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CONVERGENCE THEOREMS FOR LIPSCHITZ PSEUDOCONTRACTIVE NON-SELF MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce an iterative process and prove strong convergence result for finding the fixed point of Lipschitz pseudocontractive non-self mapping in Banach spaces more general than Hilbert spaces. In addition, strong and weak convergence of Mann type sequence to a fixed point of λ-strictly pseudocontractive non-self mapping is investigated. Moreover, a numerical example which shows the conclusion of our result is presented. Our results improve and generalize many known results in the current literature.

KEYWORDS : Fixed points, nonexpansive non-self mappings, pseudocontractive mappings, uniformly Gâteaux differentiable norm.
AMS Subject Classification: 37C20, 47H09, 47H10, 47J05.

1. INTRODUCTION

Let C be a nonempty subset of a real Banach space E with dual E^*. A mapping $T : C \longrightarrow E$ is called *L-Lipschitz* if there exits $L \geq 0$ such that

$$||Tx - Ty| \leq L||x - y||, \forall x, y \in C.$$

If $L = 1$ then T is called *nonexpansive* mapping. T is called λ*-strictly pseudocontractive mapping* if there exist $\lambda \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2 - \lambda||(I - T)x - (I - T)y||^2, \forall x, y \in C,$$

and T is called *pseudocontractive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2, \text{ for all } x, y \in C, \tag{1.1}$$

where $J : E \to 2^{E^*}$ is the *normalized duality* mapping given by $Jx := \{f^* \in E^* : \langle x, f^*\rangle = ||x||^2 = ||f^*||^2\}$, where $\langle ., .\rangle$ denotes the generalized duality pairing. It is well known that J is single-valued whenever E is *smooth*. J is said to be weakly

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sequentially continuous if it is single valued and weak-to-weak* continuous; that is, if $x_n \rightharpoonup x$ in E, then $J(x_n) \rightharpoonup^* J(x)$ in E^*.

Due to Kato [11] inequality (1.1) is equivalent to the following inequality for all $t \geq 0$.

$$||x - y|| \leq ||x - y + t[(I - T)x - (I - T)y]||, \text{ for all } x, y \in C.$$

We note that every nonexpansive and every λ-strictly pseudocontractive mappings are Lipschitz with constants $L = 1$ and $L = \frac{1+\lambda}{\lambda}$, respectively. Moreover, we observe that the class of Lipschitz pseudocontractive mappings includes the class of nonexpansive and the class of λ- strictly pseudocontractive mappings.

Pseudocontractive mappings are also related to the important class of nonlinear operators known as *accretive* mappings. A mapping $A : C \longrightarrow E$ is called *accretive* if there exists $j(x-y) \in J(x-y)$ such that $\langle Ax - Ay, j(x-y)\rangle \geq 0$ for all $x, y \in C$.

A mapping A is accretive if and only if $T := I - A$ is pseudocontractive and thus the zero set of A, $N(A) = \{x \in C : Ax = 0\}$, is the fixed point set of T, $F(T) = \{x \in C : Tx = x\}$. It is also known that the equilibrium points of some evolution systems are the solutions of the equation $Ax = 0$, when A is accretive mapping (see e.g. [31]). Consequently, several authors have studied iterative methods for approximating fixed points of a nonexpansive or pseudocontractive mapping T (see for example [2, 4, 13, 18, 22, 32] and the references contained therein).

In 1953, Mann [13] introduced the following iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.2}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The sequence $\{x_n\}$ generated by (1.2) is called *Mann iteration sequence*. The Mann iteration has been extensively investigated for nonexpansive mappings (see, e.g., [21]). In an infinite-dimensional Hilbert space, the Mann iteration can provide only weak convergence [8].

Attempts to modify the Mann iterative method, so that strong convergence is guaranteed, have been made. In 1974, Ishikawa [9] introduced an iterative process, which in some sense is more general than that of Mann and which converges to a fixed point of a Lipschitz pseudocontractive self-mapping T of C. The following theorem is proved.

Theorem IS ([9]). If C is a compact convex subset of a Hilbert space H, $T : C \longrightarrow C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by

$$y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \; x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \; n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions:
(i) $0 \leq \alpha_n \leq \beta_n \leq 1$; (ii) $\lim\limits_{n \longrightarrow \infty} \beta_n = 0$; (iii) $\sum \alpha_n \beta_n = \infty$.

The iterative method of Theorem IS, which is now referred to as the Ishikawa iterative method has been studied extensively by various authors. But it is still an open question whether or not this method can be employed to approximate fixed points of Lipschitz pseudocontractive mappings without the compactness assumption on C or T (see, e.g., [3, 19, 20]).

In 2003, Chidume and Zegeye [4] constructed an iterative scheme which provides the conclusion of Theorem IS for an important class of Lipschitz pseudocontractive self-mapping without the requirement that C or T is compact. They proved the following Theorem.

Theorem CZ ([4]). Let C be nonempty closed convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \to C$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Suppose that every closed, convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in C$ by

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \ for\ all\ n \geq 0, \tag{1.4}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0, 1]$ satisfying certain conditions. Then, $\{x_n\}$ converges strongly to a fixed point of T.

Remark 1.1. We remark that Theorem CZ has extended the results of Ishikawa [9] and other related results to Banach spaces more general than Hilbert spaces without compactness assumption on C. However, in all the above results, the operator T remains a self-mapping of a nonempty closed convex subset C of a Banach space E. If, however, the domain of T, C, is a proper subset of E (and this is the case in several applications), and T maps C into E, then the iterative processes (1.2), (1.3) and (1.4) studied by these authors may fail to be well defined. Many researchers have made significant progress to overcome this problem by employing the concept of sunny nonexpansive mappings.

Let C be a nonempty closed convex subset of a Banach space E and let D be a nonempty subset of C. A mapping $P : C \to D$ is said to be *retraction*, if $Px = x$ for all $x \in D$. A retraction $P : C \to D$ is sunny if it satisfies the property: $P(Px + t(x - Px)) = Px$ for $x \in C$ and $t > 0$, whenever $Px + t(x - Px) \in C$. P is said to be *sunny nonexpansive* if it is both sunny and nonexpansive mapping. A subset D of C is also called *sunny nonexpansive* retract of C if there exists a sunny nonexpansive retraction of C onto D.

It is well known [1] that in a smooth Banach space E, a retraction mapping P is sunny nonexpansive if and only if the following inequality holds:

$$\langle x - Px, J(y - Px)\rangle \leq 0, x \in C, y \in D. \tag{1.5}$$

Recall that, if $E = H$, a real Hilbert space, then the nearest point metric projection $P_C : H \longrightarrow C$ is characterized by the inequality $\langle x - P_C x, y - P_C x\rangle \leq 0$, for all $y \in C$. Hence, the metric projection, P_C, is a sunny nonexpansive retraction in the setting of Hilbert spaces. However, this fact characterizes Hilbert spaces and it is not available in more general Banach spaces.

For the approximation of fixed points of non-self mappings, Matsushita and Takahashi [12] have studied and proved the following theorem.

Theorem MT ([12]). Let E be a uniformly convex Banach space whose norm is uniformly Gateâux differentiable, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping from C into E with $F(T) \neq \emptyset$. Suppose that C is a sunny nonexpansive retract of E. Let $\{\alpha_n\}$ be a sequence such that $0 \leq \alpha_n \leq 1$, $\lim\limits_{n \longrightarrow \infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$ and $\sum |\alpha_{n+1} - \alpha_n| < \infty$. Let u and x_0 be elements of C. Suppose that $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) PTx_n, for\ n = 0, 1, 2, ..., \tag{1.6}$$

where P is a sunny nonexpansive retraction from E onto C. Then $\{x_n\}$ converges strongly to $z \in F(T)$.

Several authors have studied implicit and explicit iterative schemes of the type (1.6) (see, e.g., [5, 14, 23, 24, 25, 28, 29, 33] and the references therein) for non-self mappings. However, as it has been indicated by Colao and Marino [7], calculating P is time consuming process, even in Hilbert spaces when P is a metric projection, and it may require an approximating algorithm for itself. To avoid the necessity of using an auxiliary mapping P, Colao and Marino [7] introduced a new search strategy for the coefficient α_n which makes the Krasnoselskii-Mann algorithm well defined in the Hilbert space setting. They obtained the following weak and strong convergence of the algorithm for nonexpansive non-self mappings. We shall need the following definitions.

A set $C \subset E$ is said to be *strictly convex* if it is convex and with the property that $x, y \in \partial C$ and $t \in (0, 1)$ implies that $tx + (1 - t)y \in \mathring{C}$, where ∂C and $\mathring{C}$ denotes boundary and interior of C respectively. In other words, if the boundary of C does not contain any segment.

A mapping $T : C \longrightarrow E$ is said to satisfy the *inward* condition if, for any $x \in C$, we have $Tx \in I_C(x) := \{x + c(u - x) : c \geq 1, u \in C\}$. T is said to satisfy the *weakly inward* condition if for each $x \in C$, $Tx \in \overline{I_C(x)}$, where $\overline{I_C(x)}$ is the closure of $I_C(x)$. Note that $I_C(x)$ is convex whenever C is convex.

Theorem CM ([7]). Let C be a convex, closed and nonempty subset of a Hilbert space H and let $T : C \longrightarrow H$ be a mapping. Then the algorithm

$$\begin{cases} x_0 \in C, \\ \alpha_0 = max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ \alpha_{n+1} = max\{\alpha_n, h(x_{n+1})\}, \end{cases} \tag{1.7}$$

where $h : C \longrightarrow \mathbb{R}$ is defined by $h(x) = \inf\{\lambda \geq 0 : \lambda x + (1 - \lambda)Tx \in C\}$, is well defined. Further, if C is strictly convex and T is nonexpansive mapping which satisfies the inward condition and such that $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a point $p \in F(T)$. Moreover, if $\sum \alpha_n < \infty$, then the convergence is strong.

We remark that *Theorem CM* is applicable for approximating fixed points of nonexpansive non-self mappings.

Our concern now is the following: *can we construct an iterative scheme which converges strongly to a fixed point of pseudocontractive non-self mappings, without using projection mapping, which is more general than nonexpansive mappings in*

Banach spaces?

It is our purpose in this paper to construct an iterative scheme which converges strongly to a fixed point of Lipschitz pseudocontractive non-self mappings in Banach spaces more general than Hilbert spaces. Our results provide an affirmative answer to our concern. Our results extend *Theorem CM*, *Theorem CZ* and the references therein to the more general class of Lipschitz pseudocontractive non-self mappings.

2. PRELIMINARIES

Let E be a real Banach space and let $S := \{x \in E : ||x|| = 1\}$ denote the unit sphere of E. E is said to have *Gâteaux differentiable* norm if the limit

$$\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t} \tag{2.1}$$

exists for each $x, y \in S$. Such E is called *smooth*. The space E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$, the limit (2.1) is attained uniformly for $x \in S$.

The *modulus of smoothness* of E is a function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\Big\}.$$

A Banach space E is called q*-uniformly smooth* if there exist a constant $c > 0$ and a real number $q \in (1,\infty)$ such that $\rho_E(\tau) \le c\tau^q$. E is called *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$. The Lebesgue L_p, the sequence ℓ_p and the Sobolev W_p^m spaces, for $p \in (1,\infty)$, are examples of uniformly smooth Banach spaces.

It is well known that every uniformly smooth space has uniformly Gâteaux differentiable norm (see, e.g., [6]).

Let C be a nonempty subset of E. A sequence $\{x_n\} \subset C$ is said to be *Fejèr-monotone* with respect to a set $D \subset C$ if, for any element $x \in D, ||x_{n+1} - x|| \le ||x_n - x||, \forall n \in \mathbb{N}$.

In the sequel, we shall make use of the following lemmas.

Lemma 2.1. [16] *Let E be a real normed linear space and J be the normalized duality mapping on E. Then for any given $x, y \in E$, the following inequality holds:*

$$||x+y||^2 \le ||x||^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y).$$

Lemma 2.2. [15] *Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative numbers satisfying the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_1^\infty \alpha_n = \infty$, and $\frac{\gamma_n}{\alpha_n} \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1}^2 \le \lambda_n^2 - \alpha_n \psi(\lambda_{n+1}) + \gamma_n, n = 1, 2, ..., \tag{2.2}$$

be given where $\psi : [0,\infty) \to [0,\infty)$ is a strictly increasing function such that it is positive on $(0,\infty)$ and $\psi(0) = 0$. Then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.3. [16] *Let E be a Banach space. Suppose C is a nonempty closed convex subset of E and $T : C \to E$ be a continuous pseudocontractive mapping* *satisfying the weakly inward condition. Then for $u \in C$, there exists a unique path $t \to y_t \in C,\ t \in [0,1)$, satisfying the following condition:*

$$y_t = tTy_t + (1-t)u.$$

We note that in Lemma 2.3 if, in addition, $F(T) \neq \emptyset$ then $\{y_t\}$ is bounded. Furthermore, if E is assumed to be a reflexive Banach space with uniformly Gâteaux differentiable norm and every closed convex and bounded subset of C has the fixed point property for nonexpansive self-mappings, then as $t \to 1^-$, the path converges strongly to a fixed point x^* of T, which is the unique solution of the variational inequality (see [17]):

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

Lemma 2.4. [27] *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, J(x)\rangle + 2||Ky||^2,\ \forall x, y \in E.$$

Lemma 2.5. [30] *Let E be a reflexive Banach space and let C be a closed convex subset of E. Let f be a proper convex lower semi-continuous function of C into $(-\infty, \infty]$ and suppose that $f(x_n) \to \infty$ as $||x_n|| \to \infty$. Then, there exists $x_0 \in C$ such that $f(x_0) = \inf\{f(x) : x \in C\}$.*

Lemma 2.6. [34] *Let C be a non empty subset of a real 2-uniformly smooth Banach space E with the best smooth constant $K > 0$ and let $T : C \longrightarrow E$ be a $\lambda-$strictly pseudocontractive mapping. For $\alpha \in (0,1) \cap (0, \frac{\lambda}{K^2}]$, we define $T_\alpha : C \longrightarrow E$ by $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then T_α is nonexpansive and $F(T_\alpha) = F(T)$.*

Remark 2.7. If C is convex and T satisfies the inward condition then T_α satisfies the inward condition.

Lemma 2.8. [34] *Let C be a non empty subset of a real 2-uniformly smooth Banach space E. Suppose that the normalized duality mapping $J : E \longrightarrow E^*$ is weakly sequentially continuous at zero. Let $T : C \longrightarrow E$ be a $\lambda-$strictly pseudocotractive mapping for $0 < \lambda < 1$. Then, for any $\{x_n\} \subset C$, if $x_n \rightharpoonup x$, and $x_n - Tx_n \longrightarrow y \in E$ then $x - Tx = y$.*

3. MAIN RESULTS

Lemma 3.1. *Let C be a nonempty, closed and convex subset of a real Banach space E and $T : C \longrightarrow E$ be a mapping. For any given element u in C and any arbitrarily fixed μ in $[0,1)$ define $f : C \longrightarrow \mathbb{R}$ by $f(x) = \inf\{\lambda \geq 0 : \lambda\big(\mu u + (1-\mu)x\big) + (1-\lambda)Tx \in C\}$. Then the following hold:*

1) *For any $x \in C$, $f(x) \in [0,1]$ and $f(x) = 0$ if and only if $Tx \in C$;*
2) *For any $x \in C$ and $\beta \in [f(x), 1]$, $\beta\big(\mu u + (1-\mu)x\big) + (1-\beta)Tx \in C$;*
3) *If T satisfies the inward condition, then $f(x) < 1$, for all $x \in C$;*
4) *If $Tx \notin C$, then $f(x)\big(\mu u + (1-\mu)x\big) + \big(1 - f(x)\big)Tx \in \partial C$.*

Proof. 1) Clearly $f(x) \geq 0$ for all $x \in C$. If $\lambda = 1$, by convexity of C, we have $\mu u + (1-\mu)x \in C$. Therefore, $f(x) \leq 1$ and hence $f(x) \in [0,1]$. One can also easily show that $f(x) = 0$ if and only if $Tx \in C$.
2) The proof of (2) follows directly from the definition of $f(x)$.
3) We first show that $I_C(x) \subset I_C(z)$, where $z = \mu u + (1-\mu)x$. Let $y \in I_C(x)$. Then $y = x + c(v - x)$, for some $c \geq 1$ and $v \in C$. Since $\mu < 1$, we can choose a real number $k > 1$ such that $\mu < 1 - \frac{1}{k}$. Then we have:

$$y = cv + (1-c)x$$

$$\begin{aligned}
&= z + cv + (1-c)x - z \\
&= z + kc\big[\frac{1}{k}v + \frac{(1-c)}{kc}x - \frac{1}{kc}z + z - z\big] \\
&= z + kc\big[\frac{1}{k}v + \frac{(1-c)}{kc}x + (1-\frac{1}{kc})z - z\big] \\
&= z + kc\big[\frac{1}{k}v + \frac{(1-c)}{kc}x + (1-\frac{1}{kc})(\mu u + (1-\mu)x) - z\big] \\
&= z + kc\big[\frac{1}{k}v + \frac{(\mu + (k-1)c - kc\mu)}{kc}x + \mu(1-\frac{1}{kc})u - z\big].
\end{aligned}$$

It is easy to verify that $\frac{\mu+(k-1)c-kc\mu}{kc} \in (0,1)$. Then, since C is convex, $w := \frac{1}{k}v + \frac{(\mu+(k-1)c-kc\mu)}{kc}x + \mu(1-\frac{1}{kc})u \in C$. Then, $y \in I_C(z)$ and hence $I_C(x) \subset I_C(z)$. Now if T satisfies the inward condition, then $Tx \in I_C(x) \subset I_C(z)$. Thus, $Tx = z + b(w' - z)$ for some $b \geq 1$ and $w' \in C$ which gives

$$\frac{1}{b}Tx + (1-\frac{1}{b})z = w' \in C.$$

This implies that

$$f(x) = \inf\{\lambda \geq 0 : \lambda(\mu u + (1-\mu)x) + (1-\lambda)Tx \in C\} \leq 1 - \frac{1}{b} < 1.$$

4) Let $\{\beta_n\} \subset (0, f(x))$ be a real sequence such that $\beta_n \longrightarrow f(x)$. By the definition of f, we have $z_n := \beta_n\big(\mu u + (1-\mu)x\big) + (1-\beta_n)Tx \notin C$.
Now, since $\beta_n \longrightarrow f(x)$, we have:

$$\begin{aligned}
&||z_n - \big[f(x)\big(\mu u + (1-\mu)x\big) + (1-f(x))Tx\big]|| \\
&= ||(\beta_n - f(x))\big[\mu u + (1-\mu)x - Tx\big]|| \\
&\leq |\beta_n - f(x)|\big[||\mu u + (1-\mu)x|| + ||Tx||\big] \longrightarrow 0 \ as\ n \to \infty.
\end{aligned}$$

Thus, $z_n \longrightarrow f(x)\big(\mu u + (1-\mu)x\big) + (1-f(x))Tx \in C$.
But since $z_n = \beta_n\big(\mu u + (1-\mu)x\big) + (1-\beta_n)Tx \notin C$, for all $n \geq 1$, we have

$$f(x)\big(\mu u + (1-\mu)x\big) + (1-f(x))Tx \in \partial C.$$

3.1. Convergence Theorem for Pseudocontractive Mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Banach space E and $T : C \to E$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Suppose that T satisfies the inward condition. Let $\{\mu_n\} \subset (0,1)$, u be any point in C and $\{x_n\}$ be a sequence generated from arbitrary $x_1 \in C$ by:*

$$\begin{cases} \alpha_1 = max\{\frac{1}{2}, f(x_1)\}, \\ x_{n+1} = \alpha_n\big(\mu_n u + (1-\mu_n)x_n\big) + (1-\alpha_n)Tx_n, \\ \alpha_{n+1} \in [max\{\alpha_n, f(x_{n+1})\}, 1),\ n \geq 1, \end{cases} \tag{3.1}$$

where $f(x_n) := \inf\{\lambda \geq 0 : \lambda\big(\mu_n u + (1-\mu_n)x_n\big) + (1-\lambda)Tx_n \in C\}$. Let the pair (α_n, μ_n) satisfies the following conditions:

(i) $\lim_{n\to\infty} \frac{\mu_n\alpha_n}{1-\alpha_n} = 0$; *(ii)* $\sum_{n=1}^{\infty} \alpha_n\mu_n = \infty$;

(iii) $\lim_{n\to\infty} \frac{(1-\alpha_n)^2}{\alpha_n\mu_n} = 0$; *(iv)* $\lim_{n\to\infty} \frac{\big(\frac{(1-\alpha_n)\alpha_{n-1}\mu_{n-1}}{(1-\alpha_{n-1})\alpha_n\mu_n} - 1\big)}{\alpha_n\mu_n} = 0.$

Then, algorithm (3.1) is well defined and $||x_n - Tx_n|| \to 0$ as $n \to \infty$.

Proof. Since T satisfies the inward condition by Lemma 3.1(3) we have that $\alpha_n \in [f(x_n), 1)$ for all $n \in \mathbb{N}$. Hence by Lemma 3.1(2), we have:

$$x_{n+1} = \mu_n \alpha_n u + (1-\mu_n)\alpha_n x_n + (1-\alpha_n)Tx_n \in C.$$

Thus, algorithm (3.1) is well defined. To prove the second assertion we proceed as follows:

Since $\lim_{n\to\infty} \frac{\mu_n\alpha_n}{1-\alpha_n} = 0$ and $\lim_{n\to\infty} \frac{(1-\alpha_n)^2}{\mu_n\alpha_n} = 0$, for $\epsilon := \frac{1}{2(\frac{5}{2}+L)(2+L)}$ there exists $N_0 > 0$ such that $\frac{\mu_n\alpha_n}{1-\alpha_n} \leq 1$ and $\frac{(1-\alpha_n)^2}{\mu_n\alpha_n} \leq \epsilon, \forall n \geq N_0$. Let $x^* \in F(T)$ and $r > 0$ be sufficiently large such that $x_{N_0} \in B_r(x^*)$ and $u \in B_{\frac{r}{2}}(x^*)$.

We first show by mathematical induction that $\{x_n\}$ is bounded. To this end, it suffices to show that $x_n \in B_r(x^*)$ for all $n \geq N_0$. By construction $x_{N_0} \in B_r(x^*)$. Now, assume that $x_n \in B_r(x^*)$ for any $n \geq N_0$. we need to show that $x_{n+1} \in B_r(x^*)$ for all $n \geq N_0$. For contradiction, suppose $x_{n+1} \notin B_r(x^*)$. Then $||x_{n+1} - x^*|| > r$. From (3.1) and Lemma 2.1, we have:

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||\alpha_n\big(\mu_n u + (1-\mu_n)x_n\big) + (1-\alpha_n)Tx_n - x^*||^2 \\
&= ||\mu_n\alpha_n u + (1-\mu_n)\alpha_n x_n + x_n - x_n + (1-\alpha_n)Tx_n - x^*||^2 \\
&= ||x_n - x^* - (1-\alpha_n)\big[\frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u) + x_n - Tx_n\big]||^2 \\
&\leq ||x_n - x^*||^2 - 2(1-\alpha_n)\langle (x_n - Tx_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u), \\
&\quad j(x_{n+1} - x^*)\rangle \\
&= ||x_n - x^*||^2 - 2(1-\alpha_n)\langle \frac{\mu_n\alpha_n}{1-\alpha_n}(x_{n+1} - x^*) - \frac{\mu_n\alpha_n}{1-\alpha_n} \\
&\quad \times(x_{n+1} - x^*) + x_n - Tx_n + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u), j(x_{n+1} - x^*)\rangle \\
&= ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 \\
&\quad +2(1-\alpha_n)\langle \frac{\mu_n\alpha_n}{1-\alpha_n}(x_{n+1} - x_n) - (x_n - Tx_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(u - x^*) \\
&\quad +(x_{n+1} - Tx_{n+1}) - (x_{n+1} - Tx_{n+1}), j(x_{n+1} - x^*)\rangle. \qquad (3.2)
\end{aligned}$$

Since T is pseudocontractive, we have $\langle x_{n+1} - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \geq 0$. Thus, from (3.2) and the fact that $x_n \in B_r(x^*)$, $u \in B_{\frac{r}{2}}(x^*)$ and $\frac{\mu_n\alpha_n}{1-\alpha_n} \leq 1, \forall n \geq N_0$, we obtain:

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 \\
&\quad +2(1-\alpha_n)\langle \frac{\mu_n\alpha_n}{1-\alpha_n}(x_{n+1} - x_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(u - x^*) \\
&\quad +(x_{n+1} - Tx_{n+1}) - (x_n - Tx_n), j(x_{n+1} - x^*)\rangle \\
&\leq ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 + 2(1-\alpha_n) \\
&\quad \times\Big[(2+L)||x_{n+1} - x_n|| + \frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big]||x_{n+1} - x^*|| \\
&= ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 \\
&\quad +2(1-\alpha_n)\Big[(2+L)(1-\alpha_n)||(x_n - Tx_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u)|| \\
&\quad +\frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big]||x_{n+1} - x^*||
\end{aligned}$$

$$\begin{aligned}
&\leq ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 + 2(1-\alpha_n)\Big[(2+L)(1-\alpha_n)\\
&\quad\times\big(||x_n - x^*|| + L||x^* - x_n|| + \frac{\mu_n\alpha_n}{1-\alpha_n}(||x_n - x^*|| + ||x^* - u||)\big)\\
&\quad+\frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big]||x_{n+1} - x^*||\\
&\leq ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2 + 2(1-\alpha_n)\\
&\quad\times\Big[(2+L)(1-\alpha_n)\Big((2+L)||x_n - x^*|| + \frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big)\\
&\quad+\frac{\mu_n\alpha_n}{1-\alpha_n}||u - x^*||\Big]||x_{n+1} - x^*||\\
&\leq ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2\\
&\quad+2(1-\alpha_n)\Big[(2+L)(1-\alpha_n)\Big((2+L)r + \frac{r}{2}\Big) + \frac{1}{2}r\Big]||x_{n+1} - x^*||\\
&= ||x_n - x^*||^2 - 2\mu_n\alpha_n||x_{n+1} - x^*||^2\\
&\quad+2(1-\alpha_n)\Big[(1-\alpha_n)(2+L)(\frac{5}{2} + L)r + \frac{1}{2}r\Big]||x_{n+1} - x^*||. \qquad (3.3)
\end{aligned}$$

But since $||x_{n+1} - x^*|| > ||x_n - x^*||$, (3.3) implies that

$$\begin{aligned}
2\mu_n\alpha_n||x_{n+1} - x^*||^2 &\leq 2(1-\alpha_n)\Big[(1-\alpha_n)(2+L)(\frac{5}{2} + L)r + \frac{1}{2}r\Big]\\
&\quad\times||x_{n+1} - x^*||.
\end{aligned}$$

Then the fact that $\alpha_n \in [\frac{1}{2}, 1)$, $\mu_n \in (0,1)$ and $\frac{(1-\alpha_n)^2}{\mu_n\alpha_n} \leq \frac{1}{2(\frac{5}{2}+L)(2+L)}, \forall n \geq N_0$ implies

$$\begin{aligned}
||x_{n+1} - x^*|| &\leq \frac{(1-\alpha_n)^2}{\mu_n\alpha_n}(2+L)(\frac{5}{2} + L)r + \frac{r}{2}\\
&\leq r,\ \forall n \geq N_0,
\end{aligned}$$

which is a contradiction. Therefore, $x_{n+1} \in B_r(x^*)$ for all positive integers $n \geq N_0$ and hence the sequence $\{x_n\}$ is bounded.

Next we show that $||x_n - y_n|| \to 0$ as $n \to \infty$, where $y_n := y_{t_n} = t_nTy_{t_n} + (1 - t_n)u,\ t_n := \frac{1}{1+\frac{\alpha_n\mu_n}{1-\alpha_n}},\ \forall n \in \mathbb{N}$. From (3.1) and Lemma 2.1, we have:

$$\begin{aligned}
||x_{n+1} - y_n||^2 &\leq ||x_n - y_n||^2 - 2(1-\alpha_n)\langle x_n - Tx_n + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u),\\
&\quad j(x_{n+1} - y_n)\rangle\\
&= ||x_n - y_n||^2 - 2(1-\alpha_n)\langle\frac{\mu_n\alpha_n}{1-\alpha_n}x_{n+1} - \frac{\mu_n\alpha_n}{1-\alpha_n}y_n + x_n - Tx_n\\
&\quad-\frac{\mu_n\alpha_n}{1-\alpha_n}x_{n+1} + \frac{\mu_n\alpha_n}{1-\alpha_n}y_n + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n - u), j(x_{n+1} - y_n)\rangle\\
&= ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 + 2(1-\alpha_n)\langle Tx_n - x_n +\\
&\quad\frac{\mu_n\alpha_n}{1-\alpha_n}(x_{n+1} - x_n) + \frac{\mu_n\alpha_n}{1-\alpha_n}(u - y_n), j(x_{n+1} - y_n)\rangle\\
&= ||x_n - y_n||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 + 2(1-\alpha_n)\langle\frac{\mu_n\alpha_n}{1-\alpha_n}\\
&\quad\times(x_{n+1} - x_n) + [\frac{\mu_n\alpha_n}{1-\alpha_n}(u - y_n) - (y_n - Ty_n)]\\
&\quad-[(x_{n+1} - Tx_{n+1}) - (y_n - Ty_n)] + [(x_{n+1} - Tx_{n+1})\\
&\quad-(x_n - Tx_n)], j(x_{n+1} - y_n)\rangle. \qquad (3.4)
\end{aligned}$$

On the other hand, the property of y_n implies

$$\begin{aligned} y_n - Ty_n &= t_nTy_n + (1-t_n)u - Ty_n = (1-t_n)(u - Ty_n) \\ &= \frac{\mu_n\alpha_n}{1-\alpha_n+\mu_n\alpha_n}(u - Ty_n) \\ &= \frac{\mu_n\alpha_n}{1-\alpha_n+\mu_n\alpha_n}\big[u - (\frac{1-\alpha_n+\mu_n\alpha_n}{1-\alpha_n}y_n - \frac{\mu_n\alpha_n}{1-\alpha_n}u)\big] \\ &= \frac{\mu_n\alpha_n}{1-\alpha_n}(u-y_n). \end{aligned}$$

Thus, we get that

$$\frac{\mu_n\alpha_n}{1-\alpha_n}(u-y_n) - (y_n - Ty_n) = 0. \tag{3.5}$$

Then from (3.4), (3.5) and the pseudocontractivity of T, we obtain:

$$\begin{aligned} ||x_{n+1}-y_n||^2 &\le ||x_n-y_n||^2 - 2\mu_n\alpha_n||x_{n+1}-y_n||^2 + 2(1-\alpha_n)\langle\frac{\mu_n\alpha_n}{1-\alpha_n} \\ &\quad \times(x_{n+1}-x_n) + (x_{n+1}-Tx_{n+1}) - (x_n - Tx_n), j(x_{n+1}-y_n)\rangle \\ &\le ||x_n-y_n||^2 - 2\mu_n\alpha_n||x_{n+1}-y_n||^2 + 2(1-\alpha_n)\big[\frac{\mu_n\alpha_n}{1-\alpha_n} \\ &\quad \times||x_{n+1}-x_n|| + ||x_{n+1}-x_n|| + ||Tx_{n+1}-Tx_n||\big]||x_{n+1}-y_n|| \\ &\le ||x_n-y_n||^2 - 2\mu_n\alpha_n||x_{n+1}-y_n||^2 \\ &\quad +2(1-\alpha_n)(2+L)||x_{n+1}-x_n|| \times ||x_{n+1}-y_n|| \\ &= ||x_n-y_n||^2 - 2\mu_n\alpha_n||x_{n+1}-y_n||^2 + 2(1-\alpha_n)^2(2+L) \\ &\quad \times||x_n - Tx_n + \frac{\mu_n\alpha_n}{1-\alpha_n}(x_n-u)|| \times ||x_{n+1}-y_n||. \end{aligned}$$

But since $F(T)\neq\emptyset$, by Proposition 2 of [16] we have that $\{y_n\}$ is bounded. Then there exists $M_1 > 0$ such that

$$\begin{aligned} ||x_{n+1}-y_n||^2 &\le ||x_n-y_n||^2 - 2\mu_n\alpha_n||x_{n+1}-y_n||^2 \\ &\quad +2(1-\alpha_n)^2(2+L)M_1. \end{aligned} \tag{3.6}$$

Furthermore, since T is pseudocontractive, we have that

$$\begin{aligned} ||y_{n-1}-y_n|| &\le ||y_{n-1}-y_n + \frac{1-\alpha_n}{\mu_n\alpha_n}(y_{n-1}-Ty_{n-1}-(y_n-Ty_n))|| \\ &= ||\frac{1-\alpha_n+\mu_n\alpha_n}{\mu_n\alpha_n}(y_{n-1}-y_n) + \frac{1-\alpha_n}{\mu_n\alpha_n}(Ty_n - Ty_{n-1})|| \\ &= ||\frac{1-\alpha_n+\mu_n\alpha_n}{\mu_n\alpha_n}(y_{n-1}-(1-t_n)u) \\ &\quad -\frac{1-\alpha_n}{\mu_n\alpha_n}(\frac{1+\mu_{n-1}\alpha_{n-1}-\alpha_{n-1}}{1-\alpha_{n-1}})(y_{n-1}-(1-t_{n-1})u)|| \\ &= ||\big[1-\frac{(1-\alpha_n)}{\mu_n\alpha_n}(\frac{\mu_{n-1}\alpha_{n-1}}{1-\alpha_{n-1}})\big](y_{n-1}-u)|| \\ &\le |1-\frac{(1-\alpha_n)\mu_{n-1}\alpha_{n-1}}{\mu_n\alpha_n(1-\alpha_{n-1})}|(||y_{n-1}||+||u||). \end{aligned} \tag{3.7}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded from (3.7), we have:

$$\begin{aligned} ||x_n-y_n||^2 &= ||x_n-y_{n-1}+y_{n-1}-y_n||^2 \\ &\le (||x_n-y_{n-1}||+||y_{n-1}-y_n||)^2 \\ &\le ||x_n-y_{n-1}||^2 + ||y_{n-1}-y_n||\big[2||x_n-y_{n-1}||+||y_{n-1}-y_n||\big] \end{aligned}$$

$$\leq \quad ||x_n - y_{n-1}||^2 + |1 - \frac{(1-\alpha_n)\mu_{n-1}\alpha_{n-1}}{\mu_n\alpha_n(1-\alpha_{n-1})}|M_2, \tag{3.8}$$

for some positive real number M_2.

Now, from (3.6) and (3.8) we obtain:

$$\begin{aligned} ||x_{n+1} - y_n||^2 \quad &\leq \quad ||x_n - y_{n-1}||^2 - 2\mu_n\alpha_n||x_{n+1} - y_n||^2 \\ &+|1 - \frac{(1-\alpha_n)\mu_{n-1}\alpha_{n-1}}{\mu_n\alpha_n(1-\alpha_{n-1})}|M + 2(1-\alpha_n)^2(2+L)M, \end{aligned} \tag{3.9}$$

where $M = max\{M_1, M_2\}$. Thus, by (3.9) and Lemma 2.2, we get $x_{n+1} - y_n \to 0$ as $n \to \infty$. Consequently, $||x_n - y_n|| \to 0$ as $n \to \infty$.

Finally, we show that $||x_n - Tx_n|| \to 0$ as $n \to \infty$. Since $\{y_n\}$ (and hence $\{Ty_n\}$) is bounded and $t_n \to 1^-$ as $n \to \infty$, we have $||y_n - Ty_n|| \leq (1-t_n)(||Ty_n|| + ||u||) \to 0$ as $n \to \infty$. Hence, we have

$$\begin{aligned} ||x_n - Tx_n|| \quad &\leq \quad ||x_n - y_n|| + ||y_n - Ty_n|| + ||Ty_n - Tx_n|| \\ &\leq \quad (1+L)||x_n - y_n|| + ||y_n - Ty_n|| \to 0 \text{ as } n \to \infty. \end{aligned}$$

The proof is complete. □

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \to E$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Suppose that T satisfies the inward condition and every closed convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the fixed point x^* of T, which is the unique solution of the variational inequality:*

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

Proof. As in the proof of Theorem 3.1, we have that $||x_n - y_n|| \to 0$. Then, by Theorem 2 of [17], we have $y_n \to x^* \in F(T)$, which is the unique solution of the variational inequality :

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

Consequently, $\{x_n\}$ converges strongly to x^*.

If, in Theorem 3.2, we assume that E is uniformly smooth Banach space, then E has uniformly Gâteaux differentiable norm and every closed, bounded and convex subset of C has the fixed point property for nonexpansive self-mappings (see e.g., [26]). Hence we have the following corollary.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to E$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the fixed point x^* of T, which is the unique solution of the variational inequality:*

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

If, in Theorem 3.2, we assume that T is λ-strictly pseudocontractive, then it is Lipschitz with Lipschitz constant $\frac{1+\lambda}{\lambda}$ and hence we have the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $T : C \to E$ be a λ-strictly pseudocontractive mappings. Suppose that T satisfies the inward condition, $F(T) \neq \emptyset$ and every closed convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the fixed point x^* of T, which is the unique solution of the variational inequality:*

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in F(T).$$

In Theorem 3.2, if $E = H$, a real Hilbert space, then we have the following corollary.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the fixed point x^* of T nearest to u.*

We note that the method of proof of Theorem 3.2 provides the following theorem for approximating the minimum-norm point of fixed points of Lipschitz pseudocontractive non-self mappings in the Hilbert space settings.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to the minimum-norm point x^* of $F(T)$.*

In, Theorem 3.2, if we consider an accretive mapping $A : C \to E$, then we obtain the following corollary.

Corollary 3.5. *Let C be a nonempty closed convex subset of a reflexive real Banach space E with a uniformly Gâteaux differentiable norm. Let $A : C \to E$ be a Lipschitz accretive mapping with Lipschitz constant $L' \geq 0$ and $N(A) \neq \emptyset$. Suppose that $I - A$ satisfies the inward condition and every closed convex and bounded subset of C has the fixed point property for nonexpansive self-mappings. Let $\{\mu_n\} \subset (0, 1)$, u be any point in C and $\{x_n\}$ be a sequence generated from arbitrary $x_1 \in C$ by:*

$$\begin{cases} \alpha_1 := max\{\frac{1}{2}, f(x_1)\}, \\ x_{n+1} := x_n + \alpha_n\mu_n(u - x_n)) - (1 - \alpha_n)Ax_n, \\ \alpha_{n+1} \in [max\{\alpha_n, f(x_{n+1})\}, 1), \ n \geq 1, \end{cases} \tag{3.10}$$

where $f(x_n) := \inf\{\lambda \geq 0 : \lambda\mu_n(u - x_n) + x_n - (1 - \lambda)Ax_n \in C\}$. If the pair (μ_n, α_n) satisfies conditions (i)-(iv) of Theorem 3.1, then $\{x_n\}$ converges strongly to the solution of the equation $Ax = 0$, which is the unique solution of the variational inequality:

$$\langle x^* - u, J(x^* - w)\rangle \leq 0, \forall w \in N(A).$$

Proof. Since $T := (I - A)$ is a Lipschitz pseudocontractive mapping with Lipschitz constant $L := (L' + 1)$ and the fixed point of T is the solution of the equation $Ax = 0$, the conclusion follows from Theorem 3.2. □

Remark 3.6. In Theorem 3.1, if in addition, C is bounded, then the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Therefore, the condition that $F(T) \neq \emptyset$ is not required in the proof. Hence, we have the conclusions of Theorem 3.2 and Corollary 3.2 without the assumption that $F(T) \neq \emptyset$.

For λ-strictly pseudocontractive mappings, we have also the following Krasnoselskii-Mann type method in 2-uniformly smooth Banach spaces.

3.2. **Krasnoselskii-Mann Type Algorithm for λ-strictly Pseudocontractive Mapping.**

We first prove the following lemma.

Lemma 3.7. *Let C be a closed convex nonempty subset of a 2-uniformly smooth Banach space E. Suppose that E has a weakly sequentially continuous normalized duality mapping and $\{x_n\} \subset E$ is Fejèr-monotone with respect to C. Then $\{x_n\}$ converges weakly to a point in C if C contains all weak limit points of $\{x_n\}$.*

Proof. For each $x \in C$, let $p(x) = \lim_{n\to\infty} ||x_n - x||$. Then $p(x)$ is well defined for all $x \in C$. Moreover, p is proper lower semi-continuous convex function on C and $\{x_n\}$ is bounded. Then, by Lemma 2.5, p assumes its minimum at some point $x^* \in C$. It suffices to show that $\{x_n\}$ converges weakly to x^*. Suppose that $x_{n_j} \rightharpoonup z$ as $j \longrightarrow \infty$. Let $t \in (0, 1)$. Then since C is convex and E is 2-uniformly smooth, by Lemma 2.4, we have

$$||x_{n_j} - x^*||^2 + 2t\langle x^* - z, J(x_{n_j} - x^*)\rangle + 2(Kt)^2||x^* - z||^2 \geq ||x_{n_j} - (1-t)x^* - tz||^2,$$

where K is the best smooth constant of E. Since J is weakly sequentially continuous, taking the limit as $j \longrightarrow \infty$ on both sides of the above inequality, we obtain:

$$(p(x^*))^2 + 2t\langle x^* - z, J(z - x^*)\rangle + 2(Kt)^2||x^* - z||^2 \geq (p(x^*))^2,$$

which implies that

$$(p(x^*))^2 - 2t||x^* - z||^2 + 2(Kt)^2||x^* - z||^2 \geq (p(x^*))^2,$$

or

$$K^2t||x^* - z||^2 \geq ||x^* - z||^2.$$

Now, taking the limit as $t \longrightarrow 0^+$, we get that $x^* = z$. Hence, $x_n \rightharpoonup x^*$, since $\{x_{n_j}\}$ is arbitrary subsequence of $\{x_n\}$.

Now we introduce and prove Krasnoselskii-Mann type algorithm for $\lambda-$strictly pseudocontractive mappings in a 2-uniformly smooth Banach space.

Theorem 3.4. *Let C be a closed strictly convex and nonempty subset of a 2-uniformly smooth Banach space E with the best smooth constant K. Let $T : C \longrightarrow E$ be $\lambda-$strictly pseudocontractive mapping satisfying the inward condition and $F(T) \neq \emptyset$. Suppose that E has a weakly sequentially continuous normalized duality mapping and $\{x_n\}$ be a sequence generated from arbitrary x_1 in C by:*

$$\begin{cases} x_1 \in C, \\ \alpha_1 = max\{\frac{1}{2}, f(x_1)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)[(1-\alpha)x_n + \alpha T x_n], \\ \alpha_{n+1} = max\{\alpha_n, f(x_{n+1})\}, \end{cases} \tag{3.11}$$

where $f(x) := \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)[(1-\alpha)x + \alpha Tx] \in C\}$ and $\alpha \in (0, \frac{\lambda}{K^2})$. Then, $\{x_n\}$ converges weakly to a point in $F(T)$. Moreover, if $\sum(1-\alpha_n) < \infty$, then the convergence is strong.

Proof. Define $T_\alpha : C \longrightarrow E$ by $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then by Lemma 2.6 and Remark 2.7, T_α is nonexpansive and satisfies the inward condition with $F(T) = F(T_\alpha)$. Furthermore, algorithm (3.11) could be rewritten as:

$$\begin{cases} x_1 \in C, \\ \alpha_1 = max\{\frac{1}{2}, f(x_1)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_\alpha x_n, \\ \alpha_{n+1} = max\{\alpha_n, f(x_{n+1})\}, \end{cases} \tag{3.12}$$

where $f(x) := \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)T_\alpha \in C\}$. Then, as in the proof of *Theorem CM* we get that algorithm (3.12) (and hence algorithm (3.11)) is well defined.

For convergence analysis, we consider two cases. We first assume that $\sum(1-\alpha_n) = \infty$. Then by Lemma 2 of Ishikawa [10] we have that $||x_n - T_\alpha x_n|| \to 0$ as $n \to \infty$. This together with Lemma 2.8 implies that $F(T_\alpha)$ contains all weak limit points of $\{x_n\}$. Then, by Lemma 3.7, $\{x_n\}$ converges weakly to a point in $F(T_\alpha) = F(T)$.

Now, we assume that $\sum(1-\alpha_n) < \infty$. Then following the method of the proof of Theorem 1 of Colao and Marino [7], we obtain that $\{x_n\}$ converges strongly to a point in $F(T_\alpha) = F(T)$.

Remark 3.8. In all the above results, it is not difficult to observe that the same results hold true if $\frac{1}{2}$ is replaced by arbitrarily fixed point $b \in (0,1)$.

Remark 3.9. Based on Algorithm 3.1 and the nature of the mapping T, we may use different ways of choosing the parameters $\{\mu_n\}$ and $\{\alpha_n\}$. In general, we first take $\{\mu_n\} \subset (0,1)$ which goes to zero as $n \longrightarrow \infty$. Then we choose an increasing sequence $\{\alpha_n\} \subset (0,1)$ satisfying conditions $(i)-(iv)$ of Theorem 3.1 and $\alpha_n(\mu_n u + (1-\mu_n)x_n) + (1-\alpha_n)Tx_n \in C$ for each $n \geq 1$, where the last condition shows that $\alpha_{n+1} \in \big[\max\{\alpha_n, f(x_{n+1})\}, 1\big)$ for each $n \geq 1$. Hence, the pair (α_n, μ_n) satisfies all conditions of the theorem.

Remark 3.10. In case, we have a set $D \subseteq C$, convex, such that $Tx \in D$ for all $x \in D$, we may take $u \in D$ and $x_1 \in C$, arbitrary and $\{\mu_n\} \subset (0,1)$ which goes to zero as $n \longrightarrow \infty$. Then we choose an increasing sequence $\{\alpha_n\} \subset (0,1)$ satisfying conditions $(i)-(iv)$ of Theorem 3.1 such that $\alpha_1(\mu_1 u + (1-\mu_1)x_1) + (1-\alpha_1)Tx_1 \in D$. Thus, we obtain that for each $n \geq 1$, we have $\alpha_n(\mu_n u + (1-\mu_n)x_n) + (1-\alpha_n)Tx_n \in D \subseteq C$ and hence the pair (α_n, μ_n) satisfies all conditions of the theorem.

Remark 3.11. Theorem 3.1 improves *Theorem CZ* in the sense that it extends the class of Lipschitz pseudocontractive self-mappings to the class of Lipschitz pseudocontractive non-self mappings.

Remark 3.12. Theorem 3.2 extends *Theorem CM* in the sense that it provides a convergent scheme for approximating fixed points of Lipschitz pseudocontractive non-self mappings more general than nonexpansive non-self mappings in Banach spaces more general than Hilbert spaces.

4. Numerical example

Now, we give an example of a Lipschitz pseudocontractive mapping that satisfies the conditions of Theorem 3.2 and some numerical experiment results to explain the conclusion of the theorem as follows:

Example 4.1. Let $H = \mathbb{R}$ with Euclidean norm. Let $C = [-1, 2]$ and $T : C \longrightarrow \mathbb{R}$ be defined by

$$Tx = \begin{cases} -3x, & x \in [-1, 0), \\ x, & x \in [0, 1), \\ x - (x-1)^2, & x \in [1, 2]. \end{cases} \tag{4.1}$$

Then we observe that T satisfies the inward condition and $F(T) = [0, 1]$. Moreover, $\langle x - Tx - (y - Ty), x - y\rangle \geq 0$ for all $x, y \in C$. Hence, T is pseudocontractive mapping. To show that T is a Lipschitz mapping, we consider the following cases.

Case 1: Let $x, y \in [-1, 0)$. Then we have:

$$|Tx - Ty| = |-3x + 3y| = 3|x - y|.$$

Case 2: Let $x, y \in [0, 1)$. Then we have:

$$|Tx - Ty| = |x - y|.$$

Case 3: Let $x, y \in [1, 2]$. Then we have:

$$\begin{aligned} |Tx - Ty| &= |x - (x-1)^2 - y + (y-1)^2| = |3x - 3y + y^2 - x^2| \\ &\leq 3|x - y| + 4|x - y| = 7|x - y|. \end{aligned}$$

Case 4: Let $x \in [-1, 0)$ and $y \in [0, 1)$. Then we have:

$$\begin{aligned} |Tx - Ty| = |-3x - y| &= |3x + y| \\ &= |x - y + 2x + 2y| \leq |x - y| + 2|x + y| \\ &\leq |x - y| + 2|x - y| = 3|x - y|. \end{aligned}$$

Case 5: Let $x \in [-1, 0)$ and $y \in [1, 2]$. Then we have:

$$\begin{aligned} |Tx - Ty| = |-3x - y + (y-1)^2| &\leq |3x + y| + (y-1)^2 \\ &\leq |x - y + 2x + 2y| + |x - y| \\ &\leq 2|x + y| + |x - y| + |x - y| \leq 4|x - y|. \end{aligned}$$

Case 6: Let $x \in [0, 1)$ and $y \in [1, 2]$. Then we have:

$$\begin{aligned} |Tx - Ty| = |x - y + (y-1)^2| &\leq |x - y| + (y-1)^2 \\ &\leq |x - y| + |y - x| = 2|x - y|. \end{aligned}$$

From the above Cases, we conclude that T is Lipschitz with Lipschitz constant $L = 7$.

Now, for $u = -0.8$ and $x_1 = 2$ in $C = [-1, 2]$, following Remark 3.9, we take $\mu_n = \frac{1}{(n+5)^{0.3}[(n+5)^{0.6}-1]}$ and choose $\alpha_n = 1 - \frac{1}{(n+5)^{0.6}}$. Then we see that the pair (μ_n, α_n) satisfies (i)-(iv) of the conditions of Theorem 3.2 and $\alpha_n(\mu_n u + (1 - \mu_n)x_n) + (1-\alpha_n)Tx_n \in C$, for each $n \geq 1$. Thus, Algorithm (3.1) converges strongly to $0 = P_{F(T)}(-0.8)$ in $F(T)$ (see, Figure 1).

On the other hand, in this particular example, note that T is a self mapping on $D = [0, 2] \subset C = [-1, 2]$. Now, if we consider $u = 0.6 \in D$ and $x_1 = -1 \in C$, following Remark 3.10, we may take $\mu_n = \frac{1}{(n+5)^{0.3}[(n+5)^{0.6}-1]}$, and choose $\alpha_n = 1 - \frac{1}{(n+5)^{0.6}}$. Then we obtain that the pair (μ_n, α_n) satisfies (i)-(iv) of the conditions of Theorem 3.2 and $\alpha_1(\mu_1 u + (1 - \mu_1)x_1) + (1 - \alpha_1)Tx_1 \in D$ and hence we get $\alpha_n(\mu_n u + (1 - \mu_n)x_n) + (1 - \alpha_n)Tx_n \in D \subset C$ for all $n \geq 1$. Therefore, Algorithm (3.1) converges strongly to $0.6 = P_{F(T)}(0.6)$ in $F(T)$ (see, Figure 1). The following graph is obtained using MATLAB version 7.5.0.342(R2007b).

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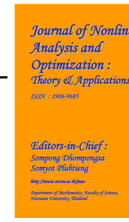

EXISTENCE, UNIQUENESS AND POSITIVITY OF SOLUTIONS FOR A NEUTRAL NONLINEAR PERIODIC DYNAMIC EQUATION ON A TIME SCALE

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ABSTRACT. Let $\mathbb{T}$ be a periodic time scale. We use Krasnoselskii's fixed point theorem, to show new results on the existence and positivity of solutions for the nonlinear periodic dynamic equation with variable delay of the form

$$x^{\triangle}(t) = -a(t)x(t) + (Q(t, x(g(t))))^{\triangle} + G(t, x(t), x(g(t))),$$
$$x(t+T) = x(t).$$

Also, by transforming the problem to an integral equation we are able, using the contraction mapping principle, to show that the periodic solution is unique.

KEYWORDS : Fixed point theory, Nonlinear neutral dynamic equation, Periodic solutions, Positivity, Time scales.
AMS Subject Classification: 34K13, 34K30, 34L30.

1. INTRODUCTION

In recent years, there have been a few papers written on the existence of periodic solutions, nontrivial periodic solutions and positive periodic solutions for several classes of functional differential and dynamic equations with delays, which arise from a number of mathematical ecological models, economical and control models, physiological and population models and other models, see the references [1]-[13] and references therein.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this paper, we are interested in the analysis of qualitative theory of periodic solutions of dynamic equations. Motivated by the papers [1, 2, 3, 4, 5, 6, 11, 12, 13] and the references therein, we consider the following nonlinear neutral periodic dynamic equation with variable delay

$$x^{\triangle}(t) = -a(t)x(t) + (Q(t, x(g(t))))^{\triangle} + G(t, x(t), x(g(t))),$$
$$x(t+T) = x(t), \tag{1.1}$$

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where $T > 0$ be fixed, the nonlinear terms Q and G are $L^1_\triangle$-Carathéodory functions and the function $a \in L^1[0, T]$ is bounded. Throughout this paper we assume that $g : \mathbb{T} \to \mathbb{T}$ is strictly increasing so that the function $x(g(t))$ is well defined over $\mathbb{T}$. Our purpose here is to use the Krasnoselskii's fixed point theorem to show the existence and positivity of solutions on time scales for periodic dynamic equation (1.1). We have to transform (1.1) into an integral equation written as a sum of two mapping; one is a contraction and the other is a completely continuous operator. After that, we use the Krasnoselskii fixed point theorem, to show the existence and positivity of periodic solutions for equation (1.1). Also, transforming equation (1.1) to an integral equation enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

A special case of equation (1.1) with $\mathbb{T} = \mathbb{R}$, in [2] we have investigated the existence, uniqueness and positivity of periodic solution for equation (1.1) by the Krasnoselskii's fixed point theorem and the contraction mapping principle. The results presented in this paper extend the main results in [2].

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [14]. Also we present the inversion of neutral nonlinear periodic dynamic equation (1.1). In Section 3, we present our main results on existence, uniqueness and positivity.

2. PRELIMINARIES

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1], [4], [6]-[10], [11], [12] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [9] and [10] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Atici et al. [7] and Kaufmann and Raffoul [11]. The following two definitions are borrowed from [7] and [11].

Definition 2.1. We say that a time scale $\mathbb{T}$ is periodic if there exist a $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive ω is called the period of the time scale.

Below are examples of periodic time scales taken from [11].

Example 2.2. The following time scales are periodic.

(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, $h > 0$ has period $\omega = 2h$.
(2) $\mathbb{T} = h\mathbb{Z}$ has period $\omega = h$.
(3) $\mathbb{T} = \mathbb{R}$.
(4) $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $\omega = 1$.

Remark 2.3 ([11])**.** All periodic time scales are unbounded above and below.

Definition 2.4. Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scales with the period ω. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.5 ([11]). If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function μ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period ω.

Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales ([9], Theorem 1.93).

Theorem 2.6 (Chain Rule). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our second theorem is the substitution rule ([9], Theorem 1.98).

Theorem 2.7 (Substitution). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t) \nu^{\triangle}(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s) \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log(1 + \mu(z) p(z)) \triangle z\right). \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t) y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.8 ([9]). *Let $p, q \in \mathcal{R}$. Then*
(i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t, s)$;
(iii) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1 + \mu(t) p(t)}$;
(iv) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;
(v) $e_p(t, s) e_p(s, r) = e_p(t, r)$;
(vi) $e_p^{\triangle}(., s) = p e_p(., s)$ *and* $\left(\dfrac{1}{e_p(., s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(., s)}$.

Theorem 2.9 ([8], Theorem 2.1). *Let $\mathbb{T}$ be a periodic time scale with period $\omega > 0$. If $p \in C_{rd}(\mathbb{T})$ is a periodic function with the period $T = n\omega$, then*

$$\int_{a+T}^{b+T} p(u) \triangle u = \int_a^b p(u) \triangle u, \ e_p(b + T, a + T) = e_p(b, a) \text{ if } p \in \mathcal{R},$$

and $e_p(t + T, t)$ is independent of $t \in \mathbb{T}$ whenever $p \in \mathcal{R}$.

Lemma 2.10 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u) \triangle u\right), \ \forall t \in \mathbb{T}.$$

Corollary 2.11 ([1]). *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u) \triangle u\right) < 1.$$

We state Krasnoselskii's fixed point theorem which enables us to prove the existence and positivity of periodic solutions to (1.1). For its proof we refer the reader to [14].

Theorem 2.12 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into S such that*
(i) $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,
(ii) A is completely continuous,
(iii) B is a contraction mapping.
Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly.

Define $P_T = \{\varphi : \mathbb{T} \to \mathbb{R} \mid \varphi \in C \text{ and } \varphi(t+T) = \varphi(t)\}$ where C is the space of continuous real-valued functions on $\mathbb{T}$. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|\varphi\| = \sup_{t \in \mathbb{T}} |\varphi(t)| = \sup_{t \in [0,T]} |\varphi(t)|.$$

We will need the following lemma whose proof can be found in [11].

Lemma 2.13. *Let $x \in P_T$. Then $\|x^\sigma\| = \|x \circ \sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

The following definition is essential in our analysis.

Definition 2.14. A function $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is an $L^1_\triangle$-Carathéodory function if it satisfies the following conditions

$(c1)$ For each $z \in \mathbb{R}^n$, the mapping $t \to F(t,z)$ is $\triangle$-measurable.

$(c2)$ For almost all $t \in [0,T]$, the mapping $t \to F(t,z)$ is continuous on $\mathbb{R}^n$.

$(c3)$ For each $r > 0$, there exists $f_r \in L^1_\triangle([0,T], \mathbb{R})$ such that for almost all $t \in [0,T]$ and for all z such that $|z| < r$, we have $|F(t,z)| \leq f_r(t)$.

In this paper we use the notation $\gamma = -a$ and assume that $\gamma \in \mathcal{R}^+$ and will assume that the following conditions hold.

$(h1)$ $\gamma \in L^1_\triangle([0,T], \mathbb{R})$ is bounded, satisfies $\gamma(t+T) = \gamma(t)$ for all t and

$$1 - e_\gamma(t, t-T) \equiv \frac{1}{\eta} \neq 0.$$

$(h2)$ $g \in P_T$.

$(h3)$ Q and G are $L^1_\triangle$-Carathéodory functions, and $Q(t+T, x) = Q(t,x)$, $G(t+T, x, y) = Q(t,x,y)$ for all t.

We have to invert equation (1.1). For this, we use the variation of parameter formula to rewrite the equation as an integral equation suitable for Krasnoselskii theorem and the contraction mapping principle.

Lemma 2.15. *Suppose (h1)-(h3) hold. If $x \in P_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= Q(t, x(g(t))) \\ &\quad + \eta \int_{t-T}^{t} \left[G(s, x(s), x(g(s))) + \gamma(s) Q(s, x(g(s)))\right] e_\gamma(t, \sigma(s)) \triangle s. \end{aligned} \tag{2.2}$$

Proof. Let $x \in P_T$ be a solution of (1.1). First we write this equation as

$$\begin{aligned} &(x(t) - Q(t, x(g(t))))^{\triangle} - \gamma(t)(x(t) - Q(t, x(g(t)))) \\ &= G(t, x(t), x(g(t))) + \gamma(t) Q(t, x(g(t))). \end{aligned}$$

Multiply both sides of the above equation by $e_{\ominus\gamma}(\sigma(t), 0)$ we get

$$\begin{aligned} &\left\{(x(t) - Q(t, x(g(t))))^{\triangle} - \gamma(t)(x(t) - Q(t, x(g(t))))\right\} e_{\ominus\gamma}(\sigma(t), 0) \\ &= \{G(t, x(t), x(g(t))) + \gamma(t) Q(t, x(g(t)))\} e_{\ominus\gamma}(\sigma(t), 0). \end{aligned}$$

Since $e_{\ominus\gamma}(t, 0)^{\triangle} = -\gamma(t) e_{\ominus\gamma}(\sigma(t), 0)$ we find

$$\begin{aligned} &\left[(x(t) - Q(t, x(g(t)))) e_{\ominus\gamma}(t, 0)\right]^{\triangle} \\ &= \left[G(t, x(t), x(g(t))) + \gamma(t) Q(t, x(g(t)))\right] e_{\ominus\gamma}(\sigma(t), 0). \end{aligned}$$

Taking the integral from $t - T$ to t, we obtain

$$\begin{aligned} &\int_{t-T}^{t} \left[(x(s) - Q(s, x(g(s)))) e_{\ominus\gamma}(s, 0)\right]^{\triangle} \triangle s \\ &= \int_{t-T}^{t} \left[G(s, x(s), x(g(s))) + \gamma(s) Q(s, x(g(s)))\right] e_{\ominus\gamma}(\sigma(s), 0) \triangle s. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &(x(t) - Q(t, x(g(t)))) e_{\ominus\gamma}(t, 0) \\ &- (x(t-T) - Q(t-T, x(g(t-T)))) e_{\ominus\gamma}(t-T, 0) \\ &= \int_{t-T}^{t} \left[G(s, x(s), x(g(s))) + \gamma(s) Q(s, x(g(s)))\right] e_{\ominus\gamma}(\sigma(s), 0) \triangle s. \end{aligned}$$

Dividing both sides of the above equation by $e_{\ominus\gamma}(t, 0)$ and using the fact that $x(t - T) = x(t)$, (h1)-(h3) and

$$\frac{e_{\ominus\gamma}(t-T, 0)}{e_{\ominus\gamma}(t, 0)} = e_\gamma(t, t-T), \quad \frac{e_{\ominus\gamma}(\sigma(s), 0)}{e_{\ominus\gamma}(t, 0)} = e_\gamma(t, \sigma(s)),$$

we obtain

$$\begin{aligned} &x(t) - Q(t, x(g(t))) \\ &= \eta \int_{t-T}^{t} \left[G(s, x(s), x(g(s))) + \gamma(s) Q(s, x(g(s)))\right] e_\gamma(t, \sigma(s)) \triangle s. \end{aligned}$$

Since each step is reversible, the converse follows easily. This completes the proof. □

3. EXISTENCE RESULTS

We present our existence results in this section. To this end, we first define the operator H by

$$(H\varphi)(t) = Q(t, \varphi(g(t))) + \eta \int_{t-T}^{t} [G(s, \varphi(s), \varphi(g(s))) + \gamma(s) Q(s, \varphi(g(s)))] e_{\gamma}(t, \sigma(s)) \triangle s, \tag{3.1}$$

From Lemma 2.15 we see that fixed points of H are solutions of (1.1) and vice versa. In order to employ Theorem 2.12 we need to express the operator H as the sum of two operators, one of which is completely continuous and the other of which is a contraction. Let $(H\varphi)(t) = (A\varphi)(t) + (B\varphi)(t)$ where

$$(A\varphi)(t) = \eta \int_{t-T}^{t} [G(s, \varphi(s), \varphi(g(s))) + \gamma(s) Q(s, \varphi(g(s)))] e_{\gamma}(t, \sigma(s)) \triangle s, \tag{3.2}$$

and

$$(B\varphi)(t) = Q(t, \varphi(g(t))). \tag{3.3}$$

Our first lemma in this section shows that $A : P_T \to P_T$ is completely continuous.

Lemma 3.1. *Suppose that conditions* $(h1) - (h3)$ *hold. Then* $A : P_T \to P_T$ *is completely continuous.*

Proof. We first show that $A : P_T \to P_T$. Clearly, if φ is continuous, then $A\varphi$ is. From (3.2) and conditions $(h1) - (h3)$, it follows trivially that $e_{\gamma}(t+T, \sigma(s)+T) = e_{\gamma}(t, \sigma(s))$ by Theorem 2.9. Consequently, we have that

$$(A\varphi)(t+T) = (A\varphi)(t),$$

by Theorem 2.7. That is, if $\varphi \in P_T$ then $A\varphi \in P_T$.

To see that A is continuous. Let $\{\varphi_i\} \subset P_T$ be such that $\varphi_i \to \varphi$ as $i \to \infty$. By the Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{i\to\infty} |(A\varphi_i)(t) - (A\varphi)(t)| \\
&\leq \lim_{i\to\infty} \eta \int_{t-T}^{t} [|G(s, \varphi_i(s), \varphi_i(g(s))) - G(s, \varphi(s), \varphi(g(s)))| \\
&+ |\gamma(s)| |Q(s, \varphi_i(g(s))) - Q(s, \varphi(g(s)))|] e_{\gamma}(t, \sigma(s)) \triangle s \\
&= \eta \int_{t-T}^{t} \Big[\lim_{i\to\infty} |G(s, \varphi_i(s), \varphi_i(g(s))) - G(s, \varphi(s), \varphi(g(s)))| \\
&+ |\gamma(s)| \lim_{i\to\infty} |Q(s, \varphi_i(g(s))) - Q(s, \varphi(g(s)))|\Big] e_{\gamma}(t, \sigma(s)) \triangle s \\
&= 0.
\end{aligned}$$

Hence $A : P_T \to P_T$ is continuous.

Finally, we show that A is completely continuous. Let $\mathfrak{B} \subset P_T$ be a closed bounded subset and let C be such that $\|\varphi\| \leq C$ for all $\varphi \in \mathfrak{B}$. then

$$\begin{aligned}
|(A\varphi)(t)| &\leq \eta \int_{t-T}^{t} [|G(s, \varphi(s), \varphi(g(s)))| + |\gamma(s)| |Q(s, \varphi(g(s)))|] e_{\gamma}(t, \sigma(s)) \triangle s \\
&\leq \eta N \left\{ \int_{t-T}^{t} g_C(s) \triangle s + \int_{t-T}^{t} |\gamma(s)| q_C(s) \triangle s \right\}
\end{aligned}$$

$$\leq \eta N\left\{\int_{t-T}^{t} g_C(s)\triangle s+\alpha\int_{t-T}^{t} q_C(s)\triangle s\right\}\equiv K,$$

where $N = \sup_{s\in[t-T,t]} e_\gamma(t,\sigma(s))$ and $\alpha = \sup_{s\in[t-T,t]} |\gamma(s)|$. And so, the family of functions $A\varphi$ is uniformly bounded.

Again, let $\varphi \in \mathfrak{B}$. Without loss of generality, we can pick $t_1 < t_2$ such that $t_2 - t_1 < T$. Then

$$\begin{aligned}
&|(A\varphi)(t_2)-(A\varphi)(t_1)|\\
&=\eta\left|\int_{t_2-T}^{t_2}[G(s,\varphi(s),\varphi(g(s)))+\gamma(s)Q(s,\varphi(g(s)))]e_\gamma(t,\sigma(s))\triangle s\right.\\
&\left.-\int_{t_1-T}^{t_1}[G(s,\varphi(s),\varphi(g(s)))+\gamma(s)Q(s,\varphi(g(s)))]e_\gamma(t,\sigma(s))\triangle s\right|.
\end{aligned}$$

We can rewrite the left hand side as the sum of three integrals. We obtain the following

$$\begin{aligned}
&|(A\varphi)(t_2)-(A\varphi)(t_1)|\\
&\leq \eta\int_{t_1}^{t_2}[|G(s,\varphi(s),\varphi(g(s)))|+|\gamma(s)||Q(s,\varphi(g(s)))|]e_\gamma(t_2,\sigma(s))\triangle s\\
&+\eta\int_{t_2-T}^{t_1}[|G(s,\varphi(s),\varphi(g(s)))|+|\gamma(s)||Q(s,\varphi(g(s)))|]\\
&\times|e_\gamma(t_2,\sigma(s))-e_\gamma(t_1,\sigma(s))|\triangle s\\
&+\eta\int_{t_1-T}^{t_2-T}[|G(s,\varphi(s),\varphi(g(s)))|+|\gamma(s)||Q(s,\varphi(g(s)))|]e_\gamma(t_1,\sigma(s))\triangle s\\
&\leq 2\eta N\left\{\int_{t_1}^{t_2} g_C(s)+\alpha q_C(s)\triangle s\right\}\\
&+\eta\int_{t_2-T}^{t_1}[g_C(s)+\alpha q_C(s)]|e_\gamma(t_2,\sigma(s))-e_\gamma(t_1,\sigma(s))|\triangle s.
\end{aligned}$$

Now $\int_{t_1}^{t_2} g_C(s)+\alpha q_C(s)\triangle s \to 0$ as $(t_2-t_1)\to 0$. Also, since

$$\begin{aligned}
&\int_{t_2-T}^{t_1}[g_C(s)+\alpha q_C(s)]|e_\gamma(t_2,\sigma(s))-e_\gamma(t_1,\sigma(s))|\triangle s\\
&\leq\int_{0}^{T}[g_C(s)+\alpha q_C(s)]|e_\gamma(t_2,\sigma(s))-e_\gamma(t_1,\sigma(s))|\triangle s,
\end{aligned}$$

and $|e_\gamma(t_2,\sigma(s))-e_\gamma(t_1,\sigma(s))| \to 0$ as $(t_2-t_1)\to 0$, then by the Dominated Convergence Theorem,

$$\int_{t_2-T}^{t_1}[g_C(s)+\alpha q_C(s)]|e_\gamma(t_2,\sigma(s))-e_\gamma(t_1,\sigma(s))|\triangle s\to 0,$$

as $(t_2-t_1)\to 0$. Thus $|(A\varphi)(t_2)-(A\varphi)(t_1)|\to 0$ as $(t_2-t_1)\to 0$ independently of $\varphi\in\mathfrak{B}$. As such, the family of functions $A\varphi$ is equicontinuous on $\mathfrak{B}$.

By the Arzelà-Ascoli Theorem, A is completely continuous and the proof is complete. □

We need the following condition on the nonlinear term Q.

$(h4)$ the function $Q(t,x)$ is Lipschitz continuous in x. That is, there exists $L>0$ such that

$$|Q(t,x)-Q(t,y)| \leq L\|x-y\|.$$

Our next lemma gives a sufficient condition under which $B: P_T \to P_T$ is a contraction.

Lemma 3.2. *Suppose that conditions $(h3)$ and $(h4)$ hold and $L<1$. Then $B: P_T \to P_T$ is a contraction.*

Proof. From (3.3) and conditions $(h2)$ and $(h3)$, we have that

$$(B\varphi)(t+T) = (B\varphi)(t).$$

That is, if $\varphi \in P_T$ then $B\varphi \in P_T$.

To see that B is a contraction. Let $\varphi, \psi \in P_T$ we have

$$\begin{aligned} \|B(\varphi)-B(\psi)\| &= \sup_{t\in[0,T]} |(B\varphi)(t)-(B\psi)(t)| \\ &\leq L \sup_{t\in[0,T]} |\varphi(g(t))-\psi(g(t))| \\ &\leq L\|\varphi-\psi\|. \end{aligned}$$

Hence $B: P_T \to P_T$ is a contraction. □

We need the following conditions on the nonlinear terms Q and G.

$(h5)$ There exists periodic functions $q_1, q_2 \in L^1_\triangle[0,T]$, with period T, such that

$$|Q(t,x)| \leq q_1(t)|x| + q_2(t),$$

for all $x \in \mathbb{R}$.

$(h6)$ There exists periodic functions $g_1, g_2, g_3 \in L^1_\triangle[0,T]$, with period T, such that

$$|G(t,x,y)| \leq g_1(t)|x| + g_2(t)|y| + g_3(t),$$

for all $x, y \in \mathbb{R}$.

Also, we now define some quantities that will be used in the following theorem. Let $\delta = \sup_{t\in[0,T]} e_\gamma(t,\sigma(s))$, $\theta = \sup_{t\in[0,T]} |Q(t,0)|$, $\lambda = \int_0^T |q_1(s)|\triangle s$, $\mu = \int_0^T |q_2(s)|\triangle s$, $\beta = \int_0^T |g_1(s)|\triangle s$, $\gamma = \int_0^T |g_2(s)|\triangle s$, $\Gamma = \int_0^T |g_3(s)|\triangle s$.

Theorem 3.3. *Suppose that conditions $(h1)-(h6)$ hold and $L<1$. Suppose there exists a positive constant J satisfying the inequality*

$$\theta + \eta\delta(\alpha\mu+\Gamma) + (L+\eta\delta(\alpha\lambda+\beta+\gamma))J \leq J.$$

Then (1.1) has a solution $\varphi \in P_T$ such that $\|\varphi\| \leq J$.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma 3.1, the operator $A: \mathbb{M} \to P_T$ is completely continuous. Since $L<1$, then by Lemma 3.2, the operator $B: \mathbb{M} \to P_T$ is a contraction. Conditions (ii) and (iii) of Theorem 2.12 are satisfied. We need to show that condition (i) is fulfilled. To this end, let $\varphi, \psi \in \mathbb{M}$. Then

$$\begin{aligned} &|(A\varphi)(t)+(B\psi)(t)| \\ &\leq |Q(t,\psi(g(t)))| + \eta\int_{t-T}^{t} |G(s,\varphi(s),\varphi(g(s)))| e_\gamma(t,\sigma(s))\triangle s \\ &+ \eta\int_{t-T}^{t} |\gamma(s)||Q(s,\varphi(g(s)))| e_\gamma(t,\sigma(s))\triangle s \\ &\leq LJ + \theta + \eta\delta(\beta J + \gamma J + \Gamma) + \eta\alpha\delta(\lambda J + \mu) \end{aligned}$$

$$= \theta + \eta\delta\left(\alpha\mu + \Gamma\right) + \left(L + \eta\delta\left(\alpha\lambda + \beta + \gamma\right)\right) J \le J.$$

Thus $\|A\varphi + B\psi\| \le J$ and so $A\varphi + B\psi \in \mathbb{M}$. All the conditions of Theorem 2.12 are satisfied and consequently the operator H defined in (3.1) has a fixed point in $\mathbb{M}$. By Lemma 2.15 this fixed point is a solution of (1.1) and the proof is complete. □

The conditions $(h5)$ and $(h6)$ are global conditions on the functions Q and G. In the next theorem we replace this conditions with the following local conditions.

$(h5^*)$ There exists periodic functions $q_1^*, q_2^* \in L^1_\triangle[0,T]$, with period T, such that

$$|Q(t,x)| \le q_1^*(t)|x| + q_2^*(t),$$

for all x with $|x| \le J$.

$(h6^*)$ There exists periodic functions $g_1^*, g_2^*, g_3^* \in L^1_\triangle[0,T]$, with period T, such that

$$|G(t,x,y)| \le g_1^*(t)|x| + g_2^*(t)|y| + g_3^*(t),$$

for all x, y with $|x| \le J$ and $|y| \le J$.
The constants λ^*, μ^* and β^*, γ^*, Γ^* are defined as before with the understanding that the functions q_1^*, q_2^* and g_1^*, g_2^*, g_3^* are those from conditions $(h5^*)$ and $(h6^*)$, respectively.

Theorem 3.4. *Suppose that conditions $(h1)-(h4)$, $(h5^*)$ and $(h6^*)$ hold and $L < 1$. Suppose there exists a positive constant J satisfying the inequality*

$$\theta + \eta\delta\left(\alpha\mu^* + \Gamma^*\right) + \left(L + \eta\delta\left(\alpha\lambda^* + \beta^* + \gamma^*\right)\right) J \le J.$$

Then (1.1) has a solution $\varphi \in P_T$ such that $\|\varphi\| \le J$.

The proof of the above theorem parallels that of Theorem 3.3.

For our next result, we give conditions for which there exists a unique solution of (1.1). We replace conditions $(h5)$ and $(h6)$ with the following conditions.

$(h5^\dagger)$ There exists periodic function $q_1^\dagger \in L^1_\triangle[0,T]$, with period T, such that

$$|Q(t,x) - Q(t,x)| \le q_1^\dagger(t)|x-y|,$$

for all $x, y \in \mathbb{R}$.

$(h6^\dagger)$ There exists periodic functions $g_1^\dagger, g_2^\dagger \in L^1_\triangle[0,T]$, with period T, such that

$$|G(t,x,y) - G(t,z,w)| \le g_1^\dagger(t)|x-z| + g_2^\dagger(t)|y-w|,$$

for all $x, y, z, w \in \mathbb{R}$.
The constants $\lambda^\dagger$ and $\beta^\dagger$, $\gamma^\dagger$ are defined as before with the understanding that the functions $q_1^\dagger$ and $g_1^\dagger$, $g_2^\dagger$ are those from conditions $(h5^\dagger)$ and $(h6^\dagger)$, respectively.

Theorem 3.5. *Suppose that conditions $(h1)-(h4)$, $(h5^\dagger)$ and $(h6^\dagger)$ hold. If*

$$L + \eta\delta\left(\alpha\lambda^\dagger + \beta^\dagger + \gamma^\dagger\right) < 1,$$

then (1.1) has a unique T-periodic solution.

Proof. Let $\varphi, \psi \in P_T$. By (3.1) we have for all t,

$$\begin{aligned}
&|(H\varphi)(t) - (H\psi)(t)| \\
&\le |Q(t,\varphi(g(t))) - Q(t,\psi(g(t)))| \\
&+ \eta\int_{t-T}^{t}\left[|G(s,\varphi(s),\varphi(g(s))) - G(s,\psi(s),\psi(g(s)))|\right. \\
&\left.+ |\gamma(s)|\,|Q(s,\varphi(g(s))) - Q(s,\psi(g(s)))|\right] e_\gamma(t,\sigma(s))\triangle s \\
&\le L\|\varphi - \psi\| + \eta\delta\left(\alpha\lambda^\dagger + \beta^\dagger + \gamma^\dagger\right)\|\varphi - \psi\|.
\end{aligned}$$

Hence, $\|H\varphi - H\psi\| \leq \left(L + \eta\delta\left(\alpha\lambda^{\dagger} + \beta^{\dagger} + \gamma^{\dagger}\right)\right)\|\varphi - \psi\|$. By the contraction mapping principle, H has a fixed point in P_T and by Lemma 2.15, this fixed point is a solution of (1.1). The proof is complete. □

For our last result, we give sufficient conditions under which there exists positive solutions of equation (1.1). We begin by defining some new quantities. Let

$$m = \min_{s\in[t-T,t]} e_{\gamma}(t, \sigma(s)), \; M = \max_{s\in[t-T,t]} e_{\gamma}(t, \sigma(s)).$$

Given constants $0 < \mathfrak{L} < \mathfrak{K}$, define the set $\mathbb{M}_1 = \{\varphi \in P_T : \mathfrak{L} \leq \varphi(t) \leq \mathfrak{K}, \; t \in [0, T]\}$.

Assume the following conditions hold.

$(h7)$ There exists constants $0 < q^* < L^*$ such that $q^*\mathfrak{L} \leq Q(t, \rho) \leq L^*\mathfrak{K}$ for all $\rho \in \mathbb{M}_1$ and $t \in [0, T]$.

$(h8)$ There exists constants $0 < \mathfrak{L} < \mathfrak{K}$ such that

$$\frac{(1 - q^*)\mathfrak{L}}{\eta m T} \leq G(s, \sigma, \rho) + \gamma(s) Q(s, \rho) \leq \frac{(1 - L^*)\mathfrak{K}}{\eta M T},$$

for all $\sigma, \rho \in \mathbb{M}_1$ and $s \in [t - T, t]$.

Theorem 3.6. *Suppose that conditions $(h1) - (h4)$, $(h7)$ and $(h8)$ hold and $L < 1$. Then there exists a positive periodic solution of (1.1).*

Proof. As in the proof of Theorem 3.3, we just need to show that condition (i) of Theorem 2.12 is satisfied. Let $\varphi, \psi \in \mathbb{M}_1$. Then

$$\begin{aligned}
&(A\varphi)(t) + (B\psi)(t) \\
&= Q(t, \psi(g(t))) + \eta\int_{t-T}^{t} \left[G(s, \varphi(s), \varphi(g(s))) + \gamma(s) Q(s, \varphi(g(s)))\right] e_{\gamma}(t, \sigma(s)) \triangle s \\
&\geq q^*\mathfrak{L} + \eta m T\frac{(1 - q^*)\mathfrak{L}}{\eta m T} = \mathfrak{L}.
\end{aligned}$$

Likewise

$$(A\varphi)(t) + (B\psi)(t) \leq L^*\mathfrak{K} + \eta M T\frac{(1 - L^*)\mathfrak{K}}{\eta M T} = \mathfrak{K}.$$

By Theorem 2.12, the operator H has a fixed point in $\mathbb{M}_1$. This fixed point is a positive solution of (1.1) and the proof is complete. □

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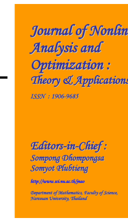

FIXED POINTS VIA wt-DISTANCE IN b-METRIC SPACES ENDOWED WITH A GRAPH

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ABSTRACT. We prove some fixed point theorems by using the concept of wt-distance in a b-metric space endowed with a graph. Our results will improve and supplement several well known comparable results in the existing literature.

KEYWORDS : b-metric; wt-distance; reflexive digraph; fixed point.
AMS Subject Classification: 54H25, 47H10

1. INTRODUCTION

Fixed point theory plays an important role in applications of many branches of mathematics such as variational and linear inequalities, linear algebra, mathematical models, optimization and the like. In 1989, Bakhtin [4] introduced the concept of b-metric spaces as a generalization of metric spaces and generalized the famous Banach contraction principle in metric spaces to b-metric spaces. Afterwards, several articles have been dedicated to the improvement of fixed point theory for single valued and multivalued mappings in b-metric spaces. In [15], Hussain et. al. introduced the concept of wt-distance on b-metric spaces and obtained some fixed point theorems in partially ordered b-metric spaces. In recent investigations, the study of fixed point theory endowed with a graph occupies a prominent place in many aspects. In 2005, Echenique [12] studied fixed point theory by using graphs. Espinola and Kirk [13] applied fixed point results in graph theory. Motivated by the idea given in some recent work on metric spaces with a graph (see [2, 3, 5, 7, 8, 17]), we reformulated some important fixed point results in metric spaces to b-metric spaces endowed with a graph by using wt-distance. As some consequences of our results, we obtain Banach Contraction Principle, Kannan fixed point theorem, Theorem 4[18] and some fixed point theorems in partially ordered b-metric spaces. Finally, some examples are provided to illustrate our results.

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2. SOME BASIC CONCEPTS

In this section, we recall some standard notations, definitions, and necessary results in b-metric spaces.

Definition 2.1. [10] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It seems important to note that if $s = 1$, then the triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above remarks.

Example 2.2. [19] Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.3. [20] Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

Definition 2.4. [6] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.
(ii) (x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 2.5. [6] In a b-metric space (X, d), the following assertions hold:

(i) A convergent sequence has a unique limit.
(ii) Each convergent sequence is Cauchy.
(iii) In general, a b-metric is not continuous.

Theorem 2.6. [1] *Let (X, d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty}\; d(x_n, y_n) \leq \limsup_{n\to\infty}\; d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty}\; d(x_n, z) \leq \limsup_{n\to\infty}\; d(x_n, z) \leq s d(x, z).$$

Definition 2.7. [15] Let (X, d) be a b-metric space with constant $s \geq 1$. Then a function $p : X \times X \to [0, \infty)$ is called a wt-distance on X if the following conditions are satisfied:

(i) $p(x, z) \leq s\,(p(x, y) + p(y, z))$ for any $x, y, z \in X$;
(ii) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is s-lower semi-continuous;
(iii) for any $\epsilon > 0$ there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

Let us recall that a real valued function f defined on a b-metric space X is said to be s-lower semi-continuous at a point x_0 in X if $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf_{x_n \to x_0} s f(x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$[16].

We now present some examples of wt-distance.

Example 2.8. [15] Let (X, d) be a b-metric space. Then d is a wt-distance on X.

Example 2.9. Let (X, d) be a b-metric space. Then a function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = c$ for every $x, y \in X$ is a wt-distance on X, where c is a positive real number.

Example 2.10. Let $X = [0, \infty)$ and $d(x, y) = (x - y)^2$ be a b-metric on X with constant $s = 2$. Then a function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = x^2 + y^2$ for every $x, y \in X$ is a wt-distance on X.

Example 2.11. Let $X = [0, \infty)$ and $d(x, y) = (x - y)^2$ be a b-metric on X with constant $s = 2$. Then a function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = y^2$ for every $x, y \in X$ is a wt-distance on X.

On wt-distance, we have the following important remark:

Remark 2.12. If p is a wt-distance on X, then
(i) $p(x, y) = p(y, x)$ does not necessarily hold for all x, $y \in X$;
(ii) $p(x, y) = 0$ if and only if $x = y$ does not necessarily hold for all x, $y \in X$.

Lemma 2.13. [15] *Let (X, d) be a b-metric space with constant $s \geq 1$ and let p be a wt-distance on X. Let (x_n) and (y_n) be sequences in X, let (α_n) and (β_n) be sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:*
(i) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;*
(ii) *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z;*
(iii) *if $p(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence;*
(iv) *if $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.*

We next review some basic notions in graph theory.

Let (X, d) be a b-metric space. We assume that G is a reflexive digraph with the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains no parallel edges. So we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [9, 11, 14]. If x, y are vertices of the digraph G, then a path in G from x to y of length n ($n \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x,\ x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1,\ 2,\ \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

3. MAIN RESULTS

In this section we assume that (X, d) is a b-metric space with the coefficient $s \geq 1$, and G is a reflexive digraph such that $V(G) = X$ and G has no parallel edges. For any mapping $f : X \to X$, C_f is the set of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for $m,\ n = 0, 1, 2, \cdots$.

Theorem 3.1. *Let (X, d) be a complete b-metric space endowed with a graph G, p a wt-distance on X and the mapping $f : X \to X$ be such that*

$$p(fx, fy) \leq k\, p(x, y) \tag{3.1}$$

for all $x, y \in X$ with $(x, y) \in E(\tilde{G})$, where $k \in [0, \frac{1}{s})$ is a constant. Suppose the triple (X, d, G) has the following property:

($*$) *If (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.*
Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, f has a unique fixed point in X if the graph G has the following property:

($**$) *If $x,\ y$ are fixed points of f in X, then $(x, y) \in E(\tilde{G})$.*

Furthermore, if $u = fu$, then $p(u, u) = 0$.

Proof. Suppose that $C_f \neq \emptyset$. We choose an $x_0 \in C_f$ and keep it fixed. We can construct a sequence (x_n) such that $x_n = fx_{n-1} = f^n x_0,\ n = 1,\ 2,\ 3, \cdots$. Evidently, $(x_n, x_m) \in E(\tilde{G})$ for $m,\ n = 0, 1, 2, \cdots$.

We now show that (x_n) is Cauchy in (X, d).
For any natural number n, we have by using condition (3.1) that

$$p(x_n, x_{n+1}) = p(fx_{n-1}, fx_n) \leq k\, p(x_{n-1}, x_n). \tag{3.2}$$

By repeated use of condition (3.2), we get

$$p(x_n, x_{n+1}) \leq k^n\, p(x_0, x_1) \tag{3.3}$$

for all $n \in \mathbb{N}$.
For $m, n \in \mathbb{N}$ with $m > n$, using condition (3.3), we have

$$\begin{aligned} p(x_n, x_m) &\leq sp(x_n, x_{n+1}) + s^2 p(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + s^{m-n-1} p(x_{m-2}, x_{m-1}) + s^{m-n-1} p(x_{m-1}, x_m) \\ &\leq \left[sk^n + s^2 k^{n+1} + \cdots + s^{m-n-1} k^{m-2} + s^{m-n-1} k^{m-1}\right] p(x_0, x_1) \\ &\leq sk^n \left[1 + sk + (sk)^2 + \cdots + (sk)^{m-n-2} + (sk)^{m-n-1}\right] p(x_0, x_1) \\ &\leq \frac{sk^n}{1 - sk}\, p(x_0, x_1). \end{aligned}$$

By Lemma 2.13 (iii), it follows that (x_n) is a Cauchy sequence in X. Since (X,d) is complete, there exists $u \in X$ such that $x_n \to u$.

Let $n \in \mathbb{N}$ be an arbitrary but fixed. Since $x_m \to u$ and $p(x_n,\cdot)$ is s-lower semi-continuous, we have

$$p(x_n,u) \leq \liminf_{m\to\infty} sp(x_n,x_m) \leq \frac{s^2k^n}{1-sk}p(x_0,x_1). \tag{3.4}$$

By property $(*)$, there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i},u) \in E(\tilde{G})$ for all $i \geq 1$.

Again, using condition (3.1), we have

$$p(x_{n_i+1},fu) = p(fx_{n_i},fu) \leq k\,p(x_{n_i},u). \tag{3.5}$$

Thus, it follows from conditions (3.4) and (3.5) that
$p(x_{n_i+1},u) \leq \frac{s^2k^{n_i+1}}{1-sk}p(x_0,x_1) \to 0$ and $p(x_{n_i+1},fu) \leq k\,p(x_{n_i},u) \to 0$ as $i \to \infty$. By Lemma 2.13(i), we obtain $fu = u$. Therefore, u is a fixed point of f.

The next is to show that the fixed point is unique. Assume that there is another fixed point v of f in X. By property $(**)$, we have $(u,v) \in E(\tilde{G})$. Then,

$$p(u,v) = p(fu,fv) \leq kp(u,v)$$

which gives that, $p(u,v) = 0$.

Again,

$$p(u,u) = p(fu,fu) \leq kp(u,u)$$

which gives that, $p(u,u) = 0$.
Now, $p(u,v) = 0$ and $p(u,u) = 0$ imply that $u = v$. Therefore, f has a unique fixed point u in X. Moreover, if $u = fu$, then $p(u,u) = 0$.

□

Corollary 3.2. *Let (X,d) be a complete b-metric space endowed with a wt-distance p and the mapping $f : X \to X$ be such that*

$$p(fx,fy) \leq k\,p(x,y)$$

for all $x,y \in X$, where $k \in [0,\frac{1}{s})$ is a constant. Then f has a unique fixed point in X. Moreover, if $u = fu$, then $p(u,u) = 0$.

Proof. The proof follows from Theorem 3.1 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. □

Remark 3.3. Theorem 3.1 is a generalization of Banach contraction theorem in metric spaces to b-metric spaces.

Corollary 3.4. *Let (X,d) be a complete b-metric space endowed with a partial ordering $\preceq$, p a wt-distance on X and the mapping $f : X \to X$ be such that*

$$p(fx,fy) \leq k\,p(x,y)$$

for all $x,y \in X$ with $x \preceq y$ or, $y \preceq x$, where $k \in [0,\frac{1}{s})$ is a constant. Suppose the triple $(X,d,\preceq)$ has the following property:

($\dagger$) *If (x_n) is a sequence in X such that $x_n \to x$ and x_n, x_{n+1} are comparable for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that x_{n_i}, x are comparable for all $i \geq 1$.*

If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \cdots$, then f has a fixed point in X. Moreover, f has a unique fixed point in X if the following property holds:

($\dagger\dagger$) *If x, y are fixed points of f in X, then x, y are comparable.*

Furthermore, if $u = fu$, then $p(u, u) = 0$.

Proof. The proof can be obtained from Theorem 3.1 by taking $G = G_2$, where the graph G_2 is defined by $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

Theorem 3.5. *Let (X, d) be a complete b-metric space endowed with a graph G, p a wt-distance on X and the mapping $f : X \to X$ be such that*

$$p(fx, f^2x) \leq k\, p(x, fx) \tag{3.6}$$

for all $x \in X$ with $(x, fx) \in E(\tilde{G})$, where $k \in [0, \frac{1}{s})$ is a constant.

Suppose that

$$inf\{p(x, y) + p(x, fx) : x \in X \text{ with } (x, fx) \in E(\tilde{G})\} > 0 \tag{3.7}$$

for every $y \in X$ with $y \neq fy$. Then f has a fixed point in X if $C_f \neq \emptyset$. Moreover, if $u = fu$, then $p(u, u) = 0$.

Proof. As in the proof of Theorem 3.1, we can construct a sequence (x_n) such that $x_n = fx_{n-1},\ n = 1, 2, 3, \cdots$ with $(x_n, x_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$ and

$$p(x_n, x_{n+1}) \leq k^n\, p(x_0, x_1) \tag{3.8}$$

for all $n \in \mathbb{N}$.

By an argument similar to that used in Theorem 3.1, it follows that (x_n) is Cauchy in (X, d). As (X, d) is complete, there exists $u \in X$ such that $x_n \to u$. Proceeding similarly to that of Theorem 3.1, we obtain

$$p(x_n, u) \leq \frac{s^2 k^n}{1 - sk}\, p(x_0, x_1).$$

Assume that $u \neq fu$. Then by using conditions (3.7) and (3.8), we have

$$\begin{aligned} 0 &< inf\{p(x_n, u) + p(x_n, x_{n+1}) : n \in \mathbb{N}\} \\ &\leq inf\{\frac{s^2 k^n}{1 - sk}\, p(x_0, x_1) + k^n p(x_0, x_1) : n \in \mathbb{N}\} \\ &= 0 \end{aligned}$$

which is a contradiction. Therefore, $u = fu$ i.e., u is a fixed point of f in X.

Moreover,

$$p(u, u) = p(fu, fu) \leq kp(u, u)$$

implies that, $p(u, u) = 0$. □

Corollary 3.6. *Let (X, d) be a complete b-metric space endowed with a wt-distance p and the mapping $f : X \to X$ be such that*

$$p(fx, f^2x) \leq k\, p(x, fx)$$

for all $x \in X$, where $k \in [0, \frac{1}{s})$ is a constant.

Suppose that

$$inf\{p(x,y)+p(x,fx): x \in X\} > 0$$

for every $y \in X$ with $y \neq fy$. Then f has a fixed point in X. Moreover, if $u = fu$, then $p(u,u) = 0$.

Proof. It can be obtained from Theorem 3.5 by taking $G = G_0$. □

Remark 3.7. Theorem 3.5 is a generalization of Theorem 4[18] in metric spaces to b-metric spaces.

Corollary 3.8. *Let (X,d) be a complete b-metric space endowed with a partial ordering $\preceq$, p a wt-distance on X and the mapping $f : X \to X$ be such that*

$$p(fx, f^2x) \leq k\, p(x, fx)$$

for all $x \in X$ with $x \preceq fx$ or, $fx \preceq x$, where $k \in [0, \frac{1}{s})$ is a constant.

Suppose that

$$inf\{p(x,y)+p(x,fx): x \in X \ with \ x \preceq fx \ or \ fx \preceq x\} > 0$$

for every $y \in X$ with $y \neq fy$. If there exists $x_0 \in X$ such that $f^n x_0, f^m x_0$ are comparable for $m, n = 0, 1, 2, \cdots$, then f has a fixed point in X. Moreover, if $u = fu$, then $p(u,u) = 0$.

Proof. The proof can be obtained from Theorem 3.5 by taking $G = G_2$. □

As an application of Theorem 3.5 we obtain the following results.

Theorem 3.9. *Let (X,d) be a complete b-metric space with constant $s \geq 1$ and let $f : X \to X$ be such that*

$$d(fx, fy) \leq \alpha\, d(x,y) + \beta\, d(x, fx) + \gamma\, d(y, fy) \tag{3.9}$$

for every $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ with $\alpha+\beta+\gamma < \frac{1}{s}$. Then f has a unique fixed point in X.

Proof. We consider the b-metric d as a wt-distance on X. We also consider $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. Then, $C_f \neq \emptyset$ and $(x, fx) \in E(G)$ for all $x \in X$. From (3.9), we have

$$d(fx, f^2x) \leq \alpha d(x, fx) + \beta d(x, fx) + \gamma d(fx, f^2x)$$

which gives that

$$d(fx, f^2x) \leq \frac{\alpha+\beta}{1-\gamma} d(x, fx). \tag{3.10}$$

Let us put $k = \frac{\alpha+\beta}{1-\gamma}$. Then $k \in [0, \frac{1}{s})$ since $s(\alpha+\beta)+\gamma \leq s(\alpha+\beta+\gamma) < 1$.

Therefore, (3.10) becomes

$$d(fx, f^2x) \leq k\, d(x, fx)$$

for every $x \in X$ with $(x, fx) \in E(\tilde{G})$.

Suppose there exists $y \in X$ with $y \neq fy$ and

$$inf\left\{d(x,y)+d(x,fx): x \in X \ with \ (x,fx) \in E(\tilde{G})\right\} = 0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty}\{d(x_n, y) + d(x_n, fx_n)\} = 0.$$

So, we get $d(x_n, y) \to 0$ and $d(x_n, fx_n) \to 0$. By Lemma 2.13, it follows that $fx_n \to y$. We also have

$$\begin{aligned} d(y, fy) &\leq s\,[d(y, fx_n) + d(fx_n, fy)] \\ &\leq s\,[d(y, fx_n) + \alpha d(x_n, y) + \beta d(x_n, fx_n) + \gamma d(y, fy)] \end{aligned}$$

for any $n \in \mathbb{N}$ and hence

$$d(y, fy) \leq s\gamma\, d(y, fy).$$

Therefore, $d(y, fy) = 0$ i.e., $y = fy$. This is a contradiction. Hence, if $y \neq fy$, then

$$inf\,\left\{d(x, y) + d(x, fx) : x \in X\ with\ (x, fx) \in E(\tilde{G})\right\} > 0.$$

Now applying Theorem 3.5, we obtain a fixed point of f in X. Clearly, f has a unique fixed point in X. □

Theorem 3.10. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and let $f : X \to X$ be such that*

$$d(fx, fy) \leq \alpha\, d(x, fy) + \beta\, d(y, fx) \tag{3.11}$$

for every $x, y \in X$, where $\alpha, \beta \geq 0$ with $\alpha s < \frac{1}{1+s}$ or $\beta s < \frac{1}{1+s}$. Then f has a fixed point in X. Moreover, if $\alpha + \beta < 1$, then f has a unique fixed point in X.

Proof. We consider the b-metric d as a wt-distance on X. We also consider $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. Then, $C_f \neq \emptyset$ and $(x, fx) \in E(G)$ for all $x \in X$. From (3.11), we have

$$d(fx, f^2x) \leq \alpha d(x, f^2x) + \beta d(fx, fx) \leq \alpha s[d(x, fx) + d(fx, f^2x)]$$

which gives that

$$d(fx, f^2x) \leq \frac{\alpha s}{1 - \alpha s} d(x, fx). \tag{3.12}$$

Let us put $k = \frac{\alpha s}{1-\alpha s}$. Then $k \in [0, \frac{1}{s})$.
Therefore, (3.12) becomes

$$d(fx, f^2x) \leq k\, d(x, fx)$$

for every $x \in X$ with $(x, fx) \in E(\tilde{G})$.
Suppose there exists $y \in X$ with $y \neq fy$ and

$$inf\,\left\{d(x, y) + d(x, fx) : x \in X\ with\ (x, fx) \in E(\tilde{G})\right\} = 0.$$

Then there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty}\{d(x_n, y) + d(x_n, fx_n)\} = 0.$$

So, we get $d(x_n, y) \to 0$ and $d(x_n, fx_n) \to 0$. By Lemma 2.13, it follows that $fx_n \to y$. We also have

$$\begin{aligned} d(y, fy) &\leq s\,[d(y, fx_n) + d(fx_n, fy)] \\ &\leq s\,[d(y, fx_n) + \alpha d(x_n, fy) + \beta d(y, fx_n)] \\ &\leq s\,[d(y, fx_n) + \alpha s d(x_n, y) + \alpha s d(y, fy) + \beta d(y, fx_n)] \end{aligned}$$

for any $n \in \mathbb{N}$ and hence

$$d(y, fy) \leq s^2\alpha\, d(y, fy).$$

Therefore, $d(y, fy) = 0$ i.e., $y = fy$. This is a contradiction. Hence, if $y \neq fy$, then

$$inf\left\{d(x,y)+d(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\right\} > 0.$$

By applying Theorem 3.5, we obtain a fixed point of f in X.
Now suppose that $\alpha + \beta < 1$. Assume that there are $u, v \in X$ such that $fu = u$ and $fv = v$. Then

$$d(u,v) = d(fu,fv) \leq \alpha\, d(u,v) + \beta\, d(v,u) = (\alpha+\beta)d(u,v).$$

This shows that $d(u, v) = 0$ i.e., $u = v$. Therefore, f has a unique fixed point in X. □

We furnish some examples in favour of our results.

Example 3.11. Let $X = [0,\infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) =\mid x-y \mid^2$ for all $x,\ y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 2$. Let $p : X \times X \to [0,\infty)$ defined by $p(x,y) = x^2+y^2$ for all $x, y \in X$ be a wt-distance on X. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{7^n}) : n = 0,\ 1,\ 2,\ \cdots\}$.

Let $f : X \to X$ be defined by

$$fx = \frac{x}{7},\ \ for\ all\ \ x \in X.$$

It is easy to check that

$$p(fx,fy) = k\, p(x,y)$$

for all $x, y \in X$ with $(x,y) \in E(\tilde{G})$, where $k = \frac{1}{49} \in [0, \frac{1}{s})$ is a constant. Obviously, $0 \in C_f$.
Also, any sequence (x_n) with the property $(x_n, x_{n+1}) \in E(\tilde{G})$ must be either a constant sequence or a sequence of the following form

$$\begin{aligned} x_n &= 0,\ \ if\ \ n\ is\ odd \\ &= \frac{1}{7^n},\ \ if\ \ n\ is\ even \end{aligned}$$

where the words 'odd' and 'even' are interchangeable. Consequently it follows that property $(*)$ holds. Thus, we have all the conditions of Theorem 3.1 and 0 is the unique fixed point of f in X with $p(0,0) = 0$.

We now examine that the condition $C_f \neq \emptyset$ in Theorem 3.5 can neither be relaxed.

Example 3.12. Let $X = \{0\} \cup \{\frac{1}{5^n} : n \geq 1\}$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) =\mid x-y \mid^2$ for all $x,\ y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 2$. Let $p : X \times X \to [0,\infty)$ defined by $p(x,y) = y^2$ for all $x, y \in X$ be a wt-distance on X. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{5^n}) : n = 1,\ 2,\ 3,\ \cdots\}$.

Let $f : X \to X$ be defined by $f(0) = \frac{1}{5}$ and $f(\frac{1}{5^n}) = \frac{1}{5^{n+1}}$ for $n \geq 1$. Then, $(x, fx) \in E(\tilde{G})$ only for $x = 0$ and it is easy to check that

$$p(fx,f^2x) = k\, p(x,fx)$$

for all $x \in X$ with $(x, fx) \in E(\tilde{G})$, where $k = \frac{1}{25} \in [0, \frac{1}{s})$ is a constant.

Moreover, $y \neq fy$ for every $y \in X$. Let $y \in X$ be arbitrary and kept it fixed. Then,

$$inf\{p(x,y)+p(x,fx) : x \in X\ with\ (x,fx) \in E(\tilde{G})\}$$

$$
\begin{aligned}
&= inf\,\{p(x,y)+p(x,fx) : x=0\}\\
&= p(0,y)+p(0,\frac{1}{5})\\
&= y^2+\frac{1}{25}\\
&> 0.
\end{aligned}
$$

Thus,

$$inf\{p(x,y)+p(x,fx) : x\in X \ with \ (x,fx)\in E(\tilde{G})\}>0$$

for every $y\in X$ with $y\neq fy$. Obviously, $C_f=\emptyset$.
Thus, we have all the conditions of Theorem 3.5 except $C_f\neq\emptyset$ and f possesses no fixed point in X.

We now supplement Theorem 3.5 by examination of conditions (3.6) and (3.7) in respect of their independence. We furnish Examples 3.13 and 3.14 below to show that these two conditions are independent in the sense that Theorem 3.5 shall fall through by dropping one in favour of the other.

Example 3.13. Let $X=\{0\}\cup\{\frac{1}{3^n} : n\geq 1\}$ and $d(x,y)=|x-y|^2$ for all $x,y\in X$. Then (X,d) is a complete b-metric space with the coefficient $s=2$. Let $p : X\times X\to[0,\infty)$ defined by $p(x,y)=y^2$ for all $x,y\in X$ be a wt-distance on X. Let G be a digraph such that $V(G)=X$ and $E(G)=\{(0,0)\}\cup\{(\frac{1}{3^n},\frac{1}{3^m}) : n,\,m=1,\,2,\,3,\cdots\}$. Let $f : X\to X$ be defined by $f(0)=\frac{1}{3}$ and $f(\frac{1}{3^n})=\frac{1}{3^{n+1}}$ for $n\geq 1$. Then, $(x,fx)\in E(\tilde{G})$ for all $x\in X\setminus\{0\}$, and it is easy to verify that

$$p(fx,f^2x)=k\,p(x,fx)$$

for all $x\in X$ with $(x,fx)\in E(\tilde{G})$, where $k=\frac{1}{9}\in[0,\frac{1}{s})$ is a constant.
Therefore, f satisfies condition (3.6).

On the other hand, $y\neq fy$ for $y=0$. But, for $y=0$ we have

$$
\begin{aligned}
&inf\{p(x,y)+p(x,fx) : x\in X \ with \ (x,fx)\in E(\tilde{G})\}\\
&= inf\,\{p(x,0)+p(x,fx) : x\in X\setminus\{0\}\}\\
&= inf\,\{(fx)^2 : x\in X\setminus\{0\}\}\\
&= inf\left\{\frac{1}{3^{2n+2}} : n\geq 1\right\}\\
&= 0.
\end{aligned}
$$

Thus, condition (3.7) does not hold. Clearly, $\frac{1}{3}\in C_f$ but f possesses no fixed point in X. We note that Theorem 3.5 does not hold without condition (3.7).

Example 3.14. Let $X=[4,\infty)\cup\{2,3\}$ and $d(x,y)=|x-y|^2$ for all $x,y\in X$. Then (X,d) is a complete b-metric space with the coefficient $s=2$. Let $p : X\times X\to[0,\infty)$ defined by $p(x,y)=x^2+y^2$ for all $x,y\in X$ be a wt-distance on X. Let G be a digraph such that $V(G)=X$ and $E(G)=\Delta\cup\{(2,3)\}$.

Define $f : X\to X$ where

$$
\begin{aligned}
fx &= 2,\ for\,x\in(X\setminus\{2\})\\
&= 3,\ for\,x=2.
\end{aligned}
$$

Then, $(x,fx)\in E(\tilde{G})$ for $x=2,\ 3$.

We note that $y \neq fy$ for every $y \in X$. Let $y \in X$ be arbitrary and kept it fixed. Then,

$$\begin{aligned} & inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} \\ &= inf\,\{p(x,y)+p(x,fx) : x = 2,\,3\} \\ &= inf\,\{p(x,y)+13 : x = 2,\,3\} \\ &> 0. \end{aligned}$$

Therefore,

$$inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} > 0$$

for every $y \in X$ with $y \neq fy$. Thus, condition (3.7) is satisfied. However, for $x = 2$, we find that $p(fx, f^2x) = p(3,2) = 13 > k\,p(x,fx)$ for any $k \in [0, \frac{1}{s})$. So, condition (3.6) does not hold. Clearly, $2,\ 3 \in C_f$ but f possesses no fixed point in X. In this case we observe that Theorem 3.5 is invalid without condition (3.6).

We now examine validity of Theorem 3.5.

Example 3.15. Let $X = [0,\infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) =\mid x - y \mid^2$ for all $x,\, y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 2$. Let $p : X \times X \to [0,\infty)$ defined by $p(x,y) = y^2$ for all $x, y \in X$ be a wt-distance on X. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(1, \frac{1}{2})\}$.

Let $f : X \to X$ be defined by

$$fx = \frac{x}{2}, \ for \ all \ \ x \in X.$$

Then, $(x,fx) \in E(\tilde{G})$ for $x = 0,\,1$.

It is easy to verify that

$$p(fx, f^2x) = k\,p(x,fx)$$

for all $x \in X$ with $(x,fx) \in E(\tilde{G})$, where $k = \frac{1}{4} \in [0, \frac{1}{s})$ is a constant.

Obviously, $0 \in C_f$.

We note that $y \neq fy$ for every $y \in X \setminus \{0\}$. Let $y \in X \setminus \{0\}$ be arbitrary and kept it fixed. Then,

$$\begin{aligned} & inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} \\ &= inf\,\{p(x,y)+p(x,fx) : x = 0,\,1\} \\ &= inf\left\{y^2+\frac{x^2}{4} : x = 0,\,1\right\} \\ &> 0. \end{aligned}$$

Therefore,

$$inf\{p(x,y)+p(x,fx) : x \in X \ with \ (x,fx) \in E(\tilde{G})\} > 0$$

for every $y \in X$ with $y \neq fy$. Thus, we have all the conditions of Theorem 3.5 and 0 is the unique fixed point of f in X with $p(0,0) = 0$.

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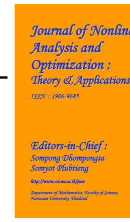

LOCAL CONVERGENCE OF A MULTI-POINT JARRATT-TYPE METHOD IN BANACH SPACE UNDER WEAK CONDITIONS

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ABSTRACT. We present a local convergence analysis of a multi-point Jarratt-type method of high convergence order in order to approximate a solution of a nonlinear equation in a Banach space. Our sufficient convergence conditions involve only hypotheses on the first Fréchet-derivative of the operator involved. In contrast to earlier studies using hypotheses up to the third Fréchet-derivative [26]. Numerical examples are also provided in this study.

KEYWORDS : Jarratt-type methods, Banach space, Local Convergence, Fréchet-derivative.
AMS Subject Classification:65H10, 65D99.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought in a form like (1.1) using mathematical modelling [3]. The solutions of equation (1.1) can rarely be found in closed form. That is why solutions of equation (1.1) are approximated by iterative methods. In particular, the practice of Numerical Functional Analysis for finding such solution is essentially connected to Newton-like methods [1]-[27]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the

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radii of convergence balls. There exist many studies which deal with the local and semilocal convergence analysis of Newton-like methods such as [1]-[27].

We present a local convergence analysis for the multi-point Jarratt-type method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} u_n &= x_n - F'(x_n)^{-1}F(x_n), \\ y_n &= u_n + \frac{1}{3}F'(x_n)^{-1}F(x_n), \\ z_n &= u_n + B_nF(x_n), \\ x_{n+1} &= z_n - A_nF(z_n), \end{aligned} \tag{1.2}$$

where $x_0 \in D$ is an initial point, $J_n = (6F'(y_n) - 2F'(x_n))^{-1}(3F'(y_n) + F'(x_n))$, $B_n = (I - J(x_n))F'(x_n)^{-1}$, and $A_n = \frac{3}{2}J_nF'(y_n)^{-1} + (I - \frac{3}{2}J_n)F'(x_n)^{-1}$.

A semilocal convergence analysis was given in [26] but the operator A_n was defined by

$$\overline{A_n} = \frac{3}{2}F'(y_n)^{-1}J_n + F'(x_n)^{-1}(I - \frac{3}{2}J_n).$$

That is, their method is defined by

$$\begin{aligned} u_n &= x_n - F'(x_n)^{-1}F(x_n), \\ y_n &= u_n + \frac{1}{3}F'(x_n)^{-1}F(x_n), \\ z_n &= u_n + B_nF(x_n), \\ x_{n+1} &= z_n - \overline{A_n}F(z_n). \end{aligned} \tag{1.3}$$

Notice that for two linear operators Q_1 and Q_2 we have that $Q_1Q_2 \neq Q_2Q_1$, so $A_n \neq \overline{A_n}$ in general. The fifth order of convergence of method (1.3) was established in [26]. These results were given in a non-affine invariant form. However, the results obtained in our paper are given in affine invariant form. The sufficient semilocal convergence conditions (given in affine invariant form) used in [26] are ($\mathcal{C}$):

($\mathcal{C}_1$): There exists $F'(x_0)^{-1} \in L(Y, X)$ and $\|F'(x_0)^{-1}\| \leq \beta$;

($\mathcal{C}_2$):
$$\|F'(x_0)^{-1}F(x_0)\| \leq \beta_1;$$

($\mathcal{C}_3$):
$$\|F'(x_0)^{-1}F''(x)\| \leq \beta_2 \quad \text{for each } \ x \in D;$$

($\mathcal{C}_4$):
$$\|F'(x_0)^{-1}F'''(x)\| \leq \beta_3 \quad \text{for each } \ x \in D;$$

($\mathcal{C}_5$):
$$\|F'(x_0)^{-1}(F'''(x) - F'''(y))\| \leq w(\|x - y\|) \quad \text{for each } \ x, y \in D$$

where $w(s)$ is a nondecreasing continuous real function for $s > 0$ with $w(0) \geq 0$.

($\mathcal{C}_6$): there exists a non-negative real function $\phi \in C[0, 1]$, with $\phi(t) \leq 1$, such that $w(ts) \leq \phi(t)w(s)$ for each $t \in [0, 1], \ s \in (0, \infty)$.

Similar conditions have been used by other authors [1]-[27], on other high convergence order methods. The corresponding conditions for the local convergence analysis are given by simply replacing x_0 by x^* in the preceding ($\mathcal{C}$) conditions.

These conditions however are very restrictive. As an academic example, let us define function F on $X = [-\frac{5}{2}, \frac{1}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then, obviously, e.g, function F cannot satisfy condition ($\mathcal{C}_4$) since $F'''(x) = 6lnx^2 + 60x^2 - 24x + 22$ is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative (see conditions (2.9)-(2.12)). This way we expand the applicability of method (1.2).

The paper is organized as follows. The local convergence of method (1.2) is given in Section 2, whereas the numerical examples are given in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Denote by $U(v, \rho), \bar{U}(v, \rho)$ the open and closed balls, respectively, in X of center v and radius $\rho > 0$.

Let $L_0 > 0$, $L > 0$ and $M > 0$ be given parameters. It is convenient for the local convergence analysis of method (1.2) that follows to define functions on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned}
g_1(r) &= \frac{Lr}{2(1 - L_0 r)}, \\
g_2(r) &= g_1(r) + \frac{M}{3(1 - L_0 r)}, \\
g_3(r) &= \frac{3}{2}\frac{L_0(1 + g_2(r))r}{1 - L_0 r}, \\
g_4(r) &= g_1(r) + \frac{3}{4}\frac{M L_0(1 + g_2(r))r}{(1 - g_3(r))(1 - L_0 r)^2}, \\
g_5(r) &= [1 + \frac{3}{4}(1 + \frac{1}{1 - g_3(r)})\frac{M}{1 - L_0 g_2(r) r} \\
&\quad + (1 + \frac{3}{4}(1 + \frac{1}{1 - g_3(r)}))\frac{M}{1 - L_0 r}] g_4(r).
\end{aligned}$$

We have that

$$\begin{aligned}
g_2(r) &= g_1(r) + \frac{M}{3(1 - L_0 r)} \\
&= \frac{Lr}{2(1 - L_0 r)} + \frac{M}{3(1 - L_0 r)}.
\end{aligned}$$

Hence, if

$$r_2 = \frac{3 - M}{3(\frac{L}{2} + L_0)}$$

and

$$M < 3, \tag{2.1}$$

then, we have that

$$0 < g_2(r) < 1 \text{ and } 0 < g_1(r) < 1 \text{ for each } r \in [0, r_2). \tag{2.2}$$

Notice that

$$r_2 < r_R := \frac{2}{3L} < r_A := \frac{1}{\frac{L}{2} + L_0} \text{ for } L_0 < L \tag{2.3}$$

and

$$r_2 < r_A = r_R \text{ for } L_0 = L. \tag{2.4}$$

Function g_3 can be written as

$$\begin{aligned} g_3(r) &= \frac{3}{2}\frac{L_0 r}{1 - L_0 r}\left(1 + \frac{Lr}{2(1-L_0 r)} + \frac{M}{3(1-L_0 r)}\right) \\ &= \frac{L_0 r}{4(1-L_0 r)^2}(6(1-L_0 r) + 3Lr + 2M). \end{aligned}$$

Define polynomial p_3 by

$$p_3(r) = L_0 r(6(1-L_0 r) + 3Lr + 2M) - 4(1-L_0 r)^2.$$

We have that $p_3(0) = -4 < 0$ and $p_3(\frac{1}{L_0}) = \frac{3L}{L_0} + 2M > 0$. It then follows from the intermediate value theorem that polynomial p_3 has roots in the interval $(0, \frac{1}{L_0})$. Let us denote by r_3 the smallest such root. Then, we have that

$$0 < p_3(r) < 1 \text{ and } 0 < g_3(r) < 1 \text{ for each } r \in [0, r_3). \tag{2.5}$$

By some algebraic manipulation we see that function g_5 can be written as

$$g_5(r) = \frac{N(r)}{D(r)},$$

where

$$\begin{aligned} N(r) = &[4(1 - L_0 g_2(r)r)(1 - g_3(r))(1 - L_0 r) + 3M(1 - g_3(r))(1 - L_0 r) \\ &+3M(1 - L_0 r) + 7M(1 - L_0 g_2(r)r)(1 - g_3(r)) + 3M(1 - L_0 g_2(r)r)] \\ &\times[4(1 - g_3(r))(1 - L_0 r)^2 g_1(r) + 3ML_0(1 + g_2(r)r)] \end{aligned}$$

and

$$D(r) = 16(1 - g_3(r))^2(1 - L_0 r)^3(1 - L_0 g_2(r)r).$$

Moreover, define function g_6 on the interval $[0, \frac{1}{L_0})$ by

$$g_6(r) = N(r) - D(r) \tag{2.6}$$

we have that $g_6(0) = -16(1 - \frac{L_0 M r}{3}) < 0$, since $L_0 r < 1$ and $M < 3$ and $g_6(r) \to \infty$ as $r \to (\frac{1}{L_0})^-$. It follows that function g_6 has zeros in the interval $(0, \frac{1}{L_0})$ (i.e., function g_5 has zeros in the interval $(0\frac{1}{L_0})$). Denote by r_5 the smallest such zero. Then, we have that

$$0 < g_5(r) < 1 \text{ and } 0 < g_4(r) < 1 \text{ for each } r \in [0, r_5). \tag{2.7}$$

Finally, set

$$r^* = \min\{r_2, r_3, r_5\}. \tag{2.8}$$

Then, clearly (2.2), (2.5) and (2.7) hold for each $r \in [0, r^*)$ (provided that (2.1) and (2.6) also hold).

Next, we present the local convergence analysis of method (1.2).

THEOREM 2.1. Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, parameters $L_0 > 0, L > 0$ and $0 < M < 3$ such that for all $x \in D$

$$F(x^*) = 0, \ F'(x^*)^{-1} \in L(Y, X), \tag{2.9}$$

$$\|F'(x^*)^{-1}(F(x) - F(x^*))\| \le L_0\|x - x^*\|, \tag{2.10}$$

$$\|F'(x^*)^{-1}(F(x) - F(x^*) - F'(x)(x - x^*)\| \le \frac{L}{2}\|x - x^*\|^2, \tag{2.11}$$

$$\|F'(x^*)^{-1}F'(x)\| \le M \tag{2.12}$$

and

$$\bar{U}(x^*, r^*) \subseteq D,$$

where r^* is given by (2.8). Then, sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r^*)$ is well defined, remains in $U(x^*, r^*)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots,$

$$\|u_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r^*, \tag{2.13}$$

$$\|y_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.14}$$

$$\|z_n - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.15}$$

$$\frac{3}{2}\|F'(x_n)^{-1}(F'(y_n) - F'(x_n))\| \leq g_3(\|x_n - x^*\|) < 1 \tag{2.16}$$

and

$$\|x_{n+1} - x^*\| \leq g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.17}$$

where the "g" functions are defined above Theorem 2.1.

Proof. Using (2.10), the definition of r^* and the hypothesis $x_0 \in U(x^*, r^*)$ we get that

$$\|F'(x^*)^{-1}(F(x_0) - F(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r^* < 1. \tag{2.18}$$

It follows from (2.18) and the Banach Lemma on invertible operators [3] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r^*}. \tag{2.19}$$

Hence, u_0, y_0 are well defined. Using method (1.2) for $n = 0$, (2.19),(2.2), (2.17), (2.11), (2.12) (for $x = x_0$) and the definition of functions g_1 and g_2, we obtain in turn that

$$\begin{aligned} u_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= F'(x_0)^{-1}[-F(x_0) + F'(x_0)(x_0 - x^*)] \\ &= -F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)] \end{aligned}$$

so,

$$\begin{aligned} \|u_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.13) for $n = 0$. Consequently from

$$y_0 - x^* = u_0 - x^* + \frac{1}{3}F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}F(x_0),$$

we get that

$$\begin{aligned} \|y_0 - x^*\| &\leq \|u_0 - x^*\| + \frac{1}{3}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + \frac{1}{3}\frac{\|\int_0^1 F'(x^*)^{-1}\int_0^1 F'(x^* + t(x^* - x_0)dt\|\|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} \\ &\leq [g_1(\|x_0 - x^*\|) + \frac{M}{3(1 - L_0\|x_0 - x^*\|)}]\|x_0 - x^*\| \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.14) for $n = 0$. We need a norm estimate on B_0. Let us start with an estimate on

$$\begin{aligned} & \frac{3}{2}\|F'(x_0)^{-1}(F'(y_0) - F'(x_0))\| \\ \leq\ & \frac{3}{2}\|F'(x_0)^{-1}F'(x^*)\|(\|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| \\ & +\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\ \leq\ & \frac{3}{2}\frac{L_0(\|y_0 - x^*\| + \|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} \\ \leq\ & \frac{3}{2}\frac{L_0(1 + g_2(\|x_0 - x^*\|)\|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} \\ =\ & g_3(\|x_0 - x^*\|) < g_3(r^*) < 1, \end{aligned}$$

(by (2.5)) which shows (2.16) for $n = 0$, $(I + \frac{3}{2}F'(x_0)^{-1}(F'(y_0) - F'(x_0)))^{-1} \in L(Y, X)$ and

$$\|(I + \frac{3}{2}F'(x_0)^{-1}(F'(y_0) - F'(x_0)))^{-1}\| \leq \frac{1}{1 - g_3(\|x_0 - x^*\|)}.$$

Then, we have the estimate

$$\begin{aligned} & \frac{1}{2}(I + \frac{3}{2}F'(x_0)^{-1}(F'(y_0) - F'(x_0)))^{-1}F'(x_0)^{-1} \\ =\ & [2F'(x_0)(I + \frac{3}{2}F'(x_0)^{-1}(F'(y_0) - F'(x_0)))^{-1} \\ =\ & [3(F'(y_0) - F'(x_0)) + 2F'(x_0)]^{-1} \\ =\ & (3F'(y_0) - F'(x_0))^{-1}. \end{aligned}$$

Then, B_0 can now be written as

$$B_0 = \frac{3}{4}(I + \frac{3}{2}F'(x_0)^{-1}(F'(y_0) - F'(x_0)))^{-1}F'(x_0)^{-1}(F'(y_0) - F'(x_0))F'(x_0)^{-1},$$

so,

$$\begin{aligned} \|B_0\| \leq\ & \frac{3}{4}\frac{1}{1 - g_3(\|x_0 - x^*\|)} \\ & \times\frac{L_0(\|x_0 - x^*\| + \|y_0 - x^*\|)}{1 - L_0\|x_0 - x^*\|}\frac{M\|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} \\ \leq\ & \frac{3}{4}\frac{ML_0(1 + g_2(\|x_0 - x^*\|))\|x_0 - x^*\|^2}{(1 - g_3(\|x_0 - x^*\|))(1 - L_0\|x_0 - x^*\|)^2}. \end{aligned} \tag{2.20}$$

Moreover from (2.20) and (2.7) we get that

$$\begin{aligned} \|z_0 - x^*\| \leq\ & \|u_0 - x^*\| + \|B_0F(x_0)\| \\ \leq\ & [g_1(\|x_0 - x^*\|) + \frac{3}{4}\frac{ML_0(1 + g_2(\|x_0 - x^*\|))\|x_0 - x^*\|}{(1 - g_3(\|x_0 - x^*\|))(1 - L_0\|x_0 - x^*\|)^2}]\|x_0 - x^*\| \\ =\ & g_4(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.15) for $n = 0$.

Next, we need an estimate on J_0 and A_0. We have that

$$\begin{aligned} J_0 =\ & \frac{1}{2}(3F'(y_0) - F'(x_0))^{-1}(3F'(y_0) - F'(x_0) + 2F'(x_0)) \\ =\ & \frac{1}{2}(I + 2(3F'(y_0) - F'(x_0))^{-1}F'(x_0)) \end{aligned}$$

so,

$$\|J_0\| \leq \frac{1}{2}(1 + \frac{1}{1 - g_3(\|x_0 - x^*\|)})$$

and

$$\begin{aligned}
\|A_0 F(z_0)\| &\leq [\frac{3}{2}\|J_0\|\|F'(y_0)^{-1}F'(x^*)\| \\
&\quad + \|I - \frac{3}{2}J_0\|\|F'(x_0)^{-1}F'(x^*)\|]\|F'(x^*)^{-1}F(z_0)\| \\
&\leq \left\{\frac{3}{4}(1 + \frac{1}{1 - g_3(\|x_0 - x^*\|)})\frac{M}{1 - L_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|}\right. \\
&\quad \left. + (1 + \frac{3}{4}(1 + \frac{1}{1 - g_3(\|x_0 - x^*\|)}))\frac{M}{1 - L_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|}\right\}\|z_0 - x^*\|,
\end{aligned}$$

so, from the preceding estimate,

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|z_0 - x^*\| + \|A_0 F(z_0)\| \\
&\leq g_4(\|x_0 - x^*\|)\|x_0 - x^*\| + \|A_0 F(z_0)\| \\
&\leq \left\{1 + \frac{3}{4}(1 + \frac{1}{1 - g_3(\|x_0 - x^*\|)})\frac{M}{1 - L_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|}\right. \\
&\quad \left. + (1 + \frac{3}{4}(1 + \frac{1}{1 - g_3(\|x_0 - x^*\|}))\frac{M}{1 - L_0\|x_0 - x^*\|}\right\} g_4(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_5(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&< \|x_0 - x^*\| < r^*,
\end{aligned}$$

(by (2.7)) which shows (2.17) for $n = 0$. By simply replacing u_0, y_0, z_0, x_1 by u_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at estimates (2.13)-(2.17). Finally, from the estimates $\|x_{k+1} - x^*\| < \|x_k - x^*\|$ we obtain $\lim_{k\to\infty} x_k = x^*$.

□

REMARK 2.2. 1. In view of (2.10) and the estimate

$$\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\
&\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\|
\end{aligned}$$

condition (2.12) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t.$$

Moreover, condition (2.11) can be replaced by the popular but stronger conditions

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\| \text{ for each } x, y \in D \tag{2.21}$$

or

$$\|F'(x^*)^{-1}(F'(x^* + t(x - x^*)) - F'(x))\| \leq L(1 - t)\|x - x^*\| \text{ for each } x, y \in D \text{ and } t \in [0, 1].$$

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3, 4].
4. The radius r_A given in (2.3) was shown by us to be the convergence radius of Newton's method [3, 4]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \ldots \tag{2.22}$$

under the conditions (2.10) and (2.21). It follows from (2.3) and (2.8) that the convergence radius r^* of the three-step method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.22). As already noted in [3, 4] r_A is at least as large as the convergence ball given by Rheinboldt [3, 4]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ \ as \ \ \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [3, 4].
5. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger ($\mathcal{C}$) conditions used in [26]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds given in [26] involving estimates up to the third Fréchet derivative of operator F.

3. Numerical Examples

We present numerical examples in this section.

EXAMPLE 3.1. Let $X = Y = \mathbb{R}^2, D = \bar{U}(0,1), x^* = (0,0)^T$ and define function F on D by

$$F(x) = (\sin x, \frac{1}{3}(e^x + 2x - 1))^T.$$

Then, using (2.10)-(2.12), we get $L_0 = L = 1$, $M = \frac{1}{3}(e+2)$. Then, by (2.8) we obtain

$$r^* = 0.0270 < r_R = r_A = 0.6667$$

EXAMPLE 3.2. Let $X = Y = \mathbb{R}^3,\ D = \overline{U}(0,1)$. Define F on D for $v = (x, y, z)^T$ by

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)^T$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1 < L = e$, $M = e$. Then, by (2.8) we obtain

$$r^* = 0.0045 < r_R = 0.2453 < r_A = 0.3249.$$

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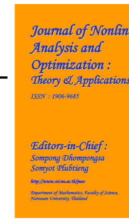

GENERALISED CESÀRO-ORLICZ DOUBLE SEQUENCE SPACES OVER N-NORMED SPACES

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ABSTRACT. This article is an attempt to highlight wide-ranging Cesàro-Orlicz double difference sequence spaces over n-normed spaces. The aim here lies in analyzing some topological properties and inclusion relations between these spaces.

KEYWORDS : Double sequence; Orlicz function; Difference sequence; Paranormed space; n-Normed spaces; Cesàro sequence space.
AMS Subject Classification: 40A05, 40A25, 46A30

1. INTRODUCTION, PRELIMINARIES AND NOTATIONS

Let $\mathbb{N}$, $\mathbb{R}$, w and w^2 denote the sets of positive integers, real numbers, single real sequences and double real sequence respectively in the entire paper. For $1 \leq p < \infty$, the *Cesàro sequence space* Ces_p is defined by

$$Ces_p = \Big\{x \in w : \sum_{j=1}^{\infty} \Big(\frac{1}{j}\sum_{i=1}^{j} |x_i|\Big)^p < \infty\Big\},$$

equipped with the norm

$$\|x\| = \Big(\sum_{j=1}^{\infty} \Big(\frac{1}{j}\sum_{i=1}^{j} |x_i|\Big)^p\Big)^{\frac{1}{p}}.$$

Beginning with the first premise of Shiue [26], the concept of space played a very significant role in the theory of matrix operators and others. In the advent, Sanhan and Suantai studied a generalized Cesàro sequence space Ces_p, where $p = (p_j)$ symbolized a bounded sequence of positive real numbers (see [25]). Later, this spaces was studied by many authors in ([8], [10], [15]).
A *double sequence* on a normed linear space X is a function x from $\mathbb{N} \times \mathbb{N}$ into X and briefly denoted by $x = (x_{kl})$. A double sequence (x_{kl}) is said to converge (in

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terms of Pringsheim) to $a \in X$ [19], if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x_{kl} - a\|_X < \varepsilon$ whenever $k, l > n_\varepsilon$.

A double series $\sum_{k,l=1}^{\infty} x_{kl}$ is *convergent* if and only if its sequence of partial sums s_{nm} is convergent (see [1], [2]), where $s_{nm} = \sum_{k=1}^{n}\sum_{l=1}^{m} x_{kl}$ for all $m, n \in \mathbb{N}$.

A double sequence $x = (x_{kl})$ is said to be *bounded* if $\|x\|_{(\infty,2)} = \sup_{k,l} |x_{kl}| < \infty$. The space of all bounded double sequences is denoted by l^2_∞.

Initially introduced by Kızmaz [9], the notion of *difference sequence spaces* was conceptualized as $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. Further, the notion was generalized by Et and Çolak [3] as they familiarized the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$. Let m, n be non-negative integers, then for Z a given sequence space, we have

$$Z(\triangle^n_m) = \{x = (x_k) \in w : (\triangle^n_m x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\triangle^n_m x = (\triangle^n_m x_k) = (\triangle^{n-1}_m x_k - \triangle^{n-1}_m x_{k+m})$ and $\triangle^0_m = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n_m x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

If $m = 1$, we get the spaces $l_\infty(\triangle^n), c(\triangle^n)$ and $c_0(\triangle^n)$ studied by Et and Çolak [3].

If $m = n = 1$, we get the spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ introduced and studied by Kizmaz [9]. Likewise, the difference operators on double sequence spaces can be examined as:

$$\begin{aligned} \Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}, \end{aligned}$$

$$\Delta^n x_{k,l} = \Delta^{n-1} x_{k,l} - \Delta^{n-1} x_{k,l+1} - \Delta^{n-1} x_{k+1,l} + \Delta^{n-1} x_{k+1,l+1}$$

and

$$\Delta^n_m x_{k,l} = \Delta^{n-1}_m x_{k,l} - \Delta^{n-1}_m x_{k,l+1} - \Delta^{n-1}_m x_{k+1,l} + \Delta^{n-1}_m x_{k+1,l+1}.$$

For further details about sequence spaces one can refer to ([16], [17], [20], [21], [22], [24]) and references therein.

An *Orlicz function* $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. The function is said to be *modulus function* if the convexity of Orlicz function is substituted by $M(x + y) \le M(x) + M(y)$. Lindenstrauss and Tzafriri [11] used the conception of Orlicz function to describe the following sequence space,

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

termed as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \le 1\Big\}.$$

Prior, [11] indicates that every Orlicz sequence space ℓ_M comprises of a subspace isomorphic to $\ell_p (p \ge 1)$. An Orlicz function M can always be imputed in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is a right differentiable for $t \geq 0,\ \eta(0) = 0,\ \eta(t) > 0,\ \eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (see [12, 13]). Complementary function where $\mathcal{N} = (N_k)$, defined as

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is derived from the Musielak-Orlicz function $\mathcal{M}$.

The sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ for a given Musielak-Orlicz function $\mathcal{M}$, can be specified as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ as a convex modular can be described as

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

A Musielak-Orlicz function $\mathcal{M} = (M_k)$ is said to be Δ_2-*condition* if there exist constants $a,\ K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in l_+^1$ (the positive cone of l^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in \mathbb{R}^+$, whenever $M_k(u) \leq a$.

2. The Spaces of Double Sequences over $n-$ Normed Spaces

This section brings to limelight Cesàro-Orlicz double difference sequence spaces over n-normed spaces with the help of Musielak-Orlicz functions. Before proceeding futher, first we recall the notion of paranormed space as follows:

A linear topological space X over the real field $\mathbb{R}$ (the set of real numbers) is said to be a *paranormed space* if there is a subadditive function $g : X \longrightarrow \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \longrightarrow 0$ and $g(x_n - x) \longrightarrow 0$ imply $g(\alpha_n x_n - \alpha x) \longrightarrow 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X. A paranorm g for which $g(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, g) is called a total paranormed space. The metric of any linear metric space is given by some total paranorm (see [27], Theorem 10.4.2, pp. 183).

In the mid of 1960's, Gähler [4] introduced the concept of 2-normed spaces while Misiak [14] propounded the n-normed spaces. This concept was further surveyed by critics like Gunawan ([5], [6]) and Gunawan and Mashadi [7] who studied it and obtained various results. Let $n \in \mathbb{N}$ and X be a linear space over the field of real numbers $\mathbb{R}$ of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n substantiates the following four conditions:

(i) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,
(ii) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,
(iii) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and
(iv) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is said to be n-*normed space* over the field $\mathbb{R}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$.
Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} as defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

is called an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to *converge* to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be *Cauchy* if

$$\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

X is said to be *complete* with respect to the n-norm if every Cauchy sequence in X converges to some $L \in X$. Thereby, any complete n-normed space is said to be n-*Banach space*.

Suppose $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $w(n-X)$ denotes the space of X-valued double sequences. Let $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function, that is, $\mathcal{M}$ is a sequence of Orlicz functions, $p = (p_{nm})$ be a bounded double sequence of positive real numbers and $u = (u_{nm})$ be a double sequence of strictly positive real numbers. In this paper we have analysed the following sequence spaces:

$$Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] =$$
$$\Big\{x \in w(n-X) : \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \ \text{for some } \rho > 0\Big\}.$$

Let us consider a few special cases of the above sequence spaces:

(i) If $\mathcal{M} = M_{nm}(x) = I$ for all $n, m \in \mathbb{N}$, then we have
$Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] = Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$.

(ii) If $u = (u_{nm}) = 1$, for all $n, m \in \mathbb{N}$ then we have
$Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] = Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, p, \|\cdot, \cdots, \cdot\|]$.

If we take $u = (u_{nm}) = 1$, $M_{nm}(x) = M(x)$ for all $n, m \in \mathbb{N}$, $\Delta^m_n = \Delta$ and X

is a normed space, then we get the spaces $Ces_M^{(2)}[\Delta, p]$ which were introduced and studied by Oğur and Duyar [18].

The following inequality will be used throughout the paper. If $0 \leq p_{nm} \leq \sup p_{nm} = H, K = \max(1, 2^{H-1})$ then

$$|a_{nm} + b_{nm}|^{p_{nm}} \leq K\{|a_{nm}|^{p_{nm}} + |b_{nm}|^{p_{nm}}\} \tag{2.1}$$

for all n, m and $a_{nm}, b_{nm} \in \mathbb{C}$. Also $|a|^{p_{nm}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The paper is an endeavor to introduce the new sequence spaces $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. The focus here is on some topological properties and inclusion relations between these sequence spaces.

3. MAIN RESULTS

Theorem 3.1. *In order to prove the double sequence $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$ is a linear space over the real field $\mathbb{R}$, let us suppose $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function, $p = (p_{nm})$ be a bounded double sequence of positive real numbers and $u = (u_{nm})$ be a double sequence of strictly positive real numbers.*

Proof. Suppose $x = (x_{ij})$ and $y = (y_{ij}) \in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$ and $\alpha, \beta \in \mathbb{R}$. Then based on the presumption there exist positive numbers ρ_1, ρ_2 such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho_1 > 0,$$

and

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{nm})$ is a non-decreasing and convex so by using inequality (2.1), we have

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\alpha\Delta_n^m x_{ij} + \beta\Delta_n^m y_{ij}}{\rho_3}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{|\alpha|}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_3}, z_1, \cdots, z_{n-1}\Big\| + \frac{|\beta|}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_3}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{2nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\| + \frac{1}{2nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$\leq K\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$+ K\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$< \infty.$$

Thus $\alpha x+\beta y \in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. This proves that $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$ is a linear space. □

Theorem 3.2. *Let $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function, $p = (p_{nm})$ be a bounded double sequence of positive real numbers and $u = (u_{nm})$ be a double sequence of strictly positive real numbers. Then the double sequence $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$ is a paranormed space with the paranorm*

$$g(x) = \inf\Big\{\rho^{\frac{pqr}{R}} > 0 : \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \le 1; \quad q, r \in \mathbb{N}\Big\}$$

where $0 < p_{nm} \le \sup p_{nm} = H < \infty$ *and* $R = \max(1, H)$.

Proof. (i) Clearly $g(x) \ge 0$ for $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. Since $M_{nm}(0) = 0$, we get $g(0) = 0$.
(ii) $g(-x) = g(x)$
(iii) Let $x = (x_{ij}), y = (y_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$ there exist positive numbers ρ_1 and ρ_2 such that

$$\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \le 1$$

and

$$\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \le 1.$$

Let $\rho_3 = 2^{\frac{R}{h}}(\rho_1 + \rho_2)$, where $h = \inf p_{nm} > 0$. Since $\mathcal{M}$ ia a non-decreasing convex function, we have

$$\begin{aligned}
&\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij} + \Delta^m_n y_{ij}}{\rho_3}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&\le \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{2^{\frac{R}{h}}(\rho_1+\rho_2)}, z_1, \cdots, z_{n-1}\Big\| \\
&+ \frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{2^{\frac{R}{h}}(\rho_1+\rho_2)}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&\le \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[\frac{\rho_1}{2^{\frac{R}{h}}(\rho_1+\rho_2)} M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\| \\
&+ \frac{\rho_2}{2^{\frac{R}{h}}(\rho_1+\rho_2)} M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&\le \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[\frac{1}{2^{\frac{R}{h}}} M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&+ \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[\frac{1}{2^{\frac{R}{h}}} M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&= \frac{1}{2}\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \\
&+ \frac{1}{2}\Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n y_{ij}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}}
\end{aligned}$$

$\leq \quad 1.$

Since ρ_1, ρ_2 and ρ_3 are positive real numbers, we get

$$g(x+y)$$

$$\inf\Big\{\rho_3^{\frac{p_{qr}}{R}} > 0 : \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}+\Delta_n^m x_{ij}}{\rho_3}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \leq 1;\ \ q,r\in\mathbb{N}\Big\}$$

$$\leq \inf\Big\{\rho_1^{\frac{p_{qr}}{R}} > 0 : \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho_1}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \leq 1;\ \ q,r\in\mathbb{N}\Big\}$$

$$+\inf\Big\{\rho_2^{\frac{p_{qr}}{R}} > 0 : \Big(\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m y_{ij}}{\rho_2}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\Big)^{\frac{1}{R}} \leq 1;\ \ q,r\in\mathbb{N}\Big\}$$

$$= g(x)+g(y).$$

Let $(x^n) = \{x_{ij}^n\}$ be any sequence in the space $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot,\cdots,\cdot\|]$ such that $g(x^n - x) \to 0$, as $n \to \infty$ and (λ_n) is a sequence of reals with $\lambda_n \to \lambda$, as $n \to \infty$. Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by subadditivity of the function g, $\{g(x^n)\}$ is bounded. Taking into account this fact we therefore derive the inequality

$$g(\lambda_n x^n - \lambda x) \leq |\lambda_n - \lambda| g(x^n) + |\lambda| g(x^n - x)$$

which tends to zero as $n \to \infty$. Hence, the scalar multiplication is continuous follows from the above inequality and thus proving the theorem. □

Theorem 3.3. *Let $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function, $p = (p_{nm})$ be a bounded double sequence of positive real numbers and $u = (u_{nm})$ be a double sequence of strictly positive real numbers. Then the space $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot,\cdots,\cdot\|]$ is complete with respect to its paranorm.*

Proof. Let $(x^s) = \{x_{ij}^s\}$ be any Cauchy sequence in the space $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot,\cdots,\cdot\|]$. Since (x^s) is a Cauchy sequence, we have $g(x^s - x^t) \to 0$ as $s,t \to \infty$. Then, we have

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}^s - \Delta_n^m x_{ij}^t}{\rho}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \to 0$$

as $s,t \to \infty$ for all $i,j \in \mathbb{N}$. Then, we have $\{x_{ij}^s\}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed $i,j \in \mathbb{N}$. Since $\mathbb{R}$ is complete as $t \to \infty$, we have $x_{ij}^s \to x_{ij}$ as $s \to \infty$ for each (i,j) and $\mathcal{M} = (M_{nm})$ is continuous. For $\epsilon > 0$, there exists a natural number N such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\sum_{s,t>N}\Big\|\frac{\Delta_n^m x_{ij}^s - \Delta_n^m x_{ij}^t}{\rho}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \varepsilon.$$

Since for any fixed natural number M, we have

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j\leq M \quad s,t>N}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}^s - \Delta_n^m x_{ij}^t}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \varepsilon,$$

by letting $t \to \infty$ in the above expression we obtain

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j\leq M \quad s,t>N}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}^s - \Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \varepsilon.$$

Since M is arbitrary, by letting $M \to \infty$ we obtain

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}^s - \Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \varepsilon.$$

Then $g(x^s - x) \to 0$ as $t \to \infty$. Since $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$ is linear space, we get $x = \{x_{ij}\} \in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. This completes the proof. □

Theorem 3.4. *If* $0 < p_{nm} \leq q_{nm} < \infty$ *for each* n *and* m*, then we have*

$$Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, q, \|\cdot, \cdots, \cdot\|].$$

Proof. Let $x \in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. Then there exists $\rho > 0$ such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty.$$

This implies that

$$u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < 1,$$

for sufficiently large values of n and m. Since M_{nm} is non-decreasing, we get

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}} \\ \leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ < \infty.\end{aligned}$$

Thus $x \in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, q, \|\cdot, \cdots, \cdot\|]$. This completes the proof. □

Theorem 3.5. *Suppose* $\mathcal{M} = (M_{mn})$ *be a Musielak-Orlicz function,* $p = (p_{mn})$ *be a bounded double sequence of positive real numbers and* $u = (u_{mn})$ *be a double sequence of strictly positive real numbers. Then*
(a) If $0 < \inf p_{mn} < p_{mn} \leq 1$*. Then* $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, \|\cdot, \cdots, \cdot\|]$.
(b) If $1 \leq p_{mn} \leq \sup p_{mn} < \infty$*. Then* $Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, \|\cdot, \cdots, \cdot\|] \subset Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$

Proof. (a) Let $x = (x_{ij}) \in Ces_{\mathcal{M}}^{(2)}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. Since $0 < \inf p_{mn} \leq 1$, we obtain the following

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] \\ \leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}\end{aligned}$$

$$< \quad \infty.$$

and hence $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, \|\cdot, \cdots, \cdot\|]$.

(b) Let $p_{nm} \geq 1$ for each n and m and $\sup p_{nm} < \infty$. Let $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, \|\cdot, \cdots, \cdot\|]$. Then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] \leq \epsilon < 1 \ \text{ for all } n, m \geq N.$$

This implies that

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ \leq \quad \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] \\ < \quad \infty.\end{aligned}$$

Therefore, $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. This completes the proof. □

Theorem 3.6. *Let $0 < p_{nm} \leq q_{nm}$ for all $n, m \in \mathbb{N}$ and $(\frac{q_{nm}}{p_{nm}})$ be bounded. Then we have $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, q, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$.*

Proof. Let $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. Then

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}} < \infty, \quad \text{for some } \rho > 0.$$

Let $s_{nm} = \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}}$ and $\lambda_{nm} = \frac{p_{nm}}{q_{nm}}$.

Since $p_{nm} \leq q_{nm}$, we have $0 \leq \lambda_{nm} \leq 1$. Take $0 < \lambda < \lambda_{nm}$.

Define

$$u_{nm} = \begin{cases} s_{nm} & \text{if } s_{nm} \geq 1 \\ 0 & \text{if } s_{nm} < 1 \end{cases}$$

and

$$v_{nm} = \begin{cases} 0 & \text{if } s_{nm} \geq 1 \\ s_{nm} & \text{if } s_{nm} < 1 \end{cases}$$

$s_{nm} = u_{nm} + v_{nm}$, $\quad s^{\lambda_{nm}}_{nm} = u^{\lambda_{nm}}_{nm} + v^{\lambda_{nm}}_{nm}$. It follows that $u^{\lambda_{nm}}_{nm} \leq u_{nm} \leq s_{nm}$, $v^{\lambda_{nm}}_{nm} \leq v^{\lambda}_{nm}$. Since $s^{\lambda_{nm}}_{nm} = u^{\lambda_{nm}}_{nm} + v^{\lambda_{nm}}_{nm}$, then $s^{\lambda_{nm}}_{nm} \leq s_{nm} + v^{\lambda}_{nm}$

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[\Big(M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big)^{q_{nm}}\Big]^{\lambda_{nm}} \\ \leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}}\end{aligned}$$

$$\Longrightarrow \sum_{n,m=1}^{\infty} u_{nm}\Big[\Big(M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big)^{q_{nm}}\Big]^{p_{nm}/q_{nm}}$$

$$\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}}$$

$$\Longrightarrow \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}}$$

$$\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}},$$

but

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{q_{nm}} < \infty \text{ for some } \rho > 0.$$

Therefore,

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty \text{ for some } \rho > 0.$$

Hence $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. Thus, we get $Ces^{(2)}_{\mathcal{M}}[\Delta_n^m, u, q, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. □

Theorem 3.7. *Let $\mathcal{M}' = (M'_{nm})$ and $\mathcal{M}'' = (M''_{nm})$ be two Musielak-Orlicz functions satisfying Δ_2-condition. Then*
(a) $Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}''\circ\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$,

(b) $Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \cap Ces^{(2)}_{\mathcal{M}''}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}'+\mathcal{M}''}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$.

Proof. (a) Let $x \in Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. Then there exists $\rho > 0$ such that

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty.$$

Since $\mathcal{M}' = (M'_{nm})$ is a continuous function, we can find a real number δ with $0 < \delta < 1$ such that $M'_{nm}(t) < \varepsilon$. Let $y_{nm} = M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)$.

Hence we write

$$\sum_{n,m=1}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}} = \sum_{y_{nm}\leq\delta}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}} + \sum_{y_{nm}>\delta}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}}$$

so we have

$$\sum_{y_{nm}\leq\delta}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}} \leq \max\{1, M''_{nm}(1)^H\} \sum_{y_{nm}\leq\delta}^{\infty} u_{nm}[y_{nm}]^{p_{nm}} \quad (3.1)$$

For $y_{mn} > \delta$, we use the fact $y_{mn} < \frac{y_{mn}}{\delta} < 1 + \frac{y_{mn}}{\delta}$. Since $\mathcal{M}'' = (M''_{nm})$ is non-decreasing and convex it follows that

$$M''_{nm}(y_{nm}) < M''_{nm}(1 + \frac{y_{nm}}{\delta}) < \frac{1}{2}M''_{nm}(2) + \frac{1}{2}\Big(\frac{2y_{nm}}{\delta}\Big).$$

Since $\mathcal{M}'' = (M''_{nm})$ satisfying the Δ_2-condition and $\frac{y_{mn}}{\delta} > 1$, there exists $T > 0$ such that

$$M''_{nm}(y_{nm}) < \frac{1}{2}T\frac{y_{mn}}{\delta}M''_{nm}(2) + \frac{1}{2}T\frac{y_{mn}}{\delta}M''_{nm}(2) = T\frac{y_{mn}}{\delta}M''_{nm}(2).$$

Therefore, we have

$$\sum_{y_{nm}>\delta}^{\infty} u_{nm}[M''_{nm}(y_{nm})]^{p_{nm}} \leq \max\left\{1, \left(T\frac{M''_{nm}(2)}{\delta}\right)^{H}\right\} \sum_{y_{nm}>\delta}^{\infty} u_{nm}[y_{nm}]^{p_{nm}} \quad (3.2)$$

Hence by the equation (3.1) and (3.2), we have

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[(M''_{nm}\circ M'_{nm})\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ = \sum_{n,m=1}^{\infty} u_{nm}[M''_{nm}y_{nm}]^{p_{nm}} \\ \leq D\sum_{y_{nm}\leq\delta}^{\infty} u_{nm}[(y_{nm})]^{p_{nm}} \\ +G\sum_{y_{nm}>\delta}^{\infty} u_{nm}[y_{nm}]^{p_{nm}}\end{aligned}$$

where $D = \max\{1, M''_{nm}(1)^H\}$ and $G = \max\left\{1, \left(T\frac{M''_{nm}(2)}{\delta}\right)^H\right\}$.
Hence $Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}_{\mathcal{M}''\circ\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$.

(b) Let $x \in Ces^{(2)}_{\mathcal{M}'}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|] \cap Ces^{(2)}_{\mathcal{M}''}[\Delta_n^m, u, p, \|\cdot, \cdots, \cdot\|]$. Then

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho > 0$$

and

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M''_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \infty, \text{ for some } \rho > 0.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. The result follows from the inequality

$$\begin{aligned}\sum_{n,m=1}^{\infty} u_{nm}\Big[(M'_{nm}+M''_{nm})\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ = \sum_{n,m=1}^{\infty} u_{nm}\Big[M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ + \sum_{n,m=1}^{\infty} u_{nm}\Big[M''_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ \leq K\sum_{n,m=1}^{\infty} u_{nm}\Big[M'_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ + K\sum_{n,m=1}^{\infty} u_{nm}\Big[M''_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta_n^m x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\ < \infty,\end{aligned}$$

where $K = \{\max 1, 2^{H-1}\}$. Therefore, $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}'+\mathcal{M}''}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. □

Theorem 3.8. *Let $\mathcal{M} = (M_{nm})$ be a Musielak-Orlicz function and Suppose that $\beta = \lim_{t\to\infty} \frac{M_{nm}(t)}{t} < \infty$. Then $Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] = Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$.*

Proof. In order to prove that $Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] = Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. It is adequate to show that $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|] \subset Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. Now, let $\beta > 0$. By definition of β, we have $M_{nm}(t) \geq \beta t$ for all $t \geq 0$. Since $\beta > 0$, we have $t \leq \frac{1}{\beta} M_{nm}(t)$ for all $t \geq 0$. Let $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. Thus, we have

$$\begin{aligned}
&\sum_{n,m=1}^{\infty} u_{nm}\Big[\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\
&\leq \frac{1}{\beta}\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\
&< \infty,
\end{aligned}$$

which implies that $x = (x_{ij}) \in Ces^{(2)}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$. This completes the proof. □

Theorem 3.9. *The double sequence space $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$ is solid.*

Proof. Suppose $x = (x_{ij}) \in Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$

$$\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} < \infty, \text{ for some } \rho > 0.$$

Let (α_{ij}) be a double sequence of scalars such that $|\alpha_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$. Then we get

$$\begin{aligned}
&\sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n \alpha_{ij} x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\
&\leq \sum_{n,m=1}^{\infty} u_{nm}\Big[M_{nm}\Big(\frac{1}{nm}\sum_{i,j=1}^{n,m}\Big\|\frac{\Delta^m_n x_{ij}}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_{nm}} \\
&< \infty.
\end{aligned}$$

This completes the proof. □

Theorem 3.10. *The double sequence space $Ces^{(2)}_{\mathcal{M}}[\Delta^m_n, u, p, \|\cdot, \cdots, \cdot\|]$ is monotone.*

Proof. The proof is insignificant so we exclude it. □

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ON THE EXISTENCE OF SOLUTIONS FOR A HADAMARD-TYPE FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION

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ABSTRACT. We study a class of fractional integro-differential inclusions with nonlocal fractional integral boundary conditions and we establish a Filippov type existence result in the case of nonconvex set-valued maps.

KEYWORDS : Differential inclusion; Fractional derivative; Boundary value problem.
AMS Subject Classification: 34A60, 34A08

1. INTRODUCTION

This note is concerned with the following problem

$$D^q x(t) \in F(t, x(t), I^\gamma x(t)) \quad a.e. \ ([1,e]), \tag{1.1}$$

$$x(1) = 0, \quad \sum_{i=1}^{m} \lambda_i I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n} \mu_i (I^{\beta_j} x(e) - I^{\beta_j} x(\xi_j)), \tag{1.2}$$

where D^q is the Hadamard fractional derivative of order q, $q \in (1,2]$, I^γ is the Hadamard integral of order γ, $\gamma > 0$, $\alpha_i, \beta_j > 0$, $\eta_i, \xi_j \in (1,e)$, $\lambda_i \in \mathbf{R}$, $\mu_j \in \mathbf{R}$, $i = \overline{1,m}$, $j = \overline{1,n}$, $\eta_1 < \eta_2 < ... < \eta_m$, $\xi_1 < \xi_2 < ... < \xi_n$ and $F : [1,e] \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathcal{P}(\mathbf{R})$ is a set-valued map.

If F is single-valued and does not depend on the last variable, fractional inclusion (1.1) reduces to the fractional equation

$$D^q x(t) = f(t, x(t)), \tag{1.3}$$

where $f : [1,e] \times \mathbf{R} \longrightarrow \mathbf{R}$.

In the last years we may see a strong development of the study of boundary value problems associated to fractional differential equations and inclusions. Most of the results in this framework are obtained for problems defined by Riemann-Liouville or Caputo fractional derivatives. Another type of fractional derivative is the one introduced by Hadamard ([6]) which differs from the others in the sense

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that the kernel of the integral contains a logarithmic function of arbitrary exponent. Recently, several papers were devoted to fractional differential equations and inclusions defined by Hadamard fractional derivative [1,2,4,9] etc.

The present note is motivated by a recent paper of Thiramanus, Ntouyas and Taribon ([9]) where existence results for problem (1.3)-(1.2) are obtained using fixed point techniques.

Our aim is to extend the study in [9] to the set-valued framework; moreover, our right-hand side contains an integral term. We show that Filippov's ideas ([5]) can be suitably adapted in order to obtain the existence of solutions for problem (1.1)-(1.2). Recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([5]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained. In this way we extend an existence result in [4].

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our result.

2. PRELIMINARIES

Let (X, d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\},\ d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

Let $I = [1,e]$, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t\in I} |x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \longrightarrow \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_1^e |u(t)|dt$.

The Hadamard fractional integral of order $q > 0$ of a Lebesgue integrable function $f : [1,\infty) \longrightarrow \mathbf{R}$ is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\ln \frac{t}{s}\right)^{q-1} \frac{f(s)}{s} ds$$

provided the integral exists and Γ is the (Euler's) Gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}dt$.

The Hadamard fractional derivative of order $q > 0$ of a function $f : [1,\infty) \longrightarrow \mathbf{R}$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\ln \frac{t}{s}\right)^{n-q-1} \frac{f(s)}{s} ds,$$

where $n = [q] + 1$, $[q]$ is the integer part of q.

Details and properties of Hadamard fractional derivative may be found in [8,9].

The next technical result is proved in [9]. Set

$$\Lambda := \sum_{i=1}^m \lambda_i \frac{\Gamma(q)}{\Gamma(q+\alpha_i)} (\ln \eta_i)^{q+\alpha_i-1} - \sum_{j=1}^n \mu_i \frac{\Gamma(q)}{\Gamma(q+\beta_j)} (1 - (\ln \xi_j)^{q+\beta_j-1}).$$

Lemma 2.1. *Assume that* $\Lambda \neq 0$. *For a given* $f(.) \in C(I, \mathbf{R})$, *the unique solution* $x(.)$ *of problem* $D^q x(t) = f(t)\quad a.e.\ ([1,e])$ *with boundary conditions (1.2) is given by*

$$x(t) = I^q f(t) + \frac{(\ln t)^{q-1}}{\Lambda} [\sum_{j=1}^n \mu_j (I^{q+\beta_j} f(e) - I^{q+\beta_j} f(\xi_j)) - \sum_{i=1}^m \lambda_i I^{q+\alpha_i} f(\eta_i)].$$

Remark 2.2. If we denote $A(t,s)=\frac{1}{\Gamma(q)}(\ln\frac{t}{s})^{q-1}\frac{1}{s}\chi_{[1,t]}(s)$, $B(t,s)=\frac{(\ln t)^{q-1}}{\Lambda}\cdot\sum_{j=1}^{n}\frac{\mu_j}{\Gamma(q+\beta_j)}(\ln\frac{e}{s})^{q+\beta_j-1}\frac{1}{s}$, $C_j(t,s)=-\frac{(\ln t)^{q-1}}{\Lambda}\frac{\mu_j}{\Gamma(q+\beta_j)}(\ln\frac{\xi_j}{s})^{q+\beta_j-1}\frac{1}{s}\chi_{[1,\xi_j]}(s)$, $j=\overline{1,n}$, $D_i(t,s)=-\frac{(\ln t)^{q-1}}{\Lambda}\frac{\lambda_i}{\Gamma(q+\alpha_i)}(\ln\frac{\eta_i}{s})^{q+\alpha_i-1}\frac{1}{s}\chi_{[1,\eta_i]}(s)$, $i=\overline{1,m}$, and $G(t,s)=A(t,s)+B(t,s)+\sum_{j=1}^{n}C_j(t,s)+\sum_{i=1}^{m}D_i(t,s)$, where $\chi_S(\cdot)$ is the characteristic function of the set S, then the solution $x(\cdot)$ in Lemma 2.1 may be written as

$$x(t)=\int_1^e G(t,s)f(s)ds. \tag{2.1}$$

Using the fact that, for fixed t, the function $g(s)=(\ln\frac{t}{s})^{q-1}\frac{1}{s}$ is decreasing and $g(1)=(\ln t)^{q-1}$ we deduce that, for any $t,s\in I$,

$$|A(t,s)|\leq\frac{1}{\Gamma(q)}(\ln t)^{q-1}\leq\frac{1}{\Gamma(q)},$$

$$|B(t,s)|\leq\sum_{j=1}^{n}\frac{|\mu_j|}{|\Lambda|\Gamma(q+\beta_j)}(\ln t)^{q-1}\leq\sum_{j=1}^{n}\frac{|\mu_j|}{|\Lambda|\Gamma(q+\beta_j)},$$

$$|C_j(t,s)|\leq\frac{(\ln t)^{q-1}}{|\Lambda|}\frac{|\mu_j|}{\Gamma(q+\beta_j)}(\ln\xi_j)^{q+\beta_j-1}\leq\frac{|\mu_j|}{|\Lambda|\Gamma(q+\beta_j)}(\ln\xi_j)^{q+\beta_j-1},$$

$$|D_i(t,s)|\leq\frac{(\ln t)^{q-1}}{|\Lambda|}\frac{|\lambda_i|}{\Gamma(q+\alpha_i)}(\ln\eta_i)^{q+\alpha_i-1}\leq\frac{|\lambda_i|}{|\Lambda|\Gamma(q+\alpha_i)}(\ln\eta_i)^{q+\alpha_i-1},$$

and therefore,

$$|G(t,s)|\leq\frac{1}{\Gamma(q)}+\sum_{j=1}^{n}\frac{|\mu_j|}{|\Lambda|\Gamma(q+\beta_j)}(1+(\ln\xi_j)^{q+\beta_j-1})+\sum_{i=1}^{m}\frac{|\lambda_i|}{|\Lambda|\Gamma(q+\alpha_i)}(\ln\eta_i)^{q+\alpha_i-1}=:M_1\quad\forall\, t,s\in I.$$

Definition 2.3. A function $x(.)\in C(I,\mathbf{R})$ with its Hadamard derivative of order q existing on $[1,e]$ is a solution of problem (1.1)-(1.2) if there exists a function $f(.)\in L^1(I,\mathbf{R})$ that satisfies $f(t)\in F(t,x(t),I^\gamma x(t))$ *a.e.* (I), $D^q x(t)=f(t)$ a.e. (I) and conditions (1.2) are satisfied.

3. THE MAIN RESULT

First we recall a selection result ([3]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 3.1. *Consider X a separable Banach space, B is the closed unit ball in X, $H:I\longrightarrow\mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g:I\longrightarrow X, L:I\longrightarrow\mathbf{R}_+$ are measurable functions. If*

$$H(t)\cap(g(t)+L(t)B)\neq\emptyset\quad a.e.(I),$$

then the set-valued map $t\longrightarrow H(t)\cap(g(t)+L(t)B)$ has a measurable selection.

In order to prove our results we need the following hypotheses.

Hypothesis H1. i) $F(.,.):I\times\mathbf{R}\times\mathbf{R}\longrightarrow\mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I)\otimes\mathcal{B}(\mathbf{R}\times\mathbf{R})$ *measurable.*

ii) *There exists $L(.)\in L^1(I,(0,\infty))$ such that, for almost all $t\in I$, $F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2))\leq L(t)(|x_1-x_2|+|y_1-y_2|)\ \forall\, x_1,x_2,y_1,y_2\in\mathbf{R}.$$

We use next the following notations

$$M(t) := L(t)(1 + \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s} \right)^{\gamma-1} \frac{1}{s} ds) = L(t)(1 + \frac{(\ln t)^\gamma}{\Gamma(\gamma+1)}), \tag{3.1}$$

$$M_0 = \int_1^e M(t)dt. \tag{3.2}$$

Theorem 3.1. *Assume that Hypothesis H1 is satisfied and* $M_1 M_0 < 1$. *Consider* $y(.) \in C(I, \mathbf{R})$ *with its Hadamard derivative of order* q *existing on* $[1, e]$ *such that* $y(1) = 0$, $\sum_{i=1}^m \lambda_i I^{\alpha_i} y(\eta_i) = \sum_{j=1}^n \mu_i (I^{\beta_j} y(e) - I^{\beta_j} y(\xi_j))$ *and there exists* $p(.) \in L^1(I, \mathbf{R}_+)$ *verifying* $d(D^q y(t), F(t, y(t), I^\gamma y(t))) \le p(t)$ *a.e.* (I).

Then there exists $x(.)$ *a solution of problem (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \frac{M_1}{1 - M_1 M_0} \int_1^e p(t)dt. \tag{3.3}$$

Proof. The set-valued map $t \longrightarrow F(t, y(t), I^\gamma y(t))$ is measurable with closed values and

$$F(t, y(t), I^\gamma y(t)) \cap \{D^q y(t) + p(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t, y(t), I^\gamma y(t))$ $a.e.$ (I) such that

$$|f_1(t) - D^q y(t)| \le p(t) \quad a.e.\ (I) \tag{3.4}$$

Define $x_1(t) = \int_1^e G(t,s) f_1(s) ds$ and one has

$$|x_1(t) - y(t)| \le M_1 \int_1^e 1 p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \ge 1$ with the following properties

$$x_n(t) = \int_1^e G(t,s) f_n(s) ds, \quad t \in I, \tag{3.5}$$

$$f_n(t) \in F(t, x_{n-1}(t), I^\gamma x_{n-1}(t)) \quad a.e.\ (I), \tag{3.6}$$

$$|f_{n+1}(t) - f_n(t)| \le L(t)(|x_n(t) - x_{n-1}(t)| + \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s} \right)^{\gamma-1} \frac{1}{s} |x_n(s) - x_{n-1}(s)| ds) \tag{3.7}$$

for almost all $t \in I$.

If this construction is realized then from (3.4)-(3.7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \le M_1 (M_1 M_0)^n \int_1^e p(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$|x_{n+1}(t) - x_n(t)| \le \int_1^e |G(t, t_1)|.|f_{n+1}(t_1) - f_n(t_1)| dt_1 \le$$

$$M_1 \int_1^e L(t_1)[|x_n(t_1) - x_{n-1}(t_1)| + \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left(\ln \frac{t_1}{s} \right)^{\gamma-1} \frac{1}{s} |x_n(s) - x_{n-1}(s)| ds)$$

$$\le M_1 \int_0^1 L(t_1)(1 + \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left(\ln \frac{t_1}{s} \right)^{\gamma-1} \frac{1}{s} ds) dt_1 . M_1^n M_0^{n-1} \int_1^e p(t)dt =$$

$$= M_1 (M_1 M_0)^n \int_1^e p(t)dt$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$, hence converging uniformly to some $x(.) \in C(I,\mathbf{R})$. Therefore, by (3.7), for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$. Moreover, one has

$$\begin{array}{c} |x_n(t)-y(t)| \leq |x_1(t)-y(t)| + \sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)| \leq \\ M_1\int_1^e p(t)dt + \sum_{i=1}^{n-1}(M_1\int_1^e p(t)dt)(M_1M_0)^i = \frac{M_1\int_1^e p(t)dt}{1-M_1M_0}. \end{array} \tag{3.8}$$

On the other hand, from (3.4), (3.7) and (3.8) we obtain for almost all $t\in I$

$$\begin{array}{l} |f_n(t)-D^q y(t)| \leq \\ \sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)| + +|f_1(t)-D^q y(t)| \leq L(t)\frac{M_1\int_1^e p(t)dt}{1-M_1M_0} + p(t) \end{array}$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.)\in L^1(I,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.5), (3.6) we deduce that $x(.)$ is a solution of (1.1). Finally, passing to the limit in (3.8) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (3.5)-(3.7). The construction will be done by induction.

Since the first step is already realized, assume that for some $N\geq 1$ we already constructed $x_n(.)\in C(I,\mathbf{R})$ and $f_n(.)\in L^1(I,\mathbf{R})$, $n=1,2,...N$ satisfying (3.5), (3.7) for $n=1,2,...N$ and (3.6) for $n=1,2,...N-1$. The set-valued map $t\longrightarrow F(t,x_N(t),I^\gamma x_N(t))$ is measurable. Moreover, the map $t\longrightarrow L(t)(|x_N(t)-x_{N-1}(t)| + \frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}|x_N(s)-x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.,.)$ we have that for almost all $t\in I$

$$\begin{array}{l} F(t,x_N(t),I^\gamma x_N(t))\cap\{f_N(t)+L(t)(|x_N(t)-x_{N-1}(t)|+ \\ \frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}|x_N(s)-x_{N-1}(s)|ds)[-1,1]\}\neq\emptyset. \end{array}$$

Lemma 3.1 yields that there exists a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.), I^\gamma x_N(.))$ such that for almost all $t\in I$

$$\begin{array}{l} |f_{N+1}(t)-f_N(t)| \leq \\ L(t)(|x_N(t)-x_{N-1}(t)| + \frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{1}{s}|x_N(s)-x_{N-1}(s)|ds). \end{array}$$

We define $x_{N+1}(.)$ as in (3.5) with $n=N+1$. Thus $f_{N+1}(.)$ satisfies (3.6) and (3.7) and the proof is complete. □

The assumption in Theorem 3.1 is satisfied, in particular, for $y(.)=0$ and therefore with $p(.)=L(.)$. We obtain the following consequence of Theorem 3.1.

Corollary 3.2. *Assume that Hypothesis H1 is satisfied,* $d(0,F(t,0,0)\leq L(t)$ *a.e.* (I) *and* $M_1M_0<1$. *Then there exists* $x(.)$ *a solution of problem (1.1)-(1.2) satisfying for all* $t\in I$

$$|x(t)| \leq \frac{M_1}{1-M_1M_0}\int_1^e L(t)dt.$$

If F does not depend on the last variable, Hypothesis H1 became

Hypothesis H2. i) $F(.,.) : I\times\mathbf{R}\longrightarrow\mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I)\otimes\mathcal{B}(\mathbf{R})$ *measurable.*

ii) *There exists* $L(.) \in L^1(I,(0,\infty))$ *such that, for almost all* $t \in I$, $F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x_1),F(t,x_2)) \le L(t)|x_1 - x_2| \quad \forall\, x_1,x_2 \in \mathbf{R}.$$

Denote $L_0 = \int_1^e L(t)dt$. and consider the fractional differential inclusion

$$D^q x(t) \in F(t,x(t)) \quad a.e.\ ([1,e]), \tag{3.9}$$

Corollary 3.3. *Assume that Hypothesis H2 is satisfied,* $d(0,F(t,0) \le L(t)$ *a.e.* (I) *and* $M_1L_0 < 1$*. Then there exists* $x(.)$ *a solution of problem (3.9)-(1.2) satisfying for all* $t \in I$

$$|x(t)| \le \frac{M_1L_0}{1 - M_1L_0}.$$

Remark 3.4. If in (1.2) $\lambda_i = 0$, $i = \overline{1,m}$, $j = 1$, $\mu_1 = 1$, then Theorem 3.1 yields Theorem 3.1 in [4].

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ON THE SEMILOCAL CONVERGENCE OF A TWO STEP NEWTON METHOD UNDER THE $\gamma-$CONDITION

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ABSTRACT. We present a semilocal convergence analysis of a two-step Newton method using the $\alpha-$theory in order to approximate a locally unique solution of an equation in a Banach space setting. The new idea uses a combination of center-γ as well as a $\gamma-$ condition in the convergence analysis. This convergence criteria are weaker than the corresponding ones in the literature even in the case of the single step Newton method [3, 14, 15, 16, 17, 18, 19, 20]. Numerical examples involving a nonlinear integral equation where the older convergence criteria are not satisfied but the new convergence criteria are satisfied, are also presented in the paper.

KEYWORDS : Two-step Newton method, Newton method, Banach space, $\alpha-$ theory, semi-local convergence, γ-condition, Fréchet-derivative.
AMS Subject Classification:65G99, 65J15, 47H17, 49M15.

1. Introduction

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. Let $U(x,r)$ and $\overline{U}(x,r)$ stand, respectively, for the open and closed ball in $\mathcal{X}$ with center x and radius $r > 0$. Denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. In the present paper we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet continuously differentiable operator defined on $\overline{U}(x_0, R)$ for some $R > 0$ with values in $\mathcal{Y}$.

A lot of problems from Computational Sciences and other disciplines can be brought in the form of equation (1.1) using Mathematical Modelling [5, 7, 8, 12, 13, 16, 17, 20]. The solution of these equations can rarely be found in closed

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form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [1]–[21]. The study about convergence matter of Newton methods is usually centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of Newton methods; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. We find in the literature several studies on the weakness and/or extension of the hypothesis made on the underlying operators. There is a plethora on local as well as semil-local convergence results, we refer the reader to [1]–[21]. The most famous among the semi-local convergence of iterative methods is the celebrated Kantorovich theorem for solving nonlinear equations. This theorem provides a simple and transparent convergence criterion for operators with bounded second derivatives F'' or the Lipschitz continuous first derivatives [2, 7, 10, 12, 13, 21]. Another important theorem inaugurated by Smale at the International Conference of Mathematics (cf. [16, 17]), where the concept of an approximate zero was proposed and the convergence criteria were provided to determine an approximate zero for analytic function, depending on the information at the initial point. Wang [19] generalized Smale's result by introducing the γ-condition (see $(\mathcal{H}_2)$). For more details on Smale's theory, the reader can refer to the excellent Dedieu's book [9, Chapter 3.3].

The two-step Newton's method defined by

$$\begin{aligned} &x_0 \text{ is an initial point} \\ &y_n = x_n - F'(x_n)^{-1}\, F(x_n), \\ &x_{n+1} = y_n - F'(x_n)^{-1}\, F(y_n) \qquad \text{for each} \quad n = 0, 1, 2, \cdots \end{aligned} \tag{1.2}$$

is the most popular cubically convergent iterative process for generating a sequence $\{x_n\}$ approximating $x^\star$. Here, $F'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the Fréchet-derivative of F at $x \in \overline{U}(x_0, R)$ [2, 5, 7]. However, the convergence domain of (1.2) is usually very small. That is why it is important to extend the convergence domain without additional hypotheses.

In the present paper, motivated by the preceding observation and optimization considerations, we expand the applicability of Newton's method under the γ-condition by introducing the notion of the center γ_0-condition (to be precised in Definition $(\mathcal{H}_3)$) for some $\gamma_0 \leq \gamma$. This way we obtain tighter upper bounds on the norms of $\| \; F'(x)^{-1}\, F'(x_0) \; \|$ for each $x \in \overline{U}(x_0, R)$ (see $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$) leading to tighter majorizing sequences and more precise information on the location of the solution x^* than in earlier studies such as [3, 14, 15, 16, 17, 18, 19, 20] (see in particular, (3.3)), (3.7), Remark 3.4, Theorem 3.5 and the numerical examples in Section 4). The approach of introducing center-Lipschitz condition has already been fruitful for expanding the applicability of Newton's method under the Kantorovich-type theory [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 21].

The rest of the paper is organized as follows: section 2 contains results on majorizing sequences. In section 3 we present the semi-local convergence analysis of (1.2). Applications and numerical examples are given in the concluding section 4.

2. Majorizing Sequences

In this section we introduce some scalar sequences that shall be shown to be majorizing for Newton's method in Section 3.

Let $\beta > 0$, $\gamma_0 > 0$ and $\gamma > 0$ be given. Define functions f_0 on $[0, \frac{1}{\gamma_0}]$ and f on $[0, \frac{1}{\gamma}]$ by

$$f_0(t) = \beta - t + \frac{\gamma_0 t^2}{1 - \gamma_0 t} \tag{2.1}$$

and

$$f(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}. \tag{2.2}$$

Moreover, define scalar sequences $\{t_n\}$ and $\{s_n\}$ for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} &t_0 = 0, \\ &s_n = t_n - f'(t_n)^{-1} f(t_n), \\ &t_{n+1} = s_n - f'(t_n)^{-1} f(s_n) \qquad \text{for each} \quad n = 0, 1, 2, \cdots . \end{aligned} \tag{2.3}$$

Notice that by direct algebraic manipulation these sequences can equivalently be written as

$$\begin{aligned} s_0 = 0, \qquad & s_0 = \beta,\ \ t_1 = s_0 - f'(t_0)^{-1} f(s_0), \\ s_{n+1} \quad &= \quad t_{n+1} - \frac{f(t_{n+1}) - f(s_n) - f'(t_n)(t_{n+1} - s_n)}{f'(t_{n+1})}, \\ &= \quad t_{n+1} + \frac{\gamma(t_{n+1} - s_n)^2}{(2 - \frac{1}{(1-\gamma t_{n+1})^2})(1 - \gamma t_{n+1})(1 - \gamma s_n)^2} \\ t_{n+2} \quad &= \quad s_{n+1} - \frac{f(s_{n+1}) - f(t_{n+1}) - f'(t_{n+1})(s_{n+1} - t_{n+1})}{f'(t_{n+1})}, \\ &= \quad s_{n+1} + \frac{\gamma(s_{n+1} - t_{n+1})^2}{(2 - \frac{1}{(1-\gamma t_{n+1})^2})(1 - \gamma s_{n+1})(1 - \gamma t_{n+1})^2}. \end{aligned}$$

Then, we can show the first result for majorizing sequences.

LEMMA 2.1. Suppose that

$$\alpha := \beta\gamma \leq 3 - 2\sqrt{2}. \tag{2.4}$$

Then, the following items hold

(a) [19] Function f has two real zeros given by

$$t^* = \frac{1 + \alpha - \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}, \quad t^{**} = \frac{1 + \alpha + \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}$$

which satisfy

$$\beta \leq t^* \leq (1 + \frac{1}{\sqrt{2}})\beta \leq (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma} \leq t^{**} \leq \frac{1}{2\gamma}.$$

(b) Sequences $\{t_n\}$ and $\{s_n\}$ are increasingly convergent to t^* and satisfy for each $n = 0, 1, 2, \cdots$

$$0 \leq t_n \leq s_n \leq t_{n+1} < t^*.$$

Proof. (b) Let us define functions g and g_1 by

$$g(x) = x - \frac{f(x)}{f'(x)} \text{ and } g_1(x) = g(x) - \frac{f(g(x))}{f'(x)}.$$

Then, it follows that $g(x)$ is strictly increasing on $[0, t^*)$, since

$$g'(x) = \frac{f(x)f''(x)}{f'(x)^2}, \ \ f(x) > 0, \ \ f'(x) < 0,$$

$f''(x) > 0$ and $f''(x)$ is strictly increasing for $x \in [0, t^*)$. We also have that

$$\begin{aligned} g_1'(x) &= g'(x) - \frac{f'(g(x))g'(x)f'(x) - f(g(x))f''(x)}{f'(x)^2} \\ &= \frac{f(x)^2 f''(x) f''(\bar{x}) + f'(x)^2 f(g(x)) f''(x)}{f'(x)^4} > 0 \end{aligned}$$

for $\bar{x} \in (x, g(x))$, $x < g(x) < g(t^*) = t^*$, if $x \in [0, t^*)$. Therefore, g_1 is strictly increasing on $[0, t^*)$ and $g(x) < g_1(x) < g_1(t^*) = t^*$. The result now follows by induction if we set $t_0 = 0 < t^*$, $s_n = g(t_n)$ and $t_{n+1} = g_1(t_n)$ for each $n = 0, 1, 2 \cdots$.

□

Moreover, we define scalar sequences $\{q_n\}$, $\{r_n\}$ by

$$\begin{aligned} q_0 = 0, \qquad & r_0 = \beta, \quad q_1 = r_0 - f_0'(q_0)^{-1} f(r_0), \\ r_{n+1} &= q_{n+1} - \frac{f(q_{n+1}) - f(r_n) - f'(q_n)(q_{n+1} - r_n)}{f_0'(q_{n+1})}, \\ &= q_{n+1} + \frac{\gamma (q_{n+1} - r_n)^2}{(2 - \frac{1}{(1-\gamma_0 q_{n+1})^2})(1 - \gamma q_{n+1})(1 - \gamma r_n)^2} \\ q_{n+2} &= r_{n+1} - \frac{f(r_{n+1}) - f(q_{n+1}) - f'(q_{n+1})(r_{n+1} - q_{n+1})}{f_0'(q_{n+1})}, \\ &= r_{n+1} + \frac{\gamma (r_{n+1} - q_{n+1})^2}{(2 - \frac{1}{(1-\gamma_0 q_{n+1})^2})(1 - \gamma r_{n+1})(1 - \gamma q_{n+1})^2}. \end{aligned} \tag{2.5}$$

Next, we compare sequences $\{s_n\}$, $\{t_n\}$ with $\{q_n\}$, $\{r_n\}$ under convergence criterion (2.4).

LEMMA 2.2. Suppose that (2.4) and

$$\gamma_0 \le \gamma \tag{2.6}$$

hold. Then, the following items hold for each $n = 0, 1, 2, \cdots$

$$0 \le q_n \le t_n \tag{2.7}$$
$$0 \le r_n \le s_n \tag{2.8}$$
$$q_n \le r_n \le q_{n+1} \tag{2.9}$$

and

$$q^* := \lim_{n\to\infty} q_n \le t^*. \tag{2.10}$$

Moreover, strict inequality holds if $\gamma_0 < \gamma$ for each $n = 1, 2, 3, \cdots$ in (2.7) and (2.9) and for each $n = 2, 3, \cdots$ in (2.8).

Proof. Using a simple induction argument and the definition of these sequences estimates (2.7)- (2.9) follows. We then have that sequences $\{q_n\}$ and $\{r_n\}$ are increasing, bounded above by t^* and as such they converge to their unique least upper bound denoted by q^* which satisfies (2.10).

□

Notice that sequences $\{s_n\}$, $\{t_n\}$ appear in the study of two step Newton methods in connection to the $\gamma-$theory and criterion (2.4) [1]-[9], [14]-[19]. So far we showed that sequences $\{q_n\}$, $\{r_n\}$ are tighter than $\{s_n\}$, $\{t_n\}$ under criterion (2.4).

However, a direct approach to the study of the convergence of sequences $\{q_n\}$, $\{r_n\}$ leads to weaker convergence criterion than (2.4). The proof of the next result can be found in [8, Theorem 2.1 (i)].

LEMMA 2.3. Let $\lambda = \frac{\gamma_0}{\gamma}$. Denote by $\rho,\ \rho_1$ the small zeroes in $(0,1)$ of the polynomials $P_\lambda(t) = 2\sqrt{2}\lambda^3 t^4 + (3-7\sqrt{2}-2\sqrt{2}\lambda)\lambda^2 t^3 + (7\sqrt{2}-6+2(3\sqrt{2}-1)\lambda)\lambda t^2 - (2(\sqrt{2}-1)+(5\sqrt{2}-4)\lambda)t + \sqrt{2}-1$ and $P^1_\lambda(t) = 2\lambda^2 t^4 - (4+5\lambda)\lambda t^3 + (1+10\lambda+2\lambda^2)t^2 - (3+4\lambda)t + 1$, respectively. Suppose that

$$\alpha \leq \begin{cases} \rho_1, \ \ if \ \frac{\gamma_0}{\gamma} \leq 1 - \frac{1}{\sqrt{2}} \\ \rho_2 := \min\{(1-\frac{1}{\sqrt{2}})\frac{\gamma}{\gamma_0}, \rho\}, \ \ if \ \frac{\gamma_0}{\gamma} > 1 - \frac{1}{\sqrt{2}} \end{cases} \tag{2.11}$$

Inequality (2.11) must be strict if $(1-\frac{1}{\sqrt{2}})\frac{\gamma}{\gamma_0} \leq \rho$. Then, scalar sequences $\{q_n\},\ \{r_n\}$ are increasingly convergent,

$$0 \leq q_n \leq r_n \leq q_{n+1}$$

and

$$\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = q^*.$$

Next, we compsre the right hsnd side of inequality (2.11) for $\lambda \in [0.0001, 1]$ to the right hand side of inequality (2.4). We observe from Table 1 that our approach

λ	Right hand side of (2.11)	Right hand side of (2.4) $\delta_0 = 3 - 2\sqrt{2}$
0.0001	0.3819529609564926	0.17157287525380990
0.001	0.3818354384029206	0.17157287525380990
0.01	0.3806532896793318	0.17157287525380990
0.1	0.3681420045094538	0.17157287525380990
0.15	0.3606611353927973	0.17157287525380990
0.2	0.3528242051436774	0.17157287525380990
0.25	0.3446613843095571	0.17157287525380990
0.26	0.3429931090685413	0.17157287525380990
0.27	0.3413137650216341	0.17157287525380990
0.28	0.3396237410848502	0.17157287525380990
0.29	0.3379234409140188	0.17157287525380990
$1-\frac{1}{\sqrt{2}}$	0.3374296493260468	0.17157287525380909
0.3	0.33447804873307100	0.17157287525380990
0.4	0.29722914975127396	0.17157287525380990
0.5	0.26682799202395086	0.17157287525380990
0.6	0.24178390124881075	0.17157287525380990
0.7	0.22090983862630980	0.17157287525380990
0.8	0.20330124076393735	0.17157287525380990
0.9	0.18827676080151223	0.17157287525380990
0.99	0.17653626898768845	0.17157287525380990
0.999	0.17544263627916407	0.17157287525380990
1	0.17157287525380990	0.17157287525380990

TABLE 1. Comparison Table

extends the applicability of the two step Newton's method (1.2).

3. SEMILOCAL CONVERGENCE

We present semilocal convergence results for the two step Newton-like method (1.2) in this section.

First, we need an auxiliary Ostrowski-type representation for operator F [2, 7, 11, 12].

LEMMA 3.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $D \subset \mathcal{X}$ be open and convex and $F : D \to \mathcal{Y}$ be twice continuously Fréchet differentiable. Moreover, suppose that sequences $\{x_n\}$ and $\{y_n\}$ generated by two step Newton-method (1.2) are well defined. Then, the following items hold for each $n = 0, 1, 2, \cdots$.

$$F(x_{n+1}) = \int_0^1 \int_0^1 F''(x_n+\theta(y_n-x_n+t(x_{n+1}-y_n)))d\theta(y_n-x_n+t(x_{n+1}-y_n))dt(x_{n+1}-y_n) \tag{3.1}$$

and

$$F(y_n) = \int_0^1 F''(x_n + t(y_n - x_n))(1 - t)dt(y_n - x_n)^2. \tag{3.2}$$

Proof. Using two-step Newton method (1.2), we get in turn that

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(y_n) - F'(x_n)(x_{n+1} - y_n) \\ &= \int_0^1 F'(y_n + t(x_{n+1} - y_n))dt(x_{n+1} - y_n) - \int_0^1 F'(x_n)dt(x_{n+1} - y_n) \\ &= \int_0^1 [F'(y_n + t(x_{n+1} - y_n)) - F'(x_n)]dt(x_{n+1} - y_n) \\ &= \int_0^1 \int_0^1 F''(x_n + \theta(y_n - x_n + t(x_{n+1} - y_n)))d\theta(y_n - x_n + t(x_{n+1} - y_n))dt(x_{n+1} - y_n), \end{aligned}$$

which shows (3.1). Similarly, we get that

$$\begin{aligned} F(y_n) &= F(y_n) - F(x_n) - F'(x_n)(y_n - x_n) \\ &= \int_0^1 F'(x_n + t(y_n - x_n))dt(y_n - x_n) - F'(x_n)(y_n - x_n) \\ &= \int_0^1 F''(x_n + t(y_n - x_n))(1 - t)dt(y_n - x_n)^2, \end{aligned}$$

which shows (3.2).

□

We shall show the main semilocal convergence result for the two-step Newton method (1.2) under conditions:

($\mathcal{H}$) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $R > 0$, $D \subseteq \mathcal{X}$ open and convex and $F : D \to \mathcal{Y}$ be continuously twice- Fréchet-differentiable on $int(D)$. Let $x_0 \in int(D)$ with $F'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X})$. Let $\gamma_0 > 0$, $\gamma > 0$ with $\gamma_0 \leq \gamma$ and $\beta > 0$. Set $R_0 = \min\{\frac{1}{\gamma}, (1 - \frac{1}{\sqrt{2}}\frac{1}{\gamma_0}\}$. Suppose:

($\mathcal{H}_1$): $R_0 \leq R$, $\quad U(x_0, R) \subseteq D$
and
$\left\|F'(x_0)^{-1}F(x_0)\right\| \leq \beta$;

($\mathcal{H}_2$): Operator F satisfies the γ-Lipschitz condition at x_0
$\left\|F'(x_0)^{-1}F''(x)\right\| \leq \dfrac{2\gamma}{(1 - \gamma \left\|x - x_0\right\|)^3} = f''(\|x - x_0\|)$ for each $x \in U(x_0, R_0)$;

($\mathcal{H}_3$): Operator F satisfies the γ_0-Lipschitz condition at x_0
$\left\|F'(x_0)^{-1}(F'(x) - F'(x_0))\right\| \leq \dfrac{\gamma_0(2 - \gamma_0 \left\|x - x_0\right\|) \left\|x - x_0\right\|}{(1 - \gamma_0 \left\|x - x_0\right\|)^2} = 1 + f_0'(\|x - x_0\|)$ for each $x \in U(x_0, R_0)$;

($\mathcal{H}_4$): Condition (2.11) holds.

We need the following Banach Lemma on invertible operators.

LEMMA 3.2. Suppose that condition $(\mathcal{H}_3)$ holds. Then $F'(x)^{-1} \in L(\mathcal{Y}, \mathcal{X})$ and

$$\|F'(x)^{-1}F'(x_0)\| \leq -f_0'(\|x - x_0\|)^{-1}. \tag{3.3}$$

Proof. Using $(\mathcal{H}_3)$ and (2.1) we have in turn that

$$\begin{aligned} \|F'(x_0)^{-1}(F'(x) - F'(x_0))\| &\leq \frac{\gamma_0(2 - \gamma_0\|x - x_0\|)\|x - x_0\|}{(1 - \gamma_0\|x - x_0\|)^2} \\ &= \int_0^1 f_0''(t\|x - x_0\|)dt\|x - x_0\| \qquad (3.4) \\ &= f_0'(\|x - x_0\|) - f_0'(0) \\ &= f_0'(\|x - x_0\|) + 1 < 1, \end{aligned}$$

since $f_0'(t) < 0$, if $0 \leq t < R_0$. It then follows from (3.4) and the Banach Lemma on invertible operators [2, 7, 11, 12] that $F'(x)^{-1} \in L(\mathcal{Y}, \mathcal{X})$ so that (3.3) holds.

□

Using the above auxiliary results and notation, we can show the main semilocal convergence result for two step Newton method (1.2) under the $(\mathcal{H})$ conditions.

THEOREM 3.3. Suppose that the $(\mathcal{H})$ conditions hold. Then, sequence $\{x_n\}$ generated by two-step Newton method (1.2) is well defined, remains in $\overline{U}(x_0, q^*)$ for each $n = 0, 1, 2, \cdots$ and converges to a unique solution x^* of equation $F(x) = 0$ in $\overline{U}(x_0, (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma_0})$. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$

$$\|x_n - x^*\| \leq q^* - r_n, \tag{3.5}$$

where q^* and $\{r_n\}$ were defined in (2.5).

Proof. We shall show the following items using induction

$$\begin{aligned} \|x_k - x_0\| &\leq q_k, \\ \|F'(x_k)^{-1}F'(x_0)\| &\leq -f_0'(q_k)^{-1}; \\ \|y_k - x_k\| &\leq r_k - q_k, \\ \|y_k - x_0\| &\leq r_k, \\ \|x_{k+1} - y_k\| &\leq q_{k+1} - r_k. \end{aligned}$$

The preceding items hold for $k = 0$ by the initial conditions. Suppose these estimates hold for all $n \leq k$. Then, we have that

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - y_k\| + \|y_k - x_0\| \leq q_{k+1} - r_k + r_k = q_{k+1}.$$

Using Lemma 3.2, we have that

$$\|F'(x_{k+1}^{-1}F'(x_0)\| \leq -f_0'(\|x_{k+1} - x_0\|)^{-1} \leq -f_0'(q_{k+1})^{-1}.$$

Using $(\mathcal{H}_2)$ and the definitions of function f and the sequences we get in turn that

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{k+1})\| &\leq \int_0^1\int_0^1 \|F'(x_0)^{-1}F''(x_k + \theta(y_k - x_k + t(x_{k+1} - y_k)))\|d\theta \\ &\quad \|(y_k - x_k + t(x_{k+1} - y_k))\|dt\|(x_{k+1} - y_k)\| \\ &\leq \int_0^1\int_0^1 f''(\|x_k - x_0 + \theta(y_k - x_k + t(x_{k+1} - y_k)))\|d\theta \\ &\quad \|(y_k - x_k + t(x_{k+1} - y_k))\|dt\|(x_{k+1} - y_k)\| \\ &\leq \int_0^1\int_0^1 \|f''(q_k + \theta(r_k - q_k + t(q_{k+1} - r_k)))d\theta \end{aligned}$$

$$\begin{aligned} & (r_k - q_k + t(q_{k+1} - r_k))dt(q_{k+1} - r_k) \\ = & \int_0^1 f'(r_k + t(q_{k+1} - r_k)) - f'(q_k)(q_{k+1} - r_k) \\ = & f(q_{k+1}) - f(r_k) - f'(q_k)(q_{k+1} - r_k). \end{aligned} \tag{3.6}$$

Then, the preceding estimates and (1.2) give that

$$\begin{aligned} \|y_{k+1} - x_{k+1}\| &= \| - F'(x_{k+1})^{-1}F(x_{k+1})\| \\ &\le \|F'(x_{k+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\| \\ &\le -f_0'(q_{k+1})^{-1}f(q_{k+1}) = r_{k+1} - q_{k+1} \\ \|y_{k+1} - x_0\| &\le \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \le r_{k+1}. \end{aligned}$$

Moreover, we also have that

$$\begin{aligned} \|F'(x_0)^{-1}F(y_k)\| &\le \int_0^1 \|F'(x_0)^{-1}F''(x_k + t(y_k - x_k)\|(1-t)dt\|y_k - x_k\|^2 \\ &\le \int_0^1 f''(q_k + t(r_k - q_k)(1-t)dt(r_k - q_k)^2 \\ &= -f'(q_k)(r_k - q_k) + \int_0^1 f'(q_k + t(r_k - q_k)dt(r_k - q_k) \\ &= -f'(q_k)(r_k - q_k) + f(r_k) - f(q_k). \end{aligned}$$

Consequently, we deduce that

$$\begin{aligned} \|x_{k+1} - y_k\| &= \|F'(x_k)^{-1}F(y_k)\| \\ &\le \|F'(x_k)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_k)\| \\ &\le f_0'(q_k)^{-1}f(r_k) \\ &= q_{k+1} - r_k. \end{aligned}$$

By Lemma 2.3 sequences $\{x_n\},\ \{y_n\}$ are complete in the Banach space $\mathcal{X}$ and as such they converge to some $x^* \in \overline{U}(x_0, q^*)$ (since $\overline{U}(x_0, q^*)$ is a closed set). Estimate (3.5) follows from the preceding or by using standard majorization techniques [2, 7, 11, 12]. Moreover, by letting $k \to \infty$ in (3.6), we obtain that $F(x^*) = 0$. Furthermore, to show uniqueness, let $y^* \in U(x_0, (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma_0})$ be such that $F(y^*) = 0$. Then, we have by $(\mathcal{H}_3)$ that

$$\begin{aligned} & \|F'(x_0)^{-1}\int_0^1 F'(x^* + t(y^* - x^*))dt - I\| \\ \le & \|F'(x_0)^{-1}\int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_0)]dt\| \\ = & \int_0^1 f_0'(\|x^* - x_0 + t(y^* - x^*)\|)dt - f_0'(0) \\ = & \int_0^1 f_0'(\|(1-t)(x^* - x_0) + t(y^* - x_0)\|)dt + 1 < 1. \end{aligned}$$

It follows by the Banach Lemma, that the inverse of $\int_0^1 F'(x^* + t(y^* - x^*))dt$ exists. Using the identity,

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*),$$

we deduce that $y^* = x^*$.

□

REMARK 3.4. (a) It follows from the proof of Theorem 3.3 that in the estimates for the upper bounds on $\|y_1 - x_1\|,\ \|x_2 - y_1\|$ condition $(\mathcal{H}_3)$ can be used instead of the less precise (for $\gamma_0 < \gamma$) condition $(\mathcal{H}_2)$. This observation motivates us to define more precise majorizing sequences $\{\overline{q_n}\},\ \{\overline{r_n}\}$ than $\{q_n\},\ \{r_n\}$, respectively by

$$\overline{q_0} = 0, \qquad \overline{r_0} = \beta,\ \ \overline{q_1} = \overline{r_0} - f'(\overline{q_0})^{-1} f(\overline{r_0}),$$

$$\overline{r_1} = \overline{q_1} - \frac{f_0(\overline{q_1}) - f_0(\overline{r_0}) - f_0'(\overline{q_0})(\overline{q_1} - \overline{r_0})}{f_0'(\overline{q_1})},$$

$$\overline{q_2} = \overline{r_1} - \frac{f_0(\overline{r_1}) - f_0(\overline{q_1}) - f_0'(\overline{q_1})(\overline{r_1} - \overline{q_1})}{f_0'(\overline{q_1})},$$

$$\overline{r_{n+1}} = \overline{q_{n+1}} - \frac{f(\overline{q_{n+1}}) - f(\overline{r_n}) - f'(\overline{q_n})(\overline{q_{n+1}} - \overline{r_n})}{f_0'(\overline{q_{n+1}})},$$

and

$$\overline{q_{n+2}} = \overline{r_{n+1}} - \frac{f(\overline{r_{n+1}}) - f(\overline{q_{n+1}}) - f'(\overline{q_{n+1}})(\overline{r_{n+1}} - \overline{q_{n+1}})}{f_0'(\overline{q_{n+1}})}$$

Clearly, $\{\overline{q_n}\},\ \{\overline{r_n}\}$ can replace $\{q_n\},\ \{r_n\}$ in Theorem 3.3. We also have that

$$\overline{q_n} \le q_n, \qquad \overline{q_{n+1}} - \overline{r_n} \le q_{n+1} - r_n$$
$$\overline{r_n} \le r_n, \qquad \overline{r_{n+1}} - \overline{q_{n+1}} \le r_{n+1} - q_{n+1}$$

and

$$\overline{q^*} = \lim_{n\to\infty} \overline{q_n} \le q^*.$$

(b) Notice that $(\mathcal{H}_2)$ implies $(\mathcal{H}_3)$ but not necessarily vice versa. The results in the literature [8, 14, 15, 16, 17, 18, 19] use the estimate (under $(\mathcal{H}_2)$)

$$\left\|F'(x_{n+1})^{-1}F'(x_0)\right\| \le \left(2 - \frac{1}{(1-\gamma t_{n+1})^2}\right)^{-1}$$

which is less precise than the one obtained in our Theorem if $\gamma_0 < \gamma$ given by

$$\|F(x_{n+1})^{-1}F'(x_0)\| \le (2 - \frac{1}{(1-\gamma_0 s_{n+1})^2})^{-1}$$

(see (3.3)). This observation is the motivation for the introduction of more precise majorizing sequences. Notice also that $(\mathcal{H}_3)$ is not an additional to $(\mathcal{H}_2)$ hypothesis, since in practice the computation of constant γ requires the computation of constant γ_0 as a special case.

(c) Concerning to the choice of constants γ_0 and γ, let us suppose that the following Lipschitz conditions hold

$(\mathcal{H}_2)$': Operator F satisfies the L-Lipschitz condition at x_0

$$\left\|F'(x_0)^{-1}\left[F'(x) - F'(y)\right]\right\| \le L\,\|x-y\| \quad \text{for each } x,\, y \in U(x_0, R_0).$$

$(\mathcal{H}_3)$': Operator F satisfies the L_0-Lipschitz condition at x_0

$$\left\|F'(x_0)^{-1}\left[F'(x) - F'(x_0)\right]\right\| \le L_0\,\|x-x_0\| \quad \text{for each } x \in U(x_0, R_0).$$

Then, $(\mathcal{H}_3)$' implies $(\mathcal{H}_3)$ for $\gamma_0 = \dfrac{L_0}{2}$. Moreover, if F is continuously twice-Fréchet-differentiable, then $(\mathcal{H}_2)$' implies $(\mathcal{H}_2)$ and we can set $\gamma = \dfrac{L}{2}$. Therefore, the conclusions of Theorem 3.3 hold with $(\mathcal{H}_2)$', $(\mathcal{H}_3)$' replacing $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, respectively. Examples, where $L_0 < L$

(i.e. $\gamma_0 < \gamma$) can be found in [2, 3, 4, 5, 6, 7] (see also the numerical examples in Section 4).

(d) If F is an analytic operator, then a choice for γ_0 (or γ) is given by $\gamma_0 = \sup_{n>1} \left\| \frac{F'(x_0)^{-1}F^{(n)}(x_0)}{n!} \right\|^{\frac{1}{n-1}}$. This choice is due to Smale [16] (see also [14, 15, 17, 18, 19]).

We complete this section with a useful and obvious extension of Theorem 3.3.

THEOREM 3.5. Suppose: there exists an integer $N \geq 1$ such that

$$q_0 < r_0 < q_1 < \ldots < q_N < R_0;$$

Let $\alpha_N = \gamma\beta_N$, where $\beta_N = r_N - q_N$. Conditions $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ are satisfied for α_N replacing α in Condition $(\mathcal{H}_4)$. Then, the conclusions of Theorem 3.3 hold. Consequently, the conclusions of Theorem 3.3 also hold for sequence $\{\overline{q_n}\}$, $\{\overline{r_n}\}$. Notice also if $N = 0$ Theorem 3.5 reduces to Theorem 3.3.

4. Numerical Examples

We present examples where the older convergence criterion (2.4) is not satisfied but the new convergence criterion (2.11) satisfied.

EXAMPLE 4.1. Let $\mathcal{C}[0,1]$ stand for the space of continuous functions defined on interval $[0,1]$ and be equipped with the max-norm. Let also $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0,1]$ and $\mathcal{D} = U(0,r)$ for some $r > 1$. Define F on $\mathcal{D}$ by

$$F(x)(s) = x(s) - y(s) - \mu \int_0^1 \mathcal{G}(s,t)\, x^3(t)\, dt, \quad x \in \mathcal{C}[0,1],\ s \in [0,1].$$

$y \in \mathcal{C}[0,1]$ is given, μ is a real parameter and the Kernel G is the Green's function defined by

$$\mathcal{G}(s,t) = \begin{cases} (1-s)\,t & if \quad t \leq s \\ s\,(1-t) & if \quad s \leq t. \end{cases}$$

Then, the Fréchet-derivative of F is defined by

$$(F'(x)\,(w))(s) = w(s) - 3\,\mu \int_0^1 \mathcal{G}(s,t)\, x^2(t)\, y(t)\, dt, \quad w \in \mathcal{C}[0,1],\ s \in [0,1].$$

Let us choose $x_0(s) = y(s) = 1$ and $|\mu| < 8/3$. Then, we have that

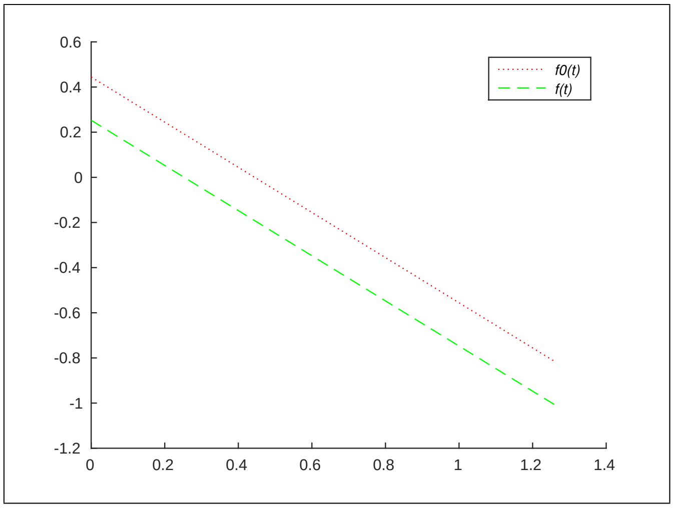
$$\| \mathcal{I} - F'(x_0) \| < \frac{3}{8}\,|\mu|, \quad F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y},\mathcal{X}),$$

$$\| F'(x_0)^{-1} \| \leq \frac{8}{8-3\,|\mu|}, \quad \beta = \frac{|\mu|}{8-3\,|\mu|}, \quad L_0 = \frac{12\,|\mu|}{8-3\,|\mu|},$$

$$L = \frac{6\,r\,|\mu|}{8-3\,|\mu|}, \quad \gamma_0 = \frac{L_0}{2} and \quad \gamma = \frac{L}{2}.$$

In Table 2, we pick some values of r and we show the values of μ for which condition (2.11) is satisfied but (2.4) is not satisfied. Hence, the new sufficient semilocal convergence criteria are satisfied but the old in [3, 14, 15, 16, 17, 18, 19, 20] are not satisfied.

FIGURE 1. Plots for f_0 and f for $r = 2.5$ and $\mu = 0.755$



r	α	Interval of μ
2	$\frac{6\mu}{(8-3\mu)^2}$	$(0.849668, 0.859174)$
2.25	$\frac{6.75\mu}{(8-3\mu)^2}$	$(0.798444, 0.807768)$
2.5	$\frac{7.5\mu}{(8-3\mu)^2}$	$(0.753551, 0.762683)$
2.75	$\frac{8.25\mu}{(8-3\mu)^2}$	$(0.713806, 0.722742)$
3	$\frac{9\mu}{(8-3\mu)^2}$	$(0.678318, 0.68706)$

TABLE 2. Comparison Table

5. CONCLUSION

We studied the semilocal convergence of a two step method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting using the $\alpha-$ theory. The novelty of our paper lies in the introduction of a center Lipschitz condition that leads to more precise upper bounds on the norms $\|F'(x_n)^{-1}F'(x_0)\|$ yielding to a tighter convergence analysis (see Remark 3.4) and even weaker sufficient convergence criteria (see Theorem 3.5) than in earlier studies such as [3, 14, 15, 16, 17, 18, 19, 20]. The theoretical results are illustrated using munerical examples to show that our new convergence criteria are satisfied but the old ones are not. Moreover, we show that the new error bounds are tighter than the old ones.

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OPERATOR THEORETIC TECHNIQUES IN THE THEORY OF NONLINEAR ORDINARY HYBRID DIFFERENTIAL EQUATIONS WITH MAXIMA

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ABSTRACT. In this paper author proves the algorithms for the existence as well as approximation of the solutions for a couple of initial value problems of nonlinear p^{th} order ordinary differential equations with maxima using the operator theoretic techniques in a partially ordered normed linear space. The main results rely on some recent hybrid fixed point theorems of Dhage (2013) in a partially ordered normed linear space and the approximation of the solutions of the considered nonlinear differential equations with maxima are obtained under weaker mixed partial continuity and partial Lipschitz conditions. Our hypotheses and results are also illustrated by some numerical examples.

KEYWORDS : Hybrid differential equation; Hybrid fixed point theorem; Dhage iteration method; Existence and approximation theorems.
AMS Subject Classification: 34A12, 34H34, 47H07, 47H10

1. INTRODUCTION

The operator theory is a vast developed branch of the subject of mathematics which has important applications to the problems of several areas of mathematics. The area of differential equations with maxima is not an exception. More specifically the area of nonlinear differential equations with maxima totally depends upon nonlinear operator theory and applications. It is known that even for a simple nonlinear differential equation there is no method to solve and obtain the exact solution. However, if we use nonlinear operator theory, then we can have more information about the solutions of the nonlinear problems such as existence, uniqueness, stability, attractivity, positivity, monotonicity and multiplicity results to mention a few. Therefore, there is considerable development of this area under the title nonlinear analysis and has been discussed all over the world. Again, fixed point theory is an important branch of nonlinear analysis which concerns with the solutions of the operator equation $\mathcal{T}x = x$, where $\mathcal{T}$ is a nonlinear operator in an abstract space under

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consideration, and the celebrated mathematicians Schauder(1914), Banach(1922) and Tarski(1940) proved three basic fixed point principles and provided the foundation stones for the subject of nonlinear analysis (cf. Granas and Dugundji [19] and the references therein). Each of these three basic fixed point theorems has some advantages and disadvantages over the others. There are several extensions and generalizations of the above three basic fixed point principles, called the fixed point theoretic techniques or operator theoretic techniques and have been widely used in the literature on nonlinear equations for proving the different aspects of the solutions. There are nonlinear equations related to some dynamic systems for which the above fixed point theorems are not applicable but the fixed point theorems with mixed arguments from these theorems may be applicable for proving the existence as well as some other information about the solutions. Therefore, a new area of operator theoretic techniques under the title hybrid fixed point theory is developed (see Dhage [3, 4, 6, 7, 8, 9, 10] and the references therein). Actually the origin of hybrid fixed point theory lies in the works of Krasnoselskii [21] and Dhage [3, 4], however the theory gained momentum after the publication of the papers of Ran and Reurings [24] and Dhage [6]. Like Picard iteration method, the operator theoretic technique involved in the hybrid fixed pint theorem of Dhage [5] is commonly known as Dhage iteration method and a few details of the hybrid fixed point theory may be found in Dhage [6, 7, 8] and Dhage and Dhage [12, 13]. In the present paper we employ some hybrid fixed point theorems in the study of initial value problems of nonlinear higher order ordinary differential equations with maxima .

The rest of the paper will be organized as follows. In Section 2 we give some preliminaries and key fixed point theorem that will be used in subsequent part of the paper. In Section 3 we discuss the existence result for the initial value problems and in Section 4 we discuss the existence result for initial value problems of hybrid differential differential equations with maxima with linear perturbation of first type.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers is preserved by $\preceq$. The details of such spaces appear in Dhage [4] and the references therein. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Deimling [2] and Heikkilä and Lakshmikantham [20] and the references therein.

We need the following definitions (see Dhage [4, 5, 6] and the references therein) in what follows.

Definition 2.1. A mapping $\mathcal{T} : E \longrightarrow E$ is called **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either monotone nondecreasing or monotone nonincreasing on E.

The following terminologies may be found in any book on nonlinear analysis and applications. See Deimling [2], Granas and Dugundji [19], Zeidler [25] and the references therein.

Definition 2.2. An operator $\mathcal{T}$ on a normed linear space E into itself is called **compact** if $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a relatively compact subset of E. If $\mathcal{T}$ is continuous and totally bounded, then it is called **completely continuous** on E.

Definition 2.3 (Dhage [5])**.** A mapping $\mathcal{T} : E \longrightarrow E$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa.

Definition 2.4. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called **uniformly partially bounded** if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called **partially compact** if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.5. Note that every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications do not hold. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded, but the converse may not be true.

Definition 2.6. The order relation $\preceq$ and the metric d on a non-empty set E are said to be **$\mathcal{D}$-compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nondecreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are **$\mathcal{D}$-compatible**. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property. In general every finite dimensional Banach space with a standard norm and an order relation is a **Janhavi Banach space**.

The essential idea of **Dhage iteration principle** may be described as " **the monotonic convergence of the sequence of successive approximations to**

the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation" and it is a very powerful tool in the existence theory of nonlinear analysis. The procedure involved in the application of Dhage iteration principle to nonlinear equation is called the " **Dhage iteration method**." It is clear that Dhage iteration method embodied in hybrid fixed point theorems is different for different nonlinear problems and also different from the usual Picard's successive iteration method. A few other hybrid fixed point theorems involving the Dhage iteration method may be found in Dhage [8, 9, 10].

Theorem 2.1 (Dhage [7])**.** *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C in E is Janhavi. Let $\mathcal{T} : E \longrightarrow E$ be a partially continuous, nondecreasing and partially compact operator. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $\mathcal{T}x_0 \preceq x_0$, then the operator equation $\mathcal{T}x = x$ has a solution x^* in E and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*.*

Remark 2.7. The regularity of E in above Theorem 2.1 may be replaced with a stronger continuity condition of the operator $\mathcal{T}$ on E which is a result proved in Dhage [6].

The Dhage iteration method involved in the following hybrid fixed point theorems are employed for proving the existence and uniqueness of the solutions for the IVP considered in the subsequent section of the paper. Before stating these results, we need the following definitions (see Dhage [4, 5, 6] and the references therein) in what follows.

Definition 2.8. An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. An operator $\mathcal{T} : E \longrightarrow E$ is called partially nonlinear $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $\psi(r) < r$ for $r > 0$. In particular if $\psi(r) = k\,r$, $\mathcal{T}$ is a partially linear contraction on E with a contraction constant k.

Theorem 2.2 (Dhage [7])**.** *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \longrightarrow E$ be a nondecreasing and partial nonlinear $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Theorem 2.3 (Dhage [8])**.** *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear space such that every compact chain C in E is Janhavi. Let $\mathcal{A}, \mathcal{B} : E \longrightarrow E$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.*

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,..., converges monotonically to x^*.

Remark 2.9. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact has been utilized to prove the main results of this paper.

Note that the Dhage iteration method presented in the above hybrid fixed point theorems have been employed in Dhage and Dhage [12, 13, 14] and Dhage *et.al.* [17] for approximating the solutions of initial value problems of nonlinear first order ordinary differential equation under some natural hybrid conditions. In the following section we approximate the solutions of certain IVPs of nonlinear higher order ordinary differential equations with maxima via successive approximations.

3. Initial Value Problems

Given a closed and bounded interval $J=[t_0,t_0+a]$ of the real line $\mathbb{R}$ for some $t_0, a\in\mathbb{R}$ with $t_0\geq 0$ and $a>0$ and given a positive integer p, consider the initial value problem (in short IVP) of p^{th} order ordinary nonlinear hybrid differential equation with maxima,

$$\left.\begin{aligned} x^{(p)}(t) &= f(t,x(t),X(t)),\ t\in J,\\ x(t_0)=\alpha_0, x'(t_0) &= \alpha_1,\dots,x^{(p-1)}(t_0)=\alpha_{p-1},\end{aligned}\right\} \tag{3.1}$$

where $f: J\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function and $X(t)=\max\limits_{t_0\leq\xi\leq t} x(\xi)$ for $t\in J$.

By a solution of the IVP (3.1) we mean a function $x\in C^p(J,\mathbb{R})$ that satisfies equation (3.1), where $C^p(J,\mathbb{R})$ is the space of p - times continuously differentiable real-valued functions defined on J.

The differential equations with maxima occur in the regulated systems of automatic control and a few details of such equations appears in Bainov and Hristova [1]. The study of first order ordinary nonlinear differential equations is exploited in Dhage [11], Dhage and Dhage [15, 16] and Dhage and Otrocol [18] via Dhage iteration method for existence and approximation theorems. Therefore it is desirable to extend the Dhage iteration method to other nonlinear higher order differential equations. The IVP (3.1) is new to the literature and not discussed for existence as well as any other aspects of the solutions. In the present paper it is proved that the existence of the solutions may be proved under weaker partial continuity and partial compactness type conditions.

The equivalent integral form of the IVP (3.1) is considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J}|x(t)| \tag{3.2}$$

and

$$x\leq y \iff x(t)\leq y(t) \tag{3.3}$$

for all $t\in J$. Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset S of $C(J,\mathbb{R})$ is Janhavi, i.e., $\|\cdot\|$ and $\leq$ are compatible in every compact chain C in S.*

Proof. The lemma mentioned in Dhage [6, 8], but the proof appears in Dhage [10, 11] and Dhage and Dhage [12, 13, 14]. Since the proof is not well-known, we give the details of the proof for completeness. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots, \tag{$*$}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x point-wise on J. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.2. A function $u \in C^{(p)}(J,\mathbb{R})$ is said to be a lower solution of the IVP (3.1) if it satisfies

$$\left.\begin{aligned} u^{(p)}(t) &\leq f(t,u(t),U(t)),\ t\in J,\\ u(t_0) \leq \alpha_0,\ u'(t_0) &\leq \alpha_1,\dots,u^{(p-1)}(t_0)\leq \alpha_{p-1}, \end{aligned}\right\} \tag{$*$}$$

for all $t \in J$, where $U(t) = \max\limits_{t_0\leq\xi\leq t} u(\xi)$ for $t \in J$. Similarly, an upper solution $\in C^{(p)}(J,\mathbb{R})$ to the IVP (3.1) is defined on J by the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

- (H_1) There exists a constant $M_f > 0$ such that $|f(t,x,y)| \leq M_f$ for all $t \in J$ and $x, y \in \mathbb{R}$.
- (H_2) The function $f(t,x,y)$ is monotone nondecreasing in x and y for each $t \in J$.
- (H_3) The IVP (3.1) has a lower solution $u \in C^p(J,\mathbb{R})$.

Lemma 3.3. *For a given integrable function $h : J \longrightarrow \mathbb{R}$, a function $u \in C^p(J,\mathbb{R})$ is a solution of the IVP*

$$\left.\begin{aligned} x^{(p)}(t) &= h(t),\ t\in J,\\ x(t_0) = \alpha_0, x'(t_0) &= \alpha_1,\dots,x^{(p-1)}(t_0) = \alpha_{p-1}, \end{aligned}\right\} \tag{3.4}$$

if and only if it is a solution of the nonlinear integral equation,

$$x(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, h(s)\,ds,\ t\in J. \tag{3.5}$$

Theorem 3.1. *Assume that the hypotheses (H_1) through (H_3) hold. Then the IVP (3.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$
$$x_{n+1}(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x_n(s), X_n(s))\, ds, \tag{3.6}$$

for all $t \in \mathbb{R}$, converges monotonically to x^*.

Proof. By Lemma 3.3, the IVP (3.1) is equivalent to the nonlinear integral equation

$$x(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds, \ t \in J. \tag{3.7}$$

Set $E = C(J, \mathbb{R})$ and define the operator $\mathcal{T}$ by

$$\mathcal{T}x(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds, \ t \in J. \tag{3.8}$$

From the continuity of the integral, it follows that $\mathcal{T}$ defines the map $\mathcal{T} : E \longrightarrow E$. Then, the IVP (3.1) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t), \ \ t \in J. \tag{3.9}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: *$\mathcal{T}$ is nondecreasing on E.*

Let $x, y \in E$ be such that $x \geq y$. Then $x(t) \geq y(t)$ for all $t \in J$. Since y is continuous on $[t_0, t]$, there exists a $\xi^* \in [t_0, t]$ such that $y(\xi^*) = \max_{t_0 \leq \xi \leq t} y(\xi)$. By definition of $\leq$, one has $x(\xi^*) \geq y(\xi^*)$. Consequently, we obtain

$$\max_{t_0 \leq \xi \leq t} x(\xi) \geq x(\xi^*) \geq y(\xi^*) = \max_{t_0 \leq \xi \leq t} y(\xi).$$

Then by hypothesis (H_2), we obtain

$$\begin{aligned} \mathcal{T}x(t) &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \\ &\geq \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, y(s), Y(s))\, ds \\ &= \mathcal{T}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{T}$ is nondecreasing operator on E into E.

Step II: *$\mathcal{T}$ is partially continuous on E.*

Let $\{x_n\}$ be a sequence of points of an arbitrary chain C in E such that $x_n \longrightarrow x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \longrightarrow \infty} \mathcal{T}x_n(t) &= \lim_{n \longrightarrow \infty} \left[\sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \right] \\ &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} \left[\lim_{n \longrightarrow \infty} f(s, x_n(s), X_n(s)) \right] ds \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \\
&= \mathcal{T}x(t),
\end{aligned}
$$

for all $t \in J$. This shows that $\{\mathcal{T}x_n\}$ converges to $\mathcal{T}x$ pointwise on J.

Next, we will show that $\{\mathcal{T}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$
\begin{aligned}
|\mathcal{T}x_n(t_2) - \mathcal{T}x_n(t_1)| &\leq \left| \sum_{i=0}^{p-1} \frac{\alpha_i (t_2-t_0)^i}{i!} - \sum_{i=0}^{p-1} \frac{\alpha_i (t_1-t_0)^i}{i!} \right| \\
&\quad + \left| \int_{t_0}^{t_2} \frac{(t_2-s)^{p-1}}{(p-1)!} f(s, x_n(s), X_n(s))\, ds \right. \\
&\qquad \left. - \int_{t_0}^{t_1} \frac{(t_1-s)^{p-1}}{(p-1)!} f(s, x_n(s), X_n(s))\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_2-s)^{p-1}\, ds - \int_{t_0}^{t_1} (t_1-s)^{p-1}\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_2-s)^{p-1}\, ds - \int_{t_0}^{t_2} (t_1-s)^{p-1}\, ds \right| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_2} (t_1-s)^{p-1}\, ds - \int_{t_0}^{t_1} (t_1-s)^{p-1}\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_0+a} \big| (t_2-s)^{p-1} - (t_1-s)^{p-1} \big|\, ds \right| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_1}^{t_2} \big| (t_1-s)^{p-1} \big|\, ds \right| \\
&\leq \sum_{i=0}^{p-1} \frac{|\alpha_i|}{i!} \big| (t_2-t_0)^i - (t_1-t_0)^i \big| \\
&\quad + \frac{M_f}{(p-1)!} \left| \int_{t_0}^{t_0+a} \big| (t_2-s)^{p-1} - (t_1-s)^{p-1} \big|\, ds \right| \\
&\quad + \frac{M_f\, a^{p-1}}{(p-1)!} |t_1 - t_2| \\
&\longrightarrow 0 \quad as \quad t_1 \longrightarrow t_2,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{T}x_n \longrightarrow \mathcal{T}x$ is uniform and hence $\mathcal{T}$ is partially continuous on E.

Step III: *$\mathcal{T}$ is partially compact on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{T}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{T}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq \sum_{i=0}^{p-1}\left|\frac{\alpha_i(t-t_0)^i}{i!}\right| + \left|\int_{t_0}^{t}\frac{(t-s)^{p-1}}{(p-1)!}f(s,x(s),X(s))\,ds\right| \\
&\leq \sum_{i=0}^{p-1}\frac{|\alpha_i|\,a^i}{i!} + \left|\int_{t_0}^{t}\frac{(t-s)^{p-1}}{(p-1)!}\big|f(s,x(s),X(s))\big|\,ds\right| \\
&\leq \sum_{i=0}^{p-1}\frac{|\alpha_i|\,a^i}{i!} + \frac{a^p}{p!}M_f \\
&= r,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{T}x\| \leq r$ for all $x \in C$. Hence $\mathcal{T}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{T}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned}
|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)| &\leq \left|\sum_{i=0}^{p-1}\frac{\alpha_i(t_2-t_0)^i}{i!} - \sum_{i=0}^{p-1}\frac{\alpha_i(t_1-t_0)^i}{i!}\right| \\
&\quad + \left|\int_{t_0}^{t_2}\frac{(t_2-s)^{p-1}}{(p-1)!}f(s,x(s),X(s))\,ds\right. \\
&\qquad\qquad \left. - \int_{t_0}^{t_1}\frac{(t_1-s)^{p-1}}{(p-1)!}f(s,x(s),X(s))\,ds\right| \\
&\leq \sum_{i=0}^{p-1}\frac{|\alpha_i|}{i!}\big|(t_2-t_0)^i - (t_1-t_0)^i\big| \\
&\quad + \frac{M_f}{(p-1)!}\left|\int_{t_0}^{t_2}(t_2-s)^{p-1}\,ds - \int_{t_0}^{t_1}(t_1-s)^{p-1}\,ds\right| \\
&\leq \sum_{i=0}^{p-1}\frac{|\alpha_i|}{i!}\big|(t_2-t_0)^i - (t_1-t_0)^i\big| \\
&\quad + \frac{M_f}{(p-1)!}\left|\int_{t_0}^{t_2}(t_2-s)^{p-1}\,ds - \int_{t_0}^{t_2}(t_1-s)^{p-1}\,ds\right| \\
&\quad + \frac{M_f}{(p-1)!}\left|\int_{t_0}^{t_2}(t_1-s)^{p-1}\,ds - \int_{t_0}^{t_1}(t_1-s)^{p-1}\,ds\right| \\
&\leq \sum_{i=0}^{p-1}\frac{|\alpha_i|}{i!}\big|(t_2-t_0)^i - (t_1-t_0)^i\big| \\
&\quad + \frac{M_f}{(p-1)!}\left|\int_{t_0}^{t_0+a}\big|(t_2-s)^{p-1} - (t_1-s)^{p-1}\big|\,ds\right| \\
&\quad + \frac{M_f}{(p-1)!}\left|\int_{t_1}^{t_2}(t_1-s)^{p-1}\,ds\right| \\
&\leq \sum_{i=0}^{p-1}\frac{|\alpha_i|}{i!}\big|(t_2-t_0)^i - (t_1-t_0)^i\big| \\
&\quad + \frac{M_f}{(p-1)!}\left|\int_{t_0}^{t_0+a}\big|(t_2-s)^{p-1} - (t_1-s)^{p-1}\big|\,ds\right|
\end{aligned}$$

$$+\frac{M_f\, a^{p-1}}{(p-1)!}|t_1-t_2|$$
$$\longrightarrow 0 \quad as \quad t_1 \longrightarrow t_2,$$

uniformly for all $x \in C$. Hence $\mathcal{T}(C)$ is compact subset of E and consequently $\mathcal{T}$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \le \mathcal{T}u$.*

Since the hypothesis (H_3) holds, u is a lower solution of (3.1) defined on J. Then

$$u^{(p)}(t) \le f(t,u(t),U(t)),\ t \in J, \tag{3.10}$$

satisfying,

$$u(t_0) \le \alpha_0,\, u'(t_0) \le \alpha_1, \dots, u^{(p-1)}(t_0) \le \alpha_{p-1}, \tag{3.11}$$

for all $t \in J$.

Integrating (3.10) from t_0 to t, we obtain

$$u^{(p-1)}(t) \le \alpha_{p-1} + \int_{t_0}^{t} f(s,u(s),U(s)),\, ds. \tag{3.12}$$

for all $t \in J$. Again, integrating (3.12) from t_0 to t,

$$u^{(p-2)}(t) \le \alpha_{p-2} + \alpha_{p-1}(t-t_0) + \int_{t_0}^{t} \frac{(t-s)^2}{2} f(s,u(s),U(s))\, ds$$

for all $t \in J$.

Proceeding in this way, by induction, we obtain

$$u(t) \le \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s,u(s),U(s))\, ds = \mathcal{T}u(t)$$

for all $t \in J$. This show that u is a lower solution of the operator equation $x = \mathcal{T}x$.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 in view of Remark 2.9 and we apply to conclude that the operator equation $\mathcal{T}x = x$ has a solution. Consequently the integral equation and the IVP (3.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.6) converges monotonically to x^*. This completes the proof. □

Remark 3.4. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (H_3) with the following one:

(H_3') The IVP (3.1) has an upper solution $v \in C^p(J,\mathbb{R})$.

Example 3.5. Given a closed and bounded interval $J = [0,1]$, consider the IVP,

$$\left.\begin{aligned} x^{(p)}(t) &= \tanh x(t) + \tanh X(t),\ t \in J,\\ x(0) = 0,\, x'(0) = 1, \dots, x^{(p-1)}(0) &= \frac{1}{p-1}, \end{aligned}\right\} \tag{3.13}$$

where $X(t) = \max\limits_{0 \le \xi \le t} x(\xi)$.

Here, $f(t,x,y) = \tanh x + \tanh y$. Clearly, the functions f is continuous on $J \times \mathbb{R} \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $M_f = 2$. Moreover, the function $f(t,x,y) = \tanh x + \tanh y$ is nondecreasing in x and y for each $t \in J$ and so the hypothesis (H_2) is satisfied.

Finally, the IVP (3.1) has a lower solution

$$u(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} - 2\int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, ds$$

defined on J. Thus all the hypotheses of Theorem 3.1 are satisfied. Hence we conclude that the IVP (3.13) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} - 2\int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, ds, \\ x_{n+1}(t) &= \sum_{i=1}^{p-1} \frac{1}{i}\cdot\frac{t^i}{i!} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\big[\tanh x_n(s) + \tanh x_n(s)\big]\, ds, \end{aligned} \tag{3.14}$$

for all $t \in J$, converges monotonically to x^*. □

Next, we prove the uniqueness theorem for the IVP (3.1) under weak Lipschitz condition. We need the following hypothesis in what follows.

(H_4) There exists a $\mathcal{D}$-function φ such that

$$0 \le f(t, x_1, x_2) - f(t, y_1, y_2) \le \varphi(\max\{x_1 - y_1\, ,\, x_2 - y_2\})$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, $x_1 \ge y_1$, $x_2 \ge y_2$. Moreover, $\frac{a^p}{p!}\varphi(r) < r$ for $r > 0$.

Theorem 3.2. *Assume that hypotheses (H_3) and (H_4) hold. Then the IVP (3.1) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by* (3.6) *converges monotonically to x^*.*

Proof. Set $E = C(J,\mathbb{R})$. Clearly, E is a lattice w.r.t. the order relation $\le$ and so the lower and the upper bound for every pair of elements in E exist. Define the operator $\mathcal{T}$ by (3.7). Then, the IVP (3.1) is equivalent to the operator equation (3.9). We shall show that $\mathcal{T}$ satisfies all the conditions of Theorem 2.2 in E.

Clearly, $\mathcal{T}$ is a nondecreasing operator on E into itself. We shall simply show that the operator $\mathcal{T}$ is a partial nonlinear $\mathcal{D}$-contraction on E. Then, we have

$$|x(t) - y(t)| \le |X(t) - Y(t)|$$

and that

$$\begin{aligned} |X(t) - Y(t)| &= X(t) - Y(t) \\ &= \max_{t_0 \le \xi \le t} x(\xi) - \max_{t_0 \le \xi \le t} y(\xi) \\ &\le \max_{t_0 \le \xi \le t} [x(\xi) - y(\xi)] \\ &= \max_{t_0 \le \xi \le t} |x(\xi) - y(\xi)| \\ &\le \|x - y\| \end{aligned}$$

for each $t \in J$. As a result, we obtain by hypothesis (H_4),

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}y(t)| &\le \int_{t_0}^{t} \left| \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s)) - \frac{(t-s)^{p-1}}{(p-1)!} f(s, y(s), Y(s)) \right| ds \\ &\le \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} \left| f(s, x(s), X(s)) - f(s, y(s), Y(s)) \right| ds \end{aligned}$$

$$\leq \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\,\varphi\Big(\max\{|x(s)-y(s)|\,,|X(t)-Y(t)|\}\Big)\,ds$$
$$\leq \frac{(t-t_0)^p}{p!}\,\varphi(\|x-y\|)$$
$$\leq \psi(\|x-y\|) \tag{3.15}$$

for all $t \in J$, where $\psi(r) = \dfrac{a^p}{p!}\varphi(r) < r$, $r > 0$.

Taking the supremum over t, we obtain

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x-y\|)$$

for all $x, y \in E$, $x \geq y$. As a result $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-contraction on E. Now a direct application of Theorem 2.2 yields that the IVP (3.1) has a unique solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (3.6) converges monotonically to x^*. □

Remark 3.6. The conclusion of Theorem 3.2 also remains true if we replace the hypothesis (H_3) with the following one:

(H_3') The IVP (3.1) has an upper solution $v \in C^p(J, \mathbb{R})$.

Example 3.7. Given a closed and bounded interval $J = [0, 1]$, consider the IVP,

$$\left.\begin{aligned} x^{(p)}(t) &= \frac{1}{2}\Big[\tan^{-1} x(t) + \tan^{-1} X(t)\Big],\ t \in J,\\ x(0) = 0,\ x'(0) &= 1, \ldots, x^{(p-1)}(0) = \frac{1}{p-1}, \end{aligned}\right\} \tag{3.16}$$

where $X(t) = \max\limits_{0 \leq \xi \leq t} x(\xi)$, $t \in J$.

Here, $f(t, x, y) = \tan^{-1} x + \tan^{-1} y$. Clearly, the functions f is continuous on $J \times \mathbb{R} \times \mathbb{R}$. The function f satisfies the hypothesis (H_1) with $M_f = \dfrac{\pi}{2}$. We show that f satisfies the hypothesis (H_4) on $J \times \mathbb{R} \times \mathbb{R}$. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$. Then, we have

$$\begin{aligned} 0 &\leq f(t, x_1, x_2) - f(t, y_1, y_2)\\ &\leq \frac{1}{2}\big[\tan^{-1} x_1 - \tan^{-1} y_1\big] + \frac{1}{2}\big[\tan^{-1} x_2 - \tan^{-1} y_2\big]\\ &\leq \varphi\Big(\max\{x_1 - y_1\,,\, x_2 - y_2\}\Big) \end{aligned}$$

for all $t \in J$, where φ is a $\mathcal{D}$-function defined by $\varphi(r) = \dfrac{r}{1+\xi^2} < r$, $0 < \xi < r$.

Finally, the IVP (3.1) has a lower solution

$$u(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} - 2\int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\,ds$$

defined on J. Thus, if $\dfrac{a^p}{p!} \cdot \dfrac{1}{1+\xi^2} < 1$ for each $\xi > 0$, then all the hypotheses of Theorem 3.2 are satisfied. Hence we conclude that the IVP (3.16) has a unique solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} - 2\int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!}\, ds, \\ x_{n+1}(t) &= \sum_{i=1}^{p-1} \frac{1}{i}\cdot\frac{t^i}{i!} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, \tan^{-1} x_n(s)\, ds, \end{aligned} \tag{3.17}$$

for all $t \in J$, converges monotonically to x^*.

4. HDE of Linear Perturbations of First Type

Next, with usual notation, we consider the following nonlinear hybrid differential equation with maxima,

$$\left.\begin{aligned} x^{(p)}(t) &= f(t, x(t), X(t)) + g(t, x(t), X(t)),\ t \in J, \\ x(t_0) &= \alpha_0, x'(t_0) = \alpha_1, \dots, x^{(p-1)}(t_0) = \alpha_{p-1}, \end{aligned}\right\} \tag{4.1}$$

where $f, g \;:\; J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and $X(t) = \max\limits_{t_0 \le \xi \le t} x(\xi)$.

By a *solution* of the IVP (4.1) we mean a function $x \in C^p(J, \mathbb{R})$ that satisfies equation (4.1), where $C^p(J, \mathbb{R})$ is the space of p-times continuously differentiable real-valued functions defined on J.

The IVP (4.1) is a hybrid differential equation with a linear perturbation of first type. See Dhage [5], Krasnoselskii [21] and the references therein. The IVP (4.1) is new to the literature and not discussed for existence as well asany other aspects of the solutions. In the present discussion, it is proved that the existence and approximation of the solutions may be proved under mixed partial Lipschitz and partial compactness type conditions.

We need the following definition in what follows.

Definition 4.1. A function $u \in C^{(p)}(J, \mathbb{R})$ is said to be a lower solution of the IVP (4.1) if it satisfies

$$\left.\begin{aligned} u^{(p)}(t) &\le f(t, u(t), U(t)) + g(t, u(t), U(t)),\ t \in J, \\ u(t_0) &\le \alpha_0,\ u'(t_0) \le \alpha_1, \dots, u^{(p-1)}(t_0) \le \alpha_{p-1}, \end{aligned}\right\} \tag{$**$}$$

where $U(t) = \max\limits_{t_0 \le \xi \le t} u(\xi)$ for $t \in J$. Similarly, an upper solution v to the IVP (4.1) is defined on J, by the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

- (H_5) There exists a constant $M_g > 0$ such that $|g(t, x, y)| \le M_g$ for all $t \in J$ and $x, y \in \mathbb{R}$.
- (H_6) The mapping $g(t, x, y)$ is monotone nondecreasing in x and y for each $t \in J$.
- (H_7) The IVP (4.1) has a lower solution $u \in C^p(J, \mathbb{R})$.

Our main existence and approximation theorem for the IVP (4.1) is as follows.

Theorem 4.1. *Assume that the hypotheses (H_1) and (H_4) through (H_7) hold. Then the IVP (4.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive* *approximations defined by*

$$
\begin{aligned}
x_0 &= u, \\
x_{n+1}(t) &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x_n(s), X_n(s))\, ds \\
&\quad + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} g(s, x_n(s), X_n(s))\, ds,
\end{aligned}
\tag{4.2}
$$

for all $t \in J$*, converges monotonically to* x^**.*

Proof. By Lemma 3.3, the IVP (4.1) is equivalent to the nonlinear integral equation

$$
\begin{aligned}
x(t) &= \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds \\
&\quad + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} g(s, x_n(s), X(s))\, ds, \ t \in J.
\end{aligned}
\tag{4.3}
$$

Set $E = C(J, \mathbb{R})$ and define the operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$
\mathcal{A}x(t) = \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} f(s, x(s), X(s))\, ds, \ t \in J, \tag{4.4}
$$

and

$$
\mathcal{B}x(t) = \sum_{i=0}^{p-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{p-1}}{(p-1)!} g(s, x(s), X(s))\, ds, \ t \in J. \tag{4.5}
$$

From the continuity of the integrals, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \longrightarrow E$. Now, the IVP (4.1) is equivalent to the operator equation

$$
\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ t \in J. \tag{4.6}
$$

Then following the arguments similar to those given in Theorems 3.1 and 3.2, it can be shown that the operator $\mathcal{A}$ is partially bounded and partial nonlinear $\mathcal{D}$-contraction and $\mathcal{B}$ is partially continuous and partially compact operator on E into E. Now by a direct application of Theorem 2.3 we conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^*. Consequently the IVP (4.1) has a solution x^* and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (4.2) converges monotonically to x^*. This completes the proof. □

The conclusion of Theorems 4.1 also remains true if we replace the hypothesis (H_7) with the following one:

(H_7') The IVP (4.1) has an upper solution $v \in C^p(J, \mathbb{R})$.

Example 4.2. Given a closed and bounded interval $J = [0, 1]$, consider the IVP,

$$
\left.
\begin{aligned}
x^{(p)}(t) = \tan^{-1} x(t) + g(t, x(t), X(t)), \ t \in J, \\
x(0) = 0, \, x'(0) = 1, \ldots, x^{(p-1)}(0) = \frac{1}{p-1},
\end{aligned}
\right\}
\tag{4.7}
$$

where $X(t) = \max\limits_{0 \le \xi \le t} x(\xi)$ and $g : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function defined by

$$
g(t, x, y) = \begin{cases} 1, & \text{if } y \le 0, \\ 1 + \dfrac{y}{1+y}, & \text{if } x > 0. \end{cases}
$$

for $t \in J$.

Here, $f(t,x,y) = \tan^{-1} x$. The function f satisfies the hypothesis (H_1) with $M_f = \frac{\pi}{2}$. Moreover, f satisfies (H_4) with $\varphi(r) = \frac{r}{1+\xi^2} < r$, $0 < \xi < r$. Next, g satisfies (H_5) with $M_g = 2$. Again, the function $y \mapsto g(t,x,y)$ is nondecreasing in y for each $t \in J$ and so the hypothesis (H_6) is satisfied. Finally, the function

$$u(t) = \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} - 2 \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds,$$

for all $t \in J$ is a lower solution of the IVP (4.7) on J. Hence, if $\frac{a^p}{p!} \cdot \frac{1}{1+\xi^2} < 1$ for each $\xi > 0$, we apply Theorem 4.1 and conclude that the IVP (4.7) has a solution x^* on J and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$\begin{aligned} x_1 &= \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} - 2 \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds, \\ x_{n+1}(t) &= \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \tan^{-1} x_n(s))\, ds \\ &\quad + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} g(s, x_n(s), X_n(s))\, ds, \end{aligned}$$

for each $t \in J$ converges monotonically x^*.

Example 4.3. Given a closed and bounded interval $J = [0,1]$ and given a positive integer p, consider the IVP,

$$\left.\begin{aligned} x^{(p)}(t) &= \tan^{-1} x(t) + \tanh X(t),\ t \in J, \\ x(0) = 0,\ x'(0) &= 1, \dots, x^{(p-1)}(0) = \frac{1}{p-1}. \end{aligned}\right\} \tag{4.8}$$

where $X(t) = \max_{0 \le \xi \le t} x(\xi)$.

Here $f(t,x,y) = \tan^{-1} x$ and $g(t,x,y) = \tanh y$ for all $t \in J$ and $x, y \in \mathbb{R}$. Then proceeding with the arguments that given in Examples 3.5 and 3.7, it is proved that the function f satisfies the hypotheses (H_1) and (H_4) and g satisfies the hypotheses (H_5)-(H_6) on $J \times \mathbb{R} \times \mathbb{R}$. Similarly, the hypothesis (H_7) is held with a lower solution u given by

$$u(t) = \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} - 2 \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds - \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds,$$

for all $t \in J$. Hence, if $\frac{a^p}{p!} \cdot \frac{1}{1+\xi^2} < 1$ for each $\xi > 0$, we apply Theorem 4.1 and prove that the IVP (4.8) has a solution x^* on J and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$\begin{aligned} x_1 &= \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} - 2 \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds - \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds, \\ x_{n+1}(t) &= \sum_{i=1}^{p-1} \frac{1}{i} \cdot \frac{t^i}{i!} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \tan^{-1} x_n(s))\, ds \\ &\quad + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \tanh x_n(s))\, ds, \end{aligned} \tag{4.9}$$

for each $t \in J$ converges monotonically x^*.

Remark 4.4. The study of the present paper may be extended with essentially the same approach but under appropriate modifications to the general higher order IVPs of the type

$$\left.\begin{array}{c}\mathcal{L}x(t) = \mathcal{N}x(t)),\ t \in J,\\ x(t_0) = \alpha_0, x'(t_0) = \alpha_1, \ldots, x^{(n-1)}(t_0) = \alpha_{n-1},\end{array}\right\} \tag{4.10}$$

where $\mathcal{L}$ is a linear n^{th} order differential operator of the type

$$\mathcal{L} = a_0 D^n + a_1 D^{n-1} + \cdots + a_n$$

and $\mathcal{N}$ is a Nemytsky operator defined by

$$\mathcal{N}x(t) = f\big(t, x(t), x'(t), \ldots, x^{n-1}(t), X(t)\big),\ t \in J,$$

for some constants $a_0, \ldots, a_n$, where $X(t) = \max\limits_{t_0 \le \xi \le t} x(\xi)$.

5. Conclusion

From the foregoing discussion it is clear that unlike Schauder fixed point principle, the proofs of Theorems 3.1 and 4.1 do not invoke the construction of a nonempty, closed, convex and bounded subset of the Banach space of navigation which is mapped into itself by the operators related to the given differential equations with maxima. The convexity hypothesis is altogether omitted from the discussion and still we have proved the existence and approximation of the solutions for the differential equations with maxima considered in this paper with stronger conclusion. Similarly, unlike the use of Banach fixed point theorem, Theorems 3.1 and 4.1 do not make any use of usual Lipschitz condition on the nonlinearity involved in the differential equations with maxima (3.1) and (4.1), but even then the algorithms for the solutions of the differential equations with maxima (3.1) and (4.1) are proved in terms of the new Dhage iteration scheme which is different from Picard iterations. Again, unlike Tarski fixed point theorem, we do not need the partially ordered space under consideration to be a complete lattice, however one needs the additional condition of regularity together with partially continuity of the mappings on it. But the advantage of our results over Tarski lies in the fact that we obtain the algorithms for the solutions of the considered nonlinear problems. Unlike Picard iteration method for the nonlinear differential equations with maxima, the new so called Dhage iteration method does not start with the initial data but starts at the given lower or upper solution of the related problems and uses compactness type arguments instead of Lipschitz condition which is usually needed for the Picard iteration scheme. The advantage of hybrid fixed point theoretic techniques over classical ones is that we obtain the algorithms along with the existence of solutions with strong conclusion of monotone convergence of the algorithms to the solutions. The nature of the convergence of the algorithms is not geometrical and so we are not able to obtain the rate of convergence of the algorithms to the solutions of the related problems. However, in a way we have been able to prove the existence as well as approximation results for the IVPs (3.1) and (4.1) under much weaker conditions with a stronger conclusion of the monotone convergence of successive approximations to the solution than those proved in the existing literature on nonlinear differential equations with maxima. The study of this paper may be extended to other nonlinear higher order differential equations such as higher order boundary value problems with appropriate modifications and some of the results in this directions will be reported elsewhere.

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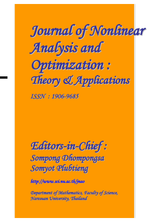

SET-VALUED PREŠIĆ-REICH TYPE CONTRACTIONS IN CONE METRIC SPACES AND FIXED POINT THEOREMS

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ABSTRACT. The purpose of this paper is to prove some fixed point theorems for set-valued mappings satisfying Prešić-Reich type contractive condition in cone metric spaces, without assuming the normality of cone. Our results generalize some known results in metric and cone metric spaces.

KEYWORDS : Prešić-Reich type; Point-set-cone metric; Cone metric space; Fixed point.
AMS Subject Classification: 47H10; 54H25

1. Introduction

Let (X, d) be a metric space and f be a self-map on X. The mapping f is called a Banach contraction if, there exists $\lambda \in [0, 1)$ such that

$$d(fx, fy) \leq \lambda d(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

The Banach contraction principle states that every Banach contraction on a complete metric space has a fixed point, i.e., there exists a point $x^* \in X$ such that $fx^* = x^*$.

Let X be a nonempty set, 2^X the collection of all possible subsets of X and $f\colon X \longrightarrow 2^X$ be a mapping. Then, f is called a set-valued mapping. Let $x \in X$ be such that $fx \neq \emptyset$, then x is called a fixed point of f if $x \in fx$.

Let A be any nonempty subset of a metric space (X, d). For $x \in X$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let $CB(X)$ denotes the set of all nonempty closed bounded subset of X. For $A, B \in CB(X)$, define

$$\delta(A, B) = \sup\{d(x, B) : x \in A\},$$
$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

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Then H is a metric on $CB(X)$ and called Pompeiu-Hausdorff (or Hausdorff) metric. A mapping $f\colon X \longrightarrow CB(X)$ is called a Nadler contraction (or a set-valued Banach contraction), if there exists $\lambda \in [0,1)$ such that

$$H(fx, fy) \leq \lambda d(x,y) \text{ for all } x, y \in X. \tag{1.2}$$

In 1969, Nadler [10] generalized the famous Banach contraction principle for the set-valued mappings defined from a complete metric space X into the set $CB(X)$. Nadler [10] proved the following theorem:

Theorem 1.1. *Let (X,d) be a complete metric space and let f be a set-valued Banach contraction. Then there exists a point $x^* \in X$ such that $x^* \in fx^*$, i.e., f has a fixed point in X.*

On the other hand, for mappings $f : X \to X$ Kannan [6] introduced the contractive condition:

$$d(fx, fy) \leq \lambda[d(x, fx) + d(y, fy)] \tag{1.3}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$ is a constant, and proved a fixed point theorem using (1.3) instead of contractive condition (1.1). The conditions (1.3) and (1.1) are independent, as it was shown by two examples in [7].

Reich [14], generalized the fixed point theorems of Banach and Kannan, using contractive condition: for all $x, y \in X$,

$$d(fx, fy) \leq \alpha d(x,y) + \beta d(x, fx) + \gamma d(y, fy) \tag{1.4}$$

where α, β, γ are nonnegative reals with $\alpha+\beta+\gamma < 1$. An example in [14] shows that the Reich's contractive condition is a proper generalization of contractive conditions of Banach and Kannan.

In 1965, Prešić [12, 13] generalized the Banach contraction principle in product spaces and proved the following theorem.

Theorem 1.2. *Let (X,d) be a complete metric space, k a positive integer and $f\colon X^k \to X$ be a mapping satisfying the following contractive type condition:*

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}), \tag{1.5}$$

for every $x_1, x_2, \ldots, x_k, x_{k+1} \in X$, where $q_1, q_2, \ldots, q_k$ are nonnegative constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then there exists a unique point $x \in X$ such that $f(x, x, \ldots, x) = x$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = f(\lim x_n, \lim x_n, \ldots, \lim x_n)$.

A mapping $f\colon X^k \to X$ is called a Prešić type contraction if it satisfies (1.5). The mapping f is called a Prešić-Kannan type contraction if,

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq a \sum_{i=0}^{k} d(x_i, f(x_i, \ldots, x_i)), \tag{1.6}$$

for all $x_0, x_1, \ldots, x_k \in X$, where the real constant a is such that $0 \leq ak(k+1) < 1$.

In a similar manner to that used by S.B. Prešić [12, 13], when extending Banach contractions to product spaces, Păcurar [11] generalized the Kannan's theorem in product spaces and proved a fixed point theorem for Prešić-Kannan type contractions.

f is called a Prešić-Reich type contraction if,

$$d(f(x_0,\dots,x_{k-1}),f(x_1,\dots,x_k)) \leq \sum_{i=1}^{k}\alpha_i d(x_{i-1},x_i) + \sum_{i=0}^{k}\beta_i d(x_i,f(x_i,\dots,x_i)), \tag{1.7}$$

for all $x_0,x_1,\dots,x_k \in X$, where α_i,β_i are nonnegative constants such that

$$\sum_{i=1}^{k}\alpha_i + k\sum_{i=0}^{k}\beta_i < 1.$$

Note that, for $k=1$ the above definition reduces into the definition due to Reich [14]. Also, Prešić-Banach type contraction (i.e., a mapping f satisfying (1.5)) and Prešić-Kannan type contraction are particular cases of Prešić-Reich type contractions. Malhotra et al. [9] first introduced the notion of Prešić-Reich type contractions (for single-valued case) in cone metric spaces and proved some common fixed point and fixed point results for such mappings.

In 2011, Wardowski [16] introduced the set-valued mappings in cone metric spaces and proved the cone metric version of the result of Nadler [10] (see also [1, 8, 15]). In this paper, we introduced the notion of set-valued Prešić-Reich type contractions in cone metric spaces and prove some fixed point results for such mappings, using the definitions due to Wardowski [16]. Our results generalize and extend the results of Nadler [10], Kannan [6], Reich [14], Prešić [12], Malhotra et al. [9] and Wardowski [16] in the setting of cone metric spaces for set-valued mappings. An example is provided which illustrate the main theorem of this paper.

2. Preliminaries

We use the following definitions and results, consistent with [2] and [3].

Definition 2.1. [3] Let E be a real Banach space and P be a subset of E. The set P is called a cone if

(i) P is closed, nonempty and $P \neq \{\theta\}$, here θ is the zero vector of E;
(ii) $a,b \in \mathbb{R},\ a,b \geq 0,\ x,y \in P \implies ax+by \in P$;
(iii) $x \in P$ and $-x \in P \implies x=\theta$.

Given a cone $P \subset E$, we define a partial ordering " $\preceq$ " with respect to P by $x \preceq y$ if and only if $y-x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. While $x \ll y$ if and only if $y-x \in P^\circ$, where P° denotes the interior of P.
Let P be a cone in a real Banach space E, then P is called normal, if there exist a constant $K>0$ such that for all $x,y \in E$,

$$\theta \preceq x \preceq y \ \text{ implies } \ \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P. A cone P is called solid if $P^\circ \neq \emptyset$.

Definition 2.2. [3] Let X be a nonempty set, E be a real Banach space with cone P. Suppose that the mapping $d: X\times X \to E$ satisfies:

1. $\theta \preceq d(x,y)$, for all $x,y \in X$ and $d(x,y)=\theta$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$ for all $x,y \in X$;
3. $d(x,y) \preceq d(x,z)+d(y,z)$, for all $x,y,z \in X$.

Then d is called a *cone metric* on X, and (X, d) is called a cone metric space. If the underlying cone is normal, then (X, d) is called a normal cone metric space.

The following lemma will be useful in the sequel.

Lemma 2.3. [4, 5] *Let E be a real Banach space, P a solid cone in E. Then:*

(i) *If $\{a_n\}$ is a sequence in P, $a_n \to \theta$ then, for every $c \in P^\circ$ there exists $n \in \mathbb{N}$ such that, $a_n \ll c$ for all $n > n_0$.*
(ii) *If $u, v, w \in P$ and $u \preceq v$, $v \ll w$ then $u \ll w$.*
(iii) *If $u, v, w \in P$ and $u \ll v$, $v \preceq w$ then $u \ll w$.*
(iv) *If $u \in P$ and $u \ll c$ for each $c \in P^\circ$, then $u = \theta$.*

Let (X, d) be a cone metric space with cone P. A subset $A \subset X$ is called closed if for any sequence $\{x_n\} \subset A$ convergent to x, we have $x \in A$.

Denote by $N(X)$ a collection of all nonempty subsets of X and by $C(X)$ a collection of all nonempty closed subsets of X.

The following definitions can be found in [16].

Definition 2.4. [16] Let (X, d) be a cone metric space and let $\mathcal{A}$ be a collection of nonempty subsets of X. A map $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ is called a $H-$cone metric with respect to d if for any $A_1, A_2 \in \mathcal{A}$ the following conditions hold:

(H1) $H(A_1, A_2) = \theta \implies A_1 = A_2$;
(H2) $H(A_1, A_2) = H(A_2, A_1)$;
(H3) $\forall_{c\in E,\theta\ll c}\ \forall_{x\in A_1}\ \exists_{y\in A_2}\quad d(x, y) \preceq H(A_1, A_2) + c$;
(H4) One of the following is satisfied:
  (i) $\forall_{c\in E,\theta\ll c}\ \exists_{x\in A_1}\ \forall_{y\in A_2}\ H(A_1, A_2) \preceq d(x, y) + c$;
  (ii) $\forall_{c\in E,\theta\ll c}\ \exists_{x\in A_2}\ \forall_{y\in A_1}\ H(A_1, A_2) \preceq d(x, y) + c$.

The following are some examples of $H-$cone metrics.

Example 2.5. [16] Let (X, d) be a cone metric space and let $\mathcal{A} = \{\{x\} \colon x \in X\}$. Define the mapping $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ by the formula

$$H(\{x\}, \{y\}) = d(x, y) \text{ for all } x, y \in X,$$

is a $H-$cone metric with respect to d.

Example 2.6. [16] Let (X, d) be a metric space and let $\mathcal{A}$ be the family of all nonempty, closed bounded subsets of X. Then the mapping $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}^+$ given by the formula

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A), A, B \in \mathcal{A}\right\}$$

which is called a Hausdorff metric, is a $H-$cone metric with respect to d.

The following lemma shows that a $H-$cone metric with respect to the cone metric d, is itself a cone metric when $\mathcal{A} \subset N(X)$.

Lemma 2.7. [16] *Let (X, d) be a cone metric space and let $\mathcal{A} \subset N(X)$, $\mathcal{A} \neq \emptyset$. If $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ is a $H-$cone metric with respect to d then a pair $(\mathcal{A}, H)$ is a cone metric space.*

Wardowski [16] proved the following cone metric version of result of Nadler [10].

Theorem 2.1. [16] *Let (X, d) be a complete cone metric space with a normal cone P with a normal constant K, $\mathcal{A}$ be a nonempty collection of nonempty closed subsets* *of X and let $H\colon \mathcal{A}\times\mathcal{A}\longrightarrow E$ be a $H-$cone metric with respect to d. If for a map $f\colon X\longrightarrow\mathcal{A}$ there exists $\lambda\in(0,1)$ such that*

$$H(fx,fy)\preceq\lambda d(x,y) \text{ for all } x,y\in X, \tag{2.1}$$

then the set of all fixed points of f is nonempty.

3. Main results

In this section, we introduce various types of set-valued Prešić type contractions, the point-set-cone metric and prove some fixed point results for set-valued Prešić type contractions in cone metric spaces. In further discussion, we assume that the cones under consideration are solid cones, i.e., $P^\circ\neq\emptyset$.

First, we define the point-set-cone metric between a point and a subset of cone metric spaces which is an extension and generalization of the distance of point from a set in ordinary metric spaces.

Definition 3.1. Let (X,d) be a cone metric space and let $\mathcal{A}$ be a nonempty collection of nonempty subsets of X. A map $d_s\colon X\times\mathcal{A}\longrightarrow E$ is called the point-set-cone metric with respect to d if for all $x\in X$, $A\in\mathcal{A}$ the following conditions hold:

(PS1) $\theta\preceq d_s(x,A)$ and $d_s(x,A)=\theta\implies x\in A$;
(PS2) $\forall_{a\in A}\ d_s(x,A)\preceq d(x,a)$.

Let us observe that for each cone metric d the family of point-set-cone metrics with respect to d is nonempty and each point-set-cone metric depends on the shape of the family $\mathcal{A}$. See the following examples:

Example 3.2. Let (X,d) be a cone metric space and let

$$\mathcal{A}=\{\{x\}\colon x\in X\}.$$

Then, the mapping $d_s\colon X\times\mathcal{A}\longrightarrow E$ defined by the formula

$$d_s(x,\{y\})=d(x,y) \text{ for all } x,y\in X,$$

is a point-set-cone metric with respect to d.

Example 3.3. Let (X,d) be a metric space and let $\mathcal{A}$ be the family of all nonempty, closed and bounded subsets of X. Then the mapping $d_s\colon X\times\mathcal{A}\longrightarrow\mathbb{R}^+$ given by the formula

$$d_s(x,A)=\inf\{d(x,a)\colon a\in A\}$$

which is called the distance of point x from the set A, is a point-set-cone metric with respect to d.

Example 3.4. Let $E=\mathbb{R}^2$, the Euclidean plane, $P=\{(x,y)\in\mathbb{R}^2\colon x,y\geq 0\}$ be the cone in E and $X=\{(x,0)\in\mathbb{R}^2\colon 0\leq x\leq 1\}\cup\{(0,x)\in\mathbb{R}^2\colon 0\leq x\leq 1\}$ Let $X=\{(x,0),(0,x)\colon 0\leq x\leq 1\}$ and the mapping $d\colon X\times X\longrightarrow E$ by defined by

$$d((x,0),(y,0))=\mid x-y\mid(0,1)\,, d((0,x),(0,y))=\mid x-y\mid(1,0)\,,$$

$$d((x,0),(0,y))=d((0,y),(x,0))=(y,x)\,.$$

Then (X,d) is a cone metric space. Let $\mathcal{A}=\{\{(0,0),(x,0),(0,x)\}\colon 0\leq x\leq 1\}$, then the mapping $d_s\colon X\times\mathcal{A}\longrightarrow\mathbb{R}^+$ given by the formula

$$d_s\big((z,0),\{(0,0),(x,0),(0,x)\}\big)=\big(0,[z\cdot\mid z-x\mid]^p\big),$$

$$d_s((0,z),\{(0,0),(x,0),(0,x)\})=\big([z\cdot\mid z-x\mid]^p,0\big)$$

where $p\in\mathbb{N}$, is a point-set-cone metric with respect to d.

Remark 3.5. It is obvious from (PS2) that if $x \in A$, then $d_s(x, A) = \theta$. Therefore, from (PS1) we conclude the double implication: $d_s(x, A) = \theta \iff x \in A$.

Lemma 3.6. *Let (X, d) be a cone metric space and let $\mathcal{A}$ be a collection of nonempty subsets of X. Let H be a $H-$cone metric and d_s be a point-set-cone metric with respect to d. Then*

$$\forall_{A,B\in\mathcal{A}}\ \forall_{a\in A}\ d_s(a, B) \preceq H(A, B).$$

Proof. Let $A, B \in \mathcal{A}$ and $a \in A$. Suppose, $\{c_n\}$ be a sequence in P° such that $c_n \longrightarrow \theta$ as $n \longrightarrow \infty$ and $\theta \ll c_n$ for all $n \in \mathbb{N}$. Then, by (H3) we have there exists $b \in B$ such that

$$d(a, b) \preceq H(A, B) + c_n \text{ for all } n \in \mathbb{N}.$$

By (PS2) we have $d_s(a, B) \preceq d(a, b)$, so by the above inequality we obtain $d_s(a, B) \preceq H(A, B) + c_n$ for all $n \in \mathbb{N}$, i.e., $H(A, B) + c_n - d_s(a, B) \in P$ for all $n \in \mathbb{N}$. Since P is closed, by choice of the sequence $\{c_n\}$ we have $H(A, B) - d_s(a, B) \in P$, i.e., $d_s(a, B) \preceq H(A, B)$. □

Let (X, d) be a cone metric space and $\mathcal{A}$ be a nonempty collection of nonempty subsets of X. In further discussion, $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ will represent the $H-$cone metric and $d_s\colon X \times \mathcal{A} \longrightarrow E$ will represent the point-set-cone metric with respect to d.

Now we can define various set-valued Prešić type contractions in cone metric spaces.

Let (X, d) be a cone metric space, k a positive integer, $\mathcal{A}$ a nonempty collection of nonempty closed subsets of X and let $f\colon X^k \to \mathcal{A}$ be a mapping. Then, f is said to be Lipschitzian on X if there exist nonnegative constants α_i such that

$$H(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \preceq \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i), \tag{3.1}$$

for all $x_0, x_1, \ldots, x_k \in X$. If $\sum\limits_{i=1}^{k} \alpha_i < 1$, then the mapping f is said to be a set-valued Prešić type contractions on X.

The mapping f is called a set-valued Prešić-Kannan type contraction on X if,

$$H(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \preceq a \sum_{i=0}^{k} d_s(x_i, f(x_i, \ldots, x_i)) \tag{3.2}$$

for all $x_0, x_1, \ldots, x_k \in X$, where the real constant a is such that $0 \le ak(k+1) < 1$.

The mapping f is called a set-valued Prešić-Reich type contraction on X if,

$$\begin{aligned} H(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \preceq & \sum_{i=1}^{k} \alpha_i d(x_{i-1}, x_i) \\ & + \sum_{i=0}^{k} \beta_i d_s(x_i, f(x_i, \ldots, x_i)), \end{aligned} \tag{3.3}$$

for all $x_0, x_1, \ldots, x_k \in X$, where α_i, β_i are nonnegative constants such that

$$\sum_{i=1}^{k} \alpha_i + k \sum_{i=0}^{k} \beta_i < 1. \tag{3.4}$$

We denote the set of all fixed points of f by Fix$f$ and

$$\text{Fix}f = \{x \in X : x \in f(x, \ldots, x)\}.$$

The following theorem is the main result of this paper.

Theorem 3.1. *Let (X, d) be a complete cone metric space, k a positive integer and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If $f\colon X^k \longrightarrow \mathcal{A}$ be a set-valued Prešić-Reich type contraction, then Fix$f \neq \emptyset$.*

Proof. Let $\{c_n\}$ be an arbitrary sequence in E which satisfies $\theta \ll c_n$ for all $n \in \mathbb{N}$. Let x_0 be an arbitrary point of X. Because $f(x_0, \ldots, x_0) \in \mathcal{A}$, let $x_1 \in f(x_0, \ldots, x_0)$. From (H3) there exists $x_2 \in f(x_1, \ldots, x_1)$ such that

$$d(x_1, x_2) \preceq H(f(x_0, \ldots, x_0), f(x_1, \ldots, x_1)) + c_1.$$

Similarly, there exists $x_3 \in f(x_2, \ldots, x_2)$ such that

$$d(x_2, x_3) \preceq H(f(x_1, \ldots, x_1), f(x_2, \ldots, x_2)) + c_2.$$

Continuing this procedure we obtain $x_{n+1} \in f(x_n, \ldots, x_n)$ and

$$d(x_n, x_{n+1}) \preceq H(f(x_{n-1}, \ldots, x_{n-1}), f(x_n, \ldots, x_n)) + c_n \tag{3.5}$$

for all $n \in \mathbb{N}$.
As H is a metric on $N(X)$, for any $n \in \mathbb{N}$ it follows from (3.5) that

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq H(f(x_{n-1}, \ldots, x_{n-1}), f(x_n, \ldots, x_n)) + c_n \\ &\preceq H(f(x_{n-1}, \ldots, x_{n-1}), f(x_{n-1}, \ldots, x_{n-1}, x_n)) \\ &\quad + H(f(x_{n-1}, \ldots, x_{n-1}, x_n), f(x_{n-1}, \ldots, x_{n-1}, x_n, x_n)) \\ &\quad + \cdots + H(f(x_{n-1}, x_n, \ldots, x_n), f(x_n, \ldots, x_n)) + c_n. \end{aligned}$$

Since f is a set-valued Prešić-Reich type contraction, it follows from the above inequality that

$$\begin{aligned} d(x_n, x_{n+1}) \preceq\ & \alpha_k d(x_{n-1}, x_n) + \beta_0 d_s(x_{n-1}, f(x_{n-1}, \ldots, x_{n-1})) + \cdots \\ & + \beta_{k-1} d_s(x_{n-1}, f(x_{n-1}, \ldots, x_{n-1})) + \beta_k d_s(x_n, f(x_n, \ldots, x_n)) \\ & + \alpha_{k-1} d(x_{n-1}, x_n) + \beta_0 d_s(x_{n-1}, f(x_{n-1}, \ldots, x_{n-1})) + \cdots \\ & + \beta_{k-1} d_s(x_n, f(x_n, \ldots, x_n)) + \beta_k d_s(x_n, f(x_n, \ldots, x_n)) \\ & + \cdots + \alpha_1 d(x_{n-1}, x_n) + \beta_0 d_s(x_{n-1}, f(x_{n-1}, \ldots, x_{n-1})) \\ & + \beta_1 d_s(x_n, f(x_n, \ldots, x_n)) + \cdots + \beta_k d_s(x_n, f(x_n, \ldots, x_n)) \\ & + c_n. \end{aligned}$$

Since $x_n \in f(x_{n-1}, \ldots, x_{n-1})$ for all $n \in \mathbb{N}$, it follows from the definition of point-set-cone metric and the above inequality that

$$\begin{aligned} d(x_n, x_{n+1}) \preceq & \left[\sum_{i=1}^{k} \alpha_i\right] d(x_{n-1}, x_n) + \beta_0 d(x_{n-1}, x_n) + \cdots \\ & + \beta_{k-1} d(x_{n-1}, x_n) + \beta_k d(x_n, x_{n+1}) + \beta_0 d(x_{n-1}, x_n) + \cdots \\ & + \beta_{k-1} d(x_n, x_{n+1}) + \beta_k d(x_n, x_{n+1}) + \cdots + \beta_0 d(x_{n-1}, x_n) \\ & + \beta_1 d(x_n, x_{n+1}) + \cdots + \beta_k d(x_n, x_{n+1}) + c_n. \end{aligned}$$

Rearranging the terms in the above expression, we obtain

$$d(x_n, x_{n+1}) \quad \preceq \quad \left[\sum_{i=1}^{k} \alpha_i + \sum_{i=0}^{k-1} (k-i)\beta_i\right] d(x_{n-1}, x_n) + \left[\sum_{i=1}^{k} i\beta_i\right] d(x_n, x_{n+1}) + c_n.$$

Thus, we have

$$d(x_n, x_{n+1}) \preceq \frac{\sum_{i=1}^{k}\alpha_i + \sum_{i=0}^{k}(k-i)\beta_i}{1 - \sum_{i=0}^{k} i\beta_i} d(x_{n-1}, x_n) + \frac{1}{1 - \sum_{i=0}^{k} i\beta_i} c_n. \tag{3.6}$$

For simplicity, set $A = \sum_{i=1}^{k}\alpha_i, B = k\sum_{i=0}^{k}\beta_i, C = \sum_{i=0}^{k} i\beta_i$ and $\lambda = \frac{A+B-C}{1-C}$, then in view of (3.4) we have,

$$A + B = \sum_{i=1}^{k}\alpha_i + k\sum_{i=0}^{k}\beta_i < 1, \quad C < 1, \text{ also } C \leq B,$$

and so, $0 \leq \lambda < 1$. Thus, from (3.6) it follows that

$$d(x_n, x_{n+1}) \preceq \lambda d(x_{n-1}, x_n) + \frac{c_n}{1-C} \text{ for all } n \in \mathbb{N}. \tag{3.7}$$

From the successive applications of the inequality (3.7) we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq \lambda d(x_{n-1}, x_n) + \frac{c_n}{1-C} \\ &\preceq \lambda\left[\lambda d(x_{n-2}, x_{n-1}) + \frac{c_{n-1}}{1-C}\right] + \frac{c_n}{1-C} \\ &= \lambda^2 d(x_{n-2}, x_{n-1}) + \lambda\frac{c_{n-1}}{1-C} + \frac{c_n}{1-C} \\ &\preceq \lambda^2\left[\lambda d(x_{n-3}, x_{n-2}) + \frac{c_{n-2}}{1-C}\right] + \lambda\frac{c_{n-1}}{1-C} + \frac{c_n}{1-C} \\ &= \lambda^3 d(x_{n-3}, x_{n-2}) + \lambda^2\frac{c_{n-2}}{1-C} + \lambda\frac{c_{n-1}}{1-C} + \frac{c_n}{1-C}, \end{aligned}$$

which yields

$$d(x_n, x_{n+1}) \preceq \lambda^n d(x_0, x_1) + \frac{1}{1-C}\sum_{i=0}^{n-1}\lambda^i c_{n-i}.$$

Let $\omega \in P^\circ$, i.e., $\omega \in E, \theta \ll \omega$ be given. Since the sequence $\{c_n\}$ was arbitrary, choose c_n such that $\theta \ll c_n \ll \lambda^n\omega$ for all $n \in \mathbb{N}$. Therefore, it follows from the above inequality that

$$d(x_n, x_{n+1}) \ll \lambda^n d(x_0, x_1) + \frac{n\lambda^n}{1-C}\omega. \tag{3.8}$$

Let $n, m \in \mathbb{N}$ be such that $m > n$, then using inequality (3.8) we obtain

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &\ll \sum_{j=n}^{m-1}\left[\lambda^j d(x_0, x_1) + \frac{j\lambda^j}{1-C}\omega\right] \\ &= d(x_0, x_1)\sum_{j=n}^{m-1}\lambda^j + \frac{\omega}{1-C}\sum_{j=n}^{m-1} j\lambda^j. \end{aligned}$$

Since $0 \leq \lambda < 1$, therefore both the series $\sum\limits_{n=1}^{\infty} \lambda^n$ and $\sum\limits_{n=1}^{\infty} n\lambda^n$ are convergent series of nonnegative terms, and so, we have $\sum\limits_{j=n}^{m-1} \lambda^j \longrightarrow 0$ and $\sum\limits_{j=n}^{m-1} j\lambda^j \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, the quantity on the right of the above inequality must tends to θ as $n \longrightarrow \infty$. Now, using Lemma 2.3 and the last inequality we obtain, for each $c \in P^\circ$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n > n_0$. Therefore, $\{x_n\}$ is a Cauchy sequence.

By completeness of X, there exists $x^* \in X$ such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. We shall show that x^* is a fixed point of f. Using similar calculations to the previous one we obtain

$$\begin{aligned} & H(f(x_n, \dots, x_n), f(x^*, \dots, x^*)) \\ & \preceq Ad(x_n, x^*) + (B - C)d_s(x_n, f(x_n, \dots, x_n)) + Cd_s(x^*, f(x^*, \dots, x^*)) \end{aligned}$$

which with the fact $x_{n+1} \in f(x_n, \dots, x_n)$ and the definition of point-set-cone metric gives

$$\begin{aligned} H(f(x_n, \dots, x_n), f(x^*, \dots, x^*)) \preceq\ & Ad(x_n, x^*) + (B - C)d_s(x_n, x_{n+1}) \\ & + Cd_s(x^*, f(x^*, \dots, x^*)). \end{aligned} \tag{3.9}$$

Suppose $c \in P^\circ$ be given, then since $x_{n+1} \in f(x_n, \dots, x_n)$, by (H3) for all $n \in \mathbb{N}$, there exists $y_n \in f(x^*, \dots, x^*)$ such that

$$d(x_{n+1}, y_n) \preceq H(f(x_n, \dots, x_n), f(x^*, \dots, x^*)) + c'_n, \tag{3.10}$$

where $\{c'_n\}$ is a sequence in P° such that $c'_n \ll \frac{(1-C)c}{4}$ for all $n \in \mathbb{N}$. Again, since $y_n \in f(x^*, \dots, x^*)$ we have

$$d_s(x^*, f(x^*, \dots, x^*)) \preceq d(x^*, y_n) \preceq d(x^*, x_{n+1}) + d(x_{n+1}, y_n)$$

which with (3.9) and (3.10) gives

$$\begin{aligned} d_s(x^*, f(x^*, \dots, x^*)) \preceq\ & d(x^*, x_{n+1}) + Ad(x_n, x^*) + (B - C)d(x_n, x_{n+1}) \\ & + Cd_s(x^*, f(x^*, \dots, x^*)) + c'_n. \end{aligned}$$

Since $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$ and $c'_n \ll \frac{(1-C)c}{4}$ for all $n \in \mathbb{N}$, we can choose $n_1 \in \mathbb{N}$ such that $d(x^*, x_{n+1}) \ll \frac{(1-C)c}{4}$, $d(x_n, x^*) \ll \frac{(1-C)c}{4A}$ and $d(x_n, x_{n+1}) \ll \frac{(1-C)c}{4(B-C)}$ for all $n > n_1$. Therefore, the above inequality yields

$$d_s(x^*, f(x^*, \dots, x^*)) \ll c \quad \text{for all} \quad n > n_1.$$

Therefore, it follows from Lemma 2.3 and the above inequality that

$$d_s(x^*, f(x^*, \dots, x^*)) = \theta.$$

Thus, $x^* \in f(x^*, \dots, x^*)$, i.e., x^* if a fixed point of f. □

Taking $k = 1$ in the above theorem, we obtain the following fixed point theorem which generalize the result of Wardowski [16] without assuming the normality of the underlying cone.

Corollary 3.7. *Let (X, d) be a complete cone metric space and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If for a map $f \colon X \longrightarrow \mathcal{A}$ there exist nonnegative constants α, β_0, β_1 such that $\alpha + \beta_0 + \beta_1 < 1$ and*

$$H(fx, fy) \preceq \alpha d(x, y) + \beta_0 d_s(x, fx) + \beta_1 d_s(y, fy)$$

for all $x, y \in X$, then Fix$f \neq \emptyset$.

Corollary 3.8. *Let (X, d) be a complete cone metric space and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If for a map $f\colon X \longrightarrow \mathcal{A}$ there exist nonnegative constants β_0, β_1 such that $\beta_0 + \beta_1 < 1$ and*

$$H(fx, fy) \preceq \beta_0 d_s(x, fx) + \beta_1 d_s(y, fy)$$

for all $x, y \in X$, then Fix$f \neq \emptyset$.

Corollary 3.9. *Let (X, d) be a complete cone metric space and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If for a map $f\colon X \longrightarrow \mathcal{A}$ there exists $\alpha \in [0, 1)$ such that*

$$H(fx, fy) \preceq \alpha d(x, y)$$

for all $x, y \in X$, then Fix$f \neq \emptyset$.

Corollary 3.10. *Let (X, d) be a complete cone metric space, k a positive integer and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If $f\colon X^k \longrightarrow \mathcal{A}$ be a set-valued Prešić-Kannan type contraction, then Fix$f \neq \emptyset$.*

Proof. Taking $\alpha_i = 0$ for $i = 1, 2, \ldots, k$ and $\beta_i = a$ (say) for $i = 0, 1, \ldots, k$ in Theorem 3.1, we obtain the desired result. □

Corollary 3.11. *Let (X, d) be a complete cone metric space, k a positive integer and $\mathcal{A}$ be a nonempty collection of nonempty closed subsets of X. If $f\colon X^k \longrightarrow \mathcal{A}$ be a set-valued Prešić type contraction, then Fix$f \neq \emptyset$.*

Proof. Taking $\beta_i = 0$ for $i = 0, 1, \ldots, k$ in Theorem 3.1, we obtain the desired result. □

Example 3.12. Let $X = [0, 1]$, $E = C^1_{\mathbb{R}}[0, 1]$ with the norm $\|\psi\| = \|\psi\|_\infty + \|\psi'\|_\infty$ and $P = \{\psi \in E\colon \psi(t) \geq 0, t \in [0, 1]\}$. Define $d\colon X \times X \longrightarrow E$ by

$$d(x, y) = \mid x - y \mid \phi(t) \text{ for all } x, y \in X,$$

where $\phi(t) = e^t$, $t \in [0, 1]$. Then, (X, d) is a complete cone metric space. Let $\mathcal{A}$ be the family of subsets of X of the form $\mathcal{A} = \{[0, x]\colon x \in X\} \cup \{\{x\}\colon x \in X\}$, and define the functions $H\colon \mathcal{A} \times \mathcal{A} \longrightarrow E$ and $d_s\colon X \times \mathcal{A} \longrightarrow E$ by

$$H(A, B) = \begin{cases} \mid x - y \mid \cdot e^t, & \text{for } A = [0, x], B = [0, y]; \\ \mid x - y \mid \cdot e^t, & \text{for } A = \{x\}, B = \{y\}; \\ \max\{y, \mid x - y \mid\} \cdot e^t, & \text{for } A = [0, x], B = \{y\}; \\ \max\{x, \mid x - y \mid\} \cdot e^t, & \text{for } A = \{x\}, B = [0, y] \end{cases}$$

and

$$d_s(x, A) = \min\{\mid x - a \mid\colon a \in A\} \cdot e^t,\ t \in [0, 1] \text{ for all } x \in X.$$

Then, H is a $H-$cone metric and d_s is a point-set-cone metric with respect to d. For $k = 2$, define a mapping $f\colon X^2 \longrightarrow \mathcal{A}$ as follows:

$$f(x, y) = \begin{cases} \left[0, \frac{1}{5}(x + y - 1)^2\right], & \text{if } x, y \in \left(\frac{1}{2}, 1\right]; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Now, by some routine calculations one can see that the mapping f is a set-valued Prešić-Reich type contraction on X^2 with $\alpha_1 = \alpha_2 = \frac{2}{5}$ and $\beta_1 = \beta_2 = \beta_3 = \frac{1}{36}$. All the conditions of Theorem 3.1 are satisfied and $0 \in$ Fix$f$.

In the next theorem, we replace the completeness of cone metric space by an additional condition on the set-valued Prešić-Reich type contractions.

Theorem 3.2. *Let (X, d) be a cone metric space, k a positive integer, $\mathcal{A}$ a nonempty collection of nonempty closed subsets of X and $f\colon X^k \longrightarrow \mathcal{A}$ be a set-valued Prešić-Reich type contraction. Suppose there exists $x^* \in X$ such that*

$$d_s(x^*, f(x^*, \dots, x^*)) \preceq d_s(x, f(x, \dots, x)) \text{ for all } x \in X.$$

Then Fix$f \neq \emptyset$.

Proof. Let $D(x) = d_s(x, f(x, \dots, x))$ for all $x \in X$. Then by assumption we have

$$D(x^*) \preceq D(x) \text{ for all } x \in X. \tag{3.11}$$

If $x^* \in f(x^*, \dots, x^*)$, then $u \in$ Fix$f$. Suppose $x^* \notin f(x^*, \dots, x^*)$, then $D(x^*) = d_s(x^*, f(x^*, \dots, x^*)) \neq \theta$. Let $x_0 = x^*$, then following similar arguments to those in Theorem 3.1, the sequence $\{x_n\}$, where $x_n \in f(x_{n-1}, \dots, x_{n-1})$ for all $n \in \mathbb{N}$ is a Cauchy sequence in X. Now, by Lemma 3.6, we have

$$\begin{aligned}
D(x_n) &= d_s(x_n, f(x_n, \dots, x_n)) \\
&\preceq H(f(x_{n-1}, \dots, x_{n-1}), f(x_n, \dots, x_n)) \\
&\preceq H(f(x_{n-1}, \dots, x_{n-1}), f(x_{n-1}, \dots, x_{n-1}, x_n)) \\
&\quad + H(f(x_{n-1}, \dots, x_{n-1}, x_n), f(x_{n-1}, \dots, u, x_n, x_n)) \\
&\quad + \cdots + H(f(x_{n-1}, x_n, \dots, x_n), f(x_n, \dots, x_n)) \\
&\preceq Ad(x_{n-1}, x_n) + (B - C)d_s(x_{n-1}, f(x_{n-1}, \dots, x_{n-1})) \\
&\quad + Cd_s(x_n, f(x_n, \dots, x_n)),
\end{aligned}$$

where $A = \sum_{i=1}^{k} \alpha_i$, $B = k \sum_{i=0}^{k} \beta_i$ and $C = \sum_{i=0}^{k} i\beta_i$.
Since $x_n \in f(x_{n-1}, \dots, x_{n-1})$ for all $n \in \mathbb{N}$, by (PS2) we have

$$d_s(x_{n-1}, f(x_{n-1}, \dots, x_{n-1})) \preceq d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}.$$

Therefore, it follows from the above inequality that

$$D(x_n) \preceq \frac{A + B - C}{1 - C}\, d(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}.$$

As, $A + B < 1, C \leq B$, and $\{x_n\}$ is a Cauchy sequence, for each $c \in P$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_{n-1}, x_n) \ll \dfrac{(1 - C)c}{A + B - C}$ for all $n > n_0$. So, it follows from the above inequality that $D(x_n) \ll c$ for all $n > n_0$. Using the inequality (3.11) and the Remark 2.3 we have

$$D(x^*) \ll c \text{ for all } n \in \mathbb{N}.$$

Therefore, we must have $D(x^*) = \theta$, i.e., $d_s(x^*, f(x^*, \dots, x^*)) = \theta$, or, $x^* \in f(x^*, \dots, x^*)$. Thus $x^* \in$ Fix$f$. □

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SOME FIXED POINT RESULTS FOR ĆIRIĆ OPERATORS IN b-METRIC SPACES

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ABSTRACT. Fixed points for Ćirić operators satisfying some generalized contractive type conditions are obtained in the setting of b-metric spaces. Moreover, we investigate that these operators satisfy property P. Finally, strength of hypothesis made in main theorem has been weighed through an illustrative example.

KEYWORDS : b-metric; Ćirić operator; property P; fixed point.
AMS Subject Classification: 54H25, 47H10

1. INTRODUCTION

Fixed point theory plays an important role in applications of many branches of mathematics and applied sciences. The study of metric fixed point theory has been at the centre of vigorous research activity. There has been a number of generalizations of the usual notion of a metric space(see [2, 5, 12, 16, 17, 18, 21, 22, 23]). One such generalization is a b-metric space introduced and studied by Bakhtin[4] and Czerwik [10]. After that a series of articles have been dedicated to the improvement of fixed point theory for single valued and multivalued operators in b-metric spaces and cone b-metric spaces(see [3, 6, 7, 9, 11, 13, 14, 19, 20, 24, 25, 26, 27]). In this paper, we introduce the concept of Ćirić operators in b-metric spaces. In fact, a Ćirić operator in b-metric spaces may not possess a fixed point and attempts have been in progress to frame appropriate conditions to be satisfied by a Ćirić operator to attract a fixed point. The aim of this work is to invite some kind of contractive conditions to be satisfied by the operator just appropriate to possess its fixed point.

Let X be a nonempty set and $T : X \longrightarrow X$ be an operator with nonempty fixed point set $F(T)$. Then T is said to satisfy property P if $F(T) = F(T^n)$ for each

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$n \in \mathbb{N}$. We prove that Ćirić operators in b-metric spaces satisfy property P.

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2. PRELIMINARIES

In this section we need to recall some basic notations, definitions, and necessary results from existing literature.

Definition 2.1. [10] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

Observe that if $s = 1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example 2.2. Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.3. [15] Let $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}^+$ be such that

$$d(x, y) = \mid x - y \mid^2 \ \ for\ any\ x, y \in X.$$

Then (X, d) is a b-metric space with $s = 2$, but not a metric space.

Definition 2.4. [8] Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.
(ii) (x_n) is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Definition 2.5. Let (X, d) be a b-metric space and let $T : X \to X$ be a given mapping. We say that T is continuous at $x_0 \in X$ if for every sequence (x_n) in X, we have $x_n \to x_0$ as $n \to \infty \Longrightarrow T(x_n) \to T(x_0)$ as $n \to \infty$. If T is continuous at each point $x_0 \in X$, then we say that T is continuous on X.

Theorem 2.6. [1]*Let (X, d) be a b-metric space and suppose that (x_n) and (y_n) converge to $x, y \in X$, respectively. Then, we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

3. MAIN RESULTS

In this section, we will present some fixed point theorems for Ćirić operators satisfying some contractive type conditions in the setting of b-metric spaces. Furthermore, we will give an example to examine the strength of hypothesis made in main theorem.

We begin with the following definition.

Definition 3.1. Let (X, d) be a b-metric space with coefficient $s \geq 1$. An operator $T : X \to X$ is said to be a Ćirić operator if there are non-negative real valued functions q and δ over $X \times X$ satisfying

$$d(T^n(x), T^n(y)) \leq q^n(x,y)\delta(x,y), \ n = 1, 2, \cdots,$$

for all $x, y \in X$, where $q(x,y) < 1$ with $\sup\limits_{x,y\in X} q(x,y) = 1$.

Let $CI(X)$ denote the set of all Ćirić operators on X.

Theorem 3.2. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, and $T \in CI(X)$ be such that*

$$d(T(x), T(y)) \leq \alpha\, d(x, T(x)) + \beta\, max \left\{ \begin{array}{c} d(x,y) + d(y, T(y)), \\ \\ d(x, T(y)),\ d(y, T(x)) \end{array} \right\} \tag{3.1}$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^2}$. Then T has a unique fixed point (say u) in X and T is continuous at u. Moreover, T has property P.

Proof. Let $x_0 \in X$ be arbitrary and define sequence (x_n) by $x_n = T^n(x_0)$. Then for all $n, m \in \mathbb{N}$, we have by using (3.1) that

$$\begin{aligned} d(x_n, x_m) &\leq \alpha\, d(x_{n-1}, x_n) + \beta\, max \left\{ \begin{array}{l} d(x_{n-1}, x_{m-1}) + d(x_{m-1}, x_m), \\ \\ d(x_{n-1}, x_m),\ d(x_{m-1}, x_n) \end{array} \right\} \\ &\leq \alpha\, d(x_{n-1}, x_n) + \beta\, max \left\{ \begin{array}{l} sd(x_{n-1}, x_n) + s^2 d(x_n, x_m) \\ \\ +s^2 d(x_m, x_{m-1}) + d(x_{m-1}, x_m), \\ \\ sd(x_{n-1}, x_n) + sd(x_n, x_m), \\ \\ sd(x_{m-1}, x_m) + sd(x_m, x_n) \end{array} \right\} \\ &= \alpha\, d(x_{n-1}, x_n) \\ &\quad + \beta \left\{ sd(x_{n-1}, x_n) + s^2 d(x_n, x_m) + (1+s^2) d(x_{m-1}, x_m) \right\}. \end{aligned}$$

Since $0 \leq \beta < \frac{1}{s^2}$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \frac{\alpha + \beta s}{1 - \beta s^2}\, d(x_{n-1}, x_n) + \frac{\beta(1+s^2)}{1-\beta s^2}\, d(x_{m-1}, x_m) \\ &\leq \frac{\alpha + \beta s}{1 - \beta s^2}\, q^{n-1}(x_0, T(x_0))\, \delta(x_0, T(x_0)) \end{aligned}$$

$$+\frac{\beta(1+s^2)}{1-\beta s^2}\, q^{m-1}(x_0,T(x_0))\,\delta(x_0,T(x_0))$$
$$\longrightarrow 0 \ as\ m,n \longrightarrow \infty.$$

Therefore, (x_n) becomes a Cauchy sequence in (X,d). By completeness of (X,d), it follows that the sequence (x_n) is convergent. So, there exists $u \in X$ such that $\lim_{n\to\infty} d(x_n,u)=0$. Then

$$\begin{aligned}
d(u,T(u)) &\leq s[d(u,x_n)+d(x_n,T(u))]\\
&\leq sd(u,x_n)+s\alpha\, d(x_{n-1},x_n)\\
&\quad +s\beta\, max\left\{\begin{array}{l} d(x_{n-1},u)+d(u,T(u)),\\ \\ d(x_{n-1},T(u)),\ d(u,x_n)\end{array}\right\}\\
&\leq sd(u,x_n)+s\alpha\, q^{n-1}(x_0,T(x_0))\,\delta(x_0,T(x_0))\\
&\quad +s\beta\, max\left\{\begin{array}{l} d(x_{n-1},u)+d(u,T(u)),\\ \\ sd(x_{n-1},u)+sd(u,T(u)),\ d(u,x_n)\end{array}\right\}.
\end{aligned}$$

Taking the limit as $n\to\infty$, it follows from above that

$$d(u,T(u)) \leq \beta s^2 d(u,T(u)).$$

Since $0\leq\beta<\frac{1}{s^2}$, we have $d(u,T(u))=0$ and so, $u=T(u)$.

For uniqueness of u, suppose that $T(v)=v$ for some $v\in X$. Then

$$\begin{aligned}
d(u,v)=d(T^n(u),T^n(v)) &\leq q^n(u,v)\,\delta(u,v)\\
&\longrightarrow 0,\ \ as\ n\to\infty.
\end{aligned}$$

This gives that $u=v$. Therefore, T has a unique fixed point u in X.

To show that T is continuous at u, let (y_n) be any sequence in X such that (y_n) is convergent to u. For $n\in\mathbb{N}$, we have

$$\begin{aligned}
d(u,T(y_n)) &= d(T(u),T(y_n))\\
&\leq \alpha\, d(u,u)+\beta\, max\left\{\begin{array}{l} d(u,y_n)+d(y_n,T(y_n)),\\ \\ d(u,T(y_n)),\ d(y_n,T(u))\end{array}\right\}\\
&\leq \beta\,[d(u,y_n)+sd(y_n,u)+sd(u,T(y_n))],
\end{aligned}$$

which implies that

$$d(u,T(y_n)) \leq \frac{(1+s)\beta}{1-\beta s} d(u,y_n).$$

Taking the limit as $n\to\infty$, we see that $d(u,y_n)\to 0$ and so, $(T(y_n))$ is convergent to $u=T(u)$. Therefore T is continuous at u.

To show that T has property P, let $u\in F(T)$. Then, $u\in F(T^n)$ for each $n\in\mathbb{N}$. Therefore, $F(T)\subseteq F(T^n)$ for each $n\in\mathbb{N}$.

Since T has a fixed point, $F(T^n)\neq\emptyset$ for each $n\in\mathbb{N}$. Let $n>1$ be fixed and assume that $p\in F(T^n)$. Using (3.1), we have

$$\begin{aligned}
d(p,T(p)) &= d(T^n(p),T^{n+1}(p))\\
&\leq \alpha\, d(T^{n-1}(p),T^n(p))
\end{aligned}$$

$$
\begin{aligned}
&+\beta\, max \left\{ \begin{array}{l} d(T^{n-1}(p), T^n(p)) + d(T^n(p), T^{n+1}(p)), \\ \\ d(T^{n-1}(p), T^{n+1}(p)),\ d(T^n(p), T^n(p)) \end{array} \right\} \\
\leq\ & \alpha\, d(T^{n-1}(p), T^n(p)) + \beta s \left\{ d(T^{n-1}(p), T^n(p)) + d(T^n(p), T^{n+1}(p)) \right\}.
\end{aligned}
$$

Thus, it must be the case that

$$
\begin{aligned}
d(p, T(p)) &\leq \frac{\alpha + \beta s}{1 - \beta s}\, d(T^{n-1}(p), T^n(p)) \\
&\leq \frac{\alpha + \beta s}{1 - \beta s}\, q^{n-1}(p, T(p))\, \delta(p, T(p)) \\
&\longrightarrow 0 \ as\ n \to \infty
\end{aligned}
$$

which gives that, $d(p, T(p)) = 0$ and so, $T(p) = p$. Therefore, $p \in F(T)$ and T has property P. □

Corollary 3.3. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Suppose that for some $m \in \mathbb{N}$, $T^m \in CI(X)$ satisfying*

$$
d(T^m(x), T^m(y)) \leq \alpha\, d(x, T^m(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T^m(y)), \\ \\ d(x, T^m(y)),\ d(y, T^m(x)) \end{array} \right\}
$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^2}$. Then T has a unique fixed point (say u) in X and T^m is continuous at u. Moreover, T^m has property P.

Proof. It follows from Theorem 3.2 that T^m has a unique fixed point (say u) in X and T^m is continuous at u. Moreover, T^m has property P. We have

$$
T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u)).
$$

This shows that $T(u)$ is also a fixed point of T^m. By uniqueness of u, we get $T(u) = u$. □

Remark 3.4. It is worth mentioning that Theorem 3.2 is a generalization of Theorem 1[28]. In [28], the authors considered the following class of mappings in a metric space (X, d):

$$
d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))] + \beta d(x, y) + \gamma\, max\{d(x, T(y)), d(y, T(x))\} \tag{3.2}
$$

for all $x, y \in X$, where $\alpha,\ \beta,\ \gamma \geq 0$ with $max\{\alpha,\ \beta\} + \gamma < 1$.

The class of mappings satisfying condition (3.1) is strictly larger than the class satisfying condition (3.2).

Theorem 3.5. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, and $T \in CI(X)$ be such that*

$$
d(T(x), T(y)) \leq \alpha\, d(x, T(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T(y)), \\ \\ d(x, T(y)) + d(y, T(x)) \end{array} \right\} \tag{3.3}
$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s(s+1)}$. Then T has a unique fixed point (say u) in X and T is continuous at u. Moreover, T has property P.

Proof. The proof is similar to the Theorem 3.2. □

Theorem 3.6. *Let (X, d) be a b-metric space with coefficient $s \geq 1$, and $T \in CI(X)$ be such that*

$$d(T(x), T(y)) \leq \alpha\, d(x, T(x)) + \beta\, max \left\{ \begin{array}{l} d(x,y) + d(y, T(y)), \\ \\ d(x, T(y)) + d(y, T(x)) \end{array} \right\} \tag{3.4}$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^2}$. If $(T^n(x_0))$ for some $x_0 \in X$, has a subsequence $(T^{n_k}(x_0))$ with $\lim_k T^{n_k}(x_0) = u \in X$, then u is the unique fixed point of T in X and $\lim_n T^n(x_0) = u$. Moreover, T is continuous at u and T has property P.

Proof. Let $\lim_k T^{n_k}(x_0) = u \in X$. Then

$$\begin{array}{rcl} d(u, T(u)) & \leq & s[d(u, T^{n_k+1}(x_0)) + d(T^{n_k+1}(x_0), T(u))] \\ & \leq & sd(u, T^{n_k+1}(x_0)) + s\alpha\, d(T^{n_k}(x_0), T^{n_k+1}(x_0)) \\ & & +s\beta\, max \left\{ \begin{array}{l} d(T^{n_k}(x_0), u) + d(u, T(u)), \\ \\ d(T^{n_k}(x_0), T(u)) + d(u, T^{n_k+1}(x_0)) \end{array} \right\} \\ & \leq & s^2 d(u, T^{n_k}(x_0)) + s^2 d(T^{n_k}(x_0), T^{n_k+1}(x_0)) \\ & & +s\alpha\, d(T^{n_k}(x_0), T^{n_k+1}(x_0)) \\ & & +s\beta\, max \left\{ \begin{array}{l} d(T^{n_k}(x_0), u) + d(u, T(u)), \\ \\ sd(T^{n_k}(x_0), u) + sd(u, T(u)) \\ \\ +sd(u, T^{n_k}(x_0)) + sd(T^{n_k}(x_0), T^{n_k+1}(x_0)) \end{array} \right\} \\ & \leq & s^2 d(u, T^{n_k}(x_0)) + (s^2 + s\alpha)\, q^{n_k}(x_0, T(x_0))\, \delta(x_0, T(x_0)) \\ & & +s\beta\{2sd(T^{n_k}(x_0), u) + sd(u, T(u)) + sq^{n_k}(x_0, T(x_0))\, \delta(x_0, T(x_0))\}. \end{array}$$

Taking the limit as $k \to \infty$, we have

$$d(u, T(u)) \leq \beta s^2\, d(u, T(u)).$$

Since $0 \leq \beta < \frac{1}{s^2}$, this implies that $d(u, T(u)) = 0$ and so, $u = T(u)$.

For uniqueness of u, suppose that $T(v) = v$ for some $v \in X$. Then

$$\begin{array}{rcl} d(u, v) = d(T^n(u), T^n(v)) & \leq & q^n(u, v)\, \delta(u, v) \\ & & \longrightarrow 0,\ as\ n \to \infty. \end{array}$$

This gives that $u = v$. Therefore, T has a unique fixed point u in X.

Finally,

$$d(u, T^n(x_0)) = d(T^n(u), T^n(x_0)) \leq q^n(u, x_0)\, \delta(u, x_0).$$

Thus,

$$d(u, T^n(x_0)) \longrightarrow 0\ as\ n \to \infty.$$

Consequently, it follows that $\lim_n T^n(x_0) = u$.

By an argument similar to that used in Theorem 3.2, we can show that T is continuous at u and T has property P.

□

Remark 3.7. Theorem 3.6 is a generalization of Theorem 2[28].

As an application of Theorem 3.6, we have the following Corollary.

Corollary 3.8. *Let (X, d) be a b-metric space with coefficient $s \geq 1$, and $T \in CI(X)$ be such that*

$$d(T(x), T(y)) \leq \alpha\, d(x, T(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T(y)), \\ \\ d(x, T(y)),\ d(y, T(x)) \end{array} \right\}$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^2}$. If $(T^n(x_0))$ for some $x_0 \in X$, has a subsequence $(T^{n_k}(x_0))$ with $\lim_k T^{n_k}(x_0) = u \in X$, then u is the unique fixed point of T in X and $\lim_n T^n(x_0) = u$. Moreover, T is continuous at u and T has property P.

As an application of Theorem 3.2, we have the following result.

Theorem 3.9. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$, and $T_j \in CI(X)$ satisfy*

$$d(T_j(x), T_j(y)) \leq \alpha\, d(x, T_j(x)) + \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T_j(y)), \\ \\ d(x, T_j(y)),\ d(y, T_j(x)) \end{array} \right\} \tag{3.5}$$

for all $x, y \in X$, where $\alpha \geq 0$ and $0 \leq \beta < \frac{1}{s^5}$ with fixed points u_j $(j = 1, 2, \cdots)$. Suppose that $T^n(x) = \lim_{j\to\infty} T_j^n(x)$ for all $x \in X (n = 1, 2, \cdots)$. Then T has the unique fixed point u in X if and only if $u = \lim_j u_j$. Moreover, T is continuous at u and T has property P.

Proof. Let $T^n(x) = \lim_{j\to\infty} T_j^n(x)$ for $x \in X (n = 1, 2, \cdots)$.

For a positive integer n,

$$\begin{aligned} d(T^n(x), T^n(y)) &\leq sd(T^n(x), T_j^n(x)) + s^2 d(T_j^n(x), T_j^n(y)) + s^2 d(T_j^n(y), T^n(y)) \\ &\leq sd(T^n(x), T_j^n(x)) + s^2 q^n(x, y)\, \delta(x, y) + s^2 d(T_j^n(y), T^n(y)). \end{aligned}$$

Taking the limit as $j \to \infty$, we have

$$d(T^n(x), T^n(y)) \leq q^n(x, y)\, \delta_1(x, y),\ \ n = 1, 2, \cdots,$$

for all $x, y \in X$, where $\delta_1(x, y) = s^2 \delta(x, y)$. Therefore, $T \in CI(X)$.
Now, using (3.5), we obtain

$$\begin{aligned} d(T(x), T(y)) &\leq sd(T(x), T_j(x)) + s^2 d(T_j(x), T_j(y)) + s^2 d(T_j(y), T(y)) \\ &\leq sd(T(x), T_j(x)) + s^2 d(T_j(y), T(y)) + s^2 \alpha\, d(x, T_j(x)) \\ &\quad + s^2 \beta\, max \left\{ \begin{array}{l} d(x, y) + d(y, T_j(y)), \\ \\ d(x, T_j(y)),\ d(y, T_j(x)) \end{array} \right\} \\ &\leq sd(T(x), T_j(x)) + s^2 d(T_j(y), T(y)) \\ &\quad + s^3 \alpha\, d(x, T(x)) + s^3 \alpha d(T(x), T_j(x)) \\ &\quad + s^2 \beta\, max \left\{ \begin{array}{l} d(x, y) + sd(y, T(y)) + sd(T(y), T_j(y)), \\ \\ sd(x, T(y)) + sd(T(y), T_j(y)), \\ \\ sd(y, T(x)) + sd(T(x), T_j(x)) \end{array} \right\}. \end{aligned}$$

Taking the limit as $j \to \infty$, we have

$$\begin{aligned} d(T(x),T(y)) &\leq s^3\alpha\, d(x,T(x)) + s^2\beta\, max \left\{ \begin{array}{l} d(x,y)+sd(y,T(y)), \\ \\ sd(x,T(y)),\ sd(y,T(x)) \end{array} \right\} \\ &\leq s^3\alpha\, d(x,T(x)) + s^2\beta\, max \left\{ \begin{array}{l} sd(x,y)+sd(y,T(y)), \\ \\ sd(x,T(y)),\ sd(y,T(x)) \end{array} \right\} \\ &\leq s^3\alpha\, d(x,T(x)) + s^3\beta\, max \left\{ \begin{array}{l} d(x,y)+d(y,T(y)), \\ \\ d(x,T(y)),\ d(y,T(x)) \end{array} \right\} \end{aligned}$$

for all $x, y \in X$, where $s^3\alpha \geq 0$ and $0 \leq s^3\beta < \frac{1}{s^2}$. So by Theorem 3.2, T has a unique fixed point (say u) in X and T is continuous at u. Moreover, T has property P.

Now u is the unique fixed point of T and $u_j = T_j(u_j)$, $j = 1, 2, \cdots$. Then, we have

$$\begin{aligned} d(u,u_j) &= d(T(u),T_j(u_j)) \\ &\leq sd(T(u),T_j(u)) + sd(T_j(u),T_j(u_j)) \\ &\leq sd(T(u),T_j(u)) + s\alpha\, d(u,T_j(u)) \\ &\quad +s\beta\, max \left\{ \begin{array}{l} d(u,u_j)+d(u_j,T_j(u_j)), \\ \\ d(u,T_j(u_j)),\ d(u_j,T_j(u)) \end{array} \right\} \\ &\leq sd(T(u),T_j(u)) + s\alpha\, d(u,T_j(u)) \\ &\quad +s\beta\, max \{d(u,u_j),\ sd(u_j,u)+sd(u,T_j(u))\} \\ &\leq sd(T(u),T_j(u)) + s\alpha\, d(u,T_j(u)) \\ &\quad +s\beta\{sd(u_j,u)+sd(u,T_j(u))\}. \end{aligned}$$

This gives that

$$\begin{aligned} d(u,u_j) &\leq \frac{s(1+\alpha+s\beta)}{1-s^2\beta} d(T(u),T_j(u)) \\ &\longrightarrow 0 \ as \ j \to \infty. \end{aligned}$$

Thus, $u = \lim_j u_j$.

Conversely, suppose that $u = \lim_j u_j$. Then

$$\begin{aligned} d(u,T(u)) &\leq sd(u,u_j) + s^2d(u_j,T_j(u)) + s^2d(T_j(u),T(u)) \\ &= sd(u,u_j) + s^2d(T_j(u),T(u)) + s^2d(T_j(u_j),T_j(u)) \\ &\leq sd(u,u_j) + s^2d(T_j(u),T(u)) + s^2\alpha d(u_j,T_j(u_j)) \\ &\quad +s^2\beta\, max\{d(u_j,u)+d(u,T_j(u)), d(u_j,T_j(u)), d(u,T_j(u_j))\} \\ &\leq sd(u,u_j) + s^2d(T_j(u),T(u)) \\ &\quad +s^2\beta\, max\{d(u_j,u)+d(u,T_j(u)), sd(T_j(u),u)+sd(u,u_j), d(u_j,u)\} \\ &= sd(u,u_j) + s^2d(T_j(u),T(u)) + s^3\beta[d(u,T_j(u))+d(u_j,u)] \\ &\leq sd(u,u_j) + s^2d(T_j(u),T(u)) \\ &\quad +s^3\beta[d(u_j,u)+sd(T_j(u),T(u))+sd(T(u),u)] \end{aligned}$$

$$\begin{aligned} &\leq sd(u,u_j)+s^2d(T_j(u),T(u)) \\ &\quad +s^3\beta[sd(u_j,u)+sd(T_j(u),T(u))+sd(T(u),u)] \\ &= sd(u,u_j)+s^2d(T_j(u),T(u)) \\ &\quad +s^4\beta[d(u_j,u)+d(T_j(u),T(u))+d(T(u),u)]. \end{aligned}$$

Taking the limit as $j\to\infty$, we have

$$d(u,T(u))\leq s^4\beta\, d(u,T(u)).$$

Since $0\leq s^4\beta<\frac{1}{s}$, we have $d(u,T(u))=0$ and so, $u=T(u)$.

□

Remark 3.10. Theorem 3.9 is a generalization of Theorem 3[28] in metric spaces to b-metric spaces.

As an application of Theorem 3.5, we have the following result which can be obtained by the argument similar to that used in Theorem 3.9.

Theorem 3.11. *Let (X,d) be a complete b-metric space with coefficient $s\geq 1$, and $T_j\in CI(X)$ satisfy*

$$d(T_j(x),T_j(y))\leq \alpha\, d(x,T_j(x))+\beta\, max\left\{\begin{array}{l} d(x,y)+d(y,T_j(y)), \\ \\ d(x,T_j(y))+d(y,T_j(x)) \end{array}\right\}$$

for all $x,y\in X$, where $\alpha\geq 0$ and $0\leq\beta<\frac{1}{s^4(s+1)}$ with fixed points u_j $(j=1,2,\cdots)$. Suppose that $T^n(x)=\lim\limits_{j\to\infty}T_j^n(x)$ for all $x\in X(n=1,2,\cdots)$. Then T has the unique fixed point u in X if and only if $u=\lim\limits_{j}u_j$. Moreover, T is continuous at u and T has property P.

We now examine the strength of hypothesis made in Theorem 3.2. In fact, we furnish Example 3.12 below to show that Theorem 3.2 shall fall through by dropping the condition that $T\in CI(X)$.

Example 3.12. Let $X=\left\{0,\ \frac{1}{6},\ \frac{1}{4},\ \frac{1}{2}\right\}\cup[1,\infty)$ with b-metric d defined by

$$d(x,y)=\mid x-y\mid^2$$

for all $x,y\in X$.

Then (X,d) is a complete b-metric space with coefficient $s=2$. Define $T:X\to X$ by

$$\begin{aligned} T(x) &= 0,\ \ for\, x\in X\setminus\{\frac{1}{4},0\} \\ &= 1,\ \ for\, x\in\{\frac{1}{4},0\}. \end{aligned}$$

It is easy to verify that T satisfies condition (3.1) for $\alpha=36$, $\beta=0$.
But T is not a Ćirić operator, because, for $x\in X\setminus\{\frac{1}{4},0\}$ and $y=\frac{1}{4}$, we have

$$\begin{aligned} d(T^n(x),T^n(y)) &= d(T^{n-1}(0),T^{n-1}(1)) \\ &= d(T^{n-2}(1),T^{n-2}(0)) \\ &\ \ \vdots \end{aligned}$$

$$\begin{aligned} &= d(1,0) \\ &= 1 > q^n(x,y)\,\delta(x,y) \;for\ large\ n, \end{aligned}$$

where q and δ are non-negative real valued functions over $X \times X$ with $q(x,y) < 1\,(x \neq y)$ and $\sup\limits_{x,y\in X} q(x,y) = 1$. Clearly, T possesses no fixed point in X. We note that Theorem 3.2 does not hold without the condition that $T \in CI(X)$.

The following example supports our main result.

Example 3.13. Let $X = [0,\infty)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) =\mid x - y \mid^2$ for all $x, y \in X$. Then (X, d) is a complete b-metric space with coefficient $s = 2$. Let $T : X \to X$ be defined by

$$T(x) = 0,\ for\ all\ x \in X\ except\ x = \frac{1}{5^i},\ i = 1,2,3,\cdots$$

and

$$T(\frac{1}{5^i}) = \frac{1}{5^{i+1}},\ i \geq 1.$$

Let us take $q(x,y) = 1 - \frac{1}{xy+2}$ and $\delta(x,y) = 5 + xy$ for all $x, y \in X$. Then $q(x,y) < 1$ with $\sup\limits_{x,y\in X} q(x,y) = 1$. It is to be noted that $q(x,y) \geq \frac{1}{2}$. We now verify that $T \in CI(X)$.

Case-I If $x \neq \frac{1}{5^i},\ y = \frac{1}{5^i},\ i = 1,2,\cdots$, then

$$\begin{aligned} d(T^n(x), T^n(y)) &= d(0, \frac{1}{5^{n+i}}) \\ &= \frac{1}{5^{2n+2i}} \\ &< \frac{1}{2^n} \\ &< q^n(x,y)\,\delta(x,y). \end{aligned}$$

Case-II If $x = \frac{1}{5^i},\ y = \frac{1}{5^j},\ j \geq i$, then

$$\begin{aligned} d(T^n(x), T^n(y)) &= d(\frac{1}{5^{n+i}}, \frac{1}{5^{n+j}}) \\ &= \frac{1}{5^{2n+2i}}(1 - \frac{1}{5^{j-i}})^2 \\ &< \frac{1}{2^n} \\ &< q^n(x,y)\,\delta(x,y). \end{aligned}$$

Case-III If $x, y \in X$ with $x, y \neq \frac{1}{5^i},\ i = 1,2,\cdots$, then

$$\begin{aligned} d(T^n(x), T^n(y)) &= d(0,0) \\ &= 0 \\ &< \frac{1}{2^n} \\ &< q^n(x,y)\,\delta(x,y). \end{aligned}$$

Thus,

$$d(T^n(x), T^n(y)) < q^n(x,y)\delta(x,y),\ n = 1,2,\cdots,$$

for all $x, y \in X$, where $q(x,y) < 1$ with $\sup\limits_{x,y\in X} q(x,y) = 1$ and so $T \in CI(X)$.

We now verify that T satisfies condition (3.1) for any $\alpha \geq 0,\ \beta = \frac{1}{16} < \frac{1}{s^2}$. If $x \neq \frac{1}{5^i},\ y = \frac{1}{5^i},\ i = 1, 2, \cdots$, then

$$\begin{aligned} d(T(x), T(y)) &= d(0, \frac{1}{5^{i+1}}) \\ &= \frac{1}{5^{2+2i}} \\ &= \frac{1}{25}.\frac{1}{5^{2i}} \\ &= \frac{1}{16}.\frac{16}{25}.\frac{1}{5^{2i}} \\ &= \beta d(y, T(y)) \\ &< \beta\{d(y, T(y)) + d(x, y)\} \\ &\leq \alpha\, d(x, T(x)) + \beta\, max \left\{ \begin{array}{l} d(x,y) + d(y, T(y)), \\ \\ d(x, T(y)),\ d(y, T(x)) \end{array} \right\}. \end{aligned}$$

The other cases may be treated similarly. Thus, T satisfies condition (3.1) for any $\alpha \geq 0,\ \beta = \frac{1}{16} < \frac{1}{s^2}$. We see that all the conditions of Theorem 3.2 are satisfied and 0 is the unique fixed point of T in X, T is continuous at 0 and T has property P.

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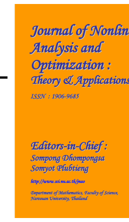

HYPERSTABILITY OF A CAUCHY FUNCTIONAL EQUATION

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ABSTRACT. The aim of this paper is to offer hyperstability results for the Cauchy functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i)$$

in Banach spaces. Namely, we show that a function satisfying the equation approximately must be actually a solution to it.

KEYWORDS : Hyperstability, Cauchy equation, Fixed point theorem.
AMS Subject Classification: Primary 39B82, 39B62; Secondary 47H14, 47H10.

1. INTRODUCTION

Let X and Y be Banach spaces. A mapping $X :\longrightarrow Y$ is called, additive function, if it satisfies the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \qquad \text{for all} \qquad x, y \in X.$$

In 1940, S. M. Ulam [15] raised the question concerning the stability of group homomorphisms: "when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?". The first answer to Ulams question, concerning the Cauchy equation, was given by D. H. Hyers [10]. Thus we speak about the Hyers-Ulam stability. This terminology is also applied to the case of other functional equations. Th. M. Rassias [14] generalized the theorem of Hyers for approximately linear mappings [14]. The stability phenomena that was proved by Th. M. Rassias [14] is called the Hyers-Ulam-Rassias stability. The modified Ulams stability problem with the generalized control function was proved by P. Găvruta [8].

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In 1994, J. M. Rassias [13] studied the Ulams problem of the following equation

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) \tag{1.1}$$

for all $x_1, x_2, ..., x_n \in X$.

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ approximately is a true solution of $\mathfrak{D}$. It seems that the first hyperstability result was published in [2] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [11]. Quite often the hyperstability is confused with superstability, which admits also bounded functions.

The hyperstability problem of various types of functional equations have been investigated by a number of authors, we refer, for example, to [1], [6], [4], [5], [9] and [12]. Throughout this paper, we present the hyperstability results for the additive functional equation (1.1).

The method of the proofs used in the main results is based on a fixed point result that can be derived from [3, Theorem 1]. To present it we need the following three hypotheses:

(H1) X is a nonempty set, Y is a Banach space, $f_1, ..., f_k : X \longrightarrow Y$ and $L_1, ..., L_k : X \longrightarrow \mathbb{R}_+$ are given.

(H2) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\|\xi\left(f_i(x)\right) - \mu\left(f_i(x)\right)\|, \qquad \xi, \mu \in Y^X, \quad x \in X.$$

(H3) $\Lambda : \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta\left(f_i(x)\right), \qquad \delta \in \mathbb{R}_+^X, \quad x \in X.$$

The following theorem is the basic tool in this paper. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

Theorem 1.1. *Let hypotheses (**H1**)-(**H3**) be valid and functions $\varepsilon : X \longrightarrow \mathbb{R}_+$ and $\varphi : X \longrightarrow Y$ fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \qquad x \in X,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in X.$$

Then there exits a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \qquad x \in X.$$

Moreover,

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in X.$$

Numerous papers on this subject have been published and we refer to [1], [6], [4], [5], [9], [12].

2. Hyperstability results

The following theorems and corollaries are the main results in this paper and concern the hyperstability of equation (1.1).

Theorem 2.1. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right\| \leq c \left(\sum_{i=1}^{n} \|x_i\|^p \right) \tag{2.1}$$

for all $x_1, x_2, ..., x_n \in X \setminus \{0\}$ where n is an integer with $n \geq 2$. Then f is additive on $X \setminus \{0\}$.

Proof. We study two cases as follows:
Case 1: n is even
In this case, let $n = 2r + 2$ where $r \in \mathbb{N}$. Then, the inequality (2.1) can be written as follows

$$\left\| f\left(\sum_{i=1}^{2r+2} x_i\right) - \sum_{i=1}^{2r+2} f(x_i) \right\| \leq c \left(\sum_{i=1}^{2r+2} \|x_i\|^p \right) \tag{2.2}$$

where $r \in \mathbb{N}$.
Replacing $x_{(2r+2)}$ by $((2r+1)m+1)\,x$, x_i by $\left(-m - \frac{i}{2}\right) x$ where $i = 2, 4, ..., 2r$ and x_j by $\left(-m + \frac{j-1}{2}\right)$ where $j = 1, 3, ..., (2r+1)$ and $m \in \mathbb{N}$ in (2.2), we obtain that

$$\left\| f(x) - f\Big(\big((2r+1)m+1\big)x\Big) - f(-mx) - \sum_{\ell=1}^{r} f\big((-m+\ell)x\big) - \sum_{\ell=1}^{r} f\big((-m-\ell)x\big) \right\|$$
$$\leq c \left(\Big((2r+1)m+1\Big)^p + m^p + \sum_{\ell=1}^{r} \big|\ell - m\big|^p + \sum_{\ell=1}^{r} \big|\ell + m\big|^p \right) \|x\|^p \tag{2.3}$$

for all $x \in X \setminus \{0\}$.
Further put

$$\mathcal{T}_m \xi(x) := \xi\Big(\big((2r+1)m+1\big)x\Big) + \xi(-mx) + \sum_{\ell=1}^{r} \xi\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} \xi\big((-m-\ell)x\big)$$

and

$$\varepsilon_m(x) := c \left(\Big((2r+1)m+1\Big)^p + m^p + \sum_{\ell=1}^{r} \big|\ell - m\big|^p + \sum_{\ell=1}^{r} \big|\ell + m\big|^p \right) \|x\|^p$$

for all $x \in X \setminus \{0\}$ and all $\xi \in Y^{X \setminus \{0\}}$. The inequality (2.3) now takes the following form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

The following operator

$$\Lambda_m \delta(x) := \delta\Big(\big((2r+1)m+1\big)x\Big) + \delta(-mx) + \sum_{\ell=1}^{r} \delta\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} \delta\big((-m-\ell)x\big)$$

for all $x \in X \setminus \{0\}$ and all $\delta \in \mathbb{R}_+^{X \setminus \{0\}}$, has the form described in (**H3**) with $k = 2r+2$, and

$$\begin{aligned} f_i(x) &= (-m \pm i)x, \quad i = 1, 2, ..., r, \\ f_{(2r+1)}(x) &= \big((2r+1)m+1\big)x, \\ f_{(2r+2)}(x) &= -mx, \\ L_i(x) &= 1, \quad i = 1, 2, ..., (2r+2). \end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X \setminus \{0\}}$

$$\begin{aligned} \Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\Big\| &= \Big\|\xi\Big(\big((2r+1)m+1\big)x\Big) + \xi(-mx) \\ &+ \sum_{\ell=1}^{r} \xi\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} \xi\big((-m-\ell)x\big) \\ &- \mu\Big(\big((2r+1)m+1\big)x\Big) - \mu(-mx) \\ &- \sum_{\ell=1}^{r} \mu\big((-m+\ell)x\big) - \sum_{\ell=1}^{r} \mu\big((-m-\ell)x\big)\Big\| \\ &\le \Big\|(\xi-\mu)\Big(\big((2r+1)m+1\big)x\Big)\Big\| + \|(\xi-\mu)(-mx)\| \\ &+ \sum_{\ell=1}^{r} \big\|(\xi-\mu)\big((-m+\ell)x\big)\big\| + \sum_{\ell=1}^{r} \big\|(\xi-\mu)\big((-m-\ell)x)\big)\big\| \\ &= \sum_{i=1}^{(2r+2)} L_i(x) \left\|\xi\big(f_i(x)\big) - \mu\big(f_i(x)\big)\right\|, \end{aligned}$$

and so **(H2)** is valid. Next, we can find $m_0 \in \mathbb{N}$ such that

$$\alpha_m = \Big((2r+1)m+1\Big)^p + m^p + \sum_{\ell=1}^{r} \big|\ell - m\big|^p + \sum_{\ell=1}^{r} \Big(m+\ell\Big)^p < 1$$

for all $m \ge m_0$. Therefore, we have

$$\begin{aligned} \varepsilon_m^*(x) &:= \sum_{s=0}^{\infty} \Lambda_m^s \varepsilon_m(x) \\ &= c\,\alpha_m \sum_{s=0}^{\infty} \alpha_m^s \|x\|^p \\ &= \frac{c\,\alpha_m}{1-\alpha_m}\|x\|^p, \qquad x \in X \setminus \{0\}, m \ge m \ge m_0. \end{aligned}$$

Thus, according to Theorem 1.1, for each $m \ge m_0$ there exists a unique solution $F_m : X \setminus \{0\} \longrightarrow Y$ of the equation

$$F_m(x) = F_m\Big(\big((2r+1)m+1\big)x\Big) + F_m(-mx) + \sum_{\ell=1}^{r} F_m\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} F_m\big((-m-\ell)x\big)$$

such that

$$\|f(x) - F_m(x)\| \le \frac{c\,\alpha_m}{1-\alpha_m}\|x\|^p, \qquad x \in X \setminus \{0\}, m \ge m \ge m_0.$$

Moreover,

$$F_m(x) := \lim_{s \to \infty} \mathcal{T}_m^s f(x), \qquad x \in X \setminus \{0\}.$$

To prove that $F_m(x)$ satisfies the Cauchy equation (1.1) on $X \setminus \{0\}$ observe that

$$\left\| \mathcal{T}_m^s f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} \mathcal{T}_m^s f(x_i) \right\| \le c\, \alpha_m^s \left(\sum_{i=1}^{n} \|x_i\|^p\right) \tag{2.4}$$

for every $x_1, x_2, ..., x_n \in X \setminus \{0\}$ and $s \in \mathbb{N}_0$.
Indeed, if $s = 0$, then (2.4) is simply (2.1). So, take $t \in \mathbb{N}_0$ and suppose that (2.4) holds for $s = t$ and $x_1, x_2, ..., x_n \in X \setminus \{0\}$. Then

$$\begin{aligned}
&\left\| \mathcal{T}_m^{t+1} f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} \mathcal{T}_m^{t+1} f(x_i) \right\| = \Bigg\| \mathcal{T}_m^t f\left(\sum_{i=1}^{n} \Big((2r+1)m+1\Big) x_i\right) \\
&+\mathcal{T}_m^t f\left(\sum_{i=1}^{n} (-m x_i)\right) + \sum_{\ell=1}^{r} \mathcal{T}_m^t f\left(\sum_{i=1}^{n} \Big(-m+\ell\Big) x_i\right) + \sum_{\ell=1}^{r} \mathcal{T}_m^t f\left(\sum_{i=1}^{n} \Big(-m-\ell\Big) x_i\right) \\
&- \sum_{i=1}^{n} \mathcal{T}_m^t f\Big(\big((2r+1)m+1\big) x_i\Big) - \sum_{i=1}^{n} \mathcal{T}_m^t f(-m x_i) \\
&- \sum_{\ell=1}^{r} \left(\sum_{i=1}^{n} \mathcal{T}_m^t f\Big((-m+\ell) x_i\Big)\right) - \sum_{\ell=1}^{r} \left(\sum_{i=1}^{n} \mathcal{T}_m^t f\Big(\big(-m-\ell\big) x_i\Big)\right) \Bigg\| \\
&\le \left\| \mathcal{T}_m^t f\left(\sum_{i=1}^{n} \Big((2r+1)m+1\Big) x_i\right) - \sum_{i=1}^{n} \mathcal{T}_m^t f\Big(\big((2r+1)m+1\big) x_i\Big) \right\| \\
&+ \left\| \mathcal{T}_m^t f\left(\sum_{i=1}^{n} (-m x_i)\right) - \sum_{i=1}^{n} \mathcal{T}_m^t f(-m x_i) \right\| \\
&+ \left\| \sum_{\ell=1}^{r} \mathcal{T}_m^t f\left(\sum_{i=1}^{n} (-m+\ell) x_i\right) - \sum_{\ell=1}^{r} \left(\sum_{i=1}^{n} \mathcal{T}_m^t f\Big((-m+\ell) x_i\Big)\right) \right\| \\
&+ \left\| \sum_{\ell=1}^{r} \mathcal{T}_m^t f\left(\sum_{i=1}^{n} (-m-\ell) x_i\right) - \sum_{\ell=1}^{r} \left(\sum_{i=1}^{n} \mathcal{T}_m^t f\Big((-m-\ell) x_i\Big)\right) \right\| \\
&\le c\, \alpha_m^t \left(\big((2r+1)m+1\big)^p + m^p + \sum_{\ell=1}^{r} \big|\ell-m\big|^p + \sum_{\ell=1}^{r} \big|-m-\ell\big|^p \right) \sum_{i=1}^{n} \|x_i\|^p \\
&= c\, \alpha_m^{t+1} \sum_{i=1}^{n} \|x_i\|^p.
\end{aligned}$$

By induction, we have shown that (2.4) holds for all $x_1, x_2, ..., x_n \in X \setminus \{0\}$ and $s \in \mathbb{N}_0$. Letting $s \longrightarrow \infty$ in (2.4), we obtain that

$$F_m\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} F_m(x_i), \qquad x_1, x_2, ..., x_n \in X \setminus \{0\}.$$

Thus, we have proved that for every $m \ge m_0$ there exists a unique function $F_m : X \setminus \{0\} \longrightarrow Y$ such that F_m is a solution of the Cauchy equation (1.1) on $X \setminus \{0\}$ and

$$\|f(x) - F_m(x)\| \le \frac{c\, \alpha_m}{1-\alpha_m} \|x\|^p, \qquad x \in X \setminus \{0\}.$$

Since $p < 0$, the sequence

$$\left\{ \frac{c\, \alpha_m}{1-\alpha_m} \|x\|^p \right\}_{m \ge m_0}$$

tends to zero when $m \longrightarrow \infty$. Consequently, f satisfies the Cauchy equation (1.1) on $X \setminus \{0\}$ as the pointwise of $(F_m)_{m \geq m_0}$.

Case 2: n is odd

Letting $n = 2r+1$ where $r \in \mathbb{N}$, we can rewrite the inequality (2.1) as follows

$$\left\| f\left(\sum_{i=1}^{2r+1} x_i\right) - \sum_{i=1}^{2r+1} f(x_i) \right\| \leq c\left(\sum_{i=1}^{2r+1} \|x_i\|^p\right). \tag{2.5}$$

Replacing x_{2r+1} by $(2rm+1)x$, x_i by $(-m-\frac{i}{2})x$ where $i = 2, 4, ..., 2r$ and x_j by $(-m+\frac{j+1}{2})$ where $j = 1, 3, ..., (2r-1)$ and $m \in \mathbb{N}$ in (2.5), we get that

$$\begin{aligned}\Bigg\| f(x) - f\Big((2rm+1)x\Big) - \sum_{\ell=1}^{r} f\Big((-m+\ell)x\Big) - \sum_{\ell=1}^{r} f\Big((-m-\ell)x\Big)\Bigg\| \\ \leq c\left(\big|2rm+1\big|^p + \sum_{\ell=1}^{r}\big|-m+\ell\big|^p + \sum_{\ell=1}^{r}\big|m+\ell\big|^p\right)\|x\|^p \end{aligned} \tag{2.6}$$

for all $x \in X \setminus \{0\}$.

Further put

$$\mathcal{T}_m\xi(x) := \xi\Big((2rm+1)x\Big) + \sum_{\ell=1}^{r}\xi\Big((-m+\ell)x\Big) + \sum_{\ell=1}^{r}\xi\Big((-m-\ell)x\Big)$$

and

$$\varepsilon_m(x) := c\left(\big|2rm+1\big|^p + \sum_{\ell=1}^{r}\big|-m+\ell\big|^p + \sum_{\ell=1}^{r}\big|m+\ell\big|^p\right)\|x\|^p$$

for all $x \in X \setminus \{0\}$ and all $\xi \in Y^{X\setminus\{0\}}$. Then the inequality (2.6) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

The following operator

$$\Lambda_m\delta(x) := \delta\Big((2rm+1)x\Big) + \sum_{\ell=1}^{r}\delta\Big((-m+\ell)x\Big) + \sum_{\ell=1}^{r}\delta\Big((-m-\ell)x\Big)$$

for all $x \in X\setminus\{0\}$ and all $\delta \in \mathbb{R}_+^{X\setminus\{0\}}$, has the form described in (**H3**) with $k = 2r+1$, and

$$\begin{aligned} f_i(x) &= (-m \pm i)x, \qquad i = 1, 2, ..., r, \\ f_{2r+1}(x) &= (2rm+1)x, \\ L_i(x) &= 1, \qquad i = 1, 2, ..., (2r+1). \end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$

$$\begin{aligned}\big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\big\| &= \Bigg\|\xi\big((2rm+1)x\big) + \sum_{\ell=1}^{r}\xi\big((-m+\ell)x\big) + \sum_{\ell=1}^{r}\xi\big((-m-\ell)x\big) \\ &\quad -\mu\big((2rm+1)x\big) - \sum_{\ell=1}^{r}\mu\big((-m+\ell)x\big) - \sum_{\ell=1}^{r}\mu\big((-m-\ell)x\big)\Bigg\| \\ &\leq \Big\|(\xi-\mu)\big((2rm+1)x\big)\Big\| + \sum_{\ell=1}^{r}\Big\|(\xi-\mu)\big((-m+\ell)x\big)\Big\| + \sum_{\ell=1}^{r}\Big\|(\xi-\mu)\Big((-m-\ell)x\Big)\Big\|\end{aligned}$$

$$= \sum_{i=1}^{(2r+1)} L_i(x) \Big\| \xi\big(f_i(x)\big) - \mu\big(f_i(x)\big) \Big\|,$$

so (**H2**) is valid. Now,we can find $m_0 \in \mathbb{N}$ such that

$$\alpha_m = \big|2rm+1\big|^p + \sum_{\ell=1}^{r} \big|-m+\ell\big|^p + \sum_{\ell=1}^{r} \big|m+\ell\big|^p < 1$$

for all $m \geq m_0$. Therefore, we have

$$\begin{aligned} \varepsilon_m^*(x) &:= \sum_{s=0}^{\infty} \Lambda_m^s \varepsilon_m(x) \\ &= c\, \alpha_m \sum_{s=0}^{\infty} \alpha_m^s \|x\|^p \\ &= \frac{c\, \alpha_m}{1-\alpha_m} \|x\|^p, \qquad x \in X \setminus \{0\}, m \geq m_0. \end{aligned}$$

The rest of the proof is similar to the proof of case 1. □

Corollary 2.2. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big) \tag{2.7}$$

for all $x, y \in X \setminus \{0\}$ where $n \in \mathbb{N}_0$. Then f is additive on $X \setminus \{0\}$.

Theorem 2.3. *Let X be a normed space, Y be a Banach space, $c \geq 0$, $p_i \in \mathbb{R}$ with $\sum_{i=1}^n p_i < 0$ and let $f : X \longrightarrow Y$ satisfy*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right\| \leq c \left(\prod_{i=1}^{n} \|x_i\|^{p_i} \right) \tag{2.8}$$

for all $x_1, x_2, ..., x_n \in X \setminus \{0\}$ where $n \in \mathbb{N}_0$. Then f is additive on $X \setminus \{0\}$.

Proof. Since $\sum_{i=1}^n p_i < 0$, some of p_i must be negative. Assume that these are $p_j < 0$ where $1 \leq j \leq n$. By using the same technic of the proof of Theorem 2.1, we study two cases as follows:

Case 1: n is even

In this case, suppose that $n = 2r+2$ where $r \in \mathbb{N}$. Then the inequality (2.8) can be written as follows

$$\left\| f\left(\sum_{i=1}^{2r+2} x_i\right) - \sum_{i=1}^{2r+2} f(x_i) \right\| \leq c \left(\prod_{i=1}^{2r+2} \|x_i\|^{p_i} \right). \tag{2.9}$$

Replacing $x_{(2r+2)}$ by $((2r+1)m+1)\, x$, x_i by $\left(-m - \frac{i}{2}\right) x$ where $i = 2, 4, ..., 2r$ and x_j by $\left(-m + \frac{j-1}{2}\right) x$ where $j = 1, 3, ..., (2r+1)$ and $m \in \mathbb{N}$ in (2.9), we obtain that

$$\left\| f(x) - f\Big(\big((2r+1)m+1\big)x\Big) - f(-mx) - \sum_{\ell=1}^{r} f\Big((-m+\ell)x\Big) - \sum_{\ell=1}^{r} f\Big((-m-\ell)x\Big) \right\|$$

$$\leq c|m|^{p_1} \cdot \big|(2r+1)m+1\big|^{p_{2r+2}} \cdot \prod_{\ell=1}^{r} \Big(\big|m+\ell\big|^{p_{2\ell}} \cdot \big|-m+\ell\big|^{p_{2\ell+1}} \Big) \|x\|^{\beta} \tag{2.10}$$

for all $x \in X \setminus \{0\}$ where $\beta = \sum_{i=1}^n p_i$.

Further put

$$\mathcal{T}_m\xi(x) := \xi\Big(\big((2r+1)m+1\big)x\Big) + \xi(-mx) + \sum_{\ell=1}^{r} \xi\Big((-m+\ell)x\Big) + \sum_{\ell=1}^{r} \xi\Big((-m-\ell)x\Big)$$

and

$$\varepsilon_m(x) := \theta|m|^{p_1} \cdot \big|(2r+1)m+1\big|^{p_{2r+2}} \cdot \prod_{\ell=1}^{r} \Big(\big|m+\ell\big|^{p_{2\ell}} \cdot \big|-m+\ell\big|^{p_{2\ell+1}}\Big)\|x\|^{\beta}$$

for all $x \in X \setminus \{0\}$ and all $\xi \in Y^X$. Thus, the inequality (2.10) takes the following form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

The following operator

$$\Lambda_m\delta(x) := \delta\Big(\big((2r+1)m+1\big)x\Big) + \delta(-mx) + \sum_{\ell=1}^{r} \delta\big((-m+\ell)x\big) + \sum_{\ell=1}^{r} \delta\big((-m-\ell)x\big)$$

for all $x \in X\setminus\{0\}$ and all $\delta \in \mathbb{R}_+^{X\setminus\{0\}}$, has the form described in **(H3)** with $k = 2r+2$ and

$$\begin{aligned}
f_i(x) &= (-m \pm i)x, \qquad i = 1, 2, ..., r,\\
f_{(2r+1)}(x) &= \big((2r+1)m+1\big)x,\\
f_{(2r+2)}(x) &= -mx,\\
L_i(x) &= 1, \qquad i = 1, 2, ..., (2r+2).
\end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$

$$\begin{aligned}
\Big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\Big\| &= \Bigg\|\xi\Big(\big((2r+1)m+1\big)x\Big) + \xi(-mx) + \sum_{\ell=1}^{r} \xi\big((-m+\ell)x\big)\\
&\quad + \sum_{\ell=1}^{r} \xi\big((-m-\ell)x\big) - \mu\Big(\big((2r+1)m+1\big)x\Big) - \mu(-mx)\\
&\quad - \sum_{\ell=1}^{r} \mu\big((-m+\ell)x\big) - \sum_{\ell=1}^{r} \mu\big((-m-\ell)x\big)\Bigg\|\\
&\leq \Big\|(\xi-\mu)\Big(\big((2r+1)m+1\big)x\Big)\Big\| + \big\|(\xi-\mu)(-mx)\big\|\\
&\quad + \sum_{\ell=1}^{r} \big\|(\xi-\mu)\big((-m+\ell)x\big)\big\| + \sum_{\ell=1}^{r} \big\|(\xi-\mu)\big((-m-\ell)x\big)\big\|\\
&= \sum_{i=1}^{(2r+2)} L_i(x)\,\Big\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big)\Big\|,
\end{aligned}$$

and so **(H2)** is valid. Next, we can find $m_0 \in \mathbb{N}$ such that

$$\lambda_m = |m|^{p_1} \cdot \big|(2r+1)m+1\big|^{p_{2r+2}} \cdot \prod_{\ell=1}^{r} \Big(\big|m+\ell\big|^{p_{2\ell}} \cdot \big|-m+\ell\big|^{p_{2\ell+1}}\Big) < 1$$

for all $m \geq m_0$. Therefore, we have

$$\varepsilon_m^*(x) := \sum_{s=0}^{\infty} \Lambda_m^s \varepsilon_m(x)$$

$$= \frac{\theta\,\lambda_m}{1-\lambda_m}\|x\|^{\beta},$$

for all $x \in X \setminus \{0\}$ where $m \geq m_0$.

The rest of the proof is similar to the proof of Theorem 2.1.

Case 2: n is odd

Let $n = 2r+1$ where $r \in \mathbb{N}$. Then, we can rewrite the inequality (2.8) as follows

$$\left\| f\left(\sum_{i=1}^{2r+1} x_i\right) - \sum_{i=1}^{2r+1} f(x_i)\right\| \leq c\left(\prod_{i=1}^{2r+1}\|x_i\|^{p}\right). \tag{2.11}$$

Replacing x_{2r+1} by $(2rm+1)x$, x_i by $(-m-\frac{i}{2})x$ where $i = 2,4,...,2r$ and x_j by $(-m+\frac{j+1}{2})x$ where $j = 1,3,...,(2r-1)$ and $m \in \mathbb{N}$ in (2.13), we get that

$$\begin{aligned}&\left\| f(x) - f\big((2rm+1)x\big) - \sum_{\ell=1}^{r} f\big((-m+\ell)x\big) - \sum_{\ell=1}^{r} f\big((-m-\ell)x\big)\right\| \\ &\leq c\,\left|2rm+1\right|^{p_{2r+1}} \cdot \prod_{\ell=1}^{r}\Big(\left|m+\ell\right|^{p_{2\ell}} \cdot \left|-m+\ell\right|^{p_{2\ell+1}}\Big)\|x\|^{\beta}\end{aligned} \tag{2.12}$$

where $\sum_{i=1}^{2r+1} p_i = \beta$ for all $x \in X \setminus \{0\}$.
Further put

$$\mathcal{T}_m\xi(x) := \xi\big((2rm+1)x\big) + \sum_{\ell=1}^{r}\xi\big((-m+\ell)x\big) + \sum_{\ell=1}^{r}\xi\big((-m-\ell)x\big)$$

and

$$\varepsilon_m(x) := \theta\,\left|2rm+1\right|^{p_{2r+1}} \cdot \prod_{\ell=1}^{r}\Big(\left|m+\ell\right|^{p_{2\ell}} \cdot \left|-m+\ell\right|^{p_{2\ell+1}}\Big)\|x\|^{\beta}$$

for all $x \in X \setminus \{0\}$ and all $\xi \in Y^{X\setminus\{0\}}$. Then the inequality (2.12) takes the form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \qquad x \in X \setminus \{0\}.$$

The following operator

$$\Lambda_m\delta(x) := \delta\big((2rm+1)x\big) + \sum_{\ell=1}^{r}\delta\big((-m+\ell)x\big) + \sum_{\ell=1}^{r}\delta\big((-m-\ell)x\big)$$

for all $x \in X\setminus\{0\}$ and all $\delta \in \mathbb{R}_{+}^{X\setminus\{0\}}$, has the form described in (**H3**) with $k = 2r+1$, and

$$\begin{aligned} f_i(x) &= (-m \pm i)x, \qquad i = 1,2,...,r,\\ f_{2r+1}(x) &= (2rm+1)x,\\ L_i(x) &= 1, \qquad i = 1,2,...,(2r+1).\end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X\setminus\{0\}}$

$$\begin{aligned}\big\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\big\| &= \Bigg\|\xi\Big((2rm+1)x\Big) + \sum_{\ell=1}^{r}\xi\Big((-m+\ell)x\Big) + \sum_{\ell=1}^{r}\xi\Big((-m-\ell)x\Big)\\ &\quad -\mu\Big((2rm+1)x\Big) - \sum_{\ell=1}^{r}\mu\Big((-m+\ell)x\Big)\end{aligned}$$

$$
\begin{aligned}
&-\sum_{\ell=1}^{r}\mu\Big((-m-\ell)x\Big)\Big\| \\
&\le \Big\|(\xi-\mu)\Big((2rm+1)x\Big)\Big\| + \sum_{\ell=1}^{r}\Big\|(\xi-\mu)\Big((-m+\ell)x\Big)\Big\| \\
&+\sum_{\ell=1}^{r}\Big\|(\xi-\mu)\Big((-m-\ell)x\Big)\Big\| \\
&=\sum_{i=1}^{(2r+1)} L_i(x)\Big\|\xi\Big(f_i(x)\Big)-\mu\Big(f_i(x)\Big)\Big\|,
\end{aligned}
$$

and so (**H2**) is valid. Now,we can find $m_0 \in \mathbb{N}$ such that

$$\lambda_m = \big|2rm+1\big|^{p_{2r+1}} \cdot \prod_{\ell=1}^{r}\Big(\big|m+\ell\big|^{p_{2\ell}}\cdot\big|-m+\ell\big|^{p_{2\ell+1}}\Big) < 1$$

for all $m \ge m_0$. Therefore, we have

$$
\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{s=0}^{\infty}\Lambda_m^s\varepsilon_m(x) \\
&= c\left(\big|2rm+1\big|^{p_{2r+1}}\cdot\prod_{\ell=1}^{r}\Big(\big|m+\ell\big|^{p_{2\ell}}\cdot\big|-m+\ell\big|^{p_{2\ell+1}}\Big)\right)\sum_{s=0}^{\infty}\Lambda_m^s\|x\|^{\beta} \\
&= \frac{c\,\lambda_m}{1-\lambda_m}\|x\|^{\beta}, \qquad x\in X\setminus\{0\}, m\ge m_0.
\end{aligned}
$$

The rest of the proof is similar to the proof of case 1. □

Corollary 2.4. *Let X be a normed space, Y be a Banach space, $c \ge 0$, $p,q\in\mathbb{R}$, $p+q<0$ and let $f : X \longrightarrow Y$ satisfy*

$$\|f(x+y)-f(x)-f(y)\| \le c\big(\|x\|^p\cdot\|y\|^q\big) \tag{2.13}$$

for all $x,y\in X\setminus\{0\}$ where $n\in\mathbb{N}_0$. Then f is additive on $X\setminus\{0\}$.

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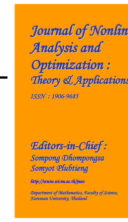

BEST PROXIMITY PAIRS IN CONE METRIC SPACES

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ABSTRACT. In this paper we introduce cone metric space and give some contraction about existence best proximity pairs and best proximity points. Also, we prove some fixed point theorems on cone metric space.

KEYWORDS : Cone metric space; Fixed point; Normal cone; Cone-cyclic contraction.
AMS Subject Classification: 46A32, 46M05, 41A17

1. INTRODUCTION

Let E be a normed linear space. the subset P of E is called a cone if
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap -P = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leqslant y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $int\ P$ denotes the interior of P. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leqslant x \leqslant y$ implies $\|x\| \leq M\|y\|$.

The least positive number satisfying the above is called the normal constant of P.

Definition 1.1. Let X be a non-empty set, $(E, \|.\|)$ a normed space that ordered by a normal cone P with constant normal $M = 1$, $intP \neq \emptyset$ and $\leqslant$ is a partial ordering with respect to P and the mapping $d : X \times X \to E$ satisfies
(i)$0 \leqslant d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$
(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 1.2. (i) Suppose $E = R^2$, $P = \{(x, y) \in E : x,\ y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is a cone metric

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space.
(ii) Suppose $E = C_R[0,1]$, $P = \{\varphi \in E : \varphi \geq 0\}, X = R$ and $d : X \times X \to E$ defined by $d(x,y) = |x-y|\varphi$ where $\varphi : [0,1] \to R$ such that $\varphi(t) = e^t$. It easy to see that d is a cone metric space.

Definition 1.3. (X,d) a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$. Then $\{x_n\}$ is said to be convergent to x. We denote this by

$$\lim_{n\to\infty} x_n = x.$$

Also sequence $\{x_n\}$ is said to be bounded if there is $M \gg 0$ such that for all $n \in \mathbb{N}$ we have

$$d(0, x_n) \ll M.$$

Definition 1.4. Let (X,d) be a cone metric space. If every Cauchy sequence be convergence, then X is called a complete cone metric space.

Definition 1.5. Let (X,d) be a cone metric space. If there is a pair $(x_0, y_0) \in A \times B$ such that

$$\|d(x_0,y_0)\| = \inf_{(a,b)\in A\times B} \|d(a,b)\|$$

the pair (x_0, y_0) is called a best proximity pair for A and B and said that the pair (A,B) has best proximity pair in X. Put $Prox(A,B)$ the set of all best proximity pairs for the pair (A,B).

Fixed point theory is an important tool for solving equations $T(x) = x$. However, if T does not have fixed points, then one often tries to find an element x which is in some sense closest to $T(x)$. A classical result in this direction is a best approximation theorem due to Ky Fan [4].
a best proximity pair evolves as a generalization of the best approximation considered by Sahney and Singh [6], Singer [7] and Xu [8], of exploring some the sufficient conditions for the non-empty of the set $Prox(A,B)$.

In this paper we consider sufficient conditions that ensure the existence of an element $x \in X$ for two subsets A, B in cone metric space X such that $\|d(x,Tx)\| = d(A,B)$ for $T : A \cup B \to A \cup B$, where

$$d(A,B) = \inf_{(a,b)\in A\times B} \|d(a,b)\|.$$

It is clear that if $d(A,B) = 0$, then T has fixed point. In continue we consider some fixed point theorems on cone metric space. It is notable we use of results in [1-3].

2. MAIN RESULTS

In this section at first we give a new definition and use for present new results.

Definition 2.1. Let A and B be nonempty subsets of a cone metric space (X,d) and P a cone of normed space E. A map $T : A \cup B \to A \cup B$ is a cone-cyclic contraction map if there exists a continuous map $\varphi : P \to [0,1)$ such that

i) $d(Tx,Ty) \leqslant \varphi(d(x,y))d(x,y) + (1 - \varphi(d(x,y)))dist(A,B)$
ii) $\lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0$, where, $x_{n+1} = T^{n+1}x_0, x_0 \in A \cup B$
iii) $T(A) \subset B$ and $T(B) \subset A$.
where $dist(A,B) = inf\{d(a,b) : a \in A, b \in B\}$.

Proposition 2.2. *Let A and B be nonempty subsets of a cone metric space X, P a normal cone with normal constant K, $T : A \cup B \to A \cup B$ a cone-cyclic contraction map and $x_{n+1} = T^{n+1}x_0$ for $x_0 \in A \cup B$. Then $d(x_n, Tx_n) \to dist(A, B)$.*

Proof. Choose $x_0 \in A \cup B$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, \cdots$ and $\varphi_n := \varphi(d(x_{n-1}, x_n))$. By definition of cone-cyclic contraction map we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant \varphi_n d(x_{n-1}, x_n) + (1 - \varphi_n)dist(A, B) \\ &\leqslant \varphi_n\varphi_{n-1}d(x_{n-2}, x_{n-1}) + (1 - \varphi_n\varphi_{n-1})dist(A, B) \\ &\leqslant \\ &\vdots \\ &\leqslant \varphi_n\varphi_{n-1}\dots\varphi_1 d(x_0, x_1) + (1 - \varphi_n\varphi_{n-1}\dots\varphi_1)dist(A, B). \end{aligned}$$

Therefore

$$d(x_n, x_{n+1}) - (1 - \varphi_n\varphi_{n-1}\dots\varphi_1)dist(A, B) \leqslant \varphi_n\varphi_{n-1}\dots\varphi_1 d(x_0, x_1)$$

and so since P is normal we obtain,

$$\|d(x_n, x_{n+1}) - (1 - \varphi_n\varphi_{n-1}\dots\varphi_1)dist(A, B)\| \leqslant K\varphi_n\varphi_{n-1}\dots\varphi_1\|d(x_0, x_1)\|.$$

Thus $d(x_n, x_{n+1}) \to dist(A, B)$. □

Theorem 2.3. *Let A and B be nonempty subsets of a complete cone metric space X, P be a normal cone with normal constant K. Let $T : A \cup B \to A \cup B$ is a cone-cyclic contraction map, let $x_0 \in A$ and define $x_{n+1} = Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists x in A such that $d(x, Tx) = dist(A, B)$.*

Proof. Suppose $\{x_{2n_k}\}$ is a subsequence of $\{x_{2n}\}$ converging to some $x \in A$. Now

$$dist(A, B) \leqslant d(x, x_{2n_k-1}) \leqslant d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}).$$

Thus, we have $d(x, x_{2n_k-1})$ converges to $dist(A, B)$. Since

$$dist(A, B) \leqslant d(x_{2n_k}, Tx) \leqslant d(x_{2n_k-1}, x),$$

$d(x, Tx) = dist(A, B)$. □

Proposition 2.4. *Let A and B be nonempty subsets of a cone metric space X, $T : A \cup B \to A \cup B$ a cone-cyclic contraction map, $x_0 \in A \cup B$ and $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$. Then the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded.*

Proof. Suppose $x_0 \in A$ then since by Proposition 2.2 $d(x_{2n}, x_{2n+1})$ converges to $dist(A, B)$. It is enough to prove that $\{x_{2n+1}\}$ is bounded. Suppose $\{x_{2n+1}\}$ is not bounded, then there exists $n_0 \in \mathbb{N}$ such that

$$d(x_2, x_{2n_0+1}) \gg M \text{ and } d(x_2, x_{2n_0-1}) \ll M,$$

where $M \gg max\{\frac{2d(x_0,Tx_0)}{1/k^2-1} + dist(A, B), d(Tx_2, Tx_0)\}$. By the cone-cyclic contraction property of T,

$$\begin{aligned} \frac{M - dist(A, B)}{k^2} + dist(A, B) &\ll d(x_0, x_{2n_0-1}) \\ &\leqslant d(x_0, x_2) + d(x_2, x_{2n_0-1}) \\ &\leqslant 2d(x_0, Tx_0) + M. \end{aligned}$$

Thus $M \ll \frac{2d(x_0,Tx_0)}{1/k^2-1} + dist(A, B)$, hence

$$\frac{2d(x_0, Tx_0)}{1/k^2 - 1} + dist(A, B) - M \in intP, \frac{2d(x_0, Tx_0)}{1/k^2 - 1} + dist(A, B) - M \in int(-P),$$

which is a contradiction, since $intP \cap int(-P) = \emptyset$. The proof when x_0 in B is similar. □

The Proposition 2.4 leads us to an existence result when one of the sets is boundedly compact. We remember that a set A is boundedly compact if every bounded sequence in A has a subsequence convergence.

Corollary 2.5. *Let A and B be nonempty closed subsets of a complete cone metric space X and $T : A \cup B \to A \cup B$ a cone-cyclic contraction map. If either A or B is boundedly compact, then there exists $x \in A \cup B$ with $d(x, Tx) = dist(A, B)$.*

Proof. It follows directly from Theorem 2.3 and Proposition 2.4 □

Proposition 2.6. *Let A and B be nonempty compact subsets of a cone metric space X, P a normal cone with normal constant $K = 1$. Then Prox(A,B) is nonempty.*

Proof. Define $f : A \times B \to [0, \infty)$ by $f(a, b) = \|d(a, b)\|$. Now choose $0 \ll c$ such that $\|c\| < \frac{\epsilon}{2}$. Suppose $z \in B(a, c)$ and $w \in B(b, c)$. Since

$$d(a, b) \leqslant d(a, z) + d(z, b),$$

$$d(z, b) \leqslant d(z, w) + d(w, b),$$

hence

$$d(a, b) - d(z, w) \leqslant d(a, z) + d(w, b),$$

$$d(z, w) - d(a, b) \leqslant d(a, z) + d(w, b).$$

Therefore

$$\|d(a, b)\| - \|d(z, w)\| < \epsilon,$$

$$\|d(z, w)\| - \|d(a, b)\| < \epsilon$$

and so $|f(a, b) - f(z, w)| < \epsilon$. Then f is continues and since A, B are compact, $f(A, B)$ is compact and so there exists $(a_0, b_0) \in A \times B$ such that

$$\|d(a_0, b_0)\| = \inf_{(a,b) \in A \times B} \|d(a, b)\|.$$

□

Definition 2.7. Let A and B be nonempty subsets of a cone metric space (X, d) and P a cone of normed space E. A map $T : A \cup B \to A \cup B$ is a cone contractive map if
i) $\|d(Tx, Ty)\| \leq \|d(x, y)\|$
ii) $\|d(Tx, Ty)\| < \|d(x, y)\|, \ if \ d(x, y) = d(A, B)$
iii) $T(A) \subseteq B \ and \ T(B) \subseteq A.$

In the following we give different version of the main result in [5].

Theorem 2.8. *Let A and B be nonempty compact subsets of a cone metric space X, P a normal cone with normal constant $K = 1$ and $T : A \cup B \to A \cup B$ cone contractive map. Then there exists $(a_0, b_0) \in A \times B$ such that $\|d(a_0, b_0)\| = \|d(a_0, Ta_0)\| = \|d(b_0, Tb_0)\| = d(A, B)$.*

Proof. Define $f : A \times B \to [0, \infty)$ by $f(a, b) = \|d(Ta, b)\| + \|d(Tb, a)\|$. Since f is continues (similar to proof Proposition 2.5) and A and B are compact, it attains minimum at some element, say (a_0, b_0), in $A \times B$. If $Ta_0 \neq b_0$, since

$$\begin{aligned} f(Ta_0, Tb_0) &= \|d(TTa_0, Tb_0)\| + \|d(Ta_0, TTb_0)\| \\ &< \|d(Ta_0, b_0)\| + \|d(a_0, Tb_0)\| \\ &= f(a_0, b_0) \end{aligned}$$

which is a contrary to the fact that f attains minimum at (a_0, b_0). Then $Ta_0 = b_0$. A similar argument can be given to show that $a_0 = Tb_0$. If $d(A, B) < \|d(a_0, b_0)\|$, then $\|d(a_0, b_0)\| = \|d(Ta_0, Tb_0)\| < \|d(a_0, b_0)\|$ which is contradiction. □

Corollary 2.9. *Let A and B be nonempty compact subsets of a cone metric space X, P a normal cone with normal constant $K = 1$ and $T : A \cup B \to A \cup B$ cone contractive map. Then T^2 has a fixed point.*

Proof. By Theorem 2.7 $Ta_0 = b_0$ and $a_0 = Tb_0$ and so $T^2 a_0 = a_0$. □

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