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Contact

Narin Petrot (narinp@nu.ac.th)
Center of Excellence in Nonlinear Analysis and Optimization,
Department of Mathematics, Faculty of Science,
Naresuan University, Phitsanulok, 65000, Thailand.

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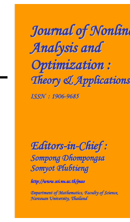

OPTIMALITY CONDITIONS FOR WEAKLY EFFICIENT SOLUTION OF VECTOR EQUILIBRIUM PROBLEM WITH CONSTRAINTS IN TERMS OF SECOND-ORDER CONTINGENT DERIVATIVES

TRAN VAN SU*

Department of Mathematics, Quangnam University, Tamky, Vietnam

ABSTRACT. In this paper, we present second-order necessary and sufficient optimality conditions for weakly efficient solution of a vector equilibrium problem with constraints (in short, VEPC) in terms of second-order contingent derivative and second-order asymptotic contingent derivative. With this purpose, we impose the objective functions, either all them are twice Fréchet differentiable at optimal point or the Fréchet derivatives are calm at optimal point or the profile mappings has the cone-Aubin properties. Besides, we also can invoke constraint qualifications of the Kurcyusz - Robinson - Zowe (KRZ) type. Our paper point out new improvements from the known results of Gutierrez, Jiménez and Novo (2010) and Khanh and Tung (2015); see [8], [10] in cases of single-valued optimization and give some discusses about it.

KEYWORDS : Second-order optimality conditions; Second-order contingent derivatives; Kurcyusz-Robinson-Zowe type constraint qualification; Weakly efficient solutions.
AMS Subject Classification: 90C29, 90C46, 90C48, 49J52

1. INTRODUCTION

The vector equilibrium problems generalize many well-known problems in the optimization theory as vector complementarity problems, vector saddle point problems, vector optimization problems and vector variational inequality problems. On second-order optimality conditions involving the above problems have been widely investigated by many researchers, see, for instance, Aubin and Frankowska [1]; Jiménez and Novo [2], [6], [7]; Guerraggio et al. [3], [4]; Luu [5]; Gutierrez et al. [8]; Khanh et al. [9], [10]; Clarke [13]; Morgan and Romaniello [15]; Su [16], [17], [18] and the references therein. On using set-valued radial second order directional derivatives, Gutierrez, Jiménez and Novo [8] obtained second order necessary optimality conditions in primal forms through second order derivatives and second order sufficient optimality conditions in dual forms with "envelope- like effect"

* *Corresponding author.*
Email address : tranuu63@gmail.com.
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through Fritz John type Lagrange multiplier rule. On using multivalued second order contingent derivatives, Khanh and Tung [10] recived Karush-Kuhn-Tucker (KRZ) second order optimality conditions for the multivalued vector optimization problems with attention to the "envelope- like effect". Luu [5] obtained higher order necessary and sufficient optimality conditions for strict local pareto minima via the higher order Studniarski's derivatives. In case the functions considered in $C^{1,1}$, the authors Guerraggio and Luc [2], [3] have established optimality conditions for vector optimization problems in terms of the second-order contingent cones. Khanh and Tuan [9] have dealt with necessary and sufficient optimality conditions for both weak efficiency and firm efficiency in multivalued optimization problems in terms of the second order Hadamard directional derivatives.

We can apply the obtained result in Khanh and Tung [10] to the (local) weak efficiency of vector equilibrium problem with constraints. Therefore, we need discuss to improve the obtained result in [10] for (local) weakly efficient solution in case of single-valued is very necessary. Motivated by this arguments, in this papers we consider the vector equilibrium problem with constraints VEPC with datas are single-valued, in which Fréchet derivatives at an optimal point are calm at that point or objective functions are twice Fréchet differentiable at optimal point that. We use the second-order contingent derivatives for functions to establish the necessary and sufficient optimality conditions for (local) weakly efficient solution to the problem VEPC with attention to the "envelope-like effect".

In this article, the following vector equilibrium problem with constraints VEPC is considered: let X, Y and Z be real Banach spaces, C be nonempty subset of X, $Q \subset Y$ be a closed convex cone with its interior nonempty, which defines a partial order on Y, where cone Q is not necessarily pointed, and let $S \subset Z$ be a convex cone in Z. Given a bifunction $F : X \times X \to Y$ and a constraints function $g : X \to Z$ such that $F(x, x) = 0 \ \forall x \in X$. Our problem here is finding $\overline{x} \in K$ satisfying

$$F(\overline{x}, x) \notin -intQ \quad \forall x \in K, \tag{VEPC}$$

where

$$K = \{x \in C \,:\, g(x) \in -S\}$$

is called a feasible set of problem VEPC. A vector $\overline{x}$ solved (VEPC) is called a weakly efficient solution of VEPC. If there exists a neighborhood U of $\overline{x}$ such that

$$F(\overline{x}, x) \notin -intQ \quad \forall x \in K \cap U,$$

then vector $\overline{x}$ is called a local weakly efficient solution of VEPC. If $\overline{x}$ is a local weakly efficient solution or a weakly efficient solution of VEPC then we write $\overline{x}$ is a (local) weakly efficient solution of VEPC.

The remainder of this paper is organized as follows. After some preliminaries and definitions, Sect. 3 deals with the second-order necessary optimality conditions for efficient solutions of problem VEPC in terms of contingent derivatives. Besides, we also give some discusses. In Sect. 4, we present the second-order sufficient optimality conditions using Fritz John type Lagrange multiplier rule for efficient solutions of problem VEPC.

2. Preliminaries and Definitions

From now on, if not otherwise stated, $X = \mathbb{R}^n, Y = \mathbb{R}^m$ and $Z = \mathbb{R}^p$, where $\mathbb{R}$ ($\mathbb{N}$, respectively) denotes the space of all real (natural, respectively) numbers. For $A \subset X$, as usual, we denote $intA, clA$ and bdA instead of the interior, closure and boundary of A, respectively. The cone generated by A is given as $coneA = \{ta \,:\, t \geq$

$0,\ a \in A\}$. The dual cone of A is defined as $A^+ = \{\xi \in X^* \mid \langle \xi, q\rangle \geq 0 \quad \forall\, a \in Q\,\}$, where $\langle\,.\,,\,.\,\rangle$ denotes the coupling between the space X and the dual space X^* of X, B_X denotes the open unit ball of X, and $B_X(x,\delta)$ denotes the open ball centered at $x \in X$ and radius $\delta > 0$ (similarly for other spaces). We also denote by $S_{g(\overline{x})} := cone(S + g(\overline{x}))$, where $\overline{x} \in X$ such that $g(\overline{x}) \in -S$. Then it is not difficult to see that $S^+_{g(\overline{x})} = \{\eta \in S^+ : \langle \xi, g(\overline{x})\rangle = 0\}$. Let $F : X \to 2^Y$ be a set-valued mapping from X into 2^Y, where 2^Y indicates the family of all subsets of Y. The effective domain, graph and epigraph of a set-valued mapping F are given respectively as

$$\begin{aligned}&\text{dom}F = \{x \in X \,|\, F(x) \neq \emptyset\},\\ &\text{graph}F = \{(x,y) \in X \times Y \,|\, y \in F(x)\},\\ &\text{epi}F = \{(x,y) \in X \times Y \,:\, x \in \text{dom}F,\ y \in F(x) + Q\}.\end{aligned}$$

We denote by $F(A) = \bigcup_{a \in A} F(a)$ and the profile mapping $F_+ : X \to 2^Y$ is defined as $F_+(x) = F(x) + Q$ $(\forall\, x \in X)$. If F is a single-valued mapping then we write f_+ instead of $f + Q$. Let the mappings $f : X \to Y,\ g : X \to Z$, let us define a new profile mapping $(f_+, g_+)(x) = (f(x)+Q) \times (g(x)+S)$ $(\forall\, x \in X)$, where Z is partially ordered by S, and moreover $(f,g)(x) = (f(x), g(x))$ $(\forall\, x \in X)$. Recall (see [10]) that f_+ is said to be $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ if and only if there exists neighborhoods U of $\overline{x}$, V of $f(\overline{x})$, and $L > 0$ satisfying

$$(f_+(x) \cap V) \subset f_+(x') + Q + L\|x - x'\|\, clB_Y \ \ \forall\, x, x' \in U.$$

The profile mapping g_+ has property $S-$ Aubin at $(\overline{x}, g(\overline{x}))$ is similarly defined.

Next, let us provide the definitions about the contingent sets, which will be needed in this paper

Definition 2.1. ([8, 10, 17]) Let M be a subset of X and let $\overline{x}, u \in X$.
(i) The contingent cone (resp., adjacent cone and interior tangent cone) of M at $\overline{x}$ is

$$T(M,\overline{x}) = \{x \in X : \exists\, t_n \to 0^+, \exists x_n \to x \text{ such that } \overline{x} + t_n x_n \in M \ \forall\ n \in \mathbb{N}\},$$

$$\Big(resp., A(M,\overline{x}) = \{x \in X : \forall\, t_n \to 0^+, \exists x_n \to x \text{ such that } \overline{x} + t_n x_n \in M \ \forall\ n \in \mathbb{N}\},$$

$$IT(M,\overline{x}) = \{x \in X : \forall\, t_n \to 0^+, \forall x_n \to x \text{ such that } \overline{x} + t_n x_n \in M \ \forall\, n \text{ large}\}\Big).$$

(ii) The second-order contingent set (resp., adjacent set and interior tangent set) of M at $\overline{x}$ in direction u is

$$T^2(M,\overline{x},u) = \{x \in X : \exists\, t_n \to 0^+, \exists x_n \to x \text{ such that } \overline{x} + t_n u + \frac{1}{2}t_n^2 x_n \in M \ \forall\ n \in \mathbb{N}\},$$

$$\Big(resp., A^2(M,\overline{x},u) = \{x \in X : \forall\, t_n \to 0^+, \exists x_n \to x \text{ such that } \overline{x} + t_n u + \frac{1}{2}t_n^2 x_n \in M \ \forall\ n \in \mathbb{N}\},$$

$$IT^2(M,\overline{x},u) = \{x \in X : \forall\, t_n \to 0^+, \forall x_n \to x \text{ such that } \overline{x} + t_n u + \frac{1}{2}t_n^2 x_n \in M \ \forall\, n \text{ large}\}\Big).$$

(iii) The asymptotic second-order contingent set of M at $\overline{x}$ in direction u is

$$T^{''}(M,\overline{x},u) = \{x \in X : \exists\,(t_n, r_n) \to (0^+, 0^+),\ \frac{t_n}{r_n} \to 0^+,\ \exists x_n \to x \text{ such that } \overline{x} + t_n u + \frac{1}{2}t_n r_n x_n \in M \ \forall\ n \in \mathbb{N}\}.$$

We say that $M \subset X$ is a second-order derivative at $(\overline{x}, u)$ if and only if

$$T^2(M, \overline{x}, u) = A^2(M, \overline{x}, u).$$

It is well known that if $\overline{x} \notin clM$ then all the above tangent sets are null. Moreover, if $u \notin T(M, \overline{x})$ then all the above second-order tangent sets are null.

Proposition 2.2. *([12], Proposition 2.2) Let $M \subset X$ be a convex set, $\overline{x} \in M$ and $v \in T(M, \overline{x})$, then*

$$T^2(M, \overline{x}, v) + T(T(M, \overline{x}), v) \subset T^2(M, \overline{x}, v). \tag{2.1}$$

Additionally, if $0 \in T^2(M, \overline{x}, v)$ (in particular, when M is polyhedral), then

$$T^2(M, \overline{x}, v) = T(T(M, \overline{x}), v).$$

We suppose that $A^2(M, \overline{x}, v) \neq \emptyset$. As X is reflexible space, making use of Proposition 2.1 (iv) [10], it follows that $clIT^2(M, \overline{x}, v) = A^2(M, \overline{x}, v)$ and

$$A^2(M, \overline{x}, v) + T(T(M, \overline{x}), v) \subset A^2(M, \overline{x}, v). \tag{2.2}$$

Definition 2.3. ([1, 17]) Let $f : X \to Y$ be a single-valued mapping and let $\overline{x} \in X$, $(u, v) \in X \times Y$.
(i) The contingent derivative of f (resp., f_+) at a point $\overline{x}$ is defined as

$$\text{graph}\Big(D_c f(\overline{x})\Big) = T(\text{graph}(f), (\overline{x}, f(\overline{x})))$$
$$\big(resp., \ \text{graph}\Big(D_c(f_+)(\overline{x}, f(\overline{x}))\Big) = T(\text{epi}(f), (\overline{x}, f(\overline{x}))\big).$$

(ii) The second-order contingent derivative of f (resp., f_+) at a point $\overline{x}$ in direction (u, v) is defined as

$$\text{graph}\Big(D^2_c f(\overline{x}, f(\overline{x}), u, v)\Big) = T^2(\text{graph}(f), (\overline{x}, f(\overline{x})), (u, v))$$
$$\big(resp., \ \text{graph}\Big(D^2_c(f_+)(\overline{x}, f(\overline{x}), u, v)\Big) = T^2(\text{epi}(f), (\overline{x}, f(\overline{x})), (u, v))\big).$$

The second-order adjacent derivatives $D^{b,2}_c f(\overline{x}, f(\overline{x}), u, v)$ and $D^{b,2}_c(f_+)(\overline{x}, f(\overline{x}), u, v)$ are similar, with A^2 replacing T^2.
(iii) The second-order asymptotic contingent derivative of f (resp., f_+) at a point $\overline{x}$ in direction (u, v) is defined as

$$\text{graph}\Big(D^{''}_c f(\overline{x}, f(\overline{x}), u, v)\Big) = T^{''}(\text{graph}(f), (\overline{x}, f(\overline{x})), (u, v))$$
$$\big(resp., \ \text{graph}\Big(D^{''}_c(f_+)(\overline{x}, f(\overline{x}), u, v)\Big) = T^{''}(\text{epi}(f), (\overline{x}, f(\overline{x})), (u, v))\big).$$

By definitions, it can easily be seen that for each $(u, v, w) \in X \times Y \times Z$, we obtain

$$D^2_c f(\overline{x}, f(\overline{x}), u, v)(x) + Q \subset D^2_c(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \quad \forall x \in X, \tag{2.3}$$

which means that

$$dom\big(D^2_c f(\overline{x}, f(\overline{x}), u, v)\big) \subset dom\big(D^2_c(f_+)(\overline{x}, f(\overline{x}), u, v)\big).$$

The case of the profile map (f_+, g_+), one has the following inclusion holds for all $x \in X$

$$\begin{aligned} D^2_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x) \subset & D^2_c(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \\ & \times D^2_c(g_+)(\overline{x}, g(\overline{x}), u, w)(x). \end{aligned} \tag{2.4}$$

Definition 2.4. ([14]) A mapping $f : X \longrightarrow Y$ is said to be m-calm at $\overline{x}$ if there exist a neighborhood U of $\overline{x}$ and $L > 0$ such that

$$\|f(x) - f(\overline{x})\| \leq L\|x - \overline{x}\|^m \quad \forall\, x \in U,$$

where $\| \,.\, \|$ denotes a norm in Banach spaces.

Of course, if f is m-calm then f is continuous at that point. When $m = 1$, $1-$ calmness is called simply calmness. If f is 1-calm then it also can be called "calm" or "stable" (see [2]). If, in addition, $||f(x) - f(x')|| \leq L||x - x'|| \quad \forall\, x, x' \in U$, then we say that f is Lipschitz around $\overline{x}$. If for each $\overline{x} \in X$ there exists a neighborhood U of $\overline{x}$ such that f is Lipschitz around $\overline{x}$, we will say that f is locally Lipschitz on X. If f is Lipschitz around $\overline{x}$, making use of the obtained result in [2], then one gets f is steady at $\overline{x}$, which yields that f is calm at $\overline{x}$.

Finally, let us denote by $t_n \to 0^+$ instead of a sequence of positive numbers with limit 0 and for each $\overline{x} \in K$, the mapping $f = F_{\overline{x}} : X \to Y$.

Proposition 2.5. *([9], Lemma 2.3) Let $\overline{x} \in X$ and assume, in addition, that S closed convex. If $g(\overline{x}) \in -S$ and $\lim\limits_{t_n \longrightarrow 0^+} \frac{z_n - g(\overline{x})}{t_n} \in -int\, S_{g(\overline{x})}$ then $z_n \in -S$ for n large enough.*

It is not hard to see that Proposition 2.5 is still holds if the closedness of cone S is deleted.

3. Second-order Necessary Optimality Conditions

In this subsection, we establish some second-order necessary optimality conditions in dual and primal forms for (local) weakly efficient solution of VEPC in terms of second-order contingent (or adjacent) derivatives.

Proposition 3.1. *Let $\overline{x} \in C$ be a (local) weakly efficient solution to the problem VEPC. Then, for every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (-S)$, we have*

$$D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x) \cap IT(-Q, v) \times -int(S_{g(\overline{x})}) = \emptyset \tag{3.1}$$

for all $x \in IT^2(C, \overline{x}, u)$. Furthermore:

(i) If, in addition, (f, g) has Fréchet derivative $(\triangledown f(\overline{x}), \triangledown g(\overline{x}))$ which is stable at $\overline{x}$ then for all $x \in A^2(C, \overline{x}, v)$, (3.1) is fulfilled.

(ii) If, in addition, f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ and one of the following two conditions is satisfied

(I) g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$;

(II) g satisfies (i).

Then one has

$$\begin{aligned} D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \times D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)(x) \\ \cap\, IT(-Q, v) \times -int(S_{g(\overline{x})}) = \emptyset \;\; \forall\, x \in A^2(C, \overline{x}, u). \end{aligned} \tag{3.2}$$

Proof. We fixed $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u)$. Assume, to the contrary, that there exists $x \in IT^2(C, \overline{x}, u)$ such that the left-hand side of (3.1) is nonempty. On finds $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x)$ such that $y \in IT(-Q, v)$ and $z \in -int(S_{g(\overline{x})})$. By definitions it holds that

$$(x, y, z) \in T^2(epi(f, g), (\overline{x}, (f, g)(\overline{x})), u, (v, w)),$$

which is equivalent to $\exists t_n \to 0^+$, $\exists x_n \to x$ and $\exists (y_n, z_n) \to (y, z)$ satisfying

$$f(\overline{x}) + t_n v + \frac{1}{2}t_n^2 y_n \in f_+(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n),$$

$$g(\overline{x}) + t_n w + \frac{1}{2}t_n^2 z_n \in g_+(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n).$$

Taking $s_n \in S$ and $z'_n := g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) + s_n - t_n w$ such that $z_n = \frac{z'_n - g(\overline{x})}{\frac{1}{2}t_n^2} \to z \in -int(S_{g(\overline{x})})$ as $2^{-1}t_n^2 \to 0^+$. Making use of Proposition 2.5, we get $z'_n \in -S$ for n sufficiently large, which yields that $g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) \in -S$ for n sufficiently large. Note that $x \in IT^2(C, \overline{x}, u)$, it implies that

$$\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n \in K \quad \text{for n large enough.} \tag{3.3}$$

By a similar argument as in the proof of Proposition 3.1 [10], we deduce that

$$f(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) \in -intQ \quad \text{for n large enough.} \tag{3.4}$$

Combining (3.3)-(3.4), it yields that $\overline{x} \in K$ is not a (local) weakly efficient solution for VEPC. In view of the initial assumptions, it follows that (3.1) holds for all $x \in IT^2(C, \overline{x}, u)$.

Case (i). If, in addition, (f, g) has Fréchet derivative $(\nabla f(\overline{x}), \nabla g(\overline{x}))$ which is stable at $\overline{x}$, then by an argument analogous to that used for the above results, with A^2 replacing IT^2, we deduce that, for the preceding sequence t_n, there exists $x'_n \to x$ such that

$$\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n \in C \quad \forall n \in \mathbb{N}.$$

We put $z''_n := g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n) + s_n - t_n w$. According to the initial assumption it follows that $\nabla g(\overline{x})$ is continuous at $\overline{x}$ and hence g Lipschitz around $\overline{x}$. Then there exists $L_g > 0$ such that

$$\begin{aligned}
\lim_{n \longrightarrow \infty} \left\| \frac{z'_n - g(\overline{x})}{2^{-1}t_n^2} - \frac{z''_n - g(\overline{x})}{2^{-1}t_n^2} \right\| &= \lim_{n \longrightarrow \infty} \left\| \frac{g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) - g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n)}{2^{-1}t_n^2} \right\| \\
&\leq \lim_{n \longrightarrow \infty} L_g \left\| \frac{(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) - (\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n)}{2^{-1}t_n^2} \right\| \\
&= \lim_{n \longrightarrow \infty} L_g \|x_n - x'_n\| \\
&\leq \lim_{n \longrightarrow \infty} L_g (\|x_n - x\| + \|x - x'_n\|) = 0.
\end{aligned}$$

Hence, $\frac{z''_n - g(\overline{x})}{2^{-1}t_n^2} \longrightarrow z \in -int(S_{g(\overline{x})})$, since $\frac{z'_n - g(\overline{x})}{2^{-1}t_n^2} \longrightarrow z$. A consequence is

$$\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n \in K \quad \text{for n large enough.} \tag{3.5}$$

Again choosing sequence $(q_n)_{n \geq 1} \subset Q$ such that

$$y_n = \frac{f(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) + q_n - t_n v}{2^{-1}t_n^2} \longrightarrow y \in IT(-Q, v).$$

By repeating the above proofs, with f replacing g, we obtain as follows (note that $IT(-Q, v) = IT(-intQ, v)$)

$$y'_n := \frac{f(\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n) + q_n - t_n v}{2^{-1}t_n^2} \longrightarrow y \in IT(-intQ, v).$$

Therefore for n large enough,

$$t_n v + \frac{1}{2}t_n^2 y_n' \in -intQ,$$

which implies that

$$f(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n') \in -intQ \quad \text{for sufficiently large n.}$$

This together with (3.5), it yields that $\overline{x}$ is not a (local) weakly efficient solution to the VEPC. From here we conclude that (3.2) holds for all $x \in A^2(C, \overline{x}, u)$.

Case (ii). Let us denote by $M = \{x \in X \,:\, g(x) \in -S\}$, and assume that f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ and g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$. Then for every $x \in A^2(C, \overline{x}, u)$, two cases can occur as follows:

Case 1. $x \in IT^2(M, \overline{x}, u)$ then by direct using result (I) of Proposition 3.2 ([10], p. 74), with $\{f\}$ replacing F, yields that $D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \cap IT(-Q, v) = \emptyset$. Thus condition (3.2) is valid.

Case 2. $x \notin IT^2(M, \overline{x}, u)$ and let us may be assumed to the contrary that there exists $y \in D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)(x)$ sao cho $y \in -int(S_{g(\overline{x})})$. By definitions, we have $(x, y) \in A^2(epi(g), (\overline{x}, g(\overline{x})), u, w)$ and this leads to the following result: $\forall\, t_n \to 0^+$, $\exists\, (x_n, y_n) \to (x, y)$ and $\exists\, s_n \in S$ such that for all $n \geq 1$,

$$g(\overline{x}) + t_n w + \frac{1}{2}t_n^2 y_n \in g_+(\overline{x} + t_n w + \frac{1}{2}t_n^2 x_n).$$

As g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$, thus $\exists U, \exists V$ (neighborhoods of $\overline{x}$ and $g(\overline{x})$, resp.,) and $\exists L > 0$ such that

$$g_+(x) \cap V \subset g(x') + L||x - x'||cl\, B_Z + S \quad \forall\, x, x' \in U.$$

For every $x_n' \longrightarrow x$, there exists $N > 0$ such that, for all $n \geq N$, $\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n, \overline{x} + t_n u + \frac{1}{2}t_n^2 x_n' \in U$, $g(\overline{x}) + t_n w + \frac{1}{2}t_n^2 y_n \in V$, and moreover

$$g_+(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) \cap V \subset g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n') + \frac{1}{2}t_n^2||x_n - x_n'||cl\, B_Z + S.$$

Consequently, for all $n \geq N$,

$$g(\overline{x}) + t_n w + \frac{1}{2}t_n^2 y_n \in g(\overline{x} + t_n w + \frac{1}{2}t_n^2 x_n') + \frac{1}{2}t_n^2||x_n - x_n'||cl\, B_Z + S,$$

which is equivalent to, for sufficiently large n, there exists $b_n \in clB_Z$ and $s_n' \in S$ such that

$$y_n - \|x_n - x_n'\|b_n = \frac{g(\overline{x} + t_n w + \frac{1}{2}t_n^2 x_n') + s_n - t_n w - g(\overline{x})}{2^{-1}t_n^2} \longrightarrow y.$$

By repeating the preceding proofs, we obtain

$$g(\overline{x} + t_n w + \frac{1}{2}t_n^2 x_n') \in -S \quad \text{for n large enough,}$$

or for large n, $\overline{x} + t_n w + \frac{1}{2}t_n^2 x_n' \in M$. By the definition of IT^2, $x \in IT^2(M, \overline{x}, u)$, and this is a contradiction. Finally, we consider g has Fréchet differentiable $\nabla g(\overline{x})$ which is stable at $\overline{x}$, then by repeating the proof of cases (i) and (ii), we arrive at the contradiction.

As was to be shown. □

Proposition 3.2. *Consider problem VEPC with X, Y and Z are real Banach spaces and $\overline{x} \in K$ is a (local) weakly efficient solution of VEPC. Assume, in addition, that (f, g) has Fréchet derivative $(\nabla f(\overline{x}), \nabla g(\overline{x}))$ which is stable at $\overline{x}$. Then, for every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (-cl\, S_{g(\overline{x})})$, we have*

$$D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x) \cap IT(-Q, v) \times IT^2(-S, g(\overline{x}), w) = \emptyset$$
$$\forall\, x \in A^2(C, \overline{x}, u).$$

Proof. Repeat the proof of Proposition 3.1 in the case (i), with $IT^2(-S, g(\overline{x}), w)$ replacing $-int(S_{\overline{x}})$ and $A^2(C, \overline{x}, u)$ replacing $IT^2(C, \overline{x}, u)$, we conclude.

As was to be shown. □

Note 3.3. (i) Proposition 3.1 is still true if X and Y are real Banach spaces. Because in the proof we only use to the result of Proposition 2.5 in sense that Z is an finite-dimensional space. Moreover, if we replace the profile mapping (f_+, g_+) with a single-valued mapping (f, g), then the statements in Propositions 3.1, 3.1 (i) and 3.2 are still not changed.

(ii) If both f and g are twice Fréchet differentiable at $\overline{x}$, then (i) in Proposition 3.1 and the statement in Proposition 3.2 are still valid. Since in this case $\nabla f(\overline{x})$ and $\nabla g(\overline{x})$ are stable at $\overline{x}$.

(iii) If f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ and g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$ then by making use of Proposition 3.2 of Khanh (2015) et al. ([10], p. 74), we deduce that for all $x \in A^2(C, \overline{x}, u)$,

$$D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \times D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)(x)$$
$$\cap\, IT(-Q, v) \times IT^2(-S, g(\overline{x}), w) = \emptyset.$$

In particular, under suitable assumptions, with $u = 0, v = 0, w = 0$, we obtain the first-order necessary optimality conditions in the primal forms respectively as follows:

$$D_c(f_+, g_+)(\overline{x}, (f,g)(\overline{x}))(x) \cap \big(-int(Q \times S_{g(\overline{x})})\big) = \emptyset;$$

$$D_c(f_+)(\overline{x}, f(\overline{x}))(x) \times D_c(g_+)(\overline{x}, g(\overline{x}))(x) \cap \big(-int(Q \times S_{g(\overline{x})})\big) = \emptyset.$$

Notice that in many well known second-order necessary conditions, such a critical direction w is not mentioned. For example, w is only in $-cone(S + g(\overline{x}))$; see [9], [10], etc.

(iv) Since $IT^2(C, \overline{x}, u) \subset A^2(C, \overline{x}, u)$, hence Proposition 3.2 improves Proposition 3.1 (i) in ([10], p. 73) in the case of single-valued optimization. It should be noted here that $-int(S_{g(\overline{x})})$ and $IT^2(-S, g(\overline{x}), w)$ in Propositions 3.1 and 3.2 play an important role in establishing necessary optimality conditions in the dual form, see, for instance, Theorems 3.1-3.3 below.

Theorem 3.1. *Consider problem VEPC with X, Y and Z are real Banach spaces and $\overline{x} \in K$ is a (local) weakly efficient solution of VEPC. Assume, in addition, that (f, g) has Fréchet derivative $(\nabla f(\overline{x}), \nabla g(\overline{x}))$ which is stable at $\overline{x}$. Then, for every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (P)$, the following assertions are holds:*

(i) If $P = -S$ and suppose, furthermore, that $dim(Z) < +\infty$ then for all $x \in A^2(C, \overline{x}, u)$ and for all $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$, there exist $(\lambda, \eta) \in$

$Q^+ \times N(-S, g(\overline{x})) \setminus \{(0,0)\}$ *satisfying*

$$\langle \lambda, v\rangle = 0;$$
$$\langle \lambda, y\rangle + \langle \eta, z\rangle \ \geq \ 0.$$

(ii) If $P = -cl\, S_{g(\overline{x})}$ *and* Z *is reflexible Banach space, then for all* $x \in A^2(C, \overline{x}, u)$ *and for all* $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x)$*, there exist* $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x})) \setminus \{(0,0)\}$ *satisfying*

$$\langle \lambda, v\rangle = \langle \eta, w\rangle = 0;$$
$$\langle \lambda, y\rangle + \langle \eta, z\rangle \ \geq \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle .$$

Proof. (i) We observe Proposition 3.1 (i) see that $\forall\ x \in A^2(C, \overline{x}, u)$,

$$D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x) \cap IT(-Q, v) \times -int(S_{g(\overline{x})}) = \emptyset.$$

Therefore, for all $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x)$, it yields that

$$(y, z) \notin IT(-Q, v) \times -int(S_{g(\overline{x})}).$$

By the standart separation theorem, one finds $(\lambda, \eta) \in (Y \times Z)^* \setminus \{0\}$ satisfying

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \geq \langle \lambda, a\rangle + \langle \eta, b\rangle \quad \forall\, a \in IT(-Q, v)\ \forall\, b \in -int(S_{g(\overline{x})}). \tag{3.6}$$

It can be seen that $\langle \lambda, a\rangle \leq 0$ for all $a \in IT(-Q, v) = -int(Q_v)$. Since λ is a continuous linear mapping on Y, hence $\langle \lambda, a\rangle \leq 0$ for all $a \in -cl\, int(Q_v) = -cl\, cone(Q+v)$. This leads to $\lambda \in Q^+$ and $\langle \lambda, v\rangle = 0$. Similarly, one obtains $\eta \in N(-S, g(\overline{x}))$. Again taking the closures of $IT(-Q, v)$ and $-int(S_{g(\overline{x})})$ in (3.6), and then taking $a = b = 0$, we obtain the result.

(ii) In the similar way as above, one obtains the following inequality

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \geq \langle \lambda, a\rangle + \langle \eta, b\rangle \quad \forall\, a \in IT(-Q, v)\ \forall\, b \in IT^2(-S, g(\overline{x}), w),$$

which yields that $\lambda \in Q^+,\ \langle \lambda, v\rangle = 0$. By repeating the proof in Theorem 3.1 of Khanh (2015) et al. ([10], p. 78) and obtain the remains results, and the claim follows. □

An example is provided to illustrate for the obtained results, which can be stated as follows.

Example 3.4. Let $X = Y = \mathbb{R}^2, Z = \mathbb{R}, C = Q = \mathbb{R}^2_+, S = \mathbb{R}_+, \overline{x} = (0,0)$, and the mappings f, g be given respectively as

$f(x, y) = (y^2 - x^2, y - x)$ for all $(x, y) \in X.$,

$g(x, y) = x - y$ for all $(x, y) \in X$.

Then the feasible set of VEPC is $K = \{(x, y)\ :\ y \geq x \geq 0\}$. It is clear to verify that $\overline{x} = (0,0)$ is a weakly efficient solution to the VEPC. One gets (f, g) is Fréchet differentiable at $\overline{x}$ and its Fréchet derivatives $\left(\nabla f(\overline{x}) = \begin{pmatrix} 0 & -1 \\ 0 & 1\end{pmatrix}, \nabla g(\overline{x}) = \begin{pmatrix} 1 \\ -1\end{pmatrix}\right)$, which does stable at $\overline{x}$. For every $u \in X,\ v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (-S)$, by directly calculating, we obtain the result $u = (a, a), v = w = 0$ for all $a \in \mathbb{R}$. Two cases can occur as follows:

Case 1. Consider $a = 0$ and this implies that $A^2(C, \overline{x}, u) = \mathbb{R}^2_+$. For all $x \in \mathbb{R}^2_+$, for all $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x)$, it follows that $y_1 \geq 0,\ y_2 \geq x_2 - x_1$ and $z \geq x_1 - x_2$. We pick $(\lambda, \eta) = ((1,0), 0) \in Q^+ \times N(-S, g(\overline{x})) \setminus \{0\}$, then $\langle \lambda, y\rangle + \langle \eta, z\rangle \geq y_1 \geq 0$.

Case 2. Consider $a \neq 0$ and this implies that $A^2(C, \overline{x}, u) = \mathbb{R}^2$. For all $x \in \mathbb{R}^2$, for all $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x)$, it follows that $y_1 \geq 0$, $y_2 \geq x_2 - x_1$ and $z \geq x_1 - x_2$. Let (λ, η) be given as in case 1 and the desired conclusion follows.

As $g(\overline{x}) = 0$ thus $-cl(S_{g(\overline{x})}) = -S$. $dim(Z) = 1$ yields Z is reflexible. In this sense, we have $A^2(-S, g(\overline{x}), w) = -S$ and $\eta \in N(-S, g(\overline{x}))$ yields $\eta \in S^+$ and this leads to

$$\sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a \rangle \leq 0.$$

From the assumption (i) it leads to (ii) be completely checked.

Theorem 3.2. *Under the assumptions of Theorem 3.1 and assume, furthermore, that there exist $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (P)$ such that $D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(A^2(C, \overline{x}, u))$ is a convex set. Then the following assertions hold:*

(i) If $P = -S$ and $dim(Z) < +\infty$ then there exists a common $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0, 0)$ satisfying $\langle \lambda, v \rangle = 0$ and

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \geq 0$$

$$\forall\, x \in A^2(C, \overline{x}, u), \quad \forall\, (y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x).$$

Moreover, $\lambda \neq 0$ if the following qualification condition of the KRZ type is satisfied:

$$\Big\{ z \in Z \,:\, (y, z) \in cone\big(D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(A^2(C, \overline{x}, u))\big) \Big\} + S_{g(\overline{x})} = Z.$$

(ii) If $P = -cl\, S_{g(\overline{x})}$ and Z is reflexible Banach space, then there exists a common $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0, 0)$ satisfying $\langle \lambda, v \rangle = \langle \eta, w \rangle = 0$ and

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \geq \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a \rangle\,,$$

$$\forall\, x \in A^2(C, \overline{x}, u), \quad \forall\, (y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x).$$

Moreover, $\lambda \neq 0$ if the following qualification condition of the KRZ type is satisfied:

$$\Big\{ z \in Z \,:\, (y, z) \in cone\big(D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(A^2(C, \overline{x}, u)) - \{0\} \times A^2(-S, g(\overline{x}), w)\big) \Big\} + S_{g(\overline{x})} = Z.$$

Proof. (i) By taking into account Proposition 3.1 (i), we get

$$D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(A^2(C, \overline{x}), u) \cap \big(-int(Q_v)\big) \times \big(-int(S_{g(\overline{x})})\big) = \emptyset.$$

By using a separation theorem, one finds $(\lambda, \eta) \in (Y \times Z)^* \setminus \{0\}$ such that

$$\inf_{(y,z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(A^2(C, \overline{x}), u)} \Big(\langle \lambda, y \rangle + \langle \eta, z \rangle \Big) \geq \sup_{(a,b) \in \big(-int(Q_v)\big) \times \big(-int(S_{g(\overline{x})})\big)} \Big(\langle \lambda, a \rangle + \langle \eta, b \rangle \Big).$$

Similar to the proof of Theorem 3.1 (i), we ensure that $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x})) \setminus \{(0, 0)\}$ satisfying $\langle \lambda, v \rangle = 0$ and

$$\inf_{(y,z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(A^2(C, \overline{x}), u)} \Big(\langle \lambda, y \rangle + \langle \eta, z \rangle \Big) \geq 0,$$

which leads to a conclusion.

Let us next show that $\lambda \neq 0$ under the qualification condition of the KRZ type. In fact, if it were not so, then we have $\eta \in N(-S, g(\overline{x})) \setminus \{0\}$ and moreover

$$\langle \eta, z\rangle \geq 0 \quad \forall\, (y,z) \in cone\big(D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(A^2(C, \overline{x}, u))\big).$$

It is not difficult to see that

$$\langle \eta, z_0\rangle \geq 0 \quad \forall\, z_0 \in Z.$$

A consequence is $\eta = 0$ and this is a contradiction. So, we have shown that $\lambda \neq 0$.

(ii) It is processed similar as in the proof of (i), and the claim follows. □

Theorem 3.3. *Let $X, Y, Z, C, K, \overline{x}$ and f be given as in Theorem 3.1. Then, for every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (P)$, the following assertions are holds:*
(i) If $P = -S$ and suppose, furthermore, that $dim(Z) < +\infty$, f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ and (I) or (II) in Proposition 3.1 (ii) is fulfilled, then for all $x \in A^2(C, \overline{x}, u)$ and for all $(y,z) \in D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \times D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)(x)$, there exist $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\langle \lambda, v\rangle = 0 \;\; and \;\; \langle \lambda, y\rangle + \langle \eta, z\rangle \;\geq\; 0.$$

In particular, for (u, v, w) such that

$$\Big(D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v), D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)\Big)(A^2(C, \overline{x}), u)$$

is convex, there exists a common $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\langle \lambda, v\rangle = 0 \;\; and \;\; \langle \lambda, y\rangle + \langle \eta, z\rangle \;\geq\; 0$$

for all (x, y, z) mentioned in (i) above.

Furthermore, $\lambda \neq 0$ if the following qualification condition of the KRZ type is satisfied:

$$\Big\{z \in Z \,:\, (y,z) \in cone\big(D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(A^2(C, \overline{x}, u))\big)\Big\} + S_{g(\overline{x})} = Z.$$

(ii) If $P = -cl\, S_{g(\overline{x})}$ and suppose, furthermore, that Z is reflexible Banach space, f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$, g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$, then for all $x \in A^2(C, \overline{x}, u)$ and for all $(y,z) \in D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \times D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)(x)$, there exist $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\langle \lambda, v\rangle = \langle \eta, w\rangle = 0 \;\; and \;\; \langle \lambda, y\rangle + \langle \eta, z\rangle \geq \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle\,.$$

In particular, for (u, v, w) such that

$$\Big(D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v), D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)\Big)(A^2(C, \overline{x}), u)$$

is convex, there exists a common $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\langle \lambda, v\rangle = \langle \eta, w\rangle = 0 \;\; and \;\; \langle \lambda, y\rangle + \langle \eta, z\rangle \geq \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle$$

for all (x, y, z) mentioned in (ii) above.

Furthermore, $\lambda \neq 0$ if the following qualification condition of the KRZ type is satisfied:

$$\Big\{z \in Z \,:\, (y,z) \in cone\big(D_c^2(f_+,g_+)(\overline{x},(f,g)(\overline{x}),u,(v,w))(A^2(C,\overline{x},u)) - \{0\} \times A^2(-S,g(\overline{x}),w)\big)\Big\} + S_{g(\overline{x})} = Z.$$

Proof. Case (i): It is proved similarly as in preceding Theorems 3.1 (i) and 3.2 (i). Case (ii) is a direct consequence of Theorem 3.2 ([10], p. 78), with (f_+, g_+) replacing (F_+, G_+), where F, G are set-valued mappings. The proof is completed. □

Note 3.5. (i) The result in article [10] of Khanh and Tung (2015) is extended from the result is well known in references of [10]. However, the case $w \in -S$ is not mentioned in article [10]. Consider the case $w \in -cl\, S_{g(\overline{x})}$, we provide assumptions, which involving f and g, such as f and g are Fréchet differentiable at $\overline{x}$ whose its Fréchet derivatives at $\overline{x}$ stable at that point, then Theorems 3.1 (ii) and 3.2 (ii) in our article are better than Theorem 3.1 of Khanh (2015) et al. ([10], p. 77) in case $F = \{f\}$. It also can be seen as a good improvement one of the results in this article when we considering problem VEPC with data is single-valued functions.

(ii) Since $IT^2(C,\overline{x},u) \subset A^2(C,\overline{x},u)$, hence in single-valued optimization, Theorem 3.3 improves Theorem 3.2 in [10] in the case, either g has Fréchet derivative $\nabla g(\overline{x})$ which is stable at $\overline{x}$, or g is twice Fréchet differentiable at $\overline{x}$.

4. Second-order Sufficient Optimality Conditions

In this subsection, we establish second-order sufficient optimality conditions for local weakly efficient solution for VEPC, where Q and S are closed cones (possibly nonconvex with empty interior). We set

$$u^{\perp} = \{w \in X \;: < u, w >= 0\};$$

$$\Delta(\overline{x}) = \Big\{(\lambda,\eta) \in Y \times Z \;:\; \lambda_0 \nabla f(\overline{x}) + \eta_0 \nabla g(\overline{x}) = 0, \quad \lambda \in Q^+,\;\; \eta \in N(-S, g(\overline{x})),\;\; (\lambda,\eta) \neq (0,0)\Big\}.$$

Theorem 4.1. *Let $\overline{x} \in C$, $f = F(\overline{x}, .)$, Q be pointed $(Q \cap (-Q) = \{0\})$. Suppose that $\nabla f(\overline{x})$ and $\nabla g(\overline{x})$ are calm at $\overline{x}$ and all the following conditions are fulfilled:*

(i) For all $u \in T(C,\overline{x}) \cap \{u \in X \,:\, \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $D_c f(\overline{x})(u) \cap (-intQ) = \emptyset$ and $D_c g(\overline{x})(u) \cap T(-S, g(\overline{x})) \neq \emptyset$;

(ii) For all $u \in T(C,\overline{x}) \cap \{u \in X \,:\, \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $v \in D_c f(\overline{x})(u) \cap (-bdQ)$ and $w \in D_c g(\overline{x})(u) \cap T(-S, g(\overline{x}))$, we have

(a) $\forall\, x \in T^2(C,\overline{x},u)$, $\forall\,(y,z) \in D_c^2(f,g)(\overline{x},(f,g)(\overline{x}),u,(v,w))(x)$ such that $z \in T^2(-S,g(\overline{x}),w)$, there exists $(\lambda,\eta) \in \Delta(\overline{x})$ satisfying

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \;>\; 0;$$

(b) $\forall\, x \in T^{''}(C,\overline{x},u) \cap u^{\perp} \setminus \{0\}$, $\forall\,(y,z) \in D_c^{''}(f,g)(\overline{x},(f,g)(\overline{x}),u,(v,w))(x)$ such that $z \in T^{''}(-S,g(\overline{x}),w)$, there exists $(\lambda,\eta) \in \Delta(\overline{x})$ satisfying

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \;>\; 0.$$

Then $\overline{x}$ is a local weakly efficient solution to the VEPC.

Proof. Assume to the contrary, that $\overline{x} \in K$ is not a local weakly efficient solution to the VEPC. Then there exists sequence $(x_n)_{n\geq 1} \subset K \setminus \{\overline{x}\} \subset C \setminus \{\overline{x}\}$ such that $x_n \to \overline{x}$ and

$$f(x_n) - f(\overline{x}) \in -intQ \quad \forall n \geq 1; \tag{4.1}$$

$$g(x_n) - g(\overline{x}) \in -S - g(\overline{x}) \quad \forall n \geq 1. \tag{4.2}$$

Making use of Lemma 4.1 (i) ([10], p. 83), we get $t_n = \|x_n - \overline{x}\| \to 0^+$ and

$$u_n := \frac{x_n - \overline{x}}{t_n} \to u \in T(C, \overline{x}) \cap \{u \in X \,:\, \|u\| = 1\}.$$

We pick $v = \nabla f(\overline{x})(u), w = \nabla g(\overline{x})(u)$. It is not difficult to see that $v \in D_c f(\overline{x})(u) \cap (-Q)$ and $w \in T(-S, g(\overline{x})) \cap D_c g(\overline{x})(u)$. Thus we have shown that $u \in T(C, \overline{x}) \cap \{u \in X \,:\, \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $v \in D_c f(\overline{x})(u)$ and $D_c g(\overline{x})(u) \cap T(-S, g(\overline{x})) \neq \emptyset$. By the hypotheses of (i), $v \notin -intQ$, which yields that $v \in D_c f(\overline{x})(u) \cap (-bdQ)$.
In other words, we pick sequence $(w_n)_{n\geq 1}$, where

$$w_n = \frac{x_n - \overline{x} - t_n u}{2^{-1} t_n^2}, \quad \forall n \geq 1.$$

Two cases can occur as follows:

(I) $(w_n)_{n\geq 1}$ is bounded. As $dim(X) < +\infty$, thus (taking a subsequence if necessary) there exists the limit of sequence $(w_n)_{n\geq 1}$, and assuming that $w_n \to x \in X$. We setting

$$(y_n, z_n) := \frac{(f,g)(x_n) - (f,g)(\overline{x}) - t_n(v,w)}{2^{-1} t_n^2}, \ n \geq 1.$$

In view of the proof of Proposition 2 ([8], p. 204), we deduce that $(y_n, z_n)_{n\geq 1}$ is bounded, and therefore, there exists a subsequence, denoted in the same way $(y_n, z_n)_{n\geq 1}$ converging to some (y, z). By definitions, we have $x \in T^2(C, \overline{x}, u)$ because $x_n \in C$ $(\forall n)$, $z \in T^2(-S, g(\overline{x}), w)$ is due to $g(x_n) \in -S$ $(\forall n)$, and moreover $(y, z) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$. On the other hand, by hypotheses (a), there exists $(\lambda, \eta) \in \Delta(\overline{x})$ such that

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \ > \ 0. \tag{4.3}$$

It follows from Equation (16) ([8], p. 210) that

$$\langle \lambda, v \rangle = 0, \ \ \langle \eta, w \rangle = 0.$$

Furthermore,

$$(4.1) \Longleftrightarrow \frac{f(x_n) - f(\overline{x})}{2^{-1} t_n^2} \in -intQ \ \ \forall n \geq 1,$$

which is equivalent to

$$y_n + \frac{2\, v}{t_n} \in -intQ \ \ \forall n \geq 1.$$

Because $\langle \lambda, v \rangle = 0$ and $\lambda \in Q^+ \cap Y^*$, hence

$$\lim_{n \longrightarrow +\infty} \langle \lambda, y_n \rangle = \langle \lambda, y \rangle \leq 0. \tag{4.4}$$

Similar to (4.1), one has

$$\frac{g(x_n) - g(\overline{x}) - t_n w}{2^{-1} t_n^2} + \frac{2w}{t_n} \in -S_{g(\overline{x})} \ \ \forall n \geq 1,$$

which is equivalent to

$$z_n + \frac{2\, w}{t_n} \in -S_{g(\overline{x})} \ \ \forall n \geq 1.$$

Because $\eta \in N(-S, g(\overline{x}))$ yields that $\eta \in (S_{g(\overline{x})})^+$. This together with the fact that $\langle \eta, w\rangle = 0$, and by taking limit, we have $\langle \eta, z\rangle \le 0$. Therefore (see Eq. 4.4) $\langle \lambda, y\rangle + \langle \eta, z\rangle \le 0$, which conflicts with (4.3).

(II) $(w_n)_{n\ge 1}$ is unbounded: Let us may assume that $\|w_n\| \to +\infty$ and

$$W_n = \frac{w_n}{\|w_n\|} \to x_1 \in X \cap \{u \in X \ \|u\| = 1\}.$$

For the preceding sequence t_n, we choose a new sequence $r_n = t_n\|w_n\|\forall\, n$, then easy to check that $r_n \to 0^+$, $\frac{t_n}{r_n} \to 0^+$ and moreover

$$x_n = \overline{x} + t_n u + \frac{1}{2} r_n t_n W_n \quad \forall\, n \ge 1.$$

Considering sequences

$$(y_n', z_n') := \frac{(f,g)(x_n) - (f,g)(\overline{x}) - t_n(v,w)}{2^{-1} t_n r_n} \to (y_1, z_1).$$

In the similar way as in (I) (can seen in the proof case (ii) in Theorem 3 ([8], p. 217-218), we conclude that $x_1 \in T^{''}(C, \overline{x}, u) \cap u^{\perp} \setminus \{0\}$ and moreover, $(y_1, z_1) \in D_c^{''}(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x_1)$ and $z_1 \in T^{''}(-S, g(\overline{x}), w)$. Therefore there exists $(\lambda_1, \eta_2) \in \Delta(\overline{x})$ satisfying

$$\langle \lambda_1, y_1\rangle + \langle \eta_1, z_1\rangle \ > \ 0.$$

Similarly to the preceding proof, $\langle \lambda_1, v\rangle = 0,\ \ \langle \eta_1, w\rangle = 0$, $y_n' + \frac{2v}{r_n} \in -intQ \ \ \forall\, n \ge 1$, $z_n' + \frac{2w}{r_n} \in -S_{g(\overline{x})} \ \ \forall\, n \ge 1$, which leads to a contradiction. From there we conclude that the proof of Theorem 4.1 is complete. □

Corollary 4.1. *Let $\overline{x} \in C$, $f = F(\overline{x}, .)$, Q be pointed $(Q \cap (-Q) = \{0\})$. Suppose that $\nabla f(\overline{x})$ and $\nabla g(\overline{x})$ are calm at $\overline{x}$ and $(v, w) = \Big(\nabla f(\overline{x})(u), \nabla g(\overline{x})(u)\Big)$. Then $\overline{x}$ is a local weakly efficient solution to the VEPC if*

(i) $\forall\, x \in T^2(C, \overline{x}, u)$, $\forall\, (y,z) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$ such that $z \in T^2(-S, g(\overline{x}), w)$, there exists $(\lambda, \eta) \in \Delta(\overline{x})$ satisfying $\langle \lambda, y\rangle + \langle \eta, z\rangle \ > \ 0$;

(ii) $\forall\, x \in T^{''}(C, \overline{x}, u) \cap u^{\perp} \setminus \{0\}$, $\forall\, (y,z) \in D_c^{''}(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$ such that $z \in T^{''}(-S, g(\overline{x}), w)$, there exists $(\lambda, \eta) \in \Delta(\overline{x})$ satisfying $\langle \lambda, y\rangle + \langle \eta, z\rangle \ > \ 0$.

Proof. It is directly inferred from the proof of Theorem 4.1 and the claim follows. □

Note 4.2. (i) Because $\eta \in S_{g(\overline{x})}$ and $\langle \eta, w\rangle = 0$, hence $\eta \in \Big(cone(cone(S + g(\overline{x})) + w)\Big)^+$. Furthermore, it is well known that

$$A^2(-S, g(\overline{x}), w) \subset cl\Big(cone(cone(-S - g(\overline{x})) - w)\Big) = -cl\Big(cone(cone(S + g(\overline{x})) + w)\Big),$$

which yields that $\sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle \le 0$. From the fact that $\langle \lambda, y\rangle + \langle \eta, z\rangle > 0$, it implies that

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \ > \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle\,. \tag{4.5}$$

When $(-S)$ is second-order derivative at $(g(\overline{x}), \nabla g(\overline{x})u)$, i.e., $T^2(-S, g(\overline{x}), w) = A^2(-S, g(\overline{x}), w)$ for all $u \in X$, the necessary optimality conditions in above Theorem 4.1 only differs from (4.5) in the substitution of " $>$ " for " $\ge$ ". In this case, it will be said that second-order sufficient optimality conditions are very close to the second-order optimality necessary conditions.

(ii) The set $\Delta(\overline{x})$ in Theorem 4.1 is also called the set of all Fritz John type multipliers.

Theorem 4.2. *Let $\overline{x} \in C$, $f = F(\overline{x}, .)$, Q be pointed $(Q \cap (-Q) = \{0\})$. Suppose that f and g are twice Fréchet differentiable at $\overline{x}$ and all the following conditions are fulfilled:*
(i) For all $u \in T(C, \overline{x}) \cap \{u \in X : \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $D_c f(\overline{x})(u) \cap (-intQ) = \emptyset$ and $D_c g(\overline{x})(u) \cap T(-S, g(\overline{x})) \neq \emptyset$;
(ii) For all $u \in T(C, \overline{x}) \cap \{u \in X : \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $v \in D_c f(\overline{x})(u) \cap (-bdQ)$ and $w \in D_c g(\overline{x})(u) \cap T(-S, g(\overline{x}))$, we have
(a) $\forall x \in T^2(C, \overline{x}, u)$ for which $(\nabla f(\overline{x})(x)+\nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x)+\nabla^2 g(\overline{x})(u,u)) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$ such that $\nabla g(\overline{x})(x)+\nabla^2 g(\overline{x})(u,u) \in T^2(-S, g(\overline{x}), w)$, there exists $(\lambda, \eta) \in \Delta(\overline{x})$ satisfying

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \right\rangle > 0;$$

(b) $\forall x \in T^{''}(C, \overline{x}, u) \cap u^{\perp} \setminus \{0\}$ for which $(\nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u)) \in D_c^{''}(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$ such that $\nabla g(\overline{x})(x)+\nabla^2 g(\overline{x})(u,u) \in T^{''}(-S, g(\overline{x}), w)$, there exists $(\lambda, \eta) \in \Delta(\overline{x})$ satisfying

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \right\rangle > 0.$$

Then $\overline{x}$ is a local weakly efficient solution to the VEPC.

Proof. It is well-known that, when f and g are twice Fréchet differentiable at $\overline{x}$, then for $(v, w) = (\nabla f(\overline{x})(u), \nabla g(\overline{x})(u))$, one has (see [8])

$$D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x) = \{(\nabla f(\overline{x})(x)+\nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x)+\nabla^2 g(\overline{x})(u,u))\}.$$

Repeat the proof Theorem 4.1, and the claim follows. □

Corollary 4.3. *Let $\overline{x} \in C$, $f = F(\overline{x}, .)$, Q be pointed $(Q \cap (-Q) = \{0\})$. Suppose that f and g are twice Fréchet differentiable at $\overline{x}$ and $(v, w) = \Big(\nabla f(\overline{x})(u), \nabla g(\overline{x})(u)\Big)$. Then $\overline{x}$ is a local weakly efficient solution to the VEPC if*
(i) $\forall x \in T^2(C, \overline{x}, u)$ for which

$$(\nabla f(\overline{x})(x)+\nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x)+\nabla^2 g(\overline{x})(u,u)) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$$

such that $\nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \in T^2(-S, g(\overline{x}), w)$, there exists $(\lambda, \eta) \in \Delta(\overline{x})$ satisfying

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \right\rangle > 0;$$

(ii) $\forall x \in T^{''}(C, \overline{x}, u) \cap u^{\perp} \setminus \{0\}$ for which

$$(\nabla f(\overline{x})(x)+\nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x)+\nabla^2 g(\overline{x})(u,u)) \in D_c^{''}(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$$

such that $\nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \in T^{''}(-S, g(\overline{x}), w)$, there exists $(\lambda, \eta) \in \Delta(\overline{x})$ satisfying

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \right\rangle > 0.$$

Proof. Repeat the proof Theorem 4.2 and we arrive at the conclusion. □

Note 4.4. Note that the condition

$$\left\langle \lambda, \triangledown f(\overline{x})(x) + \triangledown^2 f(\overline{x})(u,u)\right\rangle + \left\langle \eta, \triangledown g(\overline{x})(x) + \triangledown^2 g(\overline{x})(u,u)\right\rangle > 0$$

is implied by the following condition

$$\left\langle \lambda, \triangledown f(\overline{x})(x) + \triangledown^2 f(\overline{x})(u,u)\right\rangle + \left\langle \eta, \triangledown g(\overline{x})(x) + \triangledown^2 g(\overline{x})(u,u)\right\rangle > \sup_{a \in T^2(-S, g(\overline{x}), w)} \langle \eta, a \rangle .$$

Note 4.5. In case the objective functions are considered with vector values, the difference between Theorem 4.1 [10] and our results lies in $\Delta(\overline{x})$ and $\Delta^2(x)$, where

$$\Delta^2(x) = \Big\{(y,z) \,:\, (y,z) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x),\ (x,y,z) \in (u,v,w)^{\perp}\Big\}.$$

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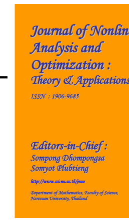

LOCAL CONVERGENCE FOR JARRATT-LIKE ITERATIVE METHODS IN BANACH SPACE UNDER WEAK CONDITIONS

IOANNIS K. ARGYROS AND SANTHOSH GEORGE*

Ioannis K. Argyros, Cameron University, Department of Mathematical Sciences, Lawton, OK 73505, USA, Email:iargyros@cameron.edu

Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-757 025, Email: sgeorge@nitk.edu.in

ABSTRACT. We study the method considered in Sharma and Arora(2014), for solving systems of nonlinear equations, modified suitably to include the nonlinear equations in Banach spaces. Our conditions are weaker than the conditions used in earlier studies. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

KEYWORDS : Jarratt-like method; radius of convergence; local convergence; restricted convergence domains.
AMS Subject Classification:65D10, 65D99, 65J20, 49M15, 74G20, 41A25

1. Introduction

In this study, we consider the problem of approximating the solution x^* of nonlinear equation

$$H(x) = 0 \tag{1.1}$$

where $H : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a continuous differentiable operator in the sense of Fréchet between the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ and Ω is a convex set. We consider the following method considered in [15] for increasing the order of convergence of iterative methods to solve (1.1).

$$\begin{aligned} y_n &= x_n - \frac{2}{3}H'(x_n)^{-1}H(x_n) \\ z_n &= x_n - [\frac{23}{8}I - H'(x_n)^{-1}H'(y_n)(3I - \frac{9}{8}H'(x_n))^{-1}H'(y_n)] \\ x_{n+1} &= z_n - (\frac{5}{2}I - \frac{3}{2}H'(x_n)^{-1}H'(y_n))H'(x_n)^{-1}H(z_n), \end{aligned} \tag{1.2}$$

where $x_0 \in \Omega$ is an initial point. Let $U(a, \rho) := \{x \in \mathcal{B}_1 : \|x - a\| < \rho\}$ and let $\bar{U}(a, \rho)$ be its closure.

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* *Corresponding author.*
Email address : sgeorge@nitk.edu.in.
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Due to its wide applications, finding solution for the equation (1.1) and improving the order of convergence of iterative method for solving (1.1) is an important problem in mathematics. In [15] the existence of the Fréchet derivative of H of order up to the sixth was used for the convergence analysis of method (1.2) although only the first derivative appears in the method for the special case $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^m$. This assumption on the higher order Fréchet derivatives of the operator H restricts the applicability of method (1.2). For example consider the following:

EXAMPLE 1.1. Let $X = C[0,1], D = \bar{U}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [2, 3, 13] defined by

$$x(s) = \int_0^1 G(s,t)\frac{x(t)^2}{2}dt,$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.1), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\frac{x(t)^2}{2}dt.$$

Notice that

$$\| \int_0^1 G(s,t)dt\| \leq \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)x(t)dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}\|x - y\|.$$

One can see that, higher order derivatives of F do not exist in this example.

Our goal is to weaken the assumptions in [15] and apply the method for solving equation (1.1) in Banach spaces, so that the applicability of the method (1.2) can be extended. The technique introduced in this study can be applied to other iterative methods [1–17].

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result. Special cases and numerical examples are given in the last section.

2. Local Convergence analysis

The local convergence analysis of method (1.2) that follows is based on some scalar functions and parameters. Let function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and non-decreasing with $w_0(0) = 0$. Define the parameter r_0 by

$$r_0 = \sup\{t \geq 0 : w_0(t) < 1\}. \tag{2.1}$$

Let $w : [0, r_0) \longrightarrow [0, +\infty)$, $v : [0, r_0) \longrightarrow [0, +\infty)$ be continuous and nondecreasing functions with $w(0) = 0$. Define functions $g_i, h_i, i = 1, 2$ on the interval $[0, r_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta + \frac{1}{3}\int_0^1 v(\theta t)d\theta}{1 - w_0(t)},$$

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)} \\ &+ \left[\frac{15}{8}\frac{w_0(t) + w_0(g_1(t)t)}{1-w_0(t)}\right. \\ &+ \left.\frac{9}{8}\frac{w_0(t) + w_0(g_1(t)t)\int_0^1 v(\theta g_1(t)t)d\theta}{(1-w_0(t))^2}\right]\frac{\int_0^1 v(\theta t)d\theta}{1-w_0(t)} \end{aligned}$$

and

$$h_i(t) = g_i(t) - 1, i = 1, 2.$$

Suppose that

$$v(0) < 3. \tag{2.2}$$

We have that $h_1(0) = \frac{v(0)}{3} - 1 < 0$(by (2.2) and $h_1(t) \longrightarrow +\infty$ as $t \longrightarrow r_0^-$, $h_2(0) = -1 < 0$ and $h_2(t) \longrightarrow +\infty$ as $t \longrightarrow r_0^-$. It follows from the intermediate value theorem that functions h_i have zeros in the interval $(0, r_0)$. Denote by r_i the smallest such zeros of functions h_i, respectively. Define parameter $\bar{r}_0$ by

$$\bar{r}_0 = \max\{t \in [0, r_0] : w_0(g_2(t)t) < 1\}. \tag{2.3}$$

Define functions g_3 and h_3 on the interval $[0, \bar{r}_0)$ by

$$\begin{aligned} g_3(t) &= \left[\frac{\int_0^1 w((1-\theta)g_2(t)t)d\theta}{1-w_0(g_2(t)t)}\right. \\ &+ \frac{(w_0(t) + w_0(g_2(t)t))\int_0^1 v(\theta g_2(t)t)d\theta}{(1-w_0(t))(1-w_0(g_2(t)t))} \\ &+ \left.\frac{3}{2}\frac{(w_0(t) + w_0(g_1(t)t)\int_0^1 v(\theta g_2(t)t)d\theta}{(1-w_0(t))^2}\right] g_2(t) \end{aligned}$$

and

$$h_3(t) = g_3(t) - 1.$$

We have that $h_3(0) = -1 < 0$ and $h_3(t) \longrightarrow +\infty$ as $t \longrightarrow \bar{r}_0^-$. Denote by r_3 the smallest zero of h_3 in the interval $(0, \bar{r}_0)$. Define the radius of convergence r by

$$r = \min\{r_i\} \quad i = 1, 2, 3. \tag{2.4}$$

Then, for each $t \in [0, r)$

$$0 \leq g_i(t) < 1, \tag{2.5}$$

$$0 \leq w_0(t) < 1, \tag{2.6}$$

and

$$0 \leq w_0(g_2(t)t) < 1. \tag{2.7}$$

Next, the local convergence analysis of method (1.2) is shown using the preceding notation.

THEOREM 2.1. Let $H : \Omega \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose:

there exist $x^* \in \Omega$ and function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ continuous and nondecreasing with $w_0(0) = 0$ such that for each $x \in \Omega$

$$H(x^*) = 0, \quad H'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{2.8}$$

and

$$\|H'(x^*)^{-1}(H'(x) - H'(x^*)\| \leq w_0(\|x - x^*\|); \tag{2.9}$$

Let $\Omega_0 = \Omega \cap U(x^*, r_0)$. There exist functions $w : [0, r_0) \longrightarrow [0, +\infty), v : [0, r_0) \longrightarrow [0, +\infty)$ continuous and nondecreasing with $w(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|H'(x^*)^{-1}(H'(x) - H'(y)\| \leq w(\|x - y\|), \tag{2.10}$$

$$\|H'(x^*)^{-1}H'(x)\| \leq v(\|x - y\|), \tag{2.11}$$

and

$$\bar{U}(x^*, r) \subseteq \Omega, \tag{2.12}$$

where the convergence radii r_0 and r are given by (2.1) and (2.4), respectively. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined in $U(x^*, r)$, stays in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.13}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.15}$$

where the functions $g_i, i = 1, 2, 3$ are defined previously. Furthermore, if there exists $R^* \geq r$ satisfies

$$\int_0^1 w_0(\theta R) d\theta < 1, \tag{2.16}$$

then the limit point x^* is the only solution of equation $H(x) = 0$ in $\Omega_1 = \Omega \cap \bar{U}(x^*, R)$.

Proof. Estimates (2.13)-(2.15) shall be shown using mathematical induction on the integer k Let $x \in U(x^*, r)$. Using (2.4), (2.6), (2.8) and (2.9), we have that

$$\|H'(x^*)^{-1}(H'(x) - H'(x^*)\| \leq w_0(\|x - x^*\|) \leq w_0(r) < 1. \tag{2.17}$$

Hence by (2.17) and the Banach Lemma on invertible operators [2, 4, 7] we get that $H'(x)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|H'(x)^{-1}H'(x^*)\| \leq \frac{1}{1 - w_0(\|x - x^*\|)}. \tag{2.18}$$

In particular, (2.18) holds for $x = x_0$, since $x_0 \in U(x^*, r)$ and points y_0, z_0 and x_1 are well defined by method (1.2) for $n = 0$. We can write by (2.8) that

$$H(x_0) = H(x_0) - H(x^*) = \int_0^1 H'(x^* + \theta(x_0 - x^*))(x_0 - x^*) d\theta. \tag{2.19}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0, 1]$. In view of (2.11) and (2.19), we obtain that

$$\|H'(x^*)^{-1}H'(x_0)\| \leq \int_0^1 v(\theta\|x_0 - x^*\|) d\theta \|x_0 - x^*\|. \tag{2.20}$$

Using (2.4), (2.5) (for $i = 1$), (2.8), (2.10), (2.18), (2.20) and method (1.2), we have in turn that

$$\begin{aligned}
\|y_0 - x^*\| &\leq \|x_0 - x^* - H'(x_0)^{-1}H'(x_0)\| + \frac{1}{3}\|H'(x_0)^{-1}H'(x^*)\| \\
&\leq \|H'(x_0)^{-1}H'(x^*)\|\|\int_0^1 H'(x^*)^{-1}(H'(x^* + \theta(x_0 - x^*)) - H'(x_0))(x_0 - x^*)d\theta\| \\
&\quad + \frac{1}{3}\|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(x_0)\| \\
&\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\
&\quad + \frac{1}{3}\frac{\int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (2.21)
\end{aligned}$$

which shows (2.13) for $n = 0$ and $y_0 \in B(x^*, r)$. By the second substep of method (1.2) for $n = 0$, we can write

$$\begin{aligned}
z_0 - x^* &= x_0 - x^* - [\frac{23}{8}I - H'(x_0)^{-1}H'(y_0)(3I - \frac{9}{8}H'(x_0)^{-1}H'(y_0))]H'(x_0)^{-1}H(x_0) \\
&= x_0 - x^* - H'(x_0)H(x_0) + \left[\frac{15}{8}H'(x_0)^{-1}(H'(y_0) - H'(x^*)) + (H'(x^*) - H'(x_0))\right. \\
&\quad \left. + \frac{9}{8}H'(x_0)^{-1}H'(y_0)H'(x_0)^{-1}(H'(x_0) - H'(y_0))\right] H'(x_0)^{-1}H(x_0). \qquad (2.22)
\end{aligned}$$

Using (2.4), (2.5) (for $i = 2$), (2.18), (2.20) (for $y_0 = x_0$), (2.21) and (2.22) we get in turn that

$$\begin{aligned}
\|z_0 - x^*\| &\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - w_0(\|x_0 x -^* \|} \\
&\quad + \left[\frac{15}{8}\frac{w_0(\|x_0 - x^*\|) + w_0(\|y_0 - x^*\|)}{1 - w_0(\|x_0 - x^*\|)}\right. \\
&\quad \left. + \frac{9}{8}\frac{(w_0(\|x_0 - x^*\|) + w_0(\|y_0 - x^*\|))\int_0^1 v(\theta\|y_0 - x^*\|)d\theta}{(1 - w_0(\|x_0 - x^*\|))^2}\right] \\
&\quad \times \frac{\int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 x -^* \|}{1 - w_0(\|x_0 - x^*\|)} \\
&\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (2.23)
\end{aligned}$$

which shows (2.14) for $n = 0$ and $z_0 \in U(x^*, r)$. Notice also that (2.18) holds for $x = z_0$. Next, by the third substep of method (1.2) for $n = 0$, we can write

$$\begin{aligned}
x_1 - x^* &= z_0 - x^* - (\frac{5}{2}I - \frac{3}{2}H'(x_0)^{-1}H'(y_0))H'(x_0)^{-1}H(z_0) \\
&= z_0 - x^* - H'(z_0)^{-1}H(z_0) + H'(z_0)^{-1}(H'(x_0) - H'(z_0))H'(x_0)^{-1}H(x_0) \\
&\quad + \frac{3}{2}H'(x_0)^{-1}(H'(x_0) - H'(y_0))H'(x_0)^{-1}H(z_0). \qquad (2.24)
\end{aligned}$$

Then, using (2.4), (2.5) (for $i = 3$) (2.18) (for $x = x_0$ and $x = z_0$), (2.21), (2.23) and (2.24), we obtain in turn that

$$\|x_1 - x^*\| \leq \frac{\int_0^1 w((1-\theta)\|z_0 - x^*\|)d\theta\|z_0 - x^*\|}{1 - w_0(\|z_0 - x^*\|)}$$

$$
\begin{aligned}
&+\frac{(w_0(\|x_0-x^*\|)+w_0(\|z_0-x^*\|))\int_0^1 v(\theta\|z_0-x^*\|)d\theta\|z_0-x^*\|}{(1-w_0(\|x_0-x^*\|))(1-w_0(\|z_0-x^*\|))}\\
&+\frac{3}{2}\frac{(w_0(\|x_0-x^*\|)+w_0(\|y_0-x^*\|))\int_0^1 v(\theta\|z_0-x^*\|)d\theta\|z_0-x^*\|}{(1-w_0(\|x_0-x^*\|))^2}\\
\leq\ & g_3(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < r, \qquad (2.25)
\end{aligned}
$$

which shows (2.15) for $n=0$ and $x_1 \in U(x^*, r)$. The induction for estimates (2.13)-(2.15) can be finished by replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates. Then, from the estimates

$$\|x_{k+1}-x^*\| \leq c\|x_k-x^*\| < r, \qquad (2.26)$$

where $c = g_3(\|x_0-x^*\|) \in [0,1)$, we conclude that $\lim_{k\longrightarrow\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally to show the uniqueness part, let $y^* \in \Omega_2$ with $H(y^*) = 0$. Define the linear operator $T = \int_0^1 H'(x^*+\theta(y^*-x^*))d\theta$. Using (2.8), we obtain that

$$
\begin{aligned}
\|H'(x^*)^{-1}(T-H'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^*-y^*\|)d\theta\\
&\leq \int_0^1 w_0(\theta R^*)d\theta < 1,
\end{aligned} \qquad (2.27)
$$

Hence, we have that $T^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity $0 = H(y^*) - H(x^*) = T(y^*-x^*)$, we conclude that $x^* = y^*$. □

REMARK 2.2. (a) In the case when $w_0(t) = L_0 t, w(t) = Lt$ and $\Omega_0 = \Omega$, the radius $r_A = \frac{2}{2L_0+L}$ was obtained by Argyros in [2] as the convergence radius for Newton's method under condition (2.7)-(2.9). Notice that the convergence radius for Newton's method given independently by Rheinboldt [14] and Traub [17] is given by

$$\rho = \frac{2}{3L} < r_A.$$

As an example, let us consider the function $H(x) = e^x - 1$. Then $x^* = 0$. Set $\Omega = B(0,1)$. Then, we have that $L_0 = e-1 < L = e$, so $\rho = 0.24252961 < r_A = 0.324947231$.

Moreover, the new error bounds [2] are:

$$\|x_{n+1}-x^*\| \leq \frac{L}{1-L_0\|x_n-x^*\|}\|x_n-x^*\|^2,$$

whereas the old ones [5, 7]

$$\|x_{n+1}-x^*\| \leq \frac{L}{1-L\|x_n-x^*\|}\|x_n-x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r_3 to be larger than r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2–7].

(c) The results can be also be used to solve equations where the operator H' satisfies the autonomous differential equation [2–4]:

$$H'(x) = P(H(x)),$$

where $P : \mathcal{B}_2 \longrightarrow \mathcal{B}_2$ is a known continuous operator. Since $H'(x^*) = P(H(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $H(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [15]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } \ n = 0, 1, 2, \ldots.$$

(e) In view of (2.4) and the estimate

$$\begin{aligned} \|H'(x^*)^{-1}H'(x)\| &= \|H'(x^*)^{-1}(H'(x) - H'(x^*)) + I\| \\ &\leq 1 + \|H'(x^*)^{-1}(H'(x) - H'(x^*))\| \leq 1 + w_0(\|x - x^*\|) \end{aligned}$$

condition (2.6) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t)$$

or

$$v(t) = 1 + w_0(r_0),$$

since $t \in [0, r_0)$.

(f) Let us choose $\alpha = 1$ and $\varphi(x, y) = y - F'(y)^{-1}F(y)$. Then, we have in (2.22) with x_k replaced by y_k

$$\|\varphi(x_k, y_k) - x^*\| \leq \frac{\int_0^1 w((1-\theta)\|y_k - x^*\|)d\theta\|y_k - x^*\|}{1 - w_0(g_1(\|x_k - x^*\|)\|x_k - x^*\|)},$$

so we can choose $p = 1$ and

$$\psi(t) = \frac{\int_0^1 w((1-\theta)g_1(t)t)d\theta g_1(t)}{1 - w_0(t)}.$$

(h) Condition $v(0) < 3$ can be dropped as follows. Define $R_0 = g_1(r)r$ and replace (2.12) by

$$\bar{U}(x^*, R_1) \subseteq D, \tag{2.28}$$

where $R_1 = \max\{R_0, r\}$. We must also replace (2.13) by

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_0 - x^*\| \leq R_1. \tag{2.29}$$

Then, the conclusions of Theorem 2.1 hold in this setting without the restrictive condition $v(0) < 3$.

3. Numerical Examples

We present two examples in this section. We choose $\alpha = 1, p = 1$ and ψ as in Remark 2.2 (g) in both examples.

EXAMPLE 3.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function H on D for $w = (x,y,z)^T$ by

$$H(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$H'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using (2.5)–(2.7), we can choose $w_0(t) = L_0 t, w(t) = e^{\frac{1}{L_0}} t, v(t) = e^{\frac{1}{L_0}}, L_0 = e - 1$.

Then, the radius of convergence r is given by

$$r_1 = 0.1544,\ r_2 = 0.0340,\ r_3 = 0.0128 = r.$$

EXAMPLE 3.2. Returning back to the motivational example given at the introduction of this study, we can choose (see also Remark 2.2 (5) for function v)$w_0(t,s) = w(t,s) = \frac{t+s}{16}$ and $v_0(t) = 1 + w_0(r_0) \backsimeq 1.4142$. Then, the radius of convergence r is given by

$$r_1 = 5.6384,\ r_2 = 0.8761,\ r_3 = 0.7651 = r.$$

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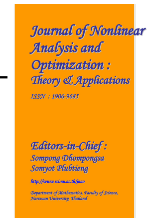

THE CHEBYSHEV WAVELETS OPERATIONAL MATRIX OF INTEGRATION AND PRODUCT OPERATION MATRIX FOR STURM-LIOUVILLE PROBLEM

R. DARZI[1*], B. AGHELI[2]

[1] Department of Mathematics, Neka Branch, Islamic Azad University, Neka, Iran
[2] Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran

ABSTRACT. In this research paper, we present the Chebyshev wavelets operational matrix of integration and product operation matrix. These matrices have been applied to find a solution for Sturm-Liouville problem. We have provided numerical examples to indicate that operational matrix of integration and product operation matrix are applicable for Chebyshev wavelets.

KEYWORDS: Chebyshev wavelets; Operational matrix; Product operational matrix; Sturm-Liouville problem.
AMS Subject Classification: 34Bxx; 34B24; 65T60.

1. INTRODUCTION

In recent years, wavelets have been applied in many different fields of science and engineering. For example, Haar wavelet operational matrix has been extensively used in system analysis [1,2], system identification [3,4], optimal control [5,6], and numerical solution of integral and differential equations [7-15]. Moreover the application of Legendre wavelets [16,17], Hybrid functions [18,19] has received special attention among researchers. In this paper, we have presented Chebyshev wavelets operational matrix and product operation matrix to find a solution for Sturm-Liouville problem. A linear Sturm-Liouville operator has the form:

$$Ky(t) := Ly(t) = \lambda r(t), \tag{1.1}$$

where

$$L := -\frac{d}{dt}[p(t)\frac{d}{dt}] + q(t), \quad t \in I := [a, b].$$

Related to with differential equation (1) are the separated homogeneous boundary conditions $ay(0) + by'(0) = 0$ and $cy(1) + dy'(1) = 0$, in which a, b, c and d, are arbitrary constants. The values of λ for which the boundary value problem has a

* *Corresponding author.*
Email address : r.darzi@iauneka.ac.ir.
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nontrivial solutions are named eigenvalues. To simplify the issue, we will assume that $p(t), p'(t), q(t)$, and $r(t)$ are continuous and $p(t) > 0$ and $r(t) > 0$ for all $t \in I$, for simplicity.
This paper consist of the following section: in section 2, we briefly review basic definitions fractional calculus. In section 3, we demonstrate how we can derive Haar wavelet operational matrix of fractional order integration. We have provided some illustrative examples in section 4 to demonstrate the application of operational matrix of integration for Haar wavelets.
Wavelets are a family of functions which are formed from delation and translation of a single function called the mother wavelet. When the the delation parameter a and translation parameter b vary continuously, the result will be following family of continuous wavelets as [8],

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi(\frac{t-b}{a}), \quad a, b \in R, \quad a \neq 0.$$

If we limit the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$ and $b_0 > 0$, where n and k are positive integers, the family of discrete wavelets are defined as

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}(a_0^k t - nb_0)$$

in which $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular when $a_0 = 2$ and $b_0 = 1$ forms as orthogonal basis. Chebyshev wavelets $\psi_{n,m}$, an the interval $[0, 1)$ are defined as [8]:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}}\widetilde{T}_m(2^{k+1}t - 2n + 1) & \frac{n-1}{2^k} \leq t < \frac{n}{2^k} \\ 0 & otherwise, \end{cases} \tag{1.2}$$

where

$$\widetilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0, \\ \frac{2}{\sqrt{\pi}}T_m(t) & m > 0, \end{cases}$$

and $m = 0, 1, ..., M - 1$, $n = 0, 1, ..., 2^k,$, k is any positive integer and $T_m(t)$ are Chebyshev polynomial of the first kind of degree m which are orthogonal with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-t^2}}$ on the interval $[-1, 1]$ and $T_m(t)$ can be determined by the following recurrence formula:
$T_0(t) = 1$, $T_1(t) = t$, $T_{m+}(t) = 2tT_m(t) - T_{m-1}(t)$, $m = 1, 2, , ...$
The set of Chebyshev wavelets are an orthogonal set with respect to the weight function $\omega_n(t) = \omega(2^{k+1}t - 2n + 1)$.
A function $f(t)$ defined over $[0, 1)$ may be expanded as:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{nm}\psi_{nm}, \tag{1.3}$$

where $f_{nm} =< f(t), \psi_{nm} >$ in which $< \cdot >$ denotes the inner product. If the infinite series in Eq.(3) is shortened, then Eq.(3) can be written as:

$$f(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm} = F^T\psi(t), \tag{1.4}$$

where F and ψ are $2^{k-1}M \times 1$ matrics given by

$$F = [f_{10}, f_{11}, ..., f_{1M-1}, f_{20}, f_{21}, ..., f_{2M-1}, ..., f_{2^k0}, f_{2^k1}, ..., f_{2^kM-1}]^T \tag{1.5}$$

$$\psi(t) = [\psi_{10}, \psi_{11}, ..., \psi_{1M-1}, \psi_{20}, \psi_{21}, ..., \psi_{2M-1}, ..., \psi_{2^k0}, \psi_{2^k1}, ..., \psi_{2^kM-1}]^T. \tag{1.6}$$

2. OPERATIONAL MATRIX OF INTEGRATION

The integration of the vector $\psi(t)$ can be determined as

$$\int_0^t \psi(r)dr = P\psi(t), \tag{2.1}$$

where P is a $(2^k M) \times (2^k M)$ matrix given by:

$$P = \begin{pmatrix} C & S & S & . & . & . & S \\ 0 & C & S & . & . & . & S \\ 0 & 0 & C & . & . & . & S \\ . & . & . & . & & & \\ . & . & . & & . & & \\ . & . & . & & & . & S \\ 0 & 0 & 0 & . & . & . & C \end{pmatrix} \tag{2.2}$$

where s and c are $M \times M$ matrices given by

$$S = \frac{\sqrt{2}}{2^k} \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ -\frac{1}{3} & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ -\frac{1}{15} & 0 & 0 & . & . & . & 0 \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ -\frac{1}{M(M-2)} & 0 & 0 & . & . & . & 0 \end{pmatrix} \tag{2.3}$$

$$C = \frac{1}{2^k} \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & . & . & . & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{8} & 0 & . & . & . & 0 & 0 & 0 \\ -\frac{1}{6\sqrt{2}} & -\frac{1}{4} & 0 & \frac{1}{12} & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & & & . & . & . \\ . & . & . & . & & . & & . & . & . \\ . & . & . & . & & & . & . & . & . \\ -\frac{1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & . & . & . & -\frac{1}{4(M-3)} & 0 & -\frac{1}{4(M-1)} \\ -\frac{1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & . & . & . & 0 & -\frac{1}{4(M-2)} & 0 \end{pmatrix} \tag{2.4}$$

3. CHEBYSHEV WAVELETS PRODUCT OPERATION MATRIX

The following property of the product of two Chebyshev wavelet function vectors will also be applied:

$$\psi(t)\psi^T(t)F \simeq \widetilde{F}\psi(t). \tag{3.1}$$

In this formula, F is given in Eq.(5), $\psi(t)$ can be obtained similarly to Eq.(6) and $\widetilde{F}$ is a $2^k M \times 2^k M$ matrix. Using $\psi(t)$, for $M = 3$ and $k = 1$ we obtain

$$\psi\psi^T = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 2\psi_{10} & 2\psi_{11} & 2\psi_{12} & 0 & 0 & 0 \\ 2\psi_{11} & 2\psi_{10} + \sqrt{2}\psi_{12} & \sqrt{2}\psi_{11} & 0 & 0 & 0 \\ 2\psi_{12} & \sqrt{2}\psi_{11} & 2\psi_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\psi_{20} & 2\psi_{21} & 2\psi_{22} \\ 0 & 0 & 0 & 2\psi_{21} & 2\psi_{20} + \sqrt{2}\psi_{22} & \sqrt{2}\psi_{21} \\ 0 & 0 & 0 & 2\psi_{22} & \sqrt{2}\psi_{21} & 2\psi_{20} \end{pmatrix}. \tag{3.2}$$

Therefore the 6×6 matrix $\widetilde{F}$ in Eq.(11) can be written as

$$\widetilde{F} = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}. \tag{3.3}$$

where B_i, $i = 1, 2$, are 3×3 matrices given by

$$B_i = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 2f_{i0} & 2f_{i1} & 2f_{i2} \\ 2f_{i1} & 2f_{i0} + \sqrt{2}f_{i2} & \sqrt{2}f_{i1} \\ 2f_{i2} & \sqrt{2}f_{i1} & 2f_{i0} \end{pmatrix}. \tag{3.4}$$

In general case, $\widetilde{F}$ is a $2^k M \times 2^k M$ matrix in the form

$$\widetilde{F} = \begin{pmatrix} B_1 & 0 & . & . & . & 0 \\ 0 & B_2 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & B_{2^k} \end{pmatrix}, \tag{3.5}$$

where B_i, $i = 1, 2, ..., 2^k$ are similar to those in Eq.(13).

4. EXAMPLES

Example 4.1. Consider the Sturm-Liouville problem

$$-y''(t) - \lambda y(t) = 0, \tag{4.1}$$

subject to

$$y(0) = -1, \qquad y'(0) = 4. \tag{4.2}$$

The exact solution of system(16-17) is

$$y(t, \lambda) = -\cos(\sqrt{\lambda}t) + \frac{4}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t). \tag{4.3}$$

Now, we solve the same problem using chebyshev wavelets, with $M = 3$ and $k = 1$. Let us suppose

$$y''(t) = Y^T \psi(t), \tag{4.4}$$

where

$$Y = [y_{10}, y_{11}, y_{12}, y_{20}, y_{21}, y_{22}]^T, \tag{4.5}$$

$$\psi = [\psi_{10}, \psi_{11}, \psi_{12}, \psi_{20}, \psi_{21}, \psi_{22}]^T. \tag{4.6}$$

From (19) and (17), we get

$$y'(t) = \int_0^t y''(s)ds + y'(0) = Y^T P\psi(t) + 4, \tag{4.7}$$

and

$$y(t) = \int_0^t y'(s)ds + y(0) = Y^T P^2\psi(t) + 4t - 1. \tag{4.8}$$

Now, if we assume

$$D = [0 \ \frac{\sqrt{\pi}}{2\sqrt{2}} \ 0 \ \sqrt{\pi} \ \frac{\sqrt{\pi}}{2\sqrt{2}} \ 0]^T, \tag{4.9}$$

then

$$y(t) = (Y^T P^2 + D^T)\psi(t). \tag{4.10}$$

Substituting Eqs.(19) and (24-25) into Eq.(16), we obtain

$$-Y^T\psi(t) - \lambda(Y^TP^2 + D^T)\psi(t) = 0. \tag{4.11}$$

Therefore

$$-\psi^T(t)Y - \lambda\psi^T(t)P^{2^T}Y = -\lambda\psi^T(t)D. \tag{4.12}$$

Consequently

$$(I + \lambda P^{2^T})Y = -\lambda D. \tag{4.13}$$

Hence, we can get the same $y(t)$ as the exact solution to the problem.

Example 4.2. (*Airy equation*) Consider the following Airy differential equation

$$y''(t) - ty(t) = 0, \tag{4.14}$$

plus initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{4.15}$$

The exact solution of Eq.(29-30) is demonstrated by

$$y(t) = 1 + \sum_{n=1}^{\infty} \frac{t^{3n}}{(3n)(3n-1)(3n-3)(3n-4)...(3)(2)}. \tag{4.16}$$

Now, we can solve the same problem by applying chebyshev wavelets, with $M = 3$ and $k = 1$.
We suppose that the unknown function $y''(t)$ is given by

$$y''(t) = Y^T\psi(t), \tag{4.17}$$

where

$$Y = [y_{10}, y_{11}, y_{12}, y_{20}, y_{21}, y_{22}]^T, \tag{4.18}$$

Therefore

$$y'(t) = \int_0^t y''(s)ds + y'(0) = Y^TP\psi(t), \tag{4.19}$$

and

$$y(t) = \int_0^t y'(s)ds + y(0) = (Y^TP^2 + D)\psi(t), \tag{4.20}$$

where

$$D_1 = [\frac{\sqrt{\pi}}{2}\ 0\ 0\ \frac{\sqrt{\pi}}{2}\ 0\ 0]^T. \tag{4.21}$$

To addition, we can denote t as

$$t = [\frac{\sqrt{\pi}}{2}\ \frac{\sqrt{\pi}}{8\sqrt{2}}\ 0\ \frac{3\sqrt{\pi}}{8}\ \frac{\sqrt{\pi}}{8\sqrt{2}}\ 0]^T\psi(t) = H^T\psi(t). \tag{4.22}$$

Now, if we substitute Eqs.(32) and (35-37) into Eq.(29), we obtain

$$Y^T\psi(t) - H^T\psi(t)(Y^TP^2 + D_1^T)\psi(t) = 0. \tag{4.23}$$

Therefore

$$\psi^T(t)Y - H^T\psi(t)\psi^T(t)P^{2^T}Y = H^T\psi(t)\psi(t)^TD_1. \tag{4.24}$$

Now, from Eq.(11) we have

$$\psi^TY - \psi^T\widetilde{H}P^{2^T}Y = \psi^T\widetilde{H}D_1, \tag{4.25}$$

or

$$(I - \widetilde{H}P^{2^T})Y = \widetilde{H}D_1, \tag{4.26}$$

where $\widetilde{H}$ can be calculated similarly to Eq.(13). Thus, we can get the same $y(t)$ as the exact solution.

Example 4.3. (*Quantum mechanical harmonic oscillator problem*) Consider The Quantum mechanical harmonic oscillator problem

$$-y''(t) + (t^2 - \lambda)y(t) = 0, \qquad t \in (-\infty, \infty). \tag{4.27}$$

The singular eigenvalue problem (42) possesses the exact analytical solutions of the form

$$y_n^{\infty}(t) = A_n e^{\frac{-x^2}{2}} H_n(t), \qquad \lambda_n^{\infty} = 2n + 1, \qquad n = 0, 1, 2, ... \tag{4.28}$$

where $H_n(t)$ indicates the Hermit polynomials and A_n are normalization constants. Now, suppose the following system (42) on a truncated domain $0 \leq t \leq l$ for all $l > 0$:

$$-y''(t) + (t^2 - \lambda)y(t) = 0, \tag{4.29}$$

subject to

$$y(0) = \sqrt{\pi}, \quad y'(0) = 0, \tag{4.30}$$

featuring boundary values (Dirichlet boundary conditions)

$$y(-l) = y(l) = 0. \tag{4.31}$$

In the same way the previous examples can be set as Setting

$$y''(t) = Y^T \psi(t). \tag{4.32}$$

And using initial conditions, we get

$$y'(t) = \int_0^t y''(s)ds + y'(0) = Y^T P \psi(t), \tag{4.33}$$

and

$$y(t) = \int_0^t y'(s)ds + y(0) = (Y^T P^2 + D_1)\psi(t), \tag{4.34}$$

where

$$D_2 = [\frac{\pi}{4} \ 0 \ 0 \ \frac{\pi}{4} \ 0 \ 0]^T. \tag{4.35}$$

We can also express t^2 as

$$t^2 = [\frac{11\sqrt{\pi}}{64} \ 0 \ \frac{\sqrt{\pi}}{64\sqrt{2}} \ 0 \ \frac{\sqrt{\pi}}{16\sqrt{2}} \ 0]^T \psi(t) = H_1^T \psi(t). \tag{4.36}$$

Substituting Eqs.(45) and (48-50) into Eq.(44), we obtain

$$-Y^T \psi(t) + (H_1^T \psi(t) - \lambda)(Y^T P^2 + D_2^T)\psi(t) = 0. \tag{4.37}$$

Therefore

$$-\psi^T(t)Y + H_1^T \psi(t)\psi^T(t)P^{2^T} Y + H_1^T \psi(t)\psi^T(t)D_2 - \lambda\psi^T(t)P^{2^T} Y - \lambda\psi^T(t)D_2 = 0 \tag{4.38}$$

Table1 : Eigenvalues of Eq.(44)

n	l	λ	$\lvert\lambda - \lambda_n^\infty\rvert$	λ_n^∞
0	2	1.08231643774281	0.08231643774281	1
	π	1.39521365408429	0.39521365408429	
	4.5	1.27710535363838	0.27710535363838	
1	2	1.08231643774291	1.91768356225709	3
	π	3.16650693566247	0.16650693566247	
	4.5	3.29162140783956	0.29162140783656	

Now, from Eq.(11), we will get

$$-\psi^T(t)Y - \psi^T(t)\widetilde{H}_1 P^{2^T} Y + \psi^T(t)\widetilde{H}_1 D_2 - \lambda\psi^T(t)P^{2^T} Y = \lambda\psi^T(t)D_2. \qquad (4.39)$$

Thus

$$(I + \widetilde{H}_1 P^{2^T} - \widetilde{H}_1 D_2 + \lambda P^{2^T})Y = -\lambda D_2, \qquad (4.40)$$

where $\widetilde{H}_1$ can be calculated in the same way as Eq.(13). Equation (53) is a set of algebraic equation which can be solved for Y with parameter λ.
In table 1, by applying Dirichlet boundary conditions $y(-l) = y(l) = 0$, we present the approximate eigenvalues of system (44) for different values of l. The obtained eigenvalues are comparable to the exact eigenvalues of the harmonic oscillator: $\lambda_n^\infty = 2n + 1$ for $n = 0$ and $n = 1$.

5. CONCLUSION

We presented Chebyshev wavelets operational matrix of integration and product operation matrix. These matrices have been applied to find a solution for Sturm-Liouville problem. The approximate examples used in this paper consequently display the efficiency of the present method. Also, the examples provided and all approximate calculations in the present study have been performed on a *PC*, applying programs written in Mathematica.

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COUPLED FIXED POINT THEOREMS FOR GENERALIZED α-ψ-CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED METRIC-TYPE SPACES

NGUYEN TRUNG HIEU*[1] AND VO THI LE HANG[2]

[1]Faculty of Mathematics and Information Technology Teacher Education, Dong Thap University, Cao Lanh City, Dong Thap Province, Vietnam
[2]Journal of Science, Dong Thap University, Cao Lanh City, Dong Thap Province, Vietnam

ABSTRACT. In this paper, we state some coupled fixed point theorems for generalized α-ψ-contractive mappings in partially ordered metric-type spaces. In addition, some particular cases and consequences of our theorems are given. Moreover, we give some examples to illustrate the obtained results.

KEYWORDS: Coupled fixed point; metric-type space; α-ψ-contractive mapping.
AMS Subject Classification: Primary 47H10, 54H25; Secondary 54D99, 54E99.

1. INTRODUCTION

The notion of a metric-type space was introduced by Khamsi in [5] as follows.

Definition 1.1 ([5], Definition 2.7)**.** Let X be a non-empty set, $K \geq 1$ be a real number and $D : X \times X \longrightarrow [0, \infty)$ be a mapping satisfying the following.

(i) $D(x, y) = 0$ if and only if $x = y$;
(ii) $D(x, y) = D(y, x)$ for all $x, y \in X$;
(iii) For all $x, y_1, y_2, \dots, y_n, z \in X$, we have

$$D(x, z) \leq K[D(x, y_1) + D(y_1, y_2) + \dots + D(y_n, z)].$$

Then D is called a *metric-type* on X and (X, D, K) is called a *metric-type space.*

Remark 1.2. (X, d) is a metric space if and only if $(X, d, 1)$ is a metric-type space.

Some other authors in [2], [3] and [4] considered another metric-type space, where the condition (3) in Definition 1.1 is replaced by

$$D(x, y) \leq K[D(x, z) + D(z, y)]$$

* *Corresponding author.*
Email address : ngtrunghieu@dthu.edu.vn (Nguyen Trung Hieu), vtlhang@dthu.edu.vn (Vo Thi Le Hang).
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for all $x, y, z \in X$ and proved several other fixed point and common fixed point results in this metric-type space. In this paper, we consider the metric-type space in the sense of Definition 1.1.

The convergence and the completeness in the metric type-spaces were defined as follows.

Definition 1.3 ([5], Definition 2.8)**.** Let (X, D, K) be a metric-type space and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is said to *converge* to $x \in X$, written as $\lim\limits_{n\to\infty} x_n = x$, if $\lim\limits_{n\to\infty} D(x_n, x) = 0$.

(ii) $\{x_n\}$ is said to be *Cauchy* if $\lim\limits_{n,m\to\infty} D(x_n, x_m) = 0$.

(iii) (X, D, K) is said to be *complete* if every Cauchy sequence is a convergent sequence.

Remark 1.4. On the metric-type space, we always use the topology induced by its convergence.

In [1], Bhaskar and Lakshmikantham introduced the notions of mixed monotone property and coupled fixed point for contractive mapping $F : X \times X \longrightarrow X$, where X is a partially ordered metric space as follows.

Definition 1.5 ([1], Definition 1.1)**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ be a mapping. Then F is said to *have the mixed monotone property* if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \text{ implies } F(x, y_1) \succeq F(x, y_2).$$

Definition 1.6 ([1], Definition 1.2)**.** An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $F : X \times X \longrightarrow X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Moreover, in [1], the authors proved some coupled fixed point theorems for a mixed monotone mapping, see [1, Theorem 2.1], [1, Theorem 2.2] and [1, Theorem 2.4]. Afterwards, in [7], the authors established coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces which extend the results of [1]. For more details on coupled fixed point theory, we also refer the reader to [8, 11, 14] and references therein.

Recently, Samet *et al.* [15] introduced the α-ψ-contractive and the α-admissible mapping with $\alpha : X \times X \longrightarrow [0, \infty)$ and proved fixed point theorems for mappings in complete metric spaces. After that, some authors studied fixed point results for a new α-ψ-contractive and various classes of mappings which are based on α-admissible mappings, see for example [6, 12, 13] and references therein. Most recently, Mursaleen *et al.* [9] introduced the notions of α-admissible mapping with $\alpha : X^2 \times X^2 \longrightarrow [0, \infty)$ and α-ψ-contractive as follows.

Denote by Ψ the family of non-decreasing functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ satisfying:

(i) $\psi^{-1}(0) = 0$;

(ii) $\psi(t) < t$ for all $t > 0$;

(iii) $\lim\limits_{r\to t^+} \psi(r) < t$ for all $t > 0$.

Lemma 1.7 ([9], Lemma 3.1)**.** *If $\psi : [0, \infty) \longrightarrow [0, \infty)$ is non-decreasing and right continuous, then $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \geq 0$ if and only if $\psi(t) < t$ for all $t > 0$.*

Definition 1.8 ([10], Definition 3.2)**.** Let $(X, d, \preceq)$ be a partially ordered metric space and $F : X \times X \longrightarrow X$ be a mapping. Then F is said to be *α-ψ-contractive* if there exist two functions $\alpha : X^2 \times X^2 \longrightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha\big((x, y), (u, v)\big)\frac{d\big(F(x, y), F(u, v)\big) + d\big(F(y, x), F(v, u)\big)}{2} \leq \psi\Big(\frac{d(x, u) + d(y, v)}{2}\Big)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$.

Definition 1.9 ([9], Definition 3.3)**.** Let $F : X \times X \longrightarrow X$ and $\alpha : X^2 \times X^2 \longrightarrow [0, \infty)$ be two mappings. Then F is said to be *α-admissible* if

$$\alpha\big((x, y), (u, v)\big) \geq 1 \text{ implies } \alpha\Big(\big(F(x, y), F(y, x)\big), \big(F(u, v), F(v, u)\big)\Big) \geq 1$$

for all $x, y, u, v \in X$.

Furthermore, in [9], the authors established some coupled fixed point results on partially ordered metric spaces which are generalizations of the main results in [1], see [9, Theorem 3.4], [9, Theorem 3.5] and [9, Theorem 3.6].

The aim of this paper is to state some coupled fixed point theorems for generalized α-ψ-contractive mappings in partially ordered metric-type spaces. In addition, some particular cases and consequences of our theorems are given. Moreover, we give some examples to illustrate the obtained results.

2. MAIN RESULTS

We start with an example about a non-continuous metric-type as follows.

Example 2.1. Let $X = \Big\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\Big\}$ and $D : X \times X \longrightarrow [0, \infty)$ be defined by

$$D(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \text{ and } x, y \in \{0, 1\} \\ |x - y| & \text{if } x, y \in \Big\{0, \frac{1}{n}, \frac{1}{m}\Big\}, \text{ where } n, m \geq 2 \\ \frac{1}{3} & \text{if } x, y \in \Big\{1, \frac{1}{n}\Big\}, \text{ where } n \geq 2. \end{cases}$$

Then, D is a non-continuous metric-type with $K = 3$.

Proof. For all $x, y \in X$, we have $D(x, y) \geq 0$, $D(x, y) = 0$ if and only if $x = y$ and $D(x, y) = D(y, x)$. For all $x, y_1, \ldots, y_k, y \in X, k \geq 1$, we will show that

$$D(x, y) \leq 3[D(x, y_1) + D(y_1, y_2) + \ldots + D(y_k, y)]. \tag{2.1}$$

Put

$$\sigma = D(x, y_1) + D(y_1, y_2) + \ldots + D(y_k, y).$$

We only consider three following cases.

Case 1. $D(x, y) = D(0, 1) = 1$ or $D(x, y) = D\big(1, \frac{1}{n}\big) = \frac{1}{3}$ for $n \geq 2$. Then $\sigma \geq \frac{1}{3}$.

Case 2. $D(x,y) = D\big(0,\frac{1}{n}\big) = \frac{1}{n}$ for $n \geq 2$. Then $\sigma \geq \frac{1}{3}$ if there exists $i \in \{1,\ldots,k\}$ such that $y_i = 1$ and $\sigma \geq \frac{1}{n}$ if $y_i \neq 1$ for all $i = 1,\ldots,k$.

Case 3. $D(x,y) = D\big(\frac{1}{n},\frac{1}{m}\big) = \Big|\frac{1}{n}-\frac{1}{m}\Big|$. Then $\sigma \geq \frac{1}{3}$ if there exists $i \in \{1,\ldots,k\}$ such that $y_i = 1$ and $\sigma \geq \Big|\frac{1}{n}-\frac{1}{m}\Big|$ if $y_i \neq 1$ for all $i = 1,\ldots,k$.

From the above cases, we conclude that (2.1) holds. This proves that D is a metric-type on X with $K = 3$.

Now, we have $\lim\limits_{n\to\infty} D\big(\frac{1}{n},0\big) = \lim\limits_{n\to\infty}\frac{1}{n} = 0$. However, $\lim\limits_{n\to\infty} D\big(\frac{1}{n},1\big) = \frac{1}{3} \neq 1 = D(0,1)$. This proves that D is non-continuous. □

Next, we introduce the notion of a generalized α-ψ-contractive mapping in a partially ordered metric-type space as follows.

Definition 2.2. Let $(X,D,K,\preceq)$ be a partially ordered metric-type space and $F : X\times X \longrightarrow X$ be a mapping. Then F is said to be *generalized α-ψ-contractive* if there exist two functions $\alpha : X^2\times X^2 \longrightarrow [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha\big((x,y),(u,v)\big)\frac{D\big(F(x,y),F(u,v)\big)+D\big(F(y,x),F(v,u)\big)}{2} \leq \psi\Big(\frac{M(x,y,u,v)}{2}\Big) \tag{2.2}$$

for all $x,y,u,v \in X$ with $x \preceq u$ and $y \succeq v$, where

$$\begin{aligned} M(x,y,u,v) \;=\; \max\Big\{ & D(u,F(x,y))+D(v,F(y,x)), D(x,F(x,y))+D(y,F(y,x)),\\ & D(x,u)+D(y,v), \frac{D(u,F(u,v))+D(v,F(v,u))}{2K},\\ & \frac{D(x,F(u,v))+D(y,F(v,u))}{2K}\Big\}. \end{aligned}$$

Our first result is the following.

Theorem 2.3. *Let $(X,D,K,\preceq)$ be a partially ordered and complete metric-type space and $F : X\times X \longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *F is generalized α-ψ-contractive;*
(ii) *F is α-admissible;*
(iii) *F is continuous;*
(iv) *There exist $x_0,y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$, $y_0 \succeq F(y_0,x_0)$ and*
$$\alpha\big((x_0,y_0),\big(F(x_0,y_0),F(y_0,x_0)\big)\big) \geq 1.$$

Then F has a coupled fixed point.

Proof. Let $x_0,y_0 \in X$ be such that $x_0 \preceq F(x_0,y_0)$, $y_0 \succeq F(y_0,x_0)$ and

$$\alpha\big((x_0,y_0),\big(F(x_0,y_0),F(y_0,x_0)\big)\big) \geq 1.$$

Let $x_1,y_1 \in X$ be such that $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. Let $x_2,y_2 \in X$ be such that $F(x_1,y_1) = x_2$ and $F(y_1,x_1) = y_2$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n,y_n) \text{ and } y_{n+1} = F(y_n,x_n)$$

for all $n \in \mathbb{N}$. We will show that

$$x_n \preceq x_{n+1}, \quad y_n \succeq y_{n+1} \tag{2.3}$$

for all $n \in \mathbb{N}$ by the mathematical induction.

Let $n = 0$. We have $x_0 \preceq F(x_0, y_0) = x_1$ and $y_0 \succeq F(y_0, x_0) = y_1$. Thus, (2.3) holds for $n = 0$. Now, suppose that (2.3) holds for some fixed $n \in \mathbb{N}$. Then, since $x_n \preceq x_{n+1}$, $y_n \succeq y_{n+1}$ and the mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = y_{n+1}.$$

From the above, we have $x_{n+1} \preceq x_{n+2}$ and $y_{n+1} \succeq y_{n+2}$. Therefore, by the mathematical induction, we conclude that (2.3) holds for all $n \in \mathbb{N}$.

If there exists some $n \in \mathbb{N}$ such that $x_{n+1} = x_n$ and $y_{n+1} = y_n$, then $F(x_n, y_n) = x_n$ and $F(y_n, x_n) = y_n$, that is, F has a coupled fixed point. Now, we assume that $x_{n+1} \neq x_n$ or $y_{n+1} \neq y_n$ for all $n \in \mathbb{N}$. Since F is α-admissible and

$$\alpha\big((x_0, y_0), (x_1, y_1)\big) = \alpha\big((x_0, y_0), \big(F(x_0, y_0), F(y_0, x_0)\big)\big) \geq 1,$$

we get $\alpha\big(\big(F(x_0, y_0), F(y_0, x_0)\big), \big(F(x_1, y_1), F(y_1, x_1)\big)\big) \geq 1$. Thus,

$$\alpha\big((x_1, y_1), (x_2, y_2)\big) \geq 1.$$

By the mathematical induction, we have

$$\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1 \tag{2.4}$$

for all $n \in \mathbb{N}$. Since F is generalized α-ψ-contractive and using (2.3), (2.4), we get

$$\begin{aligned}
&\frac{D(x_n, x_{n+1}) + D(y_n, y_{n+1})}{2} \\
= \;& \frac{D\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) + D\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big)}{2} \\
\leq \;& \alpha\big((x_{n-1}, y_{n-1}), (x_n, y_n)\big) \times \\
& \quad \times \frac{D\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) + D\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big)}{2} \\
\leq \;& \psi\Big(\frac{M(x_{n-1}, y_{n-1}, x_n, y_n)}{2}\Big)
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
& M(x_{n-1}, y_{n-1}, x_n, y_n) \\
= \;& \max\Big\{ D(x_n, F(x_{n-1}, y_{n-1})) + D(y_n, F(y_{n-1}, x_{n-1})), \\
& \quad D(x_{n-1}, F(x_{n-1}, y_{n-1})) + D(y_{n-1}, F(y_{n-1}, x_{n-1})), \\
& \quad D(x_{n-1}, x_n) + D(y_{n-1}, y_n), \frac{D(x_n, F(x_n, y_n)) + D(y_n, F(y_n, x_n))}{2K}, \\
& \quad \frac{D(x_{n-1}, F(x_n, y_n)) + D(y_{n-1}, F(y_n, x_n))}{2K} \Big\} \\
= \;& \max\Big\{ D(x_n, x_n) + D(y_n, y_n)), D(x_{n-1}, x_n) + D(y_{n-1}, y_n), \\
& \quad D(x_{n-1}, x_n) + D(y_{n-1}, y_n), \frac{D(x_n, x_{n+1}) + D(y_n, y_{n+1})}{2K}, \\
& \quad \frac{D(x_{n-1}, x_{n+1}) + D(y_{n-1}, y_{n+1})}{2K} \Big\}
\end{aligned}$$

$$\begin{aligned}
&\le \max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1}),\\
&\qquad \frac{D(x_{n-1},x_n)+D(y_{n-1},y_n)+D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2}\Big\}\\
&= \max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1})\Big\}. \qquad (2.6)
\end{aligned}$$

From (2.5) and (2.6), we have

$$\begin{aligned}
&\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2}\\
&\le \psi\Big(\frac{\max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1})\Big\}}{2}\Big). \qquad (2.7)
\end{aligned}$$

If there exists some $n \ge 1$ such that

$$\max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1})\Big\} = D(x_n,x_{n+1})+D(y_n,y_{n+1}),$$

then (2.7) becomes

$$\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2} \le \psi\Big(\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2}\Big).$$

It is a contradiction to $\psi(t) < t$ for all $t > 0$. Therefore,

$$\max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1})\Big\} = D(x_{n-1},x_n)+D(y_{n-1},y_n)$$

for all $n \ge 1$. Then, (2.7) becomes

$$\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2} \le \psi\Big(\frac{D(x_{n-1},x_n)+D(y_{n-1},y_n)}{2}\Big). \qquad (2.8)$$

Repeating the above process, we get

$$\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2} \le \psi^n\Big(\frac{D(x_0,x_1)+D(y_0,y_1)}{2}\Big) \qquad (2.9)$$

for all $n \ge 1$. For $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n\ge n(\varepsilon)} \psi^n\Big(\frac{D(x_0,x_1)+D(y_0,y_1)}{2}\Big) < \frac{\varepsilon}{2K}. \qquad (2.10)$$

Let $n, m \in \mathbb{N}$ be such that $m > n > n(\varepsilon)$. Then, by using (2.10), we have

$$\begin{aligned}
&\frac{D(x_n,x_m)+D(y_n,y_m)}{2}\\
&\le K\sum_{k=n}^{m-1}\frac{D(x_k,x_{k+1})+D(y_k,y_{k+1})}{2}\\
&\le K\sum_{k=n}^{m-1}\psi^k\Big(\frac{D(x_0,x_1)+D(y_0,y_1)}{2}\Big)\\
&\le K\sum_{n\ge n(\varepsilon)}\psi^n\Big(\frac{D(x_0,x_1)+D(y_0,y_1)}{2}\Big) < \frac{\varepsilon}{2}. \qquad (2.11)
\end{aligned}$$

It implies that $D(x_n,x_m)+D(y_n,y_m) < \varepsilon$. Therefore,

$$D(x_n,x_m) \le D(x_n,x_m)+D(y_n,y_m) < \varepsilon$$

and

$$D(y_n,y_m) \le D(x_n,x_m)+D(y_n,y_m) < \varepsilon.$$

This implies $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in (X, D, K). Since X is a complete metric-type space, we have $\{x_n\}$ and $\{y_n\}$ are convergent in (X, D, K). Then there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y \tag{2.12}$$

Since F is continuous and $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$, taking the limit as $n \to \infty$, we get

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y, x),$$

that is, $F(x, y) = x$ and $F(y, x) = y$. Therefore, F has a coupled fixed point. □

In the next theorem, we omit continuous hypothesis of F.

Theorem 2.4. *Let $(X, D, K, \preceq)$ be a partially ordered and complete metric-type space, where D is continuous in each variable and $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *F is generalized α-ψ-contractive;*
(ii) *F is α-admissible;*
(iii) *If $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha\big((x_n, y_n), (x, y)\big) \geq 1$;*
(iv) *There exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $\quad y_0 \succeq F(y_0, x_0)$ and*
$$\alpha\big((x_0, y_0), \big(F(x_0, y_0), F(y_0, x_0)\big)\big) \geq 1.$$

Then F has a coupled fixed point.

Proof. Following the proof of Theorem 2.3, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \tag{2.13}$$

and

$$\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1 \tag{2.14}$$

for all $n \in \mathbb{N}$. By using (2.13), (2.14) and hypothesis (3), we get

$$\alpha\big((x_n, y_n), (x, y)\big) \geq 1 \tag{2.15}$$

for all $n \in \mathbb{N}$. Since F is generalized α-ψ-contractive and using (2.15), we get

$$\begin{aligned}
&\frac{D(F(x,y),x) + D(F(y,x),y)}{2} \\
\leq\ & K\frac{D(F(x,y),F(x_n,y_n)) + D(F(y,x),F(y_n,x_n))}{2} \\
&\quad + K\frac{D(F(x_n,y_n),x) + D(F(y_n,x_n),y)}{2} \\
=\ & K\frac{D(F(x,y),F(x_n,y_n)) + D(F(y,x),F(y_n,x_n))}{2} \\
&\quad + K\frac{D(x_{n+1},x) + D(y_{n+1},y)}{2} \\
\leq\ & K\alpha\big((x_n,y_n),(x,y)\big)\frac{D(F(x,y),F(x_n,y_n)) + D(F(y,x),F(y_n,x_n))}{2} \\
&\quad + K\frac{D(x_{n+1},x) + D(y_{n+1},y)}{2}
\end{aligned}$$

$$
\begin{aligned}
&\le K\psi\Big(\frac{M(x_n,y_n,x,y)}{2}\Big)+K\frac{D(x_{n+1},x)+D(y_{n+1},y)}{2}\\
&\le K\frac{M(x_n,y_n,x,y)}{2}+K\frac{D(x_{n+1},x)+D(y_{n+1},y)}{2}, \qquad (2.16)
\end{aligned}
$$

where

$$
\begin{aligned}
& M(x_n,y_n,x,y)\\
= &\max\Big\{D(x,F(x_n,y_n))+D(y,F(y_n,x_n)), D(x_n,F(x_n,y_n))+D(y_n,F(y_n,x_n)),\\
&D(x_n,x)+D(y_n,y), \frac{D(x,F(x,y))+D(y,F(y,x))}{2K},\\
&\frac{D(x_n,F(x,y))+D(y_n,F(y,x))}{2K}\Big\}\\
= &\max\Big\{D(x,x_{n+1})+D(y,y_{n+1}), D(x_n,x_{n+1})+D(y_n,y_{n+1}),\\
&D(x_n,x)+D(y_n,y), \frac{D(x,F(x,y))+D(y,F(y,x))}{2K},\\
&\frac{D(x_n,F(x,y))+D(y_n,F(y,x))}{2K}\Big\}. \qquad (2.17)
\end{aligned}
$$

Letting $n\to\infty$ in (2.17), using (2.12) and the continuity in each variable property of D, we get

$$\lim_{n\to\infty} M(x_n,y_n,x,y) = \frac{D(x,F(x,y))+D(y,F(y,x))}{2K}. \qquad (2.18)$$

Letting $n\to\infty$ in (2.16), using (2.13) and (2.18), we obtain

$$D(x,F(x,y))+D(y,F(y,x)) \le \frac{D(x,F(x,y))+D(y,F(y,x))}{2}.$$

It implies $D(x,F(x,y))+D(y,F(y,x))=0$. Hence, $D(x,F(x,y))=D(y,F(y,x))=0$. Therefore, $F(x,y)=x$ and $F(y,x)=y$. Thus, F has a coupled fixed point. □

In the following theorem, we will prove the uniqueness of the coupled fixed point. If $(X,\preceq)$ is a partially ordered set, then we endow the product $X\times X$ with the partially ordered relation as follows.

$$(x,y)\preceq(u,v) \Longleftrightarrow x\preceq u, \quad y\succeq v$$

for all $(x,y),(u,v)\in X\times X$.

Theorem 2.5. *In addition to the hypothesis of Theorem 2.3 or Theorem 2.4, suppose that for every $(x,y),(s,t)$ in $X\times X$, there exists (u,v) in $X\times X$ such that (u,v) is comparable to (x,y), (s,t) and*

$$\alpha\big((x,y),(u,v)\big)\ge 1, \quad \alpha\big((s,t),(u,v)\big)\ge 1.$$

Then F has a unique coupled fixed point.

Proof. Following the proof of Theorem 2.3 and Theorem 2.4, F has a coupled fixed point. Suppose that (x,y) and (s,t) are two coupled fixed points of F. By the assumption, there exists (u,v) in $X\times X$ such that (u,v) is comparable to (x,y) and (s,t) and

$$\alpha\big((x,y),(u,v)\big)\ge 1, \quad \alpha\big((s,t),(u,v)\big)\ge 1. \qquad (2.19)$$

We define two sequences $\{u_n\}$ and $\{v_n\}$ as follows

$$u_0=u, \quad v_0=v, \quad u_{n+1}=F(u_n,v_n), \quad v_{n+1}=F(v_n,u_n)$$

for all $n \in \mathbb{N}$.

Since (u,v) is comparable to (x,y), we may assume that $(x,y) \preceq (u,v) = (u_0, v_0)$. By using the mathematical induction and the mixed monotone property of F, we can show that $x \preceq u_n$ and $y \succeq v_n$ for all $n \in \mathbb{N}$.

If $u_n = x$ and $v_n = y$ for all $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} u_n = x$ and $\lim_{n\to\infty} v_n = y$. Now, we assume that $u_n \neq x$ or $v_n \neq y$ for some $n \in \mathbb{N}$. Since F is α-admissible and using (2.19), we have

$$\alpha\big((F(x,y), F(y,x)), (F(u,v), F(v,u))\big) \geq 1.$$

Since $u_0 = u$ and $v_0 = v$, we get

$$\alpha\big((F(x,y), F(y,x)), (F(u_0,v_0), F(v_0,u_0))\big) \geq 1.$$

Thus, $\alpha\big((x,y),(u_1,v_1)\big) \geq 1$. Therefore, by the mathematical induction, we obtain

$$\alpha\big((x,y),(u_n,v_n)\big) \geq 1 \tag{2.20}$$

for all $n \in \mathbb{N}$. Since F is generalized α-ψ-contractive and (2.20), we get

$$\begin{aligned}
&\frac{D(x,u_{n+1}) + D(y,v_{n+1})}{2}\\
=\ & \frac{D(F(x,y),F(u_n,v_n)) + D(F(y,x),F(v_n,u_n))}{2}\\
\leq\ & \alpha\big((x,y),(u_n,v_n)\big)\frac{D(F(x,y),F(u_n,v_n)) + D(F(y,x),F(v_n,u_n))}{2}\\
\leq\ & \psi\Big(\frac{M(x,y,u_n,v_n)}{2}\Big)
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
& M(x,y,u_n,v_n)\\
=\ & \max\Big\{D(u_n,F(x,y)) + D(v_n,F(y,x)), D(x,F(x,y)) + D(y,F(y,x)),\\
& \qquad D(x,u_n) + D(y,v_n), \frac{D(u_n,F(u_n,v_n)) + D(v_n,F(v_n,u_n))}{2K},\\
& \qquad \frac{D(x,F(u_n,v_n)) + D(y,F(v_n,u_n))}{2K}\Big\}\\
=\ & \max\Big\{D(u_n,x) + D(v_n,y), D(x,x) + D(y,y),\\
& \qquad D(x,u_n) + D(y,v_n), \frac{D(u_n,u_{n+1}) + D(v_n,v_{n+1})}{2K},\\
& \qquad \frac{D(x,u_{n+1}) + D(y,v_{n+1})}{2K}\Big\}\\
=\ & \max\Big\{D(x,u_n) + D(y,v_n), \frac{D(u_n,u_{n+1}) + D(v_n,v_{n+1})}{2K},\\
& \qquad \frac{D(x,u_{n+1}) + D(y,v_{n+1})}{2K}\Big\}\\
\leq\ & \max\Big\{D(x,u_n) + D(y,v_n), \frac{D(x,u_n) + D(y,v_n) + D(x,u_{n+1}) + D(y,v_{n+1})}{2},\\
& \qquad D(x,u_{n+1}) + D(y,v_{n+1})\Big\}\\
=\ & \max\Big\{D(x,u_n) + D(y,v_n), D(x,u_{n+1}) + D(y,v_{n+1})\Big\}.
\end{aligned} \tag{2.22}$$

From (2.21) and (2.22), we have

$$\begin{aligned}&\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}\\ \leq\ &\psi\Big(\frac{\max\big\{D(x,u_n)+D(y,v_n),D(x,u_{n+1})+D(y,v_{n+1})\big\}}{2}\Big).\end{aligned} \tag{2.23}$$

If there exists some $n\in\mathbb{N}$ such that

$$\max\big\{D(x,u_n)+D(y,v_n),D(x,u_{n+1})+D(y,v_{n+1})\big\}=D(x,u_{n+1})+D(y,v_{n+1}),$$

then (2.23) becomes

$$\begin{aligned}\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2} &\leq \psi\Big(\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}\Big)\\ &< \frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}.\end{aligned} \tag{2.24}$$

It is a contradiction. Therefore,

$$\max\big\{D(x,u_n)+D(y,v_n),D(x,u_{n+1})+D(y,v_{n+1})\big\}=D(x,u_n)+D(y,v_n)$$

for all $n\in\mathbb{N}$, then (2.23) becomes

$$\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}\leq\psi\Big(\frac{D(x,u_n)+D(y,v_n)}{2}\Big).$$

Repeating the above process, we get

$$\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}\leq\psi^n\Big(\frac{D(x,u_1)+D(y,v_1)}{2}\Big) \tag{2.25}$$

for $n\geq 1$. Letting $n\to\infty$ in (2.25) and using Lemma 1.7, we get

$$\lim_{n\to\infty}\big(D(x,u_{n+1})+D(y,v_{n+1})\big)=0.$$

This implies that $\lim_{n\to\infty}D(x,u_{n+1})=\lim_{n\to\infty}D(y,v_{n+1})=0$. Thus, $\lim_{n\to\infty}u_n=x$ and $\lim_{n\to\infty}v_n=y$. Therefore, from the above, we have

$$\lim_{n\to\infty}u_n=x,\quad \lim_{n\to\infty}v_n=y. \tag{2.26}$$

Similarly, we can show that

$$\lim_{n\to\infty}u_n=s,\quad \lim_{n\to\infty}v_n=t. \tag{2.27}$$

From (2.26) and (2.27), we conclude that $x=s$ and $y=t$. Hence, F has a unique coupled fixed point. □

Since every metric space (X,d) is a metric-type space $(X,d,1)$, from Theorem 2.3, Theorem 2.4 and Theorem 2.5, we get two following corollaries.

Corollary 2.6. *Let $(X,d,\preceq)$ be a partially ordered and complete metric space and $F:X\times X\longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *There exist two functions $\alpha:X^2\times X^2\longrightarrow[0,\infty)$ and $\psi\in\Psi$ such that*

$$\alpha\big((x,y),(u,v)\big)\frac{d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)}{2}\quad\leq\quad\psi\Big(\frac{N(x,y,u,v)}{2}\Big)$$

for all $x,y,u,v\in X$ with $x\succeq u$ and $y\preceq v$, where

$$\begin{aligned}N(x,y,u,v)\ =\ \max\Big\{&d(u,F(x,y))+d(v,F(y,x)),d(x,F(x,y))+d(y,F(y,x)),\\ &d(x,u)+d(y,v),\frac{d(u,F(u,v))+d(v,F(v,u))}{2},\end{aligned}$$

$$\left.\frac{d(x,F(u,v))+d(y,F(v,u))}{2}\right\};$$

(ii) *F is α-admissible;*
(iii) *Suppose either*
 (a) *F is continuous or*
 (b) *If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha\big((x_n, y_n), (x, y)\big) \geq 1$;*
(iv) *There exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$ and*

$$\alpha\big((x_0, y_0), \big(F(x_0, y_0), F(y_0, x_0)\big)\big) \geq 1.$$

Then F has a coupled fixed point.

Corollary 2.7. *In addition to the hypothesis of Corollary 2.6, suppose that for every $(x, y), (s, t)$ in $X \times X$, there exists (u, v) in $X \times X$ such that (u, v) is comparable to (x, y), (s, t) and*

$$\alpha\big((x, y), (u, v)\big) \geq 1, \quad \alpha\big((s, t), (u, v)\big) \geq 1.$$

Then F has a unique coupled fixed point.

Remark 2.8. We see that [9, Theorem 3.4] and [9, Theorem 3.5] are two direct consequences of Corollary 2.6, [9, Theorem 3.6] is a direct consequence of Corollary 2.7.

By using similar arguments as in the proofs of [15, Theorem 3.4], [15, Theorem 3.5] and [15, Theorem 3.6], from Theorem 2.3, Theorem 2.4 and Theorem 2.5, we get following results.

Proposition 2.9. *Let $(X, D, K, \preceq)$ be a partially ordered and complete metric-type space and $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *There exists $\lambda \in [0, 1)$ such that*

$$\begin{aligned}
& D\big(F(x,y),F(u,v)\big)+D\big(F(y,x),F(v,u)\big)\\
\leq\ & \lambda\max\Big\{D(u,F(x,y))+D(v,F(y,x)),D(x,F(x,y))+D(y,F(y,x)),\\
& \qquad D(x,u)+D(y,v),\frac{D(u,F(u,v))+D(v,F(v,u))}{2K},\\
& \qquad\qquad \frac{D(x,F(u,v))+D(y,F(v,u))}{2K}\Big\},
\end{aligned}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$;
(ii) *F is continuous;*
(iii) *There exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$.*

Then F has a coupled fixed point.

Proposition 2.10. *Let $(X, D, K, \preceq)$ be a partially ordered and complete metric-type space where D is continuous in each variable and $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *There exists $\lambda \in [0, 1)$ such that*

$$\begin{aligned}
& D\big(F(x,y),F(u,v)\big)+D\big(F(y,x),F(v,u)\big)\\
\leq\ & \lambda\max\Big\{D(u,F(x,y))+D(v,F(y,x)),D(x,F(x,y))+D(y,F(y,x)),\\
& \qquad D(x,u)+D(y,v),\frac{D(u,F(u,v))+D(v,F(v,u))}{2K},
\end{aligned}$$

$$\frac{D(x,F(u,v))+D(y,F(v,u))}{2K}\Big\},$$

for all $x,y,u,v\in X$ *with* $x\succeq u$ *and* $y\preceq v$*;*

(ii) X *has the following properties: If* $\{x_n\}$ *is a non-decreasing sequence in* X *such that* $\lim\limits_{n\to\infty} x_n = x$ *and* $\{y_n\}$ *is a non-increasing sequence in* X *such that* $\lim\limits_{n\to\infty} y_n = y$*, then* $x_n \preceq x$ *and* $y_n \succeq y$ *for all* $n\in\mathbb{N}$*;*

(iii) *There exist* $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0,y_0)$ *and* $y_0 \succeq F(y_0,x_0)$*.*

Then F *has a coupled fixed point.*

Proposition 2.11. *In addition to the hypothesis of Corollary 2.9 or Corollary 2.10, suppose that for every* $(x,y),(s,t)$ *in* $X\times X$*, there exists* (u,v) *in* $X\times X$ *such that* (u,v) *is comparable to* (x,y) *and* (s,t)*. Then* F *has a unique coupled fixed point.*

Finally, in order to support the useability of our results, let us introduce some following examples.

Example 2.12. Let $X=\{1,2,3\}$ with the partially ordered relation as follows.

$$x\succeq y \text{ if and only if } x\geq y \text{ and } x,y\in\{1,2\}.$$

Define a function $D: X\times X \longrightarrow [0,\infty)$ such that

$$D(1,1)=D(2,2)=D(3,3)=0,$$

$$D(1,2)=D(2,1)=D(1,3)=D(3,1)=1,$$

$$D(2,3)=D(3,2)=4.$$

Then, (X,D,K) is a complete metric-type space with $K=2$. Consider a mapping $F: X\times X\longrightarrow X$ by

$$F(1,1)=F(2,2)=F(2,1)=F(1,2)=1,$$

$$F(3,3)=F(3,1)=F(1,3)=F(2,3)=F(3,2)=2.$$

Define a function $\alpha: X^2\times X^2 \longrightarrow [0,\infty)$ by

$$\alpha\big((x,y),(u,v)\big)=\begin{cases} 1 & \text{if } x=y=u=v=1,\\ \dfrac{1}{2} & \text{if otherwise.}\end{cases}$$

Then, for all $(x,y),(u,v)\in X\times X$ with $x\succeq u, y\preceq v$, we have

$$\begin{aligned} & \alpha\big((x,y),(u,v)\big)\frac{D\big(F(x,y),F(u,v)\big)+D\big(F(y,x),F(v,u)\big)}{2}\\ = \ & \alpha\big((x,y),(u,v)\big)\frac{D(1,1)+D(1,1)}{2}\\ = \ & 0\\ \leq \ & \psi\Big(\frac{M(x,y,u,v)}{2}\Big).\end{aligned}$$

Therefore, (2.2) holds for all $\psi\in\Psi$, and also the hypothesis of Theorem 2.3 are fulfilled. Therefore, there exists a coupled fixed point of F. In this case, $(1,1)$ is a coupled fixed point of F.

The following example show that Corollary 2.6 is proper generalization of some results in [9].

Example 2.13. Let $X = \{0, 1, 2\}$ with the usual order $\leq$ on $\mathbb{R}$ and d be defined by

$$d(0,0) = d(1,1) = d(2,2) = 0, d(1,2) = d(2,1) = 4,$$
$$d(0,1) = d(1,0) = d(0,2) = d(2,0) = 2.$$

Define a mapping $F : X \times X \longrightarrow X$ as follows

$$F(0,1) = F(1,1) = F(2,1) = 1,$$
$$F(0,0) = F(1,0) = F(2,0) = 2,$$
$$F(0,2) = F(1,2) = F(2,2) = 0.$$

Consider a function $\psi(t) = \dfrac{t}{2}$ for all $t \geq 0$ and a function $\alpha : X^2 \times X^2 \longrightarrow [0, \infty)$ such that

$$\alpha\big((x,y),(u,v)\big) = \begin{cases} 1 & \text{if } x = y = u = v = 1, \\ \dfrac{3}{10} & \text{if otherwise.} \end{cases}$$

Then (X, d) is a complete metric space. For all $(x, y), (u, v) \in X \times X$ with $x \preceq u, y \succeq v$, we put

$$\sigma_1 = (u, F(x,y)) + d(v, F(y,x)), \quad \sigma_2 = d(x, F(x,y)) + d(y, F(y,x)),$$
$$\sigma_3 = \frac{d(u, F(u,v)) + d(v, F(v,u))}{2}, \quad \sigma_4 = \frac{d(x, F(u,v)) + d(y, F(v,u))}{2},$$
$$\sigma_5 = d(x,u) + d(y,v), \quad N = \max\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\},$$
$$L = \frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2}.$$

Then, we have the following table.

u	v	x	y	L	σ_4	σ_3	σ_2	σ_1	σ_5	N
0	0	0	0	0	2	2	4	4	0	4
0	0	0	1	2	3	2	6	4	2	6
0	0	0	2	1	1	2	0	2	2	2
0	1	0	1	0	3	3	6	6	0	6
0	1	0	2	1	1	3	0	4	4	4
0	2	0	2	0	0	0	0	0	0	0
1	0	0	0	2	2	3	4	6	2	6
1	0	0	1	4	1	3	6	2	4	6
1	0	0	2	3	3	3	0	4	4	4
1	1	0	1	2	1	0	6	4	2	6
1	1	0	2	3	3	0	0	6	6	6
1	2	0	2	2	2	3	0	2	2	3
1	0	1	0	0	3	3	6	6	0	6
1	0	1	1	2	2	3	0	2	2	3
1	0	1	2	1	4	3	6	4	2	6
1	1	1	1	0	0	0	0	0	0	0
1	1	1	2	1	2	0	6	4	4	6
1	2	1	2	0	3	3	6	4	0	6
2	0	0	0	1	1	0	4	2	2	4
2	0	0	1	3	2	0	6	6	4	6
2	0	0	2	2	2	0	0	4	4	4
2	1	0	1	1	2	3	6	6	2	6
2	1	0	2	2	2	3	0	6	6	6
2	2	0	2	1	1	2	0	2	2	2

2	0	1	0	1	2	0	6	2	4	6
2	0	1	1	3	3	0	0	6	6	6
2	0	1	2	2	3	0	6	4	6	6
2	1	1	1	1	1	3	0	4	4	4
2	1	1	2	2	1	3	6	2	8	8
2	2	1	2	1	2	2	6	6	4	6
2	0	2	0	0	0	0	0	0	0	0
2	0	2	1	2	1	0	6	4	2	6
2	0	2	2	1	1	0	4	2	2	4
2	1	2	1	0	3	3	6	6	0	6
2	1	2	2	1	3	3	4	4	4	4
2	2	2	2	0	2	2	4	4	0	4

Now, let $(x, y, u, v) = (1, 0, 0, 1)$, we have

$$\alpha\big((x,y),(u,v)\big)\frac{d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)}{2}$$
$$= \frac{3}{10}.4 = \frac{6}{5} > 1 = \psi(2) = \psi\Big(\frac{d(1,0)+d(0,1)}{2}\Big).$$

Therefore, [9, Theorem 3.4] and [9, Theorem 3.5] are not applicable to F, (X, d), α and ψ. Otherwise, the above calculations show that assumption (1) of Corollary 2.6 holds. Moreover, the assumptions of Corollary 2.6 are fulfilled. Therefore, there exists a coupled fixed point of F. In this case, $(1, 1)$ is a coupled fixed point of F.

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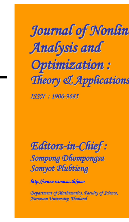

AN EXISTENCE RESULT FOR AN ELLIPTIC PROBLEM INVOLVING A FOURTH ORDER OPERATOR

ANASS OURRAOUI*

Department of Mathematics, ENSAH, University of Mohamed First, Oujda

ABSTRACT. In this paper, we prove the existence of solutions for a $p-$Biharmonic in bounded domain, by applying the Bohnenblust-Karlin fixed point theorem. The regularity of a such solution is also established.

KEYWORDS : $p-$Biharmonic, Fixed point Theorem.
AMS Subject Classification: 35J30 , 35J60, 35J92.

1. INTRODUCTION

We consider in this paper the critical situation, which is devoted to the study of the $p-$Biharmonic problem

$$(\mathcal{P}) \begin{cases} \Delta_p^2 u = V(x) \mid u \mid^{p^*-2} u + f(x,u) \text{ in } \Omega, \\ u \in \mathcal{D}_0^{2,p}(\Omega), \ \frac{N}{2} > p, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the $p-$Biharmonic operator with $\Delta u = div\,(\nabla u)$ is the Laplace operator, $1 < p < \frac{N}{2}$, $p^* = Np/(N-2p), V \in L^\infty(\Omega), V > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function where $f(x,0) \neq 0$.

The nonlinear boundary value problem involving the $p-$Biharmonic operator appears in physics and related sciences such as quantum mechanics, surface diffusion on solids, flow in Hele-Shaw cells and also furnishes a model for studying traveling wave in suspension bridges (cf.[12, 15]).

There are many results relating to these problems which have been widely studied in bounded domains. For example we just refer to [2, 4, 9, 10, 12, 13, 14, 19]... This work is motivated by the papers [1] and [8]. Our problem aroused an interesting result because of the lack of compactness, so we could not use the standard variational methods, here by means of the point fixed due to Bohnenblust-Karlin, we shall prove the existence of solution.

* *Corresponding author.*
Email address : anas.our@hotmail.com.
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Let us record the following definition,

Definition 1.1. We say that u is a weak solution for problem $(\mathcal{P})$ if

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx - \int_\Omega V(x)|u|^{p^*-2}uvdx - \int_\Omega f(x,u)vdx = 0, \forall v \in \mathcal{D}_0^{2,p}(\Omega).$$

We recall that $\mathcal{D}_0^{2,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with the norm

$$\| u \| = \left(\int_\Omega |\Delta u|^p dx\right)^{\frac{1}{p}}.$$

Our main theorem is stated below.

Theorem 1.1. *Under the standing hypothesis*
(F) $|f(x,t)| \leq C[1+|t|^{r-1}], C > 0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$ *with* $r \in (1,p)$*, then problem* $(\mathcal{P})$ *has a weak solution. Furthermore, this solution belongs to* $L^\infty(\Omega)$.

We state the Bohnenblust-Karlin Theorem which provide a platform to establish the main result of the paper.

Theorem 1.2. *(cf.[5, 6, 17]) Let* D *be a nonempty subset of a Banach space* X*, which is bounded, closed and convex. Suppose that* $L : D \to 2^X \setminus \{0\}$ *be an upper semi-continuous set-valued mapping with convex and closed values such that* $L(D) \subset D$ *and* $L(D) = \bigcup_{x \in D} L(x)$ *is relatively compact. Then* L *has a fixed point.*

Recall (cf. [16]) that L is said to be a convex if the inclusion

$$\lambda L(x) + (1-\lambda)L(y) \subset L(\lambda x + (1-\lambda)y)$$

holds for all $x, y \in D$ and for every $\lambda \in [0,1]$.

We say that L has closed values if $L(x)$ is a closed set for every $x \in D$.

2. Proof of the main result

Consider the Sobolev space

$$X = \mathcal{D}_0^{2,p}(\Omega)$$

with the norm

$$\| u \| = \left(\int_\Omega |\Delta u|^p dx\right)^{\frac{1}{p}}.$$

Define two operators A and B from X into X^* by

$$A(u).v = \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx,$$

$$B(u).v = \int_\Omega f(x,u)vdx + \int_\Omega V(x)|u|^{p^*-2}uvdx$$

where X^* is the dual of X.

Proof of Theorem *1.1.* We have the following properties,

(1) A is monotone, hemicontinuous, coercive.

In view of [7], we have the following inequality for $p \geq 2$,

$$|x-y|^p \leq (|x|^{p-2}x - |y|^{p-2}y).(x-y), \forall x, y \in \mathbb{R}^N.$$

Thus,

$$\begin{aligned}\langle A(u)-A(v),u-v\rangle &= \int_\Omega |\Delta u|^{p-2}\Delta u\Delta(u-v)dx-\int_\Omega |\Delta v|^{p-2}\Delta v\Delta(u-v)dx\\ &= \int_\Omega (|\Delta u|^{p-2}\Delta u-|\Delta v|^{p-2}\Delta v)(\Delta u-\Delta v)dx\\ &\geq \int_\Omega |\Delta u-\Delta v|^p dx = \| u-v \|^p, \end{aligned} \quad (2.1)$$

and then A is monotone. On the other hand, since A is the derivative operator of the functional $u \to \frac{1}{p}\int_\Omega |\Delta u|^p dx$ which is of class C^1, then the continuity of the operator A holds, so it is hemicontinuous.
Moreover, it is clear that A is coercive since $A(u).u = \| u \|^p$.

(2) The operator B is compact.
Let $(u_n)_n$ be a bounded sequence in X. Up to a subsequence denoted also by $(u_n)_n$, we have

$$u_n \rightharpoonup u \; in \; X,$$

by the compact embedding $\mathcal{D}^{2,p}(\Omega)$ into $L^p(\Omega)$, we have

$$u_n(x) \to u(x) \; a.e. \; in \; \Omega.$$

Since f is Carathéodory function which also verifies the condition (F),

$$f(x,u_n)u_n \to f(x,u)u, \; a.e \; in \; \Omega.$$

By using Hölder's inequality and Sobolev's embedding and according to Dominated convergence theorem, we obtain

$$B(u_n) \to B(u).$$

Remark 2.1. Let $(u_n)_n \subset X$ and $u \in X$ such that

$$u_n \rightharpoonup u, \; A(u_n) \to A(u),$$

then $A(u_n).u_n \to A(u).u$, which yields $\| u_n \|^p \to \| u \|^p$. Because X is uniformly convex so it follows that $u_n \to u$.

(3) In our next step, let $D \subset X$ be a bounded closed convex. Define the operator L by

$$L(v) = \{u : \; A(u) = B(v)\}.$$

It has a closed. Indeed, let $v_n \to v$ in X, $u_n \in L(v)$ and $u_n \to u$, so it would like to show that $u \in L(v)$.

We know that A and B are demicontinuous operators which imply that

$$A(u_n) \rightharpoonup A(u), B(v_n) \rightharpoonup B(v).$$

As we have $A(u_n) = B(v_n)$, so it yields $A(u) = B(v)$ (due to the uniqueness of the limit) then $u \in L(v)$.

Next it will be shown that $L(D) = \bigcup_{v\in D} L(v)$ is relatively compact. Let $(u_n)_n \subset \bigcup_{v\in D} L(v)$ and $v_n \in D$ with $A(u_n) = B(v_n)$. Since D is bounded domain and B is compact, hence $B(D)$ is relatively compact. Afterwards, there is $h \in X^*$ such that

$$A(u_n) = B(u_n) \to h,$$

whence, $A(u_n)$ is bounded, which means that $(u_n)_n$ is a bounded sequence, so we may choose a subsequence denoted also by $(u_n)_n$ where $u_n \rightharpoonup u$. As the operator A is monotone, we have

$$\langle A(v) - A(u_n), v - u_n \rangle \geq 0, \forall v \in X.$$

Therefore,

$$\langle A(v) - h, v - u \rangle \geq 0,$$

in view of proposition of Minty (Proposition 2.2) in [18], we get

$$A(v) = h.$$

From Remark 2.1, it follows that $u_n \to u$ in X.

Now, let $\mathbb{B}_R$ the ball of radius R, we are to prove that $L(\mathbb{B}_R) \subset \mathbb{B}_R$. Suppose that $A(u) = B(v)$ and $\| v \| \leq R$, then

$$\begin{aligned} \| u \|^p &= \int_\Omega |\Delta u|^p dx = \int_\Omega V(x)|v|^{p^*-2} vudx + \int_\Omega f(x,v)vudx \\ &\leq c_1 \| v \|^{p^*-1} \| u \| + c_2 \| u \| \| v \|^{r-1}, \end{aligned} \tag{2.2}$$

with c_1 and c_2 are two positive constants. Therefore,

$$\| u \|^{p-1} \leq c_1 \| v \|^{p^*-1} + c_2 \| v \|^{r-1} .$$

Because $r \leq p \leq p^*$, we may find $R > 0$ such that

$$\begin{aligned} c_1 \| v \|^{p^*-1} + c_2 \| v \|^{r-1} &\leq c_1 R^{p^*-1} + c_2 R^{r-1} \\ &\leq R^{p-1}, \end{aligned} \tag{2.3}$$

from which we obtain

$$\| u \| \leq R.$$

We can see that all the assumptions of Bohnenblust-Karlin Theorem are satisfied, hence L has a fixed point which is a solution of the problem $(\mathcal{P})$.

In the sequel, one proceeds as in [3], so we sketch the regularity property of this solution. Let u be a solution of $(\mathcal{P})$. We set

$$\Lambda_\lambda = \{x \in \Omega \ : \ u(x) \geq \lambda\}.$$

For $k > 0$ fixed, putting $\omega_k = u - k$ if $u(x) \geq k$ and $\omega_k = 0$ else. From Cavalieri's principle we have

$$\int_{-\infty}^{\infty} |\Lambda_\lambda| d\lambda = \int_k^\infty |\Lambda_\lambda| d\lambda = \int_\Omega \omega_k dx.$$

We point out that when $k > 1$ is greater enough, we have

$$|u|^{p^*-1} = 0, \ a.e. in \ [|u| \geq k],$$

since $u \in L^{p^*}(\Omega)$.

Using the Hölder inequality, for $k > k_0 > 0$, we entail that

$$\begin{aligned} \| \omega_k \|^p &\leq \int_\Omega V(x)|u|^{p^*-1}\omega_k dx + \int_\Omega | f(x,\omega_k)u | \omega_k dx \\ &\leq c_1 \int_{[|u|\leq 1]\cap\Lambda_k} |u|^{p^*-1}\omega_k dx + c_2 \int_{[|u|\geq 1]\cap\Lambda_k} \omega_k dx \end{aligned}$$

$$\leq \quad c_3(|\Lambda_k|^{1-\frac{1}{p^*}})(\int_\Omega |\omega_k|^{p^*}dx)^{\frac{1}{p^*}}, \tag{2.4}$$

moreover, we have

$$\| \omega_k \|^p \quad \geq \quad S(\int_\Omega |\omega_k|^{p^*}dx)^{\frac{p}{p^*}}, \tag{2.5}$$

where S is the best Sobolev constant for the embedding

$$\mathcal{D}_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

defined by

$$S = \inf_{u\in\mathcal{D}^{2,p}(\Omega), u\neq 0} \frac{\int_\Omega |\Delta u|^p dx}{\left(\int_\Omega |u|^{p^*}dx\right)^{\frac{p}{p^*}}},$$

so we get

$$\left(\int_\Omega |\omega_k|^{p^*}dx\right)^{\frac{1}{p^*}} \leq c\left(|\Lambda_k|^{(1-\frac{1}{p^*})(\frac{1}{p-1})}\right).$$

By the Cavalieri's principle [11] and the last inequality, for $k \geq k_0$

$$\begin{aligned}\int_k^\infty |\Lambda_\lambda| d\lambda &= \int_\Omega \omega_k dx \\ &\leq |\Lambda_k|^{1-\frac{1}{p^*}}(\int_\Omega |\omega_k|^{p^*}dx)^{\frac{1}{p^*}} \\ &\leq c(|\Lambda_k|^{1+\frac{1}{p^*}\frac{p^*-1}{p-1}}).\end{aligned} \tag{2.6}$$

Accordingly, since

$$1+\frac{1}{p^*}\frac{p^*-1}{p-1} > 1,$$

then easly we get

$$|\Lambda_k| = 0$$

and thus there is $M > 0$ such that

$$|u|_\infty \leq M.$$

□

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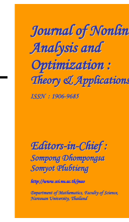

A MATHEMATICAL ANALYSIS OF THE TRANSMISSION DYNAMICS OF EBOLA VIRUS DISEASES

MITUN KUMAR MOMDAL[1], MUHAMMAD HANIF[2] AND MD. HAIDER ALI BISWAS[1*]

[1]Mathematics Discipline, Science Engineering and Technology School, Khulna University, Khulna, Bangladesh; [2]Department of Applied Mathematics, Noakhali Science and Technology University, Noakhali, Bangladesh

ABSTRACT. A mathematical model to investigate the transmission dynamics of Ebola virus disease (EVD), which causes acute viral haemorrhagic fever, is established in this paper. Based on the mechanism and characteristic of EVD transmission, we propose a susceptible-exposed-infectious-recovered-susceptible (SEIRS) epidemic model with the understanding that the recovered individuals can become infected again. The equilibria of the model and their stability are discussed in detail. Basic reproduction number (R_0) is obtained by using the next generation approach and proved that the disease free equilibrium (DFE) of our system is locally asymptotically stable if $R_0 < 1$, which means that the disease can be eradicated under such condition in finite time and unstable if $R_0 > 1$. When the associated reproduction number, $R_0 > 1$ then the endemic equilibrium is stable, otherwise unstable. We contemplate our proposed model numerically and compare the results with existing literature.

KEYWORDS : Ebola Virus; SEIRS model; Equilibria; Basic reproduction number; Stability analysis.
AMS Subject Classification: 92D30, 93A30, 49K15, 35B35

1. INTRODUCTION

Ebola virus disease (EVD) also known as Ebola hemorrhagic fever (EHF) or simply Ebola, is a disease of humans and other primates caused by Ebolavirus. EVD is actually an important problem of public health, especially in West African regions. The disease was first identified in 1976 in two simultaneous outbreaks, one in Nzara, Sudan and the other in Yambuku, Democratic Republic of Congo (formerly Zaire). The latter occurred in a village near the Ebola River, from which the disease takes its name. The virus family Filoviridae includes three genera: Cuevavirus, Marburvirus and Ebolavirus. According to Pringle [15], there are five species that have been identified: Zaire, Bundibugyo, Sudan, Reston and Tai Forest.

* *Corresponding author.*
Email address : mhabiswas@yahoo.com.
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The transmission of Ebola virus can spread from human to human by direct contact with body fluids such as blood, saliva, mucus, vomit, sweat, tears, breast milk, urine and semen of an infected human or other animal, which were described by Kuhn et al. [9]. Pattyn et al. [14] stated that the spread of the disease through the air between primates, including humans, has not been documented in either laboratory or natural conditions. It is mentioned by Bowen et al. [6] semen or breast milk of a person after recovery from EVD may still carry the virus for several weeks to months. Fruit bats are believed to be the normal carrier in nature, able to spread the virus without being affected by it. Other diseases such as meningitis, malaria, typhoid fever and other viral hemorrhagic fevers may resemble EVD. By nature, Ebola is highly contagious and deadly and there are no drugs or proven Ebola virus-specific treatment at moment. Ebola can kill up to 90% of patients, although in this outbreak, the death rate has dropped below 50%, which gives credence to the fact that early detection and good medical care could be synonymous to possible recovery, which is not permanent since with no immunity, the recovered become susceptible again. Between 1976 and 2013, World Health Organization (WHO) reports a total of 24 outbreaks involving 1,716 cases. Ndanusa et al. [11] reported that the largest outbreak is the ongoing epidemic in West Africa, still affecting Liberia, Guinea and Sierra Leone. As of 11 August 2015, this outbreak has 27,984 reported cases resulting in 11,298 deaths.

In the light of the foregoing, there is an urgent and serious need to have coordinated responses from all angles in order to combat against the EVD effectively. As part of this coordinated approach, we use the mathematical modeling for describing the EVD transmission, because mathematical modeling is an equation or set of equations that successfully describes the physical problem or phenomenon. Already many authors have developed mathematical models to improve our understanding of the dynamics and spread of this gigantic infectious disease. An EVD transmission model to the reported daily numbers of incident cases and deaths during the outbreak in Nigeria is fitted by Althaus et al. [1]. Osemwinyen et al. [13] introduced a modified SIR model in which they included a quarantined group to refine their proposed model and equally used it to simulate the transmission dynamics of the EVD. Chowell et al. [7] modeled the course of the outbreaks of ebola disease via an SEIR epidemic model by including a smooth transition in the transmission rate after control interventions were put in place. A simple ebola virus transmission model is developed by Deepa [8]. For more details on EVD transmission we refer readers to the references within as well as for some recent developments of other infectious diseases in [2, 3, 4, 5, 10, 12].

In this paper, we attempt to propose a SEIRS model that can be used to study how to reduce the spread of ebola epidemics. We propose this model with the understanding that the recovered individuals can become infected again. The first section of this paper formulates a SEIRS model for transmission of EVD, the second section analyzes the model and the last section simulates the model. The simulation is compared to the theoretical calculation by using ODE45 solvers written in MATLAB programming language.

2. MODEL FORMULATION

In this section, we formulate a SEIRS compartmental model to hit off the transmission dynamics of EVD. In order to describe the model equations, we assume that the total population (N) is divided into four different classes, which are the susceptible, (S) are people that have never come into contact with ebola virus, the

TABLE 1. Parameters used in the model with their description and values

Parameter	Description	Value (day^{-1})
A	Recruitment rate	9863
β	Effective contact rate with infectious individuals	0.90
μ	Natural death removal rate	0.0000548
γ	Rate at which exposed individuals become infectious	0.083
δ	Recovery rate	0.15
σ	Death removal rate due to EVD	0.025
ρ	Rejoin rate from recovered to susceptible class	0.023

exposed, (E) are people who have come into contact with the disease but are not yet infective, the infectious, (T) are people who have become infected with ebola virus and are able to transmit the virus, and recovered, (R) are people that have recovered from EVD. The model parameters are defined in TABLE 1 and the transmission of EVD are shown in FIGURE 1.

The susceptible class S is increased by birth or immigration at the rate A. It is decreased by infection following contact with infected individuals at the rate $\frac{\beta I}{N}$, and diminished by natural death of people at the rate μ. To determine the rate at which people become exposed in the population, the first term is considered in the susceptible class which is the rate at which susceptible, by meeting infective, become exposed is $\frac{\beta I}{N}$. This class is decreased by the rate γ at which the exposed individuals become infectious and diminished by natural death at the rate μ.The infectious class I is generated by breakthrough of exposed individuals at the rate γ. This class is decreased by recovery from infection at the rate δ and diminished by natural death and disease induced death at a rate μ and σ. The recovered class R is increased by those that recover from the infection at the rate δ and reduced by the number of people that rejoin to the susceptible class at the rate ρ and by natural death rate μ.

The model is illustrated by the following schematic diagram

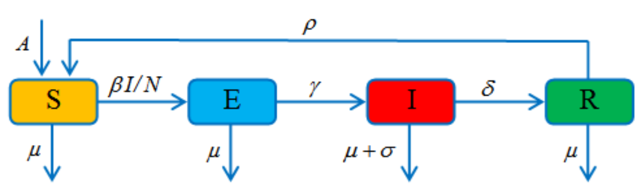

FIGURE 1. Diagram showing the compartmental model for SEIRS

The corresponding mathematical equations can be described by the following system of ordinary differential equations:

$$\frac{dS}{dt} = A - \frac{\beta IS}{N} + \rho R - \mu S \tag{2.1}$$

$$\frac{dE}{dt} = \frac{\beta IS}{N} - \gamma E - \mu E \tag{2.2}$$

$$\frac{dI}{dt} = \gamma E - \delta I - \sigma I - \mu I \tag{2.3}$$

$$\frac{dR}{dt} = \delta I - \rho R - \mu R \tag{2.4}$$

with, $N = S + E + I + R$.

3. MODEL ANALYSIS

In this section we find out the disease free equilibrium (DFE), endemic equilibrium (EE), basic reproduction number and the stability of equilibrium points.

3.1. **Disease free equilibrium.** For equilibrium point, we set the equations (2.1)-(2.4) equal to zero, that is

$$A - \frac{\beta IS}{N} + \rho R - \mu S = 0 \tag{3.1}$$

$$\frac{\beta IS}{N} - \gamma E - \mu E = 0 \tag{3.2}$$

$$\gamma E - \delta I - \sigma I - \mu I = 0 \tag{3.3}$$

$$\delta I - \rho R - \mu R = 0 \tag{3.4}$$

To determine the disease free equilibrium, substitute $E = I = R = 0$ then the equation (3.1) stands,

$$A - \mu S = 0$$
$$\therefore S = \frac{A}{\mu}$$

Hence the disease free equilibrium is $(\frac{A}{\mu}, 0, 0, 0)$.

3.2. **Endemic equilibrium.** From the equations (3.1)-(3.4) we obtain,

$$S = \frac{AN + \rho RN}{\beta I + \mu N} \tag{3.5}$$

$$E = \frac{\beta IS}{N(\gamma + \mu)} \tag{3.6}$$

$$I = \frac{\gamma E}{\delta + \sigma + \mu} \tag{3.7}$$

$$R = \frac{\delta I}{\rho + \mu} \tag{3.8}$$

Using equations (3.6) and (3.7) we have,

$$E = \frac{\beta S}{N(\gamma + \mu)} \frac{\gamma E}{\delta + \sigma + \mu}$$

$$\therefore S = \frac{N(\gamma+\mu)(\delta+\sigma+\mu)}{\beta\gamma} \tag{3.9}$$

Substituting the value of S and R in (3.5) we obtain

$$I = \frac{\beta\gamma AN - \mu N^2(\gamma+\mu)(\delta+\sigma+\mu)}{\beta N(\gamma+\mu)(\delta+\sigma+\mu) - \frac{\beta\gamma\rho\delta}{\rho+\mu}} \tag{3.10}$$

Hence from (3.6) and (3.8) we obtain,

$$E = \frac{(\delta+\sigma+\mu)\beta\gamma AN - \mu N^2(\gamma+\mu)(\delta+\sigma+\mu)^2}{\beta N(\gamma+\mu)(\delta+\sigma+\mu) - \frac{\beta\gamma\rho\delta}{\rho+\mu}} \tag{3.11}$$

and

$$R = \frac{\beta\gamma\delta AN - \mu\delta N^2(\gamma+\mu)(\delta+\sigma+\mu)}{\beta N(\rho+\mu)(\gamma+\mu)(\delta+\sigma+\mu) - \beta\gamma\rho\delta} \tag{3.12}$$

If the endemic equilibrium point is (S^*, E^*, I^*, R^*) then,

$$S^* = \frac{N(\gamma+\mu)(\delta+\sigma+\mu)}{\beta\gamma},$$

$$E^* = \frac{(\delta+\sigma+\mu)\beta\gamma AN - \mu N^2(\gamma+\mu)(\delta+\sigma+\mu)^2}{\beta N(\gamma+\mu)(\delta+\sigma+\mu) - \frac{\beta\gamma\rho\delta}{\rho+\mu}},$$

$$I^* = \frac{\beta\gamma AN - \mu N^2(\gamma+\mu)(\delta+\sigma+\mu)}{\beta N(\gamma+\mu)(\delta+\sigma+\mu) - \frac{\beta\gamma\rho\delta}{\rho+\mu}} \text{ and}$$

$$R^* = \frac{\beta\gamma\delta AN - \mu\delta N^2(\gamma+\mu)(\delta+\sigma+\mu)}{\beta N(\rho+\mu)(\gamma+\mu)(\delta+\sigma+\mu) - \beta\gamma\rho\delta}.$$

3.3. **Basic reproduction number.** The basic reproduction number (R_0) is an important part of epidemiological model. It is defined as the expected numbers of secondary cases produced by an infected individual during its entire period of infectiousness. If $R_0 < 1$, then throughout the infectious period, each infective will produce less than one new infective on the average. This in turn implies that the disease will die out and if$R_0 > 1$, then throughout the infectious period, each infective will produce more than one new infective on the average. This in turn implies that the disease will persist. It is obtained from the largest eigenvalue of the next generation matrix, FV^{-1}.
Here,

$$F = \begin{bmatrix} \frac{\partial F_1}{\partial E} & \frac{\partial F_1}{\partial I} \\ \frac{\partial F_2}{\partial E} & \frac{\partial F_2}{\partial I} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\beta S}{N} \\ 0 & 0 \end{bmatrix}; \text{ where, } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{\beta IS}{N} \\ 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \frac{\partial V_1}{\partial E} & \frac{\partial V_1}{\partial I} \\ \frac{\partial V_2}{\partial E} & \frac{\partial V_2}{\partial I} \end{bmatrix} = \begin{bmatrix} \gamma+\mu & 0 \\ -\gamma & \gamma+\delta+\mu \end{bmatrix}; \text{ where, } \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} (\gamma+\mu)E \\ -\gamma E + (\delta+\sigma+\mu)I \end{bmatrix}$$

Now,

$$V^{-1} = \frac{1}{(\gamma+\mu)(\delta+\sigma+\mu)} \begin{bmatrix} \delta+\sigma+\mu & 0 \\ \gamma & \gamma+\mu \end{bmatrix}. \tag{3.13}$$

At disease free equilibrium,

$$F = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}. \tag{3.14}$$

Therefore at disease free equilibrium,

$$FV^{-1} = \frac{1}{(\gamma+\mu)(\delta+\sigma+\mu)} \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta+\sigma+\mu & 0 \\ \gamma & \gamma+\mu \end{bmatrix}$$

$$= \frac{1}{(\gamma+\mu)(\delta+\sigma+\mu)} \begin{bmatrix} \beta\gamma & \beta(\gamma+\mu) \\ 0 & 0 \end{bmatrix}. \tag{3.15}$$

The largest eigenvalue of FV^{-1} is $\frac{\beta\gamma}{(\gamma+\mu)(\delta+\sigma+\mu)}$. Therefore, the basic reproduction number, $R_0 = \frac{\beta\gamma}{(\gamma+\mu)(\delta+\sigma+\mu)}$.

3.4. **Stability of the equilibrium points.** To investigate the stability at the equilibrium points, we present the following theorems.

Theorem 3.1. *The disease free equilibrium* $(\frac{A}{\mu}, 0, 0, 0)$ *of the system (2.1)-(2.4) is locally asymptotically stable if* $R_0 < 1$ *and unstable if* $R_0 > 1$.

Proof. The Jacobian matrix of (2.1)-(2.4) is

$$J(S,E,I,R) = \begin{bmatrix} -\frac{\beta I}{N} - \mu & 0 & -\frac{\beta S}{N} & \rho \\ \frac{\beta I}{N} & -(\gamma+\mu) & \frac{\beta S}{N} & 0 \\ 0 & \gamma & -(\delta+\sigma+\mu) & 0 \\ 0 & 0 & \delta & -(\rho+\mu) \end{bmatrix}, \tag{3.16}$$

at disease free equilibrium point the matrix becomes

$$J_{DFE} = \begin{bmatrix} -\mu & 0 & -\beta & \rho \\ 0 & -(\gamma+\mu) & \beta & 0 \\ 0 & \gamma & -(\delta+\sigma+\mu) & 0 \\ 0 & 0 & \delta & -(\rho+\mu) \end{bmatrix} \tag{3.17}$$

Now equating the characteristic equation to zero for the eigenvalue ω we get,

$$(\mu+\omega)(\rho+\mu+\omega)(a_0\omega^2 + a_1\omega + a_2) = 0 \tag{3.18}$$

with, $a_0 = 1 > 0, a_1 = \gamma + \delta + \sigma + 2\mu$ and $a_2 = \dfrac{1 - R_0}{R_0}$.

The first two eigenvalues have negative real parts and the other two eigenvalues have negative real parts if and only if $a_2 > 0$, i.e., $R_0 < 1$. Hence completes the proof. □

Theorem 3.2. *The endemic equilibrium of the system (2.1)-(2.4) is locally asymptotically stable if $R_0 > 1$, otherwise unstable.*

Proof. We present the following Jacobian matrix of (2.1)-(2.4) for proving the theorem,

$$J(S, E, I, R) = \begin{bmatrix} -\dfrac{\beta I}{N} - \mu & 0 & -\dfrac{\beta S}{N} & \rho \\ \dfrac{\beta I}{N} & -(\gamma + \mu) & \dfrac{\beta S}{N} & 0 \\ 0 & \gamma & -(\delta + \sigma + \mu) & 0 \\ 0 & 0 & \delta & -(\rho + \mu) \end{bmatrix}, \tag{3.19}$$

at endemic equilibrium point the matrix becomes,

$$J_{EE} = \begin{bmatrix} -\dfrac{\beta X}{N} - \mu & 0 & -\dfrac{\beta}{R_0} & \rho \\ \dfrac{\beta X}{N} & -(\gamma + \mu) & \dfrac{\beta}{R_0} & 0 \\ 0 & \gamma & -(\delta + \sigma + \mu) & 0 \\ 0 & 0 & \delta & -(\rho + \mu) \end{bmatrix} \tag{3.20}$$

where, $X = \dfrac{\frac{R_0 A - \mu N}{N}}{\beta - \dfrac{R_0 \rho \delta}{\rho + \mu}}$.

The characteristic polynomial of J_{EE} is,

$$P(\omega) = a_0\omega^4 + a_1\omega^3 + a_2\omega^2 + a_3\omega + a_4 \tag{3.21}$$

where, $a_0 = 1 > 0$, $a_1 = X + \rho + \gamma + \delta + \sigma + 4\mu$, $a_2 = (X + \mu)(\rho + \mu) + (X + \rho + 2\mu)(\gamma + \delta + \sigma + 2\mu)$, $a_3 = (X + \mu)(\rho + \mu)(\gamma + \delta + \sigma + 2\mu) + \dfrac{\beta\gamma X}{R_0}$ and $a_4 = X(\gamma + \mu)(\delta + \sigma + \mu)(\rho + \mu + R_0\delta)$.

Note that all coefficients of $P(\omega)$ are positive for $R_0 > 1$. Thus by the Routh-Hurwitz criteria, all roots of $P(\omega)$ have negative real parts if $a_1a_2 > a_3$ and $a_1a_2 > a_3 + a_1^2a_4$. It is possible only when $R_0 > 1$, thus the endemic equilibrium point is stable if $R_0 > 1$, otherwise unstable. □

4. NUMERICAL RESULTS

For the purpose of model validation as well as in order to ensure that the proposed model is agreement with reality, numerical simulations are carried out using the data provided in TABLE 1. Varying the values of parameters, the graphs are plotted to investigate the effect of some parameters on the transmission dynamics of EVD. The results are displayed in FIGURE 2-4. The result for the value of parameters (see TABLE 1) is displayed in FIGURE 2.

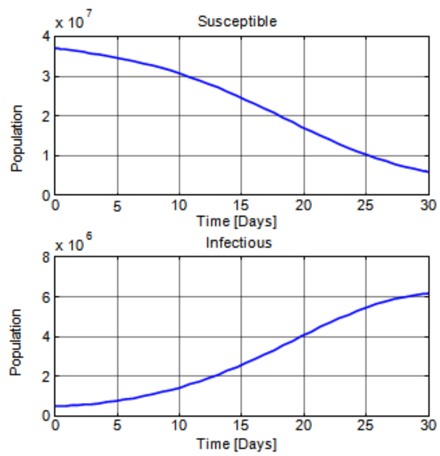

FIGURE 2. Variation of the population with time (30 days) for $R_0 = 5.65 > 1$.

According to FIGURE 2 we notice a gradual drop of susceptible group with the increasing of time. The number of exposed begins to rise from its initial state and after rising at highest point the group falls down gradually. The infected also begins to increase from its initial position and the recovered people gradually increasing over time. Again if we reduce the effective contact rate between infected and susceptible as well as the rate at which exposed individuals become infectious, as: $\beta = 0.75,\ \gamma = 0.063,\ \delta = 0.45,\ \sigma = 0.15,\ \rho = 0.023$ and $\mu = 0.0000548$. Then we get the new result which is shown in FIGURE 3.

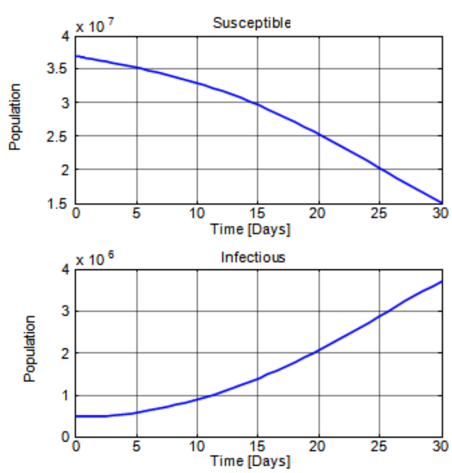

FIGURE 3. Variation of the population with time (30 days) for $R_0 = 1.25 > 1$.

FIGURE 3 shows relatively similar curves but with little change over FIGURE 2. Here we see a gradual dropping of susceptible group and gradual rising of recovered group. The change of exposed group is not like as FIGURE 2. In FIGURE 2 the exposed group reaches at a highest point then falls down but in FIGURE 3 the exposed group is always increasing with the increase of time. Also for the values of $\beta = 0.55$, $\gamma = 0.043$, $\delta = 0.45$, $\sigma = 0.15$, $\rho = 0.023$ and $\mu = 0.0000548$, the new result is shown in FIGURE 4.

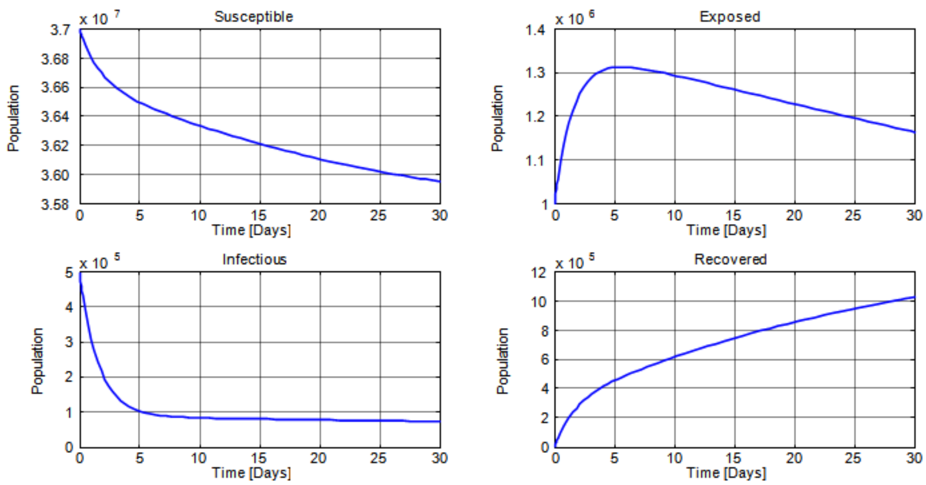

FIGURE 4. Variation of the population with time (30 days) for $R_0 = 0.91 < 1$.

The results displayed in FIGURE 4, indicates a more interesting result than in FIGURE 2 and 3. Here the susceptible group is reducing as like as FIGURE 2 and 3 but infected group is reducing from its initial state which is more changed than FIGURE 2 and 3. The exposed group is increasing first and reach at a peak then falls down gradually. The recovered always increasing over time. Therefore a greater percentage of the population is alive, though still susceptible.

5. CONCLUSIONS

The world is currently having a difficult time fighting against Ebola virus. In spite of that, this deadly virus must be dealt with. This is not merely fighting against a disease; it is an event where different nations and companies cooperate for the common cause. In this paper, we came up with a mathematical model that can be applied to the fight against EVD. First, we fitted a compartmental SEIRS model to describe the transmission dynamics of EVD, then analyzed the model briefly and at last simulated shortly the model numerically with the help of known nonlinear solver coded.

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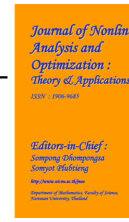

AN APPROACH FOR SIMULTANEOUSLY DETERMINING THE OPTIMAL TRAJECTORY AND CONTROL OF REDUCE THE SPREAD OF COMPUTER VIRUSES

H. R. SAHEBI*

Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran.

ABSTRACT. In the recent decade, a considerable number of optimal control problems have been solved successfully based on the properties of the measures. Even the method, has many useful benefits, in general, it is not able to determine the optimal trajectory and control at the same time; moreover, it rarely uses the advantages of the classical solutions of the involved systems. In this article, for a Susceptible-Infected-Removed-Antidotal (SIRA) model for viruses in computer, we are going to present a new solution algorithm. First, by considering all necessary conditions, the problem is represented in a variational format in which the trajectory is shown by a trigonometric series with the unknown coefficients. Then the problem is converted into a new one that the unknowns are the mentioned coefficients and a positive Radon measure. It is proved that the optimal solution is exited and it is also explained how the optimal pair would be identified from the results deduced by a finite linear programming problem. A numerical examples is also given.

KEYWORDS : viruses Computer; Optimal Control; Measure Theory; radon Measure; SIRA Model.
AMS Subject Classification: 49QJ20, 49J45, 49M25, 76D33

1. INTRODUCTION

In recent, computer viruses are an important risk to computational systems endangering either corporation systems of all sizes or personal computers used for simple applications as accessing bank accounting or even consulting entertainment activities schedules. The viruses are being developed simultaneously with the computer systems and the use of internet facilities increases the number of damaging virus incidents. Since the first trials on studying how to combat viruses, biological analogies were established because biological organisms and computer networks share many characteristics as, for example, large number of connections among large number of simple components creating complex system [3]. Local systems

* *Corresponding author.*
Email address : sahebi@aiau.ac.ir.
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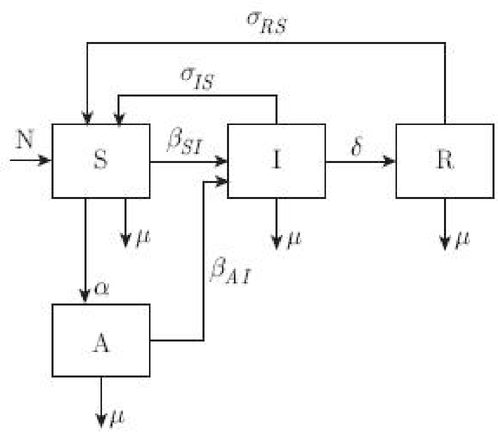
in a computer network can be attacked generating malfunctions that, spreading along the network, produce network-wide disorders following a similar qualitative model of disease spreading for a biological system. This is the main reason for designating attacks against networks by biological terms as worms and viruses. Using these ideas, it is important to consider that computer viruses have two different levels for being studied: microscopic and macroscopic [11].The microscopic level has been the subject of several studies. For instance, [1], [2] establishes theoretical principles about how to kill the new viruses created every day. Following the virus development, computer immunology is a new discipline capable of creating efficient anti-virus strategies as programs that are being sold all over the world guaranteeing protection to individual users of a global network [9]. However, the macroscopic approach has not been receiving the same attention in spite of epidemiology analogies being an important tool in order to establish the policies to preventing infections by giving figures about how to update the anti-virus programs. The interesting but simple model considering exponential variation in the number of computer viruses, proposed by [20], could not be considered realistic because the lack of limits for the growth, which is a natural phenomenon either in biological or in computer systems. There is vast catalog of Mathematical Biology models indicated for epidemiology [12]. One of them, called SIR (Susceptible-Infected-Removed) model, was originally proposed by [12]. Here, we employ a modified version of such a model in order to obtain parameter combinations representing situations with asymptotically stable disease-free solutions. The relations among network parameters can provide some hints about how to prevent infections in networks. An expression for the maximum infection rate of computers equipped with anti-virus to avoid the propagation of new infections is given. If this number is known, an updating plan for anti-virus programs in a computer network can be elaborated. According to an idea of L. C. Young, by transferring the problem in to a theoretical measure optimization, in 1986 Rubio introduced a powerful method for solving optimal control problems [17]. The important properties of the method (globality, automatic existence theorem a linear treatment even for extremely nonlinear problems, ...) caused it to be applied for the wide variety of problems. Even the method has been used frequency for solving several kinds of problems, like [4], [6] and [7] but at least two important points were not considered in applying the method yet. Generally the method can not be able to produce the acceptable optimal trajectory and control directly at the same time; and moreover, the classical format of the system solution, usually is not taken into account. Therefore, there is no any possibility to use this important fact and their related literatures in the analysis of the system. In this article, we try to bring attention these two facts; for these purposes, an optimal control problem governed by a classical epidemiological models for studying computer virus system (SIRA) with initial and boundary conditions and an integral criterion is considered as a sample. Regarding a general format of the classical solution, the problem is presented in a variational format and then by a doing deformation it is converted into a measure theoretical one with some positive coefficient. Next, extending the underlying space, using the density properties and applying some discretization scheme cause to approximate the optimal pair as a result of a finite linear programming. The approach would be improved if the number of constraints and nods are exceeded . In this manner, the optimal trajectory and control is determined at the same time.

2. The Dynamic System of SIRA

Due to the high similarity between computer virus and biological virus, some models for the spread of computer virus have been proposed . Piqueira and Navarro [15] suggested a modification of the SIRA epidemiological model for computer virus. In this section, we use the SIRA model for computer virus spread presented by Piqueira and Navarro [15] to set our control problem. In this model, they considered that the individuals in a computer system can switch between the Susceptible, Infected, Recovered and Antidotal states The system of differential equations with



time delay is defined by:

$$\begin{cases} \dot{S} = -\alpha SA - \beta_{SI} SI + \sigma_{IS} I + \sigma_{RS} R, \\ \dot{I} = \beta_{SI} SI + \beta_{AI} AI - \sigma_{IS} I - \delta I, \\ \dot{R} = \delta I - \sigma_{RS} R, \\ \dot{A} = \alpha SA - \beta_{AI} AI, \end{cases}$$

The parameters in the model are defined as follows:
δ: removing rate of infected computers;
β_{SI}: infection rate of susceptible computers;
β_{AI}: infection rate of antidotal computers due to the onset of new virus;
σ_{IS}: recovering rate of infected computers;
σ_{RS}: recovering rate of removed computers, with an operator intervention;
α: conversion of susceptible computers into antidotal ones, occurring when susceptible computers establish effective communication with antidotal ones and the antidotal install.
For $\dot{S}+\dot{I}+\dot{R}+\dot{A}=0$, then $S+A+I+R=T$, a constant for any time t. By using the optimal control theory developed by Pontryagin, we can set an optimal control problem in the $SIRA$ model to control the spread of computer virus. The main goal of this problem is to investigate an effective strategy to control the computer virus, which means that we can find an optimal strategy such that the infected nodes can meet the minimum within a specified time period. To set an optimal control problem, for given constants $\Lambda, T > 0$, we choose the following as our control class:

$$U = \{u(t) \in L^2(0,T) : 0 \leq u(t) \leq \Lambda,\ 0 \leq t \leq T\}.$$

In this problem, the meaning of the control variable is that low levels of the number of infected nodes build contact to the susceptible nodes and better antivirus software. In case of high contact rate, the number of infected nodes increases while the number of susceptible, recovered and antidotal nodes decreases. With better antivirus software and lower contact rate, susceptible nodes begin to build again and more nodes are recovered from infection. Therefore, by an optimal control strategy $u(t)$, a fraction $u(t)I(t)$ of infected nodes moved from class I to class S, class R and class A. So, our optimal control problem is given by the following. The optimal control problem is formulated as:

$$\begin{aligned} min\ J(u) = & \int_0^T [I(t) + \tfrac{\epsilon u^2(t)}{2}]dt \\ \dot{S} = & -\alpha SA - \beta_{SI} SI + \sigma_{IS} I + \sigma_{RS} R + \omega u(t) I, \\ \dot{I} = & \beta_{SI} SI + \beta_{AI} AI - \sigma_{IS} I - \delta I - u(t) I, \\ \dot{R} = & \delta I - \sigma_{RS} R + (1-\omega) u(t) I, \\ \dot{A} = & \alpha SA - \beta_{AI} AI, \end{aligned} \tag{2.1}$$

with initial conditions $S(0) = S_0, I(0) = I_0, R(0) = R_0$ and $A(0) = A_0$ .Here $\epsilon \in [0,1]$ is a positive constant which represents the weight on the size of infected nodes and systemic cost. Note that for $\epsilon = 1$, the infected ones move to the susceptible class, while for $\epsilon = 0$, the infected nodes move to the recovered class at rate of control variable $u(t)$.

3. New Representation of the Problem

Setting $S = x_1, I = x_2, R = x_3$ and $A = x_4$. We define the function $f_0 : J \times S \times I \times R \times A \times U \to R$ as following where S, I, R, A and U are compact subsets of R.

$$f_0(t, S(t), I(t), R(t), A(t), u(t)) = f_0(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)) = x_2(t) + \frac{\epsilon u^2(t)}{2} \tag{3.1}$$

then we write the problem (2.1) in the following form:

$$\begin{aligned} min\ \Xi(x_1(.), x_2(.), x_3(.), x_4(.), u(.)) = & \int_0^T f_0(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t))dt \\ \dot{x_1} = & f_1(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)), \\ \dot{x_2} = & f_2(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)), \\ \dot{x_3} = & f_3(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)), \\ \dot{x_4} = & f_4(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)), \end{aligned} \tag{3.2}$$

where
$x_1(0) = x_{10}, x_2(0) = x_{20}, x_3(0) = x_{30}, x_4(0) = x_{40}$ and $x_1(T), x_2(T), x_3(T), x_4(T)$ are not specifed. Also,

$$\begin{aligned} f_1(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)) &= -\alpha x_1 x_4 - \beta_{SI} x_1 x_2 + \sigma_{IS} x_2 + \sigma_{RS} x_3 + \omega u(t) x_2, \\ f_2(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)) &= \beta_{SI} x_1 x_2 \beta_{AI} x_4 x_2 - \sigma_{IS} x_2 - \delta x_2 - u(t) x_2, \\ f_3(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)) &= \delta x_2 - \sigma_{RS} x_4 + (1-\omega) u(t) x_2, \\ f_4(t, x_1(t), x_2(t), x_3(t), x_4(t), u(t)) &= \alpha x_1 x_4 - \beta_{AI} x_4 x_2, \end{aligned}$$

Let us consider $A = A_1 \times A_2 \times A_3 \times A_4$ and $\Omega = J \times A \times U$, where $J = [0,T], U = [0,1]$ and $A_i, i = 1,2,3,4$ closed and bounded subset of R^n. Suppose that $X(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$,consider the differential equation

$$\acute{X}(t) = f(t, X(t), u(t)), t \in J^0 = (0, 1) \tag{3.3}$$

where $g : \Omega \to R^n$ a continuous function, and the trajectory function $t \in J \to X(t) \in A$ is absolutely continuous and the control function $t \in J \to u(t) \in U$ is Lebesgue-measurable. We say that a trajectory control pair $w = [X(.), u(.)]$ is admissible if the following conditions hold :
(a) : $x(t) \in A, t \in J$
(b) : $u(t) \in U, t \in J$
(c) : The boundary conditions $X(0) = X_0$ is satisfied,
(d) : The pair w satisfies the differential equation (3.3) a.e. on J^0. We assume that the set of all admissible pairs is non-empty and denote it by W. Our control problem consists of finding the pair $w = [X(.), u(.)] \in W$, which minimizes the functional

$$I[X(.), u(.)] = \int_0^T f_0(t, X(t), u(t))dt \tag{3.4}$$

where $f_0 \in C(\Omega)$, the space of continuous functions on Ω, with topology of uniform convergence. This control problem may or may not have a solution in W.

4. Infinite-Dimensional Linear Programming

We may transform the above control problems to an infinite-dimensional linear programming problem. Let $w = [X(.), u(.)]$ be an admissible pair, and B an open ball in R^{n+1} containing $J \times A$ and $\acute{C}(B)$ be the space of real valued continuously differentiable functions on B. Let $\phi \in \acute{C}(B)$, and define function ϕ^i as follows:

$$\phi^{(i)}_{f_i}(t, X(t), u(t)) = \phi_X(t, X(t)).f_i(t, X(t), u(t)) + \phi_t(t, X(t)), i = 1, 2, 3, 4. \tag{4.1}$$

for all $(, X(t), u(t)) \in \Omega$, note that $\phi_X(t)$ is n-vector, and that the first term in the right-hand side of (4.1) is their inner product. The function ϕ^i is in the space $C(\Omega)$ the continuous functions on the compact set Ω. Since $w = [X(.), u(.)]$ is an admissible pair, we have

$$\begin{aligned} \int_0^T \phi^{(i)}_{f_i}(t, X(t), u(t))dt = & \\ = & \int_0^T \phi_X(t, X(t)).f_i(t, X(t), u(t)) + \phi_t(t, X(t)) \\ = & \int_0^T \acute{\phi}(t, X(t))dt \\ = & \phi(T, \acute{X}(T)) - \phi(0, \acute{X}(0)) = \Delta\phi_i \end{aligned} \tag{4.2}$$

for all $\phi \in \acute{C}(B)$.Let $D(J^0)$ be the space of infinitely differentiable real-valued functions with compact support in J^0. For $i = 1, 2, 3, 4$, we define

$$\psi_i(t, X(t), u(t)) = X(t)\acute{\psi}(t) + f_i(t, X(t), u(t)).\psi(t) \tag{4.3}$$

for all $\psi \in D(J^0)$, then for $\psi \in D(J^0)$ we have:

$$\begin{aligned} \int_0^T \psi_i(t, X(t), u(t))dt = & \int_0^T X(t)\acute{\psi}(t)dt + \int_0^T f_i(t, X(t), u(t)).\psi(t)dt \\ = & X(t)\psi(t)|_J - \int_0^T (\acute{X}(t) - f_i(t, X(t), u(t)))\psi(t) = 0 \end{aligned}$$

since the trajectory and control function are an admissible pair satisfying (4.3) a.e. on J^0, and since the function ψ has compact support in J^0,$\psi(0) = \psi(T) = 0$, also by choosing a variable t, we have

$$\int_0^T f_i(t, X(t, u(t)))dt = a_f,\ f \in C_1(\Omega)$$

where $C_1(\Omega)$ is subspace of the space $C(\Omega)$ of all continuous function on Ω depending only on the variable t.
Now, The mapping

$$\Lambda_W : F \to \int_J F(t, X(t), u(t))dt,\ F \in C(\Omega)$$

defines a positive linear functional on $C(\Omega)$. By the Riesz representation theorem [19] there exist a unique positive Radon measure μ on Ω such that

$$\int_J F(t, X(t), u(t))dt = \int_\Omega F\ d\mu = \mu(F),\ F \in C(\Omega)$$

Thus, the minimization of the functional Ξ in (3.2) over Ω is equivalent to the minimization of

$$\Xi(\mu) = \int_\Omega f_0 d\mu = \mu(f_0) \in R \tag{4.4}$$

over the set of positive measures μ corresponding to admissible pairs w, which satisfy

$$\begin{aligned} \mu(\phi_f^{(i)}) &= \Delta\phi_i, \phi \in \acute{C}(B) \\ \mu(\psi_i) &= 0,\ \psi \in D(J^0) \\ \mu(g) &= a_g\ g \in C_1(\Omega). \end{aligned} \tag{4.5}$$

Define the set of all positive Radon measures on Ω satisfying (4.5) as Σ. Also we assume $M^+(\Omega)$ be the set of all positive Radon measures on Ω. Now if we topologize the space $M^+(\Omega)$ by the weak*- topology, it can be shown that $\Sigma$is compact [18]. In the sense of this topology,the functional $\Xi : \Sigma \to R$ define by (4.4) is a linear continuous functional on a compact set Σ, thus it attains its minimum on Σ, and so the measure theoretical problem, which consist of finding the minimum of the functional (4.4), over the subset of $M^+(\Omega)$, possesses a minimizing solution, μ^*, in Σ, [18].

5. Metamorphosis

We now estimate the optimal control by a nearly-optimal piecewise constant control. The problem (4.4) and (4.5) are an infinite dimensional linear programming problem, because all the functionals in (4.4) and (4.5) are linear in the variable μ, and furthermore μ is required to be positive. First we consider the minimization of (4.4) not only over the set Σ but over a subset of it defined by requiring that only a finite number of constraints in (4.5) be satisfied.

Theorem 5.1. *Let $\Sigma(M_1, M_2, L)$ be a subset of $M^+(\Omega)$consists of all measures which satisfy the*

$$\mu(\phi_f^{(i)}) = \Delta\phi_i,\ i = 1, 2, ..., M_1\ \phi_i \in C_1(\Omega)$$

$$\mu(\psi_r) = 0,\ r = 1, 2, ..., M_2\ \psi_r \in D(J^0)$$

$$\mu(g_s) = a_{g_s},\ s = 1, 2, ..., L\ g_s \in C_1(\Omega)$$

As M_1, M_2 and L tend to infinity, $\eta(M_1, M_2, L) = inf_{\Sigma(M_1,M_2,L)\mu(f_0)}$ tends to $\eta = inf_\Sigma \mu(f_0)$.

Proof. see [18].
The first stage of the approximation is completed successfully. As the second stage, it is possible to develop a finite-dimensional, linear program whose solution can be used to construct the solution of the infinite-dimensional linear program (4.4) and (4.5). From Theorem (A.5) of [18], we can characterize a measure, say μ^*, in the set $\Sigma(M_1, M_2, L)$ at which the functional $\mu \to \mu(f)$ attains its minimum, it follows from a result of [16].

Theorem 5.2. *The measure μ^* in the set $\Sigma(M_1, M_2, L)$ at which the function $\mu \to \mu(f)$ attains its minimum has the form*

$$\mu^* = \sum_{k=1}^{M_1+M_2+L} \alpha_k^* \delta(z_k^*),$$

where $z_k^* \in \Omega$; the coefficients $\alpha_k^* \geq 0,\ k = 1, 2, ..., M_1 + M_2 + L$.

Here $\delta(z)$ defines a unitary atomic measure, characterized by $\delta(z)(F) = F(z)$, where $F \in C(\Omega)$. Now the measure theoretical optimization problem is equivalent to a non-linear optimization problem, in which the unknowns are the coefficients α_k^* and supports $z_k^*,\ k = 1, 2, ..., M_1 + M_2 + L$. It would be convenient if we could minimize the functional $\mu \to \mu(f)$ only with respect to the coefficients $\alpha_k^*, k = 1, 2, ..., M_1 + M_2 + L$, this would be a linear programming problem. However, we do not know the support of the optimal measure. The answer lies in approximation of this support, by introducing a dense set in Ω.
Now, we construct a piecewise constant control function corresponding to the finite-dimensional problem. Therefore in the infinite-dimensional linear programming problem (4.4) with restriction defined by (4.5), we shall consider how one can choose total functions in the constraints (4.4) and (4.5). Consider first functions ϕ^i in $\acute{C}(B)$ as follows:

$$x_1, x_2, x_3, x_4, x_2^2, x_2^3, x_1x_2, x_3x_2, x_4x_2, tx_2 \tag{5.1}$$

Trivially the linear combinations of these functions are uniformly dense in the space $C_1(B)$ [19], we choose only M_1 number of them. Also, we choose M_2 functions with compact support in the following form:

$$\psi_r(t) = \begin{cases} sin[2\pi r\left(\frac{t-0}{T-0}\right)] & r = 1, 2, ..., M_{21}, \\ 1 - cos[2\pi r\left(\frac{t-0}{T-0}\right)] & r = M_{21}+1, M_{21}+2, ..., 2M_{21}. \end{cases} \tag{5.2}$$

where, $M_2 = 2M_{21}$.
Finally, it is necessary to choose L number of functions of time only, as follows:

$$g_s(t) = \begin{cases} 1 & t \in J_s, \\ 0 & otherwise, \end{cases} \tag{5.3}$$

where $J_s = \left(\frac{0+(s-1)(T-0)}{L}, \frac{0+s(T-0)}{L}\right),\ s = 1, 2, ..., L$

The set $\Omega = J \times A \times U$ will be covered with a grid, where the grid will be defined by taking all points in Ω as $z_j = (t_j, x_{1j}, x_{2j}, x_{3j}, x_{4j}, u_j)$; the points in the grid will be numbered sequentially from 1 to N, which can be estimated numerically. Instead of the infinite-dimensional linear programming problem (4.1), we consider the following .nite dimensional linear programming problem, where $z_i \in w$ (w is a

approximately dense subset of Ω).
$Minimize\ \sum_{j=1}^{N} \alpha_j f_0(z_j)$

$subject\ to:$

$$\begin{cases} \sum_{j=1}^{N} \alpha_j \phi_f^{(i)} = \Delta\phi^i & i = 1, 2, ..., M_1, \\ \sum_{j=1}^{N} \alpha_j \psi_r(z_j) = 0 & r = 1, 2, ..., M_2, \\ \sum_{j=1}^{N} \alpha_j g_s(z_j) = a_{g_s} & s = 1, 2, ..., L. \end{cases} \quad (5.4)$$

Now, by the solution of this problem, we can get the coefficients α_j, and from the analysis in Rubio [17] we can obtain the piecewise-constant control function $u(.)$ which approximate the action of the optimal measure.

6. Numerical Simulations

In this section, the optimality system is numerically solved by applying MATLAB . The values of the parameters are presented in the following Table 1.

$Parameters$	α	β_{AI}	β_{SI}	σ_{RS}	σ_{IS}	δ	ω	ϵ
$Values$	0.5	0.2	0.4	0.4	0.3	0.2	0.5	10

We consider the initial population contain susceptible nodes $S(0) = 7$, infected nodes $I(0) = 1$, recovered nodes $R(0) = 1$ and antidotal nodes $A(0) = 1$ for numerical simulation. Let us $t \in J = [0, 25]$, and $X(t) = [x_1(t), x_2(t), x_3(t), x_4(t)] \in A = A_1 \times A_2 \times A_3 \times A_4$, where $A1 = A2 = [0, 7]$, $A_3 = [0, 5]$, $A_4 = [0, 2.5]$. Let the sets J and A_2 be divided into 10 equal subintervals, the set A_1, A_3 and A_2 are divided respectively into 4 equal subintervals, and also the set $U = [0, 1]$ divided into 4 equal subintervals, so that $\Omega = J \times A \times U$ is divided into 25600 equal subsets. We assume $Z_j = (t_j, x_{1j}, x_{2j}, x_{3j}, x_{4j}, u_j), j = 1, ..., 25600$ and

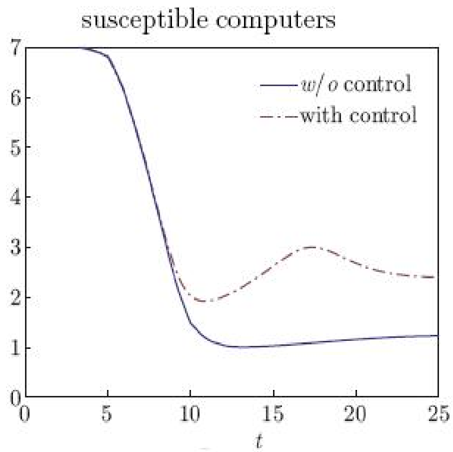
$$temp = k_1 + 4(i-1) + 16(j-1) + 64(k-1) + 640 * (f-1) + 2560(l-1)$$

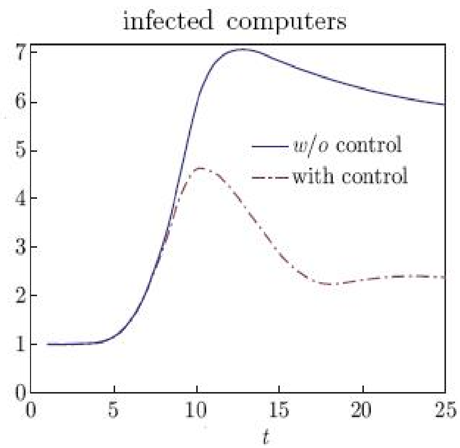
where

$$x_{1j}(temp) = -0.1 + \frac{7}{4}f;\ x_{2j}(temp) = -0.15 + \frac{7}{15}k;$$
$$x_{3j}(temp) = -0.25 + \frac{5}{4}j;\ x_{4j}(temp) = -0.35 + \frac{2.5}{4}i;$$
$$u_j(temp) = -0.45 + \frac{0.1}{4}k_1;\ t_j(temp) = 2.5l;$$
$$l = 1, \ldots 10; f = 1 \ldots 4; k = 1 \ldots 10; j = 1 \ldots 4; i = 1 \ldots 4; k_1 = 1 \ldots 4.$$

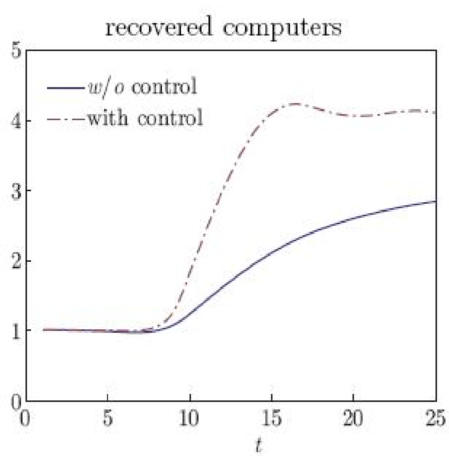
Also, we get $M_1 = 4, L = 4$ and $M_2 = 24$. And the functions $\phi_f^i\ i = 1 \ldots 4$, $h = 1 \ldots, 8$, will be chosen as the form (5.1). We have an linear programming (LP) with 25620 unknowns and 44 constrains which solved by the revised simplex code of the optimization toolbox in MATLAB. The total CPU time required on a laptop with CPU 5 GHz and 4 GB of RAM was 25.65 minutes.

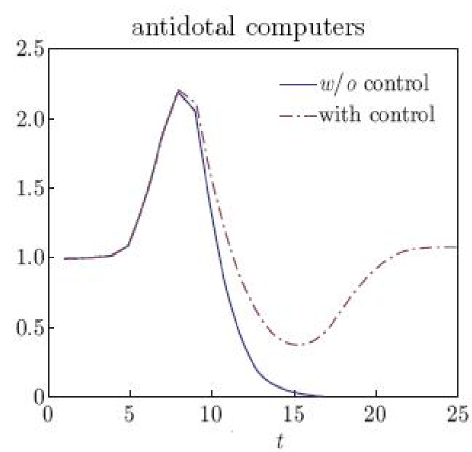
In the following figures, the population size of each individual without control is marked by solid line,while the one with control is marked by dash-dotted line. The following figure represents the population size of susceptible nodes without control and with control. The result shows that the rate of infected susceptible nodes with optimal control strategy becomes slower and smaller number of nodes is infected from the computer virus. Also, represents the infected population in both cases. The population size of infected nodes without the optimal control strategy is larger than the nodes with control strategy.





In following figures, the recovered population becomes larger after control and the antidotal population also becomes larger after the control. Thus, after the optimal control strategy is introduced into this SIRA model, the infection rate of susceptible decreases and more infected nodes are recovered or become susceptible.

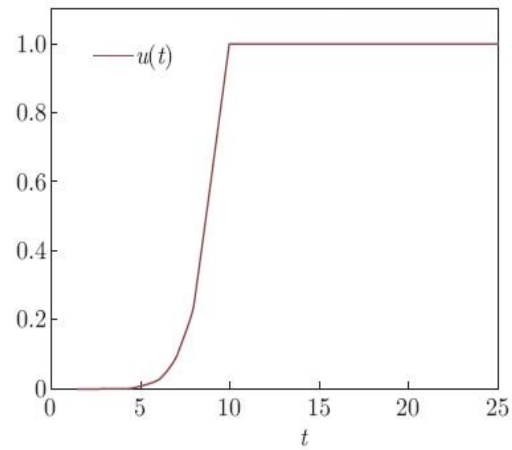




Finally, we need the optimal strategy to control the infected nodes, which is presented in following figure.

7. Conclusion

Our numerical results show that the number of infected nodes after the control is much smaller than that of infected nodes before the control. Therefore, it has



a realistic significance in the computer virus research by introducing an optimal control into an SIRA computer virus spread model. Viral attacks against computer networks are an important research area because the defense strategies need to be able to avoid infection propagation. In this work we presented the SIRA model based on epidemiological studies and conditions for the asymptotically stability of the disease free equilibrium were deduced. Some simulations were performed showing how a parameter, analogous to the epidemic basal reproduction rate, affects the dynamics of the infection propagation.

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THE GENERALIZED VON NEUMMAN-JORDAN CONSTANT AND FIXED POINTS OF MULTIVALUED NONEXPANSIVE MAPPINGS IN BANACH SPACES

M. DINARVAND*

Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, 50, Taleghani Ave., 15618, Tehran, Iran

ABSTRACT. Recently, a new geometric constant called generalized von Neumman-Jordan constant was introduced. In this work, we give some sufficient conditions for the (DL)-condition and property (D) in terms of this new constant and many coefficients namely the weakly convergent sequence coefficient, the normal structure coefficient, the coefficient of weak orthogonality and the generalized García-Falset coefficient. As consequences, we obtain several sufficient conditions which imply the existence of fixed points for multivalued nonexpansive mappings and normal structure in Banach spaces. The obtained results generalize and unify some known results in the recent literature.

KEYWORDS : Multivalued nonexpansive mapping; Fixed point; Normal structure; Generalized von Neumman-Jordan constant; Weakly convergent sequence coefficient; Normal structure coefficient.
AMS Subject Classification: 47H10, 46B20

1. INTRODUCTION

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [24] and Nadler [26]. Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, game theory, convex optimization, differential inclusion and mathematical economics. Thus, it is natural to extend the known fixed point results for singlevalued mappings to the setting of multivalued mappings. Nevertheless, the fixed point theory of multivalued nonexpansive mappings is much more complicated and difficult than the corresponding theory of singlevalued nonexpansive mappings and many problems remain unsolved in it. For instance, the celebrated Kirk's theorem [23] which states the fixed point property for singlevalued nonexpansive mappings in reflexive Banach spaces with normal structure yields to a very natural question: Do reflexive Banach spaces

* *Corresponding author.*
Email address : dinarvand_mina@yahoo.com.
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with normal structure have the fixed point property for multivalued nonexpansive mappings? Until now, the answer is unknown.

The concept of normal structure plays an important role in metric fixed point theory for nonexpansive mappings. Since under various geometric properties of a Banach space often measured by different geometric constants, normal structure of the space is guaranteed, it is natural to study if those properties imply the fixed point theory for multivalued nonexpansive mappings. Dhompongsa et al. [6, 7] introduced the Domínguez-Lorenzo condition ((DL)-condition, in short) and property (D) which imply the fixed point theory for multivalued nonexpansive mappings and normal structure in reflexive Banach spaces. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply either the (DL)-condition or property (D).

Recently, many geometric constants for a Banach space have been investigated. Moreover, many recent studies have focused on sufficient conditions for the existence of fixed points of multivalued nonexpansive mappings and normal structure of Banach spaces in terms of these constants and some well known moduli and coefficients. For more details in this direction, we refer the reader to [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 22, 25, 29, 30] and the references mentioned therein.

Throughout this paper, we assume that X be a Banach space with the closed unit ball $B_X = \{x \in X \,:\, \|x\| \leq 1\}$ and the unit sphere $S_X = \{x \in X \,:\, \|x\| = 1\}$. The following two constants of a Banach space X,

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\big(\|x\|^2 + \|y\|^2\big)} \,:\, x, y \in X,\, (x,y) \neq (0,0)\right\},$$

$$J(X) = \sup\big\{\min\big(\|x+y\|, \|x-y\|\big) \,:\, x, y \in S_X\big\}$$

are called the von Neumann-Jordan [4] and James constants [15], respectively.

Recently, a new geometric constant $C_{NJ}^{(p)}(X)$ of a Banach space X was introduced which is related to the von Neumann-Jordan constant and can be used for much better characterization of a Banach space X.

In [5], Cui et al. defined the generalized von Neumann-Jordan constant by

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}\big(\|x\|^p + \|y\|^p\big)} \,:\, x, y \in X,\, (x,y) \neq (0,0)\right\},$$

where $1 \leq p < \infty$. It is shown that $1 \leq C_{NJ}^{(p)}(X) \leq 2$.

Recall that a Banach space X is said to be uniformly nonsquare [18], in the sense of James, if there exists a positive number $\delta < 2$ such that $\frac{\|x+y\|}{2} \leq \delta$ or $\frac{\|x-y\|}{2} \leq \delta$, whenever $x, y \in S_X$. It is known that every uniformly nonsquare space is reflexive. In [5] it was proved that X is uniformly nonsquare if and only if $C_{NJ}^{(p)}(X) < 2$ for any $1 \leq p < \infty$.

The main purpose of this paper is to investigate some sufficient conditions for the (DL)-condition and property (D) in terms of the generalized von Neumman-Jordan constant, the weakly convergent sequence coefficient, the normal structure coefficient, the coefficient of weak orthogonality and the generalized García-Falset coefficient, which enable us to present several sufficient conditions for the existence of fixed points of multivalued nonexpansive mappings and normal structure in Banach spaces. The obtained results generalize and unify a number of recent well known results in this subject.

2. PRELIMINARIES

We start with some concepts and results which will be used in what follows. For a widespread discussion, the reader is directed to [1, 17].

We recall that a Banach space X is said to have normal structure (weak normal structure, respectively) [2] if for every bounded closed (weakly compact, respectively) convex subset K in X that contains more than one point, there exists a point $x_0 \in K$ such that

$$\sup\big\{\|x_0 - y\| \;:\; y \in K\big\} < \operatorname{diam}(K),$$

where $\operatorname{diam}(K) = \sup\{\|x - y\| \;:\; x, y \in K\}$ denotes the diameter of K. For a reflexive Banach space X, normal structure and weak normal structure coincide.

The weakly convergent sequence coefficient $WCS(X) \in [1, 2]$ [1] is equivalently defined by

$$WCS(X) = \inf\left\{\frac{\lim_{n\neq m}\|x_n - x_m\|}{\limsup_n \|x_n\|}\right\},$$

where the infimum is taken over all weakly (not strongly) null sequences $\{x_n\}$ with $\lim_{n\neq m}\|x_n - x_m\|$ existing.

Let C be a nonempty bounded subset of X and E be a nonempty subset of X. The Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\big\{r_x(C) \;:\; x \in E\big\},$$

where $r_x(C) = \sup\{\|x - y\| \;:\; y \in C\}$. We denote $r_C(C)$ by $r(C)$.

The normal structure coefficient $N(X) \in [1, 2]$ defined by Bynum [3] is the number

$$N(X) = \inf\left\{\frac{\operatorname{diam}(E)}{r(E)} \;:\; E \subset X \text{ bounded and convex and } \operatorname{diam}(E) > 0\right\}.$$

The WORTH property was introduced by Sims in [27]. A Banach space X has the WORTH property if

$$\lim_{n\longrightarrow\infty}\big|\|x_n + x\| - \|x_n - x\|\big| = 0$$

for all $x \in X$ and all weakly null sequences $\{x_n\}$. In [28], Sims defined the coefficient of weak orthogonality, which measures the "degree of WORTHwhileness". As in [19], we prefer to use its reciprocal, i.e. $\mu(X) \in [1, 3]$, which is defined as

$$\mu(X) = \inf\big\{\lambda \;:\; \limsup_{n\longrightarrow\infty}\|x_n + x\| \le \lambda \limsup_{n\longrightarrow\infty}\|x_n - x\|\big\},$$

where the infimum is taken over all $x \in X$ and all weakly null sequences $\{x_n\}$ in X. It is worthwhile to mention that X has the WORTH property if and only if $\mu(X) = 1$

The generalized García-Falset coefficient $R(1, X) \in [1, 2]$, introduced by Domínguez Benavides [13], is defined as

$$R(1, X) = \sup\big\{\liminf_{n\longrightarrow\infty}\|x_n + x\|\big\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$\limsup_{n\longrightarrow\infty}\big(\limsup_{m\longrightarrow\infty}\|x_n - x_m\|\big) \le 1.$$

Before going to the results, let us recall some concepts and results about multivalued mappings and ultrapowers of Banach spaces which will be needed in the sequel.

Let E be as above. We shall denote by $CB(X)$ the family of all nonempty bounded closed subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : E \longrightarrow CB(X)$ is said to be nonexpansive if

$$H(Tx,Ty) \leq \|x-y\|, \qquad x,y \in E,$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A,B) = \max\Big\{ \sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\| \Big\}, \qquad A,B \in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r\big(E,\{x_n\}\big)$ and the asymptotic center $A\big(E,\{x_n\}\big)$ of $\{x_n\}$ in E are defined by

$$r\big(E,\{x_n\}\big) = \inf\Big\{ \limsup_{n\longrightarrow\infty}\|x_n - x\| \ : \ x \in E \Big\}$$

and

$$A\big(E,\{x_n\}\big) = \Big\{ x \in E \ : \ \limsup_{n\longrightarrow\infty}\|x_n - x\| = r\big(E,\{x_n\}\big) \Big\},$$

respectively. It is known that $A\big(E,\{x_n\}\big)$ is a nonempty weakly compact convex set whenever E is (see [17]).

The sequence $\{x_n\}$ is called regular with respect to E if $r\big(E,\{x_n\}\big) = r\big(E,\{x_{n_i}\}\big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to E if $A\big(E,\{x_n\}\big) = A\big(E,\{x_{n_i}\}\big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform with respect to E if $\{x_n\}$ is regular and asymptotically uniform with respect to E.

The following two properties of Banach spaces were introduced and used to guarantee the existence of fixed points for multivalued nonexpansive mappings and normal structure in reflexive Banach spaces (see [6, 7]).

A Banach space X is said to satisfy property (D) [6] if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular asymptotically uniform with respect to E, and any sequence $\{y_n\} \subset A(E,\{x_n\})$ which is regular asymptotically uniform with respect to E,

$$r\big(E,\{y_n\}\big) \leq \lambda\, r\big(E,\{x_n\}\big).$$

A Banach space X is said to satisfy the Domínguez-Lorenzo condition [7] if there exists $\lambda \in [0,1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular with respect to E,

$$r_C\big(A\big(E,\{x_n\}\big)\big) \leq \lambda\, r\big(E,\{x_n\}\big).$$

It is clear from the definition that property (D) is weaker than the (DL)-condition. The next results show that property (D) and the (DL)-condition are stronger than weak normal structure and also imply the existence of fixed points for multivalued nonexpansive mappings (see [6, 7]).

Theorem 2.1. *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies (the (DL)-condition) property (D). Let $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.*

Theorem 2.2. *Let X be a Banach space satisfying (the (DL)-condition) property (D). Then X has weak normal structure.*

Let $\mathcal{F}$ be a filter on $\mathbb{N}$. A sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $\{A \subset \mathbb{N} : i_0 \in A\}$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial. Let $\ell_\infty(X)$ denotes the subspace of the product space $\prod_{n\in\mathbb{N}} X$ equipped with the norm $\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \big\{(x_n) \in \ell_\infty(X) \ : \ \lim_{\mathcal{U}} \|x_n\| = 0\big\}.$$

The ultrapower of X, denoted by $\widetilde{X}$, is the quotient space $\frac{\ell_\infty(X)}{N_{\mathcal{U}}}$ equipped with the quotient norm. Write $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\big\|(x_n)_{\mathcal{U}}\big\| = \lim_{\mathcal{U}} \|x_n\|.$$

Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically.

3. MAIN RESULTS

We are now in a position to formulate and prove our main results.

Theorem 3.1. *Let $1 \le p < \infty$ and X be a Banach space such that*

$$C_{NJ}^{(p)}(X) < 1 + \left(\frac{WCS(X)}{2}\right)^p.$$

Then X has property (D).

Proof. Let E be a nonempty weakly compact convex subset of X. Denote $r = r\big(E, \{x_n\}\big)$ and $A = A\big(E, \{x_n\}\big)$. We can assume that $r > 0$. Let $\{x_n\} \subset E$ and $\{y_n\} \subset A$ are regular asymptotically uniform sequences with respect to E. Passing through a subsequence of $\{y_n\}$ if necessary, we can also assume that $\{y_n\}$ is weakly convergent to a point $y \in E$ and $d := \lim_{n,m\longrightarrow\infty,\, n\neq m} \|y_n - y_m\|$ exists. By using the convexity of A and again, passing through a subsequence of $\{x_n\}$ if necessary, we assume in addition that

$$\|x_n - y_{2n}\| \le r + \frac{1}{n}, \qquad \|x_n - y_{2n+1}\| \le r + \frac{1}{n}$$

and

$$\left\|x_n - \frac{1}{2}\big(y_{2n} + y_{2n+1}\big)\right\| \ge r - \frac{1}{n}$$

for all $n \in \mathbb{N}$. Consider

$$u_n = \frac{1}{r + \frac{1}{n}}\big(x_n - y_{2n}\big) \qquad \text{and} \qquad v_n = \frac{1}{r + \frac{1}{n}}\big(x_n - y_{2n+1}\big).$$

It is easy to see that $\lim_{n\longrightarrow\infty} \|u_n + v_n\| = 2$ and $\lim_{n\longrightarrow\infty} \|u_n - v_n\| = \frac{d}{r}$. Hence, we have

$$C_{NJ}^{(p)}(X) \ge \frac{2^p + \left(\frac{d}{r}\right)^p}{2^{p-1}(1+1)} = 1 + \frac{d^p}{2^p r^p}.$$

Now, we estimate d as follows:

$$\begin{aligned} d &= \lim_{n\neq m} \|y_n - y_m\| = \lim_{n\neq m} \big\|(y_n - y) - (y_m - y)\big\| \\ &\ge WCS(X) \limsup_{n} \|y_n - y\| \end{aligned}$$

$$\geq WCS(X) r\big(E, \{y_n\}\big).$$

Therefore, we obtain

$$r\big(E, \{y_n\}\big) \leq \frac{2\big(\sqrt[p]{C_{NJ}^{(p)}(X) - 1}\big)}{WCS(X)} r.$$

Since $C_{NJ}^{(p)}(X) < 1 + \big(\frac{WCS(X)}{2}\big)^p$, it follows that X satisfies property (D). □

Since $WCS(X) \leq 2$, if $C_{NJ}^{(p)}(X) < 1 + \big(\frac{WCS(X)}{2}\big)^p$, then $C_{NJ}^{(p)}(X) < 2$ for all $1 \leq p < \infty$, which implies that X is uniformly nonsquare, and consequently X is reflexive. Thus, by applying Theorems 2.1, 2.2 and 3.1, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.2. *Let E be a nonempty bounded closed convex subset of a Banach space X such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < 1 + \left(\frac{WCS(X)}{2}\right)^p,$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.3. *Let X be a Banach space such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < 1 + \left(\frac{WCS(X)}{2}\right)^p.$$

Then X has normal structure.

Theorem 3.4. *Let E be a nonempty weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_E\big(A\big(E, \{x_n\}\big)\big) \leq \frac{2\Big(\sqrt[p]{C_{NJ}^{(p)}(X) - 1}\Big)}{N(X)} r\big(E, \{x_n\}\big)$$

for all $1 \leq p < \infty$.

Proof. Denote $r = r\big(E, \{x_n\}\big)$ and $A = A\big(E, \{x_n\}\big)$. We can assume that $r > 0$. We note that since $\{x_n\}$ is regular with respect to E, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. If $\operatorname{diam}(A) = 0$, then $r_E(A) = 0$ and hence we are done. So we assume that $\operatorname{diam}(A) > 0$. Let $\varepsilon > 0$ and $u, v \in A$ be such that $\|u - v\| \geq \operatorname{diam}(A) - \varepsilon > 0$. Convexity of A implies that $\frac{u+v}{2} \in A$. By the definition of A, we have

$$\limsup_{n\to\infty} \|x_n - u\| = \limsup_{n\to\infty} \|x_n - v\| = \limsup_{n\to\infty} \left\|x_n - \left(\frac{u+v}{2}\right)\right\| = r.$$

Since $\|u - v\| > 0$, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $x_{n'} - u$ and $x_{n'} - v$ are not both zero. Thus, we have

$$\begin{aligned}\big(\operatorname{diam}(A) - \varepsilon\big)^p + \left\|2\left(x_{n'} - \left(\frac{u+v}{2}\right)\right)\right\|^p &\leq \|u - v\|^p + \left\|2\left(x_{n'} - \left(\frac{u+v}{2}\right)\right)\right\|^p \\ &\leq C_{NJ}^{(p)}(X)\big(2^{p-1}\big(\|x_{n'} - u\|^p \\ &\quad + \|x_{n'} - v\|^p\big)\big).\end{aligned}$$

By taking the upper limit as $n' \longrightarrow \infty$ throughout, we have

$$\big(\operatorname{diam}(A) - \varepsilon\big)^p + 2^p r^p \leq C_{NJ}^{(p)}(X)\big(2^{p-1}(r^p + r^p)\big),$$

from which it follows that

$$\big(\operatorname{diam}(A)-\varepsilon\big)^p \le 2^p\big(C_{NJ}^{(p)}(X)-1\big)r^p.$$

Because ε is arbitrarily small, we get

$$\big(\operatorname{diam}(A)\big)^p \le 2^p\big(C_{NJ}^{(p)}(X)-1\big)r^p,$$

which implies that

$$\operatorname{diam}(A) \le 2\Big(\sqrt[p]{C_{NJ}^{(p)}(X)-1}\Big)r. \tag{3.1}$$

Since A is a bounded convex subset of X with $\operatorname{diam}(A)>0$, it follows that

$$r_E(A) \le r(A) \le \frac{\operatorname{diam}(A)}{N(X)}. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$r_E(A) \le \frac{2\Big(\sqrt[p]{C_{NJ}^{(p)}(X)-1}\Big)}{N(X)}\,r.$$

□

Since $N(X)\le 2$, if $C_{NJ}^{(p)}(X) < 1+\big(\frac{N(X)}{2}\big)^p$, then $C_{NJ}^{(p)}(X)<2$ for all $1\le p<\infty$, which implies that X is uniformly nonsquare, and consequently X is reflexive. Thus, by applying Theorems 2.1, 2.2 and 3.4, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.5. *Let E be a nonempty bounded closed convex subset of a Banach space X such that for some $1\le p<\infty$,*

$$C_{NJ}^{(p)}(X) < 1+\left(\frac{N(X)}{2}\right)^p,$$

and $T: E\to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.6. *Let X be a Banach space such that for some $1\le p<\infty$,*

$$C_{NJ}^{(p)}(X) < 1+\left(\frac{N(X)}{2}\right)^p.$$

Then X has normal structure.

Theorem 3.7. *Let E be a nonempty weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_C\big(A\big(E,\{x_n\}\big)\big) \le \left(\frac{\Big(2^pC_{NJ}^{(p)}(X)-\big(1+\frac{1}{\mu(X)}\big)^p\Big)^{\frac{1}{p}}}{1+\frac{1}{\mu(X)}}\right) r\big(E,\{x_n\}\big)$$

for all $1\le p<\infty$.

Proof. For convenience, we denote $r=r\big(E,\{x_n\}\big)$ and $A=A\big(E,\{x_n\}\big)$. We can assume that $r>0$. Since $\{x_n\}\subset E$ is bounded and E is a weakly compact set, we can also assume, by passing through a subsequence if necessary, that $\{x_n\}$ is weakly convergent to a point $x\in E$. We note that since $\{x_n\}$ is regular with respect to E, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$.

Let $z \in A$. Then $\limsup_n \|x_n - z\| = r$. Denote $\mu = \mu(X)$. Since $(x_n - x) \xrightarrow{\omega} 0$ and by the definition of μ, we have

$$\begin{aligned}\limsup_n \|x_n - 2x + z\| &= \limsup_n \big\|(x_n - x) + (z - x)\big\| \\ &\le \mu \limsup_n \big\|(x_n - x) - (z - x)\big\| \\ &= \mu r.\end{aligned}$$

Convexity of E implies that $\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \in E$ and thus we obtain

$$\limsup_n \left\| x_n - \left(\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z\right)\right\| \ge r.$$

On the other hand, by the weak lower semicontinuity of the norm, we have

$$\liminf_n \left\| \left(1 - \frac{1}{\mu}\right)(x_n - x) - \left(1 + \frac{1}{\mu}\right)(z - x)\right\| \ge \left(1 + \frac{1}{\mu}\right)\|z - x\|.$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| < r + \varepsilon$.

(2) $\|x_N - 2x + z\| \le \mu(r + \varepsilon)$.

(3) $\left\| x_N - \left(\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z\right)\right\| \ge r - \varepsilon$.

(4) $\left\| \left(1 - \frac{1}{\mu}\right)(x_N - x) - \left(1 + \frac{1}{\mu}\right)(z - x)\right\| \ge \left(1 + \frac{1}{\mu}\right)\|z - x\|\left(\frac{r - \varepsilon}{r}\right)$.

Consider $u = \frac{1}{r+\varepsilon}(x_N - z) \in B_X$ and $v = \frac{1}{\mu(r+\varepsilon)}(x_N - 2x + z) \in B_X$. By applying the above estimates, we obtain

$$\begin{aligned}\|u + v\| &= \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} + \frac{x_N - x}{\mu(r+\varepsilon)} + \frac{z - x}{\mu(r+\varepsilon)}\right\| \\ &= \left\| \left(\frac{1}{r+\varepsilon} + \frac{1}{\mu(r+\varepsilon)}\right)(x_N - x) - \left(\frac{1}{r+\varepsilon} + \frac{1}{\mu(r+\varepsilon)}\right)(z - x)\right\| \\ &= \left(\frac{1}{r+\varepsilon}\right)\left(1 + \frac{1}{\mu}\right)\left\|(x_N - x) - \left(\frac{1 - \frac{1}{\mu}}{1 + \frac{1}{\mu}}\right)(z - x)\right\| \\ &= \left(\frac{1}{r+\varepsilon}\right)\left(1 + \frac{1}{\mu}\right)\left\| x_N - \left(\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z\right)\right\| \\ &\ge \left(1 + \frac{1}{\mu}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right),\end{aligned}$$

$$\begin{aligned}\|u - v\| &= \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} - \frac{x_N - x}{\mu(r+\varepsilon)} - \frac{z - x}{\mu(r+\varepsilon)}\right\| \\ &= \left(\frac{1}{r+\varepsilon}\right)\left\| \left(1 - \frac{1}{\mu}\right)(x_N - x) - \left(1 + \frac{1}{\mu}\right)(z - x)\right\| \\ &\ge \left(1 + \frac{1}{\mu}\right)\left(\frac{\|z - x\|}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}C_{NJ}^{(p)}(X) &\ge \frac{\|u + v\|^p + \|u - v\|^p}{2^{p-1}(\|u\|^p + \|v\|^p)} \\ &\ge \frac{\left(1 + \frac{1}{\mu}\right)^p\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p + \left(1 + \frac{1}{\mu}\right)^p\left(\frac{\|z-x\|}{r}\right)^p\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p}{2^{p-1}(1 + 1)}.\end{aligned}$$

Letting $\varepsilon \longrightarrow 0^+$, we obtain

$$C_{NJ}^{(p)}(X) \geq \frac{\left(1+\frac{1}{\mu}\right)^p + \left(1+\frac{1}{\mu}\right)^p \left(\frac{\|z-x\|}{r}\right)^p}{2^p}.$$

This holds for arbitrary $z \in A$. Hence, we have

$$\sup_{z\in A} \|x-z\| \leq \left(\frac{\left(2^p C_{NJ}^{(p)}(X) - \left(1+\frac{1}{\mu(X)}\right)^p\right)^{\frac{1}{p}}}{1+\frac{1}{\mu(X)}}\right) r.$$

from which it follows that

$$r_E(A) \leq \left(\frac{\left(2^p C_{NJ}^{(p)}(X) - \left(1+\frac{1}{\mu(X)}\right)^p\right)^{\frac{1}{p}}}{1+\frac{1}{\mu(X)}}\right) r.$$

□

Remark 3.8. According to Theorem 3.7, if X has the WORTH property, then

$$r_C\big(A(E,\{x_n\})\big) \leq r\big(E,\{x_n\}\big) \sqrt[p]{C_{NJ}^{(p)}(X) - 1}$$

for all $1 \leq p < \infty$, since $\mu(X) = 1$.

Since $\mu(X) \geq 1$, if $C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\big(1+\frac{1}{\mu(X)}\big)^p$, then $C_{NJ}^{(p)}(X) < 2$ for all $1 \leq p < \infty$, which implies that X is uniformly nonsquare, and consequently X is reflexive. Thus, by applying Theorems 2.1, 2.2 and 3.7, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.9. *Let E be a nonempty bounded closed convex subset of a Banach space X such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{\mu(X)}\right)^p,$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.10. *Let X be a Banach space such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{\mu(X)}\right)^p.$$

Then X has normal structure.

Theorem 3.11. *Let E be a nonempty weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_C\big(A(E,\{x_n\})\big) \leq \frac{2^{p-1} C_{NJ}^{(p)}(X)}{\left(1+\frac{1}{R(1,X)}\right)^p} r\big(E,\{x_n\}\big)$$

for all $1 \leq p < \infty$.

Proof. For convenience, we denote $r = r\big(E,\{x_n\}\big)$ and $A = A\big(E,\{x_n\}\big)$. We can assume that $r > 0$. Since $\{x_n\} \subset E$ is bounded and E is a weakly compact set, we can also assume, by passing through a subsequence if necessary, that $\{x_n\}$ is weakly convergent to a point $x \in E$ and $d := \lim_{n\neq m} \|x_n - x_m\|$ exists. We note that since $\{x_n\}$ is regular with respect to E, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Observe that, since the norm is weak lower semicontinuity, we have

$$\liminf_n \|x_n - x\| \le \liminf_n \liminf_m \|x_n - x_m\| = \lim_{n\neq m} \|x_n - x_m\| = d.$$

Let $\varepsilon > 0$. Taking a subsequence if necessary, we can assume that $\|x_n - x\| < d+\varepsilon$ for all n.

Let $z \in A$, then $\limsup_n \|x_n - z\| = r$ and $\|x - z\| \le \liminf_n \|x_n - z\| \le r$. Denote $R = R(1, X)$. Since $(x_n - x) \xrightarrow{w} 0$ and by the definition of R, we have

$$R \ge \liminf_n \left\| \frac{x_n - x}{d+\varepsilon} + \frac{z - x}{r} \right\| = \liminf_n \left\| \frac{x_n - x}{d+\varepsilon} - \frac{x - z}{r} \right\|.$$

On the other hand, observe that the convexity of E implies that $\frac{R-1}{R+1}x + \frac{2}{R+1}z \in E$ and by the weak lower semicontinuity of the norm, we have

$$\begin{aligned}
&\liminf_n \left\| \frac{x_n - z}{r} + \frac{1}{R}\left(\frac{x_n - x}{d+\varepsilon} - \frac{x - z}{r} \right) \right\| \\
&= \liminf_n \left\| \left(\frac{1}{r} + \frac{1}{R(d+\varepsilon)} \right) x_n - \left(\frac{1}{R(d+\varepsilon)} + \frac{1}{Rr} \right) x - \left(\frac{1}{r} - \frac{1}{Rr} \right) z \right\| \\
&\ge \left\| \left(\frac{1}{r} - \frac{1}{Rr} \right) x + \frac{2}{Rr} z - \left(\frac{1}{r} + \frac{1}{Rr} \right) z \right\| \\
&= \left(\frac{1}{r} + \frac{1}{Rr} \right) \left\| \frac{R-1}{R+1} x + \frac{2}{R+1} z - z \right\| \\
&\ge \left(1 + \frac{1}{R} \right) \left(\frac{r_E(A)}{r} \right), \\
&\liminf_n \left\| \frac{x_n - z}{r} - \frac{1}{R}\left(\frac{x_n - x}{d+\varepsilon} - \frac{x - z}{r} \right) \right\| \\
&= \liminf_n \left\| \left(\frac{1}{r} - \frac{1}{R(d+\varepsilon)} \right) (x_n - x) - \left(\frac{1}{r} + \frac{1}{Rr} \right) (z - x) \right\| \\
&\ge \left(\frac{1}{r} + \frac{1}{Rr} \right) \|z - x\| \ge \left(1 + \frac{1}{R} \right) \left(\frac{r_E(A)}{r} \right).
\end{aligned}$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \le r + \varepsilon$.

(2) $\left\| \frac{r(x_N - x)}{d+\varepsilon} - (x - z) \right\| \le R(r+\varepsilon)$.

(3) $\frac{1}{Rr} \left\| R(x_N - z) + \frac{r(x_N - x)}{d+\varepsilon} - (x - z) \right\| \ge \left(1 + \frac{1}{R} \right) \left(\frac{r_E(A)}{r} \right) \left(\frac{r - \varepsilon}{r} \right)$.

(4) $\frac{1}{Rr} \left\| R(x_N - z) - \left(\frac{r(x_N - x)}{d+\varepsilon} - (x - z) \right) \right\| \ge \left(1 + \frac{1}{R} \right) \left(\frac{r_E(A)}{r} \right) \left(\frac{r - \varepsilon}{r} \right)$.

In the ultrapower $\widetilde{X}$ of X, we consider

$$\widetilde{u} = \left(\frac{x_N - z}{r+\varepsilon} \right)_{\mathcal{U}} \in B_X \qquad \text{and} \qquad \widetilde{v} = \frac{1}{R(r+\varepsilon)} \left(\frac{r(x_N - x)}{d+\varepsilon} - (x - z) \right)_{\mathcal{U}} \in B_X.$$

By applying the above estimates, we obtain

$$\|\widetilde{u} + \widetilde{v}\| = \frac{1}{R(r+\varepsilon)} \left\| R(x_N - z) + \frac{r(x_N - x)}{d+\varepsilon} - (x - z) \right\|$$

$$
\begin{aligned}
&\geq \left(1+\frac{1}{R}\right)\left(\frac{r_E(A)}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right),\\
\|\widetilde{u}-\widetilde{v}\| &= \frac{1}{R(r+\varepsilon)}\left\|R(x_N-z)-\left(\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\right)\right\|\\
&\geq \left(1+\frac{1}{R}\right)\left(\frac{r_E(A)}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right).
\end{aligned}
$$

Therefore, by the definition of $C_{NJ}^{(p)}(\widetilde{X})$, we have

$$
\begin{aligned}
C_{NJ}^{(p)}(\widetilde{X}) &\geq \frac{\|\widetilde{u}+\widetilde{v}\|^p+\|\widetilde{u}-\widetilde{v}\|^p}{2^{p-1}\left(\|\widetilde{u}\|^p+\|\widetilde{v}\|^p\right)}\\
&\geq \frac{2\left(1+\frac{1}{R}\right)^p\left(\frac{r_E(A)}{r}\right)^p\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p}{2^{p-1}(1+1)}\\
&\geq \frac{\left(1+\frac{1}{R}\right)^p\left(\frac{r_E(A)}{r}\right)^p\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p}{2^{p-1}}.
\end{aligned}
$$

Since the above inequality is true for every $\varepsilon > 0$ and $C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(\widetilde{X})$, we obtain

$$
r_E(A) \leq \frac{2^{p-1}C_{NJ}^{(p)}(X)}{\left(1+\frac{1}{R}\right)^p}\, r.
$$

□

Since $R(1,X) \geq 1$, if $C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{R(1,X)}\right)^p$, then $C_{NJ}^{(p)}(X) < 2$ for all $1 \leq p < \infty$, which implies that X is uniformly nonsquare, and consequently X is reflexive. Thus, by applying Theorems 2.1, 2.2 and 3.11, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.12. *Let E be a nonempty bounded closed convex subset of a Banach space X such that for some $1 \leq p < \infty$,*

$$
C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{R(1,X)}\right)^p,
$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.13. *Let X be a Banach space such that for some $1 \leq p < \infty$,*

$$
C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{R(1,X)}\right)^p.
$$

Then X has normal structure.

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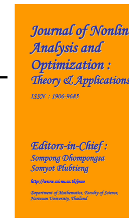

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BALL CONVERGENCE RESULTS FOR A METHOD WITH MEMORY OF EFFICIENCY INDEX 1.8392 USING ONLY FUNCTIONAL VALUES

I. K. ARGYROS AND S. GEORGE*

Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-575 025

ABSTRACT. We present a local convergence analysis for an one-step iterative method with memory of efficiency index 1.8392 to solve nonlinear equations. If the function is twice differentiable, then it was shown that the $R-$order of convergence is 1.8392. In this paper we use hypotheses up to the first derivative. This way we extend the applicability of this method. Moreover, the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples are also presented to illustrate the theoretical results.

KEYWORDS : Halley's method; local convergence; order of convergence; efficiency index.
AMS Subject Classification: 65D10, 65D99

1. Introduction

Let $f : D \subseteq S \to S$ be a nonlinear function, D is a convex subset of S where S is $\mathbb{R}$ or $\mathbb{C}$. Consider the problem of approximating a locally unique solution x^* of equation

$$f(x) = 0. \tag{1.1}$$

Newton-like methods are famous for finding solution of (1.1), these methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [3, 4, 19, 20, 23].

Third order methods such as Euler's, Halley's, super Halley's, Chebyshev's [1]-[23] require the evaluation of the second derivative f'' at each step, which in general is very expensive. That is why many authors have used higher order multipoint

* *Corresponding author.*
Email address : iargyros@cameron.edu, sgeorge@nitk.edu.in.
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methods [1]-[23]. In this paper, we study the local convergence of the one-step method with memory defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} x_{n+1} &= \frac{(x_n + x_{n-1}) f_n f_{n-1} f[x_n, x_{n-1}]}{2 f_n f_{n-1} f[x_n, x_{n-1}] - (f_{n-1}^2 f[x_n, x_{n-1}] + f_n^2 f[x_{n-1}, x_{n-2}])}, \\ & - \frac{(f_{n-1}^2 x_n f[x_n, x_{n-1}] + f_n^2 x_{n-1} f[x_{n-1}, x_{n-2}])}{2 f_n f_{n-1} f[x_n, x_{n-1}] - (f_{n-1}^2 f[x_n, x_{n-1}] + f_n^2 f[x_{n-1}, x_{n-2}])}, \end{aligned} \quad (1.2)$$

where x_{-2}, x_{-1}, x_0 are initial points , $f_n = f(x_n)$, $f[x, y]$ denotes a divided difference of order one for function f at the point x, y [3, 4, 23] defined by

$$f[x, y] = \frac{f(x) - f(y)}{x - y} \text{ if } x \neq y \quad (1.3)$$

and $f[x, x] = f'(x)$. Method (1.2) uses only one function evaluation per step, f_n and it was shown using the Herberger's matrix method in [17] that the $R-$order of convergence is $\frac{1}{3}(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}} \approx 1.8392$. Moreover, the efficiency index is

$$I = (1.8392)^{\frac{1}{1}} = 1.8392.$$

These results are obtained provided that the function f is twice differentiable. This hypotheses limits the applicability of method (1.2) although only function evaluations are needed to carry out the computation of each step. As a motivational example, let us define function f on $D = [-1, 2]$ by

$$f(x) = \begin{cases} x^2 \ln x + x^4 - x^3, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then, we have that

$$f'(x) = 2x \ln x + 4x^3 - 3x^2 + x,$$

and

$$f''(x) = 2 \ln x + 12x^2 - 6x + 3.$$

Hence, function f'' is unbounded on D. In the present paper, we study the local convergence of method (1.2) using hypotheses only on the first derivative. Moreover, we provide: the radius of convergence, computable error bounds on the distances $|x_n - x^*|$ and a uniqueness result.

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of methods (1.2). The numerical examples are given in the concluding Section 3.

2. Local convergence for method (1.2)

We present the local convergence analysis of method (1.2) in this section. We first simplify method (1.2) by using formula (1.3) to obtain that

$$x_{n+1} - x^* = \frac{A_n}{B_n}, \quad (2.1)$$

where

$$\begin{aligned} A_n &= f_{n-1}(f_n - f_{n-1})^2 (x_{n-1} - x_{n-2})(x_n - x^*) \\ & + f_n[-(f_n - f_{n-1})^2 (x_{n-1} - x_{n-2}) \\ & + f_n((f_n - f_{n-1})(x_{n-1} - x_{n-2}) \\ & - (f_{n-1} - f_{n-2})(x_n - x_{n-1})](x_{n-1} - x^*) \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} B_n &= 2f_nf_{n-1}(f_n-f_{n-1})(x_{n-1}-x_{n-2})-f_{n-1}^2(f_n-f_{n-1})(x_{n-1}-x_{n-2}) \\ &\quad -f_n^2(f_n-f_{n-1})(x_n-x_{n-1}). \end{aligned} \tag{2.3}$$

Let us also define function g by

$$g(t)=40M^3t^4+4M^3t^2-1 \text{ for some } M>0. \tag{2.4}$$

Notice that

$$r=\frac{1}{2M}\sqrt{\frac{2M}{M^2+\sqrt{M^4+10M}}} \tag{2.5}$$

is the only positive root of equation $g(t)=0$. We also have that

$$0\leq \frac{24M^3t^4}{1-4M^3t^2(1+4t^2)}<1 \text{ for each } t\in[0,r). \tag{2.6}$$

Let $U(v,\rho),\bar{U}(v,\rho)$ stand for the open and closed balls in S, respectively, with center $v\in S$ and of radius $\rho>0$.

Using the preceding notation we present the local convergence analysis of method (1.2).

THEOREM 2.1. *Let $f:D\subseteq S\to S$ be a differentiable function. Suppose that there exist $x^*\in D, M>0$ such that for each $x\in D$*

$$f(x^*)=0, \tag{2.7}$$

$$|f'(x)|\leq M, \tag{2.8}$$

and

$$\bar{U}(x^*,r)\subseteq D, \tag{2.9}$$

where r is defined by (2.5). Then, the sequence $\{x_n\}$ generated by method (1.2) for for $x_{-2},x_{-1},x_0\in U(x^*,r)$ is well defined, remains in $U(x^*,r)$ for each $n=0,1,2,\cdots$ and converges to x^*. Moreover, the following estimates hold for each $n=0,1,2,\cdots$,

$$|x_{n+1}-x^*|\leq \frac{a_n}{b_n}|x_{n-1}-x^*||x_n-x^*|<|x_n-x^*|<r, \tag{2.10}$$

where

$$a_n=2M^3(|x_n-x^*|+|x_{n-1}-x^*|)(|x_{n-1}-x^*|+|x_{n-2}-x^*|)(2|x_n-x^*|+|x_{n-1}-x^*|)$$

and

$$b_n=1-M^3(|x_n-x^*|+|x_{n-1}-x^*|)(|x_{n-1}-x^*|+|x_{n-2}-x^*|)(1+(|x_{n-1}-x^*|+|x_n-x^*|)^2).$$

Proof. By hypothesis $x_{-2},x_{-1},x_0\in U(x^*,r)$. We can write using (2.7) that

$$f_{-2}=f_{-2}-f(x^*)=\int_0^1 f'(x^*+\theta(x_{-2}-x^*))(x_{-2}-x^*)d\theta. \tag{2.11}$$

Then, by (2.11) and (2.8) we get that

$$|f_{-2}|=|\int_0^1 f'(x^*+\theta(x_{-2}-x^*))(x_{-2}-x^*)d\theta|\leq M|x_{-2}-x^*|. \tag{2.12}$$

Similarly, we have that

$$|f_{-1}|\leq M|x_{-1}-x^*|. \tag{2.13}$$

and

$$|f_0|\leq M|x_0-x^*|. \tag{2.14}$$

We shall show that B_0 given by (2.3) is invertible. We have in turn by (2.3), (2.6), (2.12)- (2.14) and the triangle inequality that

$$\begin{aligned} |B_0 - 1| &= |2f_0 f_{-1}(f_0 - f_{-1})(x_{-1} - x_{-2}) - f_{-1}^2(f_0 - f_{-1})(x_{-1} - x_{-2}) \\ &\quad - f_0^2(f_{-1} - f_{-2})(x_0 - x_{-1}) - (x_0 - x_{-1})(x_{-1} - x_{-2})| \\ &= |f_{-1}(f_0 - f_{-1})^2(x_{-1} - x_{-2}) + f_0(f_{-1}(f_0 - f_{-1})(x_{-1} - x_{-2}) \\ &\quad - f_0(f_{-1} - f_{-2})(x_0 - x_{-1})) - (x_0 - x_{-1})(x_{-1} - x_{-2})| \\ &\leq M|x_{-1} - x^*|M^2(|x_0 - x^*| + |x_{-1} - x^*|)^2(|x_{-1} - x^*| + |x_{-2} - x^*|) \\ &\quad + M|x_0 - x^*|(M|x_{-1} - x^*|M(|x_0 - x^*| + |x_{-1} - x^*|) \\ &\quad \times(|x_{-1} - x^*| + |x_{-2} - x^*|) \\ &\quad + M|x_0 - x^*|M(|x_{-1} - x^*| + |x_{-2} - x^*|)(|x_0 - x^*| + |x_{-1} - x^*|)) \\ &\quad + (|x_0 - x^*| + |x_{-1} - x^*|)(|x_{-1} - x^*| + |x_{-2} - x^*|) \\ &= 1 - b_0 < 4M^3r^2(1 + 4r^2) < 1. \end{aligned} \tag{2.15}$$

It follows from (2.15) and the Banach lemma on invertible functions [3, 4] that B_0 is invertible and

$$|B_0^{-1}| \leq \frac{1}{b_0} < \frac{1}{1 - 4M^3r^2(1 + 4r^2)}. \tag{2.16}$$

Next, we need an estimate on A_0. It follows from (2.2), (2.12)-(2.14) and triangle inequality that

$$\begin{aligned} |A_0| &\leq M|x_{-1} - x^*|M^2(|x_{-1} - x^*| + |x_0 - x^*|)^2 \\ &\quad \times(|x_{-1} - x^*| + |x_{-2} - x^*|)|x_0 - x^*| \\ &\quad + M|x_0 - x^*||x_{-1} - x^*|[M^2(|x_0 - x^*| + |x_{-1} - x^*|)^2 \\ &\quad \times(|x_{-1} - x^*| + |x_{-2} - x^*|) \\ &\quad + M|x_0 - x^*|(M(|x_0 - x^*| + |x_{-1} - x^*|)(|x_{-1} - x^*| + |x_{-2} - x^*|) \\ &\quad + M(|x_{-1} - x^*| + |x_{-2} - x^*|)(|x_0 - x^*| + |x_{-1} - x^*|)] \\ &\leq a_0|x_{-1} - x^*||x_0 - x^*| \leq 24M^3r^4|x_0 - x^*|. \end{aligned} \tag{2.17}$$

Then, using (2.1), (2.6), (2.16) and (2.17), we get that

$$\begin{aligned} |x_{n+1} - x^*| &\leq \frac{a_0|x_{-1} - x^*||x_0 - x^*|}{b_0} \\ &< \frac{24M^3r^4}{1 - 4M^3r^2(1 + 4r^2)}|x_0 - x^*| = |x_0 - x^*| < r, \end{aligned}$$

which shows (2.10) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_{-2}, x_{-1}, x_0 by x_{k-2}, x_{k-1}, x_k in the preceding estimates, we arrive at (2.10). Then from the estimate $|x_{k+1} - x^*| < |x_k - x^*| < r$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

Next, we present a uniqueness result for method (1.2).

THEOREM 2.2. *Suppose that the hypotheses of Theorem 2.1 hold and there exists $L_0 > 0$ such that for each $x \in D$, $f'(x^*) \neq 0$,*

$$|f'(x^*)^{-1}(f'(x) - f'(x^*))| \leq L_0|x - x^*| \tag{2.18}$$

and

$$L_0 r < 2. \tag{2.19}$$

Then, the limit point x^* is the only solution of equation $f(x) = 0$ in $\bar{U}(x^*, r)$.

Proof. The existence of the solution x^* in $U(x^*, r)$ has been established in Theorem 2.1. To show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ with $f(y^*) = 0$. Define $T = \int_0^1 f'(y^* + \theta(x^* - y^*)d\theta$. Using (2.18) and (2.19) we get that

$$\begin{aligned} |f'(x^*)^{-1}(T - f'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\ &\leq \int_0^1 (1-\theta)|x^* - y^*|d\theta \leq \frac{L_0}{2} r < 1. \end{aligned} \tag{2.20}$$

It follows from (2.20) that T is invertible. Finally, from the identity $0 = f(x^*) - f(y^*) = T(x^* - y^*)$, we deduce that $x^* = y^*$.

□

REMARK 2.3. *The computation of the order of convergence involves estimates of higher order derivative of operator f. So one may use the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

3. Numerical Examples

We present a numerical example in this section.

EXAMPLE 3.1. *Let $D = [-1, 1]$. Define function f of D by*

$$f(x) = e^x - 1. \tag{3.1}$$

Using (3.1) and $x^* = 0$, we get that $M = e, L_0 = e - 1, r = 0.1058$ and $\xi_1 = 1.5586$. Notice that (2.19) is satisfied, so the solution $x^* = 0$ is unique in $\bar{U}(0, 1)$.

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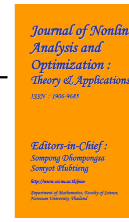

IMPROVED CONVERGENCE FOR KING-WERNER-TYPE DERIVATIVE FREE METHODS

IOANNIS K. ARGYROS AND SANTHOSH GEORGE*

Cameron University, Department of Mathematical Sciences, Lawton, OK 73505, USA,
Email:iargyros@cameron.edu

Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-757 025, Email: sgeorge@nitk.edu.in

ABSTRACT. We present an improved semilocal and local convergence analysis of some efficient King-Werner-type methods of order $1+\sqrt{2}$ free of derivatives in a Banach space setting using our new idea of restricted convergence domains. In particular, a more precise convergence domain is determined containing the iterates than in earlier studies leading to: smaller Lipschitz constants, larger radii of convergence and tighter error bounds on the distances involved. Numerical examples are presented to illustrate the theoretical results.

KEYWORDS :King's method; Werner's method; Secant-type method; Banach space; semilocal and local convergence analysis; Fréchet-derivative; efficiency index.
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1. Introduction

In [12], Argyros and Ren studied King-Werner-type methods of order $1+\sqrt{2}$ free of derivatives for approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is Fréchet-differentiable operator defined on a convex subset of a Banach space $\mathcal{B}_1$ with values in a Banach space $\mathcal{B}_2$. In the present study, we extend the applicability of the method considered in [12] using the idea of restricted convergence domains.

Precisely, in [12], Argyros and Ren studied the semilocal convergence analysis of method defined for $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1}F(x_n) \\ y_{n+1} &= x_{n+1} - A_n^{-1}F(x_{n+1}), \end{aligned} \tag{1.2}$$

* *Corresponding author.*
Email address : sgeorge@nitk.edu.in.
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where x_0, y_0 are initial points, $A_n = [x_n, y_n; F]$ and $[x, y; F]$ denotes a divided difference of order one for operator F at points $x, y \in \Omega$ [2, 5, 7, 12, 14] satisfying

$$[x, y; F](x - y) = F(x) - F(y) \quad for\ each\ x, y \in \Omega\ with\ x \neq y. \tag{1.3}$$

If F is Fréchet-differentiable on Ω, then $F'(x) = [x, x; F]$ for each $x \in \Omega$.

Method (1.2) is a useful alternative for methods involving $F'(x)$, since the computation of the inverse of $F'(x)$ may be very expensive or impossible rendering such methods useless. The local convergence analysis of method (1.2) was given in [10, 17] in the special case when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$. The convergence order of method (1.2) was shown to be $1 + \sqrt{2}$.

The paper is organized as follows: Section 2 contains the semilocal convergence analysis of method (1.4), and Section 3 contains the local convergence analysis of method (1.4). The numerical examples including favorable comparisons with earlier studies such as [9,10,13,14] are presented in the concluding Section 4.

2. Semilocal convergence of method (1.2)

We present the semilocal convergence of method (1.2) in this section. We first need an auxiliary result on majorizing sequences for method (1.2).

LEMMA 2.1. ([12, Lemma 2.1] Let $L_0 > 0$, $L > 0$, $s_0 \geq 0$, $t_1 \geq 0$ be given parameters. Denote by α the only root in the interval $(0, 1)$ of polynomial p defined by

$$p(t) = L_0 t^3 + L_0 t^2 + 2Lt - 2L. \tag{2.1}$$

Suppose that

$$0 < \frac{L(t_1 + s_0)}{1 - L_0(t_1 + s_1 + s_0)} \leq \alpha \leq 1 - \frac{2L_0 t_1}{1 - L_0 s_0}, \tag{2.2}$$

where

$$s_1 = t_1 + L(t_1 + s_0)t_1. \tag{2.3}$$

Then, scalar sequence $\{t_n\}$ defined for each $n = 1, 2, \ldots$ by

$$\begin{aligned} &t_0 = 0, s_{n+1} = t_{n+1} + \tfrac{L(t_{n+1} - t_n + s_n - t_n)(t_{n+1} - t_n)}{1 - L_0(t_n - t_0 + s_n + s_0)}, \quad for\ each\ n = 1, 2, \ldots, \\ &t_{n+2} = t_{n+1} + \tfrac{L(t_{n+1} - t_n + s_n - t_n)(t_{n+1} - t_n)}{1 - L_0(t_{n+1} - t_0 + s_{n+1} + s_0)}, \quad for\ each\ n = 0, 1, 2, \ldots \end{aligned} \tag{2.4}$$

is well defined, increasing, bounded above by

$$t^{\star\star} = \frac{t_1}{1 - \alpha} \tag{2.5}$$

and converges to its unique least upper bound $t^\star$ which satisfies

$$t_1 \leq t^\star \leq t^{\star\star}. \tag{2.6}$$

Moreover, the following estimates hold

$$s_n - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n(t_1 - t_0), \tag{2.7}$$

$$t_{n+1} - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n(t_1 - t_0) \tag{2.8}$$

and

$$t_n \leq s_n \tag{2.9}$$

for each $n = 1, 2, \ldots$.

Denote by $U(w, \xi)$, $\overline{U}(w, \xi)$, the open and closed balls in $\mathcal{B}_1$, respectively, with center $w \in \mathcal{B}_1$ and of radius $\xi > 0$. Next, we present the semilocal convergence of method (1.4) using $\{t_n\}$ as a majorizing sequence.

THEOREM 2.2. Let $F : \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exists a divided differentiable $[.,.;F]$ of order one for operator F on $\Omega \times \Omega$. Moreover, suppose that there exist $x_0, y_0 \in \Omega$, $L_0 > 0$, $L > 0$, $s_0 \geq 0$, $t_1 \geq 0$ such that

$$A_0^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1) \tag{2.10}$$

$$\|A_0^{-1}F(x_0)\| \leq t_1, \tag{2.11}$$

$$\|x_0 - y_0\| \leq s_0, \tag{2.12}$$

$$\|A_0^{-1}([x,y;F] - A_0)\| \leq L_0(\|x - x_0\| + \|y - y_0\|), \text{ for each } x, y \in \Omega, \tag{2.13}$$

$$\|A_0^{-1}([x,y;F]-[z,v;F])\| \leq L(\|x-z\|+\|y-v\|), \text{ for each } x, y, z \in \Omega \cap U(x_0, \frac{1-L_0 s_0}{2L_0}), \tag{2.14}$$

$$\overline{U}(x_0, t^\star) \subseteq \Omega \tag{2.15}$$

and hypotheses of Lemma 2.1 hold, where $A_0 = [x_0; y_0; F]$ and $t^\star$ is given in Lemma 2.1. Then, sequence $\{x_n\}$ generated by method (1.4) is well defined, remains in $\overline{U}(x_0, t^\star)$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $F(x) = 0$. Moreover, the following estimates hold for each $n = 0, 1, 2, ...$

$$\|x_n - x^\star\| \leq t^\star - t_n. \tag{2.16}$$

Furthermore, if there exists $R > t^\star$ such that

$$U(x_0, R) \subseteq \Omega \tag{2.17}$$

and

$$L_0(t^\star + R + s_0) < 1, \tag{2.18}$$

then, the point $x^\star$ is the only solution of equation $F(x) = 0$ in $U(x_0, R)$.

Proof. It follows from the corresponding proof in [12] by simply noting that the iterates x_n lie in Ω_0 which is a more precise location than Ω used in [12], since $\Omega_0 \subseteq \Omega$.

REMARK 2.3. (a) In [12] condition (2.14) holds in Ω. That is,

$$\|F'(x^\star)^{-1}([x,y;F] - [u,v;F])\| \leq \frac{L_1}{2}(\|x - y\| + \|y - v\|) \tag{2.19}$$

for all $x, y, u, v \in \Omega$. Then, we have $L \leq L_1$, since $\Omega_0 \subset \Omega$ and $L_0 \leq L_1$. If $L < L_1$, we obtain a larger radius of convergence and more precise error bounds on the distances $\|x_n - x^\star\|$ than in [11]. These advantages are obtained under the same commputational cost, since in practice the computation of L_1 requires the computation of L or L_0 as a special case.

(b) The limit point $t^\star$ can be replaced by $t^{\star\star}$ given in closed form by (2.5) in Theorem 2.1.

(c) It follows from the proof of Theorem 2.2 that hypothesis (2.13) is not needed to compute an upper bound for $\|A_0^{-1}F(x_1)\|$. Hence, we can define the more precise (than $\{t_n\}$) majorizing sequence $\{\bar{t}_n\}$ (for $\{x_n\}$) by

$$\begin{aligned} &\bar{t}_0 = 0, \bar{t}_1 = t_1, \bar{s}_0 = s_0, \bar{s}_1 = \bar{t}_1 + L_0(\bar{t}_1 + \bar{s}_0)\bar{t}_1, \\ &\bar{s}_{n+1} = \bar{t}_{n+1} + \tfrac{L(\bar{t}_{n+1} - \bar{t}_n + \bar{s}_n - \bar{t}_n)(\bar{t}_{n+1} - \bar{t}_n)}{1 - L_0(\bar{t}_n - \bar{t}_0 + \bar{s}_n + \bar{s}_0)} \quad \textit{for each } n = 1, 2, ... \end{aligned} \tag{2.20}$$

and

$$\bar{t}_{n+2} = \bar{t}_{n+1} + \tfrac{L(\bar{t}_{n+1} - \bar{t}_n + \bar{s}_n - \bar{t}_n)(\bar{t}_{n+1} - \bar{t}_n)}{1 - L_0(\bar{t}_{n+1} - \bar{t}_0 + \bar{s}_{n+1} + \bar{s}_0)} \quad \textit{for each } n = 0, 1, ...$$

Then, using a simple induction argument we have that

$$\bar{t}_n \leq t_n, \tag{2.21}$$

$$\bar{s}_n \leq s_n, \tag{2.22}$$

$$\bar{t}_{n+1} - \bar{t}_n \leq t_{n+1} - t_n, \tag{2.23}$$

$$\bar{s}_n - \bar{t}_n \leq s_n - t_n \tag{2.24}$$

and

$$\bar{t}^\star = \lim_{n\to\infty} \bar{t}_n \leq t^\star.$$

Furthermore, if $L_0 < L$, then (2.43)-(2.46) are strict for $n \geq 2$, $n \geq 1$, $n \geq 1$, $n \geq 1$, respectively. Clearly, sequence $\{\bar{t}_n\}$ increasing converges to $\bar{t}^\star$ under the hypotheses of Lemma 2.1 and can replace $\{t_n\}$ as a majorizing sequence for $\{x_n\}$ in Theorem 2.2.

3. Local convergence of method (1.2)

We present the local convergence of method (1.2) in this section. We have

THEOREM 3.1. Let $F : \Omega \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exist $x^\star \in \Omega$, $l_0 > 0$ and $l > 0$ such that

$$F(x^\star) = 0,\ F'(x^\star)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{3.1}$$

$$\|F'(x^\star)^{-1}([x, y; F] - F'(x^\star))\| \leq l_0(\|x - x^\star\| + \|y - x^\star\|) \text{ for each } x, y \in \Omega \tag{3.2}$$

$$\|F'(x^\star)^{-1}([x, y; F]-[z, u; F])\| \leq l(\|x-z\|+\|y-u\|), \text{ for each } x, y \in \Omega_1 := \Omega \cap U(x^\star, \frac{1}{2l_0}) \tag{3.3}$$

and

$$\overline{U}(x^\star, \rho) \subseteq \Omega, \tag{3.4}$$

where

$$\rho = \frac{1}{(1 + \sqrt{2})l + 2l_0}. \tag{3.5}$$

Then, sequence $\{x_n\}$ generated by method (1.4) is well defined, remains in $\overline{U}(x^\star, \rho)$ and converges to $x^\star$ with order of $1 + \sqrt{2}$ at least, provided that $x_0, y_0 \in U(x^\star, \rho)$. Moreover, the following estimates

$$\|x_{n+2} - x^\star\| \leq \frac{\sqrt{2} - 1}{\rho^2}\|x_{n+1} - x^\star\|^2\|x_n - x^\star\| \tag{3.6}$$

and

$$\|x_n - x^\star\| \leq (\frac{\sqrt{\sqrt{2} - 1}}{\rho})^{F_{n-1}}\|x_1 - x^\star\|^{F_n} \tag{3.7}$$

hold for each $n = 1, 2, \ldots$, where F_n is a generalized Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+2} = 2F_{n+1} + F_n$.

REMARK 3.2. (a) Let l_1 be the Lipschitz constant for $x, y \in \Omega$ used in [12]. We have that $l_0 \leq l$ and $l \leq l_1$. Hence, the old radius

$$\rho_1 = \frac{1}{(1 + \sqrt{2})l_1 + 2l_0} < \rho, \tag{3.8}$$

if $l < l_1$.

(b) For the special case $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, the radius of convergence ball for method (1.2) is given in [10] by

$$\rho_\star = \frac{s^\star}{M}, \tag{3.9}$$

where $s^\star \approx 0.55279$ is a constant and $M > 0$ is the upper bound for $|F(x^\star)^{-1}F''(x)|$ in the given domain Ω. Using (3.2) we have

$$\|F'(x^\star)^{-1}(F'(x) - F'(y))\| \leq 2l\|x - y\| \quad for\ any\ x, y \in \Omega. \tag{3.10}$$

That is, we can choose $l = \frac{M}{2}$. Simply set $l_0 = l$, we have from (3.5) that

$$\rho = \frac{2}{(3+\sqrt{2})M} = \frac{2(3-\sqrt{2})}{5M} \approx \frac{0.63432}{M} > \frac{s^\star}{M} = \rho_\star. \tag{3.11}$$

Therefore, even in this special case, a bigger radius of convergence ball for method (1.2) has been given in Theorem 3.1.

4. Numerical examples

We present some numerical examples in this section.

EXAMPLE 4.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $\Omega = (-1, 1)$ and define F on Ω by

$$F(x) = e^x - 1. \tag{4.1}$$

Then, $x^\star = 0$ is a solution of Eq. (1.1), and $F'(x^\star) = 1$. Note that for any $x, y, z, u \in \Omega$, we have

$$\begin{aligned}
&|F'(x^\star)^{-1}([x,y;F] - [z,u;F])| = |\int_0^1 (F'(tx+(1-t)y) - F'(tz+(1-t)u))dt| \\
&= |\int_0^1\int_0^1 (F''\big(\theta(tx+(1-t)y) + (1-\theta)(tz+(1-t)u)\big)\big(tx+(1-t)y - (tz+(1-t)u)\big)d\theta dt| \\
&= |\int_0^1\int_0^1 (e^{\theta(tx+(1-t)y)+(1-\theta)(tz+(1-t)u)}\big(tx+(1-t)y - (tz+(1-t)u)\big)d\theta dt| \\
&\le \int_0^1 e|t(x-z) + (1-t)(y-u)|dt \\
&\le \tfrac{e}{2}(|x-z| + |y-u|)
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&|F'(x^\star)^{-1}([x,y;F] - [x^\star, x^\star; F])| = |\int_0^1 F'(tx+(1-t)y)dt - F'(x^\star)| \\
&= |\int_0^1 (e^{tx+(1-t)y} - 1)dt| \\
&= |\int_0^1 (tx+(1-t)y)(1 + \tfrac{tx+(1-t)y}{2!} + \tfrac{(tx+(1-t)y)^2}{3!} + \cdots)dt| \\
&\le |\int_0^1 (tx+(1-t)y)(1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots)dt| \\
&\le \tfrac{e-1}{2}(|x-x^\star| + |y-x^\star|).
\end{aligned} \tag{4.3}$$

That is to say, the Lipschitz condition (3.2) and the center-Lipschitz condition (3.3) are true for $l_1 = \frac{e}{2}, l = \frac{e^{\frac{1}{l_0}}}{2}$ and $l_0 = \frac{e-1}{2}$, respectively. Using (3.5) in Theorem 3.1, we can deduce that the radius of convergence ball for method (1.2) is given by

$$\rho = \frac{1}{(1+\sqrt{2})e^{\frac{1}{l_0}} + 2l_0} \approx 0.17907908501857289, \tag{4.4}$$

which is bigger than the corresponding radius

$$\rho' = \frac{1}{(1+\sqrt{2})l + 2l} \approx 0.14147448123384420 \tag{4.5}$$

obtained by only using the Lipschitz condition (3.2).

Let us choose $x_0 = 0.2, y_0 = 0.199$. Suppose sequences $\{x_n\}$ and $\{y_n\}$ are generated by method (1.2). Table 1 gives a comparison results of error estimates for Example 4.1, which shows that tighter error estimates can be obtained from (3.7) by using both the Lipschitz condition (3.2) and the center-Lipschitz condition (3.3) instead of by using only the Lipschitz condition (3.2).

TABLE 1. The comparison results of error estimates for Example 4.1

n	the right-side of (3.7) by using both l and l_0	the right-side of (3.7) by using only l	the right-side of (3.7) by using both l and l_1
2	0.018640103	0.018640103	0.37609425349
3	6.70547E-05	9.65617E-05	0.000000000001
4	8.67742E-10	2.5913E-09	7.572039969341e-78
5	5.2275E-22	9.66721E-21	8.874190155368e-284

EXAMPLE 4.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$, equipped with the max norm and $\Omega = \overline{U}(0,1)$. Define function F on Ω, given by

$$F(x)(s) = x(s) - 5\int_0^1 stx^3(t)dt, \tag{4.6}$$

and the divided difference of F is defined by

$$[x, y; F] = \int_0^1 F'(tx + (1-t)y)dt. \tag{4.7}$$

Then, we have

$$[F'(x)y](s) = y(s) - 15\int_0^1 stx^2(t)y(t)dt, \quad for \;\; all \;\; y \in \Omega. \tag{4.8}$$

We have $x^\star(s) = 0$ for all $s \in [0,1]$, $l_0 = 3.75$ and $l_1 = l = 7.5$ [2]. Using Theorem 3.1, we can deduce that the radius of convergence ball for method (1.2) is given by

$$\rho = \frac{1}{(1+\sqrt{2})l + 2l_0} \approx 0.039052429, \tag{4.9}$$

which is bigger than the corresponding radius

$$\rho' = \frac{1}{(1+\sqrt{2})l + 2l} \approx 0.030205456 \tag{4.10}$$

obtained by only using the Lipschitz condition (3.2) [13, 16].

EXAMPLE 4.3. Let also $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$ equipped with the max norm and $\Omega = U(0,r)$ for some $r > 1$. Define F on Ω by

$$F(x)(s) = x(s) - y(s) - \mu\int_0^1 G(s,t)x^3(t)dt, \quad x \in C[0,1],\; s \in [0,1].$$

$y \in C[0,1]$ is given, μ is a real parameter and the Kernel G is the Green's function defined by

$$G(s,t) = \begin{cases} (1-s)t & \text{if } t \le s \\ s(1-t) & \text{if } s \le t. \end{cases}$$

Then, the Fréchet derivative of F is defined by

$$(F'(x)(w))(s) = w(s) - 3\mu\int_0^1 G(s,t)x^2(t)w(t)dt, \quad w \in C[0,1],\; s \in [0,1].$$

Let us choose $x_0(s) = y_0(s) = y(s) = 1$ and $|\mu| < \frac{8}{3}$. Then, we have that

$$\|I - A_0\| \le \tfrac{3}{8}\mu, \quad A_0^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1),$$
$$\|A_0^{-1}\| \le \tfrac{8}{8-3|\mu|}, \quad s_0 = 0, \quad t_1 = \tfrac{|\mu|}{8-3|\mu|}, \quad L_0 = \tfrac{3(1+r)|\mu|}{2(8-3|\mu|)},$$

and

$$L = \frac{3r|\mu|}{8-3|\mu|}.$$

Let us choose $r = 3$ and $\mu = \frac{1}{2}$. Then, we have that

$$t_1 = 0.076923077, \quad L_0 \approx 0.461538462, \quad L_1 = L \approx 0.692307692$$

and

$$\frac{L(t_1 + s_0)}{1 - L_0(t_1 + s_1 + s_0)} \approx 0.057441746, \quad \alpha \approx 0.711345739, \quad 1 - \frac{2L_0 t_1}{1 - L_0 s_0} \approx 0.928994083.$$

That is, condition (2.2) is satisfied and Theorem 2.2 applies.

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GENERAL FIXED POINT THEOREMS FOR PAIRS OF EXPANSIVE MAPPINGS WITH COMMON LIMIT RANGE PROPERTY IN G - METRIC SPACES

V. POPA[1] AND A.-M. PATRICIU[2]*

[1] \Vasile Alecsandri" University of Bacau, 600115, Romania

[2] Department of Mathematics and Computer Sciences, Faculty of Sciences and Environment, \Dunarea de Jos" University of Galati, 800201, Romania

ABSTRACT. In this paper two general fixed point theorems for pairs of extensive mappings with common limit range property in G - metric spaces are proved. In the last part of this paper, as applications, two general fixed point results for mappings satisfying extensive conditions of integral type are obtained.

KEYWORDS: Fixed point; G - metric space; Common limit range property; Implicit relation.
AMS Subject Classification: 54H25, 47H10.

1. INTRODUCTION

Let (X, d) be a metric space and S, T be two self mappings of X. In [11], Jungck defined S and T to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

This concept was frequently used to prove the existence theorems in fixed point theory.

Let f, g be self mappings of a nonempty set X. A point $x \in X$ is a coincidence point of f and g if $w = fx = gx$. The set of all coincidence points of f and g is denoted by $\mathcal{C}(f, g)$ and w is said a point of coincidence of f and g.

In 1994, Pant [28] introduced the notion of pointwise R - weakly commuting mappings.

* *Corresponding author.*
Email address : Alina.Patriciu@ugal.ro.
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It is proved in [29] that pointwise R – weakly commutativity is equivalent to commutativity in coincidence points.

In [12] Jungck introduced the notion of weakly compatible mappings.

Definition 1.1 ([12]). Let X be a nonempty set and f, g to be self mappings of X. f and g are weakly compatible if $fgu = gfu$ for $u \in \mathcal{C}(f, g)$.

Hence, f and g are weakly compatible if and only if f and g are pointwise R - weakly commuting.

The study of common fixed points for noncompatible mappings is also interesting, the work along this lines has been initiated by Pant in [25], [26], [27].

Aamri and El-Moutawakil [1] introduced a generalization of noncompatible mappings.

Definition 1.2 ([1]). Let S and T be two self mappings of a metric space (X, d). We say that S and T satisfy $(E.A)$ - property if there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t,$$

for some $t \in X$.

Remark 1.3. It is clear that two self mappings S and T of a metric space (X, d) will be noncompatible if there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$, but $\lim_{n\to\infty} d(TSx_n, STx_n)$ is nonzero or nonexistent. Therefore, two noncompatible self mappings of a metric space (X, d) satisfy $(E.A)$ - property.

It is proved in [30], [31] that the notions of weakly compatible mappings and mappings satisfying $(E.A)$ - property are independent.

There exists a vast literature concerning the study of fixed points for pairs of mappings satisfying $(E.A)$ - property.

In 2011, Sintunavarat and Kumam [55] introduced the idea of common limit range property.

Definition 1.4 ([55]). A pair (A, S) of self mappings of a metric space (X, d) is said to satisfy the limit range property with respect to S, denoted $CLR_{(S)}$, if there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in S(X)$.

Thus, one can infer that a pair (A, S) satisfying the $(E.A)$ - property along with the closedness of the subspace $S(X)$ always have the $CLR_{(S)}$ - property, with respect to S (see Examples 2.16, 2.17 [7]).

Some fixed point results for pairs of mappings with CLR - property are, also, obtained in [7], [8], [9], [10], [49], [54] and in other papers.

Wang et al. [58] proved some non unique fixed point theorems for expansive mappings which correspond some contractive mappings. Khan et al. [15] and Popa [32] generalized the results from [58].

Also, Rhoades [47], Taniguchi [56] generalized the results from [58] for pairs of mappings. In [33], Popa initiated the study of the unique fixed points for expansive mappings.

In [34], [35], [36] some unique fixed points theorems for two pairs of mappings are proved.

In [5], [6] Dhage introduced a new class of generalized metric space, named D – metric space. Mustafa and Sims [18], [19] proved that most of the claims concerning the fundamental topological structures on D – metric spaces are incorrect and introduced an appropriate notion of generalized metric space, named G – metric space.

In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings in G – metric spaces under certain conditions [20], [21], [22], [23], [53] and other papers.

Some fixed point theorems for expansive mappings in G - metric spaces are proved in [23], [17], [50], [53], [45], [46].

Some classical fixed point theorems and common fixed point theorems have recently unified considering a general condition by an implicit relation in [37], [38] and in other papers.

Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi – metric spaces, ultra - metric spaces, convex metric spaces, reflexive spaces, compact metric spaces, paracompact metric spaces, in two or three metric spaces, for single valued functions, hybrid pairs of mappings and set valued mappings.

Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive/extensive condition of integral type, in fuzzy metric spaces, probabilistic metric spaces and G - metric spaces.

With this method, the proofs of some fixed points theorems are more simple. Also, the method allow the study of local and global properties of fixed point structures.

The study of fixed points for mappings satisfying implicit relations in G - metric spaces is initiated in [39], [42], [43], [44].

The study of fixed points for pairs of self mappings with common limit range property in metric spaces satisfying implicit relations is initiated in [9].

The study of fixed points for a pair of self mappings with common limit range property in G - metric spaces is initiated in [3].

Definition 1.5 ([14])**.** An altering distance is a function $\psi : [0,\infty) \to [0,\infty)$ satisfying:
$(\psi_1) : \psi$ is increasing and continuous;
$(\psi_2) : \psi(t) = 0$ if and only if $t = 0$.

Fixed point theorems involving altering distances have been studied in [51], [52], [47].

Definition 1.6. An almost altering distance is a function $\psi : [0,\infty) \to [0,\infty)$ satisfying:
$(\psi'_1) : \psi$ is continuous;
$(\psi'_2) : \psi(t) = 0$ if and only if $t = 0$.

In this paper, two general fixed point theorems for pairs of self extensive mappings with common limit range property in G - metric spaces are proved.

In the last part of this paper, as applications, two general fixed point theorems for mappings satisfying extensive conditions of integral type are proved.

2. PRELIMINARIES

Definition 2.1 ([19])**.** Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:

$(G_1) : G(x, y, z) = 0$ if $x = y = z$,
$(G_2) : 0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,
$(G_3) : G(x, y, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
$(G_4) : G(x, y, z) = G(y, z, x) = G(z, x, y) = ...$ (symmetry in all three variables),
$(G_5) : G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

The function G is called a G - metric and the pair (X, G) is called a G - metric space.

Note that if $G(x, y, z) = 0$, then $x = y = z$.

Definition 2.2 ([19]). Let (X, G) be a G – metric space. A sequence (x_n) in X is said to be
a) G - convergent if for $\varepsilon > 0$, there is an $x \in X$ and $k \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $m, n \geq k$, $G(x, x_n, x_m) < \varepsilon$;
b) G - Cauchy if for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for $m, n, p \in \mathbb{N}$, $m, n, p \geq k$, $G(x_n, x_m, x_p) < \varepsilon$, that is $G(x_n, x_m, x_p) \to 0$ as $n, m, p \to \infty$;
A G - metric space (X, G) is said to be G - complete if every G - Cauchy sequence in X is G - convergent.

Lemma 2.3 ([19]). *Let (X, G) be a G - metric space. Then, the following properties are equivalent:*
1) (x_n) is G - convergent to x;
2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;
3) $G(x_n, x, x) \to 0$ as $n \to \infty$;
4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.

Lemma 2.4 ([19]). *Let (X, G) be a G - metric space. Then the following properties are equivalent:*
1) (x_n) is a G - Cauchy sequence;
2) for $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \in \mathbb{N}$, $m, n \geq k$.

Lemma 2.5 ([19]). *Let (X, G) be a G - metric space. The function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Let $\mathfrak{F}_{CL}$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$ such that:
$(F_1) : F(t, 0, t, 0, 0, t) < 0, \forall t > 0$.
$(F_2) : F(t, t, 0, 0, t, t) < 0, \forall t > 0$.

The following functions are from $\mathfrak{F}_{CL}$.

Example 2.6. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, ..., t_6\}$, where $k \in [1, \infty)$.

Example 2.7. $F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_2, t_5, t_6\}$, where $a, b, c \geq 0$ and $c > 1$.

Example 2.8. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{t_5 + t_6}{2}\}$, where $k \in [1, \infty)$.

Example 2.9. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\}$, where $k \in [2, \infty)$.

Example 2.10. $F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $\alpha \in (0, 1)$ and $a, b \geq 0$, $b > 1$.

Example 2.11. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c \min\{t_5, t_6\}$, where $a, b \geq 0, b > 1$ and $a + c > 1$.

Example 2.12. $F(t_1, ..., t_6) = t_1 - at_2 - b\frac{t_5 + t_6}{1 + t_3 + t_4}$, where $a, b \geq 0$ and $b > 1$.

Example 2.13. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $a, b, c \geq 0$ and $\max\{b, c\} > 1$.

Example 2.14. $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4, \frac{2t_4 + t_5}{3}, \frac{2t_4 + t_6}{3}, \frac{t_5 + t_6}{2}\}$, where $a, b \geq 0$ and $b > 1$.

Example 2.15. $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{2t_4 + t_5, 2t_4 + t_6, t_3 + t_5 + t_6\}$, where $a, b \geq 0$ and $b > \frac{1}{2}$.

Lemma 2.16 ([2]). *Let f and g be two weakly compatible self mappings on a nonempty set X. If f and g have an unique point of coincidence $w = fx = gx$, for some $x \in X$, then w is the unique common fixed point of f and g.*

3. MAIN RESULTS

Theorem 3.1. *Let T, S be self mappings of a G - metric space (X, G) such that*

$$\begin{gathered} F(\psi(G(Tx, Tx, Ty)), \psi(G(Sx, Sx, Sy)), \psi(G(Tx, Tx, Sx)), \\ \psi(G(Ty, Ty, Sy)), \psi(G(Sx, Sx, Ty)), \psi(G(Tx, Tx, Sy))) \geq 0 \end{gathered} \tag{3.1}$$

for all $x, y \in X$, where F satisfy property (F_2) and ψ is an almost altering distance. If there exists $u, v \in X$ such that $w = Su = Tu$ and $z = Sv = Tv$, then S and T have an unique point of coincidence.

Proof. First we prove that $Tu = Sv$. By (3.1) we obtain

$$\begin{gathered} F(\psi(G(Tu, Tu, Tv)), \psi(G(Su, Su, Sv)), \psi(G(Tu, Tu, Su)), \\ \psi(G(Tv, Tv, Sv)), \psi(G(Su, Su, Tv)), \psi(G(Tu, Tu, Sv))) \geq 0 \end{gathered}$$

which implies

$$F(\psi(G(w, w, z)), \psi(G(w, w, z)), 0, 0, \psi(G(w, w, z)), \psi(G(w, w, z))) \geq 0$$

a contradiction of (F_2) if $\psi(G(w, w, z)) \neq 0$. Hence $\psi(G(w, w, z)) = 0$ which implies $w = z$. Hence, $Tu = Sv = Su = Tv = w = z$. Therefore, z is a common point of coincidence of S and T.

Suppose that there exists two points of coincidence of T and S, $z_1 = Tu = Su$ and $z_2 = Tv = Sv$. By (3.1) we obtain

$$F(\psi(G(z_1, z_1, z_2)), \psi(G(z_1, z_1, z_2)), 0, 0, \psi(G(z_1, z_1, z_2)), \psi(G(z_1, z_1, z_2))) \geq 0,$$

a contradiction of (F_2) if $\psi(G(z_1, z_1, z_2)) \neq 0$. Hence $\psi(G(z_1, z_1, z_2)) = 0$ which implies $z_1 = z_2$. □

Theorem 3.2. *Let T, S be self mappings of a G - metric space (X, G) such that the inequality (3.1) holds for all $x, y \in X$, where $F \in \mathfrak{F}_{CL}$ and ψ is an almost altering distance. If T and S satisfies $CLR_{(S)}$ - property, then $\mathcal{C}(T, S) \neq \varnothing$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.*

Proof. Since T and S satisfies $CLR_{(S)}$ - property, there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = Su,$$

for some $u \in X$.

By (3.1) we have

$$\begin{gathered} F(\psi(G(Tu, Tu, Tx_n)), \psi(G(Su, Su, Sx_n)), \psi(G(Tu, Tu, Su)), \\ \psi(G(Tx_n, Tx_n, Sx_n)), \psi(G(Su, Su, Tx_n)), \psi(G(Tu, Tu, Sx_n))) \geq 0. \end{gathered}$$

Letting n tend to infinity we obtain

$$F(\psi(G(Tu,Tu,Su)),0,\psi(G(Tu,Tu,Su)),0,0,\psi(G(Tu,Tu,Su)))\geq 0,$$

a contradiction of (F_2) if $\psi(G(Tu,Tu,Su))\neq 0$. Hence, $\psi(G(Tu,Tu,Su))=0$, which implies $Tu=Su=z$. Hence, $\mathcal{C}(T,S)\neq\varnothing$. By Theorem 3.1, z is the unique point of coincidence of T and S. Moreover, if T and S are weakly compatible, then by Lemma 2.16, z is the unique common fixed point of T and S. □

For $\psi(t)=t$, by Theorem 3.2 we obtain

Theorem 3.3. *Let T,S be self mappings of a G - metric space (X,G) such that:*

$$\begin{gathered}F(G(Tx,Tx,Ty),G(Sx,Sx,Sy),G(Tx,Tx,Sx),\\ G(Ty,Ty,Sy),G(Sx,Sx,Ty),G(Tx,Tx,Sy))\geq 0\end{gathered}\tag{3.2}$$

for all $x,y\in X$ and $F\in\mathfrak{F}_{CL}$. If T and S satisfies $CLR_{(S)}$ - property, then $\mathcal{C}(T,S)\neq\varnothing$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Theorem 3.4. *Let T,S be self mappings of a G - metric space (X,G) such that:*

$$\begin{gathered}F(\psi(G(Tx,Ty,Ty)),\psi(G(Sx,Sy,Sy)),\psi(G(Tx,Sx,Sx)),\\ \psi(G(Ty,Sy,Sy)),\psi(G(Sx,Ty,Ty)),\psi(G(Tx,Sy,Sy)))\geq 0\end{gathered}\tag{3.3}$$

for all $x,y\in X$, $F\in\mathfrak{F}_{CL}$ and ψ is an altering distance. If T and S satisfies $CLR_{(S)}$ - property, then $\mathcal{C}(T,S)\neq\varnothing$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Proof. The proof is similar to the proof of Theorem 3.2. □

If $\psi(t)=t$, by Theorem 3.4 we obtain

Theorem 3.5. *Let T,S be self mappings of a G - metric space (X,G) such that:*

$$\begin{gathered}F(G(Tx,Ty,Ty),G(Sx,Sy,Sy),G(Tx,Sx,Sx),\\ G(Ty,Sy,Sy),G(Sx,Ty,Ty),G(Tx,Sy,Sy))\geq 0\end{gathered}\tag{3.4}$$

for all $x,y\in X$, where $F\in\mathfrak{F}_{CL}$. If T and S satisfies $CLR_{(S)}$ - property, then $\mathcal{C}(T,S)\neq\varnothing$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Remark 3.1. By Theorem 3.5 and Examples 2.6 - 2.15 we obtain particular results. For example, by 3.5 and Example 2.6 we obtain

Theorem 3.6. *Let T,S be self mappings of a G - metric space (X,G) such that*

$$\begin{gathered}G(Tx,Ty,Ty)\geq k\max\{G(Sx,Sy,Sy),G(Tx,Sx,Sx),\\ G(Ty,Sy,Sy),G(Sx,Ty,Ty),G(Tx,Sy,Sy)\},\end{gathered}\tag{3.5}$$

for all $x,y\in X$, where $k>1$. If T and S satisfies $CLR_{(S)}$ - property, then $\mathcal{C}(T,S)\neq\varnothing$. If T and S are weakly compatible, then T and S have an unique common fixed point.

4. APPLICATION: FIXED POINTS FOR MAPPINGS SATISFYING EXTENSIVE CONDITIONS OF INTEGRAL TYPE

In [4], Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying contractive/extensive conditions of integral type.

Theorem 4.1 ([4]). *Let (X, G) be a complete metric space, $c \in (0,1)$ and $f : (X,d) \to (X,d)$ such that for all $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt,$$

whenever $h : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$, such that $\int_0^{\varepsilon} h(t)dt > 0$, for each $\varepsilon > 0$. Then, f has an unique fixed point $z \in X$ such that for all $x \in X$, $z = \lim_{n\to\infty} f^n x$.

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in [16], [40], [41], [48] and in other papers.

The study of fixed points for pairs of mappings satisfying $CLR_{(S)}$ - property of integral type in G - metric spaces in initiated in [3].

Lemma 4.1. *Let $h : [0,\infty) \to [0,\infty)$ be as in Theorem 4.1. Then $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance.*

Proof. The proof it follows from Lemma 2.5 [41]. □

Theorem 4.2. *Let T, S be self mappings of a G - metric space (X, G) such that*

$$F(\int_0^{G(Tx,Tx,Ty)} h(t)dt, \int_0^{G(Sx,Sx,Sy)} h(t)dt, \int_0^{G(Tx,Tx,Sx)} h(t)dt, \\ \int_0^{G(Ty,Ty,Sy)} h(t)dt, \int_0^{G(Sx,Sx,Ty)} h(t)dt, \int_0^{G(Tx,Tx,Sy)} h(t)dt) \geq 0 \tag{4.1}$$

for all $x, y \in X$, where $F \in \mathfrak{F}_{LC}$ and $h(t)$ is as in Theorem 4.1. If T and S satisfies $CLR_{(S)}$ - property, then $\mathcal{C}(T,S) \neq \varnothing$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

Proof. By Lemma 4.1, $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance. By (4.1) we obtain

$$F(\psi(G(Tx,Tx,Ty)), \psi(G(Sx,Sx,Sy)), \psi(G(Tx,Tx,Sx)), \\ \psi(G(Ty,Ty,Sy)), \psi(G(Sx,Sx,Ty)), \psi(G(Tx,Tx,Sy))) \geq 0,$$

which is the inequality (3.1). Hence, the conditions of Theorem 3.2 are satisfied and Theorem 4.3 it follows from Theorem 3.2. □

Similarly, from Theorem 3.4 we obtain

Theorem 4.3. *Let T and S be self mappings of a G - metric space (X, G) such that*

$$F(\int_0^{G(Tx,Ty,Ty)} h(t)dt, \int_0^{G(Sx,Sy,Sy)} h(t)dt, \int_0^{G(Tx,Sx,Sx)} h(t)dt, \\ \int_0^{G(Ty,Sy,Sy)} h(t)dt, \int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Tx,Sy,Sy)} h(t)dt) \geq 0 \tag{4.2}$$

for all $x, y \in X$, where $F \in \mathfrak{F}_{LC}$ and $h(t)$ is as in Theorem 4.1. If T and S satisfies $CLR_{(S)}$ - property, then $\mathcal{C}(T,S) \neq \varnothing$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

By Theorems 4.2 and 4.3 and Examples 2.6 - 2.15 we obtain new particular results. For example, by Theorem 4.2 and Example 2.6 we obtain

Theorem 4.4. *Let T and S be self mappings of a G - metric space (X, G) such that*

$$\begin{aligned}\int_0^{G(Tx,Tx,Ty)} h(t)dt \geq k \max\{&\int_0^{G(Sx,Sx,Sy)} h(t)dt, \int_0^{G(Tx,Tx,Sx)} h(t)dt,\\ &\int_0^{G(Ty,Ty,Sy)} h(t)dt, \int_0^{G(Sx,Sx,Ty)} h(t)dt, \int_0^{G(Tx,Tx,Sy)} h(t)dt\},\end{aligned} \tag{4.3}$$

for all $x, y \in X$, where $F \in \mathfrak{F}_{LC}$ and $h(t)$ as in Theorem 4.1. If T and S satisfies $CLR_{(S)}$ - property, then $\mathcal{C}(T, S) \neq \emptyset$. Moreover, if T and S are weakly compatible, then T and S have an unique common fixed point.

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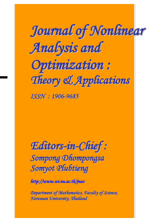

OPTIMAL SYNCHRONIZATION AND ANTI-SYNCHRONIZATION FOR A CLASS OF CHAOTIC SYSTEMS

B. NADERI*[1] H. KHEIRI[2] A. HEYDARI[1]

[1]Department of Mathematics, Payame Noor University, I. R. of Iran
[2] Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

ABSTRACT. In this study, we apply the optimal adaptive control for synchronization and anti-synchronization of chaotic T-system with complete uncertain parameters on finite and infinite time intervals. Based on the Lyapunov stability theorem and Hamilton Jacobian Bellman(HJB) technique, optimal controls and parameters estimations laws are obtained. For this aim, conditions ensuring asymptotic stability of error system and minimizing cost function are used. The derived control laws make asymptotically synchronization and anti-synchronization of two identical T-systems. Finally, numerical simulations are presented to illustrate the ability and effectiveness of the proposed method.

KEYWORDS: Lyapunov stability; Synchronization; Optimal adaptive control; Chaos; T-system.
AMS Subject Classification: 49J15, 34D06.

1. INTRODUCTION

Chaos, as an interesting phenomenon in nonlinear dynamical systems, has been studied over the last four decades [21, 28, 35, 29, 15, 34, 2]. Chaotic and hyperchaotic systems serving as nonlinear deterministic systems display complex and unpredictable behavior. In addition, these systems are sensitive with respect to initial conditions. The chaotic and hyperchaotic systems have many important applications in nonlinear sciences such as laser physics, secure communications, nonlinear circuits, control, neural networks, and active wave propagation [15, 30, 26, 19, 5, 3, 8, 13, 27].

The synchronization of chaotic systems have been investigated since their introduction in the paper by Pecora and Carrol in 1990 [29] and have been widely investigated in many fields, such as physics, chemistry, ecological sciences, and secure communications [14, 6, 34]. Various techniques and methods have been

* *Corresponding author.*
Email address : [1]b_naderi@pnu.ac.ir, [2]h-kheiri@tabrizu.ac.ir, [3]a_heidari@pnu.ac.ir .
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proposed to achieve chaos synchronization and anti-synchronization such as adaptive control [22, 33, 9, 16, 18, 17], active control [10] and nonlinear control [1]. Fortunately, some existing synchronization methods can be generalized to anti-synchronization of chaotic systems. Recently, synchronization of chaotic complex systems was studied in [25]. Furthermore, the anti-synchronization and adaptive anti-synchronization of two different chaotic systems were investigated in [24, 23].

Most methods mentioned in previous paragraph, synchronize two identical chaotic systems using adaptive methods. However, synchronization of systems with these methods is far from optimal synchronization. In general, the adaptive synchronization does not necessarily satisfy the optimality conditions.

The problem of the minimal control synthesis algorithm for controlling and synchronizing chaotic systems was studied in [12] and the optimal control for the chaos synchronization of Rössler system with complete uncertain parameters, was discussed in [11].

In this study, we present the optimal adaptive synchronization and anti-synchronization schemes between two identical chaotic T-system, with three unknown parameters. By this method, we can achieve synchronization and anti-synchronization of drive and response systems, and identify the unknown parameters. Based on the Lyapunov stability theorem and Hamilton-Jacobi- Bellman(HJB) equation in finite and infinite time intervals, optimal adaptive controllers with parameters estimation rules are designed to synchronizing and anti-synchronizing chaotic T-systems asymptotically.

The rest of this paper is organized as follows: In section 2 we introduce the chaotic T-systems briefly. In Section 3, the synchronization and anti-synchronization of two identical T-system with optimal controllers and parameters estimation rules in infinite and finite time intervals are presented. Besides, numerical simulations are computed to check the analytical expressions of optimal controllers and estimation laws. Finally, concluding remarks are given in Section 4.

2. T-system

In 2005, Tigen [32] introduced a new real chaotic nonlinear system which is called T-system, as follows

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) \\ \dot{x_2} = (c-a)x_1 - ax_1x_3 \\ \dot{x_3} = x_1x_2 - bx_3, \end{cases} \tag{2.1}$$

where x_1, x_2 and x_3 are the state variables and a, b and c are real positive parameters. Taking $a = 2.1$, $b = 0.6$ and some values of $0 < c < 40$, the positiveness of one of the Lyapunov exponents in figure 1 shows that the system (2.1) is a chaotic system. Furthermore, the negativity of sum of its Lyapunov exponents implies that the system is dissipative. figure 2 displays an attractor of T-system for some parameters and initial conditions. The attractors are bounded but not fixed points and limit cycles as a property of chaotic systems [4]. Synchronization and anti-synchronization of this system can be used for cryptography and decryption of data in secure communication.

3. Synchronization And Anti-synchronization

The drive and response systems are defined as follows

$$\dot{x} = f(x); \tag{3.1}$$

$$\dot{y} = g(y) + u, \tag{3.2}$$

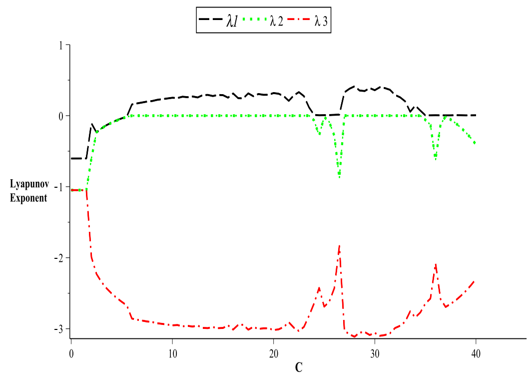


FIGURE 1. Lyapunov exponents of system (2.1), for $a = 2.1$, $b = 0.6$ and $0 < c < 40$.

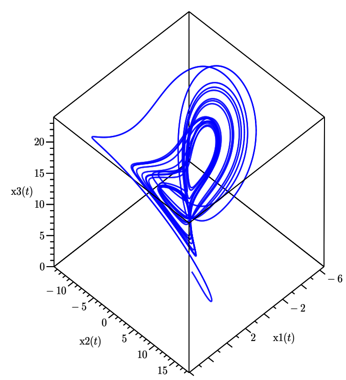


FIGURE 2. Attractor of T-system for $a = 2.1$, $b = 0.6$ and $c = 28$ with initial conditions $(x_1(0), x_2(0), x_3(0)) = (1, 3, 0)$.

where $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n$ are state vectors of the systems (3.1) and (3.2), $u = (u_1, u_2, \cdots, u_n)^T$ is a vector control and $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Synchronization error is $e = y - x$ and anti-synchronization error is $e = y + x$, therefore we can show the synchronization and the anti-synchronization errors with $e = y + kx$, for $k = -1$ and $k = 1$, respectively. Then, the error system between drive and response systems is

$$\dot{e} = \dot{y} + k\dot{x} = g(y) + kf(x) + u = g(e - kx) + kf(x) + u, \qquad k = \mp 1. \tag{3.3}$$

The goal is to design an appropriate optimal controller u such that for any initial condition y_0 and x_0, we have

$$\lim_{t \longrightarrow \infty} \|e\| = \lim_{t \longrightarrow \infty} \|y(t) + kx(t)\| = 0, \qquad k = \mp 1,$$

where $\|.\|$ is the Euclidean norm [22, 29].

Concerning synchronization and anti-synchronization of two chaotic T-system, assume that (2.1) is the drive system. The response system can be considered in two cases: First, it is identical with the drive system, but in the second instance, it can be different from the drive system. In this paper, we consider the response system to be identical with drive system, with unknown parameters. Then, the response system is defined as follow

$$\begin{cases} \dot{y_1} = \hat{a}(y_2 - y_1) + u_1 \\ \dot{y_2} = (\hat{c} - \hat{a})y_1 - \hat{a}y_1y_3 + u_2 \\ \dot{y_3} = y_1y_2 - \hat{b}y_3 + u_3, \end{cases} \tag{3.4}$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are uncertain parameters and u_1, u_2 and u_3 are control functions. Then, the error system between the drive and response systems, is

$$\begin{cases} \dot{e_1} = & \hat{a}(e_2 - e_1) + k(\hat{a} - a)(x_1 - x_2) + u_1 \\ \dot{e_2} = & (\hat{c} - \hat{a})e_1 - \hat{a}[e_1e_3 - ke_1x_3 - ke_3x_1] - (\hat{a} + ka)x_1x_3 \\ & -k(\hat{c} - c - \hat{a} + a)x_1 + u_2 \\ \dot{e_3} = & e_1e_2 - ke_1x_2 - ke_2x_1 + (k+1)x_1x_2 - \hat{b}e_3 + k(\hat{b} - b)x_3 + u_3. \end{cases} \tag{3.5}$$

Now, we discuss the optimal synchronization and anti-synchronization of (2.1), (3.4).

3.1. **Optimal synchronization.** For synchronization, let $k = -1$ in all the above mentioned relations; then, $e_i = y_i - x_i (i = 1, 2, 3)$, and the error system is

$$\begin{cases} \dot{e_1} = (\hat{a} - a)(x_2 - x_1) + U_1 \\ \dot{e_2} = (\hat{c} - c)x_1 - (\hat{a} - a)(x_1 + x_1x_3) + U_2 \\ \dot{e_3} = -(\hat{b} - b)x_3 + U_3, \end{cases} \tag{3.6}$$

where

$$\begin{cases} U_1 = \hat{a}(e_2 - e_1) + u_1 \\ U_2 = (\hat{c} - \hat{a})e_1 - \hat{a}[e_1e_3 + e_1x_3 + e_3x_1] + u_2 \\ U_3 = e_1e_2 + e_1x_2 + e_2x_1 - \hat{b}e_3 + u_3. \end{cases} \tag{3.7}$$

In the following subsection, we find the controllers and dynamics of uncertain parameters for synchronization on infinite and finite time intervals.

3.1.1. *Optimal synchronization for infinite time.* Concerning optimal synchronization on infinite time, similar to papers [11, 12], we consider the following extended dynamics for estimating the unknown parameters

$$\begin{cases} \dot{\hat{a}} = -\frac{1}{\beta_1}\left[\alpha_1e_1(x_2 - x_1) - \alpha_2e_2(x_1 + x_1x_3) + \frac{k_1}{2}(\hat{a} - a)\right] \\ \dot{\hat{b}} = \frac{1}{\beta_2}\left[\alpha_3e_3x_3 - \frac{k_2}{2}(\hat{b} - b)\right] \\ \dot{\hat{c}} = -\frac{1}{\beta_3}\left[\alpha_2e_2x_1 + \frac{k_3}{2}(\hat{c} - c)\right], \end{cases} \tag{3.8}$$

where β_i and $\alpha_i (i = 1, 2, 3)$, are real positive and $k_i (i = 1, 2, 3)$ are nonnegative constants. We design controllers $u_i (i = 1, 2, 3)$ such that the dynamic systems (3.6) and (3.8) are applied as follows when $t \to \infty$,

$$e_1 = e_2 = e_3 = 0, \hat{a} = a, \hat{b} = b, \hat{c} = c. \tag{3.9}$$

For this aim, we minimize the following cost function with respect to control vector $\vec{U} = (U_1, U_2, U_3)$

$$\begin{aligned} I &= \int_{t_0}^{\infty} \Omega(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, \vec{U})dt = \\ &= \int_{t_0}^{\infty} \{k_1(\hat{a} - a)^2 + k_2(\hat{b} - b)^2 + k_3(\hat{c} - c)^2 + \textstyle\sum_1^3 (c_ie_i^2 + \eta_iU_i^2)\}dt, \end{aligned} \tag{3.10}$$

where t_0 is a fixed time moment, c_i and $\eta_i (i = 1, 2, 3)$ are real positive constants.

Assuming the minimum of (3.10) is $\vec{U} = U^*$, we define

$$V(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, t) = \min_{\vec{U}} \int_{t_0}^{\infty} \Omega(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, \vec{U})dt. \tag{3.11}$$

In fact, the function $V(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, t)$ is the value of the performance index evaluated along the optimal trajectory of the error and updating system parameter which begins at $(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c})$. We shall show that $V(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c})$ can be selected as the Lyapunov function of our system. Now, we find the optimal controller $\vec{U}$ such that it stabilizes the state (3.9) and minimizes I on (3.10). This requirement can be met by applying HJB technique of dynamic programming problem[20, 11]; that is

$$\begin{aligned}&\tfrac{\partial V}{\partial e_1}\dot{e_1} + \tfrac{\partial V}{\partial e_2}\dot{e_2} + \tfrac{\partial V}{\partial e_3}\dot{e_3} + \tfrac{\partial V}{\partial \hat{a}}\dot{\hat{a}} + \tfrac{\partial V}{\partial \hat{b}}\dot{\hat{b}} + \tfrac{\partial V}{\partial \hat{c}}\dot{\hat{c}} + k_1(\hat{a}-a)^2\\ &+k_2(\hat{b}-b)^2 + k_3(\hat{c}-c)^2 + \textstyle\sum_1^3 (c_i e_i^2 + \eta_i U_i^{*2}) = 0.\end{aligned} \tag{3.12}$$

Substituting the relations(3.6), into the partial differential equation(3.12), we have

$$\begin{aligned}&\tfrac{\partial V}{\partial e_1}[(\hat{a}-a)(x_2-x_1) + U_1^*] + \tfrac{\partial V}{\partial e_2}[(\hat{c}-c)x_1 - (\hat{a}-a)(x_1 + x_1x_3) + U_2^*]\\ &+\tfrac{\partial V}{\partial e_3}[-(\hat{b}-b)x_3 + U_3^*] + \tfrac{\partial V}{\partial \hat{a}}\dot{\hat{a}} + \tfrac{\partial V}{\partial \hat{b}}\dot{\hat{b}} + \tfrac{\partial V}{\partial \hat{c}}\dot{\hat{c}} + k_1(\hat{a}-a)^2 + k_2(\hat{b}-b)^2\\ &+k_3(\hat{c}-c)^2 + \textstyle\sum_1^3 (c_i e_i^2 + \eta_i U_i^{*2}) = 0,\end{aligned} \tag{3.13}$$

where $\vec{U^*}$ is the optimal control. Minimizing (3.13) with respect to $\vec{U^*}$ [7], gives

$$U_i^* = -\frac{1}{2\eta_i}\frac{\partial V}{\partial e_i}(i = 1, 2, 3). \tag{3.14}$$

To ensure stability of the error and parameter estimation systems in equilibrium points, we must define the Lyaponuv function such that it be positive definite and its derivative be negative definite. For this purpose, we consider

$$\psi(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}) = \beta_1(\hat{a}-a)^2 + \beta_2(\hat{b}-b)^2 + \beta_3(\hat{c}-c)^2 + \sum_1^3 \alpha_i e_i^2, \tag{3.15}$$

Besides, in the sequel we choose its coefficient so that it has mentioned properties and is considered as a solution of (3.13). In other words, $\psi = V$ and ψ is a Lyaponuv function. From (3.14) and (3.15) we have the optimal controllers as

$$U_i^* = \frac{-\alpha_i}{\eta_i}e_i(i = 1, 2, 3).$$

Then

$$\begin{cases} u_1 = -\frac{\alpha_1}{\eta_1}e_1 - \hat{a}(e_2 - e_1)\\ u_2 = -\frac{\alpha_2}{\eta_2}e_2 - (\hat{c}-\hat{a})e_1 + \hat{a}[e_1e_3 + e_1x_3 + e_3x_1]\\ u_3 = -\frac{\alpha_3}{\eta_3}e_1 - e_1e_2 - e_1x_2 - e_2x_1 + \hat{b}e_3,\end{cases} \tag{3.16}$$

where the constants α_i, η_i are positive. A simple calculation implies with $\frac{\alpha_i}{\eta_i} = c_i$, we have

$$\dot{\psi} = -\left[k_1(\hat{a}-a)^2 + k_2(\hat{b}-b)^2 + k_3(\hat{c}-c)^2 + \sum_1^3 2c_i e_i^2\right]. \tag{3.17}$$

The relation (3.17) shows that $\dot{\psi}$ is negative definite for all nonnegative c_i and k_i. This result shows that the solution (3.9) is asymptotically stable in the Lyapunov sense via optimal control [20].

3.1.2. *Optimal synchronization in a finite time.* In this subsection, we try to finding optimal controllers $u_i (i = 1, 2, 3)$ such that the equilibrium state of error and updating parameters systems be asymptotically stable. These controllers minimize the value of the following integral performance index through the finite time interval $[0, T]$ [12]

$$\begin{array}{ll} I & = \int_0^T \Omega(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, \vec{U}, t)dt \\ & = \int_0^T \{(\hat{a}-a)^2 + (\hat{b}-b)^2 + (\hat{c}-c)^2 + \sum_1^3 (e_i^2 + lU_i^2)\}dt \end{array}. \tag{3.18}$$

Assume that function $\phi(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, t)$ is minimum of the integral performance index (3.18) along the optimal trajectory of the system that consists of both the error and update system parameters; consider

$$\phi = \min_{\vec{U}} I = \min_{\vec{U}} \int_0^T \Omega(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, \vec{U}, t)dt. \tag{3.19}$$

Using dynamical programming, there exists optimal controller $U_i^*(i = 1, 2, 3)$, such that ϕ satisfies in HJB equation

$$\begin{array}{l} \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial e_1}\dot{e_1} + \frac{\partial \phi}{\partial e_2}\dot{e_2} + \frac{\partial \phi}{\partial e_3}\dot{e_3} + \frac{\partial \phi}{\partial \hat{a}}\dot{\hat{a}} + \frac{\partial \phi}{\partial \hat{b}}\dot{\hat{b}} + \frac{\partial \phi}{\partial \hat{c}}\dot{\hat{c}} + (\hat{a}-a)^2 \\ +(\hat{b}-b)^2 + (\hat{c}-c)^2 + \sum_1^3 (e_i^2 + lU_i^{*2}) = 0, \end{array} \tag{3.20}$$

where the optimal controllers $U_i^*(i = 1, 2, 3)$ are related to the optimal Lyapunov function

$$U_i^* = -\frac{1}{2l}\frac{\partial \phi}{e_i}, (i = 1, 2, 3), \tag{3.21}$$

with

$$\phi(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, T) = 0. \tag{3.22}$$

We can choose Lyapunov function as follows so that it is satisfied in the partial differential equation (3.20)

$$\phi(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, t) = \sqrt{l}p(t)[\sum_1^3 e_i^2 + (\hat{a}-a)^2 + (\hat{b}-b)^2 + (\hat{c}-c)^2], \tag{3.23}$$

where l is positive constant and $p(t)$ is a positive function and is defined under dynamical programming. The boundary condition (3.22) implies

$$p(T) = 0. \tag{3.24}$$

By applying (3.23) in (3.20), we obtain costate equation as follow

$$\begin{cases} \sqrt{l}\frac{dp}{dt} - p^2 + 1 = 0 \\ p(T) = 0. \end{cases} \tag{3.25}$$

By solving the costate equation (3.25), we have

$$p(t) = \tanh(\frac{-t+T}{\sqrt{l}}).$$

Suppose that the estimation rules of unknown parameters are given by the following differential equations

$$\begin{cases} \dot{\hat{a}} = -e_1(x_1 - x_2) + e_2 x_1 + e_1 x_1 x_3 - \frac{p(t)}{2\sqrt{l}}(\hat{a}-a) \\ \dot{\hat{b}} = e_3 x_3 - \frac{p(t)}{2\sqrt{l}}(\hat{b}-b) \\ \dot{\hat{c}} = -e_2 x_1 - \frac{p(t)}{2\sqrt{l}}(\hat{c}-c). \end{cases} \tag{3.26}$$

TABLE 1. Initial conditions and parameters for example of synchronization

initial condition	Value	Parameter	Value	Parameter	Value
$x_1(0)$	3	α_1	12	k_1	21
$x_2(0)$	-5	α_2	40	k_2	20
$x_3(0)$	1	α_3	21	k_3	11
$y_1(0)$	-2	β_1	10	l	2
$y_2(0)$	2	β_2	1	T	40
$y_3(0)$	-4	β_3	1.2	a	2.1
$\hat{a}(0)$	0.2	η_1	10	b	0.6
$\hat{b}(0)$	-1	η_2	50	c	30
$\hat{c}(0)$	20	η_3	1	–	–

By using (3.26), (3.20) and (3.21), we get

$$\begin{cases} u_1 = -\hat{a}(e_2 - e_1) - \frac{p(t)e_1}{\sqrt{l}} \\ u_2 = -(\hat{c} - \hat{a})e_1 + \hat{a}[e_1e_3 + e_1x_3 + e_3x_1] - \frac{p(t)e_2}{\sqrt{l}} \\ u_3 = -e_1e_2 - e_1x_2 - e_2x_1 + \hat{b}e_3 - \frac{p(t)e_3}{\sqrt{l}}, \end{cases} \tag{3.27}$$

where the function $p(t)$ is positive in $t \in [0, T]$ and

$$\dot{\phi} = -[\sum_1^3 e_i^2 + (\hat{a} - a)^2 + (\hat{b} - b)^2 + (\hat{c} - c)^2] \leq 0, \tag{3.28}$$

shows that the (3.26) and (3.6) are asymptotically stable in the Lyapunov sense in finite time [12, 31].

In finite time case, the response phase trajectory doesn't track the drive phase trajectory, after $t = T$. Because the lyapunov stability condition after a finite time $t = T$, is not established.

3.1.3. *Numerical simulation of optimal synchronization.* To test the validity of the proposed scheme, we present and discuss numerical results for synchronization of chaotic T-system. For synchronization in infinite time interval, systems (2.1), (3.4) and (3.8) with controllers (3.16) are solved numerically by Maple 16 using CK45 method. In finite time interval, systems (2.1), (3.4) and (3.26) with controllers (3.27) are considered. For numerical simulation of synchronization, the initial conditions and parameters are given in table 1.

The results of chaotic synchronization of two identical chaotic T-systems (2.1) and (3.4), via optimal controllers in infinite and finite time intervals are shown in figures 3 and 4. Clearly, the response system tracks the drive system after a relatively slight time. The synchronization errors are plotted in figure 5. figure 6 shows the estimations of uncertain parameters $\hat{a}(t)$, $\hat{b}(t)$ and $\hat{c}(t)$. As expected from analytical considerations, the synchronization errors e_i and error of estimated parameter are converged to zero in infinite and finite time intervals as $t \longrightarrow \infty$ and $t \longrightarrow T$, respectively. In optimal synchronization in finite time, as after the time $t = T = 40$ the controllers will be disabled, the errors of synchronization and estimated parameters are not zero.

3.2. **Optimal anti-synchronization.** For anti-synchronization, assume $k = 1$ in the (3.3) and (3.5). Then $e_i = y_i + x_i (i = 1, 2, 3)$ and the error system between drive

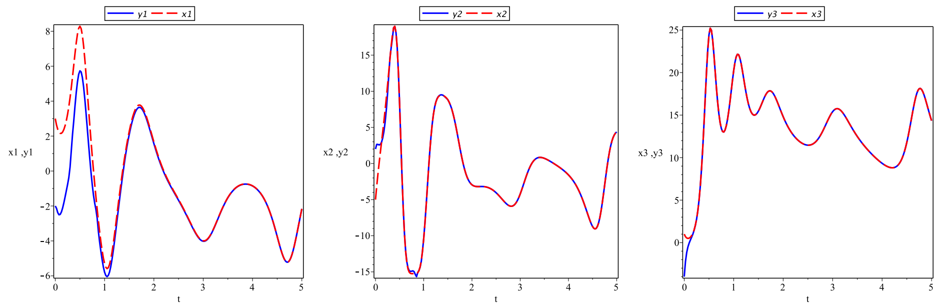


FIGURE 3. Time series trajectories for synchronization via optimal control in infinite time intervals.

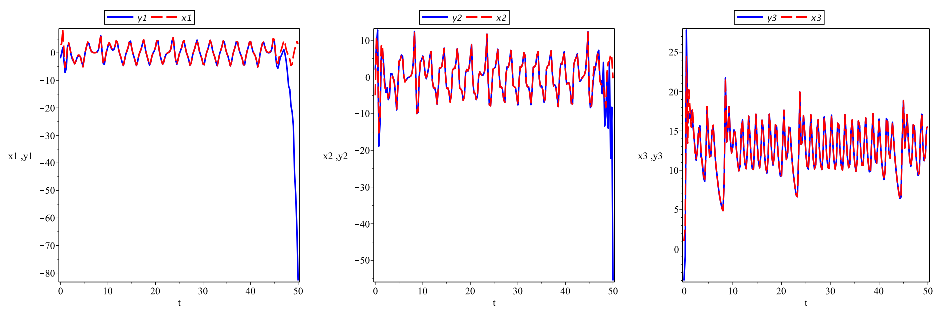
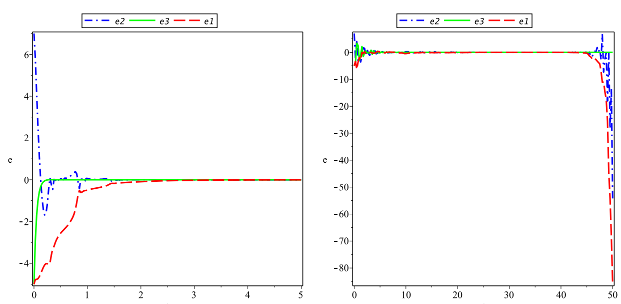


FIGURE 4. Time series trajectories for synchronization via optimal control in finite time intervals.

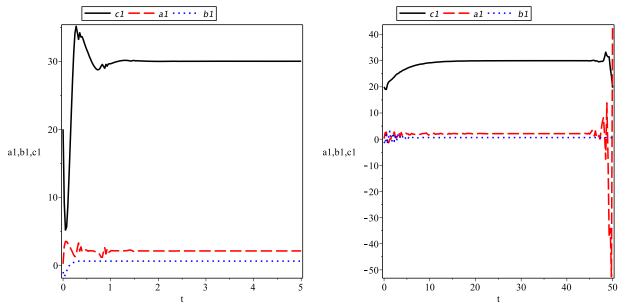


(a) (b)

FIGURE 5. The error due to synchronization. a: infinite time interval b: finite time interval.

and response systems is written as follows

$$\begin{cases} \dot{e_1} = (\hat{a} - a)(x_1 - x_2) + U_1 \\ \dot{e_2} = -(\hat{c} - c)x_1 + (\hat{a} - a)x_1 + U_2 \\ \dot{e_3} = (\hat{b} - b)x_3 + U_3, \end{cases} \tag{3.29}$$



(a) (b)

FIGURE 6. Estimation of parameters for synchronization. a: infinite time interval b: finite time interval.

where

$$\begin{cases} U_1 = \hat{a}(e_2 - e_1) + u_1 \\ U_2 = (\hat{c} - \hat{a})e_1 - \hat{a}[e_1e_3 - e_1x_3 - e_3x_1] - (\hat{a} + a)x_1x_3 + u_2 \\ \quad\;\, = (\hat{c} - \hat{a})e_1 - \hat{a}[e_1e_3 - e_1x_3 - e_3x_1] - 2\hat{a}x_1x_3 + u_2 \\ U_3 = e_1e_2 - e_1x_2 - e_2x_1 + 2x_1x_2 - \hat{b}e_3 + u_3. \end{cases} \tag{3.30}$$

In the following subsection, we obtain the controllers and dynamic of uncertain parameters that are estimated for anti-synchronization in infinite and finite time intervals.

3.2.1. *Optimal anti-synchronization on infinite time.* For optimal anti-synchronization on infinite time interval, we consider the following extended dynamics to estimate unknown parameters

$$\begin{cases} \dot{\hat{a}} = \frac{1}{\beta_1}\left[\alpha_1 e_1(x_2 - x_1) - \alpha_2 e_2 x_1 - \frac{k_1}{2}(\hat{a} - a)\right] \\ \dot{\hat{b}} = \frac{1}{\beta_2}\left[-\alpha_3 e_3 x_3 - \frac{k_2}{2}(\hat{b} - b)\right] \\ \dot{\hat{c}} = \frac{1}{\beta_3}\left[\alpha_2 e_2 x_1 - \frac{k_3}{2}(\hat{c} - c)\right], \end{cases} \tag{3.31}$$

$k_i(i = 1, 2, 3)$ is non-negative and α_i and $\beta_i(i = 1, 2, 3)$ are positive constants.

Now, we design the controllers $u_i(i = 1, 2, 3)$ such that the dynamical systems (3.29) and (3.31) have the following solution

$$e_1 = e_2 = e_3 = 0,\ \hat{a} = a,\ \hat{b} = b,\ \hat{c} = c. \tag{3.32}$$

This solution presents the equilibrium of systems (3.29) and (3.31). To this end, we consider (3.10) and (3.11) with $e_i = x_i + y_i$. In this case, the HJB for anti-synchronization will be similar to (3.12) and (3.13). Therefore, by applying HJB technique of dynamic programming problem and minimizing it, we have the optimal controller as follows

$$U_i^* = -\frac{1}{2\eta_i}\frac{\partial V}{e_i}, (i = 1, 2, 3). \tag{3.33}$$

Substituting the equation (3.33) into the HJB partial differential equation, it can be solved for the optimal Lyapunov V. The Lyapunov function (3.15) should satisfy in HJB equation of anti-synchronization. After lengthy manipulation, the optimal controllers can be obtained as follow

$$U_i^* = \frac{-\alpha_i}{\eta_i}e_i = \frac{-\alpha_i}{\eta_i}(x_i + y_i),$$

then

$$\begin{cases} u_1 = -\frac{\alpha_1}{\eta_1}e_1 - \hat{a}(e_2 - e_1) \\ u_2 = -\frac{\alpha_2}{\eta_2}e_2 - (\hat{c} - \hat{a})e_1 + \hat{a}[e_1e_3 + e_1x_3 + e_3x_1 + 2x_1x_3] \\ u_3 = -\frac{\alpha_3}{\eta_3}e_1 - e_1e_2 - e_1x_2 - e_2x_1 - 2x_1x_3 + \hat{b}e_3 \end{cases} \tag{3.34}$$

and

$$\dot{V} = -\left[k_1(\hat{a} - a)^2 + k_2(\hat{b} - b)^2 + k_3(\hat{c} - c)^2 + \sum_{0}^{3} 2c_ie_i^2\right]. \tag{3.35}$$

Hence, the time derivative of V is negative definite for all nonnegative c_i and k_i. This shows that the solution (3.32) is asymptotically stable in the lyapunov sense via optimal control [20].

3.2.2. *Optimal anti-synchronization on finite time*. In this subsection, we consider problem of finding the optimal controllers $u_i(i = 1, 2, 3)$ such that the steady state of error and updating parameters systems will be asymptotically stable. As such, assume that $e_i = y_i + x_i$ in (3.18)-(3.25). We consider (3.18) and (3.20) as cost functions and HJB equation, respectively. Now similar to calculation as in subsection 3.1.2, we obtain the optimal anti-synchronization controller $U_i(i = 1, 2, 3)$. Since the function U should be satisfied in the HJB equation (3.20), then we have

$$U_i^* = -\frac{1}{2l}\frac{\partial \phi}{\partial e_i} = -\frac{\sqrt{l}}{l}p(t)(x_i + y_i), \quad (i = 1, 2, 3), \tag{3.36}$$

where ϕ is a Lyapunov function similar to (3.23) for anti-synchronization. The ϕ is a solution of the partial differential equation (3.20) and satisfies the boundary condition (3.24). Now, by applying (3.23) in (3.20), costate equation (3.25) with solution $p(t) = \tanh(\frac{-t+T}{\sqrt{l}})$ is obtained. Hence, update rules of estimation of uncertain parameters and optimal controllers are

$$\begin{cases} \dot{\hat{a}} = -e_1(x_1 - x_2) - e_2x_1 - \frac{p(t)}{2\sqrt{l}}(\hat{a} - a) \\ \dot{\hat{b}} = -e_3x_3 - \frac{p(t)}{2\sqrt{l}}(\hat{b} - b) \\ \dot{\hat{c}} = e_2x_1 - \frac{p(t)}{2\sqrt{l}}(\hat{c} - c) \end{cases} \tag{3.37}$$

and

$$\begin{cases} u_1 = -\hat{a}(e_2 - e_1) - \frac{p(t)e_1}{\sqrt{l}} \\ u_2 = -(\hat{c} - \hat{a})e_1 + \hat{a}[e_1e_3 + e_1x_3 + e_3x_1 + x_1x_3] - \frac{p(t)e_2}{\sqrt{l}} \\ u_3 = -e_1e_2 - e_1x_2 - e_2x_1 - 2x_1x_3 + \hat{b}e_3 - \frac{p(t)e_3}{\sqrt{l}}, \end{cases} \tag{3.38}$$

where the function $p(t)$ is positive, in finite time interval $[0, T]$. The time derivative of ϕ is as follow

$$\dot{\phi} = -[\sum_{1}^{3} e_i^2 + (\hat{a} - a)^2 + (\hat{b} - b)^2 + (\hat{c} - c)^2] \leq 0. \tag{3.39}$$

The time derivative of ϕ is negative definite which shows the solution (3.9) is asymptotically stable in the Lyapunov sense via optimal control [12, 31]. This results are similar to those obtained in synchronization case.

TABLE 2. Initial conditions and parameters for example of anti-synchronization

initial condition	Value	Parameter	Value	Parameter	Value
$x_1(0)$	3	α_1	12	k_1	21
$x_2(0)$	-5	α_2	40	k_2	20
$x_3(0)$	1	α_3	21	k_3	11
$y_1(0)$	-2	β_1	10	l	2
$y_2(0)$	2	β_2	1	T	40
$y_3(0)$	-4	β_3	1.2	a	2.1
$\hat{a}(0)$	0.2	η_1	10	b	0.6
$\hat{b}(0)$	-1	η_2	50	c	30
$\hat{c}(0)$	20	η_3	1	–	–

3.2.3. *Numerical simulation of optimal anti-synchronization.* Similar to the synchronization case, to demonstrate and verify the validity of the proposed scheme, we present and discuss numerical results for anti-synchronization of chaotic T-system. For anti-synchronization in infinite time interval, systems(2.1), (3.4) and (3.31) with controllers (3.34) are solved numerically by Maple 16 with CK45. In finite time interval, systems (2.1), (3.4) and (3.37) with controllers (3.38) are considered. For numerical simulation of anti-synchronization, the initial conditions and parameters are given in table 2.

The results of chaotic anti-synchronization of two identical chaotic T-systems (2.1) and (3.4), via optimal controllers in infinite and finite time intervals are shown in FIGURES 7 and 8. As shown in these figures, it is clear that anti synchronization occur after small time.

figure 9 and figure 10 show error of anti-synchronization and estimates of uncertain parameters.

As expected from analytical considerations, the anti-synchronization errors $e_i = y_i + x_i$ and error of estimated parameter are converged to zero in infinite and finite time intervals as $t \longrightarrow \infty$ and $t \longrightarrow T$, respectively. In optimal anti-synchronization in finite time, since after the time $t = T = 15$ the controllers will be disabled, the errors of anti-synchronization and estimated parameters will not be zero.

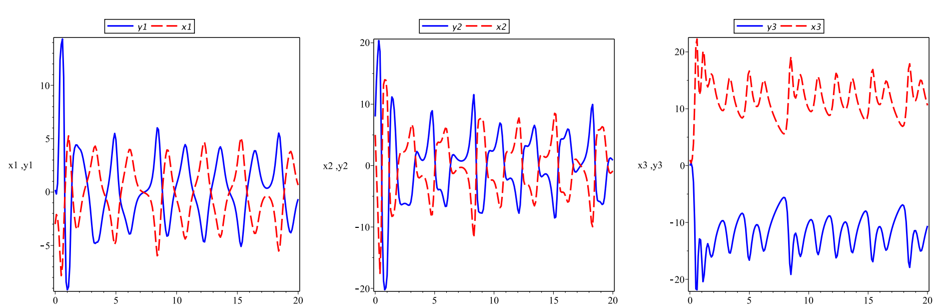


FIGURE 7. Time series trajectories for anti-synchronization via optimal control in infinite time intervals.

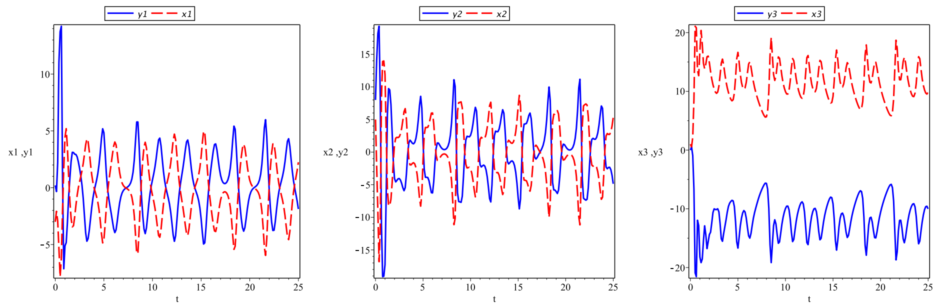
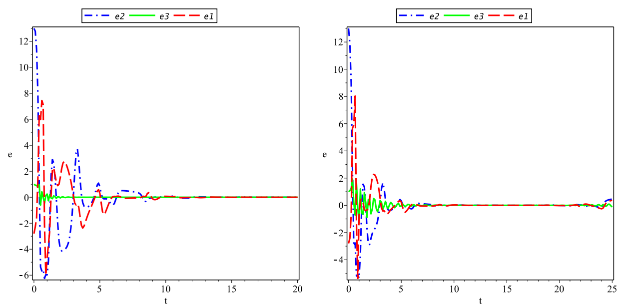
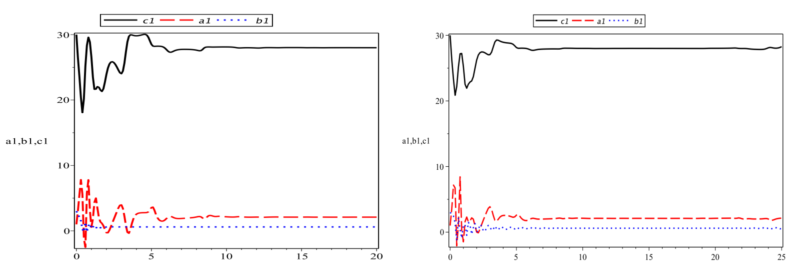


FIGURE 8. Time series trajectories for anti-synchronization via optimal control in finite time intervals.



(a) (b)

FIGURE 9. The error due to anti-synchronization. a: infinite time interval b: finite time interval.



(a) (b)

FIGURE 10. Estimation of parameters for anti-synchronization. a: infinite time interval b: finite time interval.

4. CONCLUSIONS

We designed, new optimal adaptive controller method for synchronization and anti-synchronization of two identical chaotic T-system with uncertain parameters in finite and infinite time intervals. The obtained optimal control laws and parameter estimation rules were satisfied in Lyapunov stability theorem and HJB technique. Estimation of uncertain parameters were designed successfully. The results of synchronization and anti-synchronization showed that errors and the parameter estimation are converged after a short time asymptotically. It was also noted that in finite time case, the synchronization and anti-synchronization fail after $t = T$ time. Finally, numerical examples showed the effectiveness of the proposed method.

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