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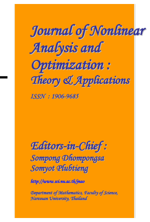

DUALITY FOR A NONDIFFERENTIABLE MULTIOBJECTIVE HIGHER-ORDER SYMMETRIC FRACTIONAL PROGRAMMING PROBLEMS WITH CONE CONSTRAINTS

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ABSTRACT. In this paper, a pair of nondifferentiable multiobjective higher-order symmetric fractional dual problem with cone constraints is formulated. For a differentiable function, we introduce the definition of higher-order (C, α, ρ, d)-convexity. Next, we prove appropriate duality relations under aforesaid assumptions.

KEYWORDS : Higher-order; symmetric duality; multiobjective fractional programming; (C, α, ρ, d)-convexity; cone constraints.
AMS Subject Classification: 90C26 . 90C30 . 90C32 . 90C46

1. INTRODUCTION

Higher-order duality is significant due to its computational importance as it provides more higher bounds whenever approximation is used. By introducing two different functions, $h : R^n \times R^n \to R$ and $k : R^n \times R^n \to R^m$, Mangasarian [8] formulated higher-order dual for a single objective nonlinear problems, $\{\min f(x), \text{ subject to } g(x) \leqq 0\}$. Inspired by this concept, many researchers have worked in this direction. Chen [1] has formulated higher-order multiobjective symmetric dual programs and established duality relations under higher-order $F-$ convexity assumptions. A higher-order vector optimization problem and its dual has been studied by Kassem [9].

In last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. In Chen [1], multiobjective fractional problem and its duality theorems have been considered under higher-order (F, α, ρ, d)- convexity. Later on, Suneja et al. [10] discussed higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order (F, ρ, σ) -type I- assumptions. Recently, Ying [12]

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has studied higher-order multiobjective symmetric fractional problem and formulated its Mond- Weir type dual. Further, duality results are obtained under higher-order (F, α, ρ, d)-convexity.

In this paper, we introduce a pair of nondifferentiable multiobjective Mond-Weir type higher-order symmetric fractional programming problems over cones. For a differentiable function $h : R^n \times R^n \longrightarrow R$, we introduce the definition of higher-order (C, α, ρ, d)-convexity, which extends some kinds of generalized convexity. Under the higher-order (C, α, ρ, d)-convexity assumptions, we prove the higher-order weak, strong and strict converse duality theorems.

2. PRELIMINARIES AND NOTATIONS

Let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. The following conventions for vectors in R^n will be used:

$$\begin{array}{lll} x < y & \text{if and only if} & x_i < y_i,\ i = 1, 2, ..., n, \\ x \leqq y & \text{if and only if} & x_i \leqq y_i,\ i = 1, 2, ..., n, \\ x \leq y & \text{if and only if} & x_i \leq y_i,\ i = 1, 2, ..., n \text{ but } x \neq y, \\ x \nleq y & \text{is the negation of} & x \leq y. \end{array}$$

For a real-valued twice differentiable function $h(x, y)$ defined on an open set in $R^n \times R^m$, denote by $\nabla_x h(\bar{x}, \bar{y})-$ the gradient vector of h with respect to x at $(\bar{x}, \bar{y})$, $\nabla_{xx} h(\bar{x}, \bar{y})-$ the Hessian matrix with respect to x at $(\bar{x}, \bar{y})$. Similarly, $\nabla_y h(\bar{x}, \bar{y})$, $\nabla_{xy} h(\bar{x}, \bar{y})$ and $\nabla_{yy} h(\bar{x}, \bar{y})$ are also defined.

Definition 2.1 [4]. Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists a $z \in R^n$ such that

$$s(y|C) \geqq s(x|C) + z^T (y - x), \forall x \in C.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For a convex set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n : y^T (z - x) \leqq 0, \forall z \in D\}.$$

When C is a compact convex set, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

Definition 2.2 [4]. The positive polar cone S^* of a cone $S \subseteq R^n$ is defined by

$$S^* = \{y \in R^n : x^T y \geqq 0, \forall x \in S\}.$$

A general multiobjective programming problem can be expressed in the following form :

(P) Minimize $f(x) = (f_1(x), f_2(x), ..., f_k(x))^T$

subject to $x \in X^0 = \{x \in X : g(x) \leqq 0\}$,

where $X \subset R^n$ is open, $f : X \to R^k$, $g : X \to R^m$, are differentiable on X.

Definition 2.3 [3]. A feasible solution $\bar{x} \in X^0$ is said to be a weakly efficient solution of (P) if there exists no $x \in X^0$ such that $f(x) < f(\bar{x})$.

Definition 2.4 [3]. A feasible solution $\bar{x} \in X^0$ is said to be an efficient (or Pareto optimal) solution of (P) if there exists no $x \in X^0$ such that $f(x) \leq f(\bar{x})$.

Definition 2.5 [7]. Let $C : X \times X \times R^n \to R$ $(X \subseteq R^n)$ be a function which satisfies $C_{x,u}(0) = 0$, $\forall (x,u) \in X \times X$. Then, the function C is said to be convex on R^n with respect to third argument iff for any fixed $(x,u) \in X \times X$,

$$C_{x,u}(\lambda x_1 + (1-\lambda)x_2) \leqq \lambda C_{x,u}(x_1) + (1-\lambda)C_{x,u}(x_2),\ \forall \lambda \in (0,1),\ \forall x_1, x_2 \in R^n.$$

Now, we introduce the definition of higher-order (C, α, ρ, d)-convex function:

Definition 2.6 A differentiable function $f : X \to R$ is said to be higher order (C, α, ρ, d) - convex at $u \in X$ with respect to $h : X \times R^n \to R$ if for all $x \in X$ and $p \in R^n$, $\exists\, \rho \in R$, a real valued function $\alpha : X \times X \to R_+ \setminus \{0\}$ and $d : X \times X \to R$ (satisfying $d(x,z) = 0 \Leftrightarrow x = z$) such that

$$\frac{1}{\alpha(x,u)}\big[f(x) - f(u) - h(u,p) + p^T \nabla_p h(u,p) - \rho d^2(x,u)\big] \geqq C_{x,u}\big[\nabla_x f(u) + \nabla_p h(u,p)\big].$$

The function f is higher-order (C, α, ρ, d)- convex over X if, $\forall\ u \in X$, it is higher (C, α, ρ, d)- convex.

3. Higher-order symmetric duality

Consider the following multiobjective fractional symmetric dual programs:

(MFP) Minimize $L(x,y,p) = (L_1(x,y,p_1), L_2(x,y,p_2), ..., L_k(x,y,p_k))^T$
subject to

$$-\sum_{i=1}^{k} \lambda_i \big[(\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i)) - L_i(x,y,p_i)(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i))\big] \in C_2^*,$$

$$y^T \left[\sum_{i=1}^{k} \lambda_i \big[(\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i)) - L_i(x,y,p_i)(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i))\big] \right] \geqq 0,$$

$$\lambda > 0,\ \lambda^T e = 1,\ x \in C_1,\ z_i \in D_i,\ r_i \in F_i,\ i = 1, 2, ..., k.$$

(MFD) Maximize $M(u,v,q) = (M_1(u,v,q_1), M_2(u,v,q_2), ..., M_k(u,v,q_k))^T$
subject to

$$\sum_{i=1}^{k} \lambda_i \big[(\nabla_x f_i(u,v) + w_i + \nabla_{q_i} \Phi_i(u,v,q_i)) - M_i(u,v,q_i)(\nabla_x g_i(u,v) - t_i + \nabla_{q_i} \Psi_i(u,v,q_i))\big] \in C_1^*,$$

$$u^T \left[\sum_{i=1}^{k} \lambda_i \big[(\nabla_x f_i(u,v) + w_i + \nabla_{q_i} \Phi_i(u,v,q_i)) - M_i(u,v,q_i)(\nabla_x g_i(u,v) - t_i + \nabla_{q_i} \Psi_i(u,v,q_i))\big] \right] \leqq 0,$$

$$\lambda > 0,\ \lambda^T e = 1,\ v \in C_2,\ w_i \in Q_i,\ t_i \in E_i,\ i = 1, 2, ..., k,$$

where

$$L_i(x,y,p_i) = \frac{f_i(x,y) + s(x|Q_i) - y^T z_i + H_i(x,y,p_i) - p_i^T \nabla_{p_i} H_i(x,y,p_i)}{g_i(x,y) - s(x|E_i) + y^T r_i + G_i(x,y,p_i) - p_i^T \nabla_{p_i} G_i(x,y,p_i)},$$

$$M_i(u,v,q_i) = \frac{f_i(u,v) - s(v|D_i) + u^T w_i + \Phi_i(u,v,q_i) - q_i^T \nabla_{q_i} \Phi_i(u,v,q_i)}{g_i(u,v) + s(v|F_i) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_{q_i} \Psi_i(u,v,q_i)},$$

where $f_i : S_1 \times S_2 \to R$; $g_i : S_1 \times S_2 \to R$; $H_i, G_i : S_1 \times S_2 \times R^m \to R$ and $\Phi_i, \Psi_i : S_1 \times S_2 \times R^n \to R$ are differentiable functions for all $i = 1,2,...,k$. $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are such that $C_1 \times C_2 \subset S_1 \times S_2$, where C_1 and C_2 are the closed convex cones in R^n and R^m, respectively. $Q_i,\ E_i$ are compact convex sets in R^n and $D_i,\ F_i$ are compact convex sets in R^m, $e = (1,1,...,1)^T \in R^k$, $p_i \in R^n,\ q_i \in R^m,\ i = 1,2,...,k,\ p = (p_1,p_2,...,p_k),\ q = (q_1,q_2,...,q_k)$. C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively. It is assumed that in the feasible regions, the numerators are nonnegative and denominators are positive.

Let $U = (U_1,U_2,...,U_k)^T$ and $V = (V_1,V_2,...,V_k)^T$. Then, we can express the programs (MFP) and (MFD) equivalently as:

(MFP)$_U$ Minimize U subject to

$$\begin{aligned}(f_i(x,y) + s(x|Q_i) - y^T z_i + H_i(x,y,p_i)) - p_i^T \nabla_{p_i} H_i(x,y,p_i)) - U_i(g_i(x,y) \\ - s(x|E_i) + y^T r_i + G_i(x,y,p_i) - p_i^T \nabla_{p_i} G_i(x,y,p_i)) = 0,\ i = 1,2,...,k.\end{aligned} \tag{3.1}$$

$$\begin{aligned}-\sum_{i=1}^k \lambda_i \big[\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i) \\ -U_i(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i))\big] \in C_2^*,\end{aligned} \tag{3.2}$$

$$\begin{aligned}y^T \Bigg[\sum_{i=1}^k \lambda_i \big[\nabla_y f_i(x,y) - z_i + \nabla_{p_i} H_i(x,y,p_i) \\ -U_i(\nabla_y g_i(x,y) + r_i + \nabla_{p_i} G_i(x,y,p_i))\big]\Bigg] \geqq 0,\end{aligned} \tag{3.3}$$

$$\lambda > 0,\ \lambda^T e = 1,\ x \in C_1,\ z_i \in D_i,\ r_i \in F_i,\ i = 1,2,...,k.$$

(MFD)$_V$ Maximize V subject to

$$\begin{aligned}(f_i(u,v) - s(v|D_i) + u^T w_i + \Phi_i(u,v,q_i) - q_i^T \nabla_{q_i} \Phi_i(u,v,q_i)) - V_i(g_i(u,v) \\ + s(v|F_i) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T \nabla_{q_i} \Psi_i(u,v,q_i)) = 0,\ i = 1,2,...,k.\end{aligned} \tag{3.4}$$

$$\begin{aligned}\sum_{i=1}^k \lambda_i \big[\nabla_x f_i(u,v) + w_i + \nabla_{q_i} \Phi_i(u,v,q_i) \\ -V_i(\nabla_x g_i(u,v) - t_i + \nabla_{q_i} \Psi_i(u,v,q_i))\big] \in C_1^*,\end{aligned} \tag{3.5}$$

$$u^T\Big[\sum_{i=1}^{k}\lambda_i\big[(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\Phi_i(u,v,q_i))$$

$$-V_i(\nabla_x g_i(u,v)-t_i+\nabla_{q_i}\Psi_i(u,v,q_i))\big]\Big]\leqq 0, \tag{3.6}$$

$$\lambda>0,\ \lambda^Te=1,\ v\in C_2,\ w_i\in Q_i,\ t_i\in E_i,\ i=1,2,...,k.$$

Next, we prove weak, strong and converse duality theorems for (MFP)$_U$ and (MFD)$_V$, which therefore are equally applicable for (MFP) and (MFD). Let $z=(z_1,z_2,...,z_k)$, $r=(r_1,r_2,...,r_k)$, $w=(w_1,w_2,...,w_k)$ and $t=(t_1,t_2,...,t_k)$.

Theorem 3.1. *(Weak duality). Let (x,y,U,z,r,λ,p) be feasible for (MFP)$_U$ and let (u,v,V,w,t,λ,q) be feasible for (MFD)$_V$. Let $\forall i\in\{1,2,...,k\}$, $f_i(.,v)+(.)^Tw_i$ be higher order (C,α,ρ_i,d_i)- convex at u with respect to $\Phi_i(u,v,q_i)$, $-(g_i(.,v)-(.)^Tt_i)$ be higher-order (C,α,ρ_i,d_i)- convex at u with respect to $-\Psi_i(u,v,q_i)$, $-(f_i(x,.)-(.)^Tz_i)$ be higher -order $(\bar{C},\bar{\alpha},\bar{\rho_i},\bar{d_i})$ -convex at y with respect to $-H_i(x,y,p_i)$ and $(g_i(x,.)+(.)^Tr_i)$ be higher -order $(\bar{C},\bar{\alpha},\bar{\rho_i},\bar{d_i})$- convex at y with respect to $G_i(x,y,p)$ where $C:R^n\times R^n\times R^n\to R$ and $\bar{C}:R^m\times R^m\times R^m\to R$. If the following conditions hold:*

$$g_i(x,v)+v^Tr_i-x^Tt_i>0,\ i=1,2,...,k, \tag{3.7}$$

$$\textit{either}\sum_{i=1}^{k}\lambda_i[(1+V_i)\rho_i d_i^2(x,u)+(1+U_i)\bar{\rho}_i\bar{d}_i^{\,2}(v,y)]\geqq 0 \ \textit{or}\ \rho_i\geqq 0$$

$$\textit{and}\quad \bar{\rho}_i\geqq 0, i=1,2,...,k, \tag{3.8}$$

$$C_{x,u}(a)+a^Tu\geqq 0,\ \forall a\in C_1^*,\ \ \bar{C}_{v,y}(b)+b^Ty\geqq 0,\ \forall b\in C_2^*. \tag{3.9}$$

Then, $U\nleq V$.

Proof. Since $\forall i\in\{1,2,...,k\}$, $f_i(.,v)+(.)^Tw_i$ and $-(g_i(.,v)-(.)^Tt_i)$ is higher-order (C,α,ρ_i,d_i) - convex in the first variable at u for fixed v, we have

$$\frac{1}{\alpha(x,u)}\big[f_i(x,v)+x^Tw_i-f_i(u,v)-u^Tw_i-\Phi_i(u,v,q_i)+q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)$$

$$-\rho_i d_i^2(x,u)\big]\geqq C_{x,u}\big(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\Phi_i(u,v,q_i)\big) \tag{3.10}$$

and

$$\frac{1}{\alpha(x,u)}\big[(-g_i(x,v)+x^Tt_i+g_i(u,v)-u^Tt_i)+(\Psi_i(u,v,q_i)-q_i^T\nabla_{q_i}\Psi_i(u,v,q_i))$$

$$-\rho_i d_i^2(x,u)\big]\geqq C_{x,u}\big((-\nabla_x g_i(u,v)+t_i)-\nabla_{q_i}\Psi_i(u,v,q_i)\big). \tag{3.11}$$

Multiplying $\dfrac{\lambda_i}{\tau}>0$ and $\dfrac{\lambda_iV_i}{\tau}\geqq 0,\ i=1,2,...,k$, where $\tau=1+\sum_{i=1}^{k}\lambda_iV_i$ together with (3.10) and (3.11), respectively, we obtain

$$\frac{\lambda_i}{\alpha(x,u)\tau}\big(f_i(x,v)+x^Tw_i-f_i(u,v)-u^Tw_i-\Phi_i(u,v,q_i)+q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)\big)$$

$$-\frac{\lambda_i}{\alpha(x,u)\tau}\rho_i d_i^2(x,u) \geqq \frac{\lambda_i}{\tau}C_{x,u}\big(\nabla_x f_i(u,v) + w_i + \nabla_{q_i}\Phi_i(u,v,q_i)\big)$$

and

$$\frac{\lambda_i V_i}{\alpha(x,u)\tau}\big[-g_i(x,v) + x^T t_i + g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T\nabla_{q_i}\Psi_i(u,v,q_i)\big]$$
$$-\frac{\lambda_i V_i}{\alpha(x,u)\tau}\rho_i d_i^2(x,u) \geqq \frac{\lambda_i V_i}{\tau}C_{x,u}\big(-\nabla_x g_i(u,v) + t_i - \nabla_{q_i}\Psi_i(u,v,q_i)\big).$$

Now, summing over i and adding the above two inequalities, using convexity of C, we have

$$\sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)\tau}\big[f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)\big]$$
$$+\sum_{i=1}^{k}\frac{\lambda_i V_i}{\alpha(x,u)\tau}\big[-g_i(x,v)+x^T t_i+g_i(u,v)-u^T t_i+\Psi_i(u,v,q_i)-q_i^T\nabla_{q_i}\Psi_i(u,v,q_i)\big]-$$
$$\sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)\tau}(1+V_i)\rho_i d_i^2(x,u) \geqq C_{x,u}\Bigg[\sum_{i=1}^{k}\frac{\lambda_i}{\tau}\Bigg(\big(\nabla_x f_i(u,v) + w_i + \nabla_{q_i}\Phi_i(u,v,q_i)\big)$$
$$-V_i(\nabla_x g_i(u,v) - t_i + \nabla_{q_i}\Psi_i(u,v,q_i))\Bigg)\Bigg]. \tag{3.12}$$

Now, from (3.5), as $\tau > 0$, we have

$$a = \sum_{i=1}^{k}\frac{\lambda_i}{\tau}[(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\Phi_i(u,v,q_i)-V_i(\nabla_x g_i(u,v)-t_i+\nabla_{q_i}\Psi_i(u,v,q_i))] \in C_1^*.$$

Hence, for this a, $C_{x,u}(a) \geqq -u^T a \geqq 0$ $\big($from (3.9)$\big)$. Using this, in (3.12), we obtain

$$\sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)\tau}\big(f_i(x,v) + x^T w_i - f_i(u,v) - u^T w_i - \Phi_i(u,v,q_i) + q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)\big) +$$
$$\sum_{i=1}^{k}\frac{\lambda_i V_i}{\alpha(x,u)\tau}\big[-g_i(x,v) + x^T t_i + g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T\nabla_{q_i}\Psi_i(u,v,q_i)\big]$$
$$\geqq \sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)\tau}(1+V_i)\rho_i d_i^2(x,u).$$

Since $v^T r_i \leqq s(v|F_i)$ and using (3.4) in above inequality, we get

$$\sum_{i=1}^{k}\lambda_i[f_i(x,v) + x^T w_i - s(v|D_i) + V_i(x^T t_i - v^T r_i - g_i(x,v))]$$
$$\geqq \sum_{i=1}^{k}\lambda_i(1+V_i)\rho_i d_i^2(x,u). \tag{3.13}$$

Similarly, by the higher-order $(\bar{C},\bar{\alpha},\bar{\rho}_i,\bar{d}_i)$ - convexity of $-f_i(x,.)+(.)^T z_i$ and $(g_i(x,.)+(.)^T r_i)$, $\forall i\in\{1,2,...,k\}$, in the second variable at y, for fixed x and from the condition (3.9), for

$$b=-\sum_{i=1}^{k}\frac{\lambda_i}{\tau}[(\nabla_y f_i(x,y)-z_i+\nabla_{p_i}H_i(x,y,p_i)-U_i(\nabla_y g_i(x,y)+r_i+\nabla_{p_i}G_i(x,y,p_i))]\in C_2^*,$$

we get

$$\sum_{i=1}^{k}\lambda_i[-f_i(x,v)+v^T z_i-s(x|Q_i)+U_i(v^T r_i-x^T t_i+g_i(x,v))]$$
$$\geqq\sum_{i=1}^{k}\lambda_i(1+U_i)\bar{\rho}_i\bar{d}_i^{\,2}(v,y). \tag{3.14}$$

Adding the inequalities (3.13), (3.14) and applying (3.8), we get

$$\sum_{i=1}^{k}\lambda_i(v^T z_i-s(v|D_i)+x^T w_i-s(x|Q_i))$$
$$+\sum_{i=1}^{k}\lambda_i(U_i-V_i)(g_i(x,v)+v^T r_i-x^T t_i)\geqq 0. \tag{3.15}$$

Since $\lambda>0$, $v^T z_i\leqq s(v|D_i)$ and $x^T w_i\leqq s(x|Q_i)$, the above inequality gives

$$\sum_{i=1}^{k}\lambda_i(U_i-V_i)(g_i(x,v)+v^T r_i-x^T t_i)\geqq 0. \tag{3.16}$$

From the fact that $\lambda>0$ and using (3.7), it follows that $U\nleq V$. This completes the proof. □

Theorem 3.2. *(Weak duality). Let* (x,y,U,z,r,λ,p) *and* (u,v,V,w,t,λ,q) *be feasible solutions of (MFP)$_U$ and (MFD)$_V$, respectively. Suppose that*

- (i) $(f_i(.,v)+(.)^T w_i)-V_i(g_i(.,v)-(.)^T t_i)$ *is higher-order* (C,α,ρ_i,d_i)*- convex at* u *with respect to* $(\Phi_i(u,v,q)-V_i\Psi_i(u,v,q))$,
- (ii) $(-f_i(x,.)+(.)^T z_i)+U_i(g_i(x,.)+(.)^T r_i)$ *is higher-order* $(\bar{C},\bar{\alpha},\bar{\rho}_i,\bar{d}_i)$ *-convex at* y *with respect to* $-H_i(x,y,p)+U_iG_i(x,y,p)$,
- (iii) *either* $\sum_{i=1}^{k}\lambda_i\left[\rho_i d_i^2(x,u)+\bar{\rho}_i\bar{d}_i^{\,2}(v,y)\right]\geqq 0$ *or* $\rho_i\geqq 0$ *and* $\bar{\rho}_i\geqq 0$, $i=1,2,...,k$,
- (iv) $C_{x,u}(a)+a^T u\geqq 0,\forall a\in C_1^*$, $\bar{C}_{v,y}(b)+b^T y\geqq 0,\forall b\in C_2^*$,
- (v) $g_i(x,v)+v^T r_i-x^T t_i>0$, $i=1,2,...,k$.

Then, $U\nleq V$.

Proof. By hypothesis (i), we have

$$\frac{1}{\alpha(x,u)}\big[f_i(x,v)+x^T w_i-V_i(g_i(x,v)-x^T t_i)-(f_i(u,v)+u^T w_i)-V_i(g_i(u,v)-u^T t_i)-$$
$$(\Phi_i(u,v,q_i)-V_i(\Psi_i(u,v,q_i))+q_i^T(\nabla_{q_i}\Phi_i(u,v,q_i)-V_i\nabla_{q_i}\Psi_i(u,v,q_i))-\rho_i d_i^2(x,u)\big]$$
$$\geqq C_{x,u}\big[\nabla_x f_i(u,v)+w_i-V_i(\nabla_x g_i(u,v)-t_i)+(\nabla_{q_i}\Phi_i(u,v,q_i)-V_i\nabla_{q_i}\Psi_i(u,v,q_i))\big].$$

Since $\lambda > 0$, we obtain

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha(x,u)}[f_i(x,v)+x^T w_i-f_i(u,v)-u^T w_i-\Phi_i(u,v,q_i)+q_i^T\nabla_{q_i}\Phi_i(u,v,q_i) +V_i(-g_i(x,v)+ x^T t_i + g_i(u,v) - u^T t_i + \Psi_i(u,v,q_i) - q_i^T\nabla_{q_i}\Psi_i(u,v,q_i)) - \rho_i d_i^2(x,u)$$

$$\geqq \sum_{i=1}^{k}\lambda_i C_{x,u}\big[(\nabla_x f_i(u,v)+w_i) - V_i(\nabla_x g_i(u,v)-t_i\big) + (\nabla_{q_i}\Phi_i(u,v,q_i) - V_i\nabla_{q_i}\Psi_i(u,v,q_i))\big].$$

Using convexity of C, we have

$$\frac{1}{\alpha(x,u)}\left[\sum_{i=1}^{k}\lambda_i\Big((f_i(x,v)+x^T w_i-f_i(u,v)-u^T w_i)-\Phi_i(u,v,q_i)+q_i^T\nabla_{q_i}\Phi_i(u,v,q_i)\Big)\right.$$
$$+\sum_{i=1}^{k}\lambda_i V_i\big(-g_i(x,v)+x^T t_i-v^T r_i\big)+\sum_{i=1}^{k}\lambda_i V_i\big(g_i(u,v)+v^T r_i-u^T t_i+\Psi_i(u,v,q_i)-$$
$$\left.q_i^T\nabla_{q_i}\Psi_i(u,v,q_i)\big)\right] - \frac{1}{\alpha(x,u)}\sum_{i=1}^{k}\lambda_i\rho_i d_i^2(x,u)$$

$$\geqq C_{x,u}\left[\sum_{i=1}^{k}\lambda_i\Big(\big(\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\Phi_i(u,v,q_i)\big) - V_i(\nabla_x g_i(u,v)-t_i-\nabla_{q_i}\Psi_i(u,v,q_i))\Big)\right].$$

From (3.5),

$$a=\sum_{i=1}^{k}\lambda_i[\nabla_x f_i(u,v)+w_i+\nabla_{q_i}\Phi_i(u,v,q_i)-V_i(\nabla_x g_i(u,v)-t_i+\nabla_{q_i}\Psi_i(u,v,q_i))] \in C_1^*$$

and using hypothesis (iv), we get

$$\frac{1}{\alpha(x,u)}\left[\sum_{i=1}^{k}\lambda_i\big(f_i(x,v)+x^T w_i-f_i(u,v)-u^T w_i\big)-\Phi_i(u,v,q_i)+q_i^T\nabla_{q_i}\Phi_i(u,v,q_i))\right.$$
$$+\sum_{i=1}^{k}\lambda_i V_i\big(-g_i(x,v)+x^T t_i-v^T r_i\big)+\sum_{i=1}^{k}\lambda_i V_i(g_i(u,v)-u^T t_i$$
$$\left.+\Psi_i(u,v,q_i)-q_i^T\nabla_{q_i}\Psi_i(u,v,q_i)\right] \geqq \sum_{i=1}^{k}\frac{\lambda_i}{\alpha(x,u)}\rho_i d_i^2(x,u). \tag{3.17}$$

Since $v^T r_i \leqq s(v|F_i)$, using (3.4) in (3.17), we get

$$\sum_{i=1}^{k}\lambda_i[(f_i(x,v)+x^T w_i-s(v|D_i))+V_i(x^T t_i-v^T r_i-g_i(x,v))]$$
$$\geqq \sum_{i=1}^{k}\lambda_i\rho_i d_i^2(x,u). \tag{3.18}$$

Similarly, by the higher-order $(\bar{C},\bar{\alpha},\bar{\rho}_i,\bar{d}_i)-$ convexity of $-f_i(x,.)+(.)^T z_i+U_i(g_i(x,.)+(.)^T r_i)$ in the second variable at y, for fixed x, we get

$$\sum_{i=1}^{k} \lambda_i[-f_i(x,v) + v^T z_i - s(x|Q_i) + U_i(v^T r_i - x^T t_i + g_i(x,v))]$$

$$\geqq \sum_{i=1}^{k} \lambda_i \bar{\rho}_i \bar{d}_i^{\,2}(v,y). \tag{3.19}$$

Using $\lambda > 0,\ v^T z_i \leqq s(v|D_i)$ and $x^T w_i \leqq s(x|Q_i)$, it follows from (3.18) and (3.19) that

$$\sum_{i=1}^{k} \lambda_i (U_i - V_i)(g_i(x,v) + v^T r_i - x^T t_i) \geqq 0. \tag{3.20}$$

Since $\lambda > 0$ and using hypothesis (v), it follows that $U \nleq V$. Hence, the result. □

Theorem 3.3. *(Strong duality). Let $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ be an efficient solution of (MFP)$_U$ and fix $\lambda = \bar{\lambda}$ in (MFD)$_V$. If the following conditions hold*

- (i) $\nabla_x H_i(\bar{x},\bar{y},0) = \nabla_x G_i(\bar{x},\bar{y},0) = 0, \nabla_{q_i}\Phi_i(\bar{x},\bar{y},0) = \nabla_{q_i}\Psi_i(\bar{x},\bar{y},0) = 0, H_i(\bar{x},\bar{y},0) = G_i(\bar{x},\bar{y},0) = 0, \Phi_i(\bar{x},\bar{y},0) = \Psi_i(\bar{x},\bar{y},0) = 0, \nabla_y H_i(\bar{x},\bar{y},0) = \nabla_y G_i(\bar{x},\bar{y},0) = 0, \nabla_{p_i} H_i(\bar{x},\bar{y},0) = \nabla_{p_i} G_i(\bar{x},\bar{y},0) = 0,\ i = 1,2,...,k,$
- (ii) *for all* $i \in \{1,2,...,k\}$*, the Hessian matrix* $\nabla_{p_i p_i} H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{U}_i \nabla_{p_i p_i} G_i(\bar{x},\bar{y},\bar{p}_i)$ *is positive or negative definite,*
- (iii) *the set of vectors* $\{\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))\}_{i=1}^k$ *is linearly independent,*
- (iv) *for* $\bar{p}_i \in R^n,\ \bar{p}_i \neq 0\ (i = 1,2,...,k)$ *implies that* $\sum_{i=1}^{k} \bar{p}_i^T[\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))] \neq 0,$
- (v) $\bar{U}_i > 0,\ \forall i \in \{1,2,...,k\}.$

Then

- (a) $\bar{p}_i = 0,\ i = 1,2,...,k,$
- (b) *there exists* $\bar{w}_i \in Q_i$ *and* $\bar{t}_i \in E_i,\ i = 1,2,...,k$ *such that* $(\bar{x},\bar{y},\bar{U},\bar{w},\bar{t},\bar{\lambda},\bar{q} = 0)$ *is a feasible solution of* $(MFD)_V$.

Furthermore, if the hypotheses in Theorem 3.1 or 3.2 are satisfied, then $(\bar{x},\bar{y},\bar{U},\bar{w},\bar{t},\bar{\lambda},\bar{q} = 0)$ *is an efficient solution of* $(MFD)_V$*, and the two objective values are equal.*

Proof. Since $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ is an efficient solution of (MFP)$_U$, therefore, by the Fritz John necessary optimality conditions [2], there exists $\alpha \in R^k$ $,\beta \in R^k,\ \gamma \in C_2,\ \delta \in R_+,\ \xi \in R^k$ $,\eta \in R,\ \bar{w}_i \in R^n$ and $\bar{t}_i \in R^n,\ i = 1,2,...,k$ such that

$$(x-\bar{x})^T\Bigg[\sum_{i=1}^{k} \beta_i(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i + \nabla_x H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i + \nabla_x G_i(\bar{x},\bar{y},\bar{p}_i))) +$$

$$(\gamma - \delta\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{yx} f_i(\bar{x},\bar{y}) - \bar{U}_i \nabla_{yx} g_i(\bar{x},\bar{y})) + \sum_{i=1}^{k} (\nabla_{p_i x} H_i(\bar{x},\bar{y},\bar{p}_i)$$

$$- \bar{U}_i(\nabla_{p_i x} G_i(\bar{x},\bar{y},\bar{p}_i))^T((\gamma - \delta\bar{y})\bar{\lambda}_i - \beta_i \bar{p}_i)\Bigg] \geqq 0, \forall x \in C_1, \tag{3.21}$$

$$\sum_{i=1}^{k} \beta_i(\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_y H_i(\bar{x},\bar{y},\bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_y G_i(\bar{x},\bar{y},\bar{p}_i))$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yy}f_i(\bar{x},\bar{y})-\bar{U}_i\nabla_{yy}g_i(\bar{x},\bar{y}))^T(\gamma-\delta\bar{y})+\sum_{i=1}^{k}(\nabla_{p_iy}H_i(\bar{x},\bar{y},\bar{p}_i)$$
$$-\bar{U}_i\nabla_{p_iy}G_i(\bar{x},\bar{y},\bar{p}_i))^T(-\beta_i\bar{p}_i+(\gamma-\delta\bar{y})\bar{\lambda}_i)-\delta\sum_{i=1}^{k}\bar{\lambda}_i[\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i$$
$$+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))]=0, \tag{3.22}$$

$$\alpha_i-\beta_i(g_i(\bar{x},\bar{y})-s(\bar{x}|E_i)+\bar{y}^T\bar{r}_i+G_i(\bar{x},\bar{y},\bar{p}_i)-\bar{p}_i^T\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))$$
$$-(\gamma-\delta\bar{y})^T\nabla_y(\bar{\lambda}_i(g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))=0,\ i=1,2,...,k, \tag{3.23}$$

$$(\gamma-\delta\bar{y})^T(\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)$$
$$-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))-\xi_i+\eta=0,\ i=1,2,...,k, \tag{3.24}$$

$$\bar{\lambda}_i(\gamma-\delta\bar{y})-\beta_i\bar{p}_i)^T(\nabla_{p_ip_i}H_i(\bar{x},\bar{y},\bar{p}_i)$$
$$-\bar{U}_i\nabla_{p_ip_i}G_i(\bar{x},\bar{y},\bar{p}_i))=0,\ i=1,2,...,k, \tag{3.25}$$

$$\beta_i\bar{y}+(\gamma-\delta\bar{y})\bar{\lambda}_i\in N_{D_i}(\bar{z}_i),\ i=1,2,...,k, \tag{3.26}$$

$$\beta_i\bar{U}_i\bar{y}+\bar{\lambda}_i\bar{U}_i(\gamma-\delta\bar{y})\in N_{F_i}(\bar{r}_i),\ i=1,2,...,k, \tag{3.27}$$

$$\gamma^T\sum_{i=1}^{k}\bar{\lambda}_i((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))$$
$$-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i)))=0, \tag{3.28}$$

$$\delta\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)$$
$$-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))=0, \tag{3.29}$$

$$\bar{\lambda}^T\xi=0, \tag{3.30}$$

$$\eta(\bar{\lambda}^Te-1)=0, \tag{3.31}$$

$$\bar{w}_i\in Q_i,\bar{t}_i\in E_i,\bar{x}^Tt_i=s(\bar{x}|E_i),\bar{x}^T\bar{w}_i=s(\bar{x}|Q_i),\ i=1,2,...,k, \tag{3.32}$$

$$(\alpha,\delta,\xi)\geqq 0,\ (\alpha,\beta,\gamma,\delta,\xi,\eta)\neq 0. \tag{3.33}$$

Since $\bar{\lambda}>0$, and $\xi\geqq 0,\ (30)$ implies that $\xi=0$.

Equation (3.22) can be re-written as

$$\sum_{i=1}^{k}(\beta_i-\delta\bar{\lambda}_i)((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))-\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i)))$$
$$+\sum_{i=1}^{k}\beta_i((\nabla_y H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{U}_i\nabla_y G_i(\bar{x},\bar{y},\bar{p}_i))-(\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{U}_i\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i\ ((\nabla_{yy}f_i(\bar{x},\bar{y})-\bar{U}_i\nabla_{yy}g_i(\bar{x},\bar{y}))^T(\gamma-\delta\bar{y})$$
$$+\sum_{i=1}^{k}((\nabla_{p_iy}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{U}_i\nabla_{p_iy}G_i(\bar{x},\bar{y},\bar{p}_i))^T(-\beta_i\bar{p}_i+(\gamma-\delta\bar{y})\bar{\lambda}_i)=0. \tag{3.34}$$

By hypothesis (ii) and (3.25), we have

$$\bar{\lambda}_i(\gamma - \delta\bar{y}) = \beta_i\bar{p}_i,\ i = 1, 2, ..., k. \tag{3.35}$$

Now, we claim that $\beta_i \neq 0,\ \forall i$. If possible, let $\beta_{t_0} = 0$ for some t_0, $1 \leq t_0 \leq k$, then from $\bar{\lambda}_{t_0} > 0$ and equation (3.35), we have

$$\gamma = \delta\bar{y}. \tag{3.36}$$

Using (3.35) and (3.36), we obtain $\beta_i\bar{p}_i = 0,\ i = 1, 2, ..., k$. Hence, by hypothesis (i), we get

$$\sum_{i=1}^{k} \beta_i((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i)$$
$$-\bar{U}_i\nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i\nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0. \tag{3.37}$$

Using (3.35)-(3.37) in (3.34), we obtain

$$\sum_{i=1}^{k}(\beta_i - \delta\bar{\lambda}_i)(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i)$$
$$-\bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0, \tag{3.38}$$

which by hypothesis (iii), follows that

$$\bar{\beta}_i - \delta\bar{\lambda}_i = 0,\ i = 1, 2, ..., k. \tag{3.39}$$

Now, for $i = t_0$, we have $\delta\bar{\lambda}_{t_0} = 0$. This implies $\delta = 0$ as $\bar{\lambda} > 0$. Hence, from (3.39), $\beta_i = 0,\ \forall\ i$. Thus, from relation (3.23) and (3.36), we get $\alpha_i = 0,\ i = 1, 2, ..., k$. Also, from relations (3.24) and (3.36), we get $\eta = 0$ and $\gamma = 0$, respectively, which contradicts the fact that $(\alpha, \beta, \gamma, \delta, \xi, \eta) \neq 0$. Hence $\beta_i \neq 0,\ i = 1, 2, ..., k$.
Now, equation (3.24) reduces to

$$(\gamma - \delta\bar{y})^T(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i)$$
$$- \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) + \eta = 0,\ i = 1, 2, ..., k, \tag{3.40}$$

Multiplying by $\bar{\lambda}_i$ and summing over i, we get

$$(\gamma - \delta\bar{y})^T \sum_{i}^{k} \bar{\lambda}_i(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i)$$
$$- \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) + \eta\ \bar{\lambda}^T e_k = 0. \tag{3.41}$$

Subtracting (3.28) and (3.29), we get

$$(\gamma - \delta\bar{y})^T \sum_{i}^{k} \bar{\lambda}_i(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i)$$
$$- \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0. \tag{3.42}$$

Using (3.42) in (3.41), we get, $\eta = 0$.

Now, equation (3.40), yield

$$(\gamma - \delta\bar{y})^T(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i)$$
$$- \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) = 0,\ i = 1, 2, ..., k, \tag{3.43}$$

Since $\bar{\lambda} > 0$, using (3.35) in (3.43), we get

$$\beta_i \bar{p}_i^T [\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) \\ -\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))] = 0,\ i = 1,2,...,k, \tag{3.44}$$

Since $\beta_i \neq 0,\ i = 1,2,...,k$, we obtain

$$\bar{p}_i^T [\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) \\ -\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))] = 0,\ i = 1,2,...,k, \tag{3.45}$$

or

$$\sum_{i=1}^{k} \bar{p}_i^T [\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x},\bar{y},\bar{p}_i) \\ -\bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x},\bar{y},\bar{p}_i))] = 0. \tag{3.46}$$

By the hypothesis (iv), we have $\bar{p}_i = 0,\ i = 1,2,...,k$. Further using, hypothesis (i), (3.35)-(3.36) in (3.21) and (3.34), respectively, we get

$$(x-\bar{x})^T \left[\sum_{i=1}^{k} \beta_i (\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)) \right] \geqq 0, \forall x \in C_1. \tag{3.47}$$

$$\sum_{i=1}^{k} (\beta_i - \delta\bar{\lambda}_i)[\nabla_y f_i(\bar{x},\bar{y}) - \bar{z}_i - \bar{U}_i(\nabla_y g_i(\bar{x},\bar{y}) + \bar{r}_i)] = 0. \tag{3.48}$$

Using hypothesis (iii) in (3.48), we have

$$\beta_i = \delta\bar{\lambda}_i,\ i = 1,2,...,k. \tag{3.49}$$

Since $\beta_i \neq 0,\ \bar{\lambda}_i > 0,\ i = 1,2,...,k$ and $\delta \geqq 0$, this implies that $\beta_i > 0,\ \forall i$. Now, using (3.49) in (3.47), we obtain

$$(x-\bar{x})^T \left[\sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)) \right] \geqq 0, \forall x \in C_1. \tag{3.50}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$, as C_1 is a closed convex cone. On substituting $x + \bar{x}$ in place of x in (3.50), we get

$$x^T \sum_{i=1}^{k} \bar{\lambda}_i \big[(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i) \big] \geqq 0, \tag{3.51}$$

which in turn implies that for all $x \in C_1$, we have

$$\sum_{i=1}^{k} \bar{\lambda}_i [(\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i))] \in C_1^*. \tag{3.52}$$

Also, by letting $x = 0$ and $x = 2\bar{x}$, simultaneously in (3.50), yields

$$\bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_x f_i(\bar{x},\bar{y}) + \bar{w}_i) - \bar{U}_i(\nabla_x g_i(\bar{x},\bar{y}) - \bar{t}_i)) = 0, \tag{3.53}$$

Using $\bar{p}_i = 0$ in (3.35), we get, $\gamma = \delta\bar{y}$ and $\delta > 0$, we have

$$\bar{y} = \frac{\gamma}{\delta} \in C_2.$$

Since $\beta > 0$ by (3.26) and the fact that $\gamma = \delta\bar{y}$, we get $\bar{y} \in N_{D_i}(\bar{z}_i), i = 1, 2, ..., k$. This implies

$$\bar{y}^T \bar{z}_i = s(\bar{y}|D_i), i = 1, 2, ..., k. \tag{3.54}$$

By (3.27) and hypothesis (v), we have $\bar{y} \in N_{F_i}(\bar{r}_i), i = 1, 2, ..., k$. Hence,

$$\bar{y}^T \bar{r}_i = s(\bar{y}|F_i), i = 1, 2, ..., k. \tag{3.55}$$

Combining (3.32), (3.54)-(3.55) and given equation (3.1), reduce to

$$\begin{aligned}&(f_i(\bar{x}, \bar{y}) + \bar{x}^T \bar{w}_i - s(\bar{y}|D_i))\\ &\qquad -\bar{U}_i(g_i(\bar{x}, \bar{y}) - \bar{x}^T \bar{t}_i - s(\bar{y}|F_i)) = 0, i = 1, 2, ..., k.\end{aligned} \tag{3.56}$$

Therefore, (3.52)-(3.53) and (3.56) shows that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(MFD)_V$.

Under the assumptions Theorems 3.1 or 3.2, if $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is not an efficient solution of $(MFD)_V$, then there exists other feasible solution $(u, v, V, w, t, \lambda, q)$, of $(MFD)_V$, such that $\bar{U} \leq V$.
Since $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ is a feasible solution of $(MFP)_U$, by Weak duality theorem, we have $\bar{U} \nleq V$, hence the contradiction implies that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is an efficient solution of $(MFD)_V$. Hence, the result. ☐

Theorem 3.4. *(Strong duality). Let $(\bar{x}, \bar{y}, \bar{U}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p})$ be efficient solution of (MFP)$_U$ and fix $\lambda = \bar{\lambda}$ in (MFD)$_V$. Suppose that*

- (i) $\nabla_x H_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_{q_i} \Phi_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \Psi_i(\bar{x}, \bar{y}, 0) = 0,\ H_i(\bar{x}, \bar{y}, 0) = G_i(\bar{x}, \bar{y}, 0) = 0, \Phi_i(\bar{x}, \bar{y}, 0) = \Psi_i(\bar{x}, \bar{y}, 0) = 0, \nabla_y H_i(\bar{x}, \bar{y}, 0) = \nabla_y G_i(\bar{x}, \bar{y}, 0) = 0, \nabla_{p_i} H_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} G_i(\bar{x}, \bar{y}, 0) = 0,\ i = 1, 2, ..., k,$
- (ii) $\bar{U}_i > 0, \forall i \in 1, 2, ..., k,$
- (iii) $\nabla_{p_i p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)$ *is nonsingular* $\forall i = 1, 2, ..., k,$
- (iv) $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{U}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ *is positive definite and* $\bar{p}_i^T \big((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))\big) \geqq 0, \quad \forall i = 1, 2, ..., k,$
  or $\sum_{i=1}^{k} \bar{\lambda}_i((\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{U}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ *is negative definite and* $\bar{p}_i^T (\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) \leqq 0, \forall i = 1, 2, ..., k,$
- (v) *the set of vectors* $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{U}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i) : i = 1, 2, ..., k\}$ *is linearly independent.*

Then $\bar{p} = 0$, and there exists $\bar{w}_i \in Q_i$ and $\bar{t}_i \in E_i$, $i = 1, 2, ..., k$ such that $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(MFD)_V$. Furthermore, if the hypotheses in theorem (3.1) or (3.2) are satisfied, then $(\bar{x}, \bar{y}, \bar{U}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q} = 0)$ is efficient solution of $(MFD)_V$, and the two objective values are equal.

Proof. It follows on the lines of Theorem 3.3. ☐

Theorem 3.5. *(Strict converse duality). Let $(\bar{u}, \bar{v}, \bar{V}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q})$ be efficient solution of (MFP)$_V$ and fix $\lambda = \bar{\lambda}$ in (MFD)$_U$. If the following conditions hold*

- (i) $\nabla_x \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_x \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{q_i} \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_{q_i} \Psi_i(\bar{u}, \bar{v}, 0) = 0, H_i(\bar{u}, \bar{v}, 0) = G_i(\bar{u}, \bar{v}, 0) = 0, \Phi_i(\bar{u}, \bar{v}, 0) = \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_y \Phi_i(\bar{u}, \bar{v}, 0) = \nabla_y \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{p_i} H_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i} G_i(\bar{u}, \bar{v}, 0) = 0,\ i = 1, 2, ..., k,$

(ii) *for all* $i \in \{1, 2, ..., k\}$, *the Hessian matrix* $\nabla_{p_i p_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_{p_i p_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)$ *is positive or negative definite,*

(iii) *the set of vectors* $\{\nabla_x f_i(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_x g_i(\bar{u}, \bar{v}) - \bar{t}_i + \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)))\}_{i=1}^{k}$ *is linearly independent,*

(iv) *for* $\bar{q}_i \in R^n$, $\bar{q}_i \neq 0$, $(i = 1, 2, ..., k)$ *implies that*
$$\sum_{i=1}^{k} \xi_i \bar{q}_i^T [\nabla_x f_i(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_x g_i(\bar{u}, \bar{v}) - \bar{t}_i + \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i))] \neq 0,$$

(v) $\bar{V}_i > 0, \ \forall i \in 1, 2, ..., k.$

Then

(a) $\bar{q}_i = 0, \ i = 1, 2, ..., k,$

(b) *there exists* $\bar{z}_i \in D_i$ *and* $\bar{r}_i \in F_i$, $i = 1, 2, ..., k$ *such that* $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ *is a feasible solution of* $(MFD)_U$.

Furthermore, if the hypotheses in Theorem 3.1 *or* 3.2 *are satisfied, then* $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ *is an efficient solution of* $(MFD)_U$, *and the two objective values are equal.*

Proof. It follows on the lines of Theorem 3.3. □

Theorem 3.6. *(Strict converse duality). Let* $(\bar{u}, \bar{v}, \bar{V}, \bar{w}, \bar{t}, \bar{\lambda}, \bar{q})$ *be efficient solution of (MFP)*$_V$ *and fix* $\lambda = \bar{\lambda}$ *in (MFD)*$_U$. *Suppose that*

(i) $\nabla_x\Phi_i(\bar{u}, \bar{v}, 0) = \nabla_x\Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, 0) = \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, 0) = 0,$ $H_i(\bar{u}, \bar{v}, 0) = G_i(\bar{u}, \bar{v}, 0) = 0, \Phi_i(\bar{u}, \bar{v}, 0) = \Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_y\Phi_i(\bar{u}, \bar{v}, 0)$ $= \nabla_y\Psi_i(\bar{u}, \bar{v}, 0) = 0, \nabla_{p_i}H_i(\bar{u}, \bar{v}, 0) = \nabla_{p_i}G_i(\bar{u}, \bar{v}, 0) = 0, \ i = 1, 2, ..., k,$

(ii) $\bar{V}_i > 0, \forall i \in 1, 2, ..., k,$

(iii) $\nabla_{q_i q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_{q_i q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)$ *is nonsingular* $\forall i = 1, 2, ..., k,$

(iv) $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{xx} f_i(\bar{u}, \bar{v}) - \bar{V}_i \nabla_{xx} g_i(\bar{u}, \bar{v}))$ *is positive definite and* $\bar{q}_i^T((\nabla_x\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_x\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) - ((\nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) \geqq 0, \forall \ i = 1, 2, ..., k,$
or $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{xx} f_i(\bar{u}, \bar{v}) - \bar{V} \nabla_{xx} g_i(\bar{u}, \bar{v}))$ *is negative definite and* $\bar{q}_i^T((\nabla_x\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_x\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) - ((\nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i \nabla_{q_i}\Phi_i\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) \leqq 0, \forall i = 1, 2, ..., k,$

(v) *the set of vectors* $\{\nabla_x f_i(\bar{u}, \bar{v}) + \bar{w}_i + \nabla_{q_i}\Phi_i(\bar{u}, \bar{v}, \bar{q}_i) - \bar{V}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i + \nabla_{q_i}\Psi_i(\bar{u}, \bar{v}, \bar{q}_i)) : \ i = 1, 2, ..., k\}$ *is linearly independent.*

Then, $\bar{q} = 0$ *and there exists* $\bar{z}_i \in D_i$ *and* $\bar{r}_i \in F_i$, $i = 1, 2, ..., k$ *such that* $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ *is a feasible solution of (MFD)*$_U$. *Furthermore, if the hypothesis in Theorems* 3.1 *or* 3.2 *are satisfied, then* $(\bar{u}, \bar{v}, \bar{V}, \bar{z}, \bar{r}, \bar{\lambda}, \bar{p} = 0)$ *is an efficient solution of (MFD)*$_U$, *and the two objective values are equal.*

Proof. It follows on the lines of Theorem 3.3. □

4. Special cases

We consider some of the special cases of the problems studied in section 3. In all the cases, $C_1 = R^n_+$ and $C_2 = R^m_+$,

(i) then, our problems $(MFP)_U$ and $(MFD)_V$ reduce to the programs studied in Ying [12].

(ii) If $k = 1$, $g_1(x, y) = 1$, $H_1(x, y, p_1) = \frac{1}{2} p_1^T \nabla_{yy} f_1(x, y) p_1$, $\Phi_1(u, v, q_1) = \frac{1}{2} q_1^T \nabla_{xx} f_1(u, v) q_1$, $g_1(u, v) = 1, F_1 = \{0\}$, $E_1 = \{0\}$, then, $(MFP)_U$ and $(MFD)_V$ reduce to the problems considered by Hou and Yang [5].

(iii) If $g_i(x,y)=1,\ E_i=\{0\},\ g_i(u,v)=1,\ F_i=\{0\}$ for all $i\in\{1,2,...,k\}$, then, $(MFP)_U$ and $(MFD)_V$ becomes the problems considered by Chen [1].

(iv) If $g_i(x,y)=1,\ E_i=\{0\},\ F_i=\{0\},\ H_i(x,y,p_i)=\frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i$, $\Phi_i(u,v,q_i)=\frac{1}{2}q_i^T\nabla_{xx}f_i(u,v)q_i$, for all $i\in\{1,2,...,k\}$ in $(MFP)_U$ and $(MFD)_V$, then, the problems reduce to the problems considered by Yang et al.[11].

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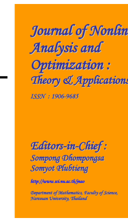

ON THE CONVERGENCE OF AN ITERATION PROCESS FOR TOTALLY ASYMPTOTICALLY I-NONEXPANSIVE MAPPINGS

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ABSTRACT. Suppose that K be a nonempty closed convex subset of a real Banach space X and $T, S : K \to K$ be two totally asymptotically I-nonexpansive mappings, where $I : K \to K$ is a totally asymptotically nonexpansive mapping. We define the iterative sequence $\{x_n\}$ by

$$\left\{ \begin{array}{c} x_0 \in K \\ x_{n+1} = (1-\alpha_n) x_n + \alpha_n S^n y_n \\ y_n = (1-\beta_n) x_n + \beta_n T^n z_n \\ z_n = (1-\gamma_n) x_n + \gamma_n I^n x_n \end{array} \right. , n \in \mathbb{N},$$

where $\{\alpha_n\}, \{\beta_n\}$ ve $\{\gamma_n\}$ are sequences in $[0,1]$ .Under some suitable conditions, the strong and weak convergence theorems of $\{x_n\}$ to a common fixed point of S,T and I are obtained.

KEYWORDS : Totally Asymptotically I-Nonexpansive Mapping; Total Asymptotically Nonexpansive Mapping; Opial Condition; $(A^{'})$ Condition; Strong and weak convergence; Common fixed point.
AMS Subject Classification: 47H09, 47J25

1. INTRODUCTION

Let K be a nonempty subset of a real normed space X and $T : K \to K$ be a mapping. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and denote by $F := F(T) \cap F(S) \cap F(I) = \{x \in K : Tx = Sx = Ix = x\}$, the set of common fixed points of the mappings S, T and I. A mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] as a generalization of the class of nonexpansive

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mappings. They proved that if K is a nonempty closed bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

A mapping $T : K \to K$ is called asymptotically nonexpansive in the intermediate sense if T is continuous and the following inequality holds:

$$\lim_{n\to\infty} \sup \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x-y\|) \leq 0.$$

Observe that if

$$a_n = \sup (\|T^n x - T^n y\| - \|x-y\|)$$

then last inequality reduces to the relation

$$\|T^n x - T^n y\| \leq \|x-y\| + a_n. \tag{1.2}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [2]. It is known [13] that if K is a nonempty closed convex bounded subset of a real uniformly convex Banach spaces E and T is a self-mapping of K which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point.

The class of asymptotically nonexpansive mappings is an important generalization of the class of nonexpansive mappings. Note that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

In 2006, Alber et al. [1] introduced a more general class of asymptotically nonexpansive mappings called total asymptotically nonexpansive mappings. Their aim is to unify various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and to prove a general convergence theorem applicable to all these classes of nonlinear mappings.

Definition 1.1. Let K be a nonempty closed subset of a real normed linear space X. A mapping $T : K \to K$ is called totally asymptotically nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x-y\| + \mu_n \phi(\|x-y\|) + \lambda_n, \text{for all } x, y \in K,\ n \geq 1. \tag{1.3}$$

Remark 1.2. If $\phi(\lambda) = \lambda$, then (1.3) reduces to $\|T^n x - T^n y\| \leq (1+\mu_n)\|x-y\| + \lambda_n, n \geq 1$. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. In addition, if $\mu_n = 0$ and $\lambda_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings becomes nonexpansive mappings. If $\mu_n = 0$ and $\lambda_n = \sigma_n = \max\{0, a_n\}$, where $a_n = \sup(\|T^n x - T^n y\| - \|x-y\|)$ for all $n \geq 1$, then (1.3) reduces to (1.2) which has been studied as mappings asymptotically nonexpansive in the intermediate sense.

The strong convergence theorems for iterative processes of finite family of total asymptotically nonexpansive mappings in Banach spaces have been studied by Chidume and Ofoedu [3, 4]. Also, Hu and Yang [11] obtained strong convergence theorems for three nonself total asymptotically nonexpansive mappings in real Banach spaces. In addition to these works, Hao [10] studied weak and strong convergence theorems in real Hilbert space.

On the other hand, in [17] an asymptotically I-nonexpansive mapping was introduced. Namely, let $T, I : K \to K$ be two mappings of a nonempty subset K of a real normed linear space X. Then T is said to be asymptotically

I-nonexpansive if there exists a sequence $\{\lambda_n\}$ with $\lim_{n\to\infty}\lambda_n = 1$ such that $\|T^n x - T^n y\| \le \lambda_n \|I^n x - I^n y\|$ for all $x, y \in K$ and $n \ge 1$.

Recently, Mukhamedov and Saburov [14] introduced the folowing iteration process for common fixed points of totally asymptotically I-nonexpansive mappings in Banach spaces. For arbitrarily chosen $x_0 \in K$, $\{x_n\}$ is define as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n T^n y_n \\ y_n = (1-\beta_n)\,x_n + \beta_n I^n x_n \end{cases}, \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. And they gave the following definition for totally asymptotically I-nonexpansive mappings.

Definition 1.3. Let $T, I : K \to K$ be two mappings of a nonempty subset K of a real normed linear space X. A mapping $T : K \to K$ is called totally asymptotically $I-$nonexpansive if there exist nonnegative real sequences $\{\mu_n\}$, $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \le \|I^n x - I^n y\| + \mu_n \phi\left(\|I^n x - I^n y\|\right) + \lambda_n, n \ge 1. \tag{1.5}$$

Note that (1.5) reduces to (1.3) when $I = Id$ (Id is the identity mapping). If $\phi(\lambda) = \lambda$, then one gets $\|T^n x - T^n y\| \le (1+\mu_n)\|I^n x - I^n y\| + \lambda_n$, which is a generalization of the asymptotically I-nonexpansive mapping. If $\phi(\lambda) = \lambda$ and $\lambda_n = 0$ for all $n \ge 1$, then total asymptotically I-nonexpansive mappings coincide with asymptotically I-nonexpansive mappings.

In this paper, we construct an explicit iterative sequence for the approximation of common fixed points of two total asymptotically I-nonexpansive mappings and a total asymptotically nonexpansive mapping in Banach spaces.

2. Preliminary

For the sake of convenience, we restate the following concepts.

A Banach space X is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$$

for all $y \in X$ with $y \ne x$, where $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. It is well know from that all Hilbert spaces and all l_p spaces for $1 < p < \infty$ have this property. However, the L_p spaces do not have this property unless $p = 2$.

Let K be a nonempty closed subset of a real Banach space X and $T : K \to K$ a mapping. The mapping T said to be demiclosed at zero if $Tx_0 = 0$ whenever $\{x_n\} \subset K, x_n \to x$ and $Tx_n \to 0$.

Recall that the mapping $T : K \to K$ with $F(T) \ne \emptyset$ is said to satisfy condition (A) [15] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \ge f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

We can modify the condition (A) for our case as follows:

Let $S, T : K \to K$ be two totally asymptotically $I-$nonexpansive mappings and $I : K \to K$ be totally asymptotically nonexpansive mapping. Then S, T and I are said to satisfy condition (A') if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$\frac{1}{3}\left(\|x - Sx\| + \|x - Tx\| + \|x - Ix\|\right) \ge f\left(d(x, F)\right)$$

for all $x \in K$. This condition is a generalization of condition of Khan and Fukharuddin [12] from two mappings to three mappings.

Next, we state the following useful lemmas.

Lemma 2.1. [18] *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1 + b_n) a_n + c_n, \ n \geq 1.$$

Suppose that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\{a_n\}$ is bounded and $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 2.2. [16] *Let X be a uniformly convex Banach space and let $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of X such that*

$$\limsup_{n\to\infty} \|x_n\| \leq d, \ \limsup_{n\to\infty} \|y_n\| \leq d \text{ and } \lim_{n\to\infty} \|(1 - t_n) x_n + t_n y_n\| = d$$

hold for some $d \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. Main Results

Now, we introduce an iteration process which can be viewed as an extension for totally asymptotically I-nonexpansive mappings of iteration process of Mukhamedov and Saburov [14].

Let K be a nonempty closed convex subset of a real Banach space X. Let $T, S : K \to K$ be two totally asymptotically I-nonexpansive mappings, where $I : K \to K$ is a totally asymptotically nonexpansive mapping. We define the iterative sequence $\{x_n\}$ by

$$\left\{ \begin{array}{c} x_0 \in K \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n S^n y_n \\ y_n = (1 - \beta_n) x_n + \beta_n T^n z_n \\ z_n = (1 - \gamma_n) x_n + \gamma_n I^n x_n \end{array} \right. , n \in \mathbb{N}, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are *sequences in* $[0, 1]$.

Remark 3.1. i. If $I = Id$ (Id is the identity mapping), then (3.1) reduces to the modified Ishikawa iterative scheme for two totally asymptotically nonexpansive mappings as follow.

$$\left\{ \begin{array}{c} x_0 \in K \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n S^n y_n \\ y_n = (1 - \beta_n) x_n + \beta_n T^n x_n \end{array} \right. , n \in \mathbb{N}, \tag{3.2}$$

where $I : K \to K$ is a totally asymptotically nonexpansive mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

ii. If $I = Id$ (Id is the identity mapping) and $\beta_n = 0$ for all $n \geq 1$, then (3.1) reduces to the modified Mann iterative scheme for one total asymptotically nonexpansive mappings as follow.

$$\left\{ \begin{array}{c} x_0 \in K \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n S^n x_n \end{array} \right. , n \in \mathbb{N}, \tag{3.3}$$

where $I : K \to K$ is a totally asymptotically nonexpansive mapping and $\{\alpha_n\}$ is *sequence in* $[0, 1]$.

iii. We remark that the iteration process (3.1) is more general than the iteration process (1.4), (3.2) and (3.3), and includes the process (3.2) and (3.3) as special cases.

Lemma 3.2. *Let K be a nonempty subset of a real Banach space X. Let $T, S : K \to K$ be two totally asymptotically I-nonexpansive mappings and let $I : K \to K$ be a totally asymptotically nonexpansive mapping. Then there exist nonnegative real sequences $\{\mu_n\}, \{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\begin{aligned} \|I^n x - I^n y\| &\leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n, \\ \|T^n x - T^n y\| &\leq \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n, \\ \|S^n x - S^n y\| &\leq \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n,\ n \geq 1. \end{aligned} \tag{3.4}$$

Proof. Since $T, S : K \to K$ are totally asymptotically I-nonexpansive mappings and $I : K \to K$ is a totally asymptotically nonexpansive mapping, then there exist nonnegative real sequences $\{\mu_n^{'}\}, \{\lambda_n^{'}\}, \{\mu_n^{''}\}, \{\lambda_n^{''}\}$ and $\{\tilde{\mu}_n\}, \{\tilde{\lambda}_n\}, n \geq 1$ with $\mu_n^{'}, \lambda_n^{'}, \mu_n^{''}, \lambda_n^{''}, \tilde{\mu}_n, \tilde{\lambda}_n \to 0$ $(n \to \infty)$ and there exist strictly increasing continuous functions $\phi_1, \phi_2, \phi_3 : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_1(0) = \phi_2(0) = \phi_3(0) = 0$ such that for all $x, y \in K$,

$$\|I^n x - I^n y\| \leq \|x - y\| + \tilde{\mu}_n \phi_1(\|x - y\|) + \tilde{\lambda}_n,$$

$$\|T^n x - T^n y\| \leq \|I^n x - I^n y\| + \mu_n^{'} \phi_2(\|I^n x - I^n y\|) + \lambda_n^{'},$$

$$\|S^n x - S^n y\| \leq \|I^n x - I^n y\| + \mu_n^{''} \phi_3(\|I^n x - I^n y\|) + \lambda_n^{''}, n \geq 1.$$

Setting

$$\begin{aligned} \mu_n &= \max\{\mu_n^{'}, \mu_n^{''}, \tilde{\mu}\}, \lambda_n = \max\{\lambda_n^{'}, \lambda_n^{''}, \tilde{\lambda}_n\}, \\ \phi(a) &= \max\{\phi_1(a), \phi_2(a), \phi_3(a)\} \text{ for } a \geq 0, \end{aligned}$$

then we obtain that there exist nonnegative real sequences $\{\mu_n\}$ and $\{\lambda_n\}$ with $\mu_n, \lambda_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|I^n x - I^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \lambda_n,$$

$$\|T^n x - T^n y\| \leq \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n,$$

$$\|S^n x - S^n y\| \leq \|I^n x - I^n y\| + \mu_n \phi(\|I^n x - I^n y\|) + \lambda_n, n \geq 1.$$

This completes the proof. □

Before proving our main results, throughout this paper we take mappings S, T and I as in (3.4).

Lemma 3.3. *Let K be a nonempty closed convex subset of a real Banach space X. Let $T, S : K \to K$ be two totally asymptotically I-nonexpansive mappings and $I : K \to K$ be a totally asymptotically nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for all $\lambda \geq M$. Let $\{x_n\}$ be the sequence as defined (3.1). If $F = F(S) \cap F(T) \cap F(I) \neq \emptyset$, then $\{x_n\}$ is bounded and $\lim_{n \to \infty} \|x_n - p\|$ exists for each $p \in F = F(S) \cap F(T) \cap F(I)$.*

Proof. Let $p \in F$. It follows from (3.1) and (3.4) that

$$\begin{aligned} \|z_n - p\| &= \|(1 - \gamma_n)(x_n - p) + \gamma_n (I^n x_n - p)\| \\ &\leq (1 - \gamma_n) \|x_n - p\| + \gamma_n \|I^n x_n - p\| \\ &\leq (1 - \gamma_n) \|x_n - p\| + \gamma_n [\|x_n - p\| + \mu_n \phi(\|x_n - p\|) + \lambda_n] \\ &\leq \|x_n - p\| + \gamma_n \mu_n \phi(\|x_n - p\|) + \lambda_n. \end{aligned} \tag{3.5}$$

Note that ϕ is an increasing function, it follows that $\phi(\lambda) \leq \phi(M)$ whenever $\lambda \leq M$ and (by hypothesis) $\phi(\lambda) \leq M^* \lambda$ if $\lambda \geq M$. In either case, we have

$$\phi(\lambda) \leq \phi(M) + M^* \lambda \tag{3.6}$$

for some $M, M^* \geq 0$. Thus, from (3.5) and (3.6), we have

$$\begin{aligned} \|z_n - p\| &\leq \|x_n - p\| + \gamma_n \mu_n [\phi(M) + M^* \|x_n - p\|] + \lambda_n \\ &\leq (1 + M^* \mu_n) \|x_n - p\| + Q_1 (\mu_n + \lambda_n) \end{aligned} \tag{3.7}$$

for some constant $Q_1 > 0$. Similarly, from (3.7) we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)(x_n - p) + \beta_n (T^n z_n - p)\| \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n \|T^n z_n - p\| \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n [\|I^n z_n - p\| + \mu_n \phi(\|I^n z_n - p\|) + \lambda_n] \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n [\|I^n z_n - p\| + \mu_n \phi(M) \\ &\quad + \mu_n M^* \|I^n z_n - p\| + \lambda_n] \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n [(1 + \mu_n M^*) \|I^n z_n - p\| + \mu_n \phi(M) + \lambda_n] \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n [(1 + \mu_n M^*) (\|z_n - p\| \\ &\quad + \mu_n \phi(\|z_n - p\|) + \lambda_n) + \mu_n \phi(M) + \lambda_n] \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n [(1 + \mu_n M^*) (\|z_n - p\| + \mu_n \phi(M) \\ &\quad + \mu_n M^* \|z_n - p\| + \lambda_n) + \mu_n \phi(M) + \lambda_n] \\ &\leq (1-\beta_n) \|x_n - p\| + \beta_n [(1 + \mu_n M^*) ((1 + \mu_n M^*) \|z_n - p\| \\ &\quad + \mu_n \phi(M) + \lambda_n) + \mu_n \phi(M) + \lambda_n] \\ &\leq (1-\beta_n) \|x_n - p\| \\ &\quad + \beta_n [(1 + \mu_n M^*) ((1 + \mu_n M^*) (1 + M^* \mu_n) \|x_n - p\| \\ &\quad + (1 + M^* \mu_n) Q_1 (\mu_n + \lambda_n) + \mu_n \phi(M) + \lambda_n) + \mu_n \phi(M) + \lambda_n] \\ &\leq \|x_n - p\| + (3 (\mu_n M^*)^2 + 3 \mu_n M^* + (\mu_n M^*)^3) \|x_n - p\| \\ &\quad + (1 + M^* \mu_n)^2 Q_1 (\mu_n + \lambda_n) + (1 + M^* \mu_n) (\mu_n \phi(M) + \lambda_n) \\ &\quad + \mu_n \phi(M) + \lambda_n \\ &\leq (1 + M_2 \mu_n) \|x_n - p\| + Q_2 (\mu_n + \lambda_n) \end{aligned} \tag{3.8}$$

for some constants $M_2, Q_2 > 0$. Hence, it follows from (3.6) and (3.8) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)(x_n - p) + \alpha_n (S^n y_n - p)\| \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n \|S^n y_n - p\| \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n [\|I^n y_n - p\| + \mu_n \phi(\|I^n y_n - p\|) + \lambda_n] \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n [\|I^n y_n - p\| + \mu_n \phi(M) \\ &\quad + \mu_n M^* \|I^n y_n - p\| + \lambda_n] \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n [(1 + \mu_n M^*) \|I^n y_n - p\| + \mu_n \phi(M) + \lambda_n] \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n [(1 + \mu_n M^*) (\|y_n - p\| \\ &\quad + \mu_n \phi(\|y_n - p\|) + \lambda_n) + \mu_n \phi(M) + \lambda_n] \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n [(1 + \mu_n M^*) (\|y_n - p\| + \mu_n \phi(M) \\ &\quad + \mu_n M^* \|y_n - p\| + \lambda_n) + \mu_n \phi(M) + \lambda_n] \\ &\leq (1-\alpha_n) \|x_n - p\| + \alpha_n [(1 + \mu_n M^*) ((1 + \mu_n M^*) \|y_n - p\| \\ &\quad + \mu_n \phi(M) + \lambda_n) + \mu_n \phi(M) + \lambda_n] \\ &\leq (1-\alpha_n) \|x_n - p\| \end{aligned}$$

$$
\begin{aligned}
& +\alpha_n[(1+\mu_n M^*)\left((1+\mu_n M^*)\left(1+M_2\mu_n\right)\|x_n-p\|\right. \\
& \left.+\left(1+M^*\mu_n\right)Q_2\left(\mu_n+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n] \\
\leq\ & \left(1+\left(M_2+2M^*\right)\mu_n+\left(2M^*M_2+(M^*)^2\right)\mu_n^2\right. \\
& \left.+M_2(M^*)^2\mu_n^3\right)\|x_n-p\|+M_2(M^*)^2\mu_n^3\|x_n-p\| \\
& +\left(1+M^*\mu_n\right)^2Q_2\left(\mu_n+\lambda_n\right) \\
& +\left(1+M^*\mu_n\right)\left(\mu_n\phi\left(M\right)+\lambda_n\right)+\mu_n\phi\left(M\right)+\lambda_n \\
\leq\ & \left(1+M_3\mu_n\right)\|x_n-p\|+Q_3\left(\mu_n+\lambda_n\right) \qquad (3.9)
\end{aligned}
$$

for some constants $M_3, Q_3 > 0$. Since $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$, by Lemma 2.1, we get $\lim_{n\to\infty}\|x_n-p\|$ exists. This completes the proof. ☐

Theorem 3.1. *Let K be a nonempty closed subset of a real Banach space X. Let $T, S : K \to K$ be two continuous totally asymptotically I-nonexpansive mappings and $I : K \to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Let $\{x_n\}$ be the sequence as defined (3.1) and $F = F(S)\cap F(T)\cap F(I) \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of S, T and I if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x_n, F) = \inf_{p\in F}\|x_n-p\|, n \geq 1$.*

Proof. The necessity of the conditions is obvious. Thus, we need only prove the sufficiency. Since S, T and I are continuous mappings, $F(T), F(S)$ and $F(I)$ are closed. Hence $F = F(S)\cap F(T)\cap F(I)$ is a nonempty closed set.

For any given $p \in F$, we have from (3.9)

$$\|x_{n+1}-p\| \leq \left(1+M_3\mu_n\right)\|x_n-p\|+Q_3\left(\mu_n+\lambda_n\right)$$

which gives

$$d\left(x_{n+1}, F\right) \leq \left(1+M_3\mu_n\right)d\left(x_n, F\right)+Q_3\left(\mu_n+\lambda_n\right). \qquad (3.10)$$

Now applying Lemma 2.1 to (3.10), we get the existence of the limit $\lim_{n\to\infty} d\left(x_n, F\right)$.

But by hypothesis $\liminf_{n\to\infty} d\left(x_n, F\right) = 0$, thus we have $\lim_{n\to\infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence in X. As $1+t \leq \exp\left(t\right)$ for all $t > 0$, from (3.9), we obtain

$$\|x_{n+1}-p\| \leq \exp\left(M_3\mu_n\right)\left(\|x_n-p\|+Q_3\left(\mu_n+\lambda_n\right)\right). \qquad (3.11)$$

Thus, for any given m, n, iterating (3.11), we obtain

$$
\begin{aligned}
\|x_{n+m}-p\| \leq\ & \exp\left(M_3\mu_{n+m-1}\right)\left(\|x_{n+m-1}-p\|+Q_3\left(\mu_{n+m-1}+\lambda_{n+m-1}\right)\right) \\
& \vdots \\
\leq\ & \exp\left(\sum_{i=n}^{n+m-1}M_3\mu_i\right)\left(\|x_n-p\|+\sum_{i=n}^{n+m-1}Q_3\left(\mu_i+\lambda_i\right)\right) \\
\leq\ & \exp\left(\sum_{i=n}^{\infty}M_3\mu_i\right)\left(\|x_n-p\|+\sum_{i=n}^{\infty}Q_3\left(\mu_i+\lambda_i\right)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x_{n+m}-x_n\| \leq\ & \|x_{n+m}-p\|+\|x_n-p\| \\
\leq\ & \left[1+\left(\exp\left(\sum_{i=n}^{\infty}M_3\mu_i\right)\right)\right]\|x_n-p\|
\end{aligned}
$$

$$+\exp(\sum_{i=n}^{\infty} M_3\mu_i)\left(\sum_{i=n}^{\infty} Q_3\left(\mu_i+\lambda_i\right)\right).$$

This imply that

$$\|x_{n+m}-x_n\| \leq D\|x_n-p\| + D\left(\sum_{i=n}^{\infty}\left(\mu_i+\lambda_i\right)\right) \tag{3.12}$$

for same constant $D>0$. Taking infimum over $p\in F$ in (3.12) gives

$$\|x_{n+m}-x_n\| \leq Dd\left(x_n,F\right) + D\left(\sum_{i=n}^{\infty}\left(\mu_i+\lambda_i\right)\right).$$

Now, since $\lim_{n\to\infty} d\left(x_n,F\right)=0$ and $\sum_{i=n}^{\infty}\left(\mu_i+\lambda_i\right)<\infty$, given $\epsilon>0$, there exists an integer $N_1>0$ such that for all $n\geq N_1$, $d\left(x_n,F\right)<\epsilon/2D$ and $\sum_{i=n}^{\infty}\left(\mu_i+\lambda_i\right)<\epsilon/2D$. Consequently, from last inequality we have $\|x_{n+m}-x_n\|<\epsilon$, which means that $\{x_n\}$ is a Cauchy sequence in X, and completeness of X yield the existence of $x^*\in X$ such that $x_n\to x^*$. We now show that $x^*\in F$. Suppose that $x^*\notin F$. Since F closed subset of X, we have that $d\left(x^*,F\right)>0$. But, for all $p\in F$, we have

$$\|x^*-p\| \leq \|x^*-x_n\| + \|x_n-p\|.$$

This implies

$$d\left(x^*,F\right) \leq \|x^*-x_n\| + d\left(x_n,F\right),$$

so we obtain $d\left(x^*,F\right)=0$ as $n\to\infty$, which contradicts $d\left(x^*,F\right)>0$. Hence, x^* is a common fixed point of T,S and I. This completes the proof. □

Lemma 3.4. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T,S:K\to K$ be two continuous totally asymptotically I-nonexpansive mappings and $I:K\to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\},\{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty}\mu_n<\infty, \sum_{n=1}^{\infty}\lambda_n<\infty$. Assume that there exist $M,M^*>0$ such that $\phi(\lambda)\leq M^*\lambda$ for all $\lambda\geq M$ and $F=F(S)\cap F(T)\cap F(I)\neq\emptyset$. Let $\{x_n\}$ be the sequence as defined (3.1), where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a,1-a]$ for some $a\in(0,1)$. Then*

$$\lim_{n\to\infty}\|x_n-S^nx_n\| = \lim_{n\to\infty}\|x_n-T^nx_n\| = \lim_{n\to\infty}\|x_n-I^nx_n\| = 0$$

and

$$\lim_{n\to\infty}\|x_n-Sx_n\| = \lim_{n\to\infty}\|x_n-Tx_n\| = \lim_{n\to\infty}\|x_n-Ix_n\| = 0.$$

Proof. Let $p\in F$. Then by Lemma 3.3, we have

$$\lim_{n\to\infty}\|x_n-p\| = d. \tag{3.13}$$

It follows from (3.1) that

$$\|x_{n+1}-p\| = \|(1-\alpha_n)(x_n-p)+\alpha_n(S^ny_n-p)\| \to d,\ (n\to\infty). \tag{3.14}$$

From (3.4) and (3.6), we obtain

$$\begin{aligned}
\|S^ny_n-p\| &\leq \|I^ny_n-p\| + \mu_n\phi\left(\|I^ny_n-p\|\right) + \lambda_n \\
&\leq \|I^ny_n-p\| + \mu_n\phi\left(M\right) + \mu_nM^*\|I^ny_n-p\| + \lambda_n \\
&\leq \left(1+\mu_nM^*\right)\|I^ny_n-p\| + \mu_n\phi\left(M\right) + \lambda_n \\
&\leq \left(1+\mu_nM^*\right)\left(\|y_n-p\| + \mu_n\phi\left(\|y_n-p\|\right) + \lambda_n\right) + \mu_n\phi\left(M\right) + \lambda_n \\
&\leq \left(1+\mu_nM^*\right)\left(\|y_n-p\| + \mu_n\phi\left(M\right) + \mu_nM^*\|y_n-p\| + \lambda_n\right) \\
&\quad + \mu_n\phi\left(M\right) + \lambda_n
\end{aligned}$$

$$\begin{aligned} &\leq (1+\mu_n M^*)\left((1+\mu_n M^*)\|y_n - p\| + \mu_n\phi(M) + \lambda_n\right) \\ &\quad +\mu_n\phi(M) + \lambda_n. \end{aligned} \tag{3.15}$$

Because of $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$, we have

$$\limsup_{n\to\infty}\|S^n y_n - p\| \leq \limsup_{n\to\infty}\|y_n - p\|$$

and from (3.8)

$$\limsup_{n\to\infty}\|S^n y_n - p\| \leq \limsup_{n\to\infty}(1+M_2\mu_n)\|x_n - p\| + Q_2(\mu_n + \lambda_n) = d. \tag{3.16}$$

Using (3.13),(3.16) and applying Lemma 2.2 to (3.14), we get

$$\lim_{n\to\infty}\|x_n - S^n y_n\| = 0. \tag{3.17}$$

Also from (3.15), we have

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - S^n y_n\| + \|S^n y_n - p\| \\ &\leq \|x_n - S^n y_n\| + (1+\mu_n M^*)\left((1+\mu_n M^*)\|y_n - p\|\right. \\ &\quad \left.+\mu_n\phi(M) + \lambda_n\right) + \mu_n\phi(M) + \lambda_n \end{aligned}$$

which implies $d \leq \lim_{n\to\infty}\|y_n - p\|$ and from (3.8), we have $\lim_{n\to\infty}\|y_n - p\| \leq d$. Thus we obtain

$$\lim_{n\to\infty}\|y_n - p\| = d. \tag{3.18}$$

On the other hand note that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\alpha_n\|x_n - S^n y_n\| = 0. \tag{3.19}$$

It follows from (3.18) that

$$\|y_n - p\| = \|(1-\beta_n)(x_n - p) + \beta_n(T^n z_n - p)\| \to d,\ n\to\infty. \tag{3.20}$$

From (3.4) and (3.6), we obtain

$$\begin{aligned} \|T^n z_n - p\| &\leq \|I^n z_n - p\| + \mu_n\phi(\|I^n z_n - p\|) + \lambda_n \\ &\leq \|I^n z_n - p\| + \mu_n\phi(M) + \mu_n M^*\|I^n z_n - p\| + \lambda_n \\ &\leq (1+\mu_n M^*)\|I^n z_n - p\| + \mu_n\phi(M) + \lambda_n \\ &\leq (1+\mu_n M^*)\left(\|z_n - p\| + \mu_n\phi(\|z_n - p\|) + \lambda_n\right) + \mu_n\phi(M) + \lambda_n \\ &\leq (1+\mu_n M^*)\left(\|z_n - p\| + \mu_n\phi(M) + \mu_n M^*\|z_n - p\| + \lambda_n\right) \\ &\quad +\mu_n\phi(M) + \lambda_n \\ &\leq (1+\mu_n M^*)\left((1+\mu_n M^*)\|z_n - p\| + \mu_n\phi(M) + \lambda_n\right) \\ &\quad +\mu_n\phi(M) + \lambda_n. \end{aligned} \tag{3.21}$$

Therefore,

$$\limsup_{n\to\infty}\|T^n z_n - p\| \leq \limsup_{n\to\infty}\|z_n - p\|$$

and from (3.7), we have

$$\limsup_{n\to\infty}\|T^n z_n - p\| \leq \limsup_{n\to\infty}(1+M^*\mu_n)\|x_n - p\| + Q_1(\mu_n + \lambda_n) = d. \tag{3.22}$$

Using (3.13), (3.22) and applying Lemma 2.2 to (3.20), we get

$$\lim_{n\to\infty}\|x_n - T^n z_n\| = 0. \tag{3.23}$$

Also from (3.21), we have

$$\begin{aligned} \|x_n - p\| &\leq \|x_n - T^n z_n\| + \|T^n z_n - p\| \\ &\leq \|x_n - T^n z_n\| + (1+\mu_n M^*)\left((1+\mu_n M^*)\|z_n - p\|\right. \\ &\quad \left.+\mu_n\phi(M) + \lambda_n\right) + \mu_n\phi(M) + \lambda_n, \end{aligned}$$

which implies $d \leq \lim_{n\to\infty} \|z_n - p\|$ and from (3.7) we have $\lim_{n\to\infty} \|z_n - p\| \leq d$. This gives

$$\lim_{n\to\infty} \|z_n - p\| = d. \tag{3.24}$$

As above, from (3.1) one can see that

$$\|z_n - p\| = \|(1-\gamma_n)(x_n - p) + \gamma_n(I^n x_n - p)\| \to d, \tag{3.25}$$

as $n \to \infty$. Besides this, we have

$$\begin{aligned} \|I^n x_n - p\| &\leq \|x_n - p\| + \mu_n \phi(\|x_n - p\|) + \lambda_n \\ &\leq (1 + M^* \mu_n) \|x_n - p\| + \mu_n \phi(M) + \lambda_n. \end{aligned}$$

Taking limit as $n \to \infty$ in last inequality, we obtain

$$\limsup_{n\to\infty} \|I^n x_n - p\| \leq d. \tag{3.26}$$

Hence from (3.13), (3.26) and (3.25), we get

$$\lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{3.27}$$

Meanwhile, note that from (3.27)

$$\begin{aligned} \|z_n - x_n\| &= \|(1-\gamma_n)(x_n - p) + \gamma_n(I^n x_n - p) - x_n\| \\ &= \gamma_n \|x_n - I^n x_n\| = 0. \end{aligned} \tag{3.28}$$

and from (3.23)

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n z_n - p) - x_n\| \\ &= \beta_n \|x_n - T^n z_n\| = 0. \end{aligned} \tag{3.29}$$

Now we show that $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$. By way of (3.21), we have

$$\begin{aligned} \|T^n z_n - T^n x_n\| \leq\ & (1 + \mu_n M^*) ((1 + \mu_n M^*) \|z_n - x_n\| \\ & + \mu_n \phi(M) + \lambda_n) + \mu_n \phi(M) + \lambda_n. \end{aligned}$$

Thus we get

$$\begin{aligned} \|x_n - T^n x_n\| &\leq \|x_n - T^n z_n\| + \|T^n z_n - T^n x_n\| \\ &\leq \|x_n - T^n z_n\| + (1 + \mu_n M^*) ((1 + \mu_n M^*) \|z_n - x_n\| \\ &\quad + \mu_n \phi(M) + \lambda_n) + \mu_n \phi(M) + \lambda_n, \end{aligned}$$

which implies

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.30}$$

Now we show that $\lim_{n\to\infty} \|x_n - S^n x_n\| = 0$. This time, by way of (3.15), we have

$$\begin{aligned} \|S^n y_n - S^n x_n\| \leq\ & (1 + \mu_n M^*) ((1 + \mu_n M^*) \|y_n - x_n\| \\ & + \mu_n \phi(M) + \lambda_n) + \mu_n \phi(M) + \lambda_n. \end{aligned}$$

Thus we get

$$\begin{aligned} \|x_n - S^n x_n\| &\leq \|x_n - S^n y_n\| + \|S^n y_n - S^n x_n\| \\ &\leq \|x_n - S^n y_n\| + (1 + \mu_n M^*) ((1 + \mu_n M^*) \|y_n - x_n\| \\ &\quad + \mu_n \phi(M) + \lambda_n) + \mu_n \phi(M) + \lambda_n. \end{aligned}$$

Taking limit in last inequality, we obtain from (3.17) and (3.29)

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0. \tag{3.31}$$

Finally, we prove

$$\lim_{n\to\infty} \|x_n - Sx_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Ix_n\| = 0.$$

Firstly, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

from (3.19) and

$$\lim_{n\to\infty} \|x_n - I^n x_n\| = 0$$

from (3.27). Hence we get that

$$\begin{aligned}
\left\|x_n - I^{n-1}x_n\right\| &\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - I^{n-1}x_{n-1}\right\| \\
&\quad + \left\|I^{n-1}x_{n-1} - I^{n-1}x_n\right\| \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - I^{n-1}x_{n-1}\right\| + \|x_n - x_{n-1}\| \\
&\quad + \mu_n \phi\left(\|x_n - x_{n-1}\|\right) + \lambda_n \\
&\leq (2 + \mu_n M^*) \|x_n - x_{n-1}\| + \left\|x_{n-1} - I^{n-1}x_{n-1}\right\| \\
&\quad + \mu_n \phi\left(M\right) + \lambda_n \\
&\to 0. \qquad (3.32)
\end{aligned}$$

Because I is continuous, therefore we have from (3.32)

$$\|I^n x_n - I x_n\| \leq \left\|I\left(I^{n-1}x_n\right) - I x_n\right\| \to 0.$$

Thus we obtain

$$\|x_n - I x_n\| \leq \|x_n - I^n x_n\| + \|I^n x_n + I x_n\| \to 0.$$

Also we have $\lim_{n\to\infty} \|x_n - S^n x_n\| = 0$ from (3.31). Thus from (3.19) and (3.27), we get that

$$\begin{aligned}
\left\|x_n - S^{n-1}x_n\right\| &\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| \\
&\quad + \left\|S^{n-1}x_{n-1} - S^{n-1}x_n\right\| \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| + \left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\| \\
&\quad + \mu_n \phi\left(\left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\|\right) + \lambda_n \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| + \left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\| \\
&\quad + \mu_n \phi\left(M\right) + \mu_n M^* \left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\| + \lambda_n \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| \\
&\quad + (1 + \mu_n M^*) \left\|I^{n-1}x_n - I^{n-1}x_{n-1}\right\| + \mu_n \phi\left(M\right) + \lambda_n \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| \\
&\quad + (1 + \mu_n M^*)[\|x_n - x_{n-1}\| + \mu_n \phi\left(\|x_n - x_{n-1}\|\right) + \lambda_n] \\
&\quad + \mu_n \phi\left(M\right) + \lambda_n \\
&\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - S^{n-1}x_{n-1}\right\| \\
&\quad + (1 + \mu_n M^*)[\|x_n - x_{n-1}\| + \mu_n \phi\left(M\right) \\
&\quad + \mu_n M^* \|x_n - x_{n-1}\| + \lambda_n] + \mu_n \phi\left(M\right) + \lambda_n \\
&\to 0. \qquad (3.33)
\end{aligned}$$

Because S is continuous, therefore we have from (3.33)

$$\|S^n x_n - S x_n\| \leq \left\|S\left(S^{n-1}x_n\right) - S x_n\right\| \to 0.$$

Thus we obtain

$$\|x_n - S x_n\| \leq \|x_n - S^n x_n\| + \|S^n x_n - S x_n\| \to 0.$$

Similarly, in the same way we get

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

This completes the proof. □

Using this lemma, we prove the following strong convergence theorems.

Theorem 3.2. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T, S : K \to K$ be two continuous totally asymptotically I-nonexpansive mappings and $I : K \to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$ and $F = F(S) \cap F(T) \cap F(I) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined (3.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If one of S, T and I is compact, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S, T and I.*

Proof. Without loss of generality, let S be compact. Then there exists a subsequence $\{S^{n_k} x_{n_k}\}$ of $\{S^n x_n\}$ such that $\{S^{n_k} x_{n_k}\}$ converges strongly to $x^* \in K$. Thus, from (3.31), $\{x_{n_k}\}$ converges strongly to x^*. Using continuity of S we have $\{S^{n_k+1} x_{n_k+1}\}$ converges strongly to Sx^*. On the other hand, according to (3.30) and (3.27), we get that $\{T^{n_k} x_{n_k}\}$, $\{I^{n_k} x_{n_k}\}$ converge strongly to x^*. Since T and I are continuous, the squences $\{T^{n_k+1} x_{n_k}\}$ and $\{I^{n_k+1} x_{n_k}\}$ converge strongly to Tx^* and Ix^*, respectively. Besides these, since $\|x_{n+1} - x_n\|$ converges to 0; definition of S, T and I imply that $\left\|S^{n_k+1} x_{n+1} - S^{n_k+1} x_n\right\|, \left\|T^{n_k+1} x_{n+1} - T^{n_k+1} x_n\right\|$ and $\left\|I^{n_k+1} x_{n+1} - I^{n_k+1} x_n\right\|$ converge to 0. Observe that

$$\begin{aligned}
\|x^* - Sx^*\| \quad &\leq \quad \|x^* - x_{n_k+1}\| + \left\|x_{n_k+1} - S^{n_k+1} x_{n_k+1}\right\| \\
&\quad + \left\|S^{n_k+1} x_{n_k+1} - S^{n_k+1} x_{n_k}\right\| + \left\|S^{n_k+1} x_{n_k+1} - S^{n_k+1} x_{n_k}\right\| \\
&\quad + \left\|S^{n_k+1} x_{n_k} - Sx^*\right\|, \\
\|x^* - Tx^*\| \quad &\leq \quad \|x^* - x_{n_k+1}\| + \left\|x_{n_k+1} - T^{n_k+1} x_{n_k+1}\right\| \\
&\quad + \left\|T^{n_k+1} x_{n_k+1} - T^{n_k+1} x_{n_k}\right\| + \left\|T^{n_k+1} x_{n_k+1} - T^{n_k+1} x_{n_k}\right\| \\
&\quad + \left\|T^{n_k+1} x_{n_k} - Tx^*\right\|
\end{aligned}$$

and

$$\begin{aligned}
\|x^* - Ix^*\| \quad &\leq \quad \|x^* - x_{n_k+1}\| + \left\|x_{n_k+1} - I^{n_k+1} x_{n_k+1}\right\| \\
&\quad + \left\|I^{n_k+1} x_{n_k+1} - I^{n_k+1} x_{n_k}\right\| + \left\|I^{n_k+1} x_{n_k+1} - I^{n_k+1} x_{n_k}\right\| \\
&\quad + \left\|I^{n_k+1} x_{n_k} - Ix^*\right\|.
\end{aligned}$$

Taking limit as $k \to \infty$ in the last inequalities, we find $x^* = Sx^*, x^* = Tx^*$ and $x^* = Ix^*$. So $x^* \in F$. However, due to Lemma 2.1, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists, $p \in F$. Hence, $\{x_n\}$ converges strongly to $x^* \in F$. This completes the proof. □

Theorem 3.3. *Let K be a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T, S : K \to K$ be two continuous totally asymptotically I-nonexpansive mappings, let $I : K \to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\}, \{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \lambda_n < \infty$. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$ and $F = F(S) \cap F(T) \cap F(I) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined (3.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If S, T and I satisfy Condition $\left(A'\right)$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S, T and I.*

Proof. By Lemma 3.4,

$$\lim_{n\to\infty}\|x_n - Sx_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - Ix_n\| = 0.$$

Using Condition $\left(A^{'}\right)$, we get

$$\lim_{n\to\infty} f\left(d\left(x,F\right)\right) = \lim_{n\to\infty}\frac{1}{3}\left(\|x - Sx\| + \|x - Tx\| + \|x - Ix\|\right) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, so it follows that

$$\lim_{n\to\infty} d\left(x,F\right) = 0.$$

Now applying the Theorem 3.1, we obtain the result. □

Our weak convergence theorem is as follows:

Theorem 3.4. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. Let $T,S : K \to K$ be two continuous totally asymptotically I-nonexpansive mappings, let $I : K \to K$ be a continuous totally asymptotically nonexpansive mapping with sequences $\{\mu_n\},\{\lambda_n\}$ defined by (3.4) such that $\sum_{n=1}^{\infty}\mu_n < \infty, \sum_{n=1}^{\infty}\lambda_n < \infty$. Let $E : X \to X$ be an identity mapping. Assume that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$ and $F = F(S)\cap F(T)\cap F(I) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined (3.1), where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If $E - S, E - T$ and $E - I$ are demiclosed at zero then the sequence $\{x_n\}$ converges weakly to a common fixed point of S, T and I.*

Proof. Let $p \in F$. Then by Lemma 3.3, $\lim_{n\to\infty}\|x_n - p\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. Since $\{x_n\}$ is a bounded sequence in a uniformly convex Banach space X, there exist two weakly convergent subsequences $\{x_{n_i}\}$ and $\left\{x_{n_j}\right\}$ of $\{x_n\}$. Let $w_1 \in K$ and $w_2 \in K$ be weak limits of the subsequences $\{x_{n_i}\}$ and $\left\{x_{n_j}\right\}$, respectively. Since S is demiclosed with respect to zero (by hypothesis) then we obtain $Sw_1 = w_1$. Similarly, $Tw_1 = w_1$ and $Iw_1 = w_1$. That is, $w_1 \in F$. In the same way, we can prove that $w_2 \in F$.

Next, we prove the uniqueness. For this, suppose that $w_1 \neq w_2$. Then, by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - w_1\| &= \lim_{i\to\infty}\|x_{n_i} - w_1\| \\
&< \lim_{i\to\infty}\|x_{n_i} - w_2\| \\
&= \lim_{n\to\infty}\|x_n - w_2\| \\
&= \lim_{j\to\infty}\left\|x_{n_j} - w_2\right\| \\
&< \lim_{j\to\infty}\left\|x_{n_j} - w_1\right\| \\
&= \lim_{n\to\infty}\|x_n - w_1\|,
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a point of F. □

Remark 3.5. Since the class of totally asymptotically I-nonexpansive mappings includes totally asymptotically nonexpansive mappings, our results improve and extend the corresponding ones announced by Ya.I. Alber et al. [1], Mukhamedov and Saburov [14], Chidume and Ofoedu [4, 3] and Gunduz [6]. Also our results genaralize corresponding results given in [7, 8, 9, 19, 20].

Remark 3.6. The iteration process (3.1) can be generalized for two finite families of totally asymptotically I_i-nonexpansive mappings $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$,where $\{I_j : j \in J\}$ is a finite family of totally asymptotically nonexpansive mappings (where $J = \{1, 2, ..., N\}$).

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STRONG CONVERGENCE OF FINITE FAMILY OF PSEUDOCONTRACTIVE MAPPINGS BY A NEW IMPLICIT ITERATION

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ABSTRACT. In this paper, we propose Mann-Kirk type implicit iteration for a finite family of pseudocontractive mappings, and prove strong convergence of proposed iteration to a common fixed point in Banach spaces. The results in the paper extend and generalize well known corresponding results.

KEYWORDS : Mann iteration; Kirk iteration; implicit iteration; pseudocontractive mapping; common fixed point.
AMS Subject Classification: 47H09,47H10.

1. INTRODUCTION

Let E be a real Banach space, K be a closed convex subset of E and let J denote the normalized duality pairing from E into 2^{E^*} given by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|\,\|f\|\,, \|x\| = \|f\|\,\}\ , \ \ \forall x \in E\,,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote elements in Jx by $j(x)$ and define $Fix(T) = \{x \in E : Tx = x\}$ to be the fixed point set of a mapping T. When $\{x_n\}$ is a sequence in E, then $x_n \longrightarrow x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x.

Let T be a mapping with domain $D(T)$ and range $R(T)$ in E. Then T is called

- Nonexpansive, if for any $x, y \in D(T)$
$$\|Tx - Ty\| \le \|x - y\|\ .$$

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- Accretive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \geq 0 \,.$$
- Pseudocontractive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 \,.$$
- Hemicontractive, if for any $x \in D(T)$ and $x^* \in Fix(T)$, there exists $j(x-x^*) \in J(x-x^*)$ such that
$$\langle Tx - x^*, j(x-x^*)\rangle \leq \|x-x^*\|^2 \,.$$
- Strongly pseudocontractive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \alpha \|x-y\|^2 \,, \quad \text{for some } 0 < \alpha < 1 \,.$$
- Strictly pseudocontractive, if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda \|(x-y) - (Tx-Ty)\|^2 \,,$$
for some $0 < \lambda < 1$.

The class of pseudocontractive mappings has close relations with the class of nonexpansive mappings and the class of accretive mappings. It is easy to see that if T is a pseudocontractive mapping, then $I - T$ is accretive.
If we define $A = (2I - T)^{-1}$, then $Fix(A) = Fix(T)$ and we have the following result :

Lemma 1.1 (Martin[18]). *A is a nonexpansive self mapping on* K.

Regarding iterative approximation of fixed points of nonexpansive mappings, it is well known that Picard (successive) iteration may fail to produce a norm convergence sequence $\{x_n\}$ for nonexpansive mappings. Thus when a fixed point of nonexpansive mappings exists, other approximation techniques are needed to approximate it. One such technique is to form a mapping
$$S_\lambda = \lambda I + (1-\lambda)T \quad (0 < \lambda < 1) \,,$$
and then show that under certain circumstances the Picard iterates of S_λ converges to a fixed point of T. The first such result was obtained by Krasnoselskii [14] in a uniformly convex Banach space for $\lambda = \frac{1}{2}$. Schaefer [22] noted that this result holds for arbitrary $\lambda \in (0, 1)$. Edelstein [9] proved corresponding result in strictly convex Banach spaces.

Kirk [13] defined a more general mapping than those of S_λ. Let K be a closed convex subset of a Banach space, and T be a nonexpansive mapping of K into itself. Define the mapping $S : K \longrightarrow K$ by
$$S = \lambda_0 I + \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_k T^k \,,$$
where $\lambda_i \geq 0$, $\lambda_1 > 0$ and $\sum_{i=1}^{k} \lambda_i = 1$.
He proved that for arbitrary $x_0 \in K$, the sequence $\{S^n x_0\}$ converges weakly to a fixed point of T in K.

Maiti and Saha [16] extended this result of Kirk and proved the strong convergence of the sequence $\{S^n x_0\}$.

Liu et al. [15] extended Kirk's idea to finite family of nonexpansive mappings.

Let $T_i : K \longrightarrow K \ \ (i = 1, 2, \dots, k)$ be nonexpansive mappings and let

$$S = \lambda_0 I + \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k \,,$$

where $\lambda_i \geq 0$, $\lambda_0 > 0, \lambda_1 > 0$, and $\sum_{i=1}^{k} \lambda_i = 1$.
They proved that for arbitrary $x_0 \in K$, the sequence $\{S^n x_0\}$ converges to a common fixed point of T_i in K.

Iterative methods for nonexpansive mappings have been extensively investigated by many researchers, see ([2, 4, 5, 10, 11, 12, 19, 26]) and references there in.

The most popular iterative scheme to approximate a fixed point of a nonexpansive mapping is the following:

$$x_0 \in K \,;\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n \,,\ \ n \geq 0 \,, \tag{1.1}$$

where $\{\alpha_n\} \subset (0, 1)$ is a real sequence satisfying appropriate conditions.
Iteration process (1.1) is known as a Mann iteration [17].

Iterative method to approximate a fixed point of a pseudocontractive mapping was initiated by Browder and Petryshyn [3] in 1965, but iterative methods for pseudocontractive mappings are far less developed than those for nonexpansive mappings. In conncection with the iterative approximation of fixed points of pseudocontractive mappings, the following question is still open [6]:

> Does the Mann iteration process always converge for continuous pseudocontractive mapping? or for even Lipschitz pseudocontractive mappings?

Chidume and Mutangadura [7], negatively resolved the above problem by providing an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which a Mann iteration process does not converge in a convex compact subset of a Hilbert space.

Rafiq [21], proposed a Mann type implicit iteration process for hemicontractive mapping T defined by:

$$x_0 \in K \,;\ x_n = (1 - \alpha_n) x_{n-1} + \alpha_n T x_n \,,\ \ n \geq 0 \tag{1.2}$$

where $\{\alpha_n\}$ is a real sequence such that $\alpha_n \in [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$.

Song [24], discovered that the iteration (1.2) for hemicontractive mappings is not well defined. He observed that for an initial point $x_0 \in K$, x_1 is defined by the equation

$$x_1 = \alpha_1 x_0 + (1 - \alpha_n) T x_1 \,,$$

but the existence of x_1 is not established, because for hemicontractive mapping T, we do not know whether mapping $S_1 = \alpha_1 x_0 + (1 - \alpha_1) T$ has a fixed point $x_1 \in K$. Similarly the existence of $x_2, x_3, \dots, x_n$ is also doubtful, but the iteration (1.2) is well defined if we consider continuous pseudocontractive mappings.

Recently, Acedo and Xu [1] defined an iteration scheme in a Hilbert space for finite family of strict pseudocontractive mappings, where the sequence $\{x_n\}$ is generated by the algorithm :

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n \,,$$

under appropriate assumptions on the sequence of weights $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$.

It is important to note that, when we establish approximation results for pseudocontractive mappings, it looks more complicated than the results for strictly pseudocontractive mappings. Because T may increase distances which is not in the case of strictly pseudocontractive mappings due to the presence of a constant $\lambda \in (0,1)$. In order to overcome this difficulty caused by increasingness of T, one need to adjust the iteration or to make some additional assumptions.

Motivated by all above facts, in this paper we define a new implicit iteration for finite family of pseudocontractive mappings where the sequence $\{x_n\}$ is generated by the algorithm

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n \,, \tag{1.3}$$

where the initial guess $x_0 \in K$ is arbitrary and $T_i \ \ (i = 1, 2, \ldots N)$ are pseudocontractive mappings, $\{\alpha_n\}$ is a real sequence in $(0,1)$ and $\{\lambda_i^{(n)}\}_{i=1}^{N}$ is a sequence of weights satisfying appropriate assumptions. We shall prove strong convergence of iteration (1.3) to a point $x \in \bigcap_{i=1}^{N} Fix(T_i)$.

2. PRELIMINARIES

A Banach space E is said to satisfy Opial's condition [20] for any sequence $\{x_n\}$ in E converging weakly to a point $x \in E$, we have $\limsup_{n\longrightarrow\infty} \|x_n - x\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

Now we establish the following result :

Proposition 2.1. *Given an integer $N \geq 1$, let $T_i : K \longrightarrow K$ be a pseudocontractive mapping for each $1 \leq i \leq N$. Define $S = \sum_{i=1}^{N} \lambda_i T_i$, where $\lambda_i > 0$ for all $1 \leq i \leq N$ such that $\sum_{i=1}^{N} \lambda_i = 1$. Then S is a pseudocontractive mapping.*

Proof. We have for $x, y \in K$,

$$\begin{aligned}
\langle Sx - Sy, j(x-y)\rangle &= \left\langle \left(\sum_{i=1}^{N} \lambda_i T_i\right) x - \left(\sum_{i=1}^{N} \lambda_i T_i\right) y, j(x-y)\right\rangle \\
&= \left\langle \sum_{i=1}^{N} \lambda_i \left(T_i x - T_i y\right), j(x-y)\right\rangle \\
&= \sum_{i=1}^{N} \lambda_i \left\langle T_i x - T_i y, j(x-y)\right\rangle \\
&\leq \sum_{i=1}^{N} \lambda_i \left\|x-y\right\|^2
\end{aligned}$$

$$= \|x-y\|^2 \sum_{i=1}^{N} \lambda_i$$
$$= \|x-y\|^2 ,$$

i.e.

$$\langle Sx - Sy, j(x-y)\rangle \le \|x-y\|^2 .$$

Hence S is a pseudocontractive mapping. □

Next, we show that the iteration (1.3), is well defined, we need the following lemma to prove it :

Lemma 2.2 (Deimling[8])**.** *Let E be a Banach space, K be a nonempty closed convex subset of E and $T : K \longrightarrow K$ be a continuous and strong pseudocontractive mapping. Then T has a unique fixed point.*

Given an integer $N \ge 1$, let $T_i : K \longrightarrow K$ be a continuous pseudocontractive mapping for each $1 \le i \le N$ with $\bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. For any fixed n, define $S_n = \sum_{i=1}^{N} \lambda_i^{(n)} T_i$, where $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all n and $\inf_{n\ge 1} \lambda_i^{(n)} > 0$ for all $1 \le i \le N$. Then for any fixed n, by Propostion 2.1, we observe that S_n is a pseudocontractive mapping which is continuous.
For any $u \in K$ and $t \in (0,1)$, define a mapping $G_t : K \longrightarrow K$ by

$$G_t x = tu + (1-t) S_n x .$$

Now,

$$\begin{aligned}\langle G_t x - G_t y, j(x-y)\rangle &= (1-t)\langle S_n x - S_n y, j(x-y)\rangle \\ &\le (1-t)\|x-y\|^2 , \quad \forall x, y \in K .\end{aligned}$$

Hence G_t is continuous strongly pseudocontractive mapping and by Lemma 2.2, it has a unique fixed point, i.e. there exists a unique $x_t \in K$ satisfying the equation

$$x_t = tu + (1-t) S_n x_t .$$

This shows that the implicit iteration scheme (1.3) is well defined and can be employed to approximate a common fixed point of a finite family of pseudocontractive self mappings on K.

Now we prove the following result, its proof is motivated from [25] :

Lemma 2.3. *Let S be a self mapping on K and $\lim_{n\longrightarrow\infty} \|x_n - Sx_n\| = 0$. If $A = (2I - S)^{-1}$, then $\lim_{n\longrightarrow\infty} \|x_n - Ax_n\| = 0$.*

Proof. We have

$$x_n - Sx_n = (2I - S)x_n - x_n = A^{-1}x_n - x_n ,$$

also,

$$A^{-1}Ax_n = x_n = AA^{-1}x_n .$$

So,

$$\begin{aligned}\|x_n - Ax_n\| &= \left\|AA^{-1}x_n - Ax_n\right\| \\ &\le \left\|A^{-1}x_n - x_n\right\| \\ &= \|x_n - Sx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty .\end{aligned}$$

Hence,

$$\lim_{n\longrightarrow\infty} \|x_n - Ax_n\| = 0\,.$$

□

3. MAIN RESULTS

Lemma 3.1. *Assume for each n, there is a finite sequence $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$ of positive numbers such that $\sum_{i=1}^{N}\lambda_i^{(n)} = 1$ for all n and $\inf_{n\geq 1}\lambda_i^{(n)} > 0$ for all $1 \leq i \leq N$. For any fixed n, set $S_n = \sum_{i=1}^{N}\lambda_i^{(n)}T_i$. Given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm (1.3). Assume that $\{\alpha_n\}$ is a real sequence satisfying $\alpha_n \in (0, b] \subset (0, 1)$ for some constant $b \in (0, 1)$, then*

(i) *for any $x^* \in \bigcap_{i=1}^{N} Fix(T_i)$, $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists,*
(ii) *$\{x_n\}$ and $\{S_n x_n\}$ are bounded.*

Proof. Since $x^* \in \bigcap_{i=1}^{N} Fix(T_i)$, we can see that $x^* \in Fix(S_n)$. Now

$$\begin{aligned}\|x_n - x^*\|^2 &= \langle \alpha_n (x_{n-1} - x^*) + (1-\alpha_n)(S_n x_n - x^*), j(x_n - x^*)\rangle\\ &= \alpha_n \langle x_{n-1} - x^*, j(x_n - x^*)\rangle + (1-\alpha_n)\langle S_n x_n - x^*, j(x_n - x^*)\rangle\\ &\leq \alpha_n \|x_{n-1} - x^*\|\, \|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|^2,\end{aligned}$$

and

$$\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|\, \|x_n - x^*\|\,. \tag{3.1}$$

Consequently, for each n,

$$\|x_n - x^*\| \leq \|x_{n-1} - x^*\|\,,$$

which implies that the sequence $\{\|x_n - x^*\|\}$ is monotone and nonincreasing. Hence $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists. Hence conclusion (i) is proved.
It follows from (i) that $\{x_n\}$ is bounded. Again from (1.3), we have

$$\begin{aligned}\|S_n x_n\| &= \left\|\frac{1}{1-\alpha_n}\,x_n - \frac{\alpha_n}{1-\alpha_n}\,x_{n-1}\right\|\\ &\leq \frac{1}{1-\alpha_n}\,\|x_n\| + \frac{\alpha_n}{1-\alpha_n}\,\|x_{n-1}\|\\ &\leq \frac{1}{1-b}\,\|x_n\| + \frac{b}{1-b}\,\|x_{n-1}\|\,.\end{aligned}$$

Hence $\{S_n x_n\}$ is bounded. This completes the proof of conclusion (ii). □

Theorem 3.2. *Let S_n and $\{x_n\}$ be as in Lemma 3.1. Assume that $\{\alpha_n\} \subset (0, 1)$ is a sequence of real numbers satisfying $\lim_{n\longrightarrow\infty}\alpha_n = 0$, then*

$$\lim_{n\longrightarrow\infty}\|x_n - S_n x_n\| = 0\,.$$

If in addition E satisfies Opial's condition and K is weakly compact convex, then the sequence $\{x_n\}$ converges weakly to a fixed point of S, where S is as in Proposition 2.1.

Proof. Since $\lim_{n\longrightarrow\infty}\alpha_n = 0$ there exists a positive integer M such that $\alpha_n \in (0, b]$ for all $n \geq M$, $b \in (0, 1)$. Hence Lemma 3.1 implies that, $\{x_n\}$ and $\{S_n x_n\}$ are bounded.
Using (1.3), we have

$$\|x_n - S_n x_n\| = \alpha_n \|x_{n-1} - S_n x_n\| \longrightarrow 0\,, \quad \text{as } n \longrightarrow \infty\,. \tag{3.2}$$

By weak compactness of K, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightharpoonup x^* \in K$ as $l \longrightarrow \infty$.
With no loss of generality, we may assume that as $l \longrightarrow \infty$

$$\lambda_i^{(n_l)} \longrightarrow \lambda_i \,, \quad 1 \le i \le N \,. \tag{3.3}$$

Now for each $\lambda_i > 0$ and $\sum_{i=1}^{N} \lambda_i = 1$, for all $x \in K$ we have,

$$S_{n_l} x \longrightarrow Sx \,, \quad \text{as} \;\; l \longrightarrow \infty \,,$$

where $S = \sum_{i=1}^{N} \lambda_i T_i$, and S is pseudocontractive by proposition 2.1.
Since

$$\begin{aligned} \|x_{n_l} - Sx_{n_l}\| &= \|x_{n_l} - S_{n_l} x_{n_l}\| + \|S_{n_l} x_{n_l} - Sx_{n_l}\| \\ &\le \|x_{n_l} - S_{n_l} x_{n_l}\| + \sum_{i=1}^{N} |\lambda_i^{n_l} - \lambda_i| \; \|T_i x_{n_l}\| \;, \end{aligned}$$

by (3.2), (3.3) and above inequality, we have

$$\|x_{n_l} - Sx_{n_l}\| \longrightarrow 0 \,, \quad \text{as } l \longrightarrow \infty \,.$$

Also, by Lemma 2.3, we have

$$\lim_{l \longrightarrow \infty} \|x_{n_l} - Ax_{n_l}\| = 0 \,,$$

where $A = (2I - S)^{-1}$.
Now we show that $x^* \in Fix(S)$. Suppose $x^* \neq Ax^*$. By nonexpansiveness of A and Opial's condition, we have

$$\begin{aligned} \limsup_{l \longrightarrow \infty} \|x_{n_l} - x^*\| &< \limsup_{l \longrightarrow \infty} \|x_{n_l} - Ax^*\| \\ &\le \limsup_{l \longrightarrow \infty} \{\|x_{n_l} - Ax_{n_l}\| + \|Ax_{n_l} - Ax^*\|\} \\ &\le \limsup_{l \longrightarrow \infty} \{\|x_{n_l} - Ax_{n_l}\| + \|x_{n_l} - x^*\|\} \\ &\le \limsup_{l \longrightarrow \infty} \|x_{n_l} - x^*\| \;, \end{aligned}$$

i.e.

$$\limsup_{l \longrightarrow \infty} \|x_{n_l} - x^*\| < \limsup_{l \longrightarrow \infty} \|x_{n_l} - x^*\| \;,$$

which is a contradiction, so $x^* = Ax^*$. Since $Fix(A) = Fix(S)$, we have $x^* \in Fix(S)$.
Next, we prove that the sequence $\{x_n\}$ converges weakly to x^*. Suppose $\{x_n\}$ does not converges weakly to x^*. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $z \neq x^*$. Since $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists and E satisfies Opial's condition, we have

$$\begin{aligned} \lim_{n \longrightarrow \infty} \|x_n - x^*\| &= \lim_{l \longrightarrow \infty} \|x_{n_l} - x^*\| < \lim_{l \longrightarrow \infty} \|x_{n_l} - z\| \\ &= \lim_{k \longrightarrow \infty} \|x_{n_k} - z\| < \lim_{k \longrightarrow \infty} \|x_{n_k} - x^*\| \\ &= \lim_{n \longrightarrow \infty} \|x_n - x^*\| \end{aligned}$$

which is a contradiction, so we must have $z = x^*$. Thus $\{x_n\}$ converges weakly to $x^* \in Fix(S)$. This completes the proof.

□

Theorem 3.3. *Let S_n and $\{x_n\}$ be as in Lemma 3.1. Assume that K be compact convex subset of E, and $\{\alpha_n\} \subset (0,1)$ is a sequence of real numbers satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of S, where S is as in Proposition 2.1.*

Proof. By Lemma 3.1, sequence $\{x_n\}$ is bounded, since K is compact, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \longrightarrow x^* \in K$. By (3.2), we have

$$\lim_{l\longrightarrow\infty} \|x_{n_l} - S_{n_l}x_{n_l}\| = 0\,,$$

by repeating the arguments in the proof of Theorem 3.2, we get that

$$\lim_{l\longrightarrow\infty} \|x_{n_l} - Sx_{n_l}\| = 0\,. \tag{3.4}$$

By the continuity of the mapping S and the norm $\|\cdot\|$, together with (3.4), we have

$$\|x^* - Sx^*\| = \lim_{l\longrightarrow\infty} \|x_{n_l} - Sx_{n_l}\| = 0\,.$$

Therefore $x^* = Sx^*$. Since $\{\|x_n - x^*\|\}$ is nonincreasing by Lemma 3.1, so x^* is the strong limit of the sequence $\{x_n\}$ itself. This completes the proof. □

We recall the following definition:

Definition 3.4 (Senter and Dotson [23]). A mapping $T : K \longrightarrow K$ with $Fix(T) \neq \emptyset$ is said to satisfy *condition* (I) on K if there exists a nondecreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0,\infty)$ such that for all $x \in K$

$$\|x - Tx\| \geq f(d(x, Fix(T)))\,,$$

where $d(x, Fix(T)) = \inf\{\|x - p\| \,:\, p \in Fix(T)\}$.

Definition 3.5. We shall say that a finite family $\{T_1, T_2, ...T_N\}$ of N self mappings of K with $F = \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ satisfies Condition (BS), if there exist f and d as in definition 3.4 such that

$$\|x - Sx\| \geq f(d(x, F))\,, \quad \forall x \in K\,,$$

where $S = \sum_{i=1}^{N} \lambda_i T_i$ and $\{\lambda_i\}_{i=1}^{N}$ is a sequence of positive number such that $\sum_{i=1}^{N} \lambda_i = 1$.

Definition 3.6. We shall say that a finite family $\{T_1, T_2, ...T_N\}$ of N self mappings of K with $F = \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ satisfies Condition (BT), if there exist f and d as in definition 3.4 such that

$$\|x - S_n x\| \geq f(d(x, F))\,, \quad \forall x \in K\,,$$

where $S_n = \sum_{i=1}^{N} \lambda_i^{(n)} T_i$ and $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$ is a sequence of positive number such that $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all n and $\inf_{n\geq 1} \lambda_i^{(n)} > 0$ for all $1 \leq i \leq N$.

Condition (BT) reduces to condition (BS) when sequence $\left\{\lambda_i^{(n)}\right\}_{i=1}^{N}$ is independent of iteration step n. Condition (BT) is identical to Condition (I) when $\lambda_i^{(n)} = 0$ for $i = 2, 3, \ldots, N$.

We now establish the main result of paper :

Theorem 3.7. *If K, S_n and $\{x_n\}$ be as in Lemma 3.1. Let T_i satisfy condition (BT), then $\{x_n\}$ converges strongly to a member of F.*

Proof. Since T_i, $i = 1, 2, \ldots, N$ satisfies condition (BT), we have

$$\|x_n - S_n x_n\| \geq f\left(d(x_n, F)\right), \quad \text{for all } n \geq 0\,.$$

Let $x^* \in F$, then by Lemma 3.1, $\|x_n - x^*\| \leq \|x_{n-1} - x^*\|$ and $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists. This implies that $d(x_n, F) \leq d(x_{n-1}, F)$, so $\{d(x_n, F)\}$ is decreasing, it follows from Theorem 3.2, that

$$\lim_{n\longrightarrow\infty} f\left(d(x_n, F)\right) = 0\,.$$

By the nature of function f and the fact that $\lim_{n\longrightarrow\infty} d(x_n, F)$ exists, we have

$$\lim_{n\longrightarrow\infty} d(x_n, F) = 0\,.$$

We can thus choose a subsequence say $\{x_{n_l}\}$ of $\{x_n\}$ such that

$$\|x_{n_l} - x_l^*\| < 2^{-l}\,,$$

for all integer $l \geq 1$ and some sequence $\{x_l^*\}$ in $Fix(T)$. Again by Lemma 3.1, we have

$$\|x_{n_l} - x_l^*\| \leq \|x_{n_l - 1} - x_l^*\| < 2^{-l}\,,$$

and hence

$$\begin{aligned}\left\|x_l^* - x_{l-1}^*\right\| &\leq \|x_l^* - x_{n_l-1}\| + \left\|x_{n_l-1} - x_{l-1}^*\right\| \\ &\leq 2^{-(l+1)} + 2^{-l} \\ &< 2^{-l+1}\,.\end{aligned}$$

Which shows that $\{x_l^*\}$ is Cauchy and therefore converges strongly to a point $x^* \in K$, since F is closed, $x^* \in F$. Since $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists, $\{x_n\}$ converges strongly to x^*. This completes the proof.

□

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Q-HYPERCONVEXITY IN QUASI-CONE METRIC SPACES AND FIXED POINT THEOREMS.

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ABSTRACT.
In this article, we introduce the concept of a q-hyperconvexity for quasi-cone metric spaces and generalise some fixed point theorems that we take from [8] and [11]. Mainly, we give some fixed point results for a pair of maps of Jungck type. Moreover we prove that quasi-cone metric spaces are topological spaces and we give a characterization of bounded sets in such spaces.

KEYWORDS :quasi-cone metric space; bicomplete space; q-Hyperconvexity.
AMS Subject Classification: 47H03, 47H10

1. INTRODUCTION AND PRELIMINARIES

Cone metric spaces were introduced in [7] and many fixed point results concerning mappings in such spaces have been established. Basically, cone metric spaces are defined by substituting, in the definition of a metric, the real line by a real Banach space that we endow with a partial order. In [14], Fawzia et al. discussed the newly introduced notion of quasi-cone metric spaces and proved some fixed point results for mappings on such spaces. Recently in [8], E. F. Kazeem et al defined a new type of completeness for quasi-cone metric spaces and the first related fixed point results. Quasi-cone metric spaces are just an asymmetric version of cone metric spaces which generalize the latter. Therefore, in light of what is done in [11], we also introduce a concept of convexity for quasi-cone metric spaces that we call q-hyperconvexity and discuss related properties. Most of the results follow the classical ones, but generalize them.

For all the key and recent results concerning cone metric spaces, the reader is advised to read [2, 5, 7, 8, 13].

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Definition 1.1. Let E be a real Banach space with norm $\|.\|$ and P be a subset of E. Then P is called a cone if and only if

(i) P is closed, nonempty and $P \neq \{\theta\}$, where θ is the zero vector in E;
(ii) for any $a, b \geq 0$, and $x, y \in P$, we have $ax + by \in P$;
(iii) for $x \in P$, if $-x \in P$, then $x = \theta$.

Given a cone P in a Banach space E, we define on E a partial order $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int(P)$ (where $Int(P)$ designates the interior of P).
The cone P is called **normal** if there is a number $C > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq C\|y\|.$$

The least positive number satisfying this inequality is called the **normal constant** of P. Therefore, we shall then say that P is a K-normal cone to indicate the fact that the normal constant is K.

Definition 1.2. Let E be a Banach space, P a cone on E and $\preceq$ the partial order defined by P. A subset $F \subset E$ is said to be **bounded from above with respect to** P if there exists $e \in E$ such that for all $f \in F,\ f \preceq e$.

Definition 1.3. (Compare [4]) A cone P is said to be **minihedral** if $x \vee y := \sup\{x, y\}$ exists for all $x, y \in E$ and **strongly minihedral** if every subset of E which is bounded from above with respect to P has a supremum.

Lemma 1.4. *(Compare[7, 12]) Let (X, q) be a quasi-cone metric space over a cone P. Then we have;*

a) *$Int(P) + Int(P) \subset Int(P)$ and $\lambda Int(P) \subset Int(P)$ for any positive real number λ.*
b) *For any given $c \gg \theta$ and $c_0 \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0}c_0 \ll c$.*
c) *If (a_n) and (b_n) are sequences in E such that $a_n \longrightarrow a$, $b_n \longrightarrow b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.*

Definition 1.5. (Compare[8]) Let X be a nonempty set. Suppose the mapping $q : X \times X \longrightarrow E$ satisfies

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
(q2) $q(x, y) = \theta = q(y, x)$ if and only if $x = y$;
(q3) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$. Then, q is called a **quasi-cone metric** on X, and (X, q) is called a **quasi-cone metric space**.
Moreover, if q satisfies
(q4) $q(x, y) = q(y, x)$ for all $x, y \in X$; then (X, q) is called a **cone metric space** in the sense of [7].

Remark 1.6. Let q be a quasi-cone metric on X, then the map q^{-1} defined by $q^{-1}(x, y) = q(y, x)$ whenever $x, y \in X$ is also a quasi-cone metric on X, called the **conjugate** of q. It can also be denoted by q^t or $\bar{q}$.

Definition 1.7. (Compare [8]) A sequence (x_n) in a quasi-cone metric space (X, q) is called

(i) Q-**Cauchy** or **bi-Cauchy** if for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that
$$\forall\, n, m \geq n_0 \quad q(x_n, x_m) \ll c;$$
(ii) **left(resp. right) Cauchy** if for every $c \in X$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that
$$\forall\, n, m : n_0 \leq m \leq n \quad q(x_m, x_n) \ll c \text{ (resp. } q(x_n, x_m) \ll c \text{)}.$$

Remark 1.8. A sequence is Q-Cauchy if and only if it is both left and right Cauchy.

Definition 1.9.
(i) In a quasi-cone metric space (X, q), we say that the sequence (x_n) **left converges** to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that for all $n > N$, $q(x_n, x) \ll c$.
(ii) Similarly, in a quasi-cone metric space (X, q), we say that a sequence (x_n) **right converges** to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that for all $n > N$, $q(x, x_n) \ll c$.
(iii) Finally, in a quasi-cone metric space (X, q), we say that the sequence (x_n) **converges** to $x \in X$ if for every $c \in E$ with $\theta \ll c$ there exists N such that for all $n > N$, $q(x_n, x) \ll c$ and $q(x, x_n) \ll c$.

Definition 1.10. (Compare [8]) A quasi-cone metric space (X, q) is called
(i) **left complete** (resp. **right complete**) if every left Cauchy(resp. right Cauchy) sequence in X left(resp. right) converges.
(ii) **bicomplete** if every Q-Cauchy sequence converges.

Remark 1.11. A quasi-cone metric space (X, q) is bicomplete if and only if it is left complete and right complete.

Definition 1.12. (Compare [8]) Let (X, q) be a quasi-cone metric space. A function $f : X \longrightarrow X$ is said to be **lipschitzian** if there exists some $\kappa \in \mathbb{R}$ such that
$$q(f(x), f(y)) \preceq \kappa q(x, y) \qquad \forall\, x, y \in X.$$
The smallest constant which satisfies the above inequality is called the **Lipschitz constant** of f and is denoted $Lip(f)$. In particular f is said to be **contractive** if $Lip(f) \in [0, 1)$ and **expansive** if $Lip(f) = 1$.

Definition 1.13. (Compare [2]) Let f and g be self maps on a set X. If $w = fx = gx$ for some $x \in X$, then x is called a **coincidence point** of f and g, and w is called the **point of coincidence** of f and g.

Definition 1.14. Let f and g be self maps on a nonempty set X. We say that f and g are **weakly compatible** if they commute at their coincidence point, that is there exists $x_0 \in X$ such that $fx_0 = gx_0$ then $gfx_0 = fgx_0$.

We also give the following proposition that we take from [2] by omitting the proof.

Proposition 1.15. *(Compare [2]) Let f and g be weakly compatible self maps on a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Lemma 1.16. *(Compare [8]) Let (X, q) be a quasi-cone metric space, P be a K-normal cone. Let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $q(x_n, x) \longrightarrow \theta$ $(n \longrightarrow \infty)$ and $q(x, x_n) \longrightarrow \theta$ $(n \longrightarrow \infty)$.*

Remark 1.17. In fact, a sequence (x_n) left-converges (resp. right-converges) to x if and only if $q(x_n, x) \longrightarrow \theta$ (resp $q(x, x_n) \longrightarrow \theta$) $(n \longrightarrow \infty)$.

Lemma 1.18. *(Compare [8]) Let (X, q) be a quasi-cone metric space and (x_n) be a sequence in X. If (x_n) converges to x, then (x_n) is a bi-Cauchy sequence.*

Lemma 1.19. *(Compare [8]) Let (X, q) be a quasi-cone metric space, P be a K-normal cone and (x_n) be a sequence in X. Then (x_n) is a bi-Cauchy sequence if and only if $q(x_n, x_m) \longrightarrow \theta$ as $n, m \longrightarrow \infty$.*

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2. Common fixed points results

Lemma 2.1. *Let (X, q) be a quasi-cone metric space, P be a K-normal cone. Let (x_n) be a sequence in X. If (x_n) converges to x and (x_n) converges to y, then $x = y$. That is the limit of (x_n) is unique.*

Proof. For any $n \geq 0$, $q(x, y) \preceq q(x, x_n) + q(x_n, y)$ which entails that $q(x, y) = \theta$. Similarly $q(y, x) = \theta$, hence by property $(q2)$ $x = y$.

□

Theorem 2.2. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq k\, q(gx, gy) \qquad \textit{for all } \ x, y \in X,$$

where $k \in (0, 1)$. If the range of g contains the range of f and $g(X)$ is bicomplete, then f and g have a unique point of coincidence. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subseteq g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(fx_{n-1}, fx_n) \preceq kq(gx_{n-1}, gx_n) \\ &\preceq k^2 q(gx_{n-2}, gx_{n-1}) \preceq \ldots \preceq \ k^n q(gx_0, gx_1). \end{aligned}$$

i.e.

$$q(gx_n, gx_{n+1}) \preceq \ k^n q(gx_0, gx_1).$$

Similarly,

$$q(gx_{n+1}, gx_n) \preceq \ k^n q(gx_1, gx_0).$$

So for $n < m$,

$$\begin{aligned} q(gx_n, gx_m) &\preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \ldots + q(gx_{m-1}, gx_m) \\ &\preceq (k^n + k^{n+1} + \ldots + k^{m-1}) q(gx_0, gx_1) \preceq \frac{k^n}{1-k} q(gx_0, gx_1). \end{aligned}$$

It entails that $\|q(gx_n, gx_m)\| \leq K \frac{k^n}{1-k} \|q(gx_0, gx_1)\| \longrightarrow 0$ as $n \longrightarrow \infty$.
Similarly for $n > m$

$$q(gx_n, gx_m) \preceq \frac{k^m}{1-k} q(gx_1, gx_0).$$

It entails that $\|q(gx_n, gx_m)\| \leq K \frac{k^m}{1-k} \|q(gx_1, gx_0)\| \longrightarrow 0$ as $m \longrightarrow \infty$. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$.
Moreover since

$$q(gx_n, fp^*) = q(fx_{n-1}, fp^*) \preceq kq(gx_{n-1}, gp^*),$$

we get that

$$\|q(gx_n, fp^*)\| \leq Kk\|q(gx_{n-1}, gp^*)\| \longrightarrow 0, \text{ as } n \longrightarrow \infty,$$

hence $q(gx_n, fp^*) \longrightarrow \theta$ as $n \longrightarrow \infty$. In the same way, we establish that $q(fp^*, gx_n) \longrightarrow \theta$ as $n \longrightarrow \infty$, to then conclude that $gx_n \longrightarrow fp^*$. The uniqueness of the limit implies that $fp^* = gp^*$. We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now

$$q(gz^*, gp^*) = q(fz^*, fp^*) \preceq kq(gz^*, gp^*),$$

which gives $q(gz^*, gp^*) = \theta$. On the other hand, by the same reasoning, it also clear that $q(gp^*, gz^*) = \theta$. By property $(q2)$, $gz^* = gp^*$. From Proposition 1.15, f and g have a unique common fixed point. □

Corollary 2.3. *Taking $g = I$(Identity map) in Theorem 2.2, we obtain Theorem 4.1 of* [8].

Remark 2.4. Let (X, q) be a bicomplete quasi-cone metric space, P be a K-normal cone. Assume $f, g : X \longrightarrow X$ satisfy the contractive condition

$$q(f^n x, f^n y) \preceq kq(gx, gy), \quad \text{for all } x, y \in X,$$

for some positive integer, where $k \in [0, 1)$ is a constant. If the range of g contains the range of f^n and $g(X)$ is a bicomplete quasi-cone metric space, then by the theorem above, there exists a unique $x^* \in X$ such that $f^n x^* = gx^* = x^*$. Observe that $f^n(fx^*) = f(f^n x^*) = fx^*$, so fx^* is also a fixed point of f^n. If f and g commute at x^*, i.e. $fgx^* = gfx^*$, then f and g have a unique common fixed point. This is easily seen by observing that $gfx^* = fgx^* = fx^*$.

Theorem 2.5. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq k\,[q(fx, gy) + q(gx, fy)] \qquad \textit{for all } x, y \in X,$$

where $k \in \left(0, \frac{1}{2}\right)$. If the range of g contains the range of f and $g(X)$ is bicomplete , then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subseteq g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} q(gx_n, gx_{n+1}) = q(fx_{n-1}, fx_n) &\preceq k[q(fx_{n-1}, gx_n) + q(gx_{n-1}, fx_n)] \\ &\preceq kq(gx_{n-1}, gx_{n+1}) \\ &\preceq k[q(gx_{n-1}, gx_n) + q(gx_n, gx_{n+1})], \end{aligned}$$

which entails that

$$q(gx_n, gx_{n+1}) \preceq \frac{k}{1-k} q(gx_{n-1}, gx_n).$$

Similarly,

$$q(gx_{n+1}, gx_n) \preceq \frac{k}{1-k} q(gx_n, gx_{n-1}).$$

So for $n < m$,

$$q(gx_n, gx_m) \preceq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \ldots + q(gx_{m-1}, gx_m)$$

$$\preceq (h^n + h^{n+1} + \ldots + h^{m-1})q(gx_0, gx_1) \preceq \frac{h^n}{1-h}q(gx_0, gx_1),$$

where $h = \frac{k}{1-k}$. It entails that $\|q(gx_n, gx_m)\| \leq K\frac{h^n}{1-h}\|q(gx_0, gx_1)\| \longrightarrow 0$ as $n \longrightarrow \infty$.
Similarly for $n > m$,

$$q(gx_n, gx_m) \preceq \frac{h^m}{1-h}q(gx_1, gx_0).$$

It entails that $\|q(gx_n, gx_m)\| \leq K\frac{h^m}{1-h}\|q(gx_1, gx_0)\| \longrightarrow 0$ as $m \longrightarrow \infty$. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$.
Moreover since

$$q(gx_n, fp^*) = q(fx_{n-1}, fp^*) \preceq k[q(fx_{n-1}, gp^*) + q(gx_{n-1}, fp^*)],$$

we get that

$$q(gp^*, fp^*) \preceq kq(gp^*, fp^*)$$

which implies that $q(gp^*, fp^*) = \theta$.
In the same way, we establish that $q(fp^*, gp^*) = \theta$, to then conclude that $fp^* = gp^*$.
We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now

$$q(gz^*, gp^*) = q(fz^*, fp^*) \preceq k[q(fz^*, gp^*) + q(gz^*, fp^*)] \preceq 2kq(gz^*, gp^*),$$

which gives $q(gz^*, gp^*) = \theta$. On the other hand, by the same reasoning, it also clear that $q(gp^*, gz^*) = \theta$. By property $(q2)$, $gz^* = gp^*$. From Proposition 1.15, f and g have a unique common fixed point. □

The results for this section therefore summarise in the following way.

Theorem 2.6. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition: for all $x, y \in X$*

$$q(fx, fy) \preceq Aq(gx, gy) + B[q(gx, fx) + q(gy, fy)] + C[q(fx, gy) + q(gx, fy)] \tag{2.1}$$

where A, B, C are non-negative real numbers with $A + 2B + 2C < 1$. If the range of $f(X) \subseteq g(X)$ and f or $g(X)$ is bicomplete , then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. The proof follows the same idea as in Theorems 2.2 and 2.5. We shall just give here the main points. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subseteq g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1}), n = 0, 1, \cdots$.

It is then very easy to derive that for $n < m$ we have

$$q(gx_n, gx_m) \preceq \frac{h^n}{1-h}q(gx_0, gx_1)$$

and for for $n > m$ we have

$$q(gx_n, gx_m) \preceq \frac{h^m}{1-h}q(gx_0, gx_1)$$

where $h = \frac{A+B+C}{1-B-C}$. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$. Moreover, observe that

$$q(gx_n, fp^*) = q(fx_{n-1}, fp^*) \quad \text{and} \quad q(fp^*, gx_n) = q(fp^*, fx_{n-1})$$

and using the condition (2.1), we obtain that $q(gz^*, gp^*) = \theta = q(gp^*, gz^*)$, i.e. f and g have a point of coincidence. Again by use of condition (2.1), we conclude that this point of coincidence is unique. Finally, from Proposition 1.15, since f and g are weakly compatible, then f and g have a unique common fixed point. □

Example 2.7. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $q(x, y) = (x * y, \alpha(x * y))$, $\alpha > 0$ where $x * y = \max\{x - y, 0\}$, whenever $x, y \in \mathbb{R}$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$, $f(x) = 2x^2 + 4x + 3$ and $g(x) = 3x^2 + 6x + 4$. Then it easy to see that

$$f(X) = g(X) = [1, \infty) \text{ is bicomplete.}$$

All the conditions of Theorem 2.6 are satisfied for

$$A \in \left[\frac{2}{3}, 1\right), B = C = 0.$$

The unique point of coincidence here is $1 = f(-1) = g(-1)$.

However, since $fg(-1) = 9 \neq 13 = gf(-1)$, f and g are not weakly compatible and therefore fail to have a common fixed point. This shows the importance of Proposition 1.15.

But if we modify f and g in the following way $f(x) = 2x^2 + 4x + 1$ and $g(x) = 3x^2 + 6x + 2$, then again all the conditions of Theorem 2.6 are satisfied, f and g become weakly compatible and we obtain a unique point of coincidence and a unique common fixed point $-1 = f(-1) = g(-1)$.

The next example explain how crucial the condition $f(X) \subseteq g(X)$ is in the statement of the theorems.

Example 2.8. Let $X = [0, \infty)$, $E = \mathbb{R}^2$, $q(x, y) = (x * y, e(x * y))$, where $x * y = \max\{x - y, 0\}$, whenever $x, y \in \mathbb{R}$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$, $f(x) = e^x$ and $g(x) = e^{x+1}$. Then it easy to see that

$$f(X) = (0, \infty) \nsubseteq g(X) = [e, \infty).$$

$$\begin{aligned} q(fx, fy) &= (e^x * e^y, e^{x+1} * e^{y+1}) \\ &= \frac{1}{e}(e^{x+1} * e^{y+1}, e^{x+2} * e^{y+2}) \\ &= \frac{1}{e} q(gx, gy). \end{aligned}$$

All the conditions of Theorem 2.6 except $f(X) \subseteq g(X)$ are satisfied for

$$A = \frac{1}{e}, B = C = 0.$$

But f and g do not have a point of coincidence.

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sectionMore fixed point results We begin with the following lemmas.

Lemma 2.9. *Let (X, q) be a quasi-cone metric space, P be a K-normal cone. Let (y_n) be a sequence in X. If (y_n) satisfies*

$$q(y_n, y_{n+1}) \leq \lambda q(y_{n-1}, y_n) \tag{2.2}$$

for some $\lambda > 0$ with $\lambda < 1$. Then (y_n) is left Cauchy.

Proof. Let $m < n \in \mathbb{N}$. From the condition $(q3)$ in the definition of a quasi-cone metric , we can write:

$$\begin{aligned} q(y_m, y_n) &\preceq q(y_m, y_{m+1}) + q(y_{m+1}, y_n) \\ &\preceq q(y_m, y_{m+1}) + q(y_{m+1}, y_{m+2}) + q(y_{m+2}, y_n) \\ &\vdots \\ &\preceq q(y_m, y_{m+1}) + q(y_{m+1}, y_{m+2}) + \cdots + q(y_{n-2}, y_{n-1}) + q(y_{n-1}, y_n). \end{aligned}$$

From (2.2) the above becomes

$$\begin{aligned} q(y_m, y_n) &\preceq (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1}) q(y_0, y_1) \\ &\preceq \frac{\lambda^m}{1-\lambda} q(y_0, y_1). \end{aligned}$$

It entails that $\|q(y_m, y_n)\| \leq K \frac{\lambda^m}{1-\lambda} \|q(y_0, y_1)\| \longrightarrow 0$ as $m \longrightarrow \infty$. It follows that (y_n) is left Cauchy.

□

Similarly,

Lemma 2.10. *Let (X, q) be a quasi-cone metric space, P be a K-normal cone. Let (y_n) be a sequence in X. If (y_n) satisfies*

$$q(y_{n+1}, y_n) \preceq \lambda q(y_n, y_{n-1}) \tag{2.3}$$

for some $\lambda > 0$ with $\lambda < 1$. Then (y_n) is right Cauchy.

Theorem 2.11. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq \lambda q(gx, gy) + \gamma q(fx, gy) \text{ for all } x, y \in X. \tag{2.4}$$

where λ, γ are positive constant such that $\lambda + 2\gamma < 1$. If the range of g contains the range of f and $g(X)$ is a bicomplete quasi-cone metric space, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take an arbitrary $x_0 \in X$. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since $f(X) \subset g(X)$. Iterating this process, once x_n is chosen in X, we can obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} q(gx_n, gx_{n+1}) = q(fx_{n-1}, fx_n) &\preceq \lambda q(gx_{n-1}, gx_n) + \gamma q(fx_{n-1}, gx_n) \\ &\preceq \lambda q(gx_{n-1}, gx_n). \end{aligned}$$

Therefore (gx_n) is a left Cauchy sequence. In a similar manner, we establish that (gx_n) is also a right Cauchy sequence. Hence (gx_n) is a bi-Cauchy sequence. Since $g(X)$ is bicomplete, there exists $x^* \in g(X)$ such that (gx_n) converges to x^*. In other words, there is a $p^* \in X$ such that (gx_n) converges to $g(p^*) = x^*$.
Moreover since

$$q(gx_n, fp^*) = q(fx_{n-1}, fp^*) \preceq \lambda q(gx_{n-1}, gp^*) + \gamma q(fx_{n-1}, gp^*)$$

we get that $q(gp^*, fp^*) = \theta$. On the other hand, by the same reasoning, it is also clear that $q(fp^*, gp^*) = \theta$. By property $(q2)$, $fp^* = gp^*$.
We finish the proof by showing that f and g have a unique point of coincidence. For this, assume $z^* \in X$ is a point such that $fz^* = gz^*$. Now

$$q(gz^*, gp^*) = q(fz^*, fp^*) \preceq \lambda q(gz^*, gp^*) + \gamma q(fz^*, gp^*) \preceq (\lambda + \gamma) q(gz^*, gp^*),$$

which gives $q(gz^*, gp^*) = \theta$. On the other hand, by the same reasoning, it also clear that $q(gp^*, gz^*) = \theta$. By property $(q2)$, $gz^* = gp^*$. From Proposition 1.15, f and g have a unique common fixed point. □

Corollary 2.12. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq \alpha[q(gx, gy) + q(fx, gy)] \text{ for all } x, y \in X. \tag{2.5}$$

where $\alpha \in \left(0, \frac{1}{3}\right)$. If the range of g contains the range of f and $g(X)$ is a bicomplete quasi-cone metric space, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Theorem 2.13. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq \lambda q(gx, gy) + \gamma q(gx, fy) \text{ for all } x, y \in X. \tag{2.6}$$

where λ, γ are positive constant such that $\lambda + 2\gamma < 1$. If the range of g contains the range of f and $g(X)$ is a bicomplete quasi-cone metric space, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

Corollary 2.14. *Let (X, q) be a quasi-cone metric space, P a K-normal cone. Suppose that mappings $f, g : X \longrightarrow X$ satisfy the contractive condition*

$$q(fx, fy) \preceq \alpha[q(gx, gy) + q(gx, fy)] \text{ for all } x, y \in X. \tag{2.7}$$

where $\alpha \in \left(0, \frac{1}{3}\right)$. If the range of g contains the range of f and $g(X)$ is a bicomplete quasi-cone metric space, then f and g have a unique coincidence point in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point.

In the following section, we shall establish some topological properties of quasi-cone metric spaces.

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sectionTopology on quasi-cone metric spaces

We begin by stating the following lemma.

Lemma 2.15. *Let (X, q) be a quasi-cone metric space. For each $c \in E$ with $c \gg \theta$, there exists $\sigma > 0$ such that $x \ll c$ whenever $\|x\| < \sigma,\ x \in E$.*

Proof. Since $c \gg \theta$, then $c \in Int(P)$. Hence, find $\sigma > 0$ such that

$$\{x \in E : \|x - c\| < \sigma\} \subset Int(P).$$

Now if $\|x\| < \sigma$ then $\|(c - x) - c\| = \| - x\| = \|x\| < \sigma$ and $(c - x) \in Int(P)$. □

Lemma 2.16. *Let (X, q) be a quasi-cone metric space over a cone P. Then for each $c_1, c_2 \in Int(P)$, there exists $c \in Int(P)$ such that $c_1 - c \in Int(P)$ and $c_2 - c \in Int(P)$.*

Proof. Since $c_2 \gg \theta$, then by Lemma 2.15, we pick $\delta > 0$ such that $\|x\| < \delta$ implies that $x \ll c_2$. Choose n_0 such that $\frac{1}{n_0} < \frac{\delta}{\|c_1\|}$. then $\|c\| = \frac{\|c_1\|}{n_0} < \delta$ and hence, $c \ll c_2$. But it is also clear that $c \gg \theta$ and $c \ll c_1$. □

Proposition 2.17. *Every quasi-cone metric space is a topological space.*

Proof. For $c \gg \theta$, and $x \in (X, q)$ let

$$B_q(x, c) = \{y \in X : q(x, y) \ll c\}$$

and

$$\mathcal{B} = \{B_q(x, c) : x \in X, c \ll \theta\}.$$

Then the collection

$$\mathcal{T} = \{U \subset X : \forall x \in U, \exists c \gg \theta, B_q(x, c) \subset U\}$$

is a topology on X. Indeed,

(i) $\emptyset$ and X belong to $\mathcal{T}$,

(ii) let $U, V \in \mathcal{T}$ and let $x \in U \cap V$. Then there exist $c_1 \gg \theta$ and $c_2 \gg \theta$ such that $B_q(x, c_1) \subset U$ and $B(x, c_2) \subset V$.
By Lemma 2.16, there exists $c \in Int(P)$ such that $c_1 - c \in Int(P)$ and $c_2 - c \in Int(P)$. Then, it is clear that $x \in B_q(x, c) \subset U \cap V$, hence $U \cap V \in \mathcal{T}$.

(iii) let $(U_\alpha)_\alpha$ be a family of sets from $\mathcal{T}$. We consider $x \in \bigcup\limits_\alpha U_\alpha$. There exists α_0 such that $x \in U_{\alpha_0}$. Hence, find $c \gg \theta$ such that

$$x \in B_q(x, c) \subset U_{\alpha_0} \subset \bigcup_\alpha U_\alpha.$$

That is $\bigcup\limits_\alpha U_\alpha \in \mathcal{T}$.
This completes the proof.

□

Definition 2.18. Let (X, q) be a quasi-cone metric space. Then a subset $F \subset X$ is called **up bounded**(resp. **bounded**) if there exists $c \gg \theta$ such that $q(x, y) \preceq c$ for all $x, y \in F$ (resp. $\delta(F) = \sup\{q(x, y) : x, y \in F\}$ exists in E).

Lemma 2.19. *Let (X, q) be a quasi-cone metric space. Then a subset $F \subset X$ is up bounded if and only if there exist $x \in X, c_1, c_2 \in Int(P)$ such that $F \subset C_q(x, c_1) \cap C_{q^{-1}}(x, c_2)$ where $C_q(x, c) = \{y \in X : q(x, y) \preceq c\}$ for any $x \in X$ and $c \gg \theta$.*

When the cone P is strongly minihedral, we have the following characterization.

Proposition 2.20. *Let (X, q) be a quasi-cone metric space over a Banach space with a K-normal strongly minihedral cone P. Then a subset $F \subset X$ is bounded if and only if the quantity $\delta'(F) = \sup_{x,y \in F} \|q(x, y)\| < \infty$.*

Proof. Suppose that F is bounded. For all $x, y \in F$, $q(x, y) \preceq \delta(F)$, which entails that $\|q(x, y)\| \leq K\|\delta(F)\| < \infty$.
Conversely, assume that $\delta'(F) = \sup_{x,y \in F} \|q(x, y)\| = M < \infty$, and fix some $c_1 \gg \theta$. By Lemma 2.15, we know that we can find $\delta > 0$ such that $\|z\| < \delta$ implies that $c_1 \gg z$. For any $x, y \in F$, we set $c_{x,y} = \frac{\delta q(x,y)}{2\|q(x,y)\|}$. Since $\|c_{x,y}\| = \delta/2 < \delta$, we have that $c_1 - c_{x,y} \in Int(P)$. By setting $\alpha_{x,y} = \frac{2\|q(x,y)\|}{\delta}$, it is then clear that $\alpha_{x,y}(c_1 - c_{x,y}) \in Int(P)$, i.e $\alpha_{x,y}c_1 - q(x, y) \in Int(P)$. In other words $q(x, y) \ll \alpha_{x,y}c_1 \preceq \frac{2M}{\delta}c_1 \in Int(P)$, from which we derive that $q(x, y) \preceq \frac{2M}{\delta}c_1$. Since P is minihedral, then F is bounded. □

We conclude this section by the following lemma.

Lemma 2.21. *Every quasi-cone metric space (X, q) is first countable.*

Proof. Let $p \in X$. Fix $c \in Int(P)$. We show that $\mathcal{B}_p = \{B_q(p, \frac{1}{n}c) : n \in \mathbb{N}\}$ is a local base at p. Let U be an open set containing p. There exists $c_1 \in Int(P)$ such that $B_q(p, c_1) \subset U$. We know by Lemma 1.4 that we can find $n_0 \in \mathbb{N}$ such that $\frac{c}{n_0} \ll c_1$. Hence $B_q(p, \frac{c}{n_0}) \subset B_q(p, c_1)$. This completes the proof. □

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3. Concept of convexity

In this section, we introduce a concept of convexity in quasi-cone metric spaces. Most of the results here are a generalization of some similar known notions in quasi-pseudometric spaces (see [10] and [11]).

Definition 3.1. Let E be a real Banach space, P be a cone on E and $\preceq$ the partial order with respect to P. An element $x \in E$ is said to be a **nonnegative vector** if $\theta \preceq x$ and a **positive vector** if $\theta \prec x$. Hence P is the set of all nonegative elements. We shall use the following notations:

- $[\theta, \longrightarrow [:= P = \{x \in E : \theta \preceq x\}$ and;
- $]\theta, \longrightarrow [:= \{x \in E : \theta \prec x\}$.

Definition 3.2. A quasi-cone metric space (X, q) will be called **Isbell-convex** or **q-hyperconvex** provided that for each family $(x_i)_{i \in I}$ of points of X and families of nonnegative vectors $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ the following condition holds:

$$\text{If } q(x_i, x_j) \preceq r_i + s_j \text{ whenever } i, j \in I, \text{ then } \bigcap_{i \in I}(C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i)) \neq \emptyset.$$

Definition 3.3. A quasi-cone metric space (X, q) will be called **metrically convex** if for any point $x, y \in X$ and nonnegative vectors r and s such that $q(x, y) \preceq r + s$, there exists $z \in X$ such that $q(x, z) \preceq r$ and $q(z, y) \preceq s$.

The following examples are basic but not trivial, and therefore, important.

Example 3.4. Let $X = \mathbb{R},\ E = \mathbb{R}^2,\ q(x,y) = (x*y, x*y)$, where the operation $*$ is defined by $x*y = \max\{x-y, 0\}$, whenever $x, y \in \mathbb{R}$ and $P = \{(x,y) : x \geq 0, y \geq 0\}$. Then $(\mathbb{R}, q)$ is metrically convex.

Example 3.5. Let $X = \mathbb{R},\ E = \mathbb{R}^2,\ q(x,y) = (x\sharp y, x\sharp y)$, where the operation $\sharp$ is defined by $x\sharp y = x - y$, if $x \geq y$ and $x\sharp y = 1$ otherwise, whenever $x, y \in \mathbb{R}$ and $P = \{(x,y) : x \geq 0, y \geq 0\}$. Then $(\mathbb{R}, q)$ is not metrically convex. Indeed, we have

$$q\left(\frac{1}{2}, 1\right) = (1,1) \preceq \left(\frac{1}{2} + \frac{1}{2}, \frac{1}{2} + \frac{1}{2}\right).$$

But there is no $z \in \mathbb{R}$ such that $q\left(\frac{1}{2}, z\right) \preceq \left(\frac{1}{2}, \frac{1}{2}\right)$ and $q\left(z, 1\right) \preceq \left(\frac{1}{2}, \frac{1}{2}\right)$, since such a z would satisfy $z \leq \frac{1}{2}$ and $z \geq 1$.

Definition 3.6. Let (X,q) be a quasi-cone metric space. A family of balls $(C_q(x_i, r_i), C_{q^t}(x_i, s_i))_{i\in I}$ with $x_i \in X$ and $r_i, s_i \in [\theta, \longrightarrow [$ whenever $i \in I$ is said to have the **mixed binary intersection property** if for all indices $i, j \in I$, $(C_q(x_i, r_i) \cap C_{q^t}(x_j, s_j) \neq \emptyset$.

Definition 3.7. A quasi-cone metric space (X,q) will be called **Isbell-complete** if every family of balls $(C_q(x_i, r_i), C_{q^t}(x_i, s_i))_{i\in I}$ with $x_i \in X$ and $r_i, s_i \in [\theta, \longrightarrow [$ whenever $i \in I$ having the mixed binary intersection property satisfies $\bigcap\limits_{i\in I} (C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i)) \neq \emptyset$.

Corollary 3.8. *If (X,q) is an Isbell-convex (resp. Isbell-complete, metrically convex) quasi-cone metric space, then (X, q^t) is Isbell-convex(resp. Isbell-complete, metrically convex).*

Proposition 3.9. *A quasi-cone metric space (X,q) is Isbell-convex if and only if it is metrically convex and Isbell-complete.*

Proof. Suppose that (X,q) is Isbell-convex. Let $x_1, x_2 \in X,\ r_1, r_2 \in [\theta, \longrightarrow [$ such that $q(x_1, x_2) \preceq r_1 + s_2$. Then set $r_2 = s_1 = q(x_2, x_1)$. By Isbell-convexity, there exists $x \in C_q(x_1, r_1) \cap C_{q^t}(x_2, s_2)$, hence (X,q) is metrically convex.
Consider now a family of balls $(C_q(x_i, r_i), C_{q^t}(x_i, s_i))_{i\in I}$ that have the mixed binary intersection property. Thus $q(x_i, x_j) \preceq r_i + s_j$ whenever $i, j \in I$. By Isbell-convexity, there exists $x \in \bigcap\limits_{i\in I} (C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i))$. So (X,q) is Isbell-complete.
For the converse, assume that (X,q) is metrically convex and Isbell-complete. Consider a family $(x_i)_{i\in I}$ of points of X and families $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ of nonnegative vectors such that $q(x_i, x_j) \preceq r_i + s_j$ whenever $i, j \in I$. Since (X,q) is metrically convex, then $(C_q(x_i, r_i), C_{q^t}(x_i, s_i))_{i\in I}$ has the mixed binary intersection property.
Therefore, there exists $x \in \bigcap\limits_{i\in I} (C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i))$ by Isbell-completeness. Thus, (X,q) is Isbell-convex. □

Example 3.10. Let $X = \mathbb{R},\ E = \mathbb{R}^2,\ q(x,y) = (x*y, x*y)$, where the operation $*$ is defined by $x*y = \max\{x-y, 0\}$, whenever $x, y \in \mathbb{R}$ and $P = \{(x,y) : x \geq 0, y \geq 0\}$. Then $(\mathbb{R}, q)$ is q-hyperconvex.

We first observe that, for any $\varepsilon = (\varepsilon_1, \varepsilon_2) \in P$, $C_q(x, \varepsilon) = [x - \tilde{\varepsilon}, \infty)$ and $C_{q^{-1}}(x, \varepsilon) = (-\infty, x + \tilde{\varepsilon}]$ whenever $x \in \mathbb{R}$ where $\tilde{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\}$. Let $(x_i)_{i\in I}$ be a family of points in $\mathbb{R}$ and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of nonnegative vectors with

$r_i = (r_i^1, r_i^2)$ and $s_i = (s_i^1, s_i^2)$ such that $q(x_i, x_j) \preceq r_i + s_j$, whenever $i, j \in I$. We recall quickly that $\tilde{r_i} = \min\{r_i^1, r_i^2\}$ and $\tilde{s_i} = \min\{s_i^1, s_i^2\}$. Suppose that

$$\bigcap_{i \in G} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) = \emptyset \text{ for some finite subset } G \text{ of } I.$$

We can assume that G is nonempty. It follows that

$$\max\{x_i - \tilde{r_i} : i \in G\} > \min\{x_i + \tilde{s_i} : i \in G\}.$$

Therefore, there are $i_0, j_0 \in G$ such that $x_{i_0} - \tilde{r_{i_0}} > x_{j_0} - \tilde{s_{i_0}}$, that is $C_q(x_{i_0}, r_{i_0}) \cap C_{q^{-1}}(x_{j_0}, s_{i_0}) = \emptyset$. In particular, $x_{i_0} > x_{j_0}$.

Thus $r_{i_0} + s_{j_0} \prec q(x_{i_0}, x_{j_0}) = (x_{i_0} - x_{j_0}, x_{i_0} - x_{j_0})$ —a contradiction. We conclude that

$$\bigcap_{i \in G} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset \text{ whenever } G \text{ is a finite subset of } I.$$

Since for any $i \in I$,$C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)$ is compact with respect to the standard topology on $\mathbb{R}$, we conclude that

$$\bigcap_{i \in G} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset.$$

Hence $(\mathbb{R}, q)$ is q-hyperconvex.

Example 3.11. Let $X = \mathbb{R},\ E = \mathbb{R}^2,\ q(x, y) = (|x - y|, |x - y|)$, whenever $x, y \in \mathbb{R}$ and $P = \{(x, y) : x \geq 0, y \geq 0\}$. Then $(\mathbb{R}, q)$ is not q-hyperconvex.

Indeed, for any $i \in [0, 1]$, set $r_i = \left(\frac{1}{4}, \frac{1}{4}\right)$ and $s_i = \left(\frac{3}{4}, \frac{3}{4}\right)$. Then for any $i, j \in [0, 1]$, $q(x, y) \preceq (1, 1) = r_i + s_i$. But

$$\begin{aligned}
\bigcap_{i \in [0,1]} (C_q(i, r_i) \cap C_{q^{-1}}(i, s_i)) &\subseteq C_q\left(0, \frac{1}{4}\right) \bigcap C_{q^{-1}}\left(1, \frac{1}{4}\right) \\
&= \left(\left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[-\frac{1}{4}, \frac{1}{4}\right]\right) \bigcap \left(\left[\frac{3}{4}, \frac{5}{4}\right] \times \left[\frac{3}{4}, \frac{5}{4}\right]\right) \\
&= \emptyset.
\end{aligned}$$

Proposition 3.12. *Let (X, q) be a q-hyperconvex quasi-cone metric space. Let $(x_i)_{i \in I}$ be a family of points in X and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of nonnegative vectors such that $q(x_i, x_j) \preceq r_i + s_j$.*

The set $D = \underset{i \in I}{\cap} (C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i))$ is nonempty and q-hyperconvex.

Proof. We first observe that since X is q-hyperconvex, then $D \neq \emptyset$. For each $\alpha \in S$, let $x_\alpha \in D$ and let r_α, s_α be nonnegative vectors such that $q(x_\alpha, x_\beta) \preceq r_\alpha + s_\beta$ whenever $\alpha, \beta \in S$.

We show that the family satisfies the hypothesis of q-hyperconvexity. Indeed, in particular, for each $\alpha \in S$ and $i \in I$, we have that $q(x_\alpha, x_i) \preceq s_i \preceq r_\alpha + s_i$ and $q(x_i, x_\alpha) \preceq r_i \preceq r_i + s_\alpha$. Hence by q-hyperconvexity of X,

$$\begin{aligned}
\emptyset &\neq \left[\bigcap_{i \in I} (C_q(x_i, r_i) \cap C_{q^t}(x_i, s_i))\right] \bigcap \left[\bigcap_{\alpha \in S} (C_q(x_\alpha, r_\alpha) \cap C_{q^t}(x_\alpha, s_\alpha))\right] \\
&= D \bigcap \left[\bigcap_{\alpha \in S} (C_q(x_\alpha, r_\alpha) \cap C_{q^t}(x_\alpha, s_\alpha))\right].
\end{aligned}$$

Hence, D is q-hyperconvex.

□

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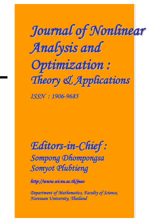

COMMON SOLUTIONS TO SOME SYSTEMS OF VECTOR EQUILIBRIUM PROBLEMS AND COMMON FIXED POINT PROBLEMS IN BANACH SPACE

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ABSTRACT. In this paper, we study some properties of set of solutions of a new generalized mixed vector equilibrium problem in Banach space. Further, we introduce an iterative method based on hybrid method and convex approximation method for finding a common element to the set of solutions of a system of unrelated generalized mixed vector equilibrium problems and the set of solutions of common fixed point problems for the two families of generalized asymptotically quasi ϕ-nonexpansive mappings in Banach space. Furthermore, we obtain a strong convergence theorem for the sequences generated by the proposed iterative scheme. Finally, we derive some consequences from our main result. The results presented in this paper extended and unify many of the previously known results in this area.

KEYWORDS : System of unrelated generalized mixed vector equilibrium problem; Common fixed-point problem; Generalized asymptotically quasi ϕ-nonexpansive mappings; Iterative schemes.
AMS Subject Classification: 49J30, 47H10, 47H17, 90C99.

1. INTRODUCTION

Throughout the paper unless otherwise stated, let E be a real Banach space with its dual space E^* and let $\langle .,. \rangle$ denote the duality pairing between E and E^* and $\|.\|$ denote the norm of E as well as of E^*. Let C be a nonempty, closed and convex subset of E and let 2^E denote the set of all nonempty subsets of E. Let Y be an ordered Banach space and let P be a pointed, proper, closed and convex cone of Y with $\text{int}P \neq \emptyset$.

In 1994, Blum and Oettli [3] introduced and studied the following equilibrium problem (in short, EP): Find $x \in C$ such that

$$F(x,y) \geq 0, \ \forall y \in C, \tag{1.1}$$

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where $F : C \times C \longrightarrow \mathbb{R}$ is a bifunction.

The EP(1.1) includes variational inequality problems, optimization problems, Nash equilibrium problems, saddle point problems, fixed point problems, complementary problems as special cases. In other words, EP(1.1) is a unified model for several problems arising in science, engineering, optimization, economics, etc.

In the last two decades, EP(1.1) has been generalized and extensively studied in many directions due to its importance; See for example [9, 11, 17] and references therein, for the literature on the existence of solution of the various generalizations of EP(1.1). Some iterative methods have been studied for solving various classes of equilibrium problems, see for example [4, 7, 12, 13, 14, 18, 25, 26, 27] and references therein.

In this paper, we introduce and study the following generalized mixed vector equilibrium problem (in short, GMVEP). Let $F : C \times C \longrightarrow Y$ and $\psi : C \times C \longrightarrow Y$ be nonlinear bimappings and let $A : C \longrightarrow B(E, Y)$, where $B(E, Y)$ is a Banach space of all continuous linear operators from E into Y, be a nonlinear mapping, then GMVEP is to find $x^* \in C$ such that

$$F(x^*, x) + \langle x - x^*, Ax^* \rangle + \psi(x, x^*) - \psi(x^*, x^*) \in P, \ \forall x \in C. \tag{1.2}$$

The solution set of GMVEP(1.2) is denoted by Sol(GMVEP(1.2)).

Example 1.1. Let $E = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy, \ \forall x, y \in \mathbb{R}$. Let $Y = \mathbb{R}$, then $P = [0, +\infty)$ and let $C = [0, 2]$. Let F and ψ be defined by $F(x, y) = x^2 - 3y - xy$ and $\psi(x, y) = x^2 - y^2 + 2y, \ \forall x, y \in C$, and $A(x) = x + 2, \ \forall x \in C$, respectively, then it is easy to observe that $\mathrm{Sol}(\mathrm{GMVEP}(1.2)) = [1, 2] \neq \emptyset$.

If $\psi = 0$ and $A = 0$, then GMVEP(1.2) reduces to the strong vector equilibrium problem (in short, SVEP): Find $x^* \in C$ such that

$$F(x^*, x) \in P, \ \forall x \in C, \tag{1.3}$$

which has been studied by Kazmi and Khan [15]. It is well known that the vector equilibrium problem provides a unified model of several problems, for example, vector optimization, vector variational inequality, vector complementary problem and vector saddle point problem [11, 17]. In recent years, the vector equilibrium problem has been intensively studied by many authors, see for example [9, 11, 15, 16, 17, 23] and the references therein.

If $Y = \mathbb{R}$, then $P = [0, +\infty)$ and hence GMVEP(1.2) reduces to the following generalized mixed equilibrium problem (in short, GMEP): Find $x \in C$ such that

$$F(x^*, x) + \langle x - x^*, Ax^* \rangle + \psi(x, x^*) - \psi(x^*, x^*) \geq 0, \ \forall x \in C, \tag{1.4}$$

where $\psi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function. The solution set of GMEP(1.4) is denoted by $\mathrm{Sol}(\mathrm{GMEP}(1.4))$. The GMEP(1.4) with $\psi(x, x^*) = \psi(x), \ \forall x, x^* \in C$ has been studied by Ceng and Yao [4] in Hilbert space.

If $Y = \mathbb{R}$ and $F = 0$, then $P = [0, +\infty)$ and hence GMVEP(1.2) reduces to the following generalized variational inequality problem (in short, GVIP): Find $x \in C$ such that

$$\langle x - x^*, Ax^* \rangle + \psi(x, x^*) - \psi(x^*, x^*) \geq 0, \ \forall x \in C, \tag{1.5}$$

where $\psi : C \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function.

Further, we consider the following new system of generalized mixed vector equilibrium problems, which we call the system of unrelated generalized mixed vector equilibrium problems (in short, SUGMVEP): For each $i = 1, 2, 3, ..., N$, let K_i be a nonempty, closed and convex set in E with $K = \cap_{i=1}^{N} K_i \neq \emptyset$; let $F_i : K_i \times K_i \longrightarrow Y$ be a bimapping, $\psi_i : K_i \times K_i \longrightarrow \mathbb{R}$ be a bifunction and $A_i : K_i \longrightarrow B(E, Y)$ be a nonlinear mapping. Then SUGMVEP is to find $x^* \in \cap_{i=1}^{N} K_i$ such that

$$F_i(x^*, y_i) + \langle y_i - x^*, A_i x^* \rangle + \psi_i(y_i, x^*) - \psi_i(x^*, x^*) \in P,\ \forall y_i \in K_i,\ i = 1, 2, 3, ..., N. \tag{1.6}$$

We denote by Sol$(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))$, the set of solution of GMVEP(1.2) corresponding to the mappings F_i, A_i, ψ_i and K_i. Then the set of solutions of SUGMVEP(1.6) is given by $\cap_{i=1}^{N} \mathrm{Sol}(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))$.

If $Y = \mathbb{R}$, then SUGMVEP(1.6) reduces to the system of unrelated generalized mixed equilibrium problems (SUGMEP) of finding $x^* \in \cap_{i=1}^{N} K_i$ such that

$$F_i(x^*, y_i) + \langle y_i - x^*, A_i x^* \rangle + \psi_i(y_i, x^*) - \psi_i(x^*, x^*) \geq 0,\ \forall y_i \in K_i,\ i = 1, 2, 3, ..., N, \tag{1.7}$$

which appears to be new.

If $E = H$, Hilbert space, $\psi_i = 0$ and $Y = \mathbb{R}$ for all i, then SUGMVEP(1.6) reduces to the system of unrelated mixed equilibrium problems introduced and studied by Djafari-Rouhani, Kazmi and Rizvi [8].

Further if $E = H,\ A_i = 0,\ \psi_i = 0$ and $Y = \mathbb{R}$ for all i, then SUGMVEP(1.6) reduces to the system of unrelated variational inequality problems considered and studied by Censor et al. [5] for set-valued version of mappings A_i.

We also observe that if $F_i = 0,\ \psi_i = 0,\ Y = \mathbb{R}$ and $E = H$ for all i, then SUGMVEP(1.6) reduces to the problem of finding a point $x \in \cap_{i=1}^{N} K_i$ which is well known convex feasibility problem. If the set K_i are fixed sets of family of operators $S_i : H \longrightarrow H$ then the convex feasibility problem is the the common fixed points problem (in short, CFPP).

Next, we recall a mapping $T : C \longrightarrow C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|,\ \forall x, y \in C$.

The fixed point problem (in short, FPP) for a nonexpansive mapping T is to:

$$\text{Find } x \in C \text{ such that } x \in \mathrm{Fix}(T), \tag{1.8}$$

where Fix(T) is the fixed point set of the nonexpansive mapping T. It is well known that if Fix$(T) \neq \emptyset$, then Fix(T) is closed and convex.

Let $U(E) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1\ \forall x, y \in U(E)$ with $x \neq y$. It is said to be *uniformly convex* if for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U(E)$, $\|x - y\| \geq \epsilon$ implies $\frac{\|x+y\|}{2} \leq 1 - \delta$. The space E is said to be *smooth* if the limit $\lim_{t \longrightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$ exists $\forall x, y \in U(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly for $x, y \in U(E)$.

The *normalized duality mapping* $J : E \longrightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},\ \forall x \in E.$$

It is well known that if E^* is strictly convex, then J is single valued and demicontinuous, i.e., if $x_n \longrightarrow x$ then $Jx_n \rightharpoonup Jx$.

It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex. Recall that E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \longrightarrow \|x\|$ then $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$. It is well known that if E is a uniformly convex Banach space then E enjoys the Kadec-Klee property.

Let E be a smooth Banach space. The *Lyapunov functional* $\phi : E \times E \longrightarrow \mathbb{R}_+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x, y \in E.$$

Observe that in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$, $\forall x, y \in H$. Alber [1] recently introduced a generalized projection operator $\prod_C$ in a Banach space E which is an analogue of the metric projection P_C in Hilbert space. Recall that the generalized projection $\prod_C : E \longrightarrow C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\prod_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

The existence and uniqueness of the operator $\prod_C$ follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. In Hilbert space, $\prod_C = P_C$. It is obvious from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle,$$

and

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \ \forall x, y \in E. \tag{1.9}$$

Note that if E is a reflexive, strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$.

Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty subset of E.

Definition 1.2. A mapping $T : C \longrightarrow C$ is said to be:

(i) *asymptotically regular* on C if for any bounded subset K of C,

$$\limsup_{n \longrightarrow \infty} \{\|T^{n+1}x - T^n x\| : x \in K\} = 0;$$

(ii) *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n \longrightarrow \infty} x_n = x_0$ and $\lim_{n \longrightarrow \infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Definition 1.3. [27] Let $T : C \longrightarrow C$ be a mapping. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges weakly to p so that $\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{\mathrm{Fix}}(T)$.

Definition 1.4. A mapping $T : C \longrightarrow C$ is said to be:

(i) [27] *relatively nonexpansive* if

$$\widehat{\mathrm{Fix}}(T) = \mathrm{Fix}(T) \neq \emptyset, \ \phi(p, Tx) \leq \phi(p, x), \ \forall x \in C, \ \forall p \in \mathrm{Fix}(T);$$

(ii) *relatively asymptotically nonexpansive* if

$$\widehat{\mathrm{Fix}}(T) = \mathrm{Fix}(T) \neq \emptyset,\ \phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x),\ \forall x \in C,\ \forall p \in \mathrm{Fix}(T),\ \forall n \geq 1,$$

where $\mu_n \subset [0, \infty)$ is a sequence such that $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$;

(iii) [29] *quasi ϕ-nonexpansive* if

$$\mathrm{Fix}(T) \neq \emptyset,\ \phi(p, Tx) \leq \phi(p, x),\ \forall x \in C,\ \forall p \in \mathrm{Fix}(T);$$

(iv) [29, 21] *asymptotically quasi ϕ-nonexpansive* if there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ such that

$$\mathrm{Fix}(T) \neq \emptyset,\ \phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x),\ \forall x \in C,\ \forall p \in \mathrm{Fix}(T),\ \forall n \geq 1;$$

(v) [22] *generalized asymptotically quasi ϕ-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and there exist two nonnegative sequences $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \longrightarrow 0$ and $\xi_n \subset [0, \infty)$ with $\xi_n \longrightarrow 0$ as $n \longrightarrow \infty$ such that

$$\phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x) + \xi_n,\ \forall x \in C,\ \forall p \in \mathrm{Fix}(T),\ \forall n \geq 1.$$

In 2007, Tada and Takahashi [25] and Takahashi and Takahashi [26] proved weak and strong convergence theorems for finding a common solution of EP(1.1) and FPP(1.8) of a nonexpansive mapping in a Hilbert space. For further related work, see Ceng and Yao [4] and Shan and Huang [23].

In 2009, Takahashi and Zembayashi [27] proved weak and strong convergence theorems for finding a common solution of EP(1.1) and FPP(1.8) of a relatively nonexpansive mapping in real Banach space. For further related work, see Kazmi and Farid [16] and the references therein. Recently, Qin and Agarwal [21] proved a strong convergence to common fixed points of the pair of asymptotic quasi ϕ-nonexpansive mappings in Banach space. Later Qin *et al.* [22] proved a strong convergence to common fixed points of a family of generalized asymptotically quasi ϕ-nonexpansive mappings. Very recently, Song and Chen [24] proved a strong convergence to common element of the set of solutions of mixed equilibrium problem and set of fixed points of FPP(1.8) of a generalized asymptotically quasi ϕ-nonexpansive mapping in Banach space.

Motivated by the work of Qin *et al.* [21, 22], Song and Chen [24], Shan and Huang [23], and by the ongoing research in this direction, we study the existence and properties of solution of a new generalized mixed vector equilibrium problem in Banach space. Further, we introduce an iterative method based on hybrid method and convex approximation method for finding a common element to the set of solutions of SUGMVEP(1.6) and the set of solutions of common fixed point problems for the two families of two generalized asymptotically quasi ϕ-nonexpansive mappings in Banach space. Furthermore, we obtain a strong convergence theorem for the sequences generated by the proposed iterative scheme. Finally, we derive some consequences from our main result. The results presented in this paper extended and unify many of the previously known results in this area, see for instance [16, 24].

2. PRELIMINARIES AND NOTATIONS

We recall some concepts and results which are needed in sequel.

Lemma 2.1. [1] *Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed and convex subset of E. Then the following conclusions hold:*

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y),\ \forall x \in C, y \in E;$

(ii) *Let $x \in E$ and $z \in C$ then*

$$z = \Pi_C(x) \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0,\ \forall y \in C.$$

Lemma 2.2. [20] *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E and let T be a relatively nonexpansive mapping from C into itself. Then $\mathrm{Fix}(T)$ is closed and convex.*

Lemma 2.3. [6] *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \mu\gamma g(\|y - z\|),$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

Definition 2.4. [19, 28] Let X and Y be two Hausdorff topological spaces and let D be a nonempty, convex subset of X and P be a pointed, proper, closed and convex cone of Y with $\mathrm{int}P \neq \emptyset$. Let 0 be the zero point of Y, $\mathbb{U}(0)$ be the neighborhood set of 0, $\mathbb{U}(x_0)$ be the neighborhood set of x_0 and $f : D \longrightarrow Y$ be a mapping.

(i) If, for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + P,\ \forall x \in U \cap D,$$

then f is called *upper P-continuous* on x_0. If f is *upper P-continuous* for all $x \in D$, then f is called *upper P-continuous* on D;

(ii) If, for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V - P,\ \forall x \in U \cap D,$$

then f is called *lower P-continuous* on x_0. If f is *lower P-continuous* for all $x \in D$, then f is called *lower P-continuous* on D;

(iii) If, for any $x, y \in D$ and $t \in [0, 1]$, the mapping f satisfies

$$f(x) \in f(tx + (1-t)y) + P \text{ or } f(y) \in f(tx + (1-t)y) + P,$$

then f is called *proper P-quasiconvex*;

(iv) If, for any $x, y \in D$ and $t \in [0, 1]$, the mapping f satisfies

$$tf(x) + (1-t)f(y) \in f(tx + (1-t)y) + P,$$

then f is called *P-convex*.

Lemma 2.5. [10] *Let X and Y be two real Hausdorff topological spaces, D be a nonempty, compact and convex subset of X and P be a pointed, proper, closed and convex cone of Y with $\mathrm{int}P \neq \emptyset$. Assume that $g : D \times D \longrightarrow Y$ and $\Phi : D \longrightarrow Y$ are two nonlinear mappings. Suppose that g and Φ satisfy*

(i) $g(x, x) \in P,\ \forall x \in D;$

(ii) *Φ is upper P-continuous on D;*

(iii) *$g(., y)$ is lower P-continuous, $\forall x \in D$;*

(iv) *$g(x, .) + \Phi(.)$ is proper P-quasiconvex, $\forall x \in D$.*

Then there exists a point $x \in D$ satisfies

$$G(x, y) \in P \setminus \{0\},\ \forall y \in D,$$

where

$$G(x, y) = g(x, y) + \Phi(y) - \Phi(x), \ \forall x, y \in D.$$

Let $F, \psi : C \times C \longrightarrow Y$ be two mappings and $A : C \longrightarrow B(E, Y)$ be a nonlinear mapping. For any $z \in E$, define a mapping $G_z : C \times C \longrightarrow Y$ as follows:

$$G_z(x, y) = F(x, y) + \langle y - x, Ax\rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r}\langle y - x, Jx - Jz\rangle, \quad (2.1)$$

where r is a positive real number and $e \in \operatorname{int}P$.

Assumption 2.6. *Let G_z, F, ψ satisfy the following conditions:*

(i) *For all $x \in C$, $F(x, x) \in P$;*
(ii) *F is P-monotone, i.e., $F(x, y) + F(y, x) \in -P, \ \forall x, y \in C$;*
(iii) *$F(., y)$ is continuous, $\forall y \in C$;*
(iv) *$F(x, .)$ is weakly continuous and P-convex, i.e.,*

$$tF(x, y_1) + (1 - t)F(x, y_2) \in F(x, ty_1 + (1 - t)y_2) + P, \ \forall x, y_1, y_2 \in C, \ \forall t \in [0, 1];$$

(v) *$G_z(., y)$ is lower P-continuous, $\forall y \in C$ and $z \in E$;*
(vi) *$\psi(., y)$ is P-convex and weakly continuous;*
(vii) *$G_z(x, .)$ is proper P-quasiconvex, $\forall x \in C$ and $z \in E$;*
(viii) *ψ is P-skew symmetric, i.e.,*

$$\psi(x, x) - \psi(x, y) - \psi(y, x) + \psi(y, y) \in P, \ \forall x, y \in C.$$

Remark 2.7. P-skew-symmetric bifunctions are natural extensions of skew-symmetric bifunctions. The skew-symmetric bifunctions have the properties that can be considered ananalogous to the monotonicity of the gradient and the non-negativity of the second derivative for convex functions. For the properties and applications of the skew-symmetric bifunctions, we refer the reader to [2].

3. MAIN RESULTS

For any $r > 0$, define a mapping $T_r : E \longrightarrow C$ as follows:

$$T_r(z) = \{x \in C : F(x, y) + \langle y - x, Ax\rangle + \psi(y, x) - \psi(x, x) + \frac{e}{r}\langle y - x, Jx - Jz\rangle \in P, \ \forall y \in C\}, \quad (3.1)$$

where $e \in \operatorname{int}P$.

Now, we study some properties of set of solutions of GMVEP(1.2) and the mapping T_r.

Theorem 3.1. *Let E be a uniformly smooth and strictly convex Banach space and let C be a nonempty, compact and convex subset of E. Assume that P is a pointed, proper, closed and convex cone of a real order Banach space Y with $\operatorname{int}P \neq \emptyset$. Let $G_z : C \times C \longrightarrow Y$ be defined by (2.1). Let $F, \psi : C \times C \longrightarrow Y$ and G_z satisfy Assumption 2.6 and $A : C \longrightarrow B(E, Y)$ be a continuous and P-monotone mapping. Let $T_r : E \longrightarrow C$ be defined by (3.1). Then the following conclusions hold:*

(i) *$T_r(z) \neq \emptyset, \ \forall z \in E$;*
(ii) *T_r is single-valued;*
(iii) *T_r is firmly nonexpansive type mapping, i.e., for all $z_1, z_2 \in E$,*

$$\langle T_r z_1 - T_r z_2, JT_r z_1 - JT_r z_2\rangle \leq \langle T_r z_1 - T_r z_2, Jz_1 - Jz_2\rangle;$$

(iv) *Fix(T_r) =Sol(GMVEP(1.2));*
(v) *Sol(GMVEP(1.2)) is closed and convex.*

Proof. (i) Let $g(x,y) = G_z(x,y)$ and $\Phi(y) = 0$ for all $x, y \in C$ and $z \in E$. It is easy to observe that $g(x,y)$ and $\Phi(y)$ satisfy all the conditions of Lemma 2.5 Then there exists a point $x \in C$ such that

$$G_z(x,y) + \Phi(y) - \Phi(x) \in P, \ \forall y \in C,$$

and thus $T_r(z) \neq \emptyset, \ \forall z \in E$.

(ii) For each $z \in E$, $T_r(z) \neq \emptyset$, let $x_1, x_2 \in T_r(z)$. Then

$$F(x_1,y)+\langle y-x_1, Ax_1\rangle+\psi(y,x_1)-\psi(x_1,x_1)+\frac{e}{r}\langle y-x_1, Jx_1-Jz\rangle \in P, \ \forall y \in C, \quad (3.2)$$

and

$$F(x_2,y)+\langle y-x_2, Ax_2\rangle+\psi(y,x_2)-\psi(x_2,x_2)+\frac{e}{r}\langle y-x_2, Jx_2-Jz\rangle \in P, \ \forall y \in C. \quad (3.3)$$

Letting $y = x_2$ in (3.2) and $y = x_1$ in (3.3), and then adding, we have

$$\begin{aligned} &F(x_1,x_2) + F(x_2,x_1) + \langle x_2 - x_1, Ax_1 - Ax_2\rangle \\ &+\psi(x_2,x_1) - \psi(x_1,x_1) + \psi(x_1,x_2) - \psi(x_2,x_2) + \tfrac{e}{r}\langle x_2 - x_1, Jx_1 - Jx_2\rangle \in P. \end{aligned}$$

Since F is P-monotone, A is P-monotone, i.e., $\langle x_2 - x_1, Ax_1 - Ax_2\rangle \in -P$ and ψ is P-skew symmetric, then we have

$$\frac{e}{r}\langle x_2 - x_1, Jx_1 - Jx_2\rangle \in P.$$

Since $e \in \mathrm{int}P$, $r > 0$ and P is closed and convex cone, we have

$$\frac{1}{r}\langle x_2 - x_1, Jx_1 - Jx_2\rangle \geq 0.$$

Since E is strictly convex, the preceding inequality implies $x_1 = x_2$. Hence T_r is single-valued.

(iii) For any $z_1, z_2 \in E$, let $x_1 = T_r(z_1)$ and $x_2 = T_r(z_2)$. Then

$$F(x_1,y) + \langle y - x_1, Ax_1\rangle + \psi(y,x_1) - \psi(x_1,x_1) + \frac{e}{r}\langle y - x_1, Jx_1 - Jz_1\rangle \in P, \ \forall y \in C, \quad (3.4)$$

and

$$F(x_2,y) + \langle y - x_2, Ax_2\rangle + \psi(y,x_2) - \psi(x_2,x_2) + \frac{e}{r}\langle y - x_2, Jx_2 - Jz_2\rangle \in P, \ \forall y \in C. \quad (3.5)$$

Letting $y = x_2$ in (3.4) and $y = x_1$ in (3.5), and then adding, we have

$$\begin{aligned} &F(x_1,x_2) + F(x_2,x_1) + \langle x_2 - x_1, Ax_1 - Ax_2\rangle \\ &+\psi(x_2,x_1) - \psi(x_1,x_1) + \psi(x_1,x_2) - \psi(x_2,x_2) + \tfrac{e}{r}\langle x_2 - x_1, Jx_1 - Jx_2 - Jz_1 + Jz_2\rangle \in P. \end{aligned}$$

By using the monotoncity of F, A and the properties of ψ and P, we have

$$\frac{1}{r}\langle x_2 - x_1, Jx_1 - Jx_2 - Jz_1 - Jz_2\rangle \geq 0.$$

Hence, we have

$$\langle x_2 - x_1, Jx_1 - Jx_2\rangle + \langle x_2 - x_1, Jz_2 - Jz_1\rangle \geq 0,$$

or,

$$\langle x_1 - x_2, Jx_1 - Jx_2\rangle \leq \langle x_1 - x_2, Jz_1 - Jz_2\rangle,$$

i.e.,

$$\langle T_r(z_1) - T_r(z_2), JT_r(z_1) - JT_r(z_2)\rangle \leq \langle T_r(z_1) - T_r(z_2), Jz_1 - Jz_2\rangle. \quad (3.6)$$

Thus T_r is firmly nonexpansive-type mapping.

(iv) Let $x \in \mathrm{Fix}(T_r)$. Then

$$F(x,y) + \langle y - x, Ax\rangle + \psi(y,x) - \psi(x,x) + \frac{e}{r}\langle y - x, Jx - Jx\rangle \in P,\ \forall y \in C,$$

and so

$$F(x,y) + \langle y - x, Ax\rangle + \psi(y,x) - \psi(x,x) \in P,\ \forall y \in C.$$

Thus $x \in \mathrm{Sol}(\mathrm{GMVEP}(1.2))$.

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and so

$$F(x,y) + \langle y - x, Ax\rangle + \psi(y,x) - \psi(x,x) + \frac{e}{r}\langle y - x, Jx - Jx\rangle \in P,\ \forall y \in C.$$

Hence $x \in \mathrm{Fix}(T_r)$. Thus $\mathrm{Fix}(T_r) = \mathrm{Sol}(\mathrm{GMVEP}(1.2))$.

(v) By the definition of ϕ, we have

$$\begin{aligned}\phi(T_r(z_1), T_r(z_2)) + \phi(T_r(z_1), T_r(z_2)) &= 2\|T_r(z_1)\|^2 - 2\langle T_r(z_1), JT_r(z_2)\rangle \\ &\quad -2\langle T_r(z_2), JT_r(z_1)\rangle + 2\|T_r(z_2)\|^2 \\ &= 2\langle T_r(z_1), JT_r(z_1) - JT_r(z_2)\rangle \\ &\quad +2\langle T_r(z_2), JT_r(z_2) - JT_r(z_1)\rangle \\ &= 2\langle T_r(z_1) - T_r(z_2), JT_r(z_1) - JT_r(z_2)\rangle,\end{aligned}$$

and

$$\phi(T_r(z_1), z_2) + \phi(T_r(z_2), z_1) - \phi(T_r(z_1), z_1) - \phi(T_r(z_2), z_2)$$

$$\begin{aligned}&= \|T_r(z_1)\|^2 - 2\langle T_r(z_1), Jz_2\rangle + \|z_2\|^2 + \|T_r(z_2)\|^2 + \|z_1\|^2 \\ &\quad -2\langle T_r(z_2), Jz_1\rangle - \|T_r(z_2)\|^2 + 2\langle T_r(z_2), Jz_2\rangle - \|z_2\|^2 \\ &\quad -\|T_r(z_1)\|^2 + 2\langle T_r(z_1), Jz_1\rangle - \|z_1\|^2 \\ &= 2\langle T_r(z_1), Jz_1 - Jz_2\rangle - 2\langle T_r(z_2), Jz_1 - Jz_2\rangle \\ &= 2\langle T_r(z_1) - T_r(z_2), Jz_1 - Jz_2\rangle.\end{aligned}$$

Thus, it follows from (3.6) and the preceding two equations that

$$\phi(T_r(z_1), T_r(z_2)) + \phi(T_r(z_2), T_r(z_1)) \leq \phi(T_r z_1, z_2) + \phi(T_r z_2, z_1) - \phi(T_r(z_1), z_1) - \phi(T_r(z_2), z_2).$$

Hence, for $z_1, z_2 \in C$, we have

$$\phi(T_r(z_1), T_r(z_2)) + \phi(T_r(z_2), T_r(z_1)) \leq \phi(T_r z_1, z_2) + \phi(T_r z_2, z_1).$$

Taking $z_2 = u \in \mathrm{Fix}(T_r)$, we have

$$\phi(u, T_r z_1) \leq \phi(u, z_1).$$

Next, we show that $\widehat{\mathrm{Fix}}(T_r) = \mathrm{Sol}(\mathrm{GMVEP}(1.2))$. Indeed, let $p \in \widehat{\mathrm{Fix}}(T_r)$. Then there exists a sequence $\{z_n\} \subset E$ such that $z_n \rightharpoonup p$ and $\lim_{n\longrightarrow\infty} \|z_n - T_r z_n\| = 0$. Moreover, we get $T_r z_n \rightharpoonup p$. Hence, we have $p \in C$. Since J is uniformly continuous on bounded sets, we have

$$\lim_{n\longrightarrow\infty} \frac{\|Jz_n - JT_r z_n\|}{r} = 0,\ r > 0. \tag{3.7}$$

From the definition of T_r, we have, $\forall y \in C$,

$$F(T_r z_n, y) + \langle y - T_r z_n, AT_r z_n\rangle + \psi(y, T_r z_n) - \psi(T_r z_n, T_r z_n) + \frac{e}{r}\langle y - T_r z_n, JT_r z_n - Jz_n\rangle \in P,$$

$$\begin{aligned}0 \in\ & F(y, T_r z_n) - \langle y - T_r z_n, AT_r z_n\rangle - (\psi(y, T_r z_n) - \psi(T_r z_n, T_r z_n)) \\ & -\tfrac{e}{r}\langle y - T_r z_n, JT_r z_n - Jz_n\rangle + P,\ \forall y \in C.\end{aligned}$$

Let $y_t = (1-t)p + ty,\ \forall t \in (0,1]$. Since $y \in C$ and $p \in C$, we get $y_t \in C$ and hence

$$\begin{array}{rcl}
0 & \in & F(y,T_rz_n) - \langle y_t - T_rz_n, AT_rz_n\rangle - (\psi(y_t,T_rz_n) - \psi(T_rz_n,T_rz_n)) \\
 & & -\frac{e}{r}\langle y_t - T_rz_n, JT_rz_n - Jz_n\rangle + P, \\
 & = & F(y_t,T_rz_n) - \langle y_t - T_rz_n, AT_rz_n\rangle - (\psi(y_t,T_rz_n) - \psi(T_rz_n,T_rz_n)) \\
 & & -e\langle y_t - T_rz_n, \frac{JT_rz_n - Jz_n}{r}\rangle + P. \\
0 & \in & F(y_t,p) - \langle y_t - p, Ap\rangle - \psi(y_t,p) + \psi(p,p) + P.
\end{array} \tag{3.8}$$

It follows from Assumption 2.6 (i), (iv) and (vi) that

$$\begin{array}{rcl}
tF(y_t,y) + (1-t)F(y_t,p) + t\psi(y,p) + (1-t)\psi(p,p) - \psi(y_t,p) & \in & F(y_t,y_t) + \psi(y_t,p) \\
 & & -\psi(y_t,p) + P \\
 & \in & P.
\end{array}$$

Now,

$$\begin{array}{l}
-t[F(y_t,y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] - \\
(1-t)[F(y_t,p) + \psi(p,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] \in -P + \langle y_t - p, Ap\rangle \\
-t[F(y_t,y)+\psi(y,p)-\psi(y_t,p)-\langle y_t-p, Ap\rangle]-(1-t)P \in -P+\langle y_t-p, Ap\rangle,\ (\text{using } (3.7)) \\
-t[F(y_t,y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] \in -P + (1-t)P + \langle y_t - p, Ap\rangle \\
-t[F(y_t,y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle] \in -tP + \langle y_t - p, Ap\rangle \\
\in -tP + \langle ty + (1-t)p - p, Ap\rangle \ \in -tP + t\langle y - p, Ap\rangle
\end{array}$$

$$F(y_t,y) + \psi(y,p) - \psi(y_t,p) - \langle y_t - p, Ap\rangle \in P - \langle y - p, Ap\rangle.$$

Letting $t \longrightarrow 0_+$, we have

$$F(p,y) + \langle y - p, Ap\rangle + \psi(y,p) - \psi(p,p) \in P.$$

Thus $p \in \mathrm{Sol}(\mathrm{GMVEP}(1.2))$. So, we get $\mathrm{Fix}(T_r) = \mathrm{Sol}(\mathrm{GMVEP}(1.2)) = \widehat{\mathrm{Fix}}(T_r)$. Therefore T_r is a relatively nonexpansive mapping. Further, it follows from Lemma 2.2 that $\mathrm{Sol}(\mathrm{GMVEP}(1.2)) = \mathrm{Fix}(T_r)$ is closed and convex. This completes the proof. □

Next, we have the following consequence of Theorem 3.1

Lemma 3.2. *Let $E,\ C,\ F,\ \psi,\ G_z$ be the same as in Theorem 3.1 and let $r > 0$. Then, for $x \in E$ and $q \in \mathrm{Fix}(T_r)$, we have*

$$\phi(q,T_rx) + \phi(T_rx,x) \leq \phi(q,x).$$

We prove a strong convergence theorem for finding a common element to the set of solutions of SUGMVEP(1.6) and set of fixed points of common fixed point problems of two families of generalized asymptotically quasi ϕ-nonexpansive mappings in Banach space.

Theorem 3.3. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. For each $i \in I := \{1,2,3,...,N\}$, let K_i be a nonempty, compact and convex subset of E such that $K = \cap_{i=1}^N K_i \neq \emptyset$. Assume that P is a pointed, proper, closed and convex cone of a real ordered Banach space Y with $\mathrm{int}P \neq \emptyset$. Let for each i, the mappings $F_i, \psi_i : K_i \times K_i \longrightarrow Y$ satisfy Assumption 2.6 and $A_i : K_i \longrightarrow B(E,Y)$ be continuous and P-monotone mapping. For each fixed i, let $S_i, T_i : K_i \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_{n,i}\}$, $\{\varsigma_{n,i}\}$ and $\{\zeta_{n,i}\}$, $\{\xi_{n,i}\}$ such that $\Gamma := (\cap_{i=1}^N \mathrm{Fix}(S_i)) \cap (\cap_{i=1}^N \mathrm{Fix}(T_i)) \cap (\cap_{i=1}^N \mathrm{Sol}(\mathrm{GMVEP}(F_i,A_i,\psi_i,K_i))) \neq \emptyset$. Assume that, for each fixed i, the sequences $\{r_{n,i}\} \subset [a,\infty)$ for some $a > 0$, $\{\alpha_{n,i}\} \subseteq (0,1)$ and $\{\beta_{n,i}^j\} \subseteq (0,1)$ $(j = 1,2,3)$ be such that*

(i) $\beta^1_{n,i} + \beta^2_{n,i} + \beta^3_{n,i} = 1$;
(ii) $\liminf_{n\longrightarrow\infty} \beta^1_{n,i}\beta^2_{n,i} > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta^1_{n,i}\beta^3_{n,i} > 0$;
(iii) $\limsup_{n\longrightarrow\infty} \alpha_{n,i} < 1$;
(iv) $\liminf_{n\longrightarrow\infty} r_{n,i} > 0$.

Let $\{x_n\}$ *be a sequence generated by the iterative scheme:*

$$\begin{array}{l} x_0 \in E, \\ C_{1,i} =: K_i,\ C_1 = \cap_{i=1}^N C_{1,i} = K, \\ x_1 = \prod_{C_1} x_0, \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_n + (1-\alpha_{n,i})Jz_{n,i}), \\ z_{n,i} = J^{-1}(\beta^1_{n,i}Jx_n + \beta^2_{n,i}JT^n_i x_n + \beta^3_{n,i}JS^n_i x_n), \\ u_{n,i} = T_{r_{n,i}}(y_{n,i}) \\ C_{n+1,i} = \{v \in C_{n,i} : \phi(v, u_{n,i}) \le \phi(v, x_n) + \delta_{n,i}M_n + \mu_{n,i}\}, \\ C_{n+1} = \cap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \prod_{C_{n+1}} x_0, \text{ for every } n \in N \cup \{0\}, \end{array}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \mathrm{int}P$; $J : E \longrightarrow E^*$ *is the normalized duality mapping with its inverse* J^{-1}; $\delta_{n,i} = \beta^2_{n,i}\zeta_{n,i} + \beta^3_{n,i}\eta_{n,i}$ *and* $\mu_{n,i} = \xi_{n,i}\beta^2_{n,i} + \varsigma_{n,i}\beta^3_{n,i}$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\prod_\Gamma x_0$.

Proof. First, we show that C_n is closed and convex for every $n \ge 1$. It suffices to show that for each $i \in I$, $C_{n,i}$ is closed and convex for every $n \ge 1$. This can be proved by induction on n. In fact, for $n = 1$, $C_{1,i} = K_i$ is closed and convex for each $i \in I$. Assume that $C_{n,i}$ is closed and convex for some $n \ge 1$ and for each $i \in I$. For $v \in C_{n+1,i}$,

$$\phi(v, u_{n,i}) \le \phi(v, x_n) + \delta_{n,i}M_n + \mu_{n,i},$$

which is equivalent to

$$2\langle v, Jx_n - Ju_{n,i}\rangle \le \|x_n\|^2 - \|u_{n,i}\|^2 + \delta_{n,i}M_n + \mu_{n,i}.$$

It is easy to see that $C_{n+1,i}$ is closed and convex for each $i \in I$. Then, for all $n \ge 1$, $C_{n,i}$ is closed and convex for each $i \in I$. Consequently, $C_n = \cap_{i=1}^N C_{n,i}$ is closed and convex for all $n \ge 1$. This shows that $\prod_{C_{n+1}} x_0$ is well defined.

Next, we prove $\Gamma \subset C_n$ for all $n \ge 1$. It suffices to show that for each $i \in I$, $\Gamma \subset C_{n,i}$. Indeed, $\Gamma \subset C_{1,i} = K_i$ is obvious. Suppose $\Gamma \subset C_{k,i}$ for some $k \ge 1$. Then, for $\forall w \in \Gamma \subset C_{k,i}$, we have

$$\begin{array}{rcl} \phi(w, z_{k,i}) & = & \phi(w, J^{-1}(\beta^1_{k,i}Jx_k + \beta^2_{k,i}JT^k_i x_k + \beta^3_{k,i}JS^k_i x_k)) \\ & \le & \|w\|^2 - 2\langle w, \beta^1_{k,i}Jx_k + \beta^2_{k,i}JT^k_i x_k + \beta^3_{k,i}JS^k_i x_k\rangle \\ & & +\|\beta^1_{k,i}Jx_k + \beta^2_{k,i}JT^k_i x_k + \beta^3_{k,i}JS^k_i x_k\|^2 \\ & \le & \|w\|^2 - 2\beta^1_{k,i}\langle w, Jx_k\rangle - 2\beta^2_{k,i}\langle w, JT^k_i x_k\rangle - 2\beta^3_{k,i}\langle w, JS^k_i x_k\rangle \\ & & +\beta^1_{k,i}\|x_k\|^2 + \beta^2_{k,i}\|T^k_i x_k\|^2 + \beta^3_{k,i}\|S_i x_k\|^2 \\ & = & \beta^1_{k,i}\phi(w, x_k) + \beta^2_{k,i}\phi(w, T^k_i x_k) + \beta^3_{k,i}\phi(w, S^k_i x_k) \\ & \le & \beta^1_{k,i}\phi(w, x^k) + \beta^2_{k,i}(1+\zeta_{k,i})\phi(w, x_k) + \xi_{k,i}\beta^2_{k,i} + \beta^3_{k,i}(1+\eta_{k,i})\phi(w, x_k) + \varsigma_{k,i}\beta^3_{k,i} \\ & \le & \phi(w, x_k) + \delta_{k,i}\phi(w, x_k) + \mu_{k,i}. \end{array} \tag{3.9}$$

Further, it follows from Theorem 3.1 that $u_{n,i} = T_{r_{n,i}} y_{n,i}$ for all $n \in N \cup \{0\}$, and $T_{r_{n,i}}$ is relatively nonexpansive. Therefore

$$
\begin{aligned}
\phi(w, u_{k,i}) &= \phi(w, T_{r_{k,i}} y_{k,i}) \\
&\leq \phi(w, y_{k,i}) \\
&= \phi(w, J^{-1}(\alpha_{k,i} J x_k + (1-\alpha_{k,i}) J z_{k,i})) \\
&= \|w\|^2 - 2\langle w, \alpha_{k,i} J x_k + (1-\alpha_{k,i}) J z_{k,i}\rangle + \|\alpha_{k,i} J x_k + (1-\alpha_{k,i}) J z_{k,i}\|^2 \\
&= \|w\|^2 - 2\alpha_{k,i}\langle w, J x_k\rangle - 2(1-\alpha_{k,i})\langle w, J z_{k,i}\rangle + \alpha_{k,i}\|x_k\|^2 + (1-\alpha_{k,i})\|z_{k,i}\|^2 \\
&= \alpha_{k,i}\phi(w, x_k) + (1-\alpha_{k,i})\phi(w, z_{k,i}) \\
&\leq \alpha_{k,i}\phi(w, x_k) + (1-\alpha_{k,i})\phi(w, x_k) - (1-\alpha_{k,i})(\beta^2_{k,i}\zeta_{k,i} + \beta^3_{k,i}\eta_{k,i})\phi(w, x_k) \\
&\quad + (1-\alpha_{k,i})(\xi_{k,i}\beta^2_{k,i} + \varsigma_{k,i}\beta^3_{k,i}) \\
&\leq \phi(w, x_k) + (1-\alpha_{k,i})[(\beta^2_{k,i}\zeta_{k,i} + \beta^3_{k,i}\eta_{k,i})\phi(w, x_k) + (\xi_{k,i}\beta^2_{k,i} + \varsigma_{k,i}\beta^3_{k,i})] \\
&\leq \phi(w, x_k) + (\beta^2_{k,i}\zeta_{k,i} + \beta^3_{k,i}\eta_{k,i})\phi(w, x_k) + (\xi_{k,i}\beta^2_{k,i} + \varsigma_{k,i}\beta^3_{k,i}) \\
&\leq \phi(w, x_k) + \delta_{k,i} M_k + \mu_{k,i}.
\end{aligned}
\tag{3.10}
$$

This shows that $w \in C_{k+1,i}$. That is, $\Gamma \subset C_{n,i}$, for all $n \geq 1$ and each $i \in I$. Therefore, $w \in C_n = \cap_{i \in I} C_{n,i}$ for all $n \geq 1$.

From Lemma 2.1, we have

$$
\begin{aligned}
\phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\
&\leq \phi(w, x_0) - \phi(w, x_n) \\
&\leq \phi(w, x_0), \text{ for each } w \in \Gamma \subset C_n \text{ and for each } n \geq 1.
\end{aligned}
$$

Therefore, the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows from (1.9) that the sequence $\{x_n\}$ is also bounded. Since E is reflexive, without loss of generality, we may assume that $x_n \rightharpoonup p$ as $n \longrightarrow \infty$. Since $C_j \subset C_n$ for $j \geq n$, we have $x_j \in C_n$ for $j \geq n$. Since C_n is closed and convex, $p \in C_n$ for all $n \geq 1$. Hence $p \in \cap_{n=1}^{\infty} C_n$. Since $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \phi(p, x_0)$, we have

$$
\phi(p, x_0) \leq \liminf_{n \longrightarrow \infty} \phi(x_n, x_0) \leq \limsup_{n \longrightarrow \infty} \phi(x_n, x_0) \leq \phi(p, x_0),
$$

which implies that $\phi(x_n, x_0) \longrightarrow \phi(p, x_0)$ as $n \longrightarrow \infty$. Hence $\|x_n\| \longrightarrow \|p\|$. By the Kadec-Klee property of E, we have $x_n \longrightarrow p \in C$ as $n \longrightarrow \infty$.

Since $x_n \longrightarrow p$ and $J : E \longrightarrow E^*$ is demicontinuous, we have $Jx_n \rightharpoonup Jp \in E^*$. Note that

$$
|\|Jx_n\| - \|Jp\|| = |\|x_n\| - \|p\|| \leq \|x_n - p\|.
$$

This implies that $\|Jx_n\| \longrightarrow \|Jp\|$. Since E is uniformly smooth, E^* is uniformly convex Banach space and hence it enjoys the Kadec-Klee property, we see that

$$
\lim_{n \longrightarrow \infty} \|Jx_n - Jp\| = 0. \tag{3.11}
$$

On the other hand, since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$
\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \text{ for all } n \geq 1.
$$

Therefore, $\{\phi(x_n, x_0)\}$ is non-decreasing. Further, it follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$
\begin{aligned}
\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\
&\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_m, x_0) - \phi(x_n, x_0).
\end{aligned}
\tag{3.12}
$$

Letting $m, n \longrightarrow \infty$ in (3.12), we have $\phi(x_m, x_n) \longrightarrow 0$.

Next, we show that $p \in (\cap_{i=1}^{n}\mathrm{Fix}(S_i)) \cap (\cap_{i=1}^{n}\mathrm{Fix}(T_i))$. By taking $m = n + 1$ in (3.12), we have

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.13}$$

Notice that $x_{n+1} \in C_{n+1}$, from the definition of C_n, for every $i \in I$, we have

$$\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) + \delta_{n,i} M_n + \mu_{n,i}. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\lim_{n \longrightarrow \infty} \phi(x_{n+1}, u_{n,i}) = 0. \tag{3.15}$$

It follows from (3.15) and inequality $0 \leq (\|x_{n+1}\| - \|u_{n,i}\|)^2 \leq \phi(x_{n+1}, u_{n,i})$ that $\|u_{n,i}\| \longrightarrow \|p\|$ and consequently, we have $\|Ju_{n,i}\| \longrightarrow \|Jp\|$. This implies that $\{J(u_{n,i})\}$ is bounded. Since E is reflexive, E^* is also reflexive. So, we may assume that $J(u_{n,i}) \rightharpoonup f_{0,i} \in E^*$.

On the other hand, in view of the reflexivity of E, we have $J(E) = E^*$, which means that for $f_{0,i} \in E^*$, there exists $e_i \in E$, such that $Je_i = f_{0,i}$. Using weakly lower semi-continuity of $\|.\|^2$, we have

$$\begin{aligned}
\liminf_{n\longrightarrow\infty} \phi(x_{n+1}, u_{n,i}) &= \liminf_{n\longrightarrow\infty}(\|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{n,i}\rangle + \|u_{n,i}\|^2)\\
&= \liminf_{n\longrightarrow\infty}(\|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_{n,i}\rangle + \|Ju_{n,i}\|^2)\\
&\geq \|p\|^2 - 2\langle p, f_{0,i}\rangle + \|f_{0,i}\|^2\\
&= \|p\|^2 - 2\langle p, Je_i\rangle + \|Je_i\|^2\\
&= \phi(p, e_i).
\end{aligned}$$

It follows from (3.15) that $\phi(p, e_i) = 0$. Hence $p = e_i$, which implies that $f_{0,i} = Jp$. Hence $Ju_{n,i} \rightharpoonup Jp \in E^*$. Since $\|Ju_{n,i}\| \longrightarrow \|Jp\|$ and by the Kadec-Klee property of E^*, we have

$$\|Ju_{n,i} - Jp\| \longrightarrow 0, \ \forall i \in I. \tag{3.16}$$

Since $J^{-1} : E^* \longrightarrow E$ is demi-continuous, therefore $u_{n,i} \rightharpoonup p$. Since $\|u_{n,i}\| \longrightarrow \|p\|$ and using the Kadec-Klee property of E^*, we have

$$u_{n,i} \longrightarrow p, \text{ as } n \longrightarrow \infty \ \forall i \in I. \tag{3.17}$$

Hence,

$$\lim_{n\longrightarrow\infty} \|x_n - u_{n,i}\| = 0, \ \forall i \in I. \tag{3.18}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\longrightarrow\infty} \|Jx_n - Ju_{n,i}\| = 0, \ \forall i \in I. \tag{3.19}$$

Now,

$$\begin{aligned}
\phi(w, x_n) - \phi(w, u_{n,i}) &= \|x_n\|^2 - \|u_{n,i}\|^2 - 2\langle w, Jx_n - Ju_{n,i}\rangle\\
&\leq \|x_n - u_{n,i}\|(\|x_n\| + \|u_{n,i}\|) + 2\|w\|\|Jx_n - Ju_{n,i}\|.
\end{aligned}$$

It follows from (3.18) and (3.19) that

$$\phi(w, x_n) - \phi(w, u_{n,i}) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.20}$$

From $u_{n,i} = T_{r_{n,i}} y_{n,i}$ and Lemma 3.2, we have

$$\begin{aligned}
\phi(u_{n,i}, y_{n,i}) &= \phi(T_{r_{n,i}} y_{n,i}, y_{n,i})\\
&\leq \phi(w, y_{n,i}) - \phi(w, u_{n,i})\\
&\leq \alpha_{n,i}\phi(w, x_n) + (1 - \alpha_{n,i})\phi(w, z_{n,i}) - \phi(w, u_{n,i})\\
&\leq \alpha_{n,i}\phi(w, x_n) + (1 - \alpha_{n,i})[\phi(w, x_n) + \delta_{n,i} M_n + \mu_{n,i}\\
&\quad -\beta^1_{n,i}\beta^3_{n,i} g(\|Jx_n - JS_i^n x_n\|)] - \phi(w, u_{n,i})\\
&\leq \phi(w, x_n) - \phi(w, u_{n,i}) + (1 - \alpha_{n,i})[\delta_{n,i} M_n + \mu_{n,i}].
\end{aligned} \tag{3.21}$$

Using (3.20) and the restrictions on the sequences in above inequality, we have

$$\lim_{n\longrightarrow\infty}\phi(u_{n,i},y_{n,i})=0. \tag{3.22}$$

Since $0\leq(\|u_{n,i}\|-\|y_{n,i})\|)^2\leq\phi(u_{n,i},y_{n,i})$, it follows from (3.17) that $\|y_{n,i}\|\longrightarrow\|p\|$ and consequently $\|Jy_{n,i}\|\longrightarrow\|Jp\|$. This implies that $\{Jy_{n,i}\}$ is bounded. Since E is reflexive, E^* is also reflexive, so we may assume that $J(y_{n,i})\rightharpoonup h_{0,i}\in E^*$.

On the other hand, in view of the reflexivity of E, we have $J(E)=E^*$, which means that for $h_{0,i}\in E^*$, there exists $d_i\in E$, such that $Jd_i=h_{0,i}$. Using lower semi-continuity of $\|.\|^2$, we have

$$\begin{aligned}
\liminf_{n\longrightarrow\infty}\phi(u_{n,i},y_{n,i}) &= \liminf_{n\longrightarrow\infty}(\|u_{n,i}\|^2-2\langle u_{n,i},Jy_{n,i}\rangle+\|y_{n,i}\|^2)\\
&= \liminf_{n\longrightarrow\infty}(\|u_{n,i}\|^2-2\langle u_{n,i},Jy_{n,i}\rangle+\|Jy_{n,i}\|^2)\\
&\geq \|p\|^2-2\langle p,h_{0,i}\rangle+\|h_{0,i}\|^2\\
&= \|p\|^2-2\langle p,Jd_i\rangle+\|d_i\|^2\\
&= \phi(p,d_i).
\end{aligned}$$

It follows from (3.22) that $\phi(p,d_i)=0$. Hence $p=d_i$, which implies that $h_{0,i}=Jp$. Hence $Jy_{n,i}\rightharpoonup Jp\in E^*$. Since $Jy_{n,i}\rightharpoonup Jp$, $\|Jy_{n,i}\|\longrightarrow\|Jp\|$ and Kadec-Klee property of E^*, we have

$$\|Jy_{n,i}-Jp\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.23}$$

Since $J^{-1}:E^*\longrightarrow E$ is demi-continuous, we have $y_{n,i}\rightharpoonup p$. Further, since $\|y_{n,i}\|\longrightarrow\|p\|$ and E has the Kadec-Klee property, we have

$$\|y_{n,i}-p\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.24}$$

It follows from (3.18) and (3.24) that

$$\|y_{n,i}-u_{n,i}\|\leq\|u_{n,i}-p\|+\|y_{n,i}-p\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty.$$

Since J is uniformly norm-to- norm continuous on bounded sets, we have

$$\|Jy_{n,i}-Ju_{n,i}\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.25}$$

It follows from (3.19) and (3.25) that

$$\|Jx_n-Jy_{n,i}\|\leq\|Jx_n-Ju_{n,i}\|+\|Ju_{n,i}-Jy_{n,i}\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.26}$$

From iterative scheme, (3.26) and $\limsup_{n\longrightarrow\infty}\alpha_{n,i}<1$, it follows that

$$\|Jz_{n,i}-Jx_n\|\leq\frac{1}{(1-\alpha_{n,i})}\|Jx_n-Jy_{n,i}\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.27}$$

Hence,

$$\|Jz_{n,i}-Jp\|\leq\|Jz_{n,i}-Jx_n\|+\|Jx_n-Jp\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.28}$$

Since $J^{-1}:E^*\longrightarrow E$ is demi-continuous, we have $z_{n,i}\rightharpoonup p$. From (3.28), we have that

$$|\|z_{n,i}\|-\|p\||=|\|Jz_{n.i}\|-\|Jp\||\leq\|Jz_{n,i}-Jp\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty.$$

This implies that $\|z_{n,i}\|\longrightarrow\|p\|$. Since E enjoys the Kadec-Klee property, we see that

$$\lim_{n\longrightarrow\infty}\|z_{n,i}-p\|=0. \tag{3.29}$$

Further,

$$\|z_{n,i}-x_n\|\leq\|z_{n,i}-p\|+\|x_n-p\|\longrightarrow 0,\ \text{as } n\longrightarrow\infty. \tag{3.30}$$

Now,

$$\begin{aligned}
\phi(w,x_n)-\phi(w,z_{n,i}) &= \|x_n\|^2-\|z_{n,i}\|^2-2\langle w,Jx_n-Jz_{n,i}\rangle\\
&\leq \|x_n-z_{n,i}\|(\|x_n\|+\|z_{n,i}\|)+2\|w\|\|Jx_n-Jz_{n,i}\|.
\end{aligned}$$

It follows from (3.27) and (3.30) that

$$\phi(w, x_n) - \phi(w, z_{n,i}) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.31}$$

Let $r = \max\{\sup_{n\geq 1}\{\|x_n\|\}, \sup_{n\geq 1}\{\|T_i^n x_n\|\}, \sup_{n\geq 1}\{\|S_i^n x_n\|\}\}$. Since E is uniformly smooth, then E^* is uniformly convex. In the light of Lemma 2.3, we have, for any fixed $w \in \mathrm{Fix}(T_i) \cap \mathrm{Fix}(S_i)$,

$$\begin{aligned}
\phi(w, z_{n,i}) &= \phi(w, J^{-1}(\beta^1_{n,i} J x_n + \beta^2_{n,i} J T_i^n x_n + \beta^3_{n,i} J S_i^n x_n)) \\
&= \|w\|^2 - 2\langle w, \beta^1_{n,i} J x_n + \beta^2_{n,i} J T_i^n x_n + \beta^3_{n,i} J S_i^n x_n\rangle \\
&\quad + \|\beta^1_{n,i} J x_n + \beta^2_{n,i} J T_i^n x_n + \beta^3_{n,i} J S_i^n x_n\|^2 \\
&\leq \|w\|^2 - 2\beta^1_{n,i}\langle w, J x_n\rangle - 2\beta^2_{n,i}\langle w, J T_i^n x_n\rangle - 2\beta^3_{n,i}\langle w, J S_i^n x_n\rangle \\
&\quad + \beta^1_{n,i}\|x_n\|^2 + \beta^2_{n,i}\|T_i^n x_n\|^2 + \beta^3_{n,i}\|S_i^n x_n\|^2 - \beta^1_{n,i}\beta^3_{n,i} g(\|J x_n - J S_i^n x_n\|) \\
&= \beta^1_{n,i}\phi(w, x_n) + \beta^2_{n,i}(1 - \zeta_{n,i})\phi(w, x_n) + \beta^2_{n,i}\xi_{n,i} \\
&\quad + \beta^3_{n,i}(1 + \eta_{n,i})\phi(w, x_n) + \beta^3_{n,i}\varsigma_{n,i} - \beta^1_{n,i}\beta^3_{n,i} g(\|J x_n - J S_i^n x_n\|) \\
&= \phi(w, x_n) + (\beta^2_{n,i}\zeta_{n,i} + \beta^3_{n,i}\eta_{n,i})\phi(w, x_n) \\
&\quad + \beta^2_{n,i}\xi_{n,i} + \beta^3_{n,i}\varsigma_{n,i} - \beta^1_{n,i}\beta^3_{n,i} g(\|J x_n - J S_i^n x_n\|) \\
&= \phi(w, x_n) + \delta_{n,i}\phi(w, x_n) + \mu_{n,i} - \beta^1_{n,i}\beta^3_{n,i} g(\|J x_n - J S_i^n x_n\|) \\
&\leq \phi(w, x_n) + \delta_{n,i} M_n + \mu_{n,i} - \beta^1_{n,i}\beta^3_{n,i} g(\|J x_n - J S_i^n x_n\|).
\end{aligned}$$

This implies that

$$\beta^1_{n,i}\beta^3_{n,i} g(\|J x_n - J S_i^n x_n\|) \leq \phi(w, x_n) - \phi(w, z_{n,i}) + \delta_{n,i} M_n + \mu_{n,i}. \tag{3.32}$$

Taking limit on both sides of (3.32) and using $\liminf_{n\longrightarrow\infty} \beta^1_{n,i}\beta^3_{n,i} > 0$ and (3.31), we have

$$g(\|J x_n - J S_i^n x_n\|) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.33}$$

Therefore, using Lemma 2.3, (3.33) implies that

$$\|J x_n - J S_i^n x_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.34}$$

Next,

$$\|J S_i^n x_n - J p\| \leq \|J S_i^n x_n - J x_n\| + \|J x_n - J p\|.$$

Using (3.34) and (3.11) in above inequality, we get

$$\lim_{n\longrightarrow\infty} \|J S_i^n x_n - J p\| = 0. \tag{3.35}$$

Since J^{-1} is demicontinuous, it follows that $S_i^n x_n \rightharpoonup p$ for each fixed i.

Using $|\|S_i^n x_n\| - \|p\|| = |\|J S_i^n x_n\| - \|J p\|| \leq \|J S_i^n x_n - J p\|$ with (3.35), we get $\|S_i^n x_n\| \longrightarrow \|p\|$ for each fixed i as $n \longrightarrow \infty$. Since E enjoys the Kadec-Klee property, we obtain

$$\lim_{n\longrightarrow\infty} \|S_i^n x_n - p\| = 0. \tag{3.36}$$

Similarly, we obtain

$$\lim_{n\longrightarrow\infty} \|T_i^n x_n - p\| = 0. \tag{3.37}$$

Notice that

$$\|S_i^{n+1} x_n - p\| \leq \|S_i^{n+1} x_n - S_i^n x_n\| + \|S_i^n x_n - p\|. \tag{3.38}$$

Since S_i^n is asymptotic regular, then from (3.36) and (3.38), we have

$$\lim_{n\longrightarrow\infty} \|S_i^{n+1} x_n - p\| = 0,$$

that is, $S_i S_i^n x_n - p \longrightarrow 0$ as $n \longrightarrow \infty$. It follows from the closedness of S_i that $S_i p = p$, for each fixed i. Hence $p \in \cap_{i=1}^n \mathrm{Fix}(S_i)$. In a similar way, we can obtain $p \in \cap_{i=1}^n \mathrm{Fix}(T_i)$. Hence $p \in (\cap_{i=1}^n \mathrm{Fix}(T_i)) \cap (\cap_{i=1}^n \mathrm{Fix}(S_i))$.

Next, we prove $p \in \cap_{i=1}^N \mathrm{Sol}(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))$.

It follows from (3.25) and $\liminf_{n\longrightarrow\infty} r_{n,i} > 0$ that

$$\lim_{n\longrightarrow\infty} \frac{\|Jy_{n,i} - JT_{r_{n,i}}y_{n,i}\|}{r_{n,i}} = 0.$$

Since $u_{n,i} = T_{r_{n,i}}y_{n,i}$, we have

$$\begin{aligned} F(T_{r_{n,i}}y_{n,i}, y_i) + \langle y_i - T_{r_{n,i}}y_{n,i}, AT_{r_{n,i}}y_{n,i}\rangle \quad &+ \quad \psi(y_i, T_{r_{n,i}}y_{n,i}) - \psi(T_{r_{n,i}}y_{n,i}, T_{r_{n,i}}y_{n,i}) \\ &+ \quad \tfrac{e}{r_{n,i}}\langle y - T_{r_{n,i}}y_{n,i}, JT_{r_{n,i}}y_{n,i} - Jy_{n,i}\rangle \in P, \ \forall y_i \in K_i, \end{aligned}$$

$$\begin{aligned} 0 \quad \in \quad & F(y_i, T_{r_{n,i}}y_{n,i}) - \langle y_i - T_{r_{n,i}}y_{n,i}, AT_{r_{n,i}}y_{n,i}\rangle - (\psi(y_i, T_{r_{n,i}}y_{n,i}) - \psi(T_{r_{n,i}}y_{n,i}, T_{r_{n,i}}y_{n,i})) \\ & -\tfrac{e}{r_{n,i}}\langle y_i - T_{r_{n,i}}y_{n,i}, JT_{r_{n,i}}y_{n,i} - Jy_{n,i}\rangle + P, \ \forall y_i \in K_i. \end{aligned}$$

Let $y_{i,t} = (1-t)p + ty_i, \ \forall t \in (0,1]$. Since $y_i \in K_i$ and $p \in K_i$, we get $y_{i,t} \in K_i$ and hence

$$\begin{aligned} 0 \quad \in \quad & F(y_i, T_{r_{n,i}}y_{n,i}) - \langle y_{i,t} - T_{r_{n,i}}y_{n,i}, AT_{r_{n,i}}y_{n,i}\rangle - (\psi(y_{i,t}, T_{r_{n,i}}y_{n,i}) - \psi(T_{r_{n,i}}y_{n,i}, T_{r_{n,i}}y_{n,i})) \\ & -\tfrac{e}{r_{n,i}}\langle y_{i,t} - T_{r_{n,i}}y_{n,i}, JT_{r_{n,i}}y_{n,i} - Jy_{n,i}\rangle + P, \\ = \quad & F(y_{i,t}, T_{r_{n,i}}y_{n,i}) - \langle y_{i,t} - T_{r_{n,i}}y_{n,i}, AT_{r_{n,i}}y_{n,i}\rangle - (\psi(y_{i,t}, T_{r_{n,i}}y_{n,i}) - \psi(T_{r_{n,i}}y_{n,i}, T_{r_{n,i}}y_{n,i})) \\ & -e\langle y_{i,t} - T_{r_{n,i}}y_{n,i}, \tfrac{JT_{r_{n,i}}y_{n,i} - Jy_{n,i}}{r_{n,i}}\rangle + P. \\ 0 \quad \in \quad & F(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle - \psi(y_{i,t}, p) + \psi(p,p) + P. \end{aligned}$$

It follows from Assumptions 2.6 (i), (iv) and (vi) that

$$\begin{aligned} tF(y_{i,t}, y_i) + (1-t)F(y_t, p) + t\psi(y_i, p) + (1-t)\psi(p,p) - \psi(y_{i,t}, p) \quad \in \quad & F(y_{i,t}, y_t) + \psi(y_{i,t}, p) \\ & -\psi(y_{i,t}, p) + P \\ \in \quad & P. \end{aligned}$$

Now,

$-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle]-$
$(1-t)[F(y_{i,t}, p) + \psi(p,p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] \in -P + \langle y_{i,t} - p, Ap\rangle$
$-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] - (1-t)P \in -P + \langle y_{i,t} - p, Ap\rangle$, (using (3.7))
$-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] \in -P + (1-t)P + \langle y_{i,t} - p, Ap\rangle$
$-t[F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle] \in -tP + \langle y_{i,t} - p, Ap\rangle$
$\in -tP + \langle ty_i + (1-t)p - p, Ap\rangle \ \in -tP + t\langle y_i - p, Ap\rangle$

$$F(y_{i,t}, y_i) + \psi(y_i, p) - \psi(y_{i,t}, p) - \langle y_{i,t} - p, Ap\rangle \in P - \langle y_i - p, Ap\rangle.$$

Letting $t \longrightarrow 0_+$, we have

$$F(p, y_i) + \psi(y_i, p) - \psi(p,p) - \langle p - p, Ap\rangle + \langle y_i - p, Ap\rangle \in P$$

$$F(p, y_i) + \langle y_i - p, Ap\rangle + \psi(y_i, p) - \psi(p,p) \in P.$$

Thus $p \in \mathrm{Sol}(\mathrm{GMVEP}(F_i, A_i, \psi_i, K_i))$. Hence $p \in \Gamma$.

Finally, we prove that $p = \prod_\Gamma x_0$. Since $\Gamma \subset C_{n+1}$ and $x_{n+1} = \prod_{C_{n+1}} x_0$, we have

$$\langle x_{n+1} - q, Jx_0 - Jx_{n+1}\rangle \geq 0, \ \forall q \in \Gamma. \tag{3.39}$$

By taking the limit in (3.39), we have

$$\langle p - q, Jx_0 - Jp\rangle \geq 0, \ \forall q \in \Gamma.$$

Hence, in view of Lemma 2.1, we see that $p = \prod_\Gamma x_0$. This completes the proof. □

Finally we give some consequences of Theorem 3.3.

Corollary 3.4. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. Let K be a nonempty, compact and convex subset of E. Assume that P is a pointed, proper, closed and convex cone of a real ordered Banach space Y with $\mathrm{int}P \neq \emptyset$. Let the mappings $F, \psi : K \times K \longrightarrow Y$ satisfy Assumption 2.6 and $A : K \longrightarrow B(E, Y)$ be continuous and P-monotone mapping. Let $S, T : K \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_n\}$, $\{\varsigma_n\}$ and $\{\zeta_n\}$, $\{\xi_n\}$ such that $\Gamma := \mathrm{Fix}(S) \cap (\mathrm{Fix}(T)) \cap (\mathrm{Sol}(\mathrm{GMVEP}(1.2))) \neq \emptyset$. Assume that, the sequences $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n^j\} \subseteq (0, 1)$ $(j = 1, 2, 3)$ be such that*

- (i) $\beta_n^1 + \beta_n^2 + \beta_n^3 = 1$;
- (ii) $\liminf_{n\longrightarrow\infty} \beta_n^1\beta_n^2 > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta_n^1\beta_n^3 > 0$;
- (iii) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;
- (iv) $\liminf_{n\longrightarrow\infty} r_n > 0$.

Let $\{x_n\}$ be a sequence generated by the iterative scheme:

$$\begin{aligned}
&x_0 \in E,\\
&x_1 = \textstyle\prod_{C_1} x_0,\\
&y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n),\\
&z_n = J^{-1}(\beta_n^1 Jx_n + \beta_n^2 JT^n x_n + \beta_n^3 JS^n x_n),\\
&u_n = T_{r_n}(y_n)\\
&C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n) + \delta_n M_n + \mu_n\},\\
&x_{n+1} = \textstyle\prod_{C_{n+1}} x_0, \text{ for every } n \in N \cup \{0\},
\end{aligned}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \mathrm{int}P$; $J : E \longrightarrow E^*$ is the normalized duality mapping with its inverse J^{-1}; $\delta_n = \beta_n^2\zeta_n + \beta_n^3\eta_n$ and $\mu_n = \xi_n\beta_n^2 + \varsigma_n\beta_n^3$. Then the sequence $\{x_n\}$ converges strongly to $\prod_\Gamma x_0$.

Proof. The proof follows by taking $i = 1$ in Theorem 3.3. □

Corollary 3.5. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. For each $i \in I := \{1, 2, 3, ..., N\}$, let K_i be a nonempty, compact and convex subset of E such that $K = \cap_{i=1}^N K_i \neq \emptyset$. Let for each i, the mappings $F_i, \psi_i : K_i \times K_i \longrightarrow \mathbb{R}$ satisfy Assumption 2.6 and $A_i : K_i \longrightarrow E^*$ be continuous and monotone mapping. For each fixed i, let $S_i, T_i : K_i \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_{n,i}\}$, $\{\varsigma_{n,i}\}$ and $\{\zeta_{n,i}\}$, $\{\xi_{n,i}\}$ such that $\Gamma := (\cap_{i=1}^N \mathrm{Fix}(S_i)) \cap (\cap_{i=1}^N \mathrm{Fix}(T_i)) \cap (\cap_{i=1}^N \mathrm{Sol}(\mathrm{SUGMEP}(1.7))) \neq \emptyset$. Assume that, for each fixed i, the sequences $\{r_{n,i}\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_{n,i}\} \subseteq (0, 1)$ and $\{\beta_{n,i}^j\} \subseteq (0, 1)$ $(j = 1, 2, 3)$ be such that*

- (i) $\beta_{n,i}^1 + \beta_{n,i}^2 + \beta_{n,i}^3 = 1$;
- (ii) $\liminf_{n\longrightarrow\infty} \beta_{n,i}^1\beta_{n,i}^2 > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta_{n,i}^1\beta_{n,i}^3 > 0$;
- (iii) $\limsup_{n\longrightarrow\infty} \alpha_{n,i} < 1$;
- (iv) $\liminf_{n\longrightarrow\infty} r_{n,i} > 0$.

Let $\{x_n\}$ be a sequence generated by the iterative scheme:

$$\begin{aligned}
&x_0 \in E,\\
&C_{1,i} =: K_i,\ C_1 = \cap_{i=1}^{N} C_{1,i} = K,\\
&x_1 = \textstyle\prod_{C_1} x_0,\\
&y_{n,i} = J^{-1}(\alpha_{n,i} J x_n + (1-\alpha_{n,i}) J z_{n,i}),\\
&z_{n,i} = J^{-1}(\beta^1_{n,i} J x_n + \beta^2_{n,i} J T_i^n x_n + \beta^3_{n,i} J S_i^n x_n),\\
&u_{n,i} = T_{r_{n,i}}(y_{n,i})\\
&C_{n+1,i} = \{v \in C_{n,i} : \phi(v, u_{n,i}) \le \phi(v, x_n) + \delta_{n,i} M_n + \mu_{n,i}\},\\
&C_{n+1} = \cap_{i\in I} C_{n+1,i},\\
&x_{n+1} = \textstyle\prod_{C_{n+1}} x_0,\ \text{for every } n \in N \cup \{0\},
\end{aligned}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \mathrm{int} P$; $J : E \longrightarrow E^*$ is the normalized duality mapping with its inverse J^{-1}; $\delta_{n,i} = \beta^2_{n,i}\zeta_{n,i} + \beta^3_{n,i}\eta_{n,i}$ and $\mu_{n,i} = \xi_{n,i}\beta^2_{n,i} + \varsigma_{n,i}\beta^3_{n,i}$. Then the sequence $\{x_n\}$ converges strongly to $\prod_\Gamma x_0$.

Proof. The proof follows by taking $Y = \mathbb{R},\ P = [0, \infty)$ in Theorem 3.3. □

Corollary 3.6. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. Let K be a nonempty, compact and convex subset of E. Let the mappings $F, \psi : K \times K \longrightarrow \mathbb{R}$ satisfy Assumption 2.6 and $A : K \longrightarrow E^*$ be continuous and monotone mapping. Let $S, T : K \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_n\}$, $\{\varsigma_n\}$ and $\{\zeta_n\}$, $\{\xi_n\}$ such that $\Gamma := \mathrm{Fix}(S) \cap (\mathrm{Fix}(T)) \cap (\mathrm{Sol(GMEP}(1.4))) \neq \emptyset$. Assume that, the sequences $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta^j_n\} \subseteq (0, 1)$ $(j = 1, 2, 3)$ be such that*

- (i) $\beta^1_n + \beta^2_n + \beta^3_n = 1$;
- (ii) $\liminf_{n\longrightarrow\infty} \beta^1_n\beta^2_n > 0$ *and* $\liminf_{n\longrightarrow\infty} \beta^1_n\beta^3_n > 0$;
- (iii) $\limsup_{n\longrightarrow\infty} \alpha_n < 1$;
- (iv) $\liminf_{n\longrightarrow\infty} r_n > 0$.

Let $\{x_n\}$ be a sequence generated by the iterative scheme:

$$\begin{aligned}
&x_0 \in E,\\
&x_1 = \textstyle\prod_{C_1} x_0,\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
&z_n = J^{-1}(\beta^1_n J x_n + \beta^2_n J T^n x_n + \beta^3_n J S^n x_n),\\
&u_n = T_{r_n}(y_n)\\
&C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n) + \delta_n M_n + \mu_n\},\\
&x_{n+1} = \textstyle\prod_{C_{n+1}} x_0,\ \text{for every } n \in N \cup \{0\},
\end{aligned}$$

where $M_n = \sup\{\phi(p, x_n) : p \in \Gamma\}$; $e \in \mathrm{int} P$; $J : E \longrightarrow E^*$ is the normalized duality mapping with its inverse J^{-1}; $\delta_n = \beta^2_n\zeta_n + \beta^3_n\eta_n$ and $\mu_n = \xi_n\beta^2_n + \varsigma_n\beta^3_n$. Then the sequence $\{x_n\}$ converges strongly to $\prod_\Gamma x_0$.

Proof. The proof follows by taking $Y = \mathbb{R},\ P = [0, \infty),\ i = 1$ in Theorem 3.3. □

The following corollary is similar to Qin and Agrawal [21].

Corollary 3.7. *Let E be a uniformly smooth and strictly convex Banach space such that E has Kadec-Klee property. Let K be a nonempty, compact and convex subset of E. Let $S, T : K \longrightarrow E$ be closed, asymptotically regular and generalized asymptotically quasi ϕ-nonexpansive mappings with the sequences $\{\eta_n\}$, $\{\varsigma_n\}$ and $\{\zeta_n\}$, $\{\xi_n\}$ such that $\Gamma := \mathrm{Fix}(S) \cap (\mathrm{Fix}(T)) \neq \emptyset$. Assume that, the sequences $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta^j_n\} \subseteq (0, 1)$ $(j = 1, 2, 3)$ be such that*

- (i) $\beta^1_n + \beta^2_n + \beta^3_n = 1$;

(ii) $\liminf_{n\longrightarrow\infty}\beta_n^1\beta_n^2>0$ *and* $\liminf_{n\longrightarrow\infty}\beta_n^1\beta_n^3>0$;
(iii) $\limsup_{n\longrightarrow\infty}\alpha_n<1$.

Let $\{x_n\}$ *be a sequence generated by the iterative scheme:*

$$\begin{aligned}
&x_0\in E,\\
&x_1=\prod\nolimits_{C_1}x_0,\\
&y_n=J^{-1}(\alpha_nJx_n+(1-\alpha_n)Jz_n),\\
&z_n=J^{-1}(\beta_n^1Jx_n+\beta_n^2JT^nx_n+\beta_n^3JS^nx_n),\\
&C_{n+1}=\{v\in C_n:\phi(v,y_n)\le\phi(v,x_n)+\delta_nM_n+\mu_n\},\\
&x_{n+1}=\prod\nolimits_{C_{n+1}}x_0,\ \text{for every } n\in N\cup\{0\},
\end{aligned}$$

where $M_n=\sup\{\phi(p,x_n):p\in\Gamma\}$; $e\in\mathrm{int}P$; $J:E\longrightarrow E^*$ *is the normalized duality mapping with its inverse* J^{-1}; $\delta_n=\beta_n^2\zeta_n+\beta_n^3\eta_n$ *and* $\mu_n=\xi_n\beta_n^2+\varsigma_n\beta_n^3$.
Then the sequence $\{x_n\}$ *converges strongly to* $\prod_\Gamma x_0$.

Proof. The proof follows by taking $Y=\mathbb{R},\ P=[0,\infty),\ i=1,$ and $F=0,\ A=0,\ \phi=0$ in Theorem 3.3. □

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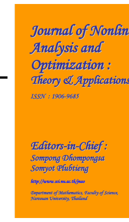

FIXED POINT THEOREMS FOR FUNDAMENTALLY NONEXPANSIVE MAPPINGS IN CAT(k) SPACES

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ABSTRACT. In this paper, we obtain fixed point theorems and $\Delta-$convergence theorems for fundamentally nonexpansive mappings on CAT(k) spaces with $k > 0$. Our results extend and improve some results of Salahifard et al. [3], and many others.

KEYWORDS : CAT(k) space; Fixed point; $\Delta-$convergence; Generalized nonexpansive mapping.
AMS Subject Classification: 47H09; 47H10.

1. INTRODUCTION

A mapping T on a subset E of Banach space X is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in E \ : \ Tx = x\}$. A mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - z\| \leq \|x - z\|$ for all $x \in E$ and $z \in F(T)$. In 2008, Suzuki [1] introduced condition (C) as follows:
A mapping T on a subset E of Banach space X is said to satisfy the condition (C) (or Suzuki's generalized nonexpansive) on E if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in E$. Moreover, he obtained some interesting fixed point theorems and convergence theorems for such mappings. In 2012, Nanjaras et al. [2] extend Suzuki's results on fixed point theorems and $\triangle-$convergence theorems for such mappings in CAT(0) spaces. In 2013, Salahifard et al. [3] introduced fundamentally nonexpansive mapping which generalizes the Suzuki's generalized nonexpansive mapping and proved some fixed point theorems for this kind of mappings in CAT(0) spaces.

Fixed point theory in CAT(k) spaces was first studied by Kirk [4, 5]. His works were followed by a series of new works by many authors, mainly focusing on CAT(0)

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spaces (see e.g., [2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]). Since any CAT(k) space is a CAT(k') space for $k' \geq k$, all results for CAT(0) spaces immediately apply to any CAT(k) space with $k \leq 0$. However, there are only a few articles that contain fixed point results in the setting of CAT(k) spaces with $k > 0$. In this paper, we extend the results of Salahifard et al. [3] to the general setting of CAT(k) spaces with $k > 0$.

2. PRELIMINARIES AND NOTATIONS

Let (X, ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $\rho\left(c(t), c(t')\right) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $\rho(x, y) = l$. The image $c\left([0, l]\right)$ of c is called a *geodesic segment* joining x and y. Write $c(\alpha 0 + (1-\alpha)l) = \alpha x \oplus (1-\alpha)y$ for $\alpha \in (0, 1)$. When it is unique this geodesic segment is denoted by $[x, y]$. This means that $z \in [x, y]$ if and only if there exists $\alpha \in [0, 1]$ such that

$$\rho(x, z) = (1-\alpha)\rho(x, y) \text{ and } \rho(y, z) = \alpha\rho(x, y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$. Let D be a positive constant. A metric space (X, ρ) is said to be a *geodesic space* (D*-geodesic space*) if every two points of X (every two points of distance smaller than D) are joined by a geodesic, and X is said to be *uniquely geodesic* (D*-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x, y \in X$ (for $x, y \in X$ with $\rho(x, y) < D$). A subset E of X is said to be *convex* if E includes every geodesic segment joining any two of its points. If this condition holds for any two points in E with distance smaller than D, E is said to be *D-convex*. The set E is said to be *bounded* if

$$\operatorname{diam}(E) := \sup\{\rho(x, y) : x, y \in E\} < \infty.$$

Now we introduce the model spaces M_k^n, for more details on these spaces the reader is referred to [21]. Let $n \in \mathbb{N}$. We denote by $\mathbb{E}^n$ the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in $\mathbb{R}^n$, that is,

$$(x|y) = x_1y_1 + ... + x_ny_n \text{ where } x = (x_1, ..., x_n),\ y = (y_1, ..., y_n).$$

Let $\mathbb{S}^n$ denote the $n-$*dimensional sphere* defined by

$$\mathbb{S}^n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : (x|x) = 1\},$$

with metric $d_{\mathbb{S}^n}(x, y) = \arccos(x|y),\ x, y \in \mathbb{S}^n$.

Let $\mathbb{E}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form which associates to vectors $u = (u_1, ..., u_{n+1})$ and $v = (v_1, ..., v_{n+1})$, the real number $\langle u|v\rangle$ is defined by

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_iv_i.$$

Let $\mathbb{H}^n$ denote the *hyperbolic* $n-$*space* defined by

$$\mathbb{H}^n = \{u = (u_1, ..., u_{n+1}) \in \mathbb{E}^{n,1} : \langle u|u\rangle = -1, u_{n+1} > 0\},$$

with metric $d_{\mathbb{H}^n}$ such that

$$\cosh d_{\mathbb{H}^n}(x, y) = -\langle x|y\rangle,\ x, y \in \mathbb{H}^n.$$

Definition 2.1. Given $k \in \mathbb{R}$, we denote by M_k^n the following metric spaces:
(i) if $k = 0$ then M_0^n is the Euclidean space $\mathbb{E}^n$;
(ii) if $k > 0$ then M_k^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $1/\sqrt{k}$;
(iii) if $k < 0$ then M_k^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance.

A *geodesic triangle* $\triangle(x, y, z)$ in a geodesic space (X, ρ) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x, y, z)$ in (X, ρ) is a triangle $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ in M_k^2 such that

$$\rho(x, y) = d_{M_k^2}(\overline{x}, \overline{y}),\ \rho(y, z) = d_{M_k^2}(\overline{y}, \overline{z}) \text{ and } \rho(z, x) = d_{M_k^2}(\overline{z}, \overline{x}).$$

If $k \le 0$ then such a comparison triangle always exists in M_k^2. If $k > 0$ then such a triangle exists whenever $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_k$, where $D_k = \pi/\sqrt{k}$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a *comparison point* for $p \in [x, y]$ if $\rho(x, p) = d_{M_k^2}(\overline{x}, \overline{p})$.

A geodesic triangle $\triangle(x, y, z)$ in X is said to satisfy the CAT(k) inequality if for any $p, q \in \triangle(x, y, z)$ and for their comparison points $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$, one has

$$\rho(p, q) \le d_{M_k^2}(\overline{p}, \overline{q}).$$

Definition 2.2. If $k \le 0$, then X is called a CAT(k) space if X is a geodesic space such that all of its geodesic triangles satisfy the CAT(k) inequality.

If $k > 0$, then X is called a CAT(k) space if X is D_k-geodesic and any geodesic triangle $\triangle(x, y, z)$ in X with $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_k$ satisfies the CAT(k) inequality.

In a CAT(0) space (X, ρ), if $x, y, z \in X$ then the CAT(0) inequality implies

$$\rho^2\left(x, \frac{1}{2}y \oplus \frac{1}{2}z\right) \le \frac{1}{2}\rho^2(x, y) + \frac{1}{2}\rho^2(x, z) - \frac{1}{4}\rho^2(y, z). \qquad \text{(CN)}$$

This is the *(CN) inequality* of Bruhat and Tits [22]. This inequality is extended by Dhompongsa and Panyanak [19] as

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \le (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - \alpha(1-\alpha)\rho^2(y, z) \qquad \text{(CN*)}$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$. In fact, if X is a geodesic space then the following statements are equivalent:
(i) X is a CAT(0) space;
(ii) X satisfies (CN);
(iii) X satisfies (CN*).

Let $R \in (0, 2]$. Recall that a geodesic space (X, ρ) is said to be $R-$convex for R (see [23]) if for any three points $x, y, z \in X$, we have

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \le (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - \frac{R}{2}\alpha(1-\alpha)\rho^2(y, z). \qquad (1)$$

It follows from (CN*) that a geodesic space (X, ρ) is a CAT(0) space if and only if (X, ρ) is $R-$convex for $R = 2$. The following lemma is a consequence of Proposition 3.1 in [23].

Lemma 2.3. *Let $k > 0$ and (X, ρ) be a CAT(k) space with* $\operatorname{diam}(X) \le \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Then (X, ρ) is $R-$convex for $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.*

The following lemma is also needed.

Lemma 2.4. ([21, p.176]) *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Then*

$$\rho\left(x, \alpha y \oplus (1-\alpha)z\right) \leq \alpha\rho(x,y) + (1-\alpha)\rho(x,z).$$

for all $x, y, z \in X$ and $\alpha \in [0,1]$.

Let $\{x_n\}$ be a bounded sequence in a CAT(k) space (X, ρ). For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \longrightarrow \infty} \rho(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 4.1 of [11] that in a CAT(k) space X with diameter smaller than $\frac{\pi}{2\sqrt{k}}$, $A(\{x_n\})$ consists of exactly one point. We now give the concept of $\triangle-$convergence and collect some of its basic properties.

Definition 2.5. ([9], [24]) A sequence $\{x_n\}$ in X is said to $\Delta-$*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the $\Delta-$limit of $\{x_n\}$.

Lemma 2.6. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Then the following statements hold:*

(i) [11, Corollary 4.4] *every sequence in X has a $\Delta-$convergence subsequence;*

(ii) [11, Proposition 4.5] *if $\{x_n\} \subset X$ and $\Delta-\lim_n x_n = x$, then $x \in \cap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, ...\}$, where $\overline{conv}(A) = \cap\{B : B \supseteq A$ and B is closed and convex$\}$.*

By the uniqueness of asymptotic centers, we can obtain the following lemma (cf. [19, Lemma 2.8]).

Lemma 2.7. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. If $\{x_n\}$ is a sequence in X with $A(\{x_n\}) = \{x\}$ and let $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{\rho(x_n, u)\}$ converges, then $x = u$.*

Definition 2.8. [3] Let E be a nonempty subset of a CAT(k) space (X, ρ). A mapping $T : E \longrightarrow E$ is said to be *fundamentally nonexpansive* if

$$\rho(T^2(x), T(y)) \leq \rho(T(x), y),$$

for all $x, y \in E$.

Proposition 2.9. [3] *Every mapping which satisfies condition (C) is fundamentally nonexpansive, but the inverse is not true.*

3. MAIN RESULTS

Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X.

In this section, we prove our main theorems.

Lemma 3.1. *Let E be a nonempty subset of a CAT(k) space (X, ρ), and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Then*

$$\rho(x, T(y)) \leq 3\rho(T(x), x) + \rho(x, y),$$

for all $x, y \in E$.

Proof. Since T is fundamentally nonexpansive, we have

$$\begin{aligned}\rho(x,T(y)) &\leq \rho(x,T(x))+\rho(T(x),T^2(x))+\rho(T^2(x),T(y))\\ &\leq 2\rho(x,T(x))+\rho(T(x),y)\\ &\leq 3\rho(x,T(x))+\rho(x,y).\end{aligned}$$

This completes the proof. □

The following lemma is a consequence of Lemma 3.4 of [16].

Lemma 3.2. *Let $k>0$ and (X,ρ) be a CAT(k) space such that* $\operatorname{diam}(X)\leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon\in(0,\pi/2)$ and let $\{z_n\}$ and $\{w_n\}$ be two sequences in X. Let $\{\beta_n\}$ be a sequence in $[0,1]$ such that $0<\liminf_n \beta_n\leq \limsup_n \beta_n<1$. Suppose that $z_{n+1}=\beta_n z_n+(1-\beta_n)w_n$ for all $n\in\mathbb{N}$ and $\limsup_n(\rho(w_{n+1},w_n)-\rho(z_{n+1},z_n))\leq 0$. Then $\lim_n \rho(w_n,z_n)=0$.*

Lemma 3.3. *Let $k>0$ and (X,ρ) be a complete CAT(k) space with* $\operatorname{diam}(X)\leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon\in(0,\pi/2)$. Let E be a nonempty closed convex subset of X, and $T:E\longrightarrow E$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1\in E$ and $x_{n+1}=\alpha_n T(x_n)\oplus(1-\alpha_n)x_n$ for all $n\in\mathbb{N}$ where $\{\alpha_n\}\subset[0,1]$ such that $0<\liminf_n \alpha_n\leq\limsup_n \alpha_n<1$. Then $\lim_{n\longrightarrow\infty}\rho(T(x_n),x_n)=0$.*

Proof. Since T is fundamentally nonexpansive, we have

$$\rho(T(x_{n+1}),T(x_n))=\alpha_n\rho(T^2(x_n),T(x_n))\leq\alpha_n\rho(T(x_n),x_n)=\rho(x_{n+1},x_n)$$

for all $n\in\mathbb{N}$ and hence

$$\rho(T(x_{n+1}),T(x_n))\leq\rho(x_{n+1},x_n).$$

This implies that

$$\limsup_{n\longrightarrow\infty}(\rho(Tx_{n+1},Tx_n)-\rho(x_{n+1},x_n))\leq 0.$$

So, by Lemma 3.2, we have

$$\lim_{n\longrightarrow\infty}\rho(T(x_n),x_n)=0.$$

This completes the proof. □

Theorem 3.1. *Let $k>0$ and (X,ρ) be a complete CAT(k) space with* $\operatorname{diam}(X)\leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon\in(0,\pi/2)$. Let E be a nonempty closed convex subset of X, and $T:E\longrightarrow E$ be a fundamentally nonexpansive mapping. Then $F(T)$ is nonempty.*

Proof. Define a sequence $\{x_n\}$ by $x_1\in E$ and $x_{n+1}=\frac{1}{2}Tx_n\oplus\frac{1}{2}x_n$ for all $n\in\mathbb{N}$. Suppose that $A(\{x_n\})=\{z\}$. Then by Lemma 2.6, $z\in E$. By Lemma 3.3, we have $\lim_n\rho(T(x_n),x_n)=0$ and by Lemma 3.1,

$$\rho(x_n,T(z))\leq 3\rho(T(x_n),x_n)+\rho(x_n,z).$$

Taking the limit superior on both sides in the above inequality, we obtain

$$\limsup_{n\longrightarrow\infty} d(x_n,T(z))\leq\limsup_{n\longrightarrow\infty} d(x_n,z).$$

Since $A(\{x_n\})=\{z\}$, it must be the case that $z=T(z)$. □

The following corollary shows that how we derive a result for CAT(0) spaces from Theorem 3.1.

Corollary 3.4. *Let (X, ρ) be a complete CAT(0) space, E be a nonempty bounded closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Then $F(T)$ is nonempty.*

Proof. It well known that every convex subset of a CAT(0) space, equipped with the included metric, is a CAT(0) space (cf. [21]). Then (E, ρ) is a CAT(0) space and hence it is a CAT(k) space for all $k > 0$. Notice also that E is $R-$convex for $R = 2$. Since E is bounded, we can choose $\varepsilon \in (0, \pi/2)$ and $k > 0$ so that $\text{diam}(E) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$. The conclusion follows from Theorem 3.1. □

Theorem 3.2. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with $\text{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping and $F(T) \neq \emptyset$. Then $F(T)$ is closed and convex.*

Proof. We can prove by following the steps of the Theorem 4.1 of [3]. □

Theorem 3.3. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with $\text{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Let $\{x_n\}$ be a sequence in E with $\lim_n \rho(T(x_n), x_n) = 0$ and $\Delta - \lim_n x_n = z$. Then $z \in E$ and $z = T(z)$.*

Proof. Since $\Delta - \lim_n x_n = z$, by Lemma 2.6, we have $z \in E$. It follows from Lemma 3.1 that

$$\rho(x_n, T(z)) \leq 3\rho(T(x_n), x_n) + \rho(x_n, z).$$

Taking the limit superior on both sides in the above inequality, we obtain

$$\limsup_{n \longrightarrow \infty} \rho(x_n, T(z)) \leq \limsup_{n \longrightarrow \infty} \rho(x_n, z).$$

By the uniqueness of asymptotic center, we obtain $z = T(z)$. □

The following corollary shows that how we derive a result for CAT(0) spaces from Theorem 3.3.

Corollary 3.5. *Let (X, ρ) be a complete CAT(0) space, E be a nonempty bounded closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Let $\{x_n\}$ be a sequence in E with $\lim_n \rho(T(x_n), x_n) = 0$ and $\Delta - \lim_n x_n = z$. Then $z \in E$ and $z = T(z)$.*

Proof. It well known that every convex subset of a CAT(0) space, equipped with the included metric, is a CAT(0) space (cf. [21]). Then (E, ρ) is a CAT(0) space and hence it is a CAT(k) space for all $k > 0$. Notice also that E is $R-$convex for $R = 2$. Since E is bounded, we can choose $\varepsilon \in (0, \pi/2)$ and $k > 0$ so that $\text{diam}(E) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$. The conclusion follows from Theorem 3.3. □

Lemma 3.6. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with $\text{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Suppose $\{x_n\}$ is a sequence in E such that $\lim_n \rho(T(x_n), x_n) = 0$ and $\{\rho(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.6, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in E$. By Theorem 3.3, $v \in F(T)$. By Lemma 2.7, $u = v$. This shows that $\omega_w(x_n) \subset F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, we have $\{\rho(x_n, u)\}$ converges. Again, by Lemma 2.7, $x = u$. This completes the proof. □

Theorem 3.4. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in E$ and $x_{n+1} = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$ where $\{\alpha_n\} \subset [0, 1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Then $\{x_n\}$ $\Delta-$ converges to a fixed point of T.*

Proof. By Lemma 3.3, we have $\lim_n \rho(T(x_n), x_n) = 0$. By Theorem 3.1, $F(T)$ is nonempty. Given $z \in F(T)$, by Lemma 3.1 we have

$$\rho(T(x_n), z) \leq 3\rho(T(z), z) + \rho(x_n, z) \leq \rho(x_n, z).$$

This implies that

$$\begin{aligned} \rho(x_{n+1}, z) &= \rho(\alpha_n T(x_n) \oplus (1-\alpha_n)x_n, z) \\ &\leq \alpha_n \rho(T(x_n), z) + (1-\alpha_n)\rho(x_n, z) \\ &\leq \rho(x_n, z). \end{aligned}$$

That is

$$\rho(x_{n+1}, z) \leq \rho(x_n, z). \tag{2}$$

Thus $\{\rho(x_n, z)\}$ is bounded and decreasing for all $z \in F(T)$, and so it is convergent. By Lemma 3.6, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This show that $\{x_n\}$ $\Delta-$ converges to an element of $F(T)$. □

Theorem 3.5. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty compact convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping. Define a sequence $\{x_n\}$ by $x_1 \in E$ and $x_{n+1} = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$ where $\{\alpha_n\} \subset [0, 1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.3, we have $\lim_{n \longrightarrow \infty} \rho(T(x_n), x_n) = 0$. Since E is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_k x_{n_k} = z$ for some $z \in E$. It follows from Lemma 3.1 that

$$\rho(x_{n_k}, T(z)) \leq 3\rho(T(x_{n_k}), x_{n_k}) + \rho(x_{n_k}, z) \text{ for all } k \in \mathbb{N}.$$

Letting $k \longrightarrow \infty$, we have $\{x_{n_k}\}$ converges to $T(z)$. This implies that $z = T(z)$, that is $z \in F(T)$. Following the proof of Theorem 3.4, we obtain $\lim_n \rho(x_n, z)$ exists for all $z \in F(T)$, it must be the case that $\lim_n \rho(x_n, z) = 0$. Therefore we obtain the desired result. □

Recall that a mapping $T : E \longrightarrow E$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $\rho(x, T(x)) \geq f(\rho(x, F(T)))$ for all $x \in E$, where $\rho(x, F(T)) = \inf_{z \in F(T)} \rho(x, z)$.

Theorem 3.6. *Let $k > 0$ and (X, ρ) be a complete CAT(k) space with* $\operatorname{diam}(X) \leq \frac{\pi/2-\varepsilon}{\sqrt{k}}$ *for some $\varepsilon \in (0, \pi/2)$. Let E be a nonempty closed convex subset of X, and $T : E \longrightarrow E$ be a fundamentally nonexpansive mapping satisfies the condition (I). Define a sequence $\{x_n\}$ by $x_1 \in E$ and $x_{n+1} = \alpha_n T(x_n) \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$ where $\{\alpha_n\} \subset [0,1]$ such that $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By condition (I), we have

$$f\left(\rho\left(x_n, F(T)\right)\right) \leq \rho(x_n, T(x_n)) \text{ for all } n \in \mathbb{N}.$$

It follows from Lemma 3.3 that

$$\lim_{n \longrightarrow \infty} f\left(\rho\left(x_n, F(T)\right)\right) = 0.$$

This implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\rho(x_{n_k}, z_k) \leq \frac{1}{2^k} \text{ for all } k \in \mathbb{N}. \tag{3}$$

Where $\{z_k\} \subset F(T)$. By (2), we have

$$\rho(x_{n_{k+1}}, z_k) \leq \rho(x_{n_k}, z_k) \leq \frac{1}{2^k}.$$

Hence

$$\begin{aligned} \rho(z_{k+1}, z_k) &\leq \rho(z_{k+1}, x_{n_{k+1}}) + \rho(x_{n_{k+1}}, z_k) \\ &\leq \frac{1}{2^{(k+1)}} + \frac{1}{2^k} < \frac{1}{2^{k-1}} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned}$$

This shows that $\{z_k\}$ is a Cauchy in $F(T)$. Since $F(T)$ is closed in X, there exists a point z in $F(T)$ such that $\lim_{k \longrightarrow \infty} z_k = z$. It follows from (3) that $\lim_{k \longrightarrow \infty} x_k = z$. Since $\lim_{n \longrightarrow \infty} \rho(x_n, z)$ exists, it must be the case that $\lim_{n \longrightarrow \infty} \rho(x_n, z) = 0$. Therefore we obtain the desired result. □

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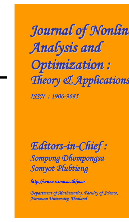

COUPLED FIXED POINTS IN PARTIALLY ORDERED METRIC SPACES BY SAMET'S METHOD AND APPLICATION

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In 2006, Bhaskar and Lakshmikantham proved a fixed point theorem for a mixed monotone mapping in a metric space endowed with partial order, using a weak contractivity type of assumption. Recently Luong and Thuan proved some results of coupled fixed point that generalized main results of them. In this paper, By using the samet's method and by using different conditions we prove some coupled fixed point theorems for mapping having mixed monotone property in partially ordered metric space. Also by considering the results of Berinde and Burcut and using the main idea of Samet and Vetro extend the concept of $\alpha-$admissibility for tripled fixed point theorems in metric spaces. As an application, we discuss the existence and solution of a nonlinear integral equation.

KEYWORDS : Coupled fixed point, Tripled fixed point, Mixed monotone mapping, Integral equation.
AMS Subject Classification: 54H25, 47H10

1. INTRODUCTION

In 1987, Guo and Lakshmikantham introduced the notion of coupled fixed points [6]. In the last decade of the previous century other authors obtained important results in this area. In 2006 Bhaskar and Lakshmikantham introduced notions of a mixed monotone mapping and a coupled fixed point [7]. They proved fixed point theorem for a mixed monotone mapping in a metric space endowed with partial order, using a weak contractivity type of assumption.
Recently Luong and Thuan proved some results of coupled fixed point that generalized main results of them [10].
Let us recall some basic definitions of mixed monotone property and α-admissiblity, [11]-[10].

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Definition 1.1. [7], Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$x_1, x_2 \in X,\ x_1 \leq x_2 \Longrightarrow F(x_1, y) \leq F(x_2, y)$

and

$y_1, y_2 \in X,\ y_1 \leq y_2 \Longrightarrow F(x, y_1) \geq F(x, y_2)$.

Definition 1.2. [7], An element $(x, y) \in X \times X$ is called coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x, y)$, and $y = F(y, x)$.

Definition 1.3. [2], An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of $F : X \times X \times X \to X$ if $F(x, y, z) = x,\ F(y, z, x) = y$, and $F(z, x, y) = z$.

Definition 1.4. [11], Let (X, d) be a metric space and $T : X \to X$ be a given mapping. we say that T is an $\alpha - \psi - contractive$ mapping, if there exist two functions $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that :

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \qquad forall \quad x, y \in X.$$

Definition 1.5. [11], Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. we say that T is $\alpha - admissible$ if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 1.6. [9] Let $f : X \to X$ and $\alpha : X \times X \to (-\infty, +\infty)$. We say that f is a triangular $\alpha-$admissible mapping if $(T1)$ $\alpha(x, y) \geq 1 \ \Rightarrow\ \alpha(fx, fy) \geq 1,\ x, y \in X$
$(T2)$

$$\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases}$$

implies $\quad \alpha(x, y) \geq 1, x, y, z \in X$.

Example 1.7. [9] Let $X = \mathbb{R}$, $fx = e^{x^7}$ and $\alpha(x, y) = \sqrt[5]{x - y} + 1$. Hence, f is a triangular $\alpha-$admissible mapping. Again, if $\alpha(x, y) = \sqrt[5]{x - y} + 1 \geq 1$ then $x \geq y$ which implies $fx \geq fy$. That is, $\alpha(fx, fy) \geq 1$. Moreover, if

$$\begin{cases} \alpha(x, z) \geq 1; \\ \alpha(z, y) \geq 1, \end{cases}$$

then $x - y \geq 0$, and hence $\alpha(x, y) \geq 1$.

Lemma 1.8. [11] *(A Coupled Fixed Point is a Fixed Point). Let $F : X \times X \to X$ be a given mapping. Define the mapping $T : X \times X \longrightarrow X \times X$ by*

$$T(x, y) = (F(x, y), F(y, x)),\ forall\ (x, y) \in X \times X.$$

Then, (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T.

Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy
(i) φ is continuous and non-decreasing,
(ii) $\varphi(t) = 0$ if and only if $t = 0$,
(iii)$\varphi(t + s) \leq \varphi(t) + \varphi(s)$, $\forall t, s \in [0, \infty)$
and Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy $lim_{t \to r}\psi(t) > 0$ for all $r > 0$ and $lim_{t \to o^+}\psi(t) = 0$.
In [10] the authors gave examples of this functions.

Theorem 1.9. *(see [11]) let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha - \psi-$contractive mapping satisfying the following condition:*

(i) *T is $\alpha-$admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 1.10. [10]*Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$x_0 \leq F(x_o, y_0)$ and $y_0 \geq F(y_0, x_0)$.

Suppose there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$\varphi(d(F(x,y), F(u,v))) \leq \frac{1}{2}\varphi(d(x,u) + d(y,v)) - \psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either

(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all n,*

then F has a coupled fixed point in X.

2. The main results

Theorem 2.1. *Let (X, d) be a complete metric space and $F : X \times X \to X$ be a mapping having the mixed monotone property on X. suppose that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ and a function $\alpha : X^2 \times X^2 \to [0, \infty)$ such that*

$$\alpha\big((x,y),(u,v)\big)\varphi\big(d\big(F(x,y),F(u,v)\big)\big) \leq \frac{1}{2}\varphi\big(d(x,u)+d(y,v)\big) - \psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big). \tag{2.1}$$

if F is $\alpha - admissible$ and there exists $(x_0, y_0) \in X \times X$ such that

$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1$ and $\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1$,

and there exists $x_0, y_0 \in X$ such that

$x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$,

and for all $x, y, u, v \in X$ Suppose either

(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,*
  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all n*

then F has a coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(x_0, y_o)$. we construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n, y_n) \; and \; y_{n+1} = F(y_n, x_n) \; for \; all \; n \geq 0 \tag{2.2}$$

We shall show that

$$x_n \leq x_{n+1} \; and \; y_n \geq y_{n+1} \; for \; all \; n \geq 0. \tag{2.3}$$

We shall use the mathematical induction.

Let $n = 0$. Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ and $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \leq x_1$ and $y_0 \geq y_1$ thus (2.28) hold for $n = 0$.

Now suppose that (2.28) hold for some fixed $n \geq 0$, then since $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$, and by mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \geq F(x_n, y_{n+1}) \geq F(x_n, y_n) = x_{n+1} \tag{2.4}$$

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_{n+1}) \leq F(y_n, x_n) = y_{n+1} \tag{2.5}$$

Now from (2.29) and (2.30), we obtain
$x_{n+1} \leq x_{n+2}$ and $y_{n+1} \geq y_{n+2}$.
Thus by the mathematical induction we conclude that (2.28) hold for all $n \geq 0$.
Since $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$, so we have

$$\begin{aligned}
&\varphi(d(x_n, x_{n+1})) = \varphi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1})))\\
&\leq \alpha\Big((x_{n-1}, y_{n-1}), \big(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})\big)\Big)\varphi\Big(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))\Big)\\
&\qquad \leq \frac{1}{2}\varphi\Big(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\big) - \psi\Big(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2}\Big).
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\varphi(d(y_n, y_{n+1})) = \varphi\Big(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))\Big)\\
&\leq \alpha\Big((F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})), (y_{n-1}, x_{n-1})\Big)\varphi\Big(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))\Big)\\
&\qquad \leq \frac{1}{2}\varphi\Big(d(y_{n-1}, y_n) + d(x_{n-1}, x_n)\Big) - \psi\Big(\frac{d(y_{n-1}, y_n) + d(x_{n-1}, x_n)}{2}\Big).
\end{aligned} \tag{2.7}$$

Adding (2.33) to (2.36), we get

$$\begin{aligned}
&\beta((\zeta_1, \zeta_2), (\eta_1, \eta_2))\Big(\varphi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) + \varphi(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)))\Big)\\
&\leq \varphi\Big(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\Big) - 2\psi\Big(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}\Big).
\end{aligned} \tag{2.8}$$

with

$$\beta\big((\zeta_1, \zeta_2), (\eta_1, \eta_2)\big) = \min\Big\{\alpha\Big((x_{n-1}, y_{n-1}), \big(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})\big)\Big), \tag{2.9}$$

$$\alpha\Big(\big(F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})\big), (x_{n-1}, y_{n-1})\Big)\Big\}. \tag{2.10}$$

If we consider $Y = X \times X$, so we can define $\beta : Y \times Y \to [0, \infty)$, such that for all $\zeta = (\zeta_1, \zeta_2), \eta = (\eta_1, \eta_2) \in Y$

$$\beta((\zeta_1, \zeta_2), (\eta_1, \eta_2)) = \min\{\alpha((\zeta_1, \zeta_2), (\eta_1, \eta_2)), \alpha((\eta_2, \eta_1), (\zeta_2, \zeta_1)).$$

and $T : Y \to Y$ is given by (1.8). let $\zeta = (\zeta_1, \zeta_2), \eta = (\eta_1, \eta_2) \in Y$ such that $\beta(\zeta, \eta) \geq 1$, we obtain immediately that $\beta(T\zeta, T\eta) \geq 1$. also there exists $(x_0, y_0) \in Y$ such that:

$$\beta\big((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))\big) \geq 1. \tag{2.11}$$

By property (iii) of φ, we have

$$\varphi\big(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\big) \leq \varphi(d(x_{n+1}, x_n)) + \varphi(d(y_n, y_{n+1})). \tag{2.12}$$

From (2.37), (2.40) and (2.41), we have

$$\varphi\big(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\big) \leq \beta((x_n, y_n), (x_{n-1}, y_{n-1}))\varphi\big(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\big)$$

$$\leq \varphi\Big(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\Big) - 2\psi\Big(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}\Big). \tag{2.13}$$

Which implies

$\varphi\big(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\big) \leq \varphi\big(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\big)$.

Using the fact that φ is non-decreasing, we get

$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq d(x_n, x_{n-1}) + d(y_n, y_{n-1})$.

Set $\delta_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ then sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$lim_{n\to\infty}\delta_n = lim_{n\to\infty}[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = \delta. \tag{2.14}$$

We shall show that $\delta = 0$. Suppose to the contrary, that $\delta > 0$, Then taking the limit as $n \to \infty$ of both sides of (2.41) and have in mind that we suppose $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and φ is continuous, we have

$$\begin{aligned}\varphi(\delta) = \lim_{n\to\infty} \varphi(\delta_n) &= \lim_{n\to\infty} \Big[\varphi(\delta_{n-1}) - 2\psi\Big(\frac{\delta_{n-1}}{2}\Big)\Big] \\ &= \varphi(\delta) - 2 \lim_{\delta_{n-1}\to\delta} \psi\big(\frac{\delta_{n-1}}{2}\big) < \varphi(\delta).\end{aligned}$$

a contradiction. Thus $\delta = 0$, that is,

$$lim_{n\to\infty}\delta_n = lim_{n\to\infty}\Big[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\Big] = 0. \tag{2.15}$$

In what follows, we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least of $\{x_n\}$ or $\{y_n\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}, \{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \geq \varepsilon. \tag{2.16}$$

Further,corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfying (2.45). Then

$$d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}) < \varepsilon. \tag{2.17}$$

Using (2.45), (2.46) and the triangle inequality, we have

$$\begin{aligned}\varepsilon \leq r_k :&= d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and using (2.44)

$$lim_{k\to\infty}r_k = lim_{k\to\infty}\Big[d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})\Big] = \varepsilon. \tag{2.18}$$

By the triangle inequality

$$\begin{aligned}r_k = d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) &\leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) \\ &+ d(x_{m(k)+1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)+1}) \\ &+ d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) \\ &= \delta_{n(k)} + \delta_{m(k)} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}).\end{aligned}$$

Using the property of φ, we have

$$\varphi(r_k) = \varphi\big(\delta_{n(k)} + \delta_{m(k)} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})\big)$$
$$\leq \varphi(\delta_{n(k)} + \delta_{m(k)}) + \varphi(d(x_{n(k)+1}, x_{m(k)+1})) + \varphi(d(y_{n(k)+1}, y_{m(k)+1})). \quad (2.19)$$

Since $n(k) > m(k)$, hence $x_{n(k)} \geq x_{m(k)}$ and $y_{n(k)} \leq y_{m(k)}$. so we have

$$\varphi\big(d(x_{n(k)+1}, x_{m(k)+1})\big) = \varphi\big(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))\big) \quad (2.20)$$
$$\leq \alpha\big((x_{n(k)}, y_{n(k)}), (x_{m(k)}, y_{m(k)})\big)\varphi\big(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))\big) \quad (2.21)$$
$$\leq \frac{1}{2}\varphi\big(d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})\big) - \psi\Big(\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}\Big) \quad (2.22)$$
$$= \tfrac{1}{2}\varphi(r_k) - \psi\Big(\tfrac{r_k}{2}\Big), \quad (2.23)$$

and

$$\varphi\big(d(y_{n(k)+1}, y_{m(k)+1})\big) = \varphi\big(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))\big)$$
$$\leq \alpha\big((y_{n(k)}, x_{n(k)}), (y_{m(k)}, x_{m(k)})\big)\varphi\big(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))\big)$$
$$\leq \frac{1}{2}\varphi\big(d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{m(k)})\big) - \psi\Big(\frac{d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{m(k)})}{2}\Big)$$
$$= \tfrac{1}{2}\varphi(r_k) - \psi\Big(\tfrac{r_k}{2}\Big). \quad (2.24)$$

From (2.48)-(2.52), we have

$$\varphi(r_k) \leq \varphi\big(\delta_{n(k)} + \delta_{m(k)}\big) + \varphi(r_k) - 2\psi\Big(\tfrac{r_k}{2}\Big).$$

a contradiction. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete metric space, there exist $x, y \in X$ such that

$$lim_{n\to\infty} x_n = x \ and \ lim_{n\to\infty} y_n = y. \quad (2.25)$$

Now, suppose F is continuous. Taking the limit as $n \to \infty$ in (2.27) and by (2.53), we get

$x = lim_{n\to\infty} x_n = lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F\big(lim_{n\to\infty} x_{n-1}, lim_{n\to\infty} y_{n-1}\big) = F(x, y)$

and

$y = lim_{n\to\infty} y_n = lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F\big(lim_{n\to\infty} y_{n-1}, lim_{n\to\infty} x_{n-1}\big) = F(y, x).$

Therefore F has coupled fixed point.
Finally, suppose that (b) holds. by assumption (b), we have $x_n \geq x$ and $y_n \leq y$ for all n. Since

$$d(x, F(x, y)) \leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y)) = d(x, x_{n+1}) + d(F(x_n, y_n), F(x, y))$$

Therefore

$$\varphi\big(d(x, F(x, y))\big) \leq \varphi\big(d(x, x_{n+1})\big) + \varphi\big(d(F(x_n, y_n), F(x, y))\big)$$
$$\leq \varphi\big(d(x, x_{n+1})\big) + \alpha\big((x_n, y_n), (x, y)\big)\varphi\big(d(F(x_n, y_n), F(x, y))\big)$$
$$\leq \varphi\big(d(x, x_{n+1})\big) + \tfrac{1}{2}\varphi\big(d(x_n, x) + d(y_n, y)\big) - \psi\Big(\tfrac{d(x_n, x) + d(y_n, y)}{2}\Big).$$

Taking the limit of above inequality, using (2.53) and the property of ψ, we get $\varphi\big(d(x, F(x,y))\big) = 0$, thus $d(x, F(x,y)) = 0$. Hence $x = F(x,y)$. Similarly, one can show that $y = F(y,x)$.
Thus we proved that F has a coupled fixed point. □

Theorem 2.1. *In addition to hypothesis of Theorem 2.1, if x_0 and y_0 are comparable then F has a Fixed point.*

Proof. By using a similar proof in Theorem (2.6) of [10] we can deduce the proof. □

Corollary 2.2. *In Theorem 2.1, if 2.1 replaced with:*

$$\alpha\big((x,y),(u,v)\big)\varphi\big(d\big(F(x,y),F(u,v)\big)\big) \leq \frac{1}{4}\varphi\big(d(x,u)+d(y,v)+d(u,F(u,v))+d(v,F(v,u))\big) -\psi\Big(\frac{d(x,u)+d(y,v)+d(u,F(u,v))+d(v,F(v,u)}{4}\Big). \tag{2.26}$$

for all $(x,y),(u,v) \in X \times X$, then F has coupled fixed point.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(x_0,y_o)$. we constract sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n,y_n) \; and \; y_{n+1} = F(y_n,x_n) \; for \; all \; n \geq 0 \tag{2.27}$$

We shall show that

$$x_n \leq x_{n+1} \; and \; y_n \geq y_{n+1} \; for \; all \; n \geq 0. \tag{2.28}$$

We shall use the mathematical induction. Let $n = 0$. Since $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$ and $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$, we have $x_0 \leq x_1$ and $y_0 \geq y_1$ thus (2.28) hold for $n = 0$.
Now suppose that (2.28) hold for some fixed $n \geq 0$, then since $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$, and by mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1},y_{n+1}) \geq F(x_n,y_{n+1}) \geq F(x_n,y_n) = x_{n+1} \tag{2.29}$$

$$y_{n+2} = F(y_{n+1},x_{n+1}) \leq F(y_n,x_{n+1}) \leq F(y_n,x_n) = y_{n+1} \tag{2.30}$$

Now from (2.29) and (2.30), we obtain
$x_{n+1} \leq x_{n+2}$ and $y_{n+1} \geq y_{n+2}$.
Thus by the mathematical induction we conclude that (2.28) hold for all $n \geq 0$.
Since $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$, so we have

$$\varphi(d(x_n,x_{n+1})) = \varphi(d(F(x_{n-1},y_{n-1}),F(x_n,y_n))) \tag{2.31}$$

$$\leq \alpha\Big((x_{n-1},y_{n-1}),\big(F(x_{n-1},y_{n-1}),F(y_{n-1},x_{n-1})\big)\Big)\varphi\Big(d(F(x_{n-1},y_{n-1}),F(x_n,y_n))\Big) \tag{2.32}$$

$$\begin{aligned}&\leq \frac{1}{4}\varphi\Big(d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(x_{n-1},F(x_{n-1},y_{n-1}))+d(y_{n-1},F(y_{n-1},x_{n-1}))\Big)\\ &-\psi\Big(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)+d(x_{n-1},F(x_{n-1},y_{n-1}))+d(y_{n-1},F(y_{n-1},x_{n-1}))}{4}\Big)\\ &\leq \frac{1}{2}\varphi\Big(d(x_{n-1},x_n)+d(y_{n-1},y_n)\Big) - \psi\Big(\frac{d(x_{n-1},x_n)+d(y_{n-1},y_n)}{2}\Big)\end{aligned} \tag{2.33}$$

$$\varphi(d(y_n,y_{n+1})) = \varphi(d(F(y_{n-1},x_{n-1}),F(y_n,x_n))) \tag{2.34}$$

$$\leq \alpha\Big(\big(F(y_{n-1},x_{n-1}),F(x_{n-1},y_{n-1})\big),(y_{n-1},x_{n-1})\Big)\varphi\Big(d(F(y_{n-1},x_{n-1}),F(y_n,x_n))\Big) \tag{2.35}$$

$$\begin{aligned}
&\leq \frac{1}{4}\varphi\Big(d(y_{n-1},y_n)+d(x_{n-1},x_n)+d(y_{n-1},F(y_{n-1},x_{n-1}))+d(x_{n-1},F(x_{n-1},y_{n-1}))\Big)\\
&-\psi\Big(\frac{d(y_{n-1},y_n)+d(x_{n-1},x_n)+d(y_{n-1},F(y_{n-1},x_{n-1}))+d(x_{n-1},F(x_{n-1},y_{n-1}))}{4}\Big)\\
&\leq \frac{1}{2}\varphi\Big(d(y_{n-1},y_n)+d(x_{n-1},x_n)\Big)-\psi\Big(\frac{d(y_{n-1},y_n)+d(x_{n-1},x_n)}{2}\Big).
\end{aligned} \tag{2.36}$$

Adding (2.33) to (2.36), we get

$$\begin{aligned}
&\beta((\zeta_1,\zeta_2),(\eta_1,\eta_2))\Big(\varphi\big(d(F(x_n,y_n),F(x_{n-1},y_{n-1}))\big)+\varphi\big(d(F(y_{n-1},x_{n-1}),F(y_n,x_n))\big)\Big)\\
&\leq \varphi\Big(d(x_n,x_{n-1})+d(y_n,y_{n-1})\Big)-2\psi\Big(\frac{d(x_n,x_{n-1})+d(y_n,y_{n-1})}{2}\Big).
\end{aligned} \tag{2.37}$$

With

$$\beta\big((\zeta_1,\zeta_2),(\eta_1,\eta_2)\big)=\min\Big\{\alpha\Big((x_{n-1},y_{n-1}),\big(F(x_{n-1},y_{n-1}),F(y_{n-1},x_{n-1})\big)\Big), \tag{2.38}$$

$$\alpha\Big(\big(F(y_{n-1},x_{n-1}),F(x_{n-1},y_{n-1})\big),(x_{n-1},y_{n-1})\Big)\Big\}. \tag{2.39}$$

If we consider $Y=X\times X$, so we can define $\beta: Y\times Y\to[0,\infty)$, such that for all $\zeta=(\zeta_1,\zeta_2),\eta=(\eta_1,\eta_2)\in Y$

$$\beta((\zeta_1,\zeta_2),(\eta_1,\eta_2))=\min\{\alpha((\zeta_1,\zeta_2),(\eta_1,\eta_2)),\alpha((\eta_2,\eta_1),(\zeta_2,\zeta_1)).$$

and $T: Y\to Y$ is given by (1.8). let $\zeta=(\zeta_1,\zeta_2),\eta=(\eta_1,\eta_2)\in Y$ such that $\beta(\zeta,\eta)\geq 1$, we obtain immediately that $\beta(T\zeta,T\eta)\geq 1$. also there exists $(x_0,y_0)\in Y$ such that:

$$\beta\big((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))\big)\geq 1. \tag{2.40}$$

By property (iii) of φ, we have

$$\varphi\big(d(x_{n+1},x_n)+d(y_{n+1},y_n)\big)\leq\varphi(d(x_{n+1},x_n))+\varphi(d(y_n,y_{n+1})). \tag{2.41}$$

From (2.37), (2.40) and (2.41), we have

$$\begin{aligned}
\varphi\big(d(x_{n+1},x_n)+d(y_{n+1},y_n)\big)&\leq\beta((x_n,y_n),(x_{n-1},y_{n-1}))\varphi\big(d(x_{n+1},x_n)+d(y_{n+1},y_n)\big)\\
&\leq\varphi\Big(d(x_n,x_{n-1})+d(y_n,y_{n-1})\Big)-2\psi\Big(\frac{d(x_n,x_{n-1})+d(y_n,y_{n-1})}{2}\Big).
\end{aligned} \tag{2.42}$$

Which implies

$$\varphi\big(d(x_{n+1},x_n)+d(y_{n+1},y_n)\big)\leq\varphi\big(d(x_n,x_{n-1})+d(y_n,y_{n-1})\big).$$

Using the fact that φ is non-decreasing, we get

$$d(x_{n+1},x_n)+d(y_{n+1},y_n)\leq d(x_n,x_{n-1})+d(y_n,y_{n-1}).$$

Set $\delta_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ then sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$lim_{n\to\infty}\delta_n = lim_{n\to\infty}[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = \delta. \tag{2.43}$$

We shall show that $\delta = 0$. Suppose to the contrary, that $\delta > 0$, Then taking the limit as $n \to \infty$ of both sides of (2.41) and have in mind that we suppose $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and φ is continuous, we have

$$\begin{aligned}\varphi(\delta) = \lim_{n\to\infty} \varphi(\delta_n) &= \lim_{n\to\infty} \Big[\varphi(\delta_{n-1}) - 2\psi\Big(\frac{\delta_{n-1}}{2}\Big)\Big] \\ &= \varphi(\delta) - 2 \lim_{\delta_{n-1}\to\delta} \psi\big(\frac{\delta_{n-1}}{2}\big) < \varphi(\delta).\end{aligned}$$

a contradiction. Thus $\delta = 0$, that is,

$$lim_{n\to\infty}\delta_n = lim_{n\to\infty}\Big[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\Big] = 0. \tag{2.44}$$

In what follows, we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least of $\{x_n\}$ or $\{y_n\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}, \{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \geq \varepsilon. \tag{2.45}$$

Further,corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfying (2.45). Then

$$d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}) < \varepsilon. \tag{2.46}$$

Using (2.45), (2.46) and the triangle inequality, we have

$$\begin{aligned}\varepsilon \leq r_k &:= d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\ &+ d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\ &+ d(x_{m(k)}, x_{m(k+1)}) + d(y_{m(k)}, y_{m(k+1)}) \\ &\leq \delta_{n(k-1)} + \varepsilon + \delta_{m(k)} \leq 2\delta_{n(k-1)} + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and using (2.44)

$$\begin{aligned}lim_{k\to\infty}r_k = lim_{k\to\infty}\Big[&d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\ &+d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)}))\Big] = \varepsilon.\end{aligned} \tag{2.47}$$

By the triangle inequality

$$\begin{aligned}r_k &= d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\ &+ d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})) \\ &\leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)+1}) \\ &+ d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) + d(x_{m(k)}, x_{m(k+1)}) + d(y_{m(k)}, y_{m(k+1)}) \\ &= \delta_{n(k)} + 2\delta_{m(k)} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}).\end{aligned}$$

Using the property of φ, we have

$$\begin{aligned}\varphi(r_k) &= \varphi\big(\delta_{n(k)} + 2\delta_{m(k)} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})\big) \\ &\leq \varphi(\delta_{n(k)} + 2\delta_{m(k)}) + \varphi(d(x_{n(k)+1}, x_{m(k)+1})) + \varphi(d(y_{n(k)+1}, y_{m(k)+1})).\end{aligned} \tag{2.48}$$

Since $n(k) > m(k)$, hence $x_{n(k)} \geq x_{m(k)}$ and $y_{n(k)} \leq y_{m(k)}$. so we have

$$\varphi\big(d(x_{n(k)+1}, x_{m(k)+1})\big) = \varphi\big(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))\big) \tag{2.49}$$

$$\leq \alpha\big((x_{n(k)}, y_{n(k)}), (x_{m(k)}, y_{m(k)})\big)\varphi\big(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))\big)$$
$$\leq \frac{1}{4}\varphi\big(d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)}))\big)$$
$$-\psi\Big(\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)}))}{4}\Big)$$
$$= \frac{1}{4}\varphi(r_k) - \psi\Big(\frac{r_k}{4}\Big) \tag{2.50}$$

$$\varphi\big(d(y_{n(k)+1}, y_{m(k)+1})\big) = \varphi\big(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))\big) \tag{2.51}$$

$$\leq \alpha\big((y_{n(k)}, x_{n(k)}), (y_{m(k)}, x_{m(k)})\big)\varphi\big(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))\big)$$
$$\leq \frac{1}{4}\varphi\big(d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{m(k)}) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})) + d(x_{m(k)}, F(x_{m(k)}, y_{m(k)}))\big)$$
$$-\psi\Big(\frac{d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{m(k)}) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})) + d(x_{m(k)}, F(x_{m(k)}, y_{m(k)}))}{4}\Big)$$
$$= \frac{1}{4}\varphi(r_k) - \psi\Big(\frac{r_k}{4}\Big). \tag{2.52}$$

From (2.48)-(2.52), we have

$$\varphi(r_k) \leq \varphi\big(\delta_{n(k)} + 2\delta_{m(k)}\big) + \tfrac{1}{2}\varphi(r_k) - 2\psi\Big(\tfrac{r_k}{4}\Big).$$

Letting $k \to \infty$ and using (2.44) and (2.47), we have

$$\varphi(\varepsilon) \leq \varphi(0) + \tfrac{1}{2}\varphi(\varepsilon) - 2lim_{k\to\infty}\psi\Big(\tfrac{r_k}{4}\Big) = \tfrac{1}{2}\varphi(\varepsilon) - 2lim_{r_k\to\varepsilon}\psi\Big(\tfrac{r_k}{4}\Big) < \varphi(\varepsilon),$$

a contradiction. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete metric space, there exist $x, y \in X$ such that

$$lim_{n\to\infty}x_n = x \ and \ lim_{n\to\infty}y_n = y. \tag{2.53}$$

Now, suppose F is continuous. Taking the limit as $n \to \infty$ in (2.27) and by (2.53), we get

$x = lim_{n\to\infty}x_n = lim_{n\to\infty}F(x_{n-1}, y_{n-1}) = F\big(lim_{n\to\infty}x_{n-1}, lim_{n\to\infty}y_{n-1}\big) = F(x, y)$

and

$y = lim_{n\to\infty}y_n = lim_{n\to\infty}F(y_{n-1}, x_{n-1}) = F\big(lim_{n\to\infty}y_{n-1}, lim_{n\to\infty}x_{n-1}\big) = F(y, x).$

Therefore F has coupled fixed point.
Finally, suppose that (b) holds. by assumption (b), we have $x_n \geq x$ and $y_n \leq y$ for all n. Since

$$d(x, F(x, y)) \leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y)) = d(x, x_{n+1}) + d(F(x_n, y_n), F(x, y)).$$

Therefore

$$\begin{aligned}
\varphi\big(d(x,F(x,y))\big) &\le \varphi\big(d(x,x_{n+1})\big)+\varphi\big(d(F(x_n,y_n),F(x,y))\big)\\
&\le \varphi\big(d(x,x_{n+1})\big)+\alpha\big((x_n,y_n),(x,y)\big)\varphi\big(d(F(x_n,y_n),F(x,y))\big)\\
&\le \varphi\big(d(x,x_{n+1})\big)+\frac{1}{4}\varphi\big(d(x_n,x)+d(y_n,y)+d(x_n,F(x_n,y_n))+d(y_n,F(y_n,x_n))\big)\\
&-\psi\Big(\frac{d(x_n,x)+d(y_n,y)+d(y_n,F(y_n,x_n))+d(x_n,F(x_n,y_n))}{4}\Big)\\
&= \varphi\big(d(x,x_{n+1})\big)+\frac{1}{4}\varphi\big(d(x_n,x)+d(y_n,y)+d(x_n,x_{n+1})+d(y_n,y_{n+1})\big)\\
&-\psi\Big(\frac{d(x_n,x)+d(y_n,y)+d(y_n,y_{n+1})+d(x_n,x_{n+1})}{4}\Big)\\
&= \varphi\big(d(x,x_{n+1})\big)+\frac{1}{4}\varphi\big(d(x_n,x)+d(y_n,y)+\delta_n\big)-\psi\Big(\frac{d(x_n,x)+d(y_n,y)+\delta_n}{4}\Big).
\end{aligned} \tag{2.54}$$

Taking the limit of above inequality, using (2.53) and the property of ψ, we get $\varphi\big(d(x,F(x,y))\big)=0$, thus $d(x,F(x,y))=0$. Hence $x=F(x,y)$. Similarly, one can show that $y=F(y,x)$.
Thus we proved that F has a coupled fixed point. □

Theorem 2.2. *Let (X,d) be a complete metric space $\alpha: X\times X\to[0,\infty)$ a function, $\psi\in\Psi$, and $T: X\to X$ be a continuous, non-decreasing triangular $\alpha-$admissible mapping such that*

$$\alpha(x,y)\psi\big(d(Tx,Ty)\big)\le\psi\Big(\frac{1}{2}\big(d(x,Ty)+d(y,Tx)\big)\Big)-\varphi\Big(d(x,Ty),d(y,Tx)\Big). \tag{2.55}$$

For all $x,y\in X$,
where $\varphi:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(x,y)=0$ if and only if $x=y$. and there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$. Then T has a fixed point.

Proof. Take $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge1$ and define sequence $\{x_n\}$ in X with $x_{n+1}=Tx_n$.
$\alpha(x_0,Tx_0)=\alpha(x_0,x_1)\ge1\Rightarrow\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\ge1$.
By continuity of this process we have $\alpha(x_n,x_{n+1})\ge1$, Thus

$$\begin{aligned}
\psi\big(d(x_n,x_{n+1})\big) &= \psi\big(d(Tx_{n-1},Tx_n)\big)\le\alpha(x_{n-1},x_n)\psi\big(d(Tx_{n-1},Tx_n)\big)\\
&\le\psi\Big(\frac{1}{2}\big(d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})\big)\Big)-\varphi\Big(d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\Big)\\
&\le\psi\Big(\frac{1}{2}\big(d(x_{n-1},x_{n+1})+d(x_n,x_n)\big)\Big)-\varphi\Big(d(x_{n-1},x_{n+1}),d(x_n,x_n)\Big)\\
&\le\psi\Big(\frac{1}{2}\big(d(x_{n-1},x_{n+1})\big)\Big)-\varphi\Big(d(x_{n-1},x_{n+1}),0\Big)\le\psi\Big(\frac{1}{2}d(x_{n-1},x_{n+1})\Big).
\end{aligned}$$

Since ψ is non-decreasing function, we get

$$\psi\big(d(x_n,x_{n+1})\big)\le\psi\big(\tfrac{1}{2}d(x_{n_1},x_{n+1})\big)\Rightarrow d(x_n,x_{n+1}\le\tfrac{1}{2}d(x_{n-1},x_{n+1}).$$

Hence $d(x_n,x_{n+1})_{n\geqslant1}$ is non-decreasing sequence and there is $r\geqslant0$ such that

$$r=\lim_{n\to\infty}d(x_n,x_{n+1}). \tag{2.56}$$

Also we have

$$d(x_n, x_{n+1}) \leq \tfrac{1}{2}d(x_{n-1}, x_{n+1}) \leq \tfrac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$$

$lim_{n\to\infty}d(x_{n-1}, x_{n+1}) = 2r$ Letting $n \to \infty$ and using (2.56), we get

$$r \leq lim_{n\to\infty}\tfrac{1}{2}d(x_{n-1}, x_{n+1}) \leq \tfrac{1}{2}(r + r).$$

Hence by using of continuity ψ, φ, we have

$$\psi(r) \leq \psi\big(\tfrac{1}{2}(2r)\big) - \varphi(2r, 0).$$

Which implies that $\varphi(2r, 0) = 0 \Rightarrow r = 0$ and $lim_{n\to\infty}d(x_n, x_{n+1}) = r = 0$. Now we show that $\{x_n\}$ is a cauchy sequence in x. Suppose to the contrary that $\{x_n\}$ is not cauchy sequence, then there exists $\varepsilon > 0$ for which we can find two subsequence $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which $n_i > m_i > 0$, $d(x_{n_i}, x_{m_i}) \geq \varepsilon$, Thus $d(x_{m_i}, x_{n_{i-1}}) < \varepsilon$, and we have

$$\begin{aligned}\varepsilon \leq d(x_{m_i}, x_{n_i}) &\leq d(x_{m_i}, x_{m_{i+1}}) + d(x_{m_{i+1}}, x_{n_{i-1}}) + d(x_{n_{i-1}}, x_{n_i})\\ &\leq 2d(x_{m_i}, x_{m_{i+1}}) + d(x_{m_i}, x_{n_i}) + 2d(x_{m_{i-1}}, x_{n_i})\\ &\leq 2d(x_{m_i}, x_{m_{i+1}}) + \varepsilon + 3d(x_{m_{i-1}}, x_{n_i}).\end{aligned}$$

Letting $i \to \infty$ we get

$$lim_{i\to\infty}d(x_{m_i}, x_{n_i}) = lim_{i\to\infty}d(x_{m_{i+1}}, x_{n_{i+1}}) = lim_{i\to\infty}d(x_{m_{i+1}}, x_{n_i}) = \varepsilon. \quad (2.57)$$

Also we have

$$\begin{aligned}\psi\big(d(x_{m_{i+1}}, x_{n_i})\big) &= \psi\big(d(Tx_{m_i}, Tx_{n_{i-1}})\big) \leq \alpha(x_{m_i}, x_{n_i})\psi\big(d(Tx_{m_i}, Tx_{n_{i-1}})\big)\\ &\leq \psi\Big(\frac{1}{2}\big(d(x_{m_i}, Tx_{n_{i-1}}) + d(x_{n_{i-1}}, Tx_{m_i})\big)\Big) - \varphi\Big(d(x_{m_i}, Tx_{n_{i-1}}), d(x_{n_{i-1}}, Tx_{m_i})\Big)\\ &= \psi\Big(\frac{1}{2}\big(d(x_{m_i}, x_{n_i}) + d(x_{n_{i-1}}, x_{m_{i+1}})\big)\Big) - \varphi\Big(d(x_{m_i}, x_{n_i}), d(x_{n_{i-1}}, x_{m_{i+1}})\Big).\end{aligned}$$

By letting $i \to \infty$ and use of continuity of φ and ψ, we get

$$\psi(\varepsilon) \leqslant \psi(\varepsilon) - \varphi(\varepsilon, \varepsilon).$$

Hence we get $\varphi(\varepsilon, \varepsilon) = 0$, and $\varepsilon = 0$, a contradiction, thus $\{x_n\}$ is cauchy sequence in X and there exists $x \in X$ such that $lim_{n\to\infty}x_n = x$.
Since T is continuous and $x_n \to x$, we obtain $x_{n+1} = Tx_n \to Tx$ and $Tx = x$. Thus T has fixed point. □

Theorem 2.3. *let $F : X \times X \times X \to X$ be a given mapping in complete metric space (X, d) and suppose that there exist $\psi \in \Psi$ and a function $\alpha : X^3 \times X^3 \to [0, \infty)$ such that*

$$\alpha\big((x, y, z), (u, v, w)\big)d\big(F(x, y, z), F(u, v, w)\big) \leqslant \frac{1}{3}\psi\big(d(x, u) + d(y, v) + d(z, w)\big) \quad (2.58)$$

for all $(x, y, z), (u, v, z) \in X \times X \times X$. suppose also that

(i) *For all $(x, y, z), (u, v, w) \in X \times X \times X$, we have $\alpha\big((x, y, z), (u, v, w)\big) \geqslant 1 \Rightarrow$*

$$\alpha\Big(\big(F(x, y, z), F(y, z, x), F(z, x, y)\big), \big(F(u, v, w), F(v, w, u), F(w, u, v)\big)\Big) \geqslant 1,$$

(ii) *there exists $(x_0, y_0, z_0) \in X \times X \times X$ such that*

$$\alpha\Big((x_0, y_0, z_0), \big(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0)\big)\Big) \geqslant 1$$

$$\alpha\Big(\big(F(y_0, z_0, x_0), F(z_0, x_0, y_0), F(x_0, y_0, z_0)\big), (y_0, z_0, x_0)\Big) \geqslant 1$$

$$\alpha\Big((z_0, x_0, y_0), \big(F(z_0, x_0, y_0,), F(x_0, y_0, z_0), F(y_0, z_0, x_0)\big)\Big) \geqslant 1$$

(iii) *and F is continuous.*
Then, F has a tripled fixed point, that is, there exists $(x^*, y^*, z^*) \in X \times X \times X$ such that $x^* = F(x^*, y^*, z^*)$ and $y^* = F(y^*, z^*, x^*)$ and $z^* = F(z^*, x^*, y^*)$.

Proof. The idea consists in transporting the problem to the complete metric space (Y, δ) where $Y = X \times X \times X$ and $\delta\big((x, y, z), (u, v, w)\big) = d(x, u) + d(y, v) + d(z, w)$ for all $(x, y, z), (u, v, w) \in X \times X \times X$. also we have

$$\alpha\big((x, y, z), (u, v, w)\big)d\big(F(x, y, z), F(u, v, w)\big) \leqslant \frac{1}{3}\psi\Big(\delta\big((x, y, z), (u, v, w)\big)\Big) \quad (2.59)$$

and

$$\alpha\big((v, w, u), (y, z, x)\big)d\big(F(v, w, u), F(y, z, x)\big) \leqslant \frac{1}{3}\psi\Big(\delta\big((v, w, u), (y, z, x)\big)\Big) \quad (2.60)$$

and

$$\alpha\big((z, x, y), (w, u, v)\big)d\big(F(z, x, y), F(w, u, v)\big) \leqslant \frac{1}{3}\psi\Big(\delta\big((z, x, y), (w, u, v)\big)\Big). \quad (2.61)$$

Now if $T : Y \to Y$ is defined by

$$T(\tau_1, \tau_2, \tau_3) = \big(F(\tau_1, \tau_2, \tau_3), F(\tau_2, \tau_3, \tau_1), F(\tau_3, \tau_1, \tau_2)\big). \quad (2.62)$$

for all $(\tau_1, \tau_2, \tau_3) \in Y$, and $\beta : Y \times Y \to [0, \infty)$ is the function defined by

$$\beta\big((\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3)\big) =$$
$$\min\big\{\alpha\big((x, y, z), (u, v, w)\big), \alpha\big((v, w, u), (y, z, x)\big), \alpha\big((z, x, y), (w, u, v)\big)\big\},$$

Then by summing up the inequalities (2.59)-(2.61), and using of (2.62) we get

$$\beta(\xi, \eta)\delta(T(x, y, z), T(u, v, w)) \leqslant \psi\big(\delta((x, y, z), (u, v, w))\big). \quad (2.63)$$

for all $\xi = (\xi_1, \xi_2, \xi_3), \eta = (\eta_1, \eta_2, \eta_3) \in Y$. Then T is continuous and $\beta-\psi-$contractive mapping and $\beta(\xi, \eta) \geqslant 1$. It is easy to check that T is $\beta-$admissible and we know that there exists $(x_0, y_0, z_0) \in Y$ such that

$$\beta\big((x_0, y_0, z_0), T(x_0, y_0, z_0)\big) \geqslant 1.$$

All the hypotheses of 1.9 are satisfied, and so we deduce the existence of a fixed point of T. □

Example 2.3. Let $X = [0, +\infty)$ equipped with the standard metric $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X,d) is complete metric space. Define the mapping $F : X \times X \times X \to X$ by

$$F(x, y, z) = \begin{cases} \frac{x-y-z}{6} & x \geq y \geq z \\ 0 & otherwise \end{cases}$$

Clearly F is continuous mapping. Define $\alpha : X^3 \times X^3 \to [0, +\infty)$ by

$$\alpha\big((x, y, z), (u, v, w)\big) = \begin{cases} 1 & u \geq v \geq w \\ 0 & otherwise \end{cases}$$

Then, (2.58) is satisfied with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Also it is easy to check that $\alpha\big((x, y, z), (u, v, w)\big) \geq 1$ implies

$$\alpha\Big(\big(F(x, y, z), F(y, z, x), F(z, x, y)\big), \big(F(u, v, w), F(v, w, u), F(w, u, v)\big)\Big) \geq 1,$$

for all $(x, y, z), (u, v, w)$
$\in X \times X \times X$. On the other hand, the condition (ii) of Theorem (2.3) is satisfied with $(x_0, y_0, z_0) = (0, 0, 0)$. All the required hypotheses of same Theorem are true

and so we deduce the existence of a tripled fixed point of F. Hence $(0,0,0)$ is a tripled fixed point of F.

Corollary 2.4. *Let (X,d) be a complete metric space and $F : X\times X\times X \to X$ be a mapping having the mixed monotone property on X. Suppose that there exist $\psi \in \Psi$ and $\varphi \in \phi$ and a function $\alpha : X^3\times X^3 \to [0,\infty)$ such that*

$$\alpha\big((x,y,z),(u,v,w)\big)\varphi\Big(d\big(F(x,y,z),F(u,v,w)\big)\Big) \leqslant \frac{1}{3}\varphi\Big(d(x,u)+d(y,v)+d(z,w)\Big)-\psi\Big(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\Big). \qquad (2.64)$$

If F is $\alpha-$admissible mapping satisfying the following conditions

(i) *For all $(x,y,z),(u,v,w)\in X\times X\times X$, we have*
$\alpha\big((x,y,z),(u,v,w)\big)\geqslant 1 \Rightarrow$

$$\alpha\Big(\big(F(x,y,z),F(y,z,x),F(z,x,y)\big),\big(F(u,v,w),F(v,w,u),F(w,u,v)\big)\Big)\geqslant 1.$$

(ii) *there exists $(x_0,y_0,z_0)\in X\times X\times X$ such that*

$$\alpha\Big((x_0,y_0,z_0),\big(F(x_0,y_0,z_0),F(y_0,z_0,x_0),F(z_0,x_0,y_0)\big)\Big)\geqslant 1,$$
$$\alpha\Big(\big(F(y_0,z_0,x_0),F(z_0,x_0,y_0),F(x_0,y_0,z_0)\big),(y_0,z_0,x_0)\Big)\geqslant 1,$$
$$\alpha\Big((z_0,x_0,y_0),\big(F(z_0,x_0,y_0,),F(x_0,y_0,z_0),F(y_0,z_0,x_0)\big)\Big)\geqslant 1.$$

(iii) *There exists $x_0,y_0,z_0\in X$ such that*
$x_0 \leqslant F(x_0,y_0,z_0)$ $y_0 \geqslant F(y_0,z_0,x_0)$ *and* $z_0 \leqslant F(z,x_0,y_0)$,
(iv) *for $x,y,z,u,v,w\in X$ with $x\geqslant u, y\leqslant v, z\geqslant w$,*
(v) *F is continuous or*
(a) *If a non-decreasing sequence $\{x_n\}\to x$, then $x_n\leqslant x$ for all n,*
(b) *If a non-increasing sequence $\{y_n\}\to y$, then $y_n \geqslant y$ for all n,*
(c) *If a non-decreasing sequence $\{z_n\}\to z$, then $z_n\leqslant z$ for all n. Then F has tripled fixed point in X.*

3. Application

In this section, we study the existence of a solution to a nonlinear integral equation, as an application to the fixed point theorem.
let Θ denote the class of those functions $\theta : [0,\infty)\to[0,\infty)$ which satisfies the following conditions:
(i) θ is increasing.
(ii) There exists $\psi\in\Psi$ such that $\theta(x)=\frac{x}{2}-\psi(\frac{x}{2})$ for all $x\in[0,\infty)$.
For example, $\theta(x)=kx$, where $0\le k\le\frac{1}{2}, \theta(x)=\frac{x^2}{2(x+1)}, \theta(x)=\frac{x}{2}-\frac{ln(x+1)}{2}$ are in Θ.
Consider the following integral equation

$$x(t)=\int_a^b (K_1(t,s)+K_2(t,s))(f(s,x(s))+g(s,x(s))ds+h(t) \qquad (3.1)$$
$$t\in I=[a,b].$$

We assume that K_1,K_2,f,g,e satisfy the following conditions (i) $K_1(t,s)\ge 0$ and $K_2(t,s)\le 0$ for all $t,s\in[a,b]$.
(ii) There exist $\lambda,\mu>0$ and $\theta\in\Theta$ such that for all $x,y\in\mathbb{R}$, $x\ge y$.
$0\le f(t,x)-f(t,y)\le\lambda\theta(x-y)$

and
$-\mu\theta(x-y) \leq g(t,x) - g(t,y) \leq 0$
and Also
$\max\{\lambda,\mu\}\sup_{t\in I}\int_a^b (K_1(t,s) - K_2(t,s))ds \leq \frac{1}{2}$.
(iii) Also there exists $(x_0, y_0) \in C(I) \times C(I)$such that for all $t \in I$, we have
$e((x_0(t), y_0(t)), (\int_a^b K_1(t,s)(f(s,x_0(s)) + g(s,y_0(s)))ds$
$+ \int_a^b K_2(t,s)(f(s,y_0(s)) + g(s,x_0(s)))ds + h(t),$
$(\int_a^b K_1(t,s)(f(s,y_0(s)) + g(s,x_0(s)))ds + \int_a^b K_2(t,s)(f(s,x_0(s)) + g(s,y_0(s)))ds + h(t)))) \geq 0.$
and
$((\int_a^b (_1(t,s)(f(s,y_0(s))+g(s,x_0(s)))ds+\int_a^b K_2(t,s)(f(s,x_0(s))+g(s,y_0(s)))ds+h(t),$
$(\int_a^b K_1(t,s)(f(s,x_0(s)) + g(s,y_0(s)))ds + \int_a^b K_2(t,s)(f(s,y_0(s))$
$+g(s,x_0(s)))ds + h(t), (y_0(t), x_0(t))))) \geq 0.$
(v) For all $t \in I, x, y \in C(I)$, $e((x(t), y(t)), (u(t), v(t))) \geq 0$ implies that
$e(\int_a^b K_1(t,s)(f(s,x(s)) + g(s,y(s)))ds + \int_a^b K_2(t,s)(f(s,y(s)) + g(s,x(s)))ds + h(t),$
$\int_a^b K_1(t,s)(f(s,y(s)) + g(s,x(s)))ds + \int_a^b K_2(t,s)(f(s,x(s)) + g(s,y(s)))ds + h(t)),$
$(\int_a^b K_1(t,s)(f(s,u(s)) + g(s,v(s)))ds + \int_a^b K_2(t,s)(f(s,v(s)) + g(s,u(s)))ds + h(t),$
$\int_a^b K_1(t,s)(f(s,v(s)) + g(s,u(s)))ds + \int_a^b K_2(t,s)(f(s,u(s)) + g(s,v(s)))ds + h(t)) \geq 0.$

Definition 3.1. An element $(\beta,\gamma) \in C(I,\mathbb{R}) \times C(I,\mathbb{R})$ is called a coupled lower and upper solution of the integral equation (3.1) if $\beta(t) \leq \gamma(t)$ and

$\beta(t) \leq \int_a^b K_1(t,s)(f(s,\beta(s)) + g(s,\gamma(s)))ds + \int_a^b K_2(t,s)(f(s,\gamma(s)) + g(s,\beta(s)))ds + h(t),$
and

$\gamma(t) \geq \int_a^b K_1(t,s)(f(s,\gamma(s)) + g(s,\beta(s)))ds + \int_a^b K_2(t,s)(f(s,\beta(s)) + g(s,\gamma(s)))ds + h(t),$
for all $t \in [a,b]$.

Theorem 3.2. *Consider the integral equation (3.1) with $K_1, K_2 \in C(I\times I,\mathbb{R})$, $f, g \in C(I\times\mathbb{R},\mathbb{R})$ and $h \in C(I\times\mathbb{R},\mathbb{R})$ and suppose that conditions of (i), (ii), (iii) and (iv) are satisfied. Then the existence of a coupled lower and upper solution for (3.1) provides the existence of a solution of (3.1) in $C(I,\mathbb{R})$.*

Proof. Let $X = C(I,\mathbb{R})$. X is a partially ordered set if we define the following order relation in X
$x, y \in C(I,\mathbb{R}) \Leftrightarrow x(t) \leq y(t)$, for all $t \in [a,b]$.
And (X,d) is a complete metric space with metric
$d(x,y) = \sup_{t\in I}|x(t) - y(t)|,\ x, y \in C(I,\mathbb{R})$
Suppose $\{u_n\}$ is a monotone non-decreasing sequence in X that converges to $u \in X$. Then for every $t \in I$, the sequence of real numbers
$u_1(t) \leq u_2(t) \leq \ldots \leq u_n(t) \leq \ldots$
converges to $u(t)$. Therefore, for all $t \in I, n \in \mathbb{N}, u_n(t) \leq u(t)$. Hence $u_n \leq u$, for all n.
Similarly, we can verify that $\lim v(t)$ of a monotone non-increasing sequence $v_n(t)$ in X is a lower bound for all the elements in the sequence. That is, $v \leq v_n$ for all n. Therefore, condition (b) of Theorem (2.1) holds. Also, $X \times X = C(I,\mathbb{R}) \times C(I,\mathbb{R})$

is a partially ordered set if we define the following order relation in $X \times X$
$(x,y),(u,v) \in X \times X, (x,y) \leq (u,v) \Leftrightarrow x(t) \leq u(t)$ and $y(t) \geq v(t),\ \forall t \in I$.
For any $x, y \in X, \max\{x(t), y(t)\}$ and $\min\{x(t), y(t)\}$, for each $t \in I$, are in X and are the upper and lower bounds of x, y, respectively. Therefore, for every $(x,y),(u,v) \in X \times X$, there exists a $(\max\{x,u\}, \min\{y,v\}) \in X \times X$ that is comparable to (x,y) and(u,v). Define $F : X \times X \to X$ by

$$F(x,y)(t) = \int_a^b K_1(t,s)(f(s,x(s))+g(s,y(s)))ds+\int_a^b K_2(t,s)(f(s,y(s))+g(s,x(s)))ds+h(t)$$

for all $t \in [a,b]$.
Now we shall show that F has the mixed monotone property. Indeed, for $x_1 \leq x_2$, that is, $x_1(t) \leq x_2(t)$, for all $t \in [a,b]$, we have
$F(x_1,y)(t) - F(x_2,y)(t) = \int_a^b K_1(t,s)(f(s,x_1(s)) + g(s,y(s)))ds$
$+ \int_a^b K_2(t,s)(f(s,y(s)) + g(s,x_1(s)))ds + h(t)$
$- \int_a^b K_1(t,s)(f(s,x_2(s)) + g(s,y(s)))ds$
$- \int_a^b K_2(t,s)(f(s,y(s)) + g(s,x_2(s)))ds - h(t)$
$= \int_a^b K_1(t,s)(f(s,x_1(s)) + f(s,x_2(s)))ds$
$+ \int_a^b K_2(t,s)(g(s,x_1(s)) + g(s,x_2(s)))ds,$
by Assumption (i) and (ii). Hence $F(x_1,y)(t) \leq F(x_2,y)(t), \forall t \in I$, that is,

$$F(x_1,y) \leq F(x_2,y).$$

Similarly, if $y_1 \geq y_2$, that is, $y_1(t) \geq y_2(t)$, for all $t \in [a,b]$, we have
$F(x,y_1)(t) - F(x,y_2)(t) = \int_a^b K_1(t,s)(f(s,x(s)) + g(s,y_1(s)))ds$
$+ \int_a^b K_2(t,s)(f(s,y_1(s)) + g(s,x(s)))ds + h(t)$
$- \int_a^b K_1(t,s)(f(s,x(s)) + g(s,y_2(s)))ds$
$- \int_a^b K_2(t,s)(f(s,y_2(s)) + g(s,x(s)))ds - h(t)$
$= \int_a^b K_1(t,s)(g(s,y_1(s)) - g(s,y_2(s)))ds$
$+ \int_a^b K_2(t,s)(f(s,y_1(s)) - f(s,y_2(s)))ds \leq 0,$
by Assumption (i) and (ii). Hence $F(x,y_1)(t) \leq F(x,y_2)(t), \forall t \in I$, that is,

$$F(x,y_1) \leq F(x,y_2).$$

Thus, $F(x,y)$ is monotone non-decreasing in x and monotone non-increasing in y.
Define the function $\alpha : C(I)^2 \times C(I)^2 \to [0,\infty)$ by

$$\alpha((x,y),(u,v)) = \begin{cases} 1 & e((x(t),y(t)),(u(t),v(t))) \geq 0,\ t \in I \\ 0 & otherwise. \end{cases}$$

Now, for $x \geq u$, $y \leq v$, that is, $x(t) \geq u(t)$, $y(t) \leq v(t)$ for all $t \in I$, we have
$\alpha((x,y),(u,v))d(F(x,y),F(u,v)) \leq d(F(x,y),F(u,v))$
$= \sup_{t\in I} |F(x,y)(t) - F(u,v)(t)|$
$= \sup_{t\in I} | \int_a^b K_1(t,s)(f(s,x(s)) + g(s,y(s)))ds$
$+ \int_a^b K_2(t,s)(f(s,y(s)) + g(s,x(s)))ds + h(t)$
$-(\int_a^b K_1(t,s)(f(s,u(s)) + g(s,v(s)))ds$
$+ \int_a^b K_2(t,s)(f(s,v(s)) + g(s,u(s)))ds + h(t))|$
$= \sup_{t\in I} | \int_a^b K_1(t,s)[(f(s,x(s)) - f(s,u(s))) + (g(s,y(s)) - g(s,v(s)))]ds$
$+ \int_a^b K_2(t,s)[(f(s,y(s)) - f(s,v(s))) + (g(s,x(s)) - g(s,u(s)))]ds|$
$= \sup_{t\in I} | \int_a^b K_1(t,s)[(f(s,x(s)) - f(s,u(s))) - (g(s,v(s)) - g(s,y(s)))]ds$

$- \int_a^b K_2(t,s)[(f(s,v(s)) - f(s,y(s))) - (g(s,x(s)) - g(s,u(s)))]ds|$
$\leq \sup_{t\in I} | \int_a^b K_1(t,s)[\lambda\theta(x(s) - u(s)) + \mu\theta(v(s) - y(s))]ds$
$- \int_a^b K_2(t,s)[\lambda\theta(v(s) - y(s)) + \mu\theta(x(s) - u(s))]ds|$
$\leq \max\{\lambda, \mu\} \sup_{t\in I} \int_a^b (K_1(t,s) - K_2(t,s))[\theta(x(s) - u(s)) + \theta(v(s) - y(s))]ds.$
As the function θ is increasing and $x \geq u, y \leq v$, then
$\theta(x(s) - u(s)) \leq \theta(d(x,u)), \theta(v(s) - y(s)) \leq \theta(d(v,y)),$
for all $s \in [a,b]$, we obtain
$d(F(x,y), F(u,v)) \leq \max\{\lambda, \mu\} \sup_{t\in I} \int_a^b (K_1(t,s) - K_2(t,s))$
$\times[\theta(x(s) - u(s)) + \theta(v(s) - y(s))]ds.$
$\leq \max\{\lambda, \mu\}.[\theta(d(x,u)) + \theta(d(v,y))]. \sup_{t\in I} \int_a^b (K_1(t,s) - K_2(t,s))ds$
$\leq \frac{1}{2}[\theta(d(x,u)) + \theta(d(v,y))]$
$\leq \theta(d(x,u)) + \theta(d(v,y))$
$= \frac{d(x,u)+d(v,y)}{2} - \psi(\frac{d(x,u)+d(v,y)}{2}).$
Therefore, for $x \geq u, y \leq v$, we have
$d(F(x,y), F(u,v)) \leq \frac{d(x,u)+d(v,y)}{2} - \psi(\frac{d(x,u)+d(v,y)}{2}).$

Now, let (β, γ) be a coupled lower and upper solution of the integral equation (3.1) then we have $\beta(t) \leq F(\beta,\gamma)(t)$ and $\gamma(t) \geq F(\gamma,\beta)(t)$ for all $t \in [a,b]$, that is, $\beta \leq F(\beta,\gamma)$ and $\gamma \geq F(\gamma,\beta)$.
From condition (iv), for all $(x,y),(u,v) \in C(I) \times C(I)$,
$\alpha((x,y),(u,v)) \geq 1 \Longrightarrow e((x(t),y(t)),(u(t),v(t))) \geq 0$
$\Longrightarrow e((F(x(t),y(t)), F(y(t),x(t))),(F(u(t),v(t)), F(v(t),u(t)))) \geq 0$
$\Longrightarrow \alpha((F(x,y), F(y,x)),(F(u,v), F(v,u))) \geq 1.$
Then F is α-admissible.
From (iii), there exists $(x_0,y_0) \in C(I) \times C(I)$ such that
$\alpha((x_0,y_0),(F((x_0,y_0)), F(y_0,x_0))) \geq 1, \alpha((F(y_0,x_0), F((x_0,y_0))),(x_0,y_0)) \geq 1.$
Finally, Theorem (3.2) give that F has a coupled fixed point (x,y). Since $\beta \leq \gamma$, then the hypothesis of Theorem (3.2) is satisfied. Therefore $x = y$, that is $x(t) = y(t)$, for all $t \in [a,b]$ implying $x = F(x,x)$ and $x(t)$ is the solution of equation (3.1). □

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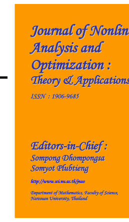

ON A NEW SYSTEM OF NONLINEAR REGULARIZED NONCONVEX VARIATIONAL INEQUALITIES IN HILBERT SPACES

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ABSTRACT. The aim of this work is to suggest a new *system of nonlinear regularized nonconvex variational inequalities in a real Hilbert spaces* and established an equivalence relation between this system and fixed point problems. By using the equivalence relation we construct a new perturbed projection iterative algorithms with mixed errors for finding a solution set of *system of nonlinear regularized nonconvex variational inequalities* and discussed the convergence of the sequences generated by an iterative algorithm.

KEYWORDS : System of nonlinear regularized nonconvex variational inequalities; uniformly r-prox regular sets; iterative sequences, convergence analysis; mixed errors.
AMS Subject Classification: 49J40, 47H06.

1. Historical Background

Variational inequalities which were introduced by Stampacchia [23] provided us with a powerful tools to study a wide class of problems arising in mechanic, physics, optimization and control theory, linear programming, economics and engineering sciences, *see* [8, 9, 12]. In recent years, several authors studied different type of system of variational inequalities and suggested iterative algorithms to find the approximate solutions of such system, *see* [7, 10, 11, 14, 15, 19, 20, 22, 26, 27]. We remark that the almost all results concerning the system of solutions of iterative scheme for solving the system of variational inequalities and related problems are being considered in the setting of convex sets. Consequently the techniques are based on the projections of operator over convex sets, which may not hold in general, when the sets are nonconvex. It is known that the unified prox-regular sets are nonconvex and included the convex sets as special cases, *see* [4, 18, 28].
Inspired by the recent works going on this fields, *see* [1, 2, 3, 5, 6, 13, 16, 17, 24, 25], in this paper, we introduced and studied a new *system of nonlinear regularized nonconvex variational inequalities in a real Hilbert spaces*. We established

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the equivalence between the *system of nonlinear regularized nonconvex variational inequalities* and the fixed point problems. By using the equivalence relation, we construct a perturbed projection iterative algorithms with mixed errors for finding a solution set of the aforementioned system. Also we proved the convergence of the defined iterative algorithms under suitable assumptions.

2. Basic Foundation

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{K}$ be a nonempty convex subset of $\mathcal{H}$, and $CB(\mathcal{H})$ denote the family of all closed and bounded subsets of $\mathcal{H}$.

Definition 2.1. The proximal normal cone of $\mathcal{K}$ at a point $u \in \mathcal{H}$ is given by

$$N^P_{\mathcal{K}}(u) = \{\zeta \in \mathcal{H} : u \in P_{\mathcal{K}}(u + \alpha\zeta)\},$$

where $\alpha > 0$ is a constant and $P_{\mathcal{K}}$ is projection operator of $\mathcal{H}$ onto $\mathcal{K}$, that is,

$$P_{\mathcal{K}}(u) = \{v \in \mathcal{K} : d_{\mathcal{K}}(u) = \|u - v\|\},$$

where $d_{\mathcal{K}}(u)$ is the usual distance function to the subset $\mathcal{K}$, that is,

$$d_{\mathcal{K}}(u) = \inf_{v\in\mathcal{K}} \|u - v\|.$$

Lemma 2.2. *Let $\mathcal{K}$ be a nonempty closed subset in $\mathcal{H}$. Then $\zeta \in N^P_{\mathcal{K}}(u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle\zeta, v - u\rangle \le \alpha\|v - u\|^2 \ \forall v \in \mathcal{K}.$$

Definition 2.3. The Clarke normal cone, denoted by $N^C_{\mathcal{K}}(u)$, is defined as

$$N^C_{\mathcal{K}}(u) = \overline{co}[N^P_{\mathcal{K}}(u)],$$

where $\overline{co}\mathcal{A}$ means the closure of the convex hull of $\mathcal{A}$. It is clear that $N^P_{\mathcal{K}}(x) \subseteq N^C_{\mathcal{K}}(x)$. The converse is not true in general. Note that $N^C_{\mathcal{K}}(x)$ is always closed and convex cone where as $N^P_{\mathcal{K}}(x)$ is always convex but may not be closed, *see* [4, 9, 18, 24].

Definition 2.4. For any $r \in (0, +\infty]$, a subset $\mathcal{K}_r$ of $\mathcal{H}$ is called the normalized uniformly prox-regular (or uniformly r-prox-regular) if every nonzero proximal normal to $\mathcal{K}_r$ can be realized by an r-ball. This mean that for all $\overline{x} \in \mathcal{K}_r$ and all $0 \neq \zeta \in N^P_{\mathcal{K}_r}(\overline{x})$ with $\|\zeta\| = 1$,

$$\langle\zeta, x - \overline{x}\rangle \le \frac{1}{2r}\|x - \overline{x}\|^2,\ x \in \mathcal{K}.$$

Lemma 2.5. [8] *A closed set $\mathcal{K} \subseteq \mathcal{H}$ is convex if and only if it is proximally smooth of radius r for every $r > 0$.*

Proposition 2.6. *Let $r > 0$ and $\mathcal{K}_r$ be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set*

$$\mathcal{U}(r) = \{u \in \mathcal{H} : 0 \le d_{\mathcal{K}_r}(u) < r\}.$$

Then the following statements are hold:

(a) *for all $x \in \mathcal{U}(r), P_{\mathcal{K}_r}(x) \neq \emptyset$;*
(b) *for all $r' \in (0, r)$, $P_{\mathcal{K}_r}$ is Lipschitz continuous mapping with constant $\frac{r}{r-r'}$ on*

$$\mathcal{U}(r') = \{u \in \mathcal{H} : 0 \le d_{\mathcal{K}_r}(u) < r'\};$$

(c) *the proximal normal cone is closed as a set valued mapping.*

From Proposition 2.6 (c), we have $N^C_{\mathcal{K}_r}(x) = N^P_{\mathcal{K}_r}(x)$. Therefore we define $N_{\mathcal{K}_r}(x) = N^C_{\mathcal{K}_r}(x) = N^P_{\mathcal{K}_r}(x)$ for a class of sets.

Definition 2.7. The single-valued operator $p : \mathcal{H} \longrightarrow \mathcal{H}$ is called

(i) monotone if

$$\langle p(x) - p(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H},$$

(ii) β-strongly monotone if there exists a constant $\beta > o$ such that

$$\langle p(x) - p(y), x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in \mathcal{H},$$

(iii) σ-Lipschitz continuous mapping if there exists a constant $\sigma > o$ such that

$$\|p(x) - p(y)\| \leq \sigma\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

Definition 2.8. Let $p : \mathcal{H} \longrightarrow \mathcal{H}$ be a single valued mapping and let $T : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a set valued mapping. Then T is said to be

(i) monotone if

$$\langle u - v, x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}, u \in T(x), v \in T(y),$$

(ii) κ-strongly monotone with respect to p if there exists a constant $\kappa > 0$ such that

$$\langle u - v, p(x) - p(y)\rangle \geq \kappa\|x - y\|^2, \quad \forall x, y \in \mathcal{H}, u \in T(x), v \in T(y).$$

Definition 2.9. A two-variable set-valued operator $T : \mathcal{H} \times \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is $\xi - \widehat{D}$-Lipschitz continuous in the first variable, if there exists a constant $\xi > o$ such that, for all $x, x' \in \mathcal{H}$,

$$\widehat{\mathcal{D}}(T(x, y), T(x', y')) \leq \xi\|x - x'\|, \quad \forall y, y' \in \mathcal{H}$$

where $\widehat{\mathcal{D}}$ is the Hausdorff pseudo metric that is for any two nonempty subsets A and B of $\mathcal{H}$

$$\widehat{\mathcal{D}}(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}.$$

3. System of Nonlinear Regularized Nonconvex Variational Inequalities

In this section, we introduce a new *system of nonlinear regularized nonconvex variational inequalities* in a real Hilbert space and investigated their relations.
Let $T_1, \dots, T_N : \mathcal{H} \times \mathcal{H} \longrightarrow CB(\mathcal{H})$ be the nonlinear set valued mappings and let $g_1, \dots, g_N$,$h_1, \dots, h_N : \mathcal{H} \longrightarrow \mathcal{H}$ be the nonlinear single valued mappings such that $\mathcal{K}_r \subseteq g_i(\mathcal{H})$ for $i = 1, \dots, N$. For any constants $\eta_1, \dots, \eta_N > 0$, we consider the problem of finding $x_1, \dots, x_N \in \mathcal{H}$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, $\dots$,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ such that $h_1(x_1), \dots, h_N(x_N) \in \mathcal{K}_r$ and

$$\begin{cases} \langle \eta_1 u_1 + h_1(x_1) - g_1(x_2), g_1(x) - h_1(x_1)\rangle + \frac{1}{2r}\|g_1(x) - h_1(x_1)\|^2 \geq 0, \\ \langle \eta_2 u_2 + h_2(x_2) - g_2(x_3), g_2(x) - h_2(x_2)\rangle + \frac{1}{2r}\|g_2(x) - h_2(x_2)\|^2 \geq 0, \\ \qquad\qquad\vdots \\ \langle \eta_{N-1} u_{N-1} + h_{N-1}(x_{N-1}) - g_{N-1}(x_N), g_{N-1}(x) - h_{N-1}(x_{N-1})\rangle \\ \quad + \frac{1}{2r}\|g_{N-1}(x) - h_{N-1}(x_{N-1})\|^2 \geq 0, \\ \langle \eta_N u_N + h_N(x_N) - g_N(x_1), g_N(x) - h_N(x_N)\rangle + \frac{1}{2r}\|g_N(x) - h_N(x_N)\|^2 \geq 0, \\ \quad \forall x \in \mathcal{K}_r : g_1(x), \dots, g_N(x) \in \mathcal{K}_r. \end{cases} \tag{3.1}$$

The problem (3.1) is called the *system of nonlinear regularized nonconvex variational inequalities.*

Lemma 3.1. *Let $\mathcal{K}_r$ be a uniformly r-prox-regular set then the problem (3.1) is equivalent to finding $x_1, \dots, x_N \in \mathcal{H}$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ such that*

$$\begin{cases} 0 \in \eta_1 u_1 + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)), \\ 0 \in \eta_2 u_2 + h_2(x_2) - g_2(x_3) + N^P_{\mathcal{K}_r}(h_2(x_2)), \\ \qquad\qquad \vdots \\ 0 \in \eta_{N-1} u_{N-1} + h_{N-1}(x_{N-1}) - g_{N-1}(x_N) + N^P_{\mathcal{K}_r}(h_{N-1}(x_{N-1})), \\ 0 \in \eta_N u_N + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{3.2}$$

where $N^P_{\mathcal{K}_r}(s)$ denotes the P-normal cone of $\mathcal{K}_r$ at s in the sense of nonconvex analysis.

Proof. Let $(x_1, \dots, x_N, u_1, \dots, u_N)$ with $x_1, \dots, x_N \in \mathcal{H}$, $h_1(x_1), \dots, h_N(x_N) \in \mathcal{K}_r$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ be a solution set of the system (3.1). If

$$\eta_1 u_1 + h_1(x_1) - g_1(x_2) = 0$$

because the vector zero always belongs to any normal cone, then

$$0 \in \eta_1 u_1 + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)).$$

If

$$\eta_1 u_1 + h_1(x_1) - g_1(x_2) \neq 0$$

then for all $x \in \mathcal{H}$ with $g_1(x) \in \mathcal{K}_r$

$$\langle -(\eta_1 u_1 + h_1(x_1) - g_1(x_2)), g_1(x) - h_1(x_1) \rangle \leq \frac{1}{2r} \|g_1(x) - h_1(x_1)\|^2. \tag{3.3}$$

From Lemma 2.2 we have

$$-(\eta_1 u_1 + h_1(x_1) - g_1(x_2)) \in N^P_{\mathcal{K}_r}(h_1(x_1))$$

and

$$0 \in \eta_1 u_1 + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)). \tag{3.4}$$

Similarly

$$\begin{cases} 0 \in \eta_2 u_2 + h_2(x_2) - g_2(x_3) + N^P_{\mathcal{K}_r}(h_2(x_2)), \\ 0 \in \eta_3 u_3 + h_3(x_3) - g_3(x_4) + N^P_{\mathcal{K}_r}(h_3(x_3)), \\ \qquad\qquad \vdots \\ 0 \in \eta_{N-1} u_{N-1} + h_{N-1}(x_{N-1}) - g_{N-1}(x_N) + N^P_{\mathcal{K}_r}(h_{N-1}(x_{N-1})), \\ 0 \in \eta_N u_N + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{3.5}$$

Conversely, if $(x_1, \dots, x_N, u_1, \dots, u_N)$ with $x_1, \dots, x_N \in \mathcal{H}$, $h_1(x_1), \dots, h_N(x_N) \in \mathcal{K}_r$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ is a solution set of the system (3.2) then from Definition 2.4, $x_1, \dots, x_N \in \mathcal{H}$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ with $h_1(x_1), \dots, h_N(x_N) \in \mathcal{K}_r$ is a solution set of the system (3.1). □

The problem (3.2) is called *system of nonlinear regularized nonconvex variational inclusions.*

4. MAIN RESULTS

Lemma 4.1. *Let $T_1,\dots,T_N$, $g_1,\dots,g_N$, $h_1,\dots,h_N$, $\eta_1,\dots,\eta_N$ be the same as in the system (3.1). Then $(x_1,\dots,x_N,u_1,\dots,u_N)$ with $x_i\in\mathcal{H}$,$h_i(x_i)\in\mathcal{K}_r$ for all $i=1,\dots,N$ and $u_1\in T_1(x_2,x_1)$, $u_2\in T_2(x_3,x_2)$, …,$u_{N-1}\in T_{N-1}(x_N,x_{N-1})$, $u_N\in T_N(x_1,x_N)$ is a solution set of the system (3.1) if and only if*

$$\begin{cases} h_1(x_1)=P_{\mathcal{K}_r}[g_1(x_2)-\eta_1u_1],\\ h_2(x_2)=P_{\mathcal{K}_r}[g_2(x_3)-\eta_2u_2],\\ \qquad\qquad\vdots\\ h_{N-1}(x_{N-1})=P_{\mathcal{K}_r}[g_{N-1}(x_N)-\eta_{N-1}u_{N-1}],\\ h_N(x_N)=P_{\mathcal{K}_r}[g_N(x_1)-\eta_Nu_N],\end{cases}\tag{4.1}$$

where $P_{\mathcal{K}_r}$ is the projection of $\mathcal{H}$ onto the uniformly r-prox-regular set $\mathcal{K}_r$.

Proof. Let $(x_1,\dots,x_N,u_1,\dots,u_N)$ with $x_i\in\mathcal{H}$,$h_i(x_i)\in\mathcal{K}_r$ for all $i=1,\dots,N$ and $u_1\in T_1(x_2,x_1)$, $u_2\in T_2(x_3,x_2)$, …,$u_{N-1}\in T_{N-1}(x_N,x_{N-1})$, $u_N\in T_N(x_1,x_N)$ is a solution set of the system (3.1). Then from Lemma 3.1, we have

$$\begin{cases} 0\in\eta_1u_1+h_1(x_1)-g_1(x_2)+N^P_{\mathcal{K}_r}(h_1(x_1)),\\ 0\in\eta_2u_2+h_2(x_2)-g_2(x_3)+N^P_{\mathcal{K}_r}(h_2(x_2)),\\ \qquad\qquad\vdots\\ 0\in\eta_{N-1}u_{N-1}+h_{N-1}(x_{N-1})-g_{N-1}(x_N)+N^P_{\mathcal{K}_r}(h_{N-1}(x_{N-1})),\\ 0\in\eta_Nu_N+h_N(x_N)-g_N(x_1)+N^P_{\mathcal{K}_r}(h_N(x_N)),\end{cases}\tag{4.2}$$

$$\Leftrightarrow\begin{cases} g_1(x_2)-\eta_1u_1\in(I+N^P_{\mathcal{K}_r})(h_1(x_1)),\\ g_2(x_3)-\eta_2u_2\in(I+N^P_{\mathcal{K}_r})(h_2(x_2)),\\ \qquad\qquad\vdots\\ g_{N-1}(x_N)-\eta_{N-1}u_{N-1}\in(I+N^P_{\mathcal{K}_r})(h_{N-1}(x_{N-1})),\\ g_N(x_1)-\eta_Nu_N\in(I+N^P_{\mathcal{K}_r})(h_N(x_N)),\end{cases}\tag{4.3}$$

$$\Leftrightarrow\begin{cases} h_1(x_1)=P_{\mathcal{K}_r}[g_1(x_2)-\eta_1u_1],\\ h_2(x_2)=P_{\mathcal{K}_r}[g_2(x_3)-\eta_2u_2],\\ \qquad\qquad\vdots\\ h_{N-1}(x_{N-1})=P_{\mathcal{K}_r}[g_{N-1}(x_N)-\eta_{N-1}u_{N-1}],\\ h_N(x_N)=P_{\mathcal{K}_r}[g_N(x_1)-\eta_Nu_N],\end{cases}\tag{4.4}$$

where I is an identity mapping and $P_{\mathcal{K}_r}=(I+N^P_{\mathcal{K}_r})^{-1}$. □

Remark 4.2. The inequality (4.1) can be written as follows

$$\begin{cases} q_1=g_1(x_2)-\eta_1u_1,\quad h_1(x_1)=P_{\mathcal{K}_r}[q_1]\\ q_2=g_2(x_3)-\eta_2u_2,\quad h_2(x_2)=P_{\mathcal{K}_r}[q_2]\\ \qquad\qquad\vdots\\ q_{N-1}=g_{N-1}(x_N)-\eta_{N-1}u_{N-1},\quad h_{N-1}(x_{N-1})=P_{\mathcal{K}_r}[q_{N-1}]\\ q_N=g_N(x_1)-\eta_Nu_N,\quad h_N(x_N)=P_{\mathcal{K}_r}[q_N],\end{cases}\tag{4.5}$$

where $\eta_i>0, i=1,\dots,N$ are constants.

The fixed point formulation (4.5) enables us to construct the following perturbed iterative algorithms with mixed errors.

Algorithm 4.1. *Let $T_1,\dots,T_N$, $g_1,\dots,g_N$, $h_1,\dots,h_N$, $\eta_1,\dots,\eta_N$ be the same as in the system (3.1) such that $h_1,\dots,h_N:\mathcal{H}\longrightarrow\mathcal{H}$ be onto operators. Let $e_1^0,\dots,e_N^0$, $r_1^0,\dots,r_N^0\in\mathcal{H}$, $\alpha_0\in\mathbb{R}$ and $\eta_0>0$. For given $q_1^0,\dots,q_N^0\in\mathcal{H}$, we let $x_1^0,\dots,x_N^0\in\mathcal{H}$, $u_1^0\in T_1(x_2^0,x_1^0)$, $u_2^0\in T_2(x_3^0,x_2^0)$, …,$u_{N-1}^0\in T_{N-1}(x_N^0,x_{N-1}^0)$, $u_N^0\in T_N(x_1^0,x_N^0)$ such that*

$$\begin{cases} h_1(x_1^0)=P_{\mathcal{K}_r}(q_1^0);\;\; q_1^1=(1-\alpha_0)q_1^0+\alpha_0(g_1(x_2^0)-\eta_0u_1^0+e_1^0)+r_1^0,\\ h_2(x_2^0)=P_{\mathcal{K}_r}(q_2^0);\;\; q_2^1=(1-\alpha_0)q_2^0+\alpha_0(g_2(x_3^0)-\eta_0u_2^0+e_2^0)+r_2^0,\\ \qquad\vdots\\ h_{N-1}(x_{N-1}^0)=P_{\mathcal{K}_r}(q_{N-1}^0);\;\; q_{N-1}^1=(1-\alpha_0)q_{N-1}^0+\alpha_0(g_{N-1}(x_N^0)-\eta_0u_{N-1}^0\\ \qquad\qquad +e_{N-1}^0)+r_{N-1}^0,\\ h_N(x_N^0)=P_{\mathcal{K}_r}(q_N^0);\;\; q_N^1=(1-\alpha_0)q_N^0+\alpha_0(g_N(x_1^0)-\eta_0u_N^0+e_N^0)+r_N^0. \end{cases} \tag{4.6}$$

We choose $x_1^1,\dots,x_1^N\in\mathcal{H}$ such that $h_1(x_1^1)=P_{\mathcal{K}_r}(q_1^1)$,…,$h_N(x_1^N)=P_{\mathcal{K}_r}(q_N^1)$. By Nadler's theorem [21], there exists

$$\begin{cases} u_1^1\in T_1(x_2^0,x_1^0);\;\; \|u_1^0-u_1^1\|\le(1+(1+n)^{-1})\widehat{\mathcal{D}}(T_1(x_2^0,x_1^0),T_1(x_2^1,x_1^1)),\\ u_2^1\in T_2(x_3^0,x_2^0);\;\; \|u_2^0-u_2^1\|\le(1+(1+n)^{-1})\widehat{\mathcal{D}}(T_2(x_3^0,x_2^0),T_2(x_3^1,x_2^1)),\\ \qquad\vdots\\ u_{N-1}^1\in T_{N-1}(x_N^0,x_{N-1}^0);\; \|u_{N-1}^0-u_{N-1}^1\|\le(1+(1+n)^{-1})\widehat{\mathcal{D}}(T_{N-1}(x_N^0,x_{N-1}^0),\\ \qquad\qquad T_{N-1}(x_N^1,x_{N-1}^1)),\\ u_N^1\in T_N(x_1^0,x_N^0);\;\; \|u_N^0-u_N^1\|\le(1+(1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^0,x_N^0),T_N(x_1^1,x_N^1)). \end{cases} \tag{4.7}$$

Continuing the above process inductively, we can obtain the sequences $\{x_1^n\}_{n=0}^\infty$, …, $\{x_N^n\}_{n=0}^\infty$, $\{u_1^n\}_{n=0}^\infty,\dots,\{u_N^n\}_{n=0}^\infty$ by using

$$\begin{cases} h_1(x_1^n)=P_{\mathcal{K}_r}(q_1^n);\;\; q_1^{n+1}=(1-\alpha_n)q_1^n+\alpha_n(g_1(x_2^n)-\eta_1u_1^n+e_1^n)+r_1^n,\\ h_2(x_2^n)=P_{\mathcal{K}_r}(q_2^n);\;\; q_2^{n+1}=(1-\alpha_n)q_2^n+\alpha_n(g_2(x_3^n)-\eta_2u_2^n+e_2^n)+r_2^n,\\ \qquad\vdots\\ h_{N-1}(x_{N-1}^n)=P_{\mathcal{K}_r}(q_{N-1}^n);\;\; q_{N-1}^{n+1}=(1-\alpha_n)q_{N-1}^n+\alpha_n(g_{N-1}(x_N^n)\\ \qquad\qquad -\eta_{N-1}u_{N-1}^n+e_{N-1}^n)+r_{N-1}^n,\\ h_N(x_N^n)=P_{\mathcal{K}_r}(q_N^n);\;\; q_N^{n+1}=(1-\alpha_n)q_N^n+\alpha_n(g_N(x_1^n)-\eta_Nu_N^n+e_N^n)+r_N^n, \end{cases} \tag{4.8}$$

and

$$\begin{cases} u_1^{n+1} \in T_1(x_2^{n+1}, x_1^{n+1}); \ \|u_1^n - u_1^{n+1}\| \leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T_1(x_2^{n+1}, x_1^{n+1})), \\ u_2^{n+1} \in T_2(x_3^{n+1}, x_2^{n+1}); \ \|u_2^n - u_2^{n+1}\| \leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_2(x_3^n, x_2^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T_2(x_3^{n+1}, x_2^{n+1})), \\ \qquad\qquad \vdots \\ u_{N-1}^{n+1} \in T_{N-1}(x_N^{n+1}, x_{N-1}^{n+1}); \ \|u_{N-1}^n - u_{N-1}^{n+1}\| \leq (1+(1+n)^{-1})\times \\ \qquad\qquad\qquad\qquad \widehat{\mathcal{D}}(T_{N-1}(x_N^n, x_{N-1}^n), T_{N-1}(x_N^{n+1}, x_{N-1}^{n+1})), \\ u_N^{n+1} \in T_N(x_1^{n+1}, x_N^{n+1}); \ \|u_N^n - u_N^{n+1}\| \leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^n, x_N^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T_N(x_1^{n+1}, x_N^{n+1})), \end{cases} \tag{4.9}$$

where $0 \leq \alpha_n \leq 1$ is a parameter and $\{e_1^n\}_{n=0}^{\infty}, \ldots, \{e_N^n\}_{n=0}^{\infty}$, $\{r_1^n\}_{n=0}^{\infty}, \ldots, \{r_N^n\}_{n=0}^{\infty}$ are sequences in $\mathcal{H}$ to take into account of a possible inexact computation of the resolvent operator satisfying the following conditions:

$$\lim_{n \longrightarrow \infty} e_i^n = \lim_{n \longrightarrow \infty} r_i^n = 0;$$

$$\sum_{n=1}^{\infty} \|e_i^n - e_i^{n-1}\| < \infty, \quad \sum_{n=1}^{\infty} \|r_i^n - r_i^{n-1}\| < \infty, \tag{4.10}$$

for all $i = 1, \ldots, N$.

Theorem 4.3. *Let T_i, g_i, h_i, η_i, for $i = 1, \ldots, N$ be the same as in the system (3.1) such that, for each $i = 1, \ldots, N$,*

(i) *T_i is κ_i-strongly monotone with respect to g_i and $\xi_i - \widehat{\mathcal{D}}$-Lipschitz continuous mapping in the first variables;*
(ii) *h_i is β_i-strongly monotone and σ_i-Lipschitz continuous mapping;*
(iii) *g_i is μ_i-Lipschitz continuous mapping.*

If the constants $\eta_i > 0$ satisfying the following conditions:

$$\begin{aligned} \left|\eta_1 - \frac{\kappa_1}{\xi_1^2}\right| &< \frac{\sqrt{r^2\kappa_1^2 - \xi_1^2\Big(r^2\mu_1^2 - (r-r')^2(1-\pi_2)^2\Big)}}{r\xi_1^2}, \\ &\vdots \\ \left|\eta_N - \frac{\kappa_N}{\xi_N^2}\right| &< \frac{\sqrt{r^2\kappa_N^2 - \xi_N^2\Big(r^2\mu_N^2 - (r-r')^2(1-\pi_1)^2\Big)}}{r\xi_N^2}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} r\kappa_1 &> \xi_1\sqrt{r^2\mu_1^2 - (r-r')^2(1-\pi_2)^2}, \\ &\vdots \\ r\kappa_N &> \xi_N\sqrt{r^2\mu_N^2 - (r-r')^2(1-\pi_1)^2}, \end{aligned} \tag{4.12}$$

$$r\mu_1 > (r-r')(1-\pi_2), \ldots, r\mu_N > (r-r')(1-\pi_1), \tag{4.13}$$

and

$$\pi_i = \sqrt{1 - 2\beta_i + \sigma_i^2}, \quad 2\pi_i < 1 + \sigma_i^2, \tag{4.14}$$

for each $i = 1, \dots, N$, where $r' \in (0, r)$, then there exists $x_1^*, \dots, x_N^* \in \mathcal{H}$ with $h_1(x_1^*), \dots, h_N(x_N^*) \in \mathcal{K}_r$ and $u_1^* \in T_1(x_2^*, x_1^*)$, $u_2^* \in T_2(x_3^*, x_2^*)$, $\dots$, $u_{N-1}^* \in T_{N-1}(x_N^*, x_{N-1}^*)$, $u_N^* \in T_N(x_1^*, x_N^*)$ such that $(x_1^*, \dots, x_N^*, u_1^*, \dots, u_N^*)$ is a solution set of system (3.1) and sequences $\{(x_1^*, \dots, x_N^*, u_1^*, \dots, u_N^*)\}_{n=0}^{\infty}$ suggested by Algorithm 4.1 converges strongly to $(x_1^*, \dots, x_N^*, u_1^*, \dots, u_N^*)$.

Proof. From (4.8), we have

$$\begin{aligned} \|q_1^{n+1} - q_1^n\| &\leq (1-\alpha_n)\|q_1^n - q_1^{n-1}\| + \alpha_n\|g_1(x_2^n) - g_1(x_2^{n-1}) - \eta_1(u_1^n - u_1^{n-1})\| \\ &\quad + \alpha_n\|e_1^n - e_1^{n-1}\| + \|r_1^n - r_1^{n-1}\|. \end{aligned} \tag{4.15}$$

Since T_1 is κ_1-strongly monotone with respect to g_1 and $\xi_1 - \widehat{\mathcal{D}}$-Lipschitz continuous mapping in the first variables and g_1 is μ_1-Lipschitz continuous, we get

$$\begin{aligned} &\|g_1(x_2^n) - g_1(x_2^{n-1}) - \eta_1(u_1^n - u_1^{n-1})\|^2 \\ &= \|g_1(x_2^n) - g_1(x_2^{n-1})\|^2 - 2\eta_1\langle u_1^n - u_1^{n-1}, g_1(x_2^n) - g_1(x_2^{n-1})\rangle \\ &\quad + \eta_1^2\|u_1^n - u_1^{n-1}\|^2 \\ &\leq \mu_1^2\|x_2^n - x_2^{n-1}\|^2 - 2\eta_1\kappa_1\|x_2^n - x_2^{n-1}\|^2 \\ &\quad + \eta_1^2(1+n^{-1})^2(\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), T_1(x_2^{n-1}, x_1^{n-1})))^2 \\ &= (\mu_1^2 - 2\eta_1\kappa_1)\|x_2^n - x_2^{n-1}\|^2 \\ &\quad + \eta_1^2(1+n^{-1})^2(\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), T_1(x_2^{n-1}, x_1^{n-1})))^2 \\ &\leq (\mu_1^2 - 2\eta_1\kappa_1)\|x_2^n - x_2^{n-1}\|^2 + \eta_1^2\xi_1^2(1+n^{-1})^2\|x_2^n - x_2^{n-1}\|^2 \\ &= (\mu_1^2 - 2\eta_1\kappa_1 + \eta_1^2\xi_1^2(1+n^{-1})^2)\|x_2^n - x_2^{n-1}\|^2. \end{aligned} \tag{4.16}$$

It follows from (4.15) and (4.16), we obtain that

$$\begin{aligned} \|q_1^{n+1} - q_1^n\| &\leq (1-\alpha_n)\|q_1^n - q_1^{n-1}\| + \alpha_n\|e_1^n - e_1^{n-1}\| + \|r_1^n - r_1^{n-1}\| \\ &\quad + \alpha_n\sqrt{\mu_1^2 - 2\eta_1\kappa_1 + \eta_1^2\xi_1^2(1+n^{-1})^2}\|x_2^n - x_2^{n-1}\|. \end{aligned} \tag{4.17}$$

Similarly, we can prove that

$$\begin{aligned} \|q_2^{n+1} - q_2^n\| &\leq (1-\alpha_n)\|q_2^n - q_2^{n-1}\| + \alpha_n\|e_2^n - e_2^{n-1}\| + \|r_2^n - r_2^{n-1}\| \\ &\quad + \alpha_n\sqrt{\mu_2^2 - 2\eta_2\kappa_2 + \eta_2^2\xi_2^2(1+n^{-1})^2}\|x_3^n - x_3^{n-1}\| \\ \|q_3^{n+1} - q_3^n\| &\leq (1-\alpha_n)\|q_3^n - q_3^{n-1}\| + \alpha_n\|e_3^n - e_3^{n-1}\| + \|r_3^n - r_3^{n-1}\| \\ &\quad + \alpha_n\sqrt{\mu_3^2 - 2\eta_3\kappa_3 + \eta_1^2\xi_3^2(1+n^{-1})^2}\|x_4^n - x_4^{n-1}\| \\ &\vdots \\ \|q_{N-1}^{n+1} - q_{N-1}^n\| &\leq (1-\alpha_n)\|q_{N-1}^n - q_{N-1}^{n-1}\| \\ &\quad + \alpha_n\sqrt{\mu_{N-1}^2 - 2\eta_{N-1}\kappa_{N-1} + \eta_{N-1}^2\xi_{N-1}^2(1+n^{-1})^2} \times \\ &\quad \|x_N^n - x_N^{n-1}\| + \alpha_n\|e_{N-1}^n - e_{N-1}^{n-1}\| + \|r_{N-1}^n - r_{N-1}^{n-1}\| \\ \|q_N^{n+1} - q_N^n\| &\leq (1-\alpha_n)\|q_N^n - q_N^{n-1}\| \\ &\quad + \alpha_n\sqrt{\mu_N^2 - 2\eta_N\kappa_N + \eta_N^2\xi_N^2(1+n^{-1})^2}\|x_1^n - x_1^{n-1}\| \\ &\quad + \alpha_n\|e_N^n - e_N^{n-1}\| + \|r_N^n - r_N^{n-1}\|. \end{aligned} \tag{4.18}$$

By using (4.8), we get that

$$\begin{aligned} \|x_1^n - x_1^{n-1}\| &\leq \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \|h_1(x_1^n) - h_1(x_1^{n-1})\| \\ &= \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \|P_{\mathcal{K}_r}(q_1^n) - P_{\mathcal{K}_r}(q_1^{n-1})\| \end{aligned}$$

$$\leq \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \tfrac{r}{r-r'}\|q_1^n - q_1^{n-1}\|. \quad (4.19)$$

Since h_1 is β_1-strongly monotone and σ_1-Lipschitz continuous, we have

$$\begin{aligned}\|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\|^2 &= \|x_1^n - x_1^{n-1}\|^2 + \|h_1(x_1^n) - h_1(x_1^{n-1})\|^2 \\ &\quad -2\langle h_1(x_1^n) - h_1(x_1^{n-1}), x_1^n - x_1^{n-1}\rangle \\ &\leq \|x_1^n - x_1^{n-1}\|^2 - 2\beta_1\|x_1^n - x_1^{n-1}\|^2 \\ &\quad +\sigma_1^2\|x_1^n - x_1^{n-1}\|^2 \\ &= (1 - 2\beta_1 + \sigma_1^2)\|x_1^n - x_1^{n-1}\|^2. \quad (4.20)\end{aligned}$$

By (4.19) and (4.20), we obtain that

$$\|x_1^n - x_1^{n-1}\| \leq \sqrt{1 - 2\beta_1 + \sigma_1^2}\|x_1^n - x_1^{n-1}\| + \frac{r}{r-r'}\|q_1^n - q_1^{n-1}\| \quad (4.21)$$

that is

$$\|x_1^n - x_1^{n-1}\| \leq \frac{r}{(r-r')(1-\sqrt{1-2\beta_1+\sigma_1^2})}\|q_1^n - q_1^{n-1}\|. \quad (4.22)$$

Similarly, we can prove that

$$\begin{aligned}\|x_2^n - x_2^{n-1}\| &\leq \frac{r}{(r-r')(1-\sqrt{1-2\beta_2+\sigma_2^2})}\|q_2^n - q_2^{n-1}\|, \\ \|x_3^n - x_3^{n-1}\| &\leq \frac{r}{(r-r')(1-\sqrt{1-2\beta_3+\sigma_3^2})}\|q_3^n - q_3^{n-1}\|, \\ &\vdots \\ \|x_{N-1}^n - x_{N-1}^{n-1}\| &\leq \frac{r}{(r-r')(1-\sqrt{1-2\beta_{N-1}+\sigma_{N-1}^2})}\|q_{N-1}^n - q_{N-1}^{n-1}\|, \\ \|x_N^n - x_N^{n-1}\| &\leq \frac{r}{(r-r')(1-\sqrt{1-2\beta_N+\sigma_N^2})}\|q_N^n - q_N^{n-1}\|. \quad (4.23)\end{aligned}$$

It follows from (4.17), (4.18), (4.22) and (4.23) that

$$\begin{aligned}\|q_1^{n+1} - q_1^n\| &\leq (1-\alpha_n)\|q_1^n - q_1^{n-1}\| + \alpha_n\frac{r\Omega_1(n)}{(r-r')(1-\pi_2)}\|q_2^n - q_2^{n-1}\| \\ &\quad +\alpha_n\|e_1^n - e_1^{n-1}\| + \|r_1^n - r_1^{n-1}\|, \\ \|q_2^{n+1} - q_2^n\| &\leq (1-\alpha_n)\|q_2^n - q_2^{n-1}\| + \alpha_n\frac{r\Omega_2(n)}{(r-r')(1-\pi_3)}\|q_3^n - q_3^{n-1}\| \\ &\quad +\alpha_n\|e_2^n - e_2^{n-1}\| + \|r_2^n - r_2^{n-1}\|, \\ &\vdots \\ \|q_{N-1}^{n+1} - q_{N-1}^n\| &\leq (1-\alpha_n)\|q_{N-1}^n - q_{N-1}^{n-1}\| + \alpha_n\frac{r\Omega_{N-1}(n)}{(r-r')(1-\pi_N)}\|q_N^n - q_N^{n-1}\| \\ &\quad +\alpha_n\|e_{N-1}^n - e_{N-1}^{n-1}\| + \|r_{N-1}^n - r_{N-1}^{n-1}\|, \\ \|q_N^{n+1} - q_N^n\| &\leq (1-\alpha_n)\|q_N^n - q_N^{n-1}\| + \alpha_n\frac{r\Omega_N(n)}{(r-r')(1-\pi_1)}\|q_1^n - q_1^{n-1}\| \\ &\quad +\alpha_n\|e_N^n - e_N^{n-1}\| + \|r_N^n - r_N^{n-1}\|, \quad (4.24)\end{aligned}$$

where

$$\Omega_i(n) = \sqrt{\mu_i^2 - 2\eta_i\kappa_i + \eta_i^2\xi_i^2(1+n^{-1})^2}, \text{ and } \pi_i = \sqrt{1-2\beta_i+\sigma_i^2},$$

for all $i = 1, 2, \ldots, N$.

Now we define $\|\cdot\|_*$ on $\underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}$ by

$$\|(x_1,\ldots,x_N)\|_* = \|x_1\|_* + \ldots + \|x_N\|_*, \quad \text{for all } (x_1,\ldots,x_N)\in\underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}.$$

It is obvious that $(\underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}, \|\cdot\|_*)$ is a Hilbert space, applying (4.24) we have

$$\begin{aligned}\|(q_1^{n+1},\ldots,q_N^{n+1}) - (q_1^n,\ldots,q_N^n)\|_* \leq\ & (1-\alpha_n)\|(q_1^n,\ldots,q_N^n)-(q_1^{n-1},\ldots,q_N^{n-1})\|_*\\ &+\alpha_n\Theta(n)\|(q_1^n,\ldots,q_N^n)-(q_1^{n-1},\ldots,q_N^{n-1})\|_* + \alpha_n\|(e_1^n,\ldots,e_N^n)\\ &-(e_1^{n-1},\ldots,e_N^{n-1})\|_* + \|(r_1^n,\ldots,r_N^n)-(r_1^{n-1},\ldots,r_N^{n-1})\|_*.\end{aligned} \tag{4.25}$$

Put

$$\Theta(n) = \max\left\{\frac{r\Omega_1(n)}{(r-r')(1-\pi_2)},\ldots,\frac{r\Omega_N(n)}{(r-r')(1-\pi_1)}\right\}. \tag{4.26}$$

Let $\Theta(n)\longrightarrow\Theta$, as $n\longrightarrow\infty$

$$\Theta = \max\left\{\frac{r\Omega_1}{(r-r')(1-\pi_2)},\ldots,\frac{r\Omega_N}{(r-r')(1-\pi_1)}\right\}. \tag{4.27}$$

By (4.11),we know that $0\leq\Theta<1$. For $\Theta=\frac{1}{2}(\Theta+1)\in(\Theta,1)$ there exists $n_0\geq 1$ such that $\Theta(n)=\widehat{\Theta}$ for each $n\geq n_0$. So it follows from (4.25) that, for each $n\geq n_0$,

$$\begin{aligned}\|(q_1^{n+1},\ldots,q_N^{n+1}) - (q_1^n,\ldots,q_N^n)\|_* \leq\ & (1-\alpha_n)\|(q_1^n,\ldots,q_N^n)-(q_1^{n-1},\ldots,q_N^{n-1})\|_*\\ &+\alpha_n\widehat{\Theta}\|(q_1^n,\ldots,q_N^n)-(q_1^{n-1},\ldots,q_N^{n-1})\|_*\\ &+\alpha_n\|(e_1^n,\ldots,e_N^n)-(e_1^{n-1},\ldots,e_N^{n-1})\|_* + \|(r_1^n,\ldots,r_N^n)-(r_1^{n-1},\ldots,r_N^{n-1})\|_*\\ =\ & (1-\alpha_n(1-\widehat{\Theta}))\|(q_1^n,\ldots,q_N^n)-(q_1^{n-1},\ldots,q_N^{n-1})\|_*\\ &+\alpha_n\|(e_1^n,\ldots,e_N^n)-(e_1^{n-1},\ldots,e_N^{n-1})\|_* + \|(r_1^n,\ldots,r_N^n)-(r_1^{n-1},\ldots,r_N^{n-1})\|_*\\ \leq\ & (1-\alpha_n(1-\widehat{\Theta}))\Big((1-\alpha_n(1-\widehat{\Theta}))\|(q_1^{n-1},\ldots,q_N^{n-1})-(q_1^{n-2},\ldots,q_N^{n-2})\|_*\\ &\quad+\alpha_n\|(e_1^{n-1},\ldots,e_N^{n-1})-(e_1^{n-2},\ldots,e_N^{n-2})\|_* + \|(r_1^{n-1},\ldots,r_N^{n-1})\\ &\quad-(r_1^{n-2},\ldots,r_N^{n-2})\|_*\Big) + \alpha_n\|(e_1^n,\ldots,e_N^n)-(e_1^{n-1},\ldots,e_N^{n-1})\|_*\\ &\quad+\|(r_1^n,\ldots,r_N^n)-(r_1^{n-1},\ldots,r_N^{n-1})\|_*\\ =\ & (1-\alpha_n(1-\widehat{\Theta}))^2\|(q_1^{n-1},\ldots,q_N^{n-1})-(q_1^{n-2},\ldots,q_N^{n-2})\|_*\\ &\quad+\alpha_n\Big((1-\alpha_n(1-\widehat{\Theta}))\|(e_1^{n-1},\ldots,e_N^{n-1})-(e_1^{n-2},\ldots,e_N^{n-2})\|_*\\ &\qquad\qquad+\|(e_1^n,\ldots,e_N^n)-(e_1^{n-1},\ldots,e_N^{n-1})\|_*\Big)\\ &\quad+(1-\alpha_n(1-\widehat{\Theta}))\|(r_1^{n-1},\ldots,r_N^{n-1})-(r_1^{n-2},\ldots,r_N^{n-2})\|_*\\ &\quad+\|(r_1^n,\ldots,r_N^n)-(r_1^{n-1},\ldots,r_N^{n-1})\|_*\\ \leq\ & \\ &\qquad\vdots\\ &\leq(1-\alpha_n(1-\widehat{\Theta}))^{n-n_0}\|(q_1^{n_0+1},\ldots,q_N^{n_0+1})-(q_1^{n_0},\ldots,q_N^{n_0})\|_*\\ &+\alpha_n\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(e_1^{n-(i-1)},\ldots,e_N^{n-(i-1)})-(e_1^{n-i},\ldots,e_N^{n-i})\|_*\end{aligned}$$

$$+\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(r_1^{n-(i-1)},\dots,r_N^{n-(i-1)})-(r_1^{n-i},\dots,r_N^{n-i})\|_*. \quad (4.28)$$

Thus, for any $m \geq n > n_0$, we get that

$$\begin{aligned}
&\|(q_1^m,\dots,q_N^m)-(q_1^n,\dots,q_N^n)\|_* \\
&\quad\leq \sum_{j=n}^{m-1}\|(q_1^{j+1},\dots,q_N^{j+1})-(q_1^j,\dots,q_N^j)\|_* \\
&\quad\leq \sum_{j=n}^{m-1}(1-\alpha_n(1-\widehat{\Theta}))^{j-n_0}\|(q_1^{n_0+1},\dots,q_N^{n_0+1})-(q_1^{n_0},\dots,q_N^{n_0})\|_* \\
&\quad+\alpha_n\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(e_1^{n-(i-1)},\dots,e_N^{n-(i-1)})-(e_1^{n-i},\dots,e_N^{n-i})\|_*
\end{aligned}$$

$$+\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(r_1^{n-(i-1)},\dots,r_N^{n-(i-1)})-(r_1^{n-i},\dots,r_N^{n-i})\|_*. \quad (4.29)$$

Since $(1-\alpha_n(1-\widehat{\Theta}))\in(0,1)$, it follows from (4.10) and (4.29) that

$$\|(q_1^m,\dots,q_N^m)-(q_1^n,\dots,q_N^n)\|_*=\|q_1^m-q_1^n\|+\ldots+\|q_N^m-q_N^n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

So $\{q_1^n\},\dots,\{q_N^n\}$ are Cauchy sequences in $\mathcal{H}$ and thus there exists $q_1^*,\dots,q_N^*\in\mathcal{H}$ such that $q_1^n\longrightarrow q_1^*$,..., $q_N^n\longrightarrow q_N^*$ as $n\longrightarrow\infty$. By (4.22) and (4.23), it follows that the sequences $\{x_1^n\}$,...,$\{x_N^n\}$ are also Cauchy sequences in $\mathcal{H}$. Hence there exists $x_1^*,\dots,x_N^*\in\mathcal{H}$ such that $x_1^n\longrightarrow x_1^*,\dots,x_N^n\longrightarrow x_N^*$ as $n\longrightarrow\infty$. Since for each $i=1,\dots,N$, T_i is $\xi_i-\widehat{\mathcal{D}}$-Lipschitz continuous mapping in the first variable, therefore it follow from (4.7) that

$$\begin{aligned}
\|u_1^n-u_1^{n+1}\| &\leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_1(x_2^n,x_1^n),T_1(x_2^{n+1},x_1^{n+1})) \\
&\leq (1+(1+n)^{-1})\xi_1\|x_2^n-x_2^{n+1}\|\longrightarrow 0, \\
\|u_2^n-u_2^{n+1}\| &\leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_2(x_3^n,x_2^n),T_2(x_3^{n+1},x_2^{n+1})) \\
&\leq (1+(1+n)^{-1})\xi_2\|x_3^n-x_3^{n+1}\|\longrightarrow 0, \\
&\;\;\vdots \\
\|u_{N-1}^n-u_{N-1}^{n+1}\| &\leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_{N-1}(x_N^n,x_{N-1}^n),T_{N-1}(x_N^{n+1},x_{N-1}^{n+1})) \\
&\leq (1+(1+n)^{-1})\xi_{N-1}\|x_N^n-x_N^{n+1}\|\longrightarrow 0, \\
\|u_N^n-u_N^{n+1}\| &\leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^n,x_N^n),T_1(x_1^{n+1},x_N^{n+1})) \\
&\leq (1+(1+n)^{-1})\xi_N\|x_1^n-x_1^{n+1}\|\longrightarrow 0, \qquad (4.30)
\end{aligned}$$

as $n\longrightarrow\infty$. Hence $\{u_1^n\},\dots,\{u_N^n\}$ are Cauchy sequences in $\mathcal{H}$ and so there exists $u_1^*,\dots,u_N^*\in\mathcal{H}$ such that $u_1^n\longrightarrow u_1^*,\dots,u_N^n\longrightarrow u_N^*$ as $n\longrightarrow\infty$. Further $u_1^n\in T_1(x_2^n,x_1^n)$ we have

$$\begin{aligned}
d(u_1^*,T_1(x_2^*,x_1^*)) &= \inf\{\|u_1^*-t\|:t\in T_1(x_2^*,x_1^*)\} \\
&\leq \|u_1^*-u_1^n\|+d(u_1^n,T_1(x_2^*,x_1^*)) \\
&\leq \|u_1^*-u_1^n\|+\widehat{\mathcal{D}}(T_1(x_2^n,x_1^n),T_1(x_2^{n+1},x_1^{n+1})) \\
&\leq \|u_1^*-u_1^n\|+\xi_1\|x_2^n-x_2^*\|\longrightarrow 0, \qquad (4.31)
\end{aligned}$$

as $n \longrightarrow \infty$. Hence $d(u_1^*, T_1(x_2^*, x_1^*)) = 0$ and so $u_1^n \longrightarrow u_1^* \in T_1(x_2^*, x_1^*)$.
By the same method, we can prove that

$$\begin{aligned}
d(u_2^*, T_2(x_3^*, x_2^*)) &\leq \|u_2^* - u_2^n\| + \xi_2\|x_3^n - x_3^*\| \longrightarrow 0,\\
d(u_3^*, T_3(x_4^*, x_3^*)) &\leq \|u_3^* - u_3^n\| + \xi_3\|x_4^n - x_4^*\| \longrightarrow 0,\\
&\vdots\\
d(u_{N-1}^*, T_{N-1}(x_N^*, x_{N-1}^*)) &\leq \|u_{N-1}^* - u_{N-1}^n\| + \xi_{N-1}\|x_N^n - x_N^*\| \longrightarrow 0,\\
d(u_N^*, T_N(x_1^*, x_N^*)) &\leq \|u_N^* - u_N^n\| + \xi_N\|x_1^n - x_1^*\| \longrightarrow 0, \qquad (4.32)
\end{aligned}$$

as $n \longrightarrow \infty$. Therefore $u_2^* \in T_2(x_3^*, x_2^*), \ldots, u_N^* \in T_N(x_1^*, x_N^*)$. Since g_i for $i = 1, \ldots, N$ is continuous, it follows from (4.8) and (4.10) that

$$q_1^* = g_1(x_2^*) - \eta_1 u_1^*, \ \ \ldots, \ \ q_N^* = g_N(x_1^*) - \eta_N u_N^*. \qquad (4.33)$$

Since $h_1, \ldots, h_N$ and $P_{\mathcal{K}_r}$ are continuous mappings, it follows from (4.8) and (4.33) that

$$\begin{aligned}
h_1(x_1^*) &= P_{\mathcal{K}_r}(q_1^*) = P_{\mathcal{K}_r}(g_1(x_2^*) - \eta_1 u_1^*),\\
h_2(x_2^*) &= P_{\mathcal{K}_r}(q_2^*) = P_{\mathcal{K}_r}(g_2(x_3^*) - \eta_2 u_2^*),\\
&\vdots\\
h_1(x_{N-1}^*) &= P_{\mathcal{K}_r}(q_{N-1}^*) = P_{\mathcal{K}_r}(g_{N-1}(x_N^*) - \eta_{N-1} u_{N-1}^*),\\
h_N(x_N^*) &= P_{\mathcal{K}_r}(q_N^*) = P_{\mathcal{K}_r}(g_N(x_1^*) - \eta_N u_N^*). \qquad (4.34)
\end{aligned}$$

Now Lemma 4.1, guarantees that $(x_1^*, \ldots, x_N^*, u_1^*, \ldots, u_N^*)$ is a solution set of system (3.1). □

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DUALITY RESULTS FOR SECOND-ORDER MULTI-OBJECTIVE PROGRAMMING PROBLEM

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ABSTRACT. The purpose of this paper is to study a class of non-differentiable multi objective fractional programming problem in which every component of objective functions contains a term involving the support function of a compact convex set. For a differentiable function, we use the definition of second-order $(C, \alpha, \rho, d) - V$-type-I convex function. Further, Wolfe type dual has been formulated for this problem and appropriate duality results have been proved under second-order $(C, \alpha, \rho, d) - V$-type-I convexity assumptions.

KEYWORDS : Duality; non-differentiable; multi-objective fractional programming; generalized convexity.
AMS Subject Classification: 90C26 . 90C30 . 90C32 . 90C46

1. INTRODUCTION

Second-order duality has even greater significance over first order duality, since it provides tighter bounds for the value of the objective function when approximation are used. Second-order duality for non-linear programming was the first to study in Mond [1]. Then, the concept of second-order duality for non-linear programming problems introduced by Mangasarian [2].

In the recent years attempts have been made by several authors to define various classes of differentiable non-convex functions and to study their duality and optimality criteria for solving such problems ([4], [5], [8], [9], [10], [11], and others). One such generalization of convex function is the invexity notion introduced by Hanson [5]. The term invex (which means invariant convex) was suggested by Craven [7]. Over the years, many generalizations of this concept have been given in the literature. For example, the concept of invexity of functions was also generalized to B-invex functions by Suneja et al.[6]. The class of (p, r)-invex functions

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with respect to η is an extension of the class of invex functions with respect to η introduced by Hanson [5].

Recently, Kim et al. [12] delineated a group of non-differentiable multi-objective fractional programs and founded necessary and sufficient optimality conditions and duality results for weakly efficient solutions of non-differentiable multi-objective fractional programming problems. Later on, Kim et al. [13] considered two pairs of non-differentiable multi-objective symmetric dual problems of second order with cone constraints over arbitrary closed convex cones, which are Mond-Weir type and Wolfe type and weak, strong, converse and self-duality theorems were founded under the assumptions of pseudo-invex functions of second order. We have to maximize the efficiency of an economic system resulting optimization problems whose objective function is a ratio. Including maximization of productivity, maximization of return on investment, maximization of risk, minimization of cost. Ojha [14] derived a pair of second order symmetric non-differentiable multi-objective fractional problems, also derived weak and strong duality theorems.

In this paper, we use the concept of second-order $(C, \alpha, \rho, d)-V$-type-I functions for a non-differentiable multi-objective second-order fractional programming problem. A numerical non-trivial example illustrates the existence of such functions. In this dual, we generalize the models already existing in the literature. Using $(C, \alpha, \rho, d) - V$-type-I function, we set up weak, strong and strict converse duality results for Wolfe-type dual program.

2. DEFINITIONS AND NUMERICAL ILLUSTRATION

Definition 2.1. A function $C : X \times X \times R^n \longrightarrow R$ $(X \subset R^n)$ is said to be convex on R^n iff for any fixed $(x,\ u) \in X \times X$ and for any $x_1,\ x_2 \in R^n$, such that

$$C_{x,u}(\lambda x_1 + (1-\lambda)x_2) \leq \lambda C_{x,u}(x_1) + (1-\lambda)C_{x,u}(x_2),\ \forall \lambda \in (0,1).$$

Suppose the real valued function $d : X \times X \to R$ satisfy the property $d(x,u) = 0 \Leftrightarrow x = u$ and let $C_{x,u}(0) = 0$, for any $(x,u) \in X \times X$.

The general multi-objective programming problem can be expressed in the following form :

(P) Minimize $f(x) = (f_1(x), f_2(x), ..., f_k(x))$

subject to $x \in X^0 = \{x \in X : g(x) \leq 0\}$,

where $X \subset R^n$ is open, $f : X \to R^k$, $g : X \to R^m$, are differentiable functions on X.

Definition 2.2 . A point $\bar{x} \in X^0$ is said to be an efficient solution of (P) if there exists no $x \in X^0$ such that $f_i(x) \leq f_i(\bar{x}),\ i = 1, 2, ..., k$.

We write the definition of second-order $(C, \alpha, \rho, d) - V$ -type-I functions. Let C be a convex function with respect to third variable.

Definition 2.3. For any $i = 1, 2, ..., k,\ j = 1, 2, ..., m$, the function (f_i, h_j) is said to be second-order $(C, \alpha, \rho, d) - V$-type-I at $u \in X$ if there exist $\alpha_i^1,\ \alpha_j^2 : X \times X \to$

$R_+ \setminus \{0\}$ and $\rho_i^1,\ \rho_j^2 \in R$, such that for each $x \in X^0$ and $p \in R^n$, we have

$$\frac{1}{\alpha_i^1(x,u)}[f_i(x) - f_i(u) + \frac{1}{2}p^T\nabla^2 f_i(u)p] \geq C_{x,u}\big(\nabla f_i(u) + \nabla^2 f_i(u)p\big) + \frac{\rho_i^1 d^2(x,u)}{\alpha_i^1(x,u)}$$

and

$$\frac{1}{\alpha_j^2(x,u)}[-h_j(u) + \frac{1}{2}p^T\nabla^2 h_j(u)p] \geq C_{x,u}\Bigg(\nabla h_j(u) + \nabla^2 h_j(u)p\Bigg) + \frac{\rho_j^2 d^2(x,u)}{\alpha_j^2(x,u)}.$$

Definition 2.4. For any $i = 1,2,...,k,\ j = 1,2,...,m$, the functions (f_i, h_j) are said to be second-order semi-strictly $(C,\alpha,\rho,d) - V$-type-I at $u \in X$ if there exist $\alpha_i^1,\ \alpha_j^2 : X \times X \to R_+ \setminus \{0\}$ and $\rho_i^1,\ \rho_j^2 \in R$, such that for each $x \in X^0$ and $p \in R^n$, we have

$$\frac{1}{\alpha_i^1(x,u)}[f_i(x) - f_i(u) + \frac{1}{2}p^T\nabla^2 f_i(u)p] > C_{x,u}\big(\nabla f_i(u) + \nabla^2 f_i(u)p\big) + \frac{\rho_i^1 d^2(x,u)}{\alpha_i^1(x,u)}$$

and

$$\frac{1}{\alpha_j^2(x,u)}[-h_j(u) + \frac{1}{2}p^T\nabla^2 h_j(u)p] \geq C_{x,u}\Bigg(\nabla h_j(u) + \nabla^2 h_j(u)p\Bigg) + \frac{\rho_j^2 d^2(x,u)}{\alpha_j^2(x,u)}.$$

Every (F,α,ρ,d) type-I function is (C,α,ρ,d) type-I. But the converse need not be true. This is seen from the following example.

Example 1. Let $X = R$. Let $f : X \to R$ and $g : X \to R$ where $f(x) = x^2 - cos2x - 1$ and $g(x) = \dfrac{cos2x + 1}{2} - 2x$. Suppose $\alpha_1^1,\ \alpha_2^1 \in R_+ \setminus \{0\}$ and $\alpha_1^1 = \dfrac{1}{20},\ \alpha_2^2 = \dfrac{1}{3}$ and $C_{x,u}(a) = \frac{a^2}{24}$. Let $d : X \times X \to R_+$ be $d(x,u) = (x-u)^2$.

Let $p = -1,\ \rho_1^1 = \frac{-1}{20},\ \rho_1^2 = -1$.

By definition of (C,α,ρ,d) type-I at $u = -0.5\pi$, we have

$$\frac{1}{\alpha_1^1(x,u)}[f(x) - f(u) + \frac{1}{2}p^T\nabla^2 f(u)p] - C_{x,u}\big(\nabla f(u) + \nabla^2 f(u)p\big) - \frac{\rho^1 d^2(x,u)}{\alpha_1^1(x,u)},$$

$$20x^2 - 40(1 + cos2x) + 60 - 5\pi^2 - \frac{1}{24}(\pi - 6)^2 + (x + 5\pi)^2 \geq 0,$$

Similarly,

$$\frac{1}{\alpha_1^2(x,u)}[-h(u) + \frac{1}{2}p^T\nabla^2 h(u)p] - C_{x,u}\Bigg(\nabla h(u) + \nabla^2 h(u)p\Bigg) - \frac{\rho_1^2 d^2(x,u)}{\alpha_1^2(x,u)} \geq 0.$$

for all $x \in X$. Hence, (f,h) is second order (C,α,ρ,d) type-I but (f,h) is not second order (F,α,ρ,d) type-I at $u = 0.5\pi$ as C is not sub-linear with respect to the third argument.

Definition 2.5. Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a sub-differential, that is, there exists a $z \in R^n$ such that

$$s(y|C) \geq s(x|C) + z^T(y - x), \forall x \in C.$$

The sub-differential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For a convex set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n : y^T(z - x) \leq 0, \forall z \in D\}.$$

When C is a compact convex set, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

3. Problem formulation

A general multi-objective fractional programming problem can be expressed in the following form :

(MFP) Min $G(x) = \left(\dfrac{f_1(x) + S(x|C_1)}{g(x) - S(x|D)}, \dfrac{f_2(x) + S(x|C_2)}{g(x) - S(x|D)}, ..., \dfrac{f_k(x) + S(x|C_k)}{g(x) - S(x|D)} \right)$

Subject to $x \in X^0 = \{x \in X : h_j(x) + S(x|E_j) \leq 0,\ j = 1, 2, ..., m\}$,

where $X \subset R^n$ is a closed convex set, $f_i,\ g : X \to R$ $(i = 1, 2, ..., k)$ and $h_j : X \to R$ $(j = 1, 2, ..., m)$ are continuously differentiable functions. $f_i(.) + S(.|C_i) \geq 0$ and $g(.) - S(.|D) > 0$, and $S(x|C_i),\ S(x|D)$ and $S(x|E_j)$ denote the support functions of compact convex sets, $C_i,\ D$ and E_j, for all $i = 1, 2, ..., k,\ j = 1, 2, ..., m$, respectively.

For every $\nu \in R^k_+$, consider the following auxiliary problem:

(MFP)$_\nu$ Minimize $G(x) = \Big(f_1(x) + S(x|C_1) - \nu_1(g(x) - S(x|D)),$

$, ..., f_k(x) + S(x|C_k) - \nu_k(g(x) - S(x|D)) \Big).$

Lemma 3[16]. Let $\bar{x} \in X$ is an efficient solution for (MFP) if and only if there exists $\bar{\nu} \in R^k_+$ such that $\bar{x}$ is an efficient solution for (MFP)$_{\bar{\nu}}$ and $\bar{\nu}_i = \dfrac{f_i(\bar{x}) + S(\bar{x}|C_i)}{g(\bar{x}) - S(\bar{x}|D)}$, $i = 1, 2, ..., k$.

Theorem 3.1 (Necessary Condition)[15]. Assume that $\bar{x}$ is an efficient solution of (MFP) and let the Slater's constraint qualification be satisfied on X. Then there exist $0 < \bar{\lambda} \in R^k$, $\sum_{i=1}^{k} \bar{\lambda}_i = 1$, $0 \leq \bar{y} \in R^m$, $\bar{z_i} \in R^n$, $\bar{v} \in R^n$ and $\bar{w_j} \in R^n$, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$ such that

$$\sum_{i=1}^{k} \bar{\lambda_i} \nabla \left(\frac{f_i(\bar{x}) + \bar{x}^T \bar{z_i}}{g_i(\bar{x}) - \bar{x}^T \bar{v}} \right) + \sum_{j=1}^{m} \bar{y_j} \nabla (h_j(\bar{x}) + \bar{x}^T \bar{w_j}) = 0,$$

$$\sum_{j=1}^{m} \bar{y_j} (h_j(\bar{x}) + \bar{x}^T \bar{w_j}) = 0,$$

$$\bar{x}^T \bar{z_i} = S(\bar{x}|C_i),\ \bar{x}^T \bar{v} = S(\bar{x}|D_i),\ \bar{x}^T \bar{w_j} = S(\bar{x}|E_j),$$

$$\bar{z_i} \in C_i,\ \bar{v} \in D_i,\ \bar{w_j} \in E_j,\ i = 1, 2, ..., k,\ j = 1, 2, ..., m.$$

4. Duality model-I

Now, in this section we study the duality for $(\mathrm{MFP})_\nu$, for some $\nu \in R^k_+$. We first consider the following auxiliary problem:

(MFD) maximize

$$\Big[f_1(u)+u^Tz_1-\nu_1(g(u)-u^Tv)-\frac{1}{2}p^T\nabla^2\Big(f_1(u)+u^Tz_1-\nu_1(g(u)$$
$$-u^Tv)\Big)p+\sum_{j=1}^m y_j(h_j(u)+u^Tw_j-\frac{1}{2}p^T\nabla^2h_j(u)p)$$
$$,...,f_k(u)+u^Tz_k-\nu_k(g(u)-u^Tv)-\frac{1}{2}p^T\nabla^2\Big(f_k(u)+u^Tz_k-\nu_k(g(u)$$
$$-u^Tv)\Big)p+\sum_{j=1}^m y_j(h_j(u)+u^Tw_j-\frac{1}{2}p^T\nabla^2h_j(u)p)\Big]$$

Subject to

$$\sum_{i=1}^k \lambda_i\nabla\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)+\sum_{j=1}^m y_j\nabla(h_j(u)+u^Tw_j)$$
$$+\sum_{i=1}^k \lambda_i\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)p+\sum_{j=1}^m y_j\nabla^2h_j(u)p=0, \tag{4.1}$$
$$z_i\in C_i,\ v\in D,\ i=1,2,...,k,\ w_j\in E_j,\ j=1,2,...,m,$$
$$y_j\geq 0,\ j=1,2,...,m,\ \lambda_i>0,\ i=1,2,...,k,\ \sum_{i=1}^k\lambda_i=1.$$

We now discuss the duality results for $(\mathrm{MFP})_\nu$ and (MFD).

Theorem 4.1 (Weak Duality Theorem). Let x be a feasible solution for $(\mathrm{MFP})_\nu$, for some $\nu \in R^k_+$ and for each feasible solution $(u,z,v,y,\lambda,\nu,w,p)$ of (MFD), for the same $\nu\in R^k_+$. Suppose that, for any $i=1,2,...,k,\ j=1,2,...,m$,

(i) $\Big[f_i(.)+(.)^Tz_i-\nu_i(g(.)-(.)^Tv),\ h_j(.)+(.)^Tw_j\Big]$ is second-order $(C,\alpha,\rho,d)-V$-type-I at u,

(ii) $\alpha_i^1(x,u)=\alpha_j^2(x,u)=\alpha(x,u)$, for all i and j,

(iii) $\sum_{i=1}^k\lambda_i\rho_i^1+\sum_{j=1}^m y_j\rho_j^2\geq 0.$

Then the following cannot hold

$$f_i(x)+S(x|C_i)-\nu_i(g(x)-S(x|D))\leq f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)$$
$$-\frac{1}{2}p^T\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)p$$
$$+\sum_{j=1}^m y_j\Big(h_j(u)+u^Tw_j-\frac{1}{2}p^T\nabla^2h_j(u)p\Big),\ \forall\ i=1,2,...,k \tag{4.2}$$

and

$$f_r(x) + S(x|C_r) - \nu_r(g(x) - S(x|D)) < f_r(u) + u^T z_r - \nu_r(g(u) - u^T v)$$
$$- \frac{1}{2}p^T\nabla^2\Big(f_r(u) + u^T z_r - \nu_r(g(u) - u^T v)\Big)p$$
$$+ \sum_{j=1}^{m} y_j\Big(h_j(u) + u^T w_j - \frac{1}{2}p^T\nabla^2 h_j(u)p\Big), \text{ for some } r = 1, 2, ..., k. \qquad (4.3)$$

Proof. Suppose that (4.2) and (4.3) hold, then using $\lambda_i > 0,\ \sum_{i=1}^{k}\lambda_i = 1,\ x^T z_i \leq S(x|C_i)$

$,\ x^T v \leq S(x|D),\ i = 1, 2, ..., k,$ we have

$$\sum_{i=1}^{k}\lambda_i\Big[f_i(x) + x^T z_i - \nu_i(g(x) - x^T v_i) - (f_i(u) + u^T z_i - \nu_i(g(u) - u^T v))$$
$$+\frac{1}{2}p^T\nabla^2\Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v)\Big)p\Big]$$
$$-\sum_{j=1}^{m} y_j\Big(h_j(u) + u^T w_j - \frac{1}{2}p^T\nabla^2 h_j(u)p\Big) < 0. \qquad (4.4)$$

For any $i = 1, 2, ..., k,\ j = 1, 2, ..., m,\ \Big[f_i(.)+(.)^T z_i-\nu_i(g(.)-(.)^T v),\ h_j(.)+(.)^T w_j\Big]$ is second-order $(C, \alpha, \rho, d) - V$-type-I, we have

$$\frac{1}{\alpha_i^1(x,u)}\Big[f_i(x) + x^T z_i - \nu_i(g(x) - x^T v) - (f_i(u) + u^T z_i - \nu_i(g(u) - u^T v))$$
$$+ \frac{1}{2}p^T\nabla^2\Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v)\Big)p\Big] - \frac{\rho_i^1 d^2(x,u)}{\alpha_i^1(x,u)}$$
$$\geq C_{x,u}\Big[\nabla\Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v)\Big) + \nabla^2\Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v)\Big)p\Big] \qquad (4.5)$$

and

$$\frac{1}{\alpha_j^2(x,u)}\Big[-(h_j(u) + u^T w_j) + \frac{1}{2}p^T\nabla^2 h_j(u)p\Big]$$
$$\geq C_{x,u}\Big[\nabla(h_j(u) + u^T w_j) + \nabla^2 h_j(u)p\Big] + \frac{\rho_j^2 d^2(x,u)}{\alpha_j^2(x,u)}. \qquad (4.6)$$

Let $\tau = 1 + \sum_{j=1}^{m} y_j > 0$. Adding the two inequalities after multiplying (4.5) by $\frac{\lambda_i}{\tau}$ and (4.6) by $\frac{y_j}{\tau}$ and summing them over all i and j, we obtain

$$\sum_{i=1}^{k}\frac{\lambda_i}{\alpha_i^1(x,u)\tau}\Big[f_i(x) + x^T z_i - \nu_i(g(x) - x^T v) - (f_i(u) + u^T z_i - \nu_i(g(u) - u^T v)) +$$
$$\frac{1}{2}p^T\nabla^2\Big(f_i(u)+u^T z_i-\nu_i(g(u)-u^T v)p\Big]-\sum_{j=1}^{m}\frac{y_j}{\alpha_j^2(x,u)\tau}\Big[h_j(u)+u^T w_j-\frac{1}{2}p^T\nabla^2 h_j(u)p\Big]$$

$$\geq \sum_{i=1}^{k} \frac{\lambda_i}{\tau} C_{x,u}\bigg\{\nabla\Big(f_i(u)+u^T z_i-\nu_i(g(u)-u^T v)\Big)+\nabla^2\Big(f_i(u)+u^T z_i-\nu_i(g(u)$$
$$-u^T v)\Big)p\bigg\}+\sum_{j=1}^{m}\frac{y_j}{\tau}C_{x,u}\Big\{\nabla(h_j(u)+u^T w_j)+\nabla^2 h_j(u)p\Big\}$$
$$+\bigg(\sum_{i=1}^{k}\frac{\lambda_i\rho_i^1}{\alpha_i^1(x,u)\tau}+\sum_{j=1}^{m}\frac{y_j\rho_j^2}{\alpha_j^2(x,u)\tau}\bigg)d^2(x,u). \tag{4.7}$$

Using hypothesis (ii) and convexity of C, it gives

$$\frac{1}{\alpha(x,u)\tau}\sum_{i=1}^{k}\lambda_i\Big[f_i(x)+x^T z_i-\nu_i(g(x)-x^T v)-(f_i(u)+u^T z_i-\nu_i(g(u)-u^T v))+$$
$$\frac{1}{2}p^T\nabla^2\Big(f_i(u)+u^T z_i-\nu_i(g(u)-u^T v\Big)p\Big]-\frac{1}{\alpha(x,u)\tau}\sum_{j=1}^{m}y_j\Big[h_j(u)+u^T w_j-\frac{1}{2}p^T\nabla^2 h_j(u)p\Big]$$
$$\geq C_{x,u}\bigg[\sum_{i=1}^{k}\frac{\lambda_i}{\tau}\bigg\{\nabla\Big(f_i(u)+u^T z_i-\nu_i(g(u)-u^T v\Big)+\nabla^2\Big(f_i(u)+u^T z_i-\nu_i(g(u)$$
$$-u^T v)\Big)p\bigg\}+\sum_{j=1}^{m}\frac{y_j}{\tau}\Big\{\nabla(h_j(u)+u^T w_j)+\nabla^2 h_j(u)p\Big\}\bigg]$$
$$+\frac{1}{\alpha(x,u)\tau}\bigg(\sum_{i=1}^{k}\lambda_i\rho_i^1+\sum_{j=1}^{m}y_j\rho_j^2\bigg)d^2(x,u). \tag{4.8}$$

Finally using feasibility condition (4.1), hypothesis (iii) and using $C_{x,u}(0)=0$, it follows that

$$\sum_{i=1}^{k}\lambda_i\bigg(f_i(x)+x^T z_i-\nu_i(g(x)-x^T v_i)-f_i(u)+u^T z_i-\nu_i(g(u)-u^T v)+\frac{1}{2}p^T\nabla^2\Big(f_i(u)+$$
$$u^T z_i-\nu_i(g(u)-u^T v\Big)p\bigg)$$
$$-\sum_{j=1}^{m}y_j\Big(h_j(u)+u^T w_j-\frac{1}{2}p^T\nabla^2 h_j(u)p\Big)\geq 0.$$

which contradict (4.4). Hence the result. □

Theorem 4.2 (Strong Duality Theorem). If $\bar{u}\in X^0$ is an efficient solution of (MFP)$_\nu$ and let the Slater constraint qualification be satisfied. Then there exist $\bar{\lambda}\in R^k$, $\bar{y}\in R^m$, $\bar{z}_i\in R^n$, $\bar{v}\in R^n$ and $\bar{w}_j\in R^n$, $i=1,2,...,k$, $j=1,2,...,m$, such that $(\bar{u},\bar{z},\bar{v},\bar{y},\bar{\lambda},\bar{\nu}_i,\bar{w},\bar{p}=0)$ is a feasible solution of (MFD) and the objective function values of (MFP)$_\nu$ and (MFD) are equal. Moreover, if the conditions of Theorem 3.1 holds for all feasible solutions of (MFP)$_\nu$ and (MFD), then $(\bar{u},\bar{z},\bar{v},\bar{y},\bar{\lambda},\bar{\nu}_i,\bar{w},\bar{p}=0)$ is an efficient solution of (MFD).

Proof. Since $\bar{u}$ is an efficient solution of (MFP)$_\nu$ and the Slater constraint qualification is satisfied, from Theorem 2.1, there exist $\bar{\mu}\in R^k,\bar{y}\in R^m,\bar{z}_i\in R^n$, $\bar{v}\in R^n$ and $\bar{w}_j\in R^n, i=1,2,...,k, j=1,2,...,m$, such that

$$\sum_{i=1}^{k}\bar{\mu}_i\nabla\bigg(\frac{f_i(\bar{u})+\bar{u}^T\bar{z}_i}{g(\bar{u})-\bar{u}^T\bar{v}}\bigg)+\sum_{j=1}^{m}\bar{y}_j\nabla(h_j(\bar{u})+\bar{u}^T\bar{w}_j)=0, \tag{4.9}$$

$$\sum_{j=1}^{m} \bar{y_j}(h_j(\bar{u}) + \bar{u}^T \bar{w_j}) = 0, \tag{4.10}$$

$$\bar{u}^T \bar{z_i} = S(\bar{u}|C_i),\ \bar{u}^T \bar{v} = S(\bar{u}|D_i),\ \bar{u}^T \bar{w_j} = S(\bar{u}|E_j), \tag{4.11}$$

$$\bar{z_i} \in C_i,\ \bar{v} \in D_i,\ \bar{w_j} \in E_j, \tag{4.12}$$

$$\bar{\mu_i} > 0,\ \bar{y_j} \geq 0,\ i = 1, 2, ..., k,\ j = 1, 2, ..., m. \tag{4.13}$$

Equation (4.9) can be written as,

$$\sum_{i=1}^{k} \frac{\bar{\mu_i}}{g(\bar{u}) - \bar{u}^T \bar{v}} \left(\nabla (f_i(\bar{u}) + \bar{u}^T \bar{z_i}) - \frac{f_i(\bar{u}) + \bar{u}^T \bar{z_i}}{g(\bar{u}) - \bar{u}^T \bar{v}} \nabla (g(\bar{u}) - \bar{u}^T \bar{v}) \right) + \sum_{j=1}^{m} \nabla \bar{y_j}(h_j(\bar{u}) + \bar{u}^T \bar{w_j}) = 0. \tag{4.14}$$

Letting $\bar{\lambda_i} = \dfrac{\bar{\mu_i}}{g(\bar{u}) - \bar{u}^T \bar{v}}$ and $\bar{\nu_i} = \dfrac{f_i(\bar{u}) + \bar{u}^T \bar{z_i}}{g(\bar{u}) - \bar{u}^T \bar{v}},\ i = 1, 2, ..., k$, we have

$$\sum_{i=1}^{k} \bar{\lambda_i} \nabla \left(f_i(\bar{u}) + \bar{u}^T \bar{z_i} - \bar{\nu_i}(g(\bar{u}) - \bar{u}^T \bar{v}) \right) + \sum_{j=1}^{m} \bar{y_j} \nabla (h_j(\bar{u}) + \bar{u}^T \bar{w_j}) = 0 \tag{4.15}$$

$$f_i(\bar{u}) + \bar{u}^T \bar{z_i} - \bar{\nu_i}(g(\bar{u}) - \bar{u}^T \bar{v}) = 0. \tag{4.16}$$

Thus, $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu_i}, \bar{w}, \bar{p} = 0)$ is feasible for (MFD) and the objective function values of (MFP)$_\nu$ and (MFD) are equal. This complete the proof of theorem 4.2. $\square$

Theorem 4.3 (Strict Converse Duality Theorem). Let $\bar{x}$ be a feasible solution for (MFP)$_\nu$ and $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ be feasible for (MFD) Suppose that, for any $i = 1, 2, ..., k,\ j = 1, 2, ..., m$,

(i) $\sum_{i=1}^{k} \bar{\lambda_i} \Big(f_i(\bar{x}) + \bar{x}^T \bar{z_i} - \bar{\nu_i}(g(\bar{x}) - \bar{x}^T \bar{v}) - (f_i(\bar{u}) + \bar{u}^T \bar{z_i} - \bar{\nu_i}(g(\bar{u}) - \bar{u}^T \bar{v}) + \frac{1}{2} p^T \nabla^2 \Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v \Big) p \Big) - \sum_{j=1}^{m} y_j \Big(h_j(u) + u^T w_j - \frac{1}{2} p^T \nabla^2 h_j(u) p \Big) \leq 0,$

(ii) $\Big[f_i(.) + (.)^T z_i - \bar{\nu_i}(g(.) - (.)^T v),\ h_j(.) + (.)^T w_j \Big]$ is second-order semi-strictly $(C, \alpha, \rho, d) - V$-type-I at $\bar{u}$,

(iii) $\alpha_i^1(\bar{x}, \bar{u}) = \alpha_j^2(\bar{x}, \bar{u}) = \alpha(\bar{x}, \bar{u})$, for all i and j,

(iv) $\sum_{i=1}^{k} \bar{\lambda}_i \rho_i^1 + \sum_{j=1}^{m} \bar{y}_j \rho_j^2 \geq 0.$

Then, $\bar{x} = \bar{u}$.

Proof. We suppose on contrary $\bar{x} \neq \bar{u}$. Since $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ is feasible solution for (MFD), we have

$$C_{\bar{x},\bar{u}} \Bigg[\sum_{i=1}^{k} \bar{\lambda}_i \nabla \left(f_i(\bar{u}) + \bar{u}^T \bar{z}_i - \bar{\nu}_i(g(\bar{u}) - \bar{u}^T \bar{v}) \right) + \sum_{j=1}^{m} \bar{y}_j \nabla (h_j(\bar{u}) + \bar{u}^T \bar{w}_j)$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v})\Big)\bar{p}+\sum_{j=1}^{m}\bar{y}_j\nabla^2 h_j(\bar{u})\bar{p}\Big]=0. \quad (4.17)$$

By hypothesis (ii), we get

$$\frac{1}{\alpha_i^1(\bar{x},\bar{u})}\Big[f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{\nu}_i(g(\bar{x})-\bar{x}^T\bar{v})-(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v})$$
$$+\frac{1}{2}\bar{p}^T\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)\bar{p}\Big]-\frac{\rho_i d^2(\bar{x},\bar{u})}{\alpha_i^1(\bar{x},\bar{u})}$$
$$>C_{\bar{x},\bar{u}}\Big[\nabla\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)+\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)\bar{p}\Big] \quad (4.18)$$

and

$$\frac{1}{\alpha_j^2(\bar{x},\bar{u})}\Big[-(h_j(\bar{u})+\bar{u}^T\bar{w}_j)+\frac{1}{2}\bar{p}^T\nabla^2 h_j(\bar{u})\bar{p}\Big]$$
$$\geq C_{\bar{x},\bar{u}}\Big[\nabla(h_j(\bar{u})+\bar{u}^T\bar{w}_j)+\nabla^2 h_j(\bar{u})\bar{p}\Big]+\frac{\rho_j d^2(\bar{x},\bar{u})}{\alpha_j^2(\bar{x},\bar{u})}. \quad (4.19)$$

Let $\bar{\tau}=1+\sum_{j=1}^{m}\bar{y}_j>0$. Multiplying (4.18) by $\frac{\bar{\lambda}_i}{\bar{\tau}}$ and (4.19) by $\frac{\bar{y}_j}{\bar{\tau}}$, summing over $i=1,2,...,k$ and $j=1,2,...,m$, we have

$$\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\alpha_i^1(\bar{x},\bar{u})\bar{\tau}}\Big[f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{\nu}_i(g(\bar{x})-\bar{x}^T\bar{v}))-(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}))+$$
$$\frac{1}{2}\bar{p}^T\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)\bar{p}\Big]-\sum_{j=1}^{m}\frac{\bar{y}_j}{\alpha_j^2(\bar{x},\bar{u})\bar{\tau}}\Big[h_j(\bar{u})+\bar{u}^T\bar{w}_j-\frac{1}{2}\bar{p}^T\nabla^2 h_j(\bar{u})\bar{p}\Big]$$
$$>\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\bar{\tau}}C_{\bar{x},\bar{u}}\Big\{\nabla\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)+\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i$$
$$-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)\bar{p}\Big\}+\sum_{j=1}^{m}\frac{\bar{y}_j}{\bar{\tau}}C_{\bar{x},\bar{u}}\Big\{\nabla(h_j(\bar{u})+\bar{u}^T\bar{w}_j)+\nabla^2 h_j(\bar{u})\bar{p}\Big\}+$$
$$\Big(\sum_{i=1}^{k}\frac{\bar{\lambda}_i\rho_i^1}{\alpha_i^1(\bar{x},\bar{u})\bar{\tau}}+\sum_{j=1}^{m}\frac{\bar{y}_j\rho_j^2}{\alpha_j^2(\bar{x},\bar{u})\bar{\tau}}\Big)d^2(\bar{x},\bar{u}).$$

Using hypothesis (iii) and convexity of C, it yield

$$\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\alpha(\bar{x},\bar{u})\bar{\tau}}\Big[f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{\nu}_i(g(\bar{x})-\bar{x}^T\bar{v}))-(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}))+$$
$$\frac{1}{2}\bar{p}^T(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}))\bar{p}\Big]-\sum_{j=1}^{m}\frac{\bar{y}_j}{\alpha(\bar{x},\bar{u})\bar{\tau}}\Big(h_j(\bar{u})+\bar{u}^T\bar{w}_j-\frac{1}{2}\bar{p}^T\nabla^2 h_j(\bar{u})\bar{p}\Big)$$
$$>C_{\bar{x},\bar{u}}\Big[\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\bar{\tau}}\Big\{\nabla\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v})\Big)+\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-$$

$$\bar{u}^T\bar{v})\Big)\bar{p}\Big\} + \sum_{j=1}^{m}\frac{\bar{y}_j}{\bar{\tau}}\Big\{\nabla(h_j(\bar{u})+\bar{u}^T\bar{w}_j)+\nabla^2 h_j(\bar{u})\bar{p}\Big\}\Big]$$

$$+\frac{1}{\tau\alpha(\bar{x},\bar{u})}\Big(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i^1+\sum_{j=1}^{m}\bar{y}_j\rho_j^2\Big)d^2(\bar{x},\bar{u}). \tag{4.20}$$

Finally, using feasibility conditions (4.1) and hypothesis (iv) in (4.20), we get

$$\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\alpha(\bar{x},\bar{u})\bar{\tau}}\Big[f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{\nu}_i(g(\bar{x})-\bar{x}^T\bar{v}))-(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}))+$$
$$\frac{1}{2}p^T\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv\Big)p\Big]-\sum_{j=1}^{m}\frac{\bar{y}_j}{\alpha(\bar{x},\bar{u})\bar{\tau}}\Big(h_j(\bar{u})+\bar{u}^T\bar{w}_j-\frac{1}{2}\bar{p}^T\nabla^2 h_j(\bar{u})\bar{p}\Big)$$
$$> C_{\bar{x},\bar{u}}\Big[\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\bar{\tau}}\Big\{\nabla\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v})\Big)+\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i$$
$$-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v})\Big)p\Big\}+\sum_{j=1}^{m}\frac{\bar{y}_j}{\bar{\tau}}[\nabla(h_j(\bar{u})+\bar{u}^T\bar{w}_j)+\nabla^2 h_j(\bar{u})\bar{p}]\Big].$$

Using (4.17), above equation gives

$$\sum_{i=1}^{k}\bar{\lambda}_i\Big(f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{\nu}_i(g(\bar{x})-\bar{x}^T\bar{v})-(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}))$$
$$+\frac{1}{2}p^T\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv\Big)p\Big)$$
$$-\sum_{j=1}^{m}y_j\Big(h_j(u)+u^Tw_j-\frac{1}{2}p^T\nabla^2 h_j(u)p\Big)>0.$$

this contradicts the hypothesis (i). Hence proved. □

5. Duality model-II

Now, in this section we study the duality for (MFP)$_\nu$, for some $\nu \in R_+^k$. We first consider the following auxiliary problem:

(MFD1)

maximize$\Big[f_1(u)+u^Tz_1-\nu_1(g(u)-u^Tv),...,f_k(u)+u^Tz_k-\nu_k(g(u)-u^Tv)\Big]$

Subject to

$$\sum_{i=1}^{k}\lambda_i\nabla\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)+\sum_{j=1}^{m}y_j\nabla(h_j(u)+u^Tw_j)$$
$$+\sum_{i=1}^{k}\lambda_i\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)p+\sum_{j=1}^{m}y_j\nabla^2 h_j(u)p=0, \tag{5.1}$$

$$\sum_{i=1}^{k}\lambda_i p^T\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)p\le 0, \tag{5.2}$$

$$p^T\nabla^2 h_j(u)p \leq 0, \tag{5.3}$$

$$z_i \in C_i,\ v \in D,\ i = 1, 2, ..., k,\ w_j \in E_j,\ j = 1, 2, ..., m,$$

$$y_j \geq 0,\ j = 1, 2, ..., m,\ \lambda_i > 0,\ i = 1, 2, ..., k,\ \sum_{i=1}^{k} \lambda_i = 1.$$

We now discuss the duality results for (MFP)$_\nu$ and (MFD1).

Theorem 5.1 (Weak Duality Theorem). Let $x \in X^0$ be a feasible solution for (MFP)$_\nu$, for some $\nu \in R_+^k$ and for each feasible solution $(u, z, v, y, \lambda, \nu, w, p)$ of (MFD1), for the same $\nu \in R_+^k$. Suppose that, for any $i = 1, 2, ..., k,\ j = 1, 2, ..., m,$

(i) $\left[f_i(.)+(.)^T z_i - \nu_i(g(.)-(.)^T v),\ h_j(.)+(.)^T w_j\right]$ is second-order $(C, \alpha, \rho, d)-V$-type-I at u,

(ii) $\alpha_i^1(x, u) = \alpha_j^2(x, u) = \alpha(x, u)$, for all i and j,

(iii) $\sum_{i=1}^{k} \lambda_i \rho_i^1 + \sum_{j=1}^{m} y_j \rho_j^2 \geq 0.$

Then the following cannot hold

$$\begin{aligned} &f_i(x) + S(x|C_i) - \nu_i(g(x) - S(x|D)) \\ &\qquad \leq f_i(u) + u^T z_i - \nu_i(g(u) - u^T v),\ \forall\ i = 1, 2, ..., k \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &f_r(x) + S(x|C_r) - \nu_r(g(x) - S(x|D)) \\ &\qquad < f_r(u) + u^T z_r - \nu_r(g(u) - u^T v),\ \text{for some } r = 1, 2, ..., k. \end{aligned} \tag{5.5}$$

Proof. The proof follows on the lines of Theorem 4.1. □

Theorem 5.2 (Strong Duality Theorem). If $\bar{u} \in X^0$ is an efficient solution of (MFP)$_\nu$ and let the Slater constraint qualification be satisfied. Then there exist $\bar{\lambda} \in R^k,\ \bar{y} \in R^m,\ \bar{z}_i \in R^n,\ \bar{v} \in R^n$ and $\bar{w}_j \in R^n,\ i = 1, 2, ..., k,\ j = 1, 2, ..., m,$ such that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu}_i, \bar{w}, \bar{p} = 0)$ is a feasible solution of (MFD1) and the objective function values of (MFP)$_\nu$ and (MFD1) are equal. Moreover, if the conditions of Theorem 3.1 holds for all feasible solutions of (MFP)$_\nu$ and (MFD1), then $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu}_i, \bar{w}, \bar{p} = 0)$ is an efficient solution of (MFD1).

Proof. The proof follows on the lines of Theorem 4.2. □

Theorem 5.3 (Strict Converse Duality Theorem). Let $\bar{x} \in X^0$ be a feasible solution for (MFP)$_\nu$ and $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ be feasible for (MFD1). Suppose that, for any $i = 1, 2, ..., k,\ j = 1, 2, ..., m,$

(i) $\sum_{i=1}^{k} \bar{\lambda}_i \left(f_i(\bar{x}) + \bar{x}^T \bar{z}_i - \bar{\nu}_i(g(\bar{x}) - \bar{x}^T \bar{v}) - (f_i(\bar{u}) + \bar{u}^T \bar{z}_i - \bar{\nu}_i(g(\bar{u}) - \bar{u}^T \bar{v}) \right) \leq 0,$

(ii) $\left[f_i(.)+(.)^T z_i - \bar{\nu}_i(g(.)-(.)^T v),\ h_j(.)+(.)^T w_j\right]$ is second-order semi-strictly $(C, \alpha, \rho, d) - V$-type-I at $\bar{u}$,

(iii) $\alpha_i^1(\bar{x}, \bar{u}) = \alpha_j^2(\bar{x}, \bar{u}) = \alpha(\bar{x}, \bar{u})$, for all i and j,

$$(iv)\ \sum_{i=1}^{k}\bar{\lambda}_i\rho_i^1+\sum_{j=1}^{m}\bar{y}_j\rho_j^2\geq 0.$$

Then, $\bar{x}=\bar{u}$.

Proof. The proof follows on the lines of Theorem 4.3. □

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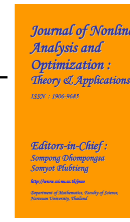

MODELLING HUMAN PERIPHERAL TISSUE TEMPERATURE BASED ON NUMERICAL COMPUTATION OF NON-LINEAR BOUNDARY VALUE PROBLEM

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ABSTRACT. A mathematical model has been established to analyse the changes in the tissue temperature on the peripheral regions of human body with respect to varying physiological conditions. The formulation is based on a parabolic heat equation with appropriate physiological parameters. The solution has been established using finite difference scheme and the simulation was done with the help of MATLAB software. It has been observed that the results have shown a considerable effect on temperature profiles due to variable physiological parameters.

KEYWORDS : Heat regulation; finite element method; bio-heat model; boundary value problem.
AMS Subject Classification(2010): 92BXX; 92CXX; 92C35; 92C50; 46N60.

1. INTRODUCTION

The heat and mass transport in human body are two important physiological processes. Any change in the homostasis disturbs the overall function of the well coordinated system. The skin is the largest organ of the human body and is responsible for all kinds of processes including temperature regulation, protecting the body from external invasion and for the manufacture of vitamin D. The body loses heat mainly through skin to its surroundings by radiation, conduction and evaporation and thus the body temperature maintains constant level under normal physiological and atmospheric conditions. Under the normal physiological and atmospheric conditions, the skin and subcutaneous tissue (SST) region of body maintains its core temperature at 37^0C. The skin is divided into two main layers epidermis and dermis. The subcutaneous or bottom layer contains muscles and fatty tissue that help keep the skin toned and firm. The dermis or main layer contains sensory nerve endings, blood and lymph vessels, hair follicles and the

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sebaceous and sweat glands. This is also where new living cells are manufactured before emerging on the surface. The epidermis, or top layer, is the visible surface of the skin, which is composed of flat, essentially dead skin cells. The unstable environmental temperature plays a great role for the disturbance in human thermoregulatory system. The effect of surrounding temperature makes its way via dermal layers and leads to hyperthermia and hypothermia in body core and other various thermal stresses to the body peripherals. From the last few years various researchers studied the distribution of temperature in the human body in relation to several environment temperatures. If T represents the temperature of the tissue at any point in the dermal layers, then the mathematical model for the distribution of temperature in the human dermal layers can be represented as:

$$k\frac{d^2T}{dr^2} + \frac{2k}{r} + \frac{dT}{dr} + Q = 0 \tag{1.1}$$

and the associated boundary conditions are

$$\lim_{r\to 0^+} \frac{\partial T}{\partial r} = 0, \quad -k\left(\frac{\partial T}{\partial r}\right)_{r=R} = E\left(T_H - T_a\right) \tag{1.2}$$

where r is the radial distance from the origin, R is the radius of the domain, E is the ambient cooling constant, T_a ambient temperature, T_H periphery temperature, Q - the heat production per unit volume and k is the thermal conductivity inside the dermal region.

To study the effect of environmental temperatures on human dermal regions various mathematical models were formulated. Earlier experimental investigations were made by Patterson[11] to obtain temperature profiles in the SST region. Some theoretical investigations have been carried out during the last few decades by Cooper and Trezek [2], Chao et. al. [1] discussed temperature distribution in SST region under normal environmental and physiological conditions. Song et. al [16] established models describing the macro and micro vascular level heat transfer in limbs. and Jas [6] studied the thermal behaviour of human organs in malignancies. Our group had also developed numerous models in this direction [7], but the thermal conductivity was mostly assumed as constant function. In the present study, we assumed the thermal conductivity as a function of temperature in the estimation of heat regulation in human peripheral tissues.

Thron [17] studied the above model to estimate the temperature distribution in human head and suggested that if there is no singularity in the differential equation(1.1), then the solution is given by

$$T(r) = T_a + \frac{QR^2}{6k}\left[1 + \frac{2k}{ER} - \left(\frac{r}{R}\right)^2\right] \tag{1.3}$$

In addition, he calculated the temperature distribution by assuming additional heat sources while the cooling of blood at peripheral regions is given by the equation (1.3) with

$$Q = Q_0 + Q_b \tag{1.4}$$

where $Q_b = Vs(T_1 - T), Q_0$ is the heat production of tissue, V is volume of the flow of blood in unit time, T_1 is the deep temperature of the human head and $s = 0.9cal/^0Ccm^3$.

Richardson and Whitelaw [14] predicted the temperature profiles and the heat conduction and skin surface as functions of surface temperature. Flesch [4] estimated the temperature distribution using the heat equation (1.3) by assuming a heat generation rate as an explicit function of the radial distance and an implicit function of the environment temperature. Khanday and Saxena [8],[9] calculated the mass and temperature distribution at multilayered skin and sub-dermal tissues by using variational finite element method with respect to various environmental temperatures. Also, they studied the conditions under which the brain maintains thermostat and also estimated the cold effect with respect to ambient temperatures.

The present paper is an attempt to study the temperature distribution at deep dermal layers of the human body with heterogeneous thermal conductivity being a function of temperature.

2. MATHEMATICAL FORMULATION

Estimation of temperature distribution in human body using mathematical techniques has gained interest among many researchers. The heat transfer in biological tissues was studied initially by Pennes' [12] and later on Perl [13]. The existing models for heat transfer in dermal regions mostly assumed thermal conductivity term k either as constant or function of displacement. The thermal conductivity of the material may also depend on the temperature, thus it is meaningful to assume thermal conductivity of the material k as temperature dependent.

Assume k as a function of temperature as $k(T) = k_0(T - T_H)^n$. Hence, a mathematical model of the heat transfer in the human tissue follows the differential equation of heat conduction as

$$\begin{aligned}\rho c\frac{\partial T}{\partial t} =& k_0(T - T_H)^n\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)\\ &+ k_0(T - T_H)^{(n-1)}\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z}\right) + Q \end{aligned} \tag{2.1}$$

where ρ, c and k_0 represent the density, specific heat of the tissue and thermal conductivity respectively.
In case of steady state, the above equation reduces to

$$k_0\frac{d^2T}{dx^2} + \frac{2k_0}{r} + \frac{nk_0}{(T - T_H)}\left(\frac{dT}{dr}\right)^2 + \frac{Q}{(T - T_H)^n} = 0 \tag{2.2}$$

Assuming the heat generation term Q as a function of temperature T and from equation (1.4) we can take $Q = q_1(37 - T)$ for some positive constant q_1, thus we have

$$k_0\frac{d^2T}{dx^2} + \frac{2k_0}{r} + \frac{nk_0}{(T - T_H)}\left(\frac{dT}{dr}\right)^2 + \frac{q_1(37 - T)}{(T - T_H)^n} = 0 \tag{2.3}$$

The boundary conditions associated with the system are given as

$$\left(\frac{dT}{dr}\right)_{r=0} = 0, \quad T(r) = T_H \tag{2.4}$$

The singular boundary value problem determining the conduction of heat in human dermal layers.

$$k_0\frac{d^2T}{dx^2} + \frac{2k_0}{r} + \frac{nk_0}{(T - T_H)}\left(\frac{dT}{dr}\right)^2 + \frac{q_1(37 - T)}{(T - T_H)^n} = 0, \quad 0 < r < R \tag{2.5}$$

$$\left(\frac{dT}{dr}\right)_{r=0} = 0, \quad T(r) = T_H \tag{2.6}$$

Using transformations

$$y = T - T_H, \quad t = r/R, \tag{2.7}$$

the boundary value problem reduces to the following system of equations

$$k_0\frac{d^2y}{dt^2}+\frac{2k_0}{y}\frac{dy}{dt}+\left(\frac{dy}{dt}\right)^2+\frac{q_1(37-y-T_H)R^2}{(y)^n}=0,\quad 0<t<1 \tag{2.8}$$

$$\left(\frac{dT}{dt}\right)_{t=0}=0,\quad y=T_H \tag{2.9}$$

By using the following substitution

$c(t)=t^2$ and $f(t,y,cy')=\dfrac{nk_0}{y}\left(\dfrac{dy}{dt}\right)^2+\dfrac{q_1(37-y-T_H)R^2}{(y)^n}$

(2.8) and (2.9) reduces to

$$\frac{1}{c(t)}\left[c(t)y'(t)\right]'+f(t,y,cy')=0 \tag{2.10}$$

$$y'(0)=0,\qquad y(1)=0$$

The solution of this singular non-linear boundary value problem exists and is unique. To compute the approximate solution by using finite difference method has been used.

3. SOLUTION AND INTERPRETATION OF THE MODEL

The temperature distribution in human dermal regions can be obtained by solving the above boundary value problem numerically discussed as

$$k_0\frac{d^2y}{dt^2}+\frac{2k_0}{y}\frac{dy}{dt}+\frac{nk_0}{y}\left(\frac{dy}{dt}\right)^2+\frac{q_1(37-y-T_H)R^2}{(y)^n}=0,\quad 0<t<1 \tag{3.1}$$

$$y'(0)=0,\qquad y(1)=0$$

Partitioning the interval $(0,1)$ into p subintervals with the length of each subinterval as $\frac{1}{p}$, then by the central differences, the equation (3.1) above for $i=0$ changes into the following form

$$2k_0y_1+\frac{q_1(37-y-T_H)R^2h^2}{(y_0)^n}-2k_0y_0=0,\quad 0<t<1 \tag{3.2}$$

and for $i=1,2,3,...(p-1)$, we have

$$\left(1+\frac{1}{i}\right)y_{i+1}+nk_0\frac{(y_{i+1}-y_{i-1})^2}{4y_i}+\frac{q_1(37-y-T_H)R^2}{(y)^n}-2k_0y_1+\left(1-\frac{1}{i}\right)y_{i-1} \tag{3.3}$$

where $p=\frac{1}{3}, q_1=0.000002, T_H^2=33.03+0.14(T_a-10)$,
$k_0=0.00009T_H(37-T_H)^{(1/3)}$ and T_a is an ambient temperature.

Making use of numerical technique to solve the resulting non-linear system of equations, the temperature distribution in human dermal regions at various environmental temperatures can be computed.

The heat generation calculated at the body core (brain and heart) is given in Figure-(3). It has been observed from Figure-(3) that the heat generation in these regions increases when the environment temperature decreases from 10^0C to 0^0C. The present solution is useful and realistic due to the fact that it maintains constant temperature at these regions.

4. DISCUSSION AND CONCLUSION

A Heat distribution model to understand the peripheral tissues temperature of human body has been formulated. The formulation is based on bio-heat equation with variable physiological parameters and appropriate boundary conditions. The boundary value problem determining the distribution of heat in the biological tissue has been solved using numerical method. The computation and simulation has been carried out by MATLAB software. The outcome of this study reflects some innovations in the existing models by means of temperature dependent. It is important to mention that the experiments have shown a great role of thermal conductivity on the thermal behaviour of the biological tissue. The numerical solution of the boundary value problem was carried out and the results were compared with the existing solutions as discussed by Thron [17]. The variation of temperature at various dermal layers of the underlying tissue is demonstrated in Figure (1). The curves (1) and (2) are due to thron[17] at two atmospheric temperatures $T_a = 0^0C, 10^0C$ while as the curves (3) and (4) are our results at the same ambient temperatures respectively. The figure reveals that there are gradual changes of temperature with radial distances. The study in this paper shows same temperature variation of tissues with that of Thron [17] with few significant differences in some results. The main reason for such differences is that Thron [17] treated the thermal conductivity as constant whereas it is temperature dependent in our case. Therefore, it may be said that the present study is comparatively more realistic. Similarly Figure-(2) is described at ambient temperatures $T_a = 21^0C, 25^0C$. It is evident from the Figures-$(1, 2)$ that the temperature variation in radial distances shows the same tendency with existing study of temperature variations on the human periphery. For temperature dependent thermal conductivity k, the present study shows some realistic values for the estimation of thermoregulation in human dermal layers. The value of thermal conductivity gradually increases from outer regions towards core with increase in temperature. Also a nonlinear singular differential equation with suitable boundary conditions was developed according to the equation-(14). The estimation of temperature distribution in human head for various ambient temperatures was done by various researchers including Khanday and Saxena[10], Thron [17]. They have realized that sub-lingual temperature of the head is not affected from the environmental temperature. Some realistic results were observed in this study while comparing these results with some experimental work carried out by Hodgson [5]. It is worthwhile to mention that the model can be used extensively in medical sciences and biomedical engineering.

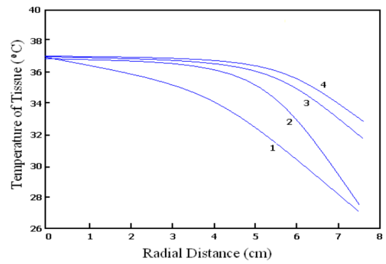

FIGURE 1. Temperature distribution in the tissue of human head versus radial distance for various values of environmental temperatures.

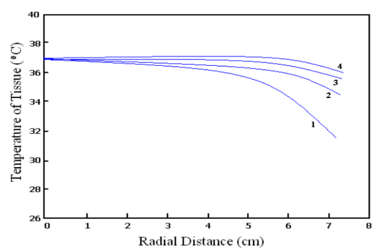

FIGURE 2. Temperature distribution in the tissue of human head versus radial distance for various values of environmental temperatures.

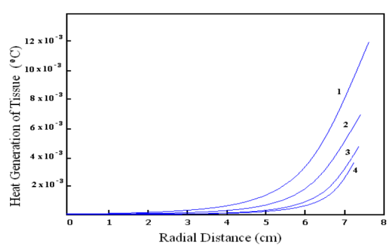

FIGURE 3. Heat generation of tissue of the human head versus radial distance for several environmental temperatures.

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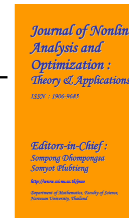

BEST APPROXIMATION AND BEST COAPPROXIMATION IN METRIC LINEAR SPACES

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ABSTRACT. In this paper, we discuss best approximation and best coapproximation in metric linear spaces. We obtain some results on the characterizations, existence and uniqueness of elements of best approximation and best coapproximation in metric linear spaces. We also study single-valuedness and linearity of metric projection and metric coprojection.

KEYWORDS : Proximinal set; coproximinal set; Chebyshev set; co-Chebyshev set; quasi-orthogonality.
AMS Subject Classification: 41A50, 41A52, 41A65.

1. INTRODUCTION AND PRELIMINARIES

A new tool in approximation theory, called best coapproximation by Papini and Singer [13], was introduced in normed linear spaces by C. Franchetti and M. Furi [1]. Subsequently, this theory has been developed to a large extent in normed linear spaces and in Hilbert spaces by C. Franchetti and M. Furi, H. Mazaheri, T.D. Narang, P.L. Papini and I. Singer, Geetha S. Rao and her coworkers, and by many others (see e.g. [1], [3], [4], [8], [13]-[16] and references cited therein). However, the situation in case of metric linear spaces and metric spaces is somewhat different. Although, some attempts have been made in this direction (see e.g. [9]-[12]) but still the theory is less developed as compared to the theory of best approximation. The present paper is also a step in this direction. This paper mainly deals with the characterizations of elements of best approximation and best coapproximation in metric linear spaces. Some results concerning the existence and uniqueness of elements of best approximation and best coapproximation have been discussed. We also study single-valuedness and linearity of metric projection and metric coprojection.

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Let G be a closed subset of a metric space (X, d). An element $g_0 \in G$ is called a *best approximation* (*best coapproximation*) to $x \in X$ if

$$d(x, g_0) \leq d(x, g)\ (d(g_0, g) \leq d(x, g))$$

for all $g \in G$. The set of all such $g_0 \in G$ is denoted by $P_G(x)(R_G(x))$. The set G is called *proximinal* (*coproximinal*) if $P_G(x)$ $(R_G(x))$ contains at least one element for every $x \in X$. If for each $x \in X$, $P_G(x)(R_G(x))$ has exactly one element, then the set G is called *Chebyshev* (*co − Chebyshev*).

We shall denote the set $\{x \in X : g_0 \in P_G(x)\}$ $(\{x \in X : g_0 \in R_G(x)\})$ by $P_G^{-1}(g_0)$ $(R_G^{-1}(g_0))$.

For a proximinal (coproximinal) subset G of X, the mapping $P_G(R_G) : X \to 2^G$($\equiv$ the collection of all subsets of G) defined by $P_G(x) = \{g_0 \in G : d(x, g_0) \leq d(x, g)\ for\ every\ g \in G\}$ $(R_G(x) = \{g_0 \in G : d(g_0, g) \leq d(x, g)\ for\ every\ g \in G\})$ is called *metric projection* (*metric coprojection*).

A linear space X together with a translation invariant metric d (i.e., $d(x + z, y + z) = d(x, y)\ for\ all\ x, y, z \in X$) such that addition and scalar multiplication are continuous in (X, d) is called a *metric linear space*.

Every normed linear space is a metric linear sapce but a metric linear space need not be normable (see [17], p.31-36).

Remarks 1.1.

(i) A proximinal subset of a metric space need not be coproximinal:
Let $X = \mathbb{R}^2$ and $G = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then G is a compact subset of $\mathbb{R}^2$ and hence proximinal. However, G is not coproximinal as $(0, 0) \in \mathbb{R}^2$ does not have any best coapproximation in G.

(ii) A coproximinal subset of a metric space need not be proximinal:
Let $X = \mathbb{R} - \{1\}$ and $M = (1, 2]$, then M is a coproximinal subset of X but is not proximinal.

(iii) A Chebyshev subset of a metric space need not be co-Chebyshev:
Let $X = \mathbb{R}$ and $G = [1, 2]$, then G is Chebyshev but not co-Chebyshev.

(iv) A co-Chebyshev subset of a metric space need not be Chebyshev:
Let $X = R^2$ with the metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ and $G = \{(x, y) \in R^2 : x = y\}$. Then G is a proximinal subset of X. We have $P_G(x, y) = \{\alpha(x, x) + (1 - \alpha)(y, y) : 0 \leq \alpha \leq 1\}$, i.e., G is not Chebyshev, but $R_G(x, y) = \{(\frac{x+y}{2}, \frac{x+y}{2})\}$, i.e., G is co-Chebyshev.

(v) The set $P_G^{-1}(g_0)(R_G^{-1}(g_0))$ is a closed set for every $g_0 \in G$.

(vi) If G is a subspace of a metric linear space (X, d) then $P_G^{-1}(0) \bigcap G = \{0\}$ and $R_G^{-1}(0) \bigcap G = \{0\}$, where $P_G^{-1}(0) = \{x \in X : 0 \in P_G(x)\}$ and $R_G^{-1}(0) = \{x \in X : 0 \in R_G(x)\}$.

(vii) If G is subspace of a metric linear space (X, d), then $g_0 \in P_G(x)$ $(g_0 \in R_G(x))$ if and only if $x - g_0 \in P_G^{-1}(0)$ $(x - g_0 \in R_G^{-1}(0))$ and $P_G(x + g) = P_G(x) + g$ $(R_G(x + g) = R_G(x) + g)$ for every $g \in G$.

(viii) If G is subspace of a metric linear space (X, d), then $d(g, 0) = d(g, R_G^{-1}(0))$ for every $g \in G$.

For a closed linear subspace G of a metric linear space (X, d), the *canonical mapping* π of X onto X/G is defined as $\pi(x) = x + G,\ x \in X$. This mapping π is linear, continuous and open (see [17], p.29).

Let (X, d) be a metric linear space and $x, y \in X$. We say that x is *orthogonal* to y, $x \perp y$ if $d(x, 0) \leq d(x, \alpha y)$ for every scalar α. For a subset G of X, we say that $G \perp x$ if $g \perp x$ for every $g \in G$.

2. MAIN RESULTS

This section mainly deals with the characterizations, existence and uniqueness of elements of best approximation and best coapproximation in metric linear spaces. We start with the following theorem which gives equivalent conditions under which coproximinal subspaces are co-Chebyshev:

Theorem 2.1. *Let G be a coproximinal subspace of a metric linear space (X, d), then the following are equivalent:*
(i) R_G is one-valued and linear.
(ii) $R_G^{-1}(0)$ is a linear subspace of X.

Proof. (i)⇒ (ii). Let $x, y \in R_G^{-1}(0)$ and $\alpha,\ \beta$ be scalars. Then $R_G(x) = \{0\}\ and\ R_G(y) = \{0\}$. Since R_G is linear, we have $R_G(\alpha x + \beta y) = \alpha R_G(x) + \beta R_G(y) = \{0\}$. This implies that $\alpha x + \beta y \in R_G^{-1}(0)$.
(ii)⇒ (i). Let $g_1, g_2 \in R_G(x)$. This gives $x - g_1, x - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is a subspace, we have $(x - g_1) - (x - g_2) \in R_G^{-1}(0)$, i.e., $g_2 - g_1 \in R_G^{-1}(0)$. This gives $g_2 - g_1 \in R_G^{-1}(0) \bigcap G = \{0\}$ and so $g_1 = g_2$. Hence R_G is one-valued.

Let $x, y \in X$ and α, β be scalars. Suppose $g_1 \in R_G(x)$ and $g_2 \in R_G(y)$. This gives $x - g_1,\ y - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is a subspace, we have $\alpha(x - g_1) + \beta(y - g_2) \in R_G^{-1}(0)$, i.e., $(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) \in R_G^{-1}(0)$. Now, $R_G(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) = R_G(\alpha x + \beta y - (\alpha g_1 + \beta g_2)) = \{0\}$, as R_G is single-valued. Hence $R_G(\alpha x + \beta y) = \alpha g_1 + \beta g_2 = \alpha R_G(x) + \beta R_G(y)$. ☐

Proceeding on similar lines, we obtain the following theorem which gives equivalent conditions under which proximinal subspaces are Chebyshev:

Theorem 2.2. *Let G be a proximinal subspace of a metric linear space (X, d), then the following are equivalent:*
(i) P_G is one-valued and linear.
(ii) $P_G^{-1}(0)$ is a linear subspace of X.

Remarks 2.3. *(i) For normed linear spaces, Theorem 2.1 was proved in* [12] *and Theorem 2.2 in* [2].
(ii) If we take G to be only a proximinal subset containing zero, then $P_G^{-1}(0)$ is a subspace but G need not be Chebyshev.

Example 2.4. *Let $X = \mathbb{R}$ with usual metric and $G = (-\infty, 1] \bigcup [2, \infty)$. Then $P_G^{-1}(0) = \{0\}$ is a subspace but G is not a Chebyshev set.*

Example 2.4 also shows that one of the main result (Theorem 2.6) proved in [5] is not valid.

We require the following lemmas proved in [12] ([10]) for our next results:

Lemma 2.5. *Let G be a linear subspace of a metric linear space (X, d), then the following are equivalent:*
(i) G is proximinal (coproximinal).
(ii) $X = G + P_G^{-1}(0)$ $(X = G + R_G^{-1}(0))$.

Lemma 2.6. *Let G be a linear subspace of a metric linear space (X, d) then the following are equivalent:*
(i) G is Chebyshev (co-Chebyshev).
(ii) $X = G \bigoplus P_G^{-1}(0)$ $(X = G \bigoplus R_G^{-1}(0))$, where $\bigoplus$ means that the sum decomposition of each $x \in X$ is unique.

The following theorem gives necessary and sufficient condition for the metric coprojection to be linear.

Theorem 2.7. *Let G be a co-Chebyshev subspace of a metric linear space (X, d), then the following are equivalent:*
(i) R_G is linear.
(ii) $R_G^{-1}(0)$ is a subspace.
(iii) $R_G^{-1}(0)$ contains a subspace N for which $X = G \bigoplus N$.

Proof. (i)$\Rightarrow$ (ii) follows from Theorem 2.1.
(ii)$\Rightarrow$(iii) is obvious by Lemma 2.6.
(iii)$\Rightarrow$ (i). Let $x, y \in X$ and α, β be scalars.. Then $x = g_1 + n_1$ and $y = g_2 + n_2$ for some $g_1, g_2 \in G$ and $n_1, n_2 \in N$. Therefore, $x - g_1, y - g_2 \in N$. Since N is a subspace, we have $\alpha(x - g_1) + \beta(y - g_2) \in N \subseteq R_G^{-1}(0)$ for all scalars α, β. This gives $(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) \in R_G^{-1}(0)$, i.e., $R_G(\alpha x + \beta y - (\alpha g_1 + \beta g_2)) = \{0\}$(as G is co-Chebyshev), i.e., $R_G(\alpha x + \beta y) = \alpha g_1 + \beta g_2 = \alpha R_G(x) + \beta R_G(y)$. □

Proceeding on similar lines, we obtain the following theorem which gives necessary and sufficient conditions for the metric projection to be linear.

Theorem 2.8. *Let G be a Chebyshev subspace of a metric linear space (X, d), then the following are equivalent:*
(i) P_G is linear.
(ii) $P_G^{-1}(0)$ is a subspace.
(iii) $P_G^{-1}(0)$ contains a subspace N for which $X = G \bigoplus N$.

Remarks 2.9. *For normed linear spaces, Theorem 2.7 was proved in* [15] *and Theorem 2.8 in* [2].

The following theorem gives necessary and sufficient conditions for the set $R_G^{-1}(0)$ to be Chebyshev:

Theorem 2.10. *Let G be a coproximinal subspace of a metric linear space (X, d). If $R_G^{-1}(0)$ is an additive group then the following are equivalent:*
(i) $R_G^{-1}(0)$ is a Chebyshev set.
(ii) $G = \{x \in X : d(x, R_G^{-1}(0)) = d(x, 0)\}$.

Proof. (i)$\Rightarrow$ (ii). Suppose $x \in X$ is such that

$$d(x, R_G^{-1}(0)) = d(x, 0). \tag{2.1}$$

Then there exist $g \in G$ such that $\breve{g} = x - g \in R_G^{-1}(0)$ (as G is a coproximinal subspace and so by Lemma 2.5, $X = G + R_G^{-1}(0)$). Therefore $d(x, \breve{g}) = d(g, 0) = d(g, R_G^{-1}(0)) = d(x - \breve{g}, R_G^{-1}(0)) = d(x, R_G^{-1}(0))$, as $R_G^{-1}(0)$ is an additive group. This implies $\breve{g} \in P_{R_G^{-1}(0)}(x)$. From (2.1), we have $0 \in P_{R_G^{-1}(0)}(x)$. Since $R_G^{-1}(0)$ is Chebyshev, we have $\breve{g} = 0$ and so $x = g \in G$. Also, by Remark 1.1 (8), we have $d(g, 0) = d(g, R_G^{-1}(0))$ for all $g \in G$. Consequently, the result follows.
(ii)$\Rightarrow$ (i). Let $x \in X$. Then there exist $g \in G$ such that $\breve{g} = x - g \in R_G^{-1}(0)$. Therefore

$$d(x, R_G^{-1}(0)) = d(x - \breve{g}, R_G^{-1}(0)) = d(x, \breve{g}) \ (as\ x - \breve{g} \in G)$$

i.e., $R_G^{-1}(0)$ is proximinal.

Suppose that for some $x \in X$, there exist $\breve{g_1}, \breve{g_2} \in P_{R_G^{-1}(0)}(x)$, i.e.,

$$d(x, \breve{g_1}) = d(x, R_G^{-1}(0)) = d(x - \breve{g_1}, R_G^{-1}(0))$$

and

$$d(x, \breve{g_2}) = d(x, R_G^{-1}(0)) = d(x - \breve{g_2}, R_G^{-1}(0))$$

Then by hypothesis, $x - \breve{g_1}, x - \breve{g_2} \in G$. Since G is a subspace, $(x - \breve{g_1}) - (x - \breve{g_2}) \in G$, i.e., $\breve{g_1} - \breve{g_2} \in G$. Also $R_G^{-1}(0)$ is an additive group, we have $\breve{g_1} - \breve{g_2} \in$

$R_G^{-1}(0)$. Therefore, $\breve{g}_1 - \breve{g}_2 \in R_G^{-1}(0) \bigcap G = \{0\}$ and so $\breve{g}_2 = \breve{g}_1$. Hence $R_G^{-1}(0)$ is Chebyshev. □

Let G be a linear subspace of a metric linear space (X, d). We say that G is a $quasi - orthogonal$ set if $G \perp P_G^{-1}(0)$, i.e., $g \perp g'$ for all $g \in G$ and $g' \in P_G^{-1}(0)$.

Concerning quasi-orthogonality of subspaces, we have

Lemma 2.11. *Let G be a quasi-orthogonal subspace of a metric linear space (X, d) then $d(g, 0) = d(g, P_G^{-1}(0))$.*

Proof. Since G is a quasi-orthogonal subspace, $G \perp P_G^{-1}(0)$, i.e., $g \perp g'$ for all $g \in G$ and $g' \in P_G^{-1}(0)$, i.e., $d(g, 0) \leq d(g, \alpha g')$ for all $g \in G,\ g' \in P_G^{-1}(0)$ and all scalars α. Taking $\alpha = 1$, we obtain $d(g, 0) \leq d(g, g')$ for all $g \in G,\ g' \in P_G^{-1}(0)$. This implies $d(g, 0) \leq \inf_{g' \in P_G^{-1}(0)} d(g, g') = d(g, P_G^{-1}(0))$ for all $g \in G$. Also, $d(g, P_G^{-1}(0)) = \inf_{g' \in P_G^{-1}(0)} d(g, g') \leq d(g, 0)$ for all $g \in G$. Consequently, $d(g, 0) = d(g, P_G^{-1}(0))$. □

Using Lemma 2.11, we have the following result which characterizes Chebyshevity of $P_G^{-1}(0)$:

Theorem 2.12. *Let G be a proximinal, quasi-orthogonal subspace of a metric linear space (X, d). If $P_G^{-1}(0)$ is an additive group then the following are equivalent:*
(i) $P_G^{-1}(0)$ is a Chebyshev set.
(ii) $G = \{x \in X : d(x, P_G^{-1}(0)) = d(x, 0)\}$.

Proof. The proof runs on similar lines as that of Theorem 2.10. □

Remarks 2.13.

(i) For normed linear spaces, Theorem 2.12 was proved in [5].
(ii) It was shown in [11] that if G is a proximinal (coproximinal) subspace of a metric linear space (X, d) and $P_G^{-1}(0)$ $(R_G^{-1}(0))$ is a convex set, then G is Chebyshev (co-Chebyshev). If we take G to be a proximinal (coproximinal) subset containing zero instead of a subspace then G need not be Chebyshev (co-Chebyshev). Example 2.4 and the following example confirm these facts.

Example 2.14. *Let $X = R$ and $G = [0, \infty)$, then $R_G^{-1}(0) = (-\infty, 0]$ and $R_G(-1) = [0, 1]$, i.e., $R_G^{-1}(0)$ is a convex set but G is not co-Chebyshev.*

Concerning the coproximinality of quotient spaces, we have

Lemma 2.15. *Let G be a closed linear subspace of a metric linear space (X, d) and F a coproximinal subspace of X containing G. Then F/G is coproximinal in X/G.*

Proof. Let $x + G \in X/G,\ x \in X$, and f be a best coapproximation to x. We prove that $f + G$ is a best coapproximation to $x + G$. Suppose it is not, then there exist $f' + G \in F/G$ such that $d(f + G, f' + G) > d(x + G, f' + G)$, $i.e.,$ $\inf_{g \in G} d(x - f', g) < d(f - f', G)$. Then there exist some $g_0 \in G$ such that

$$d(x - f', g_0) < d(f - f', G) \leq d(f - f', g_0)$$

$i.e.,$ $d(x, f' + g_0) < d(f, f' + g_0)$. Thus f is not a best coapproximation to x from F, a contradiction. Hence $f + G$ is a best coapproximation to $x + G$ and consequently, F/G is coproximinal in X/G. □

Concerning the coproximinality of F, we have

Lemma 2.16. *Let G be a proximinal subspace of a metric linear space (X, d) and F a subspace of X containing G. If F/G is coproximinal in X/G then F is coproximinal in X.*

Proof. Let $x \in X$ be arbitrary, then $x + G \in X/G$. Since F/G is coproximinal in X/G, there is some $f + G \in R_{F/G}(x + G)$, i.e., $d(f + G, f' + G) \leq d(x + G, f' + G)$ $for\ evrey$ $f' + G \in F/G$. Since G is proximinal, there exist $g_0 \in G$ such that $d(f - f', g_0) \leq d(x - f', G) \leq d(x - f', 0)$ $for\ every$ $f' \in F$. This gives $f - g_0 \in R_F(x)$. Hence F is coproximinal in X. □

Using Lemmas 2.15 and 2.16, we obtain the following result:

Theorem 2.17. *Let G be a proximinal subspace of a metric linear space (X, d) and F a coproximinal subspace of X containing G. If $\pi : X \to X/G$ is the canonical map, then $\pi(R_F(x)) = R_{F/G}(x + G)$.*

Concerning the co-Chebyshevity of quotient spaces, we have

Theorem 2.18. *Let G be a proximinal subspace of a metric linear space (X, d) and F a coproximinal subspace containing G. If $R_F^{-1}(0)$ is a convex set then F/G is a co-Chebyshev subspace of X/G.*

Proof. Using Theorem 2.17, we have $\pi(R_F(x)) = R_{F/G}(x + G)$.

In view of Remark 2.13, it is sufficient to prove that $R_{F/G}^{-1}(G)$ is convex. For this, let $x + G, y + G \in R_{F/G}^{-1}(G)$ and $0 < \lambda < 1$. Since $G \in R_{F/G}(x + G)$ and $G \in R_{F/G}(y + G)$, there exist $g \in R_F(x)$ and $h \in R_F(y)$ such that $\pi(g) = G = \pi(h)$. Therefore, $x - g, y - h \in R_F^{-1}(0)$ (as $g \in R_F(x),\ h \in R_F(y)$).

Since $R_F^{-1}(0)$ is a convex set, we have $\lambda(x - g) + (1 - \lambda)(y - h) \in R_F^{-1}(0)$ $i.e.,$ $d(0, f) \leq d(\lambda(x - g) + (1 - \lambda)(y - h), f)$ $for\ all$ $f \in F$. This implies $d(\lambda g + (1 - \lambda)h, \lambda g + (1 - \lambda)h + f) \leq d(\lambda x + (1 - \lambda)y, f + \lambda g + (1 - \lambda)h)$ $for\ all$ $f \in F$. Therefore, $\lambda g + (1 - \lambda)h \in R_F(\lambda x + (1 - \lambda)y)$.

Also $\pi(\lambda g + (1 - \lambda h)) = G$. Therefore, $G \in R_{F/G}(\lambda x + (1 - \lambda)y + G)$, i.e., $\lambda(x + G) + (1 - \lambda)(y + G) \in R_{F/G}^{-1}(G)$ and so $R_{F/G}^{-1}(G)$ is convex. Hence F/G is co-Chebyshev in X/G. □

Remarks 2.19. *For normed linear spaces, Lemmas 2.15, 2.16 and Theorems 2.17, 2.18 were proved in* [3].

Proceeding on similar lines, we obtain the following results on best approximation in quotient spaces. For normed linear spaces these results were proved in [5] and [6]:

(i) Let G be a closed linear subspace of a metric linear space (X, d) and F a proximinal subspace of X containing G. Then F/G is proximinal in X/G.
(ii) Let G be a proximinal subspace of a metric linear space (X, d) and F a subspace of X containing G. If F/G is proximinal in X/G then F is proximinal in X.
(iii) Let G be a proximinal subspace of a metric linear space (X, d) and F a proximinal subspace of X containing G. If $\pi : X \to X/G$ is the canonical map then $\pi(P_F(x)) = P_{F/G}(x + G)$.
(iv) Let G be a proximinal subspace and F a proximinal subspace of X containing G. If $P_F^{-1}(0)$ is a convex set then F/G is a Chebyshev subspace of X/G.

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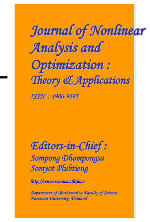

EXISTENCE AND UNIQUENESS OF COUPLED BEST PROXIMITY POINT IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. In this paper we utilize a generalized almost contractive mapping to establish some coupled best proximity point results which are global optimization results of finding the minimum distances between two sets. The results are obtained in metric spaces with a partial ordering defined therein. There is a blending of analytic and order theoretic approaches in the proofs. We illustrate the main theorem through an example.

KEYWORDS : Metric space; partial order; almost contraction; coupled best proximity point.
AMS Subject Classification: 47H10, 54H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Best proximity point results are related to the problem of finding minimum distances which is by itself a classical problem considered in many areas of mathematics. It occupies an important position in the calculus of variation [16]. In geometrical studies it is related to the concept of geodesic [3]. In our case the objects are subsets of metric spaces. The minimum distance between pairs of subsets are realized by utilizing best proximity points of non-self mappings.

Technically, $(X,\ d)$ denotes a metric space throughout the paper and $A,\ B \subseteq X$. We use the following notations.

$$D(x,\ B) = \inf\,\{d(x,\ b):\ b \in B\},\ \text{where } x \in X,$$

$$d(A,\ B) = \inf\,\{d(a,\ b) : a \in A \text{ and } b \in B\},$$

$$A_0 = \{a \in A : d(a,\ b) = d(A,\ B) \text{ for some } b \in B\},$$

$$B_0 = \{b \in B : d(a,\ b) = d(A,\ B) \text{ for some } a \in A\}.$$

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It is to be noted that for every $a \in A_0$ there exists $b \in B_0$ such that $d(a,\ b) = d(A,\ B)$ and conversely, for every $b' \in B_0$ there exists $a' \in A_0$ such that $d(a',\ b') = d(A,\ B)$.

Let $T\colon A \longrightarrow B$ be a non-self mapping. Then $x^* \in A$ is a best proximity point of T if $d(x^*,\ Tx^*) = d(A,\ B)$ [10]. A best proximity point reduces to a fixed point in case where $A = B$. A best proximity point problem can be described as the problem of finding an optimal approximate solution of the fixed point equation $x = Tx$ although the exact solution need not exist. This is an approach to the global optimization problem of finding minimum distance between two sets by minimizing globally the quantity $d(x,\ Tx)$ such that the minimum value $d(A,\ B)$ is attained at some point.

At this, point it is pertinent to point out the difference between proximity point results and best approximation results. Unlike former, the best approximation results are not necessarily optimal. For instance, the famous Ky Fan's best approximation result is not an optimality result [11].

Eldred et al [10] introduced best proximity points. Interest in results associated with this concept increased rapidly which has resulted in the publication of a good number of papers on this topic. Side by side, coupled fixed point theorems also occupied large research interest in recent times with the publication of results like [5, 6, 12, 15, 17]. Coupled mapping was utilized in research on best proximity pairs in the work of [20] and was followed by works like [13, 14, 18, 19]. Our purpose is to establish coupled best proximity point theorems in a metric space where a partial order is defined. In the sequel we use an almost contraction like inequality. These inequalities featured in the study of generalized contractions originated by Berinde [4]. This category of inequalities has been utilized in a good number of papers which are predominantly on fixed point studies, some instances being [1, 2, 7, 8, 9]. We utilize this idea for finding best proximity pairs through coupled maps. Precisely, we utilize a generalized almost contraction mapping for the purpose of obtaining coupled best proximity points. The above mentioned mapping is assumed to be defined from one set $A \times A$ to the other set B. Then under suitable conditions, by applying fixed point methodologies, we obtained a coupled best proximity point of the above mentioned mapping which realizes the minimum distance. The main result has one corollary and an illustrative example. Separate order theoretic condition is imposed to ensure the uniqueness of the coupled best proximity point in the main result.

The following are the requisite mathematical concepts for the discussions in this paper.

Throughout this paper, $(X,\ d,\ \preceq)$ denotes a partially ordered metric space where $\preceq$ is a partially order on the metric space $(X,\ d)$.

Definition 1.1 ([12]). A mapping $g\colon A \times A \longrightarrow A$ is said to have the mixed monotone property if

$$u,\ v \in A,\ \ u \preceq v \Longrightarrow g(u,\ y) \preceq g(v,\ y), \quad \text{for all } y \in A;$$

and

$$p,\ q \in A,\ \ p \preceq q \Longrightarrow g(x,\ p) \succeq g(x,\ q), \quad \text{for all } x \in A.$$

Definition 1.2 ([14]). A mapping $g\colon A \times A \longrightarrow B$ is said to have proximal mixed monotone property if for all $x,\ y \in A$

$$\left.\begin{array}{l} u \preceq v \\ d(a,\ g(u,\ y)) = d(A,\ B), \\ d(b,\ g(v,\ y)) = d(A,\ B) \end{array}\right\} \Rightarrow a \preceq b$$

and

$$\left.\begin{array}{l} p \preceq q \\ d(c,\ g(x,\ p)) = d(A,\ B), \\ d(d,\ g(x,\ q)) = d(A,\ B) \end{array}\right\} \Rightarrow c \succeq d,$$

where $u,\ v,\ p,\ q,\ a,\ b,\ c,\ d \in A$.

If $A = B$ in the above definition, the notion of the proximal mixed monotone property reduces to that of the mixed monotone property.

Definition 1.3. A mapping $F\colon A \times A \longrightarrow B$ is said to have proximal mixed monotone property on $A_0 \times A_0$ if for all $x,\ y \in A_0$

$$\left.\begin{array}{l} u \preceq v \\ d(a,\ g(u,\ y)) = d(A,\ B), \\ d(b,\ g(v,\ y)) = d(A,\ B) \end{array}\right\} \Rightarrow a \preceq b$$

and

$$\left.\begin{array}{l} p \preceq q \\ d(c,\ g(x,\ p)) = d(A,\ B), \\ d(d,\ g(x,\ q)) = d(A,\ B) \end{array}\right\} \Rightarrow c \succeq d,$$

where $u,\ v,\ p,\ q,\ a,\ b,\ c,\ d \in A_0$.

Lemma 1.4 ([14])**.** *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $A,\ B$ are nonempty subsets of X. Assume A_0 is non-empty. Let $g\colon A \times A \longrightarrow B$ be a mapping such that $g(A_0 \times A_0) \subseteq B_0$ and g has proximal mixed monotone property. Then for all $u,\ v,\ p,\ q,\ w \in A_0$*

$$\left.\begin{array}{l} u \preceq v \text{ and } p \succeq q \\ d(v,\ g(u,\ p)) = d(A,\ B), \\ d(w,\ g(v,\ q)) = d(A,\ B) \end{array}\right\} \Rightarrow v \preceq w.$$

Lemma 1.5 ([14])**.** *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $A,\ B$ are nonempty subsets of X. Assume A_0 is non-empty. Let $g\colon A \times A \longrightarrow B$ be a mapping such that $g(A_0 \times A_0) \subseteq B_0$ and g has proximal mixed monotone property. Then for all $u,\ v,\ p,\ q,\ z \in A_0$*

$$\left.\begin{array}{l} u \preceq v \text{ and } p \succeq q \\ d(q,\ g(p,\ u)) = d(A,\ B), \\ d(z,\ g(q,\ v)) = d(A,\ B) \end{array}\right\} \Rightarrow q \succeq z.$$

Definition 1.6 ([20])**.** A point $(a,\ b) \in A \times A$ is said to be a coupled best proximity point of the mapping $g\colon A \times A \longrightarrow B$ if $d(a,\ g(a,\ b)) = d(A,\ B)$ and $d(b,\ g(b,\ a)) = d(A,\ B)$.

2. MAIN RESULTS

Theorem 2.1. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $(A,\ B)$ be a pair of non-empty closed subsets of X such that A_0 is non-empty and closed. Let $F\colon A \times A \longrightarrow B$ be a mapping such that $F(A_0 \times A_0) \subseteq B_0$ and F has proximal mixed monotone property on $A_0 \times A_0$. Suppose that there exist nonnegative real numbers a, b and L with $a + b < 1$ such that for all $x,\ y,\ u,\ v,\ p,\ q \in A_0$*

$$\left.\begin{array}{l} x \preceq u \text{ and } y \succeq v, \\ d(p,\ F(x,\ y)) = d(A,\ B), \\ d(q,\ F(u,\ v)) = d(A,\ B) \end{array}\right\} \Longrightarrow d(p,\ q) \leq N(x,\ y,\ u,\ v,\ p,\ q),$$

where

$N(x,\ y,\ u,\ v,\ p,\ q) = a\, d(x,\ u) + b\, d(y,\ v) + L\, min\, \{d(p,\ u),\ d(q,\ x),\ d(p,\ x),\ d(q,\ u)\}$.
Suppose either
(a) F is continuous or
(b) X has the following properties:
(i) if a nondecreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$, for all $n \geq 0$;
(ii) if a nonincreasing sequence $\{y_n\} \longrightarrow y$, then $y \preceq y_n$, for all $n \geq 0$.
Also, suppose that there exist elements $(x_0,\ y_0)$, $(x_1,\ y_1) \in A_0 \times A_0$ such that $d(x_1,\ F(x_0,\ y_0)) = d(A,\ B)$ and $d(y_1,\ F(y_0,\ x_0)) = d(A,\ B)$ with $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Then F has a coupled best proximity point in $A_0 \times A_0$.

Proof. By the hypothesis of the theorem there exist elements $(x_0,\ y_0)$, $(x_1,\ y_1) \in A_0 \times A_0$ such that $x_0 \preceq x_1$ and $y_0 \succeq y_1$ and

$$d(x_1,\ F(x_0,\ y_0)) = d(A,\ B) \ \text{ and } \ d(y_1,\ F(y_0,\ x_0)) = d(A,\ B). \tag{2.1}$$

Since $F(A_0 \times A_0) \subseteq B_0$, there exists an element $(x_2,\ y_2) \in A_0 \times A_0$ such that

$$d(x_2,\ F(x_1,\ y_1)) = d(A,\ B) \ \text{ and } \ d(y_2,\ F(y_1,\ x_1)) = d(A,\ B). \tag{2.2}$$

Hence by the Lemmas 1.4 and 1.5, we have $x_1 \preceq x_2$ and $y_1 \succeq y_2$.

Continuing this process, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in A_0 such that

$$x_n \preceq x_{n+1} \text{ and } \ y_n \succeq y_{n+1} \ \text{ for all } \ n \geq 0 \tag{2.3}$$

and

$$d(x_{n+1},\ F(x_n,\ y_n)) = d(A,\ B) \ \text{ and } \ d(y_{n+1},\ F(y_n,\ x_n)) = d(A,\ B). \tag{2.4}$$

Now,

$$\left.\begin{array}{l} x_n \preceq x_{n+1} \text{ and } \ y_n \succeq y_{n+1}, \\ d(x_{n+1},\ F(x_n,\ y_n)) = d(A,\ B), \\ d(x_{n+2},\ F(x_{n+1},\ y_{n+1})) = d(A,\ B) \end{array}\right\} \Longrightarrow$$

$$d(x_{n+1},\ x_{n+2}) \leq N(x_n,\ y_n,\ x_{n+1},\ y_{n+1},\ x_{n+1},\ x_{n+2}), \tag{2.5}$$

where

$$\begin{aligned} N(x_n,\ y_n,\ x_{n+1},\ y_{n+1},\ x_{n+1},\ x_{n+2}) &= a\ d(x_n,\ x_{n+1}) + b\ d(y_n,\ y_{n+1}) \\ &\quad + L \min \{d(x_{n+1},\ x_{n+1}),\ d(x_{n+2},\ x_n),\ d(x_{n+1},\ x_n),\ d(x_{n+2},\ x_{n+1})\} \\ &= a\ d(x_n,\ x_{n+1}) + b\ d(y_n,\ y_{n+1}). \end{aligned}$$

So, we have

$$d(x_{n+1},\ x_{n+2}) \leq a\ d(x_n,\ x_{n+1}) + b\ d(y_n,\ y_{n+1}). \tag{2.6}$$

Similarly, it can be obtained that

$$d(y_{n+1},\ y_{n+2}) \leq a\ d(y_n,\ y_{n+1}) + b\ d(x_n,\ x_{n+1}). \tag{2.7}$$

Combination of (2.6) and (2.7) implies that

$$d(x_{n+1},\ x_{n+2}) + d(y_{n+1},\ y_{n+2}) \leq (a+b)\ [d(x_n,\ x_{n+1}) + d(y_n,\ y_{n+1})]. \tag{2.8}$$

Let $r_n = d(x_n,\ x_{n+1}) + d(y_n,\ y_{n+1})$ and $k = a + b$. By repeated application of (2.8), we get

$$0 \leq r_n \leq k\ r_{n-1} \leq k^2\ r_{n-2} \leq ... \leq k^n\ r_0. \tag{2.9}$$

Let $m,\ n \in \mathbb{N}$ with $m < n$. Then

$d(x_m,\ x_n) + d(y_m,\ y_n) \leq d(x_m,\ x_{m+1}) + d(y_m,\ y_{m+1}) + d(x_{m+1},\ x_{m+2}) + d(y_{m+1},\ y_{m+2}) + \cdots + d(x_{n-1},\ x_n) + d(y_{n-1},\ y_n)$

$$\leq r_m + r_{m+1} + ... + r_{n-1} \ \leq \ [k^m + k^{m+1} + ... + k^{n-1}] r_0$$

$$\leq \frac{k^m}{1-k}\, r_0 \longrightarrow 0 \text{ as } m \longrightarrow \infty \text{ (since } 0 \leq k < 1).$$

Then it follows that

$$\lim_{m,\, n\to\infty} d(x_m,\, x_n) = 0 \ \text{ and } \ \lim_{m,\, n\to\infty} d(y_m,\, y_n) = 0,$$

which implies that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in A_0. Since A_0 is a closed subset of the complete metric space $(X,\, d)$, A_0 is also complete. Now from the completeness of A_0, there exists $x^*,\, y^* \in A_0$ such that

$$\lim_{n\to\infty} x_n = x^*; \ \text{ that is, } \lim_{n\to\infty} d(x_n,\, x^*) = 0 \tag{2.10}$$

and

$$\lim_{n\to\infty} y_n = y^*; \ \text{ that is, } \lim_{n\to\infty} d(y_n,\, x^*) = 0. \tag{2.11}$$

Let the condition (a) holds.

Taking limit as $n \longrightarrow \infty$ in (2.4) and using (2.10), (2.11) and the continuity of F, we have

$$d(x^*,\, F(x^*,\, y^*)) = d(A,\, B) \ \text{ and } \ d(y^*,\, F(y^*,\, x^*)) = d(A,\, B).$$

Therefore, $(x^*,\, y^*)$ is a coupled best proximity point of F.

Next we suppose that the condition (b) holds.

Using the condition (b) of the theorem, (2.3), (2.10) and (2.11), we have

$$x_n \preceq x^* \ \text{ and } \ y_n \succeq y^*, \ \text{ for all } \ n \geq 0. \tag{2.12}$$

Since $x^*,\, y^* \in A_0$ and $F(A_0 \times A_0) \subseteq B_0$, there exists $u,\, v \in A_0$ such that

$$d(u,\, F(x^*,\, y^*)) = d(A,\, B) \ \text{ and } \ d(v,\, F(y^*,\, x^*)) = d(A,\, B). \tag{2.13}$$

By (2.4), (2.12) and (2.13)

$$\left.\begin{array}{l} x_n \preceq x^* \text{ and } \ y_n \succeq y^*, \\ d(x_{n+1},\, F(x_n,\, y_n)) = d(A,\, B), \\ d(u,\, F(x^*,\, y^*)) = d(A,\, B) \end{array}\right\} \Longrightarrow$$

$$d(x_{n+1},\, u) \leq N(x_n,\, y_n,\, x^*,\, y^*,\, x_{n+1},\, u), \tag{2.14}$$

where

$$\begin{aligned} N(x_n,\, y_n,\, x^*,\, y^*,\, x_{n+1},\, u) &= a\, d(x_n,\, x^*) + b\, d(y_n,\, y^*) \\ &\quad + L \min\{d(x_{n+1},\, x^*),\, d(u,\, x_n),\, d(x_{n+1},\, x_n),\, d(u,\, x^*)\}. \end{aligned}$$

Using (2.10) and (2.11), we obtain

$$\lim_{n\to\infty} N(x_n,\, y_n,\, x^*,\, y^*,\, x_{n+1},\, u) = 0. \tag{2.15}$$

Taking the limit as $n \longrightarrow \infty$ in (2.14), using (2.10) and (2.15), we have $d(x^*,\, u) \leq 0$, which implies that $d(x^*,\, u) = 0$; that is, $u = x^*$.

Again, by (2.4), (2.12) and (2.13)

$$\left.\begin{array}{l} y^* \preceq y_n \text{ and } x^* \succeq x_n, \\ d(v,\, F(y^*,\, x^*)) = d(A,\, B), \\ d(y_{n+1},\, F(y_n,\, x_n)) = d(A,\, B) \end{array}\right\} \Longrightarrow$$

$$d(v,\, y_{n+1}) \leq N(y^*,\, x^*,\, y_n,\, x_n,\, v,\, y_{n+1}), \tag{2.16}$$

where

$$\begin{aligned} N(y^*,\, x^*,\, y_n,\, x_n,\, v,\, y_{n+1}) &= a\, d(y^*,\, y_n) + b\, d(x^*,\, x_n) \\ &\quad + L \min\{d(v,\, y_n),\, d(y_{n+1},\, y^*),\, d(v,\, y^*),\, d(y_{n+1},\, y_n)\}. \end{aligned}$$

Using (2.10) and (2.11), we obtain

$$\lim_{n\to\infty} N(y^*,\, x^*,\, y_n,\, x_n,\, v,\, y_{n+1}) = 0. \tag{2.17}$$

Taking the limit as $n \longrightarrow \infty$ in (2.16), using (2.11) and (2.17), we have $d(v,\ y^*) \leq 0$ which implies that $d(v,\ y^*) = 0$; that is, $v = y^*$.
Since $u = x^*$ and $v = y^*$, we have from (2.13) that

$$d(x^*,\ F(x^*,\ y^*)) = d(A,\ B) \ \text{ and } \ d(y^*,\ F(y^*,\ x^*)) = d(A,\ B).$$

Hence $(x^*,\ y^*)$ is a coupled best proximity point of F. □

With the help of partially ordered set $(X,\ \preceq)$ we endow the product space $X \times X$ with the following partial order:
for $(x,\ y),\ (u,\ v) \in X \times X,\ \ (u,\ v) \preceq (x,\ y) \Leftrightarrow x \succeq u,\ y \preceq v.$

Theorem 2.2. *In addition to the hypotheses of Theorems 2.1, suppose that for every $(x,\ y),\ (x^*,\ y^*) \in A_0 \times A_0$ there exists $(u,\ v) \in A_0 \times A_0$ such that $(u,\ v)$ is comparable to $(x,\ y)$ and $(x^*,\ y^*)$. Then F has a unique coupled best proximity point.*

Proof. From Theorem 2.1, the set of coupled best proximity points F is non-empty. Suppose that $(x,\ y)$ and $(x^*,\ y^*)$ are coupled best proximity points of F. So

$$d(x,\ F(x,\ y)) = d(A,\ B), \quad d(y,\ F(y,\ x) = d(A,\ B), \tag{2.18}$$

and

$$d(x^*,\ F(x^*,\ y^*)) = d(A,\ B), \ \ d(y^*,\ F(y^*,\ x^*) = d(A,\ B). \tag{2.19}$$

Now, we show that $(x,\ y) = (x^*,\ y^*)$.
By the assumption, there exists $(u,\ v) \in A_0 \times A_0$ such that $(u,\ v)$ is comparable with $(x,\ y)$ and $(x^*,\ y^*)$.
Put $(u_0,\ v_0) = (u,\ v)$.
Suppose that

$$(u_0,\ v_0) \preceq (x,\ y); \text{ that is, } \ u_0 \preceq x,\ \ v_0 \succeq y \text{ (the proof is similar in other case).} \tag{2.20}$$

Since $u = u_0,\ v = v_0 \in A_0$ and $F(A_0 \times A_0) \subseteq B_0$, there exists $(u_1,\ v_1) \in A_0 \times A_0$ such that

$$d(u_1,\ F(u_0,\ v_0)) = d(A,\ B) \ \text{ and } \ d(v_1,\ F(v_0,\ u_0)) = d(A,\ B). \tag{2.21}$$

From (2.18), (2.19), (2.20) and (2.21), we have

$$\left.\begin{array}{lcl} u_0 \preceq x \text{ and } v_0 \succeq y & & u_0 \preceq x \text{ and } v_0 \succeq y \\ d(u_1,\ F(u_0,\ v_0)) = d(A,\ B) & \text{and} & d(v_1,\ F(v_0,\ u_0)) = d(A,\ B) \\ d(x,\ F(x,\ y)) = d(A,\ B), & & d(y,\ F(y,\ x)) = d(A,\ B). \end{array}\right\}$$

Since $F(A_0 \times A_0) \subseteq B_0$ and $x,\ v_0 \in A_0$, there exists $x_1 \in A_0$ such that

$$d(x_1,\ F(x,\ v_0)) = d(A,\ B).$$

Now we have

$$\left.\begin{array}{lcl} u_0 \preceq x, & & y \preceq v_0, \\ d(u_1,\ F(u_0,\ v_0)) = d(A,\ B) & \text{and} & d(x,\ F(x,\ y)) = d(A,\ B) \\ d(x_1,\ F(x,\ v_0)) = d(A,\ B), & & d(x_1,\ F(x,\ v_0)) = d(A,\ B). \end{array}\right\}$$

Using the proximal mixed monotone property of F, we have

$$u_1 \preceq x_1 \text{ and } x_1 \preceq x \text{ which implies that } u_1 \preceq x.$$

Again, since $F(A_0 \times A_0) \subseteq B_0$ and $v_0,\ x \in A_0$, there exists $y_1 \in A_0$ such that

$$d(y_1,\ F(v_0,\ x)) = d(A,\ B).$$

Now we have

$$\left.\begin{array}{lcl} u_0 \preceq x, & & y \preceq v_0, \\ d(v_1,\ F(v_0,\ u_0)) = d(A,\ B) & \text{and} & d(y,\ F(y,\ x)) = d(A,\ B) \\ d(y_1,\ F(v_0,\ x)) = d(A,\ B), & & d(y_1,\ F(v_0,\ x)) = d(A,\ B). \end{array}\right\}$$

Using the proximal mixed monotone property of F, we have

$$v_1 \succeq y_1 \text{ and } y_1 \succeq y \text{ which implies that } v_1 \succeq y.$$

Therefore, we have

$$(u_1,\ v_1) \preceq (x,\ y). \tag{2.22}$$

Continuing this process, we have sequences $\{u_n\}$ and $\{v_n\}$ in A_0 such that

$$d(u_{n+1},\ F(u_n,\ v_n)) = d(A,\ B),\ \ d(v_{n+1},\ F(v_n,\ u_n)) = d(A,\ B) \text{ and } (u_n,\ v_n) \preceq (x,\ y) \text{ for all } n \geq 0. \tag{2.23}$$

By (2.18) and (2.23)

$$\left.\begin{array}{l} u_n \preceq x \text{ and } v_n \succeq y \\ d(u_{n+1},\ F(u_n,\ v_n)) = d(A,\ B), \\ d(x,\ F(x,\ y)) = d(A,\ B) \end{array}\right\} \Longrightarrow$$

$$d(u_{n+1},\ x) \leq N(u_n,\ v_n,\ x,\ y,\ u_{n+1},\ x), \tag{2.24}$$

where

$$\begin{aligned} N(u_n,\ v_n,\ x,\ y,\ u_{n+1},\ x) &= a\ d(u_n,\ x) + b\ d(v_n,\ y) \\ &\quad + L \min\{d(u_{n+1},\ x),\ d(x,\ u_n),\ d(u_{n+1},\ u_n),\ d(x,\ x)\} \\ &= a\ d(u_n,\ x) + b\ d(v_n,\ y). \end{aligned}$$

Therefore, from (2.24), we have

$$d(u_{n+1},\ x) = a\ d(u_n,\ x) + b\ d(v_n,\ y). \tag{2.25}$$

Again, by (2.18) and (2.23)

$$\left.\begin{array}{l} y \preceq v_n \text{ and } x \succeq u_n \\ d(y,\ F(y,\ x)) = d(A,\ B), \\ d(v_{n+1},\ F(v_n,\ u_n)) = d(A,\ B) \end{array}\right\} \Longrightarrow$$

$$d(y,\ v_{n+1}) \leq N(y,\ x,\ v_n,\ u_n,\ y,\ v_{n+1}), \tag{2.26}$$

where

$$\begin{aligned} N(y,\ x,\ v_n,\ u_n,\ y,\ v_{n+1}) &= a\ d(y,\ v_n) + b\ d(x,\ u_n) \\ &\quad + L \min\{d(y,\ v_n),\ d(v_{n+1},\ y),\ d(y,\ y),\ d(v_{n+1},\ v_n)\} \\ &= a\ d(y,\ v_n) + b\ d(x,\ u_n). \end{aligned}$$

From (2.26) it follows that

$$d(y,\ v_{n+1}) \leq a\ d(y,\ v_n) + b\ d(x,\ u_n). \tag{2.27}$$

Combining (2.25) and (2.27), we have

$$d(u_{n+1},\ x) + d(v_{n+1},\ y) \leq (a+b)\ [d(u_n,\ x) + d(v_n,\ y)]. \tag{2.28}$$

Since $a + b < 1$, it follows from (2.28) that

$$d(u_{n+1},\ x) + d(v_{n+1},\ y) \leq d(u_n,\ x) + d(v_n,\ y).$$

Therefore, $\{d(u_n,\ x) + d(v_n,\ y)\}$ is a monotonically decreasing sequence of non-negative real numbers and hence there exists a $p \geq 0$ such that

$$\lim_{n\to\infty} [d(u_n,\ x) + d(v_n,\ y)] = p. \tag{2.29}$$

We show that $p = 0$. If possible, let $p > 0$.
Taking limit as $n \longrightarrow \infty$ in (2.28) and using (2.29), we have

$$p \leq (a+b)\ p < p, \quad (\text{since } (a+b) < 1)$$

which is a contradiction. Therefore, $p = 0$. Hence

$$\lim_{n\to\infty} [d(u_n,\ x) + d(v_n,\ y)] = 0, \tag{2.30}$$

which implies that

$$\lim_{n\to\infty} d(u_n,\ x) = 0 \text{ and } \lim_{n\to\infty} d(v_n,\ y) = 0. \tag{2.31}$$

Similarly, we can prove that

$$\lim_{n\to\infty} [d(u_n,\ x^*) + d(v_n,\ y^*)] = 0, \tag{2.32}$$

and hence

$$\lim_{n\to\infty} d(u_n,\ x^*) = 0 \text{ and } \lim_{n\to\infty} d(v_n,\ y^*) = 0. \tag{2.33}$$

Using triangle inequality, (2.31) and (2.33), we have

$$d(x,\ x^*)+d(y,\ y^*) \leq [d(x,\ u_n)+d(u_n,\ x^*)+d(y,\ v_n)+d(v_n,\ y^*)] \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Hence $d(x,\ x^*) + d(y,\ y^*) = 0$, which implies that $d(x,\ x^*) = 0$ and $d(y,\ y^*) = 0$; that is, $x = x^*$ and $y = y^*$; that is, $(x,\ y) = (x^*,\ y^*)$. Therefore, the coupled best proximity point of F is unique. □

Example 2.3. Let $X = R^2$ (R denotes the set of real numbers) and d be the Euclidean metric on X. We define a partial order $\preceq$ on X such that $(x,\ y) \preceq (u,\ v)$ if and only if $x \leq u$ and $y \leq v$, for all $(x,\ y),\ (u,\ v) \in X$. Let
$A = \{(2,\ 0),\ (0,\ 2)\} \cup \{(x,\ 0) : 2 \leq x \leq 3\}$,
$B = \{(-2,\ 0),\ (0,\ -2)\} \cup \{(0,\ x) : -3 \leq x \leq -2\}$,
$A_0 = \{(2,\ 0),\ (0,\ 2)\}$ and $B_0 = \{(-2,\ 0),\ (0,\ -2)\}$.
Let $F : A \times A \to B$ be defined as

$$F((x_1,\ x_2),\ (y_1,\ y_2)) = (-x_2,\ -x_1) \text{ for all } (x_1,\ x_2),\ (y_1,\ y_2) \in A \times A.$$

Let a, b and L be three nonnegative real numbers with $a + b < 1$.

Here all the conditions of theorem 2.1 are satisfied and it is seen that ((2, 0), (0, 2)) and ((0, 2), (2, 0)) are two coupled best proximity points of F.

Considering $L = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.4. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $(A,\ B)$ be a pair of non-empty closed subsets of X such that A_0 is non-empty and closed. Let $F\colon A\times A \longrightarrow B$ be a mapping such that $F(A_0\times A_0) \subseteq B_0$ and F has proximal mixed monotone property on $A_0 \times A_0$. Suppose that there exist nonnegative real numbers a and b with $a + b < 1$ such that for all $x,\ y,\ u,\ v,\ p,\ q \in A_0$*

$$\left.\begin{array}{l} x \preceq u \text{ and } y \succeq v, \\ d(p,\ F(x,\ y)) = d(A,\ B), \\ d(q,\ F(u,\ v)) = d(A,\ B) \end{array}\right\} \Longrightarrow d(p,\ q) \leq a\, d(x,\ u) + b\, d(y,\ v).$$

Suppose that the condition (a) or (b) of the theorem 2.1 holds. Also, suppose that there exist elements $(x_0,\ y_0),\ (x_1,\ y_1) \in A_0 \times A_0$ such that $d(x_1,\ F(x_0,\ y_0)) = d(A,\ B)$ and $d(y_1,\ F(y_0,\ x_0)) = d(A,\ B)$ with $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Then F has a coupled best proximity point in $A_0 \times A_0$.

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