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Table of Contents

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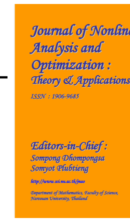

CONE METRIC TYPE SPACE AND NEW COUPLED FIXED POINT THEOREMS

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ABSTRACT. The notion of coupled fixed point was initiated in 2006 by Bhaskar and Lakshmikantham. On the other hand, Radenović and Kadelburg [S. Radenović, Z. Kadelburg, Quasi-contractions on symmetric and cone symmetric spaces, Banach J. Math. Anal. 5 (1) (2011) 38-50] defined cone metric type space and proved several fixed point theorems. In this paper we introduce the concept of a coupled fixed point for a contractive condition in cone metric type space and prove some coupled fixed point theorems.

KEYWORDS : Cone metric type space; Coupled fixed point; Cone metric space.
AMS Subject Classification: 47H10.

1. INTRODUCTION AND PRELIMINARIES

The symmetric space, as metric-like spaces lacking the triangle inequality was introduced in 1931 by Wilson [31]. Recently, a new type of spaces which they called metric type spaces are defined by Khamsi and Hussain [16] and Boriceanu [6]. Also, Jovanović et al. [14], Rahimi and Soleimani Rad [24], Bota et al. [7], Pavlović et al. [20], Singh et al. [28] and Hussain et al. [11] generalized some fixed point theorems of metric spaces by considering metric type space.

On the other hand, the cone metric space was initiated in 2007 by Huang and Zhang [12] and several fixed and common fixed point results in cone metric spaces were introduced in [2, 3, 13, 22, 23, 25, 30] and the references contained therein. In the sequel, analogously with definition of metric type space, Ćvetković et al. [9] and Radenović and Kadelburg [21] defined cone metric type space and proved several fixed and common fixed point theorems (See [17]).

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In 2006, Bhaskar and Lakshmikantham [5] introduced the concept of coupled fixed point theorem in partially ordered metric spaces. Then, some other authors generalized this concept and proved several common coupled fixed point and coupled fixed point theorems in ordered metric and ordered cone metric spaces (See [1, 4, 8, 15, 18, 19, 26, 27, 29] and the references contained therein).

In this paper we define the concept of coupled fixed point in a cone metric type space and prove some coupled fixed point theorems. Our results generalize, extend and unify several well known comparable results in the literature.

Let us start by defining some important definitions.

Definition 1.1. (See [31]). Let X be a nonempty set and the mapping $D : X \times X \to [0, \infty)$ satisfies

$$(S1) \qquad D(x, y) = 0 \Longleftrightarrow x = y;$$
$$(S2) \qquad D(x, y) = D(y, x),$$

for all $x, y \in X$. Then D is called a symmetric on X and (X, D) is called a symmetric space.

Definition 1.2. (See [10, 12]). Let E be a real Banach space and P be a subset of E. Then P is called a cone if and only if
(a) P is closed, non-empty and $P \neq \{\theta\}$;
(b) $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
(c) if $x \in P$ and $-x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \Longleftrightarrow y - x \in P.$$

We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in intP$ (where $intP$ is the interior of P). The cone P is named normal if there is a number $K > 0$ such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number satisfying the above is called the normal constant of P.

Definition 1.3. (See [12]). Let X be a nonempty set and the mapping $d : X \times X \to E$ satisfies
$(d1)$ $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$(d3)$ $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
Then, d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.4. (See [12, 22]).
(i) Let $E = \mathbf{R}^2$, $P = \{(x, y) \in E | x, y \geq 0\} \subset \mathbf{R}^2$, $X = \mathbf{R}$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.
(ii) Let $X = [0, 1]$, $E = C^2_{\mathbf{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $P = \{f \in E | f \geq 0\}$ and $d(x, y)(t) = |x - y|2^t$. Then (X, d) is a cone metric space with non-normal solid cone.

Definition 1.5. (See [16]). Let X be a nonempty set, and $K \geq 1$ be a real number. Suppose the mapping $D_m : X \times X \to [0, \infty)$ satisfies
$(D1)$ $D_m(x, y) = 0$ if and only if $x = y$;
$(D2)$ $D_m(x, y) = D_m(y, x)$ for all $x, y \in X$;
$(D3)$ $D_m(x, z) \leq K(D_m(x, y) + D_m(y, z))$ for all $x, y, z \in X$.

(X, D_m, K) is called metric type space. Obviously, for $K = 1$, metric type space is a metric space.

Definition 1.6. (See [9, 21]). Let X be a nonempty set, $K \geq 1$ be a real number and E a real Banach space with cone P. Suppose that the mapping $D : X \times X \to E$ satisfies
$(cd1)$ $\theta \preceq D(x, y)$ for all $x, y \in X$ and $D(x, y) = \theta$ if and only if $x = y$;
$(cd2)$ $D(x, y) = D(y, x)$ for all $x, y \in X$;
$(cd3)$ $D(x, z) \preceq K(D(x, y) + D(y, z))$ for all $x, y, z \in X$.
(X, D, K) is called cone metric type space. Obviously, for $K = 1$, cone metric type space is a cone metric space.

Example 1.7. (See [9]). Let $B = \{e_i | i = 1, \cdots, n\}$ be orthonormal basis of $\mathbf{R}^n$ with inner product $(.,.)$ and $p > 0$. Define

$$X_p = \Big\{[x] | x : [0, 1] \to \mathbf{R}^n, \int_0^1 |(x(t), e_j)|^p dt \in \mathbf{R}, j = 1, 2, \cdots, n\Big\},$$

where $[x]$ represents class of element x with respect to equivalence relation of functions equal almost everywhere. Let $E = \mathbf{R}^n$ and

$$P_B = \big\{y \in \mathbf{R}^n | (y, e_i) \geq 0, i = 1, 2, \cdots, n\big\}$$

be a solid cone. Define $d : X_p \times X_p \to P_B \subset \mathbf{R}^n$ by

$$d(f, g) = \sum_{i=1}^{n} e_i \int_0^1 |((f - g)(t), e_i)|^p dt, \qquad f, g \in X_p.$$

Then (X_p, d, K) is cone metric type space with $K = 2^{p-1}$.

Similarly, we define convergence in cone metric type spaces.

Definition 1.8. (See [9, 21]). Let (X, D, K) be a cone metric type space, $\{x_n\}$ a sequence in X and $x \in X$.
(i) $\{x_n\}$ converges to x if for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} x_n = x$.
(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exist $n_0 \in \mathbf{N}$ such that $d(x_n, x_m) \ll c$ for all $m, n > n_0$.

Lemma 1.9. (*See* [9, 21]). *Let (X, D, K) be a cone metric type space over ordered real Banach space E. Then the following properties are often used, particularly when dealing with cone metric type spaces in which the cone need not be normal.*
(P_1) *If $u \preceq v$ and $v \ll w$, then $u \ll w$.*
(P_2) *If $\theta \preceq u \ll c$ for each $c \in intP$, then $u = \theta$.*
(P_3) *If $u \preceq \lambda u$ where $u \in P$ and $0 \leq \lambda < 1$, then $u = \theta$.*
(P_4) *Let $x_n \to \theta$ in E and $\theta \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.*

2. MAIN RESULTS

At the first, we define the concept of the coupled fixed point in a cone metric type space. Then, we prove some fixed point theorems as generalization of Sabetghadam et al.'s works in [26] and Bhaskar and Lakshmikantham's results in [5].

Definition 2.1. Let (X, D, K) be a cone metric type space with constant $K \geq 1$. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Note that if (x, y) is a coupled fixed point of F then (y, x) is coupled fixed point of F too.

Theorem 2.2. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) \preceq \alpha D(x, x^*) + \beta D(y, y^*), \tag{2.1}$$

where α, β are nonnegative constants with $\alpha + \beta < 1/K$. Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set

$$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), \cdots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n).$$

From (2.1), we have

$$D(x_n, x_{n+1}) \preceq \alpha D(x_{n-1}, x_n) + \beta D(y_{n-1}, y_n), \tag{2.2}$$

and

$$D(y_n, y_{n+1}) \preceq \alpha D(y_{n-1}, y_n) + \beta D(x_{n-1}, x_n). \tag{2.3}$$

Let $D_n = D(x_n, x_{n+1}) + D(y_n, y_{n+1})$. From (2.2) and (2.3), we get

$$\begin{aligned} D_n &\preceq (\alpha + \beta)\big(D(x_{n-1}, x_n) + D(y_{n-1}, y_n)\big) \\ &= \lambda D_{n-1}, \end{aligned}$$

where $\lambda = \alpha + \beta < 1/K$. Thus, for all n,

$$\theta \preceq D_n \preceq \lambda D_{n-1} \preceq \lambda^2 D_{n-2} \preceq \cdots \preceq \lambda^n D_0. \tag{2.4}$$

If $D_0 = \theta$ then (x_0, y_0) is a coupled fixed point of F. Now, let $D_0 > \theta$. If $m > n$, we have

$$\begin{aligned} D(x_n, x_m) &\preceq K[D(x_n, x_{n+1}) + D(x_{n+1}, x_m)] \\ &\preceq KD(x_n, x_{n+1}) + K^2[D(x_{n+1}, x_{n+2}) + D(x_{n+2}, x_m)] \\ &\preceq \cdots \preceq KD(x_n, x_{n+1}) + K^2 D(x_{n+1}, x_{n+2}) + \cdots \\ &\quad + K^{m-n-1} D(x_{m-2}, x_{m-1}) + K^{m-n} D(x_{m-1}, x_m), \end{aligned} \tag{2.5}$$

and similarly,

$$\begin{aligned} D(y_n, y_m) &\preceq KD(y_n, y_{n+1}) + K^2 D(y_{n+1}, y_{n+2}) + \cdots \\ &\quad + K^{m-n-1} D(y_{m-2}, y_{m-1}) + K^{m-n} D(y_{m-1}, y_m). \end{aligned} \tag{2.6}$$

Adding up (2.5) and (2.6) and using (2.4). Since $\lambda < 1/K$, we have

$$\begin{aligned} D(x_n, x_m) + D(y_n, y_m) &\preceq KD_n + K^2 D_{n+1} + \cdots + K^{m-n} D_{m-1} \\ &\preceq [K\lambda^n + K^2\lambda^{n+1} + \cdots + K^{m-n}\lambda^{m-1}]D_0 \\ &\preceq \frac{K\lambda^n}{1 - K\lambda} D_0 \to \theta \quad as \quad n \to \infty. \end{aligned}$$

Now, by (P_1) and (P_4), it follows that for every $c \in intP$ there exist positive integer N such that $D(x_n, x_m) + D(y_n, y_m) \ll c$ for every $m > n > N$, so $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is a complete cone metric type space, there exist $x', y' \in X$ such that $x_n \to x'$ and $y_n \to y'$ as $n \to \infty$. Now, we prove that $F(x', y') = x'$ and $F(y', x') = y'$. From $(cd3)$ and (2.1), we have

$$\begin{aligned} D(F(x', y'), x') &\preceq K[D(F(x', y'), x_{n+1}) + D(x_{n+1}, x')] \\ &\preceq K\alpha D(x', x_n) + K\beta D(y', y_n) + KD(x_{n+1}, x'). \end{aligned}$$

Since $x_n \to x'$ and $y_n \to y'$, by using Lemma 1.9 we have $D(F(x', y'), x') = \theta$; that is, $F(x', y') = x'$. Similarly, we can get $D(F(y', x'), y') = \theta$; that is, $F(y', x') = y'$. Therefore, (x', y') is a coupled fixed point of F. Now, if (x'', y'') is another coupled fixed point of F, then

$$D(x', x'') + D(y', y'') \preceq \lambda\big(D(x', x'') + D(y', y'')\big). \tag{2.7}$$

Since $\lambda = \alpha+\beta < \frac{1}{K}$ and $K \geq 1$, (2.7) and (P_2) imply that $D(x', x'')+D(y', y'') = \theta$. Thus, $(x', y') = (x'', y'')$. This completes the proof. ☐

Corollary 2.3. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) \preceq \frac{\gamma}{2}[D(x, x^*) + D(y, y^*)], \tag{2.8}$$

where $\gamma \in [0, \frac{1}{K})$ is a constant. Then F has a unique coupled fixed point.

Proof. Corollary 2.3 follows from Theorem 2.2 by setting $\alpha = \beta = \frac{\gamma}{2}$. ☐

Theorem 2.4. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) \preceq \alpha D(F(x, y), x) + \beta D(F(x^*, y^*), x^*), \tag{2.9}$$

where α, β are nonnegative constants with $K\alpha + \beta < 1$. Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set

$$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), \cdots , x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n).$$

From (2.9), we have

$$\begin{aligned} D(x_n, x_{n+1}) &\preceq \alpha D(F(x_{n-1}, y_{n-1}), x_{n-1}) + \beta D(F(x_n, y_n), x_n) \\ &= \alpha D(x_n, x_{n-1}) + \beta D(x_{n+1}, x_n) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} D(y_n, y_{n+1}) &\preceq \alpha D(F(y_{n-1}, x_{n-1}), y_{n-1}) + \beta D(F(y_n, x_n), y_n) \\ &= \alpha D(y_n, y_{n-1}) + \beta D(y_{n+1}, y_n). \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned} D(x_n, x_{n+1}) &\preceq \lambda D(x_{n-1}, x_n), \\ D(y_n, y_{n+1}) &\preceq \lambda D(y_{n-1}, y_n), \end{aligned}$$

where $\lambda = \alpha/(1 - \beta) < 1/K$. By the analogous arguments as in Theorem 2.2 we conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is a complete cone metric type space, there exist $x', y' \in X$ such that $x_n \to x'$ and $y_n \to y'$ as $n \to \infty$. Now, we prove that $F(x', y') = x'$ and $F(y', x') = y'$. From $(cd3)$ and (2.9), we have

$$\begin{aligned} D(F(x', y'), x') &\preceq K[D(F(x', y'), x_{n+1}) + D(x_{n+1}, x')] \\ &\preceq K\alpha D(F(x', y'), x') + K\beta D(F(x_n, y_n), x_n) + KD(x_{n+1}, x'). \end{aligned}$$

Since $x_n \to x'$ and $y_n \to y'$, by using Lemma 1.9 we have $D(F(x', y'), x') = \theta$; that is, $F(x', y') = x'$. Similarly, we can get $D(F(y', x'), y') = \theta$; that is, $F(y', x') = y'$.

Therefore, (x', y') is a coupled fixed point of F. Now, if (x'', y'') is another coupled fixed point of F, then

$$D(x', x'') \preceq \alpha D(F(x', y'), x') + \beta D(F(x'', y''), x'').$$

Therefore, $D(x', x'') = \theta$; that is $x' = x''$. Similarly, we have $y' = y''$. Thus $(x', y') = (x'', y'')$. This completes the proof. □

Corollary 2.5. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) \preceq \frac{\gamma}{2}[D(F(x, y), x) + D(F(x^*, y^*), x^*)], \tag{2.12}$$

where $\gamma \in [0, \frac{2}{K+1})$ is a constant. Then F has a unique coupled fixed point.

Proof. Similar to Corollary 2.3, Corollary 2.5 follows from Theorem 2.4 by setting $\alpha = \beta = \frac{\gamma}{2}$. □

Theorem 2.6. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) \preceq \alpha D(F(x, y), x^*) + \beta D(F(x^*, y^*), x), \tag{2.13}$$

where α, β are nonnegative constants with $\alpha + \beta < 2/(K(K+1))$. Then F has a unique coupled fixed point.

Proof. Let $x_0, y_0 \in X$ and set

$$x_1 = F(x_0, y_0), y_1 = F(y_0, x_0), \cdots, x_{n+1} = F(x_n, y_n), y_{n+1} = F(y_n, x_n).$$

From (2.13), we have

$$\begin{aligned} D(x_n, x_{n+1}) &\preceq \alpha D(F(x_{n-1}, y_{n-1}), x_n) + \beta D(F(x_n, y_n), x_{n-1}) \\ &\preceq K\beta[D(x_n, x_{n-1}) + D(x_{n+1}, x_n)] \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} D(x_{n+1}, x_n) &\preceq \alpha D(F(x_n, y_n), x_{n-1}) + \beta D(F(x_{n-1}, y_{n-1}), x_n) \\ &\preceq K\alpha[D(x_n, x_{n-1}) + D(x_{n+1}, x_n)]. \end{aligned} \tag{2.15}$$

Adding up (2.14) and (2.15), we have

$$D(x_n, x_{n+1}) \preceq \lambda D(x_{n-1}, x_n),$$

where $\lambda = \frac{K(\alpha+\beta)}{2-K(\alpha+\beta)} < \frac{1}{K}$.
Similarly,

$$D(y_n, y_{n+1}) \preceq \lambda D(y_{n-1}, y_n),$$

where $\lambda = \frac{K(\alpha+\beta)}{2-K(\alpha+\beta)} < \frac{1}{K}$. By the same arguments as in Theorem 2.2 we conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is a complete cone metric type space, there exist $x', y' \in X$ such that $x_n \to x'$ and $y_n \to y'$ as $n \to \infty$. Now, we prove that $F(x', y') = x'$ and $F(y', x') = y'$. From $(cd3)$ and (2.13), we have

$$\begin{aligned} D(F(x', y'), x') &\preceq K[D(F(x', y'), x_{n+1}) + D(x_{n+1}, x')] \\ &\preceq K\alpha D(F(x', y'), x_n) + K\beta D(F(x_n, y_n), x') + KD(x_{n+1}, x'). \end{aligned}$$

Since $x_n \to x'$ and $y_n \to y'$, by using Lemma 1.9 we have $D(F(x', y'), x') = \theta$; that is, $F(x', y') = x'$. Similarly, we can get $D(F(y', x'), y') = \theta$; that is, $F(y', x') = y'$. Therefore, (x', y') is a coupled fixed point of F. By the same arguments as in Theorem 2.2 we conclude that (x', y') is unique. □

Corollary 2.7. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x,y), F(x^*,y^*)) \preceq \frac{\gamma}{2}[D(F(x,y),x^*) + D(F(x^*,y^*),x)], \tag{2.16}$$

where $\gamma \in [0, 2/(K^2+K))$ is a constant. Then F has a unique coupled fixed point.

Remark 2.8.
(i) The Theorems 2.2, 2.4 and 2.6 generalized some fixed point theorems of cone metric spaces of Sabeghadam et al.'s works in [26] by considering cone metric type spaces.
(ii) Choosing $K = 1$ from the Corollaries 2.3, 2.5 and 2.7 we get the Theorems 2.1, 2.2 and 2.4 from Bhaskar and Lakshmikantham's results in a cone metric space.

Example 2.9. Let $E = \mathbf{R}$, $P = [0, \infty)$, $X = [0, 1]$ and $D : X \times X \to [0, \infty)$ be defined by $D(x, y) = |x - y|^2$. Then (X, D) is a cone metric type space, but it is not a cone metric space since the triangle inequality is not satisfied. Starting with Minkowski inequality, we get $|x - z|^2 \leq 2(|x - y|^2 + |y - z|^2)$. Here $K = 2$.
Define the mapping $F : X \times X \to X$ by $F(x, y) = (x + y)/4$. Therefore, F satisfies the contractive condition (2.8) for $\gamma = 1/4 \in [0, 1/K)$ with $K = 2 \geq 1$; that is,

$$D(F(x,y), F(x^*,y^*)) \preceq \frac{1}{8}[D(x,x^*) + D(y,y^*)].$$

According to Corollary 2.3, F has a unique coupled fixed point. $(0, 0)$ is a unique coupled fixed point of F

Remark 2.10. Similar to previous example, one can get many examples of other coupled fixed point theorems in cone metric type spaces.

3. GENERAL APPROACH

We start with following Lemma.

Lemma 3.1. (1) *Suppose that (X, D, K) is a cone metric type space with $K \geq 1$. Then, (X^2, D_1, K) is a cone metric type space with*

$$D_1((x,y),(u,v)) = D(x,u) + D(y,v). \tag{3.1}$$

Further, (X, D, K) is complete if and only if (X^2, D_1, K).
(2) *Mapping $F : X^2 \to X$ has a coupled fixed point if and only if mapping $T_F : X^2 \to X^2$ defined by $T_F(x, y) = (F(x, y), F(y, x))$ has a fixed point in X^2.*

Proof. (1) Similar to cone metric version, one can check $(cd1)$ and $(cd2)$ conditions. Thus, we only prove $(cd3)$ condition for (X^2, D_1, K). Since (X, D, K) is a cone metric type space, we have

$$D(x,u) \preceq K(D(x,z) + D(z,u)) \tag{3.2}$$

for all $x, z, u \in X$ and

$$D(y,v) \preceq K(D(y,w) + D(w,v)) \tag{3.3}$$

for all $y, v, w \in X$. Adding up (3.2) and (3.3), we get

$$\begin{aligned} D_1((x,y),(u,v)) &= D(x,u) + D(y,v) \\ &\preceq K(D(x,z) + D(z,u)) + K(D(y,w) + D(w,v)) \\ &= K(D(x,z) + D(y,w)) + K(D(z,u) + D(w,v)) \\ &= K\big[D_1((x,y),(z,w)) + D_1((z,w),(u,v))\big]. \end{aligned}$$

Thus, (X^2, D_1, K) is a cone metric type space. The completeness proof is easy and is left to the reader.
(2) Let (x, y) be a coupled fixed point of F. In this case, $F(x, y) = x$ and $F(y, x) = y$. Thus,

$$T_F(x, y) = (F(x, y), F(y, x)) = (x, y).$$

Therefore, $(x, y) \in X^2$ is a fixed point of T_F. Conversely, suppose that $(x, y) \in X^2$ is a fixed point of T_F, then

$$T_F(x, y) = (x, y).$$

Consequently, $F(x, y) = x$ and $F(y, x) = y$. □

Now, we prove a general version of Theorem 2.2.

Theorem 3.2. *Let (X, D, K) be a complete cone metric type space with constant $K \geq 1$ and P a solid cone. Suppose $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, x^*, y^* \in X$:*

$$D(F(x, y), F(x^*, y^*)) + D(F(y, x), F(y^*, x^*)) \preceq \lambda[D(x, x^*) + D(y, y^*)], \tag{3.4}$$

where λ is a nonnegative constant with $\lambda < 1/K$. Then F has a unique coupled fixed point.

Proof. According to (3.1) and Lemma 3.1(2), the contractive condition (3.4) for all $Y = (x, y), V = (u, v) \in X^2$ become

$$D_1(T_F(Y), T_F(V)) \preceq \lambda D_1(Y, V).$$

Since $\lambda < 1/K$, the proof further follows by ([14], Theorem 3.3). □

Remark 3.3. Now, we can get Theorem 2.2 such as the result of Theorem 3.2. Also, one can prove some other theorems for general contractive version and get Theorems 2.4 and 2.6.

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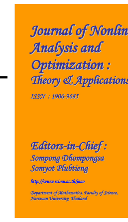

ON CHARACTERIZATION OF MULTIWAVELET PACKETS ASSOCIATED WITH A DILATION MATRIX

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ABSTRACT. We provide a complete characterization of multiwavelet packets associated with dilation matrix based on results on affine and quasi-affine frames. Moreover, these characterizations are valid without any decay assumptions on the generators of the system.

KEYWORDS : Multiresolution analysis; Multiwavelet packet; Dilation matrix; Frame; Bessel's sequence; Fourier transform.
AMS Subject Classification: 42C40 42C15 65T60.

1. INTRODUCTION

The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application. Wavelet packets, due to their nice characteristics have been widely applied to signal processing, coding theory, image compression, fractal theory and solving integral equations and so on. Coifman *et al.*[8] firstly introduced the notion of univariate wavelet packets. Chui and Li [7] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. Shen [18] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the p-wavelet packets and p-wavelet frame packets on a half-line $\mathbb{R}^+$ [13, 14, 16], higher dimensional wavelet packets with arbitrary dilation matrix [9], the orthogonal version of vector-valued wavelet packets [6] and the M-band framelet packets [17].

On the other hand, multiwavelets are natural extension and generalization of traditional wavelets. They have received considerable attention from the wavelet research communities both in the theory as well as in applications. They can be seen as vector valued-wavelets that satisfy conditions in which matrices are involved

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rather than scalars as in the wavelet case. Multiwavelets can own symmetry, orthogonality, short support and high order vanishing moments, however traditional wavelets can not possess all these properties at the same time (see [4, 10]). Yang and Cheng [20] generalized the concept of wavelet packets to the case of multiwavelet packets associated with a dilation factor a which were more flexible in applications. Subsequently, Behera [1] extended the results of Yang and Cheng to the multivariate multiwavelet packets associated with a dilation matrix A. He proved lemmas on the so-called splitting trick and several theorems concerning the Fourier transform of the multiwavelet packets and the construction of multiwavelet packets to show that their translates form an orthonormal basis of $L^2(\mathbb{R}^d)$. Recently, Sun and Li [19] have given the construction and properties of generalized orthogonal multiwavelet packets based on the results discussed in [20].

As far as the characterization of multiwavelets is concerned, Calogero studied the characterization of all multiwavelets associated with general expanding maps of $\mathbb{R}^n$ in [5]. The Calogero's work was extended by Bownik [2], taking into consideration the dilation matrices which preserves the standard lattice $\mathbb{Z}^n$ in terms of affine systems. In the same year, another characterization of multiwavelets was given by Rzeszotnik [12] for expanding dilations that preserves the lattice $\mathbb{Z}^n$. However, Bownik [3] has presented a new approach to characterize all orthonormal multiwavelets by means of basic equations in the Fourier domain. This characterization was obtained by using the results about shift invariant systems and quasi-affine systems in [11].

The characterization of multiwavelet packets associated with the general dilation matrix A has been given by Shah and Ahmad in [15] by following dual Gramian approach of Bownik [2]. In the present paper, we study the characterization of multiwavelet packets associated with expansive dilation matrices in terms of the two simple equations in the Fourier domain based on results on affine and quasi-affine frames.

In order to make the paper self-contained, we state some basic preliminaries, notations and definitions including the multiresoltion analysis associated with a dilation matrix A and corresponding multiwavelet packets in Section 2. In Section 3, we establish the characterization of multiwavelet packets associated with a dilation matrix A based on results on affine and quasi-affine frames.

2. NOTATIONS AND PRELIMINARIES

Throughout, this paper, we use the following notations. Let $\mathbb{R}$ and $\mathbb{C}$ be all real and complex numbers, respectively. $\mathbb{Z}$ and $\mathbb{Z}^+$ denote all integers and all nonnegative integers, respectively. $\mathbb{Z}^d$ and $\mathbb{R}^d$ denote the set of all d-tuples integers and d-tuples of reals, respectively. Assume that we have a lattice Γ ($\Gamma = P\mathbb{Z}^d$ for some non-degenerate $d \times d$ matrix P) of $\mathbb{R}^d$. Let A denotes a $d \times d$ dilation matrix, whose determinant is $a(a \in \mathbb{Z}, a \geq 2)$. A $d \times d$ matrix A is said to be a dilation matrix for $\mathbb{R}^d$ if $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$ and all eigenvalues λ of A satisfy $|\lambda| > 1$. Let $a = |\det A|$, $B =$ transpose of A and, if A is expanding, so is B. Let Γ^* be the dual lattice; that is,

$$\Gamma^* = \Big\{\gamma' \in \mathbb{R}^d : \forall\, \gamma \in \Gamma\ \langle \gamma, \gamma' \rangle \in \mathbb{Z}\Big\} = (P^t)^{-1}\mathbb{Z}^d.$$

By taking the transpose of $P^{-1}AP$ we observe that $B = A^t$ is a dilation preserving the dual lattice: $B\Gamma^* \subset \Gamma^*$ and let $\mathbb{S} = \Gamma^* \setminus B\Gamma^*$.

We recall the notion of higher dimensional multiresolution analysis associated with multiplicity L and orthogonal multiwavelets of $L^2(\mathbb{R}^d)$.

Definition 2.1. A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2\big(\mathbb{R}^d\big)$ is called a *multiresolution analysis* (MRA) of $L^2\big(\mathbb{R}^d\big)$ of multiplicity L associated with the dilation matrix A if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2\big(\mathbb{R}^d\big)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iii) $f \in V_j$ if and only if $f(A\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(iv) there exist L-functions $\varphi_\ell \in V_0$, such that the system of functions $\{\varphi_\ell(x-k)\}_{\ell=1,k\in\mathbb{Z}^d}^{L}$, forms an orthonormal basis for subspace V_0.

The L-functions whose existence is asserted in (iv) are called *scaling functions* of the given MRA. Given a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$, we define another sequence $\{W_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2\big(\mathbb{R}^d\big)$ by $W_j = V_{j+1} \ominus V_j,\ j \in \mathbb{Z}$. These subspaces inherit the scaling property of $\{V_j\}$, namely

$$f \in W_j \quad \text{if and only if} \quad f(A\cdot) \in W_{j+1}. \tag{2.1}$$

Further, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \Big(\bigoplus_{j\geq 0} W_j\Big). \tag{2.2}$$

A set of functions $\{\psi_\ell^r : 1 \leq \ell \leq L, 1 \leq r \leq a-1\}$ in $L^2(\mathbb{R}^d)$ is said to be a set of *basic multiwavelets* associated with the MRA of multiplicity L if the collection

$$\Big\{\psi_\ell^r(.-k) : 1 \leq r \leq a-1, 1 \leq \ell \leq L,\ k \in \mathbb{Z}^d\Big\}$$

forms an orthonormal basis for W_0. Now, in view of (2.1) and (2.2), it is clear that if $\{\psi_\ell^r : 1 \leq \ell \leq L, 1 \leq r \leq a-1\}$ is a basic set of multiwavelets, then

$$\Big\{|\det A|^{j/2}\psi_\ell^r(A^j.-k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^d, 1 \leq \ell \leq L, 1 \leq r \leq a-1\Big\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$ (see [1, 4]).

For any $n \in \mathbb{Z}^+$, we define the *basic multiwavelet packets* $\omega_\ell^n;\ 1 \leq \ell \leq L$ recursively as follows. We denote $\omega_\ell^0 = \varphi_\ell,\ 1 \leq \ell \leq L$, the scaling functions and $\omega_\ell^r = \psi_\ell^r, r \in \mathbb{Z}^+, 1 \leq \ell \leq L$ as the possible candidates for basic multiwavelets. Then, define

$$\omega_\ell^{s+ar}(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{\ell jk}^s\, a^{1/2}\,\omega_\ell^r(Ax-k),\ 1 \leq \ell \leq L, 0 \leq s \leq a-1 \tag{2.3}$$

where $\Big(h_{\ell jk}^s\Big)$ is a unitary matrix (see [1]).

Taking Fourier transform on both sides of (2.3), we obtain

$$\left(\omega_\ell^{s+ar}\right)^\wedge(\xi) = \sum_{j=1}^{L} h_{\ell j}^s(B^{-1}\xi)\left(\omega_\ell^r\right)^\wedge(B^{-1}\xi). \tag{2.4}$$

Note that (2.3) defines ω_ℓ^n for every non-negative integer n and every ℓ such that $1 \le \ell \le L$. The set of functions $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \le \ell \le L\}$ as defined above are called the *basic multiwavelet packets* corresponding to the MRA $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ of multiplicity L associated with matrix dilation A.

Definition 2.2. Let $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \le \ell \le L\}$ be the basic multiwavelet packets. The collection

$$\mathcal{P} = \left\{ |\det A|^{j/2}\omega_\ell^n(A.-\ k) : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^d \right\}$$

is called the *general multiwavelet packets* associated with MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L over matrix dilation A.

Corresponding to some orthonormal scaling vector $\Phi = \omega_\ell^0$, the family of multiwavelet packets ω_ℓ^n defines a family of subspaces of $L^2(\mathbb{R}^d)$ as follows:

$$U_j^n = \overline{\text{span}}\left\{ |\det A|^{j/2}\omega_\ell^n(A^j x - k) : k \in \mathbb{Z}^d, 1 \le \ell \le L \right\};\ j \in \mathbb{Z},\ n \in \mathbb{Z}^+. \tag{2.5}$$

Observe that

$$U_j^0 = V_j, \quad U_j^1 = W_j = \bigoplus_{r=1}^{a-1} U_j^r, \quad j \in \mathbb{Z}$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$, can be written as

$$U_{j+1}^0 = \bigoplus_{r=0}^{a-1} U_j^r. \tag{2.6}$$

A generalization of this result for other values of $n = 1, 2, \ldots$ can be written as

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_j^{an+r}, \quad j \in \mathbb{Z}. \tag{2.7}$$

The following proposition is proved in [1].

Proposition 2.3. If $j \ge 0$, then

$$W_j = \bigoplus_{r=0}^{a-1} U_j^r = \bigoplus_{r=a}^{a^2-1} U_{j-1}^r = \cdots = \bigoplus_{r=a^t}^{a^{t+1}-1} U_{j-t}^r = \bigoplus_{r=a^j}^{a^{j+1}-1} U_0^r$$

where U_j^n is defined in (2.5). Using this decomposition, we get the multiwavelet packets decomposition of subspaces $W_j,\ j \ge 0$.

Let $\{\omega_\ell^n : n \ge 0, 1 \le \ell \le L\}$ be a family of functions in $L^2(\mathbb{R}^d)$. Then, the *affine system* generated by ω_ℓ^n and associated with (A, Γ) is the collection

$$\mathcal{F}(\omega_\ell^n) = \Big\{\omega_{\ell,j,k}^n : j \in \mathbb{Z},\ k \in \Gamma, 1 \le \ell \le L, a^j \le n < a^{j+1}\Big\}, \tag{2.8}$$

where $\omega_{\ell,j,k}^n(x) = D_{A^j} T_k\, \omega_\ell^n(x) = |\det A|^{j/2} \omega_\ell^n(A^j x - k)$. The *quasi-affine system* generated by ω_ℓ^n is

$$\mathcal{F}^q(\omega_\ell^n) = \Big\{\tilde{\omega}_{\ell,j,k}^n : j \in \mathbb{Z}, k \in \Gamma,\ 1 \le \ell \le L, a^j \le n < a^{j+1}\Big\}, \tag{2.9}$$

where

$$\tilde{\omega}_{\ell,j,k}^n(x) = \begin{cases} D_{A^j} T_k\, \omega_\ell^n(x) = |\det A|^{j/2} \omega_\ell^n(A^j x - k), & j \ge 0,\ k \in \Gamma, \\ |\det A|^{j/2} T_k D_{A^j} \omega_\ell^n(x) = |\det A|^j \omega_\ell^n\big(A^j(x-k)\big), & j < 0,\ k \in \Gamma, \end{cases}$$

where $\tau_y f(x) = f(x - y)$ is translation by a vector $y \in \mathbb{R}^d$ and $D_{A^j} f(x) = |\det A|^{j/2} f(Ax)$ is dilation by the matrix A. Since A is a dilation matrix, $A^t = B$ so there exist constants $\lambda > 1$ and $c > 0$ such that

$$\left|B^j \xi\right| > c\lambda^j |\xi|, \quad \left|B^{-j}\xi\right| < 1/c\lambda^{-j}|\xi| \text{ for } j > 0. \tag{2.10}$$

The following two lemma's are proved in [2].

Lemma 2.4. Suppose $b > 0, g \in L^\infty(\mathbb{R}^d)$, $\operatorname{supp} g \subset \left\{\xi \in \mathbb{R}^d : |\xi| > b\right\}$, and $\operatorname{supp} g \subset B^{j_0} I_d + \xi_0$ for some $\xi_0 \in \mathbb{R}^d$ and $j_0 \in \mathbb{Z}$, then

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\det A|^i \Big| g(B^j\xi) g\big(B^j(\xi{+}k)\big)\Big| \le 2^d |\det A|^{j_0} M\Big((b{+}\delta)/b\Big) \|g\|_\infty^2 I_\Upsilon(\xi), a.e.\ \xi \in \mathbb{R}^d$$

where $\delta = \text{diam}(B^{j_0} I_d), \Upsilon = \bigcup_{j<J_0} B^{-j}(B^{j_0} I_d + \xi_0)$ and $I_d = (-1/2, 1/2)^d$.

Lemma 2.5. Suppose $F, G \in L^2(\mathbb{R}^d)$, and $\operatorname{supp} F,\ \operatorname{supp} G$ are bounded. Then

$$\sum_{k\in\mathbb{Z}^d} \hat{F}(k)\overline{\hat{G}(k)} = \int_{\mathbb{R}^d} \left(\sum_{\ell\in\mathbb{Z}^d} F(\xi+\ell)\right) \overline{G(\xi)}\, d\xi.$$

Definition 2.6. Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{f_k\}_{k=1}^\infty$ in $\mathbb{H}$ is called a *frame* if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\big\|f\big\|^2 \le \sum_{k=1}^{\infty} \big|\langle f, f_\alpha\rangle\big|^2 \le B\big\|f\big\|^2, \quad \text{for all } f \in \mathbb{H}. \tag{2.11}$$

The largest constant A and the smallest constant B satisfying (2.11) are called the *upper* and the *lower frame bound*, respectively. The sequence $\{f_k\}_{k=1}^\infty$ is called a *Bessel sequence* in $\mathbb{H}$ if only the right-hand side inequality in (2.11) holds. The sequence $\{f_k\}_{k=1}^\infty$ is called a *tight frame* for $\mathbb{H}$ if the upper frame bound A and the lower frame bound B coincide. A frame is called *Parseval frame* or *normalized tight frame* if $A = B = 1$ and in this case, every function $f \in \mathbb{H}$ can be written as

$$\sum_{k=1}^{\infty} \big|\langle f, f_\alpha\rangle\big|^2 = \big\|f\big\|^2. \tag{2.12}$$

The following theorem gives us an elementary characterization of tight frames.

Theorem 2.7. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in a Hilbert space $\mathbb{H}$ such that

(i) $\displaystyle\sum_{k=1}^{\infty} \left|\langle f, f_k\rangle\right|^2 = \|f\|^2, \quad \text{for all } f \in \mathbb{H}$

(ii) $\|f_k\| \geq 1, \text{ for } k \in \mathbb{Z}^+.$

Then, the sequence $\{f_k\}_{k=1}^{\infty}$ forms a Parseval's frame for $\mathbb{H}$.

We will also consider the set $\mathcal{D}$ as a dense subset of $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{D} = \left\{ f \in L^2(\mathbb{R}^d) : \hat{f} \in L^\infty(\mathbb{R}^d), \text{ supp } \hat{f} \text{ for some compact } K \subset \mathbb{R}^d \setminus \{0\} \right\}.$$

3. CHARACTERIZATION OF MULTIWAVELET PACKETS

In this section, we prove our main results concerning the characterization of multiwavelet packets associated with a dilation matrix A by means of the Fourier transform. We begin this section with the lemma which gives necessary condition for the system $\mathcal{F}(\omega_\ell^n)$ given by (2.8) to be a Bessel family.

Lemma 3.1. *Let $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \leq \ell \leq L\}$ be the basic multiwavelet packets associated with the scaling functions φ_ℓ. Then, for $f \in \mathcal{D}$ and $m \in \mathbb{Z}$, we have*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \leq m} \sum_{k \in \mathbb{Z}^d} \left|\langle f, \omega_{\ell,j,k}^n\rangle\right|^2 < \infty.$$

Moreover,

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \left|\omega_\ell^n\left(B^j\xi\right)\right|^2, \quad \textit{is locally integrable on } \mathbb{R}^d \setminus \{0\} \tag{3.1}$$

if and only if

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j,k \in \mathbb{Z}^d} \left|\langle f, \omega_{\ell,j,k}^n\rangle\right|^2 < \infty \quad \textit{for all } f \in \mathcal{D}. \tag{3.2}$$

Proof. Since $\hat{\omega}_{\ell,j,k}^n(\xi) = |\det A|^{-j/2} \omega_\ell^n(B^{-j}\xi) e^{-2\pi i \langle k, B^{-j}\xi\rangle}$. Therefore, by applying Parseval's formula, we obtain

$$\begin{aligned}
\langle f, \omega_{\ell,j,k}^n\rangle = \langle \hat{f}, \hat{\omega}_{\ell,j,k}^n\rangle &= |\det A|^{-j/2} \int_{\mathbb{R}^d} \hat{f}(\xi)\, \overline{\hat{\omega}_\ell^n\left(B^{-j}\xi\right)}\, e^{2\pi i \langle k, B^{-j}\xi\rangle}\, d\xi \\
&= |\det A|^{-j/2} \int_{\mathbb{R}^d} \hat{f}\left(B^j\xi\right) \overline{\hat{\omega}_\ell^n(\xi)}\, e^{2\pi i \langle k, \xi\rangle} |\det A|^j\, d\xi \\
&= |\det A|^{j/2} \int_{\mathbb{R}^d} \hat{f}\left(B^j\xi\right) \overline{\hat{\omega}_\ell^n(\xi)}\, e^{2\pi i \langle k, \xi\rangle}\, d\xi.
\end{aligned} \tag{3.3}$$

With the help of (3.3), we can write the series as

$$I = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\le m} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \omega^n_{\ell,j,k}\rangle\right|^2$$

$$= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\le m} \sum_{k\in\mathbb{Z}^d} |\det A|^j \left| \int_{\mathbb{R}^d} \hat f\big(B^j\xi\big)\, \overline{\hat\omega^n_\ell(\xi)}\, e^{2\pi i\langle k,\xi\rangle}\, d\xi \right|^2. \tag{3.4}$$

For any fixed $j\in\mathbb{Z}$, let $F(\xi)\equiv \hat f(B^j)\,\overline{\hat\omega^n_\ell(\xi)}$; then by Lemma 2.5 when $F=G$, we have

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^d} \left| \int_{\mathbb{R}^d} \hat f\big(B^j\xi\big)\, \overline{\hat\omega^n_\ell(\xi)}\, e^{2\pi i\langle k,\xi\rangle} d\xi \right|^2$$

$$= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \left\{ \int_{\mathbb{R}^d} \hat f\big(B^j\xi\big)\, \hat\omega^n_\ell(\xi) \left(\sum_{k\in\mathbb{Z}^d} \hat f\big(B^j(\xi+k)\big)\overline{\hat\omega^n_\ell(\xi+k)} \right) d\xi \right\}.$$

Hence

$$I = \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\le m} |\det A|^j \int_{\mathbb{R}^d} \left|\hat f\big(B^j\xi\big)\right|^2 \left|\hat\omega^n_\ell(\xi)\right|^2 d\xi + \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\le m} |\det A|^j$$

$$\times \int_{\mathbb{R}^d} \overline{\hat f\big(B^j\xi\big)}\, \hat\omega^n_\ell(\xi) \left\{ \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \hat f\big(B^j(\xi+k)\big)\, \overline{\hat\omega^n_\ell(\xi+k)} \right\} d\xi. \tag{3.5}$$

Note that for any $\ell=1,\dots,L$ and $n\in\mathbb{Z}^+$, we have

$$2\left|\hat\omega^n_\ell(\xi)\hat\omega^n_\ell(\xi+k)\right| \le \left|\hat\omega^n_\ell(\xi)\right|^2 + \left|\hat\omega^n_\ell(\xi+k)\right|^2.$$

Therefore, the second sum is absolutely convergent in $L^1(\mathbb{R}^d)$ and, thus absolutely summable for $a.e.\ \xi\in\mathbb{R}^d$ even if we extend the summation over all $j\in\mathbb{Z}$; i.e.,

$$\int_{\mathbb{R}^d} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\det A|^j \left|\hat f\big(B^j\xi\big)\, \hat\omega^n_\ell(\xi)\right| \left|\hat f\big(B^j(\xi+k)\big)\, \hat\omega^n_\ell(\xi+k)\right| d\xi$$

$$\le \frac{1}{2} \int_{\mathbb{R}^d} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\det A|^j \left[\left|\hat f\big(B^j\xi\big)\hat f\big(B^j(\xi+k)\big)\right| + \left|\hat f\big(B^j(\xi-k)\big)\hat f\big(B^j\xi\big)\right| \right]$$

$$= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \int_{\mathbb{R}^d} \left|\hat\omega^n_\ell(\xi)\right|^2 \left\{ \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\det A|^j \left|\hat f\big(B^j\xi\big)\hat f\big(B^j(\xi+k)\big)\right| \right\} d\xi$$

$$\leq C\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi<\infty, \tag{3.6}$$

where C is the constant appearing in Lemma 2.4 depending on the size and the location of supp $\hat{f}$. Furthermore, the first sum appearing in (3.5) can be estimated crudely by

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\leq m}|\det A|^j\int_{\mathbb{R}^d}\left|\hat{f}\big(B^j\xi\big)\right|^2\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi\leq\|\hat{f}\|_\infty^2\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\leq m}|\det A|^j\int_{\mathbb{R}^d}\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi$$

$$=\frac{|\det A|^{m+1}}{|\det A|-1}\|\hat{f}\|_\infty^2\|\omega_\ell^n\|^2. \tag{3.7}$$

In order to prove the second part of the theorem, we have

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j,k\in\mathbb{Z}^d}\left|\langle f,\omega_{\ell,j,k}^n\rangle\right|^2=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\det A|^j\int_{\mathbb{R}^d}\left|\hat{f}\big(B^j\xi\big)\right|^2\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi$$

$$+\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\det A|^j\int_{\mathbb{R}^d}\overline{\hat{f}\big(B^j\xi\big)}\hat{\omega}_\ell^n(\xi)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat{f}\big(B^j(\xi+k)\big)\overline{\hat{\omega}_\ell^n(\xi+k)}\right\}d\xi,$$

where the second expression in this decomposition is always finite by (3.6). Thus, the first implication follows from the fact that

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^d}|\det A|^j\int_{\mathbb{R}^d}\left|\hat{f}\big(B^j\xi\big)\right|^2\left|\hat{\omega}_\ell^n(\xi)\right|^2 d\xi=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\left|\hat{f}(\xi)\right|^2\left|\hat{\omega}_\ell^n\big(B^{-j}\xi\big)\right|^2 d\xi$$

$$\leq\|\hat{f}\|_\infty^2\int_{\mathrm{supp}\hat{f}}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}^d}\left|\hat{\omega}_\ell^n\big(B^{-j}\xi\big)\right|^2 d\xi<\infty,$$

where as the converse implication is simply the consequence of applying the above to $\hat{f}=\chi_K$ for any compact $K\subset\mathbb{R}^d\setminus\{0\}$, since we have equality (instead of inequality) in the above formula. □

Theorem 3.2. *Let* $\{\omega_\ell^n : n\in\mathbb{Z}^+, 1\leq\ell\leq L\}$ *and* $\{\tilde{\omega}_\ell^n : n\in\mathbb{Z}^+, 1\leq\ell\leq L\}$ *be the dual multiwavelet packets associated with the dilation matrix* A. *Then*

$$\lim_{m\to\infty}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\leq m}\sum_{k\in\mathbb{Z}^d}\langle f,\omega_{\ell,j,k}^n\rangle\langle\tilde{\omega}_{\ell,j,k}^n,f\rangle=\|f\|_2^2,\quad \textit{for all } f\in\mathcal{D} \tag{3.8}$$

if and only if

$$\lim_{m\to\infty}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\hat{\omega}_\ell^n\left(B^j\xi\right)\overline{\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)}=1,\quad \textit{weakly in } L^1(K),\ K\subset\mathbb{R}^d\setminus\{0\} \tag{3.9}$$

$$t_s(\xi)=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\overline{\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)}=0,\ a.e.\ \xi\in\mathbb{R}^d,\ s\in\mathbb{S}=\mathbb{Z}^d\setminus B\mathbb{Z}^d. \tag{3.10}$$

Proof. We first show that the series given by (3.8), (3.9) and (3.10) are all absolutely convergent. Since

$$2\left|\langle f,\omega_{\ell,j,k}^n\rangle\langle\tilde{\omega}_{\ell,j,k}^n,f\rangle\right|\leq\left|\langle f,\omega_{\ell,j,k}^n\rangle\right|^2+\left|\langle\tilde{\omega}_{\ell,j,k}^n,f\rangle\right|^2.$$

Therefore, the series in (3.8) is summable by Lemma 3.1. Moreover, by the polarization identity, condition (3.8) is equivalent to

$$\langle f,g\rangle=\lim_{m\to\infty}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\leq m}\sum_{k\in\mathbb{Z}^d}\langle f,\omega_{\ell,j,k}^n\rangle\langle\tilde{\omega}_{\ell,j,k}^n,g\rangle\quad\text{for all } f,g\in\mathcal{D}. \tag{3.11}$$

Thus, for $s\in\mathbb{R}^d$ and $\omega_\ell^n\in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\left|\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)\right|^2d\xi &= \int_{\mathbb{R}^d}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{-j\leq m}|\det A|^{-j}\left|\hat{\omega}_\ell^n\left(\xi+B^js\right)\right|^2d\xi\\ &= \frac{|\det A|^{m+1}}{|\det A|-1}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\left|\hat{\omega}_\ell^n(\xi)\right|^2d\xi<\infty.\end{aligned}$$

Therefore, we have

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\left|\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)\right|^2<\infty\quad\text{for } a.e.\ \xi. \tag{3.12}$$

Using the above when $s=0$ yields

$$2\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\left|\hat{\omega}_\ell^n\left(B^j\xi\right)\overline{\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)}\right|^2\leq\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq -m}\left|\hat{\omega}_\ell^n\left(B^j\xi\right)\right|^2+\left|\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\right|^2<\infty,\ a.e.\ \xi.$$

Similarly, implementation of (3.12) when $m=0$ implies

$$2\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\left|\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\overline{\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)}\right|^2\leq\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\left|\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\right|^2+\left|\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)\right|^2<\infty.$$

Next, we prove that (3.9) and (3.10) implies (3.8). To do so, let us suppose that $f, g \in \mathcal{D}$. Then, by equation (3.3), we have

$$\langle f, \omega^n_{\ell,j,k}\rangle\langle\tilde{\omega}^n_{\ell,j,k}, g\rangle = |\det A|^j \int_{\mathbb{R}^d} \hat{f}\big(B^j\xi\big)\,\overline{\hat{\omega}^n_\ell(\xi)}\,e^{2\pi i\langle k,\xi\rangle}d\xi \int_{\mathbb{R}^d} \overline{\hat{g}(B^j\xi)}\hat{\tilde{\omega}}^n_\ell(\xi)e^{-2\pi i\langle k,\xi\rangle}d\xi.$$

For any fixed $\ell = 1, \ldots, L$ and $j \in \mathbb{Z}$, let

$$F(\xi) \equiv \hat{f}(B^j\xi)\,\overline{\omega^n_\ell(\xi)}, \quad G(\xi) \equiv \hat{g}(B^j\xi)\,\overline{\tilde{\omega}^n_\ell(\xi)},\ n \in \mathbb{Z}^+.$$

Then, using the Lemma 2.5 and the above fact, we obtain

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}\langle f, \omega^n_{\ell,j,k}\rangle\langle\tilde{\omega}^n_{\ell,j,k}, g\rangle = \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\left\{\sum_{k\in\mathbb{Z}^d}\hat{f}\big(B^j(\xi+k)\big)\overline{\omega^n_\ell(\xi+k)}\right\} \times \overline{\hat{g}(B^j\xi)}\hat{\tilde{\omega}}^n_\ell(\xi)\,d\xi. \tag{3.13}$$

Hence

$$I = I(m) = \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}\sum_{k\in\mathbb{Z}^d}\langle f, \omega^n_{\ell,j,k}\rangle\langle\tilde{\omega}^n_{\ell,j,k}, g\rangle = I_1 + I_2 \tag{3.14}$$

where

$$I_1(m) = \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|detA|^j\int_{\mathbb{R}^d}\hat{f}\big(B^j\xi\big)\,\overline{\hat{g}(B^j\xi)}\,\overline{\hat{\omega}^n_\ell(\xi)}\,\hat{\tilde{\omega}}^n_\ell(\xi)\,d\xi$$

$$I_2(m) = \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|detA|^j\int_{\mathbb{R}^d}\overline{\hat{g}(B^j\xi)}\,\hat{\tilde{\omega}}^n_\ell(\xi)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat{f}\big(B^j(\xi+k)\big)\,\overline{\omega^n_\ell(\xi+k)}\right\}d\xi$$

by splitting the sum (3.13) into terms corresponding to $k = 0$ and $k \neq 0$. Moreover, we can interchange the summation and integration in I_1 and I_2, since for $h \in \mathcal{D}$, defined by $\hat{h} = \max(|\hat{f}|, |\hat{g}|)$, we have

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|detA|^j\int_{\mathbb{R}^d}\left|\hat{h}\big(B^j\xi\big)\right|^2\left|\hat{\omega}^n_\ell(\xi)\hat{\tilde{\omega}}^n_\ell(\xi)\right|d\xi < \infty$$

and

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|detA|^j\int_{\mathbb{R}^d}\left|\hat{h}\big(B^j\xi\big)\hat{\tilde{\omega}}^n_\ell(\xi)\right|\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\left|\hat{h}\big(B^j(\xi+k)\big)\hat{\omega}^n_\ell(\xi+k)\right|\right\}d\xi < \infty. \tag{3.15}$$

Now, in order to estimate (3.15), we use (3.6), (3.7) and the fact that

$$2\left|\hat{\omega}^n_\ell(\xi)\hat{\tilde{\omega}}^n_\ell(\xi)\right| \le \left|\hat{\omega}^n_\ell(\xi)\right|^2 + \left|\hat{\tilde{\omega}}^n_\ell(\xi)\right|^2 \quad\text{and}\quad 2\left|\hat{\tilde{\omega}}^n_\ell(\xi)\hat{\omega}^n_\ell(\xi+k)\right| \le \left|\hat{\tilde{\omega}}^n_\ell(\xi)\right|^2 + \left|\hat{\omega}^n_\ell(\xi+k)\right|^2.$$

Therefore, we can manipulate the sums as

$$
\begin{aligned}
I_2 &= \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}|detA|^j\int_{\mathbb{R}^d}\overline{\hat g(B^j\xi)}\,\hat\omega_\ell^n(\xi)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat f\big(B^j(\xi+k)\big)\overline{\hat\omega_\ell^n(\xi+k)}\right\}d\xi\\
&= \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}\int_{\mathbb{R}^d}\overline{\hat g(\xi)}\,\hat\omega_\ell^n\big(B^{-j}\xi\big)\left\{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\hat f\big(\xi+B^jk\big)\,\overline{\hat\omega_\ell^n(B^{-j}\xi+k)}\right\}d\xi\\
&= \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\le m}\int_{\mathbb{R}^d}\overline{\hat g(\xi)}\,\hat\omega_\ell^n\big(B^{-j}\xi\big)\sum_{r\ge0}\sum_{s\in\mathbb{S}}\hat f\big(\xi+B^jB^rs\big)\,\overline{\hat\omega_\ell^n(B^{-j}\xi+B^rs)}d\xi\\
&= \int_{\mathbb{R}^d}\overline{\hat g(\xi)}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{s\in\mathbb{S}}\sum_{r\ge0}\sum_{j\le m}\hat\omega_\ell^n\big(B^r(B^{-r-j}\xi)\big)\hat f\big(\xi+B^{j+r}s\big)\overline{\hat\omega_\ell^n\big(B^r(B^{-r-j}\xi+s)\big)}d\xi\\
&= \int_{\mathbb{R}^d}\overline{\hat g(\xi)}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{s\in\mathbb{S}}\sum_{r\ge0}\sum_{p\le m+r}\hat\omega_\ell^n\big(B^r(B^{-p}\xi\big)\overline{\hat\omega_\ell^n(B^r(B^{-p}\xi+s))}\,\hat f\big(\xi+B^ps\big)\,d\xi\\
&= \int_{\mathbb{R}^d}\overline{\hat g(\xi)}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{s\in\mathbb{S}}\sum_{r\ge0}\sum_{p\in\mathbb{Z}}\hat\omega_\ell^n\big(B^r(B^{-p}\xi)\big)\overline{\hat\omega_\ell^n\big(B^r(B^{-p}\xi+s)\big)}\,\hat f\big(\xi+B^ps\big)d\xi,
\end{aligned}
$$

for m sufficiently large so that $\hat g(\xi)\hat f(\xi+B^ps)=0$ for all $p\ge m, s\in\mathbb{S}$, i.e., $(\operatorname{supp}\hat f-\operatorname{supp}\hat g)\cap B^p\mathbb{S}=\emptyset$ for all $p\ge m$. Now, if we take, $b=\sup\Big\{|\xi|:\xi\in(\operatorname{supp}\hat f-\operatorname{supp}\hat g)\Big\}$; then, by (2.10) any $m\ge\big[\log_\lambda(b/c)\big]$ works. Therefore, for any $f,g\in\mathcal{D}$ and sufficiently large m, we have

$$I(m)=I_1(m)+I_2(m),$$

where

$$
\begin{aligned}
I_1(m) &= \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\ge -m}\int_{\mathbb{R}^d}\hat f(\xi)\,\overline{\hat g(\xi)}\,\overline{\hat\omega_\ell^n(B^j\xi)}\,\hat\omega_\ell^n(B^j\xi)\,d\xi\\
I_2(m) &= \int_{\mathbb{R}^d}\overline{\hat g(\xi)}\sum_{p\in\mathbb{Z}}\sum_{s\in\mathbb{S}}\hat f\big(\xi+B^ps\big)\,t_s\big(B^{-p}\xi\big)\,d\xi. \qquad (3.16)
\end{aligned}
$$

Here I_1 follows by a simple change of variables, and I_2 does not depend on m. Equation (3.16), combined with assumptions (3.9) and (3.10) immediately implies

$$\lim_{m\to\infty}I(m)=\lim_{m\to\infty}I_1(m)+I_2(m)=\Big\langle\hat f,\hat g\Big\rangle=\langle f,g\rangle.$$

Conversely, we shall prove that (3.8) implies (3.10). For any fixed $s_0\in\mathbb{S}$ and $q>0$, we define

$$\Omega(q)=\Big\{\xi\in\mathbb{R}^d:|\xi|>q,\,|\xi+s_0|>q\Big\}.$$

Now for any $\xi_0\in\Omega(q)$ and $j\ge0$, define

$$\hat{f}_j(\xi) = |B^{-j} I_d|^{-1/2} \arg t_{s_0}(\xi)\chi_{B^{-j} I_d+\xi_0}(\xi) \text{ and } \hat{g}_j(\xi) = |B^{-j} I_d|^{-1/2}\chi_{B^{-j} I_d+\xi_0+s_0}(\xi),$$

where for the purpose of the proof, we define, for $z \in \mathbb{C}$,

$$\arg z = \begin{cases} z/|z|, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

By separating the term corresponding to $p = 0$ and $s = s_0$ in equation (3.16) for $I_2(m), f = f_j, g = g_j$, from the rest, which we denote by $R(j)$, we have

$$I_2(m) = \frac{1}{|B^{-j} I_d|} \int_{B^{-j} I_d+\xi_0} \left|t_{s_0}(\xi)\right| d\xi + \int_{\mathbb{R}^d} \overline{\hat{g}_j(\xi)} \sum_{\substack{p\in\mathbb{Z}, s\in\mathbb{S} \\ (p,s)\neq(0,s_0)}} \hat{f}_j\big(\xi+B^p s\big) t_s\big(B^{-p}\xi\big)\, d\xi. \tag{3.17}$$

Next, if $|\hat{g}_j(\xi)\hat{f}_j(\xi+B^p s)| \neq 0$ for some $\xi \in \mathbb{R}^d$, then $(B^{-j} I_d+\xi_0)\cap(B^{-j} I_d+\xi_0+s_0-B^p s) \neq \emptyset$, hence $B^{-j}(2I_d)\cap(s_0-B^p\mathbb{S}) \neq \emptyset$ which means $2I_d\cap(B^j s_0-B^{p+j}\mathbb{S}) \neq \emptyset$. Also, if $p+j \geq 0$, then $B^j s_0 - B^{p+j}\mathbb{S} \subset \mathbb{Z}^d$, and since $2I_d \cap \mathbb{Z}^d = \{0\}$, $s_0 \notin B^p\mathbb{S}$ for $p \neq 0$, the only nonzero term happens for $p = 0$ and $s = s_0$. Therefore, the other nonzero terms can contribute only if $p + j < 0$, so we can restrict the sum in (3.17) to $p < -j$.

Using the estimate

$$2\left|t_s(\xi)\right| \leq \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{m'\geq 0} \left|\hat{\omega}_\ell^n\big(B^{m'}\xi\big)\right|^2 + \left|\hat{\omega}_\ell^n\big(B^{m'}(\xi+s)\big)\right|^2 \leq T(\xi) + T(\xi+s),$$

where

$$T(\xi) \equiv \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{m'\geq 0} \left|\hat{\omega}_\ell^n\big(B^{m'}\xi\big)\right|^2 + \left|\hat{\omega}_\ell^n\big(B^{m'}\xi\big)\right|^2, \quad \text{is locally integrable on } \mathbb{R}^d.$$

Therefore, we have

$$\begin{aligned} |R(j)| &\leq \frac{1}{2}\int_{\mathbb{R}^d} \sum_{p<-j} \sum_{s\in\mathbb{S}} |\det A|^p \left|\hat{g}_j\big(B^p\xi\big)\right|\left|\hat{f}_j\big(B^p(\xi+s)\big)\right| T(\xi) d\xi \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d} \sum_{p<-j} \sum_{s\in\mathbb{S}} |\det A|^p \left|\hat{g}_j\big(B^p\xi\big)\right|\left|\hat{f}_j\big(B^p(\xi+s)\big)\right| T(\xi+s) d\xi \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{p<-j} \sum_{s\in\mathbb{S}} |\det A|^p \left|\hat{g}_j\big(B^p\xi\big)\right|\left|\hat{f}_j\big(B^p(\xi+s)\big)\right| T(\xi) d\xi \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d} \sum_{p<-j} \sum_{s\in\mathbb{S}} |\det A|^p \left|\hat{g}_j\big(B^p(\xi-s)\big)\right|\left|\hat{f}_j\big(B^p\xi\big)\right| T(\xi) d\xi. \end{aligned} \tag{3.18}$$

Using Lemma 2.4 with the assumptions that $v > 0$, where $v = v(j) = \inf\big\{|\xi| : \xi \in B^{-j}I_d + \xi_0\big\}$, $\delta = \delta(j) = \text{diam}(B^{-j}I_d), \Upsilon = \Upsilon(j) = \bigcup_{p<-j} B^{-p}(B^{-j}I_d + \xi_0)$ and the fact that $\big|\hat{f}_j(\xi)\big| = \big|\hat{g}_j(\xi - s_0)\big|$, we obtain

$$\begin{aligned}\sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \big|\hat{g}_j\big(B^p\xi\big)\big|\big|\hat{f}_j\big(B^p(\xi+s)\big)\big| &= \sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \big|\hat{g}_j\big(B^p\xi\big)\big|\big|\hat{g}_j\big(B^p(\xi+s)-s_0\big)\big| \\ &= \sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \big|\hat{g}_j\big(B^p\xi\big)\big|\big|\hat{g}_j\big(B^p(\xi+s-B^{-p}s_0)\big)\big| \\ &\le \sum_{p<-j}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\det A|^p \big|\hat{g}_j\big(B^p\xi\big)\big|\big|\hat{g}_j\big(B^p(\xi+k)\big)\big| \\ &\le 2^d |\det A|^j M\Big((v+\delta)/v\Big)\|\hat{g}_j\|_\infty^2 \chi_\Upsilon(\xi) \\ &= 2^d M\Big((v+\delta)/v\Big)\chi_\Upsilon(\xi), \qquad (3.19)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{p<-j}\sum_{s\in\mathbb{S}} |\det A|^p \big|\hat{g}_j\big(B^p(\xi-s)\big)\big|\big|\hat{f}_j\big(B^p(\xi)\big)\big| &\le \sum_{p<-j}\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\det A|^p \big|\hat{f}_j\big(B^p\xi\big)\big|\big|\hat{f}_j\big(B^p(\xi+k)\big)\big| \\ &\le 2^d |\det A|^j M\Big((v'+\delta)/v'\Big)\|\hat{f}_j\|_\infty^2 \chi_{\Upsilon'}(\xi) \\ &= 2^d M\Big((v'+\delta)/v'\Big)\chi_{\Upsilon'}(\xi), \qquad (3.20)\end{aligned}$$

by Lemma 2.4, assuming $v' > 0$, where

$$v' = v'(j) = \inf\Big\{|\xi| : \xi \in B^{-j}I_d+\xi_0+s_0\Big\} \text{ and } \Upsilon' = \Upsilon'(j) = \bigcup_{p<-j} B^{-p}\Big(B^{-j}I_d+\xi_0+s_0\Big).$$

For any $\varepsilon > 0$, there exists $r > 0$, so that $\int_{|\xi|>r} T(\xi)\,d\xi < \varepsilon$. By (2.10), we can find $j_0 > 0$ so that $\delta(j) < q/2$ and consequently $v(j) > q/2, v'(j) > q/2$ for $j > j_0$. Furthermore, by (2.10) we can choose j_0 large enough so that for all $j > j_0$, we have

$$\inf\{|\xi| : \xi \in \Upsilon(j)\} = \inf\Big\{|\xi| : \xi \in \bigcup_{p>j} B^p(B^{-j}I_d + \xi_0)\Big\} > c\lambda^j q/2 > r, \text{ and}$$

$$\inf\{|\xi| : \xi \in \Upsilon'(j)\} = \inf\Big\{|\xi| : \xi \in \bigcup_{p>j} B^p(B^{-j}I_d + \xi_0 + s_0)\Big\} > c\lambda^j q/2 > r.$$

Substituting (3.19) and (3.20) into (3.18), we obtain

$$|R(j)| \le 2^{d-1}M(2)\int_{\Upsilon(j)} T(\xi)d\xi + 2^{d-1}\int_{\Upsilon'(j)} T(\xi)d\xi \le 2^d M(2)\int_{|\xi|>r} T(\xi)d\xi < 2^d M(2)\varepsilon \qquad (3.21)$$

for $j > j_0$ independent of the choice of $\xi_0 \in \Omega(q)$. Since the supports of $\hat{f}_j$ and $\hat{g}_j$ are disjoint $I_1(j) = 0$; moreover (3.8) (and thus (3.11)) implies

$$0 = \langle f_j, g_j\rangle = \lim_{m\to\infty} I(m) = \lim_{m\to\infty} I_2(m) = I_2.$$

Since $\varepsilon > 0$, is arbitrary, therefore (3.17) and (3.21) yields

$$\lim_{j\to\infty} \sup_{\xi_0\in\Omega(q)} \frac{1}{|B^{-j}I_d|} \int_{B^{-j}I_d+\xi_0} \left|t_{s_0}(\xi)\right| d\xi = 0. \tag{3.22}$$

Consider any ball $B(r)$ with radius $r > 0$ such that $B(r) \subset \Omega(2q)$. Let $\mathrm{Z} = \{B^{-j}k : B^{-j}(I_d + k) \cap B(r) \neq \emptyset, k \in \mathbb{Z}^d\}$. If j is sufficiently large, then $\mathrm{diam}(B^{-j}I_d) < \min(q, r)$, so

$$\tilde{Z} = \bigcup_{\xi_0\in \mathrm{Z}} (B^{-j}I_d + \xi_0) \subset \Omega(q) \cap B(2r).$$

Hence,

$$\begin{aligned}\int_{B(r)} \left|t_{s_0}(\xi)\right| d\xi &\leq \int_{\tilde{Z}} \left|t_{s_0}(\xi)\right| d\xi \\ &\leq \sum_{\xi_0\in \mathrm{Z}} \int_{B^{-j}I_d+\xi_0} \left|t_{s_0}(\xi)\right| d\xi \\ &\leq \sum_{\xi_0\in \mathrm{Z}} \left|B^{-j}I_d + \xi_0\right|\varepsilon = |\tilde{Z}|\varepsilon = 2^d|B(r)|\varepsilon\end{aligned}$$

for sufficiently large $j = j(\varepsilon)$ by (3.22). Since $\varepsilon > 0$, is arbitrary so $\int_{B(r)} |t_{s_0}(\xi)|\, d\xi = 0$, for any ball $B(r) \subset \Omega(2q)$. Therefore, $\int_{\Omega(2q)} |t_{s_0}(\xi)|\, d\xi = 0$ and since $q > 0$ is arbitrary $\int_{\mathbb{R}^d} |t_{s_0}(\xi)|\, d\xi = 0$ which implies $t_{s_0}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^d, s_0 \in \mathbb{S}$.

Finally, (3.8) implies that (3.9). Equation (3.9) follows easily from (3.10) and (3.16) since any function $h \in L^\infty(K)$ can be represented as $h = \overline{\hat{f}}\hat{g}$ for some $f, g \in \mathcal{D}$. $\square$

Theorem 3.3. *Let* $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \leq \ell \leq L\}$ *and* $\{\tilde{\omega}_\ell^n : n \in \mathbb{Z}^+, 1 \leq \ell \leq L\}$ *be the dual multiwavelet packets associated with the dilation matrix* A *such that*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\omega}_\ell^n\left(B^j\xi\right)\right|^2 \quad \text{and} \quad \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)\right|^2, \tag{3.23}$$

are locally integrable on $\mathbb{R}^d \setminus \{0\}$. *Then*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \langle f, \omega_{\ell,j,k}^n\rangle\langle\tilde{\omega}_{\ell,j,k}^n, f\rangle = \left\|f\right\|_2^2, \quad \text{for all } f \in \mathcal{D} \tag{3.24}$$

if and only if

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \hat{\omega}_\ell^n\left(B^j\xi\right) \overline{\hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right)} = 1 \qquad a.e.\ \xi \in \mathbb{R}^d \tag{3.25}$$

$$t_s(\xi) \equiv \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \hat{\tilde{\omega}}_\ell^n\left(B^j\xi\right) \overline{\hat{\omega}_\ell^n\left(B^j(\xi + s)\right)} = 0,\ a.e.\ \xi \in \mathbb{R}^d, s \in \mathbb{Z}^d \setminus B\mathbb{Z}^d. \tag{3.26}$$

Proof. By Lemma 3.1 and (3.23), the series in (3.24) is absolutely convergent. Also, by (3.23), the series in (3.25) converges absolutely in $L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ and, hence, is absolutely convergent for $a.e.\ \xi$. Therefore, under the hypothesis, $(3.23), (3.9) \Leftrightarrow (3.24)$ and $(3.10) \Leftrightarrow$ (3.25). Hence, the desired result follows from Theorem 3.2. □

Theorem 3.4. *Let $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \le \ell \le L\}$ be the basic multiwavelet packets associated with the scaling functions φ_ℓ. Then*

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \omega^n_{\ell,j,k}\rangle\right|^2 = \left\|f\right\|_2^2, \quad \textit{for all } f \in L^2(\mathbb{R}^d) \tag{3.27}$$

if and only if

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\omega_\ell^n\left(B^j\xi\right)\right|^2 = 1, \quad a.e.\ \xi \in \mathbb{R}^d \tag{3.28}$$

$$t_s(\xi) \equiv \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j=0}^{\infty} \hat{\omega}_\ell^n\left(B^j\xi\right) \overline{\hat{\omega}_\ell^n\left(B^j(\xi+s)\right)} = 0,\ a.e.\ \xi \in \mathbb{R}^d, s \in \mathbb{S} = \mathbb{Z}^d \setminus B\mathbb{Z}^d. \tag{3.29}$$

In particular, the system $\mathcal{F}(\omega_\ell^n)$ given by (2.8) forms Parseval's frame for $L^2(\mathbb{R}^d)$ if and only if (3.28), (3.29) hold and $\|\omega_\ell^n\|_2 = 1,$ for $n \in \mathbb{Z}^+, \ell = 1, ..., L$.

Proof. Using Lemma 3.1 and (3.27), we have

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \left|\hat{\omega}_\ell^n\left(B^j\xi\right)\right|^2 \in L^1_{loc}(\mathbb{R}^d \setminus \{0\}).$$

Therefore, we can apply Theorem 3.3 with $\omega_\ell^n = \tilde{\omega}_\ell^n \in L^2(\mathbb{R}^d)$ to obtain (3.28) and (3.29). Conversely, assume (3.28) and (3.29); then again by Theorem 3.3, we have

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left|\langle f, \omega^n_{\ell,j,k}\rangle\right|^2 = \left\|f\right\|_2^2, \quad \text{for all } f \in \mathcal{D}.$$

By Theorem 2.7, we have the above for all $f \in L^2(\mathbb{R}^d)$. Furthermore, the system $\mathcal{F}(\omega_\ell^n)$ forms Parseval's frame for $L^2(\mathbb{R}^d)$ if $\|\omega_\ell^n\|_2 \ge 1$ for $n \in \mathbb{Z}^+, \ell = 1, \ldots, L$. □

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SENSITIVITY ANALYSIS FOR AN OPTIMAL CONTROL PROBLEM OF PRODUCTION SYSTEM BASED ON NONLINEAR CONSERVATION LAW

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ABSTRACT. Dynamics of a production system having a large number of items can be represented through hyperbolic conservation law. Due to nonlinear dependence on the work in progress, the resulting partial differential equation becomes nonlinear. Further, occurrence of yield loss during the process makes it nonhomogeneous. In this paper, an optimal control problem has been studied incorporating hyperbolic conservation law as a constraint. One of the few ways to control the output of production system is by adjusting the influx in the system. Moreover, yield loss can also be controlled in a time dependent manner. It is well known that the solutions of nonlinear conservation laws may develop discontinuities known as shock waves that forbid the use of classical variational techniques. This paper studies sensitivity analysis with the presence of shocks. Adjoint technique has been implemented to evaluate gradients of cost functionals.

KEYWORDS : Production System; Hyperbolic Conservation Laws; Optimal Control; Shocks; Sensitivity.
AMS Subject Classification: 49K40, 35L67, 49J20

1. INTRODUCTION

State quantities such as density, velocity and energy, often give rise to nonlinear conservation laws in various fields of science and technology. One such area is production system. Armbruster et al. [1] have introduced a continuum model to study the dynamics of a production system. It is shown that the part density of the materials in a production system can be approximated by hyperbolic conservation law. Further, it has been studied by several authors [2, **?**, 10, 17, 18]. Taking into account customer satisfaction, many important aspects such as velocity form, yield loss are incorporated at macroscopic level.

The main objective behind any model of production network is to control the system in such a way that it should satisfy the demand as closely as possible. Since the demand is so stochastic over a given period of time, a manufacturing system

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needs to generate demand forecast quite frequently and functions accordingly. This motivates us to study the optimal control problem arising in a production system. However, it is well known that optimal control for hyperbolic conservation law is a difficult topic due to its considerable analytical effort as well as computational expense in practice. Only few attempts have been made to control hyperbolic equations as they can not be treated straightway with the development of elliptic and parabolic type equations.

One way is to control the production system by adjusting the inflow rate of the materials. In this context, existence of optimal control has been shown by Shang et al. [20] taking influx rate as a control variable. Further, it has been studied numerically by La Marca et al. [14]. Considering a separate equation for queue, optimal control problem with coupled system has been studied by Kirchner et al. [12]. Moreover, for this problem existence of an optimal control has been demonstrated by D'Apice et al. [8].

In order to perform numerical computations, we try to replace continuous optimization problem by discrete approximation. Subsequently, we need to develop an efficient algorithm to achieve discrete minimizer. In practice most efficient methods to approximate minimizers are gradient-based methods. The approach can be summarized as follows: we first linearize the differential equation to obtain a descent direction for the cost functional J. Then, we take the descent direction with the discrete values obtained from numerical scheme. For the linearization step, we consider a small variation of the conserved quantity with respect to the control variables. One may refer [3, 11] for general linearization technique. However, the procedure is justified only when the solution is smooth enough. For discontinuous solution, it is not justified due to the occurrence of singular terms on the linearization over the shock location. To take into account shock location, sensitivity analysis is necessary. The linearized system for the variation of the solution must be complemented with some new equations for the sensitivity of the shock position. Moreover, it is also necessary to perform the sensitivity analysis of the optimal control problem for the numerical perspective. This will be evident from the discussion in remaining sections.

In the present article, we study the sensitivity analysis of an optimal control problem for production system incorporating yield loss. Sensitivity analysis has been studied for many real life problems [16]. For Burgers equation, sensitivity analysis has been performed by Castro et al. [5, 4]. For general conservation law sensitivity analysis has been accomplished by Kowalewski et al. [13] and Ulbrich [22]. Song [21] and Godlewski et al. [9] have extended the analysis for systems. Recently, for scalar conservation law sensitivity analysis has been performed through vanishing viscosity method [15].

In this paper, we perform the sensitivity analysis for a production system by taking into account certain form of the yield loss. Further, we consider more general form of the yield loss wherein another control variable is also considered. We have carried out the analysis without the presence of shocks. In case of shocks, variation of shock position will be given by ODE's to complement the linearized equation. We evaluate the gradient through adjoint calculus. We demonstrate via numerical illustrations that the customer demand can be matched by controlling the influx in the system. Apart from the theoretical investigations, numerical illustrations for yield loss case of production system are new up to our knowledge.

The remaining part of the paper is organized as follows: In Section 2, we provide some preliminaries to describe the model of production system along with the optimal control problem. Sensitivity analysis without the presence of shocks is presented in Section 3. In Section 4, we take into account shocks in the solution and perform sensitivity analysis with the state variable. Sensitivity analysis will be carried out for cost functional in Section 5. We will end up with the numerical results of the presented optimal control problem in Section 6. Concluding remarks and further scopes of development have been pointed out in Section 7.

2. PRELIMINARIES

The nonlinear conservation law model of manufacturing system can be represented as follows:

$$\partial_t \rho(x,t) + \partial_x f(x,t,\rho) + y_l(x,t,\rho) = 0,\ x \in (0,L),\ t > 0, \tag{2.1}$$

where the flux function $f(x,t,\rho)$ is given by

$$f(x,t,\rho) = \min\{\mu(x,t), v(x,t)\rho(x,t)\}.$$

Completion of the product within the supplier is represented by continuous variable x. In (2.1), $\rho(x,t)$ represents the density of goods at stage x and time t. Raw materials entered into the suppliers are described by the parts at $x = 0$. The finished products are going out of the suppliers at $x = L$. The term $\mu(x,t)$ represents maximal capacity of the suppliers. Yield loss phenomena is expressed as $y_l(x,t,\rho)$. Influx and initial situation in the system are prescribed below.

$$\text{Initial condition: } \rho(x,0) = \rho_0(x),\ x \in [0,L]. \tag{2.2}$$

$$\text{Influx condition: } f(0,t,\rho(0,t)) = \lambda(t),\ t > 0. \tag{2.3}$$

Form of velocity function is given in [19] as follows: $v(x,t) = v(W(t))$, where $W(t)$ represents work in progress in supplier at time t. Mathematically, $W(t) = \int_0^L \rho(s,t)ds$. Similar nonlocal velocity form is also considered in [6]. Now we introduce an optimal control problem related to production system.

The profit of a manufacturing system can be affected significantly by two different aspects. One is overproduction. Producing too much of items lead to high inventory cost in the system. The other one is underproduction. Producing not sufficient number of items lead to lost sales which result into backlog cost. It is quite evident that to maximize the profit, a manufacturing system must be able to match the demand of the customers as closely as possible. We consider cost functional J as

$$J(\rho,\lambda) := \frac{1}{2}\int_0^T [y_d(t) - y(t)]^2 dt + \frac{1}{2}\int_0^T |\lambda(t)|^2 dt. \tag{2.4}$$

Here $y_d(t)$ represents the demand rate and $y(t) = v(W(t))\rho(L,t)$ measures the output of the system. The term $\lambda(t)$ provides the influx rate in the system over time T. The objective behind the choice of cost functional is to minimize the amount of influx and mismatch between the outflux and demand of the customers. We assume that the maximal capacity does not exceed the flux in the suppliers, and maximum speed of the materials denoted by V_M. Optimization problem will be studied in the subsequent sections can be formulated as follows:

min $J(\rho,\lambda)$ subject to the constraints

$$\begin{cases} \partial_t \rho(x,t) + \partial_x (v(W(t))\rho(x,t)) + y_l(x,t,\rho) = 0, \\ \rho(x,0) = \rho_0(x),\ x \in [0,L], \\ v(W(t))\rho(0,t) = \lambda(t),\ t \in (0,T], \\ v(W(t)) = \frac{V_M}{1+\int_0^L \rho(s,t)ds}. \end{cases} \tag{2.5}$$

3. Sensitivity analysis without shocks

In this section, we derive an expression for the sensitivity of the functional J with respect to influx rate by considering certain form of yield loss. After that the analysis will be carried out for general yield loss case involving another control variable.

3.1. **Control on Influx.**

We consider the yield loss form as $y_l(x,t,\rho) = -g(x,t)\rho(x,t)$, where $g(x,t)$ is a continuous function. The following theorem ensures the existence of at least one minimizer for J given in (2.4).

Theorem 3.1. *Let $U_{ad} = \{f \in L^2(0,T) : f$ is non-negative almost everywhere$\}$. Assume that $y_d \in L^2(0,T)$. Then the infimum of the functional J is achieved, i.e., there exists $\lambda_{\min} \in U_{ad}$ such that $J(\lambda_{\min}) = \inf_{\lambda \in U_{ad}} J(\rho(\lambda), \lambda)$.*

Proof of Theorem 3.1 can be carried out in the same manner as in Shang et al. [20] by considering a minimizing sequence of J in U_{ad}. Let us assume that there exist a classical solution $\rho(x,t)$ of (2.5) in $(x,t) \in [0,L] \times [0,T]$. Let $\delta\lambda$ be any possible variation of the influx rate λ. Then for $\epsilon > 0$ sufficiently small, the solution $\rho^\epsilon(x,t)$ corresponding to the influx

$$\lambda^\epsilon(t) = \lambda(t) + \epsilon\delta\lambda(t)$$

is also a solution for $(x,t) \in (0,L) \times (0,T)$ and $\rho^\epsilon(x,t)$ can be written as

$$\rho^\epsilon = \rho + \epsilon(\delta\rho) + o(\epsilon),$$

where $\delta\rho$ is the solution of the linearized equation

$$\begin{cases} \partial_t \delta\rho + \partial_x\Big(v(W)\delta\rho - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho(s,t)ds\Big) = g(x,t)\delta\rho(x,t), \\ \delta\rho(x,0) = 0, \\ v(W)\delta\rho(0,t) \approx \delta\lambda(t). \end{cases} \tag{3.1}$$

In order to find the adjoint system for the problem (2.5), we seek for first order necessary condition of (2.5). Thus, we introduce the Lagrangian function $L(\rho, p, \lambda)$ by considering $p(x,t)$ as a multiplier:

$$\begin{aligned} L(\rho,p,\lambda) := &\frac{1}{2}\int_0^T [y_d(t) - y(t)]^2 dt + \frac{1}{2}\int_0^T |\lambda(t)|^2 dt \\ &+ \int_0^L \int_0^T [\partial_t\rho(x,t) + \partial_x(v(W(t))\rho) - g(x,t)\rho(x,t)]p(x,t)dtdx. \end{aligned}$$

Taking into account the variation of Lagrangian L with respect to ρ and λ, we obtain the following adjoint system:

$$\begin{cases} \partial_t p + \partial_x(v(W)p) + g(x,t)p = \frac{v(W)^2}{V_M}\big[\rho(L,t)y_d(t) \quad - v(W)\rho(L,t)^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad - \int_0^L p(s,t)\rho_x(s,t)ds\big], \\ p(x,T) = 0, \\ p(L,t) = y_d(t) - v(W)\rho(L,t). \end{cases} \tag{3.2}$$

Multiplying the linearized equation (3.1) by $p(x,t)$ and integrating by parts, we get the following:

$$\begin{aligned}
&\int_0^L \int_0^T \Big(\partial_t p(x,t) + \partial_x (v(W)p(x,t)) + g(x,t)p(x,t)\Big)\delta\rho \; dtdx + \int_0^L p(x,0)\delta\rho(x,0)dx \\
&- \int_0^L \int_0^T \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho(s,t)ds\partial_x p(x,t)dxdt - \int_0^L p(x,T)\delta\rho(x,T)dx \\
&- \int_0^T \frac{v(W)^2}{V_M}\Big[p(0,t)\rho(0,t) - p(L,t)\rho(L,t)\Big]\int_0^L \delta\rho(s,t)dsdt \\
&- \int_0^T v(W)p(L,t)\delta\rho(L,t)dt + \int_0^T p(0,t)v(W)\delta\rho(0,t)dt = 0.
\end{aligned} \tag{3.3}$$

Making use of simple calculus it is not difficult to obtain the following:

$$\begin{aligned}
-\int_0^L \int_0^T &\frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho(s,t)ds\partial_x p(x,t)dtdx \\
&= \int_0^L \int_0^T \frac{v(W)^2}{V_M}\Big[\int_0^L \partial_x\rho(s,t)p(s,t)ds\Big]\delta\rho(x,t)dtdx \\
&\quad + \int_0^T \frac{v(W)^2}{V_M}[p(0,t)\rho(0,t) - p(L,t)\rho(L,t)]\int_0^L \delta\rho(s,t)dsdt.
\end{aligned}$$

Let δJ be the Gateaux derivative of functional J at λ in the direction $\delta\lambda$.

$$\begin{aligned}
\delta J = &\int_0^T \frac{v(W)^2}{V_M}\big(\rho(L,t)y_d(t) - v(W)\rho(L,t)^2\big)\int_0^L \delta\rho(s,t)dsdt \\
&- \int_0^T \big(v(W)^2\rho(L,t) - v(W)y_d(t)\big)\delta\rho(L,t)dt + \int_0^T \lambda(t)\delta\lambda(t)dt.
\end{aligned}$$

Taking into account adjoint system (3.2) in (3.3) we can rewrite the variation of J in the following way.

$$\delta J = \int_0^T \lambda(t)\delta\lambda dt - \int_0^T p(0,t)\delta\lambda dt = \int_0^T (\lambda(t) - p(0,t))\delta\lambda dt.$$

In order to evaluate δJ, we need to get the information from adjoint system. Adjoint state $p(x,t)$ can be computed from the prescribed influx $\lambda(t)$, $\delta\lambda(t)$, boundary data $p(L,t)$ and the terminal data $p(x,T)$. Therefore, the descent direction for the functional J can be chosen as $\delta\lambda = -d(t)$, where $d(t) = (\lambda(t) - p(0,t))$.

3.2. **Control on yield loss.** In order to achieve maximum profit, controlling the yield loss is a crucial objective for any production system. Motivated by this, we include time-dependent control variable $u(t)$ in the yield loss term of continuum model. Form of the yield loss will be considered as $y_l(x,t) = h(x,u(t),\rho(x,t))$, which is assumed to be continuous. In similar way as above, we consider small perturbations $\delta\lambda$ and δu for the influx $\lambda(t)$ and control variable $u(t)$ respectively. This results in small variation on solution $\rho(x,t)$. Let us denote the small variation by $\delta\rho(x,t)$. In this subsection, we carry out the sensitivity analysis considering $\rho(x,t)$ as a classical solution. For discontinuous solution, sensitivity analysis will be performed in the next section. We choose the cost functional as

$$J(\rho,\lambda,u) := \frac{1}{2}\int_0^T [y_d(t) - y(t)]^2 dt + \frac{1}{2}\int_0^T |u(t)|^2 dt.$$

Again $y_d(t)$ and $y(t)$ represent demand and outflux of the system respectively. The first term in the objective functional measures the difference between demand and outflux and the second term can be considered as regularization term.

The small variation $\delta\rho$ satisfies the following linearized problem

$$\begin{cases} \partial_t \delta\rho(x,t) + \partial_x\Big(v(W)\delta\rho - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho(s,t)ds\Big) \\ +\partial_\rho h(x,u(t),\rho(x,t))\delta\rho + \partial_u h(x,u(t),\rho(x,t))\delta u(t) = 0, \\ \delta\rho(x,0) = 0, \\ v(W)\delta\rho(0,t) \approx \delta\lambda(t). \end{cases} \tag{3.4}$$

By considering usual notion of Lagrangian formulation, we derive the adjoint system. Adjoint variable $p(x,t)$ satisfies the following system:

$$\begin{cases} -\partial_t p(x,t) - \partial_x(v(W)p(x,t)) + \partial_\rho h(x,u(t),\rho(x,t))p(x,t) \\ -\frac{v(W)^2}{V_M}\int_0^L p(s,t)\rho_x(s,t)ds + \frac{v(W)^2}{V_M}(\rho(L,t)y_d(t) - v(W)\rho(L,t)^2) = 0, \\ p(x,T) = 0, \\ p(L,t) = y_d(t) - v(W)\rho(L,t). \end{cases} \tag{3.5}$$

The Gateaux derivative [15] of functional J, denoted by δJ at (u,λ) in the direction $(\delta u, \delta\lambda)$ can be derived as

$$\begin{aligned} \delta J = &\int_0^T \frac{v(W)^2}{V_M}\big(\rho(L,t)y_d(t) - v(W)\rho(L,t)^2\big)\int_0^L \delta\rho(s,t)dsdt \\ &+ \int_0^T \big(v(W)^2\rho(L,t) - v(W)y_d(t)\big)\delta\rho(L,t)dt + \int_0^T u(t)\delta u(t)dt. \end{aligned}$$

From the linearized equation (3.4) multiplying by adjoint variable $p(x,t)$, we obtain

$$\begin{aligned} &\int_0^L\int_0^T \Big(-\partial_t p(x,t) - \partial_x\big(v(W)p(x,t)\big) + \partial_\rho h(x,u,\rho)p(x,t)\Big)\delta\rho(x,t)\; dtdx \\ &- \int_0^L p(x,0)\delta\rho(x,0)dx - \int_0^L\int_0^T \frac{v(W)^2}{V_M}\Big[\int_0^L \partial_x\rho(s,t)p(s,t)ds\Big]\delta\rho(x,t)dtdx \\ &+ \int_0^L p(x,T)\delta\rho(x,T)dx + \int_0^L\int_0^T \big(\partial_u h(x,u,\rho)\delta u\big)p(x,t)dtdx \\ &+ \int_0^T v(W)p(L,t)\delta\rho(L,t)dt - \int_0^T p(0,t)v(W)\delta\rho(0,t)dt = 0. \end{aligned} \tag{3.6}$$

With the help of (3.5) and (3.6), variation δJ can be reduced to

$$\begin{aligned} \delta J = &\int_0^T -p(0,t)\delta\lambda(t)dt + \int_0^T u(t)\delta u(t)dt \\ &+ \int_0^L\int_0^T (\partial_u h(x,u,\rho)\delta u(t))p(x,t)dtdx. \end{aligned}$$

Above expression for δJ provides a descent directions for functional J. Descent direction for control variable u can be chosen as $\delta u(t) = -d_1(t)$, where $d_1(t)$ is having the following expression $d_1(t) = -u(t) - \int_0^L \partial_\rho h(x,u,\rho)p(x,t)dx$. Similarly, we can choose the descent direction for influx as $\delta\lambda(t) = -d_2(t)$, where $d_2(t) = p(0,t)$. Information about adjoint variable $p(x,t)$ can be obtained by solving the system (3.5). Once we have computed adjoint state, we immediately get the descent directions.

4. Sensitivity of the state in presence shocks

The hyperbolic conservation laws may develop singularities in finite time even for the smooth input data. Therefore in practical applications, we need to consider optimal control problems in which the solutions have discontinuities. We shall study the optimal control problem of production system, described in the previous subsection in the presence of shocks. We focus on the analysis of conservation laws with a finite number of noninteracting shocks. In order to develop efficient numerical methods for the optimal control problems in the presence of shocks, we need to investigate the sensitivity of the states in production system with respect to the input data and control variable along with the infinitesimal translation of shock positions.

The conservation laws model is to be studied as in (2.5) with the yield loss and cost functional described in Section 3.2. We assume that $\rho(x,t)$ is a weak solution of conservation law model with discontinuities along Γ_j for $j = 1, 2, ..., S$, where $\Gamma_j = \{(\phi_j(t), t) : t \in [t_j^0, T]\}$. The solution $\rho(x,t)$ is defined in strong sense outside $\cup_j \Gamma_j$. The Rankine-Hugoniot condition on Γ_j can be given as following:

$$\phi_j^{'}(t)[\rho(.,t)]_{x=\phi_j(t)} = [v(W(t))\rho(.,t)]_{x=\phi_j(t)}.$$

The notation $[f]_{x_d} = f(x_d^+) - f(x_d^-)$ denotes the jump at x_d of any piecewise continuous function f with a discontinuity at $x = x_d$. We need to analyze the sensitivity of (ρ, ϕ_j, u) with respect to the variation of $\delta\lambda, \delta\phi_j$ and δu.

We recall that the linearized equation (3.4) must be interpreted in a weak sense. It is reasonable to choose the solution of the linearized equation (3.4) of the following form:

$$\delta\rho = \delta\rho_r + \sum_{j=1}^{S} q_j \chi_{\Gamma_j},$$

where $\delta\rho_r$ is the regular part and the other one is singular part at the shock locations. We observe that the regular part $\delta\rho_r$ satisfies the following linearized system in an analytical sense outside $\cup_j \Gamma_j$

$$\begin{cases} \frac{\partial}{\partial t}\delta\rho_r + \frac{\partial}{\partial x}\Big(v(W)\delta\rho_r - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho_r(s,t)ds\Big) \\ +\frac{\partial}{\partial \rho}h(x,u(t),\rho(x,t))\delta\rho_r + \frac{\partial}{\partial u}h(x,u(t),\rho(x,t))\delta u = 0, \\ \delta\rho_r(x,0) = 0, \\ v(W)\delta\rho_r(0,t) \approx \delta\lambda(t). \end{cases} \tag{4.1}$$

In order to analyze the singular part, we again get back to the linearized system (3.4). Weak formulation of the linearized system (3.4) can be expressed in the following way by considering $\psi(x,t)$ as a test function having compact support

$$\begin{aligned} &\int_0^L\int_0^T \delta\rho(\partial_t\psi + v(W)\partial_x\psi)dtdx - \int_0^L\int_0^T \psi(x,t)\Big(\partial_\rho h(x,u(t),\rho)\delta\rho \\ &+ \partial_u h(x,u(t),\rho)\delta u\Big)dtdx + \int_0^L\int_0^T \frac{v(W)^2}{V_M}\Big(\int_0^L \psi(s,t)\rho_x(s,t)ds\Big)\delta\rho dtdx \\ &+ \int_0^L \psi(x,0)\delta\rho(x,0)dx - \int_0^L \psi(x,T)\delta\rho(x,T)dx \\ &+ \int_0^T [\psi(0,t)v(W)\delta\rho(0,t) - \psi(L,t)v(W)\delta\rho(L,t)]dt = 0. \end{aligned} \tag{4.2}$$

Let D_c denotes the region $D \setminus \cup_j \Gamma_j$, where D represents the complete domain $[0, L] \times [0, T]$. Using Green's theorem and integration by parts in (4.2), we obtain

$$\begin{aligned}
&\int_{D_c} \Big(- \partial_t \delta\rho - \partial_x \big(v(W)\delta\rho - \frac{v(W)^2}{V_M} \rho(x,t) \int_0^L \delta\rho(s,t) ds \big) \Big) \psi(x,t) dt dx \\
&- \int_0^L \int_0^T \psi(x,t) \Big(\partial_\rho h(x,u(t),\rho)\delta\rho + \partial_u h(x,u(t),\rho)\delta u \Big) dt dx \\
&+ \sum_{j=1}^{S} \int_{t_j^0}^{T} \Big[\dot{\phi}_j \delta\rho - v(W)\delta\rho - \frac{v(W)^2}{V_M} \rho \int_0^L \delta\rho(s,t) ds \Big]_{x=\phi_j(t)} \psi|_{x=\phi_j(t)} = 0.
\end{aligned} \tag{4.3}$$

We would like to consider the form of $\delta\rho$ in the weak form of linearized equation (3.4). It is not difficult to derive the following expression

$$\begin{aligned}
&\int_D \sum_{j=1}^{S} q_j \chi_{\Gamma_j} \Big(\partial_t \psi + \partial_x \big(v(W)\delta\rho - \frac{v(W)^2}{V_M} \rho(x,t) \int_0^L \delta\rho(s,t) ds \big) \\
&- \partial_\rho h(x,u(t),\rho)\psi \Big) dx dt = q_j(t)|_{t=t_j^0} \psi(x,t)|_{x=\phi_j(t)} \\
&+ \sum_{j=1}^{S} \int_{t_j^0}^{T} \Big(- \frac{dq_j}{dt} - \frac{\partial}{\partial\rho} h(x,u(t),\rho(x,t))|_{x=\phi_j(t)} q_j \Big) \psi|_{x=\phi_j(t)}.
\end{aligned}$$

We observe that for $j = 1, 2, ..., S$, $q_j(t)$ are considered as the solutions of the following ODE's

$$\begin{cases}
\frac{dq_j}{dt} = & -\frac{\partial}{\partial\rho} h(x,u(t),\rho(x,t))|_{x=\phi_j(t)} q_j + \sum_{j=1}^{S} \int_{t_j^0}^{T} \Big[\dot{\phi}_j \delta\rho_r - v(W)\delta\rho_r \\
& -\frac{v(W)^2}{V_M} \rho(x,t) \int_0^L \delta\rho_r(s,t) ds \Big]_{x=\phi_j(t)}, \\
q_j(t_j^0) & = 0.
\end{cases} \tag{4.4}$$

Remark 4.1. In practice the solution of PDE (4.1) should be computed first, for instance with the method of characteristics in D_c as it is interpreted in strong sense. Then we solve the ordinary differential equations to obtain q_j's. It requires the values of $\delta\rho_r$ which will be available from the previous step.

Remark 4.2. If the discontinuities occur after certain time T_0 then the linearization can be done separately for $t \in [0, T_0)$ and $t \in [T_0, T]$. For $t \in [0, T_0)$, the linearization can be carried out as described in Section 3 since the solution is regular. After that the linearization can be done as presented above. The intermediate condition can be obtained from weak formulation of the linearized PDE by choosing appropriate test function.

5. Sensitivity of J in presence shocks

In this section, we study sensitivity of the functional J with respect to the variations of influx $\lambda(t)$ and control $u(t)$. It helps us to evaluate the gradient of cost functional and identify descent directions of the control variables. Furthermore, we describe the solution procedure of the presented optimal control problem in a concise way.

We again make use of adjoint calculus to remove the dependent variables from the variation of cost functional. The variation of cost functional J, denoted by δJ, with respect to the perturbations of (u, λ) can be derived as following

$$\begin{aligned}\delta J =& \int_0^T \frac{v(W)^2}{V_M}\big(\rho(L,t)y_d(t) - v(W)\rho(L,t)^2\big)\int_0^L \delta\rho(s,t)dsdt \\ &+ \int_0^T \big(v(W)^2\rho(L,t) - v(W)y_d(t)\big)\delta\rho(L,t)dt + \int_0^T u(t)\delta u dt,\end{aligned} \tag{5.1}$$

where the pair $(\delta\rho, \delta u)$ solves the linearized equation (3.4).

Incorporating the results of Section 4, the complete system of first variation (3.4) can be rewritten as

$$\begin{cases} \frac{\partial}{\partial t}\delta\rho_r + \frac{\partial}{\partial x}\Big(v(W)\delta\rho_r - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L \delta\rho_r(s,t)ds\Big) \\ +\frac{\partial}{\partial\rho}h(x,u(t),\rho(x,t))\delta\rho_r + \frac{\partial}{\partial u}h(x,u(t),\rho(x,t))\delta u = 0, \\ \frac{dq_j}{dt} = -\frac{\partial}{\partial\rho}h(x,u(t),\rho(x,t))|_{x=\phi_j(t)}q_j + \sum_{j=1}^S\int_{t_j^0}^T\Big[\dot{\phi}_j\delta\rho_r - v(W)\delta\rho_r \\ -\frac{v(W)^2}{V_M}\rho(x,t)\int_0^L\delta\rho_r(s,t)ds\Big]\big|_{x=\phi_j(t)}, \\ \delta\rho_r(x,0) = 0, \\ v(W)\delta\rho_r(0,t) \approx \delta\lambda(t), \\ q_j(t_j^0) = 0. \end{cases} \tag{5.2}$$

We consider adjoint state variables $p(x,t)$ and $\theta(t) = (\theta_1(t), \theta_2(t), ..., \theta_S(t))$ for $\delta\rho_r$ and $q(t) = (q_1(t), q_2(t), ..., q_S(t))$ respectively. Multiplying equations of $\delta\rho_r$ and q with adjoint variables p and θ respectively and performing integration by parts we have

$$\begin{aligned} 0 =& \int_0^T\int_0^L p(x,t)\Big(\partial_t\delta\rho_r + \partial_x\big(v(W)\delta\rho_r - \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L\delta\rho_r(s,t)ds\big) \\ &+ \partial_\rho h(x,u(t),\rho(x,t))\delta\rho_r + \frac{\partial}{\partial u}h(x,u(t),\rho(x,t))\delta u\Big)dxdt \\ &+ \sum_{j=1}^S\int_{t_j^0}^T\theta_j\Big(\frac{dq_j}{dt} + \partial_\rho h(x,u,\rho)q_j - \Big[\dot{\phi}_j\delta\rho_r - v(W)\delta\rho_r \\ &- \frac{v(W)^2}{V_M}\rho(x,t)\int_0^L\delta\rho_r(s,t)ds\Big]\big|_{x=\phi_j(t)}\Big) \end{aligned}$$

$$\begin{aligned} =& \int_0^T\int_0^L \partial_u h(x,u(t),\rho)p(x,t)\delta u dxdt + \int_0^L[p(x,T)\rho(x,T) - p(x,0)\rho(x,0)]dx \\ &+ \sum_{j=1}^S\int_{t_j^0}^T q_j\Big(-\frac{d\theta_j}{dt} + \partial_\rho h(x,u(t),\rho(x,t))|_{x=\phi_j(t)}\theta_j\Big)dt + \theta_j q_j|_{t_j^0}^T \\ &+ \sum_{j=1}^S\int_{t_j^0}^T -\theta_j\Big[\dot{\phi}_j\delta\rho_r - v(W)\delta\rho_r - \frac{v(W)^2}{V_M}\rho\int_0^L\delta\rho_r(s,t)ds\Big]\big|_{x=\phi_j(t)} \\ &+ \int_0^T[p(L,t)v(W)\delta\rho_r(L,t) - p(0,t)v(W)\delta\rho_r(0,t)]dt \\ &+ \int_0^T\int_0^L\delta\rho_r\Big(-\partial_t p - \partial_x(v(W)p) + \partial_\rho h(x,u,\rho)p(x,t)\Big)dxdt \end{aligned}$$

$$- \int_0^T \int_0^L \frac{v(W)^2}{V_M} \Big[\int_0^L p(s,t)\rho_x(s,t)ds \Big] \delta\rho_r(x,t)dxdt + \sum_{j=1}^{S} \int_{t_j^0}^{T} \big[\dot{\phi}_j(t) p(x,t)$$

$$\delta\rho_r - p(x,t)v(W)\delta\rho_r - p(x,t)\frac{v(W)^2}{V_M}\rho(x,t) \int_0^L \delta\rho_r(s,t)dx \big] |_{x=\phi_j(t)}.$$

In the process of removing dependent variables, we obtain the following system of adjoint variables

$$\begin{cases} \frac{d\theta_j}{dt} = \frac{\partial}{\partial\rho} h(x,u(t),\rho(x,t))|_{x=\phi_j(t)}\theta_j(t), \\ \theta_j(T) = 0, \\ -\partial_t p - \partial_x(v(W)p) + \partial_\rho h(x,u,\rho)p(x,t) + \frac{v(W)^2}{V_M}\big[\rho(L,t)y_d(t) \\ -\rho(L,t)^2 v(W) - \int_0^L p(s,t)\rho_x(s,t)ds\big] = 0, \\ p(x,T) = 0, \\ p(L,t) = y_d(t) - v(W)\rho(L,t), \\ p^-(\phi_j(t),t) = p^+(\phi_j(t),t) = \theta_j(t). \end{cases} \tag{5.3}$$

Taking into account (5.3), variation δJ in (5.1) can be written as

$$\delta J = \int_0^T \int_0^L \partial_u h(x,u(t),\rho)p(x,t)\delta u dxdt + \int_0^T -p(0,t)\delta\lambda dt + \int_0^T u(t)\delta u dt.$$

The above expression provides the information about the gradient of cost functional with respect to the decision variables u and λ. We can choose the descent directions as follows:

$$\delta u = -u(t) - \int_0^L \partial_\rho h(x,u,\rho)p(x,t)dx,$$
$$\delta\lambda = p(0,t),$$

where $p(x,t)$ can be obtained from (5.3).

6. Numerical Illustrations

In this section, we describe the numerical approach which is applied for an optimal control problem considering yield loss. Numerical results presented here can be considered as generalization of [14] for yield loss case. To start the process we require input values of control variable influx $\lambda(t)$ from respectable admissible set. The amount of yield loss is considered as 20 percent of density. As described in previous sections, we obtain first variation and subsequently a system with associated adjoint variable as constraints. We discretize the hyperbolic PDE's of density and adjoint variable respectively with the input values of λ. The density will be evaluated forward in time while adjoint variable is computed backward in time. We minimize the mismatch between outflux and demand over a time period $t = 10$. Then, we can evaluate the descent direction for λ as $\delta\lambda$. We update the control variable influx λ as $\lambda^{new} = \lambda^{old} + c\delta\lambda$ so that λ^{new} belong to admissible set, where $c \in (0,1)$. We proceed in this way until the gradient of cost functionals becomes sufficiently small.

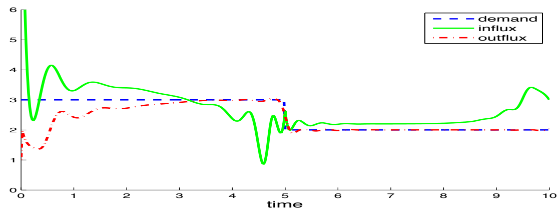

FIGURE 1. Demand, influx, outflux as a function of time with yield loss

We minimize the mismatch between outflux and demand over a time period $t = 10$. Then we can evaluate the descent direction for λ as $\delta\lambda$. We update the control variable influx λ as $\lambda^{new} = \lambda^{old} + c\delta\lambda$ so that λ^{new} belong to admissible set, where $c \in (0, 1)$. We proceed in this way until the gradient of cost functionals becomes sufficiently small.

Motivated by discontinuous nature of demand, for first experiment we have considered a demand function with a steep decrease at time $t = 5$. We start with constant influx $\lambda = 3$, initial density 1 and $v_{max} = 4$. Influx, outflux and demand are presented in Figure 1 as a function of time.

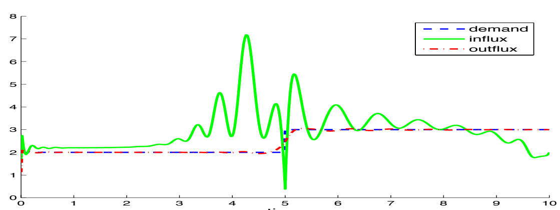

FIGURE 2. Influx, outflux with discontinuous demand

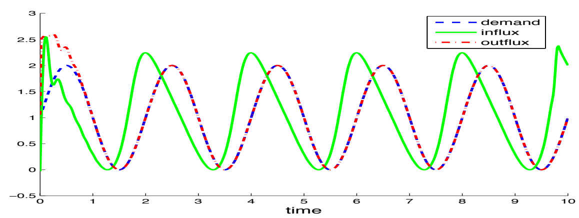

FIGURE 3. Influx, outflux with periodic demand

For second experiment, we have taken a demand function with a jump at time $t = 5$. Starting parameters except influx $\lambda = 2$, are remain same as of first case. Influx and outflux are displayed with discontinuous demand in Figure 2.

We consider a demand function with periodic nature. Demand rate and v_{max} is taken as $sin\pi t + 1$ and 5 respectively. Influx, outflux and demand are presented in Figure 3. The above figures demonstrate that we are able to generate outflux which can match the demand quite closely. It is also observed that initially there are mismatch between outflux and demand but as time progresses it has reduced significantly. Further, it is noticed that discontinuous demand leads to oscillation in influx of the system. Incorporating yield loss the presented results are quite satisfactory. In order to control the yield loss, improved optimization techniques are desirable. So we realize that several theoretical as well as numerical investigations are still to be done in this direction.

7. CONCLUSION

We have studied sensitivity analysis for an optimal control problem of production system. Special attention is given when the solution has discontinuities. By considering singular part at the shock locations, the analysis has been carried out in presence of shocks. Linearized equation is complemented by the equation of shock positions. We have discussed how to identify descent directions to find the minimizer of the optimal control problem. Numerical results are presented for yield loss case considering influx as a control variable. The presented results are new to the author knowledge. This also open several new possibilities in the area of optimal control for partial differential equation based production system models.

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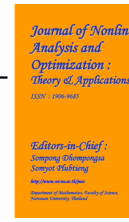

A NEW NON-LIPSCHITZIAN PROJECTION METHOD FOR SOLVING VARIATIONAL INEQUALITIES IN EUCLIDEAN SPACES

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ABSTRACT. The extragradient method introduced by Korpelevich [18] and Antipin [1] is a double projection method designed for solving variational inequalities. The double projection per iteration enable to obtain convergent under monotonicity and Lipschitz continuity while other single projection methods, for example the projected gradient method requires strong monotonicity. The subgradient extragradient method [5] is a modification of the extragradient in which the second projection onto the feasible set is replaced by a projection onto a specific constructible half-space which is actually one of the subgradient half-spaces. Still, this algorithm requires Lipschitz continuity. In this work we introduce a self-adaptive subgradient extragradient method by adopting Armijo-like searches which enables to obtain convergent under the assumption of pseudo-monotonicity and continuity.

KEYWORDS : Extragradient method; Variational inequality; Nonexpansive mapping; Armijo-Goldstein rule
AMS Subject Classification: 65K15 90C25.

1. INTRODUCTION

In this paper, we are concerned with the Variational Inequality Problem (VIP) in the Euclidean space $\mathbb{R}^n$. Let $C \subseteq \mathbb{R}^n$ be a non-empty, closed and convex set and let $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$. The VIP consists in finding a point $x^* \in C$, such that

$$\langle \mathcal{F}(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C. \tag{1.1}$$

Korpelevich [18] and Antipin [1] proposed an algorithm for solving the VIP, known as the Extragradient Method, see also Facchinei and Pang [11, Chapter 12]. In each iteration in order to get the next iterate x^{k+1}, two orthogonal projections onto C are calculated, according to the following iterative step. Given the current iterate x^k, calculate

$$y^k = P_C(x^k - \tau \mathcal{F}(x^k)), \tag{1.2}$$

$$x^{k+1} = P_C(x^k - \tau \mathcal{F}(y^k)), \tag{1.3}$$

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where τ is some positive number and P_C denotes the Euclidean nearest point projection onto C. Although convergence was proved in [18] under the assumptions of Lipschitz continuity and pseudo-monotonicity, there is still the need to calculate two projections onto the closed convex set C which might seriously affect the efficiency of the algorithm. Censor et al. [5] (see also [6, 7]) presented the Subgradient Extragradient Method (SEM), in which the second projection (1.3) onto C is replaced by a projection onto a specific constructible half-space which is actually one of the subgradient half-spaces. In order to prove convergence the authors assume that $\mathcal{F}$ is monotone on C, Lipschitz continuous on $\mathbb{R}^n$, and the Lipschitz constant L is known, so $\tau \in (0, 1/L)$.

In this paper we present a new modification of the SEM for solving the VIP (1.1) when the mapping $\mathcal{F}$ is assumed to be only continuous instead of Lipschitz. Using an Armijo-Goldstein-type rule ([2]) the step size τ is updated and convergence of the algorithm is then guaranteed under the assumptions of pseudo-monotonicity and continuity of $\mathcal{F}$. Other step size adaptations are also presented and the convergence proof can be obtain by following similar arguments. Our convergence theorem relies on the work of Khobotov [17] and Solodov and Tseng [23].

The paper is organized as follows. In Section 2 we present some preliminaries and definitions that will be needed in the sequel. Later, in Section 3 the new algorithm is presented and its convergence is analyzed. Finally, in Section 4 we illustrate the algorithm performance.

2. PRELIMINARIES

In this section we present some useful definitions and results that will be needed for our convergence theorem.

Definition 2.1. *Let $C \subset \mathbb{R}^n$ be a non-empty, closed and convex set and $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$.*

(i) The mapping $\mathcal{F}$ is called `pseudo-monotone` *if for any $x, y \in \mathbb{R}^n$ it holds*

$$\langle \mathcal{F}(y), x - y\rangle \geq 0 \;\Rightarrow\; \langle \mathcal{F}(x), x - y\rangle \geq 0. \tag{2.1}$$

Observe that by substituting $y = x^*$ in (2.1) we get

$$\langle \mathcal{F}(x), x - x^*\rangle \geq 0 \text{ for all } x \in C \text{ and for all } x^* \in \mathrm{SOL}(C, \mathcal{F}) \tag{2.2}$$

where $\mathrm{SOL}(C, \mathcal{F})$ is the solution set of (1.1).

(ii) The mapping $\mathcal{F}$ is called `Lipschitz continuous on` $\mathbb{R}^n$ *if for any $x, y \in \mathbb{R}^n$ there exists an $L \geq 0$ such that*

$$\|\mathcal{F}(x) - \mathcal{F}(y)\| \leq L\|x - y\|. \tag{2.3}$$

(iii) A sequence $\{x^k\}_{k=0}^{\infty} \subset \mathbb{R}^n$ is called `Fejer-monotone` *with respect to C if for every $u \in C$*

$$\|x^{k+1} - u\| \leq \|x^k - u\| \text{ for all } k \geq 0. \tag{2.4}$$

The following lemma is due to Gafni and Bertsekas [12] and it is central in our convergence theorem. This can also be found in Toint [24] or more recently, for example in [14] and [8].

Lemma 2.2. *Let $C \subset \mathbb{R}^n$ be a non-empty, closed and convex set. For every $x \in C$, $z \in \mathbb{R}^n$ and $\alpha > 0$, the function*

$$h(\alpha) = \frac{\|P_C(x + \alpha z) - x\|}{\alpha} \tag{2.5}$$

is monotonically non-increasing.

For each point $x \in \mathbb{R}^n$, there exists a unique nearest point in C, denoted by $P_C(x)$; that is,

$$\|x - P_C(x)\| \leq \|x - y\| \text{ for all } y \in C. \tag{2.6}$$

The mapping $P_C : \mathbb{R}^n \to C$ is called the *metric projection* of $\mathbb{R}^n$ onto C. It is well known that P_C is a *non-expansive* mapping of $\mathbb{R}^n$ onto C, i.e.,

$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \text{ for all } x, y \in \mathbb{R}^n. \tag{2.7}$$

The metric projection P_C is characterized [13, Section 3] by the following two properties:

$$P_C(x) \in C \tag{2.8}$$

and

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0 \text{ for all } x \in \mathbb{R}^n,\ y \in C, \tag{2.9}$$

and if C is a hyperplane, then (2.9) becomes an equality. It follows that

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \text{ for all } x \in \mathbb{R}^n,\ y \in C. \tag{2.10}$$

The next definition of a fixed point set of a mapping T and (2.9) give an equivalent formulation for the VIP.

Definition 2.3. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a given mapping. The* `fixed point set` *of T is defined as*

$$\mathrm{Fix}(T) := \{x \in \mathbb{R}^n \mid T(x) = x\}. \tag{2.11}$$

A well-known relation between the solution set of the VIP (1.1), $\mathrm{SOL}(C, \mathcal{F})$, and the fixed point set of the operator $P_C(I - \lambda\mathcal{F})$ is: for any $\lambda \geq 0$;

$$\mathrm{SOL}(C, \mathcal{F}) = \mathrm{Fix}\left(P_C(I - \lambda\mathcal{F})\right), \tag{2.12}$$

see e.g., Eaves [9]. By converting this relation into an iterative method for solving the VIP (1.1) we can get the well-known projected gradient method. Next we show how by using similar techniques we can recover the extragradient method ((1.2)–(1.3)).

Lemma 2.4. *Let $C \subset \mathbb{R}^n$ be non-empty, closed and convex. Let $\mathcal{F} : C \to \mathbb{R}^n$ be Lipschitz continuous with constant $L > 0$. For any $\lambda \in (0, 1/L)$, we get*

$$\mathrm{SOL}(C, \mathcal{F}) = \mathrm{Fix}\left(P_C(I - \lambda\mathcal{F}(P_C(I - \lambda\mathcal{F})))\right). \tag{2.13}$$

Proof. (i) Let $x \in \mathrm{SOL}(C, \mathcal{F})$. Applying (2.12) twice, we get

$$P_C(x - \lambda\mathcal{F}(P_C(x - \lambda\mathcal{F}(x)))) = P_C(x - \lambda\mathcal{F}(x)) = x \tag{2.14}$$

which implies that $x \in \mathrm{Fix}\left(P_C(I - \lambda\mathcal{F}(P_C(I - \lambda\mathcal{F})))\right)$.

(ii) On the other hand, let $x \in \mathrm{Fix}\left(P_C(I - \lambda\mathcal{F}(P_C(I - \lambda\mathcal{F})))\right)$. Denote by $y := P_C(x - \lambda\mathcal{F}(x))$, we get $x = P_C(x - \lambda\mathcal{F}(y))$. We now show that $x = y$. Indeed, following the non-expansiveness of the metric projection P_C and the Lipschitz continuity of $\mathcal{F}$

$$\begin{aligned} \|x - y\| &= \|P_C(x - \lambda\mathcal{F}(y)) - P_C(x - \lambda\mathcal{F}(x))\| \\ &\leq \|(x - \lambda\mathcal{F}(y)) - (x - \lambda\mathcal{F}(x))\| = \lambda\|\mathcal{F}(x) - \mathcal{F}(y)\| \\ &\leq \frac{\lambda}{L}\|x - y\|. \end{aligned} \tag{2.15}$$

following the assumption on λ we get that $x = y$, meaning that $x = y = P_C(x - \lambda\mathcal{F}(x))$, i.e., $x \in \mathrm{Sol}(\mathcal{F}, C)$. ☐

Notation 2.5. *Any closed and convex set $C \subset \mathbb{R}^n$ can be represented as*

$$C = \{x \in \mathbb{R}^n \mid c(x) \leq 0\}, \tag{2.16}$$

where $c : \mathbb{R}^n \to \mathbb{R}$ is an appropriate convex function. Take, for example, $c(x) = \operatorname{dist}(x, C)$, where dist is the distance function; see, e.g., [15, Chapter B, Subsection 1.3(c)].

We denote the *subdifferential set* of c at a point x by

$$\partial c(x) := \{\xi \in \mathbb{R}^n \mid c(y) \geq c(x) + \langle \xi, y - x\rangle \text{ for all } y \in \mathbb{R}^n\}. \tag{2.17}$$

For $z \in \mathbb{R}^n$, take any $\xi \in \partial c(z)$ and define

$$T(z) := \{w \in \mathbb{R}^n \mid c(z) + \langle \xi, w - z\rangle \leq 0\}. \tag{2.18}$$

This is a half-space the bounding hyperplane of which separates the set C from the point z if $\xi \neq 0$; otherwise $T(z) = \mathbb{R}^n$; see, e.g., [3, Lemma 7.3].

3. THE ALGORITHM

Our new modification of the subgradient extragradient algorithm without the Lipschitz assumption is given next.

Algorithm 3.1. ***The self-adaptive subgradient extragradient algorithm***

Step 0: *Select a starting point $x^0 \in \mathbb{R}^n$. Choose $\alpha_{-1} \in (0, \infty)$, $\varepsilon \in (0, 1)$, and $\beta \in (0, 1)$.*

Step 1: *Given the current iterate x^k, choose α_k to be the largest*

$$\alpha \in \{\alpha_{k-1}, \alpha_{k-1}\beta, \alpha_{k-1}\beta^2, \ldots\} \tag{3.1}$$

satisfying

$$\alpha \left\langle x^k - y^k, \mathcal{F}(x^k) - \mathcal{F}(y^k)\right\rangle \leq (1 - \varepsilon) \left\|x^k - y^k\right\|^2 \tag{3.2}$$

where

$$y^k = P_C(x^k - \alpha\mathcal{F}(x^k)). \tag{3.3}$$

Step 2: *If $x^k = y^k$ then stop. Otherwise, denote $a^k := \left(x^k - \alpha_k\mathcal{F}(x^k)\right) - y^k$ and construct the set T_k as follows*

$$T_k := \begin{cases} \{w \in \mathbb{R}^n \mid \left\langle a^k, w - y^k\right\rangle \leq 0\}, & \text{if } a^k \neq 0, \\ \mathbb{R}^n & \text{if } a^k = 0. \end{cases} \tag{3.4}$$

Calculate the next iterate

$$x^{k+1} = P_{T_k}(x^k - \alpha_k\mathcal{F}(y^k)), \tag{3.5}$$

*set $k \leftarrow (k + 1)$ and return to **Step 1**.*

Remark 3.2. *1. Observe that (3.2) can be viewed as a local approximation of the Lipschitz constant L, and then we get $\alpha < 1/L$. Indeed, if*

$$L_k = \frac{\left\langle x^k - y^k, \mathcal{F}(x^k) - \mathcal{F}(y^k)\right\rangle}{\left\|x^k - y^k\right\|^2} \tag{3.6}$$

then $\alpha \leq (1 - \varepsilon)/L_k$.

2. There exists many other techniques for the choice of α_k in Step 1, for example Khobotov [17]

$$\alpha_k = \min\left\{\alpha_{k-1}, \beta\frac{\left\|x^k - y^k\right\|}{\left\|\mathcal{F}(x^k) - \mathcal{F}(y^k)\right\|}\right\}, \tag{3.7}$$

and Marcotte [20]

$$\alpha_k = \min\left\{\frac{\alpha_{k-1}}{2}, \frac{\|x^k - y^k\|}{\sqrt{2}\,\|\mathcal{F}(x^k) - \mathcal{F}(y^k)\|}\right\}. \tag{3.8}$$

It looks like any of the above choices (and other e.g., [25]*) might work as well.*

Remark 3.3. *Observe that if c is lower semi-continuous and Gâteaux differentiable at y^k, then $\{(x^k - \tau F(x^k)) - y^k\} = \partial c(y^k) = \{\nabla c(y^k)\}$; otherwise $(x^k - \tau F(x^k)) - y^k \in \partial c(y^k)$. See* [3, Facts 7.2] *and* [10] *for more details.*

For the convergence of the algorithm, the following assumptions are needed.

Condition 3.4. *The set $\mathrm{SOL}(C, \mathcal{F})$ is non-empty.*

Condition 3.5. *The mapping $\mathcal{F}$ is pseudo-monotone on C, that is (2.2).*

Condition 3.6. *The mapping $\mathcal{F}$ is continuous on $\mathbb{R}^n$.*

Remark 3.7. *Censor et al.* [5] *(see also* [6, 7]*) introduced an extension of Korpelevich's extragradient method which is the Subgradient Extragradient Method (SEM). The general idea of the SEM is close to that in Algorithm 3.1 in which, given the current iterate x^k, the next iterate x^{k+1} is calculated as the projection onto the constructible set T_k (3.4). But while the convergence of the SEM is guaranteed under strong assumptions as monotonicity, Lipschitz continuity on $\mathbb{R}^n$ and the knowing the Lipschitz constant, Algorithm 3.1 requires only pseudo-monotonicity and continuity. This advantage is not only theoretical but also plays a central role in practice when the information regarding the Lipschitz constant in missing or when the mapping is only continues mappings; see Section 4 for numerical experiments. In addition, other step size adaptations ((3.2) in* ***Step 1****) can be chosen, for example Khobotov's* [17] *or Marcotte's* [20].

3.1. **Convergence of the self-adaptive subgradient extragradient algorithm.**

For the convergence we first show that **Step 2** is valid.

Lemma 3.8. *If for some $k \geq 0$, $x^k = y^k$ in Algorithm 3.1, then $x^k, y^k \in \mathrm{SOL}(C, \mathcal{F})$.*

Proof. Assume that $x^k = y^k$; then $x^k = P_C(x^k - \alpha_k \mathcal{F}(x^k))$, so $x^k \in C$. By the variational characterization of the projection with respect to C (2.9), we have

$$\left\langle w - x^k, (x^k - \alpha_k \mathcal{F}(x^k)) - x^k \right\rangle \leq 0, \text{ for all } w \in C, \tag{3.9}$$

which implies that

$$\alpha_k \left\langle w - x^k, \mathcal{F}(x^k) \right\rangle \geq 0, \text{ for all } w \in C. \tag{3.10}$$

Since $\alpha_k > 0$, we have that $x^k \in \mathrm{SOL}(C, \mathcal{F})$. □

From now on we assume that the algorithm generates infinite sequences $\{x^k\}_{k=0}^{\infty}$ and $\{y^k\}_{k=0}^{\infty}$. Next we prove that α_k is well defined.

Lemma 3.9. *For all $k \geq 0$, there exists $\bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha}]$ (3.2) holds. Hence α_k is well defined.*

Proof. By Condition 3.6, $\mathcal{F}$ is continuous on $\mathbb{R}^n$. Since the metric projection is also continuous on $\mathbb{R}^n$, we obtain

$$\lim_{\alpha \to 0} P_C(x^k - \alpha \mathcal{F}(x^k)) = P_C(x^k). \tag{3.11}$$

We now examine the two cases, $x^k \in C$ and $x^k \notin C$.

(i) If $x^k \in C$, then $x^k = P_C(x^k)$. By the continuity of $\mathcal{F}$ and (3.11), we get that for sufficiently small $\alpha \in (0,1]$,

$$\left\|\mathcal{F}(x^k)\right\| \left\|\mathcal{F}(x^k) - \mathcal{F}\left(P_C(x^k - \alpha\mathcal{F}(x^k))\right)\right\| \leq (1-\varepsilon)\left\|x^k - P_C(x^k - \mathcal{F}(x^k))\right\|^2. \tag{3.12}$$

Now, let $\alpha \in (0,1]$ be sufficiently small. By the Cauchy-Schwarz inequality, the non-expansivness of the metric projection and Lemma 2.2 we get

$$\begin{aligned}
&\alpha\left\langle x^k - y^k, \mathcal{F}(x^k) - \mathcal{F}(y^k)\right\rangle \\
&= \alpha\left\langle P_C(x^k) - P_C(x^k - \alpha\mathcal{F}(x^k)), \mathcal{F}(x^k) - \mathcal{F}(y^k)\right\rangle \\
&\leq \alpha\left\|P_C(x^k) - P_C(x^k - \alpha\mathcal{F}(x^k))\right\| \left\|\mathcal{F}(x^k) - \mathcal{F}(y^k)\right\| \\
&\leq \alpha^2\left\|\mathcal{F}(x^k)\right\| \left\|\mathcal{F}(x^k) - \mathcal{F}\left(y^k\right)\right\| \\
&\leq \alpha^2(1-\varepsilon)\left\|x^k - P_C(x^k - \mathcal{F}(x^k))\right\|^2 \\
&\leq (1-\varepsilon)\left\|x^k - y^k\right\|^2,
\end{aligned} \tag{3.13}$$

so (3.2) is valid.

(ii) If $x^k \notin C$, then

$$\lim_{\alpha\to 0}\alpha\left\langle x^k - y^k, \mathcal{F}(x^k) - \mathcal{F}(y^k)\right\rangle = 0 \tag{3.14}$$

while

$$\lim_{\alpha\to 0}(1-\varepsilon)\left\|x^k - y^k\right\|^2 = (1-\varepsilon)\left\|x^k - P_C(x^k)\right\|^2 > 0, \tag{3.15}$$

implying the claim. □

The next Lemma is central for the convergence theorem.

Lemma 3.10. *Let $\{x^k\}_{k=0}^{\infty}$, $\{y^k\}_{k=0}^{\infty}$ be any two sequences generated by Algorithm 3.1. Assume that Conditions 3.4--3.6* hold, and let $x^* \in \mathrm{SOL}(C,\mathcal{F})$. Then for every $k \geq 0$

$$\left\|x^{k+1} - x^*\right\|^2 \leq \left\|x^k - x^*\right\|^2 - \left\|x^k - y^k\right\|^2\left(1 - \alpha_k^2\frac{\left\|\mathcal{F}(x^k) - \mathcal{F}(y^k)\right\|^2}{\left\|x^k - y^k\right\|^2}\right). \tag{3.16}$$

Proof. Let $x^* \in \mathrm{SOL}(C,\mathcal{F})$. Since $y^k \in C$, we have by Condition 3.5

$$\left\langle \mathcal{F}(y^k), y^k - x^*\right\rangle \geq 0 \text{ for all } k \geq 0, \tag{3.17}$$

which implies that

$$\left\langle \mathcal{F}(y^k), x^{k+1} - x^*\right\rangle \geq \left\langle \mathcal{F}(y^k), x^{k+1} - y^k\right\rangle. \tag{3.18}$$

By the definition of T_k, we have

$$\left\langle x^{k+1} - y^k, \left(x^k - \alpha_k\mathcal{F}(x^k)\right) - y^k\right\rangle \leq 0 \text{ for all } k \geq 0, \tag{3.19}$$

then

$$\begin{aligned}
\left\langle x^{k+1} - y^k, (x^k - \alpha_k\mathcal{F}(y^k)) - y^k\right\rangle &= \left\langle x^{k+1} - y^k, x^k - \alpha_k\mathcal{F}(x^k) - y^k\right\rangle \\
&\quad + \alpha_k\left\langle x^{k+1} - y^k, \mathcal{F}(x^k) - \mathcal{F}(y^k)\right\rangle \\
&\leq \alpha_k\left\langle x^{k+1} - y^k, \mathcal{F}(x^k) - \mathcal{F}(y^k)\right\rangle.
\end{aligned} \tag{3.20}$$

Now by letting $z^k = x^k - \alpha_k \mathcal{F}(y^k)$ for simplicity, we obtain

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 &= \left\|P_{T_k}(z^k)-x^*\right\|^2\\ &= \left\|z^k-x^*\right\|^2+\left\|z^k-P_{T_k}(z^k)\right\|^2+2\left\langle P_{T_k}(z^k)-z^k, z^k-x^*\right\rangle. \end{aligned} \tag{3.21}$$

Since

$$\begin{aligned} &2\left\|z^k-P_{T_k}(z^k)\right\|^2+2\left\langle P_{T_k}(z^k)-z^k, z^k-x^*\right\rangle\\ &=2\left\langle z^k-P_{T_k}(z^k), x^*-P_{T_k}(z^k)\right\rangle \le 0 \text{ for all } k\ge 0, \end{aligned} \tag{3.22}$$

we get for all $k \geq 0$

$$\left\|z^k-P_{T_k}(z^k)\right\|^2+2\left\langle P_{T_k}(z^k)-z^k, z^k-x^*\right\rangle \le -\left\|z^k-P_{T_k}(z^k)\right\|^2. \tag{3.23}$$

So,

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 &\le \left\|z^k-x^*\right\|^2-\left\|z^k-P_{T_k}(z^k)\right\|^2\\ &=\left\|(x^k-\alpha_k\mathcal{F}(y^k))-x^*\right\|^2-\left\|(x^k-\alpha_k\mathcal{F}(y^k))-x^{k+1}\right\|^2\\ &=\left\|x^k-x^*\right\|^2-\left\|x^k-x^{k+1}\right\|^2+2\alpha_k\left\langle x^*-x^{k+1},\mathcal{F}(y^k)\right\rangle\\ &\le\left\|x^k-x^*\right\|^2-\left\|x^k-x^{k+1}\right\|^2+2\alpha_k\left\langle y^k-x^{k+1},\mathcal{F}(y^k)\right\rangle, \end{aligned} \tag{3.24}$$

where the last inequality follows from (3.18).
So

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 &\le \left\|x^k-x^*\right\|^2-\left\|x^k-x^{k+1}\right\|^2+2\alpha_k\left\langle y^k-x^{k+1},\mathcal{F}(y^k)\right\rangle\\ &=\left\|x^k-x^*\right\|^2-\left(\left\langle x^k-y^k+y^k-x^{k+1}, x^k-y^k+y^k-x^{k+1}\right\rangle\right)\\ &+2\alpha_k\left\langle y^k-x^{k+1},\mathcal{F}(y^k)\right\rangle\\ &=\left\|x^k-x^*\right\|^2-\left\|x^k-y^k\right\|^2-\left\|y^k-x^{k+1}\right\|^2\\ &+2\left\langle x^{k+1}-y^k, x^k-\alpha_k\mathcal{F}(y^k)-y^k\right\rangle. \end{aligned} \tag{3.25}$$

By (3.20)

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 \le \left\|x^k-x^*\right\|^2-\left\|x^k-y^k\right\|^2-\left\|y^k-x^{k+1}\right\|^2\\ +2\alpha_k\left\langle x^{k+1}-y^k,\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\rangle. \end{aligned} \tag{3.26}$$

Using Cauchy–Schwarz inequality, we have

$$2\alpha_k\left\langle x^{k+1}-y^k,\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\rangle \le 2\alpha_k\left\|x^{k+1}-y^k\right\|\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\|, \tag{3.27}$$

in addition

$$\begin{aligned}0 &\le \left(\left\|x^{k+1}-y^k\right\|-\alpha_k\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\|\right)^2\\ &=\left\|x^{k+1}-y^k\right\|^2-2\alpha_k\left\|x^{k+1}-y^k\right\|\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\|+\alpha_k^2\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\|^2, \end{aligned} \tag{3.28}$$

so

$$2\alpha_k\left\|x^{k+1}-y^k\right\|\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\| \le \left\|x^{k+1}-y^k\right\|^2+\alpha_k^2\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\|^2.$$

Combining the above inequalities yields

$$\begin{aligned}\left\|x^{k+1}-x^*\right\|^2 &\le \left\|x^k-x^*\right\|^2-\left\|x^k-y^k\right\|^2-\left\|y^k-x^{k+1}\right\|^2\\ &+\left\|x^{k+1}-y^k\right\|^2+\alpha_k^2\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\|^2\\ &=\left\|x^k-x^*\right\|^2-\left\|x^k-y^k\right\|^2+\alpha_k^2\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\|^2. \end{aligned} \tag{3.29}$$

Since $x^k \neq y^k$, we obtain

$$\left\|x^{k+1}-x^*\right\|^2 \leq \left\|x^k-x^*\right\|^2 - \left\|x^k-y^k\right\|^2 \left(1-\alpha_k^2 \frac{\left\|\mathcal{F}(x^k)-\mathcal{F}(y^k)\right\|^2}{\left\|x^k-y^k\right\|^2}\right) \quad (3.30)$$

and the desired result is obtained. □

We are now ready to present the convergence theorem of Algorithm 3.1. The outline of the proof is similar to [23, theorem 3.2].

Theorem 3.11. *Let $\{x^k\}_{k=0}^\infty$ and $\{y^k\}_{k=0}^\infty$ be any two sequences generated by Algorithm 3.1. Assume that Conditions 3.4--3.6* hold, then both sequences converge to the same point $\hat{x} \in \mathrm{SOL}(C,\mathcal{F})$.

Proof. Let $\bar{x} \in \mathrm{SOL}(C,\mathcal{F})$. According to Remark 3.2, we get from (3.30)

$$\left\|x^{k+1}-\bar{x}\right\|^2 \leq \left\|x^k-\bar{x}\right\|^2 \text{ for all } k \geq 0 \quad (3.31)$$

and $\left\|x^k-y^k\right\| \to 0$. Observe that (3.31) states that$\{x^k\}_{k=0}^\infty$ is Fejér-monotone with respect to $\mathrm{SOL}(C,\mathcal{F})$. So following [4, Theorem 5.11] it is sufficient to find a cluster point of $\{x^k\}_{k=0}^\infty$ in $\mathrm{SOL}(C,\mathcal{F})$ and obtain the desired result. According to (3.2) the sequence $\{\alpha_k\}_{k=-1}^\infty$ is non-increasing and therefore $\lim_{k\to\infty}\alpha_k = \widehat{\alpha}$. Now, we consider the following two cases: (i) $\widehat{\alpha} > 0$ and (ii) $\widehat{\alpha} = 0$.

(i) If $\widehat{\alpha} > 0$, by (3.31) the sequence $\{x^k\}_{k=0}^\infty$ is bounded, therefore there exists a subsequence $\{x^{k_j}\}_{j=0}^\infty$ of $\{x^k\}_{k=0}^\infty$ such that

$$\lim_{j\to\infty} x^{k_j} = \hat{x}, \quad (3.32)$$

since $\left\|x^k-y^k\right\| \to 0$ we also have

$$\lim_{j\to\infty} y^{k_j} = \hat{x}. \quad (3.33)$$

By the continuity of $\mathcal{F}$ (Condition 3.6) and of the metric projection

$$\hat{x} = \lim_{j\to\infty} y^{k_j} = \lim_{j\to\infty} P_C(x^{k_j} - \alpha_{k_j}\mathcal{F}(x^{k_j})) = P_C(\hat{x} - \widehat{\alpha}\mathcal{F}(\hat{x})). \quad (3.34)$$

Following similar arguments as in the proof of Lemma 3.8 it follows that $\hat{x} \in \mathrm{SOL}(C,\mathcal{F})$ and the result follows from [4, Theorem 5.11].

(ii) If $\widehat{\alpha} = 0$, we argue by contradiction by supposing that every cluster point of the sequence $\{y^k\}_{k=0}^\infty$ is not in $\mathrm{SOL}(C,\mathcal{F})$. Since $\widehat{\alpha} = 0$, there exists a subsequence of indices $\{k_l\}_{l=0}^\infty$ of $\{k\}_{k=-1}^\infty$ such that $\{\alpha_i\}_{i\in\{k_l\}_{l=0}^\infty}$ is monotonically decreasing. Taking the limit as $i \to \infty$ (passing to a subsequence if needed), we get that $\lim_{i\to\infty} y^i = \widetilde{y} \notin \mathrm{SOL}(C,\mathcal{F})$.

Since $\widetilde{y} \notin \mathrm{SOL}(C,\mathcal{F})$, then $y^i \notin \mathrm{SOL}(C,\mathcal{F})$ for all sufficiently large $i \in \{k_l\}_{l=0}^\infty$. Thus for these i and $\alpha > 0$ we have by (2.12) $y^i \neq P_C(y^i - \alpha\mathcal{F}(y^i))$ and moreover since $\{k\}_{k=-1}^\infty$ is infinite, $y^i \neq x^i$ (we did not stop at **Step 2**).

From the continuity of $\mathcal{F}$ and Lemma 3.9, we get for all of these i with $\lim_{i\to\infty}\alpha_i = 0$, that the right hand side of (3.2) goes to a positive limit while the left hand side goes to zero. Therefore inequality (3.2) holds for all sufficiently small $\alpha > 0$, in particular, it holds for $\alpha = \alpha_{i-1}$ for all $i \in \{k_l\}_{l=0}^\infty$ sufficiently large. But since α_i is chosen as the largest element in $\{\alpha_{i-1}, \alpha_{i-1}\beta, \alpha_{i-1}\beta^2, \ldots\}$ we get a contradiction to our hypothesis on $\{k_l\}_{l=0}^\infty$, that is $\alpha_i < \alpha_{i-1}$ for all $i \in \{k_l\}_{l=0}^\infty$.

Thus, there exists at least one cluster point of $\{y^k\}_{k=0}^\infty$ and also of $\{x^k\}_{k=0}^\infty$, say $\widetilde{x}$, that belongs to $\mathrm{SOL}(C,\mathcal{F})$ and again the desired result follows [4, Theorem 5.11]. □

4. NUMERICAL EXPERIMENTS

In this section we present several numerical examples to illustrate the performance of our algorithm. We choose the test problems from [19] (see also [25]). All computations were performed using MATLAB R2012a on an Intel Core i5-2348M 2.67GHz running 64-bit Windows. The cpu time is measured in seconds using the intrinsic MATLAB function cputime. The projection onto the feasible set C is performed using CVX version 1.22. The numerical results are presented in Table 4, we choose the termination criteria as $\left\|x^k - y^k\right\| \leq \delta$ for small $\delta > 0$.

Example 4.1. *We take $\mathcal{F}(x) = Mx + q$ with the matrix M randomly generated as suggested in [16], $M = AA^T + B + D$; where every entry of the n-square matrix A and of the n-skew-symmetric matrix B is uniformly generated from $(-5,5)$, and every diagonal entry of the n diagonal matrix D is uniformly generated from $(0, 0.3)$, with every entry of q uniformly generated from $(500, 0)$. The feasible set is*

$$C := \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = n\}.$$

Example 4.2. *Kojima-Shindo Nonlinear Complementarity Problem (NCP), see e.g., [22]. With $n = 4$, the feasible set is*

$$C := \{x \in \mathbb{R}^4_+ \mid x_1 + x_2 + x_3 + x_4 = 4\}$$

and $\mathcal{F}$ is given as follows.

$$\mathcal{F}(x_1, x_2, x_3, x_4) := \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}.$$

Example 4.3. *Here the feasible set $C = \mathbb{R}^5$ and $\mathcal{F}$ is given as $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5)$ where for all $i = 1, \ldots, 5$*

$$f_i = 2\left(x_i - i + 2\right) \exp\left(\sum_{i=1}^{5} \left(x_i - i + 2\right)^2\right). \tag{4.1}$$

				(1.2)–(1.3)		Algorithm 3.1	
Example	x^0	parameters	dim	iter.	time	iter.	time
4.1	$\mathbf{1}_{10}$	$\tau = \frac{0.4}{\|M\|}$, $\alpha_{-1} = 0.9$, $\varepsilon = 0.2$, $\beta = 0.5$	10	70	6.7	77	3.1
	$\mathbf{1}_{20}$		20	80	10.6	76	5.9
	$\mathbf{1}_{40}$		40	161	28	170	16
	$\mathbf{1}_{70}$		70	247	307	266	163
4.2	$\mathbf{1}_4$	$\alpha_{-1} = 0.7$, $\varepsilon = 0.2$, $\beta = 0.5$, $\tau = 0.01$	4	-	-	53	4.5
	$(\frac{1}{2}, \frac{1}{2}, 2, 1)$		4	-	-	62	6
4.3	$\mathbf{1}_5$	$\alpha_{-1} = 0.7$, $\varepsilon = 0.3$, $\beta = 0.5$, $\tau = 0.01$	5	-	-	53	2.5
	$\mathbf{0}_5$		5	-	-	62	2.1

We use the notation $\mathbf{1}_n$ and $\mathbf{0}_n$ for the unit and the zero vectors in $\mathbb{R}^n$. In Example 4.1 we generate a random data which depend on the dimension. As can be seen Korpelevich method ((1.2)–(1.3)) preforms bad compared to Algorithm 3.1

and a reasonable explanation is the fact that two projections onto C are calculated in each iteration while in Algorithm 3.1 the second projection is easily computed.

In Example 4.2 the Lipschitz constant is unknown, so one needs to guess it; but if L is very large then τ is very small and that makes the method very inefficient. So, in our experiments we stop the algorithm as we did not reached the termination criteria in a reasonable time and similarly in Example 4.3.

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SOME FIXED POINT RESULTS FOR GENERALIZED CONTRACTIONS IN PARTIALLY ORDERED CONE METRIC SPACES

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ABSTRACT. The aim of this paper is to present some fixed point theorems for generalized contractions by altering distance functions in a complete cone metric spaces endowed with a partial order. We also generalize fixed point theorems of J. Harjani, K. Sadarangani [J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Analysis 72 (2010) 1188-1197] from metric spaces to cone metric spaces.

KEYWORDS : Cone metric space; Fixed point; Partially ordered set
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1. INTRODUCTION AND PRELIMINARIES

Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a cone in E if it satisfies:

(i) P is closed, nonempty and $P \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,
(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; that is, $x \leq y$ if and only if $y - x \in P$. Also we write $x \ll y$ if $y - x \in P^o$, where P^o denotes the interior of P. A cone P is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.
In the sequel we always suppose that E is a real Banach space, P is a cone in E with nonempty interior i.e. $P^o \neq \emptyset$ and $\leq$ is the partial ordering with respect to P.

Definition 1.1. ([1]) *Let X be a nonempty set. Assume that the mapping $d : X \times X \to E$ satisfies*

(i) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *iff* $x = y$
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$

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(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d *is called a cone metric on* X, *and* (X, d) *is called a cone metric space.*

Definition 1.2. *Let* (X, d) *be a cone metric space,* $x \in X$ *and* $\{x_n\}$ *a sequence in* X. *Then*

(i) $\{x_n\}$ *is said to be convergent to* $x \in X$ *whenever for every* $c \in E$ *with* $0 \ll c$ *there is* N *such that for all* $n > N$, $d(x_n, x) \ll c$, *that is,* $\lim_{n\to\infty} x_n = x$.
(ii) $\{x_n\}$ *is called a Cauchy sequence in* X *whenever for every* $c \in E$ *with* $0 \ll c$ *there is* N *such that for all* $n, m > N$, $d(x_n, x_m) \ll c$.
(iii) (X, d) *is a complete cone metric space if every Cauchy sequence is convergent.*

The following remark will be useful in the sequel.

Remark 1.3. *([2])*

(1) *If* $u \leq v$ *and* $v \ll w$, *then* $u \ll w$.
(2) *If* $0 \leq u \ll c$ *for each* $c \in P^o$, *then* $u = 0$.
(3) *If* $a \leq b + c$ *for each* $c \in P^o$ *then* $a \leq b$.
(4) *If* $0 \leq x \leq y$, *and* $0 \leq a$, *then* $0 \leq ax \leq ay$.
(5) *If* $0 \leq x_n \leq y_n$ *for each* $n \in \mathbb{N}$, *and* $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$, *then* $0 \leq x \leq y$.
(6) *If* $0 \leq d(x_n, x) \leq b_n$ *and* $b_n \to 0$, *then* $d(x_n, x) \ll c$ *where* x_n, x *are, respectively, a sequence and a given point in* X.
(7) *If* E *is a real Banach space with a cone* P *and if* $a \leq \lambda a$ *where* $a \in P$ *and* $0 < \lambda < 1$, *then* $a = 0$.
(8) *If* $c \in P^o$, $0 \leq a_n$ *and* $a_n \to 0$, *then there exists* N *such that for all* $n > N$ *we have* $a_n \ll c$.

The altering distance functions were introduced by Khan et al. in [3] and now we define this functions on a cone. If $P := \mathbb{R}^+$ then we have the definition 1.1 in [4].

Definition 1.4. *An altering distance function is a function* $\psi : P \to P$ *which satisfies*

(a) ψ *is continuous and nondecreasing.*
(b) $\psi(x) = 0$ *if and only if* $x = 0$.

Definition 1.5. *If* $(X, \sqsubseteq)$ *is a partially ordered set and* $f : X \to X$, *we say that* f *is monotone nondecreasing if* $x, y \in X$, $x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$.

Definition 1.6. *The cone* P *is called regular if every increasing sequence which is bounded from above is convergent. That is, if* $\{x_n\}$ *is a sequence such that* $x_1 \leq x_2 \leq \cdots \leq y$ *for some* $y \in E$, *then there is* $x \in E$ *such that* $\lim_{n\to\infty} \|x_n - x\| = 0$. *Equivalently the cone* P *is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been mentioned that every regular cone is normal* [5].

Definition 1.7. P *is called minihedral cone if* $\sup\{x, y\}$ *exists for all* $x, y \in E$, *and strongly minihedral if every subset of* E *which is bounded above has a supremum* [6]. *So if cone* P *is strongly minihedral then, every subset of* P *has infimum.*

For more details and some examples about definition 1.7 and some applications on cone metric spaces refer to [7, 8].

The purpose of this paper is to present some fixed point theorems for generalized contractions involving altering distance functions that generalize the theorems of the paper [4] by Harjani and Sadarangani in the context of ordered cone metric spaces with arbitrary cones.

Existence of fixed point in partially ordered sets has been considered recently in [9]-[16].

2. MAIN RESULTS

Let $(X, \sqsubseteq)$ be a partially ordered set and suppose there exists a cone metric d in X. We define (ID) property as follows,

for all $x, y \in X$ if there exists $z \in X$ such that, $x \sqsubseteq y \sqsubseteq z$ then $d(x, y)$ and $d(y, z)$ are comparable.

Theorem 2.1. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose there exists a cone metric d in X such that (X, d) is a complete cone metric space which the (ID) property holds and if there exists a bounded decreasing sequence in P, then it converges to an element in P. Let $f : X \to X$ be a continuous and nondecreasing mapping such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)), \quad \text{for } x \sqsubseteq y, \tag{2.1}$$

where ψ and φ are altering distance functions. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$ then f has a fixed point. Further if fixed points of f are comparable, then f has a unique fixed point.

Proof. If $x_0 = fx_0$ then the proof is finished. Suppose that $x_0 \neq fx_0$. Since $x_0 \sqsubseteq fx_0$ and f is a nondecreasing function, so

$$x_0 \sqsubseteq fx_0 \sqsubseteq f^2x_0 \sqsubseteq f^3x_0 \sqsubseteq \cdots .$$

Put $x_{n+1} := fx_n = f^n x_0$ and $a_n := d(x_{n+1}, x_n)$. Then for $n \geq 1$ we have

$$\psi(d(x_{n+1}, x_n)) = \psi(d(fx_n, fx_{n-1})) \leq \psi(d(x_n, x_{n-1})) - \varphi(d(x_n, x_{n-1})),$$

therefore

$$0 \leq \psi(a_n) \leq \psi(a_{n-1}) - \varphi(a_{n-1}) \leq \psi(a_{n-1}). \tag{2.2}$$

Since $x_n \sqsubseteq x_{n+1} \sqsubseteq x_{n+2}$ by the (ID) property we have

$$a_n \leq a_{n+1} \tag{2.3}$$

or

$$a_{n+1} \leq a_n. \tag{2.4}$$

If (2.3) holds, since ψ is nondecreasing by (2.2) we have

$$0 \leq \psi(a_n) \leq \psi(a_{n-1}) - \varphi(a_{n-1}) \leq \psi(a_n) - \varphi(a_{n-1}) \leq \psi(a_n). \tag{2.5}$$

This implies that $\varphi(a_{n-1}) = 0$ and so $a_{n-1} = 0$ for $n \geq 1$ hence

$$x_n = x_{n-1} = fx_{n-1}$$

for $n \geq 1$ are fixed points of f. If (2.4) holds, since ψ and φ are nondecreasing by relation (2.2) and induction we have

$$\begin{aligned}
\varphi(a_{n+1}) &\leq \varphi(a_n) \leq \psi(a_n) \leq \psi(a_{n-1}) - \varphi(a_{n-1}) \\
&\leq \psi(a_{n-1}) - \varphi(a_n) \\
&\leq \psi(a_{n-2}) - \varphi(a_{n-2}) - \varphi(a_n) \\
&\leq \psi(a_{n-2}) - 2\varphi(a_n) \leq \cdots \\
&\leq \psi(a_0) - n\varphi(a_n),
\end{aligned}$$

so

$$0 \leq \varphi(a_n) \leq \frac{1}{n+1}\psi(a_0) \tag{2.6}$$

for all n. By Remark 1.3-5 and since $\lim_{n\to\infty} a_n$ exists by (2.4), so

$$0 \le \varphi(a_n) \le \frac{1}{n+1}\psi(a_0) \Rightarrow 0 \le \varphi(\lim_{n\to\infty} a_n) \le \lim_{n\to\infty} \frac{1}{n+1}\psi(a_0) = 0,$$

thus $\varphi(\lim_{n\to\infty} a_n) \in P \cap -P$ and we obtain $\varphi(\lim_{n\to\infty} a_n) = 0$ and since φ is altering distance function, hence $\lim_{n\to\infty} a_n = 0$ so

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.7}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $c \gg 0$ for which we can find subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \ge c. \tag{2.8}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.8). Then

$$d(x_{n_k-1}, x_{m_k}) \ll c. \tag{2.9}$$

Using (2.8), (2.9) and the triangular inequality, we have

$$\begin{aligned} c &\le d(x_{n_k}, x_{m_k}) \\ &\le d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) \\ &\ll d(x_{n_k}, x_{n_k-1}) + c. \end{aligned}$$

Letting $k \to \infty$ and using (2.7)

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = c. \tag{2.10}$$

Again, the triangular inequality gives us

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &\le d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}), \\ d(x_{n_k-1}, x_{m_k-1}) &\le d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}), \end{aligned}$$

Letting $k \to \infty$ in the above two inequalities and using (2.7) and (2.10), we have

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = c. \tag{2.11}$$

As $n_k > m_k$ and x_{n_k} and x_{m_k} are comparable (in fact, $x_{m_k-1} \sqsubseteq x_{n_k-1}$, setting $x := x_{n_k-1}$ and $y := x_{m_k-1}$ in (2.1), we obtain

$$\psi(d(x_{n_k}, x_{m_k})) \le \psi(d(x_{n_k-1}, x_{m_k-1})) - \varphi(d(x_{n_k-1}, x_{m_k-1})).$$

Letting $k \to \infty$ and taking into account (2.10) and (2.11), we have

$$\psi(c) \le \psi(c) - \varphi(c).$$

As ψ is an altering distance function, the last inequality gives us $\varphi(c) = 0$ and, consequently, $c = 0$ which is a contradiction. This implies that the sequence $\{x_n\}$ is Cauchy and since (X, d) is complete, thus there exists $x^* \in X$ such that $x_n \to x^*$ and on the other hand f is continuous and $x_{n+1} = fx_n$ so we obtain $x^* = fx^*$.

For uniqueness let $x, y \in X$ be fixed points and x is comparable to y. Hence $fx = x$ is comparable to $fy = y$ and

$$\psi(d(x, y)) = \psi(d(fx, fy)) \le \psi(d(x, y)) - \varphi(d(x, y)).$$

The last inequality gives us $\varphi(d(x, y)) = 0$ and by altering distance functions properties this implies $d(x, y) = 0$ therefore $x = y$.

□

Example 2.1. *Let* $E = (C^1([0,1],\mathbb{R}^+), \|.\|)$, *with* $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $X = \{f, g, h\} \subseteq E$, *and*

$$\sqsubseteq = \{(f,f),(g,g),(h,h),(g,h),(h,f),(g,f)\}$$

where $f(t) = 0, g(t) = e^t = 2h(t)$, *for all* $t \in [0,1]$, *so* $\sqsubseteq$ *is a partial order on* X. *Define* $d : X \times X \to E$ *by* $d(f,g) = f + g$ *and* $f \neq g$ *and* $d(f,f) = 0$. *It is easy to see that every Cauchy sequence on* X *is convergent, i.e.,* (X,d) *is a complete cone metric space, and if we put* $P = \{f \in E : f(t) \geq 0\}$, *then* P *is a non-normal cone while is not minihedral by* [7]. *Further, let* $T : X \to X$ *be* $Tf = f, Tg = h, Th = f$, $\psi(f) = f$ *and* $\varphi(f) = \frac{f}{2}$, *for all* $f \in P$. *We notice that* $g \sqsubseteq Tg$, *ID property and all conditions of Theorem 2.1 hold. Therefore* T *has a unique fixed point, i.e.,* $Tf = f$.

Example 2.2. *With hypothesis of Example 2.1, define* $X = \{f, g, h, k\} \subseteq E$, *and*

$$\sqsubseteq = \{(f,f),(g,g),(h,h),(k,k),(g,h),(h,f),(g,f)\}$$

where $f(t) = 0, g(t) = e^t = 2h(t) = 3k(t)$, *for all* $t \in [0,1]$, *so* $\sqsubseteq$ *is a partial order on* X. *Let* $T : X \to X$ *be* $Tf = f, Tg = h, Th = f, Tk = k, \psi(f) = f$ *and* $\varphi(f) = \frac{f}{2}$, *for all* $f \in P$. *Therefore* T *have two fixed points, i.e.,* $Tf = f$ *and* $Tk = k$, *where* f *and* k *aren't comparable.*

In the next theorem, we replace the (ID) property by strongly minihedrallity of the cone.

Theorem 2.2. *Let* $(X, \sqsubseteq)$ *be a partially ordered set and suppose that there exists a cone metric* d *in* X *with strongly minihedral cone* P, *such that* (X,d) *is a complete cone metric space. Let* $f : X \to X$ *be a continuous and nondecreasing mapping such that*

$$\psi(d(fx,fy)) \leq \psi(d(x,y)) - \varphi(d(x,y)),$$

for $x \sqsubseteq y$, *where* ψ *and* φ *are altering distance functions. If there exists* $x_0 \in X$ *with* $x_0 \sqsubseteq fx_0$ *then* f *has a fixed point.*

Proof. By the proof of the Theorem 2.1 the sequence $\{\psi(a_n)\}$ has infimum. Put $b = \inf_n \psi(a_n)$. So there exists $\{\psi(a_{n_k})\}_k$ such that $\psi(a_{n_k}) \to b$ as $k \to \infty$. Now by (2.2)

$$0 \leq \psi(a_{n_k}) \leq \psi(a_{n_k-1}) - \varphi(a_{n_k-1}) \leq \psi(a_{n_k-1}), \tag{2.12}$$

letting $k \to \infty$

$$b \leq b - \varphi(\lim_{k\to\infty} a_{n_k-1}) \leq b,$$

this implies that $\varphi(\lim_{k\to\infty} a_{n_k-1}) \in P \bigcap -P$ so $\varphi(\lim_{k\to\infty} a_{n_k-1}) = 0$. □

In the next corollary, we replace the (ID) property and strongly minihedrality of the cone by regularity.

Corollary 2.3. *Let* $(X, \sqsubseteq)$ *be a partially ordered set and suppose that there exists a cone metric* d *in* X *with regular cone* P *such that* (X,d) *is a complete cone metric space. Let* $f : X \to X$ *be a continuous and nondecreasing mapping such that*

$$\psi(d(fx,fy)) \leq \psi(d(x,y)) - \varphi(d(x,y)),$$

for $x \sqsubseteq y$, *where* ψ *and* φ *are altering distance functions. If there exists* $x_0 \in X$ *with* $x_0 \sqsubseteq fx_0$ *then* f *has a fixed point.*

Proof. By proofing of the Theorem 2.1 and relation (2.2) the sequence $\{\psi(a_n)\}$ is decreasing and bounded below and P is regular cone so

$$\varphi(\lim_{n\to\infty} a_n) = 0.$$

Now similar as the proof of the previous theorem the proof is completed. □

In the sequel, we prove that Theorems 2.1, 2.2 and corollary 2.3 are still valid where f is not necessarily continuous, but the following hypothesis holds in X, "if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$".

Theorem 2.3. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a cone metric d in X such that (X, d) is a complete cone metric space which the (ID) property holds. Let $f : X \to X$ be a nondecreasing mapping such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)),$$

for $x \sqsubseteq y$, where ψ and φ are altering distance functions. If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$ and X satisfies in following condition
if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$, then f has a fixed point.

Proof. Following the proof of Theorem 2.1 it is enough to prove that $fx^* = x^*$. Since $\{x_n\} \subset X$ is a nondecreasing sequence and $x_n \to x^*$ as $n \to \infty$. Now by hypothesis we conclude that $x_n \sqsubseteq x^*$ for all $n \in \mathbb{N}$ and for all $c \gg 0$ there exists N such that $d(x_n, x^*) \ll c$ and

$$\psi(d(x_{n+1}, fx^*)) = \psi(d(fx_n, fx^*)) \leq \psi(d(x_n, x^*)) - \varphi(d(x_n, x^*)) \leq \psi(c),$$

for all $n \geq N$. Since ψ and φ are altering distance function if $n \to \infty$ we have,

$$0 \leq \psi(\lim_{n\to\infty} d(x_{n+1}, fx^*)) \leq \psi(c),$$

for all $c \gg 0$. Thus $0 \leq \psi(\lim_{n\to\infty} d(x_{n+1}, fx^*)) \leq \psi(\frac{c}{m})$, for all $c \gg 0$ and every $m \in \mathbb{N}$, hence

$$\psi(\lim_{n\to\infty} d(x_{n+1}, fx^*)) = 0$$

so

$$\lim_{n\to\infty} d(x_{n+1}, fx^*) = 0.$$

Let $c \in E$ and $c \gg 0$ so there exists N such that $d(x_{n+1}, fx^*) \ll c$ for every $n \geq N$. Thus for some N we have

$$d(x^*, fx^*) \leq d(x^*, x_{n+1}) + d(x_{n+1}, fx^*) \ll c,$$

for every $n \geq N$. This implies that $0 \leq d(x^*, fx^*) \ll c$ for all $c \gg 0$. Then $d(x^*, fx^*) = 0$ and consequently $x^* = fx^*$. □

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorem 2.2 and corollary 2.3. This condition is:

"for $x, y \in X$ there exists $z \in X$ which is comparable to x and y." (2.13)

Theorem 2.4. *Adding condition (2.13) to the hypothesis of Theorem 2.2 (resp. corollary 2.3) we obtain uniqueness of the fixed point of f.*

Proof. Let $x, y \in X$ are fixed points. We distinguish two cases:
Case 1. If x is comparable to y then $fx = x$ is comparable to $fy = y$ and

$$\psi(d(x, y)) = \psi(d(fx, fy)) \leq \psi(d(x, y)) - \varphi(d(x, y)).$$

The last inequality gives us $\varphi(d(x, y)) = 0$ and by altering distance functions properties this implies $d(x, y) = 0$ therefore $x = y$.
Case 2. If x is not comparable to y then there exists $z \in X$ comparable to x and y.

Monotonicity of f implies that $f^n z$ is comparable to $f^n x = x$ and to $f^n y = y$, for $n = 0, 1, 2, \cdots$. Moreover,

$$\begin{aligned} \psi(d(x, f^n z)) &= \psi(d(f^n x, f^n z)) \\ &\leq \psi(d(f^{n-1}x, f^{n-1}z)) - \varphi(d(f^{n-1}x, f^{n-1}z)) \\ &= \psi(d(x, f^{n-1}z)) - \varphi(x, f^{n-1}z)) \leq \psi(d(x, f^{n-1}z)). \end{aligned} \quad (2.14)$$

according to regularity or strongly minihedrality of the cone P, there exists $b \in E$ such that $\psi(d(x, f^n z)) \to b$ as $n \to \infty$. Now by (2.14) and altering distance functions properties ψ and φ we have

$$\psi(d(x, f^n z)) \leq \psi(d(x, f^{n-1}z)) - \varphi(d(x, f^{n-1}z)) \leq \psi(d(x, f^{n-1}z)),$$

letting $n \to \infty$

$$b \leq b - \varphi(\lim_{n\to\infty} d(x, f^{n-1}z)) \leq b,$$

this implies that

$$\varphi(\lim_{n\to\infty} d(x, f^{n-1}z)) \in P \cap -P$$

so $\varphi(\lim_{n\to\infty} d(x, f^{n-1}z)) = 0$ thus $\lim_{n\to\infty} d(x, f^{n-1}z) = 0$. And similarly $d(y, f^n z) \to 0$. Let $c \gg 0$ and $c \in E$, so there exists N such that $d(x, f^n z) \ll c$ and $d(y, f^n z) \ll c$ for all $n \geq N$. Now by triangle inequality

$$d(x, y) \leq d(x, f^n z) + d(f^n z, y) \ll 2c,$$

for all $n \geq N$. Namely $0 \leq d(x, y) \ll c$ for all $c \gg 0$. Then $d(x, y) = 0$ so $x = y$. □

Our Theorems 2.1, 2.2 with non-normal cone and Corollary 2.3 with normal cone generalize Theorems 2.1, 2.2 [4] and also Theorem 2.4 extend Theorem 2.3 [4] to cone metric version.

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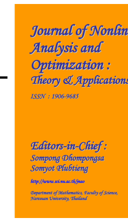

THE SPLIT EQUALITY FIXED POINT PROBLEM FOR DEMI-CONTRACTIVE MAPPINGS

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ABSTRACT. Motivated by the recent work of Moudafi (*Inverse Problems, 26 (2010), 587-600*) and inspired by Xu (*Inverse Problems, 22 (2006), 2021-2034*), Censor and Segal (*J. Convex Anal. 16 (2009), 587-600*) , and Yang (*Inverse Problems, 20 (2004), 1261-1266*), we investigate a Krasnoselskii-type iterative algorithm for solving the split equality fixed point problem recently introduced by Moudafi and Al-Shemas (*Transactions on Mathematical Programming and Applications, Vol. 1, No. 2 (2013), 1-11*). Weak and strong convergence theorems are proved for the class of demi-contractive mappings in Hilbert spaces. Our theorems extend and complement some recent results of Moudafi and a host of other recent important results.

KEYWORDS :Split equality fixed point problem, Uniform Continuity, Demicontractive mappings, iterative scheme, Fixed point.
AMS Subject Classification: 47J25, 47H06, 49J53, 90C25.

1. INTRODUCTION

The split feasibility problem arises in many areas of application such as phase retrieval, medical image reconstruction, image restoration, computer temography and radiation therapy treatment planning (see e.g., Byrne [1], Censor *et al.*[2], Censor *et al.* [3], and Censor and Elfving [4]). It takes the following form: Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as follows:

$$Find \ \ x^* \in C \ \ such \ that \ \ Ax^* \in Q. \tag{1.1}$$

The SFP was first introduced in 1994 by Censor and Elfving [4] in finite-dimensional Hilbert spaces for modelling inverse problems arising from phase retrieval and medical image reconstruction.

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Assuming that the SFP (1.1) has a solution, one can easily show that $x^* \in C$ solves SFP if and only if it solves the fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*,$$

where P_C and P_Q are the metric projections from H_1 onto C and from H_2 onto Q, respectively, where γ is a positive constant and A^* denotes the adjoint of A.

A popular algorithm used in approximating the solution of the SFP (1.1) is the CQ-algorithm of Byrne [1]:

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n,$$

for each $n \geq 1$, where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.
Based on the work of Censor and Segal [5], Moudafi [10] proposed the following scheme which does not involve the metric projections P_C and P_Q:

$$x_{n+1} = (1-\alpha_n)\Big(x_n + \gamma A^*(T-I)Ax_n\Big) + \alpha_n U\Big(x_n + \gamma A^*(T-I)Ax_n\Big), \ \ n \in \mathbb{N},$$

for approximating a solution of the split feasibility fixed point problem (1.1) and obtained a weak convergence results when U and T are *demi-contractive*.

Very recently, Moudafi and Al-Shemas [9] introduced the following *split equality fixed point problem* as a generalization of the split feasibility problem (1.1):

$$Find \ \ x \in C := F(U) \ \ and \ \ y \in Q := F(T) \ such \ that \ \ Ax = By, \tag{1.2}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, $U : H_1 \to H_1, \ \ T : H_2 \to H_2$, $F(U)$ and $F(T)$ denote the fixed point sets of U and T, respectively. Note that problem (1.2) reduces to problem (1.1) if $H_2 = H_3$ and $B = I$ (where I is the identity map on H_2) in (1.2).

In order to approximate a solution of problem (1.2), Moudafi and Al-Shemas [9] introduced the following iterative scheme:

$$\begin{cases} x_{n+1} = U(x_n - \gamma_n A^*(Ax_n - By_n)); \\ \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_n - By_n)), \ \ \forall n \geq 1, \end{cases} \tag{1.3}$$

where $U : H_1 \to H_1, \ \ T : H_2 \to H_2$ are two *firmly quasi-nonexpansive mappings*, $A : H_1 \to H_3, \ B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B, respectively, $\{\gamma_n\} \subset (\epsilon, \frac{2}{\lambda_{A^*A} + \lambda_{B^*B}} - \epsilon)$, λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B, respectively. Using the iterative scheme (1.3), Moudafi obtained a *weak convergence* result for problem (1.2).

Yuan-Fang $et \ al.$ [15] introduced the following algorithm for solving problem (1.2):

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n U(x_n - \gamma_n A^*(Ax_n - By_n)); \\ \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n T(y_n + \gamma_n B^*(Ax_n - By_n)), \ \ \forall n \geq 1, \end{cases} \tag{1.4}$$

where $U : H_1 \to H_1, \ \ T : H_2 \to H_2$ are two *firmly quasi-nonexpansive mappings*, $A : H_1 \to H_3, \ B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B, respectively, $\{\gamma_n\} \subset (\epsilon, \frac{2}{\lambda_{A^*A} + \lambda_{B^*B}} - \epsilon)$ (for ϵ small enough), where λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B, respectively and $\{\alpha_n\} \subset [\alpha, 1]$ (for some $\alpha > 0$). Under some conditions, the authors obtained strong and weak convergence results.

Motivated by the work of Moudafi [8], Moudafi and Al-Shemas [9], Moudafi [10] and Yuan-Fang $et\ al.$ [15], we define the following iterative algorithm to solve the split equality fixed point problem (1.2) in the case that U and T are *demi-contractive.*

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1-\alpha)\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) + \alpha U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big); \\ y_{n+1} = (1-\alpha)\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) + \alpha T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big), \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $U : H_1 \to H_1, \quad T : H_2 \to H_2$ are two *demi-contractive mappings*. The important class of demi-contractive mappings properly includes the class of firmly quasi-nonexpansive mappings studied by Moudafi and Al-Shemas [9]. Under suitable conditions, we prove weak and strong convergence theorems of the iterative scheme (1.5) to a solution of the split equality problem in real Hilbert spaces. Our theorems extend and complement the results of Censor and Segal [5], Maruster $et\ al.$ [7], Moudafi and Al-Shemas [9], Moudafi [10], [11], Xu [13], Yang [14], Yuan-Fang $et\ al.$ [15], and a host of other results.

2. PRELIMINARIES AND NOTATIONS

We recall some definitions and lemmas which will be needed in the proof of our main theorems.
In the sequel, we denote strong and weak convergence by " $\longrightarrow$" and " $\rightharpoonup$", respectively, the fixed point set of a mapping T by $F(T)$ and the solution set of problem (1.2) by Ω, namely,

$$\Omega := \{(x^*, y^*) \in F(U) \times F(T) \ : \ Ax^* = By^*\}.$$

Definition 2.1. Let H be a real Hilbert space.

(1) Let $T : H \to H$ be a mapping. Then, $(I - T)$ is said to be *demi-closed* at zero if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $x_n - Tx_n \longrightarrow 0$, we have $x^* = Tx^*$.

(2) A mapping $T : H \to H$ is said to be *semi-compact* if for any bounded sequence $\{x_n\} \subset H$ with $x_n - Tx_n \longrightarrow 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $x^* \in H$.

Definition 2.2. Let H be a real Hilbert space.

(1) A mapping $T : H \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \ \forall \ (x, y) \in H \times H. \tag{2.1}$$

(2) A mapping $T : H \to H$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\| \ \forall x^* \in F(T), \quad x \in H. \tag{2.2}$$

(3) A mapping $T : H \to H$ is said to be *firmly quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|x - Tx\|^2 \; \forall x^* \in F(T), \;\; x \in H. \tag{2.3}$$

(4) Let D be a nonempty subset of H. A map $T : D \to D$ is said to be *k-strictly pseudo-contractive* if there exists a constant $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \;\; \forall x, y \in D.$$

(5) $T : D \to D$ is said to be *demi-contractive* if $F(T) \neq \emptyset$ and there exists a constant $k \in (0, 1)$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2 \;\; \forall x \in D, \; x^* \in F(T).$$

Remark 2.3. The following inclusions are obvious.

Firmly quasi-nonexpansive $\subset$ *Quasi-nonexpansive* $\subset$ *Demi-contractive.*

We give examples to show that the above inclusions are proper.

Example 2.4. Let $H = l_2$; $D := \{x \in l_2 : \|x\|_2 \leq 1\}$ and $T : D \to D$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$. Then, T has a unique fixed point, zero. Clearly, T is a quasi-nonexpansive mapping which is not firmly quasi-nonexpansive.
In fact, we have:

$$\|Tx - 0\| = \|x - 0\|, \tag{$*$}$$

so T is quasi-nonexpansive, and for every $x \neq 0$, suppose

$$\|Tx - 0\|^2 \leq \|x - 0\|^2 - \|x - Tx\|^2.$$

Then, using $(*)$, we obtain that $x = 0$, which is a contradiction. Therefore, T is not firmly quasi-nonexpansive.

Example 2.5. Let $H = l_2$ and $T : l_2 \to l_2$ be defined by $T(x_1, x_2, x_3, \cdots) = -\frac{5}{2}(x_1, x_2, x_3, \cdots)$, for arbitrary $(x_1, x_2, x_3, \cdots) \in l_2$. Then, $F(T) = \{0\}$, and T is a demi-contractive mapping which is not quasi-nonexpansive.
Indeed, for each $x \in l_2$, we have

$$\|Tx - 0\|^2 = \frac{25}{4}\Big\|x - 0\Big\|^2,$$

which implies that T is not quasi-nonexpansive. We also have that

$$\|x - Tx\|^2 = \Big\|x - \Big(-\frac{5}{2}x\Big)\Big\|^2 = \frac{49}{4}\Big\|x - 0\Big\|^2,$$

so that

$$\|x - 0\|^2 = \frac{4}{49}\Big\|x - Tx\Big\|^2. \tag{$**$}$$

Thus, using $(**)$, we have:

$$\|Tx - 0\|^2 = \|x - 0\|^2 + \frac{21}{4}\Big\|x - 0\Big\|^2 = \|x - 0\|^2 + \frac{3}{7}\Big\|x - Tx\Big\|^2.$$

Hence, T is a demi-contractive mapping with constant $k = \frac{3}{7} \in (0, 1)$.

Lemma 2.6. *(Opial's Lemma [12]) Let H be a real Hilbert space and $\{\mu_n\}$ be a sequence in H such that there exists a nonempty set $W \subset H$ satisfying the following conditions:*

(i) *For every* $\mu \in W$, $\lim_{n\to\infty} \|\mu_n - \mu\|$ *exists;*
(ii) *Any weak-cluster point of the sequence* $\{\mu_n\}$ *belongs to* W.

Then, there exists $w^* \in W$ *such that* $\{\mu_n\}$ *converges weakly to* w^*.

Lemma 2.7. *(see e.g., Chidume, [6]) Let* H *be a real Hilbert space and* $\lambda \in [0,1]$. *Then, for any* $x,\ y,\ z \in H$,

$$\|\lambda x + (1-\lambda)y - z\|^2 = \lambda\|x-z\|^2 + (1-\lambda)\|y-z\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

3. MAIN RESULTS

To approximate a solution of the split equality fixed point problem (1.2), we make the following assumptions:

($\mathbf{A_1}$) $H_1,\ H_2$ and H_3 are real Hilbert spaces, $A: H_1 \to H_3$ and $B: H_2 \to H_3$ are bounded linear operators.

($\mathbf{A_2}$) $U: H_1 \to H_1,\ T: H_2 \to H_2$ are demi-contractive mappings with constants k_1 and k_2, respectively.

($\mathbf{A_3}$) $I-U$ and $I-T$ are demi-closed at zero, and U and T are uniformly continuous.

For arbitrary $x_1 \in H_1$ and $y_1 \in H_2$ define an iterative algorithm by

$$\begin{cases} x_{n+1} = (1-\alpha)\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) + \alpha U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big); \\ \\ y_{n+1} = (1-\alpha)\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) + \alpha T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\alpha \in (0, 1-k)$ and $\gamma \in \Big(0,\ \frac{2}{(\lambda_{A^*A} + \lambda_{B^*B})}\Big)$, where λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B, respectively and $k = \max\{k_1, k_2\}$.

We now prove the following theorem.

Theorem 3.1. *Suppose assumptions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold.*
If $\Omega := \{(x^*, y^*) \in F(U) \times F(T)\ :\ Ax^* = By^*\} \neq \emptyset$, *then the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges weakly to a solution of problem (1.2).*

Proof. Let $(x^*, y^*) \in \Omega$. Using lemma 2.7 and assumption $\mathbf{A_2}$, we have

$$\begin{aligned}
\Big\|x_{n+1} - x^*\Big\|^2 &= \Big\|(1-\alpha)\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) + \alpha U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - x^*\Big\|^2 \\
&= (1-\alpha)\Big\|x_n - \gamma A^*(Ax_n - By_n) - x^*\Big\|^2 + \alpha\Big\|U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - x^*\Big\|^2 \\
&- \alpha(1-\alpha)\Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\|^2 \\
&\leq (1-\alpha)\Big\|x_n - \gamma A^*(Ax_n - By_n) - x^*\Big\|^2 + \alpha\Big\|x_n - \gamma A^*(Ax_n - By_n) - x^*\Big\|^2 \\
&+ \alpha k_1\Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\|^2 \\
&- \alpha(1-\alpha)\Big\|x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big)\Big\|^2 \\
&= \Big\|x_n - \gamma A^*(Ax_n - By_n) - x^*\Big\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \quad \alpha(1-k_1-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2 \\
&\leq \quad \Big\|x_n-x^*\Big\|^2-2\gamma\langle Ax_n-By_n, Ax_n-Ax^*\rangle+\gamma^2\lambda_{A^*A}\Big\|Ax_n-By_n\Big\|^2 \\
&- \quad \alpha(1-k_1-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
$$

Similary, we have that

$$
\begin{aligned}
\Big\|y_{n+1}-y^*\Big\|^2 \quad \leq \quad & \Big\|y_n-y^*\Big\|^2+2\gamma\langle Ax_n-By_n, By_n-By^*\rangle+\gamma^2\lambda_{B^*B}\Big\|Ax_n-By_n\Big\|^2 \\
- \quad & \alpha(1-k_2-\alpha)\Big\|y_n+\gamma B^*(Ax_n-By_n)-T\Big(y_n+\gamma B^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
$$

Adding the above two inequalities and using $k=\max\{k_1,k_2\}$ and the fact that $Ax^*=By^*$, we have that

$$
\begin{aligned}
\Big\|x_{n+1}-x^*\Big\|^2+\Big\|y_{n+1}-y^*\Big\|^2 \quad \leq \quad & \Big\|x_n-x^*\Big\|^2+\Big\|y_n-y^*\Big\|^2+\gamma^2(\lambda_{A^*A}+\lambda_{B^*B})\Big\|Ax_n-By_n\Big\|^2 \\
- \quad & 2\gamma\Big\|Ax_n-By_n\Big\|^2 \\
- \quad & \alpha(1-k-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2 \\
- \quad & \alpha(1-k-\alpha)\Big\|y_n+\gamma B^*(Ax_n-By_n)-T\Big(y_n+\gamma B^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\Big\|x_{n+1}-x^*\Big\|^2+\Big\|y_{n+1}-y^*\Big\|^2 \quad \leq \quad & \Big\|x_n-x^*\Big\|^2+\Big\|y_n-y^*\Big\|^2-\gamma\Big(2-\gamma(\lambda_{A^*A}+\lambda_{B^*B})\Big)\Big\|Ax_n-By_n\Big\|^2 \\
- \quad & \alpha(1-k-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2 \\
- \quad & \alpha(1-k-\alpha)\Big\|y_n+\gamma B^*(Ax_n-By_n)-T\Big(y_n+\gamma B^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
$$

Now set $\Omega_n(x^*,y^*)=\Big\|x_n-x^*\Big\|^2+\Big\|y_n-y^*\Big\|^2$. Then, it follows that

$$
\begin{aligned}
\Omega_{n+1}(x^*,y^*) \leq \Omega_n(x^*,y^*) & -\gamma\Big(2-\gamma(\lambda_{A^*A}+\lambda_{B^*B})\Big)\Big\|Ax_n-By_n\Big\|^2 \\
& -\alpha(1-k-\alpha)\Big\|x_n-\gamma A^*(Ax_n-By_n)-U\Big(x_n-\gamma A^*(Ax_n-By_n)\Big)\Big\|^2 \\
& -\alpha(1-k-\alpha)\Big\|y_n+\gamma B^*(Ax_n-By_n)-T\Big(y_n+\gamma B^*(Ax_n-By_n)\Big)\Big\|^2.
\end{aligned}
\tag{3.2}
$$

Since $\alpha\in\Big(0,1-k\Big)$ and $\gamma\in\Big(0,\ \frac{2}{(\lambda_{A^*A}+\lambda_{B^*B})}\Big)$,

we have $2-\gamma(\lambda_{A^*A}+\lambda_{B^*B})>0$ and $1-k-\alpha>0$. It follows that

$$
\Omega_{n+1}(x^*,y^*)\leq\Omega_n(x^*,y^*).
$$

So, the sequence $\{\Omega_n(x^*, y^*)\}$ is non-increasing and bounded below, therefore, it converges. On the other hand, it follows from inequality (3.2) and the convergence of the sequence $\{\Omega_n(x^*, y^*)\}$ that

$$\lim_{n\to\infty} \Big\| Ax_n - By_n \Big\| = 0, \tag{3.3}$$

$$\lim_{n\to\infty} \Big\| x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) \Big\| = 0, \tag{3.4}$$

and

$$\lim_{n\to\infty} \Big\| y_n + \gamma B^*(Ax_n - By_n) - T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) \Big\| = 0. \tag{3.5}$$

Furthermore, since $\{\Omega_n(x^*, y^*)\}$ converges, we have that $\{x_n\}$ and $\{y_n\}$ are bounded. Let x^{**} and y^{**} be the weak-cluster points of the sequences $\{x_n\}$ and $\{y_n\}$, respectively. Then, there exists a subsequence of $\{(x_n, y_n)\}$ (without loss of generality, still denoted by $\{(x_n, y_n)\}$) such that $x_n \rightharpoonup x^{**}$ and $y_n \rightharpoonup y^{**}$. Next, we show that $Ux^{**} = x^{**}$ and $Ty^{**} = y^{**}$. Since U is uniformly continuous, it follows from (3.3) that

$$\lim_{n\to\infty} \Big\| U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - Ux_n \Big\| = 0. \tag{3.6}$$

Similarly, we have that

$$\lim_{n\to\infty} \Big\| T\Big(y_n + \gamma B^*(Ax_n - By_n)\Big) - Ty_n \Big\| = 0. \tag{3.7}$$

We now show that $\lim_{n\to\infty} \Big\| Ux_n - x_n \Big\| = 0$. Using (3.4) and (3.6), we have

$$\begin{aligned}
\Big\| Ux_n - x_n \Big\| &\leq \Big\| x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) \Big\| \\
&+ \Big\| U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - Ux_n \Big\| \\
&+ \Big\| x_n - \gamma A^*(Ax_n - By_n) - x_n \Big\| \\
&\leq \Big\| x_n - \gamma A^*(Ax_n - By_n) - U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) \Big\| \\
&+ \Big\| U\Big(x_n - \gamma A^*(Ax_n - By_n)\Big) - Ux_n \Big\| \\
&+ \gamma \Big\| A^* \Big\| \Big\| Ax_n - By_n \Big\| \longrightarrow 0 \text{ as n} \to \infty.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \Big\| Ux_n - x_n \Big\| = 0. \tag{3.8}$$

Similarly, we have that

$$\lim_{n\to\infty} \Big\| Ty_n - y_n \Big\| = 0. \tag{3.9}$$

Now, since $x_n \rightharpoonup x^{**}$, $I - U$ is demi-closed at zero, and $\lim_{n\to\infty} \Big\| Ux_n - x_n \Big\| = 0$, we have that $Ux^{**} = x^{**}$, which shows that $x^{**} \in F(U)$. Similarly, we have that $y^{**} \in F(T)$. Since A and B are bounded linear operators, and $\{x_n\}$ and $\{y_n\}$ converge weakly to x^{**} and y^{**}, respectively, we have that for arbitrary $f \in H_3^*$,

$$f(Ax_n) = (f \circ A)(x_n) \longrightarrow (f \circ A)(x^{**}) = f(Ax^{**}).$$

Similarly,

$$f(By_n) = (f \circ B)(y_n) \longrightarrow (f \circ B)(y^{**}) = f(By^{**}).$$

These convergences imply that

$$Ax_n - By_n \rightharpoonup Ax^{**} - By^{**},$$

which, in turn, implies that

$$\left\|Ax^{**} - By^{**}\right\| \leq \liminf_{n\to\infty} \left\|Ax_n - By_n\right\| = 0,$$

so that $Ax^{**} = By^{**}$. Hence, we have $(x^{**}, y^{**}) \in \Omega$.
Summing up, we have proved that:

(1) for each $(x^*, y^*) \in \Omega$, $\lim_{n\to\infty} \left(\left\|x_n - x^*\right\|^2 + \left\|y_n - y^*\right\|^2\right)$ exists;
(2) each weak cluster point of the sequence $\{(x_n, y_n)\}$ belongs to Ω.

Taking $H = H_1 \times H_2$ with the norm $\left\|(x, y)\right\| = \left(\left\|x\right\|^2 + \left\|y\right\|^2\right)^{\frac{1}{2}}$, $W = \Omega$, $\mu_n = (x_n, y_n)$, and $\mu = (x^*, y^*)$ in lemma 2.6, we have that there exists $(\bar{x}, \bar{y}) \in \Omega$ such that $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$. Hence, the sequence $\{(x_n, y_n)\}$ generated by the iterative scheme (3.1) converges weakly to a solution of problem (1.2) in Ω. This completes the proof. □

We now prove the following strong convergence theorem.

Theorem 3.2. *Suppose assumptions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold and let* $\{x_n\}$ *and* $\{y_n\}$ *be as in theorem 3.1. If* $\Omega \neq \emptyset$*, and the mappings* U *and* T *are semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges strongly to a solution of problem (1.2) in* Ω.

Proof. Since U and T are semi-compact, $\{x_n\}$ and $\{y_n\}$ are bounded (by theorem 3.1), and $\lim_{n\to\infty} \left\|(I - U)x_n\right\| = 0$, $\lim_{n\to\infty} \left\|(I - T)y_n\right\| = 0$, there exist (without loss of generality) subsequences $\{x_{n_j}\} \subset \{x_n\}$ and $\{y_{n_j}\} \subset \{y_n\}$ such that $\{x_{n_j}\}$ and $\{y_{n_j}\}$ converge strongly to some points x^* and y^*, respectively. It follows from the demi-closedness of $I - U$ and $I - T$ that $x^* \in F(U)$ and $y^* \in F(T)$.
Thus,

$$\left\|Ax^* - By^*\right\| = \lim_{j\to\infty} \left\|Ax_{n_j} - By_{n_j}\right\| = 0.$$

This implies that $Ax^* = By^*$. Hence, $(x^*, y^*) \in \Omega$. On the other hand, since $\Omega_n(x, y) = \left\|x_n - x\right\|^2 + \left\|y_n - y\right\|^2$ for any $(x, y) \in \Omega$, we know that $\lim_{j\to\infty} \Omega_{n_j}(x^*, y^*) = 0$. From theorem 3.1, we have $\lim_{n\to\infty} \Omega_n(x^*, y^*)$ exists, therefore $\lim_{n\to\infty} \Omega_n(x^*, y^*) = 0$. So, as in the proof of theorem 3.1, the iterative scheme converges strongly to a solution of problem (1.2) in Ω. The proof is complete. □

Corollary 3.1. *Suppose assumptions* $(\mathbf{A_1}) - (\mathbf{A_3})$ *hold and let* $\{x_n\}$ *and* $\{y_n\}$ *be as in theorem 3.1. If* $\Omega \neq \emptyset$*, and the mappings* U *and* T *have convex and compact domain* D*, then, the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges strongly to a solution of problem (1.2) in* Ω.

Proof. Since every map $T : D \subset H \to D$, with D compact, is semi-compact, the proof follows from theorem 3.2. □

Corollary 3.2. *Suppose assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_3})$ *hold and let* $\{x_n\}$ *and* $\{y_n\}$ *be as in theorem 3.1. If* $\Omega \neq \emptyset$*, and the mappings* U *and* T *are quasi-nonexpansive and semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges strongly to a solution of problem (1.2) in* Ω.

Corollary 3.3. *Suppose assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_3})$ *hold and let* $\{x_n\}$ *and* $\{y_n\}$ *be as in theorem 3.1. If* $\Omega \neq \emptyset$*, and the mappings* U *and* T *are firmly quasi-nonexpansive and semi-compact, then, the sequence* $\{(x_n, y_n)\}$ *generated by (3.1) converges strongly to a solution of problem (1.2) in* Ω.

Remark 3.4. Our theorems 3.1 and 3.2 extend and complement the results of Moudafi $et\ al.$ [9], Moudafi [10], and Yuan-Fang $et\ al.$ [15].

Remark 3.5. The recursion formula considered in this paper is of Krasnoselskii-type which, in general, converges as fast as a geometric progression.

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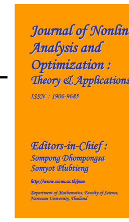

SOME GENERALIZED TRIPLE SEQUENCE SPACES OF REAL NUMBERS

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ABSTRACT. The idea of difference sequence spaces was first introduced by Kizmaz in 1981 and the idea of triple sequences was first introduced by Sahiner et.al. 2007. In this article we introduce the notion of triple sequence spaces $c_0^3(\Delta)$, $c^3(\Delta)$, and $l_\infty^3(\Delta)$ using the difference operator Δ. We study some of their algebraic and topological properties and prove some inclusion results.

KEYWORDS : Difference operator, triple sequence space, solidity.

AMS Subject Classification: 40A05, 46A45.

1. Introduction and Preliminaries

A triple sequence (real or complex) can be defined as a function $x : N \times N \times N \longrightarrow R(C)$, where N, R and C denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner, et. al. [1, 2] and Dutta, et. al. [3] and others..

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [6] as follows:

$Z(\Delta) = \{(x_n) \in w : (\Delta x_n) \in Z\}$, for $Z = c, c_0, l_\infty$, the spaces of convergent, null and bounded sequences, respectively, where $\Delta x_n = x_n - x_{n+1}$ for all $n \in N$. Later on it was further investigated by Tripathy [4] and many others. Tripathy and Sarma[5] introduced difference double sequence spaces as follows:

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$Z(\Delta) = \{(x_{mn}) \in w : (\Delta x_{mn}) \in Z\}$, for $Z = c^2, c_0^2, l_\infty^2$, the spaces of convergent, null and bounded double sequences respectively, where $\Delta x_{mn} = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbf{N}$.

Definition 1.1 [1]: A triple sequence (x_{lmn}) is said to be convergent to L in Pringsheim′s sense if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in N$ such that

$|x_{lmn} - L| < \epsilon$ whenever $l \geq \mathbf{N}$, $m \geq \mathbf{N}$, $n \geq \mathbf{N}$ and we write $lim_{l,m,n \longrightarrow \infty} x_{lmn} = L$.

Note: A triple sequence is convergent in Pringsheim′s sense may not be bounded [2].

Definition 1.2 [1]: A triple sequence (x_{lmn}) is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $\mathbf{N}(\epsilon) \in N$ such that

$|x_{lmn} - x_{pqr}| < \epsilon$ whenever $l \geq p \geq \mathbf{N}$, $m \geq q \geq \mathbf{N}$, $n \geq r \geq \mathbf{N}$.

Definition 1.3 [1]: A triple sequence (x_{lmn}) is said to be bounded if there exists $M > 0$, such that $|x_{lmn}| < M$ for all $l, m, n \in N$.

Definition 1.4 [3]: A triple sequence (x_{lmn}) is said to be converge regularly if it is convergent in Pringsheim′s sense and in addition the following limits holds:

$lim_{n \longrightarrow \infty} x_{lmn} = L_{lm}$ $(l, m \in N)$

$lim_{m \longrightarrow \infty} x_{lmn} = L_{ln}$ $(l, n \in N)$

$lim_{l \longrightarrow \infty} x_{lmn} = L_{mn}$ $(m, n \in N)$

Let w^3 denote the set of all triple sequence of real numbers. We can define the class of triple sequences as follows:

$c_0^3 = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is convergent to zero in Pringsheim′s sense }

$c^3 = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is convergent in Pringsheim ′s sense }

$l_\infty^3 = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is bounded in Pringsheim′s sense }

$c^{3R} = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is regularly convergent }

$c^{3B} = \{x = (x_{lmn}) \in w^3 : (x_{lmn})$ is convergent in Pringsheim ′s sense and bounded }

These classes are all linear spaces.

It is obvious that $c_0^3 \subset c^3$; $c^{3R} \subset c^{3B} \subset l_\infty^3$ and the inclusion is strict.

Theorem 1.1: The spaces c_0^3, c^3, l_∞^3, c^{3R} and c^{3B} are complete normed linear spaces with the normed.

$\|x\| = sup_{l,m,n}|x_{lmn}| < \infty$

Proof: simple.

Example 1.1 [1]: Let $x_{lmn} = \begin{cases} mn, & l = 3 \\ nl, & m = 5 \\ lm, & n = 7 \\ 8, & otherwise \end{cases}$

Then $(x_{lmn}) \to 8$ in Pringsheim′s sense but not bounded as well as not regularly convergent.

Example 1.2: Let $x_{lmn} = 1$, for all $l, m, n \in N$. Then (x_{lmn}) is convergent in Pringsheim′s sense, bounded and regularly convergent.

Definition 1.5[3]: A triple sequence space E is said to be solid if $(\alpha_{lmn}x_{lmn}) \in E$ whenever $(x_{lmn}) \in E$ and for all sequences (α_{lmn}) of scalars with $|\alpha_{lmn}| \leq 1$, for all $l, m, n \in \mathbf{N}$.

Definition 1.6 [3]: A triple sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 1.1 [3]: A sequence space is solid implies that it is monotone.

Definition 1.7 [3]: A triple sequence space E is said to be convergence free if $(y_{lmn}) \in E$, whenever $(x_{lmn}) \in E$ and $x_{lmn} = 0$ implies $y_{lmn} = 0$.

Definition 1.8 [3]: A triple sequence space E is said to be symmetric if $(x_{lmn}) \in E$ implies $(x_{\pi(l)\pi(m)\pi(n)}) \in E$, where π is a permutation of $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$.

Now we introduced the difference triple sequence spaces as follows:

$c_0^3(\Delta)$={$(x_{lmn}) \in w^3$: (Δx_{lmn}) is regularly null }

$c^3(\Delta)$={$(x_{lmn}) \in w^3$: (Δx_{lmn}) is convergent in Pringsheim's sense }

$c^{3R}(\Delta)$={$(x_{lmn}) \in w^3$: (Δx_{lmn}) is regularly convergent }

$l_\infty^3(\Delta)$={$(x_{lmn}) \in w^3$: (Δx_{lmn}) is bounded }

$c^{3B}(\Delta)$={$(x_{lmn}) \in w^3$: (Δx_{lmn}) is convergent in Pringsheim's sense and bounded }

Where $\Delta x_{lmn} = x_{lmn} - x_{lmn+1} - x_{lm+1n} + x_{lm+1n+1} - x_{l+1mn} + x_{l+1mn+1} + x_{l+1m+1n} - x_{l+1m+1n+1}$

2. MAIN RESULTS

Theorem 2.1: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$, $c^{3B}(\Delta)$ are linear spaces.

Proof: Obvious.

Theorem 2.2: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$, $c^{3B}(\Delta)$ are complete normed linear spaces with the norm

$$\|x\| = sup_l|x_{l11}| + sup_m|x_{1m1}| + sup_n|x_{11n}| + sup_{l,m,n}|\Delta x_{lmn}| < \infty$$

Proof: Let (x^i) be a Cauchy sequence in $l_\infty^3(\Delta)$, where $x^i = (x_{lmn}^i) \in l_\infty^3(\Delta)$ for each $i \in \mathbf{N}$.
Then we have,
$\|x^i - x^j\| = sup_l|x_{l11}^i - x_{l11}^j| + sup_m|x_{1m1}^i - x_{1m1}^j| + sup_n|x_{11n}^i - x_{11n}^j| + sup_{l,m,n}|\Delta x_{lmn}^i - \Delta x_{lmn}^j| \to 0$

as $i, j \to \infty$

Therefore, $|x_{lmn}^i - x_{lmn}^j| \to 0$, for $i, j \to \infty$ and each $l, m, n \in \mathbf{N}$

Hence (x_{lmn}^i)=$(x_{lmn}^1, x_{lmn}^2, x_{lmn}^3, \ldots \ldots \ldots \ldots \ldots \ldots)$ is a Cauchy sequence in R (Real numbers).

Whence by the completeness of R, it converges to x_{lmn} say, i.e., there exists

$lim\ x_{lmn}^i = x_{lmn}$ for each $l, m, n \in \mathbf{N}$

Further for each $\epsilon > 0$, there exists $\mathbf{N} = \mathbf{N}(\epsilon)$, such that for all $i, j \geq \mathbf{N}$, and for all $l, m, n \in \mathbf{N}$

$|x_{l11}^i - x_{l11}^j| < \epsilon$, $|x_{1m1}^i - x_{1m1}^j| < \epsilon$, $|x_{11n}^i - x_{11n}^j| < \epsilon$

$|\Delta x_{lmn}^i - \Delta x_{lmn}^j| = |(x_{l+1m+1n+1}^i - x_{l+1m+1n+1}^j) - (x_{l+1m+1n}^i - x_{l+1m+1n}^j) - (x_{l+1mn+1}^i - x_{l+1mn+1}^j) + (x_{l+1mn}^i - x_{l+1mn}^j) - (x_{lm+1n+1}^i - x_{lm+1n+1}^j) + (x_{lm+1n}^i - x_{lm+1n}^j) + (x_{lmn+1}^i - x_{lmn+1}^j) - (x_{lmn}^i - x_{lmn}^j)| < \epsilon$

and

$lim_j\ |x_{l11}^i - x_{l11}^j| = |x_{l11}^i - x_{l11}| \leq \epsilon$,

$lim_j\ |x_{1m1}^i - x_{1m1}^j| = |x_{1m1}^i - x_{1m1}| \leq \epsilon$,

$lim_j\ |x_{11n}^i - x_{11n}^j| = |x_{11n}^i - x_{11n}| \leq \epsilon$,

Now

$lim_j|\Delta x_{lmn}^i - \Delta x_{lmn}^j|$

$= |(x^i_{l+1m+1n+1} - x^j_{l+1m+1n+1}) - (x^i_{l+1m+1n} - x^j_{l+1m+1n}) - (x^i_{l+1mn+1} - x^j_{l+1mn+1}) + (x^i_{l+1mn} - x^j_{l+1mn}) - (x^i_{lm+1n+1} - x^j_{lm+1n+1}) + (x^i_{lm+1n} - x^j_{lm+1n}) + (x^i_{lmn+1} - x^j_{lmn+1}) - (x^i_{lmn} - x^j_{lmn})|$

$= |(x^i_{l+1m+1n+1} - x_{l+1m+1n+1}) - (x^i_{l+1m+1n} - x_{l+1m+1n}) - (x^i_{l+1mn+1} - x_{l+1mn+1}) + (x^i_{l+1mn} - x_{l+1mn}) - (x^i_{lm+1n+1} - x_{lm+1n+1}) + (x^i_{lm+1n} - x_{lm+1n}) + (x^i_{lmn+1} - x_{lmn+1}) - (x^i_{lmn} - x_{lmn})| \leq \epsilon$
for all $i \geq \mathbf{N}$

Since ϵ is not dependent on l, m, n

$sup_{l,m,n}\ |(x^i_{l+1m+1n+1} - x_{l+1m+1n+1}) - (x^i_{l+1m+1n} - x_{l+1m+1n}) - (x^i_{l+1mn+1} - x_{l+1mn+1}) + (x^i_{l+1mn} - x_{l+1mn}) - (x^i_{lm+1n+1} - x_{lm+1n+1}) + (x^i_{lm+1n} - x_{lm+1n}) + (x^i_{lmn+1} - x_{lmn+1}) - (x^i_{lmn} - x_{lmn})| \leq \epsilon,$

Consequently we have, $\|x^i_{lmn} - x_{lmn}\| \leq 4\epsilon$, for all $i \geq \mathbf{N}$

Hence we obtain $x^i_{lmn} \to x_{lmn}$ as $i \to \infty$ in $l^3_\infty(\Delta)$

Now we have to show that $(x_{lmn}) \in l^3_\infty(\Delta)$

$|x_{lmn} - x_{l+1m+1n+1}| = |x_{lmn} - x^N_{lmn} + x^N_{lmn} - x^N_{l+1m+1n+1} + x^N_{l+1m+1n+1} - x_{l+1m+1n+1}|$

$\leq |x^N_{lmn} - x_{l+1m+1n+1}| + \|x^N_{lmn} - x_{lmn}\| = O(1)$

This implies x = $(x_{lmn}) \in l^3_\infty(\Delta)$, (Since $l^3_\infty(\Delta)$ is a linear space.)

Hence $l^3_\infty(\Delta)$ is complete.

Similarly the others.

Theorem 2.3:

(i) $c^3_0(\Delta) \subset c^3(\Delta)$ and the inclusion is strict. .

(ii) $c^{3R}(\Delta) \subset c^3(\Delta)$ and the inclusion is strict.

Proof: The inclusion being strict follows from the following example:

Example 2.1: For theorem (i)we consider the sequence (x_{lmn}) defined by

(x_{lmn})= $-lmn$, for all $l, m, n \in \mathbf{N}$

Then $(\Delta x_{lmn}) \in c^3$, but $(\Delta x_{lmn}) \notin c^3_0$

Hence the inclusion is strict.

Example 2.2: For theorem (ii) we consider the sequence defined by

$$x_{lmn} = \begin{cases} (-1)^n lmn, & \text{for } l=1, m=1,2,3 \text{ for all } n \in \mathbf{N} \\ 1, & \text{otherwise} \end{cases}$$

Clearly $(\Delta x_{lmn}) \in c^3$, but the sequence $(\Delta x_{lmn}) \notin c^{3R}$

Hence the inclusion $c^{3R}(\Delta) \subset c^3(\Delta)$, is strict.

Theorem 2.4: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$ are not solid in general.

Proof: This is clear from the following examples:

Example 2.3: We consider the sequence (x_{lmn}) defined by

(x_{lmn})= 2, for all $l, m, n \in \mathbf{N}$

Clearly the difference triple sequence $(\Delta x_{lmn}) \in c_0^3$, c^3, c^{3R} and c^{3B}

Consider the sequence of scalars defined by

α_{lmn} = $(-1)^{l+m+n}$, for all $l, m, n \in \mathbf{N}$

Then the sequence $(\alpha_{lmn}x_{lmn})$ takes the following form

$\alpha_{lmn}x_{lmn}$ = $2.(-1)^{l+m+n}$, for all $l, m, n \in \mathbf{N}$

Clearly $(\Delta\alpha_{lmn}x_{lmn}) \notin c_0^3$, c^3, c^{3R} and c^{3B}

Hence $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$ and $c^{3B}(\Delta)$ are not solid.

Example 2.4: We consider the sequence (x_{lmn}) defined by

(x_{lmn})= lmn, for all $l, m, n \in \mathbf{N}$

Clearly the sequence $(\Delta x_{lmn}) \in l_\infty^3$

Consider the sequence of scalars defined by

α_{lmn} = $(-1)^{m+n}$, for all $l, m, n \in \mathbf{N}$

Then the sequence $(\alpha_{lmn}x_{lmn})$ takes the following form

$\alpha_{lmn}x_{lmn}$ = $(-1)^{m+n}lmn$, for all $l, m, n \in \mathbf{N}$

Clearly, $(\Delta\alpha_{lmn}x_{lmn}) \notin l_\infty^3$,

Hence $l_\infty^3(\Delta)$ are not solid.

Theorem 2.5: The spaces $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$ are not symmetric in general.

Proof: The proof is clear from the following examples:

Example 2.5: Consider the sequence (x_{lmn}) defined by

(x_{lmn})= m, for all $l, m, n \in \mathbf{N}$

Clear the sequence $(\Delta x_{lmn}) \in c_0^3$, c^3, c^{3R} and c^{3B}

Now Consider a rearrange sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn}= \begin{cases} m+1 \text{ , for } m=l, n \text{ is even} \\ m-1 \text{ , for } m=l+1, n \text{ is even} \\ m \text{ , otherwise} \end{cases}$$

Clearly $(\Delta y_{lmn}) \notin c_0^3$, c^3, c^{3R} and c^{3B}

Hence $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$ and $c^{3B}(\Delta)$ are not symmetric.

Example 2.6: Consider the sequence (x_{lmn}) defined by

(x_{lmn})= lmn, for all $l, m, n \in \mathbf{N}$

Clear the sequence $(\Delta x_{lmn}) \in l_\infty^3$

Now Consider a rearrange sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn}= \begin{cases} m+1 \text{ , for } m=l, n \text{ is even} \\ m-1 \text{ , for } m=l+1, n \text{ is even} \\ m \text{ , otherwise} \end{cases}$$

Then the sequence $(\Delta y_{lmn}) \notin l_\infty^3$

Hence $l_\infty^3(\Delta)$ are not symmetric.

Theorem 2.6: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$ are not convergence free in general.

Proof: We provide an example to prove the result.

Example 2.7: Consider the sequence defined by

$$x_{lmn} = \begin{cases} 0, & \text{if } n=1, \text{ for all } l, m \in \mathbf{N} \\ -2, & \text{otherwise} \end{cases}$$

Clearly the triple sequence $(\Delta x_{lmn}) \in c_0^3$, c^3, c^{3R}, l_∞^3 and c^{3B}

Let the sequence (y_{lmn}) be defined by

$$y_{lmn} = \begin{cases} 0, & \text{if } n \text{ is odd, for all } l, m \in \mathbf{N} \\ lmn, & \text{otherwise} \end{cases}$$

Clearly $(\Delta y_{lmn}) \notin c_0^3$, c^3, c^{3R}, l_∞^3 and c^{3B}, Hence $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$, are not convergence free.

Theorem 2.7: The classes of sequences $c_0^3(\Delta)$, $c^3(\Delta)$, $c^{3R}(\Delta)$, $l_\infty^3(\Delta)$ and $c^{3B}(\Delta)$ are all sequence algebra.

Proof: It is obvious.

Conclusion: We have introduced the notions of null, convergent and bounded triple sequence spaces based on the difference operator Δ and have investigated its different properties, which are the generalizations of null, convergent and bounded triple sequence spaces. Further generalizations may be possible based on the difference operator Δ^m.

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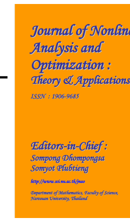

ON THE SYMMETRIC VECTOR QUASI-EQUILIBRIUM PROBLEM VIA NONLINEAR SCALARIZATION METHOD

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ABSTRACT. The aim of this paper, among other things, is, using a nonlinear scalarization function and its properties, to study an existence theorem for a solution of SVQEP in the setting of real topological vector space. One can consider this note as a new version of the reference [5] by replacing a nonlinear scalarization function by a linear functional.

KEYWORDS : Symmetric vector quasi-equilibrium problem; Properly quasi-convex; Acyclic map; Admissible set

1. INTRODUCTION

Let X and Y be real Hausdorff topological vector spaces (for short, t.v.s.), C and D be nonempty subsets of X and Y, respectively. Let Z be a real Hausdorff t.v.s. with its topological dual space Z^*. The pairing between Z and Z^* is denoted by $\langle .,. \rangle$. Let $P \subsetneqq Z$ be a convex cone with $\operatorname{int} P \neq \emptyset$, where $\operatorname{int} P$ denotes the interior of P. Let $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ be set-valued mappings and let $f,\ g : C \times D \longrightarrow Z$ be two vector-valued functions.

In 2003, Fu [8] introduced the symmetric vector quasi-equilibrium problem (for short, SVQEP) that consists in finding $(\bar{x}, \bar{y}) \in C \times D$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \notin -\operatorname{int} P, \quad \forall x \in S(\bar{x}, \bar{y}),$$
$$g(\bar{x}, y) - g(\bar{x}, \bar{y}) \notin -\operatorname{int} P, \quad \forall y \in T(\bar{x}, \bar{y}).$$

The SVQEP is a generalization of the (scalar) symmetric quasi-equilibrium problem (for short, SQEP) posed by Noor and Oettli [10] which this problem is a generalization of the equilibrium problem that, at the first, proposed by Blum and Oettli [3]. The equilibrium problem contains as special cases, for instance, optimization problems, problems of Nash equilibria, variational inequalities, and complementarity problems (see [3]).

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The aim of this paper, among other things, is, using a nonlinear scalarization function and its properties, to study an existence theorem for a solution of SVQEP in the setting real of t.v.s. This method for obtaining a solution of SVQEP is different from that which is used by Fu in [8]. Fu?s method is based on the notion of weak minimal points and well-known Kakutani-Fan- Glicksberg Fixed point theorem in locally convex Hausdorff space. Also our method enables us extends some results in [4, 8, 10, 11].

2. DEFINITIONS AND PRELIMINARIES

In the rest of this section we recall some definitions and preliminaries results which we need in the sequel.

In this paper, all topological spaces are assumed to be Hausdorff. As mentioned before, let $P \subsetneqq Z$ be a convex cone with $\operatorname{int} P \neq \emptyset$. We can define a vector ordering in Z by setting

$$x \preceq y \Leftrightarrow y - x \in P,$$

and a weak ordering by setting

$$x \prec y \Leftrightarrow y - x \in \operatorname{int} P.$$

We will denote usual ordering on real numbers by $\leq$.

It is clear that $P \cap \text{-int}\, P = \emptyset$, since $P + \operatorname{int} P \subseteq \operatorname{int} P$ and $P \neq Z$ (this fact will be used in Lemma 3.1).

Let E be a t.v.s. and $C : E \longrightarrow 2^E$ a multi-valued map and for all $x \in E$, $C(x)$ is a solid cone (that is, $intC(x)$ is non empty). Let $e : E \longrightarrow E$ be a map with $e(x) \in C(x)$ for $x \in E$. The non linear scalarization function $\xi : E \times E \longrightarrow R$ is defined as follows:

$$\xi(x, y) = \inf\{r \in R : y \in re(x) - C(x)\}.$$

Definition 2.1 [8]. Let B be a nonempty subset of Z. Element $b \in B$ is called a weak minimal point of B if $B \cap (b - \operatorname{int} P) = \emptyset$. The set of all weak minimal points of B will be denoted by $\min{}_w B$.

Lemma 2.1 [7]. Let B be a nonempty compact subset of Z. Then

(i) $\min_w B \neq \emptyset$,

(ii) $B \subset \min_w B + (\operatorname{int} P \cup \{0\})$.

In the following definition (i)-(iv) is due to Ferro [7] and (v) to Tanaka [12].

Definition 2.2. Let (Z, P) be an ordered topological vector space, and let C be a nonempty convex subset of a vector space X. Let a vector mapping $f : C \longrightarrow Z$ be given.

(i) f is called convex if for every x, $y \in C$ and $t \in [0, 1]$, one has $f(tx + (1 - t)y) \preceq tf(x) + (1 - t)f(y)$.

(ii) f is called properly quasi-convex if for every $x, y \in C$ and $t \in [0, 1]$, one has either $f(tx + (1 - t)y) \preceq f(x)$ or $f(tx + (1 - t)y) \preceq f(y)$.

(iii) f is called P-l.s.c. if, for all $z \in Z$, the set $L(z) = \{x \in C : z \not\prec f(x)\}$ is closed in C.

(iv) f is called P-u.s.c. if, for all $z \in Z$, the set $U(z) = \{x \in C : f(x) \not\prec z\}$ is closed in C.

(v) f is called natural quasi-convex if for every $x, y \in C$ and $t \in [0,1]$, there exists $\mu \in [0,1]$ such that $f(tx + (1-t)y) \preceq \mu f(x) + (1-\mu)f(y)$

Also, the function f is said to be natural quasi-concave(respectively, concave, properly quasi-concave) if $-f$ is natural quasi-convex(respectively, convex, properly quasi-convex).

Remark 2.1. Every convex or properly quasi-convex function is natural quasi-convex function (see Lemma 2.1 [14]). A vector mapping may be convex and not properly quasi-convex, and conversely (see [7]). Consequently, the class of natural quasi-convex functions is strictly larger than both the class of convex functions and the class of properly quasi-convex functions. It is easily seen that properly quasi-convexity and quasi-convexity are equivalent to each other in the scalar case, i.e., $Z = \mathbb{R}$ and $P = [0, \infty)$.

Definition 2.3. Let X and Y be two topological spaces. A set-valued mapping $T : X \longrightarrow 2^Y$ is called:

(i) **upper semi-continuous** (u.s.c.) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $t \in U$, $T(t) \subseteq V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) **lower semi-continuous** (l.s.c.) at $x \in X$ if for each open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U$, $T(t) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) **continuous** on X if it is at the same time u.s.c. and l.s.c. on X.

(iv) **closed** if the graph $G_r(T)$ of T, i.e., $\{(x, y) : x \in X, y \in T(x)\}$, is a closed set in $X \times Y$.

(v) **compact** if the closure of range T, i.e., $\overline{T(X)}$, is compact, where $T(X) = \cup_{x \in X} T(x)$.

Remark 2.2 [13]. T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$, and any net $\{x_\alpha\}$, $x_\alpha \longrightarrow x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \longrightarrow y$.

Definition 2.4. Let X be a topological space, Y be a t.v.s. A function $f : X \longrightarrow Y$ is said to be demicontinuous if

$$f^{-1}(M) = \{x \in X : f(x) \in M\}$$

is closed in X for each closed half space $M \subset Y$.

Lemma 2.2 [14]. Let X be a topological space, Z a t.v.s. and $f : X \longrightarrow Z$ be a demicontinuous function, then for any $x^* \in Z^*$, the composite function $x^* \circ f$ is continuous, where Z^* is the topological dual space of Z.

Definition 2.5 [11]. A nonempty topological space is acyclic if all of its reduced Cech homology groups over rationals vanish. Note that any convex or star-shape subset of a topological vector space is contractible, and that any contractible space is acyclic. A map $T : X \longrightarrow 2^Y$ is said to be acyclic if it is u.s.c. with compact acyclic values .

Definition 2.6 [11]. A nonempty subset X of a t.v.s. E is said to be admissible provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \longrightarrow X$ such that $x - h(x) \in V$, for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. Note that every nonempty convex subset of a locally convex t.v.s. is admissible (see [9]). Other examples of admissible t.v.s. are l^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalent class of measurable functions on $[0,1]$, the Hardy spaces H^p for $0 < p < 1$ and certain Orlicz spaces. Ultrabarrelled t.v.s. are also admissible.

We need the following theorem in the sequel.

Theorem 2.1 [11]. Let C and D be admissible convex subsets of t.v.s. X and Y, respectively. Let $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ be compact acyclic maps, and $f, g : C \times D \longrightarrow \mathbb{R}$ l.s.c. functions such that (i) The functions

$$F(x,y) = \min\{f(\xi,y) : \xi \in S(x,y)\},$$

$$G(x,y) = \min\{g(x,\eta) : \eta \in T(x,y)\}$$

are u.s.c. on $C \times D$, and

(ii) For each $(x,y) \in C \times D$, the sets

$$A(x,y) = \{\xi \in S(x,y) : f(\xi,y) = F(x,y)\},$$

$$B(x,y) = \{\eta \in T(x,y) : g(x,\eta) = G(x,y)\}$$

are acyclic.

Then there exists an $(\overline{x},\overline{y}) \in C \times D$ such that

$$\overline{x} \in S(\overline{x},\overline{y}), f(x,\overline{y}) \geq f(\overline{x},\overline{y}), \quad \text{for all } x \in S(\overline{x},\overline{y}),$$

$$\overline{y} \in T(\overline{x},\overline{y}), g(\overline{x},y) \geq g(\overline{x},\overline{y}), \quad \text{for all } y \in T(\overline{x},\overline{y}).$$

3. MAIN RESULTS

Throughout this section, let X, Y be real Hausdorff t.v.s., C and D be non empty, admissible convex subsets of X and Y, respectively. Let Z be a real Hausdorff t.v.s. with topological dual space Z^* and $P \subsetneqq Z$ a convex cone with $\operatorname{int} P \neq \emptyset$.

The following Lemma is essential tool for our main results. In the following we establish some important properties of the non linear scalarization function which generalize Propositions 2.3 and 2.4 in [4] from locally convex spaces to topological vector spaces which its proof left to the reader.

Lemma 3.1. Let Z be a t.v.s. and P be a convex cone. Let $e \in intP$ Then the following assertions, for each $r \in R$ and $y \in z$ are satisfied.

(i) $\xi_e(y) = \inf\{r \in R : y \in re - P = \min\{r \in R : y \in re - P\}$;
(ii) $\xi_e(y) \leq r \Leftrightarrow y \in re - P$
(iii) $\xi_e(y) < r \Leftrightarrow y \in re - intP$
(iv) If $y_1 \preceq y_2$, then $\xi_e(y_1) \leq \xi_e(y_2)$;

(v) The function $y \longrightarrow \xi_e(y)$ is continuous, positively homogeneous and sub additive on Z;

(vi) The function $y \longrightarrow \xi_e(y)$ is bounded on some neighborhood of zero.

Now, we are ready to prove existence theorems that extends the main result in [6], Theorems 1,2 and 3 in [8], and also is a generalization of the Theorem 1.1.

Theorem 3.1. Assume that

(i) $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ are continuous and compact; and for each $(x, y) \in C \times D,\ S(x, y),\ T(x, y)$ are nonempty, closed convex subsets;

(ii) $f, g : C \times D \to Z$ are demicontinuous;

(iii) For any fixed $y \in D,\ f(x, y)$ is natural quasi-convex in x; for any fixed $x \in C,\ g(x, y)$ is natural quasi-convex in y.

Then SVQEP has a solution.

Proof. By (ii) and Lemma 3.1 through theorem 2.2 in [6], the composite functions $\xi_e o f$ and $\xi_e o g$ are l.s.c. We claim that the real-valued continuous functions $\xi_e o f$ and $\xi_e o g$ satisfy in conditions (i) and (ii) of Theorem 2.1. Indeed, condition (i) follows from Theorem 1 in [1, p. 122].

Now for condition (ii), we must show that for any fixed $(x, y) \in C \times D$ the set A(x, y) is convex, where

$$\begin{aligned} A(x, y) &= \{u \in S(x, y) : \xi_e \circ f(u, y) = F(x, y)\} \\ F(x, y) &= \min\{\xi_e \circ f(u, y) : u \in S(x, y)\}. \end{aligned}$$

To this end, let $t \in]0, 1[$ and $u_1, u_2 \in A(x, y)$. By the definition of $A(x, y)$, $u_1, u_2 \in A(x, y)$ and convexity of the set $S(x, y)$, we get $(1 - t)u_1 + tu_2 \in S(x, y)$ and $F(x, y) = \xi_e \circ f(u_1, y) = \xi_e \circ f(u_2, y)$. Hence by (iii) there exists $\mu \in]0, 1[$ such that

$$\begin{aligned} F(x, y) &\leq \xi_e \circ f((1 - t)u_1 + tu_2, y) \\ &\leq (1 - \mu)\xi_e \circ f(u_1, y) + \mu\xi_e \circ f(u_2, y) \\ &= (1 - \mu)F(x, y) + \mu F(x, y) \\ &= F(x, y). \end{aligned}$$

In the above, the first inequality holds by the definition of $F(x, y)$ and $(1 - t)u_1 + tu_2 \in S(x, y)$, but the second inequality holds by natural quasi-convexity of the function f in the first argument (assumption (iii)) and to preserve ordering on Z by ξ_e (see, Lemma 3.1 (iv,v)). Then, $(1 - t)u_1 + tu_2 \in A(x, y)$. Similarly $\xi_e \circ g$ satisfies in conditions (i) and (ii) of Theorem 2.1. Now, by virtue of Theorem 2.1, there exists $(\overline{x}, \overline{y}) \in C \times D$ such that

$$\begin{aligned} &\overline{x} \in S(\overline{x}, \overline{y}), \xi_e \circ f(x, \overline{y}) \geq \xi_e \circ f(\overline{x}, \overline{y}),\ \forall x \in S(\overline{x}, \overline{y}), \\ &\overline{y} \in T(\overline{x}, \overline{y}), \xi_e \circ g(\overline{x}, y) \geq \xi_e \circ g(\overline{x}, \overline{y}),\ \forall y \in T(\overline{x}, \overline{y}). \end{aligned}$$

Then by Lemma 3.1 (v),

$$\begin{aligned} &\overline{x} \in S(\overline{x}, \overline{y}), \xi_e(f(x, \overline{y}) - f(\overline{x}, \overline{y})) \geq \xi_e(f(x, \overline{y})) - \xi_e(f(\overline{x}, \overline{y})) \geq 0, \forall x \in S(\overline{x}, \overline{y}), \\ &\overline{y} \in T(\overline{x}, \overline{y}), \xi_e(g(\overline{x}, y) - g(\overline{x}, \overline{y})) \geq \xi_e(g(\overline{x}, y)) - \xi_e(g(\overline{x}, \overline{y})) \geq 0, \forall y \in T(\overline{x}, \overline{y}). \end{aligned}$$

Consequently, it follows from Lemma 3.1 (iii) and the relations (1) and (2) that

$$\overline{x} \in S(\overline{x}, \overline{y}), f(x, \overline{y}) - f(\overline{x}, \overline{y}) \notin -intP,\ \forall x \in S(\overline{x}, \overline{y})$$

and

$$\overline{y} \in T(\overline{x},\overline{y}), g(\overline{x},y) - g(\overline{x},\overline{y}) \notin -intP, \ \forall y \in T(\overline{x},\overline{y}),$$

and so $(\overline{x},\overline{y})$ is a solution of the SVQEP. This completes the proof. □

The following corollary is one of the applications Theorem 3.1. which extends the existence Theorem 3.1 in [14] from locally convex topological vector spaces to topological vector space.

Corollary 3.1. Let C and D be nonempty compact admissible convex sets, and let the vector-valued function $f : C \times D \longrightarrow Z$ satisfy the following conditions

(i) The function f is demicontinuous;

(ii) For any fixed $y \in D$, $f(x,y)$ is natural quasi-convex in x; for any fixed $x \in C$, $f(x,y)$ is natural quasi-concave in y.

Then the vector-valued function f has at least one P-weak saddle point, that is, there exists $(\bar{x},\bar{y}) \in C \times D$ such that

$$f(\bar{x},\bar{y}) - f(x,\bar{y}) \notin \text{int } P \ \forall x \in C$$
$$f(\bar{x},y) - f(\bar{x},\bar{y}) \notin \text{int } P \ \forall x \in D.$$

Proof. It is enough in Theorem 3.1, we define the set-valued mappings $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ as $S(x,y) = C$, $T(x,y) = D$, and also the vector-valued function g on $C \times D$ as $g(x,y) = -f(x,y)$. □

By using Theorem 2.1 and Lemma 3.1 we can state the following theorem which is another version of Theorem 3.1 without continuity condition of the maps.

Theorem 3.2. Let $S : C \times D \longrightarrow 2^C$ and $T : C \times D \longrightarrow 2^D$ be compact acyclic maps. Suppose that $f, g : C \times D \longrightarrow Z$ and $\xi_e \in S_{-\text{int } P,P}$, be such that

(i) The composite functions $\xi_e \circ f$, $\xi_e \circ g$ are l.s.c.,

(ii) The functions

$$\begin{aligned} F(x,y) &= \min\{\xi_e(f(\xi,y)) \ : \ \xi \in S(x,y)\}, \\ G(x,y) &= \min\{\xi_e(g(x,\eta)) \ : \ \eta \in T(x,y)\} \end{aligned}$$

are u.s.c. on $C \times D$,

(iii) For each $(x,y) \in C \times D$, the sets

$$A(x,y) = \{u \in S(x,y) : \ \xi_e(f(u,y)) = F(x,y)\},$$
$$B(x,y) = \{\eta \in T(x,y) : \ \xi_e(g(x,\eta)) = G(x,y)\}$$

are acyclic.

Then SVQEP has a solution.

Remark 3.1. Let us briefly discuss assumptions (i),(iii) and convexity $C \times D$. The lower semicontinuity of $\xi_e \circ f$ and $\xi_e \circ g : C \times D \longrightarrow Z$ is ensured whenever f and g are P-l.s.c. This follows from Lemma 2.4 in [2]. We can omit compactness condition of the sets C and D in Theorem 1 in [10], by using Himmelberg's Fixed point theorem [11] instead of Berge's maximum Theorem in its proof. Then by using this form of the Theorem 1 in [10] and the property of $\xi_e \in S_{-\text{int } P,P}$, we can omit condition (iii) in Theorem 3.2, if C and D be nonempty convex subsets of real locally convex Hausdorff spaces X and Y, respectively, and S, T be u.s.c. and compact maps with nonempty closed convex values. At last convexity of $C \times D$ is not essential. In fact, $C \times D$ can be any subset of $X \times Y$ which is homomorphic to an admissible convex subset in t.v.s. $X_1 \times Y_1$ (see discussion after Theorem 1 in [11]).

The following examples show that Theorem 3.2 is sharper than Theorem 3.1.

Example 3.1. Let $C = [-1, 1],\ D = [0, 1]$. Define $T : C \times D \longrightarrow 2^D$ by $T(x, y) = [0, 1],\ S : C \times D \longrightarrow 2^C$ by

$$S(x,y) = \begin{cases} \{0\} & \text{if } x \neq 0 \\ [0,1] & \text{if } x = 0, \end{cases}$$

and $f, g : C \times D \longrightarrow \mathbb{R}$ by

$$g(x,y) = x + y, \quad f(x,y) = \begin{cases} 0 & \text{if } x \in \{\frac{-1}{n} : n \in N\} \cup \{0\} \\ 1 & \text{otherwise .} \end{cases}$$

The maps S and T are acyclic. The function f is not quasi-convex but l.s.c. and the function g is convex and continuous such that

$$F(x,y) = \min\{f(\xi, y) : \xi \in S(x,y)\} = 0, \quad \text{for all } (x,y) \in C \times D,$$

$$G(x,y) = \min\{g(x,\eta) = x + \eta : \eta \in T(x,y)\} = x, \text{ for all } (x,y) \in C \times D$$

are continuous and convex. It is clear that,

$$A(x,y) = \{\xi \in S(x,y) : f(\xi,y) = F(x,y)\} = \{0\}, \text{ for all } (x,y) \in C \times D$$

$$B(x,y) = \{\eta \in T(x,y) : g(x,\eta) = G(x,y)\} = \{0\}, \text{ for all } (x,y) \in C \times D$$ are acyclic (sets) for every $(x,y) \in C \times D$. Therefore, SVQEP has a solution by Theorem 3.2. But the example does not satisfy in the conditions of Theorem 3.1.

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