore,

$$\begin{aligned} d_q(z, z) &\leq d_q(Sz, Tz) \\ &\leq \beta d_q(z, z) + \gamma[d_q(z, Sz) + d_q(z, Tz)] \\ &\quad + \delta[d_q(z, Sz) + d_q(z, Tz)] \\ &\leq (\beta + 2\gamma + 2\delta) d_q(z, z), \end{aligned}$$

and this implies that

$$d_q(z, z) = 0.$$

□

Example 3.2. Let $X = \mathbb{Q}^+ \cup \{0\}$ and let $d_q : X^2 \times X^2 \to X$ be defined by $d_q((x_1, y_1), (x_2, y_2)) = x_1 + 4y_1 + \frac{x_2}{4} + y_2$. Then it is easy to show that (X^2, d_q) is a left K-sequentially complete dislocated quasi metric space. If $(x_0, y_0) = (4, 1), r = 28$, then

$$\overline{B_{d_q}((4, 1), 28)} = \{(x, y) \in X : x + 4y \leq 42\}.$$

In particular, $(4, 1) \in \overline{B_{d_q}((4, 1), 28)}$.

Let $S, T : X^2 \to X^2$ be defined by

$$S(x, y) = \begin{cases} \left(\dfrac{x}{7}, \dfrac{y}{7}\right), & \text{if } x + 4y \leq 42 \\ \\ (2x^2 - 2, 4x + 5), & \text{if } x + 4y > 42 \end{cases}$$

and

$$T(x, y) = \begin{cases} \left(\dfrac{x}{6}, \dfrac{y}{9}\right), & \text{if } x + 4y \leq 42 \\ \\ (3x^2 - 3, y), & \text{if } x + 4y > 42. \end{cases}$$

Also, define $\alpha : X^2 \times X^2 \to [0,\infty)$ by

$$\alpha((x_1,y_1),(x_2,y_2)) = \begin{cases} 1, & \text{if } \frac{x_1}{4} + y_1 + x_2 + y_2 \leq 42 \\ \\ 0, & \text{if } \frac{x_1}{4} + y_1 + x_2 + y_2 > 42. \end{cases}$$

Clearly, S and T are α-dominated mappings on $\overline{B_{d_q}((4,1),28)}$. Let $\beta = \frac{1}{7}$, $\gamma = \delta = \frac{1}{10}$, then $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} = \frac{3}{10} \in (0,1)$ and $(1-\lambda)r = 16$, $d_q((x_0,y_0),S(x_0,y_0) = d_q((4,1),S(4,1)) = \frac{104}{7} < 16 = (1-\lambda)r$. Observe that, for $(43,0) \notin \overline{B_{d_q}((4,1),28)}$, we have $d_q(S(43,0),T(43,0)) = d_q((3696,5),(5544,0)) = 5104$, $d_q((43,0),T(43,0)) + d_q((43,0),S(43,0)) = 2401$, and $d_q((43,0),(43,0)) = \frac{215}{4}$. Hence, there are no β,γ,δ such that $\beta+2\gamma+\delta \in (0,1)$ and (3.1) is satisfied. So the contractive condition does not hold on X^2. On the other hand, if $(x_1,y_1),(x_2,y_2) \in \overline{B_{d_q}((4,1),28)}$, then

$$\begin{aligned} d_q(S(x_1,y_1),T(x_2,y_2)) &= d_q\Big(\Big(\frac{x_1}{7},\frac{y_1}{7}\Big),\Big(\frac{x_2}{6},\frac{y_2}{9}\Big)\Big) \\ &= \frac{x_1}{7} + \frac{4y_1}{7} + \frac{x_2}{24} + \frac{y_2}{9} \\ &\leq \frac{1}{7} d_q((x_1,y_1),(x_2,y_2)) \\ &\quad + \frac{1}{10}[d_q((x_1,y_1),S(x_1,y_1)) + d_q((x_2,y_2),T(x_2,y_2))] \\ &\quad + \frac{1}{10}[d_q((x_2,y_2),S(x_1,y_1)) + d_q((x_1,y_1),T(x_2,y_2))]. \end{aligned}$$

Also,

$$\begin{aligned} d_q(T(x_1,y_1),S(x_2,y_2)) &= d_q\Big(\Big(\frac{x_1}{6},\frac{y_1}{9}\Big),\Big(\frac{x_2}{7},\frac{y_2}{7}\Big)\Big) \\ &= \frac{x_1}{6} + \frac{4y_1}{9} + \frac{x_2}{28} + \frac{y_2}{7} \\ &\leq \frac{1}{7} d_q((x_1,y_1),(x_2,y_2)) \\ &\quad + \frac{1}{10}[d_q((x_1,y_1),S(x_1,y_1)) + d_q((x_2,y_2),T(x_2,y_2))] \\ &\quad + \frac{1}{10}[d_q((x_2,y_2),S(x_1,y_1)) + d_q((x_1,y_1),T(x_2,y_2))]. \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are satisfied. Moreover, $(0,0)$ is the common fixed point of S and T.

Corollary 3.3. *Let (X,d_q) be a left K-sequentially complete dislocated quasi metric space and $S : X \to X$ be a mapping. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0,+\infty)$ such that S be an α-dominated mappings on $B_{d_q}(x_0,r)$. Suppose that $x_0 \in B_{d_q}(x_0,r)$ and there exist nonnegative real numbers β,γ,δ such that $\beta+2\gamma+\delta \in (0,1)$ and the following condition holds: if $\alpha(x,y) \geq 1$ or $\alpha(y,x) \geq 1$ and $x,y \in B_{d_q}(x_0,r)$, then*

$$d_q(Sx,Sy) \leq \beta d_q(x,y) + \gamma[d_q(x,Sx) + d_q(y,Sy)] + \delta[d_q(y,Sx) + d_q(x,Sy)]$$

and

$$d_q(x_0,Sx_0) \leq (1-\lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X,d_q) is α-regular on $\overline{B_{d_q}(x_0,r)}$. Then there exists a point $z \in \overline{B_{d_q}(x_0,r)}$ such that $z = Sz$ and $d_q(z,z) = 0$.

Proof. Letting $T = S$ in Theorem 3.1, we obtain the following result. □

Corollary 3.4. *Let (X, d) be a complete dislocated metric space and $S, T : X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mappings on $\overline{B_d(x_0, r)}$. Suppose that $x_0 \in \overline{B_d(x_0, r)}$ and there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + \delta \in (0, 1)$ and the following condition holds: if $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and $x, y \in B_d(x_0, r)$, then*

$$d(Sx, Ty) \leq \beta d(x, y) + \gamma[d(x, Sx) + d(y, Ty)] + \delta[d(y, Sx) + d(x, Ty)]$$

and

$$d(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X, d) is α-regular on $\overline{B_d(x_0, r)}$. Then there exists a point $z \in \overline{B_d(x_0, r)}$ such that $z = Sz$ and $d(z, z) = 0$.

Proof. By Theorem 3.1, we obtain the following result. □

Theorem 3.5. *Suppose that all the conditions of Theorem 3.1 are satisfied. In addition suppose that:*

(a) *The function α is a triangular function on $\overline{B_{d_q}(x_0, r)}$.*
(b) *For $x, y \in \overline{B_{d_q}(x_0, r)}$ there exists $u_0 \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(x, u_0) \geq 1, \alpha(y, u_0) \geq 1$.*
(c) *For all $u \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(Sx_0, u) \geq 1$ the following condition holds*

$$d_q(x_0, Sx_0) + d_q(u, Tu) + d_q(u, Sx_0) + d_q(x_0, Tu) \leq d_q(x_0, u) + d_q(Sx_0, Tu).$$

Then S and T have a unique common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ and $d_q(z, z) = 0$.

Proof. Define the sequence fxng as in the proof Theorem 3.1. Then, $\{x_n\}, d_q$-converges to a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ of the mappings S and T such that $\alpha(x_n, z) \geq 1$ for all $n \geq 0, (P_n)$ holds and $d_q(z, z) = 0$. In order to prove uniqueness of z, suppose that z^* is another point in $\overline{B_{d_q}(x_0, r)}$ such that $z^* = Sz^* = Tz^*$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(z^*, Sz^*) = \alpha(z^*, z^*) \geq 1$. Therefore,

$$\begin{aligned} d_q(z^*, z^*) &\leq d_q(Sz^*, Tz^*) \\ &\leq \beta d_q(z^*, z^*) + \gamma[d_q(z^*, Sz^*) + d_q(z^*, Tz^*)] \\ &\quad + \delta[d_q(z^*, Sz^*) + d_q(z^*, Tz^*)] \\ &\leq (\beta + 2\gamma + 2\delta) d_q(z^*, z^*), \end{aligned}$$

and this implies that

$$d_q(z^*, z^*) = 0.$$

By assumption, there exists a point $u_0 \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(z, u0) \geq 1$ and $\alpha(z^*, u_0) \geq 1$. Define a sequence $\{u_n\}$ in X such that,

$$u_{2k+1} = Su_{2k} \text{ and } u_{2k+2} = Tu_{2k+1}, \ \ \forall k = 0, 1, 2, \ldots.$$

By mathematical induction, we can show that

$$\begin{cases} \alpha(u_n, u_{n+1}) \geq 1, \alpha(x_n, u_n) \geq 1, \ \ \forall n \in \mathbb{N}; \\ d_q(u_n, u_{n+1}) \leq \lambda^n d_q(u_0, u_1), \ \ \forall n \in \mathbb{N}; \\ d_q(x_n, z_n) \leq \lambda^n r, \, u_n \in \overline{B_{d_q}(x_0, r)}, \ \ \forall n \in \mathbb{N}. \end{cases} \tag{P'_n}$$

Since T is α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$, we have $\alpha(u_0,Tu_0)=\alpha(u_0,u_1)\geq 1$. Since α is triangular function on $\overline{B_{d_q}(x_0,r)}$, and $\alpha(x_n,z)\geq 1,\alpha(z,u_0)\geq 1$, we have $\alpha(x_n,u_0)\geq 1$ for all $n\geq 0$. Therefore, using (c), we get that

$$\begin{aligned}d_q(x_1,u_1)&=d_q(Sx_0,Tu_0)\\&\leq \beta d_q(x_0,u_0)+\gamma[d_q(x_0,Sx_0)+d_q(u_0,Tu_0)]\\&\quad+\delta[d_q(u_0,Sx_0)+d_q(x_0,Tu_0)]\\&\leq \beta d_q(x_0,u_0)+\gamma[d_q(x_0,u_0)+d_q(Sx_0,Tu_0)]\\&\quad+\delta[d_q(x_0,u_0)+d_q(Sx_0,Tu_0)]\\&\leq \beta d_q(x_0,u_0)+\gamma[d_q(x_0,u_0)+d_q(x_1,u_1)]\\&\quad+\delta[d_q(u_0,x_0)+d_q(x_1,u_1)].\end{aligned}$$

Thus,

$$d_q(x_1,u_1)\leq \frac{\beta+\gamma+\delta}{1-\gamma-\delta}d_q(x_0,u_0)=\lambda d_q(x_0,u_0)\leq \lambda r. \tag{3.10}$$

Since $u_0\in\overline{B_{d_q}(x_0,r)}$, using (3.10), we get

$$\begin{aligned}d_q(x_0,u_1)&\leq d_q(x_0,x_1)+d_q(x_1,u_1)\\&\leq (1-\lambda)r+\lambda d_q(x_0,u_0)\\&\leq (1-\lambda)r+\lambda r\leq r\end{aligned}$$

Hence, $u_1\in\overline{B_{d_q}(x_0,r)}$. Since $\alpha(u_0,u_1)\geq 1$, by using (3.2), we get that

$$d_q(u_1,u_2)\leq \frac{\beta+\gamma+\delta}{1-\gamma-\delta}d_q(u_0,u_1)=\lambda d_q(u_0,u_1).$$

Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$, we have $\alpha(u_1,Su_1)=\alpha(u_1,u_1)\geq 1$. As, α is a triangular function on $\overline{B_{d_q}(x_0,r)}$, and $\alpha(x_1,u_0)\geq 1$, $\alpha(u_0,u_1)\geq 1$, we have $\alpha(x_1,u_1)\geq 1$. Therefore, from (P_1') holds. Since $\alpha(u_1,u_2)\geq 1$ and using (3.1), we get that

$$d_q(u_2,u_3)\leq \lambda d_q(u_1,u_2)\leq \lambda^2 d_q(u_0,u_1).$$

Since $\alpha(x_1,u_1)\geq 1$, using (3.2) that

$$\begin{aligned}d_q(x_2,u_2)&=d_q(Tx_1,Su_1)\\&\leq \beta d_q(x_1,u_1)+\gamma[d_q(x_1,Tx_1)+d_q(u_1,Su_1)]\\&\quad+\delta[d_q(u_1,Tx_1)+d_q(x_1,Su_1)]\\&\leq \beta d_q(x_1,x_2)+\gamma\lambda[d_q(x_0,Sx_0)+d_q(u_0,Tu_0)]\\&\quad+\delta\lambda[d_q(u_0,Sx_0)+d_q(x_0,Tu_0)]\end{aligned}$$

which gives wiyh (c)

$$\begin{aligned}d_q(x_2,u_2)&\leq \beta d_q(x_1,x_2)+\gamma\lambda[d_q(x_0,u_0)+d_q(Sx_0,Tu_0)]\\&\quad+\delta\lambda[d_q(u_0,x_0)+d_q(Sx_0,Tu_0)]\\&\leq (\beta+\lambda\gamma+\lambda\delta)d_q(x_1,u_1)+(\gamma\lambda+\delta\lambda)r.\end{aligned}$$

By using (3.10) and fact that $u_0\in\overline{B_{d_q}(x_0,r)}$, in above inequality we obtain

$$\begin{aligned}d_q(x_2,u_2)&\leq (\beta+\lambda\gamma+\lambda\delta)\lambda r+(\gamma\lambda+\delta\lambda)r\\&=(\beta+\lambda\gamma+\lambda\delta+\gamma+\delta)\lambda r=\lambda^2 r.\end{aligned}$$

Thus,

$$\begin{aligned} d_q(x_0, u_2) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + d_q(x_2, u_2) \\ &\leq d_q(x_0, x_1) + \lambda d_q(x_0, x_1) + \lambda^2 \leq r. \end{aligned}$$

Hence, $u_2 \in \overline{B_{d_q}(x_0, r)}$. Since T is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(u_2, u_3) \geq 1$. Therefore, from (P_2') holds. Suppose, $(P_1'), (P_2'), \ldots, (P_i')$ be the inductive hypothesis. We shall show that (P_{i+1}') holds. For this, we consider two possible cases. First, suppose that i is even. Then, since $\alpha(u_i, u_{i+1}) \geq 1$ and using (3.2), we get that

$$d_q(u_{i+1}, u_{i+2}) \leq \frac{\beta + \gamma + \delta}{1 - \gamma - \delta} d_q(u_i, u_{i+1}) = \lambda^{i+1} d_q(u_0, u_1).$$

Since $\alpha(x_i, u_i) \geq 1$, using (3.1) that

$$\begin{aligned} d_q(x_{i+1}, u_{i+1}) &= d_q(Sx_i, Tu_i) \\ &\leq \beta d_q(x_i, u_i) + \gamma[d_q(x_i, Sx_i) + d_q(u_i, Tu_i)] \\ &\quad + \delta[d_q(u_i, Sx_i) + d_q(x_i, Tu_i)] \\ &\leq \beta d_q(x_i, x_{i+1}) + \gamma\lambda[d_q(x_0, Sx_0) + d_q(u_0, Tu_0)] \\ &\quad + \delta\lambda[d_q(u_0, Sx_0) + d_q(x_0, Tu_0)] \end{aligned}$$

which gives wiyh (c) and P_i'

$$\begin{aligned} d_q(x_{i+1}, u_{i+1}) &\leq \beta d_q(x_i, u_i) + \gamma\lambda^i[d_q(x_0, u_0) + d_q(Sx_0, Tu_0)] \\ &\quad + \delta\lambda^i[d_q(u_0, x_0) + d_q(Sx_0, Tu_0)] \\ &\leq \beta\lambda^i r + \gamma\lambda^i[r + \lambda r] + \delta\lambda^i[r + \lambda r] \\ &= (\beta + \lambda\gamma + \lambda\delta + \gamma + \delta)\lambda^i r = \lambda^{i+1} r. \end{aligned}$$

Thus,

$$\begin{aligned} d_q(x_0, u_{i+1}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_i, x_{i+1}) + d_q(x_{i+1}, u_{i+1}) \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^i) d_q(x_0, x_1) + \lambda^{i+1} r \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^i)(1 - \lambda) r + \lambda^{i+1} r = r. \end{aligned}$$

Hence, $u_{i+1} \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$., we have $\alpha(u_{i+1}, Su_{i+1}) = \alpha(u_{i+1}, u_{i+2}) \geq 1$. Also, since $\alpha(x_{i+1}, u_0) \geq 1$, $\alpha(u_n, u_{n+1}) \geq 1$, $n = 0, 1, 2, \ldots, i+1$, by triangular nature of α, we have $\alpha(x_{i+1}, u_{i+1}) \geq 1$. Therefore, (P_{i+1}') holds. Similarly, one can see that if i is odd, then (P_{i+1}') holds, which completes the inductive proof. Thus, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d_q(z,u_{2n}) &= d_q(Tz,Su_{2n-1})\\
&\le \beta d_q(z,u_{2n-1}) + \gamma[d_q(z,Tz)+d_q(u_{2n-1},Su_{2n-1})]\\
&\quad + \delta[d_q(u_{2n-1},Tz)+d_q(z,Su_{2n-1})]\\
&\le \beta d_q(z,u_{2n-1}) + \gamma d_q(u_{2n-1},u_{2n})\\
&\quad + \delta[d_q(u_{2n-1},z)+d_q(z,u_{2n})]\\
&= (\beta+\delta)d_q(z,u_{2n-1}) + \gamma d_q(u_{2n-1},u_{2n}) + \delta d_q(z,u_{2n})\\
&\le (\beta+2\delta)d_q(z,u_{2n-1}) + (\gamma+\delta)d_q(u_{2n-1},u_{2n})\\
&\le (\beta+2\delta)d_q(Tz,Su_{2n-2}) + (\gamma+\delta)d_q(u_{2n-1},u_{2n})\\
&\le (\beta+2\delta)^2 d_q(z,u_{2n-2}) + (\beta+2\delta)(\gamma+\delta)d_q(u_{2n-2},u_{2n-1})]\\
&\quad + (\gamma+\delta)d_q(u_{2n-1},u_{2n})\\
&\le (\beta+2\delta)^2 d_q(Tz,Su_{2n-3}) + (\beta+2\delta)(\gamma+\delta)d_q(u_{2n-2},u_{2n-1})\\
&\quad + (\gamma+\delta)d_q(u_{2n-1},u_{2n})\\
&\le (\beta+2\delta)^3 d_q(z,u_{2n-3}) + (\beta+2\delta)^2(\gamma+\delta)d_q(u_{2n-3},u_{2n-2})\\
&\quad + (\beta+2\delta)(\gamma+\delta)d_q(u_{2n-2},u_{2n-1}) + (\gamma+\delta)d_q(u_{2n-1},u_{2n})\\
&\quad \vdots\\
&\le (\beta+2\delta)^{2n} d_q(z,u_0) + (\beta+2\delta)^{2n-1}(\gamma+\delta)d_q(u_0,u_1) + \dots\\
&\quad + (\beta+2\delta)(\gamma+\delta)d_q(u_{2n-2},u_{2n-1}) + (\gamma+\delta)d_q(u_{2n-1},u_{2n})
\end{aligned}$$

Since $\frac{\beta+2\delta}{\lambda} = \frac{(\beta+2\delta)(1-\gamma-\delta)}{\beta+\gamma+\delta} < 1$, using (P_n') in the above inequality we obtain

$$\begin{aligned}
d_q(z,u_{2n}) &\le (\beta+2\delta)^{2n} d_q(z,u_0) + (\beta+2\delta)^{2n-1}(\gamma+\delta)d_q(u_0,u_1) + \dots\\
&\quad + (\beta+2\delta)(\gamma+\delta)\lambda^{2n-2}d_q(u_0,u_1) + (\gamma+\delta)\lambda^{2n-1}d_q(u_0,u_1)\\
&= (\beta+2\delta)^{2n} d_q(z,u_0)\\
&\quad + (\gamma+\delta)\lambda^{2n-1}d_q(u_0,u_1)\Big[1+\frac{\beta+2\delta}{\lambda}+\dots+\Big(\frac{\beta+2\delta}{\lambda}\Big)^{2n-1}\Big]\\
&\le (\beta+2\delta)^{2n} d_q(z,u_0) + \frac{(\gamma+\delta)\lambda^{2n-1}d_q(u_0,u_1)}{1-\frac{\beta+2\delta}{\lambda}}
\end{aligned}$$

Since $\beta+2\delta, \lambda \in [0,1)$, it follows from the above inequality that

$$\lim_{n\to\infty} d_q(z,u_{2n}) = 0. \tag{3.11}$$

Similarly, we can show that

$$\lim_{n\to\infty} d_q(u_{2n},z) = \lim_{n\to\infty} d_q(u_{2n},z^*) = \lim_{n\to\infty} d_q(z^*,u_{2n}) = 0. \tag{3.12}$$

By using (3.11) and (3.12), we obtain

$$d_q(z,z^*) \le d_q(z,u_{2n}) + d_q(u_{2n},z^*) \to 0, \text{ as } n\to\infty.$$

$$d_q(z^*,z) \le d_q(z^*,u_{2n}) + d_q(u_{2n},z) \to 0, \text{ as } n\to\infty.$$

Hence, $d_q(z,z^*) = d_q(z^*,z) = 0$, i.e., $z = z^*$ □

Corollary 3.6. *Let (X,d_q) be a left K-sequentially complete dislocated quasi metric space and $T,S : X \to X$ be two mappings. Let $x_0 \in X, r > 0, x_0 \in \overline{B_{d_q}(x_0,r)}$ and* *there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + 2\delta \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Then there exists a unique point $z \in \overline{B_{d_q}(x_0, r)}$ such that $z = Sz = Tz$ and $d_q(z, z) = 0$. Moreover, S and T have no fixed point in $\overline{B_{d_q}(x_0, r)}$ other than z.

Proof. The proof follows by the previous results, taking $\alpha : X \times X \to [0, \infty)$ with $\alpha(x, y) = 1$ for all $x, y \in X$. □

Theorem 3.7. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space. Suppose, there exist a function $\alpha : X \times X \to [0, +\infty)$ and nonnegative constants β, γ, δ such that $\beta + 2\gamma + 2\delta \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

for all $x, y \in X$ such that $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$. If (X, d_q) is α-regular,then there exists a point z in X such that $z = Sz = Tz$ and $d_q(z, z) = 0$.

Proof. By Theorem 3.1, the condition (3.3) is imposed in order to restrict the contractive conditions (3.1) and (3.2) to $\overline{B_{d_q}(x_0, r)}$. However, the condition (3.3) can be relaxed by imposing the conditions (3.1) and (3.2) to all elements $x, y \in X$ such that $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$, we obtain the following result. □

Recall that, if $(X, \preceq)$ is a pre-ordered set and $T : X \to X$ is such that $Tx = x$ for all $x \in A \subseteq X$, then the mapping T is said to be dominated on A. Define the set ∇ by

$$\nabla = \{(x, y) \in X \times X : x \preceq y \,\text{or}\, y \preceq x\}.$$

Theorem 3.8. *Let $(X, \preceq, d_q)$ be a pre-ordered left K-sequentially complete dislocated quasi metric space, $x_0 \in X$, $r > 0$ and $S, T : X \to X$ be two dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Suppose that there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + \delta \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

for all $(x, y) \in \overline{B_{d_q}(x_0, r)} \times \overline{B_{d_q}(x_0, r)} \cap \nabla$ and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$ If for any sequence $\{x_n\} \in \overline{B_{d_q}(x_0, r)}$ such that $(x_n, x_{n+1}) \in \nabla$, $x_n \to w$ as $n \to \infty$ implies that $(w, x_n) \in \nabla$ for all $n \geq 0$, then there exists a point $z \in \overline{B_{d_q}(x_0, r)}$ such that $z = Sz = Tz$ and $d_q(z, z) = 0$. In addition, suppose that:

(a) *$(x, y), (y, z) \in \nabla$ implies $(x, z) \in \nabla$.*
(b) *For x, $y \in \overline{B_{d_q}(x_0, r)}$ there exists $u_0 \in \overline{B_{d_q}(x_0, r)}$ such that (x, u_0), $(y, u_0) \in \nabla$.*
(c) *For all $u \in \overline{B_{d_q}(x_0, r)}$ such that $(u, Sx_0) \in \nabla$ the following condition holds*

$$d_q(x_0, Sx_0) + d_q(u, Tu) + d_q(u, Sx_0) + d_q(x_0, Tu) \leq d_q(x_0, u) + d_q(Sx_0, Tu).$$

Then, z is the unique common fixed point of S and T in $\overline{B_{d_q}(x_0, r)}$.

Proof. This follows from Theorem 3.6 taking $\alpha : X \times X \to [0, +\infty)$ defined as

$$\alpha(x, y) = \begin{cases} 1, & \text{If } (x, y) \in \nabla, \\ 0, & \text{otherwise.} \end{cases}$$

□

4. Conclusions

We prov some common fixed point theorems for mappings under Hardy Rogers contractive conditions on a left K-sequentially complete dislocated quasi metric space.

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SOLVE ONE-DIMENSIONAL OPTIMIZATION PROBLEMS USING NEWTON-RAPHSON METHOD

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ABSTRACT. This research studied solve one-dimensional optimization problems using Newton – Raphson Method. This method uses first derivatives to solve solutions. It was tested with 12 function tests and tested the efficiency of the solution convergence rate Newton – Raphson method and the random search method. It was found that the Newton-Raphson method was more effective in converging rate to better answers than random search methods. The error value and the iteration are indicative.

KEYWORDS: Optimization, Newton – Raphson method, Random Search Method.
AMS Subject Classification: 47H09, 49M05.

1. Introduction

Optimization is the act of achieving the best possible result under given circumstances. In design, construction, maintenance, and engineers, etc. Have to make decisions. The goal of all such decisions is either to minimize effort or to maximize benefit. The effort or benefit can be usually expressed as a function of certain design variables. Hence, optimization is the process of finding the conditions that give the maximum or the minimum value of a function. It is obvious that if a point x^* corresponds to the minimum value of a function $f(x)$, the same point corresponds to the maximum value of the function. Thus, optimization can be taken to be minimization. There is no single method available for solving all optimization problems efficiently. Hence, a number of methods have been developed for solving different types of problems. Optimum seeking methods are also known as mathematical programming techniques, which are a branch of operations search. The Newton – Raphson (NR) method is an iterative scheme used to solve non-linear simultaneous equations. France (1991) described the application of NR method to solve various

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hydraulic problems. Martin and Peters (1963) were the first to propose the application of NR method for analysis of water distribution network (WDN) having pipes and reservoirs only. McCormick and Bellamy (1968) and Zarghamee (1971) extended its use to include other network elements such as pumps and valves. Shamir and Howard (1968) applied this method to solve for all types of unknowns in a network including pipe resistances using head equations. Epp and Fowler (1970) and Gofman and Rodeh (1981) used this method for a solution of loop equations. Some investigators proposed modifications to the classical NR method. Lam and Wolla (1972 a, b) modified the algorithm so that it does not require the Jacobian matrix or its inverse in the iterative process and also suggested a change in step size to minimize the error. The modification presented by Lemieux (1972) ensures the convergence of the algorithm irrespective of the starting assumption. Donachie (1974) suggested halving the step size at any node when oscillation occurs. Neilson (1989) compared the NR method with linear theory method and suggested starting of the NR method with a single iteration by the linear theory method followed by NR iterations. Andersen and Powell (1999) used a linear headloss formula in the NR method for the first iteration. Several computer programs are also available for analysis of WDNs using Newton-Raphson method. The application of NR method in optimal design of WDNs is however very limited. Young (1994) used NR method to solve nonlinear simultaneous equations which were generated through Lagrangian multiplier method for the optimal design of branched 3 WDNs. Johnson et al. (1995) discussed the limitations of the Lagrangian multiplier method proposed by Young (1994). Bhave (1978, 1985) developed the cost head loss ratio, criterion method for the optimal design of WDNs. This method can be used for optimal design and expansion of single as well as multi-source branched or looped networks including pumped source nodes. Herein, the cost head loss ratio criterion method is modified for faster convergence using the NR method.
The NR method is used to solve simultaneous non-linear equations iteratively. It expands the non-linear terms in Taylor's series, neglects the residues after two terms and thereby considers only the linear terms (Bhave 1991). Thus, the NR method linearizes the non-linear equations through partial differentiation and solves. Naturally, the solution is approximate and therefore is successively corrected. The iterative procedure is continued until satisfactory accuracy is reached. Thus, while applying NR method for obtaining correction in the cost head loss ratio criterion method, all correction equations would be considered simultaneously and solved at a time.

2. Preliminaries

2.1. **Optimal Conditions.**

We begin with a formal statement of the conditions which held at a minimum of a one-variable differentiable function. We have already made use of these conditions.

Definition 1. Suppose that $f(x)$ is a continuously differentiable function of the scalar variable x, and that, when $x = x^*$,

$$\frac{df}{dx} = 0 \ \ and \ \ \frac{d^2 f}{dx^2} > 0. \tag{2.1}$$

The function $f(x)$ is then said to have a *local minimum* at x^*. Conditions (1) imply that $f(x^*)$ is the smallest value of f in some region near x^*. It may also be true that $f(x^*) \leq f(x)$ for all x but condition (1) does not guarantee this.

Definition 2. If conditions (1) hold at $x = x^*$ and if $f(x^*) \leq f(x)$ for all x then x^* is said to be the *global minimum*. In practice, it is usually hard to establish that x^* is a global minimum and so we shall chiefly be concerned with methods of finding local minima. Conditions (1) are called optimality conditions.

2.2. **Optimal Solution.**

A globally optimal solution for an optimization problem is defined as the solution $x^* \in X$, where $f(x^*) \leq f(x)$ for all $x \in X$ (minimization problem). For the definition of a globally optimal solution, it is not necessary to define the structure of the search space, a metric, or a neighborhood.
Given a problem instance (X, f) and a neighborhood function N^*, a feasible solution $x^\circ \in X$ is called locally optimal (minimization problem) with respect to N^* if

$$f(x^\circ) \leq f(x) \text{ for all } x \in N^*(x^\circ)$$

Therefore, locally optimal solutions do not exist if no neighborhood is defined. Furthermore, the existence of local optima is determined by the neighborhood the definition used as different neighborhoods can result in different locally optimal solutions.

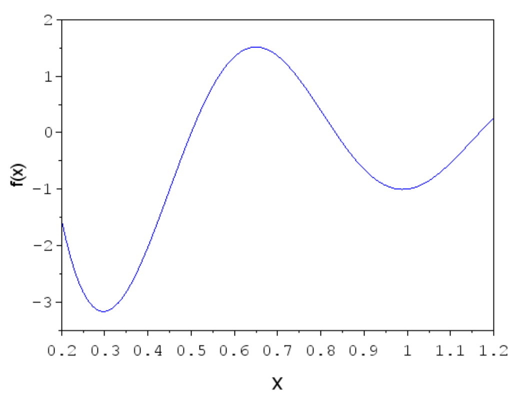

Fig.1. Locally and globally optimal solutions

Figure 1. illustrates the differences between locally and globally optimal solutions and shows how local optimal depend on the definition of N^*. We have a one-dimensional minimization problem with $x \in [a, b] \in R$. We assume an objective function f that assigns objective values to all $x \in X$. The modality of a problem describes the number of local optima in the problem. Unimodal problems have only one local optimum(which is also the global optimum), whereas muti-modal problems have multiple local optima. In general, multi-modal problems are more difficult for guided search methods to solve than unimodal problems.

2.3. **Objective function.**

The classical design procedure aims at finding an acceptable design, *i.e.* a design which satisfies the constraints. In general, there are several acceptable designs, and purpose

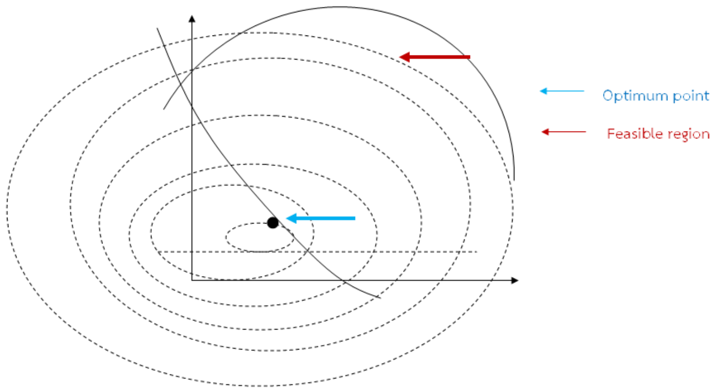

Fig.2. Design space, and optimum point

of the optimization is to single out the best possible design. Thus, a criterion has to be selected for comparing different designs. This criterion, when expressed as a function of the design variables are known as an objective function. The objective function is in general specified by physical or economic considerations. However, the selection of an objective function is not trivial, because what is the optimal design with respect to a certain criterion may be unacceptable with respect to another criterion. Typically, there is a trade-off performance-cost, performance reliability, hence the selection of the objective function is one of the most important decisions in the whole design process. If more than one criterion has to be satisfied, we have multiobjective optimization problem, that may be approximately solved considering a cost function which is a weighted sum of several objective functions.

Let $f : D \subseteq R \longrightarrow R$ is the objective function by a vector $x = (x_1, x_2, x_3, , ... , x_n)$ such as Styblinski-Tang Function $f(x) = 0.5(x^4 - 16x^2 + 5x)$ for $n = 1$. The function is usually evaluated on the hypercube $x \in [-5, 5]$ and global optimal is $f(x) = -39.16599n$

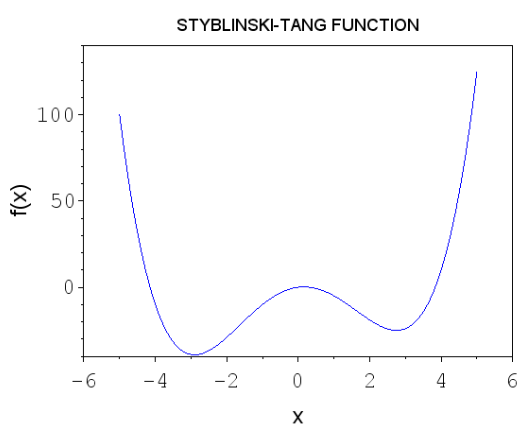

Fig.3. Styblinski-Tang Function

3. Random Search Method

This method generates trial solutions for the optimization model using random number generators for decision variables. Random search method includes random jump method and the random walk method with direction exploitation. Random jump method generates a huge number of data points for the decision variable assuming a uniform distribution for them and finds out the best solution by comparing the corresponding objective function values. Random walk method generates a trial solution with sequential improvements which is governed by a scalar step length and a unit random vector. The random walk method with direct exploitation is an improved version of a random walk method, in which, first the successful direction of generating trial solutions is found out and the maximum possible steps are taken along this successful direction. A generalized flowchart of the search algorithm in solving a nonlinear optimization with decision variable x_i, is presented in Fig.4.

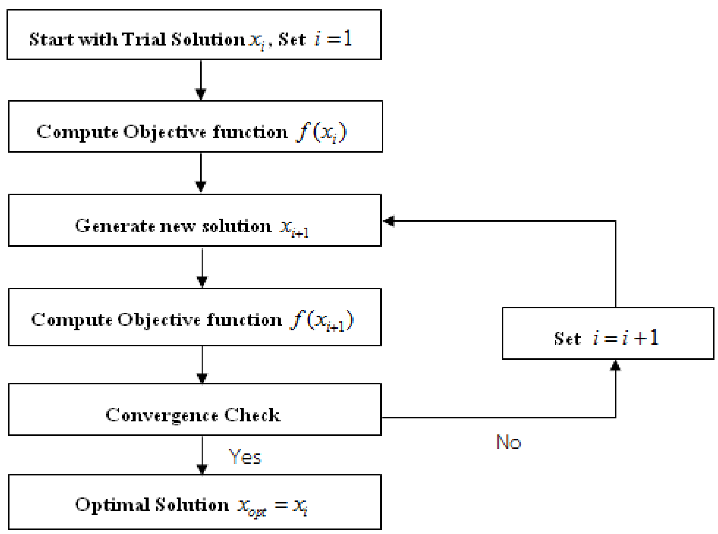

Fig.4. Flowchart Random Search Method

4. Main Results

4.1. **The Newton – Raphson Method.**

Newton – Raphson method is an open approach to find the minimum of function $f(x)$. Derivative using Taylor series recall Taylor series expansion as follows.

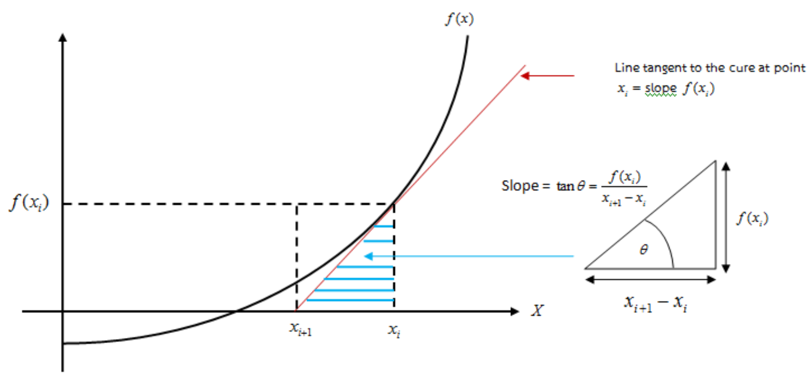

Fig.5. Newton - Raphson Method

from the previous figure 5.

$$\text{slope} = f^{'}(x_i) = \frac{df(x)}{dx}|_{x=x_i} = \frac{f(x_i)-0}{x_i-x_{i+1}}$$

or

$$f^{'}(x_i) = \frac{f(x_i)}{x_i-x_{i+1}} \qquad \text{for } i = 0, 1, 2, 3, ...$$

Thus

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{'}(x_i)}$$

This algorithm is derived by expanding $f(x)$ as a Taylor series

$$f(x_0+h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!}f^{(2)}(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + ... + \frac{h^n}{n!}f^{(n)}(x_0) + R_{n+1}$$

If we let $x_0 + h = x_i + h = x_{i+1}$ and terminate the series at its linear term, then

$$f(x_i + h) = f(x_i) + (x_{i+1} - x_i)f^{(1)}(x_i)$$

or

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)f^{'}(x_i)$$

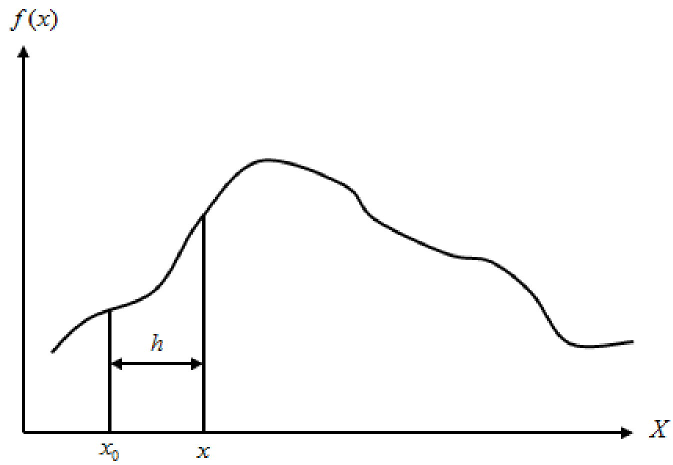

Fig.6. Interval point x and x_0 design

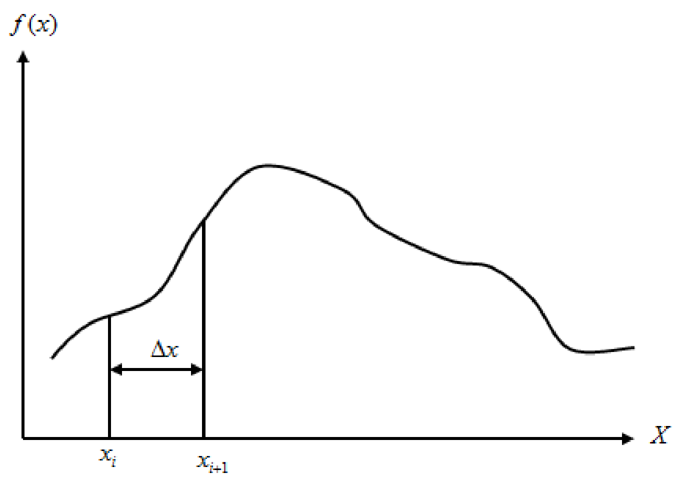

Fig.7. Interval point x_i and x_{i+1} design

Note that since the root of the function relating $f(x)$ and x is the value of x when $f(x_{i+1}) = 0$ at the intersection, hence,

$$\begin{aligned} f(x_{i+1}) &= 0 \\ f(x_i) + (x_{i+1} - x_i)f^{'}(x_i) &= 0 \\ (x_{i+1} - x_i)f^{'}(x_i) &= -f(x_i) \\ x_{i+1} &= x_i - \tfrac{f(x_i)}{f'(x_i)} \end{aligned}$$

Newton – Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{for} \quad \text{i=0,1,2,3,...} \tag{4.1}$$

where
x_i = value of the root at iteration i
x_{i+1} = a revised value of the root at iteration $i + 1$
$f(x_i)$ = value of the function at iteration i

$f^{'}(x_i)$ = derivative of $f(x)$ evaluated at iteration i

4.2. **Iterations.**

The Newton – Raphson method uses the slope(tangent) of the function $f(x)$ of the current iterative solution (x_i) to find the solution (x_{i+1}) in the next iteration. The slope at $(x_i.f(x_i))$ is given by

$$f^{'}(x_i) = \frac{f(x_i)-0}{x_i-x_{i+1}}$$

Then x_{i+1} can be solved as

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{'}(x_i)}$$

Which is known as the Newton – Raphson formula.
Relative error: $|E_{rr}| = |\frac{x_{i+1}-x_i}{x_{i+1}}|$ and Iterations stop if $E_{rr} \leq E$ is presented in Fig.8.

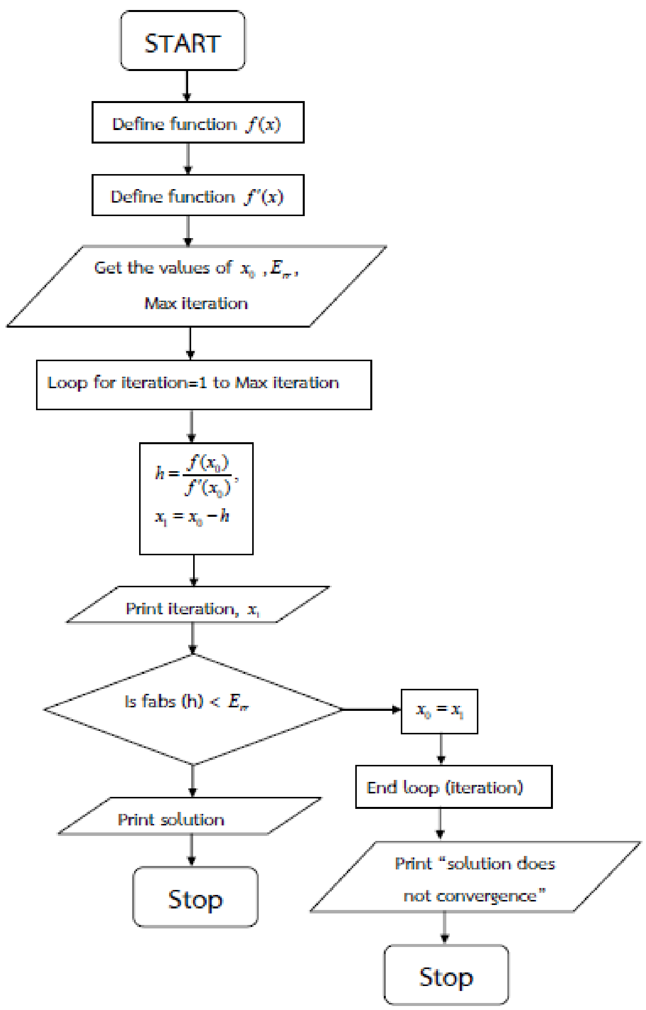

Fig.8. Flowchart of Newton – Raphon Method

Example 4.1. Use the Newton – Raphson iteration method to estimate the root of the following function employing an initial guess of $x_0 = 0 : f(x) = e^x - 5x$
Solution: Let's find the derivative of the function first,

$$f^{'}(x) = \frac{df(x)}{dx} = e^x - 5$$

The initial guess is $x_0 = 0$, hence,
$i = 0$

$$\begin{aligned} f(x) = e^x - 5x \quad &, \quad f(0) = e^0 - 5(0) = 1 \\ f^{'}(x) = e^x - 5 \quad &, \quad f(0) = e^0 - 5 = 1 \\ x_{i+1} &= x_i - \frac{f(x_i)}{f^{'}(x_i)} \\ x_1 &= x_0 - \frac{f(x_0)}{f^{'}(0)} \\ x_1 &= 0 - \frac{1}{1} \\ x_1 &= -1 \end{aligned}$$

$i = 1$ Now $x_1 = -1$, hence,

$$\begin{aligned} f(x) = e^x - 5x \quad &, \quad f(-1) = e^{-1} - 5(-1) = 5.3678794412 \\ f^{'}(x) = e^x - 5 \quad &, \quad f(-1) = e^{-1} - 5 = -4.632120559 \\ x_2 &= x_1 - \frac{f(x_1)}{f^{'}(1)} \\ x_2 &= -1 - \frac{5.3678794412}{-4.632120559} \\ x_2 &= -1 + 1.158838457 \\ x_2 &= 0.158838457 \end{aligned}$$

$i = 2$ Now $x_2 = 0.158838457$, hence,

$$\begin{aligned} f(x) = e^x - 5x \quad &, \quad f(0.158838457) = e^{0.158838457} - 5(0.158838457) \\ &= 0.3787956294 \\ f^{'}(x) = e^x - 5 \quad &, \quad f(0.158838457) = e^{0.158838457} - 5 = -3.827851421 \\ x_3 &= x_2 - \frac{f(x_2)}{f^{'}(2)} \\ x_3 &= 0.158838457 - \frac{0.3787956294}{-3.827851421} \\ x_3 &= 0.158838457 + 0.0989577671 \\ x_3 &= 0.2577962241 \end{aligned}$$

$i = 3$ Now $x_3 = 0.2577962241$, hence,

$$\begin{aligned} f(x) = e^x - 5x \quad &, \quad f(0.2577962241) = e^{0.2577962241} - 5(0.2577962241) \\ &= 0.0050939696 \\ f^{'}(x) = e^x - 5 \quad &, \quad f(0.2577962241) = e^{0.2577962241} - 5 = -3.70592491 \\ x_4 &= x_3 - \frac{f(x_3)}{f^{'}(3)} \\ x_4 &= 0.2577962241 - \frac{0.0050939696}{-3.70592491} \\ x_4 &= 0.2577962241 + 0.00137454744 \\ x_4 &= 0.25917077154 \end{aligned}$$

$i = 4$ Now $x_4 = 0.25917077154$, hence,

$$
\begin{aligned}
f(x) = e^x - 5x \quad &, \quad f(0.25917077154) = e^{0.25917077154} - 5(0.25917077154) \\
&= 1.2234E-6 \\
f^{'}(x) = e^x - 5 \quad &, \quad f(0.25917077154) = e^{0.25917077154} - 5 = -3.704144919 \\
x_5 &= x_4 - \frac{f(x_4)}{f^{'}(4)} \\
x_5 &= 0.25917077154 - \frac{1.2234E-6}{-3.704144919} \\
x_5 &= 0.25917077154 + 3.3027865e-7 \\
x_5 &= 0.2591711018
\end{aligned}
$$

Thus, the approach rapidly convergences on the true root of 0.2591 to four significant digits in table 1.

TABLE 1. Newton - Raphson iteration method $f(x) = e^x - 5x$

i	x_i	x_{i+1}	$f(x_i)$	$\lvert E_{rr}\rvert = \lvert\frac{x_{i+1}-x_i}{x_{i+1}}\rvert$
0	0	-1	1	-
1	-1	0.158838457	5.3678794412	7.29570457235
2	0.158838457	0.2577962241	0.3787956294	0.38386042097
3	0.2577962241	0.25917077154	0.0050939696	0.00530363602
4	0.25917077154	0.2591711018	1.2234E-6	0.00000127429
	0.2591696			

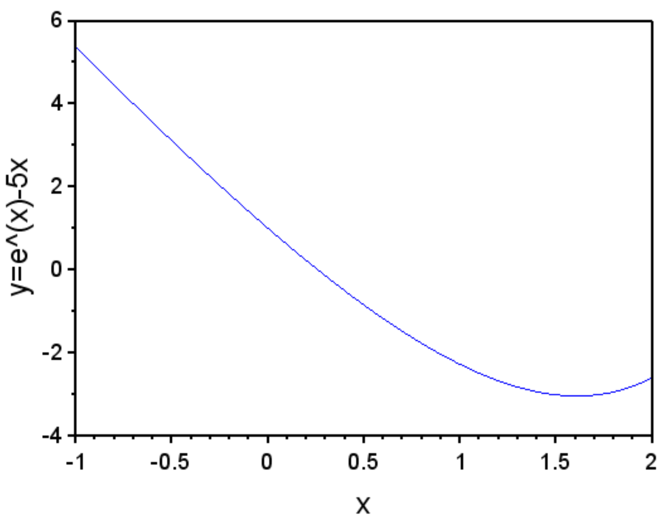

Fig.9. Results of $f(x) = e^x - 5x$

TABLE 2. The $f_1(x) - f_{12}(x)$ function is shown the convergence solution value and an error value of the function.

function	Methods	Solution	Iteration	$\|E_{rr}\|$
$f_1(x) : x^3 - 3x^2$	Newton-Raphson Method	0.0012125	8	0.995977
	Random Search method	0.0278956	9	0.924593
$f_2(x) : e^{2x} - x - 6$	Newton-Raphson Method	0.9042487	9	0.001386
	Random Search method	0.9996868	9	0.001543
$f_3(x) : y = x + tan^{-1}(x) - 1$	Newton-Raphson Method	0.5202264	4	0.000157
	Random Search method	0.8722966	5	0.003041
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5. CONCLUSION

A Newton – Raphson method is a basic tool in numerical analysis and numerous applications, including operations research. We survey the optimization problems, Random Search and Newton – Raphson method its main ideas convergence results in its global behavior. This research studied solve one-dimensional optimization problems using Newton – RaphsonMethod. This method uses first derivatives to solve solutions. It was tested with 12 function test and tested the converging rate of the solution convergence rate The Newton – Raphson method and random search method. It was found that Newton – Raphson a method was more effective in converging rate to better answers than random search methods. The error value and the iteration are indicative.

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SOLVE ONE-DIMENSIONAL OPTIMIZATION PROBLEMS USING NEWTON-RAPHSON METHOD

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ABSTRACT. This research studied solve one-dimensional optimization problems using Newton – Raphson Method. This method uses first derivatives to solve solutions. It was tested with 12 function tests and tested the efficiency of the solution convergence rate Newton – Raphson method and the random search method. It was found that the Newton-Raphson method was more effective in converging rate to better answers than random search methods. The error value and the iteration are indicative.

KEYWORDS: Optimization, Newton – Raphson method, Random Search Method.
AMS Subject Classification: 47H09, 49M05.

1. Introduction

Optimization is the act of achieving the best possible result under given circumstances. In design, construction, maintenance, and engineers, etc. Have to make decisions. The goal of all such decisions is either to minimize effort or to maximize benefit. The effort or benefit can be usually expressed as a function of certain design variables. Hence, optimization is the process of finding the conditions that give the maximum or the minimum value of a function. It is obvious that if a point x^* corresponds to the minimum value of a function $f(x)$, the same point corresponds to the maximum value of the function. Thus, optimization can be taken to be minimization. There is no single method available for solving all optimization problems efficiently. Hence, a number of methods have been developed for solving different types of problems. Optimum seeking methods are also known as mathematical programming techniques, which are a branch of operations search. The Newton – Raphson (NR) method is an iterative scheme used to solve non-linear simultaneous equations. France (1991) described the application of NR method to solve various

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hydraulic problems. Martin and Peters (1963) were the first to propose the application of NR method for analysis of water distribution network (WDN) having pipes and reservoirs only. McCormick and Bellamy (1968) and Zarghamee (1971) extended its use to include other network elements such as pumps and valves. Shamir and Howard (1968) applied this method to solve for all types of unknowns in a network including pipe resistances using head equations. Epp and Fowler (1970) and Gofman and Rodeh (1981) used this method for a solution of loop equations. Some investigators proposed modifications to the classical NR method. Lam and Wolla (1972 a, b) modified the algorithm so that it does not require the Jacobian matrix or its inverse in the iterative process and also suggested a change in step size to minimize the error. The modification presented by Lemieux (1972) ensures the convergence of the algorithm irrespective of the starting assumption. Donachie (1974) suggested halving the step size at any node when oscillation occurs. Neilson (1989) compared the NR method with linear theory method and suggested starting of the NR method with a single iteration by the linear theory method followed by NR iterations. Andersen and Powell (1999) used a linear headloss formula in the NR method for the first iteration. Several computer programs are also available for analysis of WDNs using Newton-Raphson method. The application of NR method in optimal design of WDNs is however very limited. Young (1994) used NR method to solve nonlinear simultaneous equations which were generated through Lagrangian multiplier method for the optimal design of branched 3 WDNs. Johnson et al. (1995) discussed the limitations of the Lagrangian multiplier method proposed by Young (1994). Bhave (1978, 1985) developed the cost head loss ratio, criterion method for the optimal design of WDNs. This method can be used for optimal design and expansion of single as well as multi-source branched or looped networks including pumped source nodes. Herein, the cost head loss ratio criterion method is modified for faster convergence using the NR method.
The NR method is used to solve simultaneous non-linear equations iteratively. It expands the non-linear terms in Taylor's series, neglects the residues after two terms and thereby considers only the linear terms (Bhave 1991). Thus, the NR method linearizes the non-linear equations through partial differentiation and solves. Naturally, the solution is approximate and therefore is successively corrected. The iterative procedure is continued until satisfactory accuracy is reached. Thus, while applying NR method for obtaining correction in the cost head loss ratio criterion method, all correction equations would be considered simultaneously and solved at a time.

2. Preliminaries

2.1. **Optimal Conditions.**

We begin with a formal statement of the conditions which held at a minimum of a one-variable differentiable function. We have already made use of these conditions.

Definition 1. Suppose that $f(x)$ is a continuously differentiable function of the scalar variable x, and that, when $x = x^*$,

$$\frac{df}{dx} = 0 \ \ and \ \ \frac{d^2 f}{dx^2} > 0. \tag{2.1}$$

The function $f(x)$ is then said to have a *local minimum* at x^*. Conditions (1) imply that $f(x^*)$ is the smallest value of f in some region near x^*. It may also be true that $f(x^*) \leq f(x)$ for all x but condition (1) does not guarantee this.

Definition 2. If conditions (1) hold at $x = x^*$ and if $f(x^*) \leq f(x)$ for all x then x^* is said to be the *global minimum*. In practice, it is usually hard to establish that x^* is a global minimum and so we shall chiefly be concerned with methods of finding local minima. Conditions (1) are called optimality conditions.

2.2. **Optimal Solution.**

A globally optimal solution for an optimization problem is defined as the solution $x^* \in X$, where $f(x^*) \leq f(x)$ for all $x \in X$ (minimization problem). For the definition of a globally optimal solution, it is not necessary to define the structure of the search space, a metric, or a neighborhood.
Given a problem instance (X, f) and a neighborhood function N^*, a feasible solution $x^\circ \in X$ is called locally optimal (minimization problem) with respect to N^* if

$$f(x^\circ) \leq f(x) \text{ for all } x \in N^*(x^\circ)$$

Therefore, locally optimal solutions do not exist if no neighborhood is defined. Furthermore, the existence of local optima is determined by the neighborhood the definition used as different neighborhoods can result in different locally optimal solutions.

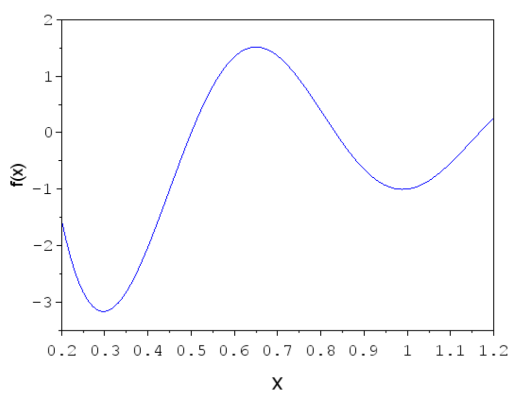

Fig.1. Locally and globally optimal solutions

Figure 1. illustrates the differences between locally and globally optimal solutions and shows how local optimal depend on the definition of N^*. We have a one-dimensional minimization problem with $x \in [a, b] \in R$. We assume an objective function f that assigns objective values to all $x \in X$. The modality of a problem describes the number of local optima in the problem. Unimodal problems have only one local optimum(which is also the global optimum), whereas muti-modal problems have multiple local optima. In general, multi-modal problems are more difficult for guided search methods to solve than unimodal problems.

2.3. **Objective function.**

The classical design procedure aims at finding an acceptable design, *i.e.* a design which satisfies the constraints. In general, there are several acceptable designs, and purpose

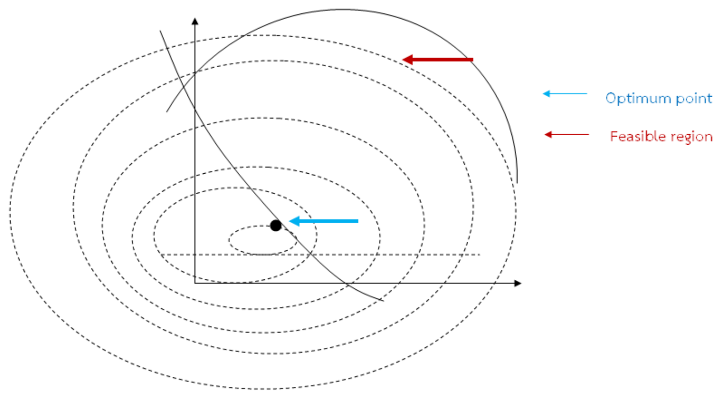

Fig.2. Design space, and optimum point

of the optimization is to single out the best possible design. Thus, a criterion has to be selected for comparing different designs. This criterion, when expressed as a function of the design variables are known as an objective function. The objective function is in general specified by physical or economic considerations. However, the selection of an objective function is not trivial, because what is the optimal design with respect to a certain criterion may be unacceptable with respect to another criterion. Typically, there is a trade-off performance-cost, performance reliability, hence the selection of the objective function is one of the most important decisions in the whole design process. If more than one criterion has to be satisfied, we have multiobjective optimization problem, that may be approximately solved considering a cost function which is a weighted sum of several objective functions.

Let $f : D \subseteq R \longrightarrow R$ is the objective function by a vector $x = (x_1, x_2, x_3, , ... , x_n)$ such as Styblinski-Tang Function $f(x) = 0.5(x^4 - 16x^2 + 5x)$ for $n = 1$. The function is usually evaluated on the hypercube $x \in [-5, 5]$ and global optimal is $f(x) = -39.16599n$

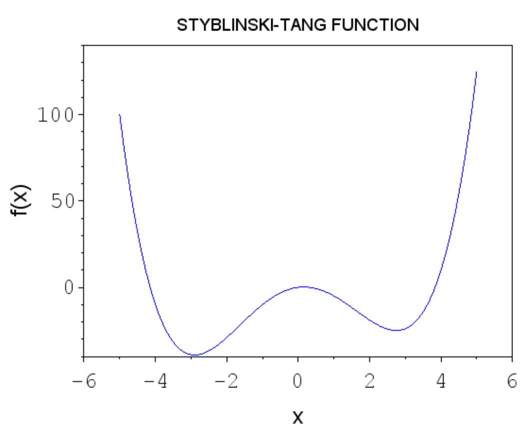

Fig.3. Styblinski-Tang Function

3. Random Search Method

This method generates trial solutions for the optimization model using random number generators for decision variables. Random search method includes random jump method and the random walk method with direction exploitation. Random jump method generates a huge number of data points for the decision variable assuming a uniform distribution for them and finds out the best solution by comparing the corresponding objective function values. Random walk method generates a trial solution with sequential improvements which is governed by a scalar step length and a unit random vector. The random walk method with direct exploitation is an improved version of a random walk method, in which, first the successful direction of generating trial solutions is found out and the maximum possible steps are taken along this successful direction. A generalized flowchart of the search algorithm in solving a nonlinear optimization with decision variable x_i, is presented in Fig.4.

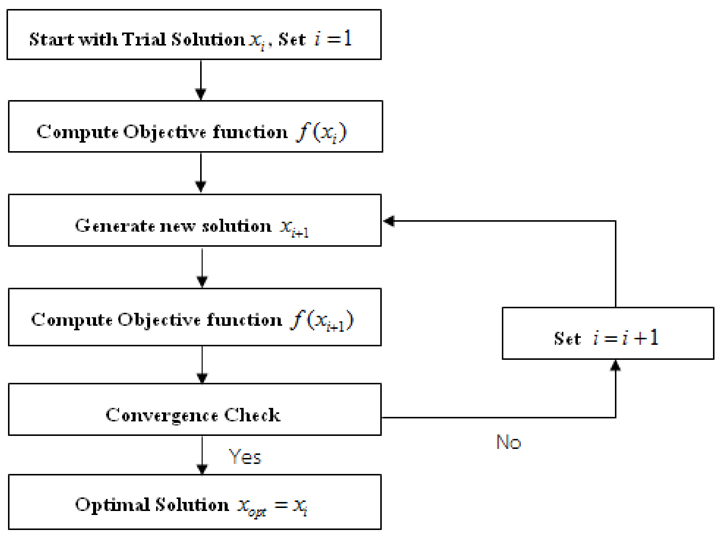

Fig.4. Flowchart Random Search Method

4. Main Results

4.1. **The Newton – Raphson Method.**

Newton – Raphson method is an open approach to find the minimum of function $f(x)$. Derivative using Taylor series recall Taylor series expansion as follows.

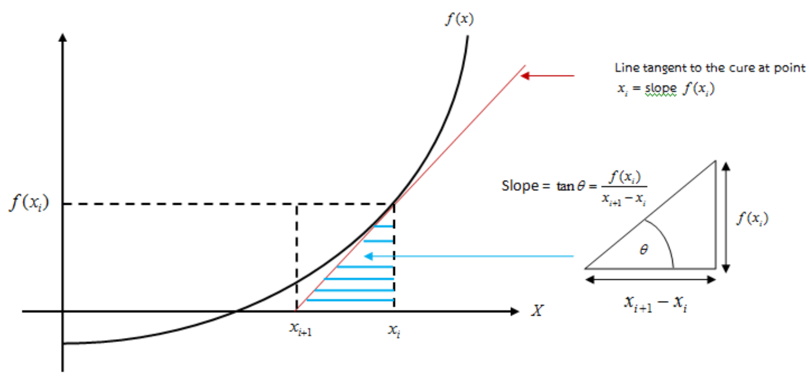

Fig.5. Newton - Raphson Method

from the previous figure 5.

$$\text{slope} = f^{'}(x_i) = \frac{df(x)}{dx}|_{x=x_i} = \frac{f(x_i)-0}{x_i-x_{i+1}}$$

or

$$f^{'}(x_i) = \frac{f(x_i)}{x_i-x_{i+1}} \qquad \text{for } i = 0, 1, 2, 3, ...$$

Thus

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{'}(x_i)}$$

This algorithm is derived by expanding $f(x)$ as a Taylor series

$$f(x_0+h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!}f^{(2)}(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + ... + \frac{h^n}{n!}f^{(n)}(x_0) + R_{n+1}$$

If we let $x_0 + h = x_i + h = x_{i+1}$ and terminate the series at its linear term, then

$$f(x_i + h) = f(x_i) + (x_{i+1} - x_i)f^{(1)}(x_i)$$

or

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)f^{'}(x_i)$$

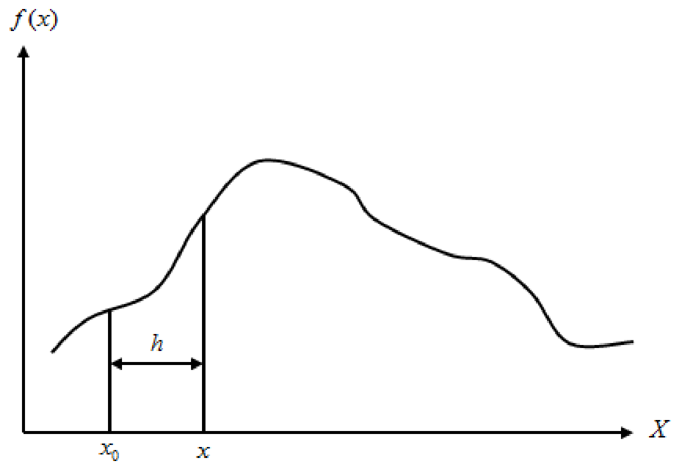

Fig.6. Interval point x and x_0 design

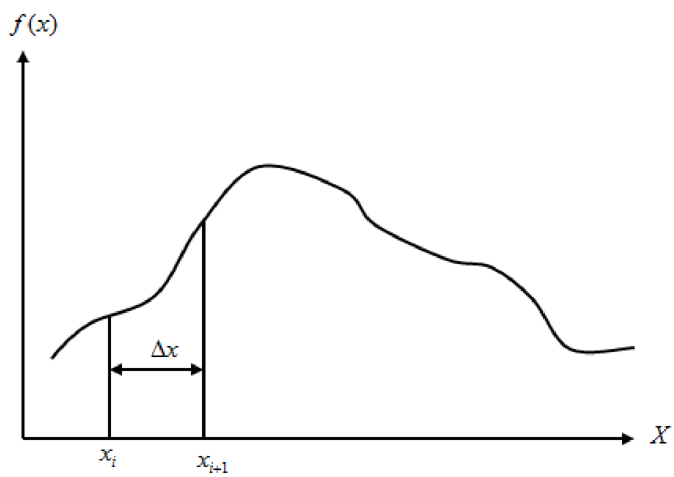

Fig.7. Interval point x_i and x_{i+1} design

Note that since the root of the function relating $f(x)$ and x is the value of x when $f(x_{i+1}) = 0$ at the intersection, hence,

$$\begin{aligned} f(x_{i+1}) &= 0 \\ f(x_i) + (x_{i+1} - x_i)f'(x_i) &= 0 \\ (x_{i+1} - x_i)f'(x_i) &= -f(x_i) \\ x_{i+1} &= x_i - \tfrac{f(x_i)}{f'(x_i)} \end{aligned}$$

Newton – Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{for} \quad i=0,1,2,3,... \tag{4.1}$$

where
x_i = value of the root at iteration i
x_{i+1} = a revised value of the root at iteration $i + 1$
$f(x_i)$ = value of the function at iteration i

$f^{'}(x_i)$ = derivative of $f(x)$ evaluated at iteration i

4.2. **Iterations.**

The Newton – Raphson method uses the slope(tangent) of the function $f(x)$ of the current iterative solution (x_i) to find the solution (x_{i+1}) in the next iteration. The slope at $(x_i.f(x_i))$ is given by

$$f^{'}(x_i) = \frac{f(x_i)-0}{x_i-x_{i+1}}$$

Then x_{i+1} can be solved as

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{'}(x_i)}$$

Which is known as the Newton – Raphson formula.
Relative error: $|E_{rr}| = |\frac{x_{i+1}-x_i}{x_{i+1}}|$ and Iterations stop if $E_{rr} \leq E$ is presented in Fig.8.

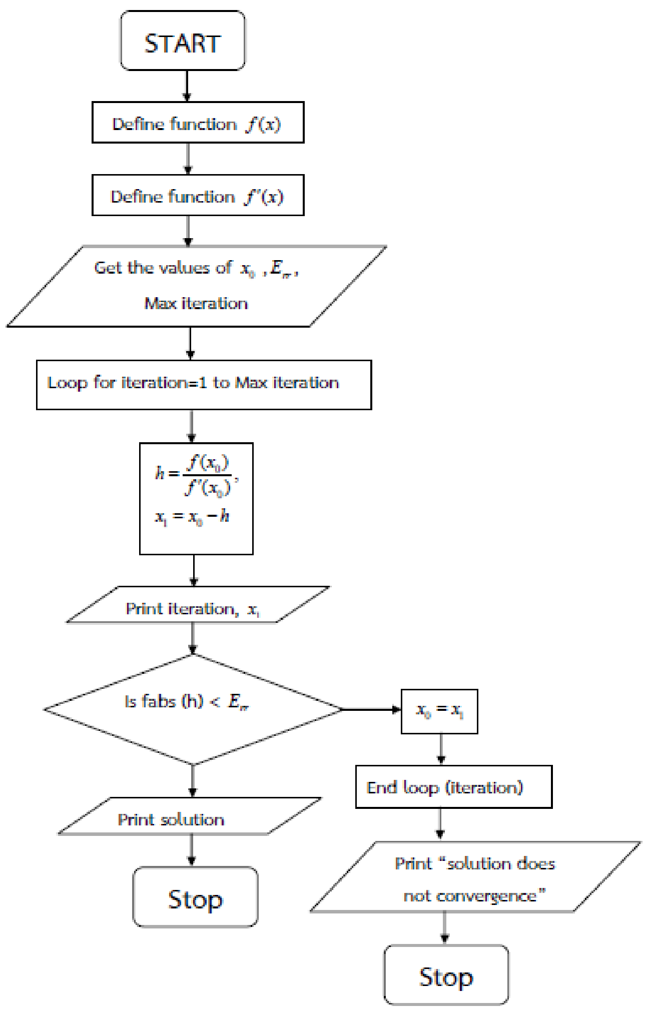

Fig.8. Flowchart of Newton – Raphon Method

Example 4.1. Use the Newton – Raphson iteration method to estimate the root of the following function employing an initial guess of $x_0 = 0 : f(x) = e^x - 5x$
Solution: Let's find the derivative of the function first,

$$f^{'}(x) = \frac{df(x)}{dx} = e^x - 5$$

The initial guess is $x_0 = 0$, hence,
$i = 0$

$$\begin{aligned} f(x) = e^x - 5x \quad &, \quad f(0) = e^0 - 5(0) = 1 \\ f^{'}(x) = e^x - 5 \quad &, \quad f(0) = e^0 - 5 = 1 \\ x_{i+1} &= x_i - \frac{f(x_i)}{f^{'}(x_i)} \\ x_1 &= x_0 - \frac{f(x_0)}{f^{'}(0)} \\ x_1 &= 0 - \frac{1}{1} \\ x_1 &= -1 \end{aligned}$$

$i = 1$ Now $x_1 = -1$, hence,

$$\begin{aligned} f(x) = e^x - 5x \quad &, \quad f(-1) = e^{-1} - 5(-1) = 5.3678794412 \\ f^{'}(x) = e^x - 5 \quad &, \quad f(-1) = e^{-1} - 5 = -4.632120559 \\ x_2 &= x_1 - \frac{f(x_1)}{f^{'}(1)} \\ x_2 &= -1 - \frac{5.3678794412}{-4.632120559} \\ x_2 &= -1 + 1.158838457 \\ x_2 &= 0.158838457 \end{aligned}$$

$i = 2$ Now $x_2 = 0.158838457$, hence,

$$\begin{aligned} f(x) = e^x - 5x \quad &, \quad f(0.158838457) = e^{0.158838457} - 5(0.158838457) \\ &= 0.3787956294 \\ f^{'}(x) = e^x - 5 \quad &, \quad f(0.158838457) = e^{0.158838457} - 5 = -3.827851421 \\ x_3 &= x_2 - \frac{f(x_2)}{f^{'}(2)} \\ x_3 &= 0.158838457 - \frac{0.3787956294}{-3.827851421} \\ x_3 &= 0.158838457 + 0.0989577671 \\ x_3 &= 0.2577962241 \end{aligned}$$

$i = 3$ Now $x_3 = 0.2577962241$, hence,

$$\begin{aligned} f(x) = e^x - 5x \quad &, \quad f(0.2577962241) = e^{0.2577962241} - 5(0.2577962241) \\ &= 0.0050939696 \\ f^{'}(x) = e^x - 5 \quad &, \quad f(0.2577962241) = e^{0.2577962241} - 5 = -3.70592491 \\ x_4 &= x_3 - \frac{f(x_3)}{f^{'}(3)} \\ x_4 &= 0.2577962241 - \frac{0.0050939696}{-3.70592491} \\ x_4 &= 0.2577962241 + 0.00137454744 \\ x_4 &= 0.25917077154 \end{aligned}$$

$i = 4$ Now $x_4 = 0.25917077154$, hence,

$$
\begin{aligned}
f(x) = e^x - 5x \quad &, \quad f(0.25917077154) = e^{0.25917077154} - 5(0.25917077154) \\
&= \quad 1.2234E - 6 \\
f^{'}(x) = e^x - 5 \quad &, \quad f(0.25917077154) = e^{0.25917077154} - 5 = -3.704144919 \\
x_5 &= x_4 - \frac{f(x_4)}{f^{'}(4)} \\
x_5 &= 0.25917077154 - \frac{1.2234E-6}{-3.704144919} \\
x_5 &= 0.25917077154 + 3.3027865e - 7 \\
x_5 &= 0.2591711018
\end{aligned}
$$

Thus, the approach rapidly convergences on the true root of 0.2591 to four significant digits in table 1.

TABLE 1. Newton - Raphson iteration method $f(x) = e^x - 5x$

i	x_i	x_{i+1}	$f(x_i)$	$\lvert E_{rr}\rvert = \lvert\frac{x_{i+1}-x_i}{x_{i+1}}\rvert$
0	0	-1	1	-
1	-1	0.158838457	5.3678794412	7.29570457235
2	0.158838457	0.2577962241	0.3787956294	0.38386042097
3	0.2577962241	0.25917077154	0.0050939696	0.00530363602
4	0.25917077154	0.2591711018	1.2234E-6	0.00000127429
	0.2591696			

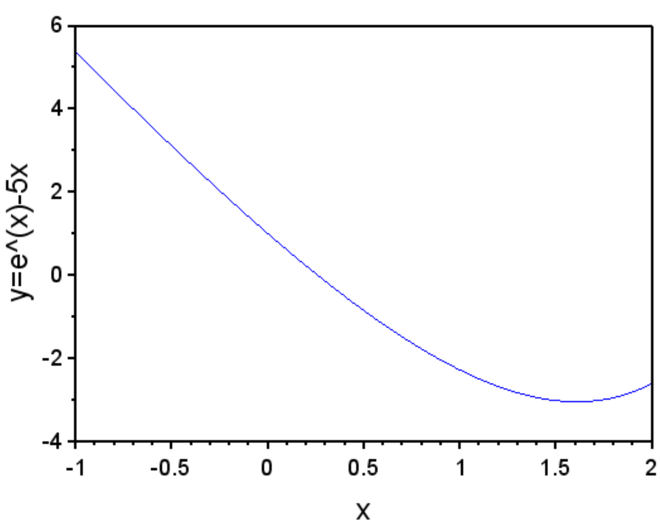

Fig.9. Results of $f(x) = e^x - 5x$

TABLE 2. The $f_1(x) - f_{12}(x)$ function is shown the convergence solution value and an error value of the function.

function	Methods	Solution	Iteration	$\|E_{rr}\|$
$f_1(x) : x^3 - 3x^2$	Newton-Raphson Method	0.0012125	8	0.995977
	Random Search method	0.0278956	9	0.924593
$f_2(x) : e^{2x} - x - 6$	Newton-Raphson Method	0.9042487	9	0.001386
	Random Search method	0.9996868	9	0.001543
$f_3(x) : y = x + tan^{-1}(x) - 1$	Newton-Raphson Method	0.5202264	4	0.000157
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MODIFY REGULARIZATION METHOD VIA PROXIMAL POINT ALGORITHMS FOR ZEROS OF SUM ACCRETIVE OPERATORS OF FIXED POINT AND INVERSE PROBLEMS

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ABSTRACT. In this paper, we investigate the regularization method via a proximal point algorithm for solving treating the sum of two accretive operators for a fixed point set and inverse problems. Strong convergence theorems are established in the framework of Banach spaces. Furthermore, we also apply our result to variational inequality and equilibrium problems.

KEYWORDS: Regularization method, proximal point algorithm, zero points, accretive operators, inverse problems.
AMS Subject Classification: 47H09, 47H17, 47J25, 49J40.

1. Introduction

Many important problems have reformulation which require finding common zero points of nonlinear operators, for instance, inverse problems, variational inequality, optimization problems and fixed point problems. In this paper,we use $A^{-1}(0)$ to denote the set of zeros point of A. A well-known method for solving zero points of maximal monotone operators is the *proximal point algorithm* (PPA). First, Martinet [1] introduced the PPA in a Hilbert space H, that is, for starting $x_0 \in H$, a sequence $\{x_n\}$ generated by

$$x_{n+1} = J^A_{r_n}(x_n) \ \forall n \in \mathbb{N}, \tag{1.1}$$

where A is maximal monotone operators, $J^A_{r_n} = (I + r_n A)^{-1}$ is the resolvent operator of A and $\{r_n\} \subset (0, \infty)$ is a regularization sequence. An iterative (1.1) is equivalent

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to

$$x_n \in x_{n+1} + r_n A x_{n+1} \ \forall n \in \mathbb{N}.$$

If $\phi(x) : H \to \mathbb{R} \cup \{\infty\}$ is a proper convex and lower semicontinuous function, then $J^A_{r_n}$ reduces to

$$x_{n+1} = \arg\min \left\{ \phi(y) + \frac{1}{2r_n}\|x_n - y\|^2, \ y \in H \right\} \ \forall n \in \mathbb{N}. \tag{1.2}$$

Later, Rockafellar [2] studied the proximal point algorithm in framework of a Hilbert space and he proved that if $\liminf_{n\longrightarrow\infty} r_n > 0$ and $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a solution of a zero point of A. Rockafellar [2] has given a more practical method which is an inexact variant of the method:

$$x_{n+1} = J^A_{r_n} x_n + e_n, \ \forall n \in \mathbb{N}, \tag{1.3}$$

where $\{e_n\}$ is an error sequence. It was shown that if $e_n \longrightarrow 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$, with $0 \in A(z)$.

In 2011, Sahu and Yao [3] also extended PPA for the zero of an accretive operator in a Banach space which has a uniformly Gâteaux differentiable norm by combining the prox-Tikhonov method and the viscosity approximation method. They introduced the iterative method to define the sequence $\{x_n\}$ as follows:

$$x_{n+1} = J^A_{r_n}((1-\alpha_n)x_n + \alpha_n f(x_n)), \ \forall n \in \mathbb{N}, \tag{1.4}$$

$$z_{n+1} = J^A_{r_n}((1-\alpha_n)z_n + \alpha_n f(z_n) + e_n), \ \forall n \in \mathbb{N}, \tag{1.5}$$

where A is an accretive operator such that $A^{-1}(0) \neq \emptyset$ and f is a contractive mapping on C and $\{e_n\}$ is an error sequence. Strong convergent were established in both algorithms. This is a source of idea about resolvent operator can be approximated by contractions.

In the same year, PPA extended to the case of sum of two monotone operators A and B by use the technique of forward-backward splitting methods. Manaka and Takahashi [4] introduced the following iterative scheme in a Hilbert space:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S J^A_{\lambda_n}(I - \lambda_n B)x_n, \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in (0,1), $\{\lambda_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping, A is a maximal monotone operator, B is an inverse strongly monotone mapping and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. They prove that a sequence $\{x_n\}$ converges weakly to some point $z \in Fix(S) \cap (A+B)^{-1}(0)$ provided that the control sequence satisfies some conditions. From [4], then we concern with the problem for finding a common element of $Fix(S) \cap (A+B)^{-1}(0)$.

In 2012, Lóṕez et al. [5] use the technique of forward-backward splitting methods for accretive operators in Banach spaces. They considered the following algorithms with errors:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J^A_{r_n}(x_n - r_n(Bx_n + a_n)) + b_n \tag{1.6}$$

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J^A_{r_n}(x_n - r_n(Bx_n + a_n)) + b_n, \tag{1.7}$$

where $u \in E$, $\{a_n\}$, $\{b_n\} \subset E$ and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. An operator A is a maximal accretive operator and B is an inverse strongly accretive. They prove that a sequence $\{x_n\}$ in equation (1.6) and (1.7) is weakly and strongly convergence, respectively.

In 2014, Cho et al. [6] introduced the following iterative scheme in a Hilbert space:

$$\begin{cases} x_1 \in C, \\ z_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ y_n = J^A_{r_n}(z_n - r_n B z_n + e_n) \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n)Sy_n), \ for\ n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are a sequences in (0, 1), $\{r_n\}$ is a positive sequence, $A : C \to H$ is an inverse strongly monotone mapping, B is a maximal monotone operator, and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. Let $S : C \to C$ is a strictly pseudo-contractive mapping with $k \in [0, 1)$, and $f : C \to C$ be a contractive mapping. They prove that a sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A + B)^{-1}(0)$ if the control sequence satisfies some restrictions.

Motivated by [3, 4, 5, 6], then we are interested in the problems for finding a common element of fixed point of nonexpansive S and element of the (quasi) variational inclusion problem as follow:

$$Find\ x \in C\ such\ that\ x \in Fix(S) \cap (A + B)^{-1}(0), \tag{1.8}$$

where A be single-valued nonlinear mapping and B be a multi-valued mapping.

The purpose of this paper is to introduce an iterative algorithm which is modify regularization method and use technique of forward-backward splitting methods for finding a common element of the set solution of nonexpansive S and the set solution of fixed point of the variational inclusion problems, where A is an m-accretive operator and B is an inverse-strongly accretive operator in the framework of Banach space with a uniformly convex and 2-uniformly smooth.

2. Preliminaries

Let E be a Banach space and let E^* be its dual. Let $\langle \cdot, \cdot \rangle$ be the pairing between E and E^*. For all $x \in E$ and $x^* \in E^*$, the value of x^* at x be denoted by $\langle x, x^* \rangle$. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\|\}$, for all $x \in E$. A *single-value* normalized duality mapping is denoted by j, which means a mapping $j : E \longrightarrow E^*$ such that, for all $u \in E$, $j(u) \in E^*$ satisfying the following:

$$\langle u, j(u) \rangle = \|u\|\|j(u)\|, \ \|j(u)\| = \|u\|.$$

If $E = H$ is a Hilbert space, then $J = I$, where I is identity mapping. If E is *smooth Banach space*, then J is single-valued j.

A Banach space E is called an *Opial's space* if for each sequence $\{x_n\}_{n=0}^{\infty}$ in E such that $\{x_n\}$ converges weakly to some x in E, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

hold for all $y \in E$ with $y \neq x$. In fact, for any normed linear space X admit the weakly sequentially continuous duality mapping implies X is Opial space. So, a Banach space with a weakly sequentially continuous duality mapping has the Opial's property; see [7].

The modulus of convexity of E is the function $\delta_E : (0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\epsilon) := \inf\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1; \|x - y\| \geq \epsilon\}.$$

E is said to be *uniformly convex* if and only if $\delta(\epsilon) > 0$, for each $\epsilon \in (0, 2]$. It known that a uniformly convex Banach space is reflexive and strictly convex.

Let $S(E)$ be the unit sphere defined by $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm $\|\cdot\|$ of E is said to be *Gâteaux differentiable norm*, if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. In this case, space E is called *smooth.* A spaces E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (2.1) exist uniformly for all $x \in S(E)$. The norm of E is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for all $x, y \in S(E)$. It is known that if the norm of E is smooth, then the duality mapping J is single-valued and norm to *weak** uniformly continuous on each bounded subset of E.

On the other hand, the *modulus of smoothness* of E is the function $\rho : [0, \infty) \to [0, \infty)$ defined by

$$\rho(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in S(E), \|x\| = 1, \|y\| \leq t\}.$$

A Banach space E be an *smooth* if $\rho_E(t) > 0$ for all $t > 0$. A Banach space E be an *uniformly smooth* if and only if $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$. A Banach space E is said to be *q-uniformly smooth*, if for $1 < q \leq 2$ be a fixed real number, there exists a constant $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. It known that every q-uniformly smooth space is smooth. In the case $\rho(t) \leq ct^2$ for $t > 0$, these is 2-uniformly smooth. The examples of uniformly convex and 2-uniformly smooth Banach space are L_p, l_p or Sobolev spaces W^p_m, where $p \geq 2$. It is well known that, Hilbert spaces are 2-uniformly convex and 2-uniformly smooth. We known that if E is a reflexive Banach space, then every bounded sequence in E has a weakly convergent subsequence. Note that all uniformly convex and 2-uniformly smooth Banach space is reflexive.

Next, we recall the definitions of some operators.

(i) Let $f : C \longrightarrow C$ be an operator. Then f is called *k-contraction* if there exists a coefficient k $(0 < k < 1)$ such that

$$\|fx - fy\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

(ii) Let $S : C \longrightarrow C$ be an operator. Then s is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(iii) Let $B : C \longrightarrow E$ be an operator. Then B is called *α-inverse-strongly accretive* if there exists a constant $\alpha > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Bx - By, j(x - y)\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

(iv) A set-valued operator $A : D(A) \subseteq E \longrightarrow 2^E$ is called *accretive* if there exists $j(x - y) \in J(x - y)$ such that $u \in A(x)$, and $v \in A(y)$,

$$\langle u - v, j(x - y)\rangle \geqslant 0, \quad \forall x, y \in D(A).$$

(v) A set-valued operator $A : D(A) \subseteq E \longrightarrow 2^E$ is called *m-accretive* if A is accretive and $R(I + rA) = E$ for some $r > 0$, where I is identity mapping.

Let C and D are nonempty subsets of a Banach space E such that C is a nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is said to be *sunny* if $Q(x + t(x - Q(x))) = Q(x)$ whenever $x + t(x - Q(x)) \in C$ for all $x \in C$ and $t \geq 0$.

A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q.

Lemma 2.1. [8] *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;
(iii) $\|(x - y) - (Qx - Qy)\|^2 \leq \|x - y\|^2 - \|Qx - Qy\|^2$
(iv) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.

Lemma 2.2. [9] *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let S be a nonexpansive mapping of C into itself with $Fix(S) \neq \emptyset$. Then, the set $Fix(S)$ is a sunny nonexpansive retract of C.*

It well known that if $E = H$ is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection P_C from E onto C, that is $Q_C = P_C$. Let C be a nonempty closed convex subset of E.

In the sequel for the proof of our main results, we also need the following lemmas.

Lemma 2.3. [10] *Let E be a Banach space and J be a normal duality mapping. Then there exits $j(x + y) \in J(x + y)$ for any given $x, y \in E$. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \; j(x + y) \in J(x + y), \tag{2.2}$$

for any given $x, y \in E$.

Lemma 2.4. [5] *Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $B : C \longrightarrow E$ be a single valued operator and α-inverse strongly accretive operator and let A is an m-accretive operator in E with $D(A) \supset C$ and $D(B) \supset C$.Then*

$$Fix\big(J_r^A(I - rB)\big) = (A + B)^{-1}(0).$$

where $J_r^A = (I + rA)^{-1}$ be a resolvent of A for $r > 0$.

Lemma 2.5. [11](*The Resolvent Identity*) *Let E be a Banach space and let A be an m-accretive operator.For $r > 0$, $s > 0$ and $x \in E$, then*

$$J_r^A x = J_s^A\left(\frac{s}{r}x + \left(1 - \frac{s}{r}\right)J_r^A x\right).$$

Lemma 2.6. [12] *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E with the 2-uniformly smooth constant K. Let the mapping $B : C \longrightarrow E$ be a α-inverse strongly accretive operator. Then, we have*

$$\|(I - rB)x - (I - rB)y\|^2 \leq \|x - y\|^2 - 2r(\alpha - K^2 r)\|Bx - By\|^2, \tag{2.3}$$

where I is identity mapping. In particular, if $r \in (0, \frac{\alpha}{K^2})$, then $(I - rB)$ is a nonexpansive.

Lemma 2.7. [13] (*Demiclosed principle*) Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and $S : C \longrightarrow E$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. Then $I - S$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Sx_n \longrightarrow 0$ implies $x = Sx$.

Lemma 2.8. [14] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.9. [15] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq (1-t_n)a_n + t_n b_n + c_n, \forall n \geq 0,$$

where $\{t_n\}$ *is a number sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} t_n = 0$ *and* $\Sigma_{n\to\infty}^{\infty} t_n = \infty$, $\{b_n\}$ *is a sequence such that* $\limsup_{n\to\infty} b_n \leq 0$ *and* $\{c_n\}$ *is a positive number sequence such that* $\Sigma_{n=0}^{\infty} c_n < \infty$. *Then,* $\lim_{n\to\infty} a_n = 0$.

3. Main results

Before prove our main result, we need the following lemma:

Lemma 3.1. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* C *be a nonempty closed convex subset of* E. *Let* $A : D(A) \subseteq C \longrightarrow 2^E$ *be a m-accretive operator and* $B : C \longrightarrow E$ *be an* α*-inverse strongly accretive operator. Let* $S : C \longrightarrow C$ *be a nonexpansive mapping and let* $f : C \longrightarrow C$ *be a contraction mapping with the constant* $k \in (0,1)$. *Let* $J_{r_n}^A = (I + r_n A)^{-1}$ *be a resolvent of* A *for* $r_n > 0$ *such that* $Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$. *If defined operator* $W_n : C \to C$ *by* $W_n := SJ_{r_n}^A ((I - r_n B)[\alpha_n f x + (1-\alpha_n)x] + e_n)$ *for all* $x \in C$, *where* $\alpha_n \in (0,1)$, $r_n > 0$. *Then* W_n *is a contraction operator and has a unique fixed point.*

Proof. Since S, $J_{r_n}^A$, and $(I - r_n B)$ are nonexpansive. Then we known that W_n is nonexpansive. Since f be a contraction mapping with coefficient $k \in (0,1)$. We have

$$\begin{aligned}
\|W_n x - W_n y\| &= \|SJ_{r_n}^A ((I - r_n B)[\alpha_n f(x) + (1-\alpha_n)x] + e_n) \\
&\quad - SJ_{r_n}^A ((I - r_n B)[\alpha_n f(y) + (1-\alpha_n)y] + e_n)\| \\
&\leq \|((I - r_n B)[\alpha_n f(x) + (1-\alpha_n)x] + e_n) \\
&\quad - ((I - r_n B)[\alpha_n f(y) + (1-\alpha_n)y] + e_n)\| \\
&\leq \|[\alpha_n f(x) + (1-\alpha_n)x] - [\alpha_n f(y) + (1-\alpha_n)y]\| \\
&\leq \|(\alpha_n f(x) + (1-\alpha_n)x) - (\alpha_n f(y) + (1-\alpha_n y)\| \\
&= \|\alpha_n (f(x) - f(y)) + (1-\alpha_n)(x-y)\| \\
&\leq \alpha_n \|f(x) - f(y)\| + (1-\alpha_n)\|x-y\| \\
&\leq \alpha_n k\|x-y\| + (1-\alpha_n)\|x-y\| \\
&= (\alpha_n k + (1-\alpha_n))\|x-y\|.
\end{aligned}$$

Since $0 < (\alpha_n k + (1-\alpha_n)) < 1$, it follows that W_n is a contraction mapping of C into it self. By Banach contraction principle, then there exist a unique fixed point, i.e., we say $\bar{x} = W_n \bar{x}$. Moreover, by use lemma 2.2, then the set $Fix(W_n)$ is sunny nonexpansive retraction of C. Hence there exist a unique fixed point $\bar{x} \in Fix(W_n) = Fix(S) \cap (A+B)^{-1}(0) := \Omega$, namely $Q_\Omega f(\bar{x}) = \bar{x} = W_n \bar{x}$. □

Theorem 3.2. *Let* E *be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let* C *be a nonempty closed convex subset of* E. *Let* $A : D(A) \subseteq C \longrightarrow 2^E$ *be a m-accretive operator and* $B : C \longrightarrow E$ *be an* α*-inverse strongly accretive operator. Let* $S : C \longrightarrow C$ *be a nonexpansive mapping and let* $f : C \longrightarrow C$ *be a contraction mapping with the constant* $k \in (0,1)$. *Let* $J_{r_n}^A = (I + r_n A)^{-1}$ *be a resolvent of* A *for* $r_n > 0$. *Assume that* $Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$.

For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by the following:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S J_{r_n}^A (y_n - r_n B y_n + e_n), \ \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$*;*
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$*;*
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A+B)^{-1}(0)$, where $\bar{x} = Q_\Omega f(\bar{})$ and $Q_F f$ is a sunny nonexpansive retraction from E onto Ω.

Proof. **Step 1** We want to show that $\{x_n\}$ is bounded. Fixed $p \in Fix(S) \cap (A + B)^{-1}(0) \neq \emptyset$. So, we have $p \in Fix(S)$ and $p \in (A+B)^{-1}(0) = Fix(J_{r_n}^A(I - r_n B))$ (see Lemma 2.4). Observe that, we consider

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - p\| \\ &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \alpha_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1-\alpha_n)\|x_n - p\| \\ &\leq \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= [\alpha_n k + (1-\alpha_n)]\|x_n - p\| + \alpha_n \|f(p) - p\| \\ &= [1 - \alpha_n(1-k)]\|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned} \tag{3.2}$$

We set $z_n := S J_{r_n}^A (y_n - r_n B y_n + e_{n+1})$. Since $J_{r_n}^A$ and $I - r_n B$ are nonexpansive, and from (3.2), it follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n x_n + (1-\beta_n) z_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &= \beta_n \|x_n - p\| + (1-\beta_n)\|S J_{r_n}^B (y_n - r_n A y_n + e_n) - Sp\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|J_{r_n}^A (y_n - r_n B y_n + e_n) - p\| \\ &= \beta_n \|x_n - p\| + (1-\beta_n)\|J_{r_n}^A (y_n - r_n B y_n + e_n) - J_{r_n}^A (I - r_n B)p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|(y_n - r_n B y_n + e_n) - (I - r_n B)p\| \\ &= \beta_n \|x_n - p\| + (1-\beta_n)\|(I - r_n B)y_n - (I - r_n B)p + e_n\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)(\|(I - r_n B)y_n - (I - r_n B)p\| + \|e_n\|) \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)[\|y_n - p\| + \|e_n\|] \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)[(1 - \alpha_n(1-k))\|x_n - p\| \\ &\quad + \alpha_n \|f(p) - p\|] + (1-\beta_n)\|e_n\| \\ &= \beta_n \|x_n - p\| + [(1-\beta_n) - \alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n \|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [\beta_n + (1-\beta_n) - \alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n \|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [1 - (1-\beta_n)\alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n \|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [1 - \lambda_n(1-k)]\|x_n - p\| + \lambda_n \|f(p) - p\| + \|e_n\|, \end{aligned}$$

where $\lambda_n := (1-\beta_n)\alpha_n$. Then, it follows that

$$\begin{aligned}\|x_{n+1}-p\| &\le \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-k}\right\}+\|e_n\|\\ &\le \max\left\{\|x_{n-1}-p\|, \frac{\|f(p)-p\|}{1-k}\right\}+\|e_{n-1}\|+\|e_n\|\\ &\le \max\left\{\|x_{n-2}-p\|, \frac{\|f(p)-p\|}{1-k}\right\}+\|e_{n-2}\|+\|e_{n-1}\|+\|e_n\|\\ &\vdots\\ &\le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-k}\right\}+\sum_{i=0}^{n}\|e_i\|<\infty.\end{aligned}$$

It follows by mathematical induction, we conclude that

$$\|x_{n+1}-p\| \le \max\left\{\|x_0-p\|, (1-k)^{-1}\|f(p)-p\|\right\}+\sum_{i=0}^{n}\|e_i\|, \ \forall n\ge 0.$$

By condition (d), this implies that $\{x_n\}$ is bounded.

From $y_n=\alpha_n f(x_n)+(1-\alpha_n)x_n$, we obtain

$$\begin{aligned}\|y_n-p\| &= \|\alpha_n f(x_n)+(1-\alpha_n)x_n-p\|\\ &\le \alpha_n\|f(x_n)-p\|+(1-\alpha_n)\|x_n-p\|.\end{aligned} \tag{3.3}$$

From (3.3) and since $\{x_n\}$ is bounded, so $\{y_n\}$ and $\{z_n\}$ are bounded too.

Step 2 We want to show that $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\|=0$. By lemma 2.8, we set $v_n := y_n - r_nAy_n + e_n$, then $z_n := SJ^B_{r_n}v_n$, it follows that

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|SJ^A_{r_{n+1}}v_{n+1}-SJ^A_{r_n}v_n\|\\ &\le \|J^A_{r_{n+1}}v_{n+1}-J^A_{r_n}v_n\|\\ &\le \|J^A_{r_{n+1}}v_{n+1}-J^A_{r_{n+1}}v_n\|+\|J^A_{r_{n+1}}v_n-J^A_{r_n}v_n\|\\ &\le \|v_{n+1}-v_n\|+\|J^A_{r_{n+1}}v_n-J^A_{r_n}v_n\|.\end{aligned} \tag{3.4}$$

Next, we compute $\|v_{n+1}-v_n\|$ that

$$\begin{aligned}\|v_{n+1}-v_n\| &= \|(y_{n+1}-r_{n+1}By_{n+1}+e_{n+1})-(y_n-r_nBy_n+e_n)\|\\ &= \|(I-r_nB)y_{n+1}-(I-r_nB)y_n+(r_n-r_{n+1})By_{n+1}+e_{n+1}-e_n\|\\ &\le \|(I-r_nB)y_{n+1}-(I-r_nB)y_n\|+\mid r_n-r_{n+1}\mid\|By_{n+1}\|+\|e_{n+1}-e_n\|\\ &\le \|y_{n+1}-y_n\|+\mid r_n-r_{n+1}\mid\|By_{n+1}\|+\|e_{n+1}\|+\|e_n\|.\end{aligned} \tag{3.5}$$

Next, we compute $\|y_{n+1}-y_n\|$ that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|(\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})(x_{n+1}))-(\alpha_n f(x_n)+(1-\alpha_n)x_n)\|\\ &= \|\alpha_{n+1}f(x_{n+1})-\alpha_n f(x_{n+1})+\alpha_n f(x_{n+1})-\alpha_n f(x_n)+(1-\alpha_{n+1})x_n\\ &\quad -(1-\alpha_{n+1})x_n-(1-\alpha_n)x_n\|\\ &= \|(\alpha_{n+1}-\alpha_n)f(x_{n+1})+\alpha_n(f(x_{n+1})-f(x_n))+(1-\alpha_{n+1})(x_{n+1}-x_n)\\ &\quad +x_n((1-\alpha_{n+1})-(1-\alpha_n))\|\\ &\le |\alpha_{n+1}-\alpha_n|\|f(x_{n+1})-x_n\|+\alpha_n\|f(x_{n+1})-f(x_n)\|+(1-\alpha_{n+1})\|x_{n+1}-x_n\|\\ &= (1-\alpha_{n+1})\|x_{n+1}-x_n\|+h_n\end{aligned}$$

$$\leq \|x_{n+1} - x_n\| + h_n, \tag{3.6}$$

where $h_n = |\alpha_{n+1} - \alpha_n|\|f(x_{n+1}) - x_n\| + \alpha_n\|f(x_{n+1}) - f(x_n)\|$.

That is

$$\|v_{n+1} - v_n\| \leq \|x_{n+1} - x_n\| + h_n + g_n, \tag{3.7}$$

where $g_n =| r_n - r_{n+1} | \|By_{n+1}\| + \|e_{n+1}\| + \|e_n\|$.

Next, we compute $\|J^A_{r_{n+1}}v_n - J^A_{r_n}v_n\|$ by the resolvent identity (see Lemma 2.5) that

$$\begin{aligned}
\|J^A_{r_{n+1}}v_n - J^A_{r_n}v_n\| &= \|J^A_{r_n}\left(\frac{r_n}{r_{n+1}}v_n + (1-\frac{r_n}{r_{n+1}})J^A_{r_{n+1}}v_n\right) - J^A_{r_n}v_n\| \\
&\leq \|\left(\frac{r_n}{r_{n+1}}v_n + (1-\frac{r_n}{r_{n+1}})J^A_{r_{n+1}}v_n\right) - v_n\| \\
&= \|(\frac{r_n}{r_{n+1}} - 1)v_n + (1-\frac{r_n}{r_{n+1}})J^A_{r_{n+1}}v_n\| \\
&= \|(1-\frac{r_n}{r_{n+1}})J^A_{r_{n+1}}v_n - (1-\frac{r_n}{r_{n+1}})v_n\| \\
&= \|\frac{r_{n+1}-r_n}{r_{n+1}}(J^A_{r_{n+1}}v_n - v_n)\| \\
&\leq \left|\frac{r_{n+1}-r_n}{r_{n+1}}\right| \|J^A_{r_{n+1}}v_n - v_n\|.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + h_n + g_n + \left|\frac{r_{n+1}-r_n}{r_{n+1}}\right| \|J^A_{r_{n+1}}v_n - v_n\|.$$

In view of the condition (a), (c), and (d), it follows that

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq 0.$$

We take lim sup, it follows that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By lemma 2.8, we conclude that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0 \tag{3.9}$$

that is $\lim_{n\to\infty}\|SJ^A_{r_n}(v_n) - x_n\| = 0$. From (4.1), we observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\beta_n x_n + (1-\beta_n)z_n - x_n\| \\
&\leq (1-\beta_n)\|z_n - x_n\|.
\end{aligned}$$

By (3.9), then we conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Step 3 To show that $\lim_{n\to\infty}\|By_n - Bp\| = 0$, $\lim_{n\to\infty}\|J^A_{r_n}(v_n) - y_n\| = 0$ and $\lim_{n\to\infty}\|SJ^A_{r_n}(v_n) - J^A_{r_n}(v_n)\| = 0$.

Step 3.1 First, we observe that $\lim_{n\to\infty}\|By_n - Bp\| = 0$. Notice that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n x_n + (1-\beta_n)SJ^A_{r_n}v_n - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|SJ^A_{r_n}v_n - p\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|v_n - (I - r_nB)p\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|(y_n - r_nBy_n + e_n) - (I - r_nB)p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|(I - r_nB)y_n - (I - r_nB)p\|^2
\end{aligned}$$

$$
\begin{aligned}
& +2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|] \\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\left(\|y_n-p\|^2-2r_n(\alpha-K^2r_n)\|By_n-Bp\|^2\right) \\
& +2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|.
\end{aligned}
$$

Set $\bar{g_n} := (1-\beta_n)2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|$, we get

$$
\begin{aligned}
& \|x_{n+1}-p\|^2 \qquad (3.11)\\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\left(\|y_n-p\|^2-2r_n(\alpha-K^2r_n)\|By_n-Bp\|^2\right)+\bar{g_n} \\
=\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2-2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2+\bar{g_n} \\
=\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_nf(x_n)+(1-\alpha_n)x_n-p\|^2 \\
& -2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2+\bar{g_n}.
\end{aligned}
$$

Set $\bar{h_n} := 2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2$, we get

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_nf(x_n)+(1-\alpha_n)x_n-p\|^2-\bar{h_n}+\bar{g_n} \\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-p\|^2 \\
& -\bar{h_n}+\bar{g_n} \\
=\ & (1-\alpha_n(1-\beta_n))\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2-\bar{h_n}+\bar{g_n}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& 2r_n(\alpha_n-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2 \\
\leq\ & (1-\alpha_n(1-\beta_n))\|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
\leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|(x_n-p)-(x_{n+1}-p)\| \\
& +(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|x_n-x_{n+1}\|+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n}.
\end{aligned}
$$

In view of the condition (a), (c), (d), and from (3.10), we conclude that $\lim_{n\longrightarrow\infty}\|By_n-Bp\|^2=0$. This implies

$$
\lim_{n\longrightarrow\infty}\|By_n-Bp\|=0. \qquad (3.12)
$$

Step 3.2 Second, we will show that $\lim_{n\to\infty}\|J^A_{r_n}(v_n)-y_n\|=0$, we observe that

$$
\begin{aligned}
& \|J^A_{r_n}(v_n)-p\|^2 \\
\leq\ & \|J^A_{r_n}(v_n)-p\|\|(y_n-r_nBy_n+e_n)-(p-r_nBp)\| \\
=\ & \frac{1}{2}\{\|J^A_{r_n}(v_n)-p\|^2+\|(y_n-r_nBy_n+e_n)-(p-r_nBp)\|^2 \\
& -\|\left(J^A_{r_n}(v_n)-p\right)-(y_n-r_nBy_n+e_n)-(p-r_nBp)\|^2\} \\
=\ & \frac{1}{2}\{\|J^A_{r_n}(v_n)-p\|^2+\|(I-r_nB)y_n-(I-r_nB)p+e_n\|^2 \\
& -\|J^A_{r_n}(v_n)-y_n-r_nBy_n-e_n+r_nBp\|^2\} \\
=\ & \frac{1}{2}\{\|J^A_{r_n}(v_n)-p\|^2+\|(I-r_nB)y_n-(I-r_nB)p\|^2+\bar{g_n} \\
& -\|\left(J^A_{r_n}(v_n)-y_n-e_n\right)-r_n(By_n-Bp)\|^2\} \\
\leq\ & \frac{1}{2}\big\{\|J^A_{r_n}(v_n)-p\|^2+\|y_n-p\|^2+\bar{g_n} \\
& -(\|J^A_{r_n}(v_n)-y_n-e_n\|^2-2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\| \\
& +\|r_nBy_n-r_nBp\|^2)\big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\big\{\|J^A_{r_n}(v_n)-p\|^2+\|y_n-p\|^2+\bar{g_n}-\|J^A_{r_n}(v_n)-y_n-e_n\|^2\\
&\quad +2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|-\|r_nBy_n-r_nBp\|^2\big\}. \qquad (3.13)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|J^A_{r_n}(v_n)-p\|^2\\
\le\ & \|y_n-p\|^2+\bar{g_n}-\|J^A_{r_n}(v_n)-y_n-e_n\|^2\\
&+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|-\|r_nBy_n-r_nBp\|^2\\
=\ & \|\alpha_n f(x_n)+(1-\alpha_n)x_n-p\|^2-\|J^A_{r_n}(v_n)-y_n-e_n\|^2\\
&-\|r_nBy_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}\\
\le\ & \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\|J^A_{r_n}(v_n)-y_n+e_n\|^2\\
&-r_n\|By_n-Bp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}. \qquad (3.14)
\end{aligned}
$$

From (3.14), this implies that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
\le\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|SJ^A_{r_n}(v_n)-p\|^2\\
\le\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|J^A_{r_n}(v_n)-p\|^2\\
\le\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\big\{\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&-\|J^A_{r_n}(v_n)-y_n+e_n\|^2-r_n\|By_n-Bp\|^2\\
&+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}\big\}\\
=\ & (1-\alpha_n)\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2\\
&-(1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2-(1-\beta_n)r_n^2\|By_n-r_nBp\|^2\\
&+(1-\beta_n)2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+(1-\beta_n)\bar{g_n}\\
\le\ & \|x_n-p\|^2+\alpha_n\|f(x_n)-p\|^2-(1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2\\
&-r_n^2\|By_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2\\
\le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|f(x_n)-p\|^2-r_n^2\|By_n-r_nBp\|^2\\
&+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}.\\
=\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+s_n\\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|(x_n-p)-(x_{n+1}-p)\|+s_n\\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|x_n-x_{n+1}\|+s_n, \qquad (3.15)
\end{aligned}
$$

where we set $s_n:=\alpha_n\|f(x_n)-p\|^2-r_n^2\|By_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}$.
From (3.15), in view of the condition (a), (c), (d), and equation (3.10), we conclude that

$$\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n)-y_n-e_n\|=0.$$

This in turn implies that

$$\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n)-y_n\|=0. \qquad (3.16)$$

Step 3.3 Lastly, we will show that $\lim_{n\to\infty}\|SJ^A_{r_n}(v_n)-J^A_{r_n}(v_n)\|=0$, we see that

$$\begin{aligned}\|y_n-x_n\| &= \|\alpha_n f(x_n)+(1-\alpha_n)x_n-x_n\| \\ &= \alpha_n\|f(x_n)-x_n\|.\end{aligned}$$

By condition (a), then

$$\lim_{n\longrightarrow\infty}\|y_n-x_n\|=0. \tag{3.17}$$

Next, from (3.16) and equation (3.17), then we see that

$$\|J^A_{r_n}(v_n)-y_n\|\le\|J^A_{r_n}(v_n)-y_n\|+\|y_n-x_n\|.$$

That is

$$\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n)-x_n\|=0. \tag{3.18}$$

From equation (3.9) and (3.18), then we see that

$$\|SJ^A_{r_n}(v_n)-J^A_{r_n}(v_n)\|\le\|SJ^A_{r_n}(v_n)-x_n\|+\|x_n-J^A_{r_n}(v_n)\|.$$

That is

$$\lim_{n\longrightarrow\infty}\|SJ^A_{r_n}(v_n)-J^A_{r_n}(v_n)\|=0. \tag{3.19}$$

Step 4 Since E is a uniformly convex and 2-uniformly smooth Banach space, then E is reflexive Banach space. By reflexive Banach space and from $\{x_n\}$, $\{y_n\}$ are bounded, then it has a weakly convergence subsequence. We may assume that $x_{n_i}\rightharpoonup\hat{x}$. In view of $\lim_{n\longrightarrow\infty}\|y_n-x_n\|=0$, then there exist a subsequence of $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to $\hat{x}$. we can say that $\{y_{n_i}\}$ also converges weakly to $\hat{x}$, i.e, $y_{n_i}\rightharpoonup\hat{x}$, without loss of generality. To show that $\hat{x}\in Fix(S)\cap(A+B)^{-1}(0)=\Omega$.

(i) First, we want to show that $\hat{x}\in Fix(S)$. Now, we have $y_{n_i}\rightharpoonup\hat{x}$. Since we known that $\{J^A_{r_n}(v_n)\}$ is bounded and form $\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n)-y_n\|=0$, then we say that $\{J^A_{r_{n_i}}(v_{n_i})\}\rightharpoonup\hat{x}$.

From (3.19), we have $\lim_{n\longrightarrow\infty}\|SJ^A_{r_{n_i}}(v_{n_i})-J^A_{r_{n_i}}(v_{n_i})\|=0$. By demiclosed principle, this implies $S\hat{x}=\hat{x}$, namely we prove that $\hat{x}\in Fix(S)$. **(ii)** Next, to show that $J^A_r(I-rB)\hat{x}=\hat{x}$. Since a Banach space with weakly continuous duality mapping has the Opial's condition, see [7]. Suppose $\hat{x}\neq J^A_r(I-rB)\hat{x}$. By the Opial's condition and condition (c), (d), then we have

$$\begin{aligned}&\liminf_{i\to\infty}\|y_{n_i}-\hat{x}\| \\ <\ &\liminf_{i\to\infty}\|y_{n_i}-J^A_r(I-rB)\hat{x}\| \\ \le\ &\liminf_{i\to\infty}\{\|y_{n_i}-J^A_{r_{n_i}}(v_{n_i})\|+\|J^A_{r_{n_i}}(v_{n_i})-J^A_{r_n}(I-r_nB)\hat{x}\|\} \\ =\ &\liminf_{i\to\infty}\{\|y_{n_i}-J^A_{r_{n_i}}(v_{n_i})\|+\|J^A_r(v_{n_i})-J^A_r(I-rB)\hat{x}\|\} \\ \le\ &\liminf_{i\to\infty}\{\|y_{n_i}-J^A_{r_{n_i}}(v_{n_i})\|+\|v_{n_i}-(I-rB)\hat{x}\|\} \\ =\ &\liminf_{i\to\infty}\{\|y_{n_i}-J^A_{r_{n_i}}(v_{n_i})\|+\|(I-rB)y_{n_i}-(I-rB)\hat{x}\|+\|e_{n_i}\|\} \\ \le\ &\liminf_{i\to\infty}\{\|y_{n_i}-J^A_{r_{n_i}}(v_{n_i})\|+\|y_{n_i}-\hat{x}\|+\|e_{n_i}\|\}.\end{aligned}$$

By (3.16) and condition (d), hence

$$\liminf_{i\to\infty}\|y_{n_i}-\bar{x}\| \quad < \quad \liminf_{i\to\infty}\|y_{n_i}-\hat{x}\|.$$

This is contradiction. Therefore, $J_r^A(I-rB)\hat{x}=\hat{x}$.
This complete the proof that $\hat{x}\in Fix(S)\cap(A+B)^{-1}(0)=\Omega$.

Step 5 We defined operator $W_n : C\to C$ by $W_n := SJ_{r_n}^A((I-r_nB)[\alpha_n fx+(1-\alpha_n)x]+e_n)$ for all $x\in C$, where $\alpha_n\in(0,1)$, $r_n>0$. From lemma 3.1 an operators W_n is a contraction operator and has a unique fixed point. Moreover, by use lemma 2.2, we known that $\bar{x}\in Fix(W_n)=Fix(S)\cap(A+B)^{-1}(0):=\Omega$, namely $Q_\Omega f(\bar{x})=\bar{x}=W_n\bar{x}$. (Now, $\hat{x}=\bar{x}$ too)

Next, we will show that $\limsup_{n\longrightarrow\infty}\langle f(\bar{x})-\bar{x},j(y_n-\bar{x})\rangle\le 0$, where $\lim_{t\longrightarrow 0}x_t=\bar{x}=Q_\Omega f(\bar{x})$ and x_t solves equation $x_t=SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t), \forall t\in(0,1)$.

(i) We want to show that $\lim_{n\longrightarrow\infty}\|W_nx_n-y_n\|=0$. Consider

$$\begin{aligned}\|W_nx_n-y_n\| &\le \|SJ_{r_n}^A((I-r_nB)[\alpha_nf(x_n)+(1-\alpha_n)x_n]+e_n)-x_n\|+\|x_n-y_n\|\\ &= \|z_n-x_n\|+\|x_n-y_n\|. \end{aligned}\tag{3.20}$$

From (3.9) and (3.17), then

$$\lim_{n\longrightarrow\infty}\|W_nx_n-y_n\|=0. \tag{3.21}$$

(ii) We want to show that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},j(y_n-\bar{x})\rangle\le 0$. We compute

$$\begin{aligned}
&\|x_t-y_n\|^2\\
=\ & \|SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-y_n\|^2\\
=\ & \langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-W_nx_n+W_nx_n-y_n,j(x_t-y_n)\rangle\\
=\ & \langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-W_nx_n,j(x_t-y_n)\rangle\\
&+\langle W_nx_n-y_n,j(x_t-y_n)\rangle\\
=\ & \langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-SJ_{r_n}^A((I-r_nB)y_n+e_n),j(x_t-y_n)\rangle\\
&+\langle W_nx_n-y_n,j(x_t-y_n)\rangle\\
\le\ & \langle (I-r_nB)(tf(x_t)+(1-t)x_t)-(I-r_nB)y_n-e_n,j(x_t-y_n)\rangle\\
&+\|W_nx_n-y_n\|\|x_t-y_n\|\\
=\ & \langle (I-r_nB)(tf(x_t)+(1-t)x_t)-(I-r_nB)y_n,j(x_t-y_n)\rangle+\langle e_n,j(x_t-y_n)\rangle\\
&+\|W_nx_n-y_n\|\|x_t-y_n\|\\
\le\ & \langle (tf(x_t)+(1-t)x_t)-x_t+x_t-y_n,j(x_t-y_n)\rangle+\|e_n\|\|x_t-y_n\|\\
&+\|W_nx_n-y_n\|\|x_t-y_n\|\\
\le\ & \langle t(f(x_t)-x_t),j(x_t-y_n)\rangle+\langle x_t-y_n,j(x_t-y_n)\rangle+\|e_n\|\|x_t-y_n\|\\
&+\|W_nx_n-y_n\|\|x_t-y_n\|\\
\le\ & t\langle f(x_t)-x_t,j(x_t-y_n)\rangle+\|x_t-y_n\|^2+\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|\\
\le\ & -t\langle f(x_t)-x_t,j(y_n-x_t)\rangle+\|x_t-y_n\|^2+\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|
\end{aligned}\tag{3.22}$$

It follows that

$$t\langle f(x_t)-x_t,j(y_n-x_t)\rangle \le \|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|.$$

Then

$$\langle f(x_t)-x_t,j(y_n-x_t)\rangle \le \frac{1}{t}\{\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|\}.$$

By virtue of (3.21) and condition (d), we found that

$$\limsup_{n\to\infty}\langle f(x_t)-x_t,j(y_n-x_t)\rangle\le 0. \tag{3.23}$$

Since $x_t \to \bar{x}$, as $t \to 0$ and the fact that j is norm-to- weak* uniformly continuous on bounded subset of E, we obtain

$$\begin{aligned}
& |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle - \langle f(x_t) - x_t, j(y_n - x_t)\rangle| \\
\leq \; & |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle - \langle f(\bar{x}) - \bar{x}, j(y_n - x_t)\rangle| \\
& + |\langle f(\bar{x}) - \bar{x}, j(y_n - x_t)\rangle - \langle f(x_t) - x_t, j(y_n - x_t)\rangle| \\
\leq \; & |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x}) - j(y_n - x_t)\rangle| + |\langle f(\bar{x}) - \bar{x} - f(x_t) + x_t, j(y_n - x_t)\rangle| \\
\leq \; & \|f(\bar{x}) - \bar{x}\| \|j(y_n - \bar{x}) - j(y_n - x_t)\| + \|f(\bar{x}) - \bar{x} - f(x_t) + x_t\| \|y_n - x_t\| \\
& \longrightarrow 0, \; as \; t \longrightarrow 0.
\end{aligned}$$

Hence, for any $\epsilon > 0$, there exist $\delta > 0$ such that $\forall t \in (0, \delta)$ the following inequality holds:

$$\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq \langle f(x_t) - x_t, j(y_n - x_t)\rangle + \epsilon.$$

Taking $\limsup_{n\to\infty}$ in the above inequality, we find that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq \limsup_{n\to\infty}\langle f(x_t) - x_t, j(y_n - x_t)\rangle + \epsilon.$$

Since ϵ is arbitrary and (3.23), we obtain that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq 0. \tag{3.24}$$

Step 6 Next, we prove that $\{x_n\}$ converges strongly to $\bar{x} = Q_\Omega f(\bar{x})$ by using the lemma 2.3 and lemma 2.9. We note that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \|\beta_n x_n + (1-\beta_n) S J_{r_n}^A(v_n) - \bar{x}\|^2 \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)\|S J_{r_n}^A(v_n) - \bar{x}\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)\|S J_{r_n}^A(v_n) - S\bar{x}\|^2 \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)\|J_{r_n}^A(v_n) - \bar{x}\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)\|J_{r_n}^A(v_n) - J_{r_n}^A(I - r_n B)\bar{x}\|^2 \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)\|v_n - (I - r_n B)\bar{x}\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)\|(y_n - r_n B y_n + e_n) - (I - r_n B)\bar{x}\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)\|(I - r_n B) y_n - (I - r_n B)\bar{x} + e_n\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)[\|(I - r_n A) y_n - (I - r_n A)\bar{x}\|^2 \\
&\quad + 2\langle e_n, j((I - r_n B) y_n - (I - r_n B)\bar{x} + e_n)\rangle \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + (1-\beta_n)\left[\|y_n - \bar{x}\|^2 + 2\|e_n\| \|(I - r_n B) y_n - (I - r_n B)\bar{x} + e_n\|\right].
\end{aligned} \tag{3.25}$$

Consider

$$\begin{aligned}
\|y_n - \bar{x}\|^2 &= \langle \alpha_n f(x_n) + (1-\alpha_n) x_n - \bar{x}, j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n (f(x_n) - \bar{x}) + (1-\alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n (f(x_n) - f(\bar{x})) + \alpha_n (f(\bar{x}) - \bar{x}) + (1-\alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n (f(x_n) - f(\bar{x})) + (1-\alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle + \langle \alpha_n (f(\bar{x}) - \bar{x}), j(y_n - \bar{x})\rangle \\
&\leq \|\alpha_n (f(x_n) - f(\bar{x})) + (1-\alpha_n)(x_n - \bar{x})\| \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&\leq [\alpha_n k\|x_n - \bar{x}\| + (1-\alpha_n)\|x_n - \bar{x}\|] \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&= [1 - \alpha_n(1-k)] \|x_n - \bar{x}\| \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&= (1 - \alpha_n(1-k)) \frac{\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2}{2} + \alpha_n \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle
\end{aligned}$$

$$= \frac{1-\alpha_n(1-k)}{2}\left(\|x_n-\bar{x}\|^2+\|y_n-\bar{x}\|^2\right)+\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle.$$

It follows that

$$\begin{aligned} & 2\|y_n-\bar{x}\|^2 \qquad (3.26)\\ \leq\ & (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+(1-\alpha_n(1-k))\|y_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle \\ \leq\ & (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+\|y_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle. \end{aligned}$$

Therefore, we obtain

$$\|y_n-\bar{x}\|^2 \quad\leq\quad (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle. \quad (3.27)$$

Replace (3.27) in (3.25) that

$$\begin{aligned} & \|x_{n+1}-\bar{x}\|^2 \\ \leq\ & \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)\left[(1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle\right] \\ & +(1-\beta_n)2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\| \\ =\ & (1-\alpha_n(1-k)(1-\beta_n))\|x_n-\bar{x}\|^2+2\alpha_n(1-\beta_n)\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle \\ & +2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\| \\ =\ & (1-\lambda_n)\|x_n-\bar{x}\|^2+\frac{2\lambda_n}{(1-k)}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle+c_n, \end{aligned}$$

where $c_n := 2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\|$, and $\lambda_n=\alpha_n(1-k)(1-\beta_n)$.

If we set $b_n=\frac{2}{(1-k)}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle$ and we have $\limsup_{n\longrightarrow\infty}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle\leq 0$, then we see that $\limsup_{n\to\infty} b_n\leq 0$, and also that $\sum_{n=0}^{\infty} c_n<\infty$.

By lemma 2.8 and condition (a), (b), and (d), we conclude that $\|x_n-\bar{x}\|^2\longrightarrow 0$, as $n\longrightarrow\infty$. This implies

$$\lim_{n\longrightarrow\infty}\|x_n-\bar{x}\|=0,$$

i.e., x_n converges strongly to $\bar{x}$. □

Next, we will utilize theorem 3.2 to study some strong convergence theorem in L_p with $2\leq p<\infty$. Since L_p, where $p\geq 2$ are uniformly convex and 2-uniformly smooth Banach space with $K=p-1$, then we consider $E=L_p$ and we derive that following theorem:

Theorem 3.3. *Let C be a nonempty closed convex subset of an L_p for $2\leq p<\infty$. Let A, B, S, f, $J^A_{r_n}$ be the same as in theorem 3.2. Let $\{\alpha_n\}$, $\{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0,\frac{\alpha}{(p-1)^2})$ and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions (a),(b) and (d) in theorem 3.2 and conditions (c) $\lim_{n\longrightarrow\infty} r_n=r$, and $r\in(0,\frac{\alpha}{(p-1)^2})$. Then the sequence $\{x_n\}$ is defined by (4.1) converges strongly to a point $\bar{x}\in Fix(S)\cap(A+B)^{-1}(0)$.*

Consider a mapping $S\equiv I$ in theorem 3.2, we can obtain the following corollary direct.

Corollary 3.4. *Let E be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let C be a nonempty closed convex subset of E. Let $A: D(A)\subseteq E\longrightarrow 2^E$ be an m-accretive operator such that the domain of A is included in C and $B: C\longrightarrow X$ be an α-inverse strongly accretive* *operator. Let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0,1)$. Let $J^A_{r_n} = (I + r_n A)^{-1}$ be a resolvent of A for $r_n > 0$ such that $(A+B)^{-1}(0) \neq \emptyset$.*

For given $x_0 \in C$, Let x_n be a sequence in the following process:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) J^A_{r_n}(y_n - r_n B y_n + e_n), \ \forall n \geq 0, \end{cases} \tag{3.28}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$;
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0)$.

Consider a mapping $S \equiv I$ and $f(x_n) \equiv u$, $\forall n \in \mathbb{N}$ in theorem 3.2, we obtain the following corollary direct.

Corollary 3.5. *Let E be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let C be a nonempty closed convex subset of E. Let $A : D(A) \subseteq E \longrightarrow 2^E$ be an m-accretive operator such that the domain of A is included in C and let $B : C \longrightarrow X$ be an α-inverse strongly accretive operator. Let $J^B_{r_n} = (I + r_n B)^{-1}$ be a resolvent of B for $r_n > 0$ such that $(A+B)^{-1}(0) \neq \emptyset$.*

For given $x_0 \in C$, Let x_n be a sequence in the following process:

$$\begin{aligned} & y_n = \alpha_n u + (1-\alpha_n)x_n, \\ & x_{n+1} = \beta_n x_n + (1-\beta_n) J^A_{r_n}(y_n - r_n B y_n + e_n), \ \forall n \geq 0, \end{aligned} \tag{3.29}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequence satisfy the following conditions:

(a) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(b) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$;
(c) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0)$.

Setting $J^A_{r_n} \equiv I$, $B \equiv 0$, $f(x_n) \equiv u$, $\forall n \in \mathbb{N}$ and $e_n \equiv 0$, then we have the following corollary of the modified Mann-Halpern iteration.

Corollary 3.6. *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(S) \neq \emptyset$. For given $x_0, u \in C$, Let x_n be a sequence in the following process:*

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S y_n, \ \forall n \geq 0, \end{cases} \tag{3.30}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$. Assume that the control sequence satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S)$.

4. Some applications

In this section, we give two applications of our main results in the framework of Hilbert spaces. Now, we consider theorem 3.2, in the framework of Hilbert spaces, it known that $K = \frac{\sqrt{2}}{2}$. Let H be a Hilbert space and let C be a nonempty closed convex subset of H.

Theorem 4.1. [6, Corollary 2.2] *Let $A : C \longrightarrow 2^H$ be a maximal monotone operators such that the domain of B which included in C and $B : C \longrightarrow H$ be an α-inverse strongly monotone operator. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0, 1)$. Let $J^A_{r_n} = (I+r_nA)^{-1}$ be a resolvent of A for $r_n > 0$ such that $Fix(S)\cap(A+B)^{-1}(0) \neq \emptyset$.*
For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)SJ^A_{r_n}(y_n - r_nBy_n + e_n), \ \forall n \geq 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a real number sequences in $(0, 2\alpha)$ and $\{e_n\}$ is a sequence in H. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$;
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A + B)^{-1}(0)$. Next, we will give some related results.

4.1. Application to projection for variational inequality.

Let C be a nonempty, close and convex subset of a Hilbert space H. The metric projection of a point $x \in H$ onto C, denoted by $P_C(x)$, is defined as the unique solution of the problem

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C, \ \forall x \in H.$$

For each $x \in H$ and $z \in C$, the metric projection P_C is satisfied

$$z = P_C(x) \Longleftrightarrow \langle y - z, x - z\rangle \leq 0, \ \forall y \in C. \tag{4.2}$$

Note that the metric projection is nonexpansive mapping.

Let $g : H \longrightarrow (-\infty, \infty]$ is a proper convex lower semicontinuous function. Then the subdifferential ∂g of g is defined as follow:

$$\partial g(x) = \{z \in H : g(y) - g(x) \geq \langle y - x, z\rangle, \ \forall y \in H\},$$

for all $x \in H$. If $g(x) = \infty$, then $\partial g(x) \neq \emptyset$, Takahashi [16] claim that ∂g is m-accretive operator. Since we know that, an m-accretive operator is maximal monotone operators in a Hilbert space, then we claim that ∂g is maximal monotone operators. Then we define the set of minimizers of g as follow:

$$argmin_{y\in H} g(y) = \{z \in H : g(z) = min_{y\in H} g(y)\}.$$

It is easy to verify that $0 \in \partial g(x)$ if and only if $g(z) = min_{y\in H} g(y)$. Let i_C be the indicator function of C by

$$i_C(x) = \begin{cases} 0, & \forall x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Then i_C is a proper lower semicontinuous convex function on H. So, we see that the subdifferential ∂i_C of i_C is maximal monotone operator; see, [16]. The resolvent J_r of ∂i_C for $r > 0$, that is $J_r x = (I + r\partial i_C)^{-1}x,\ \forall x \in H$. Next, we recall that set $N_C(u)$ is called the normal cone of C at u define by

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \leq 0,\ \forall y \in C\}.$$

Since $N_C(u) = \partial i_C(u)$. In fact, we have that for any $x \in H$ and $u \in C$,

$$\begin{aligned} u = J_r x = (I + r\partial i_C)^{-1}x &\iff x \in u + r\partial i_C u \\ &\iff x \in u + rN_C(u) \\ &\iff x - u \in rN_C(u) \\ &\iff \frac{1}{r}\langle x - u, y - u\rangle \leq 0,\ \forall y \in C \\ &\iff \langle x - u, y - u\rangle \leq 0,\ \forall y \in C \\ &\iff u = P_C x. \end{aligned} \tag{4.3}$$

Then $u = (I + r\partial i_C)^{-1}x \iff u = P_C x,\ \forall x \in H,\ u \in C$.

Now, we consider the following variational inequality problem (VIP) for B is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0,\ \forall y \in C. \tag{4.4}$$

The set of solutions of (4.4) is denoted by $VI(C, B)$.

$$VI(C, B) = \left\{\ x \in C : \langle Bx, y - x\rangle \geq 0,\ \forall y \in C\ \right\}. \tag{4.5}$$

Theorem 4.2. *Let $B : C \longrightarrow H$ be an α-inverse strongly monotone mapping. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0, 1)$. Assume that $Fix(S) \cap VI(C, B) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)SP_C(y_n - r_n By_n + e_n),\ \forall n \geq 0, \end{cases} \tag{4.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a real number sequences in $(0, 2\alpha)$ and $\{e_n\}$ is a sequence in H. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, 2\alpha)$;
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap VI(C, A)$, where $\bar{x} = P_{Fix(S)\cap VI(C,B)} f(\bar{x})$.

Proof. By lemma 2.4 we know that $Fix(J_r^A(I - rB)) = (A + B)^{-1}(0)$. Put $A = \partial i_C$, and we to show that $VI(C, B) = (\partial i_C + B)^{-1}(0)$. Note that

$$\begin{aligned} x \in (\partial i_C + B)^{-1}(0) &\iff 0 \in \partial i_C x + Bx \\ &\iff 0 \in N_C x + Bx \\ &\iff -Bx \in N_C x \\ &\iff \langle -Bx, y - x\rangle \leq 0 \\ &\iff \langle Bx, y - x\rangle \geq 0 \\ &\iff x \in VI(C, B). \end{aligned} \tag{4.7}$$

From (4.3), therefore, we can conclude the desired conclusion immediately. □

4.2. **Application for equilibrium problems.** Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for finding $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C. \tag{4.8}$$

The set of solutions of (4.8) is denoted by $EP(F)$.

For solving the equilibrium problem, we assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t \longrightarrow 0^+} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 4.3. [17] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $z \in H$. Then, there exists $x \in C$ such that*

$$F(x, y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \ \forall y \in C. \tag{4.9}$$

Lemma 4.4. [18] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $z \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(z) = \{x \in C : F(x, y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \ \forall y \in C\}, \forall z \in H. \tag{4.10}$$

Then, the following hold:

(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) $Fix(T_r) = EP(F)$;
(4) $EP(F)$ is closed and convex.

Lemma 4.5. [19] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4) and A_F be a multi-valued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, & \forall x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then $EP(F) = A_F^{-1}(0)$ and $A_F x$ is a maximal monotone operator with the domain $D(A_F) \subset C$. Furthermore, the resolvent T_r of F coincides with the resolvent of A_F, i.e.,

$$T_r x = (I + rA_F)^{-1}(x), \ \forall x \in H, \ r > 0, \tag{4.11}$$

where T_r is defined as in (4.10)

We recalled that T_r is the resolvent of A_F for $r > 0$. Since $A = A_F$, we will show that $J_r x = T_r x$. Indeed, for $x \in H$, we have

$$\begin{aligned} z \in J_r x = (I + rA_F)^{-1}(x) &\iff x \in (I + rA_F)z \\ &\iff x \in z + rA_F z \\ &\iff x - z \in rA_F z \\ &\iff \frac{1}{r}(x - z) \in A_F z \end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow \quad F(z,y) \geq \langle y-z, \frac{1}{r}(x-z)\rangle \\
&\Longleftrightarrow \quad F(z,y) \geq \langle y-z, \frac{-1}{r}(z-x)\rangle \\
&\Longleftrightarrow \quad F(z,y) \geq \frac{-1}{r}\langle y-z, z-x\rangle \\
&\Longleftrightarrow \quad F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \forall y \in C \\
&\Longleftrightarrow \quad z \in T_r x. \qquad (4.12)
\end{aligned}$$

Using lemmas 4.3, 4.4, 4.5 and theorem 4.1, we also obtain the following result.

Theorem 4.6. *Let $F : C \times C \longrightarrow \mathbb{R}$ which satisfies $(A1) - (A4)$. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0,1)$. Assume that $Fix(S) \cap EP(F) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)ST_{r_n}(y_n + e_n),\ \forall n \geq 0, \end{cases} \tag{4.13}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0,2\alpha)$ and $\{e_n\}$ is a sequence in H.

Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0,2\alpha)$;
(d) $\sum_{n=0}^{\infty} \parallel e_n \parallel < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap EP(F)$, where $\bar{x} = P_{Fix(S)\cap EP(F)} f(\bar{x})$.

Proof. Put $A \equiv A_F$ and $B \equiv 0$ in $(A+B)^{-1}(0)$ from theorem 4.1. Furthermore, for bifunction $F : C \times C \longrightarrow \mathbb{R}$, we define $A_F x$ as in lemma 4.5, we have $EP(F) = A_F^{-1}(0)$ and let T_{r_n} be the resolvent of A_F for $r_n > 0$. Therefore, we can conclude the desired conclusion immediately. ☐

5. Conclusion and remarks

Our main results extends and improves in the following:

(i) Theorem 3.2 extends and improves Theorem 3.1 of Manaka and Takahashi [4, Theorem 3.1] from a Hilbert space to a Banach space and from weak convergence to strong convergence.

(ii) Theorem 3.2 partially extends and improves Theorem 2.1 of Cho et al. [6, Theorem 2.1] from a Hilbert space to a Banach space with uniformly convex and 2-uniformly smooth.

(iii) Theorem 3.2 extends and improves Theorem 3.1 of Qing and Cho [20, Theorem 3.1] from the problems of finding an element of $A^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

(iv) Theorem 3.2 extends and improves Theorem 3.7 of Sahu and Yao [3, Theorem 3.7] from the problems of finding an element of $A^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

(v) Theorem 3.2 extends and improves Theorem 3.7 of López et al. [5, Theorem 3.7] from the problems of finding an element of $(A+B)^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

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MODIFY REGULARIZATION METHOD VIA PROXIMAL POINT ALGORITHMS FOR ZEROS OF SUM ACCRETIVE OPERATORS OF FIXED POINT AND INVERSE PROBLEMS

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ABSTRACT. In this paper, we investigate the regularization method via a proximal point algorithm for solving treating the sum of two accretive operators for a fixed point set and inverse problems. Strong convergence theorems are established in the framework of Banach spaces. Furthermore, we also apply our result to variational inequality and equilibrium problems.

KEYWORDS: regularization method, proximal point algorithm, zero points, accretive operators, inverse problems.
AMS Subject Classification: 47H09, 47H17, 47J25, 49J40.

1. Introduction

Many important problems have reformulation which require finding common zero points of nonlinear operators, for instance, inverse problems, variational inequality, optimization problems and fixed point problems. In this paper,we use $A^{-1}(0)$ to denote the set of zeros point of A. A well-known method for solving zero points of maximal monotone operators is the *proximal point algorithm* (PPA). First, Martinet [1] introduced the PPA in a Hilbert space H, that is, for starting $x_0 \in H$, a sequence $\{x_n\}$ generated by

$$x_{n+1} = J_{r_n}^A(x_n) \;\forall n \in \mathbb{N}, \tag{1.1}$$

where A is maximal monotone operators, $J_{r_n}^A = (I + r_n A)^{-1}$ is the resolvent operator of A and $\{r_n\} \subset (0, \infty)$ is a regularization sequence. An iterative (1.1) is equivalent

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to

$$x_n \in x_{n+1} + r_n A x_{n+1} \ \forall n \in \mathbb{N}.$$

If $\phi(x) : H \to \mathbb{R} \cup \{\infty\}$ is a proper convex and lower semicontinuous function, then $J^A_{r_n}$ reduces to

$$x_{n+1} = \arg\min \left\{ \phi(y) + \frac{1}{2r_n}\|x_n - y\|^2, \ y \in H \right\} \ \forall n \in \mathbb{N}. \tag{1.2}$$

Later, Rockafellar [2] studied the proximal point algorithm in framework of a Hilbert space and he proved that if $\liminf_{n\longrightarrow\infty} r_n > 0$ and $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a solution of a zero point of A. Rockafellar [2] has given a more practical method which is an inexact variant of the method:

$$x_{n+1} = J^A_{r_n} x_n + e_n, \ \forall n \in \mathbb{N}, \tag{1.3}$$

where $\{e_n\}$ is an error sequence. It was shown that if $e_n \longrightarrow 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$, with $0 \in A(z)$.

In 2011, Sahu and Yao [3] also extended PPA for the zero of an accretive operator in a Banach space which has a uniformly Gâteaux differentiable norm by combining the prox-Tikhonov method and the viscosity approximation method. They introduced the iterative method to define the sequence $\{x_n\}$ as follows:

$$x_{n+1} = J^A_{r_n}((1-\alpha_n)x_n + \alpha_n f(x_n)), \ \forall n \in \mathbb{N}, \tag{1.4}$$

$$z_{n+1} = J^A_{r_n}((1-\alpha_n)z_n + \alpha_n f(z_n) + e_n), \ \forall n \in \mathbb{N}, \tag{1.5}$$

where A is an accretive operator such that $A^{-1}(0) \neq \emptyset$ and f is a contractive mapping on C and $\{e_n\}$ is an error sequence. Strong convergent were established in both algorithms. This is a source of idea about resolvent operator can be approximated by contractions.

In the same year, PPA extended to the case of sum of two monotone operators A and B by use the technique of forward-backward splitting methods. Manaka and Takahashi [4] introduced the following iterative scheme in a Hilbert space:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S J^A_{\lambda_n}(I - \lambda_n B)x_n, \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in (0,1), $\{\lambda_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping, A is a maximal monotone operator, B is an inverse strongly monotone mapping and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. They prove that a sequence $\{x_n\}$ converges weakly to some point $z \in Fix(S) \cap (A+B)^{-1}(0)$ provided that the control sequence satisfies some conditions. From [4], then we concern with the problem for finding a common element of $Fix(S) \cap (A+B)^{-1}(0)$.

In 2012, Lóṕez et al. [5] use the technique of forward-backward splitting methods for accretive operators in Banach spaces. They considered the following algorithms with errors:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J^A_{r_n}(x_n - r_n(Bx_n + a_n)) + b_n \tag{1.6}$$

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J^A_{r_n}(x_n - r_n(Bx_n + a_n)) + b_n, \tag{1.7}$$

where $u \in E$, $\{a_n\}$, $\{b_n\} \subset E$ and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. An operator A is a maximal accretive operator and B is an inverse strongly accretive. They prove that a sequence $\{x_n\}$ in equation (1.6) and (1.7) is weakly and strongly convergence, respectively.

In 2014, Cho et al. [6] introduced the following iterative scheme in a Hilbert space:

$$\begin{cases} x_1 \in C, \\ z_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ y_n = J^A_{r_n}(z_n - r_n B z_n + e_n) \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n)S y_n), \ for\ n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are a sequences in $(0,\ 1)$, $\{r_n\}$ is a positive sequence, $A : C \to H$ is an inverse strongly monotone mapping, B is a maximal monotone operator, and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. Let $S : C \to C$ is a strictly pseudo-contractive mapping with $k \in [0, 1)$, and $f : C \to C$ be a contractive mapping. They prove that a sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A+B)^{-1}(0)$ if the control sequence satisfies some restrictions.

Motivated by [3, 4, 5, 6], then we are interested in the problems for finding a common element of fixed point of nonexpansive S and element of the (quasi) variational inclusion problem as follow:

$$Find\ x \in C\ such\ that\ x \in Fix(S) \cap (A+B)^{-1}(0), \tag{1.8}$$

where A be single-valued nonlinear mapping and B be a multi-valued mapping.

The purpose of this paper is to introduce an iterative algorithm which is modify regularization method and use technique of forward-backward splitting methods for finding a common element of the set solution of nonexpansive S and the set solution of fixed point of the variational inclusion problems, where A is an m-accretive operator and B is an inverse-strongly accretive operator in the framework of Banach space with a uniformly convex and 2-uniformly smooth.

2. Preliminaries

Let E be a Banach space and let E^* be its dual. Let $\langle \cdot, \cdot \rangle$ be the pairing between E and E^*. For all $x \in E$ and $x^* \in E^*$, the value of x^* at x be denoted by $\langle x, x^* \rangle$. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\|\}$, for all $x \in E$. A *single-value* normalized duality mapping is denoted by j, which means a mapping $j : E \longrightarrow E^*$ such that, for all $u \in E$, $j(u) \in E^*$ satisfying the following:

$$\langle u, j(u) \rangle = \|u\| \|j(u)\|,\ \|j(u)\| = \|u\|.$$

If $E = H$ is a Hilbert space, then $J = I$, where I is identity mapping. If E is *smooth Banach space*, then J is single-valued j.

A Banach space E is called an *Opial's space* if for each sequence $\{x_n\}_{n=0}^{\infty}$ in E such that $\{x_n\}$ converges weakly to some x in E, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

hold for all $y \in E$ with $y \neq x$. In fact, for any normed linear space X admit the weakly sequentially continuous duality mapping implies X is Opial space. So, a Banach space with a weakly sequentially continuous duality mapping has the Opial's property; see [7].

The modulus of convexity of E is the function $\delta_E : (0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\epsilon) := \inf\{1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1; \|x - y\| \geq \epsilon\}.$$

E is said to be *uniformly convex* if and only if $\delta(\epsilon) > 0$, for each $\epsilon \in (0, 2]$. It known that a uniformly convex Banach space is reflexive and strictly convex.

Let $S(E)$ be the unit sphere defined by $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm $\|\cdot\|$ of E is said to be *Gâteaux differentiable norm*, if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. In this case, space E is called *smooth.* A spaces E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (2.1) exist uniformly for all $x \in S(E)$. The norm of E is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for all $x, y \in S(E)$. It is known that if the norm of E is smooth, then the duality mapping J is single-valued and norm to *weak** uniformly continuous on each bounded subset of E.

On the other hand, the *modulus of smoothness* of E is the function $\rho : [0, \infty) \to [0, \infty)$ defined by

$$\rho(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in S(E), \|x\| = 1, \|y\| \leq t\}.$$

A Banach space E be an *smooth* if $\rho_E(t) > 0$ for all $t > 0$. A Banach space E be an *uniformly smooth* if and only if $\lim_{t \to 0} \frac{\rho(t)}{t} = 0$. A Banach space E is said to be *q-uniformly smooth*, if for $1 < q \leq 2$ be a fixed real number, there exists a constant $c > 0$ such that $\rho(t) \leq ct^q$ for all $t > 0$. It known that every q-uniformly smooth space is smooth. In the case $\rho(t) \leq ct^2$ for $t > 0$, these is 2-uniformly smooth. The examples of uniformly convex and 2-uniformly smooth Banach space are L_p, l_p or Sobolev spaces W_m^p, where $p \geq 2$. It is well known that, Hilbert spaces are 2-uniformly convex and 2-uniformly smooth. We known that if E is a reflexive Banach space, then every bounded sequence in E has a weakly convergent subsequence. Note that all uniformly convex and 2-uniformly smooth Banach space is reflexive.

Next, we recall the definitions of some operators.

(i) Let $f : C \longrightarrow C$ be an operator. Then f is called *k-contraction* if there exists a coefficient k $(0 < k < 1)$ such that

$$\|fx - fy\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

(ii) Let $S : C \longrightarrow C$ be an operator. Then s is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(iii) Let $B : C \longrightarrow E$ be an operator. Then B is called *α-inverse-strongly accretive* if there exists a constant $\alpha > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Bx - By, j(x - y)\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

(iv) A set-valued operator $A : D(A) \subseteq E \longrightarrow 2^E$ is called *accretive* if there exists $j(x - y) \in J(x - y)$ such that $u \in A(x)$, and $v \in A(y)$,

$$\langle u - v, j(x - y)\rangle \geqslant 0, \quad \forall x, y \in D(A).$$

(v) A set-valued operator $A : D(A) \subseteq E \longrightarrow 2^E$ is called *m-accretive* if A is accretive and $R(I + rA) = E$ for some $r > 0$, where I is identity mapping.

Let C and D are nonempty subsets of a Banach space E such that C is a nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is said to be *sunny* if $Q(x + t(x - Q(x))) = Q(x)$ whenever $x + t(x - Q(x)) \in C$ for all $x \in C$ and $t \geq 0$.

A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q.

Lemma 2.1. [8] *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;
(iii) $\|(x - y) - (Qx - Qy)\|^2 \leq \|x - y\|^2 - \|Qx - Qy\|^2$
(iv) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.

Lemma 2.2. [9] *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let S be a nonexpansive mapping of C into itself with $Fix(S) \neq \emptyset$. Then, the set $Fix(S)$ is a sunny nonexpansive retract of C.*

It well known that if $E = H$ is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection P_C from E onto C, that is $Q_C = P_C$. Let C be a nonempty closed convex subset of E.

In the sequel for the proof of our main results, we also need the following lemmas.

Lemma 2.3. [10] *Let E be a Banach space and J be a normal duality mapping. Then there exits $j(x + y) \in J(x + y)$ for any given $x, y \in E$. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \ j(x + y) \in J(x + y), \tag{2.2}$$

for any given $x, y \in E$.

Lemma 2.4. [5] *Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $B : C \longrightarrow E$ be a single valued operator and α-inverse strongly accretive operator and let A is an m-accretive operator in E with $D(A) \supset C$ and $D(B) \supset C$.Then*

$$Fix\big(J_r^A(I - rB)\big) = (A + B)^{-1}(0).$$

where $J_r^A = (I + rA)^{-1}$ be a resolvent of A for $r > 0$.

Lemma 2.5. [11](*The Resolvent Identity*) *Let E be a Banach space and let A be an m-accretive operator.For $r > 0$, $s > 0$ and $x \in E$, then*

$$J_r^A x = J_s^A\left(\frac{s}{r}x + \left(1 - \frac{s}{r}\right)J_r^A x\right).$$

Lemma 2.6. [12] *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E with the 2-uniformly smooth constant K. Let the mapping $B : C \longrightarrow E$ be a α-inverse strongly accretive operator. Then, we have*

$$\|(I - rB)x - (I - rB)y\|^2 \leq \|x - y\|^2 - 2r(\alpha - K^2 r)\|Bx - By\|^2, \tag{2.3}$$

where I is identity mapping. In particular, if $r \in (0, \frac{\alpha}{K^2})$, then $(I - rB)$ is a nonexpansive.

Lemma 2.7. [13] (*Demiclosed principle*) Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and $S : C \longrightarrow E$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. Then $I - S$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Sx_n \longrightarrow 0$ implies $x = Sx$.

Lemma 2.8. [14] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.9. [15] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq (1-t_n)a_n + t_n b_n + c_n, \forall n \geq 0,$$

where $\{t_n\}$ *is a number sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} t_n = 0$ *and* $\Sigma_{n\to\infty}^{\infty} t_n = \infty$, $\{b_n\}$ *is a sequence such that* $\limsup_{n\to\infty} b_n \leq 0$ *and* $\{c_n\}$ *is a positive number sequence such that* $\Sigma_{n=0}^{\infty} c_n < \infty$. *Then,* $\lim_{n\to\infty} a_n = 0$.

3. Main results

Before prove our main result, we need the following lemma:

Lemma 3.1. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* C *be a nonempty closed convex subset of* E. *Let* $A : D(A) \subseteq C \longrightarrow 2^E$ *be a m-accretive operator and* $B : C \longrightarrow E$ *be an* α*-inverse strongly accretive operator. Let* $S : C \longrightarrow C$ *be a nonexpansive mapping and let* $f : C \longrightarrow C$ *be a contraction mapping with the constant* $k \in (0,1)$. *Let* $J_{r_n}^A = (I + r_n A)^{-1}$ *be a resolvent of* A *for* $r_n > 0$ *such that* $Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$. *If defined operator* $W_n : C \to C$ *by* $W_n := SJ_{r_n}^A ((I - r_n B)[\alpha_n fx + (1-\alpha_n)x] + e_n)$ *for all* $x \in C$, *where* $\alpha_n \in (0,1)$, $r_n > 0$. *Then* W_n *is a contraction operator and has a unique fixed point.*

Proof. Since S, $J_{r_n}^A$, and $(I - r_n B)$ are nonexpansive. Then we known that W_n is nonexpansive. Since f be a contraction mapping with coefficient $k \in (0,1)$. We have

$$\begin{aligned}
\|W_n x - W_n y\| &= \|SJ_{r_n}^A ((I - r_n B)[\alpha_n f(x) + (1-\alpha_n)x] + e_n) \\
&\quad - SJ_{r_n}^A ((I - r_n B)[\alpha_n f(y) + (1-\alpha_n)y] + e_n) \| \\
&\leq \| ((I - r_n B)[\alpha_n f(x) + (1-\alpha_n)x] + e_n) \\
&\quad - ((I - r_n B)[\alpha_n f(y) + (1-\alpha_n)y] + e_n) \| \\
&\leq \|[\alpha_n f(x) + (1-\alpha_n)x] - [\alpha_n f(y) + (1-\alpha_n)y]\| \\
&\leq \|(\alpha_n f(x) + (1-\alpha_n)x) - (\alpha_n f(y) + (1-\alpha_n y)\| \\
&= \|\alpha_n (f(x) - f(y)) + (1-\alpha_n)(x-y)\| \\
&\leq \alpha_n \|f(x) - f(y)\| + (1-\alpha_n)\|x-y\| \\
&\leq \alpha_n k\|x-y\| + (1-\alpha_n)\|x-y\| \\
&= (\alpha_n k + (1-\alpha_n))\|x-y\|.
\end{aligned}$$

Since $0 < (\alpha_n k + (1-\alpha_n)) < 1$, it follows that W_n is a contraction mapping of C into it self. By Banach contraction principle, then there exist a unique fixed point, i.e., we say $\bar{x} = W_n \bar{x}$. Moreover, by use lemma 2.2, then the set $Fix(W_n)$ is sunny nonexpansive retraction of C. Hence there exist a unique fixed point $\bar{x} \in Fix(W_n) = Fix(S) \cap (A+B)^{-1}(0) := \Omega$, namely $Q_\Omega f(\bar{x}) = \bar{x} = W_n \bar{x}$. □

Theorem 3.2. *Let* E *be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let* C *be a nonempty closed convex subset of* E. *Let* $A : D(A) \subseteq C \longrightarrow 2^E$ *be a m-accretive operator and* $B : C \longrightarrow E$ *be an* α*-inverse strongly accretive operator. Let* $S : C \longrightarrow C$ *be a nonexpansive mapping and let* $f : C \longrightarrow C$ *be a contraction mapping with the constant* $k \in (0,1)$. *Let* $J_{r_n}^A = (I + r_n A)^{-1}$ *be a resolvent of* A *for* $r_n > 0$. *Assume that* $Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$.

For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by the following:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)SJ^A_{r_n}(y_n - r_nBy_n + e_n), \ \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$*;*
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$*;*
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A+B)^{-1}(0)$, where $\bar{x} = Q_\Omega f(\bar{})$ and $Q_F f$ is a sunny nonexpansive retraction from E onto Ω.

Proof. **Step 1** We want to show that $\{x_n\}$ is bounded. Fixed $p \in Fix(S) \cap (A + B)^{-1}(0) \neq \emptyset$. So, we have $p \in Fix(S)$ and $p \in (A+B)^{-1}(0) = Fix(J^A_{r_n}(I - r_nB))$ (see Lemma 2.4). Observe that, we consider

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - p\| \\ &\leq \alpha_n\|f(x_n) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \alpha_n\left(\|f(x_n) - f(p)\| + \|f(p) - p\|\right) + (1-\alpha_n)\|x_n - p\| \\ &\leq \alpha_n k\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= [\alpha_n k + (1-\alpha_n)]\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= [1 - \alpha_n(1-k)]\|x_n - p\| + \alpha_n\|f(p) - p\|. \end{aligned} \tag{3.2}$$

We set $z_n := SJ^A_{r_n}(y_n - r_nBy_n + e_{n+1})$. Since $J^A_{r_n}$ and $I - r_nB$ are nonexpansive, and from (3.2), it follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n x_n + (1-\beta_n)z_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &= \beta_n\|x_n - p\| + (1-\beta_n)\|SJ^B_{r_n}(y_n - r_nAy_n + e_n) - Sp\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|J^A_{r_n}(y_n - r_nBy_n + e_n) - p\| \\ &= \beta_n\|x_n - p\| + (1-\beta_n)\|J^A_{r_n}(y_n - r_nBy_n + e_n) - J^A_{r_n}(I - r_nB)p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|(y_n - r_nBy_n + e_n) - (I - r_nB)p\| \\ &= \beta_n\|x_n - p\| + (1-\beta_n)\|(I - r_nB)y_n - (I - r_nB)p + e_n\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\left(\|(I - r_nB)y_n - (I - r_nB)p\| + \|e_n\|\right) \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\left[\|y_n - p\| + \|e_n\|\right] \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)[(1 - \alpha_n(1-k))\|x_n - p\| \\ &\quad + \alpha_n\|f(p) - p\|] + (1-\beta_n)\|e_n\| \\ &= \beta_n\|x_n - p\| + [(1-\beta_n) - \alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n\|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [\beta_n + (1-\beta_n) - \alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n\|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [1 - (1-\beta_n)\alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n\|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [1 - \lambda_n(1-k)]\|x_n - p\| + \lambda_n\|f(p) - p\| + \|e_n\|, \end{aligned}$$

where $\lambda_n := (1-\beta_n)\alpha_n$. Then, it follows that

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-k}\right\} + \|e_n\| \\
&\leq \max\left\{\|x_{n-1}-p\|, \frac{\|f(p)-p\|}{1-k}\right\} + \|e_{n-1}\| + \|e_n\| \\
&\leq \max\left\{\|x_{n-2}-p\|, \frac{\|f(p)-p\|}{1-k}\right\} + \|e_{n-2}\| + \|e_{n-1}\| + \|e_n\| \\
&\vdots \\
&\leq \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-k}\right\} + \sum_{i=0}^{n}\|e_i\| < \infty.
\end{aligned}$$

It follows by mathematical induction, we conclude that

$$\|x_{n+1}-p\| \leq \max\left\{\|x_0-p\|, (1-k)^{-1}\|f(p)-p\|\right\} + \sum_{i=0}^{n}\|e_i\|, \ \forall n \geq 0.$$

By condition (d), this implies that $\{x_n\}$ is bounded.

From $y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n$, we obtain

$$\begin{aligned}
\|y_n-p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - p\| \\
&\leq \alpha_n\|f(x_n)-p\| + (1-\alpha_n)\|x_n-p\|.
\end{aligned} \tag{3.3}$$

From (3.3) and since $\{x_n\}$ is bounded, so $\{y_n\}$ and $\{z_n\}$ are bounded too.

Step 2 We want to show that $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\| = 0$. By lemma 2.8, we set $v_n := y_n - r_nAy_n + e_n$, then $z_n := SJ_{r_n}^{B}v_n$, it follows that

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|SJ_{r_{n+1}}^{A}v_{n+1} - SJ_{r_n}^{A}v_n\| \\
&\leq \|J_{r_{n+1}}^{A}v_{n+1} - J_{r_n}^{A}v_n\| \\
&\leq \|J_{r_{n+1}}^{A}v_{n+1} - J_{r_{n+1}}^{A}v_n\| + \|J_{r_{n+1}}^{A}v_n - J_{r_n}^{A}v_n\| \\
&\leq \|v_{n+1}-v_n\| + \|J_{r_{n+1}}^{A}v_n - J_{r_n}^{A}v_n\|.
\end{aligned} \tag{3.4}$$

Next, we compute $\|v_{n+1}-v_n\|$ that

$$\begin{aligned}
\|v_{n+1}-v_n\| &= \|(y_{n+1} - r_{n+1}By_{n+1} + e_{n+1}) - (y_n - r_nBy_n + e_n)\| \\
&= \|(I-r_nB)y_{n+1} - (I-r_nB)y_n + (r_n - r_{n+1})By_{n+1} + e_{n+1} - e_n\| \\
&\leq \|(I-r_nB)y_{n+1} - (I-r_nB)y_n\| + \mid r_n - r_{n+1} \mid \|By_{n+1}\| + \|e_{n+1}-e_n\| \\
&\leq \|y_{n+1}-y_n\| + \mid r_n - r_{n+1} \mid \|By_{n+1}\| + \|e_{n+1}\| + \|e_n\|.
\end{aligned} \tag{3.5}$$

Next, we compute $\|y_{n+1}-y_n\|$ that

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|(\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1})(x_{n+1})) - (\alpha_n f(x_n) + (1-\alpha_n)x_n)\| \\
&= \|\alpha_{n+1}f(x_{n+1}) - \alpha_n f(x_{n+1}) + \alpha_n f(x_{n+1}) - \alpha_n f(x_n) + (1-\alpha_{n+1})x_n \\
&\quad -(1-\alpha_{n+1})x_n - (1-\alpha_n)x_n\| \\
&= \|(\alpha_{n+1}-\alpha_n)f(x_{n+1}) + \alpha_n(f(x_{n+1}) - f(x_n)) + (1-\alpha_{n+1})(x_{n+1}-x_n) \\
&\quad +x_n((1-\alpha_{n+1}) - (1-\alpha_n))\| \\
&\leq |\alpha_{n+1}-\alpha_n|\|f(x_{n+1}) - x_n\| + \alpha_n\|f(x_{n+1}) - f(x_n)\| + (1-\alpha_{n+1})\|x_{n+1}-x_n\| \\
&= (1-\alpha_{n+1})\|x_{n+1}-x_n\| + h_n
\end{aligned}$$

$$\leq \|x_{n+1} - x_n\| + h_n, \tag{3.6}$$

where $h_n = |\alpha_{n+1} - \alpha_n| \|f(x_{n+1}) - x_n\| + \alpha_n \|f(x_{n+1}) - f(x_n)\|$.

That is

$$\|v_{n+1} - v_n\| \leq \|x_{n+1} - x_n\| + h_n + g_n, \tag{3.7}$$

where $g_n =| r_n - r_{n+1} | \|By_{n+1}\| + \|e_{n+1}\| + \|e_n\|$.

Next, we compute $\|J^A_{r_{n+1}} v_n - J^A_{r_n} v_n\|$ by the resolvent identity (see Lemma 2.5) that

$$\begin{aligned}
\|J^A_{r_{n+1}} v_n - J^A_{r_n} v_n\| &= \|J^A_{r_n}\left(\frac{r_n}{r_{n+1}} v_n + (1 - \frac{r_n}{r_{n+1}}) J^A_{r_{n+1}} v_n\right) - J^A_{r_n} v_n\| \\
&\leq \|\left(\frac{r_n}{r_{n+1}} v_n + (1 - \frac{r_n}{r_{n+1}}) J^A_{r_{n+1}} v_n\right) - v_n\| \\
&= \|(\frac{r_n}{r_{n+1}} - 1) v_n + (1 - \frac{r_n}{r_{n+1}}) J^A_{r_{n+1}} v_n\| \\
&= \|(1 - \frac{r_n}{r_{n+1}}) J^A_{r_{n+1}} v_n - (1 - \frac{r_n}{r_{n+1}}) v_n\| \\
&= \|\frac{r_{n+1} - r_n}{r_{n+1}} (J^A_{r_{n+1}} v_n - v_n)\| \\
&\leq | \frac{r_{n+1} - r_n}{r_{n+1}} | \|J^A_{r_{n+1}} v_n - v_n\|. \qquad (3.8)
\end{aligned}$$

From (3.7) and (3.8), we obtain

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + h_n + g_n + | \frac{r_{n+1} - r_n}{r_{n+1}} | \|J^A_{r_{n+1}} v_n - v_n\|.$$

In view of the condition (a), (c), and (d), it follows that

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq 0.$$

We take lim sup, it follows that

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By lemma 2.8, we conclude that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0 \tag{3.9}$$

that is $\lim_{n\to\infty} \|SJ^A_{r_n}(v_n) - x_n\| = 0$. From (4.1), we observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\beta_n x_n + (1 - \beta_n) z_n - x_n\| \\
&\leq (1 - \beta_n)\|z_n - x_n\|.
\end{aligned}$$

By (3.9), then we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Step 3 To show that $\lim_{n\to\infty} \|By_n - Bp\| = 0$, $\lim_{n\to\infty} \|J^A_{r_n}(v_n) - y_n\| = 0$ and $\lim_{n\to\infty} \|SJ^A_{r_n}(v_n) - J^A_{r_n}(v_n)\| = 0$.

Step 3.1 First, we observe that $\lim_{n\to\infty} \|By_n - Bp\| = 0$. Notice that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n x_n + (1 - \beta_n) SJ^A_{r_n} v_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|SJ^A_{r_n} v_n - p\|^2 \\
&= \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|v_n - (I - r_n B)p\|^2 \\
&= \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|(y_n - r_n B y_n + e_n) - (I - r_n B)p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n)[\|(I - r_n B) y_n - (I - r_n B)p\|^2
\end{aligned}$$

$$\begin{aligned}
&+2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|] \\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\left(\|y_n-p\|^2-2r_n(\alpha-K^2r_n)\|By_n-Bp\|^2\right) \\
&+2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|.
\end{aligned}$$

Set $\bar{g_n} := (1-\beta_n)2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|$, we get

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \qquad (3.11)\\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\left(\|y_n-p\|^2-2r_n(\alpha-K^2r_n)\|By_n-Bp\|^2\right)+\bar{g_n} \\
=\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2-2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2+\bar{g_n} \\
=\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_nf(x_n)+(1-\alpha_n)x_n-p\|^2 \\
&-2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2+\bar{g_n}.
\end{aligned}$$

Set $\bar{h_n} := 2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2$, we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_nf(x_n)+(1-\alpha_n)x_n-p\|^2-\bar{h_n}+\bar{g_n} \\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-p\|^2 \\
&-\bar{h_n}+\bar{g_n} \\
=\ & (1-\alpha_n(1-\beta_n))\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2-\bar{h_n}+\bar{g_n}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&2r_n(\alpha_n-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2 \\
\leq\ & (1-\alpha_n(1-\beta_n))\|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
\leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|(x_n-p)-(x_{n+1}-p)\| \\
&+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|x_n-x_{n+1}\|+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n}.
\end{aligned}$$

In view of the condition (a), (c), (d), and from (3.10), we conclude that $\lim_{n\longrightarrow\infty}\|By_n-Bp\|^2=0$. This implies

$$\lim_{n\longrightarrow\infty}\|By_n-Bp\|=0. \qquad (3.12)$$

Step 3.2 Second, we will show that $\lim_{n\to\infty}\|J_{r_n}^A(v_n)-y_n\|=0$, we observe that

$$\begin{aligned}
&\|J_{r_n}^A(v_n)-p\|^2 \\
\leq\ & \|J_{r_n}^A(v_n)-p\|\|(y_n-r_nBy_n+e_n)-(p-r_nBp)\| \\
=\ & \frac{1}{2}\{\|J_{r_n}^A(v_n)-p\|^2+\|(y_n-r_nBy_n+e_n)-(p-r_nBp)\|^2 \\
&-\|\left(J_{r_n}^A(v_n)-p\right)-(y_n-r_nBy_n+e_n)-(p-r_nBp)\|^2\} \\
=\ & \frac{1}{2}\{\|J_{r_n}^A(v_n)-p\|^2+\|(I-r_nB)y_n-(I-r_nB)p+e_n\|^2 \\
&-\|J_{r_n}^A(v_n)-y_n-r_nBy_n-e_n+r_nBp\|^2\} \\
=\ & \frac{1}{2}\{\|J_{r_n}^A(v_n)-p\|^2+\|(I-r_nB)y_n-(I-r_nB)p\|^2+\bar{g_n} \\
&-\|\left(J_{r_n}^A(v_n)-y_n-e_n\right)-r_n(By_n-Bp)\|^2\} \\
\leq\ & \frac{1}{2}\big\{\|J_{r_n}^A(v_n)-p\|^2+\|y_n-p\|^2+\bar{g_n} \\
&-(\|J_{r_n}^A(v_n)-y_n-e_n\|^2-2r_n\|By_n-Bp\|\|J_{r_n}^A(v_n)-y_n-e_n\| \\
&+\|r_nBy_n-r_nBp\|^2)\big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big\{\|J^A_{r_n}(v_n)-p\|^2+\|y_n-p\|^2+\bar{g_n}-\|J^A_{r_n}(v_n)-y_n-e_n\|^2\\
&\quad +2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|-\|r_nBy_n-r_nBp\|^2\big\}. \qquad (3.13)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|J^A_{r_n}(v_n)-p\|^2\\
\le\ & \|y_n-p\|^2+\bar{g_n}-\|J^A_{r_n}(v_n)-y_n-e_n\|^2\\
&+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|-\|r_nBy_n-r_nBp\|^2\\
=\ & \|\alpha_nf(x_n)+(1-\alpha_n)x_n-p\|^2-\|J^A_{r_n}(v_n)-y_n-e_n\|^2\\
&-\|r_nBy_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}\\
\le\ & \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\|J^A_{r_n}(v_n)-y_n+e_n\|^2\\
&-r_n\|By_n-Bp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}. \qquad (3.14)
\end{aligned}
$$

From (3.14), this implies that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
\le\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|SJ^A_{r_n}(v_n)-p\|^2\\
\le\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|J^A_{r_n}(v_n)-p\|^2\\
\le\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\big\{\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\
&-\|J^A_{r_n}(v_n)-y_n+e_n\|^2-r_n\|By_n-Bp\|^2\\
&+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}\big\}\\
=\ & (1-\alpha_n)\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2\\
&-(1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2-(1-\beta_n)r_n^2\|By_n-r_nBp\|^2\\
&+(1-\beta_n)2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+(1-\beta_n)\bar{g_n}\\
\le\ & \|x_n-p\|^2+\alpha_n\|f(x_n)-p\|^2-(1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2\\
&-r_n^2\|By_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&(1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2\\
\le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|f(x_n)-p\|^2-r_n^2\|By_n-r_nBp\|^2\\
&+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}.\\
=\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+s_n\\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|(x_n-p)-(x_{n+1}-p)\|+s_n\\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|x_n-x_{n+1}\|+s_n, \qquad (3.15)
\end{aligned}
$$

where we set $s_n := \alpha_n\|f(x_n)-p\|^2-r_n^2\|By_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}$.
From (3.15), in view of the condition (a), (c), (d), and equation (3.10), we conclude that

$$\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n)-y_n-e_n\|=0.$$

This in turn implies that

$$\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n)-y_n\|=0. \qquad (3.16)$$

Step 3.3 Lastly, we will show that $\lim_{n\to\infty}\|SJ^A_{r_n}(v_n) - J^A_{r_n}(v_n)\| = 0$, we see that

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - x_n\| \\ &= \alpha_n\|f(x_n) - x_n\|.\end{aligned}$$

By condition (a), then

$$\lim_{n\longrightarrow\infty}\|y_n - x_n\| = 0. \tag{3.17}$$

Next, from (3.16) and equation (3.17), then we see that

$$\|J^A_{r_n}(v_n) - y_n\| \le \|J^A_{r_n}(v_n) - y_n\| + \|y_n - x_n\|.$$

That is

$$\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n) - x_n\| = 0. \tag{3.18}$$

From equation (3.9) and (3.18), then we see that

$$\|SJ^A_{r_n}(v_n) - J^A_{r_n}(v_n)\| \le \|SJ^A_{r_n}(v_n) - x_n\| + \|x_n - J^A_{r_n}(v_n)\|.$$

That is

$$\lim_{n\longrightarrow\infty}\|SJ^A_{r_n}(v_n) - J^A_{r_n}(v_n)\| = 0. \tag{3.19}$$

Step 4 Since E is a uniformly convex and 2-uniformly smooth Banach space, then E is reflexive Banach space. By reflexive Banach space and from $\{x_n\}$, $\{y_n\}$ are bounded, then it has a weakly convergence subsequence. We may assume that $x_{n_i} \rightharpoonup \hat{x}$. In view of $\lim_{n\longrightarrow\infty}\|y_n - x_n\| = 0$, then there exist a subsequence of $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to $\hat{x}$. we can say that $\{y_{n_i}\}$ also converges weakly to $\hat{x}$, i.e, $y_{n_i} \rightharpoonup \hat{x}$, without loss of generality. To show that $\hat{x} \in Fix(S)\cap(A+B)^{-1}(0) = \Omega$.

(i) First, we want to show that $\hat{x} \in Fix(S)$. Now, we have $y_{n_i} \rightharpoonup \hat{x}$. Since we known that $\{J^A_{r_n}(v_n)\}$ is bounded and form $\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n) - y_n\| = 0$, then we say that $\{J^A_{r_{n_i}}(v_{n_i})\} \rightharpoonup \hat{x}$.

From (3.19), we have $\lim_{n\longrightarrow\infty}\|SJ^A_{r_{n_i}}(v_{n_i}) - J^A_{r_{n_i}}(v_{n_i})\| = 0$. By demiclosed principle, this implies $S\hat{x} = \hat{x}$, namely we prove that $\hat{x} \in Fix(S)$. **(ii)** Next, to show that $J^A_r(I - rB)\hat{x} = \hat{x}$. Since a Banach space with weakly continuous duality mapping has the Opial's condition, see [7]. Suppose $\hat{x} \neq J^A_r(I - rB)\hat{x}$. By the Opial's condition and condition (c), (d), then we have

$$\begin{aligned}
&\liminf_{i\to\infty}\|y_{n_i} - \hat{x}\| \\
< \; &\liminf_{i\to\infty}\|y_{n_i} - J^A_r(I - rB)\hat{x}\| \\
\le \; &\liminf_{i\to\infty}\{\|y_{n_i} - J^A_{r_{n_i}}(v_{n_i})\| + \|J^A_{r_{n_i}}(v_{n_i}) - J^A_{r_n}(I - r_nB)\hat{x}\|\} \\
= \; &\liminf_{i\to\infty}\{\|y_{n_i} - J^A_{r_{n_i}}(v_{n_i})\| + \|J^A_r(v_{n_i}) - J^A_r(I - rB)\hat{x}\|\} \\
\le \; &\liminf_{i\to\infty}\{\|y_{n_i} - J^A_{r_{n_i}}(v_{n_i})\| + \|v_{n_i} - (I - rB)\hat{x}\|\} \\
= \; &\liminf_{i\to\infty}\{\|y_{n_i} - J^A_{r_{n_i}}(v_{n_i})\| + \|(I - rB)y_{n_i} - (I - rB)\hat{x}\| + \|e_{n_i}\|\} \\
\le \; &\liminf_{i\to\infty}\{\|y_{n_i} - J^A_{r_{n_i}}(v_{n_i})\| + \|y_{n_i} - \hat{x}\| + \|e_{n_i}\|\}.
\end{aligned}$$

By (3.16) and condition (d), hence

$$\liminf_{i\to\infty}\|y_{n_i} - \bar{x}\| \quad < \quad \liminf_{i\to\infty}\|y_{n_i} - \hat{x}\|.$$

This is contradiction. Therefore, $J_r^A(I-rB)\hat{x}=\hat{x}$.
This complete the proof that $\hat{x}\in Fix(S)\cap(A+B)^{-1}(0)=\Omega$.

Step 5 We defined operator $W_n : C\to C$ by $W_n := SJ_{r_n}^A((I-r_nB)[\alpha_n fx+(1-\alpha_n)x]+e_n)$ for all $x\in C$, where $\alpha_n\in(0,1)$, $r_n>0$. From lemma 3.1 an operators W_n is a contraction operator and has a unique fixed point. Moreover, by use lemma 2.2, we known that $\bar{x}\in Fix(W_n)=Fix(S)\cap(A+B)^{-1}(0):=\Omega$, namely $Q_\Omega f(\bar{x})=\bar{x}=W_n\bar{x}$. (Now, $\hat{x}=\bar{x}$ too)

Next, we will show that $\limsup_{n\longrightarrow\infty}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle\le 0$, where $\lim_{t\longrightarrow 0}x_t=\bar{x}=Q_\Omega f(\bar{x})$ and x_t solves equation $x_t=SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t), \forall t\in(0,1)$.

(i) We want to show that $\lim_{n\longrightarrow\infty}\|W_nx_n-y_n\|=0$. Consider

$$\begin{aligned}\|W_nx_n-y_n\| &\le \|SJ_{r_n}^A((I-r_nB)[\alpha_nf(x_n)+(1-\alpha_n)x_n]+e_n)-x_n\|+\|x_n-y_n\|\\ &= \|z_n-x_n\|+\|x_n-y_n\|. \end{aligned}\tag{3.20}$$

From (3.9) and (3.17), then

$$\lim_{n\longrightarrow\infty}\|W_nx_n-y_n\|=0. \tag{3.21}$$

(ii) We want to show that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle\le 0$. We compute

$$\begin{aligned}
&\|x_t-y_n\|^2\\
=\ &\|SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-y_n\|^2\\
=\ &\langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-W_nx_n+W_nx_n-y_n, j(x_t-y_n)\rangle\\
=\ &\langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-W_nx_n, j(x_t-y_n)\rangle\\
&+\langle W_nx_n-y_n, j(x_t-y_n)\rangle\\
=\ &\langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-SJ_{r_n}^A((I-r_nB)y_n+e_n), j(x_t-y_n)\rangle\\
&+\langle W_nx_n-y_n, j(x_t-y_n)\rangle\\
\le\ &\langle (I-r_nB)(tf(x_t)+(1-t)x_t)-(I-r_nB)y_n-e_n, j(x_t-y_n)\rangle\\
&+\|W_nx_n-y_n\|\|x_t-y_n\|\\
=\ &\langle (I-r_nB)(tf(x_t)+(1-t)x_t)-(I-r_nB)y_n, j(x_t-y_n)\rangle+\langle e_n, j(x_t-y_n)\rangle\\
&+\|W_nx_n-y_n\|\|x_t-y_n\|\\
\le\ &\langle (tf(x_t)+(1-t)x_t)-x_t+x_t-y_n, j(x_t-y_n)\rangle+\|e_n\|\|x_t-y_n\|\\
&+\|W_nx_n-y_n\|\|x_t-y_n\|\\
\le\ &\langle t(f(x_t)-x_t), j(x_t-y_n)\rangle+\langle x_t-y_n, j(x_t-y_n)\rangle+\|e_n\|\|x_t-y_n\|\\
&+\|W_nx_n-y_n\|\|x_t-y_n\|\\
\le\ &t\langle f(x_t)-x_t, j(x_t-y_n)\rangle+\|x_t-y_n\|^2+\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|\\
\le\ &-t\langle f(x_t)-x_t, j(y_n-x_t)\rangle+\|x_t-y_n\|^2+\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|
\end{aligned}\tag{3.22}$$

It follows that

$$t\langle f(x_t)-x_t, j(y_n-x_t)\rangle \le \|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|.$$

Then

$$\langle f(x_t)-x_t, j(y_n-x_t)\rangle \le \frac{1}{t}\{\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|\}.$$

By virtue of (3.21) and condition (d), we found that

$$\limsup_{n\to\infty}\langle f(x_t)-x_t, j(y_n-x_t)\rangle\le 0. \tag{3.23}$$

Since $x_t \to \bar{x}$, as $t \to 0$ and the fact that j is norm-to- weak* uniformly continuous on bounded subset of E, we obtain

$$\begin{aligned}
& |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle - \langle f(x_t) - x_t, j(y_n - x_t)\rangle| \\
\leq\ & |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle - \langle f(\bar{x}) - \bar{x}, j(y_n - x_t)\rangle| \\
& + |\langle f(\bar{x}) - \bar{x}, j(y_n - x_t)\rangle - \langle f(x_t) - x_t, j(y_n - x_t)\rangle| \\
\leq\ & |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x}) - j(y_n - x_t)\rangle| + |\langle f(\bar{x}) - \bar{x} - f(x_t) + x_t, j(y_n - x_t)\rangle| \\
\leq\ & \|f(\bar{x}) - \bar{x}\|\|j(y_n - \bar{x}) - j(y_n - x_t)\| + \|f(\bar{x}) - \bar{x} - f(x_t) + x_t\|\|y_n - x_t\| \\
& \longrightarrow 0,\ as\ t \longrightarrow 0.
\end{aligned}$$

Hence, for any $\epsilon > 0$, there exist $\delta > 0$ such that $\forall t \in (0, \delta)$ the following inequality holds:

$$\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq \langle f(x_t) - x_t, j(y_n - x_t)\rangle + \epsilon.$$

Taking $\limsup_{n\to\infty}$ in the above inequality, we find that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq \limsup_{n\to\infty}\langle f(x_t) - x_t, j(y_n - x_t)\rangle + \epsilon.$$

Since ϵ is arbitrary and (3.23), we obtain that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \leq 0. \tag{3.24}$$

Step 6 Next, we prove that $\{x_n\}$ converges strongly to $\bar{x} = Q_\Omega f(\bar{x})$ by using the lemma 2.3 and lemma 2.9. We note that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \|\beta_n x_n + (1-\beta_n)SJ^A_{r_n}(v_n) - \bar{x}\|^2 \\
&\leq \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|SJ^A_{r_n}(v_n) - \bar{x}\|^2 \\
&= \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|SJ^A_{r_n}(v_n) - S\bar{x}\|^2 \\
&\leq \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|J^A_{r_n}(v_n) - \bar{x}\|^2 \\
&= \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|J^A_{r_n}(v_n) - J^A_{r_n}(I - r_nB)\bar{x}\|^2 \\
&\leq \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|v_n - (I - r_nB)\bar{x}\|^2 \\
&= \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|(y_n - r_nBy_n + e_n) - (I - r_nB)\bar{x}\|^2 \\
&= \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|(I - r_nB)y_n - (I - r_nB)\bar{x} + e_n\|^2 \\
&= \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)[\|(I - r_nA)y_n - (I - r_nA)\bar{x}\|^2 \\
&\quad + 2\langle e_n, j((I - r_nB)y_n - (I - r_nB)\bar{x} + e_n)\rangle \\
&\leq \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\left[\|y_n - \bar{x}\|^2 + 2\|e_n\|\|(I - r_nB)y_n - (I - r_nB)\bar{x} + e_n\|\right].
\end{aligned} \tag{3.25}$$

Consider

$$\begin{aligned}
\|y_n - \bar{x}\|^2 &= \langle \alpha_n f(x_n) + (1-\alpha_n)x_n - \bar{x}, j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n(f(x_n) - \bar{x}) + (1-\alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n\left(f(x_n) - f(\bar{x})\right) + \alpha_n\left(f(\bar{x}) - \bar{x}\right) + (1-\alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n(f(x_n) - f(\bar{x})) + (1-\alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle + \langle \alpha_n(f(\bar{x}) - \bar{x}), j(y_n - \bar{x})\rangle \\
&\leq \|\alpha_n(f(x_n) - f(\bar{x})) + (1-\alpha_n)(x_n - \bar{x})\|\|y_n - \bar{x}\| + \alpha_n\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&\leq [\alpha_n k\|x_n - \bar{x}\| + (1-\alpha_n)\|x_n - \bar{x}\|]\,\|y_n - \bar{x}\| + \alpha_n\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&= [1 - \alpha_n(1-k)]\|x_n - \bar{x}\|\|y_n - \bar{x}\| + \alpha_n\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&= (1 - \alpha_n(1-k))\frac{\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2}{2} + \alpha_n\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle
\end{aligned}$$

$$= \frac{1-\alpha_n(1-k)}{2}\left(\|x_n-\bar{x}\|^2+\|y_n-\bar{x}\|^2\right)+\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle.$$

It follows that

$$\begin{aligned} & 2\|y_n-\bar{x}\|^2 \\ \leq\ & (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+(1-\alpha_n(1-k))\|y_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle \\ \leq\ & (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+\|y_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle. \end{aligned} \tag{3.26}$$

Therefore, we obtain

$$\|y_n-\bar{x}\|^2 \leq (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle. \tag{3.27}$$

Replace (3.27) in (3.25) that

$$\begin{aligned} & \|x_{n+1}-\bar{x}\|^2 \\ \leq\ & \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)\left[(1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle\right] \\ & +(1-\beta_n)2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\| \\ =\ & (1-\alpha_n(1-k)(1-\beta_n))\|x_n-\bar{x}\|^2+2\alpha_n(1-\beta_n)\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle \\ & +2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\| \\ =\ & (1-\lambda_n)\|x_n-\bar{x}\|^2+\frac{2\lambda_n}{(1-k)}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle+c_n, \end{aligned}$$

where $c_n := 2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\|$, and $\lambda_n=\alpha_n(1-k)(1-\beta_n)$.

If we set $b_n=\frac{2}{(1-k)}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle$ and we have $\limsup_{n\longrightarrow\infty}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle\leq 0$, then we see that $\limsup_{n\to\infty} b_n\leq 0$, and also that $\sum_{n=0}^{\infty}c_n<\infty$.

By lemma 2.8 and condition (a), (b), and (d), we conclude that $\|x_n-\bar{x}\|^2\longrightarrow 0$, as $n\longrightarrow\infty$. This implies

$$\lim_{n\longrightarrow\infty}\|x_n-\bar{x}\|=0,$$

i.e., x_n converges strongly to $\bar{x}$. □

Next, we will utilize theorem 3.2 to study some strong convergence theorem in L_p with $2\leq p<\infty$. Since L_p, where $p\geq 2$ are uniformly convex and 2-uniformly smooth Banach space with $K=p-1$, then we consider $E=L_p$ and we derive that following theorem:

Theorem 3.3. *Let C be a nonempty closed convex subset of an L_p for $2\leq p<\infty$. Let A, B, S, f, $J_{r_n}^A$ be the same as in theorem 3.2. Let $\{\alpha_n\}$, $\{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0,\frac{\alpha}{(p-1)^2})$ and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions (a),(b) and (d) in theorem 3.2 and conditions (c) $\lim_{n\longrightarrow\infty}r_n=r$, and $r\in(0,\frac{\alpha}{(p-1)^2})$. Then the sequence $\{x_n\}$ is defined by (4.1) converges strongly to a point $\bar{x}\in Fix(S)\cap(A+B)^{-1}(0)$.*

Consider a mapping $S\equiv I$ in theorem 3.2, we can obtain the following corollary direct.

Corollary 3.4. *Let E be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let C be a nonempty closed convex subset of E. Let $A : D(A)\subseteq E\longrightarrow 2^E$ be an m-accretive operator such that the domain of A is included in C and $B : C\longrightarrow X$ be an α-inverse strongly accretive* *operator. Let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0,1)$. Let $J^A_{r_n} = (I + r_nA)^{-1}$ be a resolvent of A for $r_n > 0$ such that $(A+B)^{-1}(0) \neq \emptyset$.*

For given $x_0 \in C$, Let x_n be a sequence in the following process:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)J^A_{r_n}(y_n - r_nBy_n + e_n), \ \forall n \geq 0, \end{cases} \tag{3.28}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$;
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0)$.

Consider a mapping $S \equiv I$ and $f(x_n) \equiv u$, $\forall n \in \mathbb{N}$ in theorem 3.2, we obtain the following corollary direct.

Corollary 3.5. *Let E be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let C be a nonempty closed convex subset of E. Let $A : D(A) \subseteq E \longrightarrow 2^E$ be an m-accretive operator such that the domain of A is included in C and let $B : C \longrightarrow X$ be an α-inverse strongly accretive operator. Let $J^B_{r_n} = (I + r_nB)^{-1}$ be a resolvent of B for $r_n > 0$ such that $(A+B)^{-1}(0) \neq \emptyset$.*

For given $x_0 \in C$, Let x_n be a sequence in the following process:

$$\begin{aligned} & y_n = \alpha_n u + (1-\alpha_n)x_n, \\ & x_{n+1} = \beta_n x_n + (1-\beta_n)J^A_{r_n}(y_n - r_nBy_n + e_n), \ \forall n \geq 0, \end{aligned} \tag{3.29}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequence satisfy the following conditions:

(a) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(b) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$;
(c) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0)$.

Setting $J^A_{r_n} \equiv I$, $B \equiv 0$, $f(x_n) \equiv u$, $\forall n \in \mathbb{N}$ and $e_n \equiv 0$, then we have the following corollary of the modified Mann-Halpern iteration.

Corollary 3.6. *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(S) \neq \emptyset$. For given $x_0, u \in C$, Let x_n be a sequence in the following process:*

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, \ \forall n \geq 0, \end{cases} \tag{3.30}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$. Assume that the control sequence satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S)$.

4. Some applications

In this section, we give two applications of our main results in the framework of Hilbert spaces. Now, we consider theorem 3.2, in the framework of Hilbert spaces, it known that $K = \frac{\sqrt{2}}{2}$. Let H be a Hilbert space and let C be a nonempty closed convex subset of H.

Theorem 4.1. [6, Corollary 2.2] *Let $A : C \longrightarrow 2^H$ be a maximal monotone operators such that the domain of B which included in C and $B : C \longrightarrow H$ be an α-inverse strongly monotone operator. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0, 1)$. Let $J^A_{r_n} = (I+r_nA)^{-1}$ be a resolvent of A for $r_n > 0$ such that $Fix(S)\cap(A+B)^{-1}(0) \neq \emptyset$.*

For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)SJ^A_{r_n}(y_n - r_nBy_n + e_n), \ \forall n \geq 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a real number sequences in $(0, 2\alpha)$ and $\{e_n\}$ is a sequence in H. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$*;*
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$*;*
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A + B)^{-1}(0)$. Next, we will give some related results.

4.1. Application to projection for variational inequality.

Let C be a nonempty, close and convex subset of a Hilbert space H. The metric projection of a point $x \in H$ onto C, denoted by $P_C(x)$, is defined as the unique solution of the problem

$$\|x - P_Cx\| \leq \|x - y\|, \ \forall y \in C, \ \forall x \in H.$$

For each $x \in H$ and $z \in C$, the metric projection P_C is satisfied

$$z = P_C(x) \Longleftrightarrow \langle y - z, x - z\rangle \leq 0, \ \forall y \in C. \tag{4.2}$$

Note that the metric projection is nonexpansive mapping.

Let $g : H \longrightarrow (-\infty, \infty]$ is a proper convex lower semicontinuous function. Then the subdifferential ∂g of g is defined as follow:

$$\partial g(x) = \{z \in H : g(y) - g(x) \geq \langle y - x, z\rangle, \ \forall y \in H\},$$

for all $x \in H$. If $g(x) = \infty$, then $\partial g(x) \neq \emptyset$, Takahashi [16] claim that ∂g is m-accretive operator. Since we know that, an m-accretive operator is maximal monotone operators in a Hilbert space, then we claim that ∂g is maximal monotone operators. Then we define the set of minimizers of g as follow:

$$argmin_{y\in H} g(y) = \{z \in H : g(z) = min_{y\in H} g(y)\}.$$

It is easy to verify that $0 \in \partial g(x)$ if and only if $g(z) = min_{y\in H} g(y)$. Let i_C be the indicator function of C by

$$i_C(x) = \begin{cases} 0, & \forall x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Then i_C is a proper lower semicontinuous convex function on H. So, we see that the subdifferential ∂i_C of i_C is maximal monotone operator; see, [16]. The resolvent J_r of ∂i_C for $r > 0$, that is $J_r x = (I + r\partial i_C)^{-1}x, \ \forall x \in H$. Next, we recall that set $N_C(u)$ is called the normal cone of C at u define by

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \leq 0, \ \forall y \in C\}.$$

Since $N_C(u) = \partial i_C(u)$. In fact, we have that for any $x \in H$ and $u \in C$,

$$\begin{aligned} u = J_r x = (I + r\partial i_C)^{-1}x &\iff x \in u + r\partial i_C u \\ &\iff x \in u + rN_C(u) \\ &\iff x - u \in rN_C(u) \\ &\iff \frac{1}{r}\langle x - u, y - u\rangle \leq 0, \ \forall y \in C \\ &\iff \langle x - u, y - u\rangle \leq 0, \ \forall y \in C \\ &\iff u = P_C x. \end{aligned} \tag{4.3}$$

Then $u = (I + r\partial i_C)^{-1}x \iff u = P_C x, \ \forall x \in H, \ u \in C$.

Now, we consider the following variational inequality problem (VIP) for B is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \ \forall y \in C. \tag{4.4}$$

The set of solutions of (4.4) is denoted by $VI(C, B)$.

$$VI(C, B) = \left\{ \ x \in C : \langle Bx, y - x\rangle \geq 0, \ \forall y \in C \ \right\}. \tag{4.5}$$

Theorem 4.2. *Let $B : C \longrightarrow H$ be an α-inverse strongly monotone mapping. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0, 1)$. Assume that $Fix(S) \cap VI(C, B) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)SP_C(y_n - r_n By_n + e_n), \ \forall n \geq 0, \end{cases} \tag{4.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a real number sequences in $(0, 2\alpha)$ and $\{e_n\}$ is a sequence in H. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, 2\alpha)$;
(d) $\sum_{n=0}^{\infty} \parallel e_n \parallel < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap VI(C, A)$, where $\bar{x} = P_{Fix(S)\cap VI(C,B)} f(\bar{x})$.

Proof. By lemma 2.4 we know that $Fix(J_r^A(I - rB)) = (A + B)^{-1}(0)$. Put $A = \partial i_C$, and we to show that $VI(C, B) = (\partial i_C + B)^{-1}(0)$. Note that

$$\begin{aligned} x \in (\partial i_C + B)^{-1}(0) &\iff 0 \in \partial i_C x + Bx \\ &\iff 0 \in N_C x + Bx \\ &\iff -Bx \in N_C x \\ &\iff \langle -Bx, y - x\rangle \leq 0 \\ &\iff \langle Bx, y - x\rangle \geq 0 \\ &\iff x \in VI(C, B). \end{aligned} \tag{4.7}$$

From (4.3), therefore, we can conclude the desired conclusion immediately. □

4.2. **Application for equilibrium problems.** Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for finding $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C. \tag{4.8}$$

The set of solutions of (4.8) is denoted by $EP(F)$.

For solving the equilibrium problem, we assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t \longrightarrow 0^+} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 4.3. [17] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $z \in H$. Then, there exists $x \in C$ such that*

$$F(x, y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \ \forall y \in C. \tag{4.9}$$

Lemma 4.4. [18] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $z \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(z) = \{x \in C : F(x, y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \ \forall y \in C\}, \forall z \in H. \tag{4.10}$$

Then, the following hold:
(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) $Fix(T_r) = EP(F)$;
(4) $EP(F)$ is closed and convex.

Lemma 4.5. [19] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4) and A_F be a multi-valued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, & \forall x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then $EP(F) = A_F^{-1}(0)$ and $A_F x$ is a maximal monotone operator with the domain $D(A_F) \subset C$. Furthermore, the resolvent T_r of F coincides with the resolvent of A_F, i.e.,

$$T_r x = (I + rA_F)^{-1}(x), \ \forall x \in H, \ r > 0, \tag{4.11}$$

where T_r is defined as in (4.10)

We recalled that T_r is the resolvent of A_F for $r > 0$. Since $A = A_F$, we will show that $J_r x = T_r x$. Indeed, for $x \in H$, we have

$$\begin{aligned} z \in J_r x = (I + rA_F)^{-1}(x) &\iff x \in (I + rA_F)z \\ &\iff x \in z + rA_F z \\ &\iff x - z \in rA_F z \\ &\iff \frac{1}{r}(x - z) \in A_F z \end{aligned}$$

$$\begin{aligned}
&\iff F(z,y) \geq \langle y-z, \frac{1}{r}(x-z)\rangle \\
&\iff F(z,y) \geq \langle y-z, \frac{-1}{r}(z-x)\rangle \\
&\iff F(z,y) \geq \frac{-1}{r}\langle y-z, z-x\rangle \\
&\iff F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \forall y \in C \\
&\iff z \in T_r x. \qquad (4.12)
\end{aligned}$$

Using lemmas 4.3, 4.4, 4.5 and theorem 4.1, we also obtain the following result.

Theorem 4.6. *Let $F : C \times C \longrightarrow \mathbb{R}$ which satisfies $(A1)-(A4)$. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0,1)$. Assume that $Fix(S) \cap EP(F) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)ST_{r_n}(y_n + e_n),\ \forall n \geq 0, \end{cases} \qquad (4.13)$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, 2\alpha)$ and $\{e_n\}$ is a sequence in H.

Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, 2\alpha)$;
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap EP(F)$, where $\bar{x} = P_{Fix(S)\cap EP(F)} f(\bar{x})$.

Proof. Put $A \equiv A_F$ and $B \equiv 0$ in $(A+B)^{-1}(0)$ from theorem 4.1. Furthermore, for bifunction $F : C \times C \longrightarrow \mathbb{R}$, we define $A_F x$ as in lemma 4.5, we have $EP(F) = A_F^{-1}(0)$ and let T_{r_n} be the resolvent of A_F for $r_n > 0$. Therefore, we can conclude the desired conclusion immediately. □

5. Conclusion and remarks

Our main results extends and improves in the following:

(i) Theorem 3.2 extends and improves Theorem 3.1 of Manaka and Takahashi [4, Theorem 3.1] from a Hilbert space to a Banach space and from weak convergence to strong convergence.

(ii) Theorem 3.2 partially extends and improves Theorem 2.1 of Cho et al. [6, Theorem 2.1] from a Hilbert space to a Banach space with uniformly convex and 2-uniformly smooth.

(iii) Theorem 3.2 extends and improves Theorem 3.1 of Qing and Cho [20, Theorem 3.1] from the problems of finding an element of $A^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

(iv) Theorem 3.2 extends and improves Theorem 3.7 of Sahu and Yao [3, Theorem 3.7] from the problems of finding an element of $A^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

(v) Theorem 3.2 extends and improves Theorem 3.7 of López et al. [5, Theorem 3.7] from the problems of finding an element of $(A+B)^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

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VISCOSITY ITERATIVE SCHEME FOR SPLIT FEASIBILITY PROBLEMS

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ABSTRACT. In this paper, we intend to solve a split feasibility problem by viscosity iterative algorithm. The bounded perturbation resilience of the method is examined in Hibert spaces. As tools, averaged mappings and resolvents of maximal monotone operators are specialized procedure to simplify the proofs of the main results. Under mild conditions, we prove that our algorithms converge to a solution of the split feasibility problem. Moreover, we show the convergence and result of the algorithms by a numerical example.

KEYWORDS: Viscosity iterative algorithm, Split feasibility problem, Maximal monotone operator.
AMS Subject Classification: 47H09, 47H10.

1. Introduction

Let C and Q be nonempty closed convex subsets in real Hibert space H_1 and H_2,respectively. Let P_C be the metric projection from H_1 onto C and P_Q be the metric projection from H_2 onto Q. The problem to find

$$u^* \in C \quad with \quad Au^* \in Q \tag{1.1}$$

where A is a bounded linear operator from H_1 to H_2, if such u^* exist, this problem is called the split feasibility problem(see [1]). If problem (1.1) has a solution (say that $C \cap A^{-1}Q$ is nonempty). $u^* \in C \cap A^{-1}Q$ is equivalent to

$$u^* = P_C(I - \lambda A^*(I - P_Q)A)u^*, \tag{1.2}$$

where $\lambda > 0$ and A^* is the adjoint operator of A.

The SFP was first introduced by Censor and Elfving [2] in 1994. They used their multidistance method to obtain iterative algorithms for solving the SFP. After that,

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Byrne [3] proposed his CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I - \lambda A^*(I - P_Q)A)x_n, \quad \forall n \geq 0. \tag{1.3}$$

Let $B : H_1 \to 2^{H_1}$ be a mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for all $\lambda > 0$. Let $T : H_2 \to H_2$ be nonexpansive mapping.

In 2015, Takahashi et al. [4] proposed the following algorithm:

$$x_{n+1} = J^B_{\gamma_n}(x_n - \lambda_n A^*(I - T)Ax_n), \tag{1.4}$$

where $\{\lambda_n\}$ is a sequence in $[0, 1]$ and they proved that the sequence $\{u_n\}$ converges weakly to a point $u^* \in B^{-1}0 \cap A^{-1}Fix(T)$ in the framework of Hilbert spaces. That is this problems is to find a point $u^* \in H_1$ such that

$$0 \in Bu^* \quad and \quad Au^* \in Fix(T). \tag{1.5}$$

The set of all solution (1.5) denoted by $\Gamma = B^{-1}0 \cap A^{-1}Fix(T)$.
there are many authers have studied the SFP and its extensions by means of fixed-point methods and weak-strong convergence theorems of solutions have been established in Hilbert or Banach spaces (see [5, 6, 7, 8, 9, 10]).

Let F be an algorithm operator. Let $\{x_n\}$ be a sequence, generated by $x_{n+1} = Fx_n$, and let $\{y_n\}$ be a sequence, generated by $y_{n+1} = F(y_n + \beta_n v_n)$, where $\{\beta_n\}$ is a sequence of nonnegative real numbers and $\{v_n\}$ is a sequence in H such that

$$\sum_{n=0}^{\infty} \beta_n < \infty \quad and \quad \|v_n\| \leq M, \quad \forall n \geq 0. \tag{1.6}$$

An algorthmic operator F is call bounded perturbation resilient if the following is ture: if the sequence $\{x_n\}$ is convergent, then $\{y_n\}$ is also convergent (see [11]).

In 2017, Xu [12] presented the bounded perturbation resilience and superiorization techniques for the projected scaled gradient(PSG).The iterative method is defined as following:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C(x_n - \lambda_n D(x_n)\nabla h(x_n) + e(x_n)), \quad \forall n \geq 0 \tag{1.7}$$

where $\{\lambda_n\}$, $\{\alpha_n\}$ are a sequence in $[0, 1]$, h is a continuous differentiable and convex function, and $D(x_n)$ is a diagonal scaling matrix. The weak convergence was proved in [12].

In 2018, Guo and Chi [13] proposed the following proximal gradient algorithm with perturbations:

$$x_{n+1} = t_n f(x_n) + (1 - t_n)prox_{\lambda_n g}(1 - \lambda_n \nabla h)x_n + e(x_n), \tag{1.8}$$

where $\{\lambda_n\}$, $\{t_n\}$ are a sequence in $[0, 1]$ and f is a contractive, for solving non-smooth composite convex optimization problem. They obtained strong convergence and bounded resilience of the above method.

In 2019, Duan and Zheng [14] presented a viscosity approximation method for solving problem (1.5):

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J^B_{\lambda_n}(x_n - \tau_n A^*(1 - T)Ax_n + e(x_n)), \tag{1.9}$$

where $\{\lambda_n\}$, $\{\tau_n\}$, $\{\alpha_n\}$ are a sequence in $[0,1]$ and they gave the bounded perturbation of (1.9) yields a sequence $\{x_n\}$ generated by the iterative process:

$$\begin{aligned} y_n &= x_n + \beta_n v_n \\ x_{n+1} &= \alpha_n f(y_n) + (1-\alpha_n) J^B_{\lambda_n}(y_n - \tau_n A^*(I-T)Ay_n + e(y_n)), \end{aligned} \tag{1.10}$$

where $\{\lambda_n\}$, $\{\tau_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are a sequence in $[0,1]$

In this paper, we extend work in [14] and purpose the following process for solving problem (1.5) :

$$x_{n+1} = \alpha_n f(x_n + e(x_n)) + (1-\alpha_n) J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n + e(x_n)), \tag{1.11}$$

where $\{\lambda_n\}$, $\{\tau_n\}$, $\{\alpha_n\}$, are a sequence in $[0,1]$, f is contractive, and we give a sequence $\{x_n\}$ generated by the iterative process:

$$\begin{aligned} y_n &= x_n + \beta_n v_n \\ x_{n+1} &= \alpha_n f(y_n + e(y_n)) + (1-\alpha_n) J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)), \end{aligned} \tag{1.12}$$

where $\{\lambda_n\}$, $\{\tau_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are a sequence in $[0,1]$ and f is contractive.

After that we prove the convergence point of the iterative method which is also the unique solution of some variational inequality problem. A numerical example is also given to demonstrate the effectiveness of our iterative schemes.

2. Preliminaries

Let $\{x_n\}$ be a sequence in a real Hilbert space H. First, We give notations:

- Denote $\{x_n\}$ converging weakly to x by $x_n \rightharpoonup x$ and $\{x_n\}$ converging strongly to x by $x_n \to x$.
- Denote the set of fixed points of mapping T by $Fix(T) = \{x \in H : Tx = x\}$
- Denotetheweak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$.

Definition 2.1. A mapping $F : H \to H$ is said to be

(i) Lipschizian if there exist a positive constant L such that

$$||Fx - Fy|| \leq L||x-y||, \quad \forall x, y \in H.$$

In particular, if $L = 1$, we say that F is nonexpansive, namely,

$$||Fx - Fy|| \leq L||x-y||, \quad \forall x, y \in H,$$

if $L \in [0,1)$, we say that F is contractive.

(ii) α-averaged mapping (α-av for short) if

$$F = (1-\alpha)I + \alpha T,$$

where $\alpha \in [0,1)$ and $T : H \to H$ is nonexpansive.

Definition 2.2. A mapping $B : H \to H$ is said to be

(i) monotone if

$$\langle Bx - By, x - y\rangle \leq 0, \quad \forall x, y \in H.$$

(ii) $\eta-$strongly monotone if there exists a posive constant eta such that

$$\langle Bx - By, x - y\rangle \geq \eta||x-y||^2, \quad \forall x, y \in H.$$

(iii) α-inverse strongly monotone (for short α-ism) if there exist a positive constant α such that

$$\langle Bx - By, x - y\rangle \geq \alpha ||Bx - By||^2, \quad \forall x, y \in H.$$

In paticular, if $\alpha = 1$, we say that B is firmly nonexpansive, namely,

$$\langle Bx - By, x - y\rangle \geq ||Bx - By||^2, \quad \forall x, y \in H.$$

Definition 2.3. Let $B : H \to H$ be amonotone mapping. Then B is maximal monotone if there exists no monotone operater $A : H \to 2^H$ such that $graA$ properly contains $graB$,i.e. for every $(x, u) \in H \times H$,

$$(x, u) \in graB \Leftrightarrow \forall (y, v) \in graB, \langle x - y, u - v\rangle \geq 0.$$

Lemma 2.4. *Let H be a real Hillbert space. Theere holds the following inequality*

$$||x + y||^2 \leq ||x||^2 + 2\langle x + y, y\rangle, \quad \forall x, y \in H.$$

Lemma 2.5. *Let $f : H \to H$ be a $k \in (0, 1)$ and let $T : H \to H$ be a nonexpansive mapping.Then*

(i) *$I - f$ is (1-k)-strongly monotone:*

$$\langle (I - f)x - (I - f)y, x - y\rangle \geq (1 - \rho)||x - y||^2, \quad \forall x, y \in H.$$

(ii) *$I - T$ is monotone:*

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

Proposition 2.6. [4] *Assume that H_1 and H_2 are Hilbert space. Let $B : H_1 \to 2^H_1$ be a maximal monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then*

(i) *$A^*(I - T)A$ is $\frac{1}{2||A||^2}$-ism.*

(ii) *For $0 < \tau < \frac{1}{2||A||^2}$,*
$I - \tau A^*(I - T)A$ is $\tau||A||^2$- averaged and $J^B_\lambda(I - \tau A^*(I - T)A)$ is $\frac{1+\tau||A||^2}{2}$ averaged.

Lemma 2.7. [15] *Let B be a maximal monotone operator. Let $J^B_\gamma = (I + \gamma B)^{-1}$ and $J^B_\lambda = (I + \lambda B)^{-1}$, where $\gamma > 0$ and $\lambda > 0$ are two real numbers, be the resolvent operators of B. Then*

$$J^B_\gamma x = J^B_\lambda(\frac{\lambda}{\gamma}x + (1 - \frac{\lambda}{\gamma})J^B_\gamma x), \quad \forall x \in H.$$

Lemma 2.8. [16] *Let H be a real Hillbert space, and let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq 0$.if $\{x_n\}$ is a sequence in H weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y,then $(I - T) = y$; in particulary, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.9. [17] *Assume $\{\sigma_n\}$ is a sequence of nonnegative real numbers such that*

$$\sigma_{n+1} \leq (1 - \rho_n)\sigma_n + \rho_n\delta_n, \quad n \geq 0,$$
$$\sigma_{n+1} \leq \sigma_n - \varphi_n + \phi_n, \quad n \geq 0,$$

where $\{\rho_n\}$ is a sequence in $(0, 1)$, $\{\varphi_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\phi_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \rho_n = \infty$;

(ii) $\lim_{n\to\infty} \phi_n = 0$;

(iii) $\lim_{k\to\infty} \varphi_{n_k} = 0 \Rightarrow \limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence $(n_k) \subset (n)$.*

Then $\lim_{n\to\infty}\sigma_n = 0$.

Lemma 2.10. [4] *Let H_1 and H_2 be Hilbert space. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda^B = (I + \lambda B)^{-1}1$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $T : H_1 \to H_2$ be a bounded linear operater. Suppose that $B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. Let $\lambda, \tau > 0$. Then the following equality holds:*

$$Fix(J_\lambda^B(I - \tau A^*(I - T)A)) = (A^*(I - T)A + B)^{-1}0 = B^{-1}0 \cap A^{-1}Fix(T)$$

3. Main Results

In [1] proposed the viscosity approximation method:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)S(x_n), \quad \forall n \geq 0,$$

which converges strongly to a fixed point u^* of the nonexpansive mapping S. In [7] further proved that $u^* \in Fix(S)$ is also the unique solution of the following variational inequality problem:

$$\langle (I - f)u^*, \hat{u} - u^* \rangle \geq 0, \quad \forall \hat{u} \in Fix(S), \tag{3.1}$$

where $f : H \to H$ is a k-contraction.

In this section, we present a viscosity iterative algorithm for solving problem (1.5). Rewrite iteration (1.11) as

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n + e(x_n)) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ &= \alpha_n f(x_n) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n) + \widehat{e}_n, \quad \forall n \geq 0, \end{aligned}$$

where

$$\begin{aligned} \widehat{e}_n = {} & \alpha_n(f(x_n + e(x_n)) - f(x_n)) + (1 - \alpha_n)(J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ & - J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n)). \end{aligned}$$

Since $J_{\gamma_n}^B$ is nonexpansive and f is contractive, it is easy to get

$$\begin{aligned} \|\hat{e}_n\| &\leq \alpha_n\|f(x_n + e(x_n)) - f(x_n)\| + (1 - \alpha_n)\|J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ &\quad - J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n)\| \\ &\leq \alpha_n k\|e(x_n)\| + (1 - \alpha_n)\|e(x_n)\| \\ &= (\alpha_n k + 1 - \alpha_n)\|e(x_n)\| \\ &\leq \|e(x_n)\|. \end{aligned}$$

Theorem 3.1. *Let H_1, H_2 be two real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $B : H_1 \to 2^{H_1}$ is a maximal monotone operator and $T : H_2 \to H_2$ is a nonexpansive mapping. Assume that $\Gamma = B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. Let f be a k-contractive on H_1 with $0 \leq k < 1$. Choose $x_0 \in H_1$ arbitrarily and define a sequence $\{x_n\}$ in the following manner:*

$$x_{n+1} = \alpha_n f(x_n + e(x_n)) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \tag{3.2}$$

if the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty}\gamma_n \leq \limsup_{n\to\infty}\gamma_n < \infty$;
(iii) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < \frac{1}{L}$;
(iv) $\sum_{n=0}^{\infty}\|e(x_n)\| < \infty$.

Then $\{x_n\}$ *converges strongly to* $u^* \in \Gamma$, *which is also the unique solution of variational inequality problem* (3.1).

Proof. Let $V_{\lambda_n} = J^B_{\gamma_n}((I - \lambda_n A^*(I-T)A)$. From Proposition 2.6, it follows that $J^B_{\gamma_n}((I - \lambda_n A^*(I-T)A)$ is $\frac{1+\lambda_n L}{2} - av$ as $0 < \lambda_n < \frac{1}{L}$.
Step 1. show that $\{x_n\}$ is bounded. For any $u^* \in \Gamma$, we have

$$
\begin{aligned}
&\|x_{n+1} - u^*\| \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n + \widehat{e}_n - u^*\| \\
&= \|\alpha_n f(x_n) + V_{\lambda_n}x_n - \alpha_n V_{\lambda_n}x_n + \widehat{e}_n - u^*\| \\
&= \|(\alpha_n f(x_n) - \alpha_n u^*) + (V_{\lambda_n}x_n - u^*) - (\alpha_n V_{\lambda_n}x_n - \alpha_n u^*) + \widehat{e}_n\| \\
&= \|\alpha_n (f(x_n) - u^*) + (1-\alpha_n)(V_{\lambda_n}x_n - u^*) + \widehat{e}_n\| \\
&\le \alpha_n\|f(x_n) - u^*\| + (1-\alpha_n)\|(V_{\lambda_n}x_n - u^*)\| + \|\widehat{e}_n\| \\
&= \alpha_n\|f(x_n) - f(u^*) + f(u^*) - u^*\| + (1-\alpha_n)\|(V_{\lambda_n}x_n - u^*)\| + \|\widehat{e}_n\| \\
&\le \alpha_n\|f(x_n) - f(u^*)\| + \alpha_n\|f(u^*) - u^*\| + (1-\alpha_n)\|(V_{\lambda_n}x_n - u^*)\| + \|\widehat{e}_n\| \\
&= \alpha_n\|f(x_n) - f(u^*)\| + \alpha_n\|f(u^*) - u^*\| + (1-\alpha_n)\|(V_{\lambda_n}x_n - V_{\lambda_n}u^*)\| + \|\widehat{e}_n\| \\
&\le \alpha_n k\|x_n - u^*\| + \alpha_n\|f(u^*) - u^*\| + (1-\alpha_n)\|x_n - u^*\| + \|\widehat{e}_n\| \\
&= (1 - \alpha_n + \alpha_n k)\|x_n - u^*\| + \alpha_n(\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}) \\
&= (1 - \alpha_n(1-k))\|x_n - u^*\| + \alpha_n(1-k)\left[\frac{\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}}{1-k}\right].
\end{aligned}
$$

From condition (i), (iv) and $\alpha_n > 0$, we get $\left\{\frac{\|\widehat{e}_n\|}{\alpha_n}\right\}$ is bounded.Thus there exists $M_1 > 0$ such that $\sup\left\{\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}\right\} \le M_1$, for all $n \ge 0$. By Mathematical Induction, we get $\|x_n - u^*\| \le \max\left\{\|x_0 - u^*\|, \frac{M_1}{1-k}\right\}$, which implies that the sequence $\{x_n\}$ is bounded, so are $\{f(x_n)\}, \{V_{\lambda_n}x_n\}$ and $\{A^*(I-T)Ax_n\}$.
Step 2. Show that for any sequence$\{n_k\} \subset \{n\}$,

$$\lim_{n\to\infty} \|x_{n_k} - V_{\lambda_{n_k}} x_{n_k}\| = 0.$$

Fixing $u^* \in \Gamma$, we have

$$
\begin{aligned}
&\|x_{n+1} - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n + \widehat{e}_n - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*, \widehat{e}_n\rangle + \|\widehat{e}_n\|^2 \\
&= \|\alpha_n(f(x_n) - u^*) + (1-\alpha_n)(V_{\lambda_n}x_n - u^*)\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*, \widehat{e}_n\rangle + \|\widehat{e}_n\|^2 \\
&\le \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle \\
&\quad + 2\|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|\|\widehat{e}_n\| + \|\widehat{e}_n\|^2 \\
&= \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\|\alpha_n(f(x_n)-u^*)+(1-\alpha_n)(V_{\lambda_n}x_n-u^*)\|\|\widehat{e}_n\|+\|\widehat{e}_n\|^2\\
&\le \alpha_n^2\|f(x_n)-u^*\|^2+(1-\alpha_n)^2\|V_{\lambda_n}x_n-u^*\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n)-u^*, V_{\lambda_n}x_n-u^*\rangle\\
&\quad + 2\Big[\|\alpha_n(f(x_n)-u^*)\|+\|(1-\alpha_n)(V_{\lambda_n}x_n-u^*)\|\Big]\|\widehat{e}_n\|+\|\widehat{e}_n\|^2\\
&= \alpha_n^2\|f(x_n)-u^*\|^2+(1-\alpha_n)^2\|V_{\lambda_n}x_n-u^*\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n)-u^*, V_{\lambda_n}x_n-u^*\rangle\\
&\quad + \Big[2\alpha_n\|(f(x_n)-u^*)\|+2(1-\alpha_n)\|V_{\lambda_n}x_n-u^*\|+\|\widehat{e}_n\|\Big]\|\widehat{e}_n\|\\
&\le \alpha_n^2\|f(x_n)-u^*\|^2+(1-\alpha_n)^2\|V_{\lambda_n}x_n-u^*\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n)-u^*, V_{\lambda_n}x_n-u^*\rangle\\
&\quad + \Big(2\alpha_n\|f(x_n)-u^*\|+2(1-\alpha_n)\|x_n-u^*\|+\|\widehat{e}_n\|\Big)\|\widehat{e}_n\|\\
&\le 2\alpha_n^2\Big(\|f(x_n)-f(u^*)\|^2+\|f(u^*)-u^*\|^2\Big)+(1-\alpha_n)^2\|V_{\lambda_n}x_n-u^*\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n)-u^*, V_{\lambda_n}x_n-u^*\rangle+M_2\|\widehat{e}_n\|\\
&\le 2\alpha_n^2\Big(\|f(x_n)-f(u^*)\|^2+\|f(u^*)-u^*\|^2\Big)+(1-\alpha_n)^2\|V_{\lambda_n}x_n-u^*\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)\Big(\|f(x_n)-f(u^*)\|\|x_n-u^*\|+\langle f(u^*)-u^*, V_{\lambda_n}x_n-u^*\rangle\Big)+M_2\|\widehat{e}_n\|\\
&\le 2\alpha_n^2k\|x_n-u^*\|^2+2\alpha_n^2\|f(u^*)-u^*\|^2+(1-\alpha_n^2)\|x_n-u^*\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)k\|x_n-u^*\|^2+2\alpha_n(1-\alpha_n)\langle f(u^*)-u^*, V_{\lambda_n}x_n-u^*\rangle+M_2\|\widehat{e}_n\|\\
&= \Big(2\alpha_n^2k+(1-\alpha_n)^2+2\alpha_n(1-\alpha_n)k\Big)\|x_n-u^*\|^2+2\alpha_n^2\|f(u^*)-u^*\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(u^*)-u^*, V_{\lambda_n}x_n-u^*\rangle+M_2\|\widehat{e}_n\|\\
&= (1-\alpha_n(2-\alpha_n(1+2k^2)-2(1-\alpha_n)k))\|x_n-u^*\|^2\\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(u^*)-u^*, V_{\lambda_n}x_n-u^*\rangle+2\alpha_n^2\|f(u^*)-u^*\|^2+M_2\|\widehat{e}_n\|, \quad (3.3)
\end{aligned}
$$

where

$$M_2=\sup_{n\in\mathbb{N}}\Big\{2\alpha_n\|f(x_n)-u^*\|+2(1-\alpha_n)\|x_n-u^*\|+\|\widehat{e}_n\|\Big\}.$$

Note that

$$V_{\lambda_n}=J^B_{r_n}(I-\lambda_nA^*(I-T)A)=(1-w_n)I+w_nU_n, \tag{3.4}$$

such that $w_n=\frac{1+\lambda_nL}{2}$, and U_n is nonexpansive. By condition (iii), we get

$$\frac{1}{2}<\liminf_{n\to\infty}w_n\le\limsup_{n\to\infty}w_n<1$$

Since $u^*\in\Gamma$, then $V_{\lambda_n}u^*=u^*$. Furthermore, we have $(1-w_n)u^*+w_nU_nu^*=u^*$. It is clear that $U_nu^*=u^*$.

$$
\begin{aligned}
&\|x_{n+1}-u^*\|^2\\
&= \|\alpha_nf(x_n)+(1-\alpha_n)V_{\lambda_n}x_n+\widehat{e}_n-u^*\|^2\\
&= \|\alpha_nf(x_n)+(1-\alpha_n)V_{\lambda_n}x_n-u^*\|^2\\
&\quad + 2\langle\alpha_nf(x_n)+(1-\alpha_n)V_{\lambda_n}x_n-u^*, \widehat{e}_n\rangle+\|\widehat{e}_n\|^2\\
&\le \|\alpha_nf(x_n)+(1-\alpha_n)V_{\lambda_n}x_n-u^*\|^2\\
&\quad + 2\|\alpha_nf(x_n)+(1-\alpha_n)V_{\lambda_n}x_n-u^*\|\|\widehat{e}_n\|+\|\widehat{e}_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + \Big(2\alpha_n\|f(x_n)-u^*\| + 2(1-\alpha_n)\|x_n-u^*\| + \|\widehat{e}_n\|\Big)\|\widehat{e}_n\| \\
&\le \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 + M_2\|\widehat{e}_n\| \\
&= \|V_{\lambda_n}x_n - u^* + \alpha_n(f(x_n) - V_{\lambda_n}x_n)\|^2 + M_2\|\widehat{e}_n\| \\
&= \|V_{\lambda_n}x_n - u^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|(1-w_n)x_n + w_nU_nx_n - u^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|x_n - w_nx_n + w_nU_nx_n - (1-w_n)u^* - w_nU_nu^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|(1-w_n)(x_n-u^*) + w_n(U_nx_n - U_nu^*)\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= (1-w_n)\|x_n-u^*\|^2 + w_n\|U_nx_n - U_nu^*\|^2 - w_n(1-w_n)\|U_nx_n - x_n\|^2 \\
&\quad + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&\le \|x_n-u^*\|^2 - w_n(1-w_n)\|U_nx_n - x_n\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\|. \qquad (3.5)
\end{aligned}
$$

Furthermore, we set

$$
\begin{aligned}
&\sigma_n = \|x_n-u^*\|^2, \quad \rho_n = \alpha_n(2-\alpha_n(1+2k^2) - 2(1-\alpha_n)k), \\
&\delta_n = \frac{1}{2-\alpha_n(1+2k^2)-2(1-\alpha_n)k}\Big[2\alpha_n\|f(u^*)-u^*\|^2 + M_2\frac{\|\widehat{e}_n\|}{\alpha_n} \\
&\qquad + 2(1-\alpha_n)\langle f(u^*)-u^*, V_{\lambda_n}x_n - u^*\rangle\Big], \\
&\varphi_n = w_n(1-w_n)\|U_nx_n - x_n\|^2, and \\
&\phi_n = \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\|.
\end{aligned}
$$

Note that

$$\rho_n \to 0, \quad \sum_{n=0}^{\infty}\rho_n = \infty \quad (\lim_{n\to\infty}(2-\alpha_n(1+2k^2)-2(1-\alpha_n)k) = 2(1-k) > 0)$$

and $\phi_n \to 0 \quad (\alpha_n \to 0)$. By lemma 2.9, we have $\varphi_{n_k} \to 0 \quad (k\to\infty)$ implies that $\lim_{k\to\infty}\sup\delta_{n_k} \le 0$ for any subsequence $\{n_k\} \subset \{n\}$. Indeed, $\varphi_{n_k} \to 0 (k\to\infty)$ implies that $\|U_{n_k}x_{n_k} - x_{n_k}\| \to 0 (k\to\infty)$ due to condition (iii). From (3.3) we have

$$\|x_{n_k} - V_{\lambda_{n_k}}x_{n_k}\| = w_{n_k}\|x_{n_k} - U_{n_k}x_{n_k}\| \to 0. \qquad (3.6)$$

Step 3. Show that

$$\omega_w\{x_{n_k}\} \subset \Gamma \qquad (3.7)$$

where $\omega_w\{x_{n_k}\}$ is the set of all weak cluster points of $\{x_{n_k}\}$.
Let $\widehat{u} \in \omega_w\{x_{n_k}\}$ and $x_{n_{k_j}}$ is a subsequence of x_{n_k} weakly converging to $\widehat{u}$. We use $\{x_{n_k}\}$ to denote $x_{n_{k_j}}$ and we assume that $\lambda_{n_k} \to \lambda$.Then $0 < \lambda < \frac{1}{L}$. In the same

way, we take a subsequence $\{\gamma_{n_k}\}$ of γ_n by condition (ii) and assume that $\gamma_{n_k} \to \gamma$. Let $V_\lambda = J_\gamma^B(I - \lambda A^*(I-T))A$, we see that V is nonexpansive. Set

$$t_k = x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k}; \quad z_k = x_{n_k} - \lambda_n A^*(I-T)Ax_{n_k}.$$

By the resolvent identity, we conclude that

$$\begin{aligned}
&\|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \\
&= \|J^B_{\gamma_{n_k}}(x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k}) - J^B_\gamma(x_{n_k} - \lambda A^*(I-T)Ax_{n_k})\| \\
&= \|J^B_{\gamma_{n_k}}(t_k) - J^B_\gamma(z_k)\| \\
&= \|J^B_\gamma(\frac{\gamma}{\gamma_{n_k}} t_k + (1 - \frac{\gamma}{\gamma_{n_k}})J^B_{\gamma_{n_k}}) - J^B_{\gamma_n}(z_k)\| \\
&\le \|\frac{\gamma}{\gamma_{n_k}} t_k + (1 - \frac{\gamma}{\gamma_{n_k}})J^B_{\gamma_{n_k}}) - z_k\| \\
&= \|\frac{\gamma}{\gamma_{n_k}} t_k - \frac{\gamma}{\gamma_{n_k}} z_k + (1 - \frac{\gamma}{\gamma_{n_k}})J^B_{\gamma_{n_k}}) - z_k + \frac{\gamma}{\gamma_{n_k}} z_k\| \\
&= \|\frac{\gamma}{\gamma_{n_k}}(t_k - z_k) + (1 - \frac{\gamma}{\gamma_{n_k}})(J^B_{\gamma_{n_k}} t_k - z_k\| \\
&\le \frac{\gamma}{\gamma_{n_k}}\|t_k - z_k\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}\|x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k} - x_{n_k} + \lambda A^*(I-T)Ax_{n_k}\| \\
&\quad + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}\| - (\lambda_{n_k} - \lambda)A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}| - (\lambda_{n_k} - \lambda)|\|A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}|(\lambda_{n_k} - \lambda)|\|A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\|. \qquad (3.8)
\end{aligned}$$

Since $\gamma_{n_k} \to \gamma$ and $\lambda_{n_k} \to \lambda$ as $k \to \infty$, then $\|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \to 0$. As a result, we get

$$\|x_{n_k} - V_\lambda x_{n_k}\| \le \|x_{n_k} - V_{\lambda_{n_k}} x_{n_k}\| + \|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \to 0 \qquad (3.9)$$

From lemma 2.8, we have $\omega_w\{x_{n_k}\} \subset Fix(V_\lambda)$. It follows from lemma 2.10 that $\omega_w\{x_{n_k}\} \subset S$. We also have

$$\begin{aligned}
\limsup_{k\to\infty}\langle f(u^*) - u^*, V_{\lambda_{n_k}} x_{n_k} - u^*\rangle &= \limsup_{k\to\infty}\langle f(u^*) - u^*, x_{n_k} - u^*\rangle \\
&\quad + \limsup_{k\to\infty}\langle f(u^*) - u^*, V_{\lambda_{n_k}} x_{n_k} - x_{n_k}\rangle \qquad (3.10)
\end{aligned}$$

and

$$\limsup_{k\to\infty}\langle f(u^*) - u^*, x_{n_k} - u^*\rangle = \langle f(u^*) - u^*, \widehat{u} - u^*\rangle, \forall \widehat{u} \in \Gamma. \qquad (3.11)$$

It is easy to get from (3.10) tend to zero. Since u^* is the unique solution of variational inequality problem (3.1), we get

$$\langle f(u^*) - u^*, \widehat{u} - u^*\rangle \le 0.$$

Hence

$$\limsup_{k\to\infty} \delta_{n_k} \le 0.$$

The bounded perturbation of (3.2) by the following iterative method:

$$\begin{cases} y_n = x_n + \beta_n v_n, \\ x_{n+1} = \alpha_n f(y_n + e(y_n)) + (1-\alpha_n) J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)), \end{cases} \tag{3.12}$$

where $\{\lambda_n\}$, $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are a sequence in $[0,1]$ and f is contractive.

□

Theorem 3.2. *Let $\{\beta_n\}$ and $\{v_n\}$ be satisfied by condition (1.6). Let H_1, H_2 be two real Hilbert spaces and let A be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $\Gamma = B^{-1}0 \cap A^{-1}Fix(T) \neq 0$. Let f be k-contrative mapping on H, with $0 \leq k < 1$. Choose $x_0 \in H_1$ arbitrarily and define the sequence $\{x_n\}$ by (3.12).If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{L}$;
(iv) $\sum_{n=0}^{\infty} \|e(y_n)\| < \infty$.

Then $\{x_n\}$ converges strongly to u^*, where u^* is a solution of problem (1.5),which is also the unique solution of variational inequality problem (3.1)

Proof. we can rewrite (3.12) as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n) + \widehat{e}_n, \tag{3.13}$$

where

$$\begin{aligned} \widehat{e}_n = \alpha_n(f(y_n + e(y_n)) - f(x_n)) + (1-\alpha_n)(J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)) \\ - J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n)), \end{aligned} \tag{3.14}$$

Since $A^*(I-T)A$ is $\frac{1}{2L}$-ism, then it is $2L$-Lipschitz. Thus,

$$\begin{aligned}
&\|\widehat{e}_n\| \\
&\leq \alpha_n \|f(y_n + e(y_n)) - f(x_n)\| \qquad (3.15)\\
&\quad + (1-\alpha_n)\|J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)) - J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n)\| \\
&\leq \alpha_n k \|y_n + e(y_n) - x_n\| \\
&\quad + (1-\alpha_n)\|y_n - \lambda_n A^*(I-T)Ay_n + e(y_n) - (x_n - \lambda_n A^*(I-T)Ax_n)\| \\
&= \alpha_n k \|y_n - x_n + e(y_n)\| \\
&\quad + (1-\alpha_n)\|y_n - x_n - \lambda_n(A^*(I-T)Ay_n - A^*(I-T)Ax_n) + e(y_n)\| \\
&\leq \alpha_n k\|y_n - x_n\| + \alpha_n k \|e(y_n)\| \\
&\quad + (1-\alpha_n)\Bigg(\|y_n - x_n\| + \lambda_n\|A^*(I-T)Ay_n - A^*(I-T)Ax_n\| + e(y_n)\|\Bigg) \\
&\leq \alpha_n k\|y_n - x_n\| + \alpha_n k \|e(y_n)\| \\
&\quad + (1-\alpha_n)\Bigg(\|y_n - x_n\| + 2\lambda_n L\|y_n - x_n\| + \|e(y_n)\|\Bigg) \\
&= (\alpha_n k + 1 + 2\lambda_n L - \alpha_n - 2\alpha_n\lambda_n L)\|y_n - x_n\| + (\alpha_n k + 1 + \alpha_n)\|e(y_n)\| \\
&= (\alpha_n k + (1-\alpha_n)(1 + 2\lambda_n L))\|y_n - x_n\| + (1 + (1+k)\alpha_n)\|e(y_n)\| \\
&\leq (\alpha_n k + (1-\alpha_n)(1 + 2\lambda_n L))\beta_n\|V_n\| + (1 + (1-k)\alpha_n)\|e(y_n)\| \qquad (3.16)
\end{aligned}$$

From (1.6) and

$$\sum_{n=1}^{\infty} \|e(y_n)\| < \infty \quad and \quad \sum_{n=1}^{\infty} \|\widehat{e}_n\| < \infty$$

Thus, we find from theorem 3.1 that algorithm (3.2) is bounded perturbation resilient.

□

4. NUMERICAL RESULTS

In this section, we consider the following numerical examples to present the effectiveness, realization and convergence of Theorem 3.1.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^2$. Define $h(x) = \frac{1}{14}x$. Take $B : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ asfollows:

$$B = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}, \quad T : y = (y(1), y(2)^T \longmapsto (y(1), y(2) + \sin(y(2)))^T \text{ and } \quad e(x) = \begin{pmatrix} \frac{1}{n^4} \\ \frac{1}{n^4} \end{pmatrix}.$$

Observe that B is a positive linear operator. Then it is maximal monotone. T is $\frac{1}{2}$-av and the set of fixed points $Fix(T) = \{y \,|\, (y(1), 0)^T\}$ is nonempty. Then it is nonexpansive. Hence, we obtain the resolvent mapping $J_\gamma^B = (I + \gamma B)^{-1}$. It follows that

$$J_\gamma^B = \frac{1}{(3\gamma+1)(9\gamma+1)} \begin{pmatrix} 9\gamma+1 & 0 \\ 0 & 3\gamma+1 \end{pmatrix}$$

. Generatea 2×2 random matrix A, and compute the Lipschitz constant $L = \|A^T A\|$, where A^T represents the transpose of A. Take $\gamma_n = 0.9$, $\lambda_n = \lambda = \frac{1}{100L}$ and $\alpha_n = \frac{1}{4n+5}$.

According to the iterative process of Theorem 3.1, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \frac{1}{4n+5}(\frac{1}{14}(x_n + e(x_n)) + (1 - \frac{1}{4n+5})J_{\gamma_n}^B(x_n - \lambda_n A^T(I - T)Ax_n + e(x_n)).$$

As $n \to \infty$, we have $\{x_n\} \to u^*$. Taking random initial guess x_0 and the stopping criteria is $\|x_{n+1} - x_n\| < \epsilon$, we obtain the numerical experiment results in Table 1.

TABLE 1. $x_0 = rand(2,1)$

ϵ	$\lambda_n = \frac{1}{100L}$	n	Time	x_n	$\|x_{n+1} - x_n\|$
10^{-6}	0.006745	20	0.010811	(0.000003, 0.000001)	8.068095×10^{-7}
10^{-7}	0.022563	30	0.013420	(0.000001, 0.000000)	8.854955×10^{-8}
10^{-8}	0.011875	46	0.009761	(0.000000, 0.000000)	9.290480×10^{-8}

Next, we consider the algorithm with bounded perturbation resilience. Choose the bounded sequence $\{v_n\}$ and the summable nonnegative real sequence $\{\beta_n\}$ as follows:

$$v_n = \begin{cases} -\frac{d_n}{\|d_n\|}, & \text{if } 0 \neq d_n \in B(x_n), \\ 0, & \text{if } 0 \in B(x_n), \end{cases}$$

where $B(x_n) = (3x_n(1),\, 9x_n(2))^T,\ x_n(i),\ i = 1, 2$ denote the ith element of x_n, and $\beta_n = c^n$, for some $c \in (0, 1)$. Setting $c = 0.9$, the numerical results can be seen in Table 2.

TABLE 2. $x_0 = rand(2,1)$

ϵ	$\lambda_n = \frac{1}{100L}$	n	Time	x_n	$\|x_{n+1} - x_n\|$
10^{-6}	0.006923	33	0.015636	(0.000000, 0.000000)	6.782486×10^{-7}
10^{-7}	0.006444	60	0.015266	(0.000000, 0.000000)	8.078633×10^{-8}
10^{-8}	0.005079	83	0.015765	(0.000000, 0.000000)	7.836764×10^{-9}

5. Conclusion

We have introduced a viscosity iterative scheme and obtained the strong convergence. We also consider the bounded perturbation resilience of the proposed method and get theoretical convergence results.

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VISCOSITY ITERATIVE SCHEME FOR SPLIT FEASIBILITY PROBLEMS

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ABSTRACT. In this paper, we intend to solve a split feasibility problem by viscosity iterative algorithm. The bounded perturbation resilience of the method is examined in Hibert spaces. As tools, averaged mappings and resolvents of maximal monotone operators are the specialized procedure to simplify the proofs of the main results. Under mild conditions, we prove that our algorithms converge to a solution of the split feasibility problem. Moreover, we show the convergence and result of the algorithms by a numerical example.

KEYWORDS: Viscosity iterative algorithm, Split feasibility problem, Maximal monotone operator.
AMS Subject Classification: 47H09, 47H10.

1. Introduction

Let C and Q be nonempty closed convex subsets in real Hibert space H_1 and H_2,respectively. Let P_C be the metric projection from H_1 onto C and P_Q be the metric projection from H_2 onto Q. The problem to find

$$u^* \in C \quad with \quad Au^* \in Q \tag{1.1}$$

where A is a bounded linear operator from H_1 to H_2, if such u^* exist, this problem is called the split feasibility problem(see [1]). If problem (1.1) has a solution (say that $C \cap A^{-1}Q$ is nonempty). $u^* \in C \cap A^{-1}Q$ is equivalent to

$$u^* = P_C(I - \lambda A^*(I - P_Q)A)u^*, \tag{1.2}$$

where $\lambda > 0$ and A^* is the adjoint operator of A.

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The SFP was first introduced by Censor and Elfving [2] in 1994. They used their multidistance method to obtain iterative algorithms for solving the SFP. After that, Byrne [3] proposed his CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I - \lambda A^*(I - P_Q)A)x_n, \quad \forall n \geq 0. \tag{1.3}$$

Let $B : H_1 \to 2^{H_1}$ be a mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for all $\lambda > 0$. Let $T : H_2 \to H_2$ be nonexpansive mapping.

In 2015, Takahashi et al. [4] proposed the following algorithm:

$$x_{n+1} = J^B_{\gamma_n}(x_n - \lambda_n A^*(I - T)Ax_n), \tag{1.4}$$

where $\{\lambda_n\}$ is a sequence in $[0, 1]$ and they proved that the sequence $\{u_n\}$ converges weakly to a point $u^* \in B^{-1}0 \cap A^{-1}Fix(T)$ in the framework of Hilbert spaces. That is this problems is to find a point $u^* \in H_1$ such that

$$0 \in Bu^* \quad and \quad Au^* \in Fix(T). \tag{1.5}$$

The set of all solution (1.5) denoted by $\Gamma = B^{-1}0 \cap A^{-1}Fix(T)$.
there are many authers have studied the SFP and its extensions by means of fixed-point methods and weak-strong convergence theorems of solutions have been established in Hilbert or Banach spaces (see [5, 6, 7, 8]).

Let F be an algorithm operator. Let $\{x_n\}$ be a sequence, generated by $x_{n+1} = Fx_n$, and let $\{y_n\}$ be a sequence, generated by $y_{n+1} = F(y_n + \beta_n v_n)$, where $\{\beta_n\}$ is a sequence of nonnegative real numbers and $\{v_n\}$ is a sequence in H such that

$$\sum_{n=0}^{\infty} \beta_n < \infty \quad and \quad \|v_n\| \leq M, \quad \forall n \geq 0. \tag{1.6}$$

An algorthmic operator F is call bounded perturbation resilient if the following is ture: if the sequence $\{x_n\}$ is convergent, then $\{y_n\}$ is also convergent (see [9]).

In 2017, Xu [10] presented the bounded perturbation resilience and superiorization techniques for the projected scaled gradient(PSG).The iterative method is defined as following:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C(x_n - \lambda_n D(x_n)\triangledown h(x_n) + e(x_n)), \quad \forall n \geq 0 \tag{1.7}$$

where $\{\lambda_n\}$, $\{\alpha_n\}$ are a sequence in $[0, 1]$, h is a continuous differentiable and convex function, and $D(x_n)$ is a diagonal scaling matrix. The weak convergence was proved in [10].

In 2018, Guo and Chi [11] proposed the following proximal gradient algorithm with perturbations:

$$x_{n+1} = t_n f(x_n) + (1 - t_n)prox_{\lambda_n g}(1 - \lambda_n \triangledown h)x_n + e(x_n), \tag{1.8}$$

where $\{\lambda_n\}$, $\{t_n\}$ are a sequence in $[0, 1]$ and f is a contractive, for solving nonsmooth composite convex optimization problem. They obtained strong convergence and bounded resilience of the above method.

In 2019, Duan and Zheng [12] presented a viscosity approximation method for solving problem (1.5):

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J^B_{\lambda_n}(x_n - \tau_n A^*(1 - T)Ax_n + e(x_n)), \tag{1.9}$$

where $\{\lambda_n\},\ \{\tau_n\},\ \{\alpha_n\}$ are a sequence in $[0,1]$ and they gave the bounded perturbation of (1.9) yields a sequence $\{x_n\}$ generated by the iterative process:

$$\begin{aligned} y_n &= x_n + \beta_n v_n \\ x_{n+1} &= \alpha_n f(y_n) + (1-\alpha_n) J^B_{\lambda_n}(y_n - \tau_n A^*(I-T)Ay_n + e(y_n)), \end{aligned} \tag{1.10}$$

where $\{\lambda_n\},\ \{\tau_n\},\ \{\alpha_n\},\ \{\beta_n\}$ are a sequence in $[0,1]$

In this paper, we extend work in [12] and purpose the following process for solving problem (1.5) :

$$x_{n+1} = \alpha_n f(x_n + e(x_n)) + (1-\alpha_n) J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n + e(x_n)), \tag{1.11}$$

where $\{\lambda_n\},\ \{\tau_n\},\ \{\alpha_n\}$, are a sequence in $[0,1]$, f is contractive, and we give a sequence $\{x_n\}$ generated by the iterative process:

$$\begin{aligned} y_n &= x_n + \beta_n v_n \\ x_{n+1} &= \alpha_n f(y_n + e(y_n)) + (1-\alpha_n) J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)), \end{aligned} \tag{1.12}$$

where $\{\lambda_n\},\ \{\tau_n\},\ \{\alpha_n\},\ \{\beta_n\}$ are a sequence in $[0,1]$ and f is contractive.

After that we prove the convergence point of the iterative method which is also the unique solution of some variational inequality problem. A numerical example is also given to demonstrate the effectiveness of our iterative schemes.

2. Preliminaries

Let $\{x_n\}$ be a sequence in a real Hilbert space H. First, We give notations:

- Denote $\{x_n\}$ converging weakly to x by $x_n \rightharpoonup x$ and $\{x_n\}$ converging strongly to x by $x_n \to x$.
- Denote the set of fixed points of mapping T by $Fix(T) = \{x \in H : Tx = x\}$
- Denotetheweak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$.

Definition 2.1. A mapping $F : H \to H$ is said to be

(i) Lipschizian if there exist a positive constant L such that

$$||Fx - Fy|| \le L||x-y||, \quad \forall x, y \in H.$$

In particular, if $L = 1$, we say that F is nonexpansive, namely,

$$||Fx - Fy|| \le L||x-y||, \quad \forall x, y \in H,$$

if $L \in [0,1)$, we say that F is contractive.

(ii) α-averaged mapping (α-av for short) if

$$F = (1-\alpha)I + \alpha T,$$

where $\alpha \in [0,1)$ and $T : H \to H$ is nonexpansive.

Definition 2.2. A mapping $B : H \to H$ is said to be

(i) monotone if

$$\langle Bx - By, x - y\rangle \le 0, \quad \forall x, y \in H.$$

(ii) $\eta-$strongly monotone if there exists a posive constant eta such that

$$\langle Bx - By, x - y\rangle \ge \eta||x-y||^2, \quad \forall x, y \in H.$$

(iii) α-inverse strongly monotone (for short α-ism) if there exist a positive constant α such that

$$\langle Bx - By, x - y\rangle \geq \alpha||Bx - By||^2, \quad \forall x, y \in H.$$

In paticular, if $\alpha = 1$, we say that B is firmly nonexpansive, namely,

$$\langle Bx - By, x - y\rangle \geq ||Bx - By||^2, \quad \forall x, y \in H.$$

Definition 2.3. Let $B : H \to H$ be amonotone mapping. Then B is maximal monotone if there exists no monotone operater $A : H \to 2^H$ such that $graA$ properly contains $graB$,i.e. for every $(x, u) \in H \times H$,

$$(x, u) \in graB \Leftrightarrow \forall (y, v) \in graB, \langle x - y, u - v\rangle \geq 0.$$

Lemma 2.4. *Let H be a real Hillbert space. Theere holds the following inequality*

$$||x + y||^2 \leq ||x||^2 + 2\langle x + y, y\rangle, \quad \forall x, y \in H.$$

Lemma 2.5. *Let $f : H \to H$ be a $k \in (0, 1)$ and let $T : H \to H$ be a nonexpansive mapping.Then*

(i) *$I - f$ is (1-k)-strongly monotone:*

$$\langle (I - f)x - (I - f)y, x - y\rangle \geq (1 - \rho)||x - y||^2, \quad \forall x, y \in H.$$

(ii) *$I - T$ is monotone:*

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

Proposition 2.6. [4] *Assume that H_1 and H_2 are Hilbert space. Let $B : H_1 \to 2_1^H$ be a maximal monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then*

(i) *$A^*(I - T)A$ is $\frac{1}{2||A||^2}$-ism.*

(ii) *For $0 < \tau < \frac{1}{2||A||^2}$,*
$I - \tau A^*(I - T)A$ is $\tau||A||^2$- averaged and $J_\lambda^B(I - \tau A^*(I - T)A)$ is $\frac{1+\tau||A||^2}{2}$ averaged.

Lemma 2.7. [13] *Let B be a maximal monotone operator. Let $J_\gamma^B = (I + \gamma B)^{-1}$ and $J_\lambda^B = (I + \lambda B)^{-1}$, where $\gamma > 0$ and $\lambda > 0$ are two real numbers, be the resolvent operators of B. Then*

$$J_\gamma^B x = J_\lambda^B(\frac{\lambda}{\gamma}x + (1 - \frac{\lambda}{\gamma})J_\gamma^B x), \quad \forall x \in H.$$

Lemma 2.8. [14] *Let H be a real Hillbert space, and let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq 0$.if $\{x_n\}$ is a sequence in H weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y,then $(I - T) = y$; in particulary, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.9. [15] *Assume $\{\sigma_n\}$ is a sequence of nonnegative real numbers such that*

$$\sigma_{n+1} \leq (1 - \rho_n)\sigma_n + \rho_n\delta_n, \quad n \geq 0,$$
$$\sigma_{n+1} \leq \sigma_n - \varphi_n + \phi_n, \quad n \geq 0,$$

where $\{\rho_n\}$ is a sequence in $(0, 1)$, $\{\varphi_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\phi_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \rho_n = \infty$;
(ii) $\lim_{n\to\infty} \phi_n = 0$;
(iii) $\lim_{k\to\infty} \varphi_{n_k} = 0 \Rightarrow \limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence $(n_k) \subset (n)$.*

Then $\lim_{n\to\infty} \sigma_n = 0$.

Lemma 2.10. [4] *Let* H_1 *and* H_2 *be Hilbert space. Let* $B : H_1 \to 2^{H_1}$ *be a maximal monotone mapping and let* $J_\lambda^B = (I + \lambda B)^{-1}1$ *be the resolvent of* B *for* $\lambda > 0$. *Let* $T : H_2 \to H_2$ *be a nonexpansive mapping and let* $T : H_1 \to H_2$ *be a bounded linear operater. Suppose that* $B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. *Let* $\lambda, \tau > 0$. *Then the following equality holds:*

$$Fix(J_\lambda^B(I - \tau A^*(I - T)A)) = (A^*(I - T)A + B)^{-1}0 = B^{-1}0 \cap A^{-1}Fix(T)$$

3. Main Results

In [1] proposed the viscosity approximation method:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)S(x_n), \quad \forall n \geq 0,$$

which converges strongly to a fixed point u^* of the nonexpansive mapping S. In [5] further proved that $u^* \in Fix(S)$ is also the unique solution of the following variational inequality problem:

$$\langle (I - f)u^*, \hat{u} - u^* \rangle \geq 0, \quad \forall \hat{u} \in Fix(S), \tag{3.1}$$

where $f : H \to H$ is a k-contraction.

In this section, we present a viscosity iterative algorithm for solving problem (1.5). Rewrite iteration (1.11) as

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n + e(x_n)) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ &= \alpha_n f(x_n) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n) + \widehat{e}_n, \quad \forall n \geq 0, \end{aligned}$$

where

$$\begin{aligned} \widehat{e}_n &= \alpha_n(f(x_n + e(x_n)) - f(x_n)) + (1 - \alpha_n)(J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ &\quad - J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n)). \end{aligned}$$

Since $J_{\gamma_n}^B$ is nonexpansive and f is contractive, it is easy to get

$$\begin{aligned} \|\hat{e}_n\| &\leq \alpha_n\|f(x_n + e(x_n)) - f(x_n)\| + (1 - \alpha_n)\|J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ &\qquad - J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n)\| \\ &\leq \alpha_n k\|e(x_n)\| + (1 - \alpha_n)\|e(x_n)\| \\ &= (\alpha_n k + 1 - \alpha_n)\|e(x_n)\| \\ &\leq \|e(x_n)\|. \end{aligned}$$

Theorem 3.1. *Let* H_1, H_2 *be two real Hilbert spaces and let* $A : H_1 \to H_2$ *be a bounded linear operator with* $L = \|A^*A\|$, *where* A^* *is the adjoint of* A. *Suppose that* $B : H_1 \to 2^{H_1}$ *is a maximal monotone operator and* $T : H_2 \to H_2$ *is a nonexpansive mapping. Assume that* $\Gamma = B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. *Let* f *be a* k*-contractive on* H_1 *with* $0 \leq k < 1$. *Choose* $x_0 \in H_1$ *arbitrarily and define a sequence* $\{x_n\}$ *in the following manner:*

$$x_{n+1} = \alpha_n f(x_n + e(x_n)) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \tag{3.2}$$

if the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{L}$;
(iv) $\sum_{n=0}^{\infty} \|e(x_n)\| < \infty$.

Then $\{x_n\}$ *converges strongly to* $u^* \in \Gamma$, *which is also the unique solution of variational inequality problem* (3.1).

Proof. Let $V_{\lambda_n} = J^B_{\gamma_n}((I - \lambda_n A^*(I - T)A)$. From Proposition 2.6, it follows that $J^B_{\gamma_n}((I - \lambda_n A^*(I - T)A)$ is $\frac{1+\lambda_n L}{2} - av$ as $0 < \lambda_n < \frac{1}{L}$.
Step 1. show that $\{x_n\}$ is bounded. For any $u^* \in \Gamma$, we have

$$\begin{aligned}
&\|x_{n+1} - u^*\| \\
&= \|\alpha_n f(x_n) + (1 - \alpha_n)V_{\lambda_n} x_n + \widehat{e}_n - u^*\| \\
&= \|\alpha_n f(x_n) + V_{\lambda_n} x_n - \alpha_n V_{\lambda_n} x_n + \widehat{e}_n - u^*\| \\
&= \|(\alpha_n f(x_n) - \alpha_n u^*) + (V_{\lambda_n} x_n - u^*) - (\alpha_n V_{\lambda_n} x_n - \alpha_n u^*) + \widehat{e}_n\| \\
&= \|\alpha_n(f(x_n) - u^*) + (1 - \alpha_n)(V_{\lambda_n} x_n - u^*) + \widehat{e}_n\| \\
&\leq \alpha_n\|f(x_n) - u^*\| + (1 - \alpha_n)\|(V_{\lambda_n} x_n - u^*)\| + \|\widehat{e}_n\| \\
&= \alpha_n\|f(x_n) - f(u^*) + f(u^*) - u^*\| + (1 - \alpha_n)\|(V_{\lambda_n} x_n - u^*)\| + \|\widehat{e}_n\| \\
&\leq \alpha_n\|f(x_n) - f(u^*)\| + \alpha_n\|f(u^*) - u^*\| + (1 - \alpha_n)\|(V_{\lambda_n} x_n - u^*)\| + \|\widehat{e}_n\| \\
&= \alpha_n\|f(x_n) - f(u^*)\| + \alpha_n\|f(u^*) - u^*\| + (1 - \alpha_n)\|(V_{\lambda_n} x_n - V_{\lambda_n} u^*)\| + \|\widehat{e}_n\| \\
&\leq \alpha_n k\|x_n - u^*\| + \alpha_n\|f(u^*) - u^*\| + (1 - \alpha_n)\|x_n - u^*\| + \|\widehat{e}_n\| \\
&= (1 - \alpha_n + \alpha_n k)\|x_n - u^*\| + \alpha_n(\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}) \\
&= (1 - \alpha_n(1 - k))\|x_n - u^*\| + \alpha_n(1 - k)\left[\frac{\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}}{1 - k}\right].
\end{aligned}$$

From condition (i), (iv) and $\alpha_n > 0$, we get $\left\{\frac{\|\widehat{e}_n\|}{\alpha_n}\right\}$ is bounded.Thus there exists $M_1 > 0$ such that $\sup\left\{\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}\right\} \leq M_1$, for all $n \geq 0$. By Mathematical Induction, we get $\|x_n - u^*\| \leq \max\left\{\|x_0 - u^*\|, \frac{M_1}{1-k}\right\}$, which implies that the sequence $\{x_n\}$ is bounded, so are $\{f(x_n)\}, \{V_{\lambda_n} x_n\}$ and $\{A^*(I - T)Ax_n\}$.
Step 2. Show that for any sequence$\{n_k\} \subset \{n\}$,

$$\lim_{n\to\infty} \|x_{n_k} - V_{\lambda_{n_k}} x_{n_k}\| = 0.$$

Fixing $u^* \in \Gamma$, we have

$$\begin{aligned}
&\|x_{n+1} - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1 - \alpha_n)V_{\lambda_n} x_n + \widehat{e}_n - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1 - \alpha_n)V_{\lambda_n} x_n - u^*\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) + (1 - \alpha_n)V_{\lambda_n} x_n - u^*, \widehat{e}_n\rangle + \|\widehat{e}_n\|^2 \\
&= \|\alpha_n(f(x_n) - u^*) + (1 - \alpha_n)(V_{\lambda_n} x_n - u^*)\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) + (1 - \alpha_n)V_{\lambda_n} x_n - u^*, \widehat{e}_n\rangle + \|\widehat{e}_n\|^2 \\
&\leq \alpha_n^2\|f(x_n) - u^*\|^2 + (1 - \alpha_n)^2\|V_{\lambda_n} x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(x_n) - u^*, V_{\lambda_n} x_n - u^*\rangle \\
&\quad + 2\|\alpha_n f(x_n) + (1 - \alpha_n)V_{\lambda_n} x_n - u^*\|\|\widehat{e}_n\| + \|\widehat{e}_n\|^2 \\
&= \alpha_n^2\|f(x_n) - u^*\|^2 + (1 - \alpha_n)^2\|V_{\lambda_n} x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(x_n) - u^*, V_{\lambda_n} x_n - u^*\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ 2\|\alpha_n(f(x_n) - u^*) + (1-\alpha_n)(V_{\lambda_n}x_n - u^*)\|\|\widehat{e}_n\| + \|\widehat{e}_n\|^2 \\
&\le \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle \\
&\quad + 2\Big[\|\alpha_n(f(x_n) - u^*)\| + \|(1-\alpha_n)(V_{\lambda_n}x_n - u^*)\|\Big]\|\widehat{e}_n\| + \|\widehat{e}_n\|^2 \\
&= \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle \\
&\quad + \Big[2\alpha_n\|(f(x_n) - u^*)\| + 2(1-\alpha_n)\|V_{\lambda_n}x_n - u^*\| + \|\widehat{e}_n\|\Big]\|\widehat{e}_n\| \\
&\le \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle \\
&\quad + \Big(2\alpha_n\|f(x_n) - u^*\| + 2(1-\alpha_n)\|x_n - u^*\| + \|\widehat{e}_n\|\Big)\|\widehat{e}_n\| \\
&\le 2\alpha_n^2\Big(\|f(x_n) - f(u^*)\|^2 + \|f(u^*) - u^*\|^2\Big) + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle + M_2\|\widehat{e}_n\| \\
&\le 2\alpha_n^2\Big(\|f(x_n) - f(u^*)\|^2 + \|f(u^*) - u^*\|^2\Big) + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\Big(\|f(x_n) - f(u^*)\|\|x_n - u^*\| + \langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle\Big) + M_2\|\widehat{e}_n\| \\
&\le 2\alpha_n^2 k\|x_n - u^*\|^2 + 2\alpha_n^2\|f(u^*) - u^*\|^2 + (1-\alpha_n^2)\|x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)k\|x_n - u^*\|^2 + 2\alpha_n(1-\alpha_n)\langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle + M_2\|\widehat{e}_n\| \\
&= \Big(2\alpha_n^2 k + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)k\Big)\|x_n - u^*\|^2 + 2\alpha_n^2\|f(u^*) - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle + M_2\|\widehat{e}_n\| \\
&= (1 - \alpha_n(2 - \alpha_n(1+2k^2) - 2(1-\alpha_n)k))\|x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle + 2\alpha_n^2\|f(u^*) - u^*\|^2 + M_2\|\widehat{e}_n\|, \quad (3.3)
\end{aligned}
$$

where

$$M_2 = \sup_{n\in\mathbb{N}}\Big\{2\alpha_n\|f(x_n) - u^*\| + 2(1-\alpha_n)\|x_n - u^*\| + \|\widehat{e}_n\|\Big\}.$$

Note that

$$V_{\lambda_n} = J_{r_n}^B(I - \lambda_n A^*(I-T)A) = (1-w_n)I + w_n U_n, \tag{3.4}$$

such that $w_n = \frac{1+\lambda_n L}{2}$, and U_n is nonexpansive. By condition (iii), we get

$$\frac{1}{2} < \liminf_{n\to\infty} w_n \le \limsup_{n\to\infty} w_n < 1$$

Since $u^* \in \Gamma$, then $V_{\lambda_n}u^* = u^*$. Furthermore, we have $(1-w_n)u^* + w_n U_n u^* = u^*$. It is clear that $U_n u^* = u^*$.

$$
\begin{aligned}
&\|x_{n+1} - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n + \widehat{e}_n - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*, \widehat{e}_n\rangle + \|\widehat{e}_n\|^2 \\
&\le \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|\|\widehat{e}_n\| + \|\widehat{e}_n\|^2
\end{aligned}
$$

$$\begin{aligned}
&\le \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + \Big(2\alpha_n\|f(x_n)-u^*\| + 2(1-\alpha_n)\|x_n-u^*\| + \|\widehat{e}_n\|\Big)\|\widehat{e}_n\| \\
&\le \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 + M_2\|\widehat{e}_n\| \\
&= \|V_{\lambda_n}x_n - u^* + \alpha_n(f(x_n) - V_{\lambda_n}x_n)\|^2 + M_2\|\widehat{e}_n\| \\
&= \|V_{\lambda_n}x_n - u^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|(1-w_n)x_n + w_nU_nx_n - u^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|x_n - w_nx_n + w_nU_nx_n - (1-w_n)u^* - w_nU_nu^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|(1-w_n)(x_n-u^*) + w_n(U_nx_n - U_nu^*)\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= (1-w_n)\|x_n-u^*\|^2 + w_n\|U_nx_n - U_nu^*\|^2 - w_n(1-w_n)\|U_nx_n - x_n\|^2 \\
&\quad + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&\le \|x_n-u^*\|^2 - w_n(1-w_n)\|U_nx_n - x_n\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\|. \qquad (3.5)
\end{aligned}$$

Furthermore, we set

$$\begin{aligned}
\sigma_n &= \|x_n - u^*\|^2, \quad \rho_n = \alpha_n(2 - \alpha_n(1+2k^2) - 2(1-\alpha_n)k), \\
\delta_n &= \frac{1}{2-\alpha_n(1+2k^2) - 2(1-\alpha_n)k}\Big[2\alpha_n\|f(u^*) - u^*\|^2 + M_2\frac{\|\widehat{e}_n\|}{\alpha_n} \\
&\quad + 2(1-\alpha_n)\langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle\Big], \\
\varphi_n &= w_n(1-w_n)\|U_nx_n - x_n\|^2, and \\
\phi_n &= \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\|.
\end{aligned}$$

Note that

$$\rho_n \to 0, \quad \sum_{n=0}^{\infty}\rho_n = \infty \quad (\lim_{n\to\infty}(2 - \alpha_n(1+2k^2) - 2(1-\alpha_n)k) = 2(1-k) > 0)$$

and $\phi_n \to 0 \quad (\alpha_n \to 0)$. By lemma 2.9, we have $\varphi_{n_k} \to 0 \quad (k \to \infty)$ implies that $\lim_{k\to\infty}\sup\delta_{n_k} \le 0$ for any subsequence $\{n_k\} \subset \{n\}$. Indeed, $\varphi_{n_k} \to 0 (k \to \infty)$ implies that $\|U_{n_k}x_{n_k} - x_{n_k}\| \to 0 (k \to \infty)$ due to condition (iii). From (3.3) we have

$$\|x_{n_k} - V_{\lambda_{n_k}}x_{n_k}\| = w_{n_k}\|x_{n_k} - U_{n_k}x_{n_k}\| \to 0. \qquad (3.6)$$

Step 3. Show that

$$\omega_w\{x_{n_k}\} \subset \Gamma \qquad (3.7)$$

where $\omega_w\{x_{n_k}\}$ is the set of all weak cluster points of $\{x_{n_k}\}$.
Let $\widehat{u} \in \omega_w\{x_{n_k}\}$ and $x_{n_{k_j}}$ is a subsequence of x_{n_k} weakly converging to $\widehat{u}$. We use $\{x_{n_k}\}$ to denote $x_{n_{k_j}}$ and we assume that $\lambda_{n_k} \to \lambda$.Then $0 < \lambda < \frac{1}{L}$. In the same

way, we take a subsequence $\{\gamma_{n_k}\}$ of γ_n by condition (ii) and assume that $\gamma_{n_k} \to \gamma$. Let $V_\lambda = J_\gamma^B(I - \lambda A^*(I-T))A$, we see that V is nonexpansive. Set

$$t_k = x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k}; \quad z_k = x_{n_k} - \lambda_n A^*(I-T)Ax_{n_k}.$$

By the resolvent identity, we conclude that

$$\begin{aligned}
&\|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \\
&= \|J_{\gamma_{n_k}}^B (x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k}) - J_\gamma^B(x_{n_k} - \lambda A^*(I-T)Ax_{n_k})\| \\
&= \|J_{\gamma_{n_k}}^B(t_k) - J_\gamma^B(z_k)\| \\
&= \|J_\gamma^B(\frac{\gamma}{\gamma_{n_k}} t_k + (1 - \frac{\gamma}{\gamma_{n_k}})J_{\gamma_{n_k}}^B) - J_{\gamma_n}^B(z_k)\| \\
&\leq \|\frac{\gamma}{\gamma_{n_k}} t_k + (1 - \frac{\gamma}{\gamma_{n_k}})J_{\gamma_{n_k}}^B) - z_k\| \\
&= \|\frac{\gamma}{\gamma_{n_k}} t_k - \frac{\gamma}{\gamma_{n_k}} z_k + (1 - \frac{\gamma}{\gamma_{n_k}})J_{\gamma_{n_k}}^B) - z_k + \frac{\gamma}{\gamma_{n_k}} z_k\| \\
&= \|\frac{\gamma}{\gamma_{n_k}}(t_k - z_k) + (1 - \frac{\gamma}{\gamma_{n_k}})(J_{\gamma_{n_k}}^B t_k - z_k\| \\
&\leq \frac{\gamma}{\gamma_{n_k}}\|t_k - z_k\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J_{\gamma_{n_k}}^B t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}\|x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k} - x_{n_k} + \lambda A^*(I-T)Ax_{n_k}\| \\
&\quad + (1 - \frac{\gamma}{\gamma_{n_k}})\|J_{\gamma_{n_k}}^B t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}\| - (\lambda_{n_k} - \lambda)A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J_{\gamma_{n_k}}^B t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}| - (\lambda_{n_k} - \lambda)|\|A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J_{\gamma_{n_k}}^B t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}|(\lambda_{n_k} - \lambda)|\|A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J_{\gamma_{n_k}}^B t_k - z_k\|. \qquad (3.8)
\end{aligned}$$

Since $\gamma_{n_k} \to \gamma$ and $\lambda_{n_k} \to \lambda$ as $k \to \infty$, then $\|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \to 0$. As a result, we get

$$\|x_{n_k} - V_\lambda x_{n_k}\| \leq \|x_{n_k} - V_{\lambda_{n_k}} x_{n_k}\| + \|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \to 0 \qquad (3.9)$$

From lemma 2.8, we have $\omega_w\{x_{n_k}\} \subset Fix(V_\lambda)$. It follows from lemma 2.10 that $\omega_w\{x_{n_k}\} \subset S$. We also have

$$\begin{aligned}
\limsup_{k\to\infty}\langle f(u^*) - u^*, V_{\lambda_{n_k}} x_{n_k} - u^*\rangle &= \limsup_{k\to\infty}\langle f(u^*) - u^*, x_{n_k} - u^*\rangle \\
&\quad + \limsup_{k\to\infty}\langle f(u^*) - u^*, V_{\lambda_{n_k}} x_{n_k} - x_{n_k}\rangle \quad (3.10)
\end{aligned}$$

and

$$\limsup_{k\to\infty}\langle f(u^*) - u^*, x_{n_k} - u^*\rangle = \langle f(u^*) - u^*, \widehat{u} - u^*\rangle, \forall \widehat{u} \in \Gamma. \qquad (3.11)$$

It is easy to get from (3.10) tend to zero. Since u^* is the unique solution of variational inequality problem (3.1), we get

$$\langle f(u^*) - u^*, \widehat{u} - u^*\rangle \leq 0.$$

Hence

$$\limsup_{k\to\infty} \delta_{n_k} \leq 0.$$

The bounded perturbation of (3.2) by the following iterative method:

$$\begin{cases} y_n = x_n + \beta_n v_n, \\ x_{n+1} = \alpha_n f(y_n + e(y_n)) + (1-\alpha_n) J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)), \end{cases} \tag{3.12}$$

where $\{\lambda_n\}$, $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are a sequence in $[0,1]$ and f is contractive.

□

Theorem 3.2. *Let $\{\beta_n\}$ and $\{v_n\}$ be satisfied by condition (1.6). Let H_1, H_2 be two real Hilbert spaces and let A be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $\Gamma = B^{-1}0 \cap A^{-1}Fix(T) \neq 0$. Let f be k-contrative mapping on H, with $0 \leq k < 1$. Choose $x_0 \in H_1$arbitrarily and define the sequence $\{x_n\}$ by (3.12).If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{L}$;
(iv) $\sum_{n=0}^{\infty} \|e(y_n)\| < \infty$.

Then $\{x_n\}$ converges strongly to u^*, where u^* is a solution of problem (1.5),which is also the unique solution of variational inequality problem (3.1)

Proof. we can rewrite (3.12) as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n) + \widehat{e}_n, \tag{3.13}$$

where

$$\begin{aligned} \widehat{e}_n = {} & \alpha_n(f(y_n + e(y_n)) - f(x_n)) + (1-\alpha_n)(J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)) \\ & - J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n)), \end{aligned} \tag{3.14}$$

Since $A^*(I-T)A$ is $\frac{1}{2L}$-ism, then it is $2L$-Lipschitz. Thus,

$$\begin{aligned} & \|\widehat{e}_n\| \\ & \leq \alpha_n \|f(y_n + e(y_n)) - f(x_n)\| \qquad (3.15) \\ & \quad + (1-\alpha_n)\|J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)) - J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n)\| \\ & \leq \alpha_n k \|y_n + e(y_n) - x_n\| \\ & \quad + (1-\alpha_n)\|y_n - \lambda_n A^*(I-T)Ay_n + e(y_n) - (x_n - \lambda_n A^*(I-T)Ax_n)\| \\ & = \alpha_n k \|y_n - x_n + e(y_n)\| \\ & \quad + (1-\alpha_n)\|y_n - x_n - \lambda_n(A^*(I-T)Ay_n - A^*(I-T)Ax_n) + e(y_n)\| \\ & \leq \alpha_n k\|y_n - x_n\| + \alpha_n k \|e(y_n)\| \\ & \quad + (1-\alpha_n)\bigg(\|y_n - x_n\| + \lambda_n \|A^*(I-T)Ay_n - A^*(I-T)Ax_n\| + e(y_n)\|\bigg) \\ & \leq \alpha_n k\|y_n - x_n\| + \alpha_n k \|e(y_n)\| \\ & \quad + (1-\alpha_n)\bigg(\|y_n - x_n\| + 2\lambda_n L\|y_n - x_n\| + \|e(y_n)\|\bigg) \\ & = (\alpha_n k + 1 + 2\lambda_n L - \alpha_n - 2\alpha_n\lambda_n L)\|y_n - x_n\| + (\alpha_n k + 1 + \alpha_n)\|e(y_n)\| \\ & = (\alpha_n k + (1-\alpha_n)(1 + 2\lambda_n L))\|y_n - x_n\| + (1 + (1+k)\alpha_n)\|e(y_n)\| \\ & \leq (\alpha_n k + (1-\alpha_n)(1 + 2\lambda_n L))\beta_n\|V_n\| + (1 + (1-k)\alpha_n)\|e(y_n)\| \qquad (3.16) \end{aligned}$$

From (1.6) and

$$\sum_{n=1}^{\infty} \|e(y_n)\| < \infty \quad and \quad \sum_{n=1}^{\infty} \|\widehat{e}_n\| < \infty$$

Thus, we find from theorem 3.1 that algorithm (3.2) is bounded perturbation resilient.

□

4. NUMERICAL RESULTS

In this section, we consider the following numerical examples to present the effectiveness, realization and convergence of Theorem 3.1.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^2$. Define $h(x) = \frac{1}{14}x$. Take $B : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ asfollows:

$$B = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}, \quad T : y = (y(1), y(2)^T \longmapsto (y(1), y(2)+\sin(y(2)))^T \text{ and } \quad e(x) = \begin{pmatrix} \frac{1}{n^4} \\ \frac{1}{n^4} \end{pmatrix}.$$

Observe that B is a positive linear operator. Then it is maximal monotone. T is $\frac{1}{2}$-av and the set of fixed points $Fix(T) = \{y\,|\,(y(1), 0)^T\}$ is nonempty. Then it is nonexpansive. Hence, we obtain the resolvent mapping $J_\gamma^B = (I + \gamma B)^{-1}$. It follows that

$$J_\gamma^B = \frac{1}{(3\gamma+1)(9\gamma+1)} \begin{pmatrix} 9\gamma+1 & 0 \\ 0 & 3\gamma+1 \end{pmatrix}$$

. Generatea 2×2 random matrix A, and compute the Lipschitz constant $L = \|A^T A\|$, where A^T represents the transpose of A. Take $\gamma_n = 0.9$, $\lambda_n = \lambda = \frac{1}{100L}$ and $\alpha_n = \frac{1}{4n+5}$.

According to the iterative process of Theorem 3.1, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \frac{1}{4n+5}\left(\frac{1}{14}(x_n + e(x_n)) + (1 - \frac{1}{4n+5}\right)J_{\gamma_n}^B(x_n - \lambda_n A^T(I-T)Ax_n + e(x_n)).$$

As $n \to \infty$, we have $\{x_n\} \to u^*$. Taking random initial guess x_0 and the stopping criteria is $\|x_{n+1} - x_n\| < \epsilon$, we obtain the numerical experiment results in Table 1.

TABLE 1. $x_0 = rand(2,1)$

ϵ	$\lambda_n = \frac{1}{100L}$	n	Time	x_n	$\|x_{n+1} - x_n\|$
10^{-6}	0.006745	20	0.010811	(0.000003, 0.000001)	8.068095×10^{-7}
10^{-7}	0.022563	30	0.013420	(0.000001, 0.000000)	8.854955×10^{-8}
10^{-8}	0.011875	46	0.009761	(0.000000, 0.000000)	9.290480×10^{-8}

Next, we consider the algorithm with bounded perturbation resilience. Choose the bounded sequence $\{v_n\}$ and the summable nonnegative real sequence $\{\beta_n\}$ as follows:

$$v_n = \begin{cases} -\dfrac{d_n}{\|d_n\|}, & \text{if } 0 \neq d_n \in B(x_n), \\ 0, & \text{if } 0 \in B(x_n), \end{cases}$$

where $B(x_n) = (3x_n(1),\, 9x_n(2))^T,\ x_n(i),\ i = 1, 2$ denote the ith element of x_n, and $\beta_n = c^n$, for some $c \in (0, 1)$. Setting $c = 0.9$, the numerical results can be seen in Table 2.

TABLE 2. $x_0 = rand(2,1)$

ϵ	$\lambda_n = \frac{1}{100L}$	n	Time	x_n	$\|x_{n+1} - x_n\|$
10^{-6}	0.006923	33	0.015636	(0.000000, 0.000000)	6.782486×10^{-7}
10^{-7}	0.006444	60	0.015266	(0.000000, 0.000000)	8.078633×10^{-8}
10^{-8}	0.005079	83	0.015765	(0.000000, 0.000000)	7.836764×10^{-9}

5. Conclusion

We have introduced a viscosity iterative scheme and obtained the strong convergence. We also consider the bounded perturbation resilience of the proposed method and get theoretical convergence results.

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EQUILIBRIUM PROBLEMS AND PROXIMAL ALGORITHMS IN HADAMARD SPACES

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ABSTRACT. In this paper, we consider the equilibrium problems and also their regularized problems under the setting of Hadamard spaces. The solution to the regularized problem is represented in terms of resolvent operators. As a piece of essential machinery in the existence of an equilibrium, we first prove that the KKM principle is attained in general Hadamard spaces without assuming the compactness of the closed convex hull of a finite set. We construct the proximal algorithm based on this regularization and give convergence analysis adequately.

KEYWORDS: Equilibrium problems, Proximal algorithms, KKM principle, Hadamard space.
AMS Subject Classification: : 90C33, 65K15, 49J40, 49M30, 47H05.

1. Introduction

Equilibrium problems were originally studied in [8] as a unifying class of variational problems. Given a nonempty set K, and a bifunction $F : K \times K \to \mathbb{R}$. The equilibrium problem $EP(K,F)$ is formulated as follows:

$$\text{Find a point } \bar{x} \in K \text{ such that } F(\bar{x}, y) \geq 0 \text{ for every } y \in K. \qquad EP(K,F)$$

By assigning different settings to F, we can include, *e.g.,* minimization, minimax inequalities, variational inequalities, and fixed point problems in the class of equilibrium problems. The set of all solutions of $EP(K,F)$ is denoted by $\mathcal{E}(K,F)$. Typical studies for $EP(K,F)$ are extensively carried out in Banach or Hilbert spaces, and recently in Hadamard manifolds.

Proximal algorithm is one of the most elementary method used in solving several classes of variational problems. The main idea of the method is to perturb (or

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regularize) the original problem into a simpler and more well-behaved problem, and solve for each steps the perturbed subproblems. This method was originally proposed under the setting of Hilbert spaces by Martinet [24], and was progressively developed by Rockafellar [30].

Proximal algorithms in Hadamard spaces were recently investigated. In particular, it was introduced for minimizing convex functionals by Bačák in [5], and were extended for solving variational inequalities in [17] and for solving zeros of maximal monotone operators in [18, 10].

Equilibrium problems in Hadamard manifold were studied by [11] and were later extended to equilibrium problems for bifunctions defined on proximal pairs in [9]. Both of the results rely on different variants of the KKM lemma (consult [22] for the original version), but the latter is strongly based on Brouwer's fixed point theorem in Hadamard manifolds (see [27]). In fact, the Brouwer's theorem is proved in Hadamard spaces in [29, 28] and the KKM principle was subsequently proved in Hadamard space with the convex hull finite property (CHFP), *i.e.,* every polytope is assumed to be compact (see also the convex hull property in [15]).

Based on this KKM principle and under CHFP assumption, Kimura and Kishi [19] studied the equilibrium problem and showed the well-definedness of the resolvent associated to a bifunction. In particular, they also proved that this resolvent is firmly nonspreading and has fixed point set identical to the equilibrium points. Finally, they apply a convergence theorem from [20] to solve for an equilibrium.

In this paper, we show that the KKM principle can be extended to any Hadamard spaces without further assumptions. Using the KKM principle, we show the existence of an equilibrium of a bifunction under standard continuity, convexity, and compactness/coercivity assumptions. Then, we deduce several fundamental properties of the resolvent operator introduced by Kimura and Kishi [19]. We also deliver some comparisons with resolvents of convex functionals and of monotone vector fields. We finally define the proximal algorithm by iterating resolvent operators and provide adequate convergence analysis of the algorithm. Apart from dropping the CHFP assumption, the contents presented in this paper provide a continuation from the works of Niculescu and Rovenţa [29, 28] in the study of KKM principle, of Kimura and Kishi [19] in the study of bifunctions, their resolvents, and proximal algorithms, and of the authors [10] in the relationships between bifunctions and vector fields as well as their resolvents and convergence results.

The organization of this paper is as follows. The next section collects useful basic knowledges used in the rest of this paper. We also give in this section several auxiliary results that will be exploited to validate subsequent results. In particular, properties concerning the product $\langle \cdot, \cdot \rangle$ and the weaker convergence notions are explained. Section 3 is devoted to the discussion on the KKM theory on Hadamard spaces. This contains the key tool for proving existence theorems, which will be found in Section 4. We also deduce here the dual problem in the sense of Minty and give relationships with the primal priblem. In Section 5, we introduce the resolvent operator corresponds to a bifunction. Properties of the resolvents are thoroughly deduced, especially with the nonexpansivity of the operator. This leads to the study of Section 7, which contains the construction of the proximal algorithm and also its convergence analysis.

2. Preliminaries and Auxiliaries

We divide this section into several parts, describing each topics in brief details. This includes some basic definitions and natations up to the technical results that will be used in our main results in the next sections.

2.1. **Hadamard spaces.** A uniquely geodesic metric space (X, ρ) is a CAT(0) space if each geodesic triangle in X is at least as thin as its comparison triangle in Euclidean plane. A complete CAT(0) space is then called *Hadamard space*. The following characterization of a CAT(0) space is useful.

Proposition 2.1. *For a geodesic metric space (X, ρ), the following conditions are all equivalent:*

(i) X *is* CAT(0).

(ii) *([13]) For any $x, u, v \in X$ and $\lambda \in [0,1]$, the following inequality holds:*

$$\rho^2(x, \gamma_{u,v}(\lambda)) \le (1-\lambda)\rho^2(x,u) + \lambda\rho^2(x,v) - \lambda(1-\lambda)\rho^2(u,v). \tag{CN}$$

(iii) *([7]) For any $x, y, u, v \in X$, the following inequality holds:*

$$\rho^2(x,v) + \rho^2(y,u) \le \rho^2(x,u) + \rho^2(y,v) + 2\rho(x,y)\rho(u,v). \tag{2.1}$$

Definition 2.2. A convex set $K \subset X$ is said to be *flat* if the (CN) inequality holds as an equality for each $x, u, v \in K$.

Let (X, ρ) be an Hadamard space. For each $x, y \in X$, we write $\gamma_{x,y} : [0,1] \to X$ to denote the normalized geodesic joining x and y, *i.e.*, $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$, and $\rho(\gamma_{x,y}(t), \gamma_{x,y}(t')) = \rho(x,y)\,|t - t'|$ for all $t, t' \in [0,1]$. We also adopt the following notations: $[\![x, y]\!] := \{\gamma_{x,y}(t); t \in [0,1]\}$ and $(1-t)x \oplus ty := \gamma_{x,y}(t)$ for $t \in [0,1]$. A subset $K \subset X$ is said to be *convex* if $[\![x, y]\!] \subset K$ for any $x, y \in K$, and for a set $E \subset X$ we write $\mathrm{co}(E)$ to denote the smallest convex set that contains E. Certainly, we call $\mathrm{co}(E)$ the *convex hull* of E. A function $h : K \to \mathbb{R}$, with K being convex, is called *convex* (resp., *quasi-convex*) if $h \circ \gamma_{x,y} : [0,1] \to \mathbb{R}$ is convex (resp., *quasi-convex*) for any $x, y \in K$. Moreover, h is called *concave* (resp., *quasi-concave*) if $-h$ is convex (resp., *quasi-convex*).

If (X, d) and (X', d') are two Hadamard space, then the product $X \times X'$ is also an Hadamard space with the metric given by

$$\rho\left((x,x'),(y,y')\right) := \left(d^2(x,y) + d'^2(x',y')\right)^{\frac{1}{2}}, \quad \forall (x,x'),(y,y') \in X \times X'.$$

Unless otherwise stated, always assume throughout this paper that (X, ρ) is an Hadamard space and $K \subset X$ is nonempty, closed, and convex. Any product space is also to be understood to be an Hadamard space in the sense described above.

2.2. **Dual space and Tangent spaces.** The concept of a dual space of X was introduced in [2]. Here, we recall such a construction in brief details. Let us write $\overrightarrow{xy} := (x,y) \in X^2$. The *quasilinearization* (see [6, 7]) on X^2 is the product $\langle \cdot, \cdot \rangle : X^2 \times X^2 \to \mathbb{R}$ defined by

$$\langle \overrightarrow{uv}, \overrightarrow{xy} \rangle := \frac{1}{2}[\rho^2(u,y) + \rho^2(v,x) - \rho^2(u,x) - \rho^2(v,y)], \quad \forall \overrightarrow{uv}, \overrightarrow{xy} \in X^2.$$

For each $t \in \mathbb{R}$ and $\overrightarrow{uv} \in X^2$, we define the Lipschitzian function $\Theta(t; \overrightarrow{uv}) : X \to \mathbb{R}$ by

$$\Theta(t; \overrightarrow{uv})(x) := t\langle \overrightarrow{uv}, \overrightarrow{ux} \rangle, \quad \forall x \in X.$$

Recall that for a Lipschitzian function $h : X \to \mathbb{R}$, the Lipschitzian constant of h is the quantity $L(h) := \inf\{\frac{|h(x)-h(y)|}{\rho(x,y)} \,|\, x, y \in X,\, x \neq y\}$. Applying the Lipschizian constant to the differences of Θ's, we can define a pseudometric $\tilde{D}$ on $\mathbb{R} \times X^2$ by:

$$\tilde{D}\left((s, \overrightarrow{uv}), (t, \overrightarrow{xy})\right) := L(\Theta(s; \overrightarrow{uv}) - \Theta(t; \overrightarrow{xy})), \quad \forall (s, \overrightarrow{uv}), (t, \overrightarrow{xy}) \in \mathbb{R} \times X^2,$$

which naturally gives rise to the following equivalence relation

$$(s, \overrightarrow{uv}) \sim (t, \overrightarrow{xy}) \iff \tilde{D}\left((s, \overrightarrow{uv}), (t, \overrightarrow{xy})\right) = 0, \quad \forall (s, \overrightarrow{uv}), (t, \overrightarrow{xy}) \in \mathbb{R} \times X^2.$$

The quotient space $X^* := \mathbb{R} \times X^2/\sim$ with a metric D given by

$$D([(s, \overrightarrow{uv})]_\sim, [(t, \overrightarrow{xy})]_\sim) = \tilde{D}\left((s, \overrightarrow{uv}), (t, \overrightarrow{xy})\right), \quad \forall [(s, \overrightarrow{uv})]_\sim, [(t, \overrightarrow{xy})]_\sim \in X^*,$$

is called the *dual space* of X. For simplicity, we adopt the notation $s\overrightarrow{uv} := [(s, \overrightarrow{uv})]_\sim$. Moreover, we write $\mathbf{0} := s\overrightarrow{uu} = 0\overrightarrow{uv}$ for $s \in \mathbb{R}$ and $u, v \in X$.

In this paper, we restrict to particular subspaces of X^*. For any given $p \in X$, the *tangent space* of X at p is given by $T_pX := \{s\overrightarrow{py} \,|\, s \geq 0,\, y \in X\}$.

The following properties of the quasilinearization are fundamental.

Lemma 2.3. *For every $u, v, z \in X$, the following properties hold:*

(i) $\langle \overrightarrow{uv}, \overrightarrow{uz} \rangle + \langle \overrightarrow{vu}, \overrightarrow{vz} \rangle = \rho^2(u, v) \geq 0.$

(ii) If $\lambda \in [0, 1]$ and $x = \gamma_{z,u}(\lambda)$, it holds that $\lambda\langle \overrightarrow{zu}, \overrightarrow{zv} \rangle \leq \langle \overrightarrow{zx}, \overrightarrow{zv} \rangle$. In addition, if a convex set $K \subset X$ is flat, then the above inequality becomes equality for $u, v, z \in K$.

(iii) If a convex set $K \subset X$ is flat and $z, v \in K$, then $u \in K \mapsto \langle \overrightarrow{zu}, \overrightarrow{zv} \rangle$ is affine on K, i.e., it is both convex and concave on K.

Proof. The proof for (i) is trivial. For (ii), the (CN) inequality gives

$$\begin{aligned}
2\langle \overrightarrow{zx}, \overrightarrow{zv} \rangle &= \rho^2(z, x) + \rho^2(z, v) - \rho^2(x, v) \\
&= \lambda^2\rho^2(z, u) + \rho^2(z, v) - \rho^2(\gamma_{z,u}(\lambda), v) \\
&\geq \lambda^2\rho^2(z, u) + \rho^2(z, v) \\
&\quad - \left[(1-\lambda)\rho^2(z, v) + \lambda\rho^2(u, v) - \lambda(1-\lambda)\rho^2(z, u)\right] \\
&= \lambda\rho^2(z, u) + \lambda\rho^2(z, v) - \lambda\rho^2(u, v) \\
&= 2\lambda\langle \overrightarrow{zu}, \overrightarrow{zv} \rangle,
\end{aligned}$$

which proves the first assertion. Now, if a convex set $K \subset X$ is flat, then the (CN) inequatily becomes equality for $u, v, z \in K$, and the desired result is straightforward from the above proofline.

Next, let us show (iii). Let $p, q \in K$ and $\lambda \in [0, 1]$. It follows that

$$\begin{aligned}
2\langle \overrightarrow{z\gamma_{p,q}(\lambda)}, \overrightarrow{zv} \rangle &= \rho^2(z, \gamma_{p,q}(\lambda)) + \rho^2(z, v) - \rho^2(\gamma_{p,q}(\lambda), v) \\
&= \left[(1-\lambda)\rho^2(z, p) + \rho^2(z, q) - \lambda(1-\lambda)\rho^2(p, q)\right] + \rho^2(z, v) \\
&\quad - \left[(1-\lambda)\rho^2(v, p) + \lambda\rho^2(v, q) - \lambda(1-\lambda)\rho^2(p, q)\right] \\
&= (1-\lambda)\left[\rho^2(z, p) + \rho^2(z, v) - \rho^2(p, v)\right] \\
&\quad + \lambda\left[\rho^2(z, q) + \rho^2(z, v) - \rho^2(q, v)\right] \\
&= 2 \cdot \left[(1-\lambda)\langle \overrightarrow{zp}, \overrightarrow{zv} \rangle + \lambda\langle \overrightarrow{zq}, \overrightarrow{zv} \rangle\right].
\end{aligned}$$

The proof is thus completed. ∎

2.3. **Modes of convergence.** Convergence in the metric topology is known to be irrelevant in some situations, especially in the study of numerical algorithms in infinite dimensional spaces. In this subsection, we recall two alternative modes of convergence for bounded sequences, namely the Δ- and w-convergences. Both of the concepts are identical to weak convergence in Hilbert spaces.

Let us start with the Δ-convergence. Suppose that $(x^k) \subset X$ be a bounded sequence, and define a function $r(\cdot;(x^k)) : X \to [0,\infty)$ by

$$r(x;(x^k)) := \limsup_{k\longrightarrow\infty} \rho(x,x^k), \quad \forall x \in X.$$

The minimizer of this function is known to exists and is unique (see [12]). Following [21] (see also [23]), a bounded sequence (x^k) is said to be Δ-*convergent* to a point $\bar{x} \in X$ if $\bar{x} = \arg\min_{x\in X} r(x;(u^k))$ for any subsequence $(u^k) \subset (x^k)$. In this case, $\bar{x}$ is called the Δ-limit of (x^k). Recall that a bounded sequence is Δ-convergent to at most one point.

The topology that generates the Δ-convergence is unknown in general. However, we still adopt the topological-like notions such as Δ-accumulation points, Δ-closed sets, or Δ-continuity. For instance, a point $u \in X$ is a Δ-*accumulation point* of the sequence $(x^k) \subset X$ if it contains a subsequence that is Δ-convergent to u. A set $K \subset X$ is called Δ-*closed* if each Δ-convergent sequence in K has its Δ-limit in K. A function $f : X \to \mathbb{R}$ is called Δ-*upper semicontinuous* (briefly Δ-*usc*) if its epigraph is Δ-closed in $X \times \mathbb{R}$.

The following proposition gives two most important properties regarding the Δ-convergence that are required in our main theorems.

Proposition 2.4. *Suppose that $(x^k) \subset X$ is bounded. Then, the following properties hold:*

- *(i) ([23]) (x^k) has a Δ-convergent subsequence .*
- *(ii) ([1]) (x^k) is Δ-convergent to $\bar{x} \in X$ if and only if $\limsup_k \langle S, \overrightarrow{\bar{x}x^k}\rangle \leq 0$ for any $S \in T_{\bar{x}}X$.*

Next, let us turn to the notion of w-convergence as introduced in [1]. A bounded sequence $(x^k) \subset X$ is said to be w-*convergence* to a point $\bar{x} \in X$ if $\lim_k \langle S, \overrightarrow{\bar{x}x^k}\rangle = 0$ for any $S \in T_{\bar{x}}X$. With Proposition 2.4, we can see immediately that w-convergence implies Δ-convergence. As was noted in [1, 3], a bounded sequence does not necessarily have a w-convergent subsequence. This motivates the definition of reflexivity in Hadamard spaces, i.e., a convex set $K \subset X$ is said to be *reflexive* if each bounded sequence in K contains a w-convergent subsequence. However, it turns out that the reflexivity of K implies the equivalence between the Δ- and weak convergences. Still, the reflexivity can be useful in obtaining sharper estimates in some situations (see e.g. the proof of Lemma 7.7).

Proposition 2.5 ([4])**.** *A convex set $K \subset X$ is closed if and only if it is Δ-closed.*

Proposition 2.6. *Suppose that a convex set $K \subset X$ is reflexive and a bounded sequence $(x^k) \subset K$ is Δ-convergent. Then, it is w-convergent.*

Proof. Suppose that (x^k) is Δ-convergent to $\bar{x} \in X$. Let us assume to the contrary that (x^k) is not weakly convergent. Equivalently, there must exists $\gamma_0 \in T_{\bar{x}}X$ such that

$$\liminf_{k\longrightarrow\infty} \langle \overrightarrow{\bar{x}x^k}, \gamma_0\rangle < \limsup_{k\longrightarrow\infty} \langle \overrightarrow{\bar{x}x^k}, \gamma_0\rangle \leq 0. \tag{2.2}$$

Suppose that $(x^{k_j}) \subset (x^k)$ is a subsequence such that

$$\lim_{j\longrightarrow\infty} \langle \overrightarrow{\bar{x}x^{k_j}}, \gamma_0 \rangle = \liminf_{k\longrightarrow\infty} \langle \overrightarrow{\bar{x}x^{k}}, \gamma_0 \rangle.$$

Take into account the inequality (2.2), we get $\lim_j \langle \overrightarrow{\bar{x}x^{k_j}}, \gamma_0 \rangle < 0$, which prevents (x^k) from having a weakly convergent subsequence. Since (x^k) is bounded in K, this violates the reflexivity of K. Therefore, (x^k) must be weakly convergent to $\bar{x}$. ∎

2.3.1. *Fejér convergence.* Lastly, the notion of Fejér convergence is essential and will play a central role in the proof of our main convergence theorems. This notion encapsulates the improvement at each iteration of some approximate sequence towards a solution set.

Definition 2.7. A sequence $(x^k) \subset X$ is said to be *Fejér convergent* with respect to a nonempty set $V \subset X$ if for each $x \in V$, we have $\rho(x^{k+1}, x) \leq \rho(x^k, x)$ for all large $k \in \mathbb{N}$.

Proposition 2.8 ([10])**.** *Suppose that $(x^k) \subset X$ is Fejér convergent to a nonempty set $V \subset X$. Then, the following are true:*

- *(i)* *(x^k) is bounded.*
- *(ii)* *$(\rho(x, x^k))$ converges for any $x \in V$.*
- *(iii)* *If every Δ-accumulation point lies within V, then (x^k) is Δ-convergent to an element in V.*

3. The KKM Principle

The KKM principle was initiated in [22], and was successfully extended into topological vector spaces by Fan in [14]. Further extensions into Hadamard manifold was discussed in [11, 9]. The results can be generalized instantly also into Hadamard spaces with fixed point property for continuous mappings defined on a convex hull of finite points [28]. Here we prove the KKM principle without using such condition on the space. First, let us recall the original statement of [22] and another additional result.

Lemma 3.1 ([22])**.** *Let $C_1, \ldots, C_m$ be closed subsets of the standard $(m-1)$-simplex σ. If $\operatorname{co}(\{x_i \ ; \ i \in I\}) \subset \bigcup_{i\in I} C_i$ for each $I \subset \{1, \ldots, m\}$, then the intersection $\bigcap_{j=1}^m C_j$ is nonempty.*

Lemma 3.2. *Suppose that (x^k) is a sequence in X and define the following sequence of sets by induction:*

$$\begin{cases} D_1 := \{x_1\}, \\ D_j := \{z \in [\![x^j, y]\!] \ ; \ y \in D_{j-1}\}, \quad \text{for } j = 2, 3, \ldots \end{cases} \tag{3.1}$$

Then, D_j is compact for all $j \in \mathbb{N}$.

Proof. We shall prove the statement by using mathematical induction. It is obvious that D_1 is compact. Now, we enter the inductive step by assuming D_j is compact and show that D_{j+1} must be compact.

Suppose that (u^k) is an arbitrary sequence in D_{j+1}. By definition, there correspond sequences (v^k) in D_j and (t_k) in $[0, 1]$ such that

$$u^k = \gamma_{x^{j+1}, v^k}(t_k), \quad \forall k \in \mathbb{N}.$$

Since both D_j and $[0,1]$ are compact, we may find convergent subsequences (v^{k_i}) of (v^k) and (t_{k_i}) of (t_k) with limits $\bar{v} \in D_j$ and $\bar{t} \in [0,1]$, respectively. Set $\bar{u} := \gamma_{x^{j+1},\bar{v}}(\bar{t})$, we now show that (u^{k_i}) is in fact convergent to $\bar{u}$.

Observe that

$$\begin{aligned}
\rho(u^{k_i}, \bar{u}) &= \rho(\gamma_{x^{j+1},v^{k_i}}(t_{k_i}), \gamma_{x^{j+1},\bar{v}}(\bar{t})) \\
&\leq \rho(\gamma_{x^{j+1},v^{k_i}}(t_{k_i}), \gamma_{x^{j+1},\bar{v}}(t_{k_i})) + \rho(\gamma_{x^{j+1},\bar{v}}(t_{k_i}), \gamma_{x^{j+1},\bar{v}}(\bar{t})) \\
&\leq t_{k_i}\rho(v^{k_i}, \bar{v}) + |t_{k_i} - \bar{t}|\,\rho(x^{j+1}, \bar{v}).
\end{aligned}$$

Passing $i \longrightarrow \infty$, we obtain from the above inequalities that $u^{k_i} \longrightarrow \bar{u}$. This guaratees the compactness of D_{j+1}, and the desired conclusion is thus proved. ∎

Theorem 3.3 (The KKM Principle)**.** *Let $K \subset X$ be a closed convex set, $G : K \rightrightarrows K$ be a set-valued mapping with closed values. Suppose that for any finite subset $D := \{x_1, \ldots, x_m\} \subset K$, it holds the following inclusion:*

$$\operatorname{co}(D) \subset G(D). \tag{3.2}$$

Then, the family $\{G(x)\}_{x\in K}$ has the finite intersection property. Moreover, if $G(x_0)$ is compact for some $x_0 \in K$, then $\bigcap_{x\in K} G(x) \neq \varnothing$.

Proof. Let $D := \{x_1, \ldots, x_m\}$ be an arbitrary finite subset of K. For each $j = 1, 2, \ldots, m$, define D_j by using (3.1). Also, set $D^* := \bigcup_{j=1}^m D_j$.

Let $\sigma := \langle e_1, \ldots, e_m\rangle$ be the standard $(m-1)$-simplex in $\mathbb{R}^m$. Suppose that $\lambda_2 \in \langle e_1, e_2\rangle$, then we can represent λ_2 with a scalar $s_1 \in [0,1]$ such that $\lambda = (1-s_1)e_2 + s_1 e_1$. Now, if $\lambda_3 \in \langle e_1, e_2, e_3\rangle$, then $\lambda_3 = (1-s_2)e_3 + s_2\lambda_2$ for some $\lambda_2 \in \langle e_1, e_2\rangle$. By the earlier fact, λ_2 is represented by some $s_1 \in [0,1]$. Hence, λ_3 can be represented by a 2-dimensional vector $[s_1\ s_2]^\top \in [0,1]^2$. Likewise for $j = 3, \ldots, m$, we can see that any $\lambda_j \in \langle e_1, \ldots, e_j\rangle$ is represented by a $(j-1)$-dimensional vector $[s_1\ \ldots\ s_{j-1}]^\top \in [0,1]^{j-1}$. We shall adopt the notation $\lambda_j \equiv [s_1\ \ldots\ s_{j-1}]^\top$ for the above representation.

Define a mapping $T : \sigma \to D^*$ by induction as follows: if $\lambda_2 \equiv s_1 \in \langle e_1, e_2\rangle$, let $T(\lambda_2) := \gamma_{x_2,x_1}(s_1)$. For $j = 2, 3, \ldots, m$, if $\lambda_j \equiv [s_1\ \ldots\ s_{j-1}]^\top \in \langle e_1, \ldots, e_j\rangle \setminus \langle e_1, \ldots, e_{j-1}\rangle$, then define $T(\lambda_j) := \gamma_{x_j,T(\lambda_{j-1})}(s_{j-1})$, where $\lambda_{j-1} \equiv [s_1\ \ldots\ s_{j-2}]^\top \in \langle e_1, \ldots, e_{j-1}\rangle$.

We now show that T is continuous. Let $\lambda_m, \mu_m \in \sigma$, and $\lambda \equiv [s_1\ \ldots\ s_{m-1}]^\top$ and $\mu \equiv [t_1\ \ldots\ t_{m-1}]^\top$ respectively. For simplicity, let $\lambda_j \equiv [s_1\ \ldots\ s_{j-1}]^\top$ and $\mu_j \equiv [t_1\ \ldots\ t_{j-1}]^\top$, for $j = 1, \ldots, m$. Indeed, we have

$$\begin{aligned}
\rho(T(\lambda_m), T(\mu_m)) &= \rho(\gamma_{x_m,T(\lambda_{m-1})}(s_{m-1}), \gamma_{x_m,T(\mu_{m-1})}(t_{m-1})) \\
&\leq \rho(\gamma_{x_m,T(\lambda_{m-1})}(s_{m-1}), \gamma_{x_m,T(\lambda_{m-1})}(t_{m-1})) \\
&\quad + \rho(\gamma_{x_m,T(\lambda_{m-1})}(t_{m-1}), \gamma_{x_m,T(\mu_{m-1})}(t_{m-1})) \\
&\leq |s_{m-1} - t_{m-1}|\operatorname{diam}(D^*) + \rho(T(\lambda_{m-2}), T(\mu_{m-2})) \\
&\ \ \vdots \\
&\leq \sum_{i=1}^{m-1} |s_i - t_i|\operatorname{diam}(D^*).
\end{aligned}$$

This is sufficient to guarantee the continuity of T.

For each $j = 1, \ldots, m$, define $E_j := T^{-1}(D^* \cap G_i(x_j)) \subset \sigma$. By the continuity of T and Lemma 3.1, we may see that E_j's are closed sets. Suppose that $I \subset \{1, \ldots, m\}$,

$\lambda \in \operatorname{co}(\{e_i\ ;\ i \in I\})$ and $\lambda \equiv [s_1\ \ldots\ s_{m-1}]^\top$. Then, $s_j = 1$ if $j \notin I$. By the definition of T, we have

$$T(\lambda) \in \operatorname{co}(\{x_i\ ;\ i \in I\}) \subset \bigcup_{i \in I} G(x_i).$$

It follows that $T(\lambda) \in D^* \cap G(x_i)$ for some $i \in I$. In other words, we have $\lambda \in E_i$ and therefore $\operatorname{co}(\{e_i\ ;\ i \in I\}) \subset \bigcup_{i \in I} E_i$ for any $I \subset \{1, \ldots, m\}$. By applying the original KKM covering lemma, we get the existence of $\lambda^* \in \bigcap_{j=1}^m E_j$. Hence, we get $T(\lambda^*) \in \bigcap_{j=1}^m G(x_j)$. In fact, we have proved that the family of closed sets $\{G(x)\}_{x \in X}$ has the finite intersection property. The nonemptiness of the intersection $\bigcap_{x \in X} G(x) = \bigcap_{x \in X}(G(x) \cap G(x'))$ follows from the fact that $\{G(x) \cap G(x')\}_{x \in X}$ is a family of closed subsets with finite intersection property in a compact subspace $G(x')$. ∎

4. Solutions of Equilibrium Problems

We mainly discuss in this section two topics. First, we deduce the solvability of the problem $EP(K, F)$ under standard (semi)continuity, convexity, and compactness/coercivity assumptions. Secondly, we introduce the dual problem to $EP(K, F)$ in the sense of Minty and show the relationship between the primal and dual problems, in terms of their solutions. Note that the knowledge of the dual problem is required in the subsequent sections.

4.1. Existence theorems.

Based on the KKM principle in the previous section, we can show under some natural assumptions that the equilibrium problem $EP(K, F)$ is solvable.

Theorem 4.1. *Suppose that $K \subset X$ is closed convex, and $F : K \times K \to \mathbb{R}$ is a bifunction satisfying the following properties:*

- *(A1) $F(x, x) \geq 0$ for each $x \in K$.*
- *(A2) For every $x \in K$, the set $\{y \in K\ ;\ F(x, y) < 0\}$ is convex.*
- *(A3) For every $y \in K$, the function $x \mapsto F(x, y)$ is usc.*
- *(A4) There exists a compact subset $L \subset K$ containing a point $y_0 \in L$ such that $F(x, y_0) < 0$ whenever $x \in K \setminus L$.*

Then, the problem $EP(K, F)$ has a solution and the solution set $\mathcal{E}(K, F)$ is closed. In addition, if the function $x \mapsto F(x, y)$ is quasi-concave for every fixed $y \in K$, then $\mathcal{E}(K, F)$ is convex.

Proof. Define a set-valued mapping $G : K \rightrightarrows K$ by

$$G(y) := \overline{\operatorname{lev}}_{F(\cdot, y)}(0) = \{x \in K\ ;\ F(x, y) \geq 0\}, \quad \forall y \in K.$$

It is clear that $EP(K, F)$ has a solution if and only if $\bigcap_{y \in K} G(y) \neq \varnothing$. Thus, it is sufficient to show the nonemptiness of $\bigcap_{y \in K} G(y)$.

By (A1) and (A2), we may see that G has nonempty closed values at all $y \in K$. We will show now that G satisfies the inclusion (3.2). Let us suppose to the contrary that there exist $y_1, \ldots, y_m \in K$ such that $\operatorname{co}(\{y_1, \ldots, y_m\}) \not\subset \bigcup_{j=1}^m G(y_i)$. That is, there exists a point $y^* \in \operatorname{co}(\{y_1, \ldots, y_m\})$ such that $y^* \notin G(y_i)$ for any $i = 1, \ldots, m$. It further implies that

$$F(y^*, y_i) < 0, \quad \forall i = 1, \ldots, m.$$

Moreover, we have for all $i = 1, \ldots, m$, $y_i \in \{y \in K\ ;\ F(y^*, y) < 0\}$, which is a convex set by hypothesis. Since $y^* \in \operatorname{co}(\{y_1, \ldots, y_m\})$ and $\operatorname{co}(\{y_1, \ldots, y_m\})$ is the smallest convex set containing $y_1, \dots, y_m$, we get

$$y^* \in \operatorname{co}(\{y_1, \dots, y_m\}) \subset \{y \in K \; ; \; F(y^*, y) < 0\},$$

which further gives $F(y^*, y^*) < 0$. This contradicts hypothesis (A1), and therefore G satisfies (3.2). On the other hand, hypothesis (A4) forces $G(y_0) \subset L$, which guarantees the compactness of $G(y_0)$. Since G satisfies every conditions of Lemma 3.3, we get $\bigcap_{x\in X} G(x) \neq \varnothing$.

Next, suppose that $(x^k) \subset \mathcal{E}(K, F)$ is convergent to $\bar{x} \in K$, then (A3) gives

$$F(\bar{x}, y) \geq \limsup_{k\longrightarrow\infty} F(x^k, y) \geq 0, \quad \forall y \in K.$$

So, we have $\bar{x} \in \mathcal{E}(K, F)$ and hence the closedness of $\mathcal{E}(K, F)$.

Now, assume that $x \mapsto F(x, y)$ is quasi-concave for every fixed $y \in K$. Let $\hat{x}, \hat{y} \in \mathcal{E}(K, F)$ and $t \in [0, 1]$. Then, we have

$$F(\gamma_{\hat{x},\hat{y}}(t), y) \geq \min\{F(\hat{x}, y), F(\hat{y}, y)\} \geq 0, \quad \forall y \in K,$$

which implies the convexity of $\mathcal{E}(K, F)$. ∎

Next, we deduce an existence theorem for a compact convex domain.

Corollary 4.2. *Suppose that $K \subset X$ is compact convex, and $F : K \times K \to \mathbb{R}$ is a bifunction satisfying the following properties:*

(A1) $F(x, x) \geq 0$ for each $x \in K$.
(A2) For every $x \in K$, the set $\{y \in K \; ; \; F(x, y) < 0\}$ is convex.
(A3) For every $y \in K$, the function $x \mapsto F(x, y)$ is usc.

Then, the problem $EP(K, F)$ has a solution and the solution set $\mathcal{E}(K, F)$ is closed. In addition, if the function $x \mapsto F(x, y)$ is quasi-concave for every fixed $y \in K$, then $\mathcal{E}(K, F)$ is convex.

Proof. Apply Theorem 4.1 with $L = K$ in the hypothesis (A4). ∎

4.2. **Dual Equilibrium Problem.** It is essential to also introduce here the duality to the problem $EP(K, F)$ in the sense of Minty. To clarify the terminology, we shall sometimes refer to $EP(K, F)$ as the *primal problem*. The *dual equilibrium problem* to $EP(K, F)$, denoted by $EP^*(K, F)$, is given as follows:

$$\text{Find a point } \bar{x} \in K \text{ such that } F(y, \bar{x}) \leq 0 \text{ for every } y \in K. \qquad EP^*(K, F)$$

The solution to $EP^*(K, F)$ will also be called the *dual solution to* $EP(K, F)$, and the set of such dual solutions is then denoted by $\mathcal{E}^*(K, F)$.

Definition 4.3. F is called *monotone* if $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$.

If F is a monotone bifunction, we immediately have the inclusion $\mathcal{E}(K, F) \subset \mathcal{E}^*(K, F)$. To obtain the converse, we need additional assumptions on the bifunction F.

Proposition 4.4. *Suppose that $F : K \times K \to \mathbb{R}$ is a bifunction.*

(i) If F is monotone, then $\mathcal{E}(K, F) \subset \mathcal{E}^*(K, F)$.
(ii) If (A1), (A3) holds, and F is convex in the second variable, then $\mathcal{E}^*(K, F) \subset \mathcal{E}(K, F)$.

In particular, if F is monotone, convex in the second variable, and satisfies (A1) and (A3), then $\mathcal{E}^*(K, F) = \mathcal{E}(K, F)$.

Proof. (i) Let $\bar{x} \in \mathcal{E}(K,F)$. The monotonicity of F then gives $F(y,\bar{x}) \leq -F(\bar{x},y) \leq 0$, so that $\bar{x} \in \mathcal{E}^*(K,F)$.

(ii) Suppose that $\bar{x} \in \mathcal{E}^*(K,F)$. Let $y \in K$ be arbitrary. Take into account (A1), we have

$$0 \leq F(\gamma_{\bar{x},y}(t),\gamma_{\bar{x},y}(t)) \leq (1-t)F(\gamma_{\bar{x},y}(t),\bar{x}) + tF(\gamma_{\bar{x},y}(t),y) \leq F(\gamma_{\bar{x},y}(t),y),$$

for any $t \in [0,1]$. In view of (A3), we have $0 \leq \limsup_{t\longrightarrow 0} F(\gamma_{\bar{x},y}(t),y) \leq F(\bar{x},y)$. This gives $\bar{x} \in \mathcal{E}(K,F)$. ∎

5. Resolvents of Bifunctions

The resolvent is a fundamental notion in regularization of certain variational problems. It is understood as the solution set of particular regularized problem from the original one. Here, we introduce the resolvent of a bifunction F in relation to the equilibrium problem $EP(K,F)$.

For simplicity, we shall adopt the following perturbation $\tilde{F}$ of a given bifunction F. That is, given a bifunction $F : K \times K \to \mathbb{R}$, $\bar{x} \in X$, we define the function $\tilde{F}_{\bar{x}} : K \times K \to \mathbb{R}$ by

$$\tilde{F}_{\bar{x}}(x,y) := F(x,y) - \langle \overrightarrow{x\bar{x}}, \overrightarrow{xy} \rangle, \quad \forall x,y \in K.$$

We are now ready to construct the resolvent of a bifunction F as the unique equilibrium of a perturbed bifuntion.

Definition 5.1 ([19])**.** Suppose that $K \subset X$ is closed convex, and $F : K \times K \to \mathbb{R}$. The *resolvent* of F is the mapping $J_F : X \rightrightarrows K$ defined by

$$J_F(x) := \mathcal{E}(K,\tilde{F}_x) = \{z \in K \mid F(z,y) - \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \geq 0,\, \forall y \in K\}, \quad \forall x \in X.$$

The question well-definedness of J_F is important. Kimura and Kishi [19] showed a well-definedness result under the CHFP assumption. The use of CHFP assumption is to trigger the KKM principle of Niculescu and Rovenţa [28]. However, we can replace such KKM principle of Niculescu and Rovenţa [28] with our Theorem 3.3 and follows the same proof from [19] to obtain the well-definedness of J_F. Thus, we obtain Kimura and Kishi's theorem without CHFP.

Theorem 5.2. *Suppose that F has the following properties:*

(i) $F(x,x) = 0$ for all $x \in K$.
(ii) F is monotone.
(iii) For each $x \in K$, $y \mapsto F(x,y)$ is convex and lsc.
(iv) For each $y \in K$, $F(x,y) \geq \limsup_{t\downarrow 0} F(\gamma_{x,z}(t),y)$ for all $x,z \in K$.

Then $\operatorname{dom}(J_F) = X$ and J_F is single-valued.

Note that if we drop the assumption that $y \mapsto F(x,y)$ is lsc, we get the following result by imposing the flatness and local compactness.

Proposition 5.3. *Suppose that hypotheses (A1), (A3), and (A4) from Theorem 4.1 are satisfied. Suppose also that the following additional assumptions hold:*

(i) F is monotone.
(ii) for any fixed $x \in K$, $y \mapsto F(x,y)$ is convex.

If a nonempty convex subset $N \subset X$ is flat and locally compact, then the resolvent J_F is defined for all $x \in N$. That is, $N \subset \operatorname{dom}(J_F)$.

Proof. Fix $\tilde{x} \in N$, we define a bifunction $g_{\tilde{x}} : K \times K \to \mathbb{R}$ by

$$g_{\tilde{x}}(z, y) := F(z, y) - \langle \overrightarrow{z\tilde{x}}, \overrightarrow{zy} \rangle, \quad \forall z, y \in K.$$

Then $g_{\tilde{x}}(z, z) \geq 0$ for all $z \in K$, and $z \mapsto g_{\tilde{x}}(z, y)$ is usc for any fixed $y \in K$. Since K is flat, the mapping $y \mapsto \langle \overrightarrow{z\tilde{x}}, \overrightarrow{zy} \rangle$ is affine by (iii) of Lemma 2.3. Therefore, the set $\{y \in K \; ; \; g_{\tilde{x}}(z, y) < 0\}$ is convex for each $z \in K$.

Since all assumptions of Theorem 4.1 are satisfied, F has an equilibrium $\bar{x} \in K$. Now, using the monotonicity of F and (i) of Lemma 2.3, we get

$$\begin{aligned} g_{\tilde{x}}(z, \bar{x}) &= F(z, \bar{x}) - \langle \overrightarrow{z\tilde{x}}, \overrightarrow{z\bar{x}} \rangle \\ &\leq -F(\bar{x}, z) + \langle \overrightarrow{\bar{x}\tilde{x}}, \overrightarrow{\bar{x}z} \rangle - \rho^2(\bar{x}, z) \\ &\leq (\rho(\bar{x}, \tilde{x}) - \rho(\bar{x}, z))\rho(\bar{x}, z). \end{aligned}$$

Set $L := K \cap \operatorname{cl} A(\bar{x}; \rho(\bar{x}, \tilde{x}))$, where $\operatorname{cl}(\cdot)$ denotes the closure operator. Since K is locally compact and L is closed and bounded, we get the compactness of L. Moreover, we have $\bar{x} \in L$ and $g_{\tilde{x}}(z, \bar{x}) < 0$ for all $z \in K \setminus L$. In fact, we have just showed that $g_{\tilde{x}}$ verifies all assumptions of Theorem 4.1, and therefore $g_{\tilde{x}}$ has an equilibrium $\bar{z} \in K$. Equivalently, we have proved that $\bar{z} \in J_F(\tilde{x})$. Since $\tilde{x} \in N$ is chosen at arbitrary, the desired result is attained. ∎

The following proposition gives several valuable properties for a resolvent of a monotone bifunction.

Proposition 5.4. *Suppose that F is monotone and $\operatorname{dom}(J_F) \neq \varnothing$. Then, the following properties hold.*

(i) J_F is single-valued.
(ii) If $\operatorname{dom}(J_F) \supset K$, then J_F is nonexpansive restricted to K.
(iii) If $\operatorname{dom}(J_{\mu f}) \supset K$ for any $\mu > 0$, then $Fix(J_F) = \mathcal{E}(K, F)$.

Proof. (i) Let $x \in \operatorname{dom}(J_F)$ and suppose that $z, z' \in J_F(x)$. So, we get

$$\begin{cases} F(z, z') \geq \langle \overrightarrow{zx}, \overrightarrow{zz'} \rangle, \\ F(z', z) \geq \langle \overrightarrow{z'x}, \overrightarrow{z'z} \rangle. \end{cases}$$

By summing up the two inequalities, applying the monotonicity of F and (i) of Lemma 2.3, we obtain

$$0 \geq F(z, z') + F(z', z) \geq \langle \overrightarrow{zx}, \overrightarrow{zz'} \rangle + \langle \overrightarrow{z'x}, \overrightarrow{z'z} \rangle = \rho^2(z, z').$$

This shows $z = z'$.

(ii) Let $x, y \in K$. By the definition of J_F, we have

$$\begin{cases} F(J_F(x), J_F(y)) - \langle \overrightarrow{J_F(x)x}, \overrightarrow{J_F(x)J_F(y)} \rangle \geq 0, \\ F(J_F(y), J_F(x)) - \langle \overrightarrow{J_F(y)y}, \overrightarrow{J_F(y)J_F(x)} \rangle \geq 0. \end{cases}$$

Summing the two inequalities above yields

$$\begin{aligned} 0 &\geq \langle \overrightarrow{J_F(x)x}, \overrightarrow{J_F(x)J_F(y)} \rangle + \langle \overrightarrow{J_F(y)y}, \overrightarrow{J_F(y)J_F(x)} \rangle \\ &= \left[\rho^2(J_F(x), x) + \rho^2(J_F(x), J_F(y)) - \rho^2(x, J_F(y))\right] \\ &\quad + \left[\rho^2(J_F(y), y) + \rho^2(J_F(x), J_F(y)) - \rho^2(y, J_F(x))\right]. \end{aligned}$$

Rearraging terms in the above inequality and apply Proposition 2.1, we get

$$\rho^2(J_F(x), J_F(y)) \leq \frac{1}{2}\left[\rho^2(x, J_F(y)) - \rho^2(y, J_F(x)) - \rho^2(x, J_F(x)) - \rho^2(y, J_F(y))\right]$$

$$\leq \rho(x,y)\rho(J_F(x), J_F(y)).$$

This shows the nonexpansivity of J_F.

(iii) Let $x \in K$. Observe that

$$\begin{aligned} x \in Fix(J_F) &\iff x = J_F(x) \\ &\iff F(x,y) - \langle \overrightarrow{xx}, \overrightarrow{xy} \rangle \geq 0, \ \forall y \in K \\ &\iff F(x,y) \geq 0, \ \forall y \in K \\ &\iff x \in \mathcal{E}(K,F). \end{aligned}$$

∎

6. Bifunction Rosolvents and Other Resolvents

In this section, we consider two special cases of equilibrium problem, where the resolvent associated to a bifunction defined in Section 5 conincides with the resolvents designed for solving different variational problems. In particular, we deduce that our resolvent reduces to the Moreau-Yosida resolvent for a convex functional, and also to resolvents corresponding maximal monotone operators.

6.1. **Resolvents of convex functionals.** Always assume that $g : X \to \mathbb{R}$ is convex and lsc. The proximal of g is the operator $\text{prox}_g : X \to X$ given by

$$\text{prox}_g(x) := \arg\min_{y \in X} \left[g(y) + \frac{1}{2}\rho^2(y,x) \right], \quad \forall x \in X.$$

Also, recall (from [26, 10]) that a subdifferential of g is the set-valued vector field $\partial g : X \rightrightarrows X^*$ given by

$$\partial g(x) := \{\gamma \in T_x X \mid g(y) \geq g(x) + \langle \gamma, \overrightarrow{xy} \rangle, \ \forall y \in X\},$$

for $x \in X$. It is shown that $\text{dom}(\partial g)$ is dense in $\text{dom}(g)$ (in this case, $\text{dom}(g) = X$) [26]. Moreover, we have the following lemma.

Lemma 6.1 ([10])**.** *Given $w, z \in X$, then $\overrightarrow{zw} \in \partial g(z)$ if and only if $z = \text{prox}_g(w)$.*

It is immediate that minimizing this functional g is a particular equilibrium problem. We make the following statement explicit only for the completeness.

Lemma 6.2. *Define $F_g : K \times K \to \mathbb{R}$ by*

$$F_g(x,y) := g(y) - g(x), \quad \forall x, y \in K. \tag{6.1}$$

Then, we have $\mathcal{E}(K, F_g) = \arg\min_K g$ and $J_{F_g} = \text{prox}_g$. Moreover, we have $\text{dom}(\text{prox}_g) = X$.

Proof. The fact that $\mathcal{E}(K, F_g) = \arg\min_K g$ is obvious. Now, Lemma 6.1 implies the following relations:

$$\begin{aligned} z = J_{F_g}(x) &\iff F_g(z,y) - \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \geq 0, \quad \forall y \in K \\ &\iff g(y) \geq g(z) + \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle, \quad \forall y \in K \\ &\iff \overrightarrow{zx} \in \partial g(z) \\ &\iff z = \arg\min_{y \in X} \left[g(y) + \frac{1}{2}\rho^2(y,x) \right] = \text{prox}_g(x) \end{aligned}$$

The fact that $\text{dom}(\text{prox}_g) = X$ was proved in [16, 25]. ∎

6.2. **Resolvents of monotone operators.** Following [10], the set-valued operator $A : X \rightrightarrows X^*$ is called a *vector field* if $A(x) \subset T_xX$ for every $x \in X$. Moreover, it is said to be *monotone* if $\langle x^*, \overrightarrow{xy}\rangle \leq -\langle y^*, \overrightarrow{yx}\rangle$ for every $(x, x^*), (y, y^*) \in \operatorname{grp}(A)$. It is said to be *maximally monotone* if A is monotone and $\operatorname{grp}(A)$ is not properly contained in a graph of another monotone vector field.

We shall make an observation in this section that the following *stationary problem* (or *zero point problem*) associated to a monotone set-valued vector field $A : X \rightrightarrows X^*$:

$$\text{Find a point } \bar{x} \in K \text{ such that } \mathbf{0} \in A(\bar{x}), \tag{$A^{-1}\mathbf{0}$}$$

can be viewed in terms of an equilibrium problem for some bifuntion F_A. The solution set is naturally expressed by $A^{-1}\mathbf{0}$.

Beforehand, let us give a restatement of a result given by [10] in the language of this paper. In fact, this states the equivalence between the singularity priblem and the Minty variational inequality of A.

Proposition 6.3. *Suppose that $A : X \rightrightarrows X^*$ is a monotone vector field with $\operatorname{dom}(A) = X$. Define $F_A : X \times X \to \mathbb{R}$ by*

$$F_A(x, y) := \sup_{\xi \in A(x)} \langle \xi, \overrightarrow{xy}\rangle, \quad \forall x, y \in X, \tag{6.2}$$

Then, $A^{-1}\mathbf{0} = \mathcal{E}(X, F_A) = \mathcal{E}^*(X, F_A)$.

Proof. First, we note that the finiteness of F_A is guaranteed readily from the monotonicity of A. Moreover, the fact that $\mathcal{E}(X, F_A) \subset \mathcal{E}^*(X, F_A) = A^{-1}\mathbf{0}$ follows from the monotonicity of F_A, [10, Lemma 3.5], and [10, Lemma 3.6]. It therefore suffices to show that $A^{-1}\mathbf{0} \subset \mathcal{E}(X, F_A)$. So, let $\mathbf{0} \in A(\bar{x})$. Then

$$0 = \langle \overrightarrow{\bar{x}\bar{x}}, \overrightarrow{\bar{x}y}\rangle \leq \sup_{\bar{\xi} \in A(\bar{x})} \langle \bar{\xi}, \overrightarrow{\bar{x}y}\rangle = F_A(\bar{x}, y), \quad \forall y \in X,$$

meaning that $\bar{x} \in \mathcal{E}(X, F_A)$. ∎

Now that we have the equivalence between the three variational problems, we continue to show that their resolvents coincide, provided that A is maximally monotone. Recall from [10] that the resolvent operator of A, denoted by R_A, is defined by $R_A(x) := \{z \in X \mid \overrightarrow{zx} \in A(z)\}$, and is single-valued.

Proposition 6.4. *Suppose that $A : X \rightrightarrows X^*$ is a maximally monotone vector field with $\operatorname{dom}(A) = X$, and F_A is defined by (6.2). Then, $J_{F_A} = R_A$.*

Proof. Let $x \in X$ be given. Then, we have

$$\begin{aligned} z = R_A(x) &\iff \overrightarrow{zx} \in A(z) \\ &\implies \langle \overrightarrow{zx}, \overrightarrow{zy}\rangle \leq \sup_{\eta \in A(z)} \langle \eta, \overrightarrow{zy}\rangle, \quad \forall y \in X \qquad (6.3) \\ &\iff z \in J_{F_A}(x). \end{aligned}$$

The converse of (6.3) follows from the maximal monotonicity of A. ∎

7. Proximal Algorithms

It is very natural to ask about the proximal algorithm after defining a proper resolvent operator. Let us officially define the proximal algorithm for a bifunction $F : K \times K \to \mathbb{R}$ with $\operatorname{dom}(\mu F) \supset K$ for all $\mu > 0$. Let $(\lambda_k) \subset (0, \infty)$ be the step-size sequence. The *proximal algorithm* with step sizes (λ_k) started at an initial guess $x^0 \in K$ is the sequence $(x^k) \subset K$ generated by

$$x^k := J_{\lambda_k F}(x^{k-1}), \quad \forall k \in \mathbb{N}. \tag{Prox}$$

7.1. **Convergence in functional values.** Before we continue any further, let us give a small remark on the error measurement of this proximal algorithm. For instance, at each $k \in \mathbb{N}$, we can use the definition of the resolvent J_F and the Cauchy-Schwarz inequality to deduce:

$$\inf_{y\in K} F(x^{k+1}, y) \geq -\frac{1}{\lambda_k}\rho(x^{k+1}, x^k) \sup_{y\in K} \rho(x^{k+1}, y), \tag{7.1}$$

which is useful when as a stopping criterion for the convergence, especially when the rate of asymptotic regularity is known. Note that the above estimate (7.1) will later be made sharp and precise to guarantee the convergence of the proximal algorithm (see Theorem 7.3 and 7.8, for instance). If the asymptotic regularity does not hold or the rate is unknown but K is bounded, we otherwise have

$$\inf_{y\in K} F(x^{k+1}, y) \geq -\frac{1}{\lambda_k} \operatorname{diam}(K)^2, \tag{7.2}$$

which leads directly to the convergence of functional value presented in the next theorem.

Theorem 7.1. *Suppose that* $\operatorname{dom}(J_{\mu F}) \supset K$ *for all* $\mu > 0$ *and* K *is bounded closed convex. If* $\lambda_k \longrightarrow \infty$, *then the proximal algorithm generates a sequence* $(x^k) \subset K$ *such that* $\lim_k F(x^k, y) = 0$ *for any* $y \in K$.

Proof. Follows from (7.2). ∎

7.2. **Convergence of Proximal Algorithms.** Here, we provide a convergence theorem for proximal algorithm (Prox) for step sizes (λ_k). To prove the convergence, we first show the following 'obtuse angle' property, which gives relationship between an arbitrary point $\bar{x} \in X$, the perturbed equilibrium $\tilde{x} = J_{\mu F}(\bar{x}) \in \mathcal{E}(K, \widetilde{\mu F}_{\bar{x}})$, and the exact equilibrium $x^* \in \mathcal{E}(K, F)$.

Lemma 7.2. *Assume that* F *is monotone. Let* $\bar{x} \in X$, $\mu > 0$, $\tilde{x} \in \mathcal{E}(K, \widetilde{\mu F}_{\bar{x}})$ *and* $x^* \in \mathcal{E}(K, F)$, *then* $\langle \overrightarrow{\tilde{x}\bar{x}}, \overrightarrow{\tilde{x}x^*} \rangle \leq 0$.

Proof. Since $\tilde{x} \in \mathcal{E}(K, \tilde{F}_{\bar{x}})$, we have

$$0 \leq \widetilde{\mu F}_{\bar{x}}(\tilde{x}, x^*) = \mu F(\tilde{x}, x^*) - \langle \overrightarrow{\tilde{x}\bar{x}}, \overrightarrow{\tilde{x}x^*} \rangle,$$

which implies that $\langle \overrightarrow{\tilde{x}\bar{x}}, \overrightarrow{\tilde{x}x^*} \rangle \leq \mu F(\tilde{x}, x^*)$. Now, since $x^* \in \mathcal{E}(K, F)$ and F is monotone, we get $F(y, x^*) \leq 0$, $\forall y \in K$, and particularly $F(\tilde{x}, x^*) \leq 0$. We therefore have $\langle \overrightarrow{\tilde{x}\bar{x}}, \overrightarrow{\tilde{x}x^*} \rangle \leq 0$. ∎

Theorem 7.3. *Suppose that* F *is monotone with* $\mathcal{E}(K, F) \neq \varnothing$, Δ*-usc in the first variable, and that* $\operatorname{dom}(J_{\mu F}) \supset K$ *for all* $\mu > 0$. *Let* (λ_k) *be bounded away from* 0. *Then the proximal algorithm* (Prox) *is* Δ*-convergent to an element in* $\mathcal{E}(K, F)$ *for any initial start* $x^0 \in K$.

Proof. Let $x^0 \in K$ be an initial start and let $x^* \in \mathcal{E}(K, F)$. We can simply see that

$$\rho(x^*, x^{k+1}) = \rho(J_{\lambda_k F}(x^*), J_{\lambda_k F}(x^k)) \leq \rho(x^*, x^k),$$

which implies that (x^k) is Fejér convergent with respect to $\mathcal{E}(K, F)$. In view of Proposition 2.8, the real sequence $\rho(x^k, x^*)$ is bounded, and hence converges to some $\xi \geq 0$. Lemma 7.2 implies that

$$\rho^2(x^{k+1}, x^k) \leq \rho^2(x^k, x^*) - \rho^2(x^{k+1}, x^*).$$

Passing $k \longrightarrow \infty$, we get $\lim_k \rho(x^{k+1}, x^k) = 0$.

Now, suppose that $\hat{x} \in K$ is a Δ-accumulation point of the sequence (x^k), and also $(x^{k_j}) \subset (x^k)$ a subsequence such that $x^{k_j} \xrightarrow{\Delta} \hat{x}$. Let $y \in K$. By the construction (Prox), we have the following inequalities for any $j \in \mathbb{N}$:

$$F(x^{k_j}, y) \geq \frac{1}{\lambda_{k_j}} \langle \overrightarrow{x^{k_j} x^{k_j - 1}}, \overrightarrow{x^{k_j} y} \rangle \geq -\frac{1}{\lambda_{k_j}} \rho(x^{k_j}, x^{k_j - 1}) \rho(x^{k_j}, y). \tag{7.3}$$

Recall that (x^k) is bounded (in view of Proposition 2.8) and (λ_k) is bounded away from 0. Then (7.3) gives

$$F(x^{k_j}, y) \geq -M\rho(x^{k_j}, x^{k_j - 1}), \tag{7.4}$$

for some $M > 0$. Passing $j \longrightarrow \infty$ in (7.4) and apply the Δ-upper semiconinuity of $F(\cdot, y)$, we obtain

$$F(\hat{x}, y) \geq \limsup_{j \longrightarrow \infty} F(x^{k_j}, y) \geq -M \lim_{j \longrightarrow \infty} \rho(x^{k_j}, x^{k_j - 1}) = 0.$$

Since $y \in K$ is chosen arbitrarily, we conclude that $\hat{x} \in \mathcal{E}(K, F)$. Therefore, every Δ-accumulation point of (x^k) solves $EP(K, F)$. By Proposition 2.8, the sequence (x^k) is Δ-convergent to an element in $\mathcal{E}(K, F)$. ∎

Corollary 7.4 ([5, 10])**.** *Suppose that $g : X \to \mathbb{R}$ is convex and lsc, and that* $\arg\min g \neq \varnothing$*. Then, the proximal algorithm given by*

$$\begin{cases} x^0 \in X, \\ x^k := \operatorname{prox}_{\lambda_k g}(x^{k-1}), \quad \forall k \in \mathbb{N}, \end{cases}$$

is Δ-convergent to a minimizer of g, whenever $(\lambda_k) \subset (0, \infty)$ is bounded away from 0.

Proof. Consider the bifunction F_g as defined in Lemma 6.2. The monotonicity of F_g is immediate, and the fact that F_g is Δ-usc in the first variable follows by applying Lemma 2.5 to the epigraph of $F_g(x, \cdot)$ at each $x \in X$. The convergence is then a consequence of Theorem 7.3 applied to F_g, where the remaining requirements of F_g follows from Lemma 6.2. ∎

Let us look closer at the assumptions of Theorem 7.3, as well as the bifunction F_A given by (6.2) associated to some maximal monotone vector field A. One may notice that the Δ-usc of in the first variable of F_A is irrelevant. This motivates us to find the right mechanism to link between proximal algorithms for equilibrium problems and for monotone vector fields. Fortunately enough, we can deduce another convergence criteria that is capable to include [10, Theorem 5.2] as a certain special case. To do this, we need the following notion of skewed Δ-semicontinuity of F.

Definition 7.5. A bifunction $F : K \times K \to \mathbb{R}$ is said to be *skewed Δ-upper semicontinuous* (for short, *skewed Δ-usc*) if $-F(y, x^*) \geq \limsup_k F(x^k, y)$ for all $y \in K$, whenever $(x^k) \subset K$ is Δ-convergent to $x^* \in K$.

Remark 7.6. It is clear that if F is monotone and Δ-usc in the first variable, then F is skewed Δ-usc.

The next lemma shows that we can deduce that F_A is skewed Δ-usc for a monotone vector field $A : X \rightrightarrows X^*$.

Lemma 7.7. *Suppose that $A : X \rightrightarrows X^*$ is a monotone vector field with* $\operatorname{dom}(A) =$ *X, and F_A is defined by* (6.2)*. If X is reflexive, then F_A is skewed Δ-usc.*

Proof. Suppose that $(x^k) \subset K$ is a Δ-convergent to $x^* \in K$, and $y \in K$ be arbitrary. Then, by the monotonicity of F_A, we have

$$\begin{aligned}\limsup_{k\longrightarrow\infty} F_A(x^k, y) &\leq \limsup_{k\longrightarrow\infty}[-F_A(y, x^k)] \\ &= \limsup_{k\longrightarrow\infty}[-\sup_{\nu\in A(y)} \langle \nu, \overrightarrow{yx^k}\rangle] \\ &\leq \limsup_{k\longrightarrow\infty}[-\langle \nu_0, \overrightarrow{yx^k}\rangle], \end{aligned} \tag{7.5}$$

for any $\nu_0 \in A(y)$. Suppose that $\nu_0 = \delta\overrightarrow{yu}$ for some $u \in X$ and $\delta > 0$. Then, we have

$$\langle \nu_0, \overrightarrow{yx^k}\rangle = \langle \nu_0, \overrightarrow{yx^*}\rangle + \langle \overrightarrow{x^*u}^\delta, \overrightarrow{x^*x^k}\rangle - \langle \overrightarrow{x^*y}^\delta, \overrightarrow{x^*x^k}\rangle. \tag{7.6}$$

Combine (7.5), (7.6), and take into account the reflexivity of X and the Δ-convergence of (x^k), we obtain

$$\limsup_{k\longrightarrow\infty} F_A(x^k, y) \leq -\langle \nu_0, \overrightarrow{yx^*}\rangle.$$

Since this is true for any $\nu_0 \in A(y)$, we get

$$\limsup_{k\longrightarrow\infty} F_A(x^k, y) \leq \inf_{\nu\in T(y)}[-\langle \nu, \overrightarrow{yx^*}\rangle] = -\sup_{\nu\in T(y)} \langle \nu, \overrightarrow{yx^*}\rangle = -F_A(y, x^*). \qquad \blacksquare$$

Now that we have the motivations for skewed Δ-semicontinuity, we shall now give another convergence theorem based on this new notion of continuity. Recall that the coincidence of the primal and dual solutions of an equilibrium problem, as appeared in the following theorem, can be referenced from either Propositions 4.4 or 6.3.

Theorem 7.8. *Suppose that F is a monotone bifunction such that $\mathcal{E}(K, F) = \mathcal{E}^*(K, F) \neq \varnothing$, skewed Δ-usc, and that $\mathrm{dom}(J_{\mu F}) \supset K$ for all $\mu > 0$. Let (λ_k) be bounded away from 0. Then the proximal algorithm (Prox) is Δ-convergent to an element in $\mathcal{E}(K, F)$ for any initial start $x^0 \in K$.*

Proof. With similar proof lines to Theorem 7.3, we can also be able to obtain (7.4). Since F is skewed Δ-usc, we obtain the following inequalities by passing $j \longrightarrow \infty$:

$$-F(y, \hat{x}) \geq \limsup_{j\longrightarrow\infty} F(x^{k_j}, y) \geq 0,$$

for each $y \in K$. This means $\hat{x} \in \mathcal{E}^*(K, F) = \mathcal{E}(K, F)$, by hypothesis. Therefore, every Δ-accumulation point lies within $\mathcal{E}(K, F)$. Apply Proposition 2.8 to conclude that (x^k) is Δ-convergent to an element in $\mathcal{E}(K, F)$. $\blacksquare$

Corollary 7.9 ([10])**.** *Suppose that $A : X \rightrightarrows X^*$ is a monotone vector field with $\mathrm{dom}(A) = X$ and $\mathrm{dom}(\mu R_A) = X$ for all $\mu > 0$, and that $A^{-1}\mathbf{0} \neq \varnothing$. Assume that X is reflexive. Then, the proximal algorithm defined by*

$$\begin{cases} x^0 \in X, \\ x^k := R_{\lambda_k A}(x^{k-1}), \quad \forall k \in \mathbb{N}, \end{cases}$$

is Δ-convergent to an element in $A^{-1}\mathbf{0}$, whenever $(\lambda_k) \subset (0, \infty)$ is bounded away from 0.

Proof. It was proved in [10] that the surjectivity condition of A implies the maximal monotonicity and also gives $\mathrm{dom}(J_{\mu F_A}) = X$ (for all $\mu > 0$) in light of Proposition 6.4, where F_A defined in (6.2). The monotonicity of F_A follows from the monotonicity of A, while the skewed Δ-semicontinuity of F_A is proved in Lemma 7.7. By the

Proposition 6.3, it is feasible to apply Theorem 7.8 to F_A and obtain the desired convergence result. ∎

Remark 7.10. Amounts to (7.1) and the Fejér convergence of the proximal algorithm presented in the proofs of Theorems 7.3 and 7.8, it can be seen that we can make the convergence of functional values $\inf_{y\in K} F(x^k, y) \longrightarrow 0$ arbitrarily fast by the choosing appropriate step sizes. However, this does not ensure the speed of convergence of the sequence (x^k) even if the convergence is strong (i.e., in the metric topology). In particular, the speed enhancement of proximal algorithm for singularity problem using metric regularity conditions was developed in [10].

Conclusion and Remarks

We have provided a complete treatment of equilibrium problem situated in Hadamard spaces, from existence to approximation algorithm. We were also able to prove the KKM principle without additional assumptions, as opposed to earlier results in the literature. The existence criteria for an equilibrium is deduced under standard assumptions, which is very natural in this subject area.

In the approximation, we investigated a priori on the resolvent operator for a given bifunction. Main results here are that the resolvent operator is single-valued and firmly nonexpansive. We then define the proximal algorithm by iterating the resolvents of different bifurcating parameters. Several convergence criteria of the algorithm were proposed, including the one that involves skewed continuity, where we introduced here for the first time. Note again that the corresponding functional values converging to 0 can be achieved arbitrarily fast, but this does not imply the same for proximal algorithm itself.

Let us conclude this paper with some open questions whose answers might largely improve the applicability of the results in this present paper.

Question 7.11. Whether or not we can improve the following condition: $\operatorname{dom}(J_{\mu F}) \supset K$ for all $\mu > 0$, in order to obtain similar results regarding resolvent operators and proximal algorithms?

Question 7.12. Is it possible to drop the surjectivity condition above and still obtain the nonexpansivity of J_F (see (ii) in Proposition 5.4)?

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