Vol. 5 No. 2 (2014)

Journal of Nonlinear Analysis and Optimization: Theory & Applications

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This journal was conceived as the main scientific publication of the Center of Excellence in Nonlinear Analysis and Optimization, Naresuan University, Thailand.

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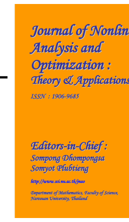
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



A NOTE ON QUASI SPLIT NULL-POINT FEASIBILITY PROBLEMS

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ABSTRACT. Inspired by the very recent work by M.-A. Noor and Kh.-I Noor [9] and given a closed convex set-valued mapping C, we propose a split algorithm for solving the problem of finding an element x^* in $C(x^*)$ such that its image, Ax^*, under a linear operator, A, is a zero of a given maximal monotone operator T in Hilbert spaces setting. Then, we present a strong convergence result and state some examples as applications.

KEYWORDS : Fixed-point; monotone operator; quasi split feasibility problem.
AMS Subject Classification: Primary, 49J53, 65K10; Secondary, 49M37, 90C25.

1. INTRODUCTION AND PRELIMINARIES

Throughout, H is a Hilbert space, $\langle \cdot, \cdot \rangle$ denotes the inner product and $\|\cdot\|$ stands for the corresponding norm. The split feasibility problem (SFP) has received much attention due to its applications in image denoising, signal processing and image reconstruction, with particular progress in intensity-modulated therapy. For a complete and exhaustive study on algorithms for solving convex feasibility problem, including comments about their applications and an excellent bibliography see, for example [1] and for split convex feasibility problem see, for instance, the excellent paper [5] and the references therein. Inspired by the idea developed in [9], our interest in this paper is on the study of the convergence of an algorithm for solving a Quasi Split Null-point Feasibility Problem, i.e., the case where the constrained set, instead of being fixed, is a set-valued mapping. Besides being a more general case, it also has many applications, see for example [2]) and is an extension of the problem introduced in [9]. At this stage, we would like to emphasize that the result in [9] is not correct. Indeed, clearly the constant θ defined by relation (16) in [9] is greater than 1. As a consequence the application F defined by relation (12) in [9] is not a contraction and thus the Banach fixed-point principle is not applicable. Actually, since the operator $A^*(I - P_C)A$ is firmly nonexpansive, it is easily seen

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Article history : Received September 04, 2014, Accepted July 21, 2013.

then that the best value of θ is in fact $\theta = 1 + \mu$ which is still greater than 1. Hence, we cannot apply again the Banach fixed-point principle. To overcome this difficulty, one way (maybe the only one) is to consider the more general problem (2.1) and assume that both the linear operator A and the set-valued mapping T are strongly monotone operators. Note that by taking $A = I$, the identity mapping and $T = \partial\phi$, the subdifferential of a strongly convex proper lower semi-continuous function, problem (2.1) reduces to minimizing the strongly convex function ϕ with respect to the implicit convex set $C(x)$. To be in a position to apply the fixed-point Banach principle and by observing that the fixed-point reformulation of the problem considered in [9] involves the projection operator over convex sets and that the techniques are strongly based on its properties which do not depend on any parameter in contrast to the resolvent and proximal mappings, we will consider a quasi split null-point feasibility problem, propose a strong convergence result and provide some applications. This will be done by taking advantage of the resolvent techniques which depend on parameters that allow more flexibility. Moreover, an appropriate choice amounts to weakened the assumptions on the data.

To begin with, let us recall that the split feasibility problem (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of a Hilbert space H_2, and $A : H_1 \to H_2$ is a bounded linear operator.

Assuming that the (SFP) is consistent (i.e (1.1) has a solution), it is no hard to see that $x \in C$ solves (1.1) if and only if it solves to fixed-point equation

$$x = P_C\big(I - \gamma A^*(I - P_Q)A\big)x,\ x \in C, \tag{1.2}$$

where P_C and P_Q are the (orthogonal) projection onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A.

To solve the (1.2), Byrne [4] proposed his CQ algorithm which generates a sequence (x_k) by

$$x_{k+1} = P_C\big(I - \gamma A^*(I - P_Q)A\big)x_k,\ k \in \mathbb{N}, \tag{1.3}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

2. MAIN RESULTS

In the sequel, we will focus our attention on the following implicit null-point feasibility problem

$$\text{find } x^* \in C(x^*) \text{ such that } Ax^* \in T^{-1}(0), \tag{2.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $T : H_1 \to H_1$ a maximal monotone operator and $C : H_1 \to 2^{H_1}$ be a set-valed map with closed convex values.

It is easy to see, using the normal cone to $C(x^*)$, that (2.1) is equivalent to the following fixed-point formulation $x^* = P_{C(x^*)}(x^* - \gamma A^*(I - J^T_{\lambda_k})Ax^*)$. To solve (2.1), the latter suggests the use of the following algorithm:

Algorithm (QSFA): *Initialization*: Let $\lambda_0 > 0$ and $x_0 \in H_1$ be arbitrary.
Iterative step:

$$x_{k+1} = P_{C(x_k)}(x_k - \gamma A^*(I - J^T_{\lambda_k})Ax_k),\ k \in \mathbb{N}, \tag{2.2}$$

where $\gamma \in]0, \frac{2\nu_k\mu^2}{L_A^2}[$ with L_A being the spectral radius of the operator A^*A, ν_k and μ will be defined in the sequel.

Lemma 2.1. *If T is strongly monotone with constant α, then (see for example [10])*

$$\|J_\lambda^T(x) - J_\lambda^T(y)\| \leq \frac{1}{1+\alpha\lambda}\|x-y\|, \quad \forall x, y.$$

A simple computation shows that its complement $I - J_\lambda^T$ (which is firmly nonexpansive) is strongly monotone. More precisely, we have

$$\langle (I - J_\lambda^T)x - (I - J_\lambda^T)y, x - y\rangle \geq \frac{\alpha\lambda}{1+\alpha\lambda}\|x-y\|^2, \quad \forall x, y.$$

We are now in a position to prove our convergence result.

Theorem 2.1. *Given a bounded linear μ-strongly positive operator $A : H_1 \to H_2$, H_1, H_2 two Hilbert spaces, $T : H_2 \to 2^{H_2}$ is a α-strongly monotone set-valued operator and $C : H_1 \to 2^{H_1}$ is a set-valued mapping with closed convex values and assume that for every x, y, z we have*

$$\|P_{C(x)}z - P_{C(y)}z\| \leq \beta\|x-y\|. \tag{2.3}$$

Then any sequence (x_k) generated by the algorithm (2.2) strongly converges to the unique solution of (2.1), provided that

$$\beta \in]0,1[, L_A\sqrt{\beta(2-\beta)} < \nu_k < L_A \text{ and } \gamma \in]0, \frac{2\nu_k\mu^2}{L_A^2}[\text{ with } \nu_k := \frac{\alpha\lambda_k}{1+\alpha\lambda_k}. \tag{2.4}$$

Proof. Let x^* be the solution to (2.2), then $P_{C(x^*)}(x^*) = x^*$, $J_{\lambda_k}^T(Ax^*) = Ax^*$. By Lemma 2.1, we know that $I - J_{\lambda_k}^T$ is nonexpansive and strongly monotone with constant ν_k. Therefore, we successively have

$$\begin{aligned}
& \langle A^*(I - J_{\lambda_k}^T)Ax_k - A^*(I - J_{\lambda_k}^T)Ax^*, x_k - x^*\rangle \\
= \; & \langle (I - J_{\lambda_k}^T)Ax_k - (I - J_{\lambda_k}^T)Ax^*, Ax_k - Ax^*\rangle \\
\geq \; & \nu_k\|Ax_k - Ax^*\|^2 \geq \nu_k\mu^2\|x_k - x^*\|^2.
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
& \|(I - \gamma A^*(I - J_{\lambda_k}^T)A)x_k - x^*\|^2 \\
= \; & \|(x_k - x^*) - \gamma(A^*(I - J_{\lambda_k}^T)Ax_k - A^*(I - J_{\lambda_k}^T)Ax^*)\|^2 \\
= \; & \|x_k - x^*\|^2 - 2\gamma\langle A^*(I - J_{\lambda_k}^T)Ax_k - A^*(I - J_{\lambda_k}^T)Ax^*, x - x^*\rangle \\
+ \; & \gamma^2\|A^*(I - J_{\lambda_k}^T)Ax_k - A^*(I - J_{\lambda_k}^T)Ax^*\|^2 \\
\leq \; & (1 - 2\nu_k\mu^2\gamma + L_A^2\gamma^2)\|x_k - x^*\|^2.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\|x_{k+1} - x^*\| \; = \; & \|P_{C(x_k)}(x_k - \gamma A^*(I - J_{\lambda_k}^T)Ax_k) - P_{C(x_k)}(x^* - \gamma A^*(I - J_{\lambda_k}^T)Ax^*)\| \\
+ \; & \|P_{C(x_k)}(x^* - \gamma A^*(I - J_{\lambda_k}^T)Ax^*) - P_{C(x^*)}(x^* - \gamma A^*(I - J_{\lambda_k}^T)Ax^*)\| \\
\leq \; & \beta\|x_k - x^*\| + \|(I - \gamma A^*(I - S)A)x_k - x^*\| \\
\leq \; & (\beta + \sqrt{1 - 2\nu_k\mu^2\gamma + L_A^2\gamma^2})\|x_k - x^*\|,
\end{aligned}$$

λ_k and γ_k were chosen judiciously such that $\theta := \beta + \sqrt{1 - 2\nu_k\mu^2\gamma + L_A^2\gamma^2} \in]0,1[$. The latter assures the strong convergence of (x_k) to x^* the unique solution of (2.1).

Remark 2.2. It is worth mentioning that we can develop the same analysis for the following quasi fixed-point feasibility problem

$$\text{find } x^* \in C(x^*) \text{ such that } Ax^* \in FixP, \tag{2.5}$$

where $P : H_2 \to H_2$ is a κ-contraction, by considering the following algorithm

Algorithm:
Initialization: Let $x_0 \in H_1$ be arbitrary.
Iterative step:

$$x_{k+1} = P_{C(x_k)}(x_k - \gamma A^*(I - P)Ax_k), \ k \in I\!N. \tag{2.6}$$

Following the same lines of the proof of the above Theorem, we obtain

Proposition 2.3. *Given a bounded linear μ-strongly positive operator $A : H_1 \to H_2$, H_1, H_2 are two Hilbert spaces, $P : H_2 \to 2^{H_2}$ a κ-contraction and $C : H_1 \to 2^{H_1}$ a set-valued mapping with closed convex values and assume that for every x, y, z we have*

$$\|P_{C(x)}z - P_{C(y)}z\| \leq \beta\|x - y\|.$$

Then any sequence (x_k) generated by the algorithm (2.6) strongly converges to the unique solution of (2.5), provided that

$\nu := 1-\kappa, \beta \in]0, 1[, L_A\sqrt{\beta(2-\beta)} < \nu\mu^2 < L_A$ *and* $|\gamma-\nu\mu^2| < \sqrt{\nu^2\mu^4 - \beta(2-\beta)L_A^2}$.

□

3. SPECIAL CASES

Now, let us consider the following special cases:

(i) **Quasi Split minimization problem:** Let $\phi : H_1 \to I\!R$ be a lower semicontinuous convex function by setting $T = \partial\phi$ in (2.1), we obtain the following Quasi Split Minimization Problem (QSMP):

$$\text{find } x^* \in C(x^*) \text{ such that } Ax^* = argmim\phi \tag{3.1}$$

and (2.2) reduces to

$$x_{k+1} = P_{C(x_k)}(x_k - \gamma A^*(I - prox_{\lambda_k\phi})Ax_k), \ k \in I\!N, \tag{3.2}$$

where $prox_{\lambda_k\phi}(x) := argmin_y\{\phi(y) + \frac{1}{2\lambda}\|x-y\|^2\}$ is the proximal mapping of ϕ. The assumption of strong monotonicity of $\partial\phi$ is equivalent to the strong convexity of ϕ.

(ii) **Split Saddle-point problem:** Let X, Y be two Hilbert spaces, a function $L : X \times Y \to I\!R \cup \{-\infty, +\infty\}$ is convex-concave if it is convex in the variable x and concave in the variable y. To such a function, Rockafellar associated the operator T_L, defined by

$$T_L = \partial_1 L \times \partial_2(-L),$$

where ∂_1 (resp. ∂_2) stands for the subdifferential of L with respect to the first (resp. the second) variable.
T_L is a maximal monotone operator if and only if L is closed and proper in Rockafellar sense (see, [10]). Moreover, it is well known that (x^*, y^*) is a saddle-point of L, namely $L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*), \ \forall(x, y) \in X \times Y$ if and only if the following monotone variational inclusion holds true $(0, 0) \in T_L(x^*, y^*)$.
Now, if in the (2.1) we set $H_1 = X_1 \times Y_1$, $H_2 = X_2 \times Y_2$, $T = T_L$ with L be a proper closed convex-concave function, then we obtain the following Quasi Split Minimax Problem (QSMMP):

$$\text{find } (x^*, y^*) \in C(x^*, y^*); \ A(x^*, y^*) = argminmax_{(x,y)\in H_1} L(x, y), \tag{3.3}$$

and (2.2) reduces to

$$(x_{k+1}, y_{k+1}) = P_{C(x_k,y_k)}((x_k, y_k) - \gamma A^*(I - prox_{\lambda_k L})A(x_k, y_k)),\ k \in I\!N, \quad (3.4)$$

where $prox_{\lambda_k L}(x, y) := argminmax_{(u,v)}\{L(u, v)) + \frac{1}{2\lambda}\|x-u\|^2 - \frac{1}{2\lambda}\|y-v\|^2\}$. The assumption of strong monotonicity of T_L is equivalent to the strong convexity of L with respect to the first variable and its strong concavity with respect to the second one.

(iii) **Quasi Split equilibrium problem:** Having in mind the connection between monotone operators and equilibrium functions, we may consider the following problem

$$A_F(x) \ni 0, \quad (3.5)$$

with A_F defined as follows $v \in A_F(x) \Leftrightarrow F(x, y) + \langle v, x - y\rangle \geq 0,\ \forall y \in D$, D is a closed convex set and $F : D \times D \to I\!R$ belongs in the class of bifunctions F verifying the following usual conditions:

(A1) $F(x, x) = 0$ for all $x, y \in D$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in D$;
(A3) $\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$ for any $x, y, z \in D$;
(A4) for each $x \in D, y \longrightarrow F(x, y)$ is convex and lower-semicontinuous.

It is well-known; see [7], that A_F is maximal monotone and that the associated resolvent operator $T_\lambda : H \longrightarrow D$ is defined by

$$T^F_\lambda(x) = \{z \in D :\ F(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0,\ \forall y \in D\}.$$

If in the (2.1) we take $T = A_F$ a monotone bifunction, then we obtain the following Quasi Split Equilibrium Problem (QSEP):

$$\text{find } x^* \in C(x^*) \text{ such that } F(Ax^*, x) \geq 0 \text{ for all } x \in C(x^*), \quad (3.6)$$

and (2.2) is nothing but

$$x_{k+1} = P_{C(x_k)}(x_k - \gamma A^*(I - T^F_{\lambda_k})Ax_k),\ k \in I\!N. \quad (3.7)$$

It is well known that in this case, the strong monotonicity of A_F is equivalent to

$$F(x, y) + F(y, x) \leq \alpha\|x - y\|^2 \text{ for all } x, y \in D.$$

(iv) **A special form of the implicit set:**
In many applications (see for example [2]) the set-valued mapping has the form $C(x) = K + \psi(x)$, where K is a fixed closed subset in H_1 and $\psi : H_1 \to H_1$ is a single-valued mapping. In this case, assumption on C is satisfied provided the mapping ψ is Lipschitz continuous. Indeed, it is not hard (using the relation below) to show that, if ψ is κ-Lipschitz then assumption (2.3) satisfies with $\beta = 2\kappa$. Using the well known relation

$$x = P_{K+v}(u) \Leftrightarrow x - v = P_K(u - v),$$

Algorithm can be rewritten in the simpler form

$$x_{k+1} = \psi(x_k) + P_K(x_k - \psi(x_k) - \gamma_k A^*(I - J^T_{\lambda_k})Ax_k),\ k \in I\!N. \quad (3.8)$$

Conclusion: Only the existence of solutions to a quasi split feasibility problem has been considered in [9] and the result is more than questionable. Also, only some algorithms are mentioned! To the best of our knowledge, nothing has been done concerning the construction of solutions in this case. Inspired by this work and to overcome the difficulties that arise in applying the Banach principle, we proposed a quasi feasibility null-point problem and study the convergence of a related algorithm. Applications to some applied nonlinear analysis problems are also provided. The techniques used in solving our problem are strongly based on the resolvent mapping which depends on a parameter. The latter allows more flexibility and an appropriate choice amounts to assume mild assumptions on the data (see Theorem 2.1).

Acknowledgment: I would like also to thank the anonymous referees for their careful reading which permitted me to improve the first version of this paper.

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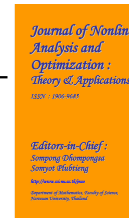
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No.2, (2014), 7-20
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



ON A GENERAL CLASS OF MULTI-VALUED STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. In this paper, a general class of multi-valued strictly pseudocontractive mappings, which properly includes the class of multi-valued $k-$strictly pseudocontractive mappings, is introduced. Furthermore, it is proved that if T belongs to this class of mappings and the set of fixed points of T is nonempty, a Krasnoselskii-type sequence is constructed and proved to be an approximate fixed point sequence of T. Finally, convergence of the sequence to a fixed point of T is proved under appropriate additional conditions.

KEYWORDS : Iterative Approximation, Multi-valued Maps, Strictly Pseudocontractive Mappings.
AMS 2010 Mathematics Subject Classification:47H04, 47H09, 47H10

1. INTRODUCTION

Let E be a metric space endowed with a metric d and K be a nonempty closed, convex subset of E. Let $CB(K)$ denote the collection of closed and bounded subsets of K and $T : K \to CB(K)$ be a multi-valued mapping.
In recent years, the study of fixed point theory for multi-valued nonlinear mappings T has attracted, and continues to attract, the interest of several well known mathematicians. Interest in this type of mappings is, perhaps, due to its many real world applications, for example, in Game Theory and Market Economy and in Non-Smooth Differential Equations (see e.g., Chidume *et al.* [7] for details).

The applications of fixed point theory for multi-valued mappings on the problem of differential equations (DEs) with discontinuous right-hand sides gave birth to the existence theory of differential inclusions (DIs). Most recent results for game theory showed that equilibrium points of games correspond to fixed points of some multi-valued mappings, under appropriate conditions. A model example of such an application, the *Nash equilibrium theorem*, (see e.g Nash [21, 22]) showed that

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Article history : Received August 08, 2014 Accepted September 29, 2014.

the existence of equilibria for non-cooperative static games is a direct consequence of fixed point theorems of Brouwer [2] or Kakutani[15].

From the point of view of social recognition, game theory is, perhaps, the most successful area of application of fixed point theory of multi-valued mappings. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a nonequilibrium point and convergent to an equilibrium solution. This is part of the problem that is being addressed by *iterative methods* for fixed point of multi-valued mappings. It is worth mentioning, at this juncture also, that iterative methods for approximating fixed points of nonexpansive mappings constitute the central tools used in signal processing and image reconstruction (see, e.g., Byrne[5]).

For early results involving fixed points of multi-valued mapping, (see, for example, Brouwer [2], Kakutani [15], Nash [21, 22], Geanakoplos [14], Downing and Kirk [10]). For details on the applications of this type of mappings in Nonsmooth Differential Equations, one may consult, for example, Chang [6], Chidume [7], Deimling [9], Erbe and Krawcewicz [11], Frigon [12], Nadler [20], Ofoedu and Zegeye [23], Reich *et al.* [25, 26, 27] and the references therein.

Fixed point problems involving a multi-valued mapping T can be reformulated as a zero problem for a multi-valued mapping A, namely;

$$\text{Find } 0 \in Ax, \qquad \text{where } A = I - T$$

and I is the identity mapping of K.

Many problems in applications can be modeled in the form of $0 \in Ax$, where, for example. $A : H \to 2^H$ is a monotone operator, that is $\langle u - v, x - y\rangle \geq 0$ for all $u \in Ax, v \in Ay, x, y \in H$. Typical examples include the equilibrum state of evolution equations and critical points of some functionals defined on Hilbert spaces. For example, let $f : H \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. It is known (see e.g., Rockafellar [29], Minty [19]) that the multi-valued mapping $T := \partial f$, *the subdifferential* of f, is maximal monotone, where for each $w \in H$,

$$\begin{aligned} w \in \partial f(x) &\Leftrightarrow f(y) - f(x) \geq \langle y - x, w\rangle, \quad \forall y \in H, \\ &\Leftrightarrow x \in Argmin(f - \langle ., w\rangle). \end{aligned}$$

In this case, a solution of the inclusion $0 \in \partial f(x)$, if any, is a critical point of f, which is precisely a minimizer of f.

The *proximal point algorithm* introduced by Martinet [18], and studied extensively by Rockafeller [28], which has also been studied by a host of other authors, is connected with the iterative algorithm for solutions of $0 \in Ax$ where A is a maximal monotone operator on a Hilbert space.

In studying the equation $Au = 0$, Browder [4], introduced an operator T given by $T := I - A$, where I is the identity mapping on H. He called such an operator a *pseudocontractive mapping*. It is easily seen that the solutions of $Ax = 0$ when A is monotone are precisely the fixed points of pseudocontractive mapping T. Every nonexpansive mapping is pseudocontractive and continuous but a pseudocontractive mapping is not neccesarily continuous. Thus the study of iterative methods

for fixed points of pseudcontractive mappings requires some continuity conditions (e.g., Lipschitz condition).
While pseudocontractive mappings are generally not continuous, a subclass of pseudocontractive mappings, the *strictly pseudocontractive mappings*, inherits Lipschitz property from their definitions. The study of fixed point theory for strictly pseudocontractive mappings helps in the study of fixed point theory for nonexpansive mappings and for Lipschitz pseudocontractive mappings. Consequently, the study by several authors of iterative methods for fixed point of multi-valued strictly pseudocontractive mappings has motivated our study of a more *general class* of multi-valued strictly pseudocontractive mappings which certainly includes the important class of multi-valued nonexpansive maps.

In this paper, we extend the notion of single-valued strictly pseudo-contractive mappings, defined by Browder and Petryshyn [3] on Hilbert spaces, and the notion of multi-valued stricly pseudocontractive mappings (see e.g., [7], [23]), to a *general class* of multi-valued strictly pseudocontractive mappings. This class is shown to properly contain the class of multi-valued strictly pseudocontractive mappings introduced by Chidume $et\ al$ [7] which itself contains the important class of multi-valued nonexpansive mappings and consequently properly contains the class of single-valued strictly pseudo-contractive maps introduced by Browder and Petryshyn [3].

Part of the novelty of this paper is that for the general class of maps considered here, convergence theorems for Krasnoselskii-type sequence, which is known to be superior to the Mann-type and Ishikawa-type sequences, are still applicable. In particular, the Krasnoselskii-type iteration sequence $\{x_n\}$ constructed is proved to be an approximate fixed point sequence of T and then, under appropriate additional conditions, convergence to a fixed point is established.

2. PRELIMINARIES AND NOTATIONS

Throughout this paper, we will adopt the following:

(i) $x_n \to x$: $\{x_n\}$ converges strongly to x.
(ii) H: a Hilbert Space with an induced norm $\|.\|$.
(iii) $F(T) := \{x \in K : x \in Tx\}$.
(iv) 2^H is the power set of H.
(v) $CB(K)$, is the family of nonempty, closed and bounded subsets of K

We recall some definitions and facts that are needed in our study.

Definition 2.1. Let $(X; d)$ be a metric space and K a nonempty subset of X. The *Hausdorf metric* on $CB(X)$ is given by

$$D(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for all $A, B \in CB(X)$.

To simplify notation, we shall denote $(D(A,B))^2$ by $D^2(A,B)$ for all $A, B \in CB(X)$.

Browder and Petryshin [3] introduced this class of single-valued mappings.

Definition 2.2. Let K be a nonempty subset of a Hilbert space H. A map $T : K \to H$ is called *strictly pseudo-contractive* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in K. \tag{2.1}$$

Definition 2.3. (Chidume *et al.* [7]) Let H be a real Hilbert space and let D be a nonempty, open and convex subset of H. Let $T : \overline{D} \to CB(\overline{D})$ be a mapping. Then, T is called *a multi-valued $k-$strictly pseudocontractive mapping* if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$, we have

$$D^2(Tx, Ty) \le \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \tag{2.2}$$

for all $u \in Tx, \ v \in Ty$.

Remark 2.4. If $k = 1$, in definition (2.3), the mapping T is called *pseudocontractive* and if $k = 0$, it is called *nonexpansive*. Notice also that the mapping becomes nonexpansive if $x \in Tx \ \ \forall x \in \overline{D}$.

Definition (2.3) is an extension of the definition of single-valued pseudo-contractive mappings to multi-valued maps.

Definition 2.5. A multi-valued mapping $T : K \subseteq H \to CB(H)$ is called Lipschitzian if there exists $L > 0$ such that

$$D(Tx, Ty) \le L\|x - y\|, \tag{2.3}$$

for each $x, y \in K$. If $L < 1$ in inequality (2.3), the mapping T is called a *contraction* and if $L = 1$, it is *nonexpansive*.

Several authors have studied the problem of approximating fixed points of *multi-valued* nonexpansive mappings (see, e.g., [1], [8], [16, 17], [24], [30], [32], and the references therein), and of their generalizations (see, e.g., [8], [13]).

Definition 2.6. A map $T : K \to CB(K)$ is said to be *hemicompact* if, for any sequence $\{x_n\}$ such that $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence, say, $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

Note that if K is compact, then every multi-valued mapping $T : K \to CB(K)$ is hemicompact.

Definition 2.7. Let H be a real Hilbert space and let T be a multi-valued mapping. The multi-valued mapping $I - T$ is said to be *strongly demiclosed* at 0 (see, e.g., [13]) if for any sequence $\{x_n\} \subseteq D(T)$ such that $x_n \to p$ and $d(x_n, Tx_n)$ converges strongly to 0, then $d(p, Tp) = 0$.

The projection mapping P_K onto a nonempty, closed and convex subset of H has the following characterization:

Lemma 2.8. *Let H be a Hilbert space, $K \subset H$ be nonempty, closed and convex, $z \in H$ and $x \in K$. Then $x = P_K z$ if and only if*

$$\langle z - x, w - x\rangle \le 0 \quad \forall w \in K.$$

The following lemma will also be used in the sequel.

Lemma 2.9. *([34]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq a_n + \sigma_n, \;\; n \geq 0,$$

such that $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then, $\lim a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence that converges to 0, then a_n converges to 0 as $n \to \infty$.

3. MAIN RESULTS

We first prove the following preliminary results.

Lemma 3.1. *Let E be a normed linear space , $B_1, B_2 \in CB(E)$ and $x_0, y_0 \in E$ arbitrary. The following hold;*

(a) $D(B_1, B_2) = D(x_0 + B_1, x_0 + B_2)$.
(b) $D(B_1, B_2) = D(-B_1, -B_2)$.
(c) $D(x_0 + B_1, y_0 + B_2) \leq \|x_0 - y_0\| + D(B_1, B_2)$.
(d) $D(\{x_0\}, B_1) = \sup_{b_1 \in B_1} \|x_0 - b_1\|$.
(e) $D(\{x_0\}, B_1) = D(0, x_0 - B_1)$.

Proof. (a) By definition, we have

$$\begin{aligned} D(x_0 + B_1, x_0 + B_2) &= \max\Big\{ \sup_{b_1 \in B_1} d(x_0 + b_1, x_0 + B_2); \sup_{b_2 \in B_2} d(x_0 + b_2, x_0 + B_2)\Big\} \\ &= \max\{ \sup_{b_1 \in B_1} d(b_1, B_2); \sup_{b_2 \in B_2} d(b_2; B_1)\} \\ &= D(B_1, B_2). \end{aligned}$$

(b) We have

$$\begin{aligned} D(-B_1, -B_2) &= \max\{ \sup_{-b_1 \in -B_1} d(-b_1; -B_2); \sup_{-b_2 \in -B_2} d(-b_2; -B_1)\} \\ &= \max\{ \sup_{b_1 \in B_1} d(b_1; B_2); \sup_{b_2 \in B_2} d(b_2; B_1)\} \\ &= D(B_1, B_2). \end{aligned}$$

(c) It is known that for any set $B \subseteq E$, $x, y \in E$ arbitrary, the inequality

$$d(x, B) \leq \|x - y\| + d(y, B)$$

holds. Using this inequality we have

$$\begin{aligned} d(x_0 + b_1, y_0 + B_2) &\leq \|(x_0 + b_1) - (y_0 + b_1\| + d(y_0 + b_1, y_0 + B_2) \\ &= \|x_0 - y_0\| + d(b_1, B_2), \end{aligned}$$

and similarly

$$d(y_0 + b_2, x_0 + B_1) \leq \|x_0 - y_0\| + d(b_2, B_1).$$

Therefore, taking sup over B_1 and B_2 respectively, we have

$$\sup_{b_1 \in B_1} d(x_0 + b_1, y_0 + B_2) \leq \|x_0 - y_0\| + \sup_{b_1 \in B_1} d(b_1, B_2),$$

and

$$\sup_{b_2 \in B_2} d(y_0 + b_2, x_0 + B_1) \leq \|x_0 - y_0\| + \sup_{b_2 \in B_2} d(b_2, B_1).$$

Thus $D(x_0 + B_1, y_0 + B_2) \leq \|x_0 - y_0\| + D(B_1, B_2)$.

(d) It is obvious that $d(x_0; B_1) = \sup_{x_0 \in \{x_0\}} d(x_0, B_1)$. On the otherhand, for any $b_1 \in B_1$, we have

$$d(b_1; \{x_0\}) = \|b_1 - x_0\| \geq d(x_0; B_1).$$

Taking sup over B_1 we have

$$\sup_{b_1 \in B_1} d(b_1, \{x_0\}) \geq d(x_0, B_1),$$

and therefore

$$D(\{x_0\}, B_1) := \max\{\sup_{b_1 \in B_1} d(b_1; \{x_0\}); \sup_{x_0 \in \{x_0\}} d(x_0; B_1)\} = \sup_{b_1 \in B_1} d(b_1, \{x_0\}).$$

(e)

$$\begin{aligned} D(\{x_0\}, B_1) &:= \max\{\sup_{b_1 \in B_1} d(b_1, \{x_0\}), d(x_0, B_1)\} \\ &= \max\{\sup_{b_1 \in B_1} \|x_0 - b_1\|), \inf_{b_1 \in B_1} \|x_0 - b_1\|\} \\ &= \max\{\sup_{b_1 \in B_1} d(0, x_0 - B_1), d(0, x_0 - B_1)\} \\ &= D(\{0\}, x_0 - B_1). \end{aligned}$$

□

We now introduce the following class of *generalized $k-$ strictly pseudocontractive multi-valued mappings*.

Definition 3.2. Let H be a real Hilbert space and let K be a nonempty subset of H. Let $T : K \to CB(K)$ be a multi-valued mapping. Then T is called **generalized $k-$strictly pseudocontractive multi-valued mapping** if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$, we have

$$D^2(Tx, Ty) \leq \|x - y\|^2 + kD^2(Ax, Ay), \quad A := I - T, \tag{3.1}$$

and I is the identity operator on K.

Remark 3.3. Definition (3.2) seems to be a more natural generalization of the single-valued definition (2.2) given by Browder and Petryshin [3] than the definition (2.3) given by Chidume *et al.* [7].

Proposition 3.4. *Let $T : K \to CB(K)$ be a multi-valued $k-$strictly pseudocontractive mapping, then T is a generalized $k-$strictly pseudocontractive multi-valued mapping.*

Proof. Given that T is a multi-valued $k-$strictly pseudocontractive mapping, we have

$$D^2(Tx, Ty) \leq \|x - y\|^2 + k \inf_{(u,v) \in (Tx, Ty)} \|(x - u) - (y - v)\|^2. \tag{3.2}$$

We now show that inequality (3.2) implies inequality (3.1).

$$\begin{aligned} D(x - Tx, y - Ty) : &= \max\left\{\sup_{u \in Tx} d(x - u; y - Ty); \sup_{v \in Ty} d(y - v; x - Tx)\right\} \\ &\geq \sup_{u \in Tx} d(x - u; y - Ty) \\ &\geq d(x - u_0; y - Ty), \qquad u_0 \in Tx. \end{aligned}$$

Now, given $\epsilon > 0$, there exist $v_\epsilon \in Ty$ such that

$$\begin{aligned} d(x-u_0; y-Ty) &\geq \|(x-u_0)-(y-v_\epsilon)\| - \epsilon \\ &\geq \inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| - \epsilon. \end{aligned}$$

Thus, for arbitrary $\epsilon > 0$, we have

$$\inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| \leq D(x-Tx, y-Ty) + \epsilon,$$

and therefore, since $\epsilon > 0$ is arbitrary, we have:

$$\inf_{(u,v)\in(Tx,Ty)} \|(x-u)-(y-v)\| \leq D(x-Tx, y-Ty). \tag{3.3}$$

We therefore obtain from (3.2) and (3.3) that:

$$D^2(Tx, Ty) \leq \|x-y\|^2 + kD^2(x-Tx, y-Ty).$$

□

Thus, every multi-valued $k-$strictly pseudocontractive mapping is also a *generalized* $k-$strictly pseudocontractive multi-valued mapping.

We now give an example to show that this inclusion is proper.

For the example, we shall need the following lemma which is trivially proved.

Lemma 3.5. *Let a, b be real numbers such that $0 \leq a \leq 4b$. Then,*

$$(a-b)^2 \leq b^2 + \frac{1}{2}a^2. \tag{3.4}$$

Example 3.6. Let H be a real Hilbert space. Define a mapping

$$T : H \to CB(H) \quad \text{by}$$

$$Tx := \begin{cases} \overline{B}(-x, \|x\|), & \|x\| > 0 \\ 0, & x = 0, \end{cases}$$

where

$$\overline{B}(-x, \|x\|) = \{u \in H : \|u + x\| \leq \|x\|\}.$$

Then, for distinct nonzero x and y, we have the following identities which follow from the definition of T:

$$\begin{aligned} x - Tx &= \overline{B}(2x, \|x\|), \\ y - Ty &= \overline{B}(2y, \|y\|), \\ Tx &= \{w \in H : \|w + x\| \leq \|x\|\}, \\ Ty \setminus Tx &= \{z \in H : \|z + y\| \leq \|y\|, \|z + x\| > \|x\|\}. \end{aligned}$$

We now establish the following equation:

$$D(Tx, Ty) = \|x - y\| + \Big|\|y\| - \|x\|\Big|. \tag{3.5}$$

First, we consider the case $\|y\| \geq \|x\|$. We proceed as follows:

Claim 1: $\forall z \in Ty \setminus Tx$, $d(z, Tx) = \|z - P_{(Tx)}z\|$, where

$$P_{(Tx)}z := -x + \frac{\|x\|}{\|x+z\|}\Big(z + x\Big). \tag{3.6}$$

Proof of Claim 1. Let $w \in Tx$. Then, $\|w + x\| \leq \|x\|$. Furthermore,

$$\begin{aligned}
\langle z - P_{(Tx)}z, w - P_{(Tx)}z\rangle &= \Big\langle z + x - \frac{\|x\|}{\|x+z\|}(z+x), w + x - \frac{\|x\|}{\|x+z\|}(z+x)\Big\rangle \\
&= \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\langle z+x, w+x\rangle - \frac{\|x\|}{\|x+z\|}\Big\langle z+x, z+x\Big\rangle\Big) \\
&= \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\langle z+x, w+x\rangle - \|x\|\|z+x\|\Big) \\
&\leq \frac{\|x+z\| - \|x\|}{\|x+z\|}\Big(\|z+x\|\|w+x\| - \|x\|\|z+x\|\Big) \\
&\leq (\|x+z\| - \|x\|)\Big(\|x\| - \|x\|\Big) \\
&= 0.
\end{aligned}$$

Thus, it follows that,

$$\langle z - P_{(Tx)}z, w - P_{(Tx)}z\rangle \leq 0,$$

and applying Lemma(2.8), the claim is proved. Also note that $P_{(Tx)}z$ is unique for each z since H is a real Hilbert space. Now, set

$$z_0 := -x + \Big(1 + \frac{\|y\|}{\|x-y\|}\Big)(x - y). \tag{3.7}$$

Clearly $z_0 \in Ty \setminus Tx$ since

$$\|z_0 + y\| = \Big\|\frac{\|y\|}{\|x-y\|}(x-y)\Big\| = \|y\|$$

and,

$$\begin{aligned}
\|z_0 + x\| = \Big(1 + \frac{\|y\|}{\|x-y\|}\Big)\|x-y\| &= \|x-y\| + \|y\| \\
&\geq \|x-y\| + \|x\| \\
&> \|x\|.
\end{aligned}$$

Moreover, from equation (3.6),

$$\begin{aligned}
P_{(Tx)}z_0 &= -x + \frac{\|x\|}{\|x+z_0\|}\Big(z_0 + x\Big) \\
&= -x + \frac{\|x\|}{\|x-y\| + \|y\|}\Big[\frac{\|x-y\| + \|y\|}{\|x-y\|}(x-y)\Big] \\
&= -x + \frac{\|x\|}{\|x-y\|}\Big(x - y\Big).
\end{aligned} \tag{3.8}$$

Therefore , using (3.8) and (3.7) we obtain that

$$d(z_0, Tx) = \|z_0 - P_{(Tx)}z_0\| = \|x-y\| + \|y\| - \|x\|, \tag{3.9}$$

establishing Claim 1.

Claim 2: $d(z_0, Tx) = \sup\limits_{v \in Ty} d(v, Tx)$.

Proof of Claim 2. Let $z \in Ty \setminus Tx$ be arbitrary. We have,

$$\|z + y\| \le \|y\|, \;\; \|z + x\| > \|x\|,$$

and so, using equation (3.9),

$$\begin{aligned}
\|z - P_{(Tx)}z\| &= \Big\| z + x - \frac{\|x\|}{\|x+z\|}\Big(z + x\Big)\Big\| \\
&= \|z + x\|\Big(1 - \frac{\|x\|}{\|x+z\|}\Big) \\
&= \|x + z\| - \|x\| \\
&\le \|x - y\| + \|z + y\| - \|x\| \\
&\le \|x - y\| + \|y\| - \|x\| \\
&= \|z_0 - P_{(Tx)}z_0\|.
\end{aligned}$$

For $z \in (Ty \cap Tx)$, we have $d(z, Tx) = 0$. Thus, we obtain that,

$$d(z, Tx) \le \|z_0 - P_{(Tx)}z_0\| \quad \forall z \in Ty.$$

Using the fact that $z_0 \in Ty$, we obtain

$$\sup_{v \in Ty} d(v, Tx) = \|z_0 - P_{(Tx)}z_0\| = \|x - y\| + \|y\| - \|x\|.$$

Thus, Claim 2 is established.

We now consider the case $\|x\| \ge \|y\|$.
For $\|x\| \ge \|y\|$, we have by interchanging the roles of x and y,

$$\sup_{u \in Tx} d(u, Ty) = \|x - y\| + \|x\| - \|y\|.$$

Therefore,

$$\max\Big\{\sup_{y \in Ty} d(y, Tx), \sup_{x \in Tx} d(x, Ty)\Big\} = \|x - y\| + \Big|\|y\| - \|x\|\Big|. \tag{3.10}$$

For $x = y$, $Tx = Ty$, and $D(Tx, Ty) = 0$. Moreover, for $x = 0, \;\; y \ne 0$, a straightforward computation gives

$$D(0, Ty) = 2\|y\| = \|0 - y\| + \Big|0 - \|y\|\Big|.$$

Thus, the identity (3.5) is fully satisfied for arbitrary $x, y \in H$.

Following a similar procedure, we obtain

$$D(x - Tx, y - Ty) = 2\|x - y\| + \Big|\|y\| - \|x\|\Big| \quad \forall x, y \in H. \tag{3.11}$$

We set

$$\begin{aligned}
a &:= D(x - Tx, y - Ty). \\
b &:= \|x - y\|.
\end{aligned}$$

Then, using equations (3.11) and (3.5), we obtain that,

$$a - b = D(Tx, Ty).$$

Clearly, by equation (3.11),

$$a = 2\|x - y\| + \Big|\|y\| - \|x\|\Big| \le 4\|x - y\| = 4b.$$

Therefore, by Lemma (3.5),

$$D^2(Tx, Ty) \le \|x - y\|^2 + \frac{1}{2}\Big(D(x - Tx, y - Ty)\Big)^2 \qquad \forall x, y \in H.$$

Therefore, T is a generalized $k-$strictly pseudocontractive multi-valued mapping with $k = \frac{1}{2}$.

We now show that T is not a multi-valued $k-$strictly pseudocontractive mapping in the sense of definition (2.3).
We establish this by contradiction. So, assume that there exists $k \in [0, 1)$ such that inequality (2.2) holds. Choose $x \in H \setminus \{0\}$. Set $y = 2x$, $u = v = 0 \in (Tx \cap Ty)$. Then,

$$||x - y|| = \|x\|,$$

$$D(Tx, Ty) = \|x - y\| + \Big|\|y\| - \|x\|\Big| = 2\|x\|,$$

and

$$\|(x - u) - (y - v)\| = \|x - y\| = \|x\|.$$

Thus,

$$4\|x\|^2 = D^2(Tx, Ty) \le \|x - y\|^2 + k\|(x - u) - (y - v)\|^2 \le 2\|x\|^2.$$

This is a contradiction to $x \in H \setminus \{0\}$. Therefore, T is not a multi-valued $k-$strictly pseudocontractive mapping for any $k \in (0, 1)$.

To prove our main theorem, we first prove the following important propositions.

Proposition 3.7. *Let K be a nonempty subset of a real Hilbert space H and $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping. Then T is Lipschitzian.*

Proof. Let $x, y \in D(T)$. Then,

$$\begin{aligned} D^2(Tx, Ty) &\le \|x - y\|^2 + kD^2(x - Tx, y - Ty) \\ &\le \|x - y\|^2 + k\Big(\|x - y\| + D(Tx, Ty)\Big)^2, \text{by Lemma (3.1), } (c), (b). \\ &\le \Big(\|x - y\| + \sqrt{k}\|x - y\| + \sqrt{k}D(Tx, Ty)\Big)^2. \end{aligned}$$

Thus,

$$D(Tx, Ty) \le (1 + \sqrt{k})\|x - y\| + \sqrt{k}D(Tx, Ty),$$

and hence,

$$D(Tx, Ty) \le \frac{1 + \sqrt{k}}{1 - \sqrt{k}}\Big\|x - y\Big\|,$$

as proposed. □

Remark 3.8. Propostion (3.7) is an improvement of Proposition 8 of [7] because it does not assume that Tx *is weakly closed for each* $x \in K$.

Proposition 3.9. *Let K be a nonempty and closed subset of a real Hilbert space H and let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping. Then, $(I - T)$ is strongly demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in K such that $x_n \to x$ and $d(x_n, Tx_n) \to 0$. For each $n \in \mathbb{N}$, take $y_n \in Tx_n$ such that $\|x_n - y_n\| \le d(x_n, Tx_n) + \frac{1}{n}$.

Then,

$$\begin{aligned} d(x,Tx) &\le \|x - x_n\| + \|x_n - y_n\| + d(y_n, Tx) \\ &\le \|x - x_n\| + d(x_n, Tx_n) + \frac{1}{n} + D(Tx_n, Tx) \\ &\le \|x - x_n\| + d(x_n, Tx_n) + \frac{1}{n} + \frac{1+\sqrt{k}}{1-\sqrt{k}}\Big\|x_n - x\Big\|. \end{aligned}$$

Thus, taking limits on both sides as $n \to \infty$, we have $d(x, Tx) = 0$. Since Tx is closed, $x \in Tx$. □

Now, using lemma (3.1)(d), we have that

$$D(\{x_n\}, Tx_n) = \sup_{y_n \in Tx_n} \|x_n - y_n\|.$$

Thus, for any given sequence $\{x_n\} \subseteq K$, the set

$$U^n := \Big\{y_n \in Tx_n : D^2(\{x_n\}, Tx_n) \le \|x_n - y_n\|^2 + \frac{1}{n^2}\Big\},$$

is nonempty.

We now prove the following theorem.

Theorem 3.1. *Let K be a nonempty, closed, convex subset of a real Hilbert space H. Let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping such that $F(T) \neq \emptyset$. Assume $Tp = \{p\}$ $\forall p \in F(T)$. Define a sequence $\{x_n\}$ by $x_0 \in K$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n \tag{3.12}$$

where $y_n \in U^n$ and $\lambda \in (0, 1-k)$. Then, $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Proof. Let $p \in F(T)$. Then, using Lemma(3.1), (d) and (e), we have

$$\begin{aligned} &\|x_{n+1} - p\|^2 = \|(1-\lambda)(x_n - p) + \lambda(y_n - p)\|^2 \\ &= (1-\lambda)\|x_n - p\|^2 + \lambda\|y_n - p\|^2 - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &\le (1-\lambda)\|x_n - p\|^2 + \lambda D^2(Tx_n, Tp) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &\le (1-\lambda)\|x_n - p\|^2 + \lambda\Big(\|x_n - p\|^2 + kD^2(x_n - Tx_n, 0)\Big) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &= (1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda k D^2(\{x_n\}, Tx_n) - \lambda(1-\lambda)\|x_n - y_n\|^2 \\ &\le (1-\lambda)\|x_n - p\|^2 + \lambda\|x_n - p\|^2 + \lambda k\Big(\|x_n - y_n\|^2 + \frac{1}{n^2}\Big) - \lambda(1-\lambda)\|x_n - y_n\|^2. \\ &= \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \lambda(1-\lambda-k)\|x_n - y_n\|^2. \end{aligned}$$

Thus,

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \lambda(1-\lambda-k)\|x_n - y_n\|^2. \tag{3.13}$$

By Lemma (2.9), the sequence $\Big\{\|x_n - p\|\Big\}$ has a limit and therefore, $\{x_n\}$ is bounded. Moreover, we have from inequality (3.13) that

$$\lambda(1-\lambda-k)\|x_n - y_n\|^2 \leq \|x_n - p\|^2 + \frac{\lambda k}{n^2} - \|x_{n+1} - p\|^2,$$

and then,

$$\begin{aligned}\lambda(1-\lambda-k)\sum_{n=1}^{m}\|x_n - y_n\|^2 &\leq \sum_{n=1}^{m}\|x_n - p\|^2 + \sum_{n=1}^{m}\frac{\lambda k}{n^2} - \sum_{n=1}^{m}\|x_{n+1} - p\|^2 \\ &= \|x_1 - p\|^2 + \sum_{n=1}^{m}\frac{\lambda k}{n^2} - \|x_{m+1} - p\|^2 \\ &\leq \|x_1 - p\|^2 + \sum_{n=1}^{m}\frac{\lambda k}{n^2}.\end{aligned}$$

This implies that

$$\lambda(1-\lambda-k)\sum_{n=1}^{\infty}\|x_n - y_n\|^2 \leq \|x_1 - p\|^2 + \sum_{n=1}^{\infty}\lambda k\frac{1}{n^2} < \infty,$$

and therefore $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Since $d(x_n, Tx_n) \leq \|x_n - y_n\|$, it follows that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

Corollary 3.10. *Let K be a nonempty, closed and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping, with $F(T) \neq \emptyset$ and assume $Tp = \{p\}$ for each $p \in F(T)$. Suppose that T is hemicompact. Then, the sequence $\{x_n\}$ defined by equation (3.12) converges strongly to a fixed point of T.*

Proof. By Theorem (3.1), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is hemicompact, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \to q$ as $n \to \infty$ and let $y_{n_k} \in Tx_{n_k}$ such that
$\|x_{n_k} - y_{n_k}\| \leq d(x_{n_k}, Tx_{n_k}) + \frac{1}{k}$.. Then

$$\begin{aligned}d(q, Tq) &\leq \|q - x_{n_k}\| + \|x_{n_k} - y_{n_k}\| + d(y_{n_k}, Tq) \\ &\leq \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \frac{1}{k} + D(Tx_{n_k}, Tq) \\ &\leq \|q - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \frac{1}{k} + \frac{1+\sqrt{k}}{1-\sqrt{k}}\Big\|x_{n_k} - q\Big\|.\end{aligned}$$

Thus, taking limits on the righthand side as $k \to \infty$, we have $d(q, Tq) = 0$. Since Tq is closed, $q \in Tq$. Moreover, $x_{n_k} \to q$ as $n \to \infty$ gives $\|x_{n_k} - q\| \to 0$ as $n \to \infty$. Thus, using inequality (3.13) and Lemma (2.9), $\lim_{n\to\infty}\|x_n - q\| = 0$. Therefore $\{x_n\}$ converges strongly to a fixed point q of T as claimed. □

Remark 3.11. Observe that we did not assume that Tx is proximinal for each $x \in K$ neither did we require any continuity assumption on T nor any compactness assumption on K. Consequenctly, Corollary(3.10) is a significant improvement on Corollary13 of [7]

Corollary 3.12. *Let K be a nonmepty, compact and convex subset of a real Hilbert space H, and let $T : K \to CB(K)$ be a generalized $k-$strictly pseudocontractive multi-valued mapping, with $F(T) \neq \emptyset$ and assume $Tp = \{p\}$ for each $p \in F(T)$. Then, the sequence $\{x_n\}$ defined by equation (3.12) converges strongly to a fixed point of T.*

Proof. Since K is compact, every map $T : K \to CB(K)$ is hemicompact. Thus, by Corollary 3.10, we have that $\{x_n\}$ converges strongly to some $p \in F(T)$.

□

Remark 3.13. Our theorem and corollaries improve convergence theorems for multi-valued nonexpansive mappings in [1], [7], [16, 17],[24, 30, 32], in the following sense:

(i) The class of mappings considered in this paper contains the class of multi-valued $k-$ strictly pseudocontractive mappings as special case, which itself properly contain the class of multi-valued nonexpansive maps.
(ii) The algorithm here is Krasnoselkii type, which is known to have a geometric order of convergence, and the theorem is proved for the much larger class of generalized multi-valued strict pseudocontractive mappings.
(iii) Inequality (3.1) of definition (3.2) is a more natural generalisation of the single-valued psudo-contractive mappings as given by inequality(2.1).
(iv) The condition that Tx be weakly closed for each $x \in K$ imposed in [7] is dispensed with here.

We conclude, by saying that the condition $T(p) = \{p\}$ for all $p \in F(P)$, which is imposed in our theorem and corollaries is not crucial. Certainly our example (3.6) satisfies the condition since $T0 = \{0\}$ is the unique fixed point of T. However, some work in the litrature shows that this condition can be replaced with another condition which does not assume that the multi-valued mapping is single-valued on the nonempty fixed point set. Deatails of this can be found, for example, in [31], [33].

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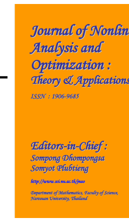
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 21-35
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



NON-STANDARD LAGRANGIANS WITH DIFFERENTIALS OPERATORS

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ABSTRACT. Nonlinear dynamics from fractional variational approach characterized by non-standard Lagrangians holding singular Weinstein and higher-order extended Euler-Poisson derivative operators are presented in this work. Many results that can enrich the theory of analytical mechanics are obtained and discussed.

KEYWORDS : fractional variational approach, non-singular Weinstein derivative operator, extended Euler-Poisson derivative operator, non-standard Lagrangians

1. INTRODUCTION

Fractional calculus of variations (FCV) is a new branch of applied mathematics which has many successful applications in physics and engineering [1] - [22]. Different formulations of FCV were introduced in literature depending on the nature of the problem under study. In fact, the FCV starts with the work of Riewe in 1996 where he formulated the problem of the calculus of variations by replacing the standard derivatives operators by a fractional one and derived the respective fractional Euler-Lagrange equations [23, 24]. The main result of Riewe's fractional approach is that non-conservative forces can be computed directly from potentials represented by fractional derivatives. It is noteworthy that the method developed by Riewe was applied to several systems by Dreisigmeyer and Young [25, 26] and by Rabei, Alhalholy and Taani [27] who also showed the limitations of the method caused by the complexity of fractional calculus. More importantly these authors also demonstrated that the resulting equations are casual and that the procedure to convert these equations into casual ones is not well-defined. In other words, past and future become linked through the fractional Euler-Lagrange equation which results on a broken causality, i.e. violation of the causality principle. Some advances in this approach are done in [28], nevertheless much work is required to settle the problem. Despite the fact that fractional causal dynamical equations may be derived from the least action principle, some of the resulting fractional functions are unclear, their physical meaning is still obscure and their physical validity is not obvious [29]. In all cases, the fractional approach derived by Riewe may represent a powerful tool to understand many hidden properties of complex dynamical systems, like

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Article history : Received November 12, 2012 Accepted June 26, 2013.

Hamiltonian chaos for example [30, 31]. The literature dealing with the FCV is immense; nevertheless much remains to be done. For a good introduction of the topic, we refer the reader to [32, 33].
One more successful form is known by the Fractional ActionLike Variational Approach (FALVA) introduced by the author in 2005 [8, 9]. Although in general, FCV and FALVA are under strong research studies, much remains to be done at both classical and quantum levels. The main purpose of this work is to extend FALVA for the case of non-standard Lagrangians (NSL) with singular derivatives operators (SDO) introduced by Weinstein, Euler and Poisson [34]. In fact, some basic equations of mathematical physics are obtained from NSL with SDO for dissipative dynamical systems and besides, it was observed that they can be applied successfully to a broad range of classical and quantum problems [35]-[39]. These NLS may increase the number of the initial data required to fix the classical trajectory and generate dynamical equations that go beyond the standard CityplaceNewton's law. In a more recent work [40], we derived a large numbers of NSL which were disregarded in the literature and which result on a number of familiar dynamical equations of motion. One chief observation is that the same equation of motion of a certain dynamical equation may be derived from dissimilar Lagrangians functionals. In reality, the significance of NSL was recognized in dissipative dynamical systems [36, 37]. Some nice examples are the Lienard type nonlinear oscillator [41, 42] and the 2^{nd} order Riccati equation [43]. Some related interesting results are found in [44]-[47].
The paper is organized as follows: the basic formalism is constructed in Sec. 2 and examples are illustrated in Sec. 3; conclusions and perspectives are given in Sec. 4.

2. BASIC FORMALISM

We start by introducing the new extended FALVA formalism with NSL and SDO:
Definition 2.1: *Consider a smooth manifold* M *and let* L *be an admissible smooth Lagrangian function* $L : \mathbb{R} \times \mathrm{TM} \longrightarrow \mathbb{R}$ *on* $\mathbb{R} \times \mathbb{R}^d \times C^d, d \geq 1$ *defined on the tangent bundle* TM*. The extended FALVA with NSL and SDO on the set of paths* $q(\tau)$*,*$0 \leq \tau \leq t$ *between two given points* $A = \mathrm{Q}(a)$ *and* $B = \mathrm{Q}(b)$*, is defined as for any piecewise smooth differentiable path* $\mathrm{Q} : [a, b] \longrightarrow \mathrm{M}$*,*

$$S[\mathrm{Q}] = \frac{1}{\Gamma(\alpha)} \int_a^b L\left(\tau, \mathrm{Q}(q(\tau)), \dot{\mathrm{Q}}(q(\tau), \dot{q}(\tau)), \ddot{\mathrm{Q}}(q(\tau), \dot{q}(\tau), \ddot{q}(\tau))\right) (t-\tau)^{\alpha-1} \, d\tau, \tag{2.1}$$

with $\mathrm{Q}(q(\tau)) = q(\tau) + \beta \int \frac{q(\tau)}{\tau - t} d\tau$*and*

$$\dot{\mathrm{Q}}(q(\tau), \dot{q}(\tau)) = \dot{q}(\tau) + \frac{\beta}{\tau - t} q(\tau), \tag{2.2}$$

$$\ddot{\mathrm{Q}}(q(\tau), \dot{q}(\tau), \ddot{q}(\tau)) = \ddot{q}(\tau) + \frac{\beta}{\tau - t}\dot{q}(\tau) - \frac{\beta}{(\tau - t)^2} q(\tau), \tag{2.3}$$

being respectively the Weinstein and the extended Euler-Poisson [**34**] *singular derivative operator*[1]*. Here* $\dot{q}(\tau) = dq(\tau)/d\tau$*,*$\ddot{q}(\tau) = d^2q(\tau)/d\tau^2$*,* α*is a real or a complex number and* $\beta \in \mathbb{R}$*;* τ*is the intrinsic time,* t *is the observer time with* $t \neq \tau$*.*

Theorem 2.1: *If* $q(\cdot)$ *are solutions of equation* (2.1)*, then* $\mathrm{Q}(\cdot)$*satisfies the following Euler-Lagrange equation:*

$$\left\{1 + \ln(\tau - t)^{\beta}\right\} \frac{\partial L}{\partial \mathrm{Q}} - \left\{\frac{1}{(t-\tau)^{\alpha-1}} + \frac{\alpha-1}{t-\tau}\right\} \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right) + \left\{\frac{\alpha - 1 - \beta}{(t-\tau)^{\alpha}} + \frac{(\alpha-1)^2}{(t-\tau)^2}\right\} \frac{\partial L}{\partial \dot{\mathrm{Q}}}$$

[1] The Euler-Poisson singular derivative operator is in reality of the form

$$\ddot{\mathrm{Q}}(\dot{q}(\tau), \ddot{q}(\tau)) = \ddot{q}(\tau) + (\beta/(\tau - t))\dot{q}(\tau)$$

and it is known as the Bessel's operator nevertheless it appeared in the work of Euler and Poisson [34]. In our work, we called equation (3) the extended Euler-Poisson operator.

$$+\left\{\frac{\beta}{(t-\tau)^{\alpha}}-\frac{2(\alpha-1)}{t-\tau}\right\}\frac{d}{d\tau}\left(\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\right)+\left\{\frac{(\alpha-1)(\alpha-2)}{(t-\tau)^{2}}-\frac{\beta(\alpha-1)}{(t-\tau)^{\alpha+1}}\right\}\frac{\partial L}{\partial\ddot{\mathrm{Q}}}+\frac{d^{2}}{d\tau^{2}}\left(\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\right)=0. \tag{2.4}$$

Proof: This problem is the same as extremizing the fractional action with Lagrangian $L(\tau,\dot{\mathrm{Q}},\dot{\mathrm{Y}})$ subject to the constraints $\mathrm{Y}=\dot{\mathrm{Q}}$. The action of the theory is now given by:

$$S[\mathrm{Q}]=\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\left\{L\left(\tau,\mathrm{Q},\dot{\mathrm{Q}},\dot{y}\right)-\lambda\left(\mathrm{Y}-\dot{\mathrm{Q}}\right)\right\}(t-\tau)^{\alpha-1}\,d\tau, \tag{2.5}$$

where λ is the Lagrange multiplier. It is easy to prove after elementary calculations that the following relation hold:

$$\frac{\partial L}{\partial q}(t-\tau)^{\alpha-1}-\frac{d}{d\tau}\left(\left(\frac{\partial L}{\partial\dot{q}}\right)(t-\tau)^{\alpha-1}+\lambda\frac{\partial\dot{\mathrm{Q}}}{\partial\dot{q}}(t-\tau)^{\alpha-1}\right)=0, \tag{2.6}$$

and $\mathrm{Y}=\dot{\mathrm{Q}}$.[2] Making use of the chain rules:

$$\frac{\partial L}{\partial q}=\frac{\partial L}{\partial\mathrm{Q}}\frac{\partial\mathrm{Q}}{\partial q}+\frac{\partial L}{\partial\dot{\mathrm{Q}}}\frac{\partial\dot{\mathrm{Q}}}{\partial q}+\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\frac{\partial\ddot{\mathrm{Q}}}{\partial q}, \tag{2.7}$$

$$\frac{\partial L}{\partial\dot{q}}=\frac{\partial L}{\partial\dot{\mathrm{Q}}}\frac{\partial\dot{\mathrm{Q}}}{\partial\dot{q}}+\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\frac{\partial\ddot{\mathrm{Q}}}{\partial\dot{q}}, \tag{2.8}$$

we obtain after simple algebraic manipulation:

$$\left(\frac{\partial L}{\partial\mathrm{Q}}\frac{\partial\mathrm{Q}}{\partial q}+\frac{\partial L}{\partial\dot{\mathrm{Q}}}\frac{\partial\dot{\mathrm{Q}}}{\partial q}+\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\frac{\partial\ddot{\mathrm{Q}}}{\partial q}\right)(t-\tau)^{\alpha-1}$$

$$-\frac{d}{d\tau}\left(\left(\frac{\partial L}{\partial\dot{\mathrm{Q}}}\frac{\partial\dot{\mathrm{Q}}}{\partial\dot{q}}+\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\frac{\partial\ddot{\mathrm{Q}}}{\partial\dot{q}}\right)(t-\tau)^{\alpha-1}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\frac{\partial\ddot{\mathrm{Q}}}{\partial\dot{q}}(t-\tau)^{\alpha-1}\right)\frac{\partial\dot{\mathrm{Q}}}{\partial\dot{q}}(t-\tau)^{\alpha-1}\right)=0. \tag{2.9}$$

Using equations (2.2) and (2.3) and the fact that

$$\frac{\partial\mathrm{Q}}{\partial q}=1+\beta\frac{\partial}{\partial q}\int\frac{q(\tau)}{\tau-t}d\tau=1+\beta\int\frac{\partial}{\partial q}\frac{q(\tau)}{\tau-t}d\tau=1+\ln(\tau-t)^{\beta}, \tag{2.10}$$

we find effortlessly:

$$\left\{1+\ln(\tau-t)^{\beta}\right\}\frac{\partial L}{\partial\mathrm{Q}}-\left\{\frac{1}{(t-\tau)^{\alpha-1}}+\frac{\alpha-1}{t-\tau}\right\}\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{\mathrm{Q}}}\right)+\left\{\frac{\alpha-1-\beta}{(t-\tau)^{\alpha}}+\frac{(\alpha-1)^{2}}{(t-\tau)^{2}}\right\}\frac{\partial L}{\partial\dot{\mathrm{Q}}}$$

$$+\left\{\frac{\beta}{(t-\tau)^{\alpha}}-\frac{2(\alpha-1)}{t-\tau}\right\}\frac{d}{d\tau}\left(\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\right)+\left\{\frac{(\alpha-1)(\alpha-2)}{(t-\tau)^{2}}-\frac{\beta(\alpha-1)}{(t-\tau)^{\alpha+1}}\right\}\frac{\partial L}{\partial\ddot{\mathrm{Q}}}+\frac{d^{2}}{d\tau^{2}}\left(\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\right)=0. \tag{2.11}$$

Remark 2.1: *When $\beta=0$and $\alpha=1$, equation (2.4) is reduced to:*

$$\frac{\partial L}{\partial q}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{q}}\right)+\frac{d^{2}}{d\tau^{2}}\left(\frac{\partial L}{\partial\ddot{q}}\right)=0. \tag{2.12}$$

For $\beta\neq0$ and $\alpha=1$, equation (2.4) is reduced to:

$$\left\{1+\ln(\tau-t)^{\beta}\right\}\frac{\partial L}{\partial\mathrm{Q}}+\frac{\beta}{\tau-t}\left\{\frac{\partial L}{\partial\dot{\mathrm{Q}}}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\right)\right\}-\frac{d}{d\tau}\left\{\left(\frac{\partial L}{\partial\dot{\mathrm{Q}}}\right)-\frac{d}{d\tau}\left(\frac{\partial L}{\partial\ddot{\mathrm{Q}}}\right)\right\}=0. \tag{2.13}$$

Illustrations and exemplifications are done in the subsequent section.

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[2] *Equation (6) is the fractional Euler-Lagrange equation obtained in FALVA approach*

3. ILLUSTRATIONS AND APPLICATIONS

Before illustrating our approach, it is noteworthy we do not claim at this stage the superiority of the FALVA method over the standard FCV method originally developed by Riewe [23, 24]. Both approaches are different nevertheless in our opinion, FALVA approach is simpler. In Riewe's approach, fractional derivatives are used to study nonconservative dynamical systems (NDS) and in that context, the generalization of Lagrangians taken into account the dissipative effects was performed. Whereas in FALVA approach we used fractional integrals to study NDS and not fractional derivatives. In general, dealing with fractional derivatives is much complicated than dealing with fractional derivatives, e.g. the fractional integration by parts require some hard mathematical restrictions on the functions involved. The form of the fractional equation of motion may be comparable with the classical one but the physical meaning of the classical derivatives is loosing the meaning in the fractional case [48]. The replacement of the classical derivative in equation (2.1) by fractional derivative renders the mathematical manipulation of the theory very hard and consequently, numerical techniques are required in order to compare FALVA approach to Riewe's approach. From another direction, by replacing the standard Lagrangians in Riewe's approach by NSL augmented by fractional SDO renders the mathematical manipulations much hard. To appreciate FALVA and the importance of NSL, let us start by discussing the following simplest example:

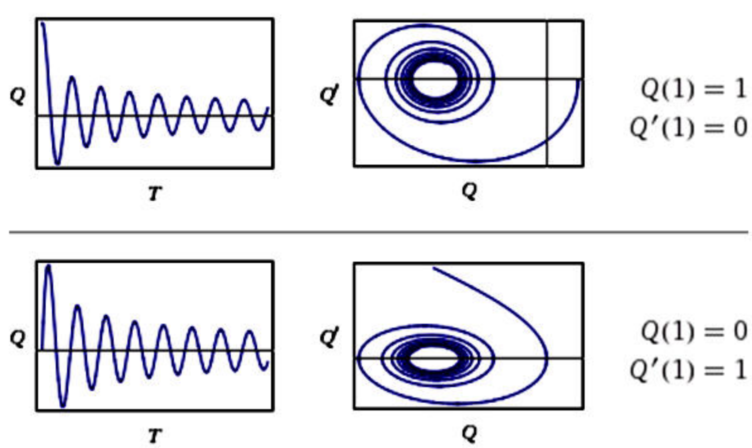
Example 1-We consider at the beginning the NSL $L = -\frac{1}{2}\{1+\ln(\tau-t)^{\beta}\}^{-1}\mathrm{Q}^2 + \frac{1}{2}\dot{\mathrm{Q}}^2 + \ddot{\mathrm{Q}}$ for $\alpha = 1$ where equation (2.13) holds accordingly. Equation (2.13) results into the following dynamical equation:

$$\ddot{\mathrm{Q}} - \frac{\beta}{\tau - t}\dot{\mathrm{Q}} + \mathrm{Q} = 0. \tag{3.1}$$

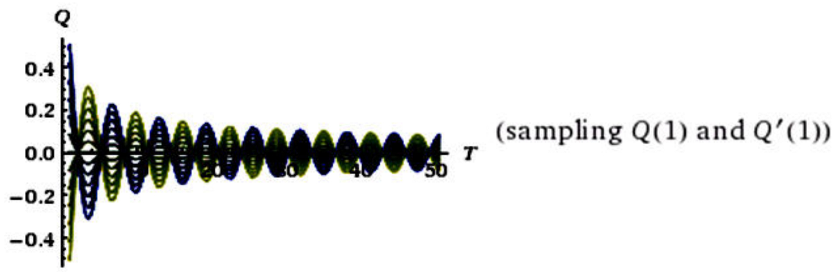
The solution is given by:

$$\mathrm{Q}(\tau) = c_1(\tau-t)^{\frac{\beta+1}{2}} J_{\frac{\beta+1}{2}}(\tau-t) + c_2(\tau-t)^{\frac{\beta+1}{2}} Y_{\frac{\beta+1}{2}}(\tau-t), \tag{3.2}$$

where c_1,c_2 are integration constants and $J_n(z)$ and $Y_n(z)$ are respectively the Bessel functions of 1^{st} and 2^{nd} kind. For $\beta = -1$, we plot in graphs 1 and 2 sample individual solutions and sample family solution of equation (3.1) respectively ($T = \tau - t$):



Graph 1: sample individual solutions of equation (3.2)



Graph 2: sample family solution of equation (3.2)

These graphs show decaying oscillations due to the presence of a positive friction term. From equation (2.2) we find:

$$q(\tau) = c_1 2^{-\frac{\beta+3}{2}} (\tau-t)^{\beta+2} \Gamma(\beta+1) {}_1\tilde{F}_2\left(\beta+1; \frac{\beta+3}{2}; \beta+2, -\frac{(\tau-t)^2}{4}\right)$$
$$+\left\{(\tau-t)^{\beta+1}\cos\left(\frac{\pi(\beta+1)}{2}\right)\Gamma(\beta+1){}_1\tilde{F}_2\left(\beta+1; \frac{\beta+3}{2}; \beta+2; -\frac{(\tau-t)^2}{4}\right)\right.$$
$$\left. -\,2^{\beta+1}\Gamma\left(\frac{\beta+1}{2}\right){}_1\tilde{F}_2\left(\frac{\beta+1}{2}; \frac{1-\beta}{2}; \frac{\beta+3}{2}; -\frac{(\tau-t)^2}{4}\right)\right\} c_2 2^{-\frac{\beta+3}{2}}(\tau-t)\sec\left(\frac{\pi\beta}{2}\right)+c_3, \tag{3.3}$$

where c_3 is another integration constant and ${}_p\tilde{F}_q\left(a_1, ..., a_p; b_1, ..., b_q; z\right)$ is the generalized regularized hypergeometric function. If for instance $\beta = -1$, then $\mathrm{Q}(\tau) = c_1 J_0(\tau-t) + c_2 Y_0(\tau-t)$and

$$q(\tau) = -c_1\frac{1}{2}(\tau-t)^2{}_1F_2\left(\frac{1}{2}; \frac{3}{2}; 2; -\frac{(\tau-t)^2}{4}\right) - c_2\frac{1}{2}(\tau-t)G^{2,1}_{2,4}\left(\frac{\tau-t}{2}, \frac{1}{2}\middle| \begin{array}{c} 1, -1 \\ -\frac{1}{2}, \frac{1}{2}, -1, 0\end{array}\right). \tag{3.4}$$

${}_pF_q\left(a_1, ..., a_p; b_1, ..., b_q; z\right)$ is the hypergeometric function and $G^{m,n}_{p,q}\left(z\middle| \begin{array}{c} a_1, ..., a_p \\ b_1, ..., b_q\end{array}\right)$ is the Meijer G-function [48]:

$$G^{m,n}_{p,q}\left(z\middle| \begin{array}{c} a_1, ..., a_p \\ b_1, ..., b_q\end{array}\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=n+1}^p \Gamma(a_j - s)\prod_{j=m+1}^q \Gamma(1-b_j+s)} x^s ds. \tag{3.5}$$

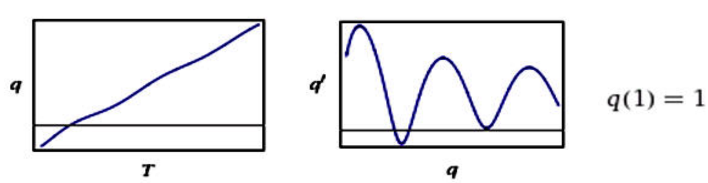
Assuming the initial conditions $\mathrm{Q}(2.1) = 1$and $\mathrm{Q}'(2.1) = 0$, then the solution of equation (3.1) which is illustrated in graph 1 is:

$$\mathrm{Q}(\tau) = \frac{J_1(1)Y_0(T) - Y_1(1)J_0(T)}{J_1(1)Y_0(1) - Y_1(1)J_0(1)}, \tag{3.6}$$

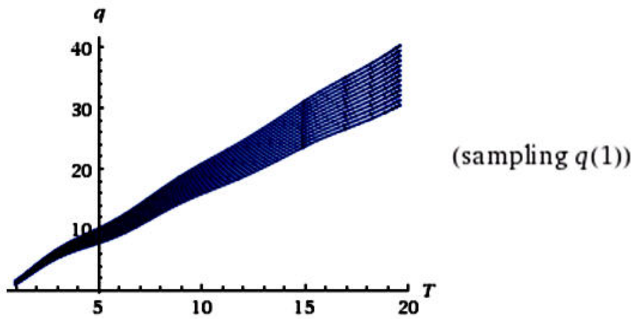
and accordingly$q(\tau)$, is equal to:

$$\frac{(\tau-t)Y_1(1)G^{2,0}_{1,3}\left(\left(\frac{\tau-t}{2}\right)^2\middle| \begin{array}{c} 1 \\ 0, 0, 0\end{array}\right)}{2J_1(1)Y_0(1) - 2J_0(1)Y_1(1)} - \frac{2J_1(1)G^{3,0}_{2,4}\left(\frac{\tau-t}{2}, \frac{1}{2}\middle| \begin{array}{c} 0, \frac{3}{2} \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\end{array}\right)}{2J_1(1)Y_0(1) - 2J_0(1)Y_1(1)} + c_3(\tau-t). \tag{3.7}$$

We plot in graphs 3 and 4 sample individual solutions and sample family solution of equation (3.7) respectively:



Graph 3: sample individual solutions of equation (3.7)



Graph 4: sample family solution of equation (3.7)

It is very hard to describe analytically the dynamical problem by replacing the ordinary derivative operator by a fractional and numerical techniques are required afterward. It is noteworthy that that the dynamical equation (3.1) may be derived in general from the standard time-dependent Lagrangian $L(\tau, \mathrm{Q}, \dot{\mathrm{Q}}) = \frac{1}{2}(\tau - t)(\dot{\mathrm{Q}}^2 - \mathrm{Q}^2)$ and using the standard Euler-Lagrange equation

$$\frac{\partial L}{\partial \mathrm{Q}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right) = 0. \tag{3.8}$$

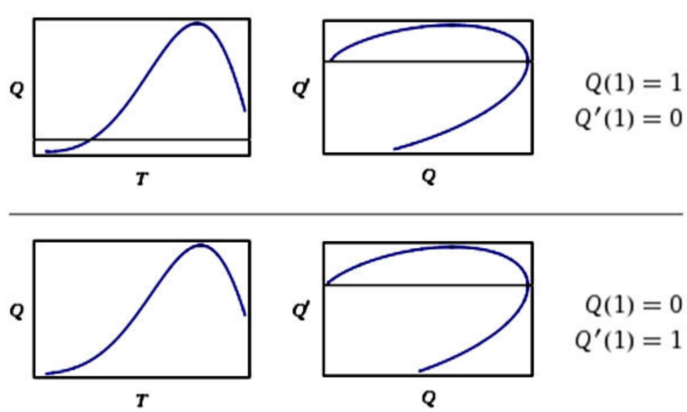
Whereas in our approach, equation (2.11) gives for $L(\tau, \mathrm{Q}, \dot{\mathrm{Q}}) = \frac{1}{2}\tau(\dot{\mathrm{Q}}^2 - \mathrm{Q}^2)$

$$\left\{\frac{1}{(t-\tau)^{\alpha-2}} + 1 - \alpha\right\}\ddot{\mathrm{Q}} + \left\{\frac{\alpha(\alpha-1)}{t-\tau} + \frac{\alpha-\beta}{(t-\tau)^{\alpha-1}}\right\}\dot{\mathrm{Q}} + \left\{1 + \ln(\tau - t)^{\beta}\right\}(\tau - t)\mathrm{Q} = 0. \tag{3.9}$$

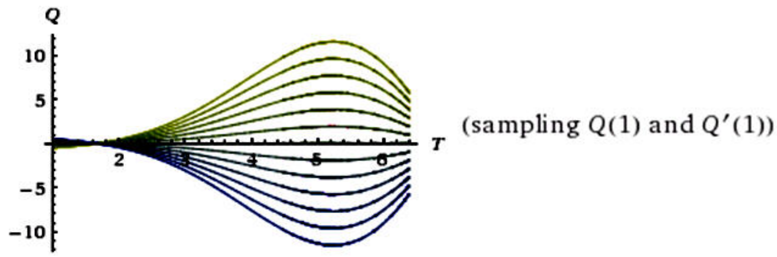
Mainly for $\alpha = 1$, we get:

$$(\tau - t)\ddot{\mathrm{Q}} + (\beta - 1)\dot{\mathrm{Q}} - \left\{1 + \ln(\tau - t)^{\beta}\right\}(\tau - t)\mathrm{Q} = 0. \tag{3.10}$$

Obviously, for $\beta = -1$ the dynamics which result from equation (3.10) is totally different from the one obtained from equation (3.1) as shown in graphs 5 and 6:



Graph 5: sample individual solutions of equation (3.10)



Graph 6: sample family solution of equation (3.10)

Comparing graphs 1 and 5, we observe that for some critical values of the parameter β, the dynamics are totally dissimilar.

Example2-We consider now the NSL $L = -\frac{1}{2}\frac{(\tau-t)^m}{1+\ln(\tau-t)^\beta}\mathrm{Q}^2 + \dot{\mathrm{Q}} + \ddot{\mathrm{Q}}$ with $\alpha - 1 = \beta$and $m \in \mathbb{R}$. Hence we get:

$$\mathrm{Q}(\tau) = \frac{(\alpha-1)(2\alpha-3)}{(t-\tau)^{2+m}} + \frac{(\alpha-1)^2}{(t-\tau)^{\alpha+m+1}}, \tag{3.11}$$

and accordingly:

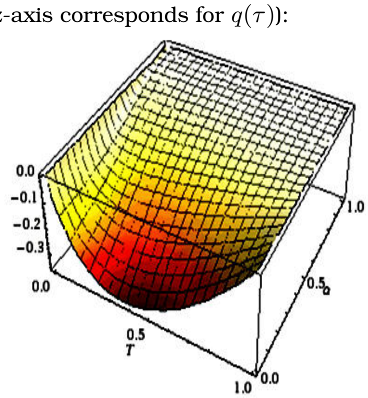
$$\dot{q} + \frac{\alpha-1}{\tau-t}q = (2+m)(\alpha-1)(2\alpha-3)\,(t-\tau)^{-3-m}$$

$$+\,(\alpha+m+1)(\alpha-1)^2\,(t-\tau)^{-\alpha-m-2}\,. \tag{3.12}$$

One particular class of solution is obtained for $m = -2$ where the solution is given by:

$$q(\tau) = (1-\alpha)^3(\tau-t)^{1-\alpha}\ln|\tau-t| + c_4. \tag{3.13}$$

c_4 is one more integration constant. It is noteworthy that the dynamics of Qis complexified for $0 < \alpha < 1$ and $T = \tau - t > 0$ whereas the dynamics of $q(\tau)$ is real. We plot in Graph 7 the variations of equations (3.13) and in Graphs 8 and 9 the variations of equation (3.11) (the z-axis corresponds for $q(\tau)$):



Graph 7: Plot of equation (3.13)

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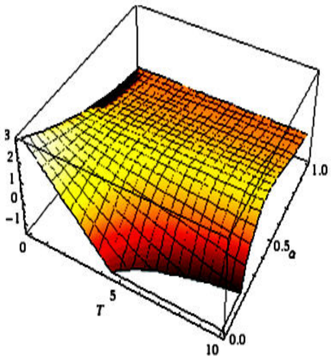
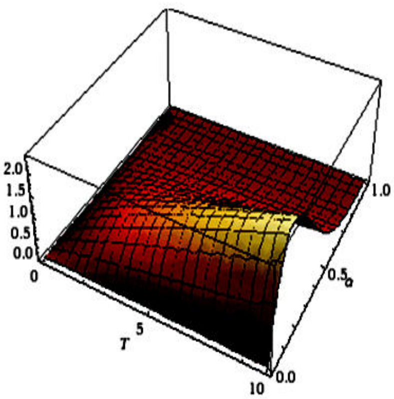


Graph 8: Plot of equation (3.13): *real part*

.



Graph 9: Plot of equation (3.13): *imaginary part*

.

We may conclude that equations (2.4) and (2.13) are motivating if we want to describe Lagrangian systems holding a NSL and starting from FALVA.

Example 3-We could at present consider the following NSL $L = (\dot{\mathrm{Q}} + f(\tau, t)\mathrm{Q})^{-1}$considered in [36] to model dissipations. Here $f(\tau, t)$is an arbitrary function of time. Equation (2.11) gives now:

$$\left\{\frac{1}{(t-\tau)^{\alpha-1}} + \frac{\alpha-1}{t-\tau}\right\}\left((\dot{\mathrm{Q}} + f(\tau,t)\mathrm{Q})^{-1}\left(\ddot{\mathrm{Q}} + \frac{df}{d\tau}\mathrm{Q} + f\dot{\mathrm{Q}}\right)\right)$$
$$-\left\{1 + \ln(\tau - t)^{\beta}\right\} f(\tau, t) + \left\{\frac{\alpha - 1 - \beta}{(t-\tau)^{\alpha}} + \frac{(\alpha-1)^2}{(t-\tau)^2}\right\} = 0. \quad (3.14)$$

For simplicity, we choose $f(\tau, t) = 1$, then equation (3.14) is simplified to:

$$\left\{\frac{1}{(t-\tau)^{\alpha-1}} + \frac{\alpha-1}{t-\tau}\right\}\left(\ddot{\mathrm{Q}} + \dot{\mathrm{Q}}\right) + \left(\left\{\frac{\alpha-1-\beta}{(t-\tau)^{\alpha}} + \frac{(\alpha-1)^2}{(t-\tau)^2}\right\} - \left\{1 + \ln(\tau-t)^{\beta}\right\}\right)(\dot{\mathrm{Q}} + \mathrm{Q}) = 0. \quad (3.15)$$

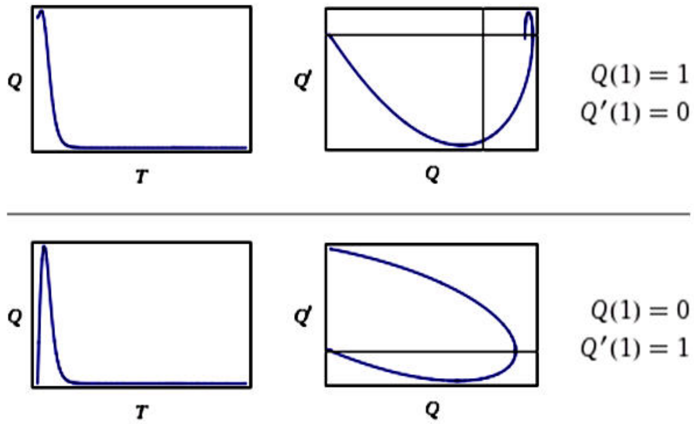
To illustrate, we choose $\beta << 1$and $\alpha = 2$; then equation (3.15) is simplified to

$$-\frac{2}{\tau-t}\ddot{\mathrm{Q}} + \left\{-\frac{2}{\tau-t} + \frac{2}{(\tau-t)^2} - 1\right\}\dot{\mathrm{Q}} + \left\{\frac{2}{(\tau-t)^2} - 1\right\}\mathrm{Q} = 0. \quad (3.16)$$

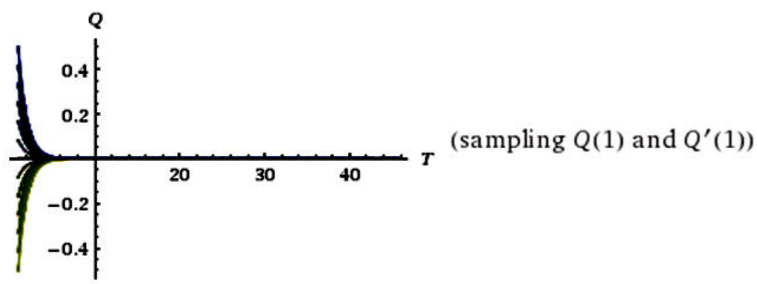
and the solution is given by:

$$\mathrm{Q}(\tau) = c_5 e^{-\frac{1}{4}(\tau-t)((\tau-t+4))}\left(\sqrt{\pi}e^{\frac{1}{4}(\tau-t)^2+1}\mathrm{erf}\left(\frac{\tau-t-2}{2}\right) - e^{\tau-t}\right) + c_6 e^{-(\tau-t)}, \quad (3.17)$$

where $\mathrm{erf}(x)$ is error function. We plot the resulting solutions in Graphs 10 and 11

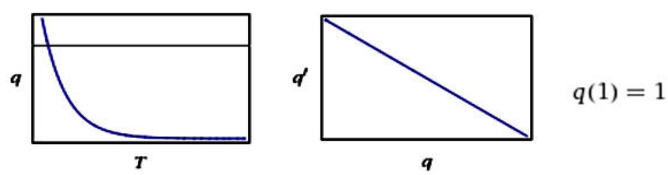


Graph 10: sample individual solutions of equation (3.17)

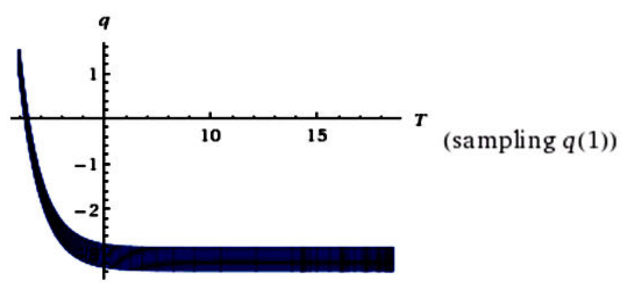


Graph 11: sample family solution of equation (3.17)

For very large time, we plot in Graphs 12 and 13 the solutions which correspond for $q(\tau)$ as deduced from equation (2.2):

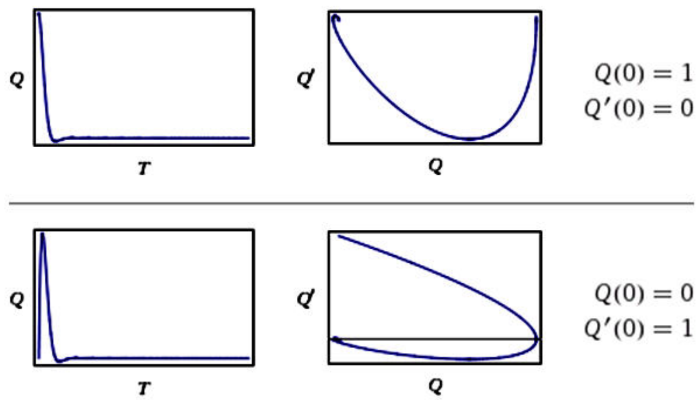


Graph 12: sample individual solutions for $q(\tau)$

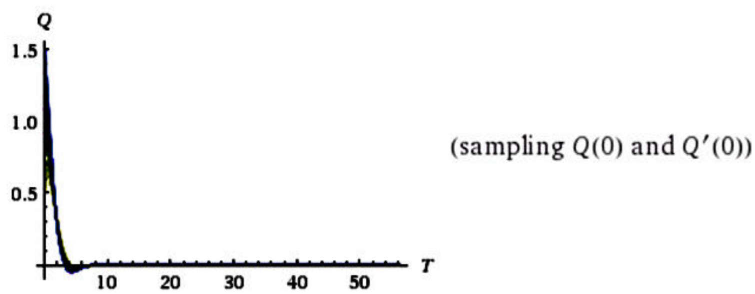


Graph 13: sample family solution of for $q(\tau)$

In [36], the resulting equation of motion which results from the NSL $L = (\dot{\mathrm{Q}}+f(\tau,t)\mathrm{Q})^{-1}$with $f(\tau,t) = 1$is: $\ddot{\mathrm{Q}} + 1.5\dot{\mathrm{Q}} + \mathrm{Q} = 0$and the solutions are plotted in Graphs 14 and 15:

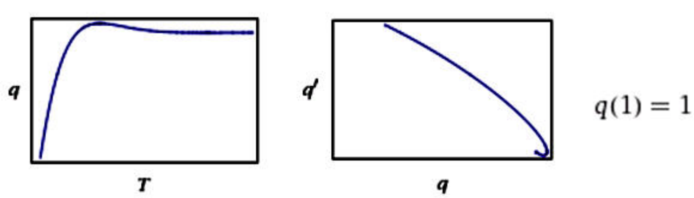


Graph 14: sample individual solutions of equation $\ddot{\mathrm{Q}} + 1.5\dot{\mathrm{Q}} + \mathrm{Q} = 0$

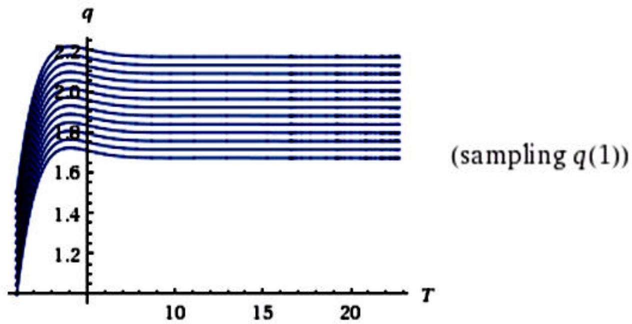


Graph 15: sample family solution of equation $\ddot{\mathrm{Q}} + 1.5\dot{\mathrm{Q}} + \mathrm{Q} = 0$

Graphs 10 and 14 are similar however in [36], we have $\beta = 0$whereas in our approach, for the particular case $\beta << 1$, e.g. $\beta = 10^{-3}$, the following graphical solutions (Graphs 16 and 17) hold for $q(\tau)$:



Graph 16: sample individual solutions of $q(\tau)$



Graph 17: sample family solution of $q(\tau)$

Remark 3.1: *If the Lagrangian is independent of* Q, *then equation* (2.13) *is reduced to*:

$$\frac{\beta}{\tau-t}\left\{\frac{\partial L}{\partial \dot{\mathrm{Q}}}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)\right\}-\frac{d}{d\tau}\left\{\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right)-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)\right\}=0. \tag{3.18}$$

It is easy to check after straightforward algebra that:

$$\left(\frac{\partial L}{\partial \dot{\mathrm{Q}}}\right)-\frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{\mathrm{Q}}}\right)=K(\tau-t)^{\beta}, \tag{3.19}$$

where K is an integration constant. This equation is similar to the Euler-Lagrange equation with a time-dependent dissipative term S in particular when $\dot{\mathrm{Q}} \longrightarrow \mathrm{Q}$*and* $S=K(\tau-t)^{\beta}$. *Here* Q*is a new generalized coordinate.*

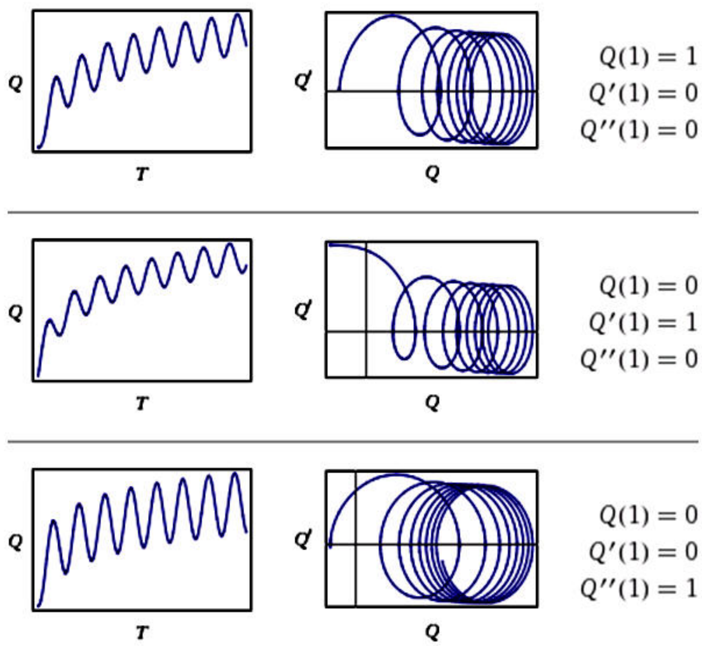
To illustrate we discuss the following examples:

Example 4-We consider the Lagrangian $L=\frac{1}{2}\dot{\mathrm{Q}}^2-\frac{1}{2}\ddot{\mathrm{Q}}^2$and we choose $\beta=-1$ Equation (3.18) gives:

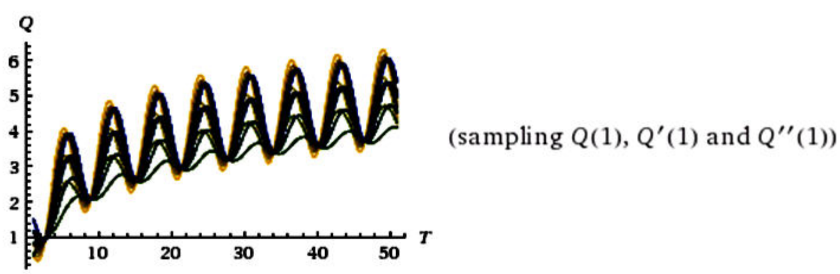
$$\mathrm{Q}(\tau)=c_5\sin(\tau-t)+c_6\cos(\tau-t)+c_7$$

$$+\mathrm{Ci}(\tau-t)(-\cos(\tau-t))-S\mathrm{i}(\tau-t)(\sin(\tau-t))+\log(\tau-t), \tag{3.20}$$

where $\mathrm{Ci}(\tau-t)$ and $\mathrm{Si}(\tau-t)$are respectively the cosine and the sine integrals [49]. We plot in Graph 18 sample individual solutions and in Graph 19 a sample family solution:



Graph 18: sample individual solutions of equation (3.20)

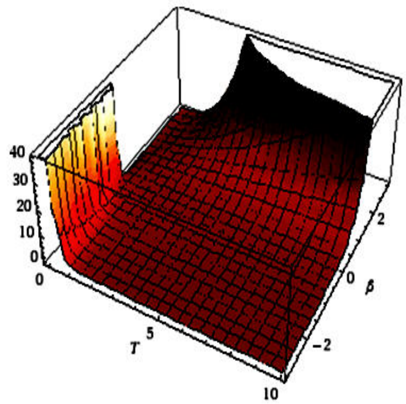


Graph 19: sample family solution of equation (3.20)

Example 5-We consider now the Lagrangian $L = \frac{1}{2}m\dot{\mathrm{Q}}^2 - U(\ddot{\mathrm{Q}})$where $U(\ddot{\mathrm{Q}})$ is a function of $\ddot{\mathrm{Q}}$and $m$is the mass of the particle. Then the equation of motion is:

$$p - \frac{d}{d\tau}\left(\frac{\partial U}{\partial \ddot{\mathrm{Q}}}\right) = K(\tau - t)^{\beta}, \tag{3.21}$$

where $p = m\dot{\mathrm{Q}}$is the momentum. In the absence of the function $U(\ddot{\mathrm{Q}})$, we get $p = K(\tau - t)^{\beta}$and hence the momentum is time-dependent. It increases with time for $\beta > 0$and decreases for $\beta < 0$. We plot in Graph 20 the variation of equation $p = K(\tau - t)^{\beta}$ (z axis corresponds for p):



Graph 20: Plot of equation $p = K(\tau - t)^{\beta}$for $K = 1$and $-3 < \beta < 3$

4. CONCLUSIONS AND PERSPECTIVES

In conclusion, we have explored nonlinear dynamics from both fractional and non-fractional FALVA holding non-standard Lagrangians. The new formalism is characterized by non-standard Lagrangians holding singular Weinstein and higher-order extended Euler-Poisson derivative operators. The examples discussed in this paper are simple, however they prove that non-standard Lagrangians are important and deserves attention. The results obtained here are very relevant to those presented by Riewe. It is notable that in 1931, Bauer proved that for a given dissipative dynamical systems with constant coefficients, the resulting dynamical equations of motion are not derived from a variational principle [50]. In Bauer' approach all the derivatives are of integer order. This is one of the main reasons why Riewe use fractional derivatives operators to model dissipative dynamical systems. Here we proved that FALVA augmented by NSL and Weinstein differential operator could be used to model nonconservative and dissipative systems without implementing fractional derivatives. In other words, the Lagrangian formulation is constructed for different kinds of dissipative systems without using the notion of fractional derivative operator. Further investigations should be carried to elucidate and understand this approach.

In literature there exist many approaches to deal with dynamical systems [51, 52, 53]; nevertheless the equations of motion are derived from the standard action principle. However, the presented approach is in fact widespread and permits obtaining miscellaneous forms of non-standard Lagrangians and equations of motion. We argue that this new formalism has important implications in many physical systems. The main advantage of the new action explored in this work is that it offers a new view of the dynamical system under study. We expect that the new arguments proposed in this work can be applied successfully to a broad range of physical problems ranging from dissipative mechanics to field theories [54]-[58]. It will be as well interesting to apply our approach to Hamiltonian and chaotic dynamics where time takes on a fractal structure [30]. Work in these directions is under progress.

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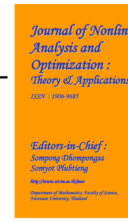
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 37-44
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



HARMONIC UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY USING SALAGEAN INTEGRAL OPERATOR

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ABSTRACT. In this paper we define and investigate a new class of harmonic functions defined by using Salagean integral operator with varying arguments. We obtain coefficient inequalities, extreme points and distortion bounds.

KEYWORDS : Harmonic functions; Analytic functions; Sense preserving; Salagean integral operator.
AMS Subject Classification: 30C45.

1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ which is defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$, $z \in D$ (see [3]).
Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f_z^{'}(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad , |b_1| < 1. \tag{1.2}$$

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Article history : Received October 14, 2012 Accepted January 14, 2013.

In 1984 Clunie and Shell-Small [3] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.
Salagean integral operator I^n is defined as follows (see [6]):

(i) $I^0 f(z) = f(z)$;
(ii) $I^1 f(z) = If(z) = \int\limits_0^z f(t)t^{-1}dt$;
....

(iii) $I^n f(z) = I(I^{n-1}f(z))$ $(n \in \mathbb{N} = \{1, 2, 3, ...\})$.

In [4], Cotirla defined Salagean integral operator for harmonic univalent functions $f(z)$ such that $h(z)$ and $g(z)$ are given by (1.2) as follows:

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}, \tag{1.3}$$

where

$$I^n h(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k \text{ and } I^n g(z) = \sum_{k=1}^{\infty} k^{-n} b_k z^k.$$

With the help of the modified Salagean integral operator we let $E_H(m, n; \gamma, \rho)$ be the family of harmonic functions $f = h + \overline{g}$, which satisfy the condition

$$Re\left\{\left(1 + \rho e^{i\alpha}\right)\frac{I^n f(z)}{I^m f(z)} - \rho e^{i\alpha}\right\} \geq \gamma \tag{1.4}$$

$$\left(\alpha \in \mathbb{R}, 0 \leq \gamma < 1, \rho \geq 0, m \in \mathbb{N}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, m > n, \text{ and } z \in U\right),$$

where $I^n f$ is defined by (1.3), we note that:
Taking $\alpha = 0$, $E_H(n+1, n; 2\beta - 1, 1) = H(n, \beta)$ $(0 \leq \beta < 1)$ (see Cotirla [4]).
Also we note that, by the special choices of α, γ, ρ, m and n, we obtain:

(i) Taking $\alpha = 0$, then $E_H(m, n, 2\beta - 1, 1) = H(m, n; \beta) = \Big\{f \in S_H :$

$$Re\left\{\frac{I^n f(z)}{I^m f(z)}\right\} > \beta\,(0 \leq \beta < 1; m \in \mathbb{N}; n \in \mathbb{N}_0; m > n; z \in U)\Big\};$$

(ii) $E_H(n+1, n; \gamma, \rho) = E_H(n; \gamma, \rho) = \Big\{f \in S_H : Re\Big\{\left(1 + \rho e^{i\alpha}\right)\dfrac{I^n f(z)}{I^{n+1} f(z)}$

$$-\rho e^{i\alpha}\Big\} \geq \gamma\,(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; n \in \mathbb{N}_0; z \in U)\Big\};$$

(iii) $E_H(1, 0; \gamma, \rho) = E_H(\gamma, \rho) = \Big\{f \in S_H : Re\Big\{\left(1 + \rho e^{i\alpha}\right)\dfrac{f(z)}{If(z)} - \rho e^{i\alpha}\Big\} \geq \gamma$

$$(\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; z \in U)\Big\}.$$

Also we define the subclass $V_{\overline{H}}(m, n; \gamma, \rho)$ consists of harmonic functions $f_n = h + \overline{g}_n$ in $E_H(m, n; \gamma, \rho)$ such that h and g_n are the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad g_n(z) = \sum_{k=1}^{\infty} b_k z^k, \tag{1.5}$$

and there exist a real number ϕ such that, mod 2π,

$$\arg(a_k) + (k-1)\phi \equiv \pi,\ k \geq 2 \text{ and } \arg(b_k) + (k+1)\phi \equiv (n-1)\pi,\ k \geq 1. \tag{1.6}$$

Also we note that, by the special choices of α, γ, m and n, we obtain:

(i)Taking $\alpha = 0$, $V_{\overline{H}}\,(n+1, n; 2\beta - 1, 1) = V_{\overline{H}}\,(n, \beta)$;
(ii) Taking $\alpha = 0$, $V_{\overline{H}}(m, n, 2\beta - 1, 1) = V_{\overline{H}}\,(m, n; \beta)$;
(iii) $V_{\overline{H}}(n+1, n; \gamma, \rho) = V_{\overline{H}}\,(n; \gamma, \rho)$;
(iv) $V_{\overline{H}}(1, 0; \gamma, \rho) = V_{\overline{H}}\,(\gamma, \rho)$.

2. MAIN RESULTS

Unless otherwise mentioned, we assume in the reminder of this paper that, $\alpha \in \mathbb{R}$, $0 \le \gamma < 1$, $\rho \ge 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in U$. We begin with a sufficient coefficient condition for functions in the class $E_H(m, n; \gamma, \rho)$.

Theorem 1. Let $f = h + \overline{g}$ be such that h and g are given by (1.2). Furthermore,

$$\sum_{k=1}^{\infty} \left[\tfrac{(1+\rho)k^{-n} - (\gamma+\rho)k^{-m}}{1-\gamma} |a_k| + \tfrac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma} |b_k| \right] \le 2, \tag{2.1}$$

where $a_1 = 1$. Then f is sense-preserving, harmonic univalent in U and $f \in E_H(m, n; \gamma, \rho)$.

Proof. If $z_1 \ne z_2$, then by using (2.1), we have

$$\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right|$$

$$\ge 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| = 1 - \left| \frac{\sum\limits_{k=1}^{\infty} b_k \left(z_1^k - z_2^k \right)}{(z_1 - z_2) + \sum\limits_{k=2}^{\infty} a_k \left(z_1^k - z_2^k \right)} \right|$$

$$\ge 1 - \frac{\sum\limits_{k=1}^{\infty} k\,|b_k|}{1 - \sum\limits_{k=2}^{\infty} k\,|a_k|} \ge 1 - \frac{\sum\limits_{k=1}^{\infty} \dfrac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)\,k^{-m}}{1-\gamma} |b_k|}{1 - \sum\limits_{k=2}^{\infty} \dfrac{(1+\rho)k^{-n} - (\gamma+\rho)\,k^{-m}}{1-\gamma} |a_k|} \ge 0,$$

which proves univalence. Also f is sense-preserving in U since

$$\begin{aligned} |h^{'}(z)| &\ge 1 - \sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} > 1 - \sum_{k=2}^{\infty} \frac{(1+\rho)k^{-n} - (\gamma+\rho)\,k^{-m}}{1-\gamma} |a_k| \\ &\ge \sum_{k=1}^{\infty} \tfrac{(1+\rho)k^{-n} - (-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma} |b_k| \ge \sum_{k=1}^{\infty} k\,|b_k|\,|z|^{k-1} \ge |g^{'}(z)|. \end{aligned}$$

Now we show that $f \in E_H(m, n; \gamma, \rho)$. We only need to show that if (2.1) holds then the condition (1.4) is satisfied, then we want to prove that

$$Re\left\{ \frac{\left(1 + \rho e^{i\alpha}\right) I^n f(z) - \rho e^{i\alpha} I^m f(z)}{I^m f(z)} \right\} = Re\frac{A(z)}{B(z)} \ge \gamma \tag{2.2}$$

Using the fact that $Re\{w\} > \gamma$ if and only if $|1 - \gamma + w| > |1 + \gamma - w|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \ge 0, \tag{2.3}$$

where $A(z) = \left(1 + \rho e^{i\alpha}\right) I^n f(z) - \rho e^{i\alpha} I^m f(z)$ and $B(z) = I^m f(z)$. Substituting for $A(z)$ and $B(z)$ in the left side of (2.3) we obtain

$$\begin{aligned}
&\left|\left(1+\rho e^{i\alpha}\right) I^n f(z) - \rho e^{i\alpha} I^m f(z) + (1-\gamma)\, I^m f(z)\right| \\
&- \left|\left(1+\rho e^{i\alpha}\right) I^n f(z) - \rho e^{i\alpha} I^m f(z) - (1+\gamma) I^m f(z)\right|
\end{aligned}$$

$$\begin{aligned}
= \ & \Bigg|(2-\gamma)\, z + \sum_{k=2}^{\infty} \left[\left(k^{-n} + (1-\gamma)\, k^{-m}\right) + \rho e^{i\alpha}\left(k^{-n} - k^{-m}\right)\right] a_k z^k \\
& - (-1)^m \sum_{k=1}^{\infty} \left[\left((\gamma-1)\, k^{-m} - (-1)^{m-n}\, k^{-n}\right) + \rho e^{i\alpha}\left(k^{-m} - (-1)^{m-n}\, k^{-n}\right)\right] \cdot \\
& \cdot \overline{b_k z^k}\Bigg| - \Bigg|\gamma z - \sum_{k=2}^{\infty} \left[\left(k^{-n} - (1+\gamma)\, k^{-m}\right) + \rho e^{i\alpha}\left(k^{-n} - k^{-m}\right)\right] a_k z^k + (-1)^m \cdot \\
& \sum_{k=1}^{\infty} \left[\left((1+\gamma)\, k^{-m} - (-1)^{m-n}\, k^{-n}\right) + \rho e^{i\alpha}\left(k^{-m} - (-1)^{m-n}\, k^{-n}\right)\right] \overline{b_k z^k}\Bigg| \\
\geq \ & (2-\gamma)\, |z| - \sum_{k=2}^{\infty} \left[(1+\rho) k^{-n} - (\gamma+\rho-1) k^{-m}\right] |a_k|\, |z|^k \\
& - \sum_{k=1}^{\infty} \left|(1+\rho) k^{-n} - (-1)^{m-n} (\gamma+\rho-1) k^{-m}\right| |b_k|\, |z|^k \\
& - \gamma\, |z| - \sum_{k=2}^{\infty} \left[(1+\rho) k^{-n} - (\gamma+\rho+1)\, k^{-m}\right] |a_k|\, |z|^k \\
& - \sum_{k=1}^{\infty} \left|(1+\rho) k^{-n} - (-1)^{m-n} (\gamma+\rho+1)\, k^{-m}\right| |b_k|\, |z|^k
\end{aligned}$$

$$= \begin{cases}
2(1-\gamma)\, |z| - 2 \sum_{k=2}^{\infty} \left[(1+\rho) k^{-n} - (\gamma+\rho)\, k^{-m}\right] |a_k|\, |z|^k \\
\quad -2 \sum_{k=1}^{\infty} \left[(1+\rho) k^{-n} + (\gamma+\rho)\, k^{-m}\right] |b_k|\, |z|^k, & \text{if n-m is odd;} \\
2(1-\gamma)\, |z| - 2 \sum_{k=2}^{\infty} \left[(1+\rho) k^{-n} - (\gamma+\rho)\, k^{-m}\right] |a_k|\, |z|^k \\
\quad -2 \sum_{k=1}^{\infty} \left[(1+\rho) k^{-n} - (\gamma+\rho)\, k^{-m}\right] |b_k|\, |z|^k, & \text{if n-m is even.}
\end{cases}$$

$$\begin{aligned}
> \ & 2\Bigg\{(1-\gamma) - \Bigg[\sum_{k=2}^{\infty} \left[(1+\rho) k^{-n} - (\gamma+\rho)\, k^{-m}\right] |a_k| \\
& + \sum_{k=1}^{\infty} \left[(1+\rho) k^{-n} - (-1)^{m-n} (\gamma+\rho)\, k^{-m}\right] |b_k|\Bigg]\Bigg\} \\
\geq \ & 0, \ \text{this by using (2.1).}
\end{aligned}$$

The harmonic univalent functions

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1-\gamma}{(1+\rho) k^{-n} - (\gamma+\rho) k^{-m}} x_k z^k + \sum_{k=1}^{\infty} \frac{1-\gamma}{(1+\rho) k^{-n} - (-1)^{m-n} (\gamma+\rho) k^{-m}} \overline{y_k z^k}, \quad (2.4)$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, shows that the coefficient bound given by (2.1) is sharp. This completes the proof of Theorem 1.

In the following theorem, it is shown that the condition (2.1) is also necessary for function $f_n = h + g_n$, where h and g_n are of the form (1.5).

Theorem 2. Let $f_n = h + g_n$, where h and g_n are given by (1.5). Then $f_n \in V_{\overline{H}}(m,n;\gamma,\rho)$, if and only if the coefficient condition (2.1) holds.

Proof. Since $V_{\overline{H}}(m,n;\gamma,\rho) \subseteq E_H(m,n;\gamma,\rho)$, we only need to prove the \only if" part of the theorem. For functions $f_n = h + g_n$, where h and g_n are given by (1.5), the inequality (1.4) with $f = f_n$ is equivalent to

$$Re\left\{\frac{(1+\rho e^{i\alpha})[z+\sum\limits_{k=2}^{\infty}k^{-n}a_kz^k+(-1)^n\sum\limits_{k=1}^{\infty}k^{-n}\overline{b}_k\overline{z}^k]}{z+\sum\limits_{k=2}^{\infty}k^{-m}a_kz^k+(-1)^m\sum\limits_{k=1}^{\infty}k^{-m}\overline{b}_k\overline{z}^k}\right\}$$

$$-Re\left\{\frac{(\gamma+\rho e^{i\alpha})[z+\sum\limits_{k=2}^{\infty}k^{-m}a_kz^k+(-1)^m\sum\limits_{k=1}^{\infty}k^{-m}\overline{b}_k\overline{z}^k]}{z+\sum\limits_{k=2}^{\infty}k^{-m}a_kz^k+(-1)^m\sum\limits_{k=1}^{\infty}k^{-m}\overline{b}_k\overline{z}^k}\right\}>0.$$

The above condition holds for all values of $\alpha \in \mathbb{R}$ and $z \in U$. Upon choosing ϕ according (1.6) and substituting $\alpha = 0$ and $z = re^{i\phi}(0 < r < 1)$, we must have

$$\frac{E}{1-\left[\sum\limits_{k=2}^{\infty}k^{-m}\left|a_k\right|-(-1)^{m+n-1}\sum\limits_{k=1}^{\infty}k^{-m}\left|b_k\right|\right]r^{k-1}}>0, \tag{2.5}$$

where

$$\begin{aligned}E &= (1-\gamma)-\left(\sum_{k=2}^{\infty}\left[(1+\rho)k^{-n}-(\gamma+\rho)\,k^{-m}\right]\left|a_k\right|\right)r^{k-1}\\ &\quad -\left(\sum_{k=1}^{\infty}\left[(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)\,k^{-m}\right]\left|b_k\right|\right)r^{k-1}.\end{aligned}$$

If the inequality (2.1) does not hold, then E is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.5) is negative. But this is a contradiction, then the proof of Theorem 2 is completed.

We now obtain the distortion bounds for functions in $V_{\overline{H}}(m,n;\gamma,\rho)$.

Theorem 3. Let $f_n = h + g_n$, where h and g_n are given by (1.5) and $f_n \in V_{\overline{H}}(m,n;\gamma,\rho)$. Then for $|z| = r < 1$, we have

$$|f_n(z)| \le (1+|b_1|)\,r+\left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\,|b_1|\right]r^2 \tag{2.6}$$

and

$$|f_n(z)| \ge (1+|b_1|)\,r-\left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\,|b_1|\right]r^2. \tag{2.7}$$

Proof. We prove the first inequality.

Let $f_n \in V_{\overline{H}}(m,n;\gamma,\rho)$, we have

$$\begin{aligned}|f_n(z)| &\le (1+|b_1|)\,r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^k\le(1+|b_1|)\,r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^2\\ &\le (1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\sum_{k=2}^{\infty}\tfrac{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}{1-\gamma}(|a_k|+|b_k|)\,r^2\\ &\le (1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}.\end{aligned}$$

$$
\begin{aligned}
&\cdot\sum_{k=2}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}\left|a_k\right|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}\left|b_k\right|\right]r^2\\
\leq\quad &(1+|b_1|)\,r+\tfrac{1-\gamma}{(1+\rho)2^{-n}-(\gamma+\rho)2^{-m}}\left[1-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}\left|b_1\right|\right]r^2\\
\leq\quad &(1+|b_1|)\,r+\left[\tfrac{1-\gamma}{(1+\rho)2^{-n}-2^{-m}(\gamma+\rho)}-\tfrac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{(1+\rho)2^{-n}-2^{-m}(\gamma+\rho)}\left|b_1\right|\right]r^2.
\end{aligned}
$$

The proof of the second inequality is similar, thus it is left.
The bounds given in Theorem 3 for functions $f_n = h + \overline{g}_n$ such that h and g_n are given by (1.6) also hold for functions $f = h + \overline{g}$ such that h and g are given by (1.2) if the coefficient condition (2.1) is satisfied.
Using the same technique used earlier by Aghalary [1] we introduce the extreme points of the class $V_{\overline{H}}(m, n; \gamma, \rho)$.

Theorem 4. The closed convex hull of the class $V_{\overline{H}}(m, n; \gamma, \rho)$ (denoted by $clcoV_{\overline{H}}(m, n; \gamma, \rho)$) is

$$
\left\{f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\overline{\sum_{k=1}^{\infty}b_kz^k}\in E_H(m,n;\gamma,\rho):\right.
$$

$$
\left.\sum_{k=1}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}\left|a_k\right|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}\left|b_k\right|\right]\leq 2\right\},
$$

where a_1=1. Set λ_k=$\frac{1-\gamma}{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}$ and μ_k=$\frac{1-\gamma}{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}$. For b_1 fixed, $|b_1|\leq\frac{1-\gamma}{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}$, the extreme points of the class $V_{\overline{H}}(m, n; \gamma, \rho)$ are

$$
\left\{z+\lambda_kxz^k+\overline{b_1z}\right\}\cup\left\{\overline{z+\mu_kxz^k+b_1z}\right\},\tag{2.8}
$$

where $k \geq 2$ and $|x| = 1 - \frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}$.

Proof. Any function $f \in V_{\overline{H}}(m, n; \gamma, \rho)$ may be expressed as

$$
f(z)=z+\sum_{k=2}^{\infty}\left|a_k\right|e^{i\beta_k}z^k+\overline{b_1z}+\overline{\sum_{k=2}^{\infty}\left|b_k\right|e^{i\delta_k}z^k},
$$

where the coefficients satisfy the inequality (2.1). Set

$$
h_1(z)=z,\;\; g_1(z)=b_1z,\;\; h_k(z)=z+\lambda_ke^{i\beta_k}z^k,\;\; g_k(z)=b_1z+\mu_ke^{i\delta_k}z^k, k=2,3,\ldots\,.
$$

Writing $X_k = \frac{|a_k|}{\lambda_k}$, $Y_k = \frac{|b_k|}{\mu_k}$, $k = 2, 3, ...$ and $X_1 = 1 - \sum\limits_{k=2}^{\infty} X_k$, $Y_1 = 1 - \sum\limits_{k=2}^{\infty} Y_k$, we have

$$
f(z)=\sum_{k=1}^{\infty}\left(X_kh_k(z)+\overline{Y_kg_k(z)}\right).
$$

In particular, setting

$$
f_1(z)=z+\overline{b_1z},
$$

and

$$
f_k(z)=z+\lambda_kxz^k+\overline{b_1z}+\overline{\mu_kyz^k},
$$

$$
\left(k\geq 2,|x|+|y|=1-\frac{(1+\rho)-(-1)^{m-n}(\gamma+\rho)}{1-\gamma}\left|b_1\right|\right),
$$

we see that extreme points of the class $V_{\overline{H}}(m, n; \gamma, \rho)$ are contained in $\{f_k(z)\}$. To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$
f_1(z)\quad=\quad\frac{1}{2}\left\{f_1(z)+\lambda\left(1-\frac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}\left|b_1\right|\right)z^2\right\}
$$

$$+\frac{1}{2}\left\{f_1(z)-\lambda\left(1-\frac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_1|\right)z^2\right\},$$

a convex linear combination of functions in the class $V_{\overline{H}}(m,n;\gamma,\rho)$. Next we will show if both $|x|\neq 0$ and $|y|\neq 0$, then f_k is not an extreme point. Without loss of generality, assume $|x|\geq|y|$. Choose $\epsilon>0$ small enough so that $\epsilon<\frac{|x|}{|y|}$. Set $A=1+\epsilon$ and $B=1-\left|\frac{\epsilon x}{y}\right|$, we then see that both

$$t_1(z)=z+\lambda_k xAz^k+\overline{b_1z+\mu_k yBz^k}$$

and

$$t_2(z)=z+\lambda_k x(2-A)z^k+\overline{b_1z+\mu_k y(2-B)z^k},$$

are in the class $V_{\overline{H}}(m,n;\gamma,\rho)$ and note that

$$f_k(z)=\frac{1}{2}\left(t_1(z)+t_2(z)\right).$$

The extremal coefficient bounds shows that functions of the form (2.8) are the extreme points for the class$V_{\overline{H}}(m,n;\gamma,\rho)$, then the proof of Theorem 4 is completed.

Now we will examine the closure properties of the class $V_{\overline{H}}(m,n;\gamma,\rho)$ under the generalized Bernardi-Libera-Livingston integral operator (see [2, 5]) $L_c(f)$ which is defined by

$$L_c(f(z))=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt\quad(c>-1). \tag{2.9}$$

Theorem 5. Let $f_n=h+g_n\in V_{\overline{H}}(m,n;\gamma,\rho)$, where h and g_n are given by (1.5). Then $L_c(f_n(z))$ belongs to the class $V_{\overline{H}}(m,n;\gamma,\rho)$.
Proof. From the representation of $L_c(f_n(z))$, it follows that

$$\begin{aligned}
L_c(f_n(z)) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left(h(t)+\overline{g}_n(t)\right)dt\\
&= \frac{c+1}{z^c}\int_0^z t^{c-1}\left\{t+\sum_{k=2}^{\infty}a_kt^k+\overline{\sum_{k=1}^{\infty}b_kt^k}\right\}dt\\
&= z+\sum_{k=2}^{\infty}A_kz^k+\overline{\sum_{k=1}^{\infty}B_kz^k},
\end{aligned}$$

where $A_k=\frac{c+1}{c+k}a_k$, $B_k=\frac{c+1}{c+k}b_k$. Therefore, we have,

$$\sum_{k=1}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}\frac{c+1}{c+k}|a_k|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}\frac{c+1}{c+k}|b_k|\right]$$

$$\leq\sum_{k=1}^{\infty}\left[\tfrac{(1+\rho)k^{-n}-(\gamma+\rho)k^{-m}}{1-\gamma}|a_k|+\tfrac{(1+\rho)k^{-n}-(-1)^{m-n}(\gamma+\rho)k^{-m}}{1-\gamma}|b_k|\right]\leq 2,$$

and the proof of Theorem 5 is completed.

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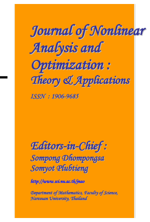
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 45-52
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



CONVERGENCE THEOREMS ON ASYMPTOTICALLY GENERALIZED Φ-PSEUDOCONTRACTIVE MAPPINGS IN THE INTERMEDIATE SENSE

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ABSTRACT. In this study, we introduce the class of asymptotically generalized Φ- pseudocontractive mappings in the intermediate sense and prove the convergence of Mann type iterative scheme to their fixed points. Our results improves and generalizes the results of Kim *et al.* [J. K. Kim, D. R. Sahu, Y. M. Nam, Convergence theorem for fixed points of nearly L-Lipschitzian mappings, Nonlinear Analysis 71 (2009) 2833-2838] and several others.

KEYWORDS : Asymptotically generalized Φ-pseudocontractive mappings in the intermediate sense, Banach spaces, Mann type iterative scheme, strong convergence, unique fixed point.
AMS Subject Classification: 47H09; 47H10.

1. INTRODUCTION

Let E be an arbitrary real normed linear space with dual E^*. We denote by J the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing.

In the sequel, we give the following definitions which will be useful in this study

Definition 1.1. Let C be a nonempty subset of real normed linear space E. A mapping $T : C \longrightarrow E$ is said to be

(1) *strongly pseudocontractive* [12] if for all $x, y \in C$, there exist constant $k \in (0, 1)$ and

$j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y) \rangle \leq k\|x - y\|^2, \tag{1.2}$$

(2) ϕ-*strongly pseudocontractive* [12] if for all $x, y \in C$, there exist strictly increasing

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Article history : Received October 09,2014 Accepted June 12, 2014.

function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|, \tag{1.3}$$

The class of ϕ-strongly pseudocontractive mappings includes the class of strongly pseudocontractive mappings by setting $\phi(s) = ks$ for all $s \in [0,\infty)$. However, the converse is not true (see, e.g. Hirano and Huang [10]).

(3) *generalized Φ-pseudocontractive* [1, 6] if for all $x, y \in C$, there exist strictly increasing

function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|), \tag{1.4}$$

(4) *asymptotically generalized Φ-pseudocontractive* [12] with sequence $\{k_n\}$ if for each $n \in \mathbb{N}$ and $x, y \in C$, there exist constant $k_n \geq 1$ with $\lim_{n\longrightarrow\infty} k_n = 1$, strictly increasing function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2 - \Phi(\|x-y\|), \tag{1.5}$$

The class of asymptotically generalized Φ-pseudocontractive was introduced by Kim *et al.* [12].

Definition 1.2. [19]. A mapping $T : C \longrightarrow C$ is said to be *asymptotically pseudocontractive mapping in the intermediate sense* if

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in C} \left(\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2\right) \leq 0, \tag{1.6}$$

where $\{k_n\}$ is a sequence in $[1,\infty)$ such that $k_n \longrightarrow 1$ as $n \longrightarrow \infty$. Put

$$\nu_n = \max\left\{0, \sup_{x,y\in C} \left(\langle T^n x - T^n y, x-y\rangle - k_n\|x-y\|^2\right)\right\}. \tag{1.7}$$

It follows that $\nu_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, (1.6) is reduced to the following:

$$\langle T^n x - T^n y, x-y\rangle \leq k_n\|x-y\|^2 + \nu_n,\ \forall n \geq 1,\ x, y \in C. \tag{1.8}$$

Qin *et al.* [19] introduced the class of asymptotically pseudocontractive mappings in the intermediate sense. They proved weak convergence theorems for this class of nonlinear mappings. They also established some strong convergence results without any compact assumption by considering the hybrid projection methods. Olaleru and Okeke [17] in 2012 proved a strong convergence of Noor type scheme for a uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the intermediate sense without assuming any form of compactness.

Motivated by the above facts, we now introduce the following class of nonlinear mappings

Definition 1.3. Let C be a nonempty subset of real normed linear space E. A mapping $T : C \longrightarrow C$ is said to be *asymptotically generalized Φ-pseudocontractive mapping in the intermediate sense* with sequence $\{k_n\}$ if for each $n \in \mathbb{N}$ and $x, y \in C$, there exists constant $k_n \geq 1$ with $\lim_{n\longrightarrow\infty} k_n = 1$ and strictly increasing function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ with $\Phi(0) = 0$ and $j(x-y) \in J(x-y)$ satisfying

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in C} \left(\langle T^n x - T^n y, j(x-y)\rangle - k_n\|x-y\|^2 + \Phi(\|x-y\|)\right) \leq 0. \tag{1.9}$$

Put

$$\tau_n = \max\left\{0, \sup_{x,y\in C} \left(\langle T^n x - T^n y, j(x-y)\rangle - k_n\|x-y\|^2 + \Phi(\|x-y\|)\right)\right\}. \tag{1.10}$$

It follows that $\tau_n \longrightarrow 0$ as $n \longrightarrow \infty$. Hence (1.9) is reduced to the following

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2 + \tau_n - \Phi(\|x-y\|). \tag{1.11}$$

We remark that if $\tau_n = 0$ for all $n \in \mathbb{N}$, the class of asymptotically generalized Φ-pseudocontractive mapping in the intermediate sense is reduced to the class of asymptotically generalized Φ-pseudocontractive.

Example 1.4. Let $E = \mathbb{R}^1$ and $C = [c, \infty)$, where $c > 0$ is any given constant. Define the mapping $T : C \longrightarrow 2^E$ by

$$Tx = \begin{cases} [0, c], \text{ if } x = c, \\ \\ \frac{k(x-c)^2}{1+(x-c)}, \text{ if } x > c, \end{cases}$$

where $k \in (0, 1)$.
Clearly, T has a unique fixed point $p = c \in C$. Define $\Phi : [0, \infty) \longrightarrow [0, \infty)$ by $\Phi(t) = \frac{t^2}{(1+t)}$. Clearly, Φ is strictly increasing and $\Phi(0) = 0$. Now, for each $x \in C$, we have

$$\begin{aligned} \langle T^n x - T^n p, j(x-p)\rangle &= \frac{k(x-c)^3}{1+(x-c)} \\ &= k^n(|x-c|^2 - \frac{|x-c|^2}{1+|x-c|}) \\ &\leq k^n|x-p|^2 - \Phi(|x-p|) + k^n. \end{aligned}$$

Hence, T is asymptotically generalized Φ-pseudocontractive mapping in the intermediate sense.

Let C be a nonempty of a normed linear space E. A mapping $T : C \longrightarrow E$ is said to be *Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\| \tag{1.12}$$

for all $x, y \in C$ and *generalized Lipschitzian* [12] if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x-y\| + 1) \tag{1.13}$$

for all $x, y \in C$. A mapping $T : C \longrightarrow C$ is called *uniformly L-Lipschitzian* [12] if for each $n \in \mathbb{N}$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\| \tag{1.14}$$

for all $x, y \in C$.

Clearly, every Lipschitzian mapping is a generalized Lipschitzian mapping. Every mapping with a bounded range is a generalized Lipschitzian mapping. The following example shows that the class of generalized Lipschitzian mappings properly contains the class of Lipschitzian mappings and that of mappings with bounded range.

Example 1.5. [4]. Let $E = (-\infty, \infty)$ and $T : E \longrightarrow E$ be defined by $Tx =$

$$\begin{cases} x - 1 & \text{if } x \in (-\infty, -1), \\ x - \sqrt{1-(x+1)^2} & \text{if } x \in [-1, 0), \\ x + \sqrt{1-(x-1)^2} & \text{if } x \in [0, 1], \\ x + 1 & \text{if } x \in (1, \infty). \end{cases}$$

Then T is a generalized Lipschitzian mapping which is not Lipschitzian and whose range is not bounded.

Sahu [20] introduced a new class of nonlinear mappings which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings.

Definition 1.6. [20]. Let C be a nonempty subset of a Banach space E and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \longrightarrow 0$.

(1) A mapping $T : C \longrightarrow C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \tag{1.15}$$

for all $x, y \in C$.

The infimum of constants k_n in (1.15) is called *nearly Lipschitz constant* and is denoted by $\eta(T^n)$.

(2) A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be *nearly uniformly L-Lipschitzian* if $k_n = L$ for all $n \in \mathbb{N}$, i.e.

$$\|T^n x - T^n y\| \leq L(\|x - y\| + a_n) \tag{1.16}$$

and *nearly asymptotically nonexpansive* if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n \longrightarrow \infty} k_n = 1$.

(3) A mapping $T : C \longrightarrow E$ will be called *generalized (M, L)-Lipschitzian* if there exist two constants $L, M > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + M) \tag{1.17}$$

for all $x, y \in C$.

Observe that the class of generalized (M, L)-Lipschitzian mappings is a generalization of the class of Lipschitzian mappings. Clearly, the class of nearly uniformly L-Lipschitzian mappings properly contains the class of generalized (M, L)-Lipschitzian mappings and the class of uniformly L-Lipschitzian mappings. We remark that every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

It has been shown by Sahu [20] that the class of nearly uniformly L-Lipschitzian is not necessarily continuous. Sahu [20] extended the results of Goebel and Kirk [8] to demicontinuous mappings and proved that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every demicontinuous nearly asymptotically nonexpansive self-mapping of C has a fixed point.

It is our purpose in this study to use the concept of nearly uniformly L- Lipschitzian (not necessarily continuous) mappings to prove a strong convergence result for the class of asymptotically generalized Φ-pseudocontractive mappings in the intermediate sense in a general Banach space. Our results is an improvement of several other results in literature.

The following Lemmas will be useful in this study

Lemma 1.1. [3]. Let E be a Banach space. Then for each $x, y \in E$, there exists $j(x + y) \in J(x + y)$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

Lemma 1.2. [18]. Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that

$$\delta_{n+1} \leq (1 + \beta_n)\delta_n + \gamma_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n \longrightarrow \infty} \delta_n$ exists.

Lemma 1.3. [14]. Let $\{\theta_n\}$ be a sequence of nonnegative real numbers and $\{\lambda_n\}$ a real sequence in $[0,1]$ such that $\sum_{n=1}^{\infty}\lambda_n=\infty$. If there exists a strictly increasing function $\phi:[0,\infty)\longrightarrow[0,\infty)$ with $\phi(0)=0$ such that

$$\theta_{n+1}^2\leq\theta_n^2-\lambda_n\phi(\theta_{n+1})+\sigma_n$$

for all $n\geq n_0$, where n_0 is some nonnegative integer and $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n=o(\lambda_n)$, then $\lim_{n\longrightarrow\infty}\theta_n=0$.

Lemma 1.4. [12]. Let $\{\delta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\sigma_n\}$ be four sequences of nonnegative numbers such that

$$\delta_{n+1}^2\leq(1+\beta_n)\delta_n^2+\gamma_n(\delta_n+\sigma_n)^2$$

for all $n\in\mathbb{N}$. If $\sum_{n=1}^{\infty}\beta_n<\infty$, $\sum_{n=1}^{\infty}\gamma_n<\infty$ and $\{\sigma_n\}$ is bounded, then $\lim_{n\longrightarrow\infty}\delta_n$ exists.

2. Main Results

We prove the following Lemma which will be needed in this study.

Lemma 2.1. Let $\{\delta_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\sigma_n\}$ and $\{\rho_n\}$ be five sequences of nonnegative numbers such that

$$\delta_{n+1}^2\leq(1+\beta_n)\delta_n^2+\gamma_n(\delta_n+\sigma_n)^2+\rho_n^2 \tag{2.1}$$

for all $n\in\mathbb{N}$. If $\sum_{n=1}^{\infty}\beta_n<\infty$, $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\rho_n<\infty$ and $\{\sigma_n\}$ is bounded, then $\lim_{n\longrightarrow\infty}\delta_n$ exists.

Proof. Using (2.1), we obtain

$$\begin{aligned}\delta_{n+1}^2 &\leq (1+\beta_n)\delta_n^2+\gamma_n(\delta_n+\sigma_n)^2+\rho_n^2\\ &\leq (1+\beta_n)\delta_n^2+2\gamma_n(\delta_n^2+\sigma_n^2)+\rho_n^2\\ &\leq (1+\beta_n+2\gamma_n)\delta_n^2+2\gamma_n\sigma_n^2+\rho_n^2.\end{aligned}\tag{2.2}$$

Since $\{\sigma_n\}$ is bounded and $\sum_{n=1}^{\infty}\rho_n<\infty$, then by Lemma 1.2, it follows that $\lim_{n\longrightarrow\infty}\delta_n$ exists. The proof of Lemma 2.1 is completed. □

Theorem 2.2. Let C be a nonempty convex subset of a real Banach space E and $T:C\longrightarrow C$ a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ and *asymptotically generalized* Φ*-pseudocontractive mapping in the intermediate sense* with sequences $\{\tau_n\}$ and $\{k_n\}$ as defined in (1.11) and $F(T)\neq\emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying the conditions:

(i) $\{\frac{a_n}{\alpha_n}\}$ is bounded,
(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$,
(iii) $\sum_{n=1}^{\infty}\alpha_n^2<\infty$ and $\sum_{n=1}^{\infty}\alpha_n(k_n-1)<\infty$.

Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1\in C$ by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_nT^nx_n,\ n\in\mathbb{N}. \tag{2.3}$$

Then the sequence $\{x_n\}$ in C defined by (2.3) converges strongly to a unique fixed point of T.

Proof. Fix $p\in F(T)$, using (1.16) and (2.3) and set
$A_n:=2\alpha_n(k_n-1)+\alpha_n^2[1+L(1+L)]$
and

$B_n := 1 - 2\alpha_n k_n - \alpha_n^2 L(1+L).$

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \alpha_n \|T^n x_n - x_n\| \\
&\leq \alpha_n (\|T^n x_n - p\| + \|x_n - p\|) \\
&\leq \alpha_n (L\|x_n - p\| + a_n) + \|x_n - p\|) \\
&\leq \alpha_n (1+L)\|x_n - p\| + a_n L. \qquad (2.4)
\end{aligned}$$

Using (1.11), (1.16), (2.3), (2.4) and Lemma 1.1, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n (T^n x_n - p)\|^2 \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle T^n x_n - p, j(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{\langle T^n x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&\quad + \langle T^n x_n - T^n x_{n+1}, j(x_{n+1} - p)\rangle\} \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{k_n \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
&\quad + \|T^n x_n - T^n x_{n+1}\| \times \|x_{n+1} - p\|\} \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{k_n \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
&\quad + L\left(\|x_{n+1} - x_n\| + a_n\right) \|x_{n+1} - p\|\} \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{k_n \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
&\quad + L\left(\alpha_n (1+L)\|x_n - p\| + a_n L + a_n\right) \|x_{n+1} - p\|\} \\
&= (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \{k_n \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|) \\
&\quad + \alpha_n L(1+L)\left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right) \|x_{n+1} - p\|\} \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \left\{k_n \|x_{n+1} - p\|^2 + \tau_n - \Phi(\|x_{n+1} - p\|)\right\} \\
&\quad + \alpha_n^2 L(1+L) \left\{ \left(\|x_n - p\| + \tfrac{a_n}{\alpha_n}\right)^2 + \|x_{n+1} - p\|^2 \right\}. \qquad (2.5)
\end{aligned}$$

From (2.5), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \left(\frac{(1-\alpha_n)^2}{1 - 2\alpha_n k_n - k_n^2 L(1+L)}\right) \|x_n - p\|^2 + \frac{2\alpha_n \tau_n}{1 - 2\alpha_n k_n - \alpha_n^2 L(1+L)} \\
&\quad - \frac{2\alpha_n \Phi(\|x_{n+1} - p\|)}{1 - 2\alpha_n k_n - \alpha_n^2 L(1+L)} + \frac{\alpha_n^2 L(1+L)}{1 - 2\alpha_n k_n - \alpha_n^2 L(1+L)} \left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2 \\
&= \left(\frac{(1-\alpha_n)^2}{B_n}\right) \|x_n - p\|^2 + \frac{2\alpha_n \tau_n}{B_n} - \frac{2\alpha_n}{B_n} \Phi(\|x_{n+1} - p\|) \\
&\quad + \frac{\alpha_n^2 L(1+L)}{B_n} \left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2. \qquad (2.6)
\end{aligned}$$

From (2.6), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \left(1 + \frac{A_n}{B_n}\right) \|x_n - p\|^2 + 2\frac{\alpha_n \tau_n}{B_n} - 2\frac{\alpha_n}{B_n} \Phi(\|x_{n+1} - p\|) \\
&\quad + \frac{2\alpha_n^2 L(1+L)}{B_n} \left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2. \qquad (2.7)
\end{aligned}$$

But $B_n = 1 - 2\alpha_n k_n - \alpha_n^2 L(1+L) \longrightarrow 1$, there exists a number $n_0 \in \mathbb{N}$ such that $\frac{1}{2} < B_n \leq 1$ for each $n \geq n_0$. From (2.7), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 + 2A_n)\|x_n - p\|^2 + 4\alpha_n \tau_n - 2\alpha_n \Phi(\|x_{n+1} - p\|) \\
&\quad + 4\alpha_n^2 L(1+L) \left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2. \qquad (2.8)
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 + 2A_n)\|x_n - p\|^2 + 4\alpha_n \tau_n \\
&\quad + 4\alpha_n^2 L(1+L) \left(\|x_n - p\| + \frac{a_n}{\alpha_n}\right)^2. \qquad (2.9)
\end{aligned}$$

From the conditions $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$. Since $\{\frac{a_n}{\alpha_n}\}$ is bounded, we have from (2.8) and Lemma 2.1 that $\lim_{n \longrightarrow \infty} \|x_n - p\|$ exists. Hence, $\{x_n\}$ is bounded. Now, we set $M_1 := \sup\{\|x_n - p\| :$

$n \in \mathbb{N}\}$, $M_2 := \sup\{\frac{a_n}{\alpha_n} : n \in \mathbb{N}\}$ and $M_3 := \sup\{\alpha_n \tau_n : n \in \mathbb{N}\}$. Then from (2.8),we have

$$\begin{aligned} \|x_{n+1} - p\|^2 \quad &\leq \quad \|x_n - p\|^2 + 4M_3 - 2\alpha_n \Phi(\|x_{n+1} - p\|) \\ &\quad +4\alpha_n^2 L(1+L)(M_1 + M_2)^2 + 2A_n M_1^2. \end{aligned} \tag{2.10}$$

Taking $\theta_n = \|x_n - p\|$, $\lambda_n = 2\alpha_n$ and $\sigma_n = 4\alpha_n^2 L(1+L)(M_1+M_2)^2 + 2A_n M_1^2 + 4M_3$, (2.10) reduces to

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \phi(\theta_{n+1}) + \sigma_n.$$

Hence from Lemma 1.3, it follows that $\|x_n - p\| \longrightarrow 0$. The proof of Theorem 2.2 is completed. □

Corollary 2.3. Let C be a nonempty convex subset of a real Banach space E and $T : C \longrightarrow C$ a nearly uniformly L-Lipschitzian mapping with sequence $\{a_n\}$ and asymptotically generalized Φ-pseudocontractive mapping with sequence $\{k_n\}$ as defined in (1.4) and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the conditions:

(i) $\{\frac{a_n}{\alpha_n}\}$ is bounded,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.

Let $\{x_n\}$ be the sequence in E generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \ n \in \mathbb{N}. \tag{2.11}$$

Then the sequence $\{x_n\}$ in C defined by (2.11) converges strongly to a unique fixed point of T.

Remark 2.4. The results of Theorem 2.2 shows that the class of asymptotically generalized Φ-pseudocontractive mappings in the intermediate sense includes the class of asymptotically generalized Φ-pseudocontractive mappings introduced by Kim *et al.* [12]. Furthermore, Our results extended the works of Qin *et al.* [19] and Zegeye *et al.* [22] from Hilbert spaces to the general Banach spaces.

Example 2.5. Let $E = \mathbb{R}$ and $C = [0, 1]$. For all $x \in C$, we define $T : C \longrightarrow C$ by

$$Tx = \begin{cases} (3 - \sqrt{x})^2 & \text{if } x \in [0,1) \\ \\ 0 & \text{if } x = 1 \end{cases}$$

It is easy to see that T is *asymptotically generalized Φ-pseudocontractive mapping in the intermediate sense* with sequence $\{k_n = 1\}$, $\Phi(t) = \frac{t^2}{3}$, $t \in [0, \infty)$ and $\{\tau\} = \frac{1}{n^2}$.

Put $\alpha_n = \frac{1}{n}$. We can see that the conditions (i), (ii) and (iii) of Theorem 2.2 are satisfied.

Acknowledgement

The author is grateful to the anonymous reviewer(s) for his(her) pertinent remarks and useful suggestions.

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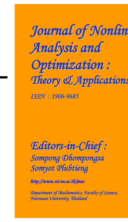
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 53-60
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



AN EXISTENCE THEOREM FOR A CLASS OF DIFFERENTIAL INCLUSIONS WITH A PRECISE ESTIMATE ON THE GRADIENT OF A SOLUTION

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ABSTRACT. In this paper, by using nonsmooth version of very recently theorem of Ricceri relating to continuously functionals, we get a new class of nonlinearities for which the Dirichlet problem has a solution, with a precise estimate on the gradient.

KEYWORDS : locally Lipschitz functions; differential inclusions; uniqueness; local minimum.
AMS Subject Classification: 47J10, 49J99, 49K99.

1. INTRODUCTION

Ricceri in [9] established a new result for sequentially weakly lower semicontinuous functionals and in [10, Theorem 1.1], applied this result for a class of continuous functions with a certain assumption and proved that problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = h(x)f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $h \in L^\infty(\Omega)$ and f is continuous function, has a weak solution and gradient of solution is lower than an estimate depends on value of a positive constant r.

In the present paper, we apply this result of Ricceri [9, Theorem 1], for a class of discontinuous functions, which obtains a new existence theorem for Dirichlet problems involving this type of functions. It is essential, in view of this study, that our nonlinearity is an upper-semicontinuous multifunction with compact convex

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Article history : Received February 05, 2014 Accepted September 08, 2014.

values. Our main theorem (Theorem 3.2) has a few assumptions on nonlinearity for which the existence of a solution is obtained with a precise estimate on the gradient of solution.

The existence of solution is proved by using variational method, following the ideas of Chang [1] relating to partial differential inclusions. we will prove that our problem admits a nonzero solution under some technical assumptions on nonlinearity.

Partial differential inclusions involving a multifunction have been studied by some authors using nonsmooth critical point theory introduced by Clark [2]. Among others, we refer the reader to [3], [4], [7] and [13].

2. PRELIMINARIES AND NOTATIONS

Now, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and also $p > 1$. On the Soboleve space $W_0^{1,p}(\Omega)$, consider the norm

$$||u|| = (\int_\Omega |\nabla u(x)|^p dx)^{\frac{1}{p}}.$$

Our assumptions on the multifunction F defined on $\mathbb{R}$ are the following:

(F_1) $F : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous with compact convex values;

(F_2) $|\xi| \leqslant e(1 + |s|^q)$ for all $s \in \mathbb{R}, \xi \in F(s)$ $(e > 0)$ such that $0 < q \leq \dfrac{pn}{n-p}$ for $n > p$ and $0 < q < +\infty$ for $n \leq p$.

Suppose that $\gamma \in L^\infty(\Omega)_+\backslash\{0\}$ and F satisfies (F_1),(F_2), consider the following Dirichlet problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) \in \gamma(x)F(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

Our aim is to prove the existence of a non-zero solution for problem (2.1). In follow-up, this section is devoted to the statement some lemmas and results of nonsmooth analysis.

Let X be a Banach space whose dual is denoted by X^*. We recall that the generalized directional derivative $\Phi^\circ(u; v)$ of a locally Lipschitz function $\Phi : X \to \mathbb{R}$ at a point $u \in X$ and in the direction $v \in X$ is defined by

$$\Phi^\circ(u; v) = \limsup_{\substack{w \to u \\ \tau \to 0^+}} \frac{\Phi(w + \tau v) - \Phi(w)}{\tau}.$$

The set $\partial\Phi(u) := \{u^* \in X^* :< u^*, v >\leq \Phi^\circ(u; v) \ for\ all\ v \in X\}$ denotes the generalized gradient of the function Φ.

Lemma 2.1. *([6, Proposition 1.1]). Let $\Phi \in C^1(X)$ be a functional. Then Φ is locally Lipschitz and*

(1) $\Phi^\circ(u; v) =< \Phi'(u), v >$ *for all* $u, v \in X$;

(2) $\partial\Phi(u) = \{\Phi'(u)\}$ *for all* $u \in X$.

Lemma 2.2. *([2, Proposition 2.2.9]). Let Φ be Lipschitz near each point of an open convex subset U of X. Then Φ is convex on U if and only if the multifunction $\partial\Phi(u)$ is monotone on U, that is, if and only if*

$$< u_1^* - u_2^*, u_1 - u_2 >\geqslant 0 \quad \forall u_i \in U,\ \forall u_i^* \in \partial\Phi(u_i)\ (i = 1, 2).$$

Lemma 2.3. *([6, Proposition 1.6]). Let $\Phi, \Psi : X \to \mathbb{R}$ be a locally Lipschitz functionals. Then*

(1) $\partial(\lambda\Phi)(u) = \lambda\partial\Phi(u)$ *for all* $u \in X, \lambda \in \mathbb{R}$;

(2) $\partial(\Phi + \Psi)(u) \subseteq \partial\Phi(u) + \partial\Psi(u)$ *for all* $u \in X$.

The following lemma helps us to relate locally Lipschitz functions to lower semicontinuous functions.

Lemma 2.4. *([5, Lemma 6]). Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional with compact gradient. Then f is sequentially weakly continuous.*

Proposition 2.5. *([6, Corollary 1.1]). If $u \in U$ is a local minimum or maximum of the locally Lipschitz function $f : U \to \mathbb{R}$ on an open set a Banach space X, then $0 \in \partial f(u)$.*
If, in addition, f is convex, then the above condition is also sufficient for u to be a global minimum.

The next theorem has proved by Ricceri[11], recall a consequence of the variational principle, and is a technical tool which obtains the estimate on solution.

Theorem 2.1. *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals, with Ψ also coercive and $\Phi(0) = \Psi(0) = 0$.*

Then, for each $\sigma > \inf\limits_X \Psi$ and each λ satisfying

$$\lambda > -\frac{\inf\limits_{\Psi^{-1}(]-\infty,\sigma])} \Phi}{\sigma},$$

the restriction of $\lambda\Psi + \Phi$ to $\Psi^{-1}(] - \infty, \sigma[)$ has a global minimum.

3. MAIN RESULTS

In the first of this section, we collect some basic notations that used in our main result and some especially results about our nonlinearity in form of some lemmas.

For each $\lambda \in [0, +\infty]$, we denote by M_λ the set of all global minima of $\lambda\psi - \varphi$ or the empty set according to whether $\lambda < +\infty$ or $\lambda = +\infty$. We adopt the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

Moreover, for a, b that are two fixed number in $[0, +\infty]$, with $a < b$, we put

$$\alpha := \max\{\inf_X \psi, \sup_{M_b} \psi\}$$

and

$$\beta := \min\{\sup_X \psi, \inf_{M_a} \psi\}.$$

From [7], by standard results of setvalued analysis, for a F satisfies (F_1) and (F_2), the mapping $\min F : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous and $\max F : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous.
Put

$$f(s) = \begin{cases} \max F(s) & \text{if } s < 0 \\ \min F(s) & \text{if } s \geqslant 0 \end{cases}$$

then $f : \mathbb{R} \to \mathbb{R}$ is a measurable selection of F. Moreover, f is lower semicontinuous on $[0, +\infty[$ and upper semicontinuous on $] - \infty, 0[$, so it is measurable in $\mathbb{R}$.

Now, we set

$$H(s) = \int_0^s f(t)dt \ \text{ for all } s \in \mathbb{R},$$

that the convexity of $F(s)$ (see (F_1)) implies the convexity of $H(s)$ for every $x \in \mathbb{R}$.
Finally, set

$$J(u) := \int_\Omega \gamma(x)H(u)dx \ \text{ for all } u \in L^p(\Omega).$$

The growth condition (F_2) implies that J is well defined on $L^p(\Omega)$, because for all $u \in L^p(\Omega)$ we have

$$\int_\Omega \gamma(x)\Big|\int_0^u f(s)ds\Big|dx \leq ||\gamma||_\infty \int_\Omega e(|u| + \frac{|u|^p}{p})dx \leq c||u||_p^p \ (c > 0).$$

Lemma 3.1. *([4, Lemma 3.2]). The functional $J : L^p(\Omega) \to \mathbb{R}$ is Lipschitz on any bounded subset of L^p. Moreover, for all $u \in L^p$ and $u^* \in \partial J(u)$, we have $u^*(x) \in \gamma(x)F(u(x))$ for a.a. $x \in \Omega$.*

Now, for the Banach space X defined before, we have the following lemma.

Lemma 3.2. *([4, Lemma 3.3]). The functional $J : X \to \mathbb{R}$ is locally Lipschitz and its gradient $\partial J : X \to 2^{X^*}$ is compact.*

Proof. Since the space $X = W_0^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$, so proof is similar to the proof of [4, Lemma 3.3] and we do not repeat it. □

Ricceri in [9, Theorem 1], proved the next theorem in a measurable space where φ and ψ were sequentially weakly lower semicontinuous. Here, by applying Lemma 2.4, we denote this theorem for locally Lipschitz functions.

Theorem 3.1. *Let X be a reflexive real Banach space, and $\psi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous functional and $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional with compact gradient such that $\sup_X \psi > 0$ and*

$$\inf_{x\in X} \frac{\psi(x)}{1 + ||x||^p} > -\infty,$$

for some $p > 0$. Moreover, assume that the functional $\lambda\psi - \varphi$ is coercive and has a unique global minimum for each $\lambda \in]a, b[$. Suppose also that $\alpha < \beta$.

Then, for each $\gamma \in L^\infty(\Omega)_+\backslash\{0\}$, and for each $r \in]\alpha, \beta[$ if we put

$$V_{\gamma,r} := \{u \in L^p(\Omega) : \int_\Omega \gamma(x)\psi(u(x))dx \leqslant r \int_\Omega \gamma(x)dx\},$$

we have

$$\sup_{u\in V_{\gamma,r}} \int_\Omega \gamma(x)\varphi(u(x))dx \leqslant \sup_{\psi^{-1}(r)} \varphi \int_\Omega \gamma(x)dx. \tag{3.1}$$

Proof. First of all by Lemma 2.4, functional φ is sequentially weakly continuous, and the rest of proof is noting else than a very particular case of [9, Theorem 1]. □

The following lemma is a particular case of [12, Theorem 1].

Lemma 3.3. *Let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ be two functions such that, for each $\lambda \in]a, b[$, the function $\lambda\psi - \varphi$ is lower semicontinuous, coercive and has a unique global minimum in $\mathbb{R}$. Assume that $\alpha < \beta$, then, for each $r \in]\alpha, \beta[$, there exists $\lambda_r \in]a, b[$, such that the unique global minimum of the function $\lambda_r\psi - \varphi$ lies in $\psi^{-1}(r)$.*

Definition 3.4. A function $u \in X$ is a weak solution of problem (2.1) if there exists $u^* \in L^q(\Omega)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) - u^* v dx = 0 \quad \text{for all } v \in X,$$

such that $u^*(x) \in \gamma(x)F(u(x))$ for a.a. $x \in \Omega$.

Moreover, if $\lambda_{1,p}$ denotes the principal eigenvalue of the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

we obtain

$$\lambda_{1,p} = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u(x)|^p dx}{\displaystyle\int_\Omega |u(x)|^p dx}.$$

We now are ready to state our main theorem.

Theorem 3.2. *Let $\gamma \in L^\infty(\Omega)_+\setminus\{0\}$ and $(F_1), (F_2)$ hold. Furthermore, we assume*

(i) $H(s) \leqslant m(1+|s|^l)$ *for* $s \in \mathbb{R}, 1 < l < p,\ m > 0$*;*

(ii) $\displaystyle\liminf_{s\to 0^+} \frac{H(s)}{s^p} > \frac{\lambda_{1,p}}{p\ \mathrm{ess\,inf}\,\gamma}$*, where* $\mathrm{ess\,inf}\,\gamma(x) > 0$*;*

(iii) *for all* $\lambda > 0$*, function* $s \to \lambda|s|^p - H(s)$ *has a unique global minimum in* $\mathbb{R}$*;*

(iv) *there is* $r > 0$ *satisfying* $\alpha < r < \beta$ *such that*

$$\sup_{|s|^p<r} H(s) < r\,(\frac{\lambda_{1,p}}{p\ \mathrm{ess\,sup}\,\gamma}), \tag{3.2}$$

then, the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) \in \gamma(x)F(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

has a non-zero weak solution satisfying

$$\int_\Omega |\nabla u(x)|^p dx < r\,(\frac{\lambda_{1,p}\displaystyle\int_\Omega \gamma(x)dx}{\mathrm{ess\,sup}\,\gamma}).$$

Proof. We are going to apply Lemma 3.3 by taking $\varphi(s) = H(s)$, $\psi(s) = |s|^p$. According to definition of $H(s)$, φ is lower semicontinuous and therefore, $\lambda\psi - \varphi$ is lower semicontinuous. Also, by hypothesis (F_2), f is bounded on any bounded subset of $\mathbb{R}$, hence H is Lipshitz on any such set with constant $L > 0$, in particular, H is a locally Lipschitz. Set $a := 0, b := +\infty$. So, let $\lambda \in]a, b[$ and from (i) one can conclude that

$$\lambda|s|^p - H(s) \geqslant \lambda|s|^p - m(1+|s|^l),$$

since $1 < l < p$, it follows that

$$\lim_{|s|\to+\infty} (\lambda|s|^p - H(s)) = +\infty,$$

this means that $\lambda\psi - \varphi$ is coercive.

Also by (iii) for all $\lambda > 0$ the function $\lambda\psi - \varphi$ has a unique global minimum. Now, we are allowed to apply Lemma 3.3, note that $\alpha = 0$ and $\beta = \inf_{s\in A} |s|^p$, where $A = \{s \in \mathbb{R} : 0 \in -\partial H(s)\}$. Set $r := \frac{1}{p}\beta$, thus, if we put

$$V_{\gamma,r} := \{u \in L^p(\Omega) : \int_\Omega \gamma(x)\psi(u(x))dx \leqslant \frac{\beta}{p}\int_\Omega \gamma(x)dx\},$$

Theorem 3.1, ensures that

$$\sup_{u\in V_{\gamma,r}} \int_\Omega \gamma(x)H(u(x))dx \leqslant \sup_{|s|^p<r} H \int_\Omega \gamma(x)dx. \tag{3.4}$$

Also, by definition of function ψ, it follows that

$$\int_\Omega \gamma(x)\psi(u(x))dx \leqslant \operatorname{ess\,sup}\gamma \int_\Omega |u(x)|^p dx, \tag{3.5}$$

and, according to the sobolev embedding theorem, one can conclude that

$$\{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u(x)|^p dx \leqslant \frac{\beta}{p}\Big(\frac{\lambda_{1,p}\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)\}$$
$$\subseteq \{u \in L^p(\Omega) : \int_\Omega |u(x)|^p dx \leqslant \frac{\beta}{p}\Big(\frac{\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)\}. \tag{3.6}$$

By setting $B := \{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u(x)|^p dx \leqslant \frac{\beta}{p}\Big(\frac{\lambda_{1,p}\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)\}$, and due to (3.5), (3.6) and definition of ψ, one can get for $u \in B$,

$$\begin{aligned}\int_\Omega \gamma(x)\psi(u(x))dx & \\ &\leqslant \operatorname{ess\,sup}\gamma \int_\Omega |u(x)|^p dx \\ &\leqslant \operatorname{ess\,sup}\gamma \frac{\beta\int_\Omega \gamma(x)dx}{p\operatorname{ess\,sup}\gamma} \\ &= \frac{\beta}{p}\int_\Omega \gamma(x)dx,\end{aligned}$$

hence $B \subseteq V_{\gamma,r}$. Consequently

$$\sup_{u\in B}\int_\Omega \gamma(x)H(u(x))dx \leqslant \sup_{u\in V_{\gamma,r}}\int_\Omega \gamma(x)H(u(x))dx. \tag{3.7}$$

Accordingly, if put $\sigma = \frac{\beta}{p}\Big(\frac{\lambda_{1,p}\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)$ in view of (3.2), (3.4) and (3.7) admits

$$\begin{aligned}\sup_{u\in B}\int_\Omega \gamma(x)H(u(x))dx & \\ &\leqslant \sup_{|s|^p<r} H\int_\Omega \gamma(x)dx \\ &\leqslant \frac{\beta\lambda_{1,p}}{p^2\operatorname{ess\,sup}\gamma}\int_\Omega \gamma(x)dx \\ &= \frac{1}{p}\sigma.\end{aligned}$$

At this point, by applying Theorem 2.1 and taking $X = W_0^{1,p}(\Omega)$, $\Psi(u) = \int_\Omega |\nabla u(x)|^p dx$ and $\Phi(u) = -J(u)$, problem has a local minimum u which is weak solution for problem (3.3) such that $\int_\Omega |\nabla u(x)|^p dx < \frac{\beta}{p}\Big(\frac{\lambda_{1,p}\int_\Omega \gamma(x)dx}{\operatorname{ess\,sup}\gamma}\Big)$.

We finally remark that 0 is not a local minimum of the energy functional. Indeed, by a classical result, there is a bounded and positive $v \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla v(x)|^p dx = \lambda_{1,p} \int_\Omega |v(x)|^p dx. \tag{3.8}$$

On the other hand, the assumption (ii) implies that there exists an element $k > 0$ such that for every $s \in]0, k[$, it follows that

$$H(s) > \frac{\lambda_{1,p}\ s^p}{p \text{ ess inf } \gamma}. \tag{3.9}$$

We deduce that for each $\eta \in]0, \dfrac{k}{\sup_\Omega v}[$ and (3.8) and (3.9),

$$\begin{aligned} I(\eta v(x)) &= \frac{1}{p}\int_\Omega \Big(|\nabla \eta v(x)|^p dx - \int_\Omega \gamma(x) H(\eta v(x))\Big) dx \\ &< \frac{1}{p}\int_\Omega \Big(|\nabla \eta v(x)|^p - \text{ess inf } \gamma(x) \frac{\lambda_{1,p}\ (\eta v(x))^p}{p \text{ ess inf } \gamma}\Big) dx \\ &= \frac{1}{p}\int_\Omega \Big(\lambda_{1,p} |\eta v(x)|^p - \text{ess inf } \gamma(x) \frac{\lambda_{1,p}\ (\eta v(x))^p}{p \text{ ess inf } \gamma}\Big) dx. \end{aligned}$$

We then get $I(\eta v(x)) < 0$. This implies that the energy functional takes negative values in each ball of $W_0^{1,p}$ centered at 0, and so 0 is not a local minimum for it, and the proof is complete. □

Corollary 3.5. *Let* $\Omega =]0, 1[, \gamma = \dfrac{1}{4}$*, by applying Theorem 3.2, the only positive solution of the problem*

$$\begin{cases} -u'' \in \gamma(x) F(u) & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases} \tag{3.10}$$

that for each $s \in \mathbb{R}$ *when* $u(x) = s$*, function*

$$F(s) = \begin{cases} \{1\} & s < 2 \\ [1, \frac{3}{2}] & s = 2 \\ \{2s - 3\} & s > 2 \end{cases}$$

satisfies the inequality

$$\int_\Omega |u'(x)|^2 dx \leqslant r\Big(\frac{\lambda_{1,2} \int_\Omega \gamma(x) dx}{\text{ess sup } \gamma}\Big). \tag{3.11}$$

In fact, for $p = 2$ *clearly the assumptions* $(F_1), (F_2)$ *and (i) and (iii) in Theorem 3.2 are verified. On the other hand,* $\liminf_{s \to 0^+} \dfrac{H(s)}{s^2} \geqslant \dfrac{\lambda_{1,2}}{2\ \text{ess inf } \gamma}$.

Also, for $\beta = \inf_{s = \frac{3}{2}} |s|^2 = \dfrac{9}{4}$*, suppose that there is* $0 < r < \dfrac{9}{4}$ *such that*

$$\sup_{|s|^2 < r} H(s) < r\Big(\frac{\lambda_{1,2}}{2 \text{ ess sup } \gamma}\Big),$$

this implies that the only weak solution of problem (3.10) satisfies (3.11).

Acknowledgment. The authors would like to thank the anonymous reviewer for his\her useful suggestions and are very grateful to Professor Ricceri for the helpful discussion on the topic and for providing a copy of [8].

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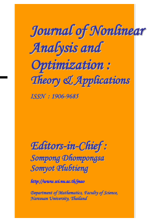

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 61-70
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

AUXILIARY PRINCIPLE TECHNIQUE AND ALGORITHM ASPECT FOR MIXED EQUILIBRIUM PROBLEMS

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ABSTRACT. In this paper, we consider a mixed equilibrium problem in real Hilbert space. By using the auxiliary principle technique, some new iterative algorithms for solving mixed equilibrium problems are suggested and analyzed. Further, we prove that the sequences generated by iterative algorithms converge weakly to a solution of mixed equilibrium problem.

KEYWORDS : Mixed equilibrium problems, nonexpansive mappings, mixed monotone mapping, regularized operator.
AMS Subject Classification: 47H06, 47H09, 65J15, 49J40, 90C33.

1. INTRODUCTION

Equilibrium problem theory is an important and interesting branch of applicable mathematics with a wide range of applications in pure and applied sciences. This theory has become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization, operation research in a general and unified way. There is a substantial number of papers on existence results for solving equilibrium problems based on different-relaxed monotonicity notions and various compactness assumptions. In 2002, Moudafi [9] considered a class of mixed equilibrium problems which includes variational inequalities as well as complementarity problems, convex optimization, saddle point problems, problems of finding a zero of a maximal monotone operator, and Nash equilibria problems as special cases. He studied sensitivity analysis and developed some iterative methods for mixed equilibrium problems. It is well-known that there are many numerical methods including projection methods, resolvent operator technique, Wiener-Hopf equations, extragradient and descent methods for

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Article history : Received February 17, 2014, Accepted September 08, 2014.

solving various variational inequality problems but there are no such methods for solving various equilibrium problems, since it is impossible to find the projection. To overcome this drawback, one uses usually the auxiliary principle technique. This technique deals with finding a suitable auxiliary problem and prove that the solution of an auxiliary problem is the solution of original problem by using fixed-point approach. Recently, Noor [11-13] and Ding [5] have used the auxiliary principle technique to suggest some iterative algorithms for solving generalized mixed variational inequality problems.

Inspired and motivated by the recent research work by [4,6,9,11,14-15], in this paper we study a new class of mixed equilibrium problems and by using the auxiliary principle, we define a class of resolvent mappings. Further, by using fixed point and resolvent methods, we give some iterative algorithms for solving mixed equilibrium problems and prove that the sequences generated by iterative algorithms converge weakly to the solution of mixed equilibrium problems. The auxiliary principle techniques and iterative methods presented in this paper generalize and improve the methods given in [11] for variational inequality problems and given in [9] for mixed equilibrium problems.

2. PROBLEM FORMULATION AND BASIC DEFINITIONS

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let $K \subset H$ be nonempty, closed, convex set; $T, A : K \longrightarrow K$ be nonlinear mappings, and $N : K \times K \longrightarrow K$ be a nonlinear mapping. If $F : K \times K \longrightarrow \mathbb{R}$ and $\phi : H \times H \longrightarrow \mathbb{R}$ are nonlinear bi-mappings, then we consider the following *mixed equilibrium problem* (for short, MEP): Find $x \in K$ such that

$$F(x,y) + \langle N(Tx, Ax), y - x \rangle + \phi(x,y) - \phi(x,x) \geq 0, \ \forall y \in K. \tag{2.1}$$

This problem include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems and equilibrium problems as special cases (see for example, [2]).

Some special cases:

(I) If $N(Tx, Ax) = B(x)$, where $B : K \longrightarrow K$, then MEP (2.1) reduces to the mixed equilibrium problem of finding $x \in K$ such that

$$F(x,y) + \langle Bx, y - x \rangle + \phi(x,y) - \phi(x,x) \geq 0, \ \forall y \in K. \tag{2.2}$$

which has been studied in [6].

(II) If $N(Tx, Ax) = B(x)$, $\phi(x,y) = 0 \ \forall x, y \in K$, where $B : K \longrightarrow K$, then MEP (2.1) reduces to the mixed equilibrium problem of finding $x \in K$ such that

$$F(x,y) + \langle Bx, y - x \rangle \geq 0, \ \forall y \in K, \tag{2.3}$$

which has been studied in [9].

(III) If $N(Tx, Ax) = B(x)$, $F(x,y) = 0$ and $\phi(x,y) = \psi(y)$, $\forall x, y \in K$, where $\psi : K \longrightarrow \mathbb{R}$, then MEP (2.1) reduces to the variational inequality problem of finding $x \in K$ such that

$$\langle Bx, y - x \rangle + \psi(y) - \psi(x) \geq 0, \ \forall y \in K. \tag{2.4}$$

This problem has been studied in [11].

(IV) If $N(Tx, Ax) = 0,\ \forall x \in K$, then MEP (2.1) reduces to the equilibrium problem of finding $x \in K$ such that

$$F(x, y) + \phi(x, y) - \phi(x, x) \geq 0,\ \forall y \in K. \tag{2.5}$$

This problem has been studied in [13].

(V) If, in (IV), $\phi(x, y) = 0,\ \forall x, y \in K$, then (2.5) reduces to the equilibrium problem of finding $x \in K$ such that

$$F(x, y) \geq 0,\ \forall y \in K. \tag{2.6}$$

This problem has been studied in [2].

The following definitions and theorem will be needed in the sequel.

Definition 2.1. Let $N : K \times K \longrightarrow K$ be a nonlinear mapping. Then N is said to be:

(a) *mixed monotone* w.r.t. T and A, if

$$\langle N(Tx, Ax) - N(Ty, Ay), x - y\rangle \geq 0,\ \forall x, y \in K;$$

(b) *mixed pseudomonotone* w.r.t. T and A, if

$$\langle N(Tx, Ax), y - x\rangle \geq 0 \text{ implies } \langle N(Ty, Ay), y - x\rangle \geq 0,\ \forall x, y \in K;$$

(c) θ- *mixed pseudomonotone* w.r.t. T and A, where θ is a real-valued multivariate function, if

$$\langle N(Tx, Ax), y-x\rangle+\theta(x, y) \geq 0 \text{ implies } \langle N(Ty, Ay), y-x\rangle+\theta(x, y) \geq 0,\ \forall x, y \in K;$$

(d) *mixed strongly monotone* w.r.t. T and A, if

$$\langle N(Tx, Ax) - N(Ty, Ay), x - y\rangle \geq \|x - y\|^2,\ \forall x, y \in K;$$

(e) *inverse mixed strongly monotone* w.r.t. T and A, if there exists a constant $\alpha > 0$ such that

$$\langle N(Tx, Ax) - N(Ty, Ay), x - y\rangle \geq \alpha\|N(Tx, Ax) - N(Ty, Ay)\|^2,\ \forall x, y \in K;$$

(f) *firmly nonexpansive* if it is inverse mixed strongly monotone with $\alpha = 1$;

(g) δ-*mixed pseudo contactive* w.r.t. T and A, if

$$\langle N(Tx, Ax) - N(Ty, Ay), x - y\rangle \geq \delta\|x - y\|^2;$$

(h) *k-Lipschitz continuous* w.r.t. T and A, if there exists a constant $k > 0$ such that

$$\|N(Tx, Ax) - N(Ty, Ay)\| \leq k\|x - y\|,\ \forall x, y \in K;$$

(i) *nonexpansive* w.r.t. T and A, if it is Lipschitz continuos with $k = 1$.

Definition 2.2[1]. A bifunction $\phi : H \times H \longrightarrow \mathbb{R}$ is said to be skew-symmetric if

$$\phi(x, x) - \phi(x, y) - \phi(y, x) + \phi(y, y) \geq 0,\ \forall x, y \in H.$$

The skew-symmetric bifunctions have the properties which can be considered an analog of monotonicity of gradient and nonnegativity of second derivative for the convex function.

Definition 2.3. Let K be a nonempty subset of a Hilbert space H and let $\{x_n\}$ be a sequence in H. Then $\{x_n\}$ is Fejer monotone with respect to K if

$$\|x_{n+1} - x\| \leq \|x_n - x\|,\ \forall x \in K.$$

Definition 2.4. Let $F : K \times K \longrightarrow \mathbb{R}$ be a real-valued function. Then F is said to be:

(a) *monotone* if $F(x, y) + F(y, x) \leq 0$, for each $x, y \in K$;
(b) *strictly monotone* if $F(x, y) + F(y, x) < 0$, for each $x, y \in K$, with $x \neq y$;
(c) *upper-hemicontinuous*, if for all $x, y, z \in K$, $\lim_{t \longrightarrow 0^+} \sup F(tz + (1-t)x,\ y) \leq F(x, y)$.

The following Theorem is a special case of Theorem 3.9.3 of Chang [3].

Theorem 2.1. If the following conditions hold true for $F : K \times K \longrightarrow \mathbb{R}$:

(i) F is monotone and upper-hemicontinuous;
(ii) $F(x, .)$ is convex and lower semicontinuous for each $x \in K$;
(iii) there exists a compact subset B of H and there exists $y_0 \in B \cap K$ such that $F(x, y_0) < 0$ for each $x \in K \backslash B$,

then the set of solutions to the following equilibrium problem of finding $x \in K$ such that $F(x, y) \geq 0,\ \forall y \in K$, is nonempty convex and compact.

Moreover, if F is strictly monotone then the solution of equilibrium problem is unique.

Lemma 2.1. MEP (2.1) has a solution x if and only if x satisfies the equation

$$x = J_r^{F,\phi}(x - rN(Tx, Ax)),\ \text{for r} > 0. \tag{2.7}$$

We now define the residue vector $R(x)$ by the relation

$$R(x) = x - J_r^{F,\phi}[x - rN(Tx, Ax)]. \tag{2.8}$$

Invoking Lemma 2.1, one can observe that $x \in K$ is a solution of MEP (2.1) if and only if $x \in K$ is a zero of the equation

$$R(x) = 0. \tag{2.9}$$

3. AUXILIARY PROBLEMS AND ITERATIVE ALGORITHMS

We consider the following auxiliary problem (in short, AP) for MEP(2.1): For $r > 0$ and for each fixed $x \in H$, find $z \in K$ such that

$$F(z, y) + \phi(z, y) - \phi(z, z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in K. \tag{3.1}$$

We remarked that when $z = x$ then the solution sets of problem (2.5) and AP(3.1) are the same.

The following lemma which gives the existence and uniqueness of solution of AP(3.1) is a special case of Lemma 3.1 due to Ding [5].

Lemma 3.1. Let $K \subset H$ be a nonempty closed and convex subset of a real Hilbert space H. Let $F : K \times K \longrightarrow \mathbb{R}$ and $\phi : H \times H \longrightarrow \mathbb{R}$ be nonlinear bifunctions and let $r > 0$. Suppose that the following conditions are satisfied:

(i) F satisfies conditions (i)-(ii) in Theorem 2.1;
(ii) ϕ is skew-symmetric, convex in second argument and continuous;

(iii) For each fixed $x \in H$, there exists a compact subset D_x of H and $y_0 \in K \cap D_x$ such that

$$F(z, y_0) + \phi(z, y_0) - \phi(z, z) + \frac{1}{r}\langle y_0 - z, z - x\rangle < 0,$$

for each $z \in K \backslash D_x$. Then for each fixed $x \in H$, AP(3.1) has a unique solution $z \in K$.

Lemma 3.2. It follows from Lemma 3.1 that for $r > 0$ and for each $x \in H$, we can write the unique solution of AP(3.1) as $z = J_r^{F,\phi}(x) \in K$. Then for all $y \in K$, we have

$$F(J_r^{F,\phi}(x), y) + \phi(J_r^{F,\phi}(x), y) - \phi(J_r^{F,\phi}(x), J_r^{F,\phi}(x)) + \frac{1}{r}\langle J_r^{F,\phi}(x) - x, y - J_r^{F,\phi}(x)\rangle \geq 0. \tag{3.2}$$

Hence $x = J_r^{F,\phi} : H \longrightarrow K$ is well defined and single-valued mapping, which is called the resolvent mapping for MEP (2.1). We observe that $x = J_r^{F,\phi}(x)$ if and only if x is a solution of problem (2.5). Further, Lemma 3.1 gives the strict proof of the assumption taken in [13] for the existence of solution of AP(3.1).

Throughout the rest of paper unless otherwise stated, we assume that the bifunctions F, ϕ satisfy all conditions of Lemma 3.1.

Lemma 3.3. The mapping $J_r^{F,\phi} : H \longrightarrow K$ is firmly nonexpansive.

Proof. Let us denote $u := J_r^{F,\phi}(x)$ and $v := J_r^{F,\phi}(y)$ for each $x, y \in H$. By Lemma 3.1 and Lemma 3.2 we have for each $x, y \in H$,

$$F(u, v) + \phi(u, v) - \phi(u, u) + \frac{1}{r}\langle v - u, u - x\rangle \geq 0,$$

$$F(v, u) + \phi(v, u) - \phi(v, v) + \frac{1}{r}\langle u - v, v - y\rangle \geq 0.$$

Adding above inequalities, we have

$$F(u, v) + F(v, u) - [\phi(u, u) - \phi(u, v) - \phi(v, u) + \phi(v, v)] + \frac{1}{r}\langle x - y, u - v\rangle \geq \frac{1}{r}\langle u - v, u - v\rangle.$$

Since F is monotone and ϕ is skew-symmetric, above inequality reduces to

$$\langle u - v, x - y\rangle \geq \|u - v\|^2$$

because $r > 0$. This completes the proof.

Remark 3.1. Lemmas 3.1, 3.2 and Lemma 3.3 generalize Lemma 2.3 due to Peng and Yao [14].

The fixed point formulation given in Lemma 2.1 for MEP (2.1) is very useful from the numerical point of views. This fixed point formulation enables us to suggest and analyze the following iterative algorithm.

Algorithm 3.1. For a given $x_0 \in K$, compute the approximate solution x_{n+1}, by the iterative scheme

$$x_{n+1} = J_r^{F,\phi}[x_n - rN(Tx_n, Ax_n)],\ n = 0, 1, 2, ...$$

Rewrite equation (2.7) in the form

$$x = J_r^{F,\phi}\big[x - rN(TJ_r^{F,\phi}[x - rN(Tx, Ax)], AJ_r^{F,\phi}[x - rN(Tx, Ax)])\big]$$

by replacing the solution. This fixed point formulation allows us to suggest the following extragradient method.

Algorithm 3.2. For a given $x_0 \in K$, compute x_{n+1}, by the iterative scheme

$$x_{n+1} = J_r^{F,\phi}\big[x_n - rN(TJ_r^{F,\phi}[x_n - rN(Tx_n, Ax_n)], AJ_r^{F,\phi}[x_n - rN(Tx_n, Ax_n)])\big],$$

where $n = 0, 1, 2...$

If $F(x, y) = \delta_k(y) - \delta_k(x)$, and $\phi(x, y) = 0$ for all $x, y \in K$, then $J_r^{F,\phi} = P_k$, the projection of H onto K and have Algorithm 3.2 reduces the extragradient method of Korpelvich [7].

Now define the residue vector $R(x)$ by the relation

$$R(x) = x - J_r^{F,\phi}\big[x - rN(TJ_r^{F,\phi}[x - rN(Tx, Ax)], AJ_r^{F,\phi}[x - rN(Tx, Ax)])\big].$$

We can easily observe that $x \in K$ is a solution of MEP (2.1) if and only if $x \in K$ is a zero of the equation

$$R(x) = 0.$$

For a constant $\gamma \in (0, 2)$, equation (2.9) can be written as

$$x + rN(Tx, Ax) = x + rN(Tx, Ax) - \gamma R(x).$$

This formulation is used to suggest a new implicit method for solving MEP (2.1).

Algorithm 3.3. For a given $x_0 \in K$, compute x_{n+1} by the iterative scheme

$$x_{n+1} = x_n + rN(Tx_n, Ax_n) - rN(Tx_{n+1}, Ax_{n+1}) - \gamma R(x_n),\ n = 0, 1, 2... \quad (3.3)$$

If $\gamma = 1$, then Algorithm 3.3 reduces to:

Algorithm 3.4. For a given $x_0 \in K$, compute x_{n+1} by the iterative scheme

$$x_{n+1} = \big(I + rN(T(.), A(.))\big)^{-1}\big[J_r^{F,\phi}[I + rN(T(.), A(.))] + rN(T(.), A(.))\big]x_n,$$

where $n = 0, 1, 2...$ and $[N(T(.), A(.))x] = N(Tx, Ax),\ \forall x \in K$, which is a variant of the Douglas-Rachford splitting algorithm studied by Lions and Mercier [8], and appears to be new for MEP (2.1).

Theorem 3.1. Let F satisfies the conditions of Theorem 2.1, and let $\bar{x} \in K$ be a solution of MEP (2.1). If N is mixed monotone with respect to T and A, then

$$\langle x - \bar{x} + r[N(Tx, Ax) - N(T\bar{x}, A\bar{x})], R(x)\rangle \geq \|R(x)\|^2,\ \forall x \in K,$$

where $R(x)$ is defined by equation (2.8).

Proof. Let $\bar{x} \in K$ be a solution of MEP (2.1), then

$$F(\bar{x}, y) + \langle N(T\bar{x}, A\bar{x}), y - \bar{x}\rangle + \phi(\bar{x}, y) - \phi(\bar{x}, \bar{x}) \geq 0,\ \forall y \in K. \quad (3.4)$$

Taking $y = x - R(x)$ in (3.4), we have

$$rF(\bar{x}, x - R(x)) + \langle rN(T\bar{x}, A\bar{x}), x - R(x) - \bar{x}\rangle + r\phi(\bar{x}, x - R(x)) - r\phi(\bar{x}, \bar{x}) \geq 0. \quad (3.5)$$

Setting: $y := \bar{x}$, $z = J_r^{F,\phi}(x) := J_r^{F,\phi}(x - rN(Tx, Ax)) = x - R(x)$ and $x := x - rN(Tx, Ax)$ in (3.2), we have

$$rF(x - R(x), \bar{x}) + r\phi(x - R(x), \bar{x}) - r\phi(x - R(x), x - R(x)) + \langle x - R(x) - (x - rN(Tx, Ax)), \bar{x} - (x - R(x))\rangle \geq 0. \quad (3.6)$$

Adding (3.5) and (3.6), we have

$$r[F(x - R(x), \bar{x}) + F(\bar{x}, (x - R(x))] + \langle rN(T\bar{x}, A\bar{x}) - rN(Tx, Ax) + R(x), x - R(x) - \bar{x}\rangle - r[\phi(\bar{x}, \bar{x}) - \phi(\bar{x}, x - R(x)) - \phi(x - R(x), \bar{x}) + \phi(x - R(x), x - R(x))] \geq 0. \quad (3.7)$$

Since F is monotone and ϕ is skew-symmetric, equation (3.7) implies that

$$\langle rN(T\bar{x}, A\bar{x}) - rN(Tx, Ax) - R(x), (x - R(x)) - \bar{x}\rangle \geq 0. \tag{3.8}$$

Since N is mixed monotone with respect to T and A from equation (3.8), we have

$$\begin{aligned}&\langle x - \bar{x} - r[N(Tx, Ax) - N(T\bar{x}, A\bar{x})], R(x)\rangle \\ &\quad = \langle R(x), R(x)\rangle + \langle R(x) - r[N(Tx, Ax) - N(T\bar{x}, A\bar{x}), x - \bar{x} - R(x)\rangle \\ &\quad +r\langle N(Tx, Ax) - N(T\bar{x}, A\bar{x}), x - \bar{x}\rangle \geq \|R(x)\|^2.\end{aligned}$$

This completes the proof.

Theorem 3.2. Let $\bar{x} \in K$ be the solution of MEP (2.1). If the mapping N is mixed monotone w.r.t. T and A then the iterative sequence x_n generated by Algorithm 3.3 is bounded.

Proof. Since $\bar{x}$ is a solution of MEP (2.1) and x_{n+1} satisfies (3.3), then using Theorem 3.1, we have

$$\begin{aligned}&\|x_{n+1} - \bar{x} + r[N(Tx_{n+1}, Ax_{n+1}) - N(T\bar{x}, A\bar{x})]\|^2 \\ &= \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\bar{x}, A\bar{x})] - \gamma R(x_n)\|^2 \\ &\leq \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\bar{x}, A\bar{x})]\|^2 - 2\gamma\|R(x_n)\|^2 + \gamma^2\|R(x_n)\|^2 \\ &= \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\bar{x}, A\bar{x})]\|^2 - \gamma(2-\gamma)\|R(x_n)\|^2. \qquad (3.9) \\ &\leq \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\bar{x}, A\bar{x})]\|^2 \qquad (3.10)\end{aligned}$$

because $\gamma \in (0, 2)$. Inequality (3.10) which gives the Fejers monotonicity of the sequence $\{(I + rN(T, A))x_n\}$ with respect to the solution set of MEP(2.1) and hence $\{(I + rN(T, A))x_n\}$ is bounded. Further it also follows from (3.10) that the sequence $\{\|(I+rN(T, A))x_n - (I+rN(T, A))\bar{x}\|^2$ is monotonically decreasing and therefore convergent.

Again since N is mixed monotone w.r.t. T and A, for any $x, y \in K$, we have

$$\begin{aligned}&\langle (I + rN(T, A))x - (I + rN(T, A))y, x - y\rangle \\ &= \|x - y\|^2 + r\langle N(Tx, Ax) - N(Ty, Ay), x - y\rangle \geq \|x - y\|^2\end{aligned}$$

which implies that the mapping $(I + rN(., .))$ is 1-strongly monotone. Hence, we have

$$\|(I + rN(T, A))x_n - (I + rN(T, A))\bar{x}\| \geq \|x_n - \bar{x}\|.$$

This implies that the sequence $\{x_n\}$ is bounded.

Theorem 3.3. Let H be a finite dimensional space. The approximate solution x_{n+1} obtained from Algorithm 3.3 converges to a solution $\bar{x}$ of MEP (2.1).

Proof. Let $\bar{x} \in K$ be the solution of MEP (2.1). From Theorem 3.2, it follows that the sequence $\{x_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \gamma(2-\gamma)\|R(x_n)\|^2 \leq \|x_0 - \bar{x} + r[N(Tx_0, Ax_0) - N(T\bar{x}, A\bar{x})]\|^2,$$

and consequently

$$\lim_{n\longrightarrow\infty} R(x_n) = 0.$$

Let $\hat{x}$ be a limit point of $\{x_n\}$. A subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which converges to $\hat{x}$. Since $R(x)$ is continuous, so

$$R(\hat{x}) = \lim_{i\longrightarrow\infty} R(x_{n_i}) = 0,$$

and hence $\hat{x}$ is the solution of MEP (2.1) and

$$\|x_{n+1} - \hat{x} + r[N(Tx_{n+1}, Ax_{n+1}) - N(T\hat{x}, A\hat{x})]\|^2$$
$$\leq \|x_n - \bar{x} + r[N(Tx_n, Ax_n) - N(T\hat{x}, A\hat{x})]\|^2.$$

It follows that the sequence $\{x_n\}$ has exactly one limit point and $\lim_{n\longrightarrow\infty} x_n = \hat{x} \in K$, satisfies the MEP (2.1).

4. RESOLVENT EQUATION TECHNIQUE

Now related to MEP (2.1), we consider the following resolvent equation (in short, RE): Find $z \in H$ such that for $x \in K$,

$$N(Tx, Ax) + A_r^{F,\phi}(z) = 0 \tag{4.1}$$

and

$$x = J_r^{F,\phi}(z), \text{ for r} > 0, \tag{4.2}$$

where $A_r^{F,\phi}$ is a regularized operator and is defined as $A_r^{F,\phi} = \frac{1}{r}(I - J_r^{F,\phi})$, I is the identity operator on H.

Lemma 4.1. MEP (2.1) has a solution x if and only if RE (4.1)-(4.2) has a solution $z \in H$ where

$$x = J_r^{F,\phi}(z) \tag{4.3}$$

and

$$z = x - rN(Tx, Ax), \text{ for r} > 0. \tag{4.4}$$

Lemma 4.1 shows that MEP (2.1) and RE (4.1)-(4.2) both have the same solution set.

Using the fact that $A_r^F = \frac{1}{r}(I - J_r^{F,\phi})$, RE (4.1)-(4.2) can be written as

$$z - J_r^{F,\phi}(z) + rN\big(TJ_r^{F,\phi}(z), AJ_r^{F,\phi}(z)\big) = 0.$$

For a step size γ, we can write above equation as

$$x = x - \gamma[z - J_r^{F,\phi}(z) + rN(TJ_r^{F,\phi}(z), AJ_r^{F,\phi}(z))] = 0.$$

This fixed point formulation allows us to suggest the following iterative algorithm for MEP (2.1).

Algorithm 4.1. For a given $x_0 \in K$, compute the approximate solution x_{n+1} by the iterative schemes

$$z_n = x_n - rN(Tx_n, Ax_n)$$
$$w_n = z_n - J_r^{F,\phi} z_n + rN(TJ_r^{F,\phi} z_n, AJ_r^{F,\phi} z_n))$$
$$x_{n+1} = x_n - \gamma w_n$$

where $n = 0, 1, 2, ..., r > 0$ and $\gamma > 0$.

Theorem 4.1. Let $\bar{x} \in K$ be the solution of MEP (2.1) and let N is θ-mixed pseudomonotone w.r.t. T and A, where $\theta(x, y) = F(x, y) + \phi(x, y) - \phi(x, x)$, $\forall x, y \in K$ and δ- mixed pseudo contractive. Then

$$\langle x - \bar{x}, R(x) - r[N(Tx, Ax) - N(Tz, Az)]\rangle \geq (1 - r\delta)\|R(x)\|^2; \ \forall x \in K,$$

where $z := x - rN(Tx, Ax)$ and $R(x)$ is defined by (2.8)

Proof. Since N is θ-mixed pseudomonotone w.r.t. T and A, where $\theta(x,y) = F(x,y)+\phi(x,y)-\phi(x,x),\ \forall x,y\in K$, then for all $z,\bar{x}\in K$,

$$\langle N(T\bar{x},A\bar{x}),z-\bar{x}\rangle+\theta(x,y)\geq 0$$

implies

$$\langle N(Tz,Az),z-\bar{x}\rangle+\theta(x,y)\geq 0$$

$$i.e.,\ F(\bar{x},z)+\langle N(Tz,Az),z-\bar{x}\rangle+\phi(\bar{x},z)-\phi(\bar{x},\bar{x})\geq 0,\ \forall z\in K.$$

Since F is monotone then above inequality implies that

$$-F(z,\bar{x})+\langle N(Tz,Az),z-\bar{x}\rangle+\phi(\bar{x},z)-\phi(\bar{x},\bar{x})\geq 0,\ \forall z\in K.$$

In particular for $z=x-R(x)$, we have

$$\begin{aligned}&-F(x-R(x),\bar{x})+\langle N(T(x-R(x)),A(x-R(x))),(x-R(x))-\bar{x}\rangle\\&+\phi(\bar{x},x-R(x))-\phi(\bar{x},\bar{x})\geq 0\end{aligned}\tag{4.5}$$

Adding (3.6) and (4.5), we have

$$\langle R(x)-rN(Tx,Ax)+rN(T(x-R(x)),A(x-R(x)),(x-R(x))-\bar{x}\rangle\geq 0,$$

where we have used skew-symmetricity of ϕ. Since N is mixed pseudo contractive w.r.t. T and A, above inequality implies

$$\begin{aligned}&\langle R(x)-rN(Tx,Ax)+rN(T(x-R(x)),A(x-R(x)),x-\bar{x}\rangle\\&\quad\geq\langle R(x)-rN(Tx,Ax)+rN(T(x-R(x)),A(x-R(x)),R(x)\rangle\\&\quad\geq\|R(x)\|^2-r\langle N(Tx,Ax)-N(T(x-R(x)),A(x-R(x)),x-(x-R(x))\rangle\\&\quad\geq(1-r\delta)\|R(x)\|^2.\end{aligned}$$

This completes the proof.

Theorem 4.2. Let $\bar{x}\in K$ be the solution of MEP (2.1) and let N is θ-mixed pseudomonotone w.r.t. T and A, where $\theta(x,y)=F(x,y)+\phi(x,y)-\phi(x,x),\ \forall x,y\in K$ and δ-Lipschitz continuous w.r.t. T and A. If $r\delta<1$ and $\gamma\in(0,2)$, then the iterative sequence $\{x_n\}$ generated by Algorithm (4.1) converges weakly to $\bar{x}$.

Proof. Let $\bar{x}\in K$ be the solution of MEP (2.1), using Algorthm (4.1), we have

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2&=\|x_n-\bar{x}-\gamma[x_n-rN(Tx_n,Ax_n)-J_r^{F,\phi}[x_n-rN(Tx_n,Ax_n]\\&\qquad+rN(TJ_r^{F,\phi})[x_n-rN(Tx_n,Ax_n),AJ_r^{F,\phi}[x_n-rN(Tx_n,Ax_n]]\|^2\\&=\|x_n-\bar{x}-\gamma[R(x_n)+rN(Tx_n-R(x_n))-rN(Tx_n,Ax_n)]\|^2\\&=\|x_n-\bar{x}\|^2-2\gamma\langle R(x_n)+rN(T(x_n-R(x_n),A(x_n-R(x_n))-rN(Tx_n,Ax_n),x_n-\bar{x}]\rangle\\&\quad+\gamma^2\|R(x_n)|+rN(T(x_n-R(x_n),A(x_n-R(x_n))-rN(Tx_n,Ax_n)\|^2\\&\leq\|x_n-\bar{x}\|^2-2\gamma\langle R(x_n)+rN(T(x_n-R(x_n),A(x_n-R(x_n))-rN(Tx_n,Ax_n),x_n-\bar{x}]\rangle\\&\quad+\gamma^2\{\|R(x_n)\|^2+\|N(T(x_n-R(x_n),A(x_n-R(x_n))-N(Tx_n,Ax_n)\|^2\}\\&\leq\|x_n-\bar{x}\|^2-\{2\gamma(1-r\delta)-\gamma^2(1+r^2\delta^2)\}\|R(x_n)\|^2\\&\leq\|x_n-\bar{x}\|^2,\end{aligned}\tag{4.6}$$

where we have used the Lipschitz continuity of N and $r\delta<1$ and $\gamma\in(0,2)$.

Inequality (4.6) gives the Fejers monotonicity of the sequence x_n with respect to the solution set of MEP(2.1) and hence x_n is bounded. Further, we observe that the sequence $\|x_n-\bar{x}\|^2$ is monotonically decreasing and therefore convergent. This completes the proof.

We remark that the iterative methods presented in this paper improve and extend the iterative methods given in [11] for variational inequality problem (2.4) in finite dimensional space and given in [9] for problem (2.3).

Acknowledgment. Second author is thankful to NBHM, Department of Atomic Energy, India, Grant no. NBHM/PDF.2/2013/291 for supporting this research work.

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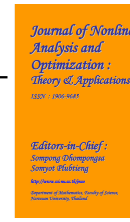
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 71-79
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



HYBRID FIXED POINT THEORY FOR NONINCREASING MAPPINGS IN PARTIALLY ORDERED METRIC SPACES AND APPLICATIONS

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ABSTRACT. We present some hybrid fixed point theorems for nonincreasing mappings in a partially ordered complete metric space and apply to prove the existence as well as an algorithm for the solutions of initial value problems of nonlinear first order ordinary differential equations. An example is also provided to illustrate the abstract theory developed in this paper.

KEYWORDS : Hybrid fixed point theorem; Nonlinear differential equation; Existence of solutions.
AMS Subject Classification: 34A12, 34A38

1. INTRODUCTION

It is well-known that the hybrid fixed point theorems which are obtained using the mixed arguments from different branches of mathematics are very rich in applications to allied areas of mathematics, particularly to the theory of nonlinear differential and integral equations (see HeikkiÏa and Lakshmikantham [6], Zeidler [9] and Dhage [2, 3, 5]). Recently, Ran and Reurings [8] initiated the study of hybrid fixed point theorems in partially ordered sets which is further continued in Nieto and Rodriguez-Lopez [7] and proved the hybrid fixed point theorems for the monotone mappings in partially ordered metric spaces using the mixed arguments from algebra, analysis and geometry. Monotone mappings include both nondecreasing and nonincreasing mappings on ordered sets. The monotone nondecreasing mappings are frequently used in nonlinear analysis whereas nincreasing mappings are rare. Very recently, a different approach for nondecreasing mappings in partially ordered sets is established in Dhage [5] which is very much useful in the existence

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Article history : Received February 25, 2014 Accepted June 02, 2014.

theory for nonlinear equations. In this paper we follow the same approach and obtain some hybrid fixed point theorems for noninreasing operators in metric spaces. The following two notions of regularity and monotone mappings are fundamental for the fixed point theory in ordered spaces.

Definition 1.1. A partially ordered metric space $(X, \preceq, d)$ is called **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in X such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$.

Definition 1.2. A mapping $\mathcal{T} : X \longrightarrow X$ is called **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in X$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in X$. A **monotone** mapping $\mathcal{T}$ is one which is either monotone nondecreasing or monotone nonincreasing on X.

Nieto and Lopez [7] introduced the following definition.

Condition (NL): A partially ordered metric space X with metric d is said to satisfy Condition(NL) if for every convergent sequence $\{x_n\}$ in X to the point x^* whose consecutive terms are comparable then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that every term is is comparable to the limit x^*.

The following hybrid fixed point theorem for nonincreasing mappings is proved in Nieto and Lopez [7].

Theorem 1.3 (Nieto and Rodriguez-Lopez [7]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d in X such that (X, d) is a complete metric space. Let $T : X \longrightarrow X$ be a monotone nonincreasing mapping such that there exists a constant $k \in [0, 1)$ such that*

$$d(\mathcal{T}x, \mathcal{T}y) \leq k\, d(x, y) \tag{1.1}$$

for all elements $x, y \in X$, $x \geq y$. Assume that either $\mathcal{T}$ is continuous or X satisfies Condition (NL). Further if there is an element $x_0 \in X$ satisfying $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point which is further unique if "every pair of elements in X has a lower and an upper bound."

Note that Condition (NL) of Theorem 1.3 is very difficult to verify in actual practice and only continuous case has been applied in Nieto and Lopez [7] to periodic BVP of first order differential equations for proving the existence of a unique solution, wherein the nonlinearity is a nonincrasing function in the unknown variable. In this paper we generalize Theorem 1.3 under a condition which is more general than Condition (NL) for the self-mappings of a partially ordered metric space satisfying a condition of nonlinear contraction which is again more general than (1.1). Our abstract result is applied to a nonlinear first order ordinary differential equations for proving the existence of unique solution under partially Lipschitz condition.

2. Hybrid Fixed Point Theory

We consider the following definition in what follows.

Condition (D): A partially ordered metric space X with metric d is said to satisfy Condition (D) if every sequence $\{x_n\}$ in X whose consecutive terms are comparable has a monotone, i.e. nondecreasing or nonincreasing subsequence.

There do exist sequences in X with Condition (D). For example, if we consider $X = \mathbb{R}$, then the sequence $\{x_n\}$ in $\mathbb{R}$ defined by $x_n = (-1)^{n+1}\dfrac{1}{n}$ has two subsequences, one is nondecreasing another is nonincreasing. Again, the sequence $\{1, -\frac{1}{2}, 3, -\frac{1}{4}, \ldots\}$ satisfies the Condition (D) but not Condition (NL).

Note that Condition (D) is more general than Condition (NL) in the sense that Condition (D) implies Condition (NL), however converse may no be true. Indeed if the Condition (D) holds and if $\{x_n\}$ is any sequence in X converging to x^* whose consecutive terms are comparable, then there is a monotone subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which also converges to x^*. By regularity of X, $x_{n_k} \preceq x^*$ or $x_{n_k} \succeq x^*$ for all $k \in \mathbb{N}$, that is, every term of $\{x_{n_k}\}$ is comparable to the limit x^*.

Let (X, d) be a metric space and let $\mathcal{T} : X \longrightarrow X$ be a mapping. Given an element $x \in X$, we define an orbit $\mathcal{O}(x; \mathcal{T})$ of $\mathcal{T}$ at x by

$$\mathcal{O}(x; \mathcal{T}) = \big\{x, \mathcal{T}x, \mathcal{T}^2x, ..., \mathcal{T}^nx, \ldots\big\}.$$

Then $\mathcal{T}$ is called T-orbitally continuous on X if for any sequence $\big\{x_n\big\} \subseteq \mathcal{O}(x; \mathcal{T})$, we have that $x_n \longrightarrow x^*$ implies $\mathcal{T}x_n \longrightarrow \mathcal{T}x^*$ for each $x \in X$. The metric space X is called $\mathcal{T}$-orbitally complete if every Cauchy sequence $\big\{x_n\big\} \subseteq \mathcal{O}(x; \mathcal{T})$ converses to a point x^* in X. Notice that continuity implies that $\mathcal{T}$-orbitally continuity and completeness implies $\mathcal{T}$-orbitally completeness of a metric space X, but the converse may not be true.

Definition 2.1 (Dhage [5]). A mapping $\mathcal{T} : X \longrightarrow X$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on X if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on X, then it is continuous on every chain C contained in X.

We frequently need a fundamental result concerning Cauchy sequence in what follows. For, we need the following definition.

Definition 2.2 (Dhage [4]). A mapping $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a **dominating function** or, in short, $\mathcal{D}$**-function** if it is an upper semi-continuous and monotonic nondecreasing function satisfying $\psi(0) = 0$.

There do exist $\mathcal{D}$-functions and commonly used $\mathcal{D}$-functions are

$$\begin{aligned}
&\psi(r) = k\,r, \quad \text{for some constant } k > 0,\\
&\psi(r) = \frac{L\,r}{K + r}, \quad \text{for some constants } L > 0, K > 0,\\
&\psi(r) = \tan^{-1} r,\\
&\psi(r) = \log(1 + r),\\
&\psi(r) = e^r - 1.
\end{aligned}$$

The above defined $\mathcal{D}$-functions have been widely used in the existence theory of nonlinear differential and integral equations.

Remark 2.3. If $\phi, \psi\ \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are two $\mathcal{D}$-functions, then i) $\phi + \psi$, ii) $\lambda\,\phi$, $\lambda > 0$, and iii) $\phi\circ\psi$ are also $\mathcal{D}$-functions on $\mathbb{R}_+$. The class of $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.

Lemma 2.4 (Dhage [4]). *Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a $\mathcal{D}$-function satisfying $\psi(r) < r$ for $r > 0$. Then $\lim_{n\longrightarrow\infty} \psi^n(t) = 0$ for each $t \in \mathbb{R}_+$ and vice versa.*

Now we are ready to state a key result in terms of $\mathcal{D}$-function characterizing the Cauchy sequences in a metric space X.

Lemma 2.5. *If $\{x_n\}$ is a sequence in a metric space (X, d) satisfying*

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \tag{2.1}$$

for all $n \in \mathbb{N}$, where ψ is a $\mathcal{D}$-function such that $\psi(r) < r$, $r > 0$, then $\{x_n\}$ is Cauchy.

Proof. The proof is well-known and may found in Dhage [5]. So we omit the details. □

Theorem 2.6. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a monotone nonincreasing mapping such that there exists a $\mathcal{D}$-function such that*

$$d(\mathcal{T}x, \mathcal{T}^2x) \leq \psi(d(x, \mathcal{T}x)) \tag{2.2}$$

for all elements $x \in X$ comparable to $\mathcal{T}x$, where $\psi(r) < r$, $r > 0$. Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). Further if there is an element $x_0 \in X$ satisfying $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{T^n x_0\}$ of iterations converges to x^*.

Proof. Define a sequence $\{x_n\}$ of successive iterations of $\mathcal{T}$ at x_0 as

$$x_{n+1} = \mathcal{T}x_n,\ n = 0, 1, \dots. \tag{2.3}$$

By nonincreasing nature of $\mathcal{T}$, $\{x_n\}$ is a sequence in X whose consecutive terms are comparable. If $x_n = x_{n+1}$ for some $\in \mathbb{N}$, then $u = x_n$ is a fixed point of T. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. If $x = x_{n-1}$ and $y = x_n$, then by condition (2.2), we obtain

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \tag{2.4}$$

for each $n = 1, 2, \dots$. Now, an application of Lemma 2.5, $\{x_n\}$ is Cauchy. Since the metric space X is $\mathcal{T}$-orbitally complete, $\{x_n\}$ converges to a unique limit x^*. If $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous, x^* is a fixed point of $\mathcal{T}$ and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^*. Next, suppose that X satisfies Condition (D). By hypothesis, the sequence $\{\mathcal{T}^n x_0\}$ of iterates of $\mathcal{T}$ at x_0 has a monotone subsequence, say $\{x_{n_k}\}$. Then $\{x_{n_k}\}$ also converges to x^* and $x_{n_k} \leq x^*$ for all $k \in \mathbb{N}$. By the partial $\mathcal{T}$-orbitally continuity of $\mathcal{T}$, we obtain

$$\mathcal{T}x^* = \mathcal{T}\left(\lim_{k \longrightarrow \infty} x_k\right) = \lim_{k \longrightarrow \infty} \mathcal{T}(\mathcal{T}^{n_k} x_0) = \lim_{k \longrightarrow \infty} x_{n_k+1} = x^*.$$

This completes the proof.‘ □

Theorem 2.7. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a monotone nonincreasing mapping such that there exists a $\mathcal{D}$-function such that*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \psi(d(x, y)) \tag{2.5}$$

for all comparable elements $x, y \in X$, where $\psi(r) < r$, $r > 0$. Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). Further if there is an element $x_0 \in X$ satisfying $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations converges to x^* which is further unique if "every pair of elements in X has a lower and an upper bound."

Proof. Now the inequality (2.5) implies that the mapping $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous on X. If we let $y = \mathcal{T}x$ in (2.5), then it reduces to (2.2). Therefore, by Theorem 2.6, $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges to x^*. The uniqueness of fixed point is proved using arguments given in Nieto and Lopez [7]. □

It is known that 'every pair of elements in X has a lower and an upper bound if it is a lattice (cf. Birkhoff [1]). As every monotone nondecreasing or monotone nonincreasing sequence always has a monotone subsequence and the limit of the sequence is the limit of the subsequence, we obtain the following general fixed point theorems for both nondecreasing as well as nonincreasing mappings on partially ordered metric spaces.

Theorem 2.8. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a monotone mapping (monotone nonincreasing or monotone nonincreasing) satisfying (2.2). Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). If there exists an $x_0 \in X$ with $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations converges to x^*.*

Theorem 2.9. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $\mathcal{T} : X \longrightarrow X$ be a monotone mapping (monotone nonincreasing or monotone nonincreasing) satisfying (2.5). Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). If there exists an $x_0 \in X$ with $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations of $\mathcal{T}$ at x_0 converges to x^* which is further unique if "every pair of elements in X has a lower and an upper bound."*

Finally, we mention that the claim made in Nieto and Lopez [7] that the continuity of the mapping $\mathcal{T}$ is not required to guarantee the existence of unique fixed point is not true. Actually we need certain kind of continuity, namely, the partial continuity of the mapping $\mathcal{T}$ which follows directly from the condition of partial contraction on X. However, the monotonicity of $\mathcal{T}$ is not essential for the existence of the fixed points, so in this context we replace this monotonicity condition by preservation of comparable elements, that is transformation of comparable elements into comparable elements. A couple of fixed point results in this direction are as follows.

Theorem 2.10. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a mapping satisfying (2.2) and maps comparable elements into comparable elements, that is,*

$$x, y \in X, x \preceq y \Rightarrow \begin{cases} \mathcal{T}x \preceq \mathcal{T}y \\ \quad or \\ \mathcal{T}x \succeq \mathcal{T}y. \end{cases}$$

Suppose that X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). If there exists an $x_0 \in X$ with x_0 is comparable to $\mathcal{T}x_0$, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations converges to x^*.

Theorem 2.11. *Let $(X, \preceq, d)$ be a partially ordered metric space. Let $\mathcal{T} : X \longrightarrow X$ be a mapping satisfying (2.5) and maps comparable elements into comparable* *elements, that is,*

$$x, y \in X, x \preceq y \Rightarrow \begin{cases} \mathcal{T}x \preceq \mathcal{T}y \\ \quad or \\ \mathcal{T}x \succeq \mathcal{T}y. \end{cases}$$

Suppose that either X is $\mathcal{T}$-orbitally complete and $\mathcal{T}$ is $\mathcal{T}$-orbitally continuous or $\mathcal{T}$ is partially $\mathcal{T}$-orbitally continuous and X is regular and satisfies Condition (D). If there exists an $x_0 \in X$ with x_0 is comparable to $\mathcal{T}x_0$, then $\mathcal{T}$ has a unique fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of iterations converges to x^*.

The proofs of Theorems 2.10 and 2.11 are similar to Theorems 2.6 and 2.7 and so we omit the details.

3. Applications to Hybrid Differential Equations

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, consider the initial value problem (in short IVP) of first order ordinary nonlinear hybrid differential equation (in short HDE)

$$\left.\begin{aligned} x'(t) &= f(t, x(t)), \;\; t \in J, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{3.1}$$

where $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function.

By a solution of the HDE (3.1) we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies equation (1.1), where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The HDE (3.1) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. The HDE (3.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.2}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.3}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$ in it. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular as well as a lattice.

We need the following definition in what follows.

Definition 3.1. A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of the HDE (1.1) if it satisfies

$$\left.\begin{aligned} u'(t) &\leq f(t, u(t)), \\ u(t_0) &\leq x_0, \end{aligned}\right\} \tag{$*$}$$

for all $t \in J$.

We consider the following set of assumptions in what follows:

(A_1) There exist constants $\lambda > 0$ and $\mu > 0$, with $\lambda \geq \mu$, such that

$$\frac{-\mu(x-y)}{1+(x-y)} \leq [f(t,x) + \lambda x] - [f(t,y) + \lambda y] \leq 0,$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \geq y$.

(A_2) The HDE (1.1) has a lower solution $u \in C^1(J, \mathbb{R})$.

Consider the IVP of the HDE

$$\left.\begin{aligned} x'(t)+\lambda x(t) &= \tilde{f}(t,x(t)),\\ x(t_0) &= x_0, \end{aligned}\right\} \tag{3.4}$$

for all $t \in J$, where $\tilde{f}, g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ and

$$\tilde{f}(t,x) = f(t,x) + \lambda x. \tag{3.5}$$

Remark 3.2. Note that the function $\tilde{f}$ is continuous on $J\times\mathbb{R}$, and so the associated superposition Nymetski operator (Fx) is integrable on J. Again, a function $u \in C^1(J,\mathbb{R})$ is a solution of the HDE (3.4) if and only if it is a solution of the HDE (1.1) on J.

Lemma 3.3. *A function $u \in C^1(J,\mathbb{R})$ is a solution of the HDE (3.4) if and only if it is a solution of the nonlinear integral equation,*

$$x(t) = c\,e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\tilde{f}(s,x(s))\,ds \tag{3.6}$$

for all $t \in J$ where c is a real number defined by $c = x_0\,e^{t_0}$.

Theorem 3.4. *Assume that hypotheses (A$_1$) and (A$_2$) hold. Then the HDE (1.1) has a unique solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_{n+1}(t) = c\,e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\tilde{f}(s,x_n(s))\,ds \tag{3.7}$$

where $x_0 = u$, converges to x^*.

Proof. Set $E = C(J,\mathbb{R})$ and define two operators $\mathcal{A}$ on E by

$$\mathcal{A}x(t) = c\,e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\tilde{f}(s,x(s))\,ds,\quad t \in J. \tag{3.8}$$

From the continuity of the integral, it follows that $\mathcal{A}$ defines the map $\mathcal{A} : E \longrightarrow E$. Now by Lemma 3.3, the HDE (3.1) is equivalent to the operator equation

$$\mathcal{A}x(t) = x(t),\quad t \in J. \tag{3.9}$$

We shall show that the operator $\mathcal{A}$ satisfies all the conditions of Theorem 2.6.

First we show that $\mathcal{A}$ is monotone nonincreasing on E. Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A$_1$), we obtain

$$\begin{aligned} \mathcal{A}x(t) &= c\,e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\tilde{f}(s,x(s))\,ds\\ &\leq c\,e^{-\lambda t} + e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\tilde{f}(s,y(s))\,ds\\ &= \mathcal{A}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{A}$ is nonincreasing operator on E into E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= \left| e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}[\tilde{f}(s,x(s)) - \tilde{f}(s,y(s))]\,ds \right|\\ &\leq e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\frac{\mu(x(s)-y(s))}{1+(x(s)-y(s))}\,ds\\ &\leq e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\lambda\frac{|x(s)-y(s)|}{1+|x(s)-y(s)|}\,ds \end{aligned}$$

$$\begin{aligned}
&\le \quad e^{-\lambda t}\int_{t_0}^{t}\frac{d}{ds}e^{\lambda s}\frac{\|x-y\|}{1+\|x-y\|}\,ds\\
&\le \quad \left[1-e^{-\lambda(t-t_0)}\right]\frac{\|x-y\|}{1+\|x-y\|}\\
&\le \quad \frac{\|x-y\|}{1+\|x-y\|},
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathcal{A}x-\mathcal{A}y\| \le \psi(\|x-y\|),$$

for all $x, y \in E$ with $x \ge y$, where ψ is a $\mathcal{D}$-function defined by $\psi(r) = \dfrac{r}{1+r} < r$, $r > 0$. Hence $\mathcal{A}$ satisfies the contraction condition (2.5) on E which further implies that $\mathcal{A}$ is a partially continuous and consequently partially $\mathcal{T}$-orbitally continuous on E.

Next, we show that u satisfies the operator inequality $u \le \mathcal{A}u$. By hypothesis (A_2), the HDE (1.1) has a lower solution u. Then we have

$$\left.\begin{aligned} u'(t) &\le f(t,u(t)),\\ u(t_0) &\le x_0, \end{aligned}\right\} \tag{3.10}$$

for all $t \in J$. Adding $\lambda u(t)$ on both sides of the first inequality in (3.10), we obtain

$$u'(t)+\lambda u(t) \le f(t,u(t))+\lambda u(t), \quad t \in J. \tag{3.11}$$

Again, multiplying the above inequality (3.11) by $e^{\lambda t}$,

$$\left(e^{\lambda t}u(t)\right)' \le e^{\lambda t}\tilde{f}(t,u(t)). \tag{3.12}$$

A direct integration of (3.12) from t_0 to t yields

$$u(t) \le c\,e^{-\lambda t}+e^{-\lambda t}\int_{t_0}^{t}e^{\lambda s}\tilde{f}(s,u(s))\,ds \tag{3.13}$$

for all $t \in J$. From definition of the operator $\mathcal{A}$ it follows that $u(t) \le \mathcal{A}u(t)$ for all $t \in J$. Hence $u \le \mathcal{A}u$. Thus $\mathcal{A}$ satisfies all the conditions of Theorem 2.7 and we apply it to conclude that the operator equation $\mathcal{A}x = x$ has a solution. Consequently the integral equation and the HDE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.7) converges to x^*. This completes the proof. □

Remark 3.5. The conclusion of Theorem 3.4 also remains true if we replace the hypothesis (A_1) with the following one:

(A_1') There exist a continuous and nondecreasing function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and the constants $\lambda > 0$ and $\mu > 0$, with $\lambda \ge \mu$, such that

$$-\mu\phi(x-y) \le [f(t,x)+\lambda x]-[f(t,y)+\lambda y] \le 0,$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \ge y$, where $\phi(r) < r$, $r > 0$.

Finally, we give a numerical example to show the realization of the abstract theory in this section.

Example 3.6. Given a closed and bounded interval $J = [0,1]$, consider the IVP of HDE,

$$\left.\begin{aligned} x'(t) &= -\tan^{-1}x(t)-x(t),\\ x(0) &= 1 \in \mathbb{R}, \end{aligned}\right\} \tag{3.14}$$

for all $t \in J$.

Here, $f(t,x) = -\tan^{-1} x - x$. Clearly, the functions f is continuous on $J \times \mathbb{R}$. The function f satisfies the hypothesis (A_1) with $\lambda = 1 = \mu$. To see this, we have

$$0 \leq \tan^{-1} x - \tan^{-1} y \leq \frac{1}{1+\xi^2}(x-y)$$

for all $x, y \in \mathbb{R}$, $x \geq y$, where $x > \xi > y$. Therefore, $\lambda = 1 = \mu$, and $\psi(r) = \dfrac{r}{1+\xi^2}$, $0 < \xi < r$, so the hypothesis (A_1') is satisfied. Finally, the HDE (3.15) has a lower solution $u(t) = -2$ defined on J and so (A_2) is held. Thus all the hypotheses of Theorem 3.4 are satisfied. Hence we apply Theorem 3.4 and conclude that the HDE (3.15) has a solution x^* defined on J and the sequence $\{x_n\}$ defined by

$$x_{n+1}(t) = e^{-t} + e^{-t} \int_0^t e^s \tan^{-1} x_n(s)\, ds, \quad t \in J, \tag{3.15}$$

where $x_0 = -2$, converges to x^*.

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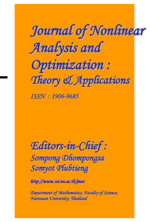
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 81-90
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



SOME DEVELOPMENT FOR NEWTON'S METHOD UNDER MILD DIFFERENTIABILITY CONDITIONS

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ABSTRACT. We develop a semilocal convergence analysis for Newton's method under mild differentiability conditions. Our sufficient convergence conditions can be weaker than conditions found in earlier studies such as [11-14,16,18–21]. Numerical examples are also provided.

KEYWORDS : Newton's method; Banach space; Hölder continuity; Lipschitz continuity; semilocal convergence; Kantorovich hypothesis.
AMS Subject Classification: 47H17, 49M15, 65H10, 65G99, 65J15

1. INTRODUCTION

In this work, we study the problem of approximating a locally unique solution $x^\star$ of the nonlinear equation

$$\mathcal{F}(x) = 0 \tag{1.1}$$

where $\mathcal{F}$ is a Fréchet–differentiable operator defined on a convex subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$.

Many problems – from nonlinear convex analysis and other disciplines – can be formulated in the form of equation (1.1) using Mathematical Modelling [6]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are usually iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to Newton-type methods [6, 13]. For iterattions $n = 0, 1, 2, \ldots$, we define Newton's method as

$$\begin{cases} \mathcal{F}'(\mathbf{x_n})\mathbf{\Delta x_n} = -\mathcal{F}(\mathbf{x_n}) \\ \quad \mathbf{x_{n+1}} = \mathbf{x_n} + \mathbf{\Delta x_n} \end{cases} \tag{1.2}$$

where x_0 is an initial guess. The study about convergence of iterative procedures is normally centered on two types: semilocal and local convergence analysis. The semilocal convergence analysis is based on the information around an initial point

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Article history : Received March 04, 2014 Accepted March 11, 2014.

to give criteria ensuring the convergence of iterative procedures. While the local analysis is based on the information around a solution to find estimates of the radii of convergence balls. There exist many studies which deals with the local and the semilocal convergence analysis of Newton-type methods such as [1-21].

In this work – particularly motivated by optimization considerations – we provide sufficient convergence conditions that can be weaker than the ones found in earlier studies such as [11-14,16,18-21] and our majorizing sequences can be finer as well. Moreover, the new advantages are obtained under the same computational cost.

The rest of the paper is organized as follows. In section 2, we present earlier results in order to make the study as self contained as possible. Section 3 contains the semilocal convergence of Newton's method. Section 4 reports numerical work.

2. Preliminaries

Let $\mathbb{U}(x_0, r)$ and $\overline{U}(x_0, r)$ stand, respectively, for the open and closed balls in $\mathbf{X}$ with center x_0 and radius $r > 0$. We denote by $L(\mathbf{X}, \mathbf{Y})$ the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$. In this Section, we assume that $p \in (0, 1]$ unless stated otherwise. Let $x_0 \in \mathbf{D}$ be such that $\mathcal{F}'(x_0)^{-1} \in L(\mathbf{Y}, \mathbf{X})$. Assume $\mathcal{F}'$ satisfies a p-center–Hölder condition at some $p \in (0, 1]$

$$\begin{gathered}\left\|\mathcal{F}'(x_0)^{-1}\left(\mathcal{F}'(x) - \mathcal{F}'(x_0)\right)\right\| \leq L_0 \left\|x - x_0\right\|^p \\ \text{for each } x \in \mathbb{U}(x_0, R) \subseteq \mathbf{D} \text{ and some } L_0 > 0, \quad R > 0\end{gathered} \tag{2.1}$$

and a p-Hölder condition at some $p \in (0, 1]$

$$\begin{gathered}\left\|\mathcal{F}'(x_0)^{-1}\left(\mathcal{F}'(x) - \mathcal{F}'(y)\right)\right\| \leq L \left\|x - y\right\|^p \\ \text{for each } x, y \in \mathbb{U}(x_0, R) \subseteq \mathbf{D} \text{ and for some } L > 0.\end{gathered} \tag{2.2}$$

We have that

$$L_0 \leq L \tag{2.3}$$

holds in general and that L/L_0 can be arbitrarily large [3, 6] (see also the Examples in §4). Note that (2.2) always implies (2.1). In practice the computation of L requires computing L_0. Hence, (2.1) is not an additional requirement to the hypothesis (2.2). For earlier results on Newton's method under conditions (2.1) or (2.2), we refer interested reader to the interesting work: [2,4,6,10–18,21 and references therein]. In particular, a new result was given by Cianciaruso in [7] which improves earlier sufficient convergence conditions for $p \in (0, 1)$, but not necessarily the error bounds [8,10,14–19]. Consider

$$c_0 = \frac{L + \sqrt{L^2 + 4\, L_0\, L\, (1+p)^p\ p^{1-p}}}{2\, L} \tag{2.4}$$

and

$$h_0(t) = \left(1 - \frac{1}{t}\right)^p \frac{1+p}{\left((L_0\,(1+p))^{\dfrac{1}{1-p}} + (L\,t\,(t-1))^{\dfrac{1}{1-p}}\right)^{1-p}}. \tag{2.5}$$

Then, the convergence condition is given by

$$\eta^p \leq h_0(c_0). \tag{2.6}$$

3. Semilocal convergence analysis

First we present four auxiliary results on majorizing sequences for Newton's method.

Lemma 3.1. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0,1]$ be given parameters. Define function g_0 on $[0,1)$ by*

$$g_0(t) = \frac{\ell}{1+p}\eta^p(t-1) + \ell_0 t^{1/p}\Big[(1+t^{1/p})^p - 1\Big]\eta^p.$$

We denote by α the minimal zero of g_0 in the interval $(0,1)$. Suppose that

$$0 < \frac{\ell\eta^p}{(1+p)(1-\ell_0\eta^p)} \leq \alpha^{1/p} \leq 1 - \ell_0^{1/p}\eta. \tag{3.1}$$

Then, the scalar sequence $\{t_n\}$ defined by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{\ell(t_{n+1}-t_n)^{1+p}}{(1+p)(1-\ell_0 t_{n+1}^p)} \quad \textit{for each} \quad n = 0,1,2,\dots \tag{3.2}$$

is increasing, bounded from above by

$$t^{\star\star} = \frac{\eta}{1-\alpha^{1/p}} \tag{3.3}$$

and converges to its unique least upper bound denoted by $t^\star$ which satisfies $t^\star \in [\eta, t^{\star\star}]$. Moreover, the following estimates hold for each $n = 0,1,2,\dots$

$$0 < t_{n+2} - t_{n+1} \leq \alpha^{1/p}(t_{n+1}-t_n) \leq \alpha^{(n+1)/p}\eta. \tag{3.4}$$

Proof. Since $g_0(1) = \ell_0\,\eta^p(2^p-1) > 0$, $g_0(0) = -\ell\,\eta^p/(1+p) < 0$ thus – it follows from the intermediate value theorem applied to the function g_0 on the interval $[0,1]$ that – the function g_0 has zeros in $(0,1)$. We denote the smallest such zero of function g_0– in the interval $(0,1)$ – by α.

Estimate (3.4) holds, if

$$0 < t_{k+2} - t_{k+1} = \frac{\ell\,(t_{k+1}-t_k)^p}{(1+p)(1-\ell_0\, t_{k+1}^p)} \leq \alpha^{1/p}. \tag{3.5}$$

Inequalities (3.4) and (3.5) hold for $k=0$ by (3.1). Let us assume that (3.4) and (3.5) hold for all $k \leq n$. Then, we obtain

$$\begin{aligned} t_{k+1} &\leq t_k + \alpha^{1/p}(t_k - t_{k-1}) \\ &\leq t_{k-1} + \alpha^{1/p}(t_{k-1}-t_{k-2}) + \alpha^{1/p}(t_k - t_{k-1}) \\ &\leq t_1 + \alpha^{1/p}(t_1-t_0) + \alpha^{1/p}(t_2-t_1) + \cdots + \alpha^{1/p}(t_k-t_{k-1}) \\ &\leq \eta + \alpha^{1/p}\eta + \alpha^{2/p}\eta + \cdots + \alpha^{k/p}\eta = \frac{1-\alpha^{k+1/p}}{1-\alpha^{1/p}}\eta < \frac{\eta}{1-\alpha^{1/p}} = t^{\star\star} \end{aligned} \tag{3.6}$$

and

$$t_{k+1} - t_k \leq \alpha^{1/p}(t_k - t_{k-1}) \leq \cdots \leq \alpha^{k/p}(t_1-t_0) = \alpha^{k/p}\eta. \tag{3.7}$$

In view of (3.6) and (3.7), estimate (3.5) holds, if

$$\ell\Big((\alpha^{1/p})^n\eta\Big)^p \leq (1+p)\alpha^{1/p}\Big[1 - \ell_0\Big(\frac{1-\alpha^{(n+1)/p}}{1-\alpha^{1/p}}\eta\Big)^p\Big]$$

or

$$\ell\alpha^n\eta^p + (1+p)\alpha^{1/p}\ell_0(1+\alpha^{1/p}+\alpha^{2/p}+\cdots+\alpha^{n/p})^p\eta^p - \alpha^{1/p}(1+p) \leq 0$$

or

$$\ell\alpha^n\eta^p + (1+p)\alpha^{1/p}\ell_0(1+\alpha^{1/p}+\alpha^{2/p}+\cdots+\alpha^{n/p})\eta - \alpha^{1/p}(1+p) \le 0. \tag{3.8}$$

Estimate (3.8) motivates us to introduce recurrent functions f_k on $(0,1)$ by

$$f_k(t) = \ell\eta^p t^k + (1+p)t^{1/p}\ell_0(1+t^{1/p}+t^{2/p}+\cdots+t^{k/p}) - t^{1/p}(1+p). \tag{3.9}$$

We need a relationship between two consecutive functions f_k. Using (3.9) we get that

$$f_{k+1} = f_k(t) + g_k(t) \tag{3.10}$$

where

$$g_k(t) = \frac{\ell}{1+p}\eta^p t^n(t-1) + \ell_0 t^{1/p}\eta^p\Big[(1+t^{1/p}+\cdots+t^{(n+1)/p})^p - (1+t^{1/p}+\cdots+t^{n/p})^p\Big]. \tag{3.11}$$

We have that

$$\begin{aligned} g_{k+1}(t) = g_k(t) + \frac{\ell}{1+p}\eta^p t^k(t-1)^2 + \ell_0 t^{1/p}\eta^p\Big[(1+t^{1/p}+\cdots+t^{(k+2)/p})^2 \\ - 2(1+t^{1/p}+\cdots+t^{(k+1)/p})^p + (1+t^{1/p}+\cdots+t^{k/p})^p\Big] \ge g_k(t). \end{aligned} \tag{3.12}$$

In particular, it follows from the definition of α, (3.10) and (3.12) that

$$f_{k+1}(\alpha) = g_k(\alpha) + f_k(\alpha) \ge g_{k-1}(\alpha) + f_k(\alpha) \ge \cdots \ge g_0(\alpha) + f_k(\alpha) = f_k(\alpha). \tag{3.13}$$

We define function f_∞ on $[0,1)$ by

$$f_\infty(t) = \lim_{t\longrightarrow\infty} f_k(t). \tag{3.14}$$

Then, using (3.8), (3.9) and (3.14) we get that

$$f_\infty(t) = (1+p)t^{1/p}\Big[\ell_0\Big(\frac{\eta}{1-\alpha^{1/p}}\Big)^p - 1\Big]. \tag{3.15}$$

Moreover, we have that

$$f_\infty(t) \ge f_k(t) \quad \text{for each} \quad k = 0,1,2,\ldots. \tag{3.16}$$

Hence, it follows from (3.8), (3.9), (3.15) and (3.16) that (3.8) holds if

$$f_\infty(\alpha) \le 0. \tag{3.17}$$

But (3.16) is implied by the right hand side of the hypothesis (3.1). The induction for (3.4) and (3.5) is complete. It follows that sequence $\{t_n\}$ is increasing, bounded from above by $t^{\star\star}$ and as such it converges to its unique least upper bound $t^\star \in [\eta, t^{\star\star}]$. $\square$

Next, we present another auxiliary result of majorizing sequences for Newton's method.

Lemma 3.2. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0,1]$ be given parameters. Suppose that*

$$\ell_0\eta^p < 1 \tag{3.18}$$

and there exists $\alpha \in (0,1)$ such that

$$\begin{aligned} \min\Big\{\frac{\ell(s_2-s_1)^p}{(1+p)(1-\ell_0 s_2^p)}, \frac{\ell}{1+p}\Big(\frac{\ell_0\eta^{1+p}}{(1+p)(1-\ell_0\eta^p)}\Big)^p\alpha \\ +\ell_0\alpha^{1/p}\Big(\eta + \frac{\ell_0\eta^{1+p}}{(1-\ell_0\eta^p)(1-\alpha^p)}\Big)^p\Big\} \le \alpha^{1/p}. \end{aligned} \tag{3.19}$$

Then, the scalar sequence $\{s_n\}$ *defined by*

$$\begin{cases} s_0 = 0, \quad s_1 = \eta, \quad s_2 = s_1 + \dfrac{\ell_0(s_1 - s_0)^{1+p}}{(1+p)(1-\ell_0 s_1^p)}, \\ s_{n+2} = s_{n+1} + \dfrac{\ell(s_{n+1} - s_n)^{1+p}}{(1+p)(1-\ell_0 s_{n+1}^p)}, \quad \text{for each} \quad n = 1, 2, \dots \end{cases} \tag{3.20}$$

is increasing, bounded from above by

$$s^{\star\star} = \left[1 + \frac{\ell_0 \eta^p}{(1-\alpha^{1/p})(1+p)(1-\ell_0\eta^p)}\right]\eta \tag{3.21}$$

and converges to its unique least upper bound $s^\star$ *which satisfies* $s^\star \in [s_2, s^{\star\star}]$. *Moreover, the following estimate holds for each* $n = 1, 2, 3, \dots$

$$0 < s_{n+2} - s_{n+1} \leq \frac{\alpha^{n/p}\ell_0\eta^{1+p}}{(1+p)(1-\ell_0\eta^p)}. \tag{3.22}$$

Proof. Estimate (3.22) holds if

$$0 < \frac{\ell(s_{k+1} - s_k)^p}{(1+p)(1-\ell_0 s_{k+1}^p)} \leq \alpha^{1/p} \quad \text{for each} \quad k = 1, 2, \dots. \tag{3.23}$$

Inequalities (3.22) and (3.23) hold for $k = 1$ by (3.19) and (3.20). From (3.23) and (3.22), we obtain

$$\begin{aligned} & s_3 - s_2 \leq \alpha^{1/p}(s_2 - s_1) \\ \Longrightarrow \quad & s_3 \leq s_2 + \alpha^{1/p}(s_2 - s_1) \\ \Longrightarrow \quad & s_3 \leq s_2 + (1+\alpha^{1/p})(s_2 - s_1) - (s_2 - s_1) \\ \Longrightarrow \quad & s_3 \leq s_1 + \frac{1-\alpha^{2/p}}{1-\alpha^{1/p}}(s_2 - s_1) < s_1 + \frac{1}{1-\alpha^{1/p}}(s_2 - s_1) = s^{\star\star} \end{aligned} \tag{3.24}$$

Suppose that (3.22) and (3.23) hold for all $k \leq n$. Then, we have

$$s_{k+2} - s_{k+1} \leq \alpha^{k/p}(s_2 - s_1) \tag{3.25}$$

and

$$s_{k+2} \leq s_1 + \frac{1-\alpha^{(k+1)/p}}{1-\alpha^{1/p}}(s_2 - s_1) < s^{\star\star}. \tag{3.26}$$

We need to show that (3.23) holds if k is replaced by $k+1$. Then – by (3.23), (3.25) and (3.26) – we must show that

$$\frac{\ell}{1+p}(s_{k+2} - s_{k+1})^p + \alpha^{1/p}\ell_0 s_{k+2}^p - \alpha^{1/p} \leq 0$$

or

$$\frac{\ell}{1+p}\Big(\alpha^{k/p}(s_2 - s_1)\Big)^p + \alpha^{1/p}\ell_0\Big(s_1 + \frac{1-\alpha^{(k+1)/p}}{1-\alpha^{1/p}}(s_2 - s_1)\Big)^p - \alpha^{1/p} \leq 0$$

or

$$\frac{\ell}{1+p}\alpha^k(s_2 - s_1)^p + \alpha^{1/p}\ell_0\Big(\eta + \frac{s_2 - s_1}{1-\alpha^{1/p}}\Big)^p - \alpha^{1/p} \leq 0$$

or since $\alpha \in (0, 1)$

$$\frac{\ell}{1+p}\alpha(s_2 - s_1)^p + \alpha^{1/p}\ell_0\Big(\eta + \frac{s_2 - s_1}{1-\alpha^{1/p}}\Big)^p \leq \alpha^{1/p}$$

which is true by (3.19). This completes the induction for (3.22) and (3.23). Hence, sequence $\{s_n\}$ is increasing, bounded from above by $s^{\star\star}$ and as such it converges to its unique least upper bound $s^\star \in [s_2, s^{\star\star}]$. $\square$

We extend Lemma 3.1 and Lemma 3.2 through the following two useful additions: Lemma 3.3 and Lemma 3.4, respectively

Lemma 3.3. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0,1]$ be given parameters. Let $N = 0,1,2,\ldots$ be fixed. We define function g_0^N on $[0,1)$ by*

$$g_0^N(t) = \frac{\ell}{1+p}(t_{N+1} - t_N)^p(t-1) + \ell_0 t^{1/p}\Big[(1+t^{1/p})^p - 1\Big](t_{N+1} - t_N)^p.$$

Let α_N denotes the minimal zero of $g_0^N(t)$ in the interval $(0,1)$. Suppose that

$$t_1 < t_2 < t_3 < \cdots < t_N < t_{N+1} < \ell_0^{-1/p} \tag{3.27}$$

and

$$0 < \frac{\ell(t_{N+1} - t_N)^p}{(1+p)(1 - \ell_0 t_{N+1}^p)} \leq \alpha_N^{1/p} \leq 1 - \ell_0^{1/p}(t_{N+1} - t_N).$$

Then, the sequence $\{t_n\}$ - defined by (3.2) - is increasing, bounded from above by $t_N^{\star\star} = (t_{N+1} - t_N)/(1 - \alpha^{1/p})$ and converges to its unique least upper bound $t_N^\star \in [t_{N+1}, t_N^{\star\star}]$. Moreover, for each $n = 1,2,\ldots$, the following estimate

$$0 < t_{N+n} - t_{N+n-1} \leq \alpha^{1/p}(t_{N+n-1} - t_{N+n-2}) \quad \textit{for each} \quad n = 0,1,2,\ldots$$

holds.

Lemma 3.4. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0,1]$ be given parameters. Let $N = 0,1,2,\ldots$ be fixed. Suppose that*

$$s_1 < s_2 < s_3 < \cdots < s_N < s_{N+1} < \ell_0^{-1/p},$$
$$\ell_0(s_{N+1} - s_N)^p < 1$$

and (3.19) holds with $s_{N+1} - s_N$ replacing η. Then, scalar sequence $\{s_n\}$ - defined by (3.20) - is increasing, bounded from above by

$$s_N^{\star\star} = \Big[1 + \frac{\ell_0(s_{N+1} - s_N)^p}{(1 - \alpha^{1/p})(1+p)(1 - \ell_0 s_{N+1}^p)}\Big](s_{N+1} - s_N)$$

and converges to its unique least upper bound $s_N^\star \in [s_{N+1}, s_N^{\star\star}]$. Moreover, for each $n = 1,2,\ldots$, the following estimate

$$0 < s_{N+n+2} - s_{N+n+1} \leq \frac{\alpha^{n/p}\ell_0(s_{N+1} - s_N)^{1+p}}{(1+p)(1 - \ell_0 s_{N+1}^p)} \quad \textit{for each} \quad n = 0,1,2,\ldots$$

holds. Notice that, for $N = 0$, the Lemma 3.3 and the Lemma 3.4 reduce to the Lemma 3.1 and the Lemma 3.2, respectively.

We state the main semilocal convergence theorem for Newton's method. The proof of which is obtained from [4, Theorem 3.3] by replacing the hypotheses of Lemma 3.1 in [4] – involving the convergence of $\{t_n\}$ – with the new Lemma 3.3 or Lemma 3.4.

Theorem 3.5. *Let $\mathcal{F}: \mathbf{D} \subseteq \mathbf{X} \longrightarrow \mathbf{Y}$ be Fréchet-differentiable. Suppose there exist a point $x_0 \in \mathbf{D}$ and parameters $\eta > 0$, $\ell_0 > 0$, $\ell > 0$, $R > 0$, $p \in (0,1]$, such that conditions (2.1), (2.2), hypotheses of Lemma 3.1 or Lemma 3.3 hold and $\overline{U}(x_0, t^\star) \subseteq U(x_0, R)$. Then, $\{x_n\}$ $(n \geq 0)$ generated by Newton's method is well* *defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $\mathcal{F}(x) = 0$. Moreover, the following estimates*

$$\|x_{n+2} - x_{n+1}\| \leq \frac{\ell\, \|x_{n+1} - x_n\|^{1+p}}{(1+p)\,(1 - \ell_0\, \|x_{n+1} - x_0\|^p)} \leq t_{n+2} - t_{n+1} \tag{3.28}$$

and

$$\|x_n - x^\star\| \leq t^\star - t_n, \tag{3.29}$$

hold for each $n = 0, 1, \ldots$, where, iteration $\{t_n\}$ and $t^\star$ are given in Lemma 3.1. Furthermore, if there exists $R \geq t^\star$ such that

$$R_0 \leq R \quad \text{and} \quad \ell_0 \int_0^1 (\theta\, t^\star + (1-\theta)\, R)^p\, d\theta < 1,$$

then, the solution $x^\star$ is unique in $\overline{U}(x_0, R_0)$. If hypotheses of Lemma 3.2 (or Lemma 3.4) hold, instead of Lemma 3.1 (or Lemma 3.3), then the preceding conclusions hold with $s^\star$, $\{s_n\}$ (or $s^\star_N$, $\{s_N\}$) replacing $t^\star$, $\{t_N\}$ (or $t^\star_N$, $\{t_N\}$), respectively.

Remark 3.6. Until now we presented convergence criteria, that can be weaker than the ones reported in Section 2 (especially criterion (2.23)), and a majorizing sequence $\{s_n\}$ which is finer than the sequence $\{t_n\}$ [11]. Under the criterion (2.23), we notice that

$$s_n \leq t_n, \tag{3.30}$$

$$s_{n+1} - s_n \leq t_{n+1} - t_n, \tag{3.31}$$

$$s^\star \leq t^\star. \tag{3.32}$$

If $\ell_0 < \ell$ then for $n \geq 2$ we observe that strict inequality applies in (3.30) and (3.32) (also see the numerical examples).

However – along the lines of Lemma 3.3 and Lemma 3.4 – we can weaken the condition (2.23) in another way too as follows:

Lemma 3.7. *Let $\ell_0 > 0$, $\ell > 0$, $\eta > 0$ and $p \in (0, 1)$. Let $N = 0, 1, 2, \ldots$ be fixed. Suppose that*

$$t_1 < t_2 < t_3 < \cdots < t_N < t_{N+1} < \ell_0^{1/p}$$

and

$$(t_{N+1} - t_N)^p \leq h_0(c_0)$$

where function h_0 is defined in (2.21). Then, sequence $\{t_n\}$ converges increasingly to $t^\star$.

Theorem 3.8. *Let $\mathcal{F} : \mathbf{D} \subseteq \mathbf{X} \to \mathbf{Y}$ be Fréchet differentiable. Suppose there exist a point $x_0 \in \mathbf{D}$ and parameters $\eta > 0$, $\ell_0 > 0$, $\ell > 0$, $p \in (0, 1)$ such that conditions (2.2), (2.3), (2.4), hypotheses of Lemma 3.7 and $\overline{U}(x_0, t^\star) \subseteq \mathbb{U}(x_0, R)$. Then, the conclusions of Theorem 3.5 hold.*

Remark 3.9. If $N = 0$, the results in Lemma 3.7 and Theorem 3.8 reduce to the corresponding onces in [11]. Moreover, if $N \geq 1$, then two preceding results constitute an improvement of the results in [11]. Clearly, sequence $\{s_n\}$ can also replace $\{t_n\}$ in Lemma 3.7 and Theorem 3.8. In practice, we shall test existing criteria and use the onces that apply and also provide the best available error estimates and uniqueness result.

4. Numerical work

Example 4.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}^{n-1}$ for natural integer $n \geq 2$. $\mathbf{X}$ and $\mathbf{Y}$ are equipped with the max-norm $\|\mathbf{x}\| = \max_{1\leq i\leq n-1} \|x_i\|$. The corresponding matrix norm is

$$\|A\| = \max_{1\leq i\leq n-1} \sum_{j=1}^{j=n-1} |a_{ij}|$$

for $A = (a_{ij})_{1\leq i,j\leq n-1}$. On the interval $[0, 1]$, we consider the following two point boundary value problem

$$\begin{cases} v'' + v^{3/2} = 0 \\ v(0) = v(1) = 0. \end{cases} \tag{4.1}$$

[?, see]]ah-WSPC,rokne. To discretize the above equation, we divide the interval $[0, 1]$ into n equal parts with length of each part: $h = 1/n$ and coordinate of each point: $x_i = i\,h$ with $i = 0, 1, 2, \ldots, n$. A second-order finite difference discretization of equation (4.1) results in the following set of nonlinear equations

$$\mathcal{F}(\mathbf{v}) := \begin{cases} v_{i-1} + h^2\, v_i^{3/2} - 2v_i + v_{i+1} = 0 \\ \text{for} \quad i = 1, 2, \ldots, (n-1) \quad \text{and from (4.1)} \quad v_0 = v_n = 0 \end{cases} \tag{4.2}$$

where $\mathbf{v} = [v_1, v_2, \ldots, v_{(n-1)}]^{\mathrm{T}}$ For the above system-of-nonlinear-equations, we provide the Fréchet derivative

$$\mathcal{F}'(\mathbf{v}) = \begin{bmatrix} \frac{3v_1^{1/2}}{2n^2} - 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \frac{3v_2^{1/2}}{2n^2} - 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \frac{3v_3^{1/2}}{2n^2} - 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{3v_{(n-1)}^{1/2}}{2n^2} - 2 \end{bmatrix}.$$

Let $n = 101$ and $x_0 = [50, 50, \ldots, 50]^{\mathrm{T}}$. To solve the linear systems (step 1 in the in Algorithm 1), we use MatLab routine "linsolve" which uses LU factorization with partial pivoting. Figure 1 plots our numerical solution. From the above expression and the inequalities (2.1), (2.2), (2.4), we obtain

$$p = \frac{1}{2}, \quad \eta = 9.15311 \times 10^{-5} \quad \text{and} \quad L = L_0 = 5.86207705 \times 10^{-4}.$$

From (2.20) and (2.21), we get

$$c_0 = 1.556421035 \quad \text{and} \quad h_0(c_0) = 883.3185142.$$

For the condition (2.23) we obtain: $0.009567188720 < 810.7294898$. Thus the condition (2.23) holds. For the Lemma 3.1, the condition (3.1) yields: $0 < 0.00008814005478 < 0.7422855001 < 0.9999971581$. Thus Lemma 3.1 is applicable. To verify the inequality (3.4) and to ascertain the properties of the sequence (3.2) we produce the Table 1: From the equation (3.3)

$$t^{\star\star} = 0.0002236156293.$$

From the Table 1 we see that the estimation (3.4) holds. Furthermore we notice that the sequence $\{t_n\}$ is increasing and bounded from above by $t^{\star\star}$.

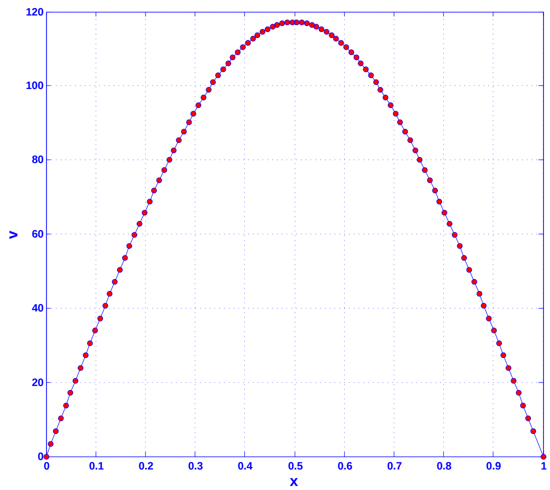

FIGURE 1. Solution of the boundary value problem (4.1).

n	t_n	$t_{n+2}-t_{n+1}$	$\alpha^{1/p}(t_{n+1}-t_n)$	$\alpha^{(n+1)/p}\eta$
0	$0.000000\times10^{+00}$	3.421069×10^{-10}	5.037923×10^{-05}	5.037923×10^{-05}
1	9.153110×10^{-05}	2.472017×10^{-18}	1.882976×10^{-10}	2.772901×10^{-05}
2	9.153144×10^{-05}	1.518400×10^{-30}	1.360612×10^{-18}	1.526221×10^{-05}
3	9.153144×10^{-05}	7.309504×10^{-49}	8.357357×10^{-31}	8.400404×10^{-06}
4	9.153144×10^{-05}	2.441409×10^{-76}	4.023192×10^{-49}	4.623630×10^{-06}
5	9.153144×10^{-05}	1.490286×10^{-117}	1.343766×10^{-76}	2.544872×10^{-06}
6	9.153144×10^{-05}	2.247571×10^{-179}	8.202620×10^{-118}	1.400712×10^{-06}
7	9.153144×10^{-05}	4.162736×10^{-272}	1.237076×10^{-179}	7.709597×10^{-07}
8	9.153144×10^{-05}	3.318005×10^{-411}	2.291193×10^{-272}	4.243406×10^{-07}
9	9.153144×10^{-05}	7.466626×10^{-620}	1.826249×10^{-411}	2.335594×10^{-07}

TABLE 1. Numerical solution of (4.1) – after 6 nonlinear iterations – at Gauss-Legendre points.

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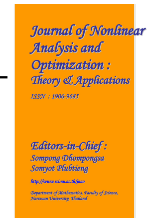
Journal of Nonlinear Analysis and Optimization
Vol. 5, No. 2, (2014), 91-103
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



KKT OPTIMALITY CONDITIONS FOR INTERVAL VALUED OPTIMIZATION PROBLEMS

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ABSTRACT. In the present paper we study the class of convex optimization problems in uncertain environment. The objective and constraint functions are assumed to be interval valued. Solution concepts are proposed under two order relations on the set of all closed intervals. Weakly continuously differentiability is employed in order to derive necessary and sufficient conditions for KKT optimality conditions. These theoretical developments are illustrated through a numerical example.

KEYWORDS : Interval valued functions; weak differentiability; KKT-optimality conditions; Type-I and Type-II solutions.
AMS Subject Classification: 90C29, 90C30.

1. INTRODUCTION

For solutions of the optimization problems the main components are theory and methods of mathematical modelling. In practice, it is usually difficult to determine the real valued coefficients of objective and/or constraint functions involved. There are two deterministic optimization models to deal with uncertain data viz. robust optimization Ben-tal et al. [2], and another is interval valued optimization Ben-Israel and Robers [1]. Many approaches have been developed to deal with these problems. Birge and Louveaux [14], Vajda [24], Stanchu-Minasian [12], Prekopa [4] provide various techniques for solving stochastic optimization problems. On the other hand the collection of papers on fuzzy optimization edited by Slowinski [21] and Delgado et al. [16] gives the main stream to the topic. Lai and Hwang [25, 26] also give useful survey. Inuiguchi and Ramik [17] give a review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problems. Slowinski and Teghem [22] provide comparison between two types of the

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Article history : Received January 05, 2014, Accepted May 15, 2014.

optimization problems for multiobjective programming problems.

Charnes et al. [3] considered the linear programming problems in which the right-hand sides of linear inequality constraints were taken as closed intervals. In a paper Stancu-Minasian and Tigan [13] obtained solutions for interval valued optimization problems. Steuer [20] proposed three algorithms, called the F-cone algorithm, E-cone algorithm and all emanating algorithm to solve the linear programming problems with interval objective functions. Ishibuchi and Tanaka [7] proposed the ordering relation between two closed intervals by considering the maximization and minimization problems separately. Mraz [6] proved algorithms to compute the exact upper bound and lower bound for linear programming problems with interval coefficients. Chanas and Kuchta [23] presented an approach to unify the solution methods proposed in Ishibuchi and Tanaka [7] and in Rommelfanger and Hanuscheck [8]. Oliveria and Antunes [5] provided an overview of multiobjective linear programming problems with interval coefficients by illustrating many numerical examples. Lai and Huang et al. [25] proposed an interval parameter fuzzy nonlinear optimization model for stream water quality management under uncertainty.

The Karush-Kuhn-Tucker optimality conditions play an important role in the area of optimization theory and have been studied for over a century. For interval valued optimization problems, the KKT optimality conditions are also studied in many recent publications. Recently Wu [10, 11] have studied KKT optimality conditions for interval valued optimization problems. Also Chalco-Cano et al. [27] studied the KKT optimality conditions of interval valued optimization problem via generalised derivative. Moreover Zhang et al. [15] derived the KKT optimality conditions for non-convex programming problems with interval valued objective functions. This paper focuses on nonlinear programming problems in which objective and constraint functions are interval valued. The main motivation for considering interval valued constraints is that the uncertainty that is imposed on objective functions is likely also to be imposed on constraints. The remaining paper is organised as: In section 2, we introduce some preliminaries of interval arithmetic and the concept of weak differentiability for intervals valued functions. Moreover by using the concept of order relations " $\preceq_{LU}$ " and " $\preceq_{UC}$ " the solution concepts for interval valued optimization problems are given. Also the concept of LU-convexity and UC-convexity are provided. In section 3, KKT optimality conditions are derived for optimization problems with interval valued objective and interval valued constraint functions. Also by invoking pseudoconvexity the same is derived. An example is also given in order to illustrate our main result. Finally section 4 is devoted to the conclusion.

2. PRELIMINARIES AND NOTATIONS

The values of objective and constraint functions in our model are closed intervals, we need to compare the closed intervals. Let us denote by I, the class of all closed and bounded intervals in R. Throughout this paper, when A is a closed interval, then it is also bounded. We also adopt the notation $A = [a^L, a^U]$, where a^L and a^U are the lower and upper end points of A, respectively.

Let $A, B \in I$. Then $A + B$ is defined by $A + B = \{a + b : a \in A$ and $b \in B\} = [a^L + b^L, a^U + b^U]$. And $-A$ is defined by $-A = -a : a \in A$. And

$A - B = A + (-B) = [a^L - b^U, a^U - b^L]$.

Further for any real number k,

$$kA = \begin{cases} [ka^L, ka^U] \text{ if } k \geq 0, \\ [ka^U, ka^L] \text{ if } k < 0; \end{cases}$$

and for $h > 0$,

$$\frac{A}{h} = \left[\frac{a^L}{h}, \frac{a^U}{h}\right].$$

The function $f : R^n \longrightarrow I$, is called an interval valued function, i.e., $f(x) = f(x_1, x_2, \cdots, x_n)$ is a closed interval in R for each $x \in R^n$. Clearly $f(x) = [f^L(x), f^U(x)]$, where f^L and f^U are real valued functions defined on R^n and satisfy $f^L(x) \leq f^U(x)$ for every $x \in R^n$.

Definition 2.1. [10] Let X be open in R^n and let $x_0 = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)}) \in X$ be fixed. An interval valued function $f : R^n \longrightarrow I$ with $f(x) = [f^L(x), f^U(x)]$ is said to be

(i) weakly differentiable at $x_0 \in X$ if the real valued functions f^L and f^U are differentiable at x_0 (in the usual sense i.e., all of the partial derivatives $\left(\frac{\partial f^L}{\partial x_i}\right)$ and $\left(\frac{\partial f^U}{\partial x_i}\right)$ exist at x_0 for $i = 1, 2, \cdots, n$).
(ii) weakly continuously differentiable at x_0 if the real valued functions f^L and f^U are continuously differentiable at x_0 (i.e., all of the partial derivatives of f^L and f^U exist on some neighborhoods of x_0 and are continuous at x_0).

Wu [10] has formulated two solution concepts for interval valued optimization problem. We may follow similar solution concepts as that used in [10]. Consider the following interval valued optimization problem.
$(IVP1)$

$$\text{Min } f(x) = [f^L(x_1, x_2, \cdots, x_n), f^U(x_1, x_2, \cdots, x_n)] = [f^L(x), f^U(x)]$$
$$\text{Subject to } x = (x_1, x_2, \cdots, x_n) \in X \subseteq R^n.$$

Since the objective function $f(x)$ is a closed interval, we need to make clear the meaning of minimization problem $(IVP1)$. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in R. We write $A \preceq_{LU} B$ if and only if $a^L \leq b^L$ and $a^U \leq b^U$, and $A \prec_{LU} B$ if and only if $A \preceq_{LU} B$ and $A \neq B$. Equivalently, $A \prec_{LU} B$ if and only if

$$\left\{ \begin{array}{c} a^L \leq b^L \\ a^U < b^U; \end{array} \right. or \left\{ \begin{array}{c} a^L < b^L \\ a^U < b^U; \end{array} \right. or \left\{ \begin{array}{c} a^L < b^L \\ a^U \leq b^U. \end{array} \right. \tag{2.1}$$

Definition 2.2. [10] Let x^* be a feasible solution of $(IVP1)$. We say that x^* is type-I solution of $(IVP1)$ if there exists no $\bar{x} \in X$, such that $f(\bar{x}) \prec_{LU} f(x^*)$.

Another solution concept follows from Ishibuchi and Tanaka [7]. Let $A = [a^L, a^U]$ be the closed interval in R. Then we can calculate the centre $a^C = \frac{1}{2}[a^L + a^U]$ and half width $a^W = \frac{1}{2}[a^U - a^L]$ of A. In this case we can use the notation $\langle a^C, a^W \rangle$ for A. i.e., $A = \langle a^C, a^W \rangle$. Ishibuchi and Tanaka [7] have proposed the ordering relation between closed intervals A and B by using minimization and maximization problem separately.

(i) For maximization we write, $A \preceq_{CW} B$ iff $a^C \leq b^C$ and $a^W \geq b^W$. i.e., the interval with higher centre and lower half width (i.e., less uncertainty) is preferred for maximization problem.
(ii) For minimization we write, $A \preceq_{CW} B$ iff $a^C \leq b^C$ and $a^W \leq b^W$. i.e., the interval with lower centre and lower half width (i.e., less uncertainty) is preferred for minimization problem.

Also we write $A \prec_{CW} B$ iff $A \preceq_{CW} B$ and $A \neq B$.

Ishibuchi and Tanaka [7] proved that

(i) $A \preceq_{UC} B$ Iff $A \preceq_{LU} B$ or $A \preceq_{CW} B$.
(ii) $A \prec_{UC} B$ Iff $A \prec_{LU} B$ or $A \prec_{CW} B$.

Definition 2.3. Let x^* be a feasible solution of $(IVP1)$. We say that x^* is type-II solution of $(IVP1)$ if there exists no $\bar{x} \in X$. s.t., $f(\bar{x}) \prec_{UC} f(x^*)$.

Remark 2.4. [10] Let x^* be a feasible solution of $(IVP1)$. If x^* is a type-I solution of $(IVP1)$ then x^* is also a type-II solution of $(IVP1)$.

For our on-going discussion we consider the following definition of convexity for interval valued functions.

Definition 2.5. [10] Let $f(x) = [f^L(x), f^U(x)]$ be an interval valued function defined on convex set $X \subseteq R^n$. We say that F is LU-convex or simply convex at x^* if

$$f(\lambda x^* + (1-\lambda)x) \preceq_{LU} \lambda f(x^*) + (1-\lambda)f(x) \tag{2.2}$$

for each $\lambda \in (0,1)$ and for each $x \in X$. f is said to be LU-convex on X if it is LU-convex on each point of X. Similarly we can define UC-convexity by using relation '$\preceq_{UC}$'.

Proposition 2.6. [10] *Let X be a convex subset of R and f be an interval valued function defined on X. Then we have the following properties.*

(i) *f is LU-convex at x^* iff f^L and f^U are convex at x^*.*
(ii) *f is UC-convex at x^* iff f^U and f^C are convex at x^*.*
(iii) *If f is LU-convex at x^*, then f is UC-convex at x^*.*

3. THE KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS

Consider the optimization problem.
$(IVP2)$

$$\text{Min } f(x) = [f^L(x), f^U(x)]$$
$$\text{Subject to } g_j(x) \leq 0, j = 1, 2, \cdots, m.$$

Where $X = \{x \in R^n : g_j(x) \leq 0, j = 1, 2, \cdots, m\}$ be feasible set of problem $(IVP2)$ and let $J(x^*) = \{j : g_j(x^*) = 0, j = 1, 2, \cdots, m\}$ be the index set of active constraints. We say that the real valued constraint functions $g_j, j = 1, 2, \cdots, m$ satisfy the Kuhn-Tucker constraint qualification at x^* when, if, $\nabla g_j(x^*)^T d \leq 0$ for all $j \in J(x^*)$, where $d \in R^n$, then there exists an n-dimensional vector function $a : [0, 1] \longrightarrow R^n$ such that a is right-differentiable at $0, a(0) = x^*, a(t) \in X$ for all $t \in [0, 1]$, and there exists a real number $\alpha > 0$ with $a'_+(0) = \alpha d$ Wu [9]. Also let $x^* \in X$, We say that $g_j, j = 1, 2, \cdots, m$ satisfy KKT-assumptions at x^* if g_j is convex on R^n and continuously differentiable at x^* for $j = 1, 2, \cdots, m$ Wu [10]. The KKT optimality conditions for $(IVP2)$ obtained in Wu [10] are as follows.

Theorem 3.1. [10] *Suppose that the real valued constraint functions $g_j : R^n \longrightarrow R, j = 1, 2, \cdots, m$ satisfies KKT-assumptions at x^* and the interval valued objective functions $f : R^n \longrightarrow I$ is LU-convex and weakly continuously differentiable at x^*. If there exist (lagrange) multipliers $\lambda = (\lambda^L, \lambda^U) > 0$ and $\mu_j \geq 0, j = 1, 2, \cdots, m$ in R, such that*

(i) $\lambda^L \nabla f^L(x^*) + \lambda^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^*) = 0$

(ii) $\mu_j g_j(x^*) = 0$*for* $j = 1, 2, \cdots, m.$

Then x^* is type-I and type-II solution of $(IVP2)$.

Next in this section we are going to obtain KKT optimality conditions for optimization problem $(IVP1)$. For this we consider the optimization problem $(IVP1)$ with feasible set $X = \{x \in R^n : g_j(x) \preceq_{LU} [0, 0], j = 1, 2, ..., m\}$, where $g_j(x) = [g_j^L(x), g_j^U(x)]$ are interval valued functions for $j = 1, 2, ..., m$, defined on R^n. That is
$(IVP3)$

$$\text{Min } f(x) = [f^L(x), f^U(x)]$$
$$\text{Subject to } g_j(x) \preceq_{LU} [0, 0], j = 1, 2, ..., m.$$

Where f and $g_j, j = 1, 2, ..., m$ are interval valued functions.

Next we define the following.

Definition 3.2. Let x^* be the feasible solution of $(IVP3)$. We say that the interval valued constraint functions $g_j, j = 1, 2, ..., m$ satisfy the Kuhn-Tucker constraint qualification at x^* if g_j^L and $g_j^U, j = 1, 2, ..., m$ satisfy the Kuhn-Tucker constraint qualification at x^*.

In the proof of the following theorem, the Motzkins theorem of alternative is required. It states that, given matrices $A \neq 0$ and C, exactly one of the following system has a solution:

System 1: $Ax < 0, Cx \leq 0$ for some $x \in R^n$.
System 2: $A^T\lambda + C^T\mu = 0$ for some $\mu \geq 0$ and $\lambda \geq 0$ with $\lambda \neq 0$.

In the following theorem we obtain necessary conditions for type-I solution.

Theorem 3.3. *(KKT optimality conditions) Suppose that x^* is type-I solution of problem $(IVP3)$ and the interval valued functions f and $g_j, j = 1, 2, \cdots, m$ are weakly differentiable at x^*. Also assume that the interval valued constraint function $g_j, j = 1, 2, \cdots, m$ satisfy Kuhn-Tucker constraint qualification at x^*. Then there exist (Lagrange) multipliers $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, \cdots, m$ and $\xi^L, \xi^U > 0$ in R , such that*

$$\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0;$$
$$\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), j = 1, 2, \cdots, m.$$

Proof. Since f is weakly differentiable at x^*, by Definition 2.1 f^L and f^U are differentiable at x^*. Let there exists $d \in R^n$, such that

$$\begin{cases} \nabla f^L(x^*)^T d < 0, \\ \nabla f^U(x^*)^T d < 0, \\ \nabla g_j^L(x^*)^T d \leq 0, j \in J(x^*), \\ \nabla g_j^U(x^*)^T d \leq 0, j \in J(x^*). \end{cases} \tag{3.1}$$

Since $g_j, j = 1, 2, \cdots, m$ satisfy Kuhn-Tucker constraint qualification at x^* and f^L is differentiable at x^*, we have

$$\begin{aligned} f^L(a(t)) &= f^L(x^*) + \nabla f^L(x^*)^T(a(t) - x^*) + \| a(t) - x^* \| \epsilon(a(t), x^*) \\ &= f^L(x^*) + \nabla f^L(x^*)^T(a(t) - a(0)) + \| a(t) - a(0) \| \epsilon(a(t), a(0)) \\ &= f^L(x^*) + t\nabla f^L(x^*)^T \frac{(a(0+t) - a(0))}{t} + \| a(t) - a(0) \| \epsilon(a(t), a(0)) \end{aligned}$$

Where $\epsilon(a(t), a(0)) \longrightarrow 0$ as $\| a(t) - a(0) \| \longrightarrow 0$. Therefore, when $t \longrightarrow 0^+$, we have $\frac{a(0+t)-a(0)}{t} \longrightarrow a'_+(0) = \alpha d$, where $\alpha > 0$.

Since $\nabla f^L(x^*)d < 0$, we have $f^L(a(t_1)) < f^L(x^*)$ for a sufficiently small $t_1 > 0$. Similarly we have $f^U(a(t_2)) < f^U(x^*)$ for a sufficiently small $t_2 > 0$. Therefore we have $f^L(a(t)) < f^L(x^*)$ and $f^U(a(t)) < f^U(x^*)$ for a sufficiently small $t < min\{t_1, t_2\}$; consequently $f(a(t)) \prec_{LU} f(x^*)$ for a sufficiently small t, which contradicts the fact that x^* is type-I solution of $(IVP3)$. Therefore system (3.1) has no solution.

Now let A be the matrix whose rows are $\nabla f^L(x^*)^T$ and $\nabla f^U(x^*)^T$ and C be the matrix whose rows are $\nabla g_j^L(x^*)^T$ and $\nabla g_j^U(x^*)^T$ for $j \in J(x^*)$. According to Motzkins theorem of alternative, since the system (3.1) has no solution, there exist multipliers $\xi^L, \xi^U > 0$ and $\mu_j^L, \mu_j^U \geq 0$ in R for $j \in J(x^*)$, such that

$$\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j \in J(x^*)} \{\mu_j^L \nabla g_j^L(x^*) + \mu_j^U \nabla g_j^U(x^*)\} = 0$$

Set $\mu_j^L = 0 = \mu_j^U$ for $j \in \{1, 2, \cdots, m\} \setminus J(x^*)$. Then we have

$$\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0.$$
$$\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), \ for \ j = 1, 2, \cdots, m.$$

and the proof is completed.

□

The following theorem states some sufficient conditions for type-I optimality.

Theorem 3.4. *Suppose that the interval valued functions f and $g_j, j = 1, 2, \cdots, m$ are LU-convex and weakly continuously differentiable at $x^* \in X$. If there exist $\xi^L, \xi^U > 0$ and $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, \cdots, m$ in R, such that*

(i) $\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(ii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*)$, *for* $j = 1, 2, \cdots, m$.

Then x^* is type-I and type-II solution of $(IVP3)$.

Proof. Since $g_j, j = 1, 2, ..., m$ are weakly continuously differentiable at x^*, by Definition 2.1 we see that the real valued functions g_j^L and $g_j^U, j = 1, 2, \cdots, m$ are continuously differentiable at x^*. Define the real valued function

$$\bar{g}_j(x) = \bar{\mu}_j^L g_j^L(x) + \bar{\mu}_j^U g_j^U(x), j = 1, 2, \cdots, m. \tag{3.2}$$

Where $\bar{\mu}_j^L, \bar{\mu}_j^U \geq 0, j = 1, 2, \cdots, m$. Since $g_j, j = 1, 2, \cdots, m$ are LU-convex at x^*, according to proposition 2.6, g_j^L and $g_j^U, j = 1, 2, \cdots, m$ are convex at x^*. Therefore $\bar{g}_j, j = 1, 2, \cdots, m$ are also convex and continuously differentiable at x^*.

Utilising (3.2), we see that

$$\begin{aligned} \mu_j^L \nabla g_j^L(x^*) + \mu_j^U \nabla g_j^U(x^*) &= \mu_j \{\bar{\mu}_j^L g_j^L(x) + \bar{\mu}_j^U g_j^U(x)\} \\ &= \mu_j \bar{g}_j(x), j = 1, 2, \cdots, m \end{aligned} \tag{3.3}$$

Where $\mu_j \bar{\mu}_j^L = \mu_j^L$ and $\mu_j \bar{\mu}_j^U = \mu_j^U$ for $j = 1, 2, \cdots, m$. Invoking (3.2) and (3.3) in (i) and (ii) of theorem we obtain.

(i) $\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j \nabla \bar{g}_j(x^*) = 0$

(ii) $\mu_j \bar{g}_j(x^*) = 0$,for $j = 1, 2, \cdots, m$.

Therefore using Theorem 3.1, x^* is a type-I solution of the problem heaving interval valued objective function $f(x)$ subject to real valued constraints $\bar{g}_j(x^*), j = 1, 2, \cdots, m$, i.e.,

$$f(x^*) \prec_{LU} f(\bar{x}), for\ each\ \bar{x}(\neq x^*) \in X. \tag{3.4}$$

Now suppose that x^* is not a type-I solution of problem $(IVP3)$. Then, based on Definition 2.2, there exists $\bar{x} \in X$, such that

$$\begin{cases} f^L(\bar{x}) \leq f^U(x^*) \\ f^L(\bar{x}) < f^U(x^*); \end{cases} or \begin{cases} f^L(\bar{x}) < f^U(x^*) \\ f^L(\bar{x}) < f^U(x^*); \end{cases} or \begin{cases} f^L(\bar{x}) < f^U(x^*) \\ f^L(\bar{x}) \leq f^U(x^*). \end{cases}$$

Therefore we see that, $f(\bar{x}) \prec_{LU} f(x^*)$. Which contradicts (3.4). This shows that x^* is type-I solution of problem $(IVP3)$ and hence by Remark 2.4, x^* is also type-II solution of problem $(IVP3)$. This proves the theorem.

□

Example 3.5. Consider the following programming problem with interval valued objective and constraint functions:

$$\begin{gathered} \text{Min } f(x) = [f^L(x), f^U(x)] = [2x_1^2 + 2x_2^2 + 3, 2x_1^2 + 2x_2^2 + 4] \\ \text{Subject to } g_1 = [g_1^L, g_1^U] = [1 - x_1 - x_2, 6 - 3x_1 - x_2] \preceq [0, 0] \\ x_1 \geq 0, x_2 \geq 0 \end{gathered}$$

Then we have

$$\begin{gathered} f^L(x_1, x_2) = 2x_1^2 + 2x_2^2 + 3, f^U(x_1, x_2) = 2x_1^2 + 2x_2^2 + 4 \\ g_1^L(x_1, x_2) = 1 - x_1 - x_2, g_1^U(x_1, x_2) = 6 - 3x_1 - x_2 \end{gathered}$$

It is easy to see that the above functions satisfy the assumptions of Theorem 3.4. We have to find x_1, x_2 and ξ^L, ξ^U and μ_1^L, μ_1^U , such that:

$$\xi^L \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} + \xi^U \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix} + \mu_1^L \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \mu_1^U \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{cases} 1 - x_1 - x_2 \leq 0, \\ 6 - 3x_1 - x_2 \leq 0, \\ \mu_1^L(1 - x_1 - x_2) = 0, \\ \mu_1^U(6 - 3x_1 - x_2) = 0, \\ \xi^L, \xi^U > 0, \mu_1^L, \mu_1^U \geq 0, x_i \geq 0, i = 1, 2. \end{cases} \tag{3.5}$$

That is, we have to find a solution for the following simultaneous equations which satisfy the conditions (3.5).

$$4\xi^L x_1 + 4\xi^U x_1 - \mu_1^L - 3\mu_1^U = 0;$$

$$4\xi^L x_2 + 4\xi^U x_2 - \mu_1^L - \mu_1^U = 0.$$

After some algebraic calculations, we obtain

$$(x_1^*, x_2^*) = (\tfrac{9}{5}, \tfrac{3}{5}), \xi^L = \tfrac{1}{4}, \xi^U = \tfrac{1}{4}, \mu_1^L = 0 \text{ and } \mu_1^U = \tfrac{6}{5}.$$

Since $g_1^U(\frac{9}{5}, \frac{3}{5}) = 0$, condition (ii) in Theorem 3.4 is satisfied. Therefore $(x_1^*, x_2^*) = (\frac{9}{5}, \frac{3}{5})$ is type-I solution. In view of the Remark 2.4, type-II solution is obvious.

Further let k is non zero integer such that $1 < k < m$, the sufficient conditions can be resorted as:

Theorem 3.6. *Under the same assumption of Theorem 3.4, let k be any integer with $1 < k < m$. If there exist (Lagrange) multipliers $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, \cdots, m$ s.t,*

(i) $\nabla f^L(x^*) + \sum\limits_{j=1}^{k} \mu_j^L \nabla g_j^L(x^*) + \sum\limits_{j=1}^{k} \mu_j^U \nabla g_j^U(x^*) = 0$

(ii) $\nabla f^U(x^*) + \sum\limits_{j=k}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum\limits_{j=k}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(iii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), for j = 1, 2, \cdots, m.$

Then x^* is type-I and type-II solution of $(IVP3)$.

Proof. Multiplying (i) by $\xi^L > 0$ and (ii) by $\xi^U > 0$, and adding then we get.

(i) $\xi^L \nabla f^L(x^*) + \xi^U \nabla f^U(x^*) + \sum\limits_{j=1}^{k} \hat{\mu}_j^L \nabla g_j^L(x^*) + \sum\limits_{j=1}^{k} \hat{\mu}_j^U \nabla g_j^U(x^*) = 0$

(ii) $\hat{\mu}_j^L g_j^L(x^*) = 0 = \hat{\mu}_j^U g_j^U(x^*), for j = 1, 2, \cdots, m.$

Where $\hat{\mu}_j^L = \xi^L \mu_j^L, \hat{\mu}_j^U = \xi^L \mu_j^U$, for $j = 1, 2, \cdots, k$ and $\hat{\mu}_j^L = \xi^U \mu_j^L, \hat{\mu}_j^U = \xi^U \mu_j^U$, for $j = k+1, \cdots, m$ Therefore by Theorem 3.4, we see that x^* is type-I and type-II solution of $(IVP3)$.

□

Next we introduce centre function $f^c = \frac{1}{2}[f^L + f^U]$ and then resort the conditions as:

Theorem 3.7. *Under the same assumption of Theorem 3.4, Let $f^C = \frac{1}{2}[f^L + f^U]$. If there exist $\xi^C, \xi^U > 0$ and $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, \cdots, m.$ s.t,*

(i) $\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) + \sum\limits_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum\limits_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(ii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), for j = 1, 2, \cdots, m.$

Then x^* is type-I and type-II solution of $(IVP3)$.

Proof. Using (3.2) and (3.3) in (i) and (ii) of this theorem we obtain.

(i) $\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) + \sum\limits_{j=1}^{m} \mu_j \nabla \bar{g}_j(x^*) = 0$

(ii) $\mu_j^L \bar{g}_j^L(x^*) = 0, forj = 1, 2, \cdots, m.$

Also since f is LU-convex and weakly continuously differentiable at x^*, therefore $f^C = \frac{1}{2}[f^L + f^U]$ is convex and continuously differentiable at x^* (by Proposition 2.6 and Definition 2.1). Now suppose x^* is not type-II solution, using similar arguments as in Theorem 3.4, we conclude that x^* is a type-II solution of problem $(IVP3)$. Since

$$\begin{aligned}\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) &= \xi^U \nabla f^U(x^*) + \frac{1}{2}\xi^C[\nabla f^L(x^*) + \nabla f^U(x^*)] \\ &= \frac{1}{2}\xi^C \nabla f^L(x^*) + \left(\frac{1}{2}\xi^C + \xi^U\right)\nabla f^U(x^*),\end{aligned}$$

We conclude that x^* is also a type-I solution by using Theorem 3.4.

□

Next we present KKT conditions for type-II solution.

Theorem 3.8. *Suppose that the interval valued functions $g_j, j = 1, 2, ..., m$ are LU-convex and weakly continuously differentiable at $x^* \in X$. Also suppose that the interval valued function f is UC-convex and weakly continuously differentiable at x^*. If there exist (Lagrange) multipliers $\xi^L, \xi^C > 0$ and $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, ..., m$ in R, such that*

(i) $\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(ii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), forj = 1, 2, \cdots, m$

Then x^* is type-II solution of $(IVP3)$.

Proof. Using (3.2) and (3.3) in (i) and (ii) of this theorem we obtain.

(i) $\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) + \sum_{j=1}^{m} \mu_j \nabla \bar{g}_j(x^*) = 0$

(ii) $\mu_j \bar{g}_j(x^*) = 0, for\ j = 1, 2, \cdots, m$

Since f is UC-convex and weakly continuously differentiable at x^*, we see that f^U and f^C are convex and continuously differentiable at x^* (by Proposition 2.6 and Definition 2.1). Using similar arguments as in Theorem 3.4, we conclude that x^* is a type-II solution of $(IVP3)$. Despite of the fact that

$$\xi^U \nabla f^U(x^*) + \xi^C \nabla f^C(x^*) = \frac{1}{2}\xi^C \nabla f^L(x^*) + \left(\frac{1}{2}\xi^C + \xi^U\right)\nabla f^U(x^*).$$

We cannot conclude that x^* is also a type-I solution in view of Theorem 3.4, since the assumptions in this theorem is different from that of Theorem 3.4 and the UC-convexity does not imply LU-convexity in general.

□

Next let us consider pseudoconvexity in order to relax the convexity assumptions of interval valued objective function.

Definition 3.9. [10] Let f be differentiable real valued function defined on non empty convex set X of R, then f is said to be pseudoconvex at x^* if $f(x) < f(x^*)$ then $\nabla f(x^*)^T(x - x^*) < 0$ for $x \in X$ and f is strictly pseudoconvex at x^* if $f(x) \leq f(x^*)$ then $\nabla f(x^*)^T(x - x^*) < 0$ for $x \in X$.

Definition 3.10. [10] Let $f(x) = [f^L(x), f^U(x)]$ be an interval valued function defined on convex set $X \subseteq R^n$. We say that f is pseudoconvex at x^* if the real valued functions f^L and f^U are pseudoconvex at x^*.

Let X be a nonempty feasible set and $x^* \in cl(X)$ (the closure of X). The cone of feasible directions of X at x^*, denoted by D, is defined by

$$D = \{d \in R^n : d \neq 0, \text{there exists a } \delta > 0 \text{ such that } x^*+\tau d \in X \text{ for all } \tau \in (0, \delta)\}.$$

Each d of D is called a feasible direction of X.

Proposition 3.11. [19] *Let $X = \{x \in R^n : g_j(x) \leq 0, j = 1, 2, ..., m\}$ be a feasible set and a point $x^* \in X$. Assume that g_j are differentiable at x^* for all $j = 1, 2, ..., m$. Let $J = \{j : g_j(x^*) = 0\}$ be the index set for the active constraints. Then*

$$D \subseteq \{d \in R^n : \nabla g_j(x^*)^T d \leq 0 \textit{ for each } j \in J\}.$$

(Note that this proposition still hold true if we just assume that g_j are continuous at x^* instead of differentiable at x^* for $j \notin J$).

In the proof of the following theorem the Tuckers theorem of the alternative is needed. It states that, given matrices A and C, exactly one of the following system has a solution:

System 1: $Ax \leq 0, Ax \neq 0, Cx \leq 0$ for some $x \in R^n$;
System 2: $A^T\lambda + C^T\mu = 0$, for some $\lambda > 0$ and $\mu \geq 0$.

Theorem 3.12. *Suppose that the interval valued functions $g_j, j = 1, 2, ..., m$ are LU-convex and weakly continuously differentiable at $x^* \in X$. and the interval valued objective function f is weakly differentiable and pseudoconvex at x^*. If there exist (Lagrange) multipliers $\mu_j^L, \mu_j^U \geq 0, j = 1, 2, ..., m$ in R, such that,*

(i) $\nabla f^L(x^*) + \sum_{j=1}^{m} \mu_j^L \nabla g_j^L(x^*) = 0$

(ii) $\nabla f^U(x^*) + \sum_{j=1}^{m} \mu_j^U \nabla g_j^U(x^*) = 0$

(iii) $\mu_j^L g_j^L(x^*) = 0 = \mu_j^U g_j^U(x^*), for\ j = 1, 2, \cdots, m$

Then x^* is type-I and type-II solution of $(IVP3)$.

Proof. We shell prove this result by contradiction. Suppose that x^* is not a type-I solution. Then by definition there exists an $\bar{x} \neq x^*$ such that $f(\bar{x}) \prec_{LU} f(x^*)$, which implies that either $f^L(\bar{x}) < f^L(x^*)$ or $f^U(\bar{x}) < f^U(x^*)$. Since f is weakly differentiable and pseudoconvex at x^*, by Definition 2.1 and 3.10, f^L and f^U are differentiable and pseudoconvex at x^*, we have either $\nabla f^L(x^*)^T(\bar{x} - x^*) < 0$ or $\nabla f^U(x^*)^T(\bar{x} - x^*) < 0$. Let us consider the case

$$\nabla f^L(x^*)^T(\bar{x} - x^*) < 0. \tag{3.6}$$

Let $d = \bar{x} - x^*$. Then $x = x^* + \tau d = \tau\bar{x} + (1-\tau)x^* \in X$ for $\tau \in (0,1)$. Since X is a convex set and $\bar{x}, x^* \in X$. This shows that $d \in D$. From Proposition 3.11, we conclude that

$$\nabla g_j^L(x^*)d \leq 0 \; for \; each \; j \in J(x^*). \tag{3.7}$$

Let A be the matrix whose rows are $\nabla f^L(x^*)^T$ and C be the matrix whose rows are $\nabla g_j^L(x^*)^T$ for $j \in J$. Therefore by using Tuckers theorem of the alternative, since System 1 has a solution $d = \bar{x} - x^*$ (see (3.6) and (3.7)), there exist no multipliers $0 < \lambda \in R$ and $0 \leq \mu_j \in R, j \in J(x^*)$, such that $\lambda \nabla f^L(x^*) + \sum_{j \in J(x^*)} \mu_j \nabla g_j^L(x^*) = 0$, or equivalently, there exist no multipliers $0 \leq \mu_j^L \in R$, $j \in J(x^*)$, such that $\nabla f^L(x^*) + \sum_{j \in J(x^*)} \mu_j^L \nabla g_j^L(x^*) = 0$, where $\mu_j^L = \frac{\mu_j}{\lambda}$. Setting $\mu_j^L = 0, j \notin J$, we get a contradiction to (i) and (iii). Similarly, If $\nabla f^U(x^*)^T(\bar{x} - x^*) < 0$, then conditions (ii) and (iii) will be violated. This shows that x^* is a type-I solution. From Remark 2.4, the proof is complete. □

4. CONCLUSIONS

Interval programming is one of the approaches to handle the uncertain optimization, in which an interval is used to model the uncertainty of variables. Most of the recent work has been done by considering interval coefficients of objective function. Although the same uncertainty is also likely to be imposed on constraints. In this paper we have successfully derived the KKT optimality conditions for programming problems with interval valued objective and interval valued constraint functions. Although the interval valued equality constraints are not considered in this paper, the similar methodology proposed in this paper can also be used to handle the interval valued equality constraints. Future research is oriented to consider the uncertain environment in order to study the optimality conditions involving Fuzzy parameters.

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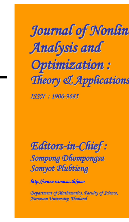
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 105-117
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPS IN INTUITIONISTIC FUZZY METRIC SPACES USING PROPERTY (S-B)

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ABSTRACT. In this note we prove a common fixed point theorem for weakly compatible maps in intuitionistic fuzzy metric spaces using property (S-B), which generalize the results of Kumar and Vats [26], Alaca, Turkoglu and Yildiz [5]

KEYWORDS :Intuitionistic fuzzy metric space, weakly compatible mappings, and (S-B) property.
AMS Subject Classification: 47H10, 54H25

1. INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh [45] in 1965. Since, then to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. A fuzzy set A in X is a function with domain X and values in [0, 1]. The concept of fuzzy set corresponds to the degree of nearness between two objects. Deng [9], Erceg [11], Fang [12], George and Veeramani [14], Kaleva and Seikkala [20], Kramosil and Michalek [23] have introduced the concept of fuzzy metric spaces in different ways. Atanassove [6] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Intuitionistic fuzzy sets deals with both degree of nearness and non-nearness. Park [32] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [14]. Further, Alaca et al. [4] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space, as park with the help of continuous t-norms and continuous t-conorms, as a generalization of fuzzy metric space due to Kramosil and Michalek [24]. Further Coker [8] introduced the concept of intuitionisitc fuzzy topological spaces. Turkoglu et. al. [42] gave a generalization of Jungck's common fixed point

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Article history : Received November 22, 2012, Accepted June 28, 2013.

theorem [17] to intuitionistic fuzzy metric spaces. They first formulated the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of Pant's theorem [31]. Recently, many authors have also studied the fixed point and common fixed point theorems in intuitionistic fuzzy metric space (See [2], [39], [40], [41]). Sharma and Bamboria [38] defined a property (S-B) for self maps and obtained some common fixed point theorems for such mappings under strict contractive conditions. The class of (S-B) maps contains the class of non compatible maps. Kamran [21] obtained some coincidence and fixed point theorems for hybrid strict contractions. Sharma and Sharma [37] proved common fixed point theorem in intuitionistic fuzzy metric spaces under (S-B) property.

2. PRELIMINARIES

The concept of triangular norms (t-norm) and triangular conorms (t- conorm) are known as the axiomatic skeletons that we use for characterization fuzzy intersections and union respectively. These concepts were originally introduced by Menger [29] in study of statistical metric spaces.

Definition 2.1:[36] A binary operation * : $[0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-norm if * is satisfying the following conditions:

(i) * is commutative and associative;
(ii) * is continuous;
(iii) $a * 1 = a \,\forall\, a \in [0,1]$;
(iv) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$; $\forall\, a, b, c, d \in [0,1]$

Definition 2.2:[36] A binary operation $\Diamond$: $[0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-conorm if $\Diamond$ is satisfying the following conditions:

(i) $\Diamond$ is commutative and associative;
(ii) $\Diamond$ is continuous;
(iii) $a \Diamond 0 = a \,\forall\, a \in [0,1]$;
(iv) $a \Diamond b \leqslant c \Diamond d$ whenever $a \leqslant c$ and $b \leqslant d$; $\forall\, a, b, c, d \in [0,1]$

Alaca et al. [4] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [24] as follows;

Definition 2.3:[4] A 5 tuple $(X, M, N, *, \Diamond)$ is said to an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions;

(i) $M(x,y,t) + N(x,y,t) \leqslant 1 \,\forall\, x, y \in X$ and $t > 0$;
(ii) $M(x,y,t) = 0 \,\forall\, x, y \in X$;
(iii) $M(x,y,t) = 1$ for all $x, y \in X$ and t > 0
(iv) $M(x,y,t) = M(y,x,t) \,\forall\, x, y \in X$ and $t > 0$;
(v) $M(x,y,t) * M(y,z,s) \leqslant M(x,z,t+s)$ for all $x, y, z \in X$ and s, t > 0;
(vi) For all $x, y \in X, M(x,y,.) : [0,\infty) \longrightarrow [0,1]$ is continuous;
(vii) $\lim_{t \longrightarrow \infty} M(x,y,t) = 1$ for all $x, y \in X$ and t > 0;
(viii) $N(x,y,0) = 1$ for all $x, y \in X$;
(ix) $N(x,y,t) = 0$ for all x, y $\in X$ and t > 0 iff $x = y$;
(x) $N(x,y,t) = N(y,x,t)$ for all $x, y \in X$ and $t > 0$;
(xi) $N(x,y,t) \Diamond N(y,z,s) \geqslant N(x,z,t+s)$ for all $x, y, z \in X$ and s, t > 0;

(xii) For all $x, y \in X, N(x, y, .) : [0, \infty) \longrightarrow [0, 1]$ is continuous;
(xiii) $\lim_{t\longrightarrow\infty} N(x, y, t) = 0$ for all $x, y \in X$;

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Example 2.1:[26] Let $X = \{1/n : n \varepsilon N\} u\{0\}$ with * continuous t-norm and $\Diamond$ continuous t-conorm defined by $a * b = ab$ and $a \Diamond b$ = min $\{1, a + b\}$ respectively for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define(M, N) by

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|} & \text{if } t > 0. \\ 0 & \text{if } t = 0. \end{cases} \; and \, N(x, y, t) = \begin{cases} \frac{|x-y|}{t+|x-y|} & \text{if } t > 0. \\ 1 & \text{if } t = 0. \end{cases}$$

Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space.

Remark 2.1:[27] Every fuzzy metric space $(X, M, *)$is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \Diamond)$ such that t-norm * and t-conorm $\Diamond$ are associated as $x \Diamond y = 1 - ((1 - x) * (1 - y)) \, \forall \, x, y \in$ X.

Remark 2.2:[26] An intuitionistic fuzzy metric spaces with continuous t-norm * and continuous t-conorm $\Diamond$ defined by $a * a \geqslant a$ and $(1-a)\Diamond(1-a) \leqslant (1-a) \, \forall \, a \in [0, 1]$. Then for all $x, y \in X, M(x, y, *)$ is non- decreasing and $N(x, y, \Diamond)$ is non-increasing.

Alaca, Turkoglu, and Yildiz [4] introduced the following notions:

Definition 2.4: Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then

(i) a sequence $\{x_n\}$ is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $\lim_{n\longrightarrow\infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n\longrightarrow\infty} N(x_{n+p}, x_n, t) = 0$

(ii) a Sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0, \lim_{n\longrightarrow\infty} M(x_n, x, t) = 1$, $\lim_{n\longrightarrow\infty} N(x_n, x, t) = 0$

Since * and $\Diamond$ are continuous, the limit is uniquely determined from (v) and (xi) of definitions 3 respectively.

Definition 2.5: An intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Turkoglu, Alaca and Yildiz [43] introduced the notions of compatible mappings in intuitionistic fuzzy metric space, akin to the concept of compatible mappings introduced by Jungck [18] in metric spaces.

Definition 2.6: A pair of self mappings (f, g) of a intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be compatible if $\lim_{n\longrightarrow\infty} M(fgx_n, gfx_n, t) = 1$ and $\lim_{n\longrightarrow\infty} N(fgx_n, gfx_n, t = 0$ for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = z$ for some $z \in X$.

Definition 2.7: A pair of self mappings (f, g) of a intuitionistic fuzzy metric space

$(X, M, N, *, \Diamond)$ is said to be non-compatible if $\lim_{n\longrightarrow\infty} M(fgx_n, gfx_n, t) \neq 1$ or non- existent and $\lim_{n\longrightarrow\infty} N(fgx_n, gfx_n, t) \neq 0$ or non- existent for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = z$ for some $z \in X$.

In 1998, Jungck and Rhoades [19] introduced the concept of weakly compatible maps as follows.

Definition 2.8: Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Example 2.2: Let $X = R$ and define $f, g : R \longrightarrow R$ by $fx = x/3$ and $gx = x^2 \,\forall\, x \in R$. Here 0 and $1/3$ are two coincidence points for the maps$f$ and g. Note that $f$and g commute at 0, i.e.$fg(0) = gf(0) = 0$.But $fg(1/3) = f(1/9) = 1/27$ and $gf(1/3) = g(1/9) = 1/81$ and so f and g are not weakly compatible maps on R.

Remark 2.3: Weakly compatible maps need not be compatible.

Aamri and Moutawakil [1] generalized the concept of non compatibility in metric spaces by defining the notion of E.A. property and proved common fixed point theorems. Using this property Sharma and Bamboria [38] defined $(S - B)$ property in fuzzy metric space.

Definition 2.9:[38] A pair of self mappings (S, T) of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to satisfy the $(S-B)$ property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\longrightarrow\infty} M(Sx_n, Tx_n, t) = 1, \lim_{n\longrightarrow\infty} N(Sx_n, Tx_n, t) = 0$.

Example 2.3: Let $X = [0, \infty)$ consider $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space, where M and N are two fuzzy sets defined by $M(x, y, t) = t/[t + d(x, y)]$ and $N(x, y, t) = d(x, y)/[t + d(x, y)]$ where d is usual metric. Define $T, S : X \longrightarrow [0, \infty)$ by $Tx = x/5$ and $Sx = 2x/5$ for all x in X. Consider $x_n = 1/n$. Now, $\lim_{n\longrightarrow\infty} M(Sx_n, Tx_n, t) = 1, \lim_{n\longrightarrow\infty} N(Sx_n, Tx_n, t) = 0$. Therefore S and T satisfy property $(S - B)$.

Now we state two lemmas which are useful in proving our main results.

Lemma 2.1:[3] Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0, 1), M(x, y, kt) \geqslant M(x, y, t)$ and $N(x, y, kt) \leqslant N(x, y, t)$ then $x = y$.

Lemma 2.2:[3] Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in X. If there exists a number $k \in (0, 1)$ such that;
$M(y_{n+2}, y_{n+1}, kt) \geqslant M(y_{n+1}, y_n, t), N(y_{n+2}, y_{n+}, kt) \leqslant N(y_{n+1}, y_n, t)$ for all $t > 0$ and $n = 1, 2, \ldots$ then $\{y_n\}$ is a Cauchy sequence in X.

Alaca et al [5] proved the following theorem.

Theorem A: Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces $(X, M, N, *, \Diamond)$ with continuous t-norm * and continuous t-conorm $\Diamond$ defined by $a * a \geqslant a$ and $(1 - a) \leqslant (1 - a) \Diamond (1 - a)$ for all $a \in [0, 1]$ satisfying the following conditions;

$(A.1)$ $A(X) \subset T(X), B(X) \subset S(X)$;
$(A.2)$ S and T are continuous;
$(A.3)$ The pairs $\{A, S\}$ and $\{B, T\}$ are compatible maps;
$(A.4)$ For all $x, y \in X, t > 0$ and $k \in (0, 1)$ such that

$M(Ax, By, kt) \geqslant M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(By, Sx, 2t) *$ $M(Ax, Ty, t)$
$N(Ax, By, kt) \leqslant N(Sx, Ty, t) \lozenge N(Ax, Sx, t) \lozenge N(By, Ty, t) \lozenge N(By, Sx, 2t) \lozenge N(Ax, Ty, t)$.

Then A, B, S and T have a unique common fixed point in X.

Kumar and Vats [26] proved the following.

Theorem B: Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces$(X, M, N, *, \lozenge)$ with continuous t-norm * and continuous t-conorm $\lozenge$ defined by $a* a \geqslant a$ and$(1-a)\lozenge(1-a) \leqslant (1-a)$ for all $a \in [0, 1]$ satisfying the following conditions;

$(B.1)$ $A(X) \subset T(X), B(X) \subset S(X)$
$(B.2)$ $\forall\, x, y \in X, t > 0$ and $k \in (0, 1)$ such that
$M(Ax, By, kt) \geqslant M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(By, Sx, 2t) *$ $M(Ax, Ty, t$
$N(Ax, By, kt) \leqslant N(Sx, Ty, t) \lozenge N(Ax, Sx, t) \lozenge N(By, Ty, t) \lozenge N(By, Sx, 2t) \lozenge N(Ax, Ty, t)$

Then
(i) A and S have a point of coincidence;
(ii) B and T have a point of coincidence.
Moreover, if the pairs$\{A, S\}$ and $\{B, T\}$ are weakly compatible maps, then A, B, S and T have a unique common fixed point in X.

3. MAIN RESULTS

Now, we prove our main result which generalizes the theorems A and B.

Theorem 3.1. *Let A, B, S and T be self maps of a intuitionistic fuzzy metric spaces $(X, M, N, *, \lozenge)$ with continuous t-norm * and continuous t-conorm $\lozenge$ defined by $a * a \geqslant a$ and $(1-a)\lozenge(1-a) \leqslant (1-a)$ for all $a \in [0, 1]$satisfying the following conditions;*
(3.1) A (X) $\subset$ T (X), B (X) $\subset$ S (X)
(3.2) pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible.
(3.3) pairs $\{A, S\}$ or $\{B, T\}$ satisfies the property (S-B)
(3.4) for all $x, y \in X, t > 0$ and $k \in (0, 1)$such that
*M (Ax, By, kt) $\geqslant$ M (Sx, Ty, t) * M (Ax, Sx, t) * M (By, Ty, t) * M (By, Sx, 2t) * M (Ax, Ty, t)*
N (Ax, By, kt) $\leqslant$ N (Sx, Ty, t) $\lozenge$ N (Ax, Sx, t) $\lozenge$ N (By, Ty, t) $\lozenge$ N (By, Sx, 2t) $\lozenge$ N (Ax, Ty, t)
(3.5) one of A(X), B(X), S(X) or T(X) is a closed subset of X.
Then A, B, S and T have a unique common fixed point in X.

Proof. Suppose that (B, T) satisfies the property $(S - B)$, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\longrightarrow\infty} Bx_n = \lim_{n\longrightarrow\infty} Tx_n = z$ for some $z \in X$.

Since$B(X) \subset S(X)$ there exists a sequence $\{x_n\} \in$ X such that

$\lim_{n\longrightarrow\infty} Bx_n = \lim_{n\longrightarrow\infty} Sy_n = z$

Now we shall show that $\lim_{n\longrightarrow\infty} Ay_n = z$

From (3.4), we have;

$M(Ay_n, Bx_n, kt) \geqslant M(Sy_n, Tx_n, t) * M(Ay_n, Sy_n, t) * M(Bx_n, Tx_n, t) * M(Bx_n, Sy_n, 2t)$
$* M(Ay_n, Tx_n, t)$
and
$N(Ay_n, Bx_n, kt) \leqslant N(Sy_n, Tx_n, t) \diamondsuit N(Ay_n, Sy_n, t) \diamondsuit N(Bx_n, Tx_n, t) \diamondsuit N(Bx_n, Sy_n, 2t) \diamondsuit N(Ay_n, Tx_n, t)$

Proceeding limit as $n \longrightarrow \infty$, we have;

$\lim_{n\longrightarrow\infty} M(Ay_n, z, kt) \geqslant \lim_{n\longrightarrow\infty}[M(z, z, t)*M(Ay_n, z, t) *M(z, z, t) *M(z, z, 2t)$
$* M(Ay_n, z, t)]$
and
$\lim_{n\longrightarrow\infty} N(Ay_n, z, kt) \leqslant \lim_{n\longrightarrow\infty}[N(z, z, t) \diamondsuit N(Ay_n, z, t) \diamondsuit N(z, z, t) \diamondsuit (z, z, 2t) \diamondsuit N(Ay_n, z, t)]$,

$\lim_{n\longrightarrow\infty} M(Ay_n, z, kt) \geqslant \lim_{n\longrightarrow\infty}[1*M(Ay_n, z, t)* 1* 1* M(Ay_n, z, t)]$
and
$\lim_{n\longrightarrow\infty} N(Ay_n, z, kt) \leqslant \lim_{n\longrightarrow\infty}[0 \diamondsuit N(Ay_n, z, t) \diamondsuit 0 \diamondsuit 0 \diamondsuit N(Ay_n, z, t)]$

It follows that

$\lim_{n\longrightarrow\infty} M(Ay_n, z, kt) \geqslant \lim_{n\longrightarrow\infty} M(Ay_n, z, t)$
and
$\lim_{n\longrightarrow\infty} N(Ay_n, z, kt) \leqslant \lim_{n\longrightarrow\infty} N(Ay_n, z, t)$

and we deduce that $\lim_{n\longrightarrow\infty} Ay_n = z$

Suppose that $S(X)$ is a closed subset of X .Then $z = Su$ for some $u \in X$.
Subsequently, we have,

$\lim_{n\longrightarrow\infty} Ay_n = \lim_{n\longrightarrow\infty} Bx_n = \lim_{n\longrightarrow\infty} Tx_n = \lim_{n\longrightarrow\infty} Sy_n = z = Su.$

Now, we shall show that $Au = Su$

From (3.4) we have,

$M(Au, Bx_n, kt) \geqslant M(Su, Tx_n, t) * M(Au, Su, t) *M(Bx_n, Tx_n, t) *M(Bx_n, Su, 2t)$
$*M(Au, Tx_n, t)$
and
$N(Au, Bx_n, kt) \leqslant N(Su, Tx_n, t) \diamondsuit N(Au, Su, t) \diamondsuit N(Bx_n, Tx_n, t) \diamondsuit N(Bx_n, Su, 2t) \diamondsuit N(Au, Tx_n, t)$

Taking the limit as $n \longrightarrow \infty$, we have,

$M(Au, Su, kt) \geqslant M(Su, Su, t) * M(Au, Su, t) * M(Su, Su, t) * M(Su, Su, 2t) * M(Au, Su, t)$
and
$N(Au, Su, kt) \leqslant N(Su, Su, t) \lozenge N(Au, Su, t) \lozenge N(Su, Su, t) \lozenge N(Su, Su, 2t) \lozenge N(Au, Su, t)$,

$M(Au, Su, kt) \geqslant 1 * M(Au, Su, t) * 1 * 1 * M(Au, Su, t)$
and
$N(Au, Su, t) \leqslant 0 * N(Au, Su, t) * 0 * 0 * N(Au, Su, t)$,

$M(Au, Su, kt) \geqslant M(Au, Su, t)$
and
$N(Au, Su, kt) \leqslant N(Au, Su, t)$

Therefore by lemma 2.1, we have; $Au = Su$.

i.e. u is a coincidence point of A and S

The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$ on the other hand since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$.

Suppose that $Tv \neq Bv$. Then (3.4) implies that.

$M(Au, Bv, kt) \geqslant M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) * M(Bv, Su, 2t) * M(Au, Tv, t)$
and
$N(Au, Bv, kt) \leqslant N(Su, Tv, t) \lozenge N(Au, Su, t) \lozenge N(Bv, Tv, t) \lozenge N(Bv, Su, 2t) \lozenge N(Au, Tv, t)$,

$M(Au, Bv, kt) \geqslant M(Au, Au, t) * M(Au, Au, t) * M(Bv, Au, t) * M(Bv, Au, 2t) * M(Au, Au, t)$
and
$N(Au, Bv, kt) \leqslant N(Au, Au, t) \lozenge N(Au, Au, t) \lozenge N(Bv, Tu, t) \lozenge N(Bv, Au, 2t) \lozenge N(Au, Au, t)$,

$M(Au, Bv, kt) \geqslant 1 * 1 * M(Bv, Au, t) * M(Bv, Au, 2t) * 1$
and
$N(Au, Bv, kt) \leqslant 0 \lozenge 0 \lozenge N(Bv, Au, t) \lozenge N(Bv, Au, 2t) \lozenge 0$

Thus, it follows that $M(Au, Bv, kt) \geqslant M(Au, Bv, t)$
and
$N(Au, Bv, kt) \leqslant N(Au, Bv, t)$

Therefore by lemma 2.1 we have; $Au = Bv$. Hence $Tv = Bv$.

i.e. v is a coincidence point of B and T.

Thus we have: $Au = Su = Tv = Bv$.

The weak compatibility of B and T implies that

$BTv = TBv$ and $TTv = TBv = BTv = BBv$.

Let us show that Au is a common fixed point of A, B, S and T.

Suppose that $AAu \neq Au$. Then using (3.4) we have;

$M(AAu, Bv, kt) \geqslant M(SAu, Tv, t) * M(AAu, SAu, t) * M(Bv, Tv, t) * M(Bv, SAu, 2t)$
$*M(AAu, Tv, t)$
and
$N(AAu, Bv, kt) \leqslant N(SAu, Tv, t) \diamondsuit N(AAu, SAu, t) \diamondsuit N(Bv, Tv, t) \diamondsuit N(Bv, SAu, 2t) \diamondsuit N(AAu, Tv, t)$,

$M(AAu, Au, kt) \geqslant M(AAu, Au, t) * M(AAu, AAu, t) * M(Bv, Bv, t) * M(Au, AAu, 2t)$
$* M(AAu, Au, t)$
and
$N(AAu, Au, kt) \leqslant N(AAu, Au, t) \diamondsuit N(AAu, AAu, t) \diamondsuit N(Bv, Bv, t) \diamondsuit N(Au, AAu, 2t) \diamondsuit N(AAu, Au, t)$,

$M(AAu, Au, kt) \geqslant M(AAu, Au, t) * 1 * 1 * M(Au, AAu, 2t) * M(AAu, Au, t)$
and
$N(AAu, Au, kt) \leqslant N(AAu, Au, t) \diamondsuit 0 \diamondsuit 0 \diamondsuit N(Au, AAu, 2t) \diamondsuit N(AAu, Au, t)$

$M(AAu, Au, kt) \geqslant M(AAu, Au, t)$
and
$N(AAu, Au, kt) \leqslant N(AAu, Au, t)$

Therefore by lemma 2.1, we have; $AAu = Au$.

Since $AAu = SAu$, therefore, $Au = AAu = SAu$

i.e. Au is a common fixed point of A and S.

Similarly, we prove that Bv is a common fixed point of B and T. Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, S and T. The proof is similar when $T(X)$ is assumed to be a closed subset of X. The cases in which $A(X)$ or $B(X)$ is a closed subset of X are similar to the cases in which $T(X)$ or $S(X)$ respectively is closed subset of X, since $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Uniqueness follows easily.

□

Theorem 3.2. *Let A, B and S be self maps of a intuitionistic fuzzy metric spaces $(X, M, N, *, \diamondsuit)$ with continuous t-norm * and continuous t-conorm $\diamondsuit$ defined by $a * a \geqslant a$ and $(1-a) \diamondsuit (1-a) \leqslant (1-a)$ for all $a \in [0,1]$ satisfying the following conditions;*
(3.21) A (X) ⊂ S (X), B (X) ⊂ S (X)
(3.22) pairs $\{A, S\}$ and $\{B, S\}$ are weakly compatible.
(3.23) pairs $\{A, S\}$ or $\{B, S\}$ satisfies the property (S-B)
(3.24) for all $x, y \in X, t > 0$ and $k \in (0,1)$ such that
$M^2(Ax, By, kt) \geqslant [M(Ax, Sx, t) * M(By, Sx, t)]^2$
$N^2(Ax, By, kt) \leqslant [N(Ax, Sx, t) \diamondsuit N(By, Sx, t)]^2$
(3.25) one of A(X), B(X), S(X) or T(X) is a closed subset of X.
Then A, B and S have a unique common fixed point in X.

Proof. Suppose that (B, S) satisfies the property $(S - B)$, then there exists a sequence $\{x_n\}$ in XSuch that $\lim_{n\longrightarrow\infty} Bx_n = \lim_{n\longrightarrow\infty} Sx_n = z$ for some $z \in X$.

Since $B(X) \subset S(X)$ there exists a sequence$\{y_n\} \in X$ such that $\lim_{n\longrightarrow\infty} Bx_n = \lim_{n\longrightarrow\infty} Sy_n = z$

Now, we shall show that $\lim_{n\longrightarrow\infty} Ay_n = z$

From (3.24), we have; (at $x = y_n, y = x_n$)

$[M(Ay_n, Bx_n, kt)]^2 \geqslant [M(Ay_n, Sy_n, t) * M(Bx_n, Sy_n, t)]^2$

$[N(Ay_n, Bx_n, kt)]^2 \leqslant [N(Ay_n, Sy_n, t) \Diamond N(Bx_n, Sy_n, t)]^2$

Proceeding limit as $n \longrightarrow \infty$, we have;

$[\lim_{n\longrightarrow\infty} M(Ay_n, z, kt)]^2 \geqslant [\lim_{n\longrightarrow\infty} M(Ay_n, z, t) * M(z, z, t)]^2$
and
$[\lim_{n\longrightarrow\infty} N(Ay_n, z, kt)]^2 \leqslant [\lim_{n\longrightarrow\infty} N(Ay_n, z, t) \Diamond N(z, z, t)]^2$

$[\lim_{n\longrightarrow\infty} M(Ay_n, z, kt)]^2 \geqslant [\lim_{n\longrightarrow\infty} M(Ay_n, z, t) * 1]^2$
and
$[\lim_{n\longrightarrow\infty} N(Ay_n, z, kt)]^2 \leqslant [\lim_{n\longrightarrow\infty} N(Ay_n, z, t) \Diamond 0]^2$

Thus, it follows that,

$\lim_{n\longrightarrow\infty} M(Ay_n, z, kt) \geqslant \lim_{n\longrightarrow\infty} M(Ay_n, z, t)$
and
$\lim_{n\longrightarrow\infty} N(Ay_n, z, kt) \leqslant \lim_{n\longrightarrow\infty} N(Ay_n, z, t)$

and we deduce that $\lim_{n\longrightarrow\infty} Ay_n = z$.

Suppose that $S(X)$ is a closed subset of X. then $z = Su$. For some $u \in X$, subsequently we have;

$\lim_{n\longrightarrow\infty} Ay_n = \lim_{n\longrightarrow\infty} Bx_n = \lim_{n\longrightarrow\infty} Sy_n = z = Su$

Now, we shall show that $Au = Su$.

From (3.24) we have (at $x = u, y = x_n$)

$[M(Au, Bx_n, kt)]^2 \geqslant [M(Au, Su, t) * M(Bx_n, Su, t)]^2$
and
$[N(Au, Bx_n, kt)]^2 \leqslant [N(Au, Su, t) \Diamond N(Bx_n, Su, t)]^2$

Taking the limit as $n \longrightarrow \infty$, we have;

$[M(Au, Su, kt)]^2 \geqslant [M(Au, Su, t) * M(Su, Su, t)]^2$
and
$[N(Au, Su, kt)]^2 \leqslant [N(Au, Su, t) \Diamond N(Su, Su, t)^2$

$[M(Au, Su, kt)]^2 \geqslant [M(Au, Su, t) * 1]^2$

and
$[N(Au, Su, kt)]^2 \leqslant [N(Au, Su, t) \diamondsuit 0]^2$

Thus, it follows that,

$M(Au, Su, kt) \geqslant M(Au, Su, t)$
and
$N(Au, Su, kt) \leqslant N(Au, Su, t)$

Therefore, by lemma 2.1, we have; $Au = Su$.

i.e. u is a coincidence point of A and S.

The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$. On the other hand, since $A(X) \subset S(X)$, there exist a point $v \in X$ such that $Au = Sv$. We claim that $Sv = Bv$.

i.e. v is a coincidence point of S and B.

Using (3.24), we have;

$[M(Au, Bv, kt)]^2 \geqslant [M(Au, Su, t) * M(Bv, Su, t)]^2$
and
$[N(Au, Bv, kt)]^2 \leqslant [N(Au, Su, t) \diamondsuit N(Bv, Su, t)]^2$

$[M(Au, Bv, kt)]^2 \geqslant [M(Au, Au, t) * M(Bv, Au, t)]^2$
and
$[N(Au, Bv, kt)]^2 \leqslant [N(Au, Au, t) \diamondsuit N(Bv, Au, t)]^2$

$[M(Au, Bv, kt)]^2 \geqslant [1 * M(Au, Bv, t)]^2$
and
$[N(Au, Bv, kt)]^2 \leqslant [0 \diamondsuit N(Au, Bv, t)]^2$

Thus, it follows that,

$M(Au, Bv, kt) \geqslant M(Au, Bv, t)$
and
$N(Au, Bv, kt) \leqslant N(Au, Bv, t)$

Therefore, by lemma 2.1, we have; $Au = Bv$

Thus $Au = Su = Sv = Bv$.

The weak compatibility of B and S implies $BSv = SBv$ and then$BBv = BSv = SBv = SSv$.

Let us show that Au is a common fixed point of A, B and S.

In view of (3.24), it follows that

$[M(AAu, Bv, kt)]^2 \geqslant [M(AAu, SAu, t) * M(Bv, SAu, t)]^2$
and
$[N(AAu, Bv, kt)]^2 \leqslant [N(AAu, SAu, t) \lozenge N(Bv, SAu, t)]^2$

$[M(AAu, Bv, kt)]^2 \geqslant [M(AAu, AAu, t) * M(Au, AAu, t)]^2$
and
$[N(AAu, Bv, kt)]^2 \leqslant [N(AAu, AAu, t) \lozenge N(Au, AAu, t)]^2$

$[M(AAu, Bv, kt)]^2 \geqslant [1 * M(Au, AAu, t)]^2$
and
$[N(AAu, Bv, kt)]^2 \leqslant [0 \lozenge N(Au, AAu, t)]^2$

That, it follows that,

$M(AAu, Au, kt) \geqslant M(Au, AAu, t)$
and
$N(AAu, Au, kt) \leqslant N(Au, AAu, t)$

Therefore, by lemma 2.1, we have; $AAu = Au$.

Thus $AAu = Au = SAu$ and Au is a common fixed point of A and S.

Similarly, we prove that Bv is a common fixed point of B and S.

Since $Au = Bv$, we conclude that Au is a common fixed point of A, B and S.

If $Az = Bz = Sz = z$ and $Aw = Bw = Sw = w$, then by (3.24), we have;

$[M(Az, Bw, kt)]^2 \geqslant [M(Az, Sz, t) * M(Bw, Sz, t)]^2$
and
$[N(Az, Bw, kt)]^2 \leqslant [N(Az, Sz, t) \lozenge N(Bw, Sz, t)]^2$

$[M(z, w, kt)]^2 \geqslant [M(z, z, t) * M(w, z, t)]^2$
and
$[N(z, w, kt)]^2 \leqslant [N(z, z, t) \lozenge N(w, z, t)]^2$

$[M(z, w, kt)]^2 \geqslant [1 * M(z, w, t)]^2$
and
$[N(z, w, kt)]^2 \leqslant [0 \lozenge N(z, w, t)]^2$

Thus, it follows that,

$M(z, w, kt) \geqslant M(z, w, t)$
and
$N(z, w, kt) \leqslant N(z, w, t)$

Therefore, by lemma 2.1, we have;$z = w$.

Hence the common fixed point is unique.

This completes the proof of the theorem.

□

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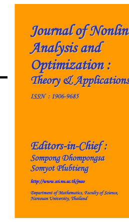
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 119-125
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



ON GENERALIZED VECTOR QUASI EQUILIBRIUM INEQUALITY PROBLEM VIA SCALARIZATION

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ABSTRACT. In this note, we consider the nonlinear scalarization function in topological vector spaces and present some properties of it and by using the nonlinear scalarization function we establish an existence theorem for a solution of a generalized quasi-vector equilibrium problems. Moreover, we show that under suitable conditions the solution set of the generalized quasi-vector equilibrium problem is compact.

KEYWORDS : Scalarization function; Topological vector space; Generalized quasi-vector equilibrium.
AMS Subject Classification: 90C47, 90C33.

1. INTRODUCTION AND PRELIMINARIES

Equilibrium problems have been extensively studied in recent years, the origin of which can be traced back to Blum and Oettli [1]. It is well known that vector equilibrium problems provide a unified model for several classes of problems, for examples, vector variational inequality problems, vector complementarity problems, vector optimization problems, and vector saddle point problems, see [1, 10] and the references therein. It is notable that vector variational inequality was first introduced and studied by Giannessi [8] in 1980.

Later on, vector variational inequality and its various extensions have been studied by Chen and Cheng [3], Chen, Huang and Yang [4] and other authors.

Recently, Chen et al. [5] introduced a nonlinear scalarization function for a variable domain structure and obtained several important properties , such as, the subadditivity and the continuity of it in the setting of a locally convex space.

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Article history : Received May 15, 2014, Accepted September 23, 2014.

Further, they applied their nonlinear scalarization function in order to study the existence of solutions for a generalized vector quasi-equilibrium problem (GVQEP, for short).

Inspired and motivated by the works mentioned above, we first consider nonlinear scalarization function for a variable domain structure and present some properties of it in the setting of topological vector spaces and then using the function we deal with an existence theorem for a solution of (GVQEP) without any continuity on the maps and show that the solution set of (GVQEP) is compact under our assumptions. It worth noting that the author of [2] introduced the weakest notion of continuity for a (scalar) equilibrium problem. The results presented in this paper generalized some corresponding results in the literature.

In the rest of this section we recall some definitions and preliminaries results which we need in the sequel.

Definition 1.1. Let X and Y be two topological spaces. A multi-valued map $T : X \longrightarrow 2^Y$ is :

(i) **upper semi-continuous** (u.s.c.) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $t \in U$, $T(t) \subseteq V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) **lower semi-continuous** (l.s.c.) at $x \in X$ if for each open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U$, $T(t) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) **continuous** on X if it is at the same time u.s.c. and l.s.c. on X.

(iv) **closed** if the graph $G_r(T)$ of T, i.e., $\{(x, y) : x \in X, y \in T(x)\}$, is a closed set in $X \times Y$.

(v) **compact** if the closure of range T, i.e., $\overline{T(X)}$, is compact, where $T(X) = \bigcup_{x \in X} T(x)$.

Lemma 1.2. ([11]) Let X and Y be topological spaces and $T : X \longrightarrow 2^Y$ be a multi-valued map. The following assertions are valid.

(i) T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$, and any net $\{x_\alpha\}$, $x_\alpha \longrightarrow x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \longrightarrow y$.

(ii) If for any $x \in X, T(x)$ is compact, then T is u.s.c. on X if and only if for any net $\{x_\alpha\}$ in X such that $x_\alpha \longrightarrow x$ $(x \in X)$ and for every $y_\alpha \in T(x_\alpha)$, there exist $y \in T(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$, such that $y_\beta \longrightarrow y$.

(iii) T is closed if and only if for any net x_α, $x_\alpha \longrightarrow x$, and any net y_α, $y_\alpha \in T(x_\alpha)$, with $y_\alpha \longrightarrow y$ one has $y \in T(x)$.

(iv) If T is closed and T is compact, then T is u.s.c..

(v) If T is u.s.c. and for each $x \in X, T(x)$ is closed set, then T is closed.

(vi) If X is compact and T is u.s.c. and for any $x \in X, T(x)$ is compact, then $T(X)$ is compact.

Theorem 1.3. ([9]) (Kakutani-Fan-Glicksberg Fixed point Theorem) Let S be a non-empty, compact and convex subset of a locally convex topological vector space. Let $F : S \longrightarrow 2^S$ be a mapping with nonempty compact and convex values. Then F has a fixed point.

2. MAIN RESULTS

In this section we first introduce a nonlinear scalarization function for a moving cone in the setting of topological vector spaces (t.v.s. for short) and then some important properties of the nonlinear scalarization function will be presented. Finally, we establish an existence theorem for a solution of (GQEP) and we show that under our assumptions the solution set (GQEP) is compact.

Let E be a topological vector space with its zero vector θ and the topological dual space E^*. By a cone $P \neq \{\theta\}$ we understand a closed convex subset of E such that $\lambda P \subseteq P$ for all $\lambda \geq 0$ and $P \cap -P = \{\theta\}$. Given a cone $P \subseteq E$, we define a *partial ordering* $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ if P has nonempty interior.

Definition 2.1. Let E be a t.v.s. and $C : E \longrightarrow 2^E$ a multi-valued map and for all $x \in E$, $C(x)$ be a solid cone (that is, $intC(x)$ is nonempty). Let $e : E \longrightarrow E$ be a map with $e(x) \in C(x)$ for all $x \in E$. The nonlinear scalarization function $\xi : E \times E \longrightarrow \mathbb{R}$ is defined as follows:

$$\xi(x, y) = \inf\{r \in \mathbb{R} : y \in re(x) - C(x)\}.$$

In the following we establish some important properties of the nonlinear scalarization function which generalize Propositions 2.3 and 2.4 in [5] from locally convex spaces to topological vector spaces by presenting a new proof. Moreover if we take $C(x) = P$ (P is a cone with $e \in intP$) and define $e(x) = e$ for all $x \in E$, then it collapses to the corresponding result given in [7]. It is worthwhile to note that, we will not assume $int(\bigcap_{x\in X} C(x)) \neq \emptyset$ as considered in [6]. The condition $int(\bigcap_{x\in X} C(x)) \neq \emptyset$ is too restrictive even for finite dimensional t.v.s., for example, take $X = \mathbb{R}$ and consider $C(x) = [0, \infty)$ for $x \in Q$(the rational numbers) and $C(x) = (-\infty, 0]$ for $x \in Q^c$(the irrational numbers) then $int(\bigcap_{x\in X} C(x)) = \emptyset$.

Proposition 2.2. *Let E be a t.v.s., $C : E \longrightarrow 2^E$ a multi-valued map and for all $x \in E$, $C(x)$ be a solid cone. Let $e : E \longrightarrow E$ be a function with $e(x) \in C(x)$, for all $x \in E$. Then the following assertions, for each $r \in \mathbb{R}$ and $y \in E$, are satisfied.*

(i) $\xi(x, y) = \inf\{r \in \mathbb{R} : y \in re(x) - C(x)\} = \min\{r \in \mathbb{R} : y \in re(x) - C(x)\}$;
(ii) $\xi(x, y) \leq r \iff y \in re(x) - C(x)$;
(iii) $\xi(x, y) < r \iff y \in re(x) - \text{int}C(x)$;
(iv) *If* $y_1 \preceq y_2$, *then* $\xi(x, y_1) \leq \xi(x, y_2)$;
(v) *For each fixed $x \in E$ the function $z \longrightarrow \xi(x, z)$ is continuous, positively homogeneous and subadditive on E;*
(vi) *For each fixed $x \in E$ the function $z \longrightarrow \xi(x, z)$ is bounded on some neighborhood of zero.*

Proof. (i) It is obvious from $e(x) \in intC(x)$ that , for each $y \in E$, the set

$$A(y) = \{r \in \mathbb{R} : y \in re(x) - C(x)\}$$

is nonempty. The nonlinear scalarization function ξ is well-defined, otherwise there exist a $y \in E$ and a sequence $\{r_n\}$ of real numbers such that

$$y \in r_n e(x) - C(x) \text{ and } r_n \longrightarrow -\infty.$$

Hence
$$\frac{1}{-r_n}y \in e(x) + C(x),$$
and so
$$-e(x) \in intC(x).$$
Thus
$$e(x) \in C(x) \cap -C(x) = \{\theta\},$$
and hence $\theta = e(x) \in intC(x)$ and so $C(x) = E$ which is contradicted by $C(x) \cap -C(x) = \{\theta\}$. Finally the infimum is attainable for each $y \in E$. Indeed, by the property of the infimum there exists a sequence $\{r_n\}$ of real numbers such that $r_n \longrightarrow \xi(x,y)$ with $y \in r_n e(x) - C(x)$. Hence it follows from the continuities of the scalar multiplication and the vector addition defined on E and the closedness of $C(x)$ that $\xi(x,y)e(x) - y \in C(x)$.

To see (ii), if $y \notin re(x) - C(x)$ then assume $r < \xi(x,y)$. The converse is obvious by the definition of $\xi(x,y)$.

To verify (iii) let $y \in re(x) - \text{int}C(x)$. Since $re(x) - \text{int}C(x)$ is an open set and the sequence $\{y + \frac{1}{n}e(x)\}$ is convergent to y, there exists a natural number n_0 such that
$$y + \frac{1}{n}e(x) \in re(x) - \text{int}C(x), \quad \forall n > n_0.$$
Then
$$y \in (r - \frac{1}{n})e(x) - \text{int}C(x), \quad \forall n > n_0,$$
and so for all $n > n_0$ we have
$$\xi(x,y) \leq r - \frac{1}{n} < r.$$
Conversely, if $\xi(x,y) < r$ and $re(x) - y \in C(x)\backslash intC(x)$, then
$$ne(x) - y \in C(x)\backslash intC(x),\ \forall n \in N,\ n > r,$$
and so
$$e(x) - \frac{1}{n}y = \frac{1}{n}(ne(x) - y) \in C(x)\backslash intC(x),\ \forall n \in N,\ n > r.$$
Hence $e(x) \in C(x)\backslash intC(x)$, which is a contradiction. This completes the proof of (iii).

To prove (iv), let $y_1 \preceq y_2$. If $re(x) - y_2 \in C(x)$, then it follows from $y_1 \preceq y_2$ that $re(x) - y_1 \in C(x)$ and so the proof of (iv) is finished. It follows from (i) and (ii) that, for all $r \in \mathbb{R}$,
$$\xi^{-1}(x,.)(r,\infty) = \{z \in E : \xi(x,z) \in (r,\infty)\} = (re(x) - C(x))^c,$$
and similarly
$$\xi^{-1}(x,.)(-\infty,r) = re(x) - \text{int}C(x),$$
are open sets and then the function $z \longrightarrow \xi(x,z)$ is continuous. Also if t is a positive real number and $y \in E$, then
$$\xi(x,ty) = \inf\{r \in \mathbb{R} : ty \in re(x) - C(x)\} = \inf\{r \in \mathbb{R} : y \in \frac{r}{t}e(x) - C(x)\}$$
$$= t\inf\{\frac{r}{t} \in \mathbb{R} : y \in \frac{r}{t}e(x) - C(x)\} = t\inf\{r^{'} \in \mathbb{R} : x \in r^{'}e(x) - C(x)\} = t\xi(x,y),$$
and so the function $z \longrightarrow \xi(x,z)$ is positively homogenous. If $y,z \in E$ with $y \in re(x) - C(x)$ and $z \in se(x) - C(x)$ then $y + z \in (r+s)e(x) - C(x)$ and so
$$\xi(x,y+z) \leq \xi(x,y) + \xi(x,z).$$

Hence the function $z \longrightarrow \xi(x,z)$ is subadditive and so the proof of (v) is completed. Finally, if we take $r = 1$ in (ii) then $\xi(x,y) \leq 1$ for all $y \in e(x) - C(x)$ and especially for some symmetric neighborhood U of zero (note that $e(x) \in intC(x)$). Hence, for each $y \in U$, by (v), we have $-\xi(x,-y) \leq \xi(x,y) \leq 1$ and so $\xi(x,-y) \geq -1$ and hence since $U = -U$ we obtain $|\xi(x,y)| \leq 1$ for each $y \in U$, and so the proof completes. □

Theorem 2.3. *Let E be a topological vector space and K be a closed subset of E. Let $C : K \longrightarrow 2^E$ be a multi-valued map such that, for each $x \in K$, $C(x)$ is a solid convex cone, and let $e : K \longrightarrow E$ be a continuous map with $e(x) \in intC(x)$ for all $x \in E$. Define $W : K \longrightarrow 2^E$ by $W(x) = E \setminus intC(x)$, for all $x \in K$.*

(i) If the multi-valued map W is closed, then the function $(x,y) \longrightarrow \xi(x,y)$ is upper semicontinuous on $K \times K$.
(ii) If the multi-valued map C is closed, then the function $(x,y) \longrightarrow \xi(x,y)$ is lower semicontinuous on $K \times K$.

Proof. (i) Let λ be an arbitrary real number. We shall prove that

$$A(\lambda) = \{(x,y) \in K \times K : \xi(x,y) \geq \lambda\}$$

is closed.

For the purpose, let $\{(x_\alpha, z_\alpha)\}_{\alpha \in I} \in A(\lambda)$ be a net and $(x_\alpha, z_\alpha) \longrightarrow (x,z)$.

Since $(x_\alpha, z_\alpha) \in A(\lambda)$, it follows from Proposition 2.2 (iii) that

$$z_\alpha \notin \lambda e(x_\alpha) - intC(x_\alpha), \quad \forall \alpha \in I,$$

and so

$$z_\alpha \in \lambda e(x_\alpha) - W(x_\alpha), \quad \forall \alpha \in I.$$

This means

$$\lambda e(x_\alpha) - z_\alpha \in W(x_\alpha), \quad \forall \alpha \in I.$$

Since W has a closed graph, the map $x \longrightarrow e(x)$ is continuous and $(x_\alpha, z_\alpha) \longrightarrow (x,z)$, we get $\lambda e(x) - z \in W(x)$ and so $\lambda e(x) - z \notin intC(x)$. Hence it follows from Proposition 2.2 (iii) that $\xi(x,z) \geq \lambda$, which shows that $A(\lambda)$ is closed.

(ii) Let λ be an arbitrary real number. We show that

$$B(\lambda) = \{(x,y) \in K \times K : \xi(x,y) \leq \lambda\}$$

is closed. For that purpose, we use a similar way as given for (i). Let $\{(x_\alpha, z_\alpha)\}_{\alpha \in I} \in B(\lambda)$ be a net and $(x_\alpha, z_\alpha) \longrightarrow (x,z)$. Now the result follows from Proposition 2.2(ii), continuity of the map e and being closed of the graph $G_r(C)$ of C. □

We need the following definition and lemma in the sequel.

Definition 2.4. ([10]) Let X and Y be linear spaces and $C \subseteq Y$ a convex cone of Y. The function $f : X \longrightarrow Y$ is called $C-$ quasi-convex if for any $y \in Y$ the set

$$\{x \in X : f(x) \in y - C\},$$

is a convex subset of X.

Lemma 2.5. *([5]) Let E, Z and X be topological vector spaces. Let $C : X \longrightarrow 2^X$ be a multi-valued map so that for every $x \in X, C(x)$ is a proper, closed and convex cone with a nonempty interior $intC(x)$. Assume that $intC(.)$ has a continuous selection $e(.)$. $Y \subset E$ and $D \subset Z$ be nonempty convex sets. Let $g : E \longrightarrow X$ and $f : Y \times D \times Y \longrightarrow X$ be two functions. If, for each $y \in Y$, $z \in Z$, the function*

$v \longrightarrow f(y,z,v)$ is $C(g(y))-$ quasi-convex, then the function $v \longrightarrow \xi(g(y), f(y,z,v))$ is $\mathbb{R}+-$ quasi-convex.

Now, we are ready, by using the nonlinear scalarization function, present the first existence theorem of (GVQEP) without any assumption of monotonicity and semi-continuity on the multi-valued functions.

Theorem 2.6. *Let E, Z and X be locally convex topological vector spaces. Let $C : X \longrightarrow 2^X$ be a multi-valued map so that for every $x \in X, C(x)$ is a proper, closed and convex cone with a nonempty interior $intC(x)$. Assume that $intC(.)$ has a continuous selection $e(.)$. Define a multi-valued map $W : X \longrightarrow 2^X$ by $W(x) = X\backslash intC(x)$, for $x \in X$. Let $Y \subset E$ and $D \subset Z$ be nonempty compact convex sets. Let $Q : Y \longrightarrow 2^Y$ be a lower semi-continuous function and $V : Y \longrightarrow 2^D$ be multi-valued function. Let $g : E \longrightarrow X$ and $f : Y \times D \times Y \longrightarrow X$ be two functions. Suppose all the following conditions are satisfied*

(i)The multi-valued functions C, W, V and Q are closed ;
(ii)f and g are continuous on $Y \times D \times Y$ and E, respectively;
(iii) For each $y \in Y$ and $z \in D$ the function $v \in f(y,z,v)$ is $C(g(y))-$ quasi-convex;
(iv) For each $y \in Y$, $V(y)$ $(Q(y)$, respectively) is a nonempty and convex subset of D (Y, respectively).

Then, there exists $\overline{y} \in Q(\overline{y})$ and $\overline{z} \in V(\overline{y})$ such that

$$f(\overline{y},\overline{z},\overline{y}) - f(\overline{y},\overline{z},y) \notin int\ C(g(\overline{y})), \quad \forall y \in Q(\overline{y}).$$

Furthermore the solution set of (GVQEP) is a closed subset of Y and so compact.

Proof. Define $F : Y \times D \longrightarrow 2^Y$ by

$$F(y,z) = \{u \in Q(y) : \xi(g(y), f(y,z,u)) = \min_{v \in Q(y)} \xi(g(y), f(y,z,v))\},$$

where ξ is the mapping given in Lemma 2.5. It follows from lemma 2.5 and (iii) that, for each $(y,z) \in Y \times D$, the set $F(y,z)$ is convex. Moreover F is closed. Indeed, let $u_\alpha \in F(y_\alpha, z_\alpha)$ and $(u_\alpha, y_\alpha, z_\alpha) \longrightarrow (u,y,z)$ we shall show that $u \in F(y,z)$. For that purpose, let $w \in Q(y)$. Since $y_\alpha \longrightarrow y$ and Q is lower semi-continuous it follows from Lemma 1.2(i) that there is a net $w_\alpha \in Q(y_\alpha)$ with $w_\alpha \longrightarrow w$ and then using the definition of the multi-valued function F and $u_\alpha \in F(y_\alpha, z_\alpha)$ we deduce

$$\xi(g(y_\alpha), f(y_\alpha, z_\alpha, u_\alpha)) \leq \xi(g(y_\alpha), f(y_\alpha, z_\alpha, w_\alpha)),$$

and by taking limit the inequality (note ξ, g, f are continuous and $(y_\alpha, z_\alpha, w_\alpha)$ is a convergent net) and applying our assumptions

$$\xi(g(y), f(y,z,u)) \leq \xi(g(y), f(y,z,w))$$

and since $w \in Q(y)$ was an arbitrary element we get

$$\xi(g(y), f(y,z,u)) = \min_{w \in Q(y)} \xi(g(y), f(y,z,v))$$

and hence $u \in F(y,z)$. So F has a closed graph and so F is upper semi-continuous, note that F has a closed graph and the values of F are closed subsets of the compact set Y (see Lemma 1.2 (iv)).

Now we define $W : Y \times D \longrightarrow 2^{Y \times D}$, for each $(y,z) \in Y \times D$ by

$$W(y,z) = F(y,z) \times V(y).$$

Since F and V are upper semi-continuous, note that V has closed graph with closed values of the compact set Y, then W is an upper semi-continuous function. Further it has closed convex values, since F and V have closed convex values.

Hence W fulfils all the assumptions of Theorem 1.3 and so there is $(\overline{y},\overline{z}) \in W(\overline{y},\overline{z})$ and then $\overline{z} \in V(\overline{y})$ and $\overline{y} \in F(\overline{y},\overline{z})$. So

$$\xi(g(\overline{y}), f(\overline{y},\overline{z},\overline{y})) = \min_{w\in Q(\overline{y})} \xi(g(\overline{y}), f(\overline{y},\overline{z},v)).$$

Then

$$\xi(g(\overline{y}), f(\overline{y},\overline{z},\overline{y})) \leq \xi(g(\overline{y}), f(\overline{y},\overline{z},v)), \quad \forall v \in Q(\overline{y}),$$

and so it follows from Proposition 2.2 (v)(in fact ξ is subadditive) that

$$0 \leq \xi(g(\overline{y}), f(\overline{y},\overline{z},\overline{y})) - \xi(g(\overline{y}), f(\overline{y},\overline{z},v)) \leq \xi(g(\overline{y}), f(\overline{y},\overline{z},v) - (g(\overline{y}), f(\overline{y},\overline{z},v)), \quad \forall v \in Q(\overline{y})$$

and so it follows from Proposition 2.2 (iii), by taking $r = 0$, that

$$f(\overline{y},\overline{z},\overline{y}) - f(\overline{y},\overline{z},v) \notin -intC(g(\overline{y})), \quad \forall v \in Q(\overline{y}).$$

Since f, g are continuous, W closed, Q lower semi-continuous through Lemma 1.2 one can see that the solution set (GQEP) is closed and so compact (note the solution set of (GQEP) is a subset of the compact set Y). This completes the proof. $\square$

Acknowledgement

This research was supported by Islamic Azad University, Kermanshah branch and the authors gratefully acknowledge the Islamic Azad University, Kermanshah branch for financial support.

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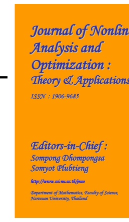
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 5, No. 2, (2014), 127-137
ISSN : 1906-9685 http://
www.math.sci.nu.ac.th



ANOTHER HYBRID CONJUGATE GRADIENT METHOD FOR UNCONSTRAINED OPTIMIZATION PROBLEMS

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ABSTRACT. Conjugate gradient method is one of the most useful method for solving large scale unconstrained optimization problems. In this article a new hybrid conjugate gradient method that satisfies the descent condition independently of the line searches is proposed. In particular, it is a hybrid of the Fletcher-Reeves (β_k^{FR}) and Polak-Ribiere-Polyak (β_k^{PRP}) methods. Convergence analysis of the new method is presented. Numerical results of the method show that the proposed hybrid algorithm is just as competitive.

KEYWORDS : hybrid Conjugate Gradient, line search, Convergence analysis.
AMS Subject Classification: 90C30, 90C06, 65K05

1. INTRODUCTION

Conjugate gradient methods (CG) are very useful in finding the optimal solution to the unconstrained optimization problem

$$\min\{f(x) : x \in \mathbb{R}^n\}, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function and is continuously differentiable. Conjugate gradient methods are the most preferred methods for solving large scale unconstrained problems because, unlike Newton and Quasi-Newton methods [4, 13, 23], they only need the first derivatives and hence less storage capacity is needed. They are also relatively simple to program.

Given an initial guess $x_0 \in \mathbb{R}^n$, the CG method generates a sequence $\{x_k\}$ for problem (1.1) as

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k = 0, 1, 2, \dots, \tag{1.2}$$

where α_k is a step length which is determined by a line search and d_k is a descent direction of f at x_k. The step length α_k is obtained by carrying out an exact or

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Article history : Received November 12, 2013 Accepted January 28, 2015.

inexact one dimensional line search. If exact line search is used, then α_k is such that

$$f(x_k + \alpha_k d_k) = \min_{\alpha \geq 0} f(x_k + \alpha d_k). \tag{1.3}$$

As for inexact line searches, we have the Amirjo condition [4, 23], which requires α_k to satisfy

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \mu \alpha_k \nabla f(x_k)^T d_k, \tag{1.4}$$

and the standard Wolfe conditions [4, 23], which require α_k to satisfy (1.4) and the curvature condition

$$\nabla f(x_k + \alpha_k d_k)^T d_k \geq \sigma \nabla f(x_k)^T d_k, \tag{1.5}$$

where $0 < \mu < \sigma < 1$. Strong Wolfe conditions have also been used in a number of papers and are given by (1.4) and

$$|\nabla f(x_k + \alpha_k d_k)^T d_k| \leq -\sigma \nabla f(x_k)^T d_k, \tag{1.6}$$

again with $0 < \mu < \sigma < 1$. The search direction d_k for CG methods is generated as

$$d_k = \begin{cases} -g_k, & \text{if} \quad k = 0 \\ -g_k + \beta_k d_{k-1}, & \text{if} \quad k \geq 1, \end{cases} \tag{1.7}$$

where $g_k = \nabla f(x_k)$ is the gradient of f at x_k and β_k is a scalar, known as the conjugate gradient coefficient. Different choices of the conjugate gradient coefficient β_k lead to different CG methods. Some of the well-known CG methods include the Hestenes-Stiefel (β_k^{HS}) method [3, 8, 18], the Polak-Ribière-Polyak (β_k^{PRP}) method [11, 15, 16, 23, 24, 26], the Fletcher-Reeves (β_k^{FR}) method [11, 13, 14, 23, 25, 28], the Liu-Storey (β_k^{LS}) method [3, 19], the conjugate descent (β_k^{CD}) method [3, 13] and the Dai-Yuan (β_k^{DY}) method[6, 8, 10, 11]. The conjugate gradient coefficient β_k for these mentioned CG methods are, respectively,

$$\beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T(g_k - g_{k-1})}, \tag{1.8}$$

$$\beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{||g_{k-1}||^2}, \tag{1.9}$$

$$\beta_k^{FR} = \frac{g_k^T g_k}{||g_{k-1}||^2}, \tag{1.10}$$

$$\beta_k^{LS} = -\frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T g_{k-1}}, \tag{1.11}$$

$$\beta_k^{CD} = -\frac{g_k^T g_k}{d_{k-1}^T g_{k-1}}, \tag{1.12}$$

$$\beta_k^{DY} = \frac{g_k^T g_k}{d_{k-1}^T g_{k-1}}, \tag{1.13}$$

where $|| \cdot ||$ represent the norm of vectors.

It has been shown in the literature that although the above formulae are equivalent for the quadratic functions, their performance strongly depends on the coefficient β_k. The CG methods β_k^{FR}, β_k^{CD} and β_k^{DY} possess strong global convergence properties [1, 5, 8, 7, 17, 23], but have less computational performance. On the other hand, the β_k^{PRP}, β_k^{HS} and β_k^{LS} methods have been shown that although they may not always converge, they often offer better computational performance [5, 15, 16, 17, 23].

In this paper, we suggest another approach to get a hybrid conjugate gradient method that combines the strengths of β_k^{PRP} and β_k^{FR} methods. This proposed method is presented in section 2. In Section 3 we present the convergence analysis of the new algorithm. Section 4 presents some numerical experiments and conclusion is given in Section 5.

2. New algorithm

In this section a hybrid of the β_k^{PRP} and β_k^{FR} methods is presented. As already mentioned, β_k^{FR} method has an attractive property as far as convergence is concerned. Its strength, that is, the global convergence property usually happens under strong Wolfe conditions. On the other hand, the β_k^{PRP} method has good computational properties and often performs better compared to other conjugate gradient methods. This method has been proved that when the function is strongly convex and the line search is exact, then the method is globally convergent. However, for general nonlinear functions, the convergence of the β_k^{PRP} method is uncertain. It appeared, after several failed attempts to prove global convergence of the β_k^{PRP} algorithm, that positiveness of β_k is crucial as far as convergence is concerned. This lead Gilbert and Nocedal [15] to modify β_k^{PRP} method as

$$\beta_k^{PRP+} = \max\{0, \beta_k^{PRP}\}$$

and proved that it is globally convergent with the standard Wolfe conditions.

There are a number of other β_k^{PRP} and β_k^{FR} hybrid conjugate gradient methods that have been proposed in the literature. One of the first hybrid conjugate gradient method of this form was introduced by Touati-Ahmed and Storey [27] where the parameter β_k is computed as

$$\beta_k^{TS} = \max\{0, \min\{\beta_k^{PRP}, \beta_k^{FR}\}\}.$$

This motivated other researchers to come up with more and improved β_k hybrids involving β_k^{PRP} and β_k^{FR}. For instance, Mo, Gu and Wei [21] proposed a β_k method which is a modification of the hybrid method proposed by Touati-Ahmed and Storey [27]. Their hybrid method computes β_k as

$$\beta_k^{MGW} = \max\{0, \min\{\beta_k^{PRP}, \beta_k^{FR}, \beta_k^{+}\}\},$$

where

$$\beta_k^{+} = \beta_k^{PRP} + \frac{2g_k^T g_{k-1}}{\| g_{k-1} \|^2}.$$

They proved that their hybrid method is globally convergent when the step size satisfies the strong Wolfe conditions. Another hybrid was that of Gilbert and Nocedal [15] who suggested a combination between the β_k^{PRP} and β_k^{FR} method as

$$\beta_k^{GN} = \max\{-\beta_k^{FR}, \min\{\beta_k^{PRP}, \beta_k^{FR}\}\}, \tag{2.1}$$

which is an extension of the Touati-Ahmed and Storey [27] method.

In this article, we propose yet another hybrid of β_k^{PRP} and β_k^{FR} that is based on the ideas of Gilbert and Nocedal [15], where their β_k given by (2.1), and those of Dai and Yuan [8], where they suggested the hybrid method

$$\beta_k^{HS-DY} = \max\{-c\beta_k^{DY}, \min\{\beta_k^{HS}, \beta_k^{DY}\}\}, \tag{2.2}$$

where $c = \frac{1-\gamma}{1+\gamma} > 0$. In particular, we propose a hybrid method which computes the parameter β_k as

$$\beta_k^{*} = \max\{\min\{-c\beta_k^{PRP}, \beta_k^{FR}\}, \min\{\beta_k^{FR}, \beta_k^{PRP}\}\}, \tag{2.3}$$

with $c = \frac{1-\gamma}{1+\gamma}$, $\gamma \in [\frac{1}{2}, 1]$ and the direction d_k defined as

$$d_k = \begin{cases} -g_k & k = 0 \\ -\theta_k g_k + \beta_k^* d_{k-1} & k \geq 1 \end{cases} \tag{2.4}$$

where $\theta_k = 1 + \beta_k^* \frac{d_{k-1}^T g_k}{\|g_k\|^2}$. The parameter θ_k, as defined, makes the direction d_k satisfy the descent condition independently of any line search. Also, from the above definition of β_k^* and the range of γ, we see that $0 < \beta_k^* \leq \beta_k^{FR}$ for all k. Now, with β_k^* and d_k defined as above, we present our new β_k^* algorithm.

Algorithm 2.1. The New β_k^* algorithm

Step 1 Given $x_0 \in \mathbb{R}^n$ and the parameters $\epsilon > 0$, $0 < \mu < \sigma < 1$, $\gamma \in [\frac{1}{2}, 1]$
set $k = 0$
compute $f(x_0)$ and $g_0 = \nabla f(x_0)$
set $d_0 = -g_0$,
if $\| g_0 \| \leq \epsilon$ then stop.

Step 2 Compute $\alpha_k > 0$ using any line search and find the next iterate

$$x_{k+1} = x_k + \alpha_k d_k.$$

compute $f(x_{k+1})$, $g_{k+1} = \nabla f(x_{k+1})$
if $||g_{k+1}|| \leq \epsilon$ then stop.

Step 3 compute β_k^* from (2.3) and generate d_k from (2.4)

Step 4 let $k = k + 1$ and go to step 2.

3. Convergence analysis

To establish the convergence of our method, we make the following basic assumptions on the objective function which have been widely used in the literature to analyze the global convergence of conjugate gradient methods.

Assumptions

(i) f is bounded below on the level set $S = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$, where x_0 is the starting point.

(ii) In some neighborhood N of S the function f is continuously differentiable and its gradient, $g(x) = \nabla f(x)$, is Lipschitz continuous, i.e. there exist a constant $L > 0$ such that $\| g(x) - g(y) \| \leq L \| x - y \|$ for all $x, y \in N$

Under Assumptions (i) and (ii) on f, we have the following lemma.

Lemma 3.1. *(**Zoutendijk**). Suppose that Assumptions (i) and (ii) hold. Consider a CG method in the form $x_{k+1} = x_k + \alpha_k d_k$ and (1.7), where d_k is a descent direction and the step length α_k satisfies the standard Wolfe conditions (1.4) and (1.5). Then we have that*

$$\sum_{k=0}^{\infty} \frac{(\nabla f(x_k)^T d_k)^2}{\| d_k \|^2} < +\infty. \tag{3.1}$$

Proof. From the Lipschitz continuity and (1.5) we have that

$$(\sigma - 1) d_k^T \nabla f(x_k) \leq d_k^T (\nabla f(x_k + \alpha_k d_k) - \nabla f(x_k)) \tag{3.2}$$

$$\leq \| \nabla f(x_k + \alpha_k d_k) - \nabla f(x_k) \| \| d_k \| \tag{3.3}$$

$$= L\alpha_k \| d_k \|^2 \tag{3.4}$$

Thus,

$$\alpha_k \geq \frac{(\sigma - 1) d_k^T \nabla f(x_k)}{L \| d_k \|^2}. \tag{3.5}$$

It follows from (1.4) that

$$f(x_k) - f(x_k + \alpha_k d_k) \geq -\mu \left(\frac{(\sigma - 1) d_k^T \nabla f(x_k)}{L \parallel d_k \parallel^2} \right) \nabla f(x_k)^T d_k, \tag{3.6}$$

which implies

$$f(x_k) - f(x_k + \alpha_k d_k) \geq C_1 \frac{(\nabla f(x_k)^T d_k)^2}{\parallel d_k \parallel^2}, \tag{3.7}$$

where $C_1 = \frac{\mu(1-\sigma)}{L} > 0$. Now,

$$f(x_0) - f(x_0 + \alpha d_0) \geq C_1 \frac{(\nabla f(x_0)^T d_0)^2}{\parallel d_0 \parallel^2} \tag{3.8}$$

$$f(x_1) - f(x_2) \geq C_1 \frac{(\nabla f(x_1)^T d_1)^2}{\parallel d_1 \parallel^2} \tag{3.9}$$

$$f(x_2) - f(x_3) \geq C_1 \frac{(\nabla f(x_2)^T d_2)^2}{\parallel d_2 \parallel^2} \tag{3.10}$$

$$\vdots \tag{3.11}$$

$$\tag{3.12}$$

$$f(x_{k-1}) - f(x_k) \geq C_1 \frac{(\nabla f(x_{k-1})^T d_{k-1})^2}{\parallel d_{k-1} \parallel^2} \tag{3.13}$$

Adding up we get

$$f(x_0) - f(x_k) \geq C_1 \sum_{i=0}^{k-1} \frac{(\nabla f(x_i)^T d_i)^2}{\parallel d_i \parallel^2} \tag{3.14}$$

Noting that f is bounded from below as $k \to \infty$, we have

$$f(x_0) - f^* \geq C_1 \sum_{k=0}^{\infty} \frac{(\nabla f(x_k)^T d_k)^2}{\parallel d_k \parallel^2}, \tag{3.15}$$

where

$$f^* = \lim_{k \to \infty} f(x_k).$$

Hence

$$\sum_{k=0}^{\infty} \frac{(\nabla f(x_k)^T d_k)^2}{\parallel d_k \parallel^2} < +\infty. \tag{3.16}$$

□

Lemma 3.2. *Let $x_{k+1} = x_k + \alpha_k d_k$ be given by Algorithm (2.1). Then the direction d_k given by (2.4) satisfies the descent condition*

$$d_k^T g_k = - \parallel g_k \parallel^2, \qquad \forall k \geq 0. \tag{3.17}$$

Proof. Let $\beta_k = \beta_k^*$. For $d_0 = -g_0$, we have

$$g_0^T d_0 = -g_0^T g_0 \tag{3.18}$$

$$= - \parallel g_0 \parallel^2 . \tag{3.19}$$

Therefore the result holds for $k = 0$.
For $k \geq 1$, we have that

$$d_k = -\theta_k g_k + \beta_k^* d_{k-1}. \tag{3.20}$$

Now, for $\beta_k = \beta_k^*$, we have

$$d_k = -(1 + \beta_k^* \frac{g_k^T d_{k-1}}{\| g_k \|^2}) g_k + \beta_k^* d_{k-1} \tag{3.21}$$

$$= \beta_k^* d_{k-1} - (1 + \beta_k^* \frac{g_k^T d_{k-1}}{\| g_k \|^2}) g_k. \tag{3.22}$$

Multiplying both sides by g_k^T we get

$$g_k^T d_k = \beta_k^* g_k^T d_{k-1} - (1 + \beta_k^* \frac{g_k^T d_{k-1}}{\| g_k \|^2}) g_k^T g_k \tag{3.23}$$

$$= \beta_k^* g_k^T d_{k-1} - \| g_k \|^2 - \beta_k^* \frac{g_k^T d_{k-1}}{\| g_k \|^2} \| g_k \|^2 \tag{3.24}$$

$$\Rightarrow \; g_k^T d_k = - \| g_k \|^2 . \tag{3.25}$$

Thus (3.17) holds for all $k \geq 1$, which concludes the proof. □

Theorem 3.3. *Suppose that Assumptions (i) and (ii) hold. Consider the conjugate gradient method of the form $x_{k+1} = x_k + \alpha_k d_k$ and d_k is given by (2.4) with α_k satisfying any line search. Then either $g_k = 0$ for some k or*

$$\lim_{k \to \infty} \inf \| g_k \| = 0. \tag{3.26}$$

Proof. If $g_k = 0$ then the statement holds. Suppose that (3.26) is not true, then there exist a constant $\varepsilon > 0$ such that

$$\| g_k \| \geq \varepsilon \; \forall k. \tag{3.27}$$

From (2.4), we have

$$d_k + \theta_k g_k = \beta_k^* d_{k-1} \tag{3.28}$$

By squaring both sides of (3.28) and applying the descent condition (3.17), we get

$$\| d_k \|^2 = (\beta_k^*)^2 \| d_{k-1} \|^2 - 2\theta_k d_k^T g_k - \theta_k^2 \| g_k \|^2 .$$

Dividing both sides by $(g_k^T d_k)^2$, and noting that $g_k^T d_k = - \| g_k \|^2$, we have

$$\frac{\| d_k \|^2}{(g_k^T d_k)^2} = (\beta_k^*)^2 \frac{\| d_{k-1} \|^2}{(g_k^T d_k)^2} + \frac{2\theta_k}{\| g_k \|^2} - \frac{\theta_k^2}{\| g_k \|^2}. \tag{3.29}$$

Since $0 < \beta_k^* \leq \beta_k^{FR}$, we have that

$$\frac{\| d_k \|^2}{(g_k^T d_k)^2} \leq (\beta_k^{FR})^2 \frac{\| d_{k-1} \|^2}{\| g_k \|^4} + \frac{2\theta_k}{\| g_k \|^2} - \frac{\theta_k^2}{\| g_k \|^2} \tag{3.30}$$

$$= \left(\frac{\| g_k \|^2}{\| g_{k-1} \|^2} \right)^2 \frac{\| d_{k-1} \|^2}{\| g_k \|^4} + \frac{2\theta_k}{\| g_k \|^2} - \frac{\theta_k^2}{\| g_k \|^2} \tag{3.31}$$

$$= \frac{\| d_{k-1} \|^2}{\| g_{k-1} \|^4} + \frac{2\theta_k}{\| g_k \|^2} - \frac{\theta_k^2}{\| g_k \|^2} \tag{3.32}$$

$$= \frac{\| d_{k-1} \|^2}{\| g_{k-1} \|^4} - \frac{1}{\| g_k \|^2} (\theta_k^2 - 2\theta_k + 1 - 1) \tag{3.33}$$

$$= \frac{\| d_{k-1} \|^2}{\| g_{k-1} \|^4} - \frac{(\theta_k - 1)^2}{\| g_k \|^2} + \frac{1}{\| g_k \|^2} \tag{3.34}$$

$$\leq \frac{\| d_{k-1} \|^2}{\| g_{k-1} \|^4} + \frac{1}{\| g_k \|^2} \tag{3.35}$$

$$= \frac{\| d_{k-1} \|^2}{(g_{k-1}^T d_{k-1})^2} + \frac{1}{\| g_k \|^2}. \tag{3.36}$$

From the above, and the fact that $g_0^T d_0 = - \| g_0 \|^2$, it follows that

$$\frac{\| d_1 \|^2}{(g_1^T d_1)^2} \leq \frac{\| d_0 \|^2}{(g_0^T d_0)^2} + \frac{1}{\| g_1 \|^2} \tag{3.37}$$

$$= \frac{1}{\| g_0 \|^2} + \frac{1}{\| g_1 \|^2} \tag{3.38}$$

$$= \sum_{i=0}^{1} \frac{1}{\| g_i \|^2}. \tag{3.39}$$

$$\frac{\| d_2 \|^2}{(g_2^T d_2)^2} \leq \frac{\| d_1 \|^2}{(g_1^T d_1)^2} + \frac{1}{\| g_2 \|^2} \tag{3.40}$$

$$\leq \frac{1}{\| g_0 \|^2} + \frac{1}{\| g_1 \|^2} + \frac{1}{\| g_2 \|^2} \tag{3.41}$$

$$= \sum_{i=0}^{2} \frac{1}{\| g_i \|^2}. \tag{3.42}$$

$$\vdots \tag{3.43}$$

$$\frac{\| d_k \|^2}{(g_k^T d_k)^2} \leq \frac{\| d_{k-1} \|^2}{(g_{k-1}^T d_{k-1})^2} + \frac{1}{\| g_k \|^2} \tag{3.44}$$

$$\leq \frac{1}{\| g_0 \|^2} + \frac{1}{\| g_1 \|^2} + \frac{1}{\| g_2 \|^2} + \cdots + \frac{1}{\| g_k \|^2} \tag{3.45}$$

$$= \sum_{i=0}^{k} \frac{1}{\| g_i \|^2}. \tag{3.46}$$

Thus,

$$\frac{\| d_k \|^2}{(g_k^T d_k)^2} \leq \sum_{i=0}^{k} \frac{1}{\| g_i \|^2}. \tag{3.47}$$

From (3.27), we have

$$\sum_{i=0}^{k} \frac{1}{\| g_i \|^2} \leq \frac{k+1}{\varepsilon^2}.$$

Therefore, the last inequality implies

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\| d_k \|^2} \geq \varepsilon^2 \sum_{k=0}^{\infty} \frac{1}{k+1} = +\infty,$$

which contradicts (3.1). Thus the proof is complete. □

4. Numerical results

In this chapter, we present numerical results of the new β_k^* algorithm. We also do a comparison of our method with other methods in the literature. These methods include the β_k^{GN} hybrid by Gilbert and Nocedal [15], the β_k^{TS} conjugate gradient hybrid method by Touati-Ahmed and Storey [27] and the β_k^{HS-DY} (Hestenes and Stiefel and Dai and Yuan) [8] hybrid conjugate gradient method. A total of 14

test problems are used to test our algorithms and have been taken from different sources, that is, Luksan and Vlcek [20], Neculai Andrei [2] and More, Garbow and Hillstrom [22].
A number of parameters used are defined. These are the tolerance, ϵ, the constants μ and σ and the step length α_k. The tolerance has been set to $\epsilon = 10^{-6}$, the constants μ and σ are set to $\sigma = 0.7$ and $\mu = 0.3$. The step length $\alpha_k > 0$ is calculated using the strong Wolfe line search. All the parameters used in testing our algorithms have been set to the same values for each algorithm. Our new algorithm is coded in MATLAB R2010a.

We first of all present our numerical results in the form of a table, Table 1, where the methods are presented as follows:

- M1: The new β_k^* hybrid method;
- M2: The Gilbert and Nocedal β_k^{GN} Hybrid method [15];
- M3: The Touti Ahmed and Storey β_k^{TS} hybrid method [27];
- M4: The Dai and Yuan β^{HS-DY} hybrid method [8].

The columns 'Problem' and 'Dim' represent the name of the test problem and the dimension of the problems, respectively. The results are denoted by $'iter/fe'$, where $iter$ and fe are the number of iterations and function evaluations, respectively. The highlighted results show the best out of the 4 methods.

		M1	M2	M3	M4
Problem	Dim	$iter/fe$	$iter/fe$	$iter/fe$	$iter/fe$
Rosenbrock	2	72/1476	68/1314	76/1486	**47/894**
Freud n Roth	2	**36/723**	71/1434	60/1205	50/1033
Beale	2	69/741	**28/288**	**28/288**	45/453
Himmelblau	2	**16/167**	18/179	21/212	23/195
White	6	52/1140	**50/1008**	66/1389	82/1760
Wood	4	165/3593	148/3004	**94/1922**	106/2134
PQuad	7	35/289	31/221	34/236	**29/205**
Power	6	**41/526**	50/596	50/596	**41/492**
Fletcher	5	42/1035	**33/798**	36/873	39/942
Trig	3	17/249	19/254	20/265	**16/220**
Powell	2	19/793	**15/632**	17/708	18/727
ExPowell	4	266/3954	**150/2165**	199/2862	253/3530
Penalty I	5	**10/31**	13/25	13/25	12/26
Broyden tri	10	**29/463**	33/514	33/514	29/449

TABLE 1. Numerical results for all the four methods

From Table 1, we see that the new β_k^* hybrid method (M1) and the Gilbert and Nocedal hybrid method (M2) requires fewer function evaluations and number of iterations for 5 problems. On the other hand Touti-Ahmed and Storey hybrid method (M3) and Dai and Yuan hybrid (M4) have fewer function evaluations and number of iterations for 2 problems and 4 problems, respectively. We also see that although M2 and M3 had the same number of iterations and function evaluations for Beale problem, it is not always the case as we see that for the Power problem, M1 and M4 have the same number of iterations but the number function evaluations is higher for M1. Generally, we can say that our new hybrid conjugate gradient method is promising and competitive.

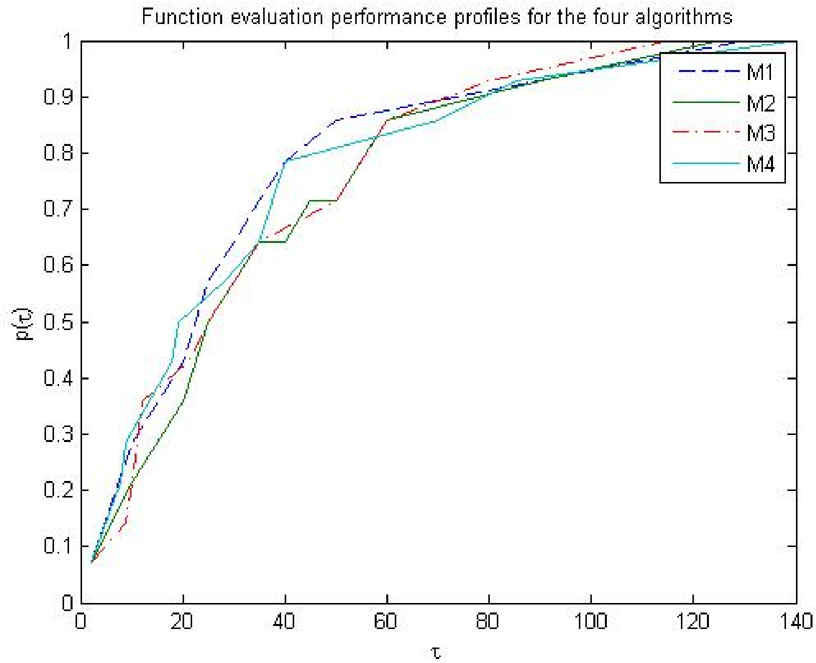

FIGURE 1. Performance Profile for Function Evaluations

To better compare the numerical performance of the 4 methods, we use performance profiles, introduced in [12]. This is reflected in Figure 1 where the performance profile for function evaluations is plotted. Letting $P = \{p_1, p_2, \ldots, p_{14}\}$ be the set of problems and $S = \{s_1, s_2, s_3, s_4\}$ be the set of the solvers M1, M2, M3, M4, respectively, we compare the performance of the solvers in S on the problems in P. Let $a_{p,s}$ denote the performance measure (e.g. function evaluations) required by solver $s \in S$ to solve problem $p \in P$. Then the performance ratio is given by

$$r_{p,s} = \frac{a_{p,s}}{\min\{a_{p,s} : s \in S\}}.$$

We assume that a parameter $r_M \geq r_{p,s}$ is chosen, for all p, s, and $r_{p,s} = r_M$ if and only if solver s does not solve problem p. To obtain an overall assessment of the performance of the solver, we define performance profile as,

$$\rho_s(\tau) = \frac{1}{n_p} size\{p \in P : r_{p,s} \leq \tau\},$$

where $\rho_s(\tau)$ is the probability for solver $s \in S$ that a performance ratio $r_{p,s}$ is within a factor $\tau \in \mathbb{R}$ of the best possible ratio and n_p is the number of problems. The function ρ_s is the cumulative distribution function for the performance ratio. Note that we always have $r_{p,s} \geq 1$. When $r_{p,s} = 1$ we have

$$a_{p,s} = \min\{a_{p,s} : s \in S\},$$

meaning that solver $s \in S$ was best for a certain problem p of all the problems.

Figure 1 shows the performance of the four methods relative to the function evaluations. We can see that all the methods successfully solved all the problems. From the figure, we see that the new method β_k^* is very much competitive with the other hybrid methods. Thus, the new method adds to the already available collection of hybrid methods that can be useful both to other researchers and people looking for solutions to optimization problems in the industries.

5. Conclusion

In this research, we have presented a new hybrid conjugate gradient algorithm in which the parameter β_k is a combination of the ideas of Dai and Yuan [8] and Gilbert and Nocedal [15]. Our new computational scheme takes advantage of the attractive features of the Fletcher Reeves (β_k^{FR}) and Polak-Ribiere-Polyak (β_k^{PRP}) methods. The direction d_k generated by our algorithm satisfies the descent condition independently of the line search used. A convergence analysis of the proposed algorithm was also carried out and we showed that the algorithm is globally convergent independently of any line search.

Furthermore, our new algorithm was compared with three other hybrid conjugate gradient methods that have been proposed in the literature. Using a set of 14 test unconstrained optimization problems, a numerical study concerning the behavior of our new algorithm has been presented. The numerical results show that our algorithm is very competitive with these other methods.

Further research will be done on developing more hybrid conjugate gradient methods for large scale unconstrained optimization problems. Although test problems of lower dimension were mainly used for testing our algorithm, we intend to extend the algorithm in future to problems with much higher dimension. Another direction would be to extend this conjugate gradient methods to constrained optimization problems, as well as optimal control problems.

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