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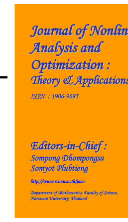

A NOTE ON A DUAL SCHEME OF A LINEAR FRACTIONAL PROGRAMMING PROBLEM

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ABSTRACT. We are interested in the duality scheme of a linear fractional programming problem proposed by Seshan [11]. The remarkable feature of the duality scheme is that the dual problem and the primal problem have the same linear fractional objective functions. Although the duality scheme is fascinating and has been introduced in literature, the steps how to build the scheme still be silent. The aim of this paper is to show that the dual problem can be obtained based upon the transformation forms given by Charnes-Cooper or by Dinkelbach with a simple change variable method.

KEYWORDS : Seshan duality; Charnes-Cooper transformation; Dinkelbach transformation.
AMS Subject Classification: 90C32, 90C05, 90C46

1. **Introduction**

Fractional programming problems were attracted by many authors early [4], [6], [7], [13]. As a generalization of linear programming problems, the following linear fractional programming problem was considered.

$$\text{(P) Max } F(x) = \frac{c^T x + c_0}{d^T x + d_0}$$
$$\text{s.t. } Ax \le b,$$
$$x \ge 0,$$

where $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, $d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$; c_0, d_0 are constants, A is an $m \times n$ real matrix ($m < n$), and $\text{rank}A = m$. There exist several methods for solving the problem (P) introduced in the books [2], [10]. Moreover, there are several dual schemes for (P) are proposed for years [1], [5], [6], [11], [12].

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It is well known that a dual problem of a linear programming problem is also a linear programming one. Naturally, it was expected that a dual problem of a linear fractional programming problem is also a linear fractional programming one. Among dual schemes in linear fractional programming, there exist some ones are linear fractional problems [3]. We are interested in the one proposed by Seshan [11]. In the paper the following dual problem for (P) is given.

$$\text{(D) Min } I(u,v) = \frac{c^T u + c_0}{d^T u + d_0} \tag{1.1}$$

$$\text{s.t. } c(d^T u) - d(c^T u) - A^T v \leq c_0 d - d_0 c, \tag{1.2}$$

$$c_0 d^T u - d_0 c^T u + b^T v \leq 0, \tag{1.3}$$

$$u \geq 0, v \geq 0. \tag{1.4}$$

The remarkable feature of the dual scheme is that the dual problem and the primal problem have the same linear fractional objective functions. For this duality scheme, the weak and strong duality theorems were established [11] and the results were also quoted in [2].

Although the dual scheme above was introduced since 1980 and was quoted in literature, as far as we know, the rule for building a dual problem (behind the construction) from the problem (P) was not introduced and the steps how to obtain the formulation (1.1)-(1.4) still be silent. In the paper [3], published in 2010, it was shown that there exist some duality schemes for (P) are equivalences. The paper only shows that the duality scheme for (P) proposed by Gol'stein [8] can derive the Seshan's scheme via the use of a Lagrange function associated with the Chanes-Cooper transformation [9].

Our aim of this paper is to clarify the way for building the Seshan's duality scheme proposed in [11]. For this purpose, firstly, we use Charnes - Cooper transformation [9] to change (P) to a linear problem. Next, by using a basic dual rule, we formulate its dual problem. Lastly, by using a simple change of variable method, we access to the Seshan's dual scheme. In addition, by using Dinkelbach transformation [2], we can see that the problem (P) is equivalent to the one in the form of linear problems. Then, we consider its dual problem. From this step, based on a simple change variable method, we can reach to the Seshan's dual problem.

The remains of the paper are organized as follows. The next section is devoted some basic results. In the last section, we introduce two way to obtain Seshan's dual scheme for (P).

2. Preliminaries and notations

Denote by $X = \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ the feasible set of (P). Suppose that $d^T x + d_0 > 0$ for all $x \in X$, X is bounded and the objective function F of (P) is not constant on X. We also denote the feasible set of (D) by Y and assume that $d^T u + d_0 > 0$ for all $(u,v) \in Y$. Using Charnes - Cooper transformation [2], the problem (P) can be changed to a linear programming (see [2, p.78]) as follows.

Let $t = \dfrac{1}{d^T x + d_0}$ and $y = tx$ we derive the following linear problem:

$$\begin{aligned} \text{(L1) Max } & G(y,t) = c^T y + c_0 t \\ \text{s.t. } & Ay - bt \leq 0, \\ & d^T y + d_0 t = 1, \\ & y \geq 0,\ t > 0. \end{aligned}$$

Denote by F_1 the feasible set of (L_1). We also note that the problem (P) is solvable if and only if the problem (L1) is solvable also. Moreover, they have the same optimal values.

Setting $\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \\ c_0 \end{pmatrix}$, $z = \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ t \end{pmatrix}$, $\beta = \begin{pmatrix} d_1 \\ \vdots \\ d_n \\ d_0 \end{pmatrix}$, $\bar{A} = [A| - b]$, the problem (L_1) can be rewritten in the following formulation:

$$\begin{aligned} \text{(L1a)} \quad & \text{Max } \alpha^T z \\ & \text{s.t. } \bar{A} z \leq 0, \\ & \beta^T z = 1, \\ & z \geq 0 \ (z_{n+1} > 0). \end{aligned}$$

We need the following results. For the problem (P), set $f(x) = c^T x + c_0$ and $g(x) = d^T x + d_0$ where $d^T x + d_0 > 0$ for all $x \in X$.

Lemma 2.1. ([2, p. 88]) *The function F defined by $F(\lambda) = \max_{x \in X}[f(x) - \lambda g(x)], \lambda \in \mathbb{R}$, is strictly decreasing in λ.*

Lemma 2.2. ([2, Theorem 3.5.3, p. 87]) *The point $x_0 \in X$ is the optimal solution of* (P) *if and only if*

$$\max_{x \in X}[f(x) - \lambda_0 g(x)] = F(\lambda_0) = 0,$$

where $\lambda_0 = \dfrac{f(x_0)}{g(x_0)}$.

3. **Main results**

3.1. **Using Charnes - Cooper transformation to derive Seshan duality scheme.**

Since the problem (L1a) is a linear programming problem, by using the duality rule for (L1a), the dual problem of (L1a) is

$$\begin{aligned} \text{(DL1) Min } & H(\xi) = \xi^T s \\ \text{s.t. } & \xi_i \geq 0, i = \overline{1, m}, \\ & \xi_{m+1} \in \mathbb{R}, \\ & \xi^T B \geq \alpha^T, \end{aligned}$$

where $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \\ \xi_{m+1} \end{pmatrix}$, $s = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, $B = \left[\begin{array}{c|c} A & -b \\ d^T & d_0 \end{array} \right]$.

For (DL1), denote a_i the i-column of A, $i = \overline{1, n}$. Set $w = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix}$ and $\lambda = \xi_{m+1}$.

Then,

$$\xi^T B = (a_1^T w + \lambda d_n, ..., a_n^T w + \lambda d_n, -b^T w + \lambda d_0).$$

Hence,

$$\xi^T B \geq \alpha^T \Leftrightarrow A^T w + \lambda d \geq c \text{ v } -b^T w + \lambda d_0 \geq c_0.$$

Note that, $\xi^T s = \xi_{m+1}$ and $\lambda = \xi_{m+1}$. The problem (DL1) becomes

$$(\mathcal{Q}_\lambda)\ \text{Min}\ \lambda \tag{3.1}$$
$$\text{s.t. } A^T w \geq c - \lambda d, \tag{3.2}$$
$$b^T w + c_0 - \lambda d_0 \leq 0, \tag{3.3}$$
$$w \geq 0, w \in \mathbb{R}^m. \tag{3.4}$$

We will show that the problem (1.1)-(1.4) can be obtained by the problem (3.1)-(3.4) via the following transformation.

Proposition 3.1. *Assume that* $d^T u + d_0 > 0$ *for all* $u \geq 0$ *and set* $\lambda = \dfrac{c^T u + c_0}{d^T u + d_0}$, $v = (d^T u + d_0)w$. *Then, the constraint* (1.2) *is equivalent to* (3.2) *and the constraint* (1.3) *is equivalent to* (3.3).

Proof. We have

$$\begin{aligned}
& A^T w \geq c - \lambda d \\
\Leftrightarrow\ & -A^T w \leq \lambda d - c \\
\Leftrightarrow\ & -A^T v \leq (\lambda d - c)(d^T u + d_0) \\
\Leftrightarrow\ & (d^T u)c - (c^T u)d - A^T v \leq (d^T u)c - (c^T u)d - (d^T u + d_0)(c - \lambda d) \\
\Leftrightarrow\ & (d^T u)c - (c^T u)d - A^T v \leq (d^T u)c - (c^T u)d - (d^T u)c - d_0 c + (c^T u + c_0)d \\
\Leftrightarrow\ & (d^T u)c - (c^T u)d - A^T v \leq c_0 d - d_0 c.
\end{aligned}$$

Hence, the constraint (1.2) is equivalent to (3.2). Furthermore, we have

$$\begin{aligned}
& b^T w + c_0 - \lambda d_0 \leq 0 \\
\Leftrightarrow\ & (d^T u + d_0) b^T w + c_0(d^T u + d_0) - d_0(c^T u + c_0) \leq 0 \\
\Leftrightarrow\ & b^T v + c_0 d^T u - d_0 c^T u \leq 0.
\end{aligned}$$

Thus, the constraint (1.3) is equivalent to (3.3). □

Note that, since $d^T u + d_0 > 0$ and $w \geq 0$, $v \geq 0$.

Remark 3.2. From Proposition 3.1 and $w \geq 0$, we say that the problem (D) can be reached by $(\mathcal{Q}_\lambda)$.

3.2. **Using Dinkelbach transformation to derive Seshan duality scheme.**

Based on Lemma 2.2, an optimal solution of (P) is also an optimal solution of the following problem:

$$\begin{aligned}
\text{(L2) Max } & \{(c^T x + c_0) - \overline{\lambda}(d^T x + d_0)\} \\
\text{s.t. } & Ax \leq b, \\
& x \geq 0,
\end{aligned}$$

where $\overline{\lambda}$ is the optimal value of (P). Note that, the problem (L2) can be rewritten by the following formulation.

$$\begin{aligned}
\text{Max } & \{(c - \overline{\lambda} d)^T x + c_0 - \overline{\lambda} d_0\} \\
\text{s.t. } & Ax \leq b, \\
& x \geq 0.
\end{aligned}$$

By applying the basic duality rule for linear programming problem to (L2), we obtain the following problem

$$\begin{aligned}
\text{(DL2) Min } & \{b^T w + c_0 - \overline{\lambda} d_0\} \\
\text{s.t. } & A^T w \geq c - \overline{\lambda} d,
\end{aligned}$$

$$w \geq 0, w \in \mathbb{R}^m.$$

Denote by F_2 the feasible set of (DL2).

Remark 3.3. The optimal value of (L2) equals to 0 and the optimal value of (DL2) does also by the strong duality.

Lemma 3.4. *The function R defined by*

$$R(\lambda) = \min_w \{b^T w + c_0 - \lambda d_0 \mid A^T w \geq c - \lambda d, w \geq 0, w \in \mathbb{R}^m\}$$

is strictly decreasing.

Proof. Suppose that $\lambda_2 > \lambda_1$. We get

$$\begin{aligned} R(\lambda_2) &= \min_w \{b^T w + c_0 - \lambda_2 d_0 \mid A^T w \geq c - \lambda_2 d, w \geq 0, w \in \mathbb{R}^m\} \\ &= b^T \bar{w} + c_0 - \lambda_2 d_0 \end{aligned}$$

where $\bar{w}$ be a optimal solution of (DL2) according to λ_2. Based on the strong duality property of linear programming, we get

$$\begin{aligned} R(\lambda_2) &= \max_x \{(c - \lambda_2 d)^T x + c_0 - \lambda_2 d_0 \mid Ax \leq b, x \geq 0\} \\ &= (c - \lambda_2 d)^T \bar{x} + c_0 - \lambda_2 d_0 \\ &< (c - \lambda_1 d)^T \bar{x} + c_0 - \lambda_1 d_0 \\ &\leq \max_x \{(c - \lambda_1 d)^T x + c_0 - \lambda_1 d_0 \mid Ax \leq b, x \geq 0\} \\ &\leq \min_w \{b^T w + c_0 - \lambda_1 d_0 \mid A^T w \geq c - \lambda_1 d, w \geq 0, w \in \mathbb{R}^m\} = R(\lambda_1). \end{aligned}$$

Hence, the function R is strictly decreasing in λ. □

Proposition 3.5. *Suppose that $\bar{\lambda}$ is the optimal value of* (P). *Then, the vector $\bar{v}$ is an optimal solution of* (DL2) *if and only if $(\bar{v}, \bar{\lambda})$ is an optimal value of* (Q_λ).

Proof. Let $\bar{v}$ be an optimal solution of (DL2). Then, by Remark 3.3, we get

$$b^T \bar{v} + c_0 - \bar{\lambda} d_0 = 0.$$

Moreover

$$A^T \bar{v} \geq c - \bar{\lambda} d \text{ v } \bar{v} \geq 0.$$

Hence,

$$R(\bar{\lambda}) = 0 \text{ v } (\bar{v}, \bar{\lambda}) \in F_2.$$

On the other hand, for any $(w, \lambda) \in F_2$, we get

$$R(\lambda) \leq 0 = R(\bar{\lambda}).$$

Since the function R is strictly decreasing, it yields $\bar{\lambda} \leq \lambda$. Hence, $\bar{\lambda}$ is the optimal value of (Q_λ), i.e., $(\bar{v}, \bar{\lambda})$ is an optimal solution of (Q_λ).

Conversely, let $(\bar{v}, \bar{\lambda})$ be an optimal solution of (Q_λ). We have

$$b^T \bar{v} + c_0 - \bar{\lambda} d_0 \leq 0,$$

$$A^T \bar{v} \geq c - \bar{\lambda} d \text{ v } \bar{v} \geq 0.$$

Since $\bar{\lambda}$ is the optimal value of (P), $\min_{w \in F_2} \{b^T w + c_0 - \bar{\lambda} d_0\} = 0$ by Remark 3.3. On the other hand, $b^T \bar{v} + c_0 - \bar{\lambda} d_0 \geq \min_{w \in F_2} \{b^T w + c_0 - \bar{\lambda} d_0\} = 0$. We obtain

$$b^T \bar{v} + c_0 - \bar{\lambda} d_0 = 0.$$

This means that $\bar{v}$ is an optimal solution of (DL2). □

Remark 3.6. The problem (DL2) equals to (Q_λ). This together with Remark 3.2 imply that the problem (D) can be reached by (DL2).

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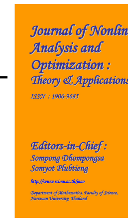

GENERALIZED VARIATIONAL-LIKE INCLUSION PROBLEM INVOLVING $(H(.,.),\eta)$-MONOTONE OPERATORS IN BANACH SPACES

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ABSTRACT. In this paper, we consider the generalized variational-like inclusion problem involving $(H(.,.),\eta)$-monotone operators in Banach spaces. Using proximal operator technique, we prove the existence of solution and suggest an iterative algorithm for solving the generalized variational-like inclusion problem. Also, we discuss the convergence analysis of the iterative algorithm. The results presented in this paper improve and generalize many known results in the literature.

KEYWORDS : $(H(.,.),\eta)$-monotone operator; Generalized η-proximal operator; Generalized variational-like inclusion problem; Iterative algorithm; Convergence analysis.
AMS Subject Classification: 47H04; 47H10; 49J40

1. PRELIMINARIES AND BASIC RESULTS

Throughout this paper unless or otherwise stated, X is a real Banach space with dual space $X^\star$, $\langle \cdot,\cdot \rangle$ is the dual pair between X and $X^\star$, 2^X denote the family of all the nonempty subsets of X. The normalized duality mapping $J : X \longrightarrow 2^{X^\star}$ is defined by

$$J(u) = \{f \in X^\star : \langle f, u\rangle = \|f\|\|u\|,\ \|f\| = \|u\|\},\ \forall u \in X.$$

A selection of the duality mapping J is a single-valued mapping $j : X \longrightarrow X^\star$ satisfying $j(u) \in J(u)$ for each $u \in X$.
Further, $J^\star : X^\star \longrightarrow X^{\star\star}$ be the normalized duality mapping on $X^\star$ defined by

$$J^\star(v) = \{f \in X^{\star\star} : \langle f, v\rangle = \|f\|\|v\|,\ \|f\| = \|v\|\},\ \forall v \in X^\star,$$

where $X^{\star\star}$ is a dual space of $X^\star$. Furthermore, $j^\star$ denotes a selection of $J^\star$

If $X \equiv \mathbf{H}$, a Hilbert space, then J and $J^\star$ are the identity mappings on $\mathbf{H}$.

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Let $CB(X)$ denotes the family of all nonempty closed and bounded subsets of X; $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$D(A,B) = \max\left\{\sup_{u\in A} d(u,B), \sup_{v\in B} d(A,v)\right\},\ A,B \in CB(X).$$

The following concepts and results are needed in the sequel:

Lemma 1.1 (10). *Let X be a complete metric space, $T : X \longrightarrow CB(X)$ be a set-valued mapping. Then for any $\epsilon > 0$ and for any $u, v \in X$, $x \in T(u)$, there exists $y \in T(v)$ such that*

$$d(x,y) \leq (1+\epsilon)D(T(u),T(v)),$$

where D is the Hausdorff metric on $CB(X)$.

Definition 1.2. Let $T : X \longrightarrow X^\star$; $A, B : X \longrightarrow X$, $N : X\times X \longrightarrow X, H : X \times X \longrightarrow X^\star$ and $\eta : X\times X \longrightarrow X$ be single-valued mappings. Then $\forall u, v, \cdot \in X$

(i) T is monotone, if

$$\left\langle Tu - Tv, u - v\right\rangle \geq 0.$$

(ii) T is strictly monotone, if

$$\left\langle Tu - Tv, u - v\right\rangle > 0,$$

and equality holds if and only if $u = v$.

(iii) T is α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\left\langle Tu - Tv, u - v\right\rangle \geq \alpha\|u - v\|^2.$$

(iv) T is γ-Lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$\|Tu - Tv\| \ \leq \ \gamma\|u - v\|.$$

(v) T is η-monotone, if

$$\left\langle Tu - Tv, \eta(u,v)\right\rangle \geq 0.$$

(vi) T is strictly η-monotone, if

$$\left\langle Tu - Tv, \eta(u,v)\right\rangle > 0,$$

and equality holds if and only if $u = v$.

(vii) A is said to be δ-strongly accretive, if there exists a constant $\delta > 0$ and $j(u-v) \in J(u-v)$ such that

$$\left\langle Au - Av, j(u-v)\right\rangle \geq \delta\|u - v\|^2,$$

where J is the normalized duality mapping.

(viii) $N(\cdot,\cdot)$ is l_1-Lipschitz continuous in the first argument, if there exists a constant $l_1 > 0$ such that

$$\|N(u,\cdot) - N(v,\cdot)\| \leq l_1\|u - v\|.$$

(ix) $N(\cdot,\cdot)$ is l_2-Lipschitz continuous in the second argument, if there exists a constant $l_2 > 0$ such that

$$\|N(\cdot,u) - N(\cdot,v)\| \leq l_2\|u - v\|.$$

(x) $H(A,\cdot)$ is α_1-strongly η-monotone with respect to A, if there exists a constant $\alpha_1 > 0$ such that

$$\left\langle H(Au,\cdot) - H(Av,\cdot), \eta(u,v)\right\rangle \geq \alpha_1\|u - v\|^2.$$

(xi) $H(\cdot, B)$ is α_2-relaxed η-monotone with respect to B, if there exists a constant $\alpha_2 > 0$ such that

$$\left\langle H(\cdot, Bu) - H(\cdot, Bv), \eta(u,v) \right\rangle \geq -\alpha_2 \|u - v\|^2.$$

(xii) $H(\cdot, \cdot)$ is h_1-Lipschitz continuous with respect to A, if there exists a constant $h_1 > 0$ such that

$$\|H(Au, \cdot) - H(Av, \cdot)\| \ \leq \ h_1 \|u - v\|.$$

(xiii) $H(\cdot, \cdot)$ is h_2-Lipschitz continuous with respect to B, if there exists a constant $h_2 > 0$ such that

$$\|H(\cdot, Bu) - H(\cdot, Bv)\| \ \leq \ h_2 \|u - v\|.$$

(xiv) η is τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(u,v)\| \ \leq \ \tau \|u - v\|.$$

Remark 1.3. If X is a Hilbert space, $\eta(u,v) = u - v, \forall u, v \in X$, then (x) and (xi) of Definition 1.2 reduces to (i) and (ii) of Definition 1.2, respectively in [12].

Definition 1.4. Let $M : X \longrightarrow 2^{X^\star}$ be a multi-valued mapping, $H : X \longrightarrow X^\star$ and $\eta : X \times X \longrightarrow X$ be single-valued mappings. Then:

(i) M is monotone, if

$$\langle x - y, u - v \rangle \geq 0, \ \forall\, u, v \in X, \ x \in M(u), y \in M(v).$$

(ii) M is η-monotone, if

$$\langle x - y, \eta(u,v) \rangle \geq 0, \ \forall\, u, v \in X, \ x \in M(u), y \in M(v).$$

(iii) M is strictly η-monotone, if

$$\langle x - y, \eta(u,v) \rangle > 0, \ \forall\, u, v \in X, \ x \in M(u), y \in M(v),$$

and equality holds if and only if $u = v$.

(iv) M is λ-strongly η-monotone, if there exists a constant $\lambda > 0$ such that

$$\langle x - y, \eta(u,v) \rangle \geq \lambda \|u - v\|^2, \ \forall\, u, v \in X, \ x \in M(u), y \in M(v).$$

(v) M is m-relaxed η-monotone, if there exists a constant $m > 0$ such that

$$\langle x - y, \eta(u,v) \rangle \geq -m \|u - v\|^2, \ \forall\, u, v \in X, \ x \in M(u), y \in M(v).$$

(vi) M is maximal monotone, if M is monotone and

$$(J + \lambda M)(X) = X^\star, \ \forall\, \lambda > 0,$$

where J is the normalized duality mapping.

(vii) M is maximal η-monotone, if M is η-monotone and

$$(J + \lambda M)(X) = X^\star, \ \forall\, \lambda > 0.$$

(viii) M is H-monotone, if M is monotone and

$$(H + \lambda M)(X) = X^\star, \ \forall\, \lambda > 0.$$

(ix) M is H-η-monotone, if M is m-relaxed η-monotone and $(H + \lambda M)(X) = X^\star, \ \forall\, \lambda > 0$.

Definition 1.5. For all $u, v, \cdot \in X$, a mapping $F : X \times X \times X \longrightarrow X^\star$ is said to be ϵ_1-Lipschitz continuous with respect to first argument, if there exists a constant $\epsilon_1 > 0$ such that

$$\|F(u, \cdot\,, \cdot) - F(v, \cdot\,, \cdot)\| \leq \epsilon_1 \|u - v\|.$$

Similarly, we can define Lipschitz continuity of F in other arguments.

Lemma 1.6 (11). *Let X be a real Banach space and $J : X \longrightarrow 2^{X^\star}$ be the normalized duality mapping. Then, for all $u, v \in X$,*

$$\|u+v\|^2 \leq \|u\|^2 + 2\Big\langle v, j(u+v)\Big\rangle,\ \forall j(u+v) \in J(u+v).$$

2. $(H(.,.),\eta)$-Monotone Operator and Formulation of the Problem

Definition 2.1. Let X be a Banach space with the dual space $X^\star$. Let $H : X \times X \longrightarrow X^\star$, $\eta : X \times X \longrightarrow X,\ A, B : X \longrightarrow X$ be single-valued mappings. Then the set-valued mapping $M : X \longrightarrow 2^{X^\star}$ is said to be $(H(.,.),\eta)$-monotone with respect to A and B, if M is m-relaxed-η-monotone and $(H(A,B)+\rho M)(X) = X^\star,\ \forall \rho > 0$.

Remark 2.2. (i) If $H(Au, Bu) = Hu,\ \forall u \in X$, then Definition 2.1 reduces to the definition of H-η-monotone operators considered in [8]. It follows that this class of operators in Banach spaces provides a unifying framework for the class of η-subdifferential operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators, (H,η)-monotone operators, G-η-monotone operators, A-monotone operators, A-η-monotone operators in Hilbert spaces and H-monotone operators, H-η-monotone operators, A-monotone operators in Banach spaces . We remark that $(H(.,.),\eta)$-monotone operator in Banach spaces acts from X to $X^\star$.
(ii) If $X \equiv \mathbf{H}$, a Hilbert space, $m = 0$ and $\eta(u,v) = u - v,\ \forall u, v \in \mathbf{H}$, then Definition 2.1 reduces to M-monotone operator studied in [12].

Now we give some properties of $(H(.,.),\eta)$-monotone operator.

Theorem 2.3. *Let $A, B : X \longrightarrow X,\ \eta : X \times X \longrightarrow X$, and $H : X \times X \longrightarrow X^\star$ be single-valued mappings and $H(A,B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : X \longrightarrow 2^{X^\star}$ be $(H(.,.),\eta)$-monotone operator with respect to A and B. If $\Big\langle x - y, \eta(u,v)\Big\rangle \geq 0,\ \forall\ (v,y) \in Graph\ (M)$, then $(u,x) \in Graph(M)$, where $Graph\ (M) = \Big\{(a,b) \in X \times X : b \in M(a)\Big\}$.*

Theorem 2.4. *Let $A, B : X \longrightarrow X,\ \eta : X \times X \longrightarrow X$ and $H : X \times X \longrightarrow X^\star$ be single-valued mappings and $H(A,B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : X \longrightarrow 2^{X^\star}$ be $(H(.,.),\eta)$-monotone operator with respect to A and B. Then $(H(A,B)+\rho M)^{-1}$ is a single-valued mapping for $0 < \rho < \dfrac{\alpha-\beta}{m}$.*

Based on Theorem 2.4, we define the generalized η-proximal operator associated with $(H(A,B),\eta)$-monotone operator as under:

Definition 2.5. Let $A, B : X \longrightarrow X,\ \eta : X \times X \longrightarrow X$ and $H : X \times X \longrightarrow X^\star$ be single-valued mappings and $H(A,B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : X \times X \longrightarrow 2^{X^\star}$ be $(H(.,.),\eta)$-monotone operator with respect to A and B. Then the generalized η-proximal operator $J^{H(.,.),\eta}_{M(\cdot,z),\rho} : X \longrightarrow X$ for fixed $z \in X$ is defined by

$$J^{H(.,.),\eta}_{M(\cdot,z),\rho}(u) = \Big(H(A,B) + \rho M(\cdot,z)\Big)^{-1}(u),\ \forall u \in X.$$

Remark 2.6. The generalized η-proximal operator associated with $(H(.,.),\eta)$-monotone operator include as special cases the corresponding proximal operators associated with maximal monotone operators, η-subdifferential operators, maximal η-monotone operators, H-monotone operators, (H,η)-monotone operators, G-η-monotone operators, A-monotone operators, A-η-monotone operators.

One of the important properties of generalized η-proximal operator is its Lipschitz continuity which is as under:

Theorem 2.7. *Let $A,B : X \longrightarrow X$, $\eta : X\times X \longrightarrow X$ and $H : X\times X \longrightarrow X^\star$ be single-valued mappings and $H(A,B)$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and $\alpha > \beta$. Let $M : X\times X \longrightarrow 2^{X^\star}$ be $(H(.,.),\eta)$-monotone operator with respect to A and B. Then the generalized η-proximal operator $J^{H(.,.),\eta}_{M(\cdot,z),\rho} : X \longrightarrow X$ for fixed $z\in X$ is k-Lipschitz continuous, where $k = \dfrac{\tau}{\alpha-\beta-m\rho}$, that is*

$$\|J^{H(.,.),\eta}_{M(\cdot,z),\rho}(u) - J^{H(.,.),\eta}_{M(\cdot,z),\rho}(v)\| \le k\|u-v\|,\ \forall u,v\in X.$$

Now we formulate our main problem:

Let X be a real Banach space. Let $S,T,G : X \longrightarrow CB(X)$ be set-valued mappings, $N,H : X\times X \longrightarrow X^\star$, $\eta : X\times X \longrightarrow X$, $F : X\times X\times X \longrightarrow X^\star$ and $A,B,p,g : X \longrightarrow X$ be single-valued mappings. Let $M : X\times X \longrightarrow 2^{X^\star}$ be set-valued mapping such that for fixed $z\in G(X)$, $M(.,z) : X\times X \longrightarrow 2^{X^\star}$ is an $(H(.,.),\eta)$-monotone operator with respect to A and B and $Range(g-p)\cap \mathrm{dom}(M(.,z)) \neq \emptyset$. For any given $f\in X^\star$, we consider the following generalized variational-like inclusion problem (in short, GVLIP): Find $u\in X$, $x\in S(u), y\in T(u), z\in G(u)$ such that

$$\theta^\star \in N\Big(g(x),A(y)\Big) + F(u,u,z) + M\Big((g-p)(u),z\Big) + f, \tag{2.1}$$

where $\theta^\star$ is the zero element in $X^\star$.

We remark that if $g-p\equiv I$ and $f\equiv 0$, then GVLIP (2.1) reduces to a variational inclusion of finding $u\in X$, $x\in S(u), y\in T(u), z\in G(u)$ such that

$$\theta^\star \in N\Big(g(x),A(y)\Big) + F(u,u,z) + M(u,z). \tag{2.2}$$

Variational inclusion (2.2) is an important generalization of variational inclusions considered by many researchers including [12,15]. For applications of such variational inclusions, see [7,8].

If $F=p=f\equiv 0$, $g\equiv I$ and $X\equiv \mathbf{H}$, a Hilbert space, then GVLIP (2.1) reduces to a generalized mixed quasi-variational-like inclusion involving $(H(\cdot,\cdot),\eta)$-monotone operators in a Hilbert space: Find $u\in\mathbf{H}$, $x\in S(u), y\in T(u), z\in G(u)$ such that

$$\theta^\star \in N\Big(x,A(y)\Big) + M(u,z). \tag{2.3}$$

Variational inclusion (2.3) is an important generalization of variational inclusions considered by Kazmi and Bhat [4,5].

We remark that for the suitable choice of mappings $A,B,S,T,G,N,H,F,M,\eta,g,p$ and the underlying space X, GVLIP (2.1) reduces to different classes of new and already known systems of variational inclusions/inequalities considered by many researchers including [6,9,13,15,18] and the related references cited therein.

3. Existence of Solution, Iterative Algorithm and Convergence Analysis

First, we give the following technical result:

Lemma 3.1. *Let $X, A, B, S, T, G, N, H, F, M, \eta, g, p$ be same as in GVLIP (2.1). Then (u, x, y, z) where $x \in S(u), y \in T(u), z \in G(u)$ is the solution of GVLIP (2.1) if and only if*

$$(g-p)(u) = J^{H(.,.),\eta}_{M(.,z),\rho}\Big[H\Big(A((g-p)(u)), B((g-p)(u))\Big)-\rho\Big\{N\Big(g(x), A(y)\Big)+F(u,u,z)+f\Big\}\Big],$$

and $J^{H(.,.),\eta}_{M(.,z),\rho}(u) = \Big(H(A,B)+\rho M(.,z)\Big)^{-1}(u)$ is the generalized η-proximal operator and $\rho > 0$ is a constant.

The above result along with Nadler's Theorem (Lemma 1.1) allow us to suggest the following iterative algorithm for solving GVLIP (2.1).

Iterative Algorithm 3.2. *For any arbitrary chosen $u_0 \in X,\ x_0 \in S(u_0),\ y_0 \in T(u_0)$ and $z_0 \in G(u_0)$, compute the sequences $\{u_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative schemes such that*

$$\begin{aligned}
(g-p)(u_{n+1}) \quad &= J^{H(.,.),\eta}_{M(.,z_n),\rho}\Big[H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) \\
&\quad -\rho\Big\{N\Big(g(x_n), A(y_n)\Big) + F(u_n, u_n, z_n) + f\Big\}\Big], \\
x_n \in S(u_n): \quad &\|x_{n+1} - x_n\| \le \Big(1 + (1+n)^{-1}\Big)D\Big(S(u_{n+1}), S(u_n)\Big); \\
y_n \in T(u_n): \quad &\|y_{n+1} - y_n\| \le \Big(1 + (1+n)^{-1}\Big)D\Big(T(u_{n+1}), T(u_n)\Big); \\
z_n \in G(u_n): \quad &\|z_{n+1} - z_n\| \le \Big(1 + (1+n)^{-1}\Big)D\Big(G(u_{n+1}), G(u_n)\Big).
\end{aligned}$$

for all $n = 0, 1, 2, \cdots$.

If $\rho = 1,\ g - p \equiv I$ and $f \equiv 0$, then the Iterative Algorithm 3.2 reduces to the following iterative algorithm.

Iterative Algorithm 3.3. *For any arbitrary chosen $u_0 \in X,\ x_0 \in S(u_0),\ y_0 \in T(u_0)$ and $z_0 \in G(u_0)$, compute the sequences $\{u_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative schemes such that*

$$u_{n+1} \quad = \quad J^{H(.,.),\eta}_{M(.,z_n)}\Big[H\Big(A(u_n), B(u_n)\Big) - \Big\{N\Big(g(x_n), A(y_n)\Big) + F(u_n, u_n, z_n)\Big\}\Big],$$

$$\begin{aligned}
x_n \in S(u_n): \quad &\|x_{n+1} - x_n\| \le \Big(1 + (1+n)^{-1}\Big)D\Big(S(u_{n+1}), S(u_n)\Big); \\
y_n \in T(u_n): \quad &\|y_{n+1} - y_n\| \le \Big(1 + (1+n)^{-1}\Big)D\Big(T(u_{n+1}), T(u_n)\Big); \\
z_n \in G(u_n): \quad &\|z_{n+1} - z_n\| \le \Big(1 + (1+n)^{-1}\Big)D\Big(G(u_{n+1}), G(u_n)\Big).
\end{aligned}$$

We remark that Iterative Algorithm 3.3 gives the approximate solution to the variational inclusion (2.2).

Now, we prove the following theorem which ensures the convergence of iterative sequences generated by the Iterative Algorithm 3.2 to the solution of GVLIP 2.1.

Theorem 3.4. *Let X be a real Banach space. Let $S, T, G : X \longrightarrow CB(X)$ be α_1-D-Lipschitz, α_2-D-Lipschitz, α_3-D-Lipschitz continuous mappings, respectively. Let $N : X \times X \longrightarrow X^\star$ be l_1-Lipschitz continuous and l_2-Lipschitz continuous with* *respect to first and second arguments, respectively.* $\eta : X \times X \longrightarrow X$ *be* τ*-Lipschitz continuous,* $F : X \times X \times X \longrightarrow X^{\star}$ *be* β_j*-Lipschitz continuous with respect to* j*th argument, for* $j = 1, 2, 3$ *and* $A, B, p, g : X \longrightarrow X$ *be single-valued mappings such that* g *is* r_1*-Lipschitz continuous,* A *is* r_2*-Lipschitz continuous,* $(g - p)$ *is* s*-Lipschitz continuous and* $(g - p - I)$ *is* λ*-strongly accretive. Let* $H : X \times X \longrightarrow X^{\star}$ *be* α*-strongly* η*-monotone with respect to* A*,* β*-relaxed* η*-monotone with respect to* B *and* h_1*-Lipschitz continuous and* h_2*-Lipschitz continuous with respect to* A *and* B*, respectively. Let* $M : X \times X \longrightarrow 2^{X^{\star}}$ *be set-valued mapping such that for fixed* $z \in G(X), M(., z) : X \times X \longrightarrow 2^{X^{\star}}$ *be* $(H(., .), \eta)$*-monotone operator with respect to* A *and* B *and* $Range(g - p) \cap dom(M(., z)) \neq \emptyset$*. In addition, suppose there exists a constant* $\sigma > 0$ *such that*

$$\left\| J^{H(.,.),\eta}_{M(.,z_{n+1}),\rho}(u) - J^{H(.,.),\eta}_{M(.,z),\rho}(u) \right\| \leq \sigma \parallel z_{n+1} - z_n \parallel . \tag{3.1}$$

Furthermore, suppose the following condition is satisfied

$$0 < Q < 1,$$

where Q *is given by,*

$$\begin{aligned} Q &= \frac{1}{\sqrt{1+2\lambda}}\Big\{k\Big[\sqrt{s^2h_1^2 - 2\rho l_1 r_1 \alpha_1(sh_1 + \rho l_1 r_1 \alpha_1)} \\ &\quad + \sqrt{s^2h_2^2 - 2\rho l_2 r_2 \alpha_2(sh_2 + \rho l_2 r_2 \alpha_2)} + \rho(\beta_1 + \beta_2 + \beta_3 \alpha_3)\Big] + \sigma\alpha_3\Big\}, \end{aligned} \tag{3.2}$$

then the sequences $\{u_n\}, \{x_n\}, \{y_n\}$ *and* $\{z_n\}$*, generated by the Iterative Algorithm* 3.2 *converge strongly to the unique solution* (u, x, y, z)*, respectively, where* $u \in X,\ x \in S(u), y \in T(u)$ *and* $z \in G(u)$ *is the solution of GVLIP (2.1).*

Proof. From Iterative Algorithm 3.2 and Lemma 2.7, we have

$$\begin{aligned} &\|(g-p)u_{n+2} - (g-p)u_{n+1}\| \\ &= \Big\| J^{H(.,.),\eta}_{M(.,z_{n+1}),\rho}\Big[H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) \\ &\quad -\rho\Big\{N\Big(g(x_{n+1}), A(y_{n+1})\Big) + F(u_{n+1}, u_{n+1}, z_{n+1}) + f\Big\}\Big] \\ &\quad -J^{H(.,.),\eta}_{M(.,z_n),\rho}\Big[H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) \\ &\quad -\rho\Big\{N\Big(g(x_n), A(y_n)\Big) + F(u_n, u_n, z_n) + f\Big\}\Big]\Big\| \\ &\leq \Big\| J^{H(.,.),\eta}_{M(.,z_{n+1}),\rho}\Big[H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) \\ &\quad -\rho\Big\{N\Big(g(x_{n+1}), A(y_{n+1})\Big) + F(u_{n+1}, u_{n+1}, z_{n+1}) + f\Big\}\Big] \\ &\quad -J^{H(.,.),\eta}_{M(.,z_{n+1}),\rho}\Big[H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) \\ &\quad -\rho\Big\{N\Big(g(x_n), A(y_n)\Big) + F(u_n, u_n, z_n) + f\Big\}\Big]\Big\| \\ &\quad +\Big\| J^{H(.,.),\eta}_{M(.,z_{n+1}),\rho}\Big[H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) \end{aligned}$$

$$
\begin{aligned}
& -\rho\Big\{N\Big(g(x_n), A(y_n)\Big) + F(u_n, u_n, z_n) + f\Big\}\Big] \\
& -J^{H(.,.),\eta}_{M(.,z_n),\rho}\Big[H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) \\
& -\rho\Big\{N\Big(g(x_n), A(y_n)\Big) + F(u_n, u_n, z_n) + f\Big\}\Big]\Big\| \\
\leq\ & k\Big\|H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) \\
& -\rho\Big\{N\Big(g(x_{n+1}), A(y_{n+1})\Big) + F(u_{n+1}, u_{n+1}, z_{n+1})\Big\} \\
& -\Big[H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) \\
& -\rho\Big\{N\Big(g(x_n), A(y_n)\Big) + F(u_n, u_n, z_n)\Big\}\Big]\Big\| + \sigma\Big\|z_{n+1} - z_n\Big\| \\
\leq\ & k\Big[\Big\|H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) \\
& -H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big) - \rho\Big\{N\Big(g(x_{n+1}), A(y_{n+1})\Big) \\
& -N\Big(g(x_n), A(y_{n+1})\Big)\Big\}\Big\| + \Big\|H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big) \\
& -H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) - \rho\Big\{N\Big(g(x_n), A(y_{n+1})\Big) \\
& -N\Big(g(x_n), A(y_n)\Big)\Big\}\Big\| + \rho\Big\|F(u_{n+1}, u_{n+1}, z_{n+1}) - F(u_n, u_n, z_n)\Big\|\Big] \\
& +\sigma\Big\|z_{n+1} - z_n\Big\|. \qquad (3.3)
\end{aligned}
$$

Since $(g-p)$ is s-Lipschitz continuous, $H(.,.)$ is h_1-Lipschitz continuous with respect to A, $N(.,.)$ is l_1-Lipschitz continuous with respect to first argument and from Lemma 1.6, we have

$$
\begin{aligned}
& \Big\|H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) - H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big) \\
& \qquad -\rho\Big\{N\Big(g(x_{n+1}), A(y_{n+1})\Big) - N\Big(g(x_n), A(y_{n+1})\Big)\Big\}\Big\|^2 \\
\leq\ & \Big\|H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) \\
& -H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big)\Big\|^2 - 2\rho\Big\langle N\Big(g(x_{n+1}), A(y_{n+1})\Big) \\
& -N\Big(g(x_n), A(y_{n+1})\Big), j^\star\Big(H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) \\
& -H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big) - \rho\Big\{N\Big(g(x_{n+1}), A(y_{n+1})\Big)
\end{aligned}
$$

$$
\begin{aligned}
& -N\Big(g(x_n), A(y_{n+1})\Big)\Big\}\Big)\Big\rangle \\
\leq\ & \Big\|H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) \\
& -H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big)\Big\|^2 \\
& -2\rho\Big\|N\Big(g(x_{n+1}), A(y_{n+1})\Big) - N\Big(g(x_n), A(y_{n+1})\Big)\Big\| \\
& \times\Big[\Big\|H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) \\
& -H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big)\Big\| \\
& +\rho\Big\|N\Big(g(x_{n+1}), A(y_{n+1})\Big) - N\Big(g(x_n), A(y_{n+1})\Big)\Big\|\Big] \\
\leq\ & s^2h_1^2\|u_{n+1}-u_n\|^2 - 2\rho l_1\|g(x_{n+1}) - g(x_n)\| \\
& \times\Big[sh_1\|u_{n+1}-u_n\| + \rho l_1\|g(x_{n+1}) - g(x_n)\|\Big]. \qquad (3.4)
\end{aligned}
$$

Since g is r_1-Lipschitz continuous and S is α_1-D-Lipschitz continuous, we have

$$
\begin{aligned}
& \Big\|N\Big(g(x_{n+1}), A(y_{n+1})\Big) - N\Big(g(x_n), A(y_{n+1})\Big)\Big\| \\
& \leq\ l_1\|g(x_{n+1}) - g(x_n)\| \\
& \leq\ l_1r_1\|x_{n+1}-x_n\| \\
& \leq\ l_1r_1\alpha_1\Big(1+(1+n)^{-1}\Big)\|u_{n+1}-u_n\|. \qquad (3.5)
\end{aligned}
$$

Using (3.5) in (3.4), we have

$$
\begin{aligned}
& \Big\|H\Big(A((g-p)(u_{n+1})), B((g-p)(u_{n+1}))\Big) - H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big) \\
& \qquad -\rho\Big\{N\Big(g(x_{n+1}), A(y_{n+1})\Big) - N\Big(g(x_n), A(y_{n+1})\Big)\Big\}\Big\| \\
\leq\ & \sqrt{s^2h_1^2 - 2\rho l_1r_1\alpha_1\Big(1+(1+n)^{-1}\Big)\times\Big[sh_1+\rho l_1r_1\alpha_1\Big(1+(1+n)^{-1}\Big)\Big]} \\
& \times\|u_{n+1}-u_n\|. \qquad (3.6)
\end{aligned}
$$

Similarly, using h_2-Lipschitz continuity of $H(.,.)$ with respect to B, l_2-Lipschitz continuity of $N(\cdot,\cdot)$ with respect to second argument, s-Lipschitz continuity of $(g-p)$ and Lemma 1.6, we have the following estimate:

$$
\begin{aligned}
& \Big\|H\Big(A((g-p)(u_n)), B((g-p)(u_{n+1}))\Big) - H\Big(A((g-p)(u_n)), B((g-p)(u_n))\Big) \\
& \qquad -\rho\Big\{N\Big(g(x_n), A(y_{n+1})\Big) - N\Big(g(x_n), A(y_n)\Big)\Big\}\Big\|^2 \\
& \leq\ s^2h_2^2\|u_{n+1}-u_n\|^2 - 2\rho l_2\|A(y_{n+1}) - A(y_n)\|
\end{aligned}
$$

$$\times\Big[sh_2\|u_{n+1}-u_n\|+\rho l_2\|A(y_{n+1})-A(y_n)\|\Big]. \tag{3.7}$$

Since A is r_2-Lipschitz continuous and T is α_2-D-Lipschitz continuous, we have

$$\begin{aligned}\Big\|A(y_{n+1})-A(y_n)\Big\| &\leq r_2\Big\|y_{n+1}-y_n\Big\| \\ &\leq r_2\alpha_2\Big(1+(1+n)^{-1}\Big)\|u_{n+1}-u_n\|. \end{aligned}\tag{3.8}$$

Using (3.8) in (3.7), we have

$$\begin{aligned}&\Big\|H\Big(A((g-p)(u_n)),B((g-p)(u_{n+1}))\Big)-H\Big(A((g-p)(u_n)),B((g-p)(u_n))\Big)\\ &\qquad -\rho\Big\{N\Big(g(x_n),A(y_{n+1})\Big)-N\Big(g(x_n),A(y_n)\Big)\Big\}\Big\|\\ &\leq \sqrt{s^2h_2^2-2\rho l_2r_2\alpha_2\Big(1+(1+n)^{-1}\Big)\times\Big[sh_2+\rho l_2r_2\alpha_2\Big(1+(1+n)^{-1}\Big)\Big]}\\ &\quad\times\|u_{n+1}-u_n\|. \end{aligned}\tag{3.9}$$

Since $F(.,.,.)$ is β_j-Lipschitz continuous in the jth argument, for $j=1,2,3$, G is α_3-D-Lipschitz continuous and using Iterative Algorithm 3.2, we have

$$\begin{aligned}&\Big\|F(u_{n+1},u_{n+1},z_{n+1})-F(u_n,u_n,z_n)\Big\|\\ &\leq \Big\|F(u_{n+1},u_{n+1},z_{n+1})-F(u_n,u_{n+1},z_{n+1})\Big\|\\ &\quad+\Big\|F(u_n,u_{n+1},z_{n+1})-F(u_n,u_n,z_{n+1})\Big\|\\ &\quad+\Big\|F(u_n,u_n,z_{n+1})-F(u_n,u_n,z_n)\Big\|\\ &\leq \beta_1\Big\|u_{n+1}-u_n\Big\|+\beta_2\Big\|u_{n+1}-u_n\Big\|+\beta_3\Big\|z_{n+1}-z_n\Big\|\\ &\leq \Big[\beta_1+\beta_2+\beta_3\alpha_3\Big(1+(1+n)^{-1}\Big)\Big]\Big\|u_{n+1}-u_n\Big\|. \end{aligned}\tag{3.10}$$

Using (3.6),(3.9) and (3.10) in (3.3), we have

$$\begin{aligned}&\|(g-p)u_{n+2}-(g-p)u_{n+1}\|\\ &\leq \Big\{k\Big[\sqrt{s^2h_1^2-2\rho l_1r_1\alpha_1\Big(1+(1+n)^{-1}\Big)\times\Big[sh_1+\rho l_1r_1\alpha_1\Big(1+(1+n)^{-1}\Big)\Big]}\\ &\quad+\sqrt{s^2h_2^2-2\rho l_2r_2\alpha_2\Big(1+(1+n)^{-1}\Big)\times\Big[sh_2+\rho l_2r_2\alpha_2\Big(1+(1+n)^{-1}\Big)\Big]}\\ &\quad+\rho\Big\{\beta_1+\beta_2+\beta_3\alpha_3\Big(1+(1+n)^{-1}\Big)\Big\}\Big]\\ &\quad+\sigma\alpha_3\Big(1+(1+n)^{-1}\Big)\Big\}\ \Big\|u_{n+1}-u_n\Big\|. \end{aligned}\tag{3.11}$$

Since $(g - p - I)$ is λ-strongly accretive, by Lemma 1.6 and (3.11), we have the following estimate:

$$\begin{aligned}\|u_{n+2} - u_{n+1}\|^2 &\le \Big\|(g-p)u_{n+2} - (g-p)u_{n+1} + u_{n+2} - u_{n+1} \\ &\quad -\Big((g-p)u_{n+2} - (g-p)u_{n+1}\Big)\Big\|^2 \\ &\le \|(g-p)u_{n+2} - (g-p)u_{n+1}\|^2 \\ &\quad -2\Big\langle (g-p-I)u_{n+2} - (g-p-I)u_{n+1}, j(u_{n+2} - u_{n+1})\Big\rangle \\ &\le \|(g-p)u_{n+2} - (g-p)u_{n+1}\|^2 - 2\lambda\|u_{n+2} - u_{n+1})\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\|u_{n+2} - u_{n+1}\| \\ &\le \frac{1}{\sqrt{1+2\lambda}}\Big\{k\Big[\sqrt{s^2h_1^2 - 2\rho l_1 r_1 \alpha_1\Big(1+(1+n)^{-1}\Big) \times \Big[sh_1 + \rho l_1 r_1 \alpha_1\Big(1+(1+n)^{-1}\Big)\Big]} \\ &\quad + \sqrt{s^2h_2^2 - 2\rho l_2 r_2 \alpha_2\Big(1+(1+n)^{-1}\Big) \times \Big[sh_2 + \rho l_2 r_2 \alpha_2\Big(1+(1+n)^{-1}\Big)\Big]} \\ &\quad + \rho\Big\{\beta_1 + \beta_2 + \beta_3\alpha_3\Big(1+(1+n)^{-1}\Big)\Big\}\Big] + \sigma\alpha_3\Big(1+(1+n)^{-1}\Big)\Big\} \times \ \|u_{n+1} - u_n\| \\ &\le \phi_{n+1}\ \|u_{n+1} - u_n\|, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned}&\phi_{n+1} \\ &= \frac{1}{\sqrt{1+2\lambda}}\Big\{k\Big[\sqrt{s^2h_1^2 - 2\rho l_1 r_1 \alpha_1\Big(1+(1+n)^{-1}\Big) \times \Big[sh_1 + \rho l_1 r_1 \alpha_1\Big(1+(1+n)^{-1}\Big)\Big]} \\ &\quad + \sqrt{s^2h_2^2 - 2\rho l_2 r_2 \alpha_2\Big(1+(1+n)^{-1}\Big) \times \Big[sh_2 + \rho l_2 r_2 \alpha_2\Big(1+(1+n)^{-1}\Big)\Big]} \\ &\quad + \rho\Big\{\beta_1 + \beta_2 + \beta_3\alpha_3\Big(1+(1+n)^{-1}\Big)\Big\}\Big] + \sigma\alpha_3\Big(1+(1+n)^{-1}\Big)\Big\}.\end{aligned}$$

Let

$$\begin{aligned}\phi &= \frac{1}{\sqrt{1+2\lambda}}\Big\{k\Big[\sqrt{s^2h_1^2 - 2\rho l_1 r_1 \alpha_1\Big(sh_1 + \rho l_1 r_1 \alpha_1\Big)} \\ &\quad + \sqrt{s^2h_2^2 - 2\rho l_2 r_2 \alpha_2\Big(sh_2 + \rho l_2 r_2 \alpha_2\Big)} + \rho\Big(\beta_1 + \beta_2 + \beta_3\alpha_3\Big)\Big] + \sigma\alpha_3\Big\}.\end{aligned}$$

Then we know that $\phi_n \longrightarrow \phi$ as $n \longrightarrow \infty$.
By condition (3.2), we know that $\phi \in (0,1)$ and hence there exist $n_0 > 0$ and $\phi_0 \in (0,1)$ such that $\phi_{n+1} \le \phi_0$ for all $n \ge n_0$. Therefore by (3.12), we have

$$\|u_{n+2} - u_{n+1}\| \le \phi_0\|u_{n+1} - u_n\|,\ \forall\, n \ge n_0.$$

This implies

$$\|u_{n+1} - u_n\| \le \phi_0^{n-n_0}\|u_{n_0+1} - u_{n_0}\|.$$

Hence, for any $m \geq n > n_0$, we have

$$\begin{aligned} \|u_m - u_n\| &\leq \sum_{t=n}^{m-1} \|u_{t+1} - u_t\| \\ &\leq \sum_{t=n}^{m-1} \phi_0^{t-n_0} \|u_{n_0+1} - u_{n_0}\|. \end{aligned}$$

It follows $\|u_m - u_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ so that $\{u_n\}$ is a Cauchy sequence in X. Then there exists $u \in X$ such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$.
Now from α_1-D-Lipschitz continuity of S and Iterative Algorithm 3.2, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \Big(1 + (1+n)^{-1}\Big) D\Big(S(u_{n+1}), S(u_n)\Big) \\ &\leq \Big(1 + (1+n)^{-1}\Big) \alpha_1 \|u_{n+1} - u_n\|. \end{aligned} \tag{3.13}$$

Since $\{u_n\}$ being Cauchy in X, (3.13) implies that $\{x_n\}$ is a Cauchy sequence in X. Thus, in general, there exist x, y, z in X such that $x_n \longrightarrow x$, $y_n \longrightarrow y$, $z_n \longrightarrow z$ as $n \longrightarrow \infty$.
Now, we show that $x \in S(u)$. Since $x_n \in S(u_n)$, we have

$$\begin{aligned} d(x, S(u)) &\leq \|x - x_n\| + d(x_n, S(u)) \\ &\leq \|x - x_n\| + D\Big(S(u_n), S(u)\Big) \\ &\leq \|x - x_n\| + \alpha_1 \|u_n - u\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Since $S(u)$ is closed, it implies that $x \in S(u)$. Similarly, we can show that $y \in T(u)$, $z \in G(u)$. By assumption (3.1), Lipschitz continuity of proximal mapping $J^{H(.,.),\eta}_{M(.,z),\rho}$, continuity of the respective mappings and Iterative Algorithm 3.2, it follows that $u \in X$, $x \in S(u)$, $y \in T(u)$, $z \in G(u)$, where $J^{H(.,.),\eta}_{M(.,z),\rho}(u) = \Big(H(A,B) + \rho M(.,z)\Big)^{-1}(u)$ and ρ are constants. By Lemma 3.1, (u, x, y, z) is the solution of GVLIP 2.1. This completes the proof. □

Finally, we give the following result which gives the convergence of the sequences generated by the Iterative Algorithm 3.3 to the solution of Problem 2.2.

Theorem 3.5. *Let X be a real Banach space. Let $S, T, G : X \longrightarrow CB(X)$ be α_1-D-Lipschitz, α_2-D-Lipschitz, α_3-D-Lipschitz continuous mappings, respectively. Let $N : X \times X \longrightarrow X^\star$ be l_1-Lipschitz continuous and l_2-Lipschitz continuous with respect to first and second arguments, respectively. $\eta : X \times X \longrightarrow X$ be τ-Lipschitz continuous, $F : X \times X \times X \longrightarrow X^\star$ be β_j-Lipschitz continuous with respect to jth argument, for $j = 1, 2, 3$ and $A, B, p, g : X \longrightarrow X$ be single-valued mappings such that g is r_1-Lipschitz continuous, A is r_2-Lipschitz continuous. Let $H : X \times X \longrightarrow X^\star$ be α-strongly η-monotone with respect to A, β-relaxed η-monotone with respect to B and h_1-Lipschitz continuous and h_2-Lipschitz continuous with respect to A and B, respectively. Let $M : X \times X \longrightarrow 2^{X^\star}$ be set-valued mapping such that for fixed $z \in G(X)$, $M(., z) : X \times X \longrightarrow 2^{X^\star}$ be $(H(.,.), \eta)$-monotone operator with respect to A and B. In addition, suppose there exists a constant $\sigma > 0$ such that*

$$\left\| J^{H(.,.),\eta}_{M(.,z_{n+1})}(u) - J^{H(.,.),\eta}_{M(.,z)}(u) \right\| \leq \sigma \parallel z_{n+1} - z_n \parallel .$$

Furthermore, suppose the following condition is satisfied

$$0 < P < 1,$$

where P is given by,

$$P = k\Big\{\sqrt{h_1^2 - 2l_1r_1\alpha_1(h_1 + l_1r_1\alpha_1)} + \sqrt{h_2^2 - 2l_2r_2\alpha_2(h_2 + l_2r_2\alpha_2)} + (\beta_1 + \beta_2 + \beta_3\alpha_3)\Big\} + \sigma\alpha_3,$$

then the sequences $\{u_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$, generated by the Iterative Algorithm 3.3 converge strongly to the unique solution (u, x, y, z), respectively, where $u \in X,\ x \in S(u), y \in T(u)$ and $z \in G(u)$ is the solution of the problem (2.2).

Remark 3.6. Using the technique developed in this paper we can extend the results of Bhat and Zahoor [1], Chang *et. al* [2], Kazmi and Bhat [3-6], Mitrovic [9], Verma [14] and the related results cited therein for the system of variational inclusions.

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GENERAL ITERATIVE METHODS FOR A FAMILY OF NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we introduce implicit and explicit iterative methods for finding a common element of the set of solutions of a variational inequality and the set of common fixed points for a countable family of nonexpansive mappings in a Hilbert space. For these methods, we prove some strong convergence theorems. These theorems improve and extend some results of Yao et al. [21] and Xu [20].

KEYWORDS : Fixed point; Nonexpansive mapping; Weak contraction; Variational inequality.
AMS Subject Classification: 47H10, 47H09

1. INTRODUCTION

Let H be a real Hilbert space and A be a bounded operator on H. In this paper, we assume A is strongly positive; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in H$. A typical problem is that of minimizing a quadratic function over the set of the fixed points of nonexpansive mapping on a real Hilbert space:

$$\min_{x\in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where b is a given point in H.

We recall a mapping T of H into itself is called nonexpansive, if $\|Tx - Ty\| \leq \|x-y\|$ for all $x, y \in H$. Let $F(T)$ denote the fixed points set of T, and a contraction on H is a self-mapping f of H such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in H$, where $\alpha \in [0, 1)$ is a constant.

Finding an optimal point in the intersection F of the fixed points set of a family of nonexpansive mappings is one that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed points set of a family of nonexpansive mappings; see, e.g., [3, 5]. The problem of finding an optimal point that minimizes a given cost function $\Theta : H \to \mathbb{R}$ over F is of wide

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interdisciplinary interest and practical importance see, e.g., [2, 4, 6, 23]. A simple algorithmic solution to the problem of minimizing a quadratic function over F is of extreme value in many applications including the set theoretic signal estimation, see, e.g., [23, 9]. The best approximation problem of finding the projection $P_F(a)$ (in the norm induced by inner product of H) from any given point a in H is the simplest case of our problem.

In 2006, Marino and Xu [10] considered an iterative method for a single nonexpansive mapping. Let f be a contraction on H and $A : H \to H$ be a strongly positive bounded linear operator. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.1}$$

where $\gamma > 0$ is a constant and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(I) $\lim_{n\to\infty} \alpha_n = 0$;
(II) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(III) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Consequently, Marino and Xu [10] proved the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution of the following variational inequality:

$$\langle (A - \gamma f)x^*, x^* - x\rangle \leq 0, \text{ for all } x \in F(T),$$

which is the optimality condition for minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf(i.e., $h'(x) = \gamma f(x)$ for $x \in H$).
In 2012, Razani and Yazdi [13] study convergence of a composite iterative scheme which generalizes iterative sequence (1.1).

In 2008, Yao et al. [21] introduced the iterative sequence

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n x_n, \text{ for all } n \geq 0, \tag{1.2}$$

where W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \dots, T_n, \dots$ and $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ such that the common fixed points set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Under very mild conditions on the parameters, it was proved the sequence $\{x_n\}$ converges strongly to $p \in F$ where p is the unique solution in F of the following variational inequality:

$$\langle (A - \gamma f)p, p - x^*\rangle \leq 0, \text{ for all } x^* \in F, \tag{1.3}$$

which is the optimality condition for minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x).$$

In this paper, motivated by Yao et al. [21] and Rhoades [14], we introduce an implicit and explicit iterative schemes for finding a common element of the set of solutions of a variational inequality and the set of common fixed points for a countable family of nonexpansive mappings in a Hilbert space. Then, we prove some strong convergence theorems which improve and extend some results of Yao et al. [21] and Xu [20].

Now, we collect some lemmas which will be used in the main result.

Lemma 1.1. [10] *Assume A is a strongly positive bounded linear operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 1.2. *Let H be a real Hilbert space. Then, for all $x, y \in H$*
(I) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(II) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 1.3. [17] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in Banach space X and $\{\beta_n\}$ a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integer $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.4. [16] *Assume $\{s_n\}$ and $\{\gamma_n\}$ are two sequences of nonnegative real numbers such that*

$$s_{n+1} \leq s_n - r_n \Psi(s_n) + \gamma_n,\ n \geq 1,$$

where Ψ is a continuous and strict increasing function on $[0,\infty)$ with $\Psi(0) = 0$ and $\{r_n\}$ is a sequence of positive numbers satisfying the conditions:
(I) $\sum_{n=1}^{\infty} r_n = \infty$;
(II) $\limsup_{n\to\infty} \frac{\gamma_n}{r_n} = 0$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 1.5. [19] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(I) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(II) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

2. MAIN RESULTS

Let H be a real Hilbert space with inner product $\langle ., .\rangle$ and the norm $\|.\|$. We denote weak convergence and strong convergence by notation $\rightharpoonup$ and $\to$, respectively. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n of H into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{2.1}$$

Such a mapping W_n is called the $W-$mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Lemma 2.1. [15] *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.2. [22] It can be known from Lemma 2.1 that if D is a nonempty bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \varepsilon.$$

Remark 2.3. [22] Using Lemma 2.1, one can define mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for all $x \in C$. Such a W is called the $W-$mapping generated by $\{T_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive.

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 2.2 that for an arbitrary $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \varepsilon.$$

This implies

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Throughout this paper, we assume $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$.

Lemma 2.4. [15] *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Definition 2.5. [18] A self-mapping $f : C \to C$ is called weak contraction with the function Ψ if there exists a continuous and nondecreasing function $\Psi : [0, \infty) \to [0, \infty)$ such that $\Psi(s) > 0$, for all $s > 0$, $\Psi(0) = 0, \lim_{s\to\infty} \Psi(s) = +\infty$ and for any $x, y \in C, \|f(x) - f(y)\| \leq \|x - y\| - \Psi(\|x - y\|)$.

Remark 2.6. Clearly a contraction with constant k must be a weak contraction, where $\Psi(s) = (1 - k)s$, but the converse is not true.

Example 2.7. [1] The mapping $Ax = \sin x$ from $[0, 1]$ to $[0, 1]$ is a weak contraction with $\Psi(s) = \frac{s^3}{8}$. But A is not a contraction. Indeed, suppose that A is a contraction with constant $k \in (0, 1)$, i.e.,

$$|\sin x - \sin y| \leq k|x - y|, \text{ for all } x, y \in [0, 1]. \tag{2.2}$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, taking $\varepsilon = 1 - k$, there exists $\delta > 0$ as $0 < x < \delta$, we have $|\frac{\sin x}{x} - 1| < 1 - k$. Therefore $k < |\frac{\sin x - \sin 0}{x - 0}|$, i.e., $k|x - 0| < |\sin x - \sin 0|$, which contradicts the assumption of (2.2). Thus A is not a contraction.

Lemma 2.8. [14] *Let (X, d) be a complete metric space and $f : X \to X$ be a weak contraction. Then f has a unique fixed point in X.*

Lemma 2.9. [7] *Let H be a real Hilbert space, C be a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges to y, then $(I - T)x = y$.*

Lemma 2.10. [8] *Let $\{T_n\}$ be a sequence of nonexpansive mapping on a closed convex subset C of H and A be a strongly positive bounded linear operator on H with coefficient $0 < \gamma \leq \overline{\gamma}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. Define a sequence $\{y_n\}$ by $y_1 \in C$ and*

$$y_{n+1} = \alpha_n \gamma u + \beta_n y_n + ((1 - \beta_n)I - \alpha_n A)T_n y_n,$$

for all $n \in \mathbb{N}$. Suppose the sequence $\{y_n\}$ converges strongly. Set $Pu = \lim_{n\to\infty} y_n$, for each $u \in C$. Then, the following hold:

(I) Pu does not depend on the initial point y_1;
(II) P is a nonexpansive mapping on C.

Lemma 2.11. [16] *Let X be a Banach space , f be a weak contraction with a function Ψ on X and T be a nonexpansive mapping on X. Then, the composite mapping Tf is a weak contraction.*

It is easy to see the following lemma.

Lemma 2.12. *Let H be a real Hilbert space, $f : H \to H$ be a weak contraction and A be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \overline{\gamma}$*

$$\langle (A - \gamma f)x - (A - \gamma f)y, x - y\rangle \geq (\overline{\gamma} - \gamma)\|x - y\|^2, \text{ for all } x, y \in C.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma$.

Let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma \leq \overline{\gamma}$ where γ is some constant. First, we give our implicit iterative scheme as follows: let $\{t_n\}$ be a sequence in $(0, 1)$ such that $t_n \leq \|A\|^{-1}$, for all $n \geq 1$ and $u \in H$. For each $n \geq 1$, define a mapping $S_{t_n} : H \to H$ by

$$S_{t_n}(x) = t_n\gamma u + (I - t_nA)W_nx,\ x \in H.$$

It is easy to see that for each $t_n \in (0, 1)$, $n \geq 1$, S_{t_n} is a weak contraction on H. Indeed, by Lemma 1.1,

$$\begin{aligned}\|S_{t_n}(x) - S_{t_n}(y)\| &\leq t_n\gamma\|u - u\| + \|(I - t_nA)(W_nx - W_ny)\| \\ &\leq (1 - t_n\overline{\gamma})\|x - y\|.\end{aligned}$$

By Banach contraction principle, for each $n \in \mathbb{N}$, there exists a unique element $z_n \in H$ of S_{t_n} such that

$$z_n = t_n\gamma u + (I - t_nA)W_nz_n, \text{ for all } n \geq 1. \tag{2.3}$$

Theorem 2.1. *Let H be a real Hilbert space and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of H into itself which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{z_n\}$ be defined by (2.3) and $t_n \in (0, 1)$ such that $\lim_{n\to\infty} t_n = 0$. Then $\{z_n\}$ converges strongly to $p \in F$ which is the unique solution of the following variational inequality:*

$$\langle Ap - \gamma u, p - x^*\rangle \leq 0, \text{ for all } x^* \in F. \tag{2.4}$$

Proof. First, we show the uniqueness of the solution of the variational inequality (2.4). In fact, if p, q are two distinct solutions of the variational inequality (2.4), then

$$\langle Ap - \gamma u, p - q\rangle \leq 0 \text{ and } \langle Aq - \gamma u, q - p\rangle \leq 0.$$

Adding up these two inequalities, we have

$$\langle (Ap - \gamma u) - (Aq - \gamma u), p - q\rangle \leq 0.$$

But the strong monotonicity of $A - \gamma u$ (Lemma 2.12) implies that $p = q$. We use $p \in F$ to denote the unique solution of variational inequality (2.4). Thus, for $p \in F$

$$z_n - p = t_n(\gamma u - Ap) + (I - t_nA)(W_nz_n - p). \tag{2.5}$$

From (2.5),

$$\begin{aligned}\|z_n - p\|^2 &= t_n\langle \gamma u - Ap, z_n - p\rangle + \langle (I - t_nA)(W_nz_n - p), z_n - p\rangle \\ &\leq t_n\langle \gamma u - Ap, z_n - p\rangle + (1 - t_n\overline{\gamma})\|z_n - p\|^2.\end{aligned} \tag{2.6}$$

Simplifying (2.6), we have

$$\|z_n - p\|^2 \leq \frac{1}{\gamma}\langle \gamma u - Ap, z_n - p\rangle. \tag{2.7}$$

Hence, $\{z_n\}$ is bounded, so are $\{AW_n z_n\}$. Therefore

$$\lim_{n\to\infty} \|z_n - W_n z_n\| = \lim_{n\to\infty} t_n \|\gamma u - AW_n z_n\| = 0. \tag{2.8}$$

Take a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma u - Ap, z_n - p\rangle = \lim_{k\to\infty} \langle \gamma u - Ap, z_{n_k} - p\rangle.$$

Since $\{z_{n_k}\}$ is bounded in H, without loss of generality, we assume $z_{n_k} \rightharpoonup z \in H$. It follows from (2.8) and Remark 2.3 that $z \in F(W)$. So

$$\limsup_{n\to\infty} \langle \gamma u - Ap, z_n - p\rangle = \langle \gamma u - Ap, z - p\rangle \leq 0.$$

From (2.7), $\lim_{n\to\infty} z_{n_k} = z$. Next, we prove z solves the variational inequality (2.4). From (2.3),

$$Az_n - \gamma u = \frac{-1}{t_n}(I - t_n A)(z_n - W_n z_n).$$

Thus, for $q \in F$

$$\begin{aligned} \langle Az_n - \gamma u, z_n - q\rangle &= \tfrac{-1}{t_n}\langle (I - t_n A)(z_n - W_n z_n), z_n - q\rangle \\ &= \tfrac{-1}{t_n}\langle (I - W_n)z_n - (I - W_n)q, z_n - q\rangle + \\ &\qquad \langle A(I - W_n)z_n, z_n - q\rangle \\ &\leq \langle A(I - W_n)z_n, z_n - q\rangle, \end{aligned} \tag{2.9}$$

since $I - W_n$ is monotone (i.e.,$\langle (I - W_n)x - (I - W_n)y, x - y\rangle \geq 0$ for $x, y \in H$. This is due to the nonexpansivity of W_n). Now, replacing z_n in (2.9) with z_{n_k} and letting $k \to \infty$. Note that $\lim_{n\to\infty} z_{n_k} = z$ which implies

$$\langle Az - \gamma u, z - q\rangle \leq 0.$$

That is, $z \in F$ is a solution of the variational inequality (2.4) and hence $z = p$ by uniqueness. Since each cluster point of $\{z_n\}$ equals p, $z_n \to p$ as $n \to \infty$. This completes the proof. □

Theorem 2.2. *Let H be a real Hilbert space, $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of H into itself which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \overline{\gamma}$ where γ is some constant. Let $\{z_n\}$ be defined by*

$$z_n = t_n \gamma f(z_n) + (I - t_n A)W_n z_n, \text{ for all } n \geq 1, \tag{2.10}$$

where $f : H \to H$ is a weak contraction with a function Ψ, $t_n \in (0, 1)$ such that $\lim_{n\to\infty} t_n = 0$. Then $\{z_n\}$ converges strongly to $p \in F$ which is the unique solution of the following variational inequality:

$$\langle (A - \gamma f)p, p - x^*\rangle \leq 0, \text{ for all } x^* \in F. \tag{2.11}$$

Proof. Define a sequence $\{u_n\}$ by

$$u_n = t_n \gamma u + (I - t_n A)W_n u_n, \text{ for all } n \geq 1,$$

for any $u \in H$. From Theorem 2.1, $\{u_n\}$ converges strongly. Set $Pu = \lim_{n\to\infty} u_n$, for each $u \in H$. It follows from Lemma 2.10 that P is nonexpansive. Then Pf is a weak contraction by Lemma 2.11. From Lemma 2.8, there exists a unique element $z \in H$ such that $z = P(f(z))$. Define a sequence $\{k_n\}$ by

$$k_n = t_n\gamma f(z) + (I - t_nA)W_nk_n, \text{ for all } n \geq 1. \tag{2.12}$$

Then, by Theorem 2.1, $\lim_{n\to\infty} k_n = P(f(z)) = z \in F(W)$. Therefore

$$\begin{aligned}\|z_n - k_n\| = \;& t_n\gamma\|f(z_n) - f(z)\| + (1 - t_n\overline{\gamma})\|W_nz_n - W_nk_n\| \\ \leq \;& t_n\gamma(\|f(z_n) - f(k_n)\| + \|f(k_n) - f(z)\|) + (1 - t_n\overline{\gamma})\|z_n - k_n\| \\ \leq \;& t_n\gamma(\|z_n - k_n\| - \psi(\|z_n - k_n\|) + \|k_n - z\| - \psi(\|k_n - z\|)) \\ & +(1 - t_n\overline{\gamma})\|z_n - k_n\| \\ \leq \;& (1 - t_n(\overline{\gamma} - \gamma))\|z_n - k_n\| + t_n\gamma\|k_n - z\|.\end{aligned}$$

Which implies

$$\|z_n - k_n\| \leq \frac{\gamma}{\overline{\gamma} - \gamma}\|k_n - z\|.$$

So $\lim_{n\to\infty} \|z_n - k_n\| = 0$ and hence $\lim_{n\to\infty} \|z_n - z\| = 0$. This complete the proof. □

Secondly, we give an explicit iterative scheme: for any given $x_0 \in H$, let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n\gamma u + \beta_nx_n + ((1 - \beta_n)I - \alpha_nA)W_nx_n, \text{ for all } n \geq 0. \tag{2.13}$$

Now, we prove the following strong convergence theorem concerning the iterative scheme (2.13).

Theorem 2.3. *Let H be a real Hilbert space, $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of H into itself which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $\|A\| \leq 1$. Let $0 < \gamma \leq \overline{\gamma}$ where γ i some constant. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $(0, 1)$ satisfying the following conditions:*
(I) $\lim_{n\to\infty} \alpha_n = 0$;
(II) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(III) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then, the sequence $\{x_n\}$ defined by (2.13) converges strongly to $p \in F$ which is the unique solution of the following variational inequality (2.4).

Proof. Let $Q = P_{\bigcap_{n=1}^{\infty} F(T_n)}$. So

$$\begin{aligned}\|Q((I - A)x + \gamma u) - Q((I - A)y + \gamma u)\| \leq \;& \|(I - A)x + \gamma u - \\ & ((I - A)y + \gamma u)\| \\ \leq \;& \|(I - A)x - (I - A)y\| \\ \leq \;& (1 - \overline{\gamma})\|x - y\|,\end{aligned}$$

for all $x, y \in H$. Therefore $Q = P_{\bigcap_{n=1}^{\infty} F(T_n)}$ is a contraction of H into itself. By Banach contraction principle there exists a unique element $p \in H$ such that $p = Q((I - A)p + \gamma u) = P_{\bigcap_{n=1}^{\infty} F(T_n)}((I - A)p + \gamma u)$ or equivalently

$$\langle Ap - \gamma u, p - x^*\rangle \leq 0, \text{ for all } x^* \in F.$$

From the condition (I), we may assume, without loss of generality, $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. Since A is strongly positive bounded linear operator on H,

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe

$$\begin{aligned}\langle((1 - \beta_n)I - \alpha_nA)x, x\rangle \;& = (1 - \beta_n) - \alpha_n\langle Ax, x\rangle \\ & \geq 1 - \beta_n - \alpha_n\|A\| \\ & \geq 0,\end{aligned}$$

that is to say $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle((1-\beta_n)I - \alpha_n A)x, x\rangle : x \in H, \|x\| = 1\} \\ &= \sup\{1-\beta_n - \alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\} \\ &\leq 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

Next, we prove $\{x_n\}$ is bounded. Indeed, for $p \in F$

$$\begin{aligned}\|x_{n+1}-p\| = \; & \|\alpha_n(\gamma u - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(W_n x_n - p)\| \\ \leq \; & (1-\beta_n-\alpha_n\overline{\gamma})\|x_n - p\| + \beta_n\|x_n - p\| + \alpha_n\|\gamma u - Ap\| \\ \leq \; & (1-\alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\|\gamma u - Ap\|.\end{aligned} \tag{2.14}$$

It follows from (2.14) that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma u - Ap\|}{\overline{\gamma}}\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded, so are $\{W_n x_n\}$.

Define

$$x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \; n \geq 0.$$

Observe from the definition of y_n,

$$\begin{aligned}y_{n+1} - y_n = \; & \tfrac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \tfrac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ = \; & \tfrac{\alpha_{n+1}\gamma u + ((1-\beta_{n+1})I - \alpha_{n+1}A)W_{n+1}x_{n+1}}{1-\beta_{n+1}} \\ & - \tfrac{\alpha_n \gamma u + ((1-\beta_n)I - \alpha_n A)W_n x_n}{1-\beta_n} \\ = \; & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma u - \tfrac{\alpha_n}{1-\beta_n}\gamma u + W_{n+1}x_{n+1} \\ & - W_n x_n + \tfrac{\alpha_n}{1-\beta_n}AW_n x_n - \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}AW_{n+1}x_{n+1} \\ = \; & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma u - AW_{n+1}x_{n+1}] + \tfrac{\alpha_n}{1-\beta_n}[AW_n x_n - \gamma u] \\ & + W_{n+1}x_{n+1} - W_{n+1}x_n + W_{n+1}x_n - W_n x_n.\end{aligned}$$

So

$$\begin{aligned}\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq \; & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma u\| + \|AW_{n+1}x_{n+1}\|) \\ & + \tfrac{\alpha_n}{1-\beta_n}(\|AW_n x_n\| + \|\gamma u\|) \\ & + \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_n x_n\| \\ & - \|x_{n+1} - x_n\| \\ \leq \; & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma u\| + \|AW_{n+1}x_{n+1}\|) \\ & + \tfrac{\alpha_n}{1-\beta_n}(\|AW_n x_n\| + \|\gamma u\|) + \|W_{n+1}x_n - W_n x_n\|.\end{aligned} \tag{2.15}$$

From (2.1), Since T_i and $U_{n,i}$ are nonexpansive, we get

$$\begin{aligned}\|W_{n+1}x_n - W_n x_n\| = \; & \|\lambda_1 T_1 U_{n+1,2}x_n - \lambda_1 T_1 U_{n,2}x_n\| \\ \leq \; & \lambda_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\ = \; & \lambda_1\|\lambda_2 T_2 U_{n+1,3}x_n - \lambda_2 T_2 U_{n,3}x_n\| \\ \leq \; & \lambda_1\lambda_2\|U_{n+1,3}x_n - U_{n,3}x_n\| \\ \leq \; & \dots \\ \leq \; & \lambda_1\lambda_2\dots\lambda_n\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\ \leq \; & M\prod_{i=1}^{n}\lambda_i,\end{aligned} \tag{2.16}$$

where $M \geq 0$ is a constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M$, for all $n \geq 0$. Substituting (2.16) into (2.15), we have

$$\begin{aligned}\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \; & \leq \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma u\| + \|AW_{n+1}x_{n+1}\|) \\ & + \tfrac{\alpha_n}{1-\beta_n}(\|AW_n x_n\| + \|\gamma u\|) + M\prod_{i=1}^{n}\lambda_i,\end{aligned}$$

which implies (noting that (I) and $0 < \lambda_i \le b < 1$, for all $i \ge 1$)

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, by Lemma 1.3,

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Consequently

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(1 - \beta_n)\|y_n - x_n\| = 0. \tag{2.17}$$

Note

$$\begin{aligned} \|x_n - W_n x_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - W_n x_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|\gamma u - AW_n x_n\| + \beta_n\|W_n x_n - x_n\|, \end{aligned} \tag{2.18}$$

which implies

$$\|x_n - W_n x_n\| \le \frac{\|x_{n+1} - x_n\| + \alpha_n\|\gamma u - AW_n x_n\|}{1 - \beta_n}.$$

It follows from (2.17) that

$$\lim_{n\to\infty} \|W_n x_n - x_n\| = 0. \tag{2.19}$$

By the same argument as in the proof of Theorem 2.1,

$$\limsup_{n\to\infty}\langle \gamma u - Ap, x_n - p\rangle \le 0, \tag{2.20}$$

where $p = P_{\bigcap_{n=1}^{\infty} F(T_n)}((I - A)p + \gamma u)$. From (2.19),

$$\limsup_{n\to\infty}\langle \gamma u - Ap, W_n x_n - p\rangle \le 0 \tag{2.21}$$

Finally, we prove $x_n \to p$ as $n \to \infty$. From (2.13),

$$\begin{aligned} \|x_{n+1} - p\|^2 = &\ \|\alpha_n(\gamma u - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A)(W_n x_n - p)\|^2 \\ = &\ \alpha_n^2\|\gamma u - Ap\|^2 + \|\beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A)(W_n x_n - p)\|^2 \\ &\ +2\beta_n\alpha_n\langle \gamma u - Ap, x_n - p\rangle \\ &\ +2\alpha_n\langle \gamma u - Ap, ((1 - \beta_n)I - \alpha_n A)(W_n x_n - p)\rangle \\ \le &\ ((1 - \beta_n - \alpha_n\overline{\gamma})\|W_n x_n - p\| + \beta_n\|x_n - p\|)^2 + \alpha_n^2\|\gamma u - Ap\|^2 \\ &\ +2\beta_n\alpha_n\langle \gamma u - Ap, x_n - p\rangle + 2(1 - \beta_n)\alpha_n\langle \gamma u - Ap, W_n x_n - p\rangle \\ &\ -2\alpha_n^2\langle \gamma u - Ap, A(W_n x_n - p)\rangle, \end{aligned}$$

which implies

$$\begin{aligned} \|x_{n+1} - p\|^2 \le &\ (1 - \alpha_n\overline{\gamma})^2\|x_n - p\|^2 + 2\beta_n\alpha_n\langle \gamma u - Ap, x_n - p\rangle \\ &\ +\alpha_n^2\|\gamma u - Ap\|^2 + 2(1 - \beta_n)\alpha_n\langle \gamma u - Ap, W_n x_n - p\rangle \\ &\ -2\alpha_n^2\langle \gamma u - Ap, A(W_n x_n - p)\rangle \\ \le &\ [1 - 2\alpha_n\overline{\gamma}]\|x_n - p\|^2 + \overline{\gamma}^2\alpha_n^2\|x_n - p\|^2 + \alpha_n^2\|\gamma u - Ap\|^2 \\ &\ +2\beta_n\alpha_n\langle \gamma u - Ap, x_n - p\rangle + 2(1 - \beta_n)\alpha_n\langle \gamma u - Ap, W_n x_n - p\rangle \\ &\ +2\alpha_n^2\|\gamma u - Ap\|\|A(W_n x_n - p)\| \\ = &\ [1 - 2\alpha_n\overline{\gamma}]\|x_n - p\|^2 + \alpha_n\{\alpha_n(\overline{\gamma}^2\|x_n - p\|^2 \\ &\ +\|\gamma u - Ap\|^2 + 2\|\gamma u - Ap\|\|A(W_n x_n - p)\|) \\ &\ +2\beta_n\langle \gamma u - Ap, x_n - p\rangle + 2(1 - \beta_n)\langle \gamma u - Ap, W_n x_n - p\rangle\}. \end{aligned}$$

Since $\{x_n\}$ and $\{W_n x_n\}$ are bounded, we can take a constant $M_1 > 0$ such that

$$\overline{\gamma}^2\|x_n - p\|^2 + \|\gamma u - Ap\|^2 + 2\|\gamma u - Ap\|\|A(W_n x_n - p)\| \le M_1,$$

for all $n \ge 0$. So

$$\|x_{n+1} - p\|^2 \le [1 - 2\alpha_n\overline{\gamma}]\|x_n - p\|^2 + \alpha_n\xi_n, \tag{2.22}$$

where

$$\xi_n = 2\beta_n\langle \gamma u - Ap, x_n - p\rangle + 2(1-\beta_n)\langle \gamma u - Ap, W_n x_n - p\rangle + \alpha_n M_1.$$

By (I), (2.20) and (2.21), we get $\limsup_{n\to\infty} \xi_n \leq 0$. Now, applying Lemma 1.5 to (2.22) concludes $x_n \to p$ as $n \to \infty$. This completes the proof. □

Theorem 2.4. *Let H be a real Hilbert space, $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of H into itself which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, $f : H \to H$ be a weak contraction with a function Ψ, A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $\|A\| \leq 1$. Let $0 < \gamma \leq \overline{\gamma}$ where γ is some constant. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $(0,1)$ satisfying the following conditions:*
(I) $\lim_{n\to\infty} \alpha_n = 0$;
(II) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(III) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
For any given $x_0 \in H$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n x_n, \text{ for all } n \geq 1, \tag{2.23}$$

converges strongly to $p \in F$ which is the unique solution of the following variational inequality (2.11).

Proof. Define a sequence $\{u_n\}$ by

$$u_{n+1} = \alpha_n \gamma u + \beta_n u_n + ((1-\beta_n)I - \alpha_n A)W_n u_n, \text{ for all } n \geq 1,$$

for any $u \in C$. From Theorem 2.3, $\{u_n\}$ converges strongly. Set $Pu = \lim_{n\to\infty} u_n$, for each $u \in C$. By the same argument as in the proof of Theorem 2.2, there exists $z = P(f(z))$. Define a sequence $\{k_n\}$ by

$$k_{n+1} = \alpha_n \gamma f(z) + \beta_n k_n + ((1-\beta_n)I - \alpha_n A)W_n k_n, \text{ for all } n \geq 1. \tag{2.24}$$

Then, by Theorem 2.3, $\lim_{n\to\infty} k_n = P(f(z)) = z \in F(W)$. Therefore

$$\begin{aligned}
\|x_{n+1} - k_{n+1}\| = \;& \alpha_n\gamma\|f(x_n) - f(z)\| + \beta_n\|x_n - k_n\| \\
& +(1-\beta_n-\alpha_n\overline{\gamma})\|W_n x_n - W_n k_n\| \\
\leq \;& \alpha_n\gamma(\|f(x_n) - f(k_n)\| + \|f(k_n) - f(z)\|) + \beta_n\|x_n - k_n\| \\
& +(1-\beta_n-\alpha_n\overline{\gamma})\|x_n - k_n\| \\
\leq \;& \alpha_n\gamma(\|x_n - k_n\| - \psi(\|x_n - k_n\|) + \|k_n - z\| - \psi(\|k_n - z\|)) \\
& +(1-\alpha_n\overline{\gamma})\|x_n - k_n\| \\
\leq \;& (1-\alpha_n(\overline{\gamma}-\gamma))\|x_n - k_n\| - \alpha_n\gamma\psi(\|x_n - k_n\|) \\
& +\alpha_n\gamma(\|k_n - z\| - \psi(\|k_n - z\|)) \\
\leq \;& \|x_n - k_n\| - \alpha_n\gamma\psi(\|x_n - k_n\|) + \alpha_n\gamma(\|k_n - z\| - \psi(\|k_n - z\|)).
\end{aligned}$$

Set $s_n = \|x_n - k_n\|$, $\gamma_n = \alpha_n\gamma(\|k_n - z\| - \psi(\|k_n - z\|))$ and $r_n = \alpha_n\gamma$. Since

$$\lim_{n\to\infty} \frac{\gamma_n}{r_n} = \|k_n - z\| - \psi(\|k_n - z\|) = 0$$

and

$$\sum_{n=0}^{\infty} r_n = \sum_{n=0}^{\infty} \alpha_n\gamma = \infty,$$

by Lemma 1.4 , $\lim_{n\to\infty} \|x_n - k_n\| = 0$. Hence $\lim_{n\to\infty} \|x_n - z\| = 0$. This complete the proof. □

Remark 2.13. Theorem 2.2 is a generalization of [21, Theorem 3.1].

Remark 2.14. Theorem 2.4 is a generalization of [20, Theorem 3.2] and [21, Theorem 3.2] with assumption $\|A\| \leq 1$.

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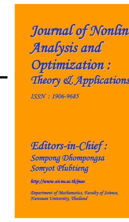

OPTIMAL PRICING POLICY FOR MANUFACTURER AND RETAILER USING ITEM PRESERVATION TECHNOLOGY FOR DETERIORATING ITEMS

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ABSTRACT. This paper optimizes the selling price, the replenishment cycle and lot size of deteriorating seasonal items. The seasonal items become useless or completely deteriorated after sale season. The production of manufacturer and stock of retailer are affected by deterioration and assuming it is reduced by preservation technology investment. Manufacturer and retailer both are invested on a preservation technology under revenue sharing. We studied the effect of preservation technology on profit of manufacture as well as for retailer. Retailer can change the strategy by reducing the selling price to generate the excess demand for limited time duration. This paper aims to develop a continuous supply chain inventory model by optimizing the selling price of seasonal items. We optimized the profit by reducing price for stock dependent price sensitive demand and have shown that the profit function is concave function of selling price. The model is simulated and illustrated with numerical examples.

KEYWORDS : Inventory, stock and price dependent demand, optimal profit, deterioration, replenishment cycle.
AMS Subject Classification: 90B05, 90B30, 90B50

1. INTRODUCTION

In the real life, deterioration of product is a common phenomenon and always occur in the nature. There are many items in the nature that deteriorate significantly such as fruits, vegetable, milks, meat fresh foods, perfumes, alcohols soft drinks gasoline etc. Also demand of such items are just for a limited time horizon such type of items known as seasonal items. Recently more and more items become deteriorating nature and seasonal simultaneously because of the business competition, instant and rapid change in the technology. Hence this will become a very difficult problem to determine the inventory if the item is both deteriorating and seasonal.

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The preservation play a vital role for decreasing deterioration rate and for increasing seasonal item's life. Recently the effective use of preservation technology become essential for manufacturers and retailers both, to improve the customer service level, increase the business profit and reduce the economic losses. In the last few years, market surveys indicated that the demand rate of most of the inventory depends on the stock levels, and influenced by selling prices. In this paper we studied the pricing strategy of deteriorating seasonal items, optimize lot size, and replenishment cycle when deterioration rate is controlled by preservation technology investment. The decision variables of this model are the market demand, the production rate, the ordering frequency. Ghare and Schrader [19], are the first mathematician who developed an inventory model in which they considered deterioration as exponential decreasing function, after then remarkable works have been done on deteriorating inventory modeling. Liu and Lian [14], as well as Nandakumar and Morton [20], studied deteriorating items which have fixed life time. Jain and silver [13] as well as Kalpakam and Sapana [24], developed inventory models for random life deteriorated items.

In the deteriorating inventory modeling deterioration plays a vital role. Raffat, Wolfe and Eldin [7], developed an inventory model with constant demand rate and finite replenishment rate for deteriorating items. Heng, Labban and Linn [12], considering exponential decay in inventory model with constant demand and finite replenishment rate. The constant demand rate is a uniform deterministic demand rate but in the real life demand rate is not always constant. Time varying demand in inventory modeling have been developed mostly considering either linearly increasing/decreasing $D(t) = (\alpha + \gamma t)$, $\alpha > 0$, $\gamma \neq 0$ or exponentially increasing/decreasing $D(t) = \alpha.e^{at}$, $\alpha > 0$, $a \neq 0$. Haiping and Wang [31], suggested a model in which they considered time proportional demand and find optimum order quantity for deteriorating items. Xu and Wang [11], presented a model in which they consider linearly time varying demand for exponentially deteriorating items. Giri and Chaudhuari [2], developed a model in which they consider deterioration rate demand rate and costs were assumed to variable of time. Jalan and Chaudhuari [1] as well as Chakraborty, Giri and Chaudhuari [25], considered deterioration rate as two parameter and three parameter weibull distribution with instantaneous supply in their inventory models.

In analysis of market survey, it is observed that in the supermarket customers attraction is based on a large pile of goods. Hence displaying each of items in large quantities may be generate extra demand. But due to large quantity of items there may be arise problem of spacing of each item and also requirement of large scale investment. The situation become more critical when the displayed items is in nature of deteriorating. Due to this reason research attracted on inventory modeling in which demand consider as stock and price dependent. Sarkar, Mukherjee and Balan [4] as well as Datta and Pal [26], developed an inventory model assuming stock and selling price dependent demand for deteriorating items with and without shortages. Kim [5], developed price dependent inventory model for considering constant rate of deterioration with infinite rate of replenishment. Wee [10], studied the joint pricing and replenishment policy for deteriorating inventory with price elastic demand rate in the decline market for time dependent deterioration.

Samanta and Roy [9], developed a continuous production control inventory model for deteriorating items with shortages. They consider deterioration rate is very small, demand and production rate is constant. Ilkeong, Giri and Byungsung [16], developed an inventory model for amelioration/deteriorating items with time varying demand pattern finite planing horizon, taking into account the effects of inflation and time value of money. Ouyang, Wu, and Cheng [15], studied an economical inventory model for deteriorating items with exponential decreasing demand. In this model shortages are allowed assuming backlogging rate is a function of waiting time for the next period.

Uthayakumar and Parvathi [22], described an inventory model in which they assumed demand is stock dependent and deterioration rate is nonlinear function of time. They also assumed that the retailer adopts the trade credit policy offered by supplier. Jain, Rathore and Sharma [17], presented an economical production quantity model for deteriorating items in which they consider price and stock dependent demand with considering shortages. Jain and Kumar [23], presented an economical order quantity model in which he consider ramp type demand, starting with and without two parameter weibull distribution deterioration rate $z(t) = \alpha\beta t^{\beta-1}$ where $\alpha(0 < \alpha << 1)$ is the scale parameter and $(\beta > 0)$ is the shape parameter.

Cachon and Lariviere [8], developed supply chain coordination with revenue sharing contracts model, in this revenue sharing contract, a retailer pays to supplier a wholesale price for each unit that he purchased and also pays a percentage of the revenue that generates retailer. Shukla and Khedlekar [6], presented time and price dependent with varying holding cost inventory model for deteriorating items in this model they considered the demand as a parametric dependent linear function of time and price both. The coefficient of time parameter and coefficient of price parameter are examined simultaneously and proved that time is dominating variable over price in term of earning more profit. It is also proved that deterioration of items in the inventory is one of the most sensitive parameter to look in to besides many others. Khedlekar and Namdeo [27], developed an inventory model for stock and price dependent demand with deterioration, but there is no explanation about deteriorating rate of products. He, Wang and Lai [33], developed production inventory model in which they consider deteriorating properties of products

Giri and Bardhan [3], presented an integrated single-manufacturer single retailer supply chain model for deteriorating item. In this model demand function is assumed to be the function of on hand stock and price furthermore manufacturer and retailer are in an agreement of lot for lot policy.The proposed model is developed under the contract that the retailer offers the manufacturers a percentage of revenue(s), he generates by selling a lot. Palani and Maragatham [21], proposed a deterministic inventory model for exponential deteriorating items in which demand rate and holding cost are quadratic and linear function of time. They also consider that the deterioration is controlled by using preservation technology investment.

Yang, Wee, Chunge and Huang [18], developed a piecewise production inventory model for a multi market deteriorating product with time varying and price varying sensitive demand. He and Huange [32], studied a kind of deteriorating products whose deterioration can be controlled by investing on the preservation efforts. Study considers the seasonal and deterioration properties simultaneously, Demand rate is assumed to be decreasing linear function of selling price and assuming resultant deterioration is decreasing exponential function of cost of preservation technology investment per unit time.

Mishra [30], developed an integrated single-retailer and single-supplier inventory model for deteriorating items under revenue sharing on preservation technology investment in which he considered the demand rate is a non negative power function of selling price and stock level,and production rate is constant. He also proposed the manufacturer offers to the retailers for a percentage of revenue sharing on preservation technology investment. Numerical and graphical illustration is given by him. Khedlekar, Namdeo and Nigwal [28], introduced disruption factor in production inventory modeling considering with shortages and time proportional demand, Khedlekar, Shukla and Namdeo [29], designed pricing strategies for declining market demand of deteriorating item introducing item preservation technology. In the previous study in this field, demand function as a price sensitive, stock dependent and constant production rate is considered by researchers. This paper presents a model in which demand rate as a exponential decreasing function of t as well as price and stock dependent production rate taken as a linear function of t and also considered preservation technology investment factor for deteriorating items.

2. Notations and Assumptions

Following notation are used in this model.

- c Purchase cost per unit for retailer and selling cost for manufacturer,
- p Retail price per unit items,
- C_s Compiling cost per lot for manufacturer,
- C_p Production cost per unit items,
- C_0 The ordering cost per order of the retailer,
- C_d Deteriorating cost per cycle [Value of deteriorated products per unit],
- κ Cost coefficient of investment in the preservation technology,
- C_h Unit inventory holding cost per unit time,
- ξ Preservation technology cost for reducing deterioration rate in order preserve the product
- θ The deterioration rate,
- ρ Consequent deterioration rate $\rho = \theta e^{-\eta\xi}$,
- $D(p, I(t))$ Market Demand rate at time t; $D(p, I(t)) = \alpha e^{-at} - \beta p + \phi I(t)$ where α demand sensitive parameter, β price sensitive parameter, and ϕ stock sensitive parameter,
- q_m Production rate which is linear function of t; we assume $q_m = q + rt$; where r is the production sensitive parameter,
- q Production scale,
- δ The subsidy proportion provided by manufacturer to the retailer for preservation technology investment,
- T The length of cycle time,
- Q Initial lot-size during a cycle of length T,
- $I_m(t)$ Inventory level at time t for the manufacturer, $0 \le t \le T$,
- $I_r(t)$ Inventory level at time t for the retailer, $0 \le t \le T$,
- AP_R Total profit per unit time for the retailer,
- AP_M Total profit per unit time for the manufacturer,
- NTP Average Total profit per unit time under integrated system,

The following assumption are made in this model:

- Market Demand of product is $D(p, I(t))$ at unit time t; we assumed demand function $D(p, I(t)) = \alpha e^{-at} - \beta p + \phi I(t)$, is nonnegative exponential function of t as well as price and stock level, where α is initial demand and β is price sensitive parameter, ϕ is stock sensitive parameter, and $\alpha > 0$, $a \ge 0$, $\beta > 0$, $\phi \ge 0$,
- Holding cost and deterioration cost are constant,
- Production rate is linear increasing function $q_m = q + rt$,

- Partly or wholly deteriorated products have no value and there is no holding cost for them.
- The deterioration rate is controlled by preservation and products which are fully preserved by preservation technology,
- Preservation technology investment is shared by manufacturer and retailer for reducing the deterioration rate, and sharing rate is δ.
- The proportion of reduced deterioration rate, $\rho = \theta e^{-\eta\xi}$, is concave increasing function of ξ,
- The deterioration cost due to deterioration and holding cost for both manufacturer and retailer are same,
- The lead time is zero, and replenishment rate is finite, however the planning horizon is finite.
- In the finite time horizon T it is considered that $e^{-aT} \approx e^{-at}$ because a is taken as very small.

3. Proposed Model for Retailer

According to the assumptions the retailer receives the stock initially from the manufacturer, at time t, $0 \leq t \leq T$. The rate of change in inventory level for retailer is equal to demand rate and deterioration rate. Thus the following first order nonlinear differential equation representing the inventory status at any time t

$$\begin{aligned} \frac{dI_r(t)}{dt} + \rho I_r(t) &= -D(p, I_r(t)), \quad \text{where} \quad 0 \leq t \leq T. \\ &= -(\alpha e^{-at} - \beta p + \phi I_r(t)) \end{aligned} \tag{3.1}$$

with boundary condition $I_r(t) = Q$, at $t = 0$ and $I_r(t) = 0$, at $t = T$
Now we derived the average profit function of retailer during a replenishment cycle interval $[0, T]$.
The average profit for retailer can be formulated as
Average Profit $= \frac{1}{T}$[Sales Revenue-Purchase Cost-Ordering Cost-Inventory Holding Cost-Deterioration Cost-Preservation Technology Investment Cost]
Equation (3.1) leads to

$$I_r(t) = \left(1 - e^{(\rho+\phi)(T-t)}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \tag{3.2}$$

The initial order lot size for retailer at time $t = 0$, where $t \in [0, T]$ is

$$I_r(0) = Q = \left(1 - e^{(\rho+\phi)T}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \tag{3.3}$$

The total sales revenue in replenishment cycle time $[0, T]$ can be formulated as

$$\begin{aligned} SR_r &= p\int_0^T D(p, I_r(t))dt \\ SR_r &= \phi p\left(T + \frac{1}{\rho+\phi} - \frac{e^{(\rho+\phi)T}}{(\rho+\phi)}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \\ &\quad + \frac{p\alpha}{a}\left(1 - e^{-aT}\right) - \beta p^2 T \end{aligned} \tag{3.4}$$

Purchase cost of retailer is

$$\begin{aligned} PC_r &= c.Q \\ PC_r &= c\left(1 - e^{(\rho+\phi)T}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \end{aligned} \tag{3.5}$$

The inventory holding cost $IHCr$ is

$$IHC_r = h\int_0^T I_r(t)dt$$

$$IHCr = h\left(T + \frac{1}{\rho+\phi} - \frac{e^{(\rho+\phi)T}}{(\rho+\phi)}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \tag{3.6}$$

Deterioration cost DC_r in the interval of length $[0,T]$ is

$$DC_r = C_d\theta e^{-\eta\xi}\int_0^T I_r(t)dt$$

$$DC_r = C_d\theta e^{-\eta\xi}\left(T + \frac{1}{\rho+\phi} - \frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \tag{3.7}$$

Preservation technology investment cost $PTIC_r$ is

$$PTIC_r = (1-\delta)\kappa\xi T \tag{3.8}$$

The ordering cost is given by

$$OC_r = C_o \tag{3.9}$$

Hence the average profit function for retailer per unit time is

$$\begin{aligned} AP_r = {} & \frac{p\alpha}{Ta}(1-e^{-aT}) - \beta p^2 - \frac{1}{T}C_o - (1-\delta)\kappa\xi \\ & + \frac{\zeta}{T}\left(T + \frac{1}{\rho+\phi} - \frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \\ & - \frac{c}{T}\left(1-e^{(\rho+\phi)T}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \end{aligned} \tag{3.10}$$

where $\zeta = \left(\phi p - h - C_d\theta e^{-\eta\xi}\right)$

4. Proposed Model for Manufacturer

Manufacturer supply the quantity at rate q_m to the retailer. At time t, on hand inventory of manufacturer is $I(t)$. Due to preservation technology the reduced deterioration value is $\rho I_m(t)$. Thus the differential equation will be

$$\begin{aligned} \frac{dI_m(t)}{dt} + \rho I_m(t) &= q_m \quad \text{,where } \ t_s \le t \le T. \\ &= q + rt \end{aligned} \tag{4.1}$$

with boundary condition $I_m(t_s) = 0$, at $t = t_m$ and $I_m(t) = Q$, at $t = T$

Now we derived the net profit function for the manufacturer during a replenishment cycle of length $[0,T]$.

The net profit function for manufacturer can be formulated as

Average Profit= $\frac{1}{T}$[Sales Revenue - Production Cost - Raw Material Ordering Cost - Holding Cost - Deterioration Cost - Preservation Technology Investment Cost]

Equation (3.11) leads to

$$I_m(t) = \left(\frac{q}{\rho} - \frac{r}{\rho^2}\right)\left(e^{\rho(t_s-t)}\right) + \frac{r}{\rho}\left(t - t_s\left(e^{\rho(t_s-t)}\right)\right) \tag{4.2}$$

Sales income in the cycle $[0,T]$ is

$$SR_m = c.Q$$

$$SR_m = c.\left(1-e^{(\rho+\phi)T}\right)\left(\frac{\beta p}{\rho+\phi} - \frac{\alpha e^{-aT}}{\rho+\phi-a}\right) \tag{4.3}$$

The production cost of product for manufacturer is

$$PC_m = C_p \int_{t_s}^{T} (q + rt)dt$$

$$PC_m = C_p(T - t_s)\left(q + \frac{r}{2}(T - t_s)\right) \tag{4.4}$$

Raw material ordering cost per lot for manufacturer is

$$RMOC = C_s \tag{4.5}$$

Preservation Technology Investment Cost is

$$PTIC = \delta\kappa\xi T \tag{4.6}$$

The Deterioration cost product per production cycle for manufacturer is

$$DC_m = C_d\theta e^{-\eta\xi} \int_{t_s}^{T} I_m(t)dt$$

$$\begin{aligned} DC_m = {} & \omega\left(\frac{q}{\rho} - \frac{r}{\rho^2}\right)(T - t_s) + \omega\left(\frac{q}{\rho} - \frac{r}{\rho^2}\right)\frac{1}{\rho}\left(e^{\rho(t_s - T)} - 1\right) \\ & + \omega\frac{r}{\rho}\left(\frac{T^2}{2} - \frac{t_s^2}{2}\right) - \omega\frac{rt_s}{\rho^2}\left(e^{\rho(t_s - T)} + 1\right) \end{aligned} \tag{4.7}$$

where, $\omega = C_d\theta e^{-\eta\xi}$
The holding cost of product per production cycle for manufacturer is

$$HC_m = h \int_{t_s}^{T} I_m(t)dt$$

$$\begin{aligned} HC_m = {} & h.\left(\frac{q}{\rho} - \frac{r}{\rho^2}\right)(T - t_s) + h\left(\frac{q}{\rho} - \frac{r}{\rho^2}\right)\frac{1}{\rho}\left(e^{\rho(t_s - T)} - 1\right) \\ & + h\frac{r}{\rho}\left(\frac{T^2}{2} - \frac{t_s^2}{2}\right) - h\frac{rt_s}{\rho^2}\left(e^{\rho(t_s - T)} + 1\right) \end{aligned} \tag{4.8}$$

Hence the average profit function for manufacturer per time unit is given by,

$$\begin{aligned} AP_m = {} & \frac{c}{T}.\left(1 - e^{(\rho+\phi)T}\right)\left(\frac{\beta p}{\rho + \phi} - \frac{\alpha e^{-aT}}{\rho + \phi - a}\right) - \frac{C_p}{T}(T - t_s)\left(q + \frac{r}{2}(T - t_s)\right) \\ & - \psi\left(\frac{q}{\rho} - \frac{r}{\rho^2}\right)(T - t_s) - \frac{\psi}{\rho}\left(\frac{q}{\rho} - \frac{r}{\rho^2}\right)\left(1 - e^{\rho(T - t_s)}\right) - \frac{\psi r}{\rho}\left(\frac{T^2}{2} - \frac{t_s^2}{2}\right) \\ & - \delta\kappa\xi - \frac{C_s}{T} + \frac{\psi r t_s}{\rho^2}\left(1 - e^{\rho(T - t_s)}\right) \end{aligned} \tag{4.9}$$

where, $\psi = \left(h + C_d\theta e^{-\eta\xi}\right)$

5. Total Profit Function

In this article we have considered that the manufacturer and retailer both are work together as a single unit and for reducing the deterioration rate of items they both are invest on preservation technology with revenue sharing. To find total profit of whole supply chain unit we formulate the total average profit function of whole supply chain inventory system as,

$$NTP = AP_r + AP_m$$

$$\begin{aligned} NTP = & \frac{p\alpha}{Ta}(1-e^{-aT}) - \beta p^2 - \frac{C_p}{T}(T-t_s)\left(q+\frac{r}{2}(T-t_s)\right) - \frac{C_0}{T} - \frac{C_s}{T} - \kappa\xi \\ & + \frac{1}{T}\left(T+\frac{1}{\rho+\phi}-\frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)\left(\frac{\beta p}{\rho+\phi}-\frac{\alpha e^{-aT}}{\rho+\phi-a}\right) - \psi\left(\frac{q}{\rho}-\frac{r}{\rho^2}\right)(T-t_s) \\ & - \frac{\psi}{\rho}\left(\frac{q}{\rho}-\frac{r}{\rho^2}\right)\left(1-e^{\rho(T-t_s)}\right) - \psi\frac{r}{\rho}\left(\frac{T^2}{2}-\frac{t_s^2}{2}\right) - \frac{\psi r t_s}{\rho^2}\left(1-e^{\rho(T-t_s)}\right) \end{aligned}$$

where, $\psi = \left(h + C_d\theta e^{-\eta\xi}\right)$ and $\zeta = \left(\phi p - h - C_d\theta e^{-\eta\xi}\right)$

Proposition 5.1. *There exist an unique optimal selling price p^* for stock dependent demand, net total profit function NTP(T, p) is maximum for fixed time horizon T and preservation cost ξ.*

Proof. The first order partial derivative of the net profit function is

$$\begin{aligned} \frac{\partial NTP(T,p)}{\partial p} = & \frac{\alpha}{Ta}(1-e^{-aT}) - 2\beta p - \frac{1}{T}\zeta\left(T+\frac{1}{\rho+\phi}-\frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)\frac{\beta p}{\rho+\phi} \\ & + \phi\left(\frac{\beta p}{\rho+\phi}-\frac{\alpha e^{-aT}}{\rho+\phi-a}\right)\left(T+\frac{1}{\rho+\phi}-\frac{e^{(\rho+\phi)T}}{\rho+\phi}\right) \end{aligned} \quad (5.1)$$

Where, $\zeta = \left(\phi p - h - C_d\theta e^{-\eta\xi}\right)$

If p^* is a optimal value p, then $\frac{\partial NTP(T,p)}{\partial p}$ must be equal to zero

i.e.

$$\frac{\partial NTP(T,p)}{\partial p} = 0$$

Solve for the optimal price p^*

we have

$$p^* = \frac{\frac{\alpha}{a}(1-e^{-aT}) - \left(T+\frac{1}{\rho+\phi}-\frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)\left(\frac{\beta(h+C_d\theta e^{-\eta\xi})}{\rho+\phi}+\frac{\phi\alpha e^{-aT}}{\rho+\phi-a}\right)}{2\beta T - \frac{2\beta\phi}{\rho+\phi}\left(T+\frac{1}{\rho+\phi}-\frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)} \quad (5.2)$$

for maximum value NTP(T, p) at point $p = p^*$, we have

$$\frac{\partial^2 NTP(T,p)}{\partial p^2} = -2\beta T + \left(T+\frac{1}{\rho+\phi}-\frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)\frac{2\beta p}{\rho+\phi} < 0 \quad (5.3)$$

for $\beta > 0$ and $\left(T+\frac{1}{\rho+\phi}-\frac{e^{(\rho+\phi)T}}{\rho+\phi}\right) < 0$, because $\left(T+\frac{1}{\rho+\phi} < \frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)$ □

Proposition 5.2. *For fixed ξ, there exist an optimal solution ($T^*.p^*$) that maximize the net profit function NTP(T, p), and also it is unique.*

Special case

When we consider demand function as a price sensitive only (i.e. $\phi = 0$) then the Equation (5.1) reduces to the following form

6. Total Profit Function

Therefore, the total average profit function whole inventory supply chain system is,

$$NTP = AP_r + AP_m$$

$$\begin{aligned} NTP = & \frac{p\alpha}{Ta}(1-e^{-aT}) - \beta p^2 - \frac{C_p}{T}(T-t_s)\left(q+\frac{r}{2}(T-t_s)\right) - \frac{C_0}{T} - \frac{C_s}{T} - \kappa\xi \\ & -\frac{1}{T}\sigma\left(T+\frac{1}{\rho}-\frac{e^{\rho T}}{\rho}\right)\left(\frac{\beta p}{\rho}-\frac{\alpha e^{-aT}}{\rho-a}\right) - \psi\left(\frac{q}{\rho}-\frac{r}{\rho^2}\right)(T-t_s) \\ & -\frac{\psi}{\rho}\left(\frac{q}{\rho}-\frac{r}{\rho^2}\right)\left(e^{\rho(t_s-T)}-1\right) - \psi\frac{r}{\rho}\left(\frac{T^2}{2}-\frac{t_s^2}{2}\right) - \frac{\psi r t_s}{\rho^2}\left(e^{\rho(t_s-T)}+1\right) \end{aligned}$$

where, $\psi = \left(h + C_d\theta e^{-\eta\xi}\right)$ and $\sigma = \left(h + C_d\theta e^{-\eta\xi}\right)$

Proposition 6.1. *There exist an unique optimal selling price p_1^* at which the net total profit function NTP (T, p) is maximum for fixed time horizon T and preservation cost ξ. where p_1^* is a price of unit item for special case.*

Proof. The first order partial derivative of the net profit function are

$$\frac{\partial NTP(T,p)}{\partial p} = \frac{\alpha}{Ta}(1-e^{-aT}) - 2\beta p - \frac{1}{T}\zeta\left(T+\frac{1}{\rho}-\frac{e^{\rho T}}{\rho}\right)\frac{\beta}{\rho} \tag{6.1}$$

Where, $\zeta = -\left(h + C_d\theta e^{-\eta\xi}\right)$
If p_1^* is a optimal value of p, then $\frac{\partial NTP(T,p)}{\partial p}$ must be equal to zero
i.e.

$$\frac{\partial NTP(T,p)}{\partial p} = 0$$

Solve for the optimal price p_1^*
we have

$$p_1^* = \frac{\frac{\alpha}{a}(1-e^{-aT}) - \left(T+\frac{1}{\rho}-\frac{e^{\rho T}}{\rho}\right)\frac{\beta}{\rho}(h + C_d\theta e^{-\eta\xi})}{2\beta T} \tag{6.2}$$

at point $p = p_1^*$, NTP(T, p) has maximum value if

$$\frac{\partial^2 NTP(T,p)}{\partial p^2} = -2\beta T < 0 \tag{6.3}$$

for $\beta > 0$. □

Proposition 6.2. *For fixed ξ, there exist an optimal solution ($T^*.p_1^*$), that maximize the net profit function NTP(T, p), and also it is unique.*

Theorem 6.1. *If p is selling price of a product with stock dependent and price sensitive demand and p_1^* is a selling price of a product with price sensitive demand, than for $aT \geq 0$, p^* is always less than equal to p_1^*.*

Proof. For this we will prove that p_1^*-$p^* > 0$
From the prepositions 1 and 3 we have

$$\left[\frac{\alpha}{a}(1-e^{-aT}) - \Delta_1\pi\frac{\beta}{\rho}\right]\left[2\beta T - \frac{2\beta\phi}{\rho+\phi}\Delta_2\right]$$
$$-2\beta T\left[\frac{\alpha}{a}(1-e^{-aT}) - \Delta_2\left(\frac{\beta\pi}{\rho+\phi} + \frac{\phi\alpha e^{-aT}}{\rho+\phi-a}\right)\right] > 0$$

$$or - \frac{\alpha}{a}(1-e^{-aT})\frac{2\beta\phi}{\rho+\phi}\Delta_2 + \Delta_1\pi\frac{\beta}{\rho}\frac{2\beta\phi}{\rho+\phi}\Delta_2$$

$$-2\beta T\Delta_2\left(\frac{\beta\pi}{\rho+\phi}+\frac{\phi\alpha e^{-aT}}{\rho+\phi-a}\right) > 0 \tag{6.4}$$

since $\pi = (h + C_d\theta e^{-\eta\xi}) > 0$, $\Delta_1 = \left(T + \frac{1}{\rho} - \frac{e^{\rho T}}{\rho}\right) < 0$
and $\Delta_2 = \left(T + \frac{1}{\rho+\phi} - \frac{e^{(\rho+\phi)T}}{\rho+\phi}\right) < 0, \quad \forall\, aT \in R^+$,
therefore, from (6.4) p_1^*-$p > 0$.
Where, $\pi = (h + C_d\theta e^{-\eta\xi})$, $\Delta_1 = \left(T + \frac{1}{\rho} - \frac{e^{\rho T}}{\rho}\right)$,
and $\Delta_2 = \left(T + \frac{1}{\rho+\phi} - \frac{e^{(\rho+\phi)T}}{\rho+\phi}\right)$. □

Corollary 6.3. *Net total profit $NTP(T,p)$ is a increasing function with respect to selling price p, i.e.*
if $p_1^* > p$, then

$$NTP(T,p_1^*) > NTP(T,p).$$

Example 6.4. In case of stock dependent and price sensitive demand the numerical example are as follow; $\alpha = 65$, $\beta = 0.5$, $\kappa = 0.5$, $\xi = 0.1$, $h = 0.09$, $\theta = 0.1$, $\phi = 0.13$, $a = 0.01$, $\eta = 0.5$, $q = 8$, $r = 0.5$, $C = 10$, $C_m = 15$, $C_p = 2$, $C_s = 1.5$, $C_d = 1.25$, then $p = 18$, $T = 1.47$, Net Total Profit= 1072.18

Example 6.5. (for special case)

In case of only price sensitive demand the numerical example are as follow; $\alpha = 65$, $\beta = 0.5$, $\kappa = 0.5$, $\xi = 0.1$, $h = 0.09$, $\theta = 0.1$, $\phi = 0$, $a = 0.01$, $\eta = 0.5$, $q = 8$ $r = 0.5$, $C = 10$ $C_m = 15$, $C_p = 2$, $C_s = 1.5$, $C_d = 1.25$,then $p = 93.39$, $T = 1.47$, Net Total Profit= 1642.66

In view of above numerical examples case second is more profitable in place of case first. In the second demand pattern manufacturer and retailer both are save the bulk revenue which are required initial investment on spacing, preservation investment and deterioration.

7. Sensitive Analysis

If the initial demand increases, the consumed order quantity, net profit and selling price of item (figure 1) increases sharply, thou time span (figure 2) decreased marginally. This reveals that initial demand boost the profit of manufacturer and as well as of retailer. If the stock dependent parameter ϕ increases then time horizon T reduces sharply (figure 3). This reveals that for highly stock dependent items keep the time horizon less as possible, and accordingly orders frequently. Moreover the net profit of manufacturer and retailer is sensitive on stock dependent parameter, for certain value of ϕ (0.01 to 0.025) the profit of both increases marginally, but thereafter $\phi \geq 0.025$, manufacturer's profit is constant and retailer's profit decreases sharply. Then, there exist an optimal value of ϕ that maximize the profit function (see figure 5)

Since β is a price sensitive parameter of demand function, and in this supply chain model, retailer may decide their items price for maximizing the total profit. On the basis of above statement for fixed $\beta = 0.03$, $\alpha = 10$, $\kappa = 0.5$, $\xi = 0.1$, $h = 0.09$, $\theta = 0.02$, $\phi = 0.13$, $a = 0.01$, $\eta = 0.5$, $q = 8$, $r = 0.5$, $c = 10$, $C_m = 15$, $C_p = 2$, $C_r = 1.5$, $C_d = 1.25$, then at the value of decision variables are $T = 0.795$, $p = 38$, NTP (Net total profit) = 320.10 (figure 4).

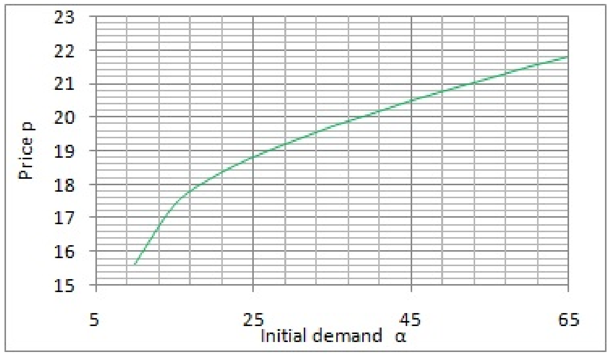

FIGURE 1. Effect of initial demand α on price p

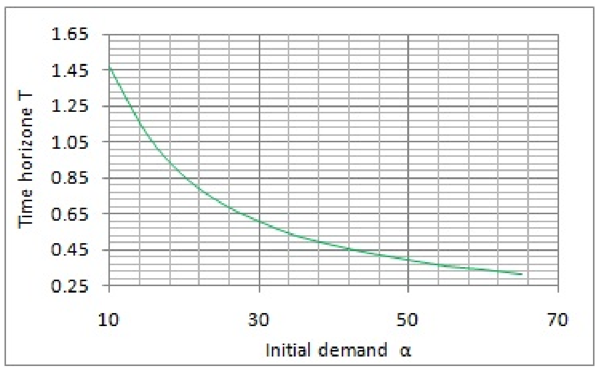

FIGURE 2. Effect of initial demand α on time horizon T

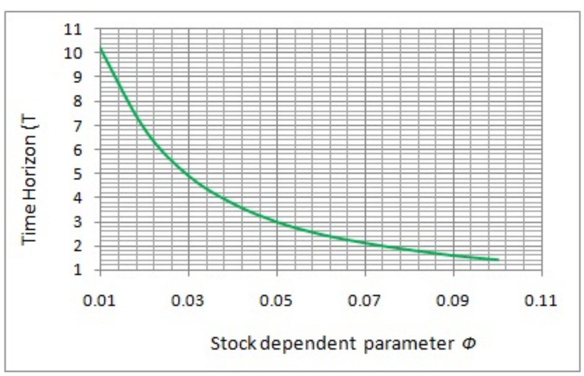

FIGURE 3. Effect of stock dependent parameter ϕ on time horizon T

Selling price of the item is highly sensitive on parameter β. Therefor those items which are highly price sensitive (for greater value of β) would be sustain in market with fluctuating price (figure 6). However high selling price does not guarantee to earn more profit (figure 4), for even business setup an optimal selling price exists for this profit is optimal for manufacturer as well as retailer. As per (figure 4) optimal selling price is $p = 38$ and for this sum of the manufacturer and retailer profit 320.10 is optimal. On increasing the parameter β, net profit NTP is

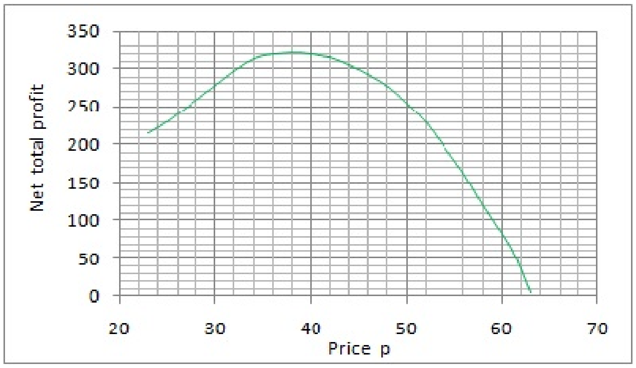

FIGURE 4. The effect of price p on NTP

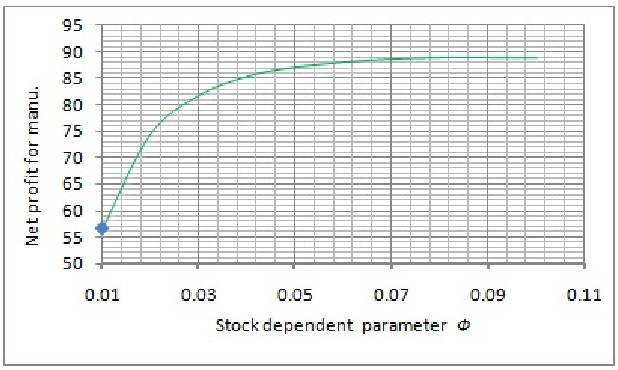

FIGURE 5. Effect of stock dependent parameter ϕ on AP_m

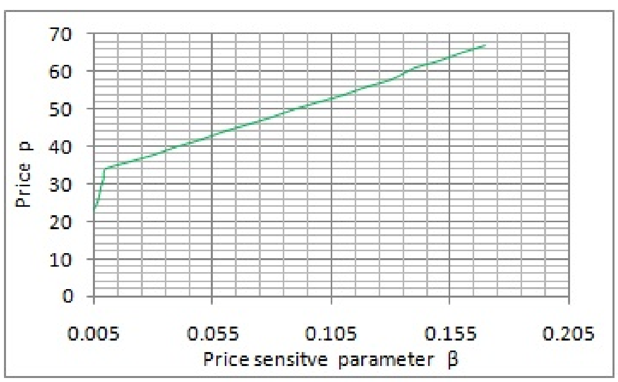

FIGURE 6. Effect of price sensitive parameter β on price p

also increases but after certain value of $\beta = 0.03$, NTP decreases.

Thus we have observed from (figure 6) selling price is linearly proportional to the parameter β and there exists an optimal price for highest profit. The above phenomenon follows for parameter β also. Hence there exists an optimal value of β that maximize the profit of manufacturer and retailer (figure 8). Above phenomenon is also applicable for time cycle, therefore there exists an optimal value of T

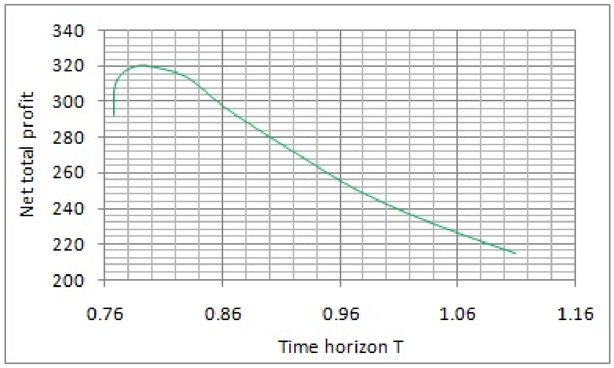

FIGURE 7. Effect of time horizon T on NTP

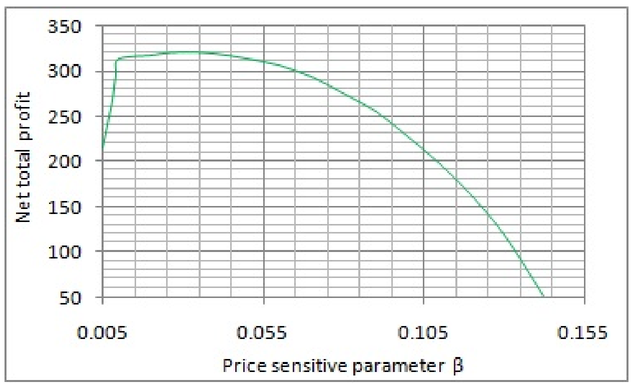

FIGURE 8. Effect of price sensitive parameter β on net total profit

(see figure 7) that maximize, the net total profit

In this research study it is found that when the production rate is time dependent linear function of t then the deterioration factor is more effective on the profit of manufacturer therefor if the deterioration rate is very low the profit of manufacturer is proportionally large and if the deterioration rate is high the profit of manufacturer is proportionally small i.e. the profit of manufacturer is inversely proportional to deteriorating rate.

8. CONCLUSION

The paper contains an inventory supply chain model for deteriorating seasonal items in which the deterioration rate can be controlled by investing on the preservation technology. By analysis, we have observed that for price sensitive and stock dependent demand pattern, deteriorating nature of products is more effective of the profit of manufacturer, therefore production management must provide to the retailer for a percentage of revenue sharing on preservation technology. We also observe that price dependent demand is more profitable than price and stock dependent demand. For a business setup we have found optimum time, price and time cycle to obtain maximum profit. It is advised to retailer to order in small lot size and small time cycle to obtain maximum profit, because the deterioration highly influence the model output. One can extend the model for multi supply chain and also for multi products. Also one can formulate the model in fuzzy environment.

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BALL CONVERGENCE FOR A TWO STEP METHOD WITH MEMORY AT LEAST OF ORDER $2+\sqrt{2}$

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ABSTRACT. We present a local convergence analysis of at least $2+\sqrt{2}$ convergence order two-step method in order to approximate a locally unique solution of nonlinear equation in a Banach space setting. In the earlier study, [6, 15] the authors of these paper did not discuss that studies. Furthermore, the order of convergence was shown using Taylor series expansions and hypotheses up to the sixth order derivative or or even higher of the function involved which restrict the applicability of the proposed scheme. However, only first order derivative appears in the proposed scheme. In order to overcome this problem, we proposed the hypotheses up to only first order derivative. In this way, we not only expand the applicability of the methods but also propose convergence domain. Finally, we present some numerical experiments where earlier studies cannot apply to solve nonlinear equations but our study does not exhibit this type of problem/restriction.

KEYWORDS : Two-step method with memory; local convergence; convergence order.
AMS Subject Classification: 65D10; 65D99

1. INTRODUCTION

There are several problems of pure and applied science which can be studied in the unified frame work of the scalar or system of nonlinear equations. In this paper, we are concerned with one of the most important and challenging task in the field of numerical analysis, is to approximate the local unique solution x^* of the equation of the form

$$F(x) = 0, \tag{1.1}$$

where F is a twice Fréchet differentiable function defined on a subset $\mathbb{D}$ of $\mathbb{R}$ with values in $\mathbb{R}$.

We can say that either lack or intractability of their analytic solutions often forces researchers from the worldwide trying their best to resort to an iterative

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method. While, using these iterative methods researchers face the problems of slow convergence, non-convergence, divergence, inefficiency or failure (for details please see Traub [14] and Petkovic et al. [12]).

The convergence analysis of iterative methods is usually divided into two categories: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. A very important problem in the study of iterative procedures is the convergence domain. Therefore, it is very important to propose the radius of convergence of the iterative methods.

We study the local convergence analysis of two-step method with memory defined for each $n = 0,\ 1\ 2,\ \ldots$ by

$$\begin{aligned} y_n &= x_n - \left(F'(x_n) + \alpha_n F(x_n)\right)^{-1} F(x_n) \\ x_{n+1} &= y_n - \left(F(x_n) + (\beta-2)F(y_n)\right)^{-1}\left(F(x_n) + \beta F(y_n)\right)\left(F'(x_n) + 2\alpha_n F(x_n)\right)^{-1} F(y_n), \end{aligned} \tag{1.2}$$

where $x_{-1},\ x_0$ are initial points, $\beta \in \mathbb{R}$, $\alpha_n = -\frac{1}{2}\frac{[x_{n-1},\, x_n;\, F']}{[x_{n-1},\, x_n;\, F]}$, $n = 0,\ 1,\ 2,\ \ldots,$ $[\cdot,\ \cdot; F']$ and $[\cdot,\ \cdot; F]$ denote divided differences of order one for functions F' and F, respectively. Method (1.2) was introduced in [6] as an alternative to the King-like method

$$\begin{aligned} y_n &= x_n - \left(F'(x_n) + aF(x_n)\right)^{-1} F(x_n) \\ x_{n+1} &= y_n - \left(F(x_n) + (\beta-2)F(y_n)\right)^{-1}\left(F(x_n) + \beta F(y_n)\right)\left(F'(x_n) + 2aF(x_n)\right)^{-1} F(y_n), \end{aligned} \tag{1.3}$$

where $a,\ \beta \in \mathbb{R}$. Method (1.2) was shown to be of order $2+\sqrt{2}$ using hypotheses up to the sixth derivative of function F [6]. Method (1.3) is of order four [15] and hypotheses up to the fourth derivative of the function. These hypotheses on the derivatives of F limit the applicability of method (1.2) and method (1.3). As a motivational example, define function F on $\mathbb{R}$, $\mathbb{D} = [-\frac{1}{\pi},\ \frac{2}{\pi}]$ by

$$F(x) = \begin{cases} x^3\log(\pi^2x^2) + x^5\sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then, we have that

$$F'(x) = 2x^2 - x^3\cos\left(\frac{1}{x}\right) + 3x^2\log(\pi^2x^2) + 5x^4\sin\left(\frac{1}{x}\right),$$

$$F''(x) = -8x^2\cos\left(\frac{1}{x}\right) + 2x(5 + 3\log(\pi^2x^2)) + x(20x^2-1)\sin\left(\frac{1}{x}\right)$$

and

$$F'''(x) = \frac{1}{x}\left[(1-36x^2)\cos\left(\frac{1}{x}\right) + x\left(22 + 6\log(\pi^2x^2) + (60x^2-9)\sin\left(\frac{1}{x}\right)\right)\right].$$

One can easily find that the function $F'''(x)$ is unbounded on $\mathbb{D}$ at the point $x = 0$. Hence, the results in [6, 15], cannot apply to show the convergence of method (1.2) and method (1.3) or its special cases requiring hypotheses on the fifth derivative of function F or higher. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations [1, 2, 3, 4, 5, 6, 15, 7, 9, 8, 10, 11, 12, 13, 14]. These results show that initial guess should be close to the required root for the convergence of the corresponding methods and same thing is also mentioned by the authors of papers [6, 15]. But, how close initial guess should be required for the convergence of the corresponding method? These local results give no information on the radius of the convergence ball for the corresponding method. The same technique can be used on other methods.

In the present study we expand the applicability of method (1.2) and method (1.3) using only hypotheses up to the second order derivative of function F. We also proposed the computable radii of convergence and error bounds based on the Lipschitz constants. We further present the range of initial guesses x_0 that tell us how close the initial guess should be required for granted convergence of the method (1.2) and method (1.3). This problem was not addressed in [6, 15]. The advantages of our approach are similar to the ones already mentioned for method (1.2) and method (1.3).

Definition 1.1. (Error Equation, Asymptotic Error Constant, Order of Convergence)

Let us consider a sequence $\{x_n\}$ converging to a root ξ of $f(x) = 0$. Let $e_n = x_n - \xi$ be the error at n^{th} iteration. If constants $p \geq 1$, $c \neq 0$ exist in such a way that $e_{n+1} = ce_n^p + O(e_n^{p+1})$ known as the error equation then p and $\eta = |c|$ are said to be the order of convergence and the asymptotic error constant, respectively. From this definition the asymptotic error constant is found to be $\eta = |c| = \lim_{n \longrightarrow \infty} \frac{|e_{n+1}|}{|e_n^p|}$. However, some researchers call $c = \lim_{n \longrightarrow \infty} \frac{e_{n+1}}{e_n^p}$ asymptotic error constant instead of $|c|$.

Definition 1.2. (Asymptotic Order of Convergence)

With the help of above definition 1.1, we can define the asymptotic order of convergence as follows:

$$p = \lim_{n \longrightarrow \infty} \frac{|e_{n+1}/\eta|}{|e_n^p|}.$$

But, the main drawback of calculating η according to the above formula is that it involves the exact root ξ and there are many real situations in which the exact root is not known in advance. To over come this problem, we can use $(x_{n+1} - x_n)$ instead of (e_{n+1}) in the above formula to calculate η.

2. Local convergence: One dimensional case

In this section, we shall define some scalar functions and parameters in order to present the local convergence of method (1.2) that follows.

Let $L_0 > 0,\ L > 0,\ M \geq 1$ and $\beta \in \mathbb{R}$ be given constants. Let us also assume some functions $p,\ h_p,\ p_1$ and h_{p_1}defined on the interval $\left[0,\ \frac{1}{L_0}\right)$ by

$$\begin{aligned} p(t) &= \left(L_0 + \frac{LM}{2(1-L_0t)}\right)t, \\ p_1(t) &= \left(L_0 + \frac{LM}{1-L_0t}\right)t, \end{aligned} \tag{2.1}$$

$h_p(t) = p(t) - 1,$ and $h_{p_1}(t) = p_1(t) - 1.$ We have $h_p(0) = h_{p_1}(0) = -1 < 0$ and $h_p(t) \longrightarrow +\infty,\ h_{p_1}(t) \longrightarrow +\infty$ as $t \longrightarrow \frac{1^{-1}}{L_0}$. Then, by the intermediate value theorem functions h_p and h_{p_1} have zeros in the interval $\left(0,\ \frac{1}{L_0}\right)$. Further, let r_p

and r_{p_1} respectively be the smallest such zeros. Then, we have that

$r_{p_1} < r_p, \;\; p(r_p) = p_1(r_{p_1}) = 1$

$0 \leq p(t) \leq 1,$

$0 \leq p_1(t) \leq 1$

and

$0 \leq p(t) \leq p_1(t)$ for each $t \in [0, r_{p_1})$.

Moreover, define functions $g_1, \; h_1, \; q$ and h_q in the interval $[0, \; r_{p_1})$ by

$$g_1(t) = \frac{Lt}{2(1-L_0t)}\left(1 + \frac{M^2}{1-p_1(t)}\right),$$

$$h_1(t) = g_1(t) - 1$$

$$q(t) = \frac{L_0}{2}t + |\beta - 2|Mg_1(t)$$

and

$$h_q(t) = q(t) - 1.$$

We get that $h_1(0) = h_q(0) = -1 < 0$ and $h_1(t) \longrightarrow +\infty$, $h_q(t) \longrightarrow +\infty$ as $t \longrightarrow r_{p_1}$. Then, it follows from the intermediate value theorem that functions h_1 and h_q have zeros in the interval $(0, \; r_{p_1})$. Denote by r_1 and r_q, respectively the smallest such zeros. Furthermore, define functions g_2 and h_2 on the interval $[0, \; r_q)$ by

$$g_2(t) = \left(1 + \frac{M^2\big(1 + |\beta|g_1(t)\big)}{\big(1-q(t)\big)\big(1-p_1(t)\big)}\right)g_1(t)$$

and

$$h_2 = g_2(t) - 1.$$

Then, we get $h_2(0) = -1$ and $h_2(t) \to +\infty$ as $t \to r_q^-$. Denote by r_2 the smallest zero of function h_2 on the interval $(0, \; r_q)$. Finally, define

$$r = \min\{r_1, \; r_2, \}. \tag{2.2}$$

Then, we have that for each $t \in [0, r)$

$$0 \leq p(t) < 1, \tag{2.3}$$

$$0 \leq p_1(t) < 1, \tag{2.4}$$

$$0 \leq p(t) < p_1(t), \tag{2.5}$$

$$0 \leq g_1(t) < 1, \tag{2.6}$$

$$0 \leq q(t) < 1 \tag{2.7}$$

and

$$0 \leq g_2(t) < 1. \tag{2.8}$$

Let $U(\gamma, \; \rho)$ and $\bar{U}(\gamma, \; \rho)$ stand, respectively for the open and closed balls in S with center $\gamma \in S$ and radius $\rho > 0$. Next, we present the local convergence analysis of method (1.2) using the preceding notations.

Theorem 2.1. *Let us consider $F : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Let us also assume $[\cdot, \cdot\,; F] : D^2 \to L(\mathbb{R})$ to be a divided difference of order one for function F. Suppose that there exist $x^* \in \mathbb{D}$ and $L_0 > 0$ such that for each $x \in \mathbb{D}$*

$$F(x^*) = 0, \quad F'(x^*) \neq 0 \tag{2.9}$$

and

$$\left|F(x^*)^{-1}\left(F'(x)-F'(x^*)\right)\right| \leq L_0|x-x^*|. \tag{2.10}$$

Moreover, suppose that there exist $L>0,\ M\geq 1$ and $\beta\in S$ such that for each $x,\ y\in U\left(x^*,\ \frac{1}{L_0}\right)\cap\mathbb{D}$

$$\left|F'(x^*)^{-1}\left(F'(x)-F'(y)\right)\right| \leq L|x-y|, \tag{2.11}$$

$$\left|F'(x^*)^{-1}F'(x)]\right| \leq M, \tag{2.12}$$

and

$$\bar{U}\left(x^*,\ r\right)\subseteq\mathbb{D}, \tag{2.13}$$

where the radius of convergence r is defined by (2.2). Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_{-1},\ x_0\in U(x^*,\ r)-\{x^*\}$ with $x_{-1}\neq x_0$ is well defined, remains in $U(x^*,\ r)$ for each $n=0,1,2,\ldots$ and converges to x^*. Moreover, the following estimates hold

$$|y_n-x^*|\leq g_1(r)|x_n-x^*|<|x_n-x^*|<r \tag{2.14}$$

and

$$|x_{n+1}-x^*|\leq g_2(r)|x_n-x^*|<|x_n-x^*|, \tag{2.15}$$

where the "g" functions are defined by previously. Furthermore, for $T\in\left[r,\ \frac{2}{L_0}\right)$, the limit point x^* is the only solution of equation $F(x)=0$ in $\bar{U}(x^*,\ r)$.

Proof. We shall show estimates (2.14) and (2.15) hold with the help of mathematical induction. First, we must show $\alpha_0\neq 0$. We can write

$$\begin{aligned}\alpha_0 &= -\frac{1}{2}\frac{\frac{F'(x_0)-F'(x_{-1})}{x_0-x_{-1}}}{\frac{F(x_0)-F(x_{-1})}{x_0-x_{-1}}}\\ &= -\frac{1}{2}\frac{\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta}{\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta},\quad \text{for } x_0\neq x_{-1}.\end{aligned} \tag{2.16}$$

Using (2.2) and (2.10) we have that

$$\begin{aligned}\left|F'(x^*)^{-1}\left[\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta-F'(x^*)\right]\right| &\leq \frac{L_0}{2}\left(|x_{-1}-x^*|+|x_0-x^*|\right)\\ &< L_0 r<1.\end{aligned} \tag{2.17}$$

Then, by (2.17) and the Banach Lemma on invertible functions [4, 13], we get that $\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta\neq 0$ and

$$\begin{aligned}\left|\left(\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta\right)^{-1}F'(x^*)\right| &\leq \frac{1}{1-\frac{L_0}{2}(|x_{-1}-x^*|+|x_0-x^*|)}\\ &\leq \frac{1}{1-L_0 r}.\end{aligned} \tag{2.18}$$

In view of (2.11), (2.16) and (2.18), we have that

$$\begin{aligned}|\alpha_0| &= \frac{1}{2}\frac{\left|\int_0^1 F'(x^*)^{-1}F''(x_{-1}+\theta(x_0-x_{-1}))d\theta\right|}{\left|\int_0^1 F'(x^*)^{-1}F'(x_{-1}+\theta(x_0-x_{-1}))d\theta\right|},\\ &\leq \frac{1}{2}\frac{L}{\left(1-\frac{L_0}{2}(|x_{-1}-x^*|+|x_0-x^*|)\right)}\leq\frac{L}{2(1-L_0 r)}.\end{aligned} \tag{2.19}$$

We must show $F'(x_0) + \alpha_0 F(x_0) \neq 0$. We can write by (2.9) that

$$F(x_0 = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.20}$$

Notice that $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r$. That is $x^* + \theta(x_0 - x^*) \in U(x^*,\ r)$. Then by (2.12) and (2.20), we get that

$$\left|F'(x^*)^{-1}F'(x_0)\right| \leq M|x_0 - x^*|. \tag{2.21}$$

Using (2.2), (2.3), (2.4), (2.19) and (2.21), we get in turn

$$\begin{aligned}
&\left|F'(x^*)^{-1}\left(F'(x_0) - F'(x^*) - \alpha_0 F(x_0)\right)\right| \\
&\leq \left|F'(x^*)^{-1}\left(F'(x_0) - F'(x^*)\right)\right| + |\alpha_0|\left|F'(x^*)^{-1}F(x_0)\right| \\
&\leq L_0|x_0 - x^*| + \frac{LM|x_0 - x^*|}{2\left(1 - \frac{L_0}{2}(|x_{-1} - x^*| + |x_0 - x^*|)\right)} \\
&\leq p(r) < p_1(r) < 1.
\end{aligned} \tag{2.22}$$

It follows from (2.22) that

$$\left|\left(F'(x_0) + \alpha_0 F(x_0)\right)^{-1} F'(x^*)\right| \leq \frac{1}{1 - p_1(r)} \tag{2.23}$$

and y_0 is well defined by the first sub step of method (1.2) for $n = 0$. As in (2.22) and (2.23), we obtain that

$$\left|\left(F'(x_0) + 2\alpha_0 F(x_0)\right)^{-1} F'(x^*)\right| \leq \frac{1}{1 - p_1(r)} \tag{2.24}$$

and

$$\left|F'(x_0)^{-1}F'(x^*)\right| \leq \frac{1}{1 - L_0(r)} \tag{2.25}$$

Using the first sub step of method (1.2) for $n = 0$, (2.2), (2.6), (2.9), (2.11), (2.21) and (2.23), we get in turn that

$$\begin{aligned}
|y_0 - x^*| &= \left|x_0 - F'(x_0)^{-1}F(x_0) + F'(x_0)^{-1}F(x_0) - \left(F'(x_0) + \alpha_0 F(x_0)\right)^{-1} F(x_0)\right| \\
&\leq \left|x_0 - x^* - F'(x_0)^{-1}F(x_0)\right| + |\alpha_0|\left|\left(F'(x_0) + \alpha_0 F(x_0)\right)^{-1} F(x_0)^2\right| \\
&\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{|\alpha_0|M^2|x_0 - x^*|^2}{1 - p(|x_0 - x^*|)} \\
&\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{LM^2|x_0 - x^*|^2}{2\left(1 - \frac{L_0}{2}(|x_{-1} - x^*| + |x_0 - x^*|)\right)(1 - p(|x_0 - x^*|))} \\
&\leq g_1(r)|x_0 - x^*| < |x_0 - x^*| < r,
\end{aligned} \tag{2.26}$$

which shows (2.14) for $n = 0$ and $y_0 \in U(x^*,\ r)$. Notice that (2.21) holds for y_0 replacing x_0, since $y_0 \in U(x^*,\ r)$. We must shows $F(x_0) + (\beta - 2)F(y_0) \neq 0$.

Using (2.2), (2.3), (2.7), (2.8), (2.9) and (2.21) (for $y_0 = x_0$), we get in turn that for $x_0 \neq x^*$

$$\begin{aligned}
&\left|\left((x_0 - x^*)F'(x^*)^{-1}\right)[F(x_0 - F(x^*) - F'(x^*)(x_0 - x^*) + (\beta - 2)F(y_0)]\right| \\
&\leq |x_0 - x^*|^{-1}\left[\left|F'(x^*)^{-1}(F(x_0) - F(x^*) - F'(x^*)(x_0 - x^*))\right| + |\beta - 2|\left|F'(x^*)^{-1}F(y_0)\right|\right] \\
&\leq |x_0 - x^*|^{-1}\left(\frac{L_0}{2}|x_0 - x^*|^2 + |\beta - 2|M|y_0 - x^*|\right) \\
&\leq \frac{L_0}{2}|x_0 - x^*| + M|\beta - 2|g_1(|x_0 - x^*|) \\
&= q(|x_0 - x^*|) < q(r) < 1.
\end{aligned} \tag{2.27}$$

Hence, we get from (2.27) that

$$\left|(F(x_0) + (\beta - 2)F(y_0))^{-1}\right| \leq \frac{1}{|x_0 - x^*|(1 - q(|x_0 - x^*|))} \tag{2.28}$$

and x_1 is well defined by the second sub step of method (1.2) for $n = 0$. Then, in view of (2.2), (2.3), (2.8), (2.21)(for $x_0 = y_0$ and $x_0 = x_0$), (2.23), (2.26) and (2.28), we obtain in turn that

$$\begin{aligned}
|x_1 - x^*| &\leq |y_0 - x^*| + \left|(F(x_0) + (\beta - 2)F(y_0))^{-1}F'(x^*)\right| \\
&\times \left[|F'(x^*)^{-1}F(x_0)| + |\beta||F'(x^*)^{-1}F(y_0)|\right] \\
&\times \left|(F(x_0) + 2\alpha_0 F(x_0))^{-1}F'(x^*)\right| |F'(x^*)^{-1}F(y_0)| \\
&\leq \left(1 + \frac{M^2(1 + |\beta|g_1(|x_0 - x^*|))|x_0 - x^*|}{|x_0 - x^*|(1 - q(|x_0 - x^*|))(1 - p_1(|x_0 - x^*|))}\right)|y_0 - x^*| \\
&\leq g_2(|x_0 - x^*|)|x_0 - x^*| < g_2(r)|x_0 - x^*| \\
&< |x_0 - x^*| < r,
\end{aligned} \tag{2.29}$$

which shows (2.15) and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0 by x_m, y_m, z_m in the preceding estimates we arrive at (2.17)-(2.19). Then, from the estimates $|x_{m+1} - x^*| < |x_m - x^*| < r$, we conclude that $\lim_{m \longrightarrow \infty} x_k = x^*$ and $x_{m+1} \in U(x^*, r)$. Finally, to show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ be such that $F(y^*) = 0$. Set $Q = \int_0^1 F'(x^* + \theta(y^* - x^*))\, d\theta$. Then, using (2.12), we get that

$$\left|F'(x^*)^{-1}(Q - F'(x^*))\right| \leq L_0 \int_0^1 \theta|x^* - y^*|d\theta = \frac{L_0}{2}T < 1. \tag{2.30}$$

Hence, $Q^{-1} \in L(Y, X)$. Then, in view of the identity $F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. □

Remark 2.1. (a) In view of (2.9) and the estimate

$$\begin{aligned}
\left|F'(x^*)^{-1}F'(x)\right| &= \left|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\right| \\
&\leq 1 + \left|F'(x^*)^{-1}(F'(x) - F'(x^*))\right| \\
&\leq 1 + L_0|x_0 - x^*|
\end{aligned}$$

condition (2.11) can be dropped and M can be replaced by

$$M = M(t) = 1 + L_0 t$$

or $M = 2$, since $t \in [0, \frac{1}{L_0})$.

(b) The results obtained here can be used for operators F satisfying the autonomous differential equation [4, 5] of the form

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x + 2$. Then, we can choose $P(x) = x - 2$.

(c) The radius $r_A = \frac{2}{2L_0 + L_1}$ was shown by us in [4, 5] to be the convergence radius for Newton's method under conditions (2.9) - (2.11). Radius r_A is at least as large as the convergence ball given by Rheinboldt [13] and Traub [14]

$$r_R = \frac{2}{3L_1}.$$

Notice that for $L_0 < L_1$,

$$r_R < r_A.$$

Moreover,

$$\frac{r_R}{r_A} \longrightarrow \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L_1} \longrightarrow 0.$$

Hence, r_A is at most three times larger than r_R. In the numerical examples, we compare r to $r_A^* = \frac{2}{2L_0 + L} \geq r_A$ and r_R. Notice that L_1 satisfies $\left|F'(x_0)^{-1}\big(F'(x) - F'(y)\big)\right| \leq L_1|x - y|$ for each $x,\ y \in D$. Then, we have that $L < L_1$ since $U\left(x^*,\ \frac{1}{L_0}\right) \cap D \subset D$

(d) It is worth noticing that method (1.2) is not changing if we use the conditions of Theorem 2.1 instead of the stronger conditions given in [6, 15]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [8]

$$\xi = \frac{ln\frac{|x_{n+2} - x^*|}{|x_{n+1} - x^*|}}{ln\frac{|x_{n+1} - x^*|}{|x_n - x^*|}}, \quad \text{for each } n = 0, 1, 2, \ldots \tag{2.31}$$

or the approximate computational order of convergence (ACOC) [8]

$$\xi^* = \frac{ln\frac{|x_{n+2} - x_{n+1}|}{|x_{n+1} - x_n|}}{ln\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}}, \quad \text{for each } n = 1, 2, \ldots \tag{2.32}$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first Fréchet derivative. Notice also that the computation of ξ^* does not involve the solution x^*.

Remark 2.2. In order to obtain the corresponding results for method (1.3), simply replace functions $p_1,\ g_1$ and g_2 by $\bar{p}_1,\ \bar{g}_1$ and $\bar{g}_2$ defined by

$$\bar{p}_1(t) = (L_0 + |a|M)t,$$
$$\bar{h}_{p_1}(t) = \bar{p}_1(t) - 1$$
$$\bar{g_1}(t) = \left(\frac{L}{2(1-L_0t)} + \frac{|a|M^2}{1-p_1(t)}\right)t,$$
$$\bar{h}_1(t) = \bar{g}_1(t) - 1$$
$$\bar{q}(t) = \frac{L_0}{2}t + M|\beta - 2|\bar{g}_1(t),$$
$$\bar{h}_q(t) = \bar{q}(t) - 1$$

and

$$\bar{g}_2(t) = \left(1 + \frac{M^2(1+|\beta|\bar{g}_1(t))}{(1-\bar{q}(t))(1-\bar{p_1}(t))}\right),$$
$$\bar{h}_2(t) = \bar{g}_2(t) - 1,$$

respectively and follow the proof of Theorem 2.1 with these changes. Let us consider that $\bar{r}_{p_1}$, $\bar{r}_1$, $\bar{r}_q$ and $\bar{r}_2$ be the smallest zeros of the functions $\bar{h}_{p_1}(t)$, $\bar{h}_1(t)$, $\bar{h}_q(t)$ and $\bar{h}_2(t)$, respectively. Notice that we have

$$\bar{r} = \min\{\bar{r_1},\ \bar{r}_2\} < r_0 = \frac{1}{L_0 + |a|M} = \bar{r}_2 < \bar{r}_q.$$

Theorem 2.2. *Under the hypotheses of Theorem 2.1, the conclusions hold for method (1.3) replacing method (1.2) and functions $\bar{p}_1$, $\bar{g}_1$ and $\bar{g}_2$ replacing functions p_1, g_1 and g_2.*

3. Numerical example and applications

This section is fully devoted to verify the validity and effectiveness of our theoretical results which we have proposed earlier. In this regard, we will consider some numerical examples in order to demonstrate the convergence behavior of the scheme proposed in [6, 15]. We will also check the applicability of our study where earlier study did not work.

Now, we employ the three special cases of method (1.2) for $\beta = 0$, $\beta = \frac{1}{2}$ and $\beta = 1$ are denoted by (M_1), (M_2) and (M_3), respectively. In addition, we also consider three special cases of method (1.3) for $\beta = 0$, $\beta = \frac{1}{2}$ and $\beta = 1$ are called by (M_4) and (M_5), (M_6), respectively to check the effectiveness and validity of the theoretical results.

For every iterative method, we require an initial approximation x_0 close to the required root which gives the guarantee for convergence of the corresponding iterative method. In this regard, first of all, we shall calculate the values of r_A, r_R, r_p, r_{p_1}, r_1, r_q, r_2 and r which are defined in the section 2, to find the convergence domain. We displayed all theses values in the Tables 1 and 4 which are corrected up to 5 significant digits. However, we have the values of these constants up to several number of significant digits. Then, we will also verify the theoretical convergence behavior of these methods on the basis of computational order of convergence and $\left|\frac{e_{n+1}}{e_n^p}\right|$.

In the Tables 3 and 6, we presented the number of iteration indexes (n), approximated zeros (x_n), residual error of the corresponding function $(|F(x_n)|)$, errors $|e_n|$

(where $e_n = x_n - x^*$), $\left|\frac{e_{n+1}}{e_n^p}\right|$ and the asymptotic error constant $\eta = \lim_{n\longrightarrow\infty}\left|\frac{e_{n+1}}{e_n^p}\right|$. Moreover, we will also present the computational order of convergence which is calculated by using the above formulas proposed by Sánchez et al. in [8]. We calculate the computational order of convergence, asymptotic error constant and other constants up to several number of significant digits (minimum 1000 significant digits) to minimize the round off error.

As we mentioned in the above paragraph that we calculate the values of all the constants and functional residuals up to several number of significant digits but due to the limited paper space, we display the values of x_n up to 15 significant digits. In addition, the values of other constants namely, $\xi(COC)$ up to 5 significant digits and the values $\left|\frac{e_n}{e_{n-1}^p}\right|$ and η are up to 10 significant digits. Moreover, the residual error in the function ($|F(x_n)|$) and the error $|e_n|$ are display up to 2 significant digits with exponent power which are mentioned in the following Tables corresponding to the test function. However, minimum 1000 significant digits are available with us for every value.

During the current numerical experiments with programming language Mathematica (Version 9), all computations have been done with multiple precision arithmetic, which minimize round-off errors.

Further, we use $\alpha_n = -\frac{1}{2}\frac{[x_{n-1},\ x_n;\ F']}{[x_{n-1},\ x_n;\ F]}$, $n = 0,\ 1,\ 2,\ \ldots$ in the method (1.2). and $a = \alpha_0$ in method (1.3).

Example 3.1. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R},\ \mathbb{D} = \bar{U}(0,\ 1)$. Define F on $\mathbb{D}$ by

$$F(x) = e^x - 1. \tag{3.1}$$

Then the derivative is given by

$$F'(x) = e^x.$$

Notice that $x^* = 0,\ L_0 = e - 1,\ L = M = e^{\frac{1}{L_0}}$ and $L_1 = e$. We obtain different radius of convergence, COC (ξ) and n in the following Table 1.

TABLE 1. (Different radius of convergence for different cases of method (1.2))

β	r_A	r_R	r_p	r_{p_1}	r_1	r_q	r_2	r
M_1	0.38269	0.24525	0.22932	0.16234	0.10455	0.050831	0.027969	0.027969
M_2	0.38269	0.24525	0.22932	0.16234	0.10455	0.061390	0.029590	0.029590
M_3	0.38269	0.24525	0.22932	0.16234	0.10455	0.077519	0.031207	0.031207

TABLE 2. (Different radius of convergence for different cases of method (1.3))

β	r_A^*	r_R	$\bar{r}_{p_1}$	$\bar{r}_1$	$\bar{r}_q$	$\bar{r}_2$	$\bar{r}$
M_4	0.38269	0.24525	0.38269	0.20602	0.083769	0.044751	0.044751
M_5	0.38269	0.24525	0.38269	0.20602	0.10341	0.047606	0.047606
M_6	0.38269	0.24525	0.38269	0.20602	0.135464	0.050500	0.050500

TABLE 3. (Convergence behavior of methods on example (3.1))

Methods; I. guesses	n	x_n	$\|f(x_n)\|$	$\|e_n\|$	ρ	$\left\|\frac{e_n}{e_{n-1}^p}\right\|$	η
M_1;	1	$-1.34790139256212e(-10)$	$1.3e(-10)^a$	$1.3e(-10)$		$3.975878133e(-5)$	$8.631045304e(-84)$
$x_0 = 0.026$;	2	$6.17956797140613e(-52)$	$6.2e(-52)$	$6.2e(-52)$	4.9997	$3.093237159e(-18)$	
$x_{-1} = 0.025$	3	$-1.25157843463683e(-258)$	$1.3e(-258)$	$1.3e(-258)$	5.0000	$8.631045304e(-84)$	
M_2;	1	$-2.37361265176050e(-10)$	$2.4e(-10)$	$2.4e(-10)$		$4.754928001e(-5)$	$7.666765294e(-82)$
$x_0 = 0.028$;	2	$1.04644581262418e(-50)$	$1.0e(-50)$	$1.0e(-50)$	4.9996	$7.587924013e(-18)$	
$x_{-1} = 0.027$	3	$-1.74281474660902e(-252)$	$1.7e(-252)$	$1.7e(-252)$	5.0000	$7.666765294e(-82)$	
M_2;	1	$-3.34959078958091e(-10)$	$3.3e(-10)$	$3.3e(-10)$		$5.301820294e(-5)$	$1.176621971e(-80)$
$x_0 = 0.030$;	2	$5.85633985970618e(-50)$	$5.9e(-50)$	$5.9e(-50)$	4.9996	$1.310170843e(-17)$	
$x_{-1} = 0.029$	3	$-9.56750866152990e(-249)$	$9.6e(-249)$	$9.6e(-249)$	5.0000	$1.176621971e(-80)$	
M_4;	1	$-1.83720876618666e(-4)$	$1.8e(-4)$	$1.8e(-4)$		$5.364665184e(-3)$	$4.037825064e(-23)$
$x_0 = 0.043$	2	$2.90723404620118e(-21)$	$2.9e(-21)$	$2.9e(-21)$	4.9861	$2.551796045e(-6)$	
$x_{-1} = 0.042$	3	$-2.88447967644456e(-105)$	$2.9e(-105)$	$2.9e(-105)$	5.0000	$4.037825064e(-23)$	
M_5;	1	$-2.53964035862797e(-4)$	$e(-)$	$e(-)$		$5.659552194e(-3)$	$2.038091513e(-22)$
$x_0 = 0.046$	2	$1.46742588925498e(-20)$	$1.5e(-20)$	$1.5e(-20)$	4.9838	$3.527502228e(-6)$	
$x_{-1} = 0.045$	3	$-9.45036076907927e(-102)$	$9.5e(-102)$	$9.5e(-102)$	5.0000	$2.038091513e(-22)$	
M_6;	1	$-3.43233270545234e(-4)$	$3.4e(-4)$	$3.4e(-4)$		$5.937295070e(-3)$	$9.190077267e(-22)$
$x_0 = 0.049$	2	$6.61685563211979e(-20)$	$6.6e(-20)$	$6.6e(-20)$	4.9811	$4.767537798e(-6)$	
$x_{-1} = 0.048$	3	$-1.76167503404114(-98)$	$1.8e(-98)$	$1.8e(-98)$	5.0000	$9.190077267e(-22)$	

[a] $1.3e(-10)$ denotes $1.3 \times 10^{(-10)}$ and [b] $4.6e(+1)$ denotes $4.6 \times 10^{(+1)}$.

Example 3.2. Returning back to the motivation example at the introduction on this paper, we have $L = L_0 = L_1 = 96.662907$, $M = 1.0631$ and our required zero is $x^* = \frac{1}{\pi}$. We obtain different radius of convergence, COC (ρ) and n in the following Table 4.

TABLE 4. (Different radius of convergence for different cases of method (1.2))

β	r_A^*	r_R	r_p	r_{p_1}	r_1	r_q	r_2	r
M_1	0.0068968	0.0068968	0.0050668	0.0038436	0.0029697	0.0021535	0.0014647	0.0014647
M_2	0.0068968	0.0068968	0.0050668	0.0038436	0.0029697	0.0024379	0.0015136	0.0015136
M_3	0.0068968	0.0068968	0.0050668	0.0038436	0.0029697	0.0028015	0.0015571	0.0015571

4. CONCLUSION

Most of time, whenever a researcher from the worldwide proposed a new or modified variant of Newton's method or Newton like method. He/she mentioned that initial guess should be very close to the required root for the granted convergence

TABLE 5. (Different radius of convergence for different cases of method (1.3))

β	r_A^*	r_R	$\bar{r}_{p_1}$	$\bar{r}_1$	$\bar{r}_q$	$\bar{r}_2$	$\bar{r}$
M_4	0.0068968	0.0068968	0.0094475	0.0063886	0.0039803	0.0026772	0.0026772
M_5	0.0068968	0.0068968	0.0094467	0.0063881	0.0045987	0.0027843	0.0027843
M_6	0.0068968	0.0068968	0.0094471	0.0063883	0.0054826	0.0028832	0.0028832

TABLE 6. (Convergence behavior of methods on example (3.2))

Methods; I. guesses	n	x_n	$\|f(x_n)\|$	$\|e_n\|$	ρ	$\left\|\frac{e_n}{e_{n-1}^p}\right\|$	η
M_1;	1	0.318309886198535	$3.5e(-12)$	$1.5e(-11)$		$5.072535235e(-2)$	$6.722422756e(-34)$
$x_0 = 0.3167$	2	0.318309886183791	$1.7e(-44)$	$7.1e(-44)$	4.0205	$6.793257776e(-7)$	
$x_{-1} = 0.3165$	3	0.318309886183791	$7.6e(-182)$	$3.2e(-181)$	4.2498	$6.722422756e(-34)$	
M_2;	1	0.318309886203752	$4.7e(-12)$	$2.0e(-11)$		$5.590350794e(-2)$	$1.922758031e(-33)$
$x_0 = 0.3166$	2	0.318309886183791	$6.0e(-40)$	$2.5e(-42)$	4.0206	$8.657309044e(-7)$	
$x_{-1} = 0.3164$	3	0.318309886183791	$1.7e(-179)$	$7.2e(-179)$	4.2499	$1.922758031e(-33)$	
M_3;	1	0.318309886202808	$4.5e(-12)$	$1.9e(-11)$		$5.325860168e(-2)$	$4.777713914e(-33)$
$x_0 = 0.3166$	2	0.318309886183791	$4.9e(-44)$	$2.1e(-43)$	4.0179	$8.413251931e(-7)$	
$x_{-1} = 0.3165$	3	0.318309886183791	$7.4e(-180)$	$3.2e(-179)$	4.2501	$1.634940044e(-33)$	
M_4;	1	0.318309886264393	$1.9e(-11)$	$1.8e(-11)$		1.292984230	3.498998586
$x_0 = 0.3155$	2	0.318309886183791	$3.5e(-40)$	$1.5e(-40)$	3.9427	3.498998654	
$x_{-1} = 0.3154$	3	0.318309886183791	$3.9e(-160)$	$1.7e(-159)$	4.0000	3.498998586	
M_5;	1	0.318309886285849	$2.4e(-11)$	$1.0e(-10)$		1.423451314	3.736693556
$x_0 = 0.3154$	2	0.318309886183791	$9.5e(-41)$	$4.1e(-41)$	3.9438	3.736693644	
$x_{-1} = 0.3153$	3	0.318309886183791	$2.4e(-158)$	$1.0e(-157)$	4.0000	3.736693556	
M_6;	1	0.318309886278453	$2.1e(-11)$	$9.5e(-11)$		1.423451314	3.736693556
$x_0 = 0.3153$	2	0.318309886183791	$6.8e(-41)$	$2.9e(-40)$	3.9438	3.736693644	
$x_{-1} = 0.3152$	3	0.318309886183791	$6.0e(-159)$	$2.6e(-158)$	4.0000	3.736693556	

of proposed scheme. But, they do not talk about the range or interval of the required root which give the grantee for the convergence of the proposed method. Therefore, we propose the computable radius of convergence and error bound by using Lipschitz conditions in this paper. Further, we also reduce the hypotheses from sixth order derivative of the involved function to only first order derivative. It is worth noticing that method (1.2) and method (1.3) are not changing if we use the conditions of Theorem 2.1 instead of the stronger conditions proposed by them. Moreover, to obtain the error bounds in practice and order of convergence, we can use the computational order of convergence which is defined in numerical section 3. Therefore,we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first order derivative.

Finally, on accounts of the results obtained in section 3, it can be concluded that the proposed study not only expand the applicability but also given the computable radius of convergence and error bound of the scheme given by the authors of [6, 15], to solve nonlinear equations.

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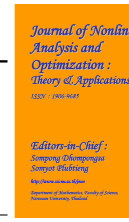

APPLICATIONS ON DIFFERENTIAL SUBORDINATION INVOLVING LINEAR OPERATOR

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ABSTRACT. In the present paper, we introduce and investigate some subclasses of strongly close-to-convex functions associated with the linear operator of meromorphic pvalently functions and study several inclusion relationships with some properties of this operator.

KEYWORDS linear operator , meromorphic functions , differential subordination, strongly close-to-convex functions, p-valently functions.

AMS Subject Classification:: Secondary 30C45.

1. INTRODUCTION

Let $\sum_p$ denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_{k+p} z^{k+p}, \tag{1.1}$$

which are analytic and p-valently in the punctured unit disk $\mathcal{U}^* = \{z : z \in \mathbb{C} : 0 < |z| < 1\} = \mathcal{U} - 0$.

If $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathcal{U}$ such that $f(z) = g(w(z))$.
A function $f(z) \in \sum_p$ is said to be p-valent meromorphic starlike of order $\alpha(0 \leq \alpha \leq p)$ if it satisfies

$$Re\Big\{ -\frac{zf'(z)}{f(z)} \Big\} > \alpha, \quad (z \in \mathcal{U}) \tag{1.2}$$

and the class of such functions is defined by $MS^*(\alpha)$.

Furthermore, a function $f(z) \in \sum_p$ is said to be p-valently meromorphic convex functions of order $\alpha(0 \leq \alpha \leq p)$ if it satisfies

$$Re\Big\{ -(1 + \frac{zf''(z)}{f'(z)}) \Big\} > \alpha, \quad (z \in \mathcal{U}) \tag{1.3}$$

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and the class of such functions is defined by $MK(\alpha)$.
Let $f(z) \in \sum_p$ and $g(z) \in MS^*(\alpha)$. Then $f(z) \in MC(\alpha, \beta)$ if and only if

$$Re\Big\{ -\frac{zf'(z)}{g(z)}\Big\} > \beta, \quad (z \in \mathcal{U}), \tag{1.4}$$

where $0 \leq \alpha < p$ and $0 \leq \beta < p$. Such functions are called close-to-convex functions of order β and type α in $\mathcal{U}$, (see for details ,[[4], [9]].

Further, a function $f(z) \in \sum_p$ is called p-valently meromorphic strongly starlike of order $\gamma(0 < \gamma \leq p)$ and type $\alpha(0 < \gamma \leq p)$ in $\mathcal{U}$ if it satisfies

$$\Big| arg(-\frac{zf'(z)}{f(z)} - \alpha)\Big| < \frac{\pi}{2}\gamma, \quad (z \in \mathcal{U}), \tag{1.5}$$

and denoted by $MS^*(\gamma, \alpha)$.

If $f(z) \in \sum_p$ satisfies

$$\Big| arg(-(1 + \frac{zf''(z)}{f'(z)}) - \alpha)\Big| < \frac{\pi}{2}\gamma, \quad (z \in \mathcal{U}),$$

for some $\gamma(0 < \gamma \leq p)$ and $\alpha(0 < \alpha \leq p)$, then f is called p-valently meromorphic strongly convex of order γ and type α in $\mathcal{U}$ and denoted by $MC(\gamma, \alpha)$. We note that the classes mentioned above are the familiar classes which have been studied by many authors (see for example,([3],[6],[9],[10]).

For a function $f(z) \in \sum_p$ given by (1), we define a linear operator D^n by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = z^{-p}(z^{p+1} f(z))' = z^{-p} + \sum_{k=0}^{\infty}(2p + k + 1)a_{k+p}z^{k+p}$$

and

$$\begin{aligned} D^n f(z) &= D(D^{n-1} f(z)) = z^{-p}(z^{p+1} D^{n-1} f(z))' \\ &= z^{-p} + \sum_{k=0}^{\infty}(2p + k + 1)^n a_{k+p}z^{k+p}. \quad (n \in \mathbb{N}) \end{aligned} \tag{1.6}$$

Using the relation (6), it is easy to verify that

$$z(D^n f(z))' = D^{n+1} f(z) - (p + 1)D^n f(z). \tag{1.7}$$

Also, we note that $D^n f(z)$ of another form of function studied by Liu and Srivastava [7] ,Srivastava and Patel [13] who introduce several inclusion relationships by using various subclasses of meromorphic p-valent function. A special cases of linear operator D^n for $p = 1$ studied by Uralegaddi and Somanatha [14], Aouf and Hossen.[1], and got interesting results by using the operator D^n.

For $n \in \mathbb{N}$, let $MC_p^{n+1}(\alpha, \beta, \gamma, A, B)$ be the class of functions $f(z) \sum_p$ satisfying the condition:

$$\Big| - arg\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} - \gamma\Big| < \frac{\pi}{2}\delta \quad (0 < \gamma \leq p, 0 \leq \delta < p; z \in \mathcal{U}), \tag{1.8}$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, where

$$S_p^{n+1}(\alpha, A, B) = \Big\{ g : \frac{1}{p+\alpha}(\frac{z(D^{n+1} g(z))'}{D^{n+1} g(z)} - \alpha) \prec \frac{1 + Az}{1 + Bz}\Big\} \tag{1.9}$$

$(0 \leq \alpha < p, -1 \leq B \leq A \leq 1, z \in \mathcal{U}$ and $g \in \sum_p)$ and the functions f belonging to this class is called strongly close-to-convex function. In this study and by using the technique of Cho[2],we find some argument properties of functions belonging to $\sum_p$ which include inclusion relationship and we obtain some interesting results for the functions class $MC_p^{n+1}(\alpha,\beta,\gamma,A,B)$ which we have defined here by the operator D^n.

To establish our main results, we shall need the following lemmas.

Lemma 1.1 [5] Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $Re\{\varepsilon h(z) + \eta\} > 0(\varepsilon, \eta \in \mathbb{C})$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\varepsilon p(z) + \eta} \prec h(z) \quad (z \in \mathcal{U}),$$

implies $p(z) \prec h(z) \quad (z \in \mathcal{U})$.

Lemma 1.2 [8]**:** Let $h(z)$ be convex univalent in $\mathcal{U}$ and $w(z)$ be analytic in $\mathcal{U}$ with $Re\{w(z) \geq 0$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = h(0)$,then

$$p(z) + w(z)zp'(z) - \frac{\pi}{2}(z) \prec h(z) \quad (z \in \mathcal{U}),$$

implies $p(z) \prec h(z) \quad (z \in \mathcal{U})$.

Lemma 1.3[9]**:** Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. If there exists two points z_1, z_2 in $\mathcal{U}$ such that

$$-\frac{\pi}{2}\alpha_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\alpha_2 \tag{1.10}$$

for some $\alpha_1, \alpha_2(\alpha_1, \alpha_2 > 0)$ and for all $z(|z| < |z_1| = |z_2|)$, then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = i\frac{\alpha_1 + \alpha_2}{2}m$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\alpha_1 + \alpha_2}{2}m, \tag{1.11}$$

where $m \geq \frac{1-|c|}{1+|c|}$ and

$$c = i\tan\frac{\pi}{4}(\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}). \tag{1.12}$$

2. MAIN RESULTS

We first derive the following with use of Lemma 1.1.

Proposition 2.1. Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $Re\{h(z)\} > 0$.
If a function $f(z) \in \sum_p$ satisfies the following condition:

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha) \prec h(z),$$

then

$$-\frac{1}{p+\alpha}(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha) \prec h(z),$$

$$(0 \leq \alpha < p; z \in \mathcal{U})$$

Proof. Let

$$p(z) = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha). \tag{2.1}$$

Then $p(z)$ is analytic function in $\mathcal{U}$ with $p(0) = 1$. By using (1.7), we obtain

$$p + 1 + \alpha + (p+\alpha)p(z) = -\frac{D^{n+1}f(z)}{D^n f(z)}. \tag{2.2}$$

Differentiating Logarithmically with resprct to z and multiplying by z, we get

$$p(z) + \frac{zp'(z)}{p+1+\alpha+(p+\alpha)p(z)} = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha).$$

Now, by using Lemma 1.1,we obtain

$$-\frac{1}{p+\alpha}(\frac{z(D^{n}f(z))'}{D^{n}f(z)} - \alpha) \prec h(z),$$

deduce that $p(z) \prec h(z)$.
Setting $h(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B \leq A \leq 1)$, in Lemma 2.1, we obtain

Corollary 2.1: For $n \in N$ and $p \in \{1, 2, ...\}$, we have

$$S_p^{n+1}(\alpha, A, B) \subset S_p^{n}(\alpha, A, B).$$

Proposition 2.2: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $Re\{h(z)\} > 0$. If $f(z) \in \sum_p$ satisfies

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha) \prec h(z),$$

then

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta f(z)} - \alpha) \prec h(z),$$

$$(0 \leq \alpha < p; z \in \mathcal{U})$$

where

$$\mathbb{I}_\theta f(z) = \frac{\theta - p}{z^\theta}\int_0^z t^{\theta-1}f(t)dt \quad (\theta \geq 0) \tag{2.3}$$

Proof. From (2.3), we have

$$z(D^{n+1}\mathbb{I}_\theta f(z))' = (\theta - p)(D^{n+1}f(z)) - \theta(D^{n+1}f(z)). \tag{2.4}$$

Let

$$p(z) = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha),$$

$p(z)$ is analytic function in $\mathcal{U}$ with $p(0) = 1$. Then from (2.4), we get

$$\theta + \alpha + (p+\alpha)p(z) = -(\theta - p)\frac{D^{n+1}f(z)}{D^{n+1}\mathbb{I}_\theta f(z)}. \tag{2.5}$$

By differentiating (2.5) logarthmically with respect to z and multiplying by z,we have

$$p(z) + \frac{zp'(z)}{\theta+\alpha+(p+\alpha)p(z)} = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha).$$

Thus, by Lemma 1.1, we get

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha) \prec h(z).$$

Taking $h(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B \leq A \leq 1)$, in Proposition 2.2, we obtain

Corollary 2.2: If $f(z) \in S_p^{n+1}(\alpha, A, B)$, then $\mathbb{I}_\theta f(z) \in S_p^{n+1}(\alpha, A, B)$. Hence on Applying Proposition 2.2, we prove the following theorem

Theorem 2.1: Let $f(z) \in \sum_p$ and $(0 < \delta_1, \delta_2 \leq p, 0 \leq \alpha < p)$. If

$$-\frac{\pi}{2}\delta < \arg(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma) < \frac{\pi}{2}\delta_2$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, then

$$-\frac{\pi}{2}\beta_1 < \arg(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma) < \frac{\pi}{2}\beta_2,$$

where β_1 and $\beta_2 (0 < \beta_1, \beta_2 \leq p)$ are the solution of the equations:

$$\delta_1 = \begin{cases} \beta_1 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2(\frac{(p+\alpha)(1+A)}{1+B}+p+1+\alpha)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_1}\right\} & B \neq -1 \\ \beta_1 & B = -1, \end{cases} \tag{2.6}$$

and

$$\delta_2 = \begin{cases} \beta_2 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2(\frac{(p+\alpha)(1+A)}{1+B}+p+1+\alpha)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_1}\right\} & B \neq -1 \\ \beta_2 & B = -1, \end{cases} \tag{2.7}$$

where c is given by (1.12) and $t_1 = \frac{2}{\pi}\sin^{-1}(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(p+1+\alpha)(1-B^2)})$.

Proof. Let

$$p(z) = -\frac{1}{p+\alpha}(\frac{z(D^n f(z))'}{D^n g(z)} - \gamma). \tag{2.8}$$

It follows from (1.7) that

$$[(p+\gamma)(p(z)-\gamma)]D^n g(z) = D^{n+1}f(z) - (p+1)D^n f(z). \tag{2.9}$$

Differentiating both sides of (2.9), and multiplying by z, we deduce that

$$(p+\gamma)zp'(z)D^n g(z) + [(p+\gamma)p(z)-\gamma]z(D^n g(z))' = z(D^{n+1}f(z))' - (p+1)z(D^n f(z))'. \tag{2.10}$$

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$, by applying Corollary 2.1, we find that $g(z) \in S_p^n(\alpha, A, B)$. Thus, by using (1.7) and put $q(z) = -\frac{1}{p+\alpha}(\frac{z(D^n g(z))'}{D^n g(z)} - \alpha)$, we immediately have

$$(p+\alpha)q(z) + \alpha + p + 1 = -\frac{D^{n+1}g(z)}{D^n g(z)}. \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma) = p(z) + \frac{zp'(z)}{(p+\alpha)q(z)+\alpha+p+1}.$$

Making use the result of Silverman and Silvia [10] ,we obtain

$$|q(z) - \frac{1-AB}{1-B^2}| < \frac{A-B}{1-B^2} \quad (z \in \mathcal{U}; B \neq -1) \tag{2.12}$$

and

$$Re\{q(z)\} > \frac{1-A}{2} \quad (z \in \mathcal{U}; B = -1) \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$(p+\alpha)q(z) + p + \alpha + 1 = re^{i\frac{\pi\phi}{2}}.$$

Now, if $B \neq -1$, we have

$$\frac{(p+\alpha)(1-A)}{1-B} + \alpha + p + 1 < r < \frac{(p+\alpha)(1+A)}{1+B} + \alpha + p + 1, \quad -t_1 < \phi < t_1,$$

and if $B = -1$, we have

$$\frac{(p+\alpha)(1-A)}{2} + \alpha + p + 1 < r < \infty, \quad -1 < \phi < 1,$$

Applying Lemma 1.2 with $w = -\frac{1}{(p+\alpha)q(z)+p+\alpha+1}$, we note that $p(z)$ is analytic with $p(0) = 1$ and $Re\{p(z)\} > 0$ in $\mathcal{U}$.

Hence by Lemma 1.3 for $z_1, z_2 \in \mathcal{U}$, such that the condition (1.10) is satisfied ,then we obtain (1.11) under the restriction (1.12). On other hand, if $B \neq -1$, we readily get

$$\arg(-(p(z_1)+\frac{z_1p'(z_1)}{(p+\alpha)q(z_1)+p+\alpha+1}))=-\frac{\pi}{2}\beta_1+\arg(1-i\frac{\beta_1+\beta_2}{2}m(re^{i\frac{\pi}{2}})^{-1})$$

$$\leq\frac{-\pi}{2}\beta_1-\tan^{-1}(\frac{(\beta_1+\beta_2)m\sin\frac{\pi}{2}(1-\phi)}{2r+(\beta_1+\beta_2)m\cos\frac{\pi}{2}(1-\phi)})$$

$$\frac{-\pi}{2}\beta_1-\tan^{-1}(\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2\frac{(p+\alpha)(1+A)}{1+B}+p+\alpha+1)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_1})=\frac{-\pi}{2}\delta_1,$$

and

$\arg(-(p(z_2)+\frac{z_2p'(z_2)}{(p+\alpha)q(z_2)+p+\alpha+1}))\geq\frac{-\pi}{2}\beta_2-\tan^{-1}(\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2\frac{(p+\alpha)(1+A)}{1+B}+p+\alpha+1)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_1})=\frac{-\pi}{2}\delta_2.$

Also, if $B=-1$, we readily get

$$\arg(-(p(z_1)+\frac{z_1p'(z_1)}{(p+\alpha)q(z_1)+p+\alpha+1}))\leq\frac{-\pi}{2}\beta_1$$

and

$$\arg(-(p(z_2)+\frac{z_2p'(z_2)}{(p+\alpha)q(z_2)+p+\alpha+1}))\geq\frac{\pi}{2}\beta_2$$

There are contradiction with a assumption. This completes the proof of Theorem 2.1

Corollary 2.3:

$$MC_p^{n+1}(\alpha,\beta,\gamma,A,B)\subset MC_p^{n}(\alpha,\beta,\gamma,A,B).$$

Setting $n=0, \delta_1=\delta_2=\delta$ in Theorem 2.1, we get:

Corollary 2.4: Let $f(z)\in\sum_p$. If

$$|\frac{z(z^{-p}(z^{p+1}f)')'}{z^{-p}(z^{p+1}g(z))'}-\gamma|<\frac{\pi}{2}\delta$$

for some $g(z)\in S_p^1$, then

where $\beta(0<\beta\leq p)$ is the solution of equation:

$$\delta=\begin{cases}\beta+\frac{2}{\pi}\tan^{-1}\left\{\frac{\beta\cos\frac{\pi}{2}t_1}{\frac{(p+\alpha)(1+A)}{1+B}+p+1+\alpha+\beta\sin\frac{\pi}{2}t_1}\right\} & B\neq-1\\ \beta & B=-1,\end{cases}$$

and $t_1=\frac{2}{\pi}\sin^{-1}(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(p+1+\alpha)(1-B^2)})$.

Theorem 2.2: Let $f(z)\in\sum_p$ and $(0<\delta_1,\delta_2\leq 1, 0\leq\gamma<1)$. If

$$\frac{-\pi}{2}\delta_1<\arg(-(\frac{z(D^{n+1}f(z))'}{D^{n+}g(z)}-\gamma))<\frac{\pi}{2}\delta_2,$$

for some $g(z)\in S_p^{n+1}(\alpha,A,B)$, then

$$\frac{-\pi}{2}\beta_1<\arg(-(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+}\mathbb{I}_\theta g(z)}-\gamma))<\frac{\pi}{2}\beta_2,$$

where $\mathbb{I}_\theta$ is defined by (2.3), and β_1,β_2, are the solutions of

$$\delta_1=\begin{cases}\beta_1+\frac{2}{\pi}\tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_2}{2(\frac{(p+\alpha)(1+A)}{1+B}+\theta+\alpha)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_2}\right\} & B\neq-1\\ \beta_1 & B=-1,\end{cases}\quad(2.14)$$

and

$$\delta_2 = \begin{cases} \beta_2 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_2}{2(\frac{(p+\alpha)(1+A)}{1+B}+\theta+\alpha)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_2}\right\} & B \neq -1 \\ \beta_2 & B = -1, \end{cases} \quad (2.15)$$

here c is given by (1.12) and $t_2 = \frac{2}{\pi}\sin^{-1}(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(\theta+\alpha)(1-B^2)})$.

Proof. Let

$$p(z) = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \gamma).$$

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$, and by using Corollary 2.2, we obtain $\mathbb{I}_\theta g(z) \in S_p^{n+1}(\alpha, \beta, \gamma, A, B)$. By using (2.5), we get

$$[(p+\gamma)(p(z)-\gamma)]D^{n+1}\mathbb{I}_\theta g(z) = (\theta - p)(D^{n+1}f(z)) - \theta D^{n+1}\mathbb{I}_\theta f(z)$$

and simplifying, we obtain

$$(p+\gamma)zp'(z) + [(p+\gamma)p(z)+\gamma][(p+\alpha)q(z)+\theta+\alpha] = (\theta-p)\frac{z(D^{n+1}f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)},$$

where

$$q(z) = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}\mathbb{I}_\theta g(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \alpha).$$

Therefore,

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \alpha) = p(z) + \frac{zp'(z)}{(p+\alpha)q(z)+\alpha+\theta}.$$

Applying a similar method as in the proof of Theorem 2.1 we get the required result and the proof is complete.

Setting $\delta_1 = \delta_2 = \delta$ in Theorem 2.2, we obtain

Corollary 2.5: Let $f(z) \in \sum_p$ and $0 \leq \gamma < p, 0 < \delta \leq 1$. If

$$\left|\arg(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, then

$$\left|\arg(-\frac{z(D^{n+1}\mathbb{I}_\theta(f)(z))'}{D^{n+1}\mathbb{I}_\theta(g)(z)} - \gamma)\right| < \frac{\pi}{2}\beta,$$

where $\mathbb{I}_\theta$ is given by (2.5), and $\beta (0 < \beta \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} \beta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\beta\cos\frac{\pi}{2}t_2}{(\frac{(p+\alpha)(1+A)}{1+B}+\theta+\alpha)+\beta\sin\frac{\pi}{2}t_2}\right\} & B \neq -1 \\ \beta & B = -1 \end{cases}$$

Corollary 2.6: If $f(z) \in MC_p^{n+1}(\gamma, \delta, \alpha, A, B)$, then $\mathbb{I}_\theta(f) \in MC_p^{n+1}(\gamma, \delta, \alpha, A, B)$.

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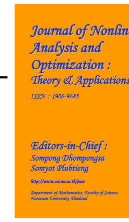

AN HYBRID EXTRAGRADIENT ALGORITHM FOR VARIATIONAL INEQUALITIES WITH PSEUDOMONOTONE EQUILIBRIUM CONSTRAINTS

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ABSTRACT. In this paper, we propose a new hybrid extragradient algorithm for solving a variational inequality problem over the solution set of an equilibrium problem in Euclidean space. By using fixed point and hybrid plane cutting techniques, we show that this problem can be solved by an explicit extragradient method. Under certain conditions on parameters, the convergence of the iteration sequences generated by the algorithm are obtained.

KEYWORDS: Variational inequalities; equilibrium problems; KyFan inequality; auxiliary subproblem principle; projection method; Armijo linesearch; pseudomonotone.
AMS Subject Classification: 47H06 47H09 47H10 47J05 47J05

1. Introduction and Motivation

Let $\mathbb{R}^n$ be a n-dimentional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Let C be a nonempty closed convex subset in $\mathbb{R}^n$ and $G : C \longrightarrow \mathbb{R}^n$ be an operator, and $f : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $f(x,x) = 0$ for every $x \in C$. We consider the following variational inequality problem over the solution set of the equilibrium problem (shortly VIEP(C, f, G)):

$$\text{Find } x^* \in S_f \text{ such that } \langle G(x^*), y - x^* \rangle \geq 0 \ \forall y \in S_f, \tag{1.1}$$

where $S_f = \{u \in C : f(u,y) \geq 0, \ \forall y \in C\}$, i.e., S_f is the solution set of the following equilibrium problems (EP(C, f) for short):

$$\text{Find } u \in C \text{ such that } f(u,y) \geq 0 \ \forall y \in C. \tag{1.2}$$

As usual, we call problem (1.1) the upper problem and (1.2) the lower one. Problem (1.1) can be consider as a special case of mathematical programs with equilibrium constraints. Sources for such problems can be found in [11, 15, 17]. Bilevel variational inequalities were considered in [1], Moudafi in [16] and Yao et al

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in [22] suggested the use of the proximal point method for monotone bilevel equilibrium problems, which contain monotone variational inequalites as a special case. Recently, Ding in [6] used the auxiliary problem principle to monotone bilevel equilibrium problems. In those papers, the lower problem is required to be monotone. In this case the subproblems to be solved are monotone.

It should be noticed that the solution set S_f of the lower problem (1.2) is convex whenever f is pseudomonotone on C. However, the main difficulty is that, even the constrained set S_f is convex, it is not given explicitly as in a standard mathematical programming problem, and therefore the available methods of convex optimization and variational inequality cannot be applied directly to problem (1.1).

In our recent paper [4] we proposed penalty and gap function methods for solving bilevel equilibrium problems which contains (1.1) as a special case. Under a certain strictly ∇-pseudomonotonicity, it has been proved that any stationary point of the gap function over C is a solution of the penalized problem. This assumption is satisfied for strict monotonicity case, but it may fail to hold for problem (1.1) when the lower equilibrium problem is pseudomonotone. The reason is that the sum of a strongly monotone and a pseudomonotone bifunction, in general, is not pseudomonotone, even not strongly monotone.

In this paper, we continue our work in [4] by further extend the hybrid extragradient-viscosity methods introduced by Maingé in [13] for solving bilevel problem (1.1) when the lower problem is pseudomonotone with respect to its solution set equilibrium problems rather than monotone variational inequalities as in [13], the later pseudomonotonicity is somewhat general than pseudomonotone. We show that the sequence of iterates generated by the proposed algorithm converges to the unique solution of the bilevel problem (1.1).

The paper is organized as follows. The next section contains some preliminaries on the Euclidean projection and equilibrium problems. The third section is devoted to presentation of the algorithm and its convergence. In the last section, we describe a special case of minimizing the Euclidean norm over the solution set of an equilibrium problem, where the bifunction is pseudomonotone with respect to its solution set. The latter problem arises from the Tikhonov regularization method for pseudomonotone equilibrium problems [8].

2. Preliminaries

Throughout the paper, by P_C we denote the projection operator on C with the norm $\|.\|$, that is

$$P_C(x) \in C: \ \|x - P_C(x)\| \leq \|y - x\| \ \forall y \in C.$$

The following well known results on the projection operator onto a closed convex set will be used in the sequel.

Lemma 2.1. *Suppose that C is a nonempty closed convex set in $\mathbb{R}^n$. Then*
(i) $P_C(x)$ is singleton and well defined for every x;
(ii) $\pi = P_C(x)$ if and only if $\langle x - \pi, y - \pi \rangle \leq 0, \forall y \in C$;
(iii) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \ \forall x, y \in C$.

We recall some well known definitions on monotonicity (see e.g., [2, 7, 9, 17, 21])

Definition 2.2. A bifunction $\varphi : C \times C \longrightarrow \mathbb{R}$ is said to be
(a) strongly monotone on C with modulus $\beta > 0$, if

$$\varphi(x, y) + \varphi(y, x) \leq -\beta\|x - y\|^2 \ \forall x, y \in C;$$

(b) monotone on C if

$$\varphi(x,y)+\varphi(y,x)\leq 0 \ \forall x,y\in C;$$

(c) pseudomonotone on C if

$$\varphi(x,y)\geq 0 \Longrightarrow \varphi(y,x)\leq 0 \ \forall x,y\in C;$$

(d) pseudomonotone on C with respect to x^* if

$$\varphi(x^*,y)\geq 0 \Longrightarrow \varphi(y,x^*)\leq 0 \ \forall y\in C.$$

We say that φ is pseudomonotone on C with respect to a set S if it is pseudomonotone on C with respect to every point $x^*\in S$.

From the definitions it follows that $(a)\Rightarrow(b)\Rightarrow(c)\Rightarrow(d)\ \forall x^*\in C$.

When $\varphi(x,y)=\langle\phi(x),y-x\rangle$, where $\phi: C\longrightarrow\mathbb{R}^n$ is an operator then the definition (a) becomes:

$$\langle\phi(x)-\phi(y),x-y\rangle\geq\beta\|x-y\|^2 \ \forall x,y\in C$$

i.e., ϕ is β-strongly monotone on C. Similarly, if φ satisfies (b) ($(c),(d)$ resp) on C then ϕ becomes monotone, (pseudomonotone, pseudomonotone with respect to x^* resp) on C.

In the sequel, we need the following blanket assumptions

$(A1)$ $f(.,y)$ is continuous on Ω for every $y\in C$;

$(A2)$ $f(x,.)$ is convex on Ω for every $x\in C$;

$(A3)$ f is pseudomonotone on C with respect to the solution set S_f of EP(C,f);

$(A4)$ G is L-Lipschitz and β-strongly monotone on C;

$(B1)$ $h(.)$ is δ-strongly convex, continuously differentiable on Ω;

$(B2)$ $\{\lambda_k\}$ is a positive sequence such that $\sum_{k=0}^{\infty}\lambda_k=\infty$ and $\sum_{k=0}^{\infty}\lambda_k^2<\infty$.

Lemma 2.3. *Suppose Problem EP(C,f) has a solution. Then under Assumptions $(A1)$, $(A2)$ and $(A3)$ the solution set S_f is closed, convex and*

$$f(x^*,y)\ \geq 0\ \forall y\in C \text{ if and only if } f(y,x^*)\leq 0\ \forall y\in C.$$

The proof of this lemma when f is pseudomonotone on C can be found, for instance, in [9, 17]. When f is pseudomonotone with respect to the solution set of EP(C,f), it can be done by the same way. So we omit it.

The following lemmas are well-known from the auxiliary problem principle for equilibrium problems.

Lemma 2.4. *([14]) Suppose that h is a continuously differentiable and strongly convex function on C with modulus $\delta>0$. Then under Assumptions $(A1)$ and $(A2)$, a point $x^*\in C$ is a solution of EP(C,f) if and only if it is a solution to the equilibrium problem:*

$$\text{Find } x^*\in C: f(x^*,y)+h(y)-h(x^*)-\langle\nabla h(x^*),y-x^*\rangle\geq 0\ \forall y\in C. \qquad (AEP)$$

The function

$$D(x,y):=h(y)-h(x)-\langle\nabla h(x),y-x\rangle$$

is called Bregman function. Such a function was used to define a generalized projection, called D-projection, which was used to develop algorithms for particular problems, see e.g., [3]. An important case is $h(x):=\frac{1}{2}\|x\|^2$. In this case D-projection becomes the Euclidean one.

Lemma 2.5. *([14]) Under Assumptions (A1), (A2), a point $x^* \in C$ is a solution of Problem (AEP) if and only if*

$$x^* = \textit{argmin}\{f(x^*, y) + h(y) - h(x^*) - \langle \nabla h(x^*), y - x^* \rangle : \ y \in C\}. \qquad (CP)$$

Note that, since $f(x,.)$ is convex and h is strongly convex, Problem (CP) is a strongly convex program.

For each $z \in C$, by $\partial_2 f(z,z)$ we denote the subgradient of the convex function $f(z,.)$ at z, i.e.,

$$\partial_2 f(z,z) := \{w \in \mathbb{R}^n : f(z,y) \geq f(z,z) + \langle w, y - z \rangle, \ \forall y \in C\}$$

$$= \{w \in \mathbb{R}^n : f(z,y) \geq \langle w, y - z \rangle, \ \forall y \in C\},$$

and we define the halfspace H_z as

$$H_z := \{x \in \mathbb{R}^n : \ \langle w, x - z \rangle \leq 0\} \qquad (2.1)$$

where $w \in \partial_2 f(z,z)$. Note that when $f(x,y) = \langle F(x), y - x \rangle$, this halfspace becomes the one introduced in [21]. The following lemma says that the hyperplane does not cut off any solution of problem EP(C, f).

Lemma 2.6. *([5]) Under Assumptions (A2) and (A3), one has $S_f \subseteq H_z$ for every $z \in C$.*

Lemma 2.7. *([5]) Under Assumptions (A1) and (A2), if $\{z^k\} \subset C$ is a sequence such that $\{z^k\}$ converges to $\bar{z}$ and the sequence $\{w^k\}$ with $w^k \in \partial_2 f(z^k, z^k)$ converges to $\bar{w}$, then $\bar{w} \in \partial_2 f(\bar{z}, \bar{z})$.*

The following lemma is in [21] (see also [5]).

Lemma 2.8. *([21], [5]) Suppose that $x \in C$ and $u = P_{C \cap H_z}(x)$. Then*

$$u = P_{C \cap H_z}(\bar{x}), \ \textit{where } \bar{x} = P_{H_z}(x).$$

Lemma 2.9. *(Lemma 3.1 [12]) Let $\{a_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that*

$$a_{k_j} < a_{k_j+1} \textit{ for all } j \geq 0$$

Also consider the sequence of integers $\{\sigma(k)\}_{k \geq k_0}$ defined by

$$\sigma(k) = \max\{j \leq k \mid a_j < a_{j+1}\}.$$

Then $\{\sigma(k)\}_{k \geq k_0}$ is a nondecreasing sequence verifying

$$\lim_{k \longrightarrow \infty} \sigma(k) = \infty$$

and, for all $k \geq k_0$, the following two estimates hold:

$$a_{\sigma(k)} \leq a_{\sigma(k)+1} \qquad (2.2)$$

$$a_k \leq a_{\sigma(k)+1} \qquad (2.3)$$

3. An Hybrid Extragradient Algorithm for VIEP(C, f, G)

Algorithm 1. Pick $x^0 \in C$ and choose two parameters $\eta \in (0,1)$, $\rho > 0$.
At each iteration $k = 0, 1, ...$ having x^k do the following steps:
Step 1. Solve the strongly convex program

$$\min \Big\{ f(x^k, y) + \frac{1}{\rho}\Big[h(y) - h(x^k) - \langle \nabla h(x^k), y - x^k\rangle\Big] :\ y \in C \Big\} \qquad CP(x^k)$$

to obtain its unique solution y^k.

If $y^k = x^k$, take $u^k = x^k$ and go to *Step 3*. Otherwise, do Step 2.

Step 2. (Armijo linesearch rule) Find m_k as the smallest positive integer number m satisfying

$$\begin{cases} z^{k,m} = (1-\eta^m)x^k + \eta^m y^k : \\ \langle w^{k,m}, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] \\ \text{with } \ w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}). \end{cases} \tag{3.1}$$

Step 3. Set $\eta_k := \eta^{m_k}$, $z^k := z^{k,m_k}$, $w^k := w^{k,m_k}$. Take

$$C_k := \{x \ \in C : \langle w^k, x - z^k\rangle \leq 0\},\ u^k := P_{C_k}(x^k). \tag{3.2}$$

Step 4. $x^{k+1} = P_C(u^k - \lambda_k G(u^k))$ and go to Step 1 with k is replaced by $k+1$.

Remark 3.1. (i) If $y^k = x^k$ then x^k is a solution to EP(C, f).

(ii) $w^k \neq 0\ \forall k$, indeed, at the begining of Step 2, $x^k \neq y^k$. By the Armijo linesearch rule and δ-strong convexity of h, we have

$$\langle w^k, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] \geq$$

$$\geq \frac{\delta}{\rho}\|x^k - y^k\|^2 > 0.$$

Now we are going to analyze the validity and convergence of the algorithm. Some parts in our proofs are based on the proof scheme in [13].

Lemma 3.2. *Under Assumptions $(A1)$, $(A2)$, $(A3)$, and $(A4)$, the linesearch rule (3.1) is well-defined in the sense that, at each iteration k, there exists an integer number $m > 0$ satisfying the inequality in (3.1) for every $w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$, then for every solution x^* of EP(C, f), one has*

$$\begin{aligned} \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \big(\frac{\eta_k \delta}{\rho\|w^k\|}\big)^2\|x^k - y^k\|^4 \\ - 2\lambda_k\langle u^k - x^*, G(u^k)\rangle + \lambda_k^2\|G(u^k)\|^2\ \forall k. \end{aligned} \tag{3.3}$$

where $\bar{x}^k = P_{H_{z^k}}(x^k)$.

Proof. First we prove that there exists a positive integer m_0 such that

$$\langle w^{k,m_0}, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big]$$

$$\forall w^{k,m_0} \in \partial_2 f(z^{k,m_0}, z^{k,m_0}).$$

Indeed, suppose by contradiction that, for every positive integer m and $z^{k,m} = (1-\eta^m)x^k + \eta^m y^k$ there exists $w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$ such that

$$\langle w^{k,m}, x^k - y^k\rangle < \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big].$$

Since $z^{k,m} \longrightarrow x^k$ as $m \longrightarrow \infty$, by Theorem 24.5 in [20], the sequence $\{w^{k,m}\}_{m=1}^{\infty}$ is bounded. Thus we may assume that $w^{k,m} \longrightarrow \bar{w}$ for some $\bar{w}$. Taking the limit as $m \longrightarrow \infty$, from $z^{k,m} \longrightarrow x^k$ and $w^{k,m} \longrightarrow \bar{w}$, by Lemma 2.7, it follows that $\bar{w} \in \partial_2 f(x^k, x^k)$ and

$$\langle \bar{w}, x^k - y^k \rangle \leq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle\Big]. \tag{3.4}$$

Since $\bar{w} \in \partial_2 f(x^k, x^k)$, we have

$$f(x^k, y^k) \geq f(x^k, x^k) + \langle \bar{w}, y^k - x^k \rangle = \langle \bar{w}, y^k - x^k \rangle.$$

Combining with (3.4) yields

$$f(x^k, y^k) + \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle\Big] \geq 0,$$

which contradicts to the fact that

$$f(x^k, y^k) + \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle\Big] < 0.$$

Thus, the linesearch is well defined.

Now we prove (3.3). For simplicity of notation, let $d^k := x^k - y^k$, $H_k := H_{z^k}$. Since $u^k = P_{C \cap H_k}(\bar{x}^k)$ and $x^* \in S_f$, by Lemma 2.6, $x^* \in C \cap H_k$, we have

$$\|u^k - \bar{x}^k\|^2 \leq \langle x^* - \bar{x}^k, u^k - \bar{x}^k \rangle$$

which together with

$$\|u^k - x^*\|^2 = \|\bar{x}^k - x^*\|^2 + \|u^k - \bar{x}^k\|^2 + 2\langle u^k - \bar{x}^k, \bar{x}^k - x^* \rangle$$

implies

$$\|u^k - x^*\|^2 \leq \|\bar{x}^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2. \tag{3.5}$$

Replacing

$$\bar{x}^k = P_{H_k}(x^k) = x^k - \frac{\langle w^k, x^k - z^k \rangle}{\|w^k\|^2} w^k$$

into (3.5) we obtain

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - 2\langle w^k, x^k - x^* \rangle \frac{\langle w^k, x^k - z^k \rangle}{\|w^k\|^2} + \frac{\langle w^k, x^k - z^k \rangle^2}{\|w\|^2}.$$

Substituting $x^k = z^k + \eta_k d^k$ into the last inequality we get

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 + \Big(\frac{\eta_k \langle w^k, d^k \rangle}{\|w^k\|}\Big)^2 - \frac{2\eta_k \langle w^k, d^k \rangle}{\|w^k\|^2}\langle w^k, x^k - x^* \rangle$$

$$= \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \Big(\frac{\eta_k \langle w^k, d^k \rangle}{\|w^k\|}\Big)^2 - \frac{2\eta_k \langle w^k, d^k \rangle}{\|w^k\|^2}\langle w^k, z^k - x^* \rangle.$$

In addition, by the Armijo linesearch rule, using the δ-strong convexity of h we have

$$\langle w^k, x^k - y^k \rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k \rangle\Big] \geq \frac{\delta}{\rho}\|x^k - y^k\|^2.$$

Note that $x^* \in H_k$ we can write

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \Big(\frac{\eta_k \delta}{\rho \|w^k\|}\Big)^2 \|x^k - y^k\|^4. \tag{3.6}$$

We have

$$\|x^{k+1}-x^*\|^2 = \|P_C(u^k-\lambda_k G(u^k)) - P_C(x^*)\|^2 \le \|u^k - x^* - \lambda_k G(u^k))\|^2$$
$$= \|u^k-x^*\|^2 - 2\lambda_k\langle u^k - x^*, G(u^k)\rangle + \lambda_k^2\|G(u^k)\|^2,$$

which together with (3.6) implies

$$\begin{aligned}\|x^{k+1}-x^*\|^2 \le \|x^k-x^*\|^2 - \|u^k-\bar{x}^k\|^2 - \Big(\frac{\eta_k\delta}{\rho\|w^k\|}\Big)^2\|x^k-y^k\|^4 \\ - 2\lambda_k\langle u^k-x^*, G(u^k)\rangle + \lambda_k^2\|G(u^k)\|^2 \ \forall k\end{aligned} \tag{3.7}$$

as desired.

□

Lemma 3.3. *The sequences* $\{x^k\}, \{u^k\}$ *generated by the Algorithm 1, are bounded under Assumptions* $(A1)$, $(A2)$, $(A3)$, *and* $(A4)$.

Proof. We have

$$\begin{aligned}\|x^{k+1}-x^*\| &= \|P_C(u^k-\lambda_k G(u^k)) - P_C(x^*)\| \le \|u^k-\lambda_k G(u^k) - x^*\| \\ &\le \|(u^k-\lambda_k G(u^k)) - (x^*-\lambda_k G(x^*))\| + \lambda_k\|G(x^*)\| \\ &= \|(1-L^2\frac{\lambda_k}{\beta})(u^k-x^*) - L^2\frac{\lambda_k}{\beta}\big[(\frac{\beta}{L^2}G-I)u^k - (\frac{\beta}{L^2}G-I)x^*\big]\| \\ &\quad + \lambda_k\|G(x^*)\| \\ &\le (1-L^2\frac{\lambda_k}{\beta})\|u^k-x^*\| + L^2\frac{\lambda_k}{\beta}T_k + \lambda_k\|G(x^*)\|,\end{aligned} \tag{3.8}$$

where $T_k = \|(\frac{\beta}{L^2}G-I)u^k - (\frac{\beta}{L^2}G-I)x^*\|$.

Since G is L-Lipschitz and β-strongly monotone, we have

$$\begin{aligned}T_k^2 &= \|\frac{\beta}{L^2}(G(u^k)-G(x^*)) - (u^k-x^*)\|^2 \\ &= \frac{\beta^2}{L^4}\|G(u^k)-G(x^*)\|^2 - 2\frac{\beta}{L^2}\langle G(u^k)-G(x^*), u^k-x^*\rangle + \|u^k-x^*\|^2 \\ &\le \frac{\beta^2}{L^2}\|u^k-x^*\|^2 - 2\frac{\beta^2}{L^2}\|u^k-x^*\|^2 + \|u^k-x^*\|^2 \\ &= (1-\frac{\beta^2}{L^2})\|u^k-x^*\|^2.\end{aligned}$$

Hence $T_k \le \sqrt{1-\frac{\beta^2}{L^2}}\|u^k-x^*\|$. Then combining with (3.8) we get

$$\begin{aligned}\|x^{k+1}-x^*\| &\le (1-\lambda_k\frac{L^2}{\beta}(1-\sqrt{1-\frac{\beta^2}{L^2}}\,))\|u^k-x^*\| + \lambda_k\|G(x^*)\| \\ &= (1-\lambda_k\frac{L^2}{\beta}\gamma)\|u^k-x^*\| + \lambda_k\|G(x^*)\| \\ &= (1-\gamma_k)\|u^k-x^*\| + \gamma_k(\frac{\beta}{L^2\gamma}\|G(x^*)\|)\end{aligned}$$

where, $\gamma = 1-\sqrt{1-\frac{\beta^2}{L^2}}$ and $\gamma_k = \lambda_k\frac{L^2}{\beta}\gamma \in (0;1)$.

By induction we get

$\|x^{k+1}-x^*\| \le \max\{\|x^k-x^*\|, \frac{\beta}{L^2\gamma}\|F(x^*)\|\} \le ... \le \max\{\|x^0-x^*\|, \frac{\beta}{L^2\gamma}\|F(x^*)\|\}$.

Hence $\{x^k\}$ is bounded, which, from (3.6), implies that $\{u^k\}$ is bounded too. □

Lemma 3.4. *There exists a subsequence $\{x^{k_i}\} \subset \{x^k\}$ converges to some $\bar{x} \in C$ such that $\{y^{k_i}\}, \{z^{k_i}\}, \{w^{k_i}\}$ are bounded.*

Proof. First, we show that there exists $M > 0$ such that $\|x^{k_i} - y^{k_i}\| \le M$ for all i large enough.

Indeed, from the δ-strong convexity of the function

$$f_{k_i}(.) = \rho f(x^{k_i}, .) + h(y) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), . - x^{k_i}\rangle$$

we have

$$\langle s(x^{k_i}) - s(y^{k_i}), x^{k_i} - y^{k_i}\rangle \ge \delta \|x^{k_i} - y^{k_i}\|^2,\ \forall s(x^{k_i}) \in \partial f_{k_i}(x^{k_i}), \forall s(y^{k_i}) \in \partial f_{k_i}(y^{k_i})$$

which implies

$$\langle s(x^{k_i}), x^{k_i} - y^{k_i}\rangle \ge \langle s(y^{k_i}), x^{k_i} - y^{k_i}\rangle + \delta\|x^{k_i} - y^{k_i}\|^2.$$

Since $y^{k_i} = \arg\min\{f_{k_i}(y) : y \in C\}$, we have $0 \in \partial f_{k_i}(y^{k_i}) + N_C(y^{k_i})$ which, by necessary and sufficient optimality condition for convex programming,is equivalent to $\langle s(y^{k_i}), y - y^{k_i}\rangle \ge 0\ \forall y \in C$, in particular, $\langle s(y^{k_i}), x^{k_i} - y^{k_i}\rangle \ge 0$. Thus $\langle s(x^{k_i}), x^{k_i} - y^{k_i}\rangle \ge \delta\|x^{k_i} - y^{k_i}\|^2$, which implies

$$\|x^{k_i} - y^{k_i}\| \le \frac{1}{\sqrt{\delta}}\|s(x^{k_i})\|,\ \forall s(x^{k_i}) \in \partial f_{k_i}(x^{k_i}). \tag{3.9}$$

Since $x^{k_i} \longrightarrow \bar{x}$ by Theorem 24.5 in [20] there exists an integer number $i_0 > 0$, large enough such that

$$\partial_2 f(x^{k_i}, x^{k_i}) \subset \partial_2 f(\bar{x}, \bar{x}) + B[0; 1],\ \forall i > i_0 \tag{3.10}$$

where $B[0; 1]$ denotes the closed unit ball of $\mathbb{R}^n$.

In addition, $s(x^{k_i}) \in \partial f_{k_i}(x^{k_i}) = \rho\partial_2 f(x^{k_i}, x^{k_i})\ \forall i$ and the set $\partial_2 f(\bar{x}, \bar{x})$ is bounded, we deduce from (3.9) and (3.10) that $\{\|x^{k_i} - y^{k_i}\|\}$ is bounded. So that, combining with Lemma 3.3 we get the boundedness of $\{y^{k_i}\}$. By definition of $z^{k_i} : z^{k_i} = (1-\eta_{k_i})x^{k_i} + \eta_{k_i} y^{k_i}$ it implies that $\{z^{k_i}\}$ is also bounded. Without loss of generality we may assume that z^{k_i} converges to some $\bar{z}$. Since $w^{k_i} \in \partial_2 f(z^{k_i}, z^{k_i})$, by again Theorem 24.5 in [20] we get the boundedness of the subsequence $\{w^{k_i}\}$. □

Lemma 3.5. *If the subsequence $\{x^{k_i}\} \subset \{x^k\}$ converges to some $\bar{x}$ and*

$$\|y^{k_i} - x^{k_i}\|^4 (\frac{\eta_{k_i}}{\|w^{k_i}\|})^2 \longrightarrow 0 \ \textit{as}\ \ i \longrightarrow \infty \tag{3.11}$$

then $\bar{x} \in S_f$.

Proof. We will consider two distinct cases:

Case 1. $\text{Inf}\frac{\eta_{k_i}}{\|w^{k_i}\|} > 0$. Then by (3.11), one has $\lim_{i\longrightarrow\infty} \|y^{k_i} - x^{k_i}\| = 0$, thus $y^{k_i} \longrightarrow \bar{x}$ and $z^{k_i} \longrightarrow \bar{x}$.

From definition of y^{k_i} we have

$$f(x^{k_i}, y) + \tfrac{1}{\rho}[h(y) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y - x^{k_i}\rangle]$$

$$\ge f(x^{k_i}, y^{k_i}) + \frac{1}{\rho}[h(y^{k_i}) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y^{k_i} - x^{k_i}\rangle],\ \forall y \in C$$

by the continuity of $h, \nabla h$, we get in the limit as $i \longrightarrow \infty$ that

$$f(\bar{x}, y) + \frac{1}{\rho}[h(y) - h(\bar{x}) - \langle \nabla h(\bar{x}), y - \bar{x}\rangle] \ge 0,\ \forall y \in C$$

this fact shows that $\bar{x} \in S_f$.

Case 2. $\text{Lim}\frac{\eta_{k_i}}{\|w^{k_i}\|} = 0$. By the linesearch rule and τ-strong convexity of h we have

$$\langle w^{k_i}, x^{k_i} - y^{k_i}\rangle \geq \frac{1}{\rho}\Big[h(y^{k_i}) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y^{k_i} - x^{k_i}\rangle\Big]$$
$$\geq \frac{\tau}{\rho}\|x^{k_i} - y^{k_i}\|^2.$$

Thus $\|y^{k_i} - x^{k_i}\| \leq \sqrt{\frac{\rho}{\tau}}\|w^{k_i}\|$.

From the boundedness of $\{w^{k_i}\}$ and (3.11) it follows $\eta_{k_i} \longrightarrow 0$, so that $z^{k_i} = (1-\eta_{k_i})x^{k_i} + \eta_{k_i} y^{k_i} \longrightarrow \bar{x}$ as $i \longrightarrow \infty$. Without loss of generality, we suppose that $w^{k_i} \longrightarrow \bar{w} \in \partial_2 f(\bar{x}, \bar{x})$ and $y^{k_i} \longrightarrow \bar{y}$ as $i \longrightarrow \infty$.

We have

$$f(x^{k_i}, y) + \tfrac{1}{\rho}[h(y) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y - x^{k_i}\rangle]$$
$$\geq f(x^{k_i}, y^{k_i}) + \frac{1}{\rho}[h(y^{k_i}) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y^{k_i} - x^{k_i}\rangle],\ \forall y \in C$$

letting $i \longrightarrow \infty$, we obtain in the limit that

$$f(\bar{x}, y) + \tfrac{1}{\rho}[h(y) - h(\bar{x}) - \langle \nabla h(\bar{x}), y - \bar{x}\rangle]$$
$$\geq f(\bar{x}, \bar{y}) + \frac{1}{\rho}[h(\bar{y}) - h(\bar{x}) - \langle \nabla h(\bar{x}), \bar{y} - \bar{x}\rangle]\ \forall y \in C.$$

In the other hand, by the linesearch rule (3.1), for $m_{k_i} - 1$ there exists $w^{k_i, m_{k_i}-1} \in \partial_2 f(z^{k_i, m_{k_i}-1}, z^{k_i, m_{k_i}-1})$ such that

$$\langle w^{m_{k_i}-1}, x^{k_i} - y^{k_i}\rangle < \frac{1}{\rho}\Big[h(y^{k_i}) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y^{k_i} - x^{k_i}\rangle\Big]. \tag{3.12}$$

Letting $i \longrightarrow \infty$ and combining with $z^{k_i, m_{k_i}-1} \longrightarrow \bar{x},\ w^{k_i, m_{k_i}-1} \longrightarrow \bar{w} \in \partial_2 f(\bar{x}, \bar{x})$ we obtain in the limit from (3.12) that

$$\langle \bar{w}, \bar{x} - \bar{y}\rangle \leq \frac{1}{\rho}\Big[h(\bar{y}) - h(\bar{x}) - \langle \nabla h(\bar{x}), \bar{y} - \bar{x}\rangle\Big].$$

Note that $\bar{w} \in \partial f(\bar{x}, \bar{y})$, it follows from the last inequality that,

$$f(\bar{x}, \bar{y}) + \frac{1}{\rho}\Big[h(\bar{y}) - h(\bar{x}) - \langle \nabla h(\bar{x}), \bar{y} - \bar{x}\rangle\Big] \geq 0.$$

Hence

$$f(\bar{x}, y) + \frac{1}{\rho}\Big[h(y) - h(\bar{x}) - \langle \nabla h(\bar{x}), y - \bar{x}\rangle\Big] \geq 0,\ \forall y \in C,$$

which shows that $\bar{x} \in S_f$. □

Now we are in a position to prove the convergence of the proposed algorithm.

Theorem 3.6. *Suppose that the solution set S_f of EP(C, f) is nonempty and that the function $h(.)$, the sequence $\{\lambda_k\}$ satisfying the conditions $(B1)$, $(B2)$ respectively. Then under Assumptions $(A1)$, $(A2)$, $(A3)$, and $(A4)$, the sequence $\{x^k\}$ generated by Algorithm 1 converges to the unique solution x^* of VIEP(C, f, G).*

Proof. By Lemma 3.2 we have

$$\|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2 + \Big(\frac{\eta_k \delta}{\rho\|w^k\|}\Big)^2\|x^k - y^k\|^4 \leq -2\lambda_k\langle u^k - x^*, G(u^k)\rangle + \lambda_k^2\|G(u^k)\|^2\ \forall k. \tag{3.13}$$

From the boundedness of $\{u^k\}$ and $\{G(u^k)\}$ it implies that, there exist positive numbers A, B such that

$$|\langle u^k - x^*, G(u^k)\rangle| \leq A,\ \|G(u^k)\|^2 \leq B\ \forall k.$$

By setting $a_k = \|x^k - x^*\|^2$, and combining with the last inequalities, (3.13) becomes

$$a_{k+1} - a_k + \left(\frac{\eta_k \delta}{\rho\|w^k\|}\right)^2 \|x^k - y^k\|^4 \leq 2\lambda_k A + \lambda_k^2 B. \tag{3.14}$$

We will consider two distinct cases:

Case 1. There exists k_0 such that $\{a_k\}$ is decreasing when $k \geq k_0$.

Then there exists $\lim_{k\longrightarrow\infty} a_k = a$, taking the limit on both sides of (3.14) we get

$$\lim_{k\longrightarrow\infty} \left(\frac{\eta_k \delta}{\rho\|w^k\|}\right)^2 \|x^k - y^k\|^4 = 0. \tag{3.15}$$

In addition,

$$\begin{aligned} \|x^{k+1} - u^k\| &= \|P_C(u^k - \lambda_k G(u^k)) - P_C(u^k)\| \\ &\leq \|u^k - \lambda_k G(u^k) - u^k\| \\ &= \lambda_k \|G(u^k)\| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned} \tag{3.16}$$

From the boundedness of $\{u^k\}$ it implies that, there exists $\{u^{k_i}\} \subset \{u^k\}$ and $u^{k_i} \longrightarrow \bar{u} \in C$ such that $\liminf \langle u^k - x^*, G(x^*)\rangle = \lim_{i\longrightarrow\infty} \langle u^{k_i} - x^*, G(x^*)\rangle$.

Combining this fact with (3.15) and (3.16) we obtain

$$x^{k_i+1} \longrightarrow \bar{u} \text{ and } \left(\frac{\eta_{k_i+1}\delta}{\rho\|w^{k_i+1}\|}\right)^2 \|x^{k_i+1} - y^{k_i+1}\|^4 \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

By Lemma 3.5 we get $\bar{u} \in S_f$. Thus

$$\liminf_{k\longrightarrow\infty} \langle u^k - x^*, F(x^*)\rangle = \lim_{i\longrightarrow\infty} \langle u^{k_i} - x^*, G(x^*)\rangle = \langle \bar{u} - x^*, G(x^*)\rangle \geq 0.$$

Since F is β-strongly monotone, one has

$$\begin{aligned} \langle u^k - x^*, G(u^k)\rangle &= \langle u^k - x^*, G(u^k) - G(x^*)\rangle + \langle u^k - x^*, G(u^*)\rangle \\ &\geq \beta\|u^k - x^*\|^2 + \langle u^k - x^*, G(u^*)\rangle. \end{aligned}$$

Taking the limit as $k \longrightarrow \infty$ and remember that $a = \lim \|u^k - x^*\|^2$ we get

$$\liminf_{k\longrightarrow\infty} \langle u^k - x^*, G(u^k)\rangle \geq \beta a. \tag{3.17}$$

If $a > 0$, then by choosing $\epsilon = \frac{1}{2}\beta a$, from (3.17) it implies that, there exists $k_0 > 0$ such that

$$\langle u^k - x^*, G(u^k)\rangle \geq \frac{1}{2}\beta a, \ \forall k \geq k_0.$$

From (3.13) we get

$$a_{k+1} - a_k \leq -\lambda_k \beta a + \lambda_k^2 B, \ \forall k \geq k_0$$

and thus summing up from k_0 to k we have

$$a_{k+1} - a_{k_0} \leq -\sum_{j=k_0}^{k} \lambda_j \beta a + B \sum_{j=k_0}^{k} \lambda_j^2$$

combining this fact with $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\sum_{k=0}^{\infty} \lambda_k^2 < \infty$ we obtain

$\liminf a_k = -\infty$, which is a contradiction.

Thus we must have $a = 0$. i.e., $\lim_{k\longrightarrow\infty} \|x^k - x^*\| = 0$.

Case 2. There exists a subsequence $\{a_{k_i}\}_{i\geq 0} \subset \{a_k\}_{k\geq 0}$ such that $a_{k_i} < a_{k_i+1}$ for all $i \geq 0$. In this situation, we consider the sequence of indices $\{\sigma(k)\}$ defined as in Lemma 2.9. It follows that $a_{\sigma(k)+1} - a_{\sigma(k)} \geq 0$, which by (3.14) amounts to

$$\left(\frac{\eta_{\sigma(k)}\delta}{\rho\|w^{\sigma(k)}\|}\right)^2 \|x^{\sigma(k)} - y^{\sigma(k)}\|^4 \leq 2\lambda_{\sigma(k)} A + \lambda_{\sigma(k)}^2 B.$$

Therefore

$$\lim_{k\longrightarrow\infty}\Big(\frac{\eta_{\sigma(k)}\delta}{\rho\|w^{\sigma(k)}\|}\Big)^2\|x^{\sigma(k)}-y^{\sigma(k)}\|^4=0.$$

From the boundedness of $\{x^{\sigma(k)}\}$, without loss of generality we may assume that $x^{\sigma(k)}\longrightarrow\bar{x}$. By Lemma 3.5 we get $\bar{x}\in S_f$.

In addition, $u^{\sigma(k)}=P_{C\cap H_{\sigma(k)}}(x^{\sigma(k)})=P_{C_{\sigma(k)}}(x^{\sigma(k)})$.

Then combining with Lemma 2.6 we have

$$\|u^{\sigma(k)}-\bar{x}\|\le\|x^{\sigma(k)}-\bar{x}\|\longrightarrow 0 \text{ as } k\longrightarrow\infty$$

so that $\lim_{k\longrightarrow\infty}u^{\sigma(k)}=\bar{x}$.

By (3.13) we get

$$2\lambda_{\sigma(k)}\langle u^{\sigma(k)}-x^*,G(u^{\sigma(k)})\rangle\le a_{\sigma(k)}-a_{\sigma(k)+1}-\Big(\frac{\eta_{\sigma(k)}\delta}{\rho\|g^{\sigma(k)}\|}\Big)^2\|x^{\sigma(k)}-y^{\sigma(k)}\|^4$$
$$+\lambda^2_{\sigma(k)}\|G(u^{\sigma(k)})\|^2\le\lambda^2_{\sigma(k)}B$$

which implies

$$\langle u^{\sigma(k)}-x^*,G(u^{\sigma(k)})\rangle\le\frac{\lambda_{\sigma(k)}}{2}B. \tag{3.18}$$

Since G is β-strongly monotone, we have

$$\begin{aligned}\beta\|u^{\sigma(k)}-x^*\|^2&\le\langle u^{\sigma(k)}-x^*,G(u^{\sigma(k)})-G(x^*)\rangle\\&=\langle u^{\sigma(k)}-x^*,G(u^{\sigma(k)})\rangle-\langle u^{\sigma(k)}-x^*,G(x^*)\rangle\end{aligned}$$

which combining with (3.18) we get

$$\|u^{\sigma(k)}-x^*\|^2\le\frac{1}{\beta}\Big[\frac{\lambda_{\sigma(k)}}{2}B-\langle u^{\sigma(k)}-x^*,G(x^*)\rangle\Big]$$

so that

$$\lim_{k\longrightarrow\infty}\|u^{\sigma(k)}-x^*\|^2\le-\langle u^{\sigma(k)}-x^*,G(x^*)\rangle\le 0$$

which amounts to

$$\lim_{k\longrightarrow\infty}\|u^{\sigma(k)}-x^*\|=0. \tag{3.19}$$

In addition,

$$\begin{aligned}\|x^{\sigma(k)+1}-u^{\sigma(k)}\|&=\|P_C(u^{\sigma(k)}-\lambda_{\sigma(k)}G(u^{\sigma(k)}))-P_(u^{\sigma(k)})\|\\&\le\lambda_{\sigma(k)}\|G(u^{\sigma(k)})\|\longrightarrow 0 \text{ as } k\longrightarrow\infty\end{aligned}$$

which together with (3.19), one has $\lim_{k\longrightarrow\infty}x^{\sigma(k)+1}=x^*$, which means that $\lim_{k\longrightarrow\infty}a_{\sigma(k)+1}=0$.

By (2.3) in Lemma 2.9 we have

$$0\le a_k\le a_{\sigma(k)+1}\longrightarrow 0 \text{ as } k\longrightarrow\infty.$$

Thus $\{x^k\}$ converges to x^*. □

4. Application to Minimizing the Euclidean Norm with Pseudomomotone Equilibrium Constraints

In this section, we consider the problem:

$$\min\{\|x-x^g\|^2 : x\in S_f\}, \qquad \text{MNEP}(C,f)$$

where $x^g\in C$ is given (plays the role of a guess-solution of EP(C,f)) and S_f is the solution set of problem EP(C,f). This problem arises in the Tikhonov regularization method for pseudomonotone equilibrium problems, see, e.g., [8]. In this case, by choosing $G(x) = x - x^g$, the problem MNEP(C, f) becomes to the one in the form of VIEP(C, f, G).

It is well known that, under Assumptions $(A1)$, $(A2)$ and $(A3)$, the solution set S_f of EP(C, f) is a closed convex set. As we have mentioned that the main difficulty in problem MNEP(C, f) is that its feasible domain S_f, although is convex, it is not given explicitly as in a standard mathematical programming problem. In the sequel, we always suppose that Assumptions (A1), (A2), and (A3) are satisfied. The algorithm for this case takes the form.

Algorithm 2. Take $x^1 := x^g \in C$ and choose parameters $\rho > 0,\ \eta,\ \in (0, 1)$.
At each iteration $k = 1, 2, ..$ having x^k do the following steps:
Step 1. Solve the strongly convex program

$$\min\Big\{f(x^k, y) + \frac{1}{\rho}\Big[h(y) - h(x^k) - \langle \nabla h(x^k), y - x^k\rangle\Big] :\ y \in C\Big\} \qquad CP(x^k)$$

to obtain its unique solution y^k. If $x^k = y^k$, take $u^k := x^k$ and go to Step 4.
Step 2. Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k : \\ \langle w^{k,m}, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] \\ \text{with } \ w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}). \end{cases} \qquad (4.1)$$

Set $\eta_k := \eta^{m_k}$, $z^k := z^{k,m_k}, w^k = w^{k,m}$.
Step 3. Take $u^k := P_{C_k}(x^k)$, where

$$C_k := \{x \in C : \langle w^k, x - z^k\rangle \leq 0\}. \qquad (4.2)$$

Step 4.

$$x^{k+1} := \lambda_k x^g + (1 - \lambda_k)u^k \qquad (4.3)$$

Repeat iteration k with k is replaced by $k + 1$.

Similar to Theorem 3.1, we have the following theorem

Theorem 4.1. *Under Assumptions $(A1)$ $(A2)$, $(A3)$, and $(B1)$, $(B2)$, the sequence $\{x^k\}$ generated by Algorithm 2 converges to the unique solution x^* of MNEP(C, f).*

Conclusion. We have proposed an explicit hybrid extragradient algorithm for solving the variational inequality problems with equilibrium problems constraint, where the bifunction is pseudomonotone with respect to its solution set. The convergence of the algorithm is obtained, and a special case of this problem is considered.

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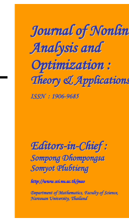

ON SOFT FIXED POINT OF PICARD-MANN HYBRID ITERATIVE SEQUENCES IN SOFT NORMED LINEAR SPACES

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ABSTRACT. In the present paper, we contribute to the development of soft set theory by introducing soft contractive-like operators and soft Picard-Mann hybrid iterative sequences. We then show that the soft Picard-Mann hybrid iterative sequences converges strongly to the unique soft fixed point for the class of soft contractive-like operators. Our results are generalization and improvement of several results on iterative schemes in literature.

KEYWORDS : soft unique fixed point; soft contractive-like operators; soft Picard-Mann hybrid iterative sequences; soft normed linear spaces.
AMS Subject Classification: 47H10, 54H25

1. INTRODUCTION

Mathematical tools have been used in the study of behaviour of different parts of systems and their subsystems. This behaviour are either usually certain or uncertain in nature. In 1999, Molodtsov [16] introduced a new concept called soft set as a mathematical tool for dealing with uncertainties arising in problems in different areas of mathematical sciences. Chief among them are problems in computer science, economics, engineering, medical sciences, and physics. He argued that soft set provides better tool for handling uncertainty than fuzzy set because of its non-restrictive parametrization and is easily applicable to real life problems.

The concept of soft topology on soft set was initiated by Cagman et al. in [6] and some important properties of soft topological spaces were considered. In 2012, Das and Samanta [7] introduced the concept of real soft set and soft real number and explained their properties. In 2013, Das and Samanta [8] also introduced the concept of soft metric using the notion in [7], they hence proved that each soft metric space is a toplological space. Wardowski [22], introduced a new notion of soft element of a soft set and establish its natural relation with soft operations and

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soft objects in soft topological spaces. They defined in a different way than in the literature, a soft mapping transforming a soft set into a soft set and provided basic properties of such mappings using the notion of soft element. They obtained the natural first fixed point results in the soft set theory using the new approach to soft mappings. Abbas et al. [1] in 2015, initiated their notion of soft contraction mapping based on the theory of soft elements of soft metric spaces and proved interesting results on fixed point of such mappings including soft Banach contraction principle.

Fixed point iterative sequences are designed to be applied in solving equations arising in physical formulation but there is no systematic study of numerical aspects of these iterative sequences. The reader can see [3, 4, 21] and other literature for contributions to research on numerical iterative schemes for approximating fixed points. Here, we shall employ the concepts of [2] and [22] and prove soft fixed point results for soft Picard-Mann hybrid iterative sequences using a soft norm version of contractive-like operators. Numerical examples will also be presented to back up our results.

We will now consider some of these schemes as they are relevant to this work. Let (X, d) be a metric space and $T : X \to X$ be a self map of X. Assume that $F_T = \{p \in X : T_p = p\}$ is the set of fixed points of T. For $x_0 \in X$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n \geq 0, \tag{1.1}$$

is called the Picard iterative scheme [21].

Let $(E, ||.||)$ be a real normed linear space and $T : E \longrightarrow E$ a self map of E. For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ is called the Mann iterative scheme [15].

If $\alpha_n = 1$ in (1.2), we have the Picard iterative scheme (1.1).

Rhoades [18, 20] perhaps for the first time used computer programs to compare the rate of convergence Mann and Ishikawa iterative procedures. He illustrated the difference in the rate of convergence for increasing and decreasing functions through examples.

These various results are worth emulating. In 2013, Khan [11], gave a different perspective to iteration procedure, he introduced the following Picard-Mann hybrid iterative scheme for a single nonexpansive mapping T. For any initial point $x_0 \in E$ the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$\begin{aligned} x_{n+1} &= Ty_n \\ y_n &= (1 - \alpha_n)x_n + \alpha_n Tx_n, n \geq 0, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1].

He showed that the hybrid scheme (Picard-Mann scheme (1.3)) converges faster than all of Picard (1.1), Mann (1.2) and Ishikawa [13] iterative schemes in the sense of Berinde [5] for contractions. He also proved strong convergence and weak convergence theorems with the help of his iterative process (1.3)for the class of nonexpansive mappings in general Banach spaces and applied it to obtain results in uniformly convex Banach spaces. Motivated by the work of Khan [11], we prove strong convergence of Picard-Mann iterative scheme for a general class of operators in a real normed space.

Osilike [17] proved several stability results which are generalizations and extensions of most of the results of Rhoades [19] using the following contractive definition: for each $x, y \in X$, there exist $a \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq ad(x, y) + Ld(x, Tx). \tag{1.4}$$

In 2003, Imoru and Olatinwo [12] proved some stability results using the following general contractive definition : for each $x, y \in X$, there exist $\delta \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + \varphi(d(x, Tx)). \tag{1.5}$$

Definition 1.1. [16] Let U be an universe and E be a set of parameters. Let $P(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\epsilon \in A$, $F(\epsilon)$ may be consider as the set of ϵ-approximate element of the soft set (F, A).

Definition 1.2. [10] For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(i) $A \subseteq B$
(ii) for all $\epsilon \in A$, $F(\epsilon) \subseteq G(\epsilon)$. We write $(F, A)\tilde{\subset}(G, B)$. (F, A) is said to be a soft superset of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A)\tilde{\supset}(G, B)$.

Definition 1.3. [9] Two soft sets (F, A) and (G, B) over a common universe U are said to be equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 1.4. [9] The complement of a soft set (F, A) is denoted by $(F, A)^c = (F^c, A)$, where $F^c : A \to P(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, for all $\alpha \in A$.

Definition 1.5. [14] A soft set (F, E) over U is said to be an absolute soft set denoted by $\check{U}$ if $\forall \epsilon \in E$, $F(\epsilon) = U$.

Definition 1.6. [14] A soft set (F, E) over U is said to be a null soft set denoted by ϕ if $\forall \epsilon \in E$,$F(\epsilon) = \emptyset$.

Definition 1.7. [7] Let X be a non-empty set and E be a non-empty parameter set. Then a function $\epsilon : E \to X$ is said to be a soft element of X. A soft element ϵ of X is said to belongs to a soft set A of X, which is denoted by $\epsilon\tilde{\in}A$, if $\epsilon(e) \leq A(e)$, $\forall e \in E$. Thus for a soft set A of X with respect to the index set E, we have $A(e) = \epsilon(e), \epsilon\tilde{\in}A, e \in E$.

It is to be noted that every singleton soft set (a soft set (F, E) for which $F(e)$ is a singleton set, $\forall e \in E$) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall e \in E$.

Definition 1.8. [7] Let R be the set of real numbers and $B(R)$ the collection of all non-empty bounded subsets of R and A taken as a set of parameters. Then a mapping $F : A \to B(R)$ is called a soft real set. It is denoted by (F, A). If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a soft real number. We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\bar{r}$, $\bar{s}$, $\bar{t}$ will denote a particular type of soft real numbers such that $\bar{r}(\lambda) = r, \forall \lambda \in A$. For instance, $\bar{0}$ is the soft real number where $\bar{0}(\lambda) = 0, \forall \lambda \in A$.

Definition 1.9. [8] Let U be a universe, A be a non-empty subset of parameters and $\check{U}$ an absolute soft set, i.e $F(\epsilon) = U$ for all $\epsilon \in A$, where $(F, A) = \check{U}$. Let $SP(\tilde{U})$ be any nonempty set of soft elements of a soft set (F, A) and $R(A)^*$ be a set of all soft real sets. A mapping $d : SP(\tilde{U}) \times SP(\tilde{U}) \to R(A)^*$ is said to be a soft metric on the soft set $\tilde{U}$ if d satisfies the following axioms:
$(M1)$. $d(\tilde{x}, \tilde{y}) \tilde{\geq} \bar{0}, \forall \tilde{x}, \tilde{y} \in \tilde{U}$.
$(M2)$. $d(\tilde{x}, \tilde{y}) = \bar{0} \iff \tilde{x} = \tilde{y}$.
$(M3)$. $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x}), \forall \tilde{x}, \tilde{y} \in \tilde{U}$.
$(M4)$. $d(\tilde{x}, \tilde{y}) \tilde{\leq} d(\tilde{x}, \tilde{z}) + d(\tilde{z}, \tilde{y}), \forall \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{U}$.

The soft set $\tilde{U}$ endowed with the soft metric d is called a soft metric space and is denoted by $(\tilde{U}, d, A)$ or $(\tilde{U}, d)$. $(M1), (M2), (M3)$ and $(M4)$ are said to be soft metric axioms.

Definition 1.10. [8] Let $\{\tilde{x}_n\}$ be a sequence of soft elements in a soft metric space $(\tilde{U}, d)$. The sequence $\{\tilde{x}_n\}$ is said to be convergent in $(\tilde{U}, d)$ if there is a soft element $\tilde{x} \in \tilde{U}$ such that $d(\tilde{x}_n, \tilde{x}) \to \bar{0}$ as $n \to \infty$. This means for every $\tilde{\epsilon} \tilde{>} \bar{0}$ chosen arbitrarily, there exists a natural number $N = N(\tilde{\epsilon})$, such that $\bar{0} \tilde{<} d(\tilde{x}_n, \tilde{x}) \tilde{<} \tilde{\epsilon}$ whenever $n > N$. We denote this by

$$\lim_{n \to \infty} \tilde{x}_n = \tilde{x}.$$

Proposition 1.11. [8] *The limit of a sequence $\{\tilde{x}_n\}$ in a soft metric space $(\tilde{U}, d)$, if it exists is unique.*

Definition 1.12. [8] A sequence $\{\tilde{x}_n\}$ of soft point in a soft metric space $(\tilde{U}, d)$ is said to be a Cauchy sequence in $(\tilde{U}, d)$ if for each $\tilde{\epsilon} \tilde{>} \bar{0}$, there exists an $m \in N$ such that $d(\tilde{x}_i, \tilde{x}_j) \tilde{<} \tilde{\epsilon}$ for all $i, j \geq m$. That is $d(\tilde{x}_i, \tilde{x}_j) \to \bar{0}$ as $i, j \to \infty$.

Proposition 1.13. [8] *Every convergent sequence $\{\tilde{x}_n\}$ in a soft metric space $(\tilde{U}, d)$ is a Cauchy sequence.*

Definition 1.14. [8] A soft metric space $(\tilde{U}, d)$ is called complete if every Cauchy sequence in it converges to some soft point of $\tilde{U}$.

Definition 1.15. [9] Let V be a vector or linear space over a field K and A a set of parameters. A soft set (F, A) where $F : A \to P(V)$ is called a soft vector or linear space over V. It is denoted by $\tilde{V}$.

Definition 1.16. [9] Let V be a vector or linear space over a field K and A a set of parameters. Let G be a soft set over V. Now G is said to be a soft vector space or a soft linear space of V over K if $G(\lambda)$ is vector or linear subspace of V for every $\lambda \in A$.

Definition 1.17. [9] Let N be the absolute soft Linear Space, i.e. $\check{N}(\lambda) = N$ for every $\lambda \in A$ and $SE(\check{N})$ be any nonempty set of soft elements of absolute soft Linear Space and $R(A)^*$ be a set of all soft real sets. Then a mapping $\|.\| : SE(\check{N}) \to R(A)^*$ is said to be a soft norm on the soft vector space $\check{N}$ if $\|.\|$ satisfies the following conditions: For all $\tilde{x}, \tilde{y} \in \check{N}$,

N1. $\|\tilde{x}\| \geq 0$,
N2. $\|\tilde{x}\| = 0 \iff x = 0$,
N3. $\|\tilde{\alpha}\tilde{x}\| = |\tilde{\alpha}|\|\tilde{x}\|$ for every soft scalar $\tilde{\alpha}$,

N4. $\|\tilde{x}+\tilde{y}\|\tilde{\leq}\|\tilde{x}\|+\|\tilde{y}\|$.

The soft vector space $\check{N}$ with the soft norm $\|.\|$ on it is called a soft normed linear space and denoted by $(\check{N},\|.\|,A)$ or $(\check{N},\|.\|)$. A complete soft normed linear space is a soft Banach space.

Theorem 1.18. [1] *Let $(\check{N},\|.\|,A)$ be a soft Banach space with a finite set A. Suppose the soft mapping $T:\check{N}\to\check{N}$ satisfies:*

$$\|T(\tilde{x})-T(\tilde{y})\|\leq\tilde{a}\|\tilde{x}-\tilde{y}\|,$$

for all $\tilde{x},\tilde{y}\in SP(\check{N})$ where $\bar{0}\leq\tilde{a}<\bar{1}$. Then T has a unique fixed point.

Theorem 1.19. [2] *Let $(\tilde{U},d,A)$ be complete soft metric space with a finite set A. Suppose the soft mapping $f:\tilde{U}\to\tilde{U}$ satisfies*

$$d(f(\tilde{x}),f(\tilde{y}))\leq cd(\tilde{x},\tilde{y}),$$

for all $\tilde{x},\tilde{y}\in SP(\tilde{U})$ where $\bar{0}\tilde{\leq}\tilde{c}\tilde{<}\bar{1}$. then f has a unique fixed point, that is, there exists a unique soft point $\tilde{x}$ such that $f(\tilde{x})=\tilde{x}$.

Theorem 1.20. [2] *Let $(\tilde{U},d,A)$ be complete soft metric space with a finite set A, suppose the soft mapping $f:\tilde{U}\to\tilde{U}$ satisfies*

$$d(f(\tilde{x}),f(\tilde{y}))\leq c[d(\tilde{x},f(\tilde{x}))+d(\tilde{y},f(\tilde{y}))]$$

for all $\tilde{x},\tilde{y}\in SP(\tilde{U})$ where $\bar{0}\tilde{\leq}\tilde{c}\tilde{<}\frac{\bar{1}}{2}$. then f has a unique fixed point.

Definition 1.21. [9] Let $(\tilde{N},\|.\|,A)$ be a soft normed linear space and $\tilde{r}\tilde{>}\bar{0}$ be soft real numbers. Then $B(\tilde{x},\tilde{r})$, $\bar{B}(\tilde{x},\tilde{r})$ and $S(\tilde{x},\tilde{r})$ are called soft open ball, soft closed ball and soft sphere respectively, where
(a) $B(\tilde{x},\tilde{r})=\{\tilde{y}\tilde{\in}\tilde{N}:\|\tilde{x}-\tilde{y}\|\tilde{<}\tilde{r}\}\subset SE(\tilde{N})$,
(b) $\bar{B}(\tilde{x},\tilde{r})=\{\tilde{y}\tilde{\in}\tilde{N}:\|\tilde{x}-\tilde{y}\|\tilde{\leq}\tilde{r}\}\subset SE(\tilde{N})$,
(c) $\tilde{S}(\tilde{x},\tilde{r})=\{\tilde{y}\tilde{\in}\tilde{N}:\|\tilde{x}-\tilde{y}\|\tilde{=}\tilde{r}\}\subset SE(\tilde{N})$.

Definition 1.22. [9] A sequence of soft element $\{\tilde{x}_n\}$ in a soft normed linear space $(\tilde{N},\|.\|,A)$ converges to a soft element $\tilde{x}$ if $\|\tilde{x}_n-\tilde{x}\|\to\bar{0}$ as $n\to\infty$. This means for every $\tilde{\epsilon}\tilde{>}\bar{0}$ chosen arbitrarily, there exists a natural number $N=N(\tilde{\epsilon})$, such that $\bar{0}\tilde{\leq}\|\tilde{x}_n-\tilde{x}\|\tilde{<}\tilde{\epsilon}$, whenever $n>N$. i.e $n>N$ implies $\tilde{x}_n\tilde{\in}B(\tilde{x},\tilde{\epsilon})$. $\tilde{x}$ is said to be the limit of the sequence $\{\tilde{x}_n\}$ as $n\to\infty$.

Example 1.23. [9] Let's consider the set R of all real number endowed with the usual norm $\|.\|$ and $(R,\|.\|,A)$ a soft normed space generated by the crisp norm $\|.\|$ where A is a non-empty set of parameters. Let $(Y,A)\subset R$ such that $Y(\lambda)=(0,1]$ in a real line, $\forall\lambda\in A$. Let's choose a sequence $\{\tilde{x}_n\}$ of soft element of (Y,A) where $\tilde{x}_n(\lambda)=\frac{1}{n},\forall n\in N,\lambda\in A$. Then there is a number $\tilde{x}\in(Y,A)$ such that $\tilde{x}_n\to\tilde{x}$ in $(Y,\|.\|,A)$. However, the sequence $\{\tilde{y}_n\}$ of soft element of (Y,A) where $\tilde{y}_n(\lambda)=\frac{1}{2},\forall n\in N,\lambda\in A$ is convergent in $(Y,\|.\|,A)$ and converges to $\frac{\bar{1}}{2}$.

Proposition 1.24. [9] *The limit of a sequence $\{\tilde{x}_n\}$ in a soft normed linear space, if it exists is unique.*

Definition 1.25. [9] A sequence $\{\tilde{x}_n\}$ of a soft element in a soft normed linear space $(\tilde{N},\|.\|)$ is said to be bounded if the set $\{\|\tilde{x}_n-\tilde{x}_m\|:n,m\in N\}$ of real numbers is bounded. i.e. if there exists $\tilde{M}\tilde{\geq}\bar{0}$ such that

$$\|\tilde{x}_n-\tilde{x}_m\|\tilde{\leq}\tilde{M}\ \ \forall n,m\in N.$$

Definition 1.26. [9] A sequence $\{\tilde{x}_n\}$ of soft element in a soft normed linear space $(\tilde{N}, \|.\|, A)$ is said to be a Cauchy sequence in $\tilde{N}$ if for every $\tilde{\epsilon}\tilde{>}\bar{0}$, there exists an $m \in N$ such that $\|\tilde{x}_i - \tilde{x}_j\|\tilde{<}\tilde{\epsilon}$ for all $i, j \geq m$. That is $\|\tilde{x}_i - \tilde{x}_j\| \to \bar{0}$ as $i, j \to \infty$.

Proposition 1.27. [9] *Every convergent sequence $\{\tilde{x}_n\}$ in a soft normed linear space is Cauchy and every Cauchy sequence is bounded.*

Definition 1.28. [9] A soft subset (Y, A) with $Y(\lambda)\neq\emptyset, \ \forall\lambda \in A$ in a soft normed linear space $(\tilde{N}\|.\|, A)$ is said to be bounded if there exists a soft real number $\bar{k}$ such that $\|\tilde{x}\|\tilde{\leq}\bar{k}, \ \forall\tilde{x}\tilde{\in}(Y, A)$.

Definition 1.29. [9] A soft normed linear space $(\tilde{N}, \|.\|, A)$ is called complete if every Cauchy sequence in it converges to a soft element of $\tilde{N}$.

Definition 1.30. [9] Let $(\tilde{N}, \|.\|, A)$ be a soft normed linear space. Then
(i) If $\tilde{x}_n \to \tilde{x}$ and $\tilde{y}_n \to \tilde{y}$, then $\tilde{x}_n + \tilde{y}_n \to \tilde{x} + \tilde{y}$
(ii) If $\tilde{x}_n \to \tilde{x}$ and $\tilde{\lambda}_n \to \tilde{\lambda}$, then $\tilde{\lambda}_n\tilde{x}_n \to \tilde{\lambda}\tilde{x}$
(iii) If $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ are Cauchy sequences in $\tilde{N}$ and $\{\tilde{\lambda}_n\}$ is a Cauchy sequence of soft scalars, then $\{\tilde{x}_n + \tilde{y}_n\}$ and $\{\tilde{x}_n + \tilde{\lambda}_n\}$ are also Cauchy sequences in $\tilde{N}$.

Definition 1.31. Let $(\tilde{N}, \|.\|, A)$ be a soft complete normed linear space with a finite set A. Suppose the soft mapping $f : \check{N} \to \check{N}$ satisfies

$$\|T(\tilde{x}) - T(\tilde{y})\| \leq \bar{b}[\|\tilde{x} - T(\tilde{x})\| + \|\tilde{y} - T(\tilde{y})\|]$$

for all $\tilde{x}, \tilde{y} \in SP(\check{N})$ where $\bar{0}\tilde{\leq}\bar{b}\tilde{<}\frac{\bar{1}}{2}$. then T has a unique fixed soft point. This is called soft Kannan contractive mapping.

Definition 1.32. Let $(\tilde{N}, \|.\|, A)$ be a soft complete normed linear space with a finite set A. Suppose the soft mapping $T : \check{N} \to \check{N}$ satisfies

$$\|T(\tilde{x}) - T(\tilde{y})\| \leq \bar{c}[\|\tilde{x} - T(\tilde{y})\| + \|\tilde{y} - T(\tilde{x})\|]$$

for all $\tilde{x}, \tilde{y} \in SP(\check{N})$ where $\bar{0}\tilde{\leq}\bar{c}\tilde{<}\frac{\bar{1}}{2}$. then T has a unique fixed soft point. This is called soft Chaterjea contractive mapping.

Proposition 1.33. *Let $(\tilde{N}, \|.\|, A)$ be a soft complete normed linear space with a finite set A . Suppose the soft mapping $f : \check{N} \to \check{N}$ satisfies:*
(SZ_1). $\|T(\tilde{x}) - T(\tilde{y})\| \leq \tilde{a}\|\tilde{x} - \tilde{y}\|$,
(SZ_2). $\|T(\tilde{x}) - T(\tilde{y})\| \leq \tilde{b}[\|\tilde{x} - T(\tilde{x})\| + \|\tilde{y} - T(\tilde{y})\|]$,
(SZ_3). $\|T(\tilde{x}) - T(\tilde{y})\| \leq \tilde{c}[\|\tilde{x} - T(\tilde{y})\| + \|\tilde{y} - T(\tilde{x})\|]$, *for all $\tilde{x}, \tilde{y} \in SP(\check{N})$ where $\bar{0}\tilde{\leq}\tilde{a}\tilde{<}\bar{1}$, $\bar{0}\tilde{\leq}\tilde{b}\tilde{<}\frac{\tilde{1}}{2}$ and $\bar{0}\tilde{\leq}\tilde{c}\tilde{<}\frac{\tilde{1}}{2}$. Then T has a unique fixed soft point if at least one of the conditions above is true. This is called soft Zamfirescu contractive mapping. It is the soft space version of the contractive mapping of Zamfirescu [23] in literature.*

We will now show that every soft Zamfirescu operator T satisfies the inequalities: $\|T(\tilde{x}) - T(\tilde{y})\| \leq \bar{\delta}\|\tilde{x} - \tilde{y}\| + 2\delta\|\tilde{x} - T(\tilde{x})\|$, where $\tilde{\delta} = max\{\tilde{a}, \frac{\tilde{b}}{1-\tilde{b}}, \frac{\tilde{c}}{1-\tilde{c}}\}\tilde{<}\tilde{1}$.

Consider (SZ_1):
$\|T(\tilde{x}) - T(\tilde{y})\| \leq \tilde{a}\|\tilde{x} - \tilde{y}\|$.

Consider (SZ_2):

$$\begin{aligned}
\|T(\tilde{x}) - T(\tilde{y})\| &\leq \tilde{b}[\|\tilde{x} - T(\tilde{x})\| + \|\tilde{y} - T(\tilde{y})\|] \\
&\leq \tilde{b}[\|\tilde{x} - T(\tilde{x})\| + \|\tilde{y} - \tilde{x} + \tilde{x} - T(\tilde{x}) + T(\tilde{x}) - T(\tilde{y})\|] \\
&\leq 2\tilde{b}\|\tilde{x} - T(\tilde{x})\| + \tilde{b}\|\tilde{x} - \tilde{y}\| + \tilde{b}\|T(\tilde{x}) - T(\tilde{y})\|
\end{aligned}$$

$$\leq \quad \frac{\tilde{b}}{1-\tilde{b}}\|\tilde{x}-\tilde{y}\| + \frac{\tilde{2}\tilde{b}}{1-\tilde{b}}\|\tilde{x}-T(\tilde{x})\|.$$

Consider (SZ_3):

$$\begin{aligned}
\|T(\tilde{x})-T(\tilde{y})\| &\leq \tilde{c}[\|\tilde{x}-T(\tilde{y})\| + \|\tilde{y}-T(\tilde{x})\|] \\
&\leq \tilde{c}[\|\tilde{x}-T(\tilde{x})+T(\tilde{x})-T(\tilde{y})\| + \|\tilde{y}-\tilde{x}+\tilde{x}-T(\tilde{x})\|] \\
&\leq 2\tilde{c}\|\tilde{x}-T(\tilde{x})\| + \tilde{c}\|\tilde{x}-\tilde{y}\| + \tilde{c}\|T(\tilde{x})-T(\tilde{y})\| \\
&\leq \frac{\tilde{c}}{1-\tilde{c}}\|\tilde{x}-\tilde{y}\| + \frac{\tilde{2}\tilde{c}}{1-\tilde{c}}\|\tilde{x}-T(\tilde{x})\|.
\end{aligned}$$

Denote $\tilde{\delta} = max\{\tilde{a}, \frac{\tilde{b}}{1-\tilde{b}}, \frac{\tilde{c}}{1-\tilde{c}}\}$. By (SZ_1), (SZ_2) and (SZ_3), we get

$$\|T(\tilde{x})-T(\tilde{y})\| \leq \bar{\delta}\|\tilde{x}-\tilde{y}\| + 2\delta\|\tilde{x}-T(\tilde{x})\|,$$

where $\bar{0}\tilde{\leq}\tilde{\delta}\tilde{<}\bar{1}$. If $\tilde{L} = 2\tilde{\delta}$, we obtain:

$$\|T(\tilde{x})-T(\tilde{y})\| \leq \bar{\delta}\|\tilde{x}-\tilde{y}\| + \tilde{L}\|\tilde{x}-T(\tilde{x})\|.$$

Suppose $\tilde{\varphi}(t) = \tilde{L}t$, we get

$$\|T(\tilde{x})-T(\tilde{y})\| \leq \bar{\delta}\|\tilde{x}-\tilde{y}\| + \tilde{\varphi}(\|\tilde{x}-T(\tilde{x})\|),$$

where $\tilde{\varphi} : R(A)^* \to R(A)^*$ is a monotone increasing function with $\tilde{\varphi}(\bar{0}) = \bar{0}$. This ends the proof.

We now consider some iterative schemes in a soft normed linear space.
Let $(\tilde{N}, \|.\|, A)$ be a soft normed linear space with A, a finite set and $f : \check{N} \to \check{N}$ a soft self mapping of $\check{N}$. Define $F_T = \{\tilde{q}\tilde{\in}\check{N} : T\tilde{q} = \tilde{q}\}$ to be the set of fixed point of T. For $\tilde{x}_0\tilde{\in}\check{N}$, the sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ defined by

$$\tilde{x}_{n+1} = T\tilde{x}_n, \tag{1.6}$$

$n\tilde{\geq}\bar{0}$ is called the soft Picard iterative scheme.

For $\tilde{x}_0\tilde{\in}\check{N}$, the sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ defined by

$$\tilde{x}_{n+1} = (1-\tilde{\alpha}_n)\tilde{x}_n + \tilde{\alpha}_n T\tilde{x}_n, \tag{1.7}$$

$n\tilde{\geq}\bar{0}$, where $\{\tilde{\alpha}_n\}_{n=1}^{\infty}$ is a soft real sequence in $[\bar{0}, \bar{1}]$ is called the soft Mann iterative scheme.

For $\tilde{x}_0\tilde{\in}\check{N}$, the sequence $\{\tilde{x}_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned}
\tilde{x}_{n+1} &= T\tilde{y}_n, \\
\tilde{y}_n &= (1-\tilde{\alpha}_n)\tilde{x}_n + \tilde{\alpha}_n T\tilde{x}_n,
\end{aligned} \tag{1.8}$$

$n\tilde{\geq}\bar{0}$, where $\{\tilde{\alpha}_n\}_{n=1}^{\infty}$ is a soft real sequence in $[\bar{0}, \bar{1}]$ is called the soft Picard-Mann hybrid iterative scheme.

We shall need the following lemma in proving our result.

Lemma 1.34. [5] *Let δ be a real number satisfying $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ a sequence of positive numbers such that $lim_{n\to\infty}\epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying $u_{n+1} \leq \delta u_n + \epsilon_n$, n=0,1,2,..., we have $lim_{n\to\infty}u_n = 0$.*

2. MAIN RESULTS

Theorem 2.1. *Let $(\tilde{N}, \|.\|, A)$ be a soft normed linear space with a finite set A and $T : \tilde{N} \to \tilde{N}$ be a soft self mapping satisfying the soft contractive-like condition*

$$\|T\tilde{x} - T\tilde{y}\| \leq \tilde{\delta}\|\tilde{x} - \tilde{y}\| + \tilde{\varphi}(\|\tilde{x} - T\tilde{x}\|), \tag{2.1}$$

for each $\tilde{x}, \tilde{y} \in SP(\tilde{N})$, $\bar{0}\tilde{\leq}\tilde{\delta} < \bar{1}$ and $\tilde{\varphi}$ is a monotone increasing function with $\tilde{\varphi}(\bar{0}) = \bar{0}$. For arbitrary $\tilde{x}_0\tilde{\in}\tilde{N}$, let $\{\tilde{x}_n\}_{n=0}^{\infty}$ be the soft Picard-Mann hybrid iterative scheme defined by (1.8), where $\{\tilde{\alpha}_n\}_{n=0}^{\infty}$ is a soft real sequence in $[\bar{0}, \bar{1}]$. Then
(i) T defined by (2.1) has a unique soft fixed point $\tilde{q}$;
(ii) the soft Picard-Mann hybrid iterative scheme (1.8) converges strongly to $\tilde{q}$ of T.

Proof. We shall first show that T has a unique fixed point.
Suppose $\tilde{q}_1, \tilde{q}_2\tilde{\in}\tilde{F}_T$ such that $\tilde{q}_1\tilde{\neq}\tilde{q}_2$

$$\begin{aligned}\|\tilde{q}_1 - \tilde{q}_2\| &= \|T\tilde{q}_1 - T\tilde{q}_2\| \\ &\leq \tilde{\delta}\|\tilde{q}_1 - \tilde{q}_2\| + \tilde{\varphi}(\|\tilde{q}_1 - T\tilde{q}_1\|) \\ &\leq \tilde{\delta}\|\tilde{q}_1 - \tilde{q}_2\| + \tilde{\varphi}(0) \\ &\leq \tilde{\delta}\|\tilde{q}_1 - \tilde{q}_2\|.\end{aligned}$$

Thus, $(1 - \tilde{\delta})\|\tilde{q}_1 - \tilde{q}_2\| \leq \bar{0}$, which implies $\|\tilde{q}_1 - \tilde{q}_2\| \leq \bar{0}$.
That is, $\tilde{q}_1 = \tilde{q}_2$. Thus, T has a unique fixed point $\tilde{q}$.

Next, we prove that $\lim_{n\to\infty} \tilde{x}_n = \tilde{q}$. That is, we show that the soft Picard-Mann hybrid iterative sequence converges strongly to $\tilde{q}$ of T.

$$\begin{aligned}\|\tilde{x}_{n+1} - \tilde{q}\| &= \|T\tilde{y}_n - T\tilde{q}\| \\ &\leq \tilde{\delta}\|\tilde{y}_n - \tilde{q}\| + \tilde{\varphi}(\|\tilde{q} - T\tilde{q}\|) \\ &\leq \tilde{\delta}\|\tilde{y}_n - \tilde{q}\| \\ &\leq \tilde{\delta}[(1 - \tilde{\alpha}_n)\|\tilde{x}_n - \tilde{q}\| + \tilde{\alpha}_n\|T\tilde{x}_n - \tilde{q}\|] \\ &\leq \tilde{\delta}[(1 - \tilde{\alpha}_n)\|\tilde{x}_n - \tilde{q}\| + \tilde{\delta}\tilde{\alpha}_n\|\tilde{x}_n - \tilde{q}\|] \\ &\leq \tilde{\delta}[1 - \tilde{\alpha}_n(\bar{1} - \tilde{\delta})]\|\tilde{x}_n - \tilde{q}\|.\end{aligned}$$

Hence, $\lim_{n\to\infty} \|\tilde{x}_n - \tilde{q}\| = \bar{0}$.
Since, $\tilde{\delta}[1 - \tilde{\alpha}_n(\bar{1} - \tilde{\delta})] \to \bar{0}$ as $n \to \infty$. Therefore $\{\tilde{x}_n\}_{n=0}^{\infty}$ converges strongly to a soft fixed point $\tilde{q}$. □

Theorem 2.2. *Let $(\tilde{N}, \|.\|, A)$ be a soft normed linear space with a finite set, A and $T : \tilde{N} \to \tilde{N}$ be a soft self mapping satisfying the soft contractive-like condition*

$$\|T\tilde{x} - T\tilde{y}\| \leq \tilde{\delta}\|\tilde{x} - \tilde{y}\| + \tilde{\varphi}(\|\tilde{x} - T\tilde{x}\|), \tag{2.2}$$

for each $\tilde{x}, \tilde{y} \in SP(\tilde{N})$, $\bar{0}\tilde{\leq}\tilde{\delta} < \bar{1}$. For arbitrary $\tilde{x}_0\tilde{\in}\tilde{N}$, let $\{\tilde{x}_n\}_{n=0}^{\infty}$ be the soft Mann iterative scheme defined by (1.7), where $\{\tilde{\alpha}_n\}_{n=0}^{\infty}$ is a soft real sequence in $[\bar{0}, \bar{1}]$. Then
(i) T defined by (2.2) has a unique soft fixed point $\tilde{q}$;
(ii) the soft Mann iterative scheme (1.7) converges strongly to $\tilde{q}$ of T.

Proof. The proof is similar to that of Theorem 2.1. □

Theorem 2.2 leads to the following corollary:

Corollary 2.3. *Let $(\tilde{N}, \|.\|, A)$ be a soft normed linear space with a finite set A and $T : \tilde{N} \to \tilde{N}$ be a soft self mapping satisfying the soft contractive-like condition*

$$\|T\tilde{x} - T\tilde{y}\| \leq \tilde{\delta}\|\tilde{x} - \tilde{y}\| + \tilde{\varphi}(\|\tilde{x} - T\tilde{x}\|), \tag{2.3}$$

for each $\tilde{x}, \tilde{y} \in SP(\tilde{N})$, $\bar{0}\tilde{\leq}\tilde{\delta} < \bar{1}$. For arbitrary $\tilde{x}_0\tilde{\in}\tilde{N}$, let $\{\tilde{x}_n\}_{n=0}^{\infty}$ be the soft Picard iterative scheme defined by (1.6). Then
(i) T defined by (2.3) has a unique soft fixed point $\tilde{q}$;
(ii) the soft Picard iterative scheme (1.6) converges strongly to $\tilde{q}$ of T.

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Competing Interest
The authors declare that there are no competing interest.

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