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BEST PROXIMITY THEOREMS FOR REICH TYPE CYCLIC ORBITAL AND REICH TYPE MEIR-KEELER CONTRACTION MAPS

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ABSTRACT. In this manuscript, Reich type cyclic orbital contraction and Reich type Meir Keeler cyclic contraction are defined and also some related best proximity point theorems are obtained. These theorems generalize some results of Kirk-Srinavasan-Veeramani and Karpagam-Agrawal.

KEYWORDS : Cyclic contraction; Best proximity points; Reich type cyclic orbital contraction; Reich type Meir Keeler cyclic contraction spaces.
AMS Subject Classification: 47H10,46T99, 54H25

1. INTRODUCTION AND PRELIMINARIES

Banach Contraction mapping principle is a fundamental result in fixed point theory. It says that every contraction in a complete metric space has a unique fixed point. In this theorem, the contraction is necessarily conditions. It is natural to consider the following question. Is it possible to prove the existence and uniqueness of a fixed point of mappings that are not continuous. One of the successive answers of this question was given by Kirk-Srinavasan-Veeramani [11] by defining cyclic contraction. We should also notice that Banach considered the fixed point of self-mapping, but the authors [11] investigated the existence and uniqueness a best proximity point of non-self mappings.

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For the sake of completeness, we recollect some definitions and basic results on the topic in the literature. Suppose that (X, d) is a metric spaces. Let A and B be non-empty subsets of X. We say that $T : A \cup B \to A \cup B$ is a *cyclic map* if $T(A) \subset B$ and $T(B) \subset A$. Furthermore, a point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$ where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

A number of authors have attracted attention to the concept of cyclic mapping and best proximity points, and published interesting results on these topics, see e.g.[1, 2, 4, 5, 6, 7, 8, 10, 13, 12, 14, 18, 19, 16].

The initial result, in this direction, was given by Kirk-Srinavasan-Veeramani [11] in 2003.

Theorem 1.1. ([11, Theorem 1.1]) *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a map satisfying $T(A) \subset B$ and $T(B) \subset A$ and there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$. Then, T has a unique fixed point in $A \cap B$.*

In this paper, we first introduce the notions of Reich type cyclic orbital contraction and Reich type Meir Keeler cyclic contraction. Later, we investigate the existence and uniqueness of a best proximity point Reich type cyclic orbital contraction and Reich type Meir Keeler cyclic contraction. Our results generalize, improve and extend some results in the literatures, such as, [10, 11].

2. MAIN RESULTS

We start to this section with following definition.

Definition 2.1. (See [10]) Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a cyclic orbital contraction if for some $x \in A$ there exists a $k_x \in (0, 1)$ such that

$$d(T^{2n}x, Ty) \leq k_x d(T^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A. \tag{2.1}$$

Definition 2.1 is generalized in the following way:

Definition 2.2. Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ said to be a Reich type cyclic orbital contraction if for some $x \in A$ there exists a $k_x \in (0, \frac{1}{3})$ such that

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)], \quad n \in \mathbb{N}, \quad y \in A. \tag{2.2}$$

Karpagam and Agrawal proved the following interesting theorem.

Theorem 2.1. ([10, Theorem 2.2]) *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital contraction. Then $A \cap B$ is non empty and T has a unique fixed point.*

The following theorem is a generalization of Theorem 2.1.

Theorem 2.2. *Suppose A and B is non-empty closed subsets of a complete metric space (X, d). Let $T : A \cup B \to A \cup B$ be a Reich type cyclic orbital contraction. Then $A \cap B$ is non empty and T has a unique fixed point.*

Proof. Take $x \in A$. By (2.2), we have

$$d(T^2x, Tx) \leq k_x[d(x, Tx) + d(T^2x, Tx) + d(Tx, x)] \tag{2.3}$$

which yields that

$$d(T^2x, Tx) \leq t_x d(Tx, x), \tag{2.4}$$

where $t_x = \frac{2k_x}{1-k_x}$. Since $k_x \in (0, \frac{1}{3})$, then $t_x \in (0,1)$. Analogously, by using substitution $Tx = u$ together with (2.2), we have

$$d(T^3x, T^2x) = d(T^2(Tx), T(Tx)) \leq k_u[d(u, Tu) + d(T^2u, Tu) + d(Tu, u)] \quad (2.5)$$

and thus

$$d(T^3x, T^2x) = d(T^2u, Tu) \leq t_u d(Tu, u) \quad (2.6)$$

where $t_u = \frac{2k_u}{1-k_u}$. Since $k_u \in (0, \frac{1}{3})$, then $t_u \in (0,1)$. Keeping (2.4) and (2.6) in the mind, we obtain that

$$\begin{aligned} d(T^3x, T^2x) &= d(T^2u, Tu) \\ &\leq t_u d(Tu, u) \\ &\leq t_u d(T(Tx), Tx) \\ &\leq t_u t_x d(Tx, x) \end{aligned}$$

where $t_u \cdot t_x < 1$. By using the same argument, we derive that

$$d(T^4x, T^3x) = d(T^3(Tx), T^2(Tx) \leq t_u^2 t_x d(Tx, x). \quad (2.7)$$

Iteratively, we have

$$d(T^{n+1}x, T^n x) \leq t_u^{n-1} t_x d(Tx, x), \quad (2.8)$$

for any $n \in \mathbb{N}$, either n or $n+1$ is even. Consequently, we obtain that

$$\sum_{n=1}^{\infty} d(T^{n+1}x, T^n x) \leq \left(\sum_{n=1}^{\infty} t_u^{n-1} \right) t_x d(Tx, x) < \infty.$$

Thus, we conclude that $\{T^n x\}$ is a Cauchy sequence. As a result, there exists a $z \in A \cup B$ such that $T^n x \to z$. Note that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B in a way that both sequences tend to same limit z. Due to the fact that A and B are closed, we observe $z \in A \cap B$. Hence, $A \cap B \neq \emptyset$.

We claim that $Tz = z$. Observe that

$$\begin{aligned} d(Tz, z) &= \lim_{n\to\infty} d(Tz, T^{2n}x) \\ &\leq k_x \lim_{n\to\infty} [d(z, T^{2n-1}x) + d(T^{2n}x, T^{2n-1}x) + d(Tz, z)] \\ &\leq k_x d(Tz, z) \end{aligned}$$

which is equivalent to $(1 - k_x)d(Tz, z) = 0$. On account of $k \in (0, \frac{1}{3})$, we conclude that $d(Tz, z) = 0$, that is, $Tz = z$.

We shall prove that z is the unique fixed point of the operator T. Suppose, on the contrary, that there exists $w \in A \cup B$ such that $z \neq w$ and $Tw = w$. Taking account into T is a cyclic mapping, we derive that $w \in A \cap B$. On the other hand, we have

$$\begin{aligned} d(z, w) &= d(z, Tw) \\ &= \lim_{n\to\infty} d(T^{2n}x, Tw) \\ &\leq \lim_{n\to\infty} k_x[d(T^{2n-1}x, w) + d(T^{2n}x, T^{2n-1}x) + d(Tw, w)] \\ &= k_x d(z, w) \leq d(z, w) \end{aligned}$$

which concludes that $(1 - k_x)d(z, w) \leq 0$ where $k_x \in (0, \frac{1}{3})$. Thus $z = w$ and hence z is the unique fixed point of T. □

As an immediate consequence of the above theorem we get the following result.

Corollary 2.3. *Suppose T is a self map on a complete metric space (X,d). If for some $x \in X$, there exists a $k_x \in (0, \frac{1}{3})$ such that*

$$d(T^{2n}x, Ty) \le k_x[d(T^{2n-1}x, y) + d(T^{2n-1}x, T^{2n}x) + d(Ty, y)], \quad n \in \mathbb{N}, \quad y \in X$$

then, T has a unique fixed point.

Example 2.4. Let $A = B = [0,1] = X$ with the metric $d(x,y) = |x-y|$.
Define $T : X \to X$ a follows:

$$T(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Fix any $x \in [\frac{1}{2}, 1]$. By setting $k_x = \frac{1}{3}$, we have $Tx = \frac{1}{2}, T^2x = \frac{1}{2}, \ldots, T^n x = \frac{1}{2}, \forall n$ and for every $y \in [0,1]$ we have

$$Ty = \begin{cases} 1 & \text{if } y \in [0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } y \in [\frac{1}{2}, 1] \end{cases}$$

$$d(T^{2n}x, Ty) = \begin{cases} \frac{1}{2} & \text{if } y \in [0, \frac{1}{2}) \\ 0 & \text{if } y \in [\frac{1}{2}, 1] \end{cases}$$

and thus, $d(T^{2n-1}x, y) = |\frac{1}{2} - y|$ and

$$d(Ty, y) = \begin{cases} |1-y| & \text{if } y \in [0, \frac{1}{2}) \\ |\frac{1}{2} - y| & \text{if } y \in [\frac{1}{2}, 1] \end{cases}$$

Therefore the Reich type cyclic orbital contraction condition

$$d(T^{2n}x, Ty) \le k_x[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$$

For each $n \in \mathbb{N}$ and for each $y \in [0,1]$ satisfies for $k_x = \frac{1}{3}$. Thus, by the Theorem 2.2, T has the unique fixed point and it is observe that $x = \frac{1}{2}$ is the unique fixed point of T.

Remark 2.5. Notice that the statement (2.2) in Definition 2.2 could not be generalized to the following condition:

$$d(T^{2n}x, Ty) \le k_x[d(T^{2n-1}x, y) + d(T^{2n}x, y) + d(Ty, x)]; n \in \mathbb{N}; y \in A \tag{2.9}$$

since both $T^{2n}x$ and y lies in A, the statement (2.9) fails to be cyclic. To avoid such cases, throughout of the manuscript we define and use the notion "opposite parity": We say that $p, q \in \mathbb{N}$ are opposite parity if either $T^p x \in A$, $T^q x \in B$ or $T^p x \in B$, $T^q x \in A$ holds.

3. REICH TYPE CYCLIC ORBITAL MEIR-KEELER CONTRACTIONS

We start to this section with the following definition.

Definition 3.1. (See [10]) Let (X,d) be a metric space, and A and B be nonempty subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic map such that, for some $x \in A$, and for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(T^{2n-1}x, y) < d(A,B) + \varepsilon + \delta \text{ implies } d(T^{2n}x, Ty) < d(A,B) + \varepsilon, \; n \in \mathbb{N}, y \in A. \tag{3.1}$$

Then T is said to be a cyclic orbital Meir-Keeler contraction.

We extend the Definition 3.1 in the following way.

Definition 3.2. Suppose that (X, d) is a metric space, and A and B are nonempty subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic map such that, for some $x \in A$, and for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} &R(T^{2n-1}x, y) < d(A, B) + \varepsilon + \delta \\ \text{implies } \ &d(T^{2n}x, Ty) < d(A, B) + \varepsilon, n \in \mathbb{N}, y \in A \end{aligned} \tag{3.2}$$

where $R(T^{2n-1}x, y) = \frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$. Then T is said to be a Reich Type cyclic orbital Meir-Keeler contraction.

Now we prove two technical results, which would be usee in the sequel.

Proposition 3.3. *Suppose that A, B are nonempty and closed subsets of a metric space X and $T : A\cup B \to A\cup B$ is a Reich type cyclic orbital Meir-Keeler contraction. If $x \in A$ satisfies condition (3.2) then $d(T^{n+1}x, T^n x) \to d(A, B)$, as $n \to \infty$.*

Proof. Assume that T is Reich type cyclic orbital Meir-Keeler contraction. Let $x \in A$ for which the expression (3.2) is satisfied. Since either n or $n+1$ is even, then for each $x \in A$, we get $\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] \geq d(A, B)$.

Now, we examine the case

$$\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] = d(A, B).$$

By regarding (3.2), we derive that

$$d(T^{n+1}x, T^n x) < d(A, B) + \varepsilon$$

which is equivalent to

$$d(T^{n+1}x, T^n x) < \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] + \varepsilon.$$

Consequently, we have

$$d(T^{n+1}x, T^n x) \leq d(T^n x, T^{n-1}x), \quad \text{as} \quad \varepsilon \to 0.$$

Let us consider the other case:

$$\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] > d(A, B).$$

We set $\varepsilon_1 = \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] - d(A, B) > 0$. For this ε_1, there exists a δ, satisfying (3.2). Also $R(T^n x, T^{n-1}x) < d(A, B) + \varepsilon_1 + \delta$. Next

$$d(T^{n+1}x, T^n x) < d(A, B) + \varepsilon_1 = \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)].$$

Hence, we get that $d(T^{n+1}x, T^n x) \leq d(T^n x, T^{n-1}x)$ for all $n \in \mathbb{N}$.

Let $s_n = d(T^{n+1}x, T^n x)$. So, the sequence $\{s_n\}$ is a non-increasing and bounded below by $d(A, B)$. As a result, $\{s_n\}$ converges to some s with $s \geq d(A, B)$.

We shall show that $s = d(A, B)$. Suppose, on the contrary, that $s > d(A, B)$. So we have $\varepsilon = s - d(A, B) > 0$. Consequently, there exists a $\delta > 0$ which satisfies (3.2). Regarding $\{d(T^{n+1}x, T^n x)\} \to s$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} s &\leq \frac{1}{3}[d(T^{n+1}x, T^n x) + d(T^{n+2}x, T^{n+1}x) + d(T^{n+1}x, T^n x)] \\ &< s + \delta = \varepsilon + d(A, B) + \delta, \end{aligned}$$

for all $n \geq n_0$. So we conclude that

$$d(T^{n+2}x, T^{n+1}x) < d(A, B) + \varepsilon = s, \ \forall n \geq n_0,$$

a contradiction. Therefore, we have $s = d(A, B)$. □

Proposition 3.4. *Suppose that A and B are nonempty and closed subsets of a metric space X and $T : A \cup B \to A \cup B$ is a Reich type cyclic orbital Meir-Keeler contraction. Suppose also that $d(A, B) = 0$. Then, for each $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and a $\delta > 0$ such that*

$$d(T^p x, T^q x) < \varepsilon + \delta \quad \textit{implies that} \quad d(T^{p+1}x, T^{q+1}x) < \varepsilon \tag{3.3}$$

where p and q are opposite parity, with $p, q \geq n_1$.

Proof. First, we take $x \in X$ for which (3.2) holds. Owing to the fact that T is a Reich type cyclic orbital Meir-Keeler contraction, for a given $\varepsilon > 0$, there exists a $\delta > 0$ satisfying (3.2). So, we have

$$\begin{gathered} \tfrac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)] < \varepsilon + \delta \\ \text{implies} \;\; d(T^{2n}x, Ty) < \varepsilon, n \in \mathbb{N}, y \in A. \end{gathered} \tag{3.4}$$

Without loss of generality we may choose

$$\delta < \varepsilon. \tag{3.5}$$

Regarding $d(A, B) = 0$ and Proposition 3.3, one can choose $n_1 \in \mathbb{N}$ in a way that

$$d(T^n x, T^{n+1}x) < \frac{\delta}{2}, \quad \text{for each} \quad n \geq n_1. \tag{3.6}$$

We shall show that $d(T^p x, T^q x) < \varepsilon + \delta$ implies that $d(T^{p+1}x, T^{q+1}x) < \varepsilon$. For this purpose, fix $n \geq n_1$ and take $p, q \in \mathbb{N}$ which are opposite parity with $p, q \geq n_1$. Assume that $d(T^p x, T^q x) < \varepsilon + \delta$. Without loss of generality we may assume $T^p x \in A$ and $T^q x \in B$ with $p = 2n$ and $q = 2m - 1$. Otherwise, we revise the indices respectively.

Therefore, we have $d(T^p x, T^q x) = d(T^{2n}x, T^{2m-1}x) < \varepsilon + \delta$, for $m \geq n$. Then, regarding (3.6) we get

$$\frac{1}{3}[d(T^{2m-1}x, T^{2n}x) + d(T^{2m}x, T^{2m-1}x) + d(T^{2n+1}x, T^{2n}x)] \leq \frac{1}{3}[\varepsilon + \delta + \frac{\delta}{2} + \frac{\delta}{2}] < \varepsilon + \delta. \tag{3.7}$$

Regarding (3.4) under the assumption $y = T^{2n}x$, the inequality (3.7) yields that

$$d(T^{2n+1}x, T^{2m}x) = d(T^{p+1}x, T^{q+1}x) < \varepsilon$$

Hence, we derive that for a given $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and a $\delta > 0$ such that

$$d(T^p x, T^q x) < \varepsilon + \delta \quad \text{implies that} \quad d(T^{p+1}x, T^{q+1}x) < \varepsilon \tag{3.8}$$

where p and q are opposite parity, with $p, q \geq n_1$.

□

Next we prove a necessary condition for a cyclic orbital Meir-Keeler contraction and we use this to prove the main result.

Lemma 3.5. *Suppose that X is a complete metric space, A and B non-empty, closed subsets of X such that $d(A, B) = 0$. Suppose also that $T : A \cup B \to A \cup B$ is a Reich Type cyclic orbital Meir-Keeler contraction. Then*

$$d(T^{2n}x, Ty) < R(T^{2n-1}x, y) \quad \textit{if} \quad T^{2n-1}x \neq y, \tag{3.9}$$

where $R(T^{2n-1}x, y) = \frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$.

Proof. To reach (3.9), it is sufficient to show that (3.2) is equivalent to the following condition: For each $\varepsilon > 0$ there exists δ such that

$$\begin{gathered} \varepsilon \leq R(T^{2n-1}x, y) < \varepsilon + \delta \\ \text{implies} \;\; d(T^{2n}x, Ty) < \varepsilon, n \in \mathbb{N}, y \in A \end{gathered} \tag{3.10}$$

where $R(T^{2n-1}x,y)=\frac{1}{3}[d(T^{2n-1}x,y)+d(T^{2n}x,T^{2n-1}x)+d(Ty,y)]$ and recall that $d(A,B)=0$.

It is evident that (3.2) implies (3.10). Now, suppose that (3.10) holds. Now, we fix $T^{2n-1}x,y\in A\cup B$ and $\varepsilon>0$. If $R(T^{2n-1}x,y)<\varepsilon$, since (3.10) we have $d(T^{2n}x,Ty)\le R(T^{2n-1}x,y)$ and consequently $d(T^{2n}x,Ty)<\varepsilon$. If $R(T^{2n-1}x,y)\ge\varepsilon$, then immediately (3.2) holds. Hence, (3.10) and (3.2) are equivalent under the condition $d(A,B)=0$.

We shall sow that if (3.10) holds then $d(T^{2n}x,Ty)\le R(T^{2n-1}x,y)$.
If $R(T^{2n-1}x,y)=0$ then $T^{2n-1}x=y$. Thus $d(T^{2n}x,Ty)\le R(T^{2n-1}x,y)$. Suppose $R(T^{2n-1}x,y)\ne 0$ and fix $\varepsilon\le R(T^{2n-1}x,y)$. Choose a $\delta>0$ such that (3.10) holds. Recall that $R(T^{2n-1}x,y)\le d(T^{2n}x,Ty)$ which contradicts with (3.10). □

Now we prove the existence, uniqueness and convergence theorem for a Reich type cyclic orbital Meir-Keeler contraction.

Theorem 3.1. *Suppose that X is a complete metric space, A and B non-empty, closed subsets of X with $d(A,B)=0$. Assume that $T:A\cup B\to A\cup B$ is a Reich type cyclic orbital Meir-Keeler contraction. Then, there exists a fixed point, say $z\in A\cap B$, such that for each satisfying (3.2), the sequence $\{T^{2n}x\}$ converges to z.*

Proof. First, we take $x\in A$. We shall show that $\{T^nx\}$ is a Cauchy sequence. Suppose, on the contrary, that the sequence $\{T^nx\}$ is not Cauchy. Then, there exists an $\varepsilon>0$ and a subsequence in $\{T^{n(i)}\}$ of $\{T^nx\}$ with

$$d(T^{n(i)}x,T^{n(i+1)}x)>2\varepsilon. \tag{3.11}$$

For this ε, there exists $\delta>0$ such that

$$R(T^{2n-1}x,y)<\varepsilon+\delta \text{ implies that } \quad d(T^{2n}x,Ty)<\varepsilon \tag{3.12}$$

where $R(T^{2n-1}x,y)=\frac{1}{3}[d(T^{2n-1}x,y)+d(T^{2n}x,T^{2n-1}x)+d(Ty,y)]$. We set $r=\min\{\varepsilon,\delta\}$ and $d_m=d(T^mx,T^{m+1}x)$. Owing to Proposition 3.3, one can choose $n_0\in\mathbb{N}$ such that

$$d^m=d(T^mx,T^{m+1}x)<\frac{r}{4}, \quad \text{for } \ m\ge n_0. \tag{3.13}$$

Let $n(i)\ge N$. Suppose that $d(T^{n(i)}x,T^{n(i+1)-1}x)\le\varepsilon+\frac{r}{2}$. Then triangle inequality implies that

$$\begin{aligned} d(T^{n(i)}x,T^{n(i+1)}x) &\le d(T^{n(i)}x,T^{n(i)-1}x)+d(T^{n(i+1)-1}x,T^{n(i+1)}x)\\ &<\varepsilon+\tfrac{r}{2}+d_{n(i+1)-1}\\ &<2\varepsilon \end{aligned} \tag{3.14}$$

which contradicts the assumption (3.11). Therefore, there are values of k with $n(i)\le k\le n(i+1)$ such that $d(T^{n(i)},T^kx)>\varepsilon+\frac{r}{2}$.opposite parity. We assume that $d(T^{n(i)}x,T^{n(i)+1}x)\ge\varepsilon+\frac{r}{2}$. Then

$$d_{n(i)}=d(T^{n(i)}x,T^{n(i)+1}x)\ge\varepsilon+\frac{r}{2}>r+\frac{r}{2}>\frac{r}{4}$$

which contradicts with (3.13). Hence, there are values of k with $n(i)\le k\le n(i+1)$ such that $d(T^{n(i)},T^kx)<\varepsilon+\frac{r}{2}$ where k and $n(i)$ are opposite parity. Choose smallest integer k with $k\ge n(i)$ such that $d(T^{n(i)}x,T^kx)\ge\varepsilon+\frac{r}{2}$. So,

$$d(T^{n(i)}x,T^{k-1}x)<\varepsilon+\frac{r}{2}. \tag{3.15}$$

Hence,

$$\begin{aligned} d(T^{n(i)}x, T^{k-1}x) &\leq d(T^{n(i)}x, T^{k-1}x) + d(T^{k-1}x, T^{k}x) \\ &< \varepsilon + \frac{r}{2} + \frac{r}{4} \\ &= \varepsilon + \frac{3r}{4}. \end{aligned}$$

Then, there exists an integer k satisfying $n(i) \leq k \leq n(i+1)$ such that

$$\varepsilon + \frac{r}{2} \leq d(T^{n(i)}x, T^{k}x) < \varepsilon + \frac{3r}{4}. \tag{3.16}$$

Owing to the facts

$$d(T^{n(i)}x, T^{k}x) < \varepsilon + \frac{3r}{4} < \varepsilon + r,$$
$$d(T^{n(i)}x, T^{n(i)+1}x) = d_{n(i)} < \frac{r}{4} < \varepsilon + r \text{ and}$$
$$d(T^{k}, T^{k+1}x) = d_k < \frac{r}{4} < \varepsilon + r.$$

We have

$$\begin{aligned} R(T^{n(i)}x, T^{k}x) &= \frac{1}{3}[d(T^{n(i)}x, T^{k}x) + d(T^{n(i)}x, T^{n(i)+1}x) + d(T^{k+1}x, T^{k}x)] \\ &\leq \frac{1}{3}[\varepsilon + r + \varepsilon + r + \varepsilon + r] = \varepsilon + r, \end{aligned}$$

which implies $d(T^{n(i)+1}, T^{k+1}x) < \varepsilon$. But,

$$\begin{aligned} d(T^{n(i)+1}x, T^{k+1}x) &\geq d(T^{n(i)}x, T^{k}x) - d(T^{n(i)}x, T^{n(i)+1}x) - d(T^{k}x, T^{k+1}x) \\ &> \varepsilon + \frac{r}{2} - \frac{r}{4} - \frac{r}{4} = \varepsilon, \end{aligned}$$

which contradicts the preceding inequality.

Hence the sequence $\{T^n x\}$ is a Cauchy. Thus, the sequence $\{T^n x\}$ converges to some $z \in A$. Hence,

$$0 \leq d(T^{2n-1}x, z) \leq d(T^{2n-1}x, T^{2n}x) + d(T^{2n}x, z) \tag{3.17}$$

tends to zero as well. Thus,

$$\lim_{n\to\infty} d(T^{2n-1}x, z) = 0. \tag{3.18}$$

Since $\{T^{2n-1}x\}$ is a sequence in B, it converges to $z \in B$. Taking into account both A and B are closed, we get $z \in A \cap B$.

We shall show that $Tz = z$.

On account of Lemma 3.5

$$\begin{aligned} d(Tz, z) &= \lim_{n\to\infty} d(T^{2n}x, Tz) \\ &< \lim_{n\to\infty} R(T^{2n-1}x, z) \\ &= \lim_{n\to\infty} \frac{1}{3}[d(T^{2n-1}x, z) + d(T^{2n}x, T^{2n-1}x) + d(Tz, z)] \end{aligned}$$

which implies that

$$d(Tz, z) < \frac{1}{3}d(Tz, z).$$

This is a contradiction. Hence, we have $Tz = z$.

Finally, we shall show that z is a unique fixed point of T. Suppose, on the contrary, that there exists a point $w \in A \cap B$ such that $z \neq w$ and $Tw = w$. Owing to Lemma 3.5

$$\begin{aligned} d(w,z) &= d(Tw,z) \\ &= \lim_{n\to\infty} d(T^{2n}x, Tw) \\ &< \lim_{n\to\infty} R(T^{2n-1}x, w) \\ &\leq \lim_{n\to\infty} \frac{1}{3}[d(T^{2n-1}x,w) + d(T^{2n}x, T^{2n-1}x) + d(Tw,w)] \\ &\leq \frac{1}{3}[d(z,w) + d(z,z) + d(Tw,w)] \\ &= \frac{1}{3}d(z,w) \end{aligned}$$

which is a contradiction. Hence, $z = w$.

□

4. CONCLUDING REMARKS

In this paper, two different types of Reich type cyclic contractions, such as Reich type cyclic orbital contractions and Reich type Meir-Keeler cyclic contractions, are introduced and thereby obtained the existence and uniqueness of best proximity for such mappings. We also gave a procedure to find the best proximity point (a type of iterative sequence that converges to the point) for a Reich type Meir-Keeler cyclic contraction. Our results generalize some results of Kirk-Srinavasan-Veeramani [11] and Karpagam-Agrawal [10].

For further research, we propose the studying the existence of best proximity points of Reich type cyclic orbital contractions and Reich type Meir-Keeler cyclic contractions on $A \cup B$ in a complete metric space in the case that $d(A,B) > 0$. Further research may also include studying the necessary condition for convergence of iterative sequence $\{T^{2n}x\}$, for any $x \in A \cup B$, that converges to the unique best proximity point for a Reich type cyclic orbital contraction or Reich type Meir-Keeler cyclic contraction on $A \cup B$ in a complete metric space in the case that $d(A,B) > 0$.

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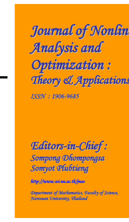

ON GENERALIZED LIPSCHITZIAN MAPPING AND EXPANSIVE LIPSCHITZ CONSTANT

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ABSTRACT. In this paper, we introduce a new uniformly generalized Lipschitzian type condition for a one-parameter semigroup of self-mappings and utilize the same to show that a uniformly generalized Lipschitzian semigroup of nonlinear self-mappings of a nonempty closed convex subset C of real Banach space X admits a common fixed point provided the semigroup has a bounded orbit and k is appropriately larger than one. Finally, we prove that a semigroup of self mappings $T = \{T(t) : t \in G\}$ defined on a nonempty weakly compact convex subset $\overline{C}$ of a Banach space X with a weak uniform normal structure satisfying $\liminf_{G\ni t\longrightarrow\infty} \Big| ||T(t)|| \Big| = \lim_{G\ni t\longrightarrow\infty} \Big| ||T(t)|| \Big| = k < WCS(X)\mu_0$ admits a common fixed point where $\mu_0 = \inf\{\mu \geq 1 : \mu(1-\delta_X(1/\mu)) \geq (1/2)\}$, and $WCS(X)$ is the weak convergent sequence coefficient of X while $\Big| ||T(t)|| \Big|$ is the exact Lipschitz constant of $T(t)$. Our such result is an extension of the corresponding results due to L.C. Ceng, H. K. Xu and J.C. Yao [5] and L. C. Zeng [29].

1. INTRODUCTION

Let X be a real Banach space equipped with uniform normal structure and C be a nonempty closed convex subset of X. A mapping $T : C \longrightarrow C$ is said to be Lipschitzian if, for each integer $n \geq 1$, there exists a constant $k_n > 0$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \text{ for all } x, y \in C.$$

A Lipschitzian mapping is said to be a k-uniformly Lipschitzian mapping if $k_n \equiv k$ for all $n \geq 1$.

A Banach space X is said to have weak normal structure if every weakly compact convex subset C of X with more than one point contains a nondiametral point, that is, $x_0 \in C$ for which

$$\sup\{||x_0 - y|| : y \in C\} < diam(C).$$

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Every Banach space equipped with normal structure also owns the weak normal structure, but the converse is not true. For reflexive Banach space these properties are equivalent.

Bynum [3] defined weak convergent sequence coefficient of X as the number

$$WCS(X) = \inf\{A(\{x_n\})/\inf\{\limsup_{n\longrightarrow\infty} ||x_n - y|| : y \in \overline{co}\{x_n\}\}\},$$

where the first infimum is taken over all weakly convergent sequences in X while $\overline{co}(A)$ denotes the closure of the convex hull of the subset $A \subset X$, and $A(\{x_n\})$ is the asymptotic diameter of $\{x_n\}$ (i.e., the number $\lim_{n\longrightarrow\infty}(\sup\{\|x_i - x_j\| : i, j \geq n\})$. It is readily seen that $1 \leq WCS(X) \leq 2$. Following [6], we say that the Banach space X has a weak uniform normal structure provided $WCS(X) > 1$.

In 1973, Goebel and Kirk [11] posed the question whether or not the constant $\gamma > 1$ satisfying the equation

$$(1 - \delta_X(1/\gamma))\gamma = 1, \tag{1.1}$$

is the largest number for which any k-uniformly Lipschitzian mapping T with $k < \gamma$ has a fixed point where δ_X denotes the modulus of convexity of X.

In 1975, Lifschits [19] proved that a k-uniformly Lipschitzian mapping defined on a Hilbert space with $k < \sqrt{2}$ has a fixed point.

Casini and Maluta [4] and Ishihara and Takahashi [14] proved that a uniformly $k-$Lipschitzian semigroup of self-mappings defined on a Banach space X has a common fixed point provided $k < \sqrt{N(X)}$ where $N(X)$ denotes the uniform normal structure coefficient.

In 1992 Jimenez-Melado [15] defined the GGLD property for a Banach space X as follows:
X is said to have GGLD provided $D[(x_n)] > 1$ where $\{x_n\}$ is any weakly null sequence such that $\lim_{n\longrightarrow\infty} ||x_n|| = 1$, and $D[(x_n)] = \limsup_{m\longrightarrow\infty} \limsup_{n\longrightarrow\infty} ||x_n - x_m||$; he defined also the coefficient $\beta(X)$ of X by $\beta(X) = \inf\{D[(x_n)] : x_n \longrightarrow 0 \text{ weakly and } ||x_n|| \longrightarrow 1\}$.

Thereafter, k-uniformly Lipschitzian mappings have extensively been investigated by many authors. Moreover, some of results proved for uniformly Lipschitzian mapping have been extended to uniformly Lipschitzian semigroups, and even to Lipschitzian semigroups (e.g. [25]-[35]).

Particularly, in 1993, Tan and Xu [25] answered the earlier mentioned question of Goebel and Kirk [11] in negative by proving the following:

Theorem 1.1. ([25], Theorem 3.5) Let C be a nonempty closed convex subset of a real uniformly convex Banach space X while $\tau = \{T_s : s \in G\}$ be a k-uniformly Lipschitzian semigroup of mappings on C with $k < \alpha$ where $\alpha > 1$ is the unique solution of the equation

$$\frac{\alpha^2}{N(X)}\delta_X^{-1}(1 - \frac{1}{\alpha}) = 1 \tag{1.2}$$

wherein $N(X) > 1$ is the normal structure coefficient of X. If there exists an $x_0 \in C$ such that the orbit $\{T_s x_0 : s \in G\}$ is bounded, then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

It is easy to prove that $\gamma < \alpha$, where γ and α are the solution of equations (1.1) and (1.2), respectively. Consequently, the constant γ satisfying equation (1.1) is not the biggest number for which every k-uniformly Lipschitzian mapping T with $k < \gamma$ has a fixed point. Indeed, the best possible number γ is still unknown even in the setting of Hilbert spaces. It is therefore an interesting question to find another constant α^* which is strictly bigger than α and for which every k-uniformly Lipschitzian mapping T with $k < \alpha^*$ has a fixed point.

In 1995, Dominguez Benavides et al. [6] applied a new method to construct a sequence which converges to a fixed point of non-expansive mappings.

Theorem 1.2 (cf. [7]). Suppose X is a Banach space which is not Schur. Then:
(i) X has the GGLD property if and only if $\limsup_{n\longrightarrow\infty} ||x_n - x_\infty|| < A(\{x_n\})$, where $\{x_n\}$ is any weakly (not strongly) convergent sequence in X with limit x_∞.
(ii) $\beta(X) = WCS(X)$.

Some years later, Zeng and Yang [32] proved a fixed point result for Lipschitzian semigroups of mappings as follows:

Theorem 1.3. ([32], Theorem 3.1) Let C be a nonempty bounded subset of a uniformly convex Banach Space X, and let $\tau = \{T_s : s \in G\}$ be a k-uniformly Lipschitzian semigroup on C with

$$\liminf_s ||T_s|| < \sqrt{\gamma_0 N(X)},$$

where

$$\gamma_0 = \inf\{\gamma : \gamma(1 - \delta_X(1/\gamma \geq 1/2)),$$

and $||T_s||$ is the exact Lipschitzian constant of T_s. Suppose also there exists a nonempty bounded closed convex subset E of C with the following properties:
(P1) $x \in E$ implies $w_w(x) \subset E$; where $w_w(x)$ is the weak w-limit set of τ at x, i.e.,

$$w_w(x) = \{y \in X : y = weak - lim_{t_\alpha} T_{t_\alpha} x \text{ for some subnet } \{t_\alpha\} \subset G\}.$$

(P2) τ is asymptotically regular on E; i.e., $\lim_t ||T_{t+s}x - T_t x|| = 0$, $\forall\ s \in G, x \in E$.
Then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

Also, Kuczumow [18] proved the following theorem:

Theorem 1.4 (cf. [18]) Let C be a nonempty convex weakly compact separable subset of a Banach space X while $T = \{T(t) : t \in G\}$ be an asymptotically regular semigroup of mappings defined on C such that $\liminf_{G\ni t\longrightarrow\infty} \Big| \, ||T(t)|| \, \Big| = k < \sqrt{WCS(X)}$. Then there exists $z \in C$ such that $T(t)z = z \ \ \forall \ \ t \in G$.

Recently, Ceng, Xu and Yao [5] studied the existence of fixed points of uniformly Lipschitzian semigroup $\tau = \{T_s : s \in G\}$ of mappings in the setting of Banach space X under conditions weaker than uniform convexity. More precisely, their improvements can be adjudged twofold:
(1) firstly, authors replaced the uniform convexity of X (in Theorem 1.1) by relatively weaker condition of uniform normal structure of X;
(2) secondly, they removed the asymptotic regularity of the semigroup $\tau = \{T_s : s \in G\}$ on the Banach space X(in Theorem 1.3).
Also, Zeng [29] proved the following fixed point result which extends previously known results due to [7, 8, 13, 18]:

Theorem 1.5 (cf. [29]) Let C be a nonempty convex weakly compact subset of a Banach space X equipped with weak uniform normal structure while $T = \{T(t) : t \in G\}$ be asymptotically regular semigroup of mappings on C such that $\liminf_{G\ni t\longrightarrow\infty} \Big| \, ||T(t)|| \, \Big| = k < WCS(X)$.
If each $T(t)$ is weakly continuous, then $F(t)$ is nonempty.

The purpose of this paper, is to extend the result due to Ceng, Xu and Yao [5] by using the new definition of uniformly generalized Lipschitzian type mappings for one-parameter semigroups of self-mappings.

Clearly, it remains a natural question whether Theorem 1.5 is true for the estimate $\liminf_{G\ni t\longrightarrow\infty}\Big|\ ||T(t)||\ \Big| = k < WCS(X)\mu_0$, where $\mu_0 \geq 1$.

The other purpose of this paper is to answer the above question as well.

2. PRELIMINARIES

In what follows, we recall some relevant definitions and results in respect of uniformly generalized Lipschitzian mappings in Banach spaces.

In 2001, Jung and Thakur [16] introduced and studied the following class of mappings.

Definition 2.1(see [16]). A mapping $T : X \longrightarrow X$ is said to be "generalized Lipschitzian mapping (in short G1-Lipschitzian)" if

$$||T^n x - T^n y|| \leq a_n||x-y|| + b_n(||x - T^n x|| + ||y - T^n y||) + c_n(||x - T^n y|| + ||y - T^n x||)$$

for each $x, y \in X$ and $n \geq 1$, where a_n, b_n and c_n are nonnegative constants such that there exists an integer n_0 such that $b_n + c_n < 1$ for all $n > n_0$. Here it may be pointed out that this class of generalized Lipschitzian mappings is relatively larger than the classes of nonexpansive, asymptotically nonexpansive, Lipschitzian, and uniformly k-Lipschitzian mappings. The earlier mentioned facts can be realized by choosing constants a_n, b_n and c_n suitably.

On other hand, in 2009, Soliman [24] defined another class of generalized Lipschitzian mappings as follows.

Definition 2.2 (see [24]). A mapping $T : X \longrightarrow X$ is said to be "generalized Lipschitzian mapping (in short G2-Lipschitzian)" if for each integer $n \geq 1$, there exists a constant $k_n > 0$ (depending on n) such that

$$||T^n x - T^n y|| \leq k_n \max\left\{||x-y||, \frac{1}{2}||x - T^n x||, \frac{1}{2}||y - T^n y||\right\}$$

for every $x, y \in X$. If $k_n = k$ for all $n \geq 1$, then T is called uniformly G2-Lipschitzian.

Now, we will define another class of generalized Lipschitzian mappings as follows.

Definition 2.3. A mapping $T : X \longrightarrow X$ is said to be "generalized Lipschitzian mapping (in short G3-Lipschitzian)" if for each integer $n \geq 1$ there exists a constant $k_n > 0$ (depending on n) such that

$$||T^n x - T^n y|| \leq k_n \max\left\{||x-y||, \frac{1}{2\rho}||x - T^n x||, \frac{1}{2\rho}||y - T^n y||, \frac{1}{2\rho}||x - T^n y||, \frac{1}{2\rho}||y - T^n x||\right\}$$

for every $x, y \in X$, If $k_n \equiv k$ for all $n \geq 1$, then T is called uniformly G3-Lipschitzian, where $\rho > k,\ \rho > 1$.

Definition 2.4. Let C be a closed convex subset of a Banach space X. Then the collection $\tau = \{T_s : s \in G\}$ of mappings of C into itself is said to be Lipschitzian semigroup on C if the following conditions are satisfied:
(i) $T_{st}x = T_sT_tx$ for all $s, t \in G$ and $x \in C$;
(ii) for each $x \in C$, the mapping $t \longrightarrow T_tx$ from G into C is continuous;
(iii)for each $t \in G$, $T_t : C \longrightarrow C$ is continuous on C;
(iv) for each $t \in G$, there exists a constant $k_t > 0$ such that

$$||T_tx - T_ty|| \leq k_t||x-y|| \quad \text{for all } x, y \in C.$$

In particular, if $k_t \equiv k$ then $\tau = \{T_s : s \in G\}$ is called k-uniformly Lipschitzian semigroup on C.

Definition 2.5. A semigroup $\tau = \{T_s : s \in G\}$ of self mappings defined on X is called a uniformly G1-Lipschitzian semigroup if

$$||T(t)x-T(t)y|| \le a(t)||x-y||+b(t)(||x-T(t)x||+||y-T(t)y||)+c(t)(||x-T(t)y||+||y-T(t)x||)$$

for each $x, y \in X$, where $a(t), b(t)$ and $c(t)$ are nonnegative constants $b(t) + c(t) < 1$, $\sup\{a(t) : t \in G\} = a < \infty$, $\sup\{b(t) : t \in G\} = b < \infty$, and $\sup\{c(t) : t \in G\} = c < \infty$ with $b + c < 1$.

The simplest uniformly G1-Lipschitzian semigroup is a semigroup of iterates of a mapping $T : X \longrightarrow X$ whenever $\sup\{a(t) : t \in G\} = a < \infty$, $\sup\{b(t) : t \in G\} = b < \infty$, and $\sup\{c(t) : t \in G\} = c < \infty$ with $b + c < 1$.

Ahmed H. Soliman [24] introduced the following definition.

Definition 2.6. A semigroup $\tau = \{T_s : s \in G\}$ of self mappings defined on X is called a uniformly G2-Lipschitzian semigroup if

$$\sup\{k(t) : t \in G\} = k < \infty,$$

where

$$||T(t)x - T(t)y|| \le k(t) \max\Big\{||x - y||, \frac{1}{2}||x - T(t)x||, \frac{1}{2}||y - T(t)y||\Big\}$$

for each $x, y \in X$ and $\max\Big\{||x - y||, \frac{1}{2}||x - T(t)x||, \frac{1}{2}||y - T(t)y||\Big\} \neq 0$.

Finally, we will introduce the following definition,

Definition 2.7. A semigroup $\tau = \{T_s : s \in G\}$ of self mappings defined on X is called a uniformly G3-Lipschitzian semigroup if

$$\sup\{k(t) : t \in G\} = k < \infty,$$

where

$$||T(t)x - T(t)y|| \le k(t)M(x, y)$$

for each $x, y \in X$ and $M(x, y) = \max\Big\{||x - y||, \frac{1}{2\rho}||x - T(t)x||, \frac{1}{2\rho}||y - T(t)y||, \frac{1}{2\rho}||x - T(t)y||, \frac{1}{2\rho}||y - T(t)x||\Big\} \neq 0$.

Remark 2.8. The class of uniformly G3-Lipschitzian semigroups is relatively larger than the other classes namely: uniformly G1-Lipschitzian semigroups, uniformly G2-Lipschitzian semigroups, and also uniformly k-Lipschitzian semigroups.

Recall that X is strictly convex if its unit sphere does not contain any line segments, that is, X is strictly convex if and only if the following implication holds:

$$x, y \in X, \quad ||x|| = ||y|| = 1 \text{ and } ||(x + y)/2|| = 1 \quad \Rightarrow x = y.$$

In order to measure the degree of convexity of X, we define its modulus of convexity $\delta_X : [0, 2] \longrightarrow [0, 1]$ by

$$\delta_X(\varepsilon) = \inf\{1 - ||(x + y)/2|| : ||x|| \le 1, ||y|| \le 1 \text{ and } ||x - y|| \ge \varepsilon\}.$$

The characteristic of convexity of X is the number $\varepsilon_0(X) = \sup\{\varepsilon : \delta_X(\varepsilon) = 0\}$. It is easy to see [10] that X is uniformly convex iff $\varepsilon_0(X) = 0$; uniformly nonsquare iff $\varepsilon_0(X) < 2$; and strictly convex iff $\delta(2) = 1$. Moreover, if $\varepsilon_0(X) < 1$; then X has a normal structure, that is, each bounded convex subset H of X containing more than one points admits a point x_0 such that $\sup\{||x_0 - x|| : x \in H\} < diam(H)$.

The following properties of modulus of convexity of X are quite well-known (see [12]):
(a) δ_X is increasing on [0,2] and moreover strictly increasing on $[\varepsilon_0, 2]$;
(b) δ_X is continuous on [0,2)(but not necessarily at $\varepsilon = 2$);
(c) $\delta_X(2) = 1$ iff X is strictly convex;
(d) $\delta_X(0) = 0$ and $\lim_{\varepsilon \longrightarrow 2^-} \delta_X(\varepsilon) = 1 - \varepsilon_0/2$

(e) $[||a-x|| \le r, ||a-y|| \le r$ and $||x-y|| \ge \varepsilon] \Rightarrow ||a-(x+y)/2|| \le r(1-\delta_X(\varepsilon/r))$.

Recall that the normal structure coefficient $N(X)$ of X is the number (see [3])

$$\inf\Big\{\frac{diamK}{r_K(K)}\Big\},$$

where the infimum is taken over all bounded closed convex subsets K of X with more than one member, and $r_K(K)$ and $diam(K)$ are Chebyshev radius of K relative to it self and the diameter of K, respectively, i.e., $r_K(K) = \inf_{x\in K}\sup_{y\in K}||x-y||$ and $diamK = \sup_{x,y\in K}||x-y||$. A Banach space X is said to have uniform normal structure if $N(X) > 1$. It is known that a Banach space with uniform normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniform normal structure (see, e.g., [35]). It is also been computed that $N(H) = \sqrt{2}$ for a Hilbert spaces H. The computations of the normal structure coefficient $N(X)$ for general Banach spaces look however complicated. No exact values of $N(X)$ are known except for some special cases (e.g., Hilbert and L^p spaces). In general, we have the following lower bound for $N(X)$ (see [3, 21, 1])

$$N(X) \ge \frac{1}{1-\delta_X(1)}.$$

Other lower bounds for $N(X)$ in terms of some Banach space parameters or constants can be found in [17, 22].

Tan and Xu [25] have also proved that if X is uniformly convex and $\gamma > 1$ is the unique solution of the equation (1.1), then $N(X) > \gamma$. Note that for a Hilbert space H, we have $N(H) = \sqrt{2}$ and $\gamma = \sqrt{5}/2$.

Suppose X is uniformly convex Banach space. Then it is easily seen that the equation

$$\alpha^2\delta_X^{-1}(1-\frac{1}{\alpha})\widetilde{N}(X) = 1 \tag{2.1}$$

has a unique solution $\alpha > 1$, where $\widetilde{N}(X) = 1/N(X)$. Tan and Xu [25] proved that if $\gamma > 1$ and $\alpha > 1$ are the solution of (1.1) and (2.1), respectively, then $\gamma < \alpha$. Note that $\gamma = \sqrt{5}/2$, and $\alpha = \frac{1}{\sqrt{\sqrt{3}-1}} > \gamma$.

We need the notation of asymptotic centers, due to Edelstein [9]. Let C be a nonempty closed convex subset of a Banach space X and let $\{x_t : t \in G\}$ be a bounded net of elements of X. Then the asymptotic radius and asymptotic center of $\{x_t\}_{t\in G}$ with respect to C are the number

$$r_C\{x_t\} = \inf_{y\in C}\limsup_t ||x_t - y||,$$

and respectively, the (possibly empty) set

$$A_C(\{x_t\}) = \{y \in C : \limsup_t ||x_t - y|| = r_C(\{x_t\}).$$

Lemma 2.1. (cf. [25]) If C is a nonempty closed convex subset of a reflexive Banach space X, then for every bounded net $\{x_t\}_{t\in G}$ of elements of X, $A_C(\{x_t\})$ is a nonempty bounded closed convex subset of C. In particular, if X is a uniformly convex Banach space, then $A_C(\{x_t\})$ consists of a single point.

The following lemma can be proved in exactly the same way as in Lim [20] for sequences and the proof is thus omitted here.

Lemma 2.2.(cf. [25]) Suppose X is a Banach space with uniformly normal structure. Then for every bounded net $\{x_t\}_{t\in G}$ of elements of X there exists $y \in \overline{co}(\{x_t : t \in G\})$ such that

$$\limsup_t ||x_t - y|| \le \widetilde{N}(X)D(\{x_t\}),$$

where $\widetilde{N}(X) = 1/N(X)$, and $\overline{co}(E)$ is the closure of the convex hull of a set $E \subset X$ and $D(\{x_t\}) = \lim_t(\sup\{||x_i - x_j|| : t \le i, j \in G\})$ is the asymptotic diameter of $\{x_t\}$.

Lemma 2.3 (cf. [34]). Let X be a Banach space, C be a nonempty weakly compact separable subset of X, and $T = \{T(t) : t \in G\}$ be a semigroup of mappings of C into it self with $k = \liminf_{G \ni t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| < +\infty$. Then there exists a positive sequence $\{t_n\} \subset G$ such that for each $x \in C$, the sequence $\{T(t_n)x\}$ converges weakly.

3. ON GENERALIZED LIPSCHITZIAN MAPPING

The following lemma plays an important role in proving our results.

Lemma 3.1. If $\{T_s x_0; s \in G\}$ is bounded for some $x_0 \in C$ and $\tau = \{T_s; s \in G\}$ is a k-uniformly generalized Lipschitzian semigroup of mappings on C, then $\{T_s x; s \in G\}$ is bounded for each $x \in C$.

We next present the first result of this paper which weakens the uniform convexity assumption in Theorem 1.1.

Theorem 3.2. Suppose C be a nonempty closed convex subset of a real Banach space X with $N(X) > \max(1, \varepsilon_0)$, while $\tau = \{T_s; s \in G\}$ be a uniformly generalized Lipschitzian (in short G3-Lipschitzian) semigroup of mappings on C with $\rho < \alpha_*$ where ε_0 is the characteristic of convexity of X and

$$\alpha_* = \sup\Big\{\alpha : \alpha^2 \delta_X^{-1}(1 - \frac{1}{\alpha})N(X)^{-1} \leq 1 \text{ and } 4\ 1 - \frac{1}{\alpha} \in (0, 1 - \frac{1}{2}\varepsilon_0)\Big\}. \tag{3.1}$$

If $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in C$, then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

Proof. Put $\widetilde{N}(X) = N(X)^{-1}$. Observe that the set

$$\Big\{\alpha : \alpha^2 \delta_X^{-1}(1 - \frac{1}{\alpha})N(X)^{-1} \leq 1 \; and \; 1 - \frac{1}{\alpha} \in (0, 1 - \frac{1}{2}\varepsilon_0)\Big\} \neq \phi. \tag{3.2}$$

Indeed, by properties (a),(b),(d) of the modulus δ_x of convexity of X, we see that the mapping

$$\delta_x : [\varepsilon_0, 2) \longrightarrow \delta_x([\varepsilon_0, 2)) = [0, 1 - \frac{1}{2}\varepsilon_0)$$

is strictly increasing and continuous, and hence a bijection. Thus, we deduce that

$$\lim_{\alpha \longrightarrow 1^+} \alpha^2 \delta_X^{-1}(1 - \frac{1}{\alpha})N(X)^{-1} = \delta_X^{-1}(0)\widetilde{N}(X) = \varepsilon_0 \widetilde{N}(X) < 1.$$

which amounts to say that there exists $\alpha_0 > 1$ such that $\alpha_0^2 \delta_X^{-1}(1 - \frac{1}{\alpha_0})\widetilde{N}(X) < 1$ and

$$1 - \frac{1}{\alpha_0} \in \delta_x([\varepsilon_0, 2)) = [0, 1 - \frac{1}{2}\varepsilon_0).$$

This verifies our assertion (5).

Since X has a uniform normal structure and X is reflexive, due to the boundedness of $\{T_s x_0 : s \in G\}$ and Lemma 2.1, we conclude that $A_C(\{T_t x_0\} t \in G)$ is nonempty bounded closed and convex subset of C. Then, we can choose $x_1 \in A_C(\{T_t x_0\}_{t \in G})$ such that

$$\limsup_t ||T_t x_0 - x_1|| = \inf_{y \in C} \limsup_t ||T_t x_0 - y||.$$

Since τ satisfies a $k-$uniformly generalized Lipschitzian property, owing to Lemma 3.1 $T_t x_1$ remains bounded.

Consequently we can choose $x_2 \in A_C(\{T_t x_1\}_{t \in G})$ such that

$$\limsup_t ||T_t x_1 - x_2|| = \inf_{y \in C} \limsup_t ||T_t x_1 - y||.$$

Continuing this process, we can construct a sequence $\{x_n\}_{n=0}^{\infty}$ in C with the following two properties:

(i) for each $n \geq 0$, $\{T_t x_n\}_{t \in G}$ is bounded;

(ii) for each $n \geq 0$, $x_{n+1} \in A_C(\{T_t x_n\}_{t \in G})$; that is x_{n+1} is a point in C such that

$$\lim_t ||T_t x_n - x_{n+1}|| = \inf_{y \in C} \lim_t ||T_t x_n - y||.$$

Write $r_n = r_C(\{T_t x_n\}_{t\in G})$. Then by Lemma 2.2 we have

$$\begin{aligned} r_n &= \limsup_t \|T_t x_n - x_{n+1}\| \\ &\le \widetilde{N}(X) D(\{T_t x_n\}_{t\in G}) \\ &= \widetilde{N}(X) \lim_t (\sup\{\|T_i x_n - T_j x_n\| : t \le i, j \in G\}) \\ &\le \widetilde{N}(X) k \lim_t (\sup \max\{\|x_n - T_{j-i} x_n\|, \frac{1}{2\rho}\|x_n - T_i x_n\|, \frac{1}{2\rho}\|T_j x_n - T_{j-i} x_n\|, \\ &\frac{1}{2\rho}\|x_n - T_j x_n\|, \frac{1}{2\rho}\|T_{j-i} x_n - T_i x_n\|\} \\ &\le \widetilde{N}(X) k \lim_t (\sup \max\{d(x_n), \frac{1}{2\rho} d(x_n), \frac{1}{\rho} d(x_n), \frac{1}{2\rho} d(x_n), \frac{1}{\rho} d(x_n)\} \\ &\le \widetilde{N}(X).k.d(x_n), \end{aligned}$$

that is,

$$r_n \le \widetilde{N}(X).k.d(x_n) \le \rho.\widetilde{N}(X).d(x_n). \tag{3.3}$$

where

$$d(x_n) = \sup\{\|x_n - T_t x_n\| : t \in G\}.$$

We may assume that $d(x_n) > 0$ for all $n \ge 0$ (otherwise x_n is a common fixed point of the semigroup τ and the proof is over). Let $n \ge 0$ be fixed and let $\varepsilon > 0$ be small enough. We can choose $j \in G$ such that

$$\|T_j x_{n+1} - x_{n+1}\| > d(x_{n+1}) - \varepsilon$$

and then choose $s_0 \in G$ so large that

$$\|T_s x_n - x_{n+1}\| < r_n + \varepsilon \le \rho(r_n + \varepsilon)$$

for all $s \ge s_0$. Now, for $s \ge s_0 + j$,

$$\begin{aligned} \|T_s x_n - T_j x_{n+1}\| &\le k \max\{\|T_{s-j} x_n - x_{n+1}\|, \frac{1}{2\rho}\|T_s x_n - T_{s-j} x_n\|, \frac{1}{2\rho}\|x_{n+1} - T_j x_{n+1}\|, \\ &\frac{1}{2\rho}\|T_{s-j} x_n - T_j x_{n+1}\|, \frac{1}{2\rho}\|x_{n+1} - T_s x_n\|\} \\ &\le k \max\{\|T_{s-j} x_n - x_{n+1}\|, \frac{1}{2\rho}\|T_s x_n - T_{s-j} x_n\|, \frac{1}{2\rho}\limsup_t \|x_{n+1} - T_{j+t} x_n\|, \\ &\frac{1}{2\rho}\limsup_t \|T_{s-j} x_n - T_{j+t} x_n\|, \frac{1}{2\rho}\|x_{n+1} - T_s x_n\|\} \\ &\le k \max\{r_n + \varepsilon, \frac{1}{\rho}(r_n + \varepsilon), \frac{1}{2\rho}(r_n + \varepsilon), \frac{1}{\rho}(r_n + \varepsilon), \frac{1}{2\rho}(r_n + \varepsilon)\} \end{aligned}$$

so that

$$\|T_s x_n - T_j x_{n+1}\| \le k(r_n + \varepsilon) \le \rho(r_n + \varepsilon).$$

Then owing to property (e), it follows that

$$\|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \quad \le \rho(r_n + \varepsilon)\Big(1 - \delta_X\Big(\frac{d(x_{n+1} - \varepsilon)}{\rho(r_n + \varepsilon)}\Big)\Big)$$

for $s \ge s_0 + j$ and hence

$$r_n \le \limsup_s \|T_s x_n - \frac{1}{2}(x_{n+1} + T_j x_{n+1})\| \le \rho(r_n + \varepsilon)\Big(1 - \delta_X\Big(\frac{d(x_{n+1} - \varepsilon)}{\rho(r_n + \varepsilon)}\Big)\Big).$$

Taking the limit as $\varepsilon \longrightarrow 0$, we obtain

$$r_n \le \rho r_n \Big(1 - \delta_X\Big(\frac{d(x_{n+1})}{\rho r_n}\Big)\Big)$$

which implies that

$$\delta_X\Big(\frac{d(x_{n+1})}{\rho r_n}\Big) \le 1 - \frac{1}{\rho} \tag{3.4}$$

or

$$d(x_{n+1}) \leq \rho r_n \delta_X^{-1}(1-\frac{1}{\rho}). \tag{3.5}$$

Indeed, if $d(x_{n+1})/(\rho r_n) \in [0,\varepsilon_0)$, then noticing that $\delta_X : [\varepsilon_0, 2) \longrightarrow [0, 1-\varepsilon_0/2)$ is a bijection and that $1-\frac{1}{\rho}$ lies in $[0,1-\varepsilon_0/2)$. By assumption $k<\rho<\alpha_*$, we have $\delta_X^{-1}(1-\frac{1}{\rho}) \geq \varepsilon_0$; hence $d(x_{n+1})/(\rho r_n) \leq \delta_X^{-1}(1-\frac{1}{\rho})$ and (3.5) follows. If $d(x_{n+1})/(\rho r_n) \in [\varepsilon_0, 2]$, then it is clear that $d(x_{n+1})/(\rho r_n) \leq \delta_X^{-1}(1-\frac{1}{\rho})$. This also shows that (3.5) is true. Therefore, utilizing (3.3) and (3.5), we obtain

$$d(x_{n+1}) \leq \rho^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\rho})d(x_n). \tag{3.6}$$

Write $A = \rho^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\rho})$. Then $A<1$. Indeed, from the assumption that $\rho<\alpha_*$ it follows that there exists an $\tilde{\alpha}>\rho$ such that

$$\tilde{\alpha}^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\tilde{\alpha}}) \leq 1 \text{ and } (1-\frac{1}{\tilde{\alpha}}) \in \delta_X((\varepsilon_0, 2)).$$

It then turns out that $\delta_X^{-1}(1-\frac{1}{\rho}) < \delta_X^{-1}(1-\frac{1}{\tilde{\alpha}})$, and

$$A = \rho^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\rho}) < \tilde{\alpha}^2 \widetilde{N}(X)\delta_X^{-1}(1-\frac{1}{\tilde{\alpha}}) \leq 1.$$

Hence, it is follows from (3.6) that

$$d(x_n) \leq Ad(x_{n-1}) \leq ... \leq A^n d(x_0). \tag{3.7}$$

Since

$$\begin{aligned} ||x_{n+1}-x_n|| \quad &\leq \limsup_t ||T_t x_n - x_{n+1}|| + \limsup_t ||T_t x_n - x_n|| \\ &\leq r_n + d(x_n) \leq 2d(x_n). \end{aligned}$$

We get from (3.7) that $\sum_{n=1}^{\infty} ||x_{n+1}-x_n|| < \infty$, and hence $\{x_n\}$ is a norm-Cauchy. Let $z = ||.|| - \lim_n x_n$. Finally, we have for each $s \in G$,

$$\begin{aligned} ||z - T_s z|| \quad &= \lim_{n\longrightarrow\infty} ||x_n - T_s x_n|| \\ &\leq \lim_{n\longrightarrow\infty} d(x_n) = 0 \end{aligned}$$

so that $T_s z = z$ for all $s \in G$. This completes the proof of the theorem.

Corollary 3.4. Suppose that X is a real Banach space with $N(X) > \max(1, \varepsilon_0)$, and C is a nonempty closed convex subset of X, while $\tau = \{T_s; s \in G\}$ is a uniformly generalized Lipschitzian (in short G2-Lipschitzian) semigroup on C with $\rho < \alpha_*$. Also, ε_0 is the characteristic of convexity of X and

$$\alpha_* = \sup\Big\{\alpha : \alpha^2\delta_X^{-1}(1-\frac{1}{\alpha})N(X)^{-1} \leq 1 \textit{ and } 1-\frac{1}{\alpha} \in (0, 1-\frac{1}{2}\varepsilon_0)\Big\}.$$

If $\{T_s x_0 : s \in G\}$ is bounded for some $x_0 \in C$, then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

Theorem 3.5. Let C be a nonempty bounded subset of a uniformly convex Banach space X, and $\tau = \{T_s : s \in G\}$ be a k-uniformly generalized Lipschitzian (in short G3-Lipschitzian) semigroup of mappings on C such that

$$\rho < \sqrt{\gamma_0 N(X)}, \text{ where } \gamma_0 = \inf\{\gamma \geq 1 : \gamma(1-\delta_X(1/\gamma)) \geq 1/2. \tag{3.8}$$

Also, there exists a nonempty bounded closed convex subset E of C with the following property $(\mathbb{R})$:

$(\mathbb{R})$ $x \in E$ implies $w_w(x) \subset E$.

Then there exists $z \in E$ such that $T_s z = z$ for all $s \in G$.

Proof. Take an $x_0 \in E$ and, consider for $t \in G$, the bounded net $\{T_s x_0 : t \leq s \in G\}$. Owing to Lemma 2.2, we have a $y_t \in \overline{co}\{T_s x_0 : t \leq s \in G\}$ such that

$$\limsup_s ||T_s x_0 - y_t|| \leq \widetilde{N}(X) D(\{T_s x_0\} t \leq s \in G), \tag{3.9}$$

where $\widetilde{N}(X) = 1/N(X)$ and $D(\{T_s x_0\} t \leq s \in G)$ denotes the asymptotic diameter of the net $\{z_l\}$ i.e, the number

$$\lim_t (sup\{||z_i - z_j|| : t \leq i, j \in G\}).$$

Since X is reflexive, $\{y_t\}$ admits a subnet $\{y_{t_\beta}\}$ converging weakly to some $x_1 \in X$. From (3.9) and the weak lower semicontinuity of the functional $\limsup_t ||T_t x_0 - y||$, it follows that

$$\limsup_t ||T_t x_0 - x_1|| \leq \widetilde{N}(X) D(\{T_t x_0\} t \in G). \tag{3.10}$$

It is also seen that $x_1 \in \bigcap_{t \in G} \overline{co}\{T_s x_0 : t \leq s \in G\}$ and

$$||z - x_1|| \leq \limsup_t ||z - T_t x_0|| \text{ for all } z \in X. \tag{3.11}$$

Owing to Property ($\mathbb{R}$) and the fact that $\bigcap_{t \in G} \overline{co}\{T_s x_0 : t \leq s \in G\} = \overline{co}\{w_w(x_0)\}$ which is easy to prove by using the Separation Theorem (see [2]). As, we know that x_1 lies in E, we can repeat the above process and obtain a sequence $\{x_n\}_{n=0}^{\infty}$ in E with the properties: (for all nonnegative integers $n \geq 0$),

$$\limsup_t ||T_t x_n - x_{n+1}|| \leq \widetilde{N}(X) D(\{T_t x_n\} t \in G) \tag{3.12}$$

and

$$||z - x_{n+1}|| \leq \limsup_t ||z - T_t x_n|| \text{ for all } z \in X. \tag{3.13}$$

Write $r_n = \limsup_t ||T_t x_n - x_{n+1}||$ and $d(x_n) = \sup\{||x_n - T_t x_n|| : t \in G\}$. Thus in view of (3.12), we have

$$\begin{aligned} r_n &= \limsup_t ||T_t x_n - x_{n+1}|| \\ &\leq \widetilde{N}(X) D(\{T_t x_n\}_{t \in G}) \\ &= \widetilde{N}(X) \lim_t (\sup\{||T_i x_n - T_j x_n|| : t \leq i, j \in G\}) \\ &\leq \widetilde{N}(X) k \lim_t (\sup \max\{||x_n - T_{j-i} x_n||, \frac{1}{2\rho}||x_n - T_i x_n||, \frac{1}{2\rho}||T_j x_n - T_{j-i} x_n||, \frac{1}{2\rho}||x_n - T_j x_n|| \\ &, \frac{1}{2\rho}||T_{j-i} x_n - T_i x_n||\} \\ &\leq \widetilde{N}(X) k \lim_t (\sup \max\{d(x_n), \frac{1}{2\rho} d(x_n), \frac{1}{\rho} d(x_n), \frac{1}{2\rho} d(x_n), \frac{1}{\rho} d(x_n)\} \\ &\leq \widetilde{N}(X).k.d(x_n), \end{aligned}$$

so that

$$r_n \leq \widetilde{N}(X).k.d(x_n) \leq \rho.\widetilde{N}(X).d(x_n). \tag{3.14}$$

We may assume that $d(x_n) > 0$ for all $n \geq 0$. Let $n \geq 0$ be fixed and let $\varepsilon > 0$ be small enough. First choose $j \in G$ such that

$$||T_j x_{n+1} - x_{n+1}|| > d(x_{n+1}) - \varepsilon$$

and then choose $s_0 \in G$ so large that

$$||T_s x_n - x_{n+1}|| < r_n + \varepsilon \leq \rho(r_n + \varepsilon)$$

for all $s \geq s_0$. Now, for $s \geq s_0 + j$,

$$||T_s x_n - T_j x_{n+1}|| \leq k \max\{||T_{s-j} x_n - x_{n+1}||, \frac{1}{2\rho}||T_s x_n - T_{s-j} x_n||, \frac{1}{2\rho}||x_{n+1} - T_j x_{n+1}||,$$

$$\frac{1}{2\rho}||T_{s-j}x_n - T_jx_{n+1}||, \frac{1}{2\rho}||x_{n+1} - T_sx_n||\}$$
$$\leq k\max\{||T_{s-j}x_n - x_{n+1}||, \frac{1}{2\rho}||T_sx_n - T_{s-j}x_n||, \frac{1}{2\rho}\limsup_t ||x_{n+1} - T_{j+t}x_n||,$$
$$\frac{1}{2\rho}\limsup_t ||T_{s-j}x_n - T_{j+t}x_n||, \frac{1}{2\rho}||x_{n+1} - T_sx_n||\}$$
$$\leq k\max\{r_n+\varepsilon, \frac{1}{\rho}(r_n+\varepsilon), \frac{1}{2\rho}(r_n+\varepsilon), \frac{1}{\rho}(r_n+\varepsilon), \frac{1}{2\rho}(r_n+\varepsilon)\}.$$

so that

$$||T_sx_n - T_jx_{n+1}|| \leq k(r_n+\varepsilon) \leq \rho(r_n+\varepsilon)$$

Then, it follows from property (e) that (for $s \geq s_0 + j$),

$$||T_sx_n - \frac{1}{2}(x_{n+1} + T_jx_{n+1})|| \qquad \leq \rho(r_n+\varepsilon)\Big(1 - \delta_X\Big(\frac{d(x_{n+1}-\varepsilon)}{\rho(r_n+\varepsilon)}\Big)\Big).$$

Hence from (3.13) (taking $z := (x_{n+1} + T_jx_{n+1})/2$), we obtain

$$\begin{aligned}\frac{1}{2}(d(x_{n+1}-\varepsilon)) \quad &< ||\frac{1}{2}(T_jx_{n+1} - x_{n+1})|| \\ &\leq ||T_jx_{n+1} - \frac{1}{2}(x_{n+1} + T_jx_{n+1})|| \\ &\leq \limsup_t ||T_tx_n - \frac{1}{2}(x_{n+1} + T_jx_{n+1})|| \\ &\leq \rho(r_n+\varepsilon)\Big(1 - \delta_X\Big(\frac{d(x_{n+1}-\varepsilon)}{\rho(r_n+\varepsilon)}\Big)\Big). \end{aligned} \tag{3.15}$$

Taking the limit as $\varepsilon \longrightarrow 0$, we have

$$\frac{1}{2}d(x_{n+1}) \leq \rho r_n\Big(1 - \delta_X\Big(\frac{d(x_{n+1})}{\rho r_n}\Big)\Big). \tag{3.16}$$

On the other hand, using (3.13)we easily find (for each $j \in G$),

$$||T_jx_{n+1} - x_{n+1})|| \leq \limsup_t ||T_{j+t}x_n - x_{n+1}|| = r_n \leq \rho r_n.$$

It turns out that

$$d(x_{n+1}) \leq \rho r_n \tag{3.17}$$

Combining (3.16) and (3.17) and using the definition of γ_0 in (3.8), we infer that $(\rho r_n)/d(x_{n+1}) \geq \gamma_0$. It turns out from (3.14) that

$$d(x_{n+1}) \leq \frac{\rho}{\gamma_0}r_n \leq \frac{\rho^2}{\gamma_0 N(X)}d(x_n).$$

Consequently, we obtain

$$d(x_n) \leq Ad(x_{n-1} \leq A^n d(x_0)),$$

where $A = \rho^2[\gamma_0 N(X)]^{-1} < 1$ by assumption. Noticing that

$$\begin{aligned}||x_{n+1} - x_n|| \quad &\leq \limsup_t ||T_tx_n - x_{n+1}|| + \limsup_t ||T_tx_n - x_n|| \\ &\leq r_n + d(x_n) \\ &\leq (1 + k\widetilde{N}(X))d(x_n) \\ &\leq (1 + k\widetilde{N}(X))A^n d(x_0),\end{aligned}$$

so that the series $\sum_{n=1}^{\infty} ||x_{n+1} - x_n||$ is convergent. This implies that $\{x_n\}$ is strongly convergent. Let $z = ||.|| - \lim_n x_n$. Then, we have (for each $s \in G$)

$$\begin{aligned}||z - T_sz|| \quad &= \lim_{n\longrightarrow\infty} ||x_n - T_sx_n|| \\ &\leq \lim_{n\longrightarrow\infty} d(x_n) = 0\end{aligned}$$

so that $T_sz = z$ for all $s \in G$ and this concludes the proof.

Corollary 3.6. Let C be a nonempty bounded subset of a uniformly convex Banach space X and $\tau = \{T_s : s \in G\}$ be a k-uniformly generalized Lipschitzian (in short G2-Lipschitzian) semigroup of mappings on C such that

$$\rho < \sqrt{\gamma_0 N(X)}, \text{ where } \gamma_0 = \inf\{\gamma \geq 1 : \gamma(1 - \delta_X(1/\gamma)) \geq 1/2.$$

Suppose that there exists a nonempty bounded closed convex subset E of C with the following property ($\mathbb{R}$):
($\mathbb{R}$) $x \in E$ implies $w_w(x) \subset E$.
Then there exists $z \in E$ such that $T_s z = z$ for all $s \in G$.

4. ON EXPANSIVE LIPSCHITZ CONSTANT

We prove our final result as follows.

Theorem 4.1. Let C be a nonempty convex weakly compact subset of a Banach space X equipped with weak uniform normal structure and $T = \{T(t) : t \in G\}$ be an asymptotically regular $k-$uniformly Lipschitzian semigroup of mappings on C such that

$$\liminf_{G \ni t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| = \lim_{G \ni t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| = k < WCS(X)\mu_0,$$

where $\mu_0 = \inf\{\mu \geq 1 : \mu(1 - \delta_X(1/\mu)) \geq (1/2)\}$.
If each $T(t)$ is weakly continuous, then $F(t)$ is nonempty.

Proof. Firstly, let us choose a sequence of positive real numbers $\{t_n\} \subset G$ which increase monotonically to $+\infty$ such that

$$\liminf_{t \longrightarrow \infty} \Big| \, ||T(t)|| \, \Big| = \lim_{n \longrightarrow \infty} \Big| \, ||T(t_n)|| \, \Big| = k < WSC(X).$$

Since one can construct (cf. [23]) a nonempty convex closed separable subset C_0 of C which is invariant under $T(t_n)$ (i.e., $(T(t_n)C_0 \subset C_0 \quad \forall \; n = 0, 1, 2, ...)$), we may assume for a while that C itself is separable. In view of lemma 2.1, by passing to a subsequence it is possible to assume that for each $x \in C$ the sequence $\{T(t_n)x\}$ is weakly convergent. Now we define a sequence $\{x_n\}_{n=1}^{\infty}$ in C as

$$x_0 \in C \quad \text{arbitrary}, \qquad x_{m+1} = w - \lim_{n \longrightarrow \infty} T(t_n)x_m, \; m \geq 0.$$

Now, it is easy to show that $x_{m+1} = w - \lim_{j \longrightarrow \infty} T(t_j + s)x_m \quad \forall \; s \in G, \; m \geq 0$. Define

$$R_m = \limsup_{j \longrightarrow \infty} ||T(t_j)x_m - x_{m+1}|| \;\; \forall \;\; m \geq 0.$$

Now, we prove that $R_m \leq WCS(X)^{-1} D[(T(t_n)x_m)] \quad \forall \; m \geq 0$. Indeed, let $R_m \neq 0$, and let $\{t_{n_i}\}$ be a subsequence of $\{t_n\}$ such that

$$R_m = \limsup_{n \longrightarrow \infty} ||T(t_n)x_m - x_{m+1}|| = \lim_{j \longrightarrow \infty} ||T(t_{n_j})x_m - x_{m+1}||,$$

Also, define a sequence $\{y_j\}$ as

$$y_j = (T(t_{n_j})x_m - x_{m+1})/R_m \qquad \forall \;\; j \geq 1$$

then $||y_j|| \longrightarrow 1$ and $y_j \longrightarrow 0$.
Owing to Lemma 2.1, we have

$$\begin{aligned} WCS(X) = \beta(X) \quad & \leq D[(y_j)] = \limsup_{j \longrightarrow \infty} \limsup_{i \longrightarrow \infty} || \frac{T(t_{n_j})x_m - x_{m+1}}{R_m} - \frac{T(t_{n_i})x_m - x_{m+1}}{R_m} || \\ & \leq \frac{1}{R_m} \limsup_{j \longrightarrow \infty} \limsup_{i \longrightarrow \infty} ||T(t_{n_j})x_m - T(t_{n_i})x_m|| \\ & \leq \frac{1}{R_m} D[(T(t_{n_j})x_m)] \leq \frac{1}{R_m} D[(T(t_n)x_m)] \end{aligned}$$

so that

$$R_m \leq WCS(X)^{-1} D[(T(t_n)x_m)]. \tag{4.1}$$

Let $n \geq 0$ be fixed and $\varepsilon > 0$ be small enough. Choose $t_s, t_a \in \{t_n\}$ such that

$$||T(t_n)x_{m+1} - T(t_a)x_{m+1}|| > D[(T(t_n)x_{m+1})] - \varepsilon,$$

so that for every $s > n$,

$$\begin{aligned} ||T(t_s)x_m - w - \lim_{n\longrightarrow\infty} T(t_n)x_{m+1}|| &= w - \lim_{n\longrightarrow\infty} ||T(t_n)T(t_s - t_n)x_m - T(t_n)x_{m+1}|| \\ &\leq w - \lim_{n\longrightarrow\infty} \Big| \, ||T(t_n)|| \, \Big| \, ||T(t_s - t_n)x_m - x_{m+1}|| \\ &\leq (k+\epsilon)(R_m + \varepsilon). \end{aligned}$$

Also; for all $s > a$,

$$\begin{aligned} ||T(t_s)x_m - w - \lim_{a\longrightarrow\infty} T(t_a)x_{m+1}|| &\leq w - \lim_{a\longrightarrow\infty} \Big| \, ||T(t_a)|| \, \Big| \, ||T(t_s - t_a)x_m - x_{m+1}|| \\ &\leq (k+\epsilon)(R_m + \varepsilon). \end{aligned}$$

Then; in view of the property (e):

$$\begin{aligned} w - \lim_{n,a\longrightarrow\infty} ||T(t_s)x_m &- \frac{1}{2}(T(t_a)x_{m+1} + T(t_n)x_{m+1})|| \\ &\leq (k+\epsilon)(R_m - \varepsilon)\Big[1 - \delta_X(\frac{D[(T(t_n)x_{m+1})] - \varepsilon}{k(R_m - \varepsilon)})\Big]. \end{aligned} \tag{4.2}$$

Since, each $T(t)$ is weakly continuous,

$$\begin{aligned} \tfrac{1}{2}(D[(T(t_n)x_{m+1})] - &\varepsilon) < ||\frac{1}{2}(T(t_n)x_{m+1} - T(t_a)x_{m+1})|| \\ &\leq ||T(t_n)x_{m+1} - \frac{1}{2}(T(t_n)x_{m+1} + T(t_a)x_{m+1})|| \\ &\leq \lim_{l\longrightarrow\infty} ||T(t_n + t_l)x_m - \frac{1}{2}(T(t_n)x_{m+1} + T(t_a)x_{m+1})|| \end{aligned} \tag{4.3}$$

so that in view of (4.2) and (4.3), we have

$$\tfrac{1}{2}(D[(T(t_n)x_{m+1})] - \varepsilon) \leq (k+\epsilon)(R_m - \varepsilon)\Big[1 - \delta_X(\tfrac{D[(T(t_n)x_{m+1})] - \varepsilon}{(k+\epsilon)(R_m - \varepsilon)})\Big].$$

Taking the limit as $\varepsilon \longrightarrow \infty$, we have

$$\tfrac{1}{2}D[(T(t_n)x_{m+1})] \leq kR_m\Big[1 - \delta_X(\tfrac{D[(T(t_n)x_{m+1})]}{kR_m})\Big]. \tag{4.4}$$

Since, each $T(t)$ is weakly continuous, therefore $T(t_i)x_m = w - \lim_{l\longrightarrow\infty} T(t_i + t_l)x_{m-1}$ for all $i \geq 1$, so that

$$\begin{aligned} D[(T(t_n)x_{m+1})] &= \limsup_{n\longrightarrow\infty} \limsup_{l\longrightarrow\infty} ||T(t_n)x_{m+1} - T(t_l)x_{m+1}|| \\ &\leq \limsup_{n\longrightarrow\infty} \limsup_{l\longrightarrow\infty} \lim_{i\longrightarrow\infty} ||T(t_n + t_i)x_m - T(t_l)x_{m+1}|| \\ &\leq k \limsup_{n\longrightarrow\infty} \limsup_{l\longrightarrow\infty} \lim_{i\longrightarrow\infty} ||T(t_n + t_i - t_l)x_m - x_{m+1}|| \\ &\leq kR_m. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) and using the definition of μ_0, we get $(kR_m)/D[(T(t_n)x_{m+1})] \geq \mu_0$. Owing to (4.1), we have

$$\begin{aligned} D[(T(t_n)x_{m+1})] &\leq \frac{k}{\mu_0}R_m \\ &\leq \frac{k}{\mu_0 WCS(X)}D[(T(t_n)x_m)] \\ &\leq A.D[(T(t_n)x_m)] \leq ... \leq A^m.D[(T(t_n)x_0)] \end{aligned}$$

where $A = \frac{k}{\mu_0 WCS(X)} < 1$ by assumption. It follows from the weak lower semicontinuity of the norm $||.||$ of X that,

$$\begin{aligned} ||x_{m+1} - x_m|| &\leq \limsup_{j\longrightarrow\infty} \Big[||x_m - T(t_j)x_m|| + ||T(t_j)x_m - x_{m+1}||\Big] \\ &\leq \limsup_{j\longrightarrow\infty} \liminf_{i\longrightarrow\infty} ||T(t_i)x_{m-1} - T(t_j)x_m|| + R_m \end{aligned}$$

$$\begin{aligned}
&\le \limsup_{j\longrightarrow\infty}\liminf_{i\longrightarrow\infty}||T(t_j)T(t_i-t_j)x_{m-1}-T(t_j)x_m||+R_m\\
&\le \limsup_{j\longrightarrow\infty}\limsup_{i\longrightarrow\infty}||T(t_j)T(t_i-t_j)x_{m-1}-T(t_j)x_m||+R_m\\
&\le \limsup||T(t_j)||\limsup_{j\longrightarrow\infty}\limsup_{i\longrightarrow\infty}||T(t_i-t_j)x_{m-1}-x_m||+R_m\\
&\le kR_{m-1}+R_m\\
&\le k\frac{D[(T(t_n)x_{m-1})]}{WCS(X)}+\frac{D[(T(t_n)x_m)]}{WCS(X)}\\
&\le \frac{kA^{m-1}+A^m}{WCS(X)}D[(T(t_n)x_0)].
\end{aligned}$$

Hence, $\{x_m\}$ is a Cauchy sequence. Let $x_\infty=\lim_{m\longrightarrow\infty}x_m$. Then, $x_\infty\in C$, and

$$\begin{aligned}
||T(t_j)x_\infty-x_\infty|| &= ||T(t_j)x_\infty-\lim_{m\longrightarrow\infty}x_m||\\
&\le \lim_{m\longrightarrow\infty}\limsup_{k\longrightarrow\infty}||T(t_j)x_m-T(t_k)x_{m-1}||\\
&\le ||T(t_j)||\lim_{m\longrightarrow\infty}\limsup_{k\longrightarrow\infty}||x_m-T(t_k-t_j)x_{m-1}||\\
&\le ||T(t_j)||\lim_{m\longrightarrow\infty}R_{m-1}=0.
\end{aligned}$$

Obviously, $T(t_j)x_\infty=x_\infty$ for all $j\ge 1$. Now we prove that $T(s)x_\infty=x_\infty$ for all $s\in G$,

$$\begin{aligned}
||T(s)x_\infty-x_\infty|| &= \lim_{m\longrightarrow\infty}||T(s)x_m-x_m||\\
&\le \lim_{m\longrightarrow\infty}\lim_{j\longrightarrow\infty}||T(s+t_j)x_{m-1}-x_m||\\
&\le \lim_{m\longrightarrow\infty}\lim_{k\longrightarrow\infty}||T(t_k)x_{m-1}-x_m||\\
&\le \lim_{m\longrightarrow\infty}R_{m-1}=0
\end{aligned}$$

so that $T(s)x_\infty=x_\infty$. This completes the proof.

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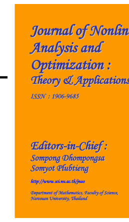

TWO GENERAL FIXED POINT RESULTS ON WEAK PARTIAL METRIC SPACE

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ABSTRACT. In this work, we obtain two fixed point results on weak partial metric space. Our results are extend and generalize some previous results.

KEYWORDS : Fixed point; Partial metric; Weak partial metric.
AMS Subject Classification: 54H25, 47H10

1. INTRODUCTION

The concept of partial metric p on a nonempty set X was introduced by Matthews [8]. One of the most interesting properties of a partial metric is that $p(x, x)$ may not be zero for $x \in X$. Also, each partial metric p on a nonempty set X generates a T_0 topology on X. After the definition of partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Then many authors gave some generalizations of this result on this space (See [1, 3, 7, 9, 10, 11, 12]). Recently, Chi, Karapınar and Thanh [4] obtained a fixed point theorem using a new type contractive condition, which is quite different from usual contractive conditions.

On the other hand, Heckman defined the concept of weak partial metric space and viewed some topological properties of it. Then Altun and Durmaz [2] proved the fundamental fixed point theorem on this space. Also, Durmaz et al [5], obtained some generalization of the result of [2]. In this work, we continue to study on fixed point theory in weak partial metric space. For this, we use Chi, Karapınar and Thanh type contractive condition.

2. PRELIMINARIES

In this section, we recall partial metric and weak partial metric space and some properties of them.

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Definition 2.1 ([8]). A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) such that for all $x, y, z \in X$:

(i) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$ (T_0-separation axiom),

(ii) $p(x,x) \leq p(x,y)$ (small self-distance axiom),

(iii) $p(x,y) = p(y,x)$ (symmetry),

(iv) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (modified triangular inequality).

A partial metric space (for short PMS) is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Example 2.2. A mapping $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by

$$p(x,y) = \begin{cases} |x-y| & , \quad x, y \in [0,1) \\ \max\{x,y\} & , \quad \text{otherwise} \end{cases}$$

is a partial metric on $\mathbb{R}^+$.

Example 2.3. Let $p : \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ be defined by

$$p(x,y) = \begin{cases} 0 & , \quad x = y \geq 0 \\ 2^{-|x|} & , \quad x \neq 0 \text{ and } y = 0 \\ 2^{-|y|} & , \quad x = 0 \text{ and } y \neq 0 \\ 2^{-\min\{|x|,|y|\}} & , \quad otherwise \end{cases}$$

is a partial metric on $\mathbb{N} \cup \{0\}$.

Example 2.4. Let $P(\mathbb{N})$ be the set all subsets of $\mathbb{N}$. If

$$p(x,y) = 1 - \sum_{n \in x \cap y} 2^{-n}$$

for all $x, y \in P(\mathbb{N})$, then p is a partial metric on $P(\mathbb{N})$.

If p is a partial metric on X, then the functions $p^s, p^w : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

and

$$\begin{aligned} p^w(x,y) &= \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \\ &= p(x,y) - \min\{p(x,x), p(y,y)\} \end{aligned}$$

are ordinary metrics on X. It is easy to see that p^s and p^w are equivalent metrics on X. For example, let $X = \mathbb{R}^+$ and $p(x,y) = \max\{x,y\}$, then $p^s(x,y) = |x-y| = p^w(x,y)$.

Note that each partial metric p on X generates a T_0-topology τ_p with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$. Since τ_p may not be Hausdorff, then if there exists the limit of a sequence may not be unique, too.

Remark 2.5. A sequence $\{x_n\}$ in a PMS (X,p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. Indeed, let $\{x_n\}$ converges to $x \in X$, with respect to τ_p, then there for all $\varepsilon > 0$, exists a positive integer n_0 such that $x_n \in B_p(x,\varepsilon)$ for $n \geq n_0$. Therefore, considering the small self distance property we have $p(x,x) \leq p(x_n,x) < p(x,x) + \varepsilon$ for $n \geq n_0$ and so letting limit $n \to \infty$, we have $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. The converse may be shown similarly.

Example 2.6. Let $X = \mathbb{R}^+$ and $p(x, y) = \max\{x, y\}$. Define a sequence in X by $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to any point of X.

Definition 2.7. (X, p) is a partial metric space. Then

(i) A sequence $\{x_n\}$ in X is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.

(ii) (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

It is well known that, every convergent sequence on an ordinary metric space is Cauchy, but this is not true on partial metric space. For example, let $X = \mathbb{R}^+$ and $p(x, y) = \max\{x, y\}$. Define a sequence $\{x_n\}$ by $\{x_n\} = \{0, 1, 0, 1, \cdots\}$, then it converges to any point of $[1, \infty)$, but it is not a Cauchy sequence. Also, we know that an ordinary metric is continuous and so sequentially continuous, but this is not true as shown in Example 2.2 for a partial metric.

The following lemma have an important role in the proof of our main result.

Lemma 2.8. *Assume that $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

According to [8], a sequence $\{x_n\}$ in X converges, with respect to τ_{p^s}, to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

By omitting the small self distance axiom, Heckmann [6] introduced the concept of weak partial metric space (for short WPMS), which is generalized version of Matthews' partial metric space. That is, the function $p : X \times X \to \mathbb{R}^+$ is called weak partial metric on X if it satisfies T_0 seperation axiom, symmetry and modified triangular inequality. Heckmann also shows that, if p is weak partial metric on X, then for all $x, y \in X$ we have the following weak small self-distance property

$$p(x, y) \geq \frac{p(x, x) + p(y, y)}{2}. \tag{2.1}$$

Weak small self-distance property shows that WPMS are not far from small self-distance axiom. It is clear that PMS is a WPMS, but the converse may not be true. A basic example of a WPMS but not a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \dfrac{x + y}{2}$ for all $x, y \in \mathbb{R}^+$. For another example, for $x, y \in \mathbb{R}$ the function $p(x, y) = \dfrac{e^x + e^y}{2}$ is a non partial metric but weak partial metric on $\mathbb{R}$.

The concepts of convergence of a sequence, Cauchy sequence and completeness in WPMS are defined as in PMS. Following Heckmann, in [2, 5] gave some fundamental fixed point results on weak partial metric space such that:

Theorem 2.1. *([2])Let (X, p) be a complete WPMS and let $F : X \to X$ be a map such that*

$$\begin{aligned} p(Fx, Fy) \quad &\leq \quad ap(x, y) + bp(x, Fx) + cp(y, Fy) + \\ &\qquad dp(x, Fy) + ep(y, Fx) \end{aligned}$$

for all $x, y \in X$, where $a, b, c, d, e \geq 0$ and, if $d \geq e$, then $a + b + c + 2d < 1$, if $d < e$, then $a + b + c + 2e < 1$. Then F has a unique fixed point.

Theorem 2.2. *([5])Let (X, p) be a complete WPMS, $\alpha \in [0, 1)$ and $T : X \to X$ a mapping. Suppose that for each $x, y \in X$ the following condition holds:*

$$p(Tx, Ty) \leq \max\{\alpha p(x, y), \min\{p(x, x), p(y, y)\}\}$$

Then:

(1) *the set* $X_p = \{x \in X : p(x,x) = \inf\{p(y,y) : y \in X\}\}$ *is nonempty,*

(2) *there is a unique* $u \in X_p$ *such that* $u = Tu$,

(3) *for each* $x \in X_p$ *the sequence* $\{T^n x\}$ *converges with respect to the metric* p^w *to* u.

3. THE MAIN RESULT

Theorem 3.1. *Let* (X,p) *be a complete weak partial metric space and* $T : X \to X$ *be a mapping such that for all* $x, y \in X$

$$p(Tx,Ty) \leq \max\left\{\begin{array}{l} ap(x,y), bp(x,Tx), cp(y,Ty), \\ d\{p(x,Ty)+p(y,Tx)\}, \\ \min\{p(x,x),p(y,y)\} \end{array}\right\} \tag{3.1}$$

for some $a, b, c \in [0,1)$ *and* $d \in [0, \frac{1}{2})$. *Then*

(a) $X_p = \{x \in X : p(x,x) = \inf\{p(y,y) : y \in X\}\}$ is nonempty,

(b) There is a unique $u \in X_p$ such that $u = Tu$.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be the sequence defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. First, we will prove that X_p is nonempty. For this, by taking $x = x_{n-1}$ and $y = x_n$ in (3.1) and then

$$\begin{array}{rcl} p(Tx_{n-1},Tx_n) & \leq & \max\left\{\begin{array}{l} ap(x_{n-1},x_n), bp(x_{n-1},Tx_{n-1}), cp(x_n,Tx_n), \\ d\{p(x_{n-1},Tx_n)+p(x_n,Tx_{n-1})\}, \\ \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\} \end{array}\right\} \\ & \leq & \max\left\{\begin{array}{l} ap(x_{n-1},x_n), bp(x_{n-1},Tx_{n-1}), cp(x_n,Tx_n), \\ d\{p(x_{n-1},x_n)+p(x_n,x_{n+1})\}, \\ \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\} \end{array}\right\} \\ & \leq & \max\left\{\begin{array}{l} ap(x_{n-1},x_n), bp(x_{n-1},Tx_{n-1}), cp(x_n,Tx_n), \\ 2dp(x_{n-1},x_n), 2dp(x_n,x_{n+1}), \\ \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\} \end{array}\right\}. \end{array}$$

We suppose that $\alpha = \max\{a,b,c,2d\}$, then

$$p(x_n,x_{n+1}) \leq \max\left\{\begin{array}{c} \alpha p(x_{n-1},x_n), \alpha p(x_n,x_{n+1}), \\ \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\} \end{array}\right\} \tag{3.2}$$

So we consider this in two cases:

Case I:

$$\max\{\alpha p(x_{n-1},x_n), \alpha p(x_n,x_{n+1}), \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\}\} = \alpha p(x_n,x_{n+1})$$

then we obtain

$$p(x_n,x_{n+1}) \leq \alpha p(x_n,x_{n+1})$$

since $\alpha \in [0,1)$, we say that $p(x_n,x_{n+1}) = 0$ and then $x_n = Tx_n$. Since $p(x_n,x_n) \leq 2p(x_n,x_{n+1})$, we obtain $p(x_n,x_n) = 0$. This implies that X_p is nonempty.

Case II:

$$\max\{\alpha p(x_{n-1},x_n), \alpha p(x_n,x_{n+1}), \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\}\} \neq \alpha p(x_n,x_{n+1})$$

for all $n \in \mathbb{N}$, then from (3.2), we obtain

$$\begin{array}{rcl} p(x_n,x_{n+1}) & \leq & \max\{\alpha p(x_{n-1},x_n), \min\{p(x_{n-1},x_{n-1}),p(x_n,x_n)\}\} \\ & \leq & \max\left\{\alpha p(x_{n-1},x_n), \dfrac{p(x_{n-1},x_{n-1})+p(x_n,x_n)}{2}\right\} \end{array} \tag{3.3}$$

$$\leq \quad p(x_{n-1}, x_n).$$

Hence $\{p(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. It follows that, there exist $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r.$$

If $r = 0$, then $p(x_n, x_n) \leq 2p(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. So $\lim_{n\to\infty} p(x_n, x_n) = 0$. Now, we consider the case $r > 0$. To do this, we set

$$r_n = \max\{\alpha p(x_{n-1}, x_n), \min\{p(x_{n-1}, x_{n-1}), p(x_n, x_n)\}\}$$

for all $n \in \mathbb{N}$. From (3.3) and $\lim_{n\to\infty} p(x_n, x_{n+1}) = r$ we have $\lim_{n\to\infty} r_n = r$.

We shall prove that $r_n = \alpha p(x_n, x_{n-1})$ for finite n. If $r_n = \alpha p(x_n, x_{n-1})$ for infinitely many n then there exists a sequence $\{n_k\}$ of positive integers such that

$$r_{n_k} = \alpha p(x_{n_k}, x_{n_k - 1}).$$

Letting $n_k \to \infty$ we obtain $r = \alpha r$. This is a contradiction with $\alpha \in [0, 1)$ and $r > 0$. Hence $r_n = \alpha p(x_n, x_{n-1})$ for finite n. Combining this fact with the definition of r_n , we can deduce that

$$\lim_{n\to\infty} p(x_n, x_n) = r.$$

Now for each $n = 1, 2, ...$ by $(P4)$ in the definition of weak partial metric space , we have

$$\begin{aligned}
\min\{p(x_n, x_n), p(x_{n+2}, x_{n+2})\} &\leq \frac{p(x_n, x_n) + p(x_{n+2}, x_{n+2})}{2} \\
&\leq p(x_n, x_{n+2}) \\
&\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}).
\end{aligned}$$

It follows from the above inequalities and

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{n\to\infty} p(x_n, x_n) = r$$

that

$$\lim_{n\to\infty} p(x_n, x_{n+2}) = r.$$

By induction we infer that

$$\lim_{n\to\infty} p(x_n, x_{n+s}) = r$$

for every positive integers s that is equivalent to saying that $\lim_{n,m\to\infty} p(x_n, x_m) = r$. Hence $\{x_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is complete there exist $u \in X$ such that $\{x_n\}$ converges to u as $n \to \infty$ that is

$$r = p(u, u) = \lim_{n\to\infty} p(x_n, u) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Let us prove

$$p(u, Tu) \leq p(u, u).$$

For each n, we have

$$\begin{aligned}
\min\{p(u, u), p(Tu, Tu)\} &\leq \frac{p(u, u) + p(Tu, Tu)}{2} \\
&\leq p(u, Tu) \\
&\leq p(u, x_n) + p(x_n, Tu) - p(x_n, x_n). \qquad (3.4)
\end{aligned}$$

Now we need some computations for $p(x_n, Tu)$. So from (3.1) we have

$$p(Tu, x_n) = p(Tu, Tx_{n-1})$$

$$\begin{aligned}
&\le \max\left\{\begin{array}{l} ap(u,x_{n-1}), bp(u,Tu), cp(x_{n-1},Tx_{n-1}), \\ d\left\{p(u,Tx_{n-1})+p(x_{n-1},Tu)\right\}, \\ \min\left\{p(u,u),p(x_{n-1},x_{n-1})\right\} \end{array}\right\} \\
&\le \max\left\{\begin{array}{l} ap(u,x_{n-1}), bp(u,Tu), cp(x_{n-1},x_n), \\ 2dp(u,x_n), 2dp(x_{n-1},Tu), p(u,x_{n-1}) \end{array}\right\} \\
&\le \max \begin{array}{l} p(u,x_{n-1}), bp(u,Tu), cp(x_{n-1},x_n), \\ 2dp(u,x_n), 2dp(x_{n-1},Tu) \end{array} \\
&\le \max\left\{\begin{array}{l} p(u,x_{n-1}), bp(u,Tu), cp(x_{n-1},x_n), 2dp(u,x_n), \\ 2d\left[p(x_{n-1},u)+p(u,Tu)-p(u,u)\right] \end{array}\right\} \\
&\le \max\left\{p(u,u), bp(u,Tu), 2dp(u,Tu)\right\} \\
&\le \max\left\{p(u,u), \alpha p(u,Tu)\right\} \qquad (3.5)
\end{aligned}$$

and we get $\{p(Tu,x_n)\}$ is bounded sequence. Thus it has a convergent subsequence $\{p(Tu,x_{n_k})\}$. Taking the limits from (3.5) as $n_k \to \infty$ and we get

$$\lim_{n_k\to\infty} p(Tu,x_{n_k}) \le \max\left\{p(u,u), \alpha p(u,Tu)\right\}.$$

Also letting $n_k \to \infty$ in (3.4) and combining with the above fact , we have

$$\begin{aligned}
p(u,Tu) &\le p(u,u) + \max\left\{p(u,u), \alpha p(u,Tu)\right\} - p(u,u) \\
&\le \max\left\{p(u,u), \alpha p(u,Tu)\right\} \\
&\le p(u,u).
\end{aligned}$$

Set

$$\rho_p = \inf\left\{p(y,y) : y \in X\right\}$$

For each $k = 1, 2, ...$ we can fix $x^k \in X$ such that

$$p(x^k,x^k) \le \rho_p + \frac{1}{k}.$$

By what we have proved for each $k = 1, 2, ...$ we can seek u^k such that $T^n x^k \to u^k$ as $n \to \infty$ and

$$p(Tu^k,u^k) \le p(u^k,u^k) = r_{u^k}.$$

We shall show that

$$\lim_{n,m\to\infty} p(u^n,u^m) = \rho_p.$$

Given $\epsilon > 0$ and put $n_0 := \left[\frac{3}{\varepsilon(1-\alpha)}\right] + 1$. If $k \ge n_0$ then using (3.1) we have

$$\begin{aligned}
\rho_p &\le p(Tu^k,Tu^k) \\
&\le \max\left\{\begin{array}{l} ap(u^k,u^k), bp(u^k,Tu^k), cp(u^k,Tu^k), \\ d\left\{p(u^k,Tu^k)+p(u^k,Tu^k)\right\}, \\ \min\left\{p(u^k,u^k),p(u^k,u^k)\right\} \end{array}\right\} \\
&\le \max\left\{\alpha p(u^k,Tu^k), p(u^k,u^k)\right\} \\
&\le p(u^k,u^k)
\end{aligned}$$

and so

$$\begin{aligned}
\rho_p &\le p(Tu^k,Tu^k) \le p(u^k,u^k) = r_{u^k} \le p(x^k,x^k) \\
&\le \rho_p + \frac{1}{k} \le \rho_p + \frac{1}{n_0} < \rho_p + \left[\frac{\varepsilon(1-\alpha)}{3}\right].
\end{aligned}$$

This implies that

$$U_k \quad : \quad = p(x^k,x^k) - p(Tx^k,Tx^k)$$

$$< \quad \rho_p + \left[\frac{\varepsilon(1-\alpha)}{3}\right] - \rho_p = \left[\frac{\varepsilon(1-\alpha)}{3}\right].$$

Also if $k \geq n_0$ then

$$p(u^k, u^k) = r_{u_k} \leq p(x^k, x^k) < \rho_p + \frac{1}{k} < \rho_p + \frac{1}{n_0}$$

implies that

$$p(u^k, u^k) < \rho_p + \left[\frac{\varepsilon(1-\alpha)}{3}\right].$$

Now, for each $m, n > n_0$, it follows from $p(u^k, Tu^k) \leq p(u^k, u^k)$ for all $k = 1, 2, \ldots$ that

$$\begin{aligned}
p(u^n, u^m) &\leq p(u^m, Tu^m) + p(Tu^n, u^n) + p(Tu^m, Tu^n) \\
&\quad -p(Tu^m, Tu^m) - p(Tu^n, Tu^n) \\
&\leq p(u^m, u^m) + p(u^n, u^n) + p(Tu^m, Tu^n) \\
&\quad -p(Tu^m, Tu^m) - p(Tu^n, Tu^n) \\
&= U_m + U_n + p(Tu^m, Tu^n) \\
&\leq 2\left[\frac{\varepsilon(1-\alpha)}{3}\right] + p(Tu^m, Tu^n). \qquad (3.6)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
p(Tu^m, Tu^n) &\leq \max\left\{\begin{array}{l} ap(u^m, u^n), bp(u^m, Tu^m), cp(u^n, Tu^n), \\ d\left\{p(u^m, Tu^n) + p(u^n, Tu^m)\right\}, \\ \min\left\{p(u^m, u^m), p(u^n, u^n)\right\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} ap(u^m, u^n), bp(u^m, Tu^m), cp(u^n, Tu^n), \\ d\left\{\begin{array}{l} p(u^m, u^n) + p(u^n, Tu^n) - p(u^n, u^n) + \\ p(u^n, u^m) + p(u^m, Tu^m) - p(u^m, u^m) \end{array}\right\}, \\ \min\left\{p(u^m, u^m), p(u^n, u^n)\right\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} ap(u^m, u^n), bp(u^m, Tu^m), cp(u^n, Tu^n), \\ d\left\{\begin{array}{l} p(u^m, u^n) + p(u^n, u^n) - p(u^n, u^n) + \\ p(u^n, u^m) + p(u^m, u^m) - p(u^m, u^m) \end{array}\right\}, \\ \min\left\{p(u^m, u^m), p(u^n, u^n)\right\} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} ap(u^m, u^n), bp(u^m, Tu^m), cp(u^n, Tu^n), \\ 2dp(u^m, u^n), \min\left\{p(u^m, u^m), p(u^n, u^n)\right\} \end{array}\right\} \\
&\leq \max\left\{\alpha p(u^m, u^n), p(u^m, u^m), p(u^n, u^n)\right\}.
\end{aligned}$$

By combining the above inequality with (3.6) we get

$$\begin{aligned}
p(u^n, u^m) &\leq 2\left[\frac{\varepsilon(1-\alpha)}{3}\right] + p(Tu^m, Tu^n) \\
&\leq 2\left[\frac{\varepsilon(1-\alpha)}{3}\right] + \max\left\{\alpha p(u^m, u^n), p(u^m, u^m), p(u^n, u^n)\right\}.
\end{aligned}$$

This implies that

$$p(u^n, u^m) \leq \max\left\{\begin{array}{l} \alpha p(u^m, u^n) + 2\left[\frac{\varepsilon(1-\alpha)}{3}\right], \\ p(u^m, u^m) + 2\left[\frac{\varepsilon(1-\alpha)}{3}\right], p(u^n, u^n) + 2\left[\frac{\varepsilon(1-\alpha)}{3}\right] \end{array}\right\}.$$

Thus

$$\rho_p \leq p(u^n, u^m)$$

$$
\begin{aligned}
&\le \max\left\{\begin{array}{l} \frac{2}{3}\varepsilon, p(u^m,u^m)+2\left[\frac{\varepsilon(1-\alpha)}{3}\right], \\ p(u^n,u^n)+2\left[\frac{\varepsilon(1-\alpha)}{3}\right] \end{array}\right\} \\
&\le \max\left\{\begin{array}{l} \frac{2}{3}\varepsilon, \rho_p+\left[\frac{\varepsilon(1-\alpha)}{3}\right]+2\left[\frac{\varepsilon(1-\alpha)}{3}\right], \\ \rho_p+\left[\frac{\varepsilon(1-\alpha)}{3}\right]+2\left[\frac{\varepsilon(1-\alpha)}{3}\right] \end{array}\right\} \\
&\le \max\left\{\frac{2}{3}\varepsilon, \rho_p+(1-\alpha)\varepsilon\right\} \\
&\le \max\left\{\frac{2}{3}\varepsilon, \rho_p+\varepsilon\right\} \\
&= \rho_p+\varepsilon.
\end{aligned}
$$

Therefore $\lim\limits_{n,m\to\infty} p(u^n,u^m)=\rho_p$, hence $\{u^n\}$ is Cauchy sequence. Since (X,p) is complete, there exist $y\in X$ such that $u^n\to y$ as $n\to\infty$ that is

$$p(y,y)=\lim_{n\to\infty} p(u^n,y)=\lim_{n,m\to\infty} p(u^n,u^m)=\rho_p.$$

Hence $y\in X_p$ or $X_p\neq\emptyset$. In this way (a) is proved.

Now if $y\in X_p$ then there exist $u\in X$ such that

$$p(u,Tu)\le p(u,u)=r_y$$

where

$$u=\lim_{n\to\infty} T^n y.$$

We have

$$\rho_p\le p(Tu,Tu) \text{ and } \rho_p\le p(u,u)=p(u,Tu)$$

and

$$\rho_p\le \frac{p(Tu,Tu)+p(u,u)}{2} \le p(u,Tu)=p(u,u)=r_y\le p(y,y)=\rho_p$$

so

$$p(u,u)=p(Tu,u)=p(Tu,Tu)$$

or $u=Tu$. To finish the proof we have to show that if $u,v\in X_p$ are both fixed point of T then $u=v$. Indeed it follows from $Tu=u$,$Tv=v$ and $p(u,u)=p(v,v)=\rho_p$ that

$$
\begin{aligned}
p(u,v) &= p(Tu,Tv) \\
&\le \max\left\{\begin{array}{l} ap(u,v), bp(u,Tu), cp(v,Tv), \\ d\{p(u,Tv)+p(v,Tu)\}, \\ \min\{p(u,u),p(v,v)\} \end{array}\right\} \\
&\le \max\{\alpha p(u,v), p(u,u), p(v,v)\}.
\end{aligned}
$$

This implies that $(1-\alpha)p(u,v)\le 0$ or $p(u,v)\le p(u,u)=p(v,v)=\rho_p$. If

$$(1-\alpha)p(u,v)\le 0$$

then $p(u,v)=0$, that is $u=v$. If

$$p(u,v)\le p(u,u)=p(v,v)=\rho_p$$

then $p(u,v)=p(u,u)=p(v,v)$ that is $u=v$.

□

Corollary 3.1. *Let (X, p) be a complete weak partial metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leq \max \left\{ \begin{array}{l} ap(x, y), bp(x, Tx), cp(y, Ty), \\ d\{p(x, Ty) + p(y, Tx)\} \end{array} \right\}$$

for some $a, b, c \in [0, 1)$ and $d \in [0, \frac{1}{2})$. Then X_p is nonempty and there is a unique $u \in X_p$ such that $u = Tu$.

Corollary 3.2. *Let (X, p) be a complete weak partial metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leq \max\{ap(x, y), \min\{p(x, x), p(y, y)\}\}$$

for some $a \in [0, 1)$. Then X_p is nonempty and there is a unique $u \in X_p$ such that $u = Tu$.

Corollary 3.3. *Let (X, p) be a complete weak partial metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$*

$$p(Tx, Ty) \leq ap(x, y) + bp(x, Tx) + cp(y, Ty) + d\{p(x, Ty) + p(y, Tx)\}$$

for some $a + b + c + d < 1$ and $a, b, c, d \geq 0$. Then T has a unique fixed point.

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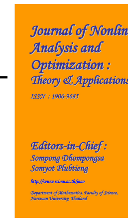

HYBRID FIXED POINT THEOREMS WITH PPF DEPENDENCE IN BANACH ALGEBRAS WITH APPLICATIONS

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ABSTRACT. In this paper, a couple of hybrid fixed point theorems with PPF dependence are proved in a Banach algebra and they are then applied to some nonlinear hybrid functional differential equations of delay and neutral type for proving the existence of PPF dependent solutions under some mixed Lipschitz and compactness type conditions.

KEYWORDS : Fixed point theorem; PPF dependence; Banach space; Functional differential equations.
AMS Subject Classification: 47H10; 34K10

1. INTRODUCTION

In recent papers [2, 7], the authors proved some fundamental fixed point theorems for nonlinear operators in Banach spaces satisfying the condition of linear contraction, wherein the domain and range of the operators are not same. The fixed point theorems of this kind are called PPF dependent fixed point theorems and are useful for proving the existence (and uniqueness) of solutions of nonlinear functional differential and integral equations which may depend upon the past, present and future. The properties of a special minimal or Razumikhin class of functions are employed in the development of existence theory of PPF solutions for certain nonlinear equations in abstract spaces. A study along this line is further continued in Dhage [5, 6], Agarwal *et. al.* [1], Kutbi and Sintunavarat [10] and Sintunavarat and Kumam [12] and proved some PPF dependent fixed point theorems in Banach spaces. In this paper, we prove some hybrid fixed point theorems with PPF dependence in Banach algebras and discuss some of their applications to nonlinear functional hybrid differential equations for proving the existence of PPF dependent solutions.

Given a Banach space E with norm $\|\cdot\|_E$ and given a closed interval $I = [a, b]$ in $\mathbb{R}$, the set of real numbers, let $E_0 = C(I, E)$ be the Banach space of continuous

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E-valued continuous functions defined on I. We equip the space E_0 with the supremum norm $\|\cdot\|_{E_0}$ defined as

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E. \tag{1.1}$$

Let $c \in I$ arbitrarily fixed. The ***minimal*** or ***Razumikhin class*** or $\mathcal{D}$-***class*** functions (cf. [2, 7]) is defined as

$$\mathcal{R}_c = \big\{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\big\}. \tag{1.2}$$

A Razumikhin class of functions $\mathcal{R}_c$ is said to be algebraically closed w.r.t. difference if $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Similarly, $\mathcal{R}_c$ is topologically closed if it is closed in the topology of of E_0 generated by the norm $\|\cdot\|_{E_0}$. Similarly, other notions such as compactness and connectedness for $\mathcal{R}_c$ may be defined.

Let $T : E_0 \longrightarrow E$. A point $\phi^* \in E_0$ is called a PPF dependent fixed point of T if $T\phi^* = \phi^*(c)$ for some $c \in I$ and any statement that guarantees the existence of PPF dependent fixed point is called a fixed point theorem with PPF dependence for the mapping T .

As mentioned in Bernfield *et al.* [2], the **Razumikhin class** of functions plays a significant role in proving the existence of PPF-fixed points with different domain and range of the operators. Very recently, generalizing a fixed point theorem of Bernfield *et al.* [2], the present author in Dhage [5] proved a first fixed point theorems with PPF dependence in the setting of nonlinear contraction of the operators in Banach spaces.

Definition 1.1. A nonlinear operator $T : E_0 \longrightarrow E$ is called a nonlinear $(\mathcal{B}, \mathcal{W})$-contraction if there exists an upper continuous function from the right $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\|T\phi - T\xi\|_E \le \psi(\|\phi - \xi\|_{E_0}) \tag{1.3}$$

for all $\phi, \xi \in E_0$, where $\psi(r) < r, r > 0$. T is called $\mathcal{B}$-contraction if there exists a nondecreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is continuous from right and satisfies (1.3). Finally, T is called $\mathcal{M}$-contraction if there exists a nondecreasing function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that satisfies satisfies (1.3), where $\lim_{n\longrightarrow\infty} \psi^n(t) = 0$, $t > 0$. We say T is nonlinear contraction if it is either a nonlinear $(\mathcal{D}, \mathcal{W})$ or $\mathcal{B}$ or $\mathcal{M}$-contraction on E_0 into E.

Note that every contraction is a nonlinear $(\mathcal{D}, \mathcal{W})$-contraction and every nonlinear $\mathcal{B}$-contraction is $\mathcal{M}$-contraction. However, the converse of the above statements may not be true. The details of different types of contractions appear in the monographs of Krasnoselskii [9], Browder [4], Boyd and Wong [3], Granas and Dugundji [8] and Mathowski [11]. The following fixed point theorem is a slight generalization of a fixed point theorem proved in Dhage [5] with PPF dependence.

Theorem 1.1. *Suppose that $T : E_0 \longrightarrow E$ is a nonlinear contraction. Then the following statements hold in E_0.*

(a) *If $\mathcal{R}_c$ is algebraically closed with respect to difference, then every sequence $\{\phi_n\}$ of successive iterates of T at each point $\phi_0 \in E_0$ converges to a PPF dependent fixed point of T.*
(b) *If $\mathcal{R}_c$ is topologically closed, then ϕ^* is the only fixed point of T in $\mathcal{R}_c$.*

In this paper, we prove two hybrid fixed point theorems with PPF dependence in a Banach algebra and apply them to hybrid differential equations of functional differential equations of delay and neutral type for proving the existence of solutions with PPF dependence.

2. PPF DEPENDENT HYBRID FIXED POINT THEORY

Throughout subsequent part of this paper, unless otherwise specified, let E denote a Banach algebra with norm $\|\cdot\|_E$. Then $E_0 = C(I, E)$ becomes a Banach algebra with respect to the norm (1.1) and the multiplication " $\cdot$ " defined by

$$(x \cdot y)(t) = x(t) \cdot y(t) = x(t)y(t)$$

for all $t \in I$, whenever $x, y \in E_0$. When there is no confusion, we simply write xy instead of $x \cdot y$.

While working on fixed point theorems in abstract algebras, the present author introduced a class of $\mathcal{D}$-functions to define the growth of the operators in question. We mention that $\mathcal{D}$-functions are in line with the the growth functions mentioned in definition 1.1 and are useful in practical applications to nonlinear differential equations. Here also we employ same notations and terminologies in what follows.

Definition 2.1. A mapping $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a **dominating function** or, in short, **$\mathcal{D}$-function** if it is continuous and nondecreasing function satisfying $\psi(0) = 0$. A mapping $Q : E_0 \longrightarrow E$ is called **strong $\mathcal{D}$-Lipschitz** if there is a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying

$$\|Q\phi - Q\xi\|_E \le \psi(\|\phi(c) - \xi(c)\|_E) \tag{2.1}$$

for all $\phi, \xi \in E$. The function ψ is called a $\mathcal{D}$-function of Q on E. If $\psi(r) = k\,r$, $k > 0$, then Q is called **strong Lipschitz** with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a **strong contraction** on X with the contraction constant k. Further, if $\psi(r) < r$ for $r > 0$, then Q is called **strong nonlinear $\mathcal{D}$-contraction** and the function ψ is called $\mathcal{D}$-function of Q on X.

There do exist $\mathcal{D}$-functions and commonly used $\mathcal{D}$-functions are

$$\begin{aligned}
&\psi(r) = k\,r, \quad \text{for some constant } k > 0,\\
&\psi(r) = \frac{L\,r}{K + r}, \quad \text{for some constants } L > 0, K > 0,\\
&\psi(r) = \tan^{-1} r,\\
&\psi(r) = \log(1 + r),\\
&\psi(r) = e^r - 1.
\end{aligned}$$

The above defined $\mathcal{D}$-functions have been widely used in the existence theory of nonlinear differential and integral equations.

Remark 2.2. If $\phi, \psi\ \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are two $\mathcal{D}$-functions, then i) $\phi + \psi$, ii) $\lambda\,\phi$, $\lambda > 0$, and iii) $\phi \circ \psi$ are also $\mathcal{D}$-functions on $\mathbb{R}_+$.

Another notion that we need in the sequel is the following definition.

Definition 2.3. An operator Q on a Banach space E into itself is called compact if $Q(E)$ is a relatively compact subset of E. Q is called totally bounded if for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called completely continuous on E.

Our main hybrid fixed point theorem with PPF dependence is the following result in a Banach algebra E.

Theorem 2.1. *Let $A, C : E_0 \longrightarrow E$ and $B : E \longrightarrow E$ be three operators such that*

(a) *A is bounded and strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_A,*
(b) *C is strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_C,*

(c) *B is continuous and compact, and*
(d) $M\psi_A(r) + \psi_C(r) < r$ *if* $r > 0$, *where* $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a, b]$ the operator equation

$$A\phi\, B\phi(c) + C\phi = \phi(c) \tag{2.2}$$

has a PPF dependent solution.

Proof. Let $\xi \in E_0$ be fixed and let $c \in [a, b]$ be given. Define an operator $T_{\xi(c)} : E_0 \longrightarrow E$ by

$$T_{\xi(c)}(\phi) = A\phi\, B\xi(c) + C\phi. \tag{2.3}$$

Clearly, $T_{\xi(c)}$ is a strong nonlinear $\mathcal{B}$-contraction on E_0. To see this, let $\phi_1, \phi_2 \in E_0$. Then,

$$\begin{aligned}\|T_{\xi(c)}(\phi_1) - T_{\xi(c)}(\phi_2)\|_E &\le \|A\phi_1 - A\phi_2\|_E\, \|B\xi(c)\|_E + \|C\phi_1 - C\phi_2\|_E \\ &\le \|B(E)\|_E\, \psi_A(\|\phi_1(c) - \phi_2(c)\|_E) + \psi_C(\|\phi_1(c) - \phi_2(c)\|_E) \\ &\le M\, \psi_A(\|\phi_1(c) - \phi_2(c)\|_E) + \psi_C(\|\phi_1(c) - \phi_2(c)\|_E).\end{aligned}$$

This shows that $T_{\xi(c)}$ is a strong nonlinear $\mathcal{D}$-contraction and hence nonlinear $\mathcal{D}$-contraction on E_0. By Theorem 1.1, there is a unique PPF dependent fixed point $\phi^* \in E_0$ such that

$$T_{\xi(c)}(\phi^*) = \phi^*(c) \quad \text{or} \quad A\phi^*\, B\xi(c) + C\phi^* = \phi^*(c).$$

Next, we define a mapping $Q : E \longrightarrow E$ by

$$Q\xi(c) = \phi^*(c) = A\phi^*\, B\xi(c) + C\phi^*. \tag{2.4}$$

It then follows that

$$\begin{aligned}\|Q\xi_1(c) - Q\xi_2(c)\|_E &= \|A\phi_1^*\, B\xi_1(c) - A\phi_2^*\, B\xi_2(c)\|_E \\ &\qquad + \|C\phi_1^* - C\phi_2^*\|_E \\ &\le \|A\phi_1^* - A\phi_2^*\|_E\, \|B\xi_1\|_E + \|A\phi_2\|_E\, \|B\xi_1(c) - B\xi_2(c)\|_E \\ &\qquad + \|C\phi_1^* - C\phi_2^*\|_E \\ &\le M\, \psi_A(\|\phi_1^*(c) - \phi_2^*(c)\|_E) + k\, \|B\xi_1(c) - B\xi_2(c)\|_E \\ &\qquad + \psi_C(\|\phi_1^*(c) - \phi_2^*(c)\|_E) \\ &\le M\, \psi_A(\|\phi_1^*(c) - \phi_2^*(c)\|_E) + \psi_C(\|\phi_1^*(c) - \phi_2^*(c)\|_E) \\ &\qquad + k\, \|B\xi_1(c) - B\xi_2(c)\|_E \end{aligned} \tag{2.5}$$

where k is a bound of A on E_0.

Since B is compact, if $\{B\xi_n(c)\}$ is any sequence in E, then $\{B\xi_n(c)\}$ has a convergent subsequence. Without loss of generality, we may assume that $\{B\xi_n(c)\}$ is convergent. Hence, $\{B\xi_n(c)\}$ is a Cauchy sequence. From inequality (2.5), we obtain

$$\begin{aligned}\|Q\xi_m(c) - Q\xi_n(c)\|_E \le M\, \psi(\|\phi_m^*(c) - \phi_n^*(c)\|_E) + \psi_C(\|\phi_m^*(c) - \phi_n^*(c)\|_E) \\ + k\, \|B\xi_m(c) - B\xi_n(c)\|_E.\end{aligned}$$

Taking the limit superior in above inequality yields

$$\begin{aligned}&\limsup_{m,n\longrightarrow\infty} \|Q\xi_m(c) - Q\xi_n(c)\|_E \\ &\qquad \le M \limsup_{m,n\longrightarrow\infty} \psi_A(\|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E)\end{aligned}$$

$$+ \limsup_{m,n\longrightarrow\infty} \psi_C(\|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E)$$
$$+ k \limsup_{m,n\longrightarrow\infty} \|B\xi_m(c) - B\xi_n(c)\|_E$$
$$\leq M\,\psi_A\left(\limsup_{m,n\longrightarrow\infty} \|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E\right)$$
$$+ \psi_C\left(\limsup_{m,n\longrightarrow\infty} \|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E\right).$$

Hence,

$$\lim_{m,n\longrightarrow\infty} \|Q\xi_m(c) - Q\xi_n(c)\|_E = 0.$$

As a result, $\{Q\xi_n(c)\}$ is a Cauchy sequence. Since E is complete, $\{Q\xi_n(c)\}$ has a convergent subsequence. Now a direct application of Schauder fixed point principle yields that there is a point $\xi \in E_0$ such that $Q\xi^*(c) = \xi^*(c)$. Consequently $A\xi^*\,B\xi^*(c) + C\xi^* = \xi^*(c)$. This completes the proof. □

Theorem 2.2. *Let $A : E_0 \longrightarrow E$ and $B, C : E \longrightarrow E$ be three operators such that*

(a) *A is bounded and strong $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ_A,*
(b) *B is continuous and compact,*
(c) *C is continuous and compact, and*
(d) *$M\psi_A(r) < r$ if $r > 0$, where $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.*

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is algebraically closed with respect to difference and topologically closed, then for a given $c \in [a, b]$ the operator equation

$$A\phi\,B\phi(c) + C\phi(c) = \phi(c) \tag{2.6}$$

has a PPF dependent solution.

Proof. The proof is similar to Theorem 2.2 with appropriate modifications. □

Remark 2.4. If we consider Theorems 2.1 and 2.1 in a closed, convex and bounded subset of the Banach space E, then condition of the boundedness of the operator A is not required because in that case the boundedness of A follows immediately from the strong Lipschitz condition.

Remark 2.5. If we take $\psi_{\mathcal{A}}(r) = \frac{L_1 r}{K+r}$ and $\psi_{\mathcal{C}}(r) = L_2\, r$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $\frac{L_1 M}{K+r} + L_2 < 1$ for each real number $r > 0$. Similarly, if $\psi_{\mathcal{A}}(r) = L_1\, r$, and $\psi_{\mathcal{C}}(r) = \frac{L_2 r}{K+r}$, then hypothesis (d) of the above hybrid fixed point theorem takes the form $M\,L_1 + \frac{L_2 M}{K+r} < 1$ for each real number $r > 0$.

In view of above remark, we obtain the following results as special cases of Theorems 2.1 and 2.2 as corollaries.

Corollary 2.3. *Let $A, C : E_0 \longrightarrow E$ and $B : E \longrightarrow E$ be three operators such that*

(a) *A is bounded and strong Lipschitz with the Lipschitz constant L_1, and*
(b) *C is strong Lipschitz with the Lipschitz with the Lipschitz constant L_2,*
(c) *B is continuous and compact, and*
(d) *$ML_1 + L_2 < 1$, where $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.*

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is topologically and algebraically closed with respect to difference, then for a given $c \in [a, b]$ the operator equation (2.2) has a PPF dependent solution.

Corollary 2.4. *Let $A : E_0 \longrightarrow E$ and $B, C : E \longrightarrow E$ be three operators such that*

(a) *A is bounded and strong Lipschitz with the Lipschitz constant L_1,*
(b) *B is continuous and compact,*
(c) *C is continuous and compact, and*
(d) *$ML_1 < 1$, where $M = \|B(E)\| = \sup\{\|Bx\| : x \in E\}$.*

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is algebraically closed with respect to difference and topologically closed, then for a given $c \in [a, b]$ the operator equation (2.6) has a PPF dependent solution.

3. APPLICATIONS

In this section, we apply the abstract result of the previous section to functional differential equations for proving the existence of solutions under a weaker Lipschitz condition. Given a closed interval $I_0 = [-r, 0]$ in $\mathbb{R}$ for some real number $r > 0$, let $\mathcal{C}$ denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with supremum norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|\phi\|_{\mathcal{C}} = \sup_{\theta \in I_0} |\phi(\theta)|. \tag{3.1}$$

It is clear that $\mathcal{C}$ is a Banach space with this norm called the history space of the problem under consideration.

Given the closed and bounded interval $J = [-r, T]$ in $\mathbb{R}$, let $C(J, \mathbb{R})$ denote the Banach space of continuous and real-valued functions defined on J with the usual supremum norm $\|\cdot\|$. Given a function $x \in C(J, \mathbb{R})$, for each $t \in I = [0, T]$, define a function $t \longrightarrow x_t \in \mathcal{C}$ by

$$x_t(\theta) = x(t + \theta),\ \theta \in I_0, \tag{3.2}$$

where the argument θ represents the delay in the argument of solutions.

Now we are equipped with the necessary details to study the nonlinear problems of functional differential equations.

3.1. FUNCTIONAL DIFFERENTIAL EQUATION OF DELAY TYPE.

Given a function $\phi \in \mathcal{C}$, consider the perturbed or a hybrid differential equation of functional differential equations of delay type (in short HDE),

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t) - k(t, x(t))}{f(t, x(t))}\right] &= g(t, x_t), \\ x_0 &= \phi, \end{aligned}\right\} \tag{3.3}$$

for all $t \in I$, where $f : I \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ and $g : I \times \mathcal{C} \longrightarrow \mathbb{R}$ are continuous.

By a solution x of the HDE (3.3) we mean a function $x \in C(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{x - k(t, x)}{f(t, x)}$ is continuous in I for each $x \in \mathbb{R}$, and
(ii) x satisfies the equations in (3.3) on J,

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

We consider the following hypotheses in what follows.

(H_1) There exist real numbers $L > 0$ and $K > 0$ such that

$$|g(t, x) - g(t, y)| \leq \frac{L|x(0) - y(0)|}{K + |x(0) - y(0)}$$

for all $t \in I$ and $x, y \in \mathcal{C}$.

(H_2) The function f is uniformly continuous and there exists a real number $M_f > 0$ such that
$$0 < |f(t,x)| \leq M_f$$
for all $t \in I$ and $x \in \mathbb{R}$.

(H_3) The function k is uniformly continuous and there exists a real number $M_k > 0$ such that
$$|k(t,x)| \leq M_k$$
for all $t \in I$ and $x \in \mathbb{R}$.

Remark 3.1. If $L < K$ in hypothesis (H_1), then it reduces to the usual Lipschitz condition of g, namely,
$$|g(t,x) - g(t,y)| \leq (L/K)|x(0) - y(0)|,$$
for all $t \in I$ and $x, y \in \mathcal{C}$.

Theorem 3.1. *Assume that the hypotheses (H_1) through (H_3) hold. Furthermore, if $LT \max\{M_f, 1\} \leq K$, then the HDE (3.3) has a solution defined on J.*

Proof. Set $E = C(J,\mathbb{R})$. Then E is a Banach algebra with respect to the usual supremum norm $\|\cdot\|_E$ and the multiplication " $\cdot$ " defined by
$$(x \cdot y)(t) = x(t) \cdot y(t) = x(t)y(t)$$
for all $t \in I$, whenever $x, y \in E$.

Define a set of functions
$$\widehat{E} = \{\hat{x} = (x_t)_{t\in I} : x_t \in \mathcal{C}, x \in C(I,\mathbb{R}) \text{ and } x_0 = \phi\}. \tag{3.4}$$
Define a norm $\|\hat{x}\|_{\widehat{E}}$ in $\widehat{E}$ by
$$\|\hat{x}\|_{\widehat{E}} = \sup_{t\in I} \|x_t\|_{\mathcal{C}}. \tag{3.5}$$

Clearly, $\hat{x} \in C(I_0,\mathbb{R}) = \mathcal{C}$. Next we show that $\widehat{E}$ is a Banach space. Consider a Cauchy sequence $\{\hat{x}_n\}$ in $\hat{E}$. Then, $\{(x_t^n)_{t\in I}\}$ is a Cauchy sequence in $\mathcal{C}$ for each $t \in I$. This further implies that $\{x_t^n(s)\}$ is a Cauchy sequence in $\mathbb{R}$ for each $s \in [-r, 0]$. Then $\{x_t^n(s)\}$ converges to $x_t(s)$ for each $t \in I_0$. Since $\{x_t^n\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $x_t(s)$ is also continuous in $s \in [-r, 0]$. Hence the sequence $\{\hat{x}_n\}$ converges to $\hat{x} \in \hat{E}$. As a result, $\widehat{E}$ is Banach space.

Now the HDE (3.3) is equivalent to the nonlinear hybrid integral equation (in short HIE)
$$x(t) = \begin{cases} k(t,x(t)) + \big[f(t,x(t))\big]\left(\dfrac{\phi(0)}{f(0,\phi(0))} + \displaystyle\int_0^t g(s,x_s)\,ds\right), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.6}$$

Consider three operators $A : \widehat{E} \longrightarrow \mathbb{R}$, $B : C(J,\mathbb{R}) \longrightarrow \mathbb{R}$ and $C : C(J,\mathbb{R}) \longrightarrow \mathbb{R}$ defined by
$$A\hat{x} = A(x_t)_{t\in I} = \begin{cases} \dfrac{\phi(0)}{f(0,\phi(0))} + \displaystyle\int_0^t g(s,x_s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3.7}$$
$$Bx(t) = \begin{cases} f(t,x(t)), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \tag{3.8}$$

and

$$Cx(t) = \begin{cases} k(t, x(t)), & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{3.9}$$

Then the HIE (3.6) is equivalent to the operator equation

$$A\hat{x}\, B\hat{x}(0) + C\hat{x}(0) = \hat{x}(0). \tag{3.10}$$

We shall show that the operators A, B and C satisfy all the conditions of Theorem 2.2. First we show that A is a bounded operator on $\widehat{E}$ into E. Now for any $\hat{x} \in \widehat{E}$, one has

$$\begin{aligned} \|A\hat{x}\|_E &\leq \|A0\|_E + \|A(x_t)_{t\in I} - A0\|_E \\ &\leq \|A0\|_E + \left| \int_0^t g(s, x_s)\, ds - \int_0^t g(s, 0)\, ds \right| \\ &\leq \|A0\|_E + \int_0^t \frac{L|x_s(0) - 0|}{K + |x_t(0) - 0|}\, ds \\ &\leq \|A0\|_E + \int_0^t \frac{L\|\hat{x}(0)\|_E}{K + \|\hat{x}(0)\|_E}\, ds \\ &\leq \|A0\|_E + LT \end{aligned}$$

which shows that A is a bounded operator on $\widehat{E}$ with bound $\|A0\|_E + LT$.

Next, we prove that A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$. Then,

$$\begin{aligned} \|A\hat{x} - A\hat{y}\|_E &= \|A(x_t)_{t\in I} - A(y_t)_{t\in I}\| \\ &= \left| \int_0^t g(s, x_s)\, ds - \int_0^t g(s, y_s)\, ds \right| \\ &\leq \int_0^t \frac{L|x_s(0) - y_s(0)|}{K + |x_s(0) - y_s(0)|}\, ds \\ &\leq \int_0^t \frac{L\|\hat{x}(0) - \hat{y}(0)\|_E}{K + \|\hat{x}(0) - \hat{y}(0)\|_E}\, ds \\ &= \psi_A(\|\hat{x}(0) - \hat{y}(0)\|_E) \end{aligned}$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi_A(r) = \dfrac{LTr}{K + r}$. Hence, A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function ψ_A.

Next, we show that B is compact and continuous operator on $C(J, \mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J, \mathbb{R})$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Then by continuity of f,,

$$\lim_{n\longrightarrow\infty} Bx_n(t) = \lim_{n\longrightarrow\infty} f(s, x_n(s)) = f(s, x(s)) = Bx(t)$$

for all $t \in I$. Similarly, if $t \in I_0$, then $\lim_{n\longrightarrow\infty} Bx_n(t) = 1 = Bx(t)$. This shows that $\{Bx_n(t)\}$ converges to $Bx(t)$ point-wise on J. But $\{Bx_n(t)\}$ is a sequence of uniformly continuous functions on J, So $Bx_n \longrightarrow Bx$ uniformly. Hence, B is a continuous operator on E into itself.

Secondly, we show that B is compact. To finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$|Bx(t)| \leq |f(s, x(s))| \leq M_f$$

for all $t \in J$, and $|Bx(t)| \leq 1$ for all $t \in I_0$. From this it follows that

$$|Bx(t)| \leq \max\{M_f\, , 1\} = M^*$$

for all $t \in J$, whence B is uniformly bounded on E.

To show equi-continuity, let $t, \tau \in I$. Then, from the uniform continuity of f it follows that

$$|Bx(t) - Bx(\tau)| \leq |f(t, x(t)) - f(\tau, x(\tau))| < \epsilon$$

for all $x \in C(J, \mathbb{R})$. If $\tau \in I_0$ and $t \in I$, then $\tau \longrightarrow 0$ and $t \longrightarrow 0$ whenever, $|\tau - t| \longrightarrow 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \leq |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \longrightarrow 0 \text{ as } t \longrightarrow \tau$$

for all $x \in C(J, \mathbb{R})$. From this, it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzella-Ascoli theorem yields that B is a compact operator on E into itself. Similarly, it can be shown that the operators C is also a compact and continuous operator on E into itself.

Finally,

$$M\psi_A(r) = \frac{LT \max\{M_f, 1\}\, r}{K + r} < r$$

for all $r > 0$ and so, all the conditions of Theorem 2.1 are satisfied. Moreover, here the Razumikhin class $\mathcal{R}_0$, $0 \in [-r, T]$ is $C([0, T], \mathbb{R})$ which is topologically and algebraically closed with respect to difference. Hence, an application of Theorem 2.2 yields that integral equation (3.6) has a solution on J with PPF dependence. This further implies that the HDE (3.3) has a PPF dependent solution on J. This completes the proof. □

3.2. **FUNCTIONAL DIFFERENTIAL EQUATION OF NEUTRAL TYPE.** Given a function $\phi \in \mathcal{C}$, consider the perturbed or a hybrid functional differential equation of neutral type (in short HDE)

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t) - k(t, x_t)}{f(t, x_t)}\right] &= g(t, x(t)), \\ x_0 &= \phi, \end{aligned}\right\} \tag{3.11}$$

for all $t \in I$, where $f : I \times \mathcal{C} \longrightarrow \mathbb{R} \setminus \{0\}$, $k : I \times \mathcal{C} \longrightarrow \mathbb{R}$ and $g : I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

By a solution x of the FDE (3.11) we mean a function $x \in C(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \dfrac{y - k(t, y)}{f(t, y)}$ is continuous in I for all $y \in \mathcal{C}$, and
(ii) x satisfies the equations in (3.11) on J,

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J = I_0 \cup I$.

We consider the following hypotheses in what follows.

(H_4) There exist real numbers $L_1 > 0$ and $K_1 > 0$ such that

$$|f(t, x) - f(t, y)| \leq \frac{L_1 |x(0) - y(0)|}{K_1 + |x(0) - y(0)|}$$

for all $x, y \in \mathcal{C}$.

(H_5) There exists a real number $M_g > 0$ such that

$$|g(t, x)| \leq M_g$$

for all $t \in I$ and $x \in \mathbb{R}$.

(H_6) There exist real numbers $L_2 > 0$ and $K_2 > 0$ such that

$$|k(t,x) - k(t,y)| \leq \frac{L_2 |x(0) - y(0)|}{K_2 + |x(0) - y(0)|}$$

for all $x, y \in \mathcal{C}$.

Theorem 3.2. *Assume that the hypotheses (H_4) through (H_6) hold. Furthermore, if*

$$L_1 \left[\|\phi\|_{\mathcal{C}} + M_g T\right] + L_2 \leq \min\{K_1, K_2\},$$

then the HDE (3.11) has a solution defined on J.

Proof. Set $E = C(J, \mathbb{R})$. Clearly, E is a Banach algebra with respect to the norm and the multiplication as defined in the proof of Theorem 3.1. Define a set of functions $\widehat{E}$ by (3.4) which is equipped with the norm $\|\hat{x}\|_{\widehat{E}}$ defined by (3.5) Clearly, $\hat{x} \in C(I_0, \mathbb{R}) = \mathcal{C}$. It can be shown as in Theorem 3.1 that $\widehat{E}$ is Banach space.

Now the HDE (3.10) is equivalent to the nonlinear hybrid integral equation (in short HIE)

$$x(t) = \begin{cases} k(t, x_t) + \left[f(t, x_t)\right] \left(\phi(0) + \displaystyle\int_0^t g(s, x(s))\, ds\right), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{3.12}$$

Consider three operators $A, B : \widehat{E} \longrightarrow \mathbb{R}$, $B : C(J, \mathbb{R}) \longrightarrow \mathbb{R}$ and $C : C(J, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$A\hat{x} = A(x_t)_{t \in I} = \begin{cases} f(t, x_t), & \text{if } t \in I, \\ 1, & \text{if } t \in I_0, \end{cases} \tag{3.13}$$

$$Bx(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t g(s, x(s))\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3.14}$$

and

$$C\hat{x} = C(x_t)_{t \in I} = \begin{cases} k(t, x_t), & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{3.15}$$

Then the HIE (3.11) is equivalent to the operator equation

$$A\hat{x}\, B\hat{x}(0) + C\hat{x} = \hat{x}(0). \tag{3.16}$$

We shall show that the operators A, B and C satisfy all the condition of Theorem 2.1. First we show that A is bounded on $\widehat{E}$.

$$\begin{aligned} |A\hat{x}| &\leq |A0| + |A(x_t)_{t \in I} - A0| \\ &\leq |f(t, 0)| + |f(t, x_t) - f(s, 0)| \\ &\leq F_0 + \frac{L|x_t(0) - 0|}{K + |x_t(0) - 0|} \\ &\leq F_0 + \frac{L\|\hat{x}(0)\|_E}{K + \|\hat{x}(0)\|_E} \\ &= F_0 + L, \end{aligned}$$

for all $\hat{x} \in \widehat{E}$, where $F_0 = \sup_{t \in I} |f(t, 0)|$. Hence, A is bounded on $\widehat{E}$ with bound $F_0 + L$.

Next, we show that a strong $\mathcal{B}$-Lipschitz on $\widehat{E}$. Then,

$$\|A\hat{x} - A\hat{y}\|_E = |A(x_t)_{t \in I} - A(y_t)_{t \in I}|$$

$$\begin{aligned}
&= |f(t,x_t) - f(t,y_t)| \\
&\le \frac{L_1|x_t(0)-y_t(0)|}{K_1+|x_t(0)-y_t(0)|} \\
&\le \frac{L_1\|\hat{x}(0)-\hat{y}(0)\|_E}{K_1+\|\hat{x}(0)-\hat{y}(0)\|_E} \\
&= \psi_A(\|\hat{x}(0)-\hat{y}(0)\|_E)
\end{aligned}$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi_A(r) = \dfrac{L_1 r}{K_1 + r}$. Hence, A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function ψ_A. Similarly, it can be shown that C is also a A is a strong $\mathcal{D}$-Lipschitz on $\widehat{E}$ with $\mathcal{D}$-function $\psi_C(r) = \dfrac{L_2 r}{K_2 + r}$.

Next, we show that B is compact and continuous operator on $C(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J,\mathbb{R})$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{n\longrightarrow\infty} Bx_n(t) &= \phi(0) + \lim_{n\longrightarrow\infty}\int_0^t g(s,x_n(s))\,ds \\
&= \phi(0) + \int_0^t \lim_{n\longrightarrow\infty} g(s,x_n(s))\,ds \\
&= Bx(t)
\end{aligned}$$

for all $t \in I$. Similarly, if $t \in I_0$, then $\lim\limits_{n\longrightarrow\infty} Bx_n(t) = \phi(t) = Bx(t)$. This shows that $\{Bx_n(t)\}$ converges to $Bx(t)$ point-wise on J. But $\{Bx_n(t)\}$ is a sequence of uniformly continuous functions on J, So $Bx_n \longrightarrow Bx$ uniformly. Hence, B is a continuous operator on E into itself.

Secondly, we show that B is compact. To finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$|Bx(t)| \le |\phi(0)| + \int_0^t |g(s,x(s))|\,ds \le \|\phi\|_{\mathcal{C}} + M_g T$$

for all $t \in J$ which shows that $B(E)$ is uniformly bounded set in E. To show equi-continuity, let $t, \tau \in I$. Then,

$$|Bx(t) - Bx(\tau)| \le \left|\int_\tau^t |g(s,x(s))|\,ds\right| \le M_g\,|t-\tau|.$$

If $\tau \in I_0$ and $t \in I$, then $\tau \longrightarrow 0$ and $t \longrightarrow 0$ whenever, $|\tau - t| \longrightarrow 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \le |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \le M_g\,|t-\tau|.$$

From the above inequalities it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzelá-Ascoli theorem yields that B is a compact operator on E into itself. Finally,

$$\begin{aligned}
M\psi_A(r) + \psi_C(r) &= \frac{L_1\left[\|\phi\|_{\mathcal{C}} + M_g T\right] r}{K_1 + r} + \frac{L_2 r}{K_2 + r} \\
&\le \frac{\left[L_1\left(\|\phi\|_{\mathcal{C}} + M_g T\right) + L_2\right] r}{\min\{K_1, K_2\} + r} \\
&< r
\end{aligned}$$

for all $r > 0$ and so, all the conditions of Theorem 2.1 are satisfied. Again, here the Razumikhin class $\mathcal{R}_0$, $0 \in [-r, T]$ is $C([0,T], \mathbb{R})$ which is topologically and algebraically closed with respect to difference. Hence, an application of Theorem 2.1 yields that the integral equation (3.12) has a solution on J with PPF dependence. This further implies that the HDE (3.11) has a PPF dependent solution defined on J. This completes the proof. □

Remark 3.2. Finally, we conclude this paper with the remark that the functional differential equations considered here are of simple nature, however, other complex nonlinear functional differential equations involving the arguments of past, present and future can also be considered and studied for the existence theorems on similar lines with appropriate modifications.

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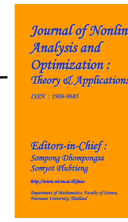

COMMON FIXED POINT OF PRESIĆ TYPE CONTRACTION MAPPINGS IN PARTIAL METRIC SPACES

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ABSTRACT. Presić (Publ. de L'Inst. Math. Belgrade, 5 (19), 75-78) introduced the concept of a kth-order Banach type contraction mapping and obtained fixed point of such mappings on metric spaces. Ćirić and Presić (Acta Math. Univ. Comenian. LXXVI (2) (2007), 143-147) extended the notion to kth-order Ciric type contraction mappings on a metric space. On the other hand, Matthews (Ann. New York Acad. Sci. 728 (1994), 183-197) introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks. He gave a modified version of the Banach contraction principle, more suitable in this context. In this paper, we study the common fixed points of kth-order Ciric type contractions in the framework of partial metric spaces. We also present an example to validate our result.

KEYWORDS: Partial metric space; Common fixed point; Presić type contraction.
AMS Subject Classification: 54H25, 47H10, 15A24, 65H05.

1. INTRODUCTION AND PRELIMINARIES

Banach contraction principle [7] is a simple and powerful result with a wide range of applications, including iterative methods for solving linear, nonlinear, differential, integral, and difference equations. There are several generalizations and extensions of the Banach contraction principle in the existing literature.

Banach contraction principle reads as follows:

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Theorem 1.1. [7] *Let (X, d) be a complete metric space and mapping $f : X \to X$ satisfies*

$$d(fx, fy) \leq kd(x, y) \text{ for all } x, y \in X.$$

where $k \in [0, 1)$. Then, there exists a unique x in X such that $x = fx$. Moreover, for any $x_0 \in X$, the iterative sequence $x_{n+1} = fx_n$ converges to x.

Let $f : X^k \to X$, where $k \geq 1$ is a positive integer. A point $x^* \in X$ is called a fixed point of f if $x^* = f(x^*, ..., x^*)$. Consider the k-th order nonlinear difference equation

$$x_{n+1} = f(x_{n-k+1}, x_{n-k+2}, ..., x_n) \text{ for } n = k-1, k, k+1, ... \tag{1.1}$$

with the initial values $x_0, x_1, ..., x_{k-1} \in X$.

Equation (1.1) can be studied by means of fixed point theory in view of the fact that x in X is a solution of (1.1) if and only if x is a fixed point of f. One of the most important results in this direction is obtained by Presić [20] in the following way.

Theorem 1.2. [20] *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$. Suppose that*

$$d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}) \tag{1.2}$$

holds for all $x_1, ..., x_{k+1}$ in X, where $q_i \geq 0$ and $\sum_{i=1}^{k} q_i \in [0, 1)$. Then f has a unique fixed point x^*. Moreover, for any arbitrary points $x_1, ..., x_{k+1}$ in X, sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1} ..., x_{n+k-1})$, for all $n \in \mathbb{N}$ converges to x^*.

It is easy to show that for $k = 1$, Theorem 1.1 reduces to the Banach contraction principle.

Ćirić and Presić [11] generalized above theorem as follows.

Theorem 1.3. [11] *Let (X, d) be a complete metric space, k a positive integer and $f : X^k \to X$. Suppose that*

$$d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}, \tag{1.3}$$

holds for all $x_1, ..., x_{k+1}$ in X, where $\lambda \in [0, 1)$. Then f has a fixed point $x^* \in X$. Moreover, for any arbitrary points $x_1, ..., x_{k+1} \in X$, the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1} ..., x_{n+k-1})$, for all $n \in \mathbb{N}$ converges to x^*. Moreover, if

$$d(f(u, ..., u), f(v, ..., v)) < d(u, v),$$

holds for all $u, v \in X$, with $u \neq v$, then x^* is unique fixed point of f.

The applicability of the above result to the study of global asymptotic stability of the equilibrium for the nonlinear difference equation (1.1) can be found in [10]. For further work in this direction, we refer to [2, 16, 19, 23].

On the other hand, partial metric space is a generalized metric space in which each object does not necessarily have to have a zero distance from itself [17]. A motivation behind introducing the concept of a partial metric was to obtain appropriate mathematical models in the theory of computation [13, 17, 22, 24, etc]. Altun and Simsek [4], Oltra and Valero [18] and Valero [25] established potential generalizations of the results in [17]. Romaguera [21] proved a Caristi type fixed point theorem on partial metric spaces. Further results in this direction were proved in [1, 3, 5, 6, 8, 9, 14, 15].

Recently, Geroge et al. [12] proved generalized fixed point theorem of Presic type in cone metric spaces and gave its application to Markov process.

The aim of this paper is to study the common fixed point results for mappings satisfying Presić type contractive conditions in the setup of partial metric spaces.

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$ and N will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integer numbers, respectively.

Consistent with [4] and [17], the following definitions and results will be needed in the sequel.

Definition 1.4. Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold true:

- **($\mathbf{P}_1$):** $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;
- **($\mathbf{P}_2$):** $p(x,x) \leq p(x,y)$;
- **($\mathbf{P}_3$):** $p(x,y) = p(y,x)$;
- **($\mathbf{P}_4$):** $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then called a partial metric space.
If $p(x,y) = 0$, then (P_1) and (P_2) imply that $x = y$. But the converse does not hold always.

A trivial example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $p(x,y) = \max\{x,y\}$.

Example 1.5. [17] If $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$, then $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

For some more examples of partial metric spaces, we refer to [4, 9, 21, 24].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$,for all $x \in X$ and $\varepsilon > 0$.

Observe (see [17, p. 187]) that a sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

If p is a partial metric on X, then the function $p^S : X \times X \to \mathbb{R}^+$ given by $p^S(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, defines a metric on X.

Furthermore, a sequence $\{x_n\}$ converges in (X,p^S) to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x).$$

Definition 1.6. [17]. Let (X,p) be a partial metric space.

- **(a):** A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
- **(b):** (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim\limits_{n\to\infty} p(x,x_n) = p(x,x)$. In this case, we say that the partial metric p is complete.

Lemma 1.7. [4, 17] *Let (X,p) be a partial metric space. Then:*

- **(a):** *A sequence $\{x_n\}$ in X is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in metric space (X,p^S).*
- **(b):** *A partial metric space (X,p) is complete if and only if the metric space (X,p^S) is complete.*

2. INTRODUCTION AND PRELIMINARIES

In this section, we obtain some common fixed point results for self maps satisfying Presić type contractions defined on a complete partial metric space. We begin with the following theorem.

Theorem 2.1. *Let (X,p) be a complete partial metric space. Suppose that $f,g : X^k \to X$ be two mappings satisfy*

$$p(f(x_1,...,x_k), g(x_2,...,x_{k+1})) \leq \lambda \max\{p(x_i,x_{i+1}) : 1 \leq i \leq k\}, \tag{2.1}$$

for all $x_1, ..., x_{k+1}$ in X, where $\lambda \in [0, 1)$, k a positive integer. Then f has a unique fixed point x^*. Moreover, for any arbitrary points $x_1, ..., x_{k+1}$ in X, sequence $\{x_n : n \in \mathbb{N}\}$ defined by $x_{n+k} = f(x_n, x_{n+1} ..., x_{n+k-1})$ converges to x^*.

Proof. Let $x_1, ..., x_{k+1}$ be arbitrary k elements in X. Define

$$\begin{aligned} x_{2n+k} &= f(x_{2n}, x_{2n+1}, ..., x_{2n+k-1}) \text{ and} \\ x_{2n+1+k} &= g(x_{2n+1}, x_{2n+2}, ..., x_{2n+k}) \end{aligned}$$

for all $n = 1, 2, ...$ First, we prove that the following inequalities holds for each $n \in \mathbb{N}$:

$$p(x_{2n}, x_{2n+1}) \leq \lambda^{\frac{2n}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{2i}{k}}} : 1 \leq i \leq k\right\}. \tag{2.2}$$

It is obvious to note that (2.2) is valid for $n = 1, 2, 3, ..., k$. Now let the following k inequalities:

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \lambda^{\frac{2n}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{2i}{k}}} : 1 \leq i \leq k\right\}, \\ p(x_{n+1}, x_{n+2}) &\leq \lambda^{\frac{2(n+1)}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{2i}{k}}} : 1 \leq i \leq k\right\}, \\ &\quad \cdot \cdot \cdot, \\ p(x_{n+k-1}, x_{n+k}) &\leq \lambda^{\frac{2(n+k-1)}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{2i}{k}}} : 1 \leq i \leq k\right\} \end{aligned}$$

by the induction hypotheses. Then we have

$$\begin{aligned} p(x_{2n+k}, x_{2n+k+1}) &= p\left(f(x_{2n}, ..., x_{2n+k-1}), g(x_{2n+1}, ..., x_{2n+k})\right) \\ &\leq \lambda \max\left\{p(x_i, x_{i+1}) : 2n \leq i \leq 2n + k - 1\right\} \\ &\leq \lambda . \lambda^{\frac{2n}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{2i}{k}}} : 1 \leq i \leq k\right\} \\ &= \lambda^{\frac{2n+k}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{2i}{k}}} : 1 \leq i \leq k\right\} \end{aligned}$$

and the inductive proof of (2.2) is complete. In similar way, we obtain

$$p(x_{2n+k+1}, x_{2n+k+2}) \leq \lambda^{\frac{2n+k+1}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{2i}{k}}} : 1 \leq i \leq k\right\}.$$

Hence

$$p(x_{n+k}, x_{n+k+1}) \leq \lambda^{\frac{n+k}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{i}{k}}} : 1 \leq i \leq k\right\}$$

for all $n = 1, 2, ...$ Now we have

$$\begin{aligned} p_s(x_{n+k}, x_{n+k-1}) &= 2p(x_{n+k}, x_{n+k-1}) - p(x_{n+k}, x_{n+k}) - p(x_{n+k-1}, x_{n+k-1}) \\ &\leq 2p(x_{n+k}, x_{n+k-1}) + p(x_{n+k}, x_{n+k}) + p(x_{n+k-1}, x_{n+k-1}) \\ &\leq 4p(x_{n+k}, x_{n+k-1}) \\ &\leq 4\lambda^{\frac{n+k-1}{k}} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^{\frac{i}{k}}} : 1 \leq i \leq k\right\}. \end{aligned}$$

So we have

$$\begin{aligned} p_s(x_{n+k}, x_n) &\leq p_s(x_{n+k}, x_{n+k-1}) + ... + p_s(x_{n+1}, x_n) \\ &\leq 4\lambda^n[\lambda^{k-1} + ... + \lambda^1] \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^i} : 1 \leq i \leq k\right\} \\ &\leq \frac{4\lambda^n}{1-\lambda} \max\left\{\frac{p(x_i, x_{i+1})}{\lambda^i} : 1 \leq i \leq k\right\}. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in (X, p_s). By Lemma 1.7, $\{x_n\}$ is a Cauchy sequence in (X, p). Now, since (X, p) is complete, there exists u in X such that $x_n \to u$ as $n \to \infty$. So that

$$\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(x_n, u) = p(u, u).$$

Now, for any integer n we have

$$\begin{aligned} & p(u, g(u, u, ..., u)) \\ \leq\ & p(u, x_{2n+k}) + p(x_{2n+k}, g(u, u, ..., u)) - p(x_{2n+k}, x_{2n+k}) \\ =\ & p(u, x_{2n+k}) + p(f(x_{2n}, x_{2n+1} ..., x_{2n+k-1}), g(u, u, ..., u)) - p(x_{2n+k}, x_{2n+k}) \\ \leq\ & p(u, x_{2n+k}) + \lambda \max\{p(x_{2n}, u), p(x_{2n+1}, u), ..., p(x_{2n+k-1}, u)\} - p(x_{2n+k}, x_{2n+k}). \end{aligned}$$

On taking limit as $n \to \infty$, we obtain

$$p(u, g(u, u, ..., u)) \leq \lambda p(x_{2n}, u),$$

implies $u = g(u, u, ..., u)$. Again

$$\begin{aligned} & p(f(u, u, ..., u), u) \\ \leq\ & p(f(u, u, ..., u), x_{2n+k+1}) + p(x_{2n+k+1}, u) - p(x_{2n+k+1}, x_{2n+k+1}) \\ =\ & p(f(u, u, ..., u), g(x_{2n+1}, x_{2n+2} ..., x_{2n+k})) + p(u, x_{2n+k+1}) - p(x_{2n+k+1}, x_{2n+k+1}) \\ \leq\ & \lambda \max\{p(u, x_{2n+1}), p(u, x_{2n+2}), ..., p(u, x_{2n+k})\} + p(u, x_{2n+k+1}) - p(x_{2n+k+1}, x_{2n+k+1}) \end{aligned}$$

and on taking limit as $n \to \infty$, we get

$$p(f(u, u, ..., u), u) \leq \lambda p(u, u),$$

which implies $f(u, u, ..., u) = u$. Hence u is the common fixed point of f and g.

Now, to prove the uniqueness of u, let v be another point in X such that $v = f(v, v, ..., v) = g(v, v, ..., v)$. Then, we have

$$\begin{aligned} p(u, v) &= p(f(u, u, ..., u), g(v, v, ..., v)) \\ &\leq \lambda p(u, v), \end{aligned}$$

implies $u = v$. So, u is the unique common fixed point of f and g in X.

□

Example 2.2. Let $X = [0, 2]$. Let $p : X \times X \to \mathbb{R}^+$ defined by $p(x, y) = |x - y|$ if $x, y \in [0, 1)$, and $p(x, y) = \max\{x, y\}$ otherwise. It is easily seen that (X, p) is a complete partial metric space. For a positive integer k, we define $f, g : X^k \to X$ by

$$f(x_1, ..., x_k) = \begin{cases} \dfrac{x_2 + x_3 + x_k}{6}, & \text{if } x_1, ..., x_k \in [0, 1) \\ 0, & \text{otherwise,} \end{cases}$$

$$g(x_1, ..., x_k) = \begin{cases} \dfrac{x_1 + x_2 + x_k}{6}, & \text{if } x_1, ..., x_k \in [0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

Now for all $x_1, x_2, ..., x_{k+1} \in [0, 1)$ and $\lambda = 1/2$, we have

$$\begin{aligned} p(f(x_1, ..., x_k), g(x_2, ..., x_{k+1})) &= \left| \frac{x_2 + x_3 + x_k}{6} - \frac{x_2 + x_3 + x_{k+1}}{6} \right| \\ &= \frac{1}{6} |x_k - x_{k+1}| \\ &\leq \frac{1}{2} \max\{p(x_i, x_{i+1}) : 1 \leq i \leq k\} \\ &= \lambda \max\{p(x_i, x_{i+1}) : 1 \leq i \leq k\}. \end{aligned}$$

If for $x_1, x_2, ..., x_k, \in [0, 1)$ and $x_{k+1} \in [1, 2]$, then

$$\begin{aligned} p(f(x_1, ..., x_k), g(x_2, ..., x_{k+1})) &= \frac{1}{6} (x_2 + x_3 + x_k) \\ &\leq \frac{1}{2} x_{k+1} = \lambda \max\{p(x_i, x_{i+1}) : 1 \leq i \leq k\}. \end{aligned}$$

When some $x_{j's} \in [1, 2]$, and $x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_{k+1} \in [0, 1)$, then we obtain $p(f(x_1, ..., x_k), g(x_2, ..., x_{k+1})) = 0$ and (2.1) is satisfied obviously. Thus the conditions of Theorem 2.1 are satisfied and there exist a unique $u = 0$ in X such that $f(u, u, ..., u) = g(u, u, ..., u) = u$.

Corollary 2.3. *Let (X, p) be a complete partial metric space, k a positive integer and $f, g : X^k \to X$. Suppose that*

$$p(f(x_1, ..., x_k), g(x_2, ..., x_{k+1})) \leq \sum_{i=1}^{k} \lambda_i p(x_i, x_{i+1}), \tag{2.3}$$

holds for all $x_1, ..., x_{k+1}$ in X, where $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$. Then f and g have a unique fixed point x^*.

In Theorem 2.1, take $f = g$ to obtain the following corollary which extends and generalizes the corresponding results of [20].

Corollary 2.4. *Let (X, p) be a complete partial metric space. Suppose that a mapping $f : X^k \to X$ satisfies*

$$p(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \leq \lambda \max\{p(x_i, x_{i+1}) : 1 \leq i \leq k\}, \tag{2.4}$$

for all $x_1, ..., x_{k+1}$ in X, where $\lambda \in [0, 1)$, k a positive integer. Then f has a fixed point x^*. Moreover, for any arbitrary points $x_1, ..., x_{k+1}$ in X, sequence $\{x_n : n \in \mathbb{N}\}$ defined by $x_{n+k} = f(x_n, x_{n+1} ..., x_{n+k-1})$ converges to x^*.

If we take $f = g$ in Theorem 2.1, then the following corollary is obtained which extends and generalizes the comparable results of [11].

Corollary 2.5. *Let (X, p) be a complete partial metric space, k a positive integer and $f : X^k \to X$. Suppose that*

$$p(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \leq \sum_{i=1}^{k} \lambda_i p(x_i, x_{i+1}), \tag{2.5}$$

holds for all $x_1, ..., x_{k+1}$ in X, where $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$. Then f has a unique fixed point x^*. Moreover, for any arbitrary points $x_1, ..., x_{k+1}$ in X, sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1} ..., x_{n+k-1})$, for all $n \in \mathbb{N}$ converges to x^*.

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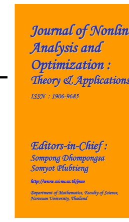

NONLINEAR ERGODIC THEOREMS FOR WIDELY MORE GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this paper, using strongly asymptotically invariant nets, we first obtain some properties of widely more generalized hybrid mappings in a Hilbert space. Then, using the idea of mean convergence by Shimizu and Takahashi [24, 25], we prove a nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space. This generalizes the Kawasaki and Takahashi nonlinear ergodic theorem.

KEYWORDS : Banach limit; generalized hybrid mapping; Hilbert space; nonexpansive mapping; nonspreading mapping; strongly asymptotically invariant net
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1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. For a mapping $T : C \longrightarrow C$, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \longrightarrow C$ is called nonexpansive [28] if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \longrightarrow C$ is called nonspreading [19], hybrid [29] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2,$$
$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$, respectively; see also [12] and [33]. These three mappings are independent and they are deduced from a firmly nonexpansive mapping in a Hilbert space; see [29]. A mapping $F : C \longrightarrow H$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

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for all $x, y \in C$; see, for instance, Goebel and Kirk [7]. The class of nonspreading mappings was first defined in a strictly convex, smooth and reflexive Banach space. The resolvents of a maximal monotone operator are nonspreading mappings; see [19] for more details. These three classes of nonlinear mappings are important in the study of the geometry of infinite dimensional spaces. Indeed, by using the fact that the resolvents of a maximal monotone operator are nonspreading mappings, Takahashi, Yao and Kohsaka [34] solved an open problem which is related to Ray's theorem [23] in the geometry of Banach spaces. Motivated by these mappings, Kocourek, Takahashi and Yao [17] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings and hybrid mappings. A mapping $T : C \longrightarrow C$ is said to be *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such a mapping an (α, β)-generalized hybrid mapping. An (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [18] and Iemoto and Takahashi [12]. Moreover, they proved the following nonlinear ergodic theorem which generalizes Baillon's theorem [2].

Theorem 1.1 ([17]). *Let H be a real Hilbert space, let C be a nonempty, closed and convex subset of H, let T be a generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Very recently Kawasaki and Takahashi [15] introduced a class of nonlinear mappings in a Hilbert space which covers contractive mappings [3] and generalized hybrid mappings. A mapping $T : C \longrightarrow C$ is called *widely more generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0\end{aligned}$$

for any $x, y \in C$; see also Kawasaki and Takahashi [14]. A mapping $T : C \longrightarrow C$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. In [15], they extended the nonlinear ergodic theorem of [17] to widely more generalized hybrid mappings.

In this paper, using strongly asymptotically invariant nets, we first obtain some properties of widely more generalized hybrid mappings in a Hilbert space. Then, using the idea of mean convergence by Shimizu and Takahashi [24, 25], we prove a nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space. This generalizes the Kawasaki and Takahashi nonlinear ergodic theorem.

2. PRELIMINARIES

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known [28] that for any $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|y\|^2 - \|x\|^2 \leq 2\langle y - x, y \rangle, \tag{2.1}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.2}$$

Furthermore, we have that

$$2\langle x - y, z - w \rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.3}$$

for any $x, y, z, w \in H$. Let C be a nonempty subset of H. It is well-known that if C is closed and convex and $T : C \longrightarrow C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Ito and Takahashi [13]. For a simpler proof of such a result in a Hilbert space, see, for example, [16]. Let D be a nonempty, closed and convex subset of H and $x \in H$. Then we know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is called the metric projection of H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u \rangle \geq 0$$

for any $x \in H$ and $u \in D$; see [28] for more details. For proving a nonlinear ergodic theorem in this paper, we also need the following lemma proved by Takahashi and Toyoda [31].

Lemma 2.1. *Let D be a nonempty, closed and convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$ (the dual space of ℓ^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_0, x_1, x_2, \ldots) \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then for $f = (x_0, x_1, x_2, \ldots) \in \ell^\infty$,

$$\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.$$

In particular, if $f = (x_0, x_1, x_2, \ldots) \in \ell^\infty$ and $x_n \longrightarrow a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [27] for the proof of existence of a Banach limit and its other elementary properties. For $f \in \ell^\infty$, define $\ell_1 : \ell^\infty \longrightarrow \ell^\infty$ as follows:

$$\ell_1 f(k) = f(1 + k), \quad \forall k \in \mathbb{N} \cup \{0\}.$$

A net $\{\mu_\alpha\}$ of means on ℓ^∞ is said to be *strongly asymptotically invariant* if

$$\|\mu_\alpha - \ell_1^* \mu_\alpha\| \longrightarrow 0,$$

where ℓ_1^* is the adjoint operator of ℓ_1. See [6] for more details. The following definition which was introduced by Takahashi [26] is crucial in the fixed point theory. Let h be a bounded function of $\mathbb{N} \cup \{0\}$ into H. Then, for any mean μ on ℓ^∞, there exists a unique element $h_\mu \in H$ such that

$$\langle h_\mu, z \rangle = (\mu)_k \langle h(k), z \rangle, \quad \forall z \in H.$$

Such a h_μ is contained in $\overline{\text{co}}\{h(k)\}$, where $\overline{\text{co}}A$ is the closure of convex hull of A. In particular, let T be a mapping of a subset C of a Hilbert space H into itself such that $\{T^k x\}$ is bounded for some $x \in C$. Putting $h(k) = T^k x$ for all $k \in \mathbb{N} \cup \{0\}$, we have that there exists $z_0 \in H$ such tat

$$\mu_k \langle T^k x, y \rangle = \langle z_0, y \rangle, \quad \forall y \in H.$$

We denote such z_0 by $S_\mu x$.

From Kawasaki and Takahashi [15], we also know the following fixed point theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 2.1 ([15]). *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself, i.e., there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that*

$$\begin{aligned} \alpha\|Tx - Ty\|^2 &+ \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+ \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0 \end{aligned}$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\alpha + \beta + \zeta + \eta > 0$*;*

(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ under the conditions (1) *and* (2).

3. NONLINEAR ERGODIC THEOREMS

In this section, using the technique developed by Takahashi [26], we prove a mean convergence theorem for widely more generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following three lemmas. The following lemma was proved by Kawasaki and Takahashi [15].

Lemma 3.1. *Let H be a real Hilbert space, let C be a closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\zeta + \eta \ge 0$ *and* $\alpha + \beta > 0$*;*

(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\varepsilon + \eta \ge 0$ *and* $\alpha + \gamma > 0$.

Then T is quasi-nonexpansive.

The following two lemmas are crucial in the proof of our main theorem.

Lemma 3.2. *Let C be a non-empty, closed and convex subset of a real Hilbert space H. Let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$. Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma > 0$, $\varepsilon + \eta \ge 0$ *and* $\alpha + \beta + \zeta + \eta \ge 0$*;*

(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta > 0$, $\zeta + \eta \ge 0$ *and* $\alpha + \gamma + \varepsilon + \eta \ge 0$.

Let $\{\mu_\nu\}$ be a srongly asymptotically invariant net of means on ℓ^∞. For any $x \in C$, define $S_{\mu_\nu} x$ as follows:

$$\langle S_{\mu_\nu} x, y \rangle = (\mu_\nu)_k \langle T^k x, y \rangle, \quad \forall y \in H.$$

Then $\lim_\nu \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0$. *In addition, if* C *is bounded, then*

$$\lim_\nu \sup_{x\in C} \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0.$$

Proof. Let $x \in C$. Since $F(T)$ is nonempty and $T : C \longrightarrow C$ is quasi-nonexpansive from Lemma 3.1, we obtain that

$$\|T^{n+1}x - y\| \le \|T^n x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and $y \in F(T)$. Then $\{T^n x\}$ is bounded. Furthermore, we have that for any $x \in C$ and $y \in F(T)$

$$\begin{aligned}
\|S_{\mu_\nu}x - y\|^2 &= \langle S_{\mu_\nu}x - y, S_{\mu_\nu}x - y\rangle \\
&= (\mu_\nu)_k \langle T^k x - y, S_{\mu_\nu}x - y\rangle \\
&\le \|\mu_\nu\| \sup_k |\langle T^k x - y, S_{\mu_\nu}x - y\rangle| \\
&\le \sup_k \|T^k x - y\| \cdot \|S_{\mu_\nu}x - y\| \\
&\le \sup_k \|x - y\| \cdot \|S_{\mu_\nu}x - y\| \\
&= \|x - y\| \cdot \|S_{\mu_\nu}x - y\|
\end{aligned}$$

and hence

$$\|S_{\mu_\nu}x - y\| \le \|x - y\|. \tag{3.1}$$

Therefore, $\{S_{\mu_\nu}x\}$ is also bounded. Since T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself, we obtain that

$$\begin{aligned}
&\alpha\|Tz - T^{k+1}x\|^2 + \beta\|z - T^{k+1}x\|^2 + \gamma\|Tz - T^k x\|^2 + \delta\|z - T^k x\|^2 \\
&\quad + \varepsilon\|z - Tz\|^2 + \zeta\|T^k x - T^{k+1}x\|^2 + \eta\|(z - Tz) - (T^k x - T^{k+1}x)\|^2 \le 0
\end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$ and $z \in C$. By (2.3) we obtain that

$$\begin{aligned}
&\|(z - Tz) - (T^k x - T^{k+1}x)\|^2 \\
&\quad = \|z - Tz\|^2 + \|T^k x - T^{k+1}x\|^2 - 2\langle z - Tz, T^k x - T^{k+1}x\rangle \\
&\quad = \|z - Tz\|^2 + \|T^k x - T^{k+1}x\|^2 + \|z - T^k x\|^2 + \|Tz - T^{k+1}x\|^2 \\
&\qquad - \|z - T^{k+1}x\|^2 - \|Tz - T^k x\|^2.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
&(\alpha + \eta)\|Tz - T^{k+1}x\|^2 + (\beta - \eta)\|z - T^{k+1}x\|^2 + (\gamma - \eta)\|Tz - T^k x\|^2 \\
&\quad + (\delta + \eta)\|z - T^k x\|^2 + (\varepsilon + \eta)\|z - Tz\|^2 + (\zeta + \eta)\|T^k x - T^{k+1}x\|^2 \le 0.
\end{aligned}$$

From

$$\begin{aligned}
(\gamma - \eta)\|Tz - T^k x\|^2 &= (\gamma + \alpha)\|Tz - T^k x\|^2 - (\alpha + \eta)\|Tz - T^k x\|^2 \\
&= (\alpha + \gamma)(\|z - Tz\|^2 + \|z - T^k x\|^2 - 2\langle z - Tz, z - T^k x\rangle) \\
&\quad - (\alpha + \eta)\|Tz - T^k x\|^2,
\end{aligned}$$

we have that

$$\begin{aligned}
&(\alpha + \eta)\|Tz - T^{k+1}x\|^2 + (\beta - \eta)\|z - T^{k+1}x\|^2 \\
&\quad + (\alpha + \gamma)(\|z - Tz\|^2 + \|z - T^k x\|^2 - 2\langle z - Tz, z - T^k x\rangle) \\
&\quad - (\alpha + \eta)\|Tz - T^k x\|^2 + (\delta + \eta)\|z - T^k x\|^2 \\
&\quad + (\varepsilon + \eta)\|z - Tz\|^2 + (\zeta + \eta)\|T^k x - T^{k+1}x\|^2 \le 0
\end{aligned}$$

and hence

$$\begin{aligned}&(\alpha+\eta)(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)+(\beta-\eta)\|z-T^{k+1}x\|^2\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-T^kx\rangle+(\alpha+\gamma+\delta+\eta)\|z-T^kx\|^2\\&\quad+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2+(\zeta+\eta)\|T^kx-T^{k+1}x\|^2\le 0.\end{aligned}$$

By $\alpha+\beta+\gamma+\delta\ge 0$, we have that

$$-(\beta-\eta)=-(\beta+\delta)+\delta+\eta\le\alpha+\gamma+\delta+\eta.$$

From this inequality and $\zeta+\eta\ge 0$ we obtain that

$$\begin{aligned}&(\alpha+\eta)(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)\\&\quad+(\beta-\eta)(\|z-T^{k+1}x\|^2-\|z-T^kx\|^2)\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-T^kx\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0\end{aligned}$$

for any $k\in\mathbb{N}\cup\{0\}$ and $z\in C$. We apply μ_ν to both sides of this inequality. We have that

$$\begin{aligned}&(\alpha+\eta)(\mu_\nu)_k(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)\\&\quad+(\beta-\eta)(\mu_\nu)_k(\|z-T^{k+1}x\|^2-\|z-T^kx\|^2)\\&\quad-2(\alpha+\gamma)(\mu_\nu)_k\langle z-Tz,z-T^kx\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0\end{aligned}$$

and hence

$$\begin{aligned}&-|\alpha+\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|Tz-T^kx\|^2\\&\quad-|\beta-\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|z-T^kx\|^2\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-S_{\mu_\nu}x\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0.\end{aligned}\tag{3.2}$$

Replacing z by $S_{\mu_\nu}x$ in (3.2), we have that

$$\begin{aligned}&-2(\alpha+\gamma)\langle S_{\mu_\nu}x-TS_{\mu_\nu}x,S_{\mu_\nu}x-S_{\mu_\nu}x\rangle+(\alpha+\gamma+\varepsilon+\eta)\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|^2\\&\quad\le|\alpha+\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|TS_{\mu_\nu}x-T^kx\|^2\\&\quad\quad+|\beta-\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|S_{\mu_\nu}x-T^kx\|^2\end{aligned}$$

and hence

$$\begin{aligned}&(\alpha+\gamma+\varepsilon+\eta)\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|^2\\&\quad\le|\alpha+\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|TS_{\mu_\nu}x-T^kx\|^2\\&\quad\quad+|\beta-\eta|\|\mu_\nu-\ell_1^*\mu_\nu\|\sup_{k\in\mathbb{N}}\|S_{\mu_\nu}x-T^kx\|^2.\end{aligned}$$

Since $\{TS_{\mu_\nu}x\}$, $\{S_{\mu_\nu}x\}$ and $\{T^nx\}$ are bounded and $\|\mu_\nu-\ell_1^*\mu_\nu\|\longrightarrow 0$, we have that

$$(\alpha+\gamma+\varepsilon+\eta)\limsup_{n\longrightarrow\infty}\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|^2\le 0.$$

Since $\alpha+\gamma+\varepsilon+\eta>0$, we have that $\lim_{n\longrightarrow\infty}\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|=0$. In addition, if C is bounded, then

$$\limsup_{n\longrightarrow\infty}\sup_{x\in C}\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|\le 0$$

and hence $\lim_{n\longrightarrow\infty}\sup_{x\in C}\|S_{\mu_\nu}x-TS_{\mu_\nu}x\|=0$.

Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta\ge 0$, $\alpha+\beta+\zeta+\eta>0$, $\alpha+\gamma>0$ and $\varepsilon+\eta\ge 0$. This completes the proof. □

Lemma 3.3. *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta + \eta > 0$*.*

If $x_\nu \rightharpoonup z$ and $x_\nu - Tx_\nu \longrightarrow 0$, then $z \in F(T)$.

Proof. We give the proof for the case of (2). Since $T : C \longrightarrow C$ is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping, we have that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned} \tag{3.3}$$

for any $x, y \in C$. Replacing x by x_ν in (3.3), we have that

$$\begin{aligned}\alpha\|Tx_\nu - Ty\|^2 + \beta\|x_\nu - Ty\|^2 + \gamma\|Tx_\nu - y\|^2 + \delta\|x_\nu - y\|^2 \\ + \varepsilon\|x_\nu - Tx_\nu\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x_\nu - Tx_\nu) - (y - Ty)\|^2 \leq 0.\end{aligned} \tag{3.4}$$

From this inequality, we have that

$$\begin{aligned}&\alpha(\|Tx_\nu - x_\nu\|^2 + \|x_\nu - Ty\|^2 + 2\langle Tx_\nu - x_\nu, x_\nu - Ty\rangle) + \beta\|x_\nu - Ty\|^2 \\ &\quad + \gamma(\|Tx_\nu - x_\nu\|^2 + \|x_\nu - y\|^2 + 2\langle Tx_\nu - x_\nu, x_\nu - y\rangle) + \delta\|x_\nu - y\|^2 \\ &\qquad + \varepsilon\|x_\nu - Tx_\nu\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x_\nu - Tx_\nu) - (y - Ty)\|^2 \leq 0.\end{aligned}$$

From $\|x_\nu - Ty\|^2 = \|x_\nu - y\|^2 + \|y - Ty\|^2 + 2\langle x_\nu - y, y - Ty\rangle$, we also have

$$\begin{aligned}&(\alpha + \beta + \gamma + \delta)\|x_\nu - y\|^2 \\ &\quad + (\alpha + \beta + \zeta)\|y - Ty\|^2 + 2(\alpha + \beta)\langle x_\nu - y, y - Ty\rangle \\ &\qquad + (\alpha + \gamma + \varepsilon)\|Tx_\nu - x_\nu\|^2 + 2\alpha\langle Tx_\nu - x_\nu, x_\nu - Ty\rangle \\ &\qquad\quad + 2\gamma\langle Tx_\nu - x_\nu, x_\nu - y\rangle + \eta\|(x_\nu - Tx_\nu) - (y - Ty)\|^2 \leq 0.\end{aligned}$$

From $\alpha + \beta + \gamma + \delta \geq 0$ we obtain that

$$\begin{aligned}&(\alpha + \beta + \zeta)\|y - Ty\|^2 + 2(\alpha + \beta)\langle x_\nu - y, y - Ty\rangle \\ &\quad + (\alpha + \gamma + \varepsilon)\|Tx_\nu - x_\nu\|^2 + 2\alpha\langle Tx_\nu - x_\nu, x_\nu - Ty\rangle \\ &\qquad + 2\gamma\langle Tx_\nu - x_\nu, x_\nu - y\rangle + \eta\|(x_\nu - Tx_\nu) - (y - Ty)\|^2 \leq 0.\end{aligned}$$

Since $x_\nu \rightharpoonup z$ and $x_\nu - Tx_\nu \longrightarrow 0$, we have that

$$(\alpha + \beta + \zeta + \eta)\|y - Ty\|^2 + 2(\alpha + \beta)\langle z - y, y - Ty\rangle \leq 0.$$

Putting $y = z$, we have that

$$(\alpha + \beta + \zeta + \eta)\|z - Tz\|^2 \leq 0.$$

Since $\alpha + \beta + \zeta + \eta > 0$, we have that $z \in F(T)$.

Similarly, by replacing the variables x and y in (3.3), we can obtain the desired result for the case when $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma + \varepsilon + \eta > 0$. This completes the proof. □

Now we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$. Suppose that T satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma>0$, $\varepsilon+\eta \geq 0$ *and* $\alpha+\beta+\zeta+\eta>0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta>0$, $\zeta+\eta \geq 0$ *and* $\alpha+\gamma+\varepsilon+\eta>0$.

Let $\{\mu_\nu\}$ *be a srongly asymptotically invariant net of means on* ℓ^∞ *and let* P *be the metric projection of* H *onto* $F(T)$. *Then for any* $x \in C$, *the net* $\{S_{\mu_\nu}x\}$ *converges weakly to a fixed point* p *of* T, *where* $p=\lim_{n\to\infty} PT^n x$.

Proof. Let $x \in C$. As in the proof of Lemma 3.2, we have that $\{T^n x\}$ is bounded and $\{S_{\mu_\nu}x\}$ is bounded. Therefore, there exist a subnet $\{S_{\mu_{\nu_\omega}}x\}$ of $\{S_{\mu_\nu}x\}$ and $p \in H$ such that $\{S_{\mu_{\nu_\omega}}x\}$ converges weakly to p. Using $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma>0$, $\varepsilon+\eta \geq 0$ and $\alpha+\beta+\zeta+\eta>0$, we have from Lemma 3.2 that

$$\lim_\nu \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0. \tag{3.5}$$

We have from Lemma 3.3 that $p \in F(T)$. Since $F(T)$ is closed and convex from Lemma 3.1, the metric projection P from H onto $F(T)$ is well-defined. By Lemma 2.1, there exists $q \in F(T)$ such that $\{PT^n x : n \in \mathbb{N}\}$ converges strongly to q. To complete the proof, we show that $q=p$. Note that the metric projection P satisfies

$$\langle z-Pz, Pz-u\rangle \geq 0$$

for any $z \in H$ and for any $u \in F(T)$; see [27]. Therefore, we have that

$$\langle T^k x - PT^k x, PT^k x - y\rangle \geq 0$$

for any $k \in \mathbb{N}\cup\{0\}$ and $y \in F(T)$. From the properties of the metric projection P and $PT^{n-1}x \in F(T)$, we obtain that

$$\begin{aligned}\|T^k x - PT^k x\| &\leq \|T^k x - PT^{k-1}x\| \\ &\leq \|T^{k-1}x - PT^{k-1}x\|.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}\langle T^k x - PT^k x, y-q\rangle &\leq \langle T^k x - PT^k x, PT^k x - q\rangle \\ &\leq \|T^k x - PT^k x\| \cdot \|PT^k x - q\| \\ &\leq \|x-Px\| \cdot \|PT^k x - q\|.\end{aligned}$$

We apply μ_ν to both sides of this inequality. Then we obtain that

$$(\mu_\nu)_k \langle T^k x - PT^k x, y-q\rangle \leq \|x-Px\| (\mu_\nu)_k \|PT^k x - q\|. \tag{3.6}$$

Replacing ν by ν_ω in (3.6), we have that

$$(\mu_{\nu_\omega})_k \langle T^k x - PT^k x, y-q\rangle \leq \|x-Px\| (\mu_{\nu_\omega})_k \|PT^k x - q\|.$$

Since $\{\mu_{\nu_\omega}\}$ has a subnet converging a Banach limit λ in the weak* topology, we obtain that

$$\lambda_k \langle T^k x - PT^k x, y-q\rangle \leq \|x-Px\| \lambda_k \|PT^k x - q\|.$$

Since $\{S_{\mu_{\nu_\omega}}x\}$ converges weakly to p, $\{PT^n x\}$ converges strongly to q and λ is a Banach limit, we obtain that

$$\langle p-q, y-q\rangle \leq 0.$$

Putting $y=p$, we obtain $\|p-q\|^2 \leq 0$ and hence $q=p$. This completes the proof.

Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta>0$, $\zeta+\eta \geq 0$ and $\alpha+\gamma+\varepsilon+\eta>0$. □

Using Theorem 3.1, we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space which was proved by Kawasaki and Takahashi [14].

Theorem 3.2. *Let H be a real Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, $\zeta + \eta \geq 0$ *and* $\alpha + \beta > 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\varepsilon + \eta \geq 0$ *and* $\alpha + \gamma > 0$.

Then for any $x \in C$ the Cesàro means

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to a fixed point p of T and $p = \lim_{n\to\infty} PT^n x$, where P is the metric projection of H onto $F(T)$.

Proof. For any $f = (x_0, x_1, x_2, \dots) \in \ell^\infty$, define

$$\mu_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} x_k, \quad \forall n \in \mathbb{N}.$$

Then $\{\mu_n : n \in \mathbb{N}\}$ is a strongly asymptotically invariant sequence of means on ℓ^∞; see [27, p.78]. Furthermore, we have that for any $x \in C$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x.$$

Therefore, we have the desired result from Theorem 3.1. □

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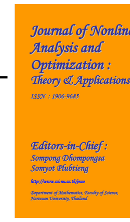

A COMPROMISE ALLOCATION IN MULTIVARIATE STRATIFIED SAMPLING IN PRESENCE OF NON-RESPONSE

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ABSTRACT. In this paper, we have considered the problem of determining the optimum allocation of sample sizes and sampling fractions for all the characteristics, which minimizes the coefficients of variance among the non-respondent to various characteristics under study. The problem is formulated as a Multi objective Programming Problem (MNLPP) and Multi objective Goal Programming Problem. A solution procedure is developed by using Lagrange Multiplier's Technique (LMT) and Goal Programming technique. A numerical example is presented to illustrate the computational details.

KEYWORDS : Compromise allocation; Compromise criterion; Non-response; Lagrange Multiplier's Technique; Multivariate Stratified Sampling; Non-linear programming problem; Goal programming.
AMS Subject Classification:

1. INTRODUCTION

While conducting a mailed survey, the problem of non-response generally occurs. In such cases, Hansen and Hurwitz [18] have suggested the method of sub-sampling from non-respondents to provide an estimator for population mean. They selected a preliminary sample and mailed the questionnaires to all the selected units. Non-respondents are identified and a second attempt was made by interviewing a sub-sample of non-respondents. They constructed the estimate of the population mean by combining the data from the two attempts and derived the expression for the sampling variance of the estimate. The optimum sampling fraction among the non-respondents is also obtained. El-Badry [11] has extended Hansen and Hurwitz's technique based on the experience that an appreciable in response rates to mail questionnaires can be secured by sending waves of questionnaires to the non-respondents group. Foradari [13] has generalized El-Badry's approach and has also studied the uses of Hansen and Hurwitz's technique under

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different designs. Srinath [37] described the selection of subsamples by making several attempts in non-respondents group.

Stratified sampling design is most widely sampling design for estimating the population parameters of a heterogeneous population. It deals with the properties of the estimates constructed from a stratified random sample and with the optimum choice of the sizes of the samples to be selected from various strata either to maximize the precision of the constructed estimate for a fixed cost or to minimize the cost of the survey for a fixed precision of the estimate. The sample sizes allocated according to either of the above criteria is called an "Optimum allocation". Khare [26] discussed the problem of optimum allocation in stratified sampling in presence of non-response for fixed cost as well as for fixed precision of the estimate. Khan et al. [25], has worked on optimum allocation and optimum size of subsamples to various strata in multivariate stratified sampling in presence of non-response by Lagrange Multiplier's Techniques.

For a population the coefficient of variation (C.V.) is represented by the ratio of population standard deviation to the population mean. It is useful statistic for comparing the degree of variation from one data series to another, even if the means are drastically different from each other. It also eliminates the effect of the scale of the observations.

The "optimum allocation" in stratified random sampling and its solution is well known in sampling literature for univariate population (see Cochran, [6]; Shukhatme et al., [38]. In multivariate populations where more than one characteristic are to be studied on every selected unit of the population. The problem of finding an optimum allocation becomes more complex due to complicating behavior of characteristics. Various another's such as Dalenius [7, 8], Ghosh [14], Yates [42], Aoyama [3], Gren [16, 17], Folks and Antle [12], Hertley [19], Kokan and Khan [28], Chromy [5], Wywial [41], Bethel [4], Kreienbrock [30], Jahan et al. [22], Khan et al. [23, 24], Ahsan et al. [1], Díaz-García and Ulloa [9, 10], Ansari et al. [2] etc. used different compromise criteria to work out a compromise allocation that is optimum for all characteristics in some sense.

When some auxiliary information is available, it can be used to increase the precision of estimates. Ige and Tripathi [21], Rao [35], Tripathi and Bahl [39] and some other discussed the use of auxiliary information in stratified sampling using double sampling technique.

The problem of optimum allocation, where the strata weights are unknown and non-response also occures have been studied by some authors. Okafor [34], solved the above problem for stratified population in univariate case using a double sampling strategy (DDS). The same problem was also formulated by Najmussehar and Bari [33] using dynamic programming technique to obtain a solution. A comparative study has also been done by Varshney et al. [40] by developing a goal programming to solve the problem.

In sampling literature, the allocation based on compromise criteria is termed as the "compromise allocation". Holmberg [20], discussed three compromise criteria to work out a compromise allocation.

- **(i):** Minimizing the sum of sampling variances of the estimators of the population parameters of various characteristics.
- **(ii):** Minimizing the sum of squared coefficients of variation of the estimator over the characteristics.
- **(iii):** Minimizing the sum of efficiency losses for not using the individual optimum allocation.

Kozak [29], used the concept of minimizing some function of squared coefficient of variations as an objective function. Latest Ghufran et al. [15] also discussed this concept.

In the present manuscript, we have developed a method to work out the compromise allocation in multivariate stratified sampling in presence of non-response using the criterion "Minimizing the sum of squared coefficients of variations of the estimators over the characteristics". It is assumed that the estimation of population means is of interest. The problem is formulated as a Multi-objective Non-Linear Programming Problem (MNLPP) and Multi-objective Non Linear Goal Programming Problem (MNLGPP). The Lagrange Multipliers Technique and goal programming are used to obtain a solution of formulated problem.

2. SAMPLING STRATEGIES AND ESTIMATION PROCEDURE

Let us consider a population consisting of N units divided into k strata. Let $N_i,\ \bar{Y}_{ji},\ S^2_{ji}$ and $p_i = \frac{N_i}{N}(i = 1, 2, \ldots, k, j = 1, 2, \ldots, p)$ denote respectively, the stratum size, stratum mean, stratum variance and stratum weight of j^{th} character in i^{th} stratum, where p is the number of variables under study. Assume that every stratum is divided into two disjoint groups of respondents and non-respondents, with N_{i1} and $N_{i2} = N - N_{i1}$ as the sizes of respondents and non-respondents in the i^{th} stratum respectively. We decide to select a sample of size n from the entire population in such a way that n_i units are selected from the N_i units in the i^{th} stratum with $\sum_{i=1}^{k} n_i = n$. Let $p \geq 2$, characteristics be defined on each population unit and the estimation of the p unknown population means $\bar{Y}_j;\ j = 1, 2, \ldots, p$ are of interest.
The population mean $\bar{Y}_j$ of the j^{th} characteristic is given by:

$$\bar{Y}_j = \frac{1}{N}\sum_{i=1}^{k}\sum_{h=1}^{N_i} y_{jih};\ j = 1, 2, \ldots, p \tag{2.1}$$

$$= \sum_{i=1}^{k} p_i \bar{Y}_{ji} \tag{2.2}$$

where y_{jih} is the value of the j^{th} variable (characteristic) of the h^{th} element in the i^{th} stratum, $j = 1, 2, \ldots, p;\ i = 1, 2, \ldots, k;\ h = 1, 2, \ldots, N_i$
$\bar{Y}_{ji} = \frac{1}{N_i}\sum_{h=1}^{N_i} y_{jih}$ is the i^{th} stratum mean for the j^{th} characteristic
Let n_{i1} be the number of units of the sample in the i^{th} stratum that provide the data sought and n_{i2} be the number of units of non-respondents. By extensive efforts, the data are later obtained from a random sample of u_{i2} out of n_{i2} unit such that

$$n_{i2} = K_i u_{i2}(K_i > 1) \tag{2.3}$$

where $\frac{1}{K_i}$, denotes the sampling fraction among non-respondents in the i^{th} stratum.
Now we have

$$E\left(\frac{n_{i1}}{N_{i1}}\right) = E\left(\frac{n_{i2}}{N_{i2}}\right) = K_i E\left(\frac{u_{i2}}{N_{i2}}\right) \tag{2.4}$$

Let $\bar{y}^*_{ji}$ be the unbiased estimator of population mean $\bar{Y}_{ji}$ in i^{th} stratum for j^{th} characteristics which is given by

$$\bar{y}^*_{ji} = \frac{n_{i1}\bar{y}_{jn_{i1}} + n_{i2}\bar{y}_{ju_{i2}}}{n_i} \tag{2.5}$$

where $\bar{y}_{jn_{i1}}$ and $\bar{y}_{ju_{i2}}$ are the means based on n_{i1} units of response group and u_{i2} units of sub-sample of non-response group respectively for the j^{th} characteristics in i^{th} stratum.
Using Hansen-Hurwitz technique, an unbiased estimator of population mean $\bar{Y}_j$ for the j^{th} characteristics is given by

$$\bar{y}^*_{jst} = \sum_{i=1}^{k} p_i \bar{y}^*_{ji} \forall j = 1, 2, \ldots, p. \tag{2.6}$$

$$i.e., \; E(\bar{y}^*_{jst}) = \bar{Y}_j \forall j = 1, 2, \ldots, p. \tag{2.7}$$

Ignoring the terms independent of n_i and K_i of the variance of the estimator is given by the expression

$$V(\bar{y}^*_{jst}) = \sum_{i=1}^{k} \frac{p_i^2 S_{ji}^2}{n_i} + \sum_{i=1}^{k} \left(\frac{k_i - 1}{n_i}\right) W_{i2} p_i^2 S_{ji2}^2 \forall j = 1, 2, \ldots, p. \tag{2.8}$$

where S_{ji}^2 are the population mean-square errors of entire group for the j^{th} characteristic in the i^{th} stratum. And S_{ji2}^2 are mean-square errors of non-response group for the j^{th} characteristic in the i^{th} stratum. The $W_{i2} = \frac{N_{i2}}{N_i}$ is the Non-Response rate in the i^{th} stratum.It is assumed that information on all the units of sub sample selected from the non-response group is available.
The values of n_i and K_i are to be chosen so as to give the maximum precision for fixed cost. Let C_{i0} be the cost per unit of selecting n_i units, $C_{i1} = \sum_{j=1}^{p} C_{ji1}$ be the cost per unit in enumerating n_{i1} units and $C_{i2} = \sum_{j=1}^{p} C_{ji2}$ be the cost per unit in enumerating u_{i2} units from non-respondent group. Then the total cost for the i^{th} stratum is given by

$$C_i = C_{i0} n_i + C_{i1} n_{i1} + C_{i2} u_{i2}, \forall i = 1, 2, \ldots, k. \tag{2.9}$$

Since the values of n_{i1} and n_{i2} are not known until the first attempt is made, the expected cost is used in planning the sample. The expected value of n_{i1} and u_{i2} are $W_{i1} n_i$ and $\frac{W_{i2} n_i}{K_i}$ respectively.
The average cost for the i^{th} stratum is

$$E(C_i) = \left(C_{i0} + C_{i1} W_{i1} + C_{i2} \frac{W_{i2}}{K_i}\right) n_i \tag{2.10}$$

where $W_{i1} = \frac{N_{i1}}{N_i}$ is the Response rate in the i^{th} stratum.
Thus the total cost over all the strata is given by

$$C_0 = \sum_{i=1}^{k} E(C_i) = \sum_{i=1}^{k} \left[C_{i0} + C_{i1} W_{i1} + C_{i2} \frac{W_{i2}}{K_i}\right] n_i \tag{2.11}$$

3. STATEMENT OF THE PROBLEM

Obviously, the best compromise allocation will be that which minimizes all the p variances given by (2.8) simultaneously. Thus, for finding the optimum compromise allocation we need to solve the following Multi-objective Non-linear Programming

Problem (MNLPP):

$$\left.\begin{aligned} &Minimize \begin{pmatrix} V(\bar{y}^*_{1st}) \\ \vdots \\ V(\bar{y}^*_{pst}) \end{pmatrix} \\ &Subject\ to\ \sum_{i=1}^{k}\left(C_{i0}+C_{i1}W_{i1}+C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ &\qquad K_i>1,\ 2\le n_i\le N_i \\ &\qquad and\ n_i\ are\ integers; i=1,2,\dots,k \end{aligned}\right\} \tag{3.1}$$

Kish [27], Rios et al. [36], Miettinen [32], Khan et al. [24] studied the problem of multi-objective optimization in detailed. Khan et al. [24] and García and Ulloa [10] expressed the problem (3.1) under the value function technique as

$$\left.\begin{aligned} &Minimize\ \vartheta(V(\bar{y}^*_{jst})) \\ &Subject\ to\ \sum_{i=1}^{k}\left(C_{i0}+C_{i1}W_{i1}+C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ &\qquad K_i>1,\ 2\le n_i\le N_i \\ &\qquad and\ n_i\ are\ integers; i=1,2,\dots,k \end{aligned}\right\} \tag{3.2}$$

where $\vartheta(.)$ is a scalar function that summarizes the importance of each of the variance of the p characteristics. Usually, ϑ is taken as weighted sum of the p variances that is

$$\vartheta=\sum_{j=1}^{p} w_j V(\bar{y}^*_{jst}) \tag{3.3}$$

where $\sum_{j=1}^{p} w_j = 1,\ w_j \ge 0,\ j=1,2,\dots,p$
w_j are the weights according to the importance of each characteristics.
Many authors object on using the weighted sum of variances because variances are not unit free thus cannot be added. In this paper, instead of variances we use the squared coefficient of variation that is unit free and positive. Thus, our problem under the value function technique may be expressed as:

$$\left.\begin{aligned} &Minimize\ Z=\sum_{j=1}^{p} w_j (CV)_j^2 \\ &Subject\ to\ \sum_{i=1}^{k}\left(C_{i0}+C_{i1}W_{i1}+C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ &\qquad K_i>1,\ 2\le n_i\le N_i \\ &\qquad and\ n_i\ are\ integers; i=1,2,\dots,k \end{aligned}\right\} \tag{3.4}$$

where $(CV)_j = CV(\bar{y}^*_{j,st}) = \frac{SD(\bar{y}^*_{j,st})}{\bar{Y}_j};\ j=1,2,\dots,p$

$$(CV)_j^2=\frac{V(\bar{y}^*_{j,st})}{\bar{Y}_j^2};\ j=1,2,\dots,p \tag{3.5}$$

$\bar{Y}_j$, $\bar{y}^*_{jst}$ and $V(\bar{y}^*_{jst})$ are as defined in (2.2), (2.6) and (2.8), respectively. AINLPP (3.4) may now be restated as

$$\left.\begin{array}{l} Minimize\ Z = \sum_{j=1}^{p} \bar{Y}_j^{-2}\{\sum_{i=1}^{k}\left(\frac{N_i-n_i}{N_i n_i}\right)p_i^2 S_{ji}^2 + \sum_{i=1}^{k}\left(\frac{k_i-1}{n_i}\right)W_{i2}p_i^2 S_{ji2}^2\} \\ Subject\ to\ \sum_{i=1}^{k}\left(C_{i0}+C_{i1}W_{i1}+C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ \qquad K_i > 1,\ 2 \le n_i \le N_i \\ \qquad and\ n_i\ are\ integers; i = 1,2,\dots,k \end{array}\right\} \tag{3.6}$$

ignoring the terms independent of n_i and K_i the equation (3.6) can be rewritten as:

$$\left.\begin{array}{l} Minimize\ Z = \sum_{i=1}^{k}\frac{p_i^2 a_i^2}{n_i} + \sum_{i=1}^{k}\left(\frac{k_i-1}{n_i}\right)W_{i2}p_i^2 b_i^2 \\ Subject\ to\ \sum_{i=1}^{k}\left(C_{i0}+C_{i1}W_{i1}+C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ \qquad K_i > 1,\ 2 \le n_i \le N_i \\ \qquad and\ n_i\ are\ integers; i = 1,2,\dots,k \end{array}\right\} \tag{3.7}$$

where

$$a_i^2 = \sum_{j=1}^{p}\bar{Y}_j^{-2}S_{ji}^2,\ b_i^2 = \sum_{j=1}^{p}\bar{Y}_j^{-2}S_{ji2}^2 \tag{3.8}$$

and all the p weights are assumed to be equal, that is $w_1 = w_2 = \cdots = w_p = \frac{1}{p}$ without loss of generality the common factor $\frac{1}{p}$ may be dropped from the objective function.

4. LAGRANGE MULTIPLIER'S TECHNIQUE

Taking care of the integer restrictions by rounding off the continuous solution the NLPP (3.7) (AINLPP (3.7) without integer restrictions) may be solved by Lagrange Multipliers Technique. For applying Lagrange Multipliers Technique (LMT), the restrictions $2 \le n_i \le N_i;\ i = 1,2,\dots,k$ are ignored. We take cost constraint as an equation. To determine the optimum values of n_i and K_i for the cost function (2.11), we consider the function

$$\phi = \vartheta(V(\bar{y}^*_{jst})) + \mu(C_0) \tag{4.1}$$

where μ is the Lagrange Multiplier.

$$\phi(n_i,K_i,\mu) = \sum_{i=1}^{k}\left(\frac{1}{n_i}-\frac{1}{N_i}\right)p_i^2 a_i^2 + \sum_{i=1}^{k}\frac{(k_i-1)W_{i2}p_i^2 b_i^2}{n_i} + \mu\left[\sum_{i=1}^{k}\left(C_{i0}+C_{i1}W_{i1}+C_{i2}\frac{W_{i2}}{K_i}\right)n_i - C_0\right] \tag{4.2}$$

Now differentiating $\phi(n_i,K_i,\mu)$ w.r.t. K_i and equating to zero, noting that $\frac{\partial^2\phi}{\partial K_i^2} > 0$, we have

$$\frac{\partial\phi}{\partial K_i} = \frac{W_{i2}^2 p_i^2 b_i^2}{n_i} - \frac{\mu n_i C_{i2} W_{i2}}{K_i^2} = 0 \tag{4.3}$$

$$n_i = \frac{k_i p_i b_i}{\sqrt{\mu C_{i2}}} \tag{4.4}$$

again differentiating $\phi(n_i, K_i, \mu)$ w.r.t. n_i and equating to zero, noting that $\frac{\partial^2 \phi}{\partial n_i^2} > 0$, we have

$$\frac{\partial \phi}{\partial n_i} = -\frac{p_i^2 a_i^2}{n_i^2} - \frac{(k_i - 1) W_{i2} p_i^2 b_i^2}{n_i^2} + \mu \left(C_{i0} + C_{i1} W_{i1} + C_{i2} \frac{W_{i2}}{K_i} \right) = 0 \tag{4.5}$$

Now eliminating μ from (4.5) by putting its value from (4.4), we have

$$K_i = \sqrt{\frac{C_{i2}(a_i^2 + W_{i2} b_i^2)}{b_i^2 (C_{i0} + C_{i1} W_{i1})}} \tag{4.6}$$

which shows that K_i increases with the increase in C_{i2} and subsequently the number of units to be repeated from non-response group of the i^{th} stratum decreases. The total cost C_0 is fixed. So we have

$$C_0 = \sum_{i=1}^{k} \left(C_{i0} + C_{i1} W_{i1} + C_{i2} \frac{W_{i2}}{K_i} \right) n_i \tag{4.7}$$

Now putting the value of n_i from (4.4) in (4.7) and using the values of K_i from (4.6), we have

$$\frac{1}{\sqrt{\mu}} = \frac{C_0}{\sum_{i=1}^{k} p_i \left[\{ \sqrt{(C_{i0} + C_{i1} W_{i1}) * (a_i^2 - W_{i2} b_i^2)} \} + W_{i2} b_i \sqrt{C_{i2}} \right]} \tag{4.8}$$

Hence the value of n_i is given by

$$n_i = \frac{p_i C_0 \sqrt{(a_i^2 + W_{i2} b_i^2)/(C_{i0} + C_{i1} W_{i1})}}{\sum_{i=1}^{k} p_i \left[\{ \sqrt{(C_{i0} + C_{i1} W_{i1}) * (a_i^2 - W_{i2} b_i^2)} \} + W_{i2} b_i \sqrt{C_{i2}} \right]} \tag{4.9}$$

If the compromise allocation (4.9) satisfies the restriction $2 \leq n_i \leq N_i;\ i = 1, 2, \ldots, k$ then it will provide a continuous solution of NLPP (3.7). The continuous solution, rounded off to the nearest integer values of n_i, will then give an integer solution. After rounding off one has to be careful in rechecking that the rounded off values satisfy the cost constraints $\sum_{i=1}^{k} \left(C_{i0} + C_{i1} W_{i1} + C_{i2} \frac{W_{i2}}{K_i} \right) n_i \leq C_0$. If the cost constraints are violated by the rounded off solution we may use some integer non linear programming technique. Ignoring the finite population correction (fpc) n_i given by (4.9) satisfies the restrictions restriction $2 \leq n_i \leq N_i;\ i = 1, 2, \ldots, k$.
However, if any $n_i > N_i$ it can be taken as equal to N_i and for the remaining *(k-1)* strata n_i are recalculated. Similarly, if $n_i < 2$ it can be put equal to 2 and the remaining n_i are recalculated.
As an alternative, the constraints $2 \leq n_i \leq N_i;\ i = 1, 2, \ldots, k$ may also be included and the Integer Non- Linear Programming Problem (3.7) may also be solved by other integer NLPP technique. Softwares are also available to solve AINLPP. One such software is LINGO. LINGO is a user's friendly package for constrained optimization developed by LINDO Systems Inc. A user's guide LINGO [31] is also available. For more information one can visit the site http://www.lindo.com.

5. THE GOAL PROGRAMMING TECHNIQUE

The problem (3.4) may be stated separately for all the p characteristics as

$$\left.\begin{array}{l} Minimize\ Z = [(CV)_1^2, (CV)_2^2, \ldots, (CV)_p^2] \\ Subject\ to\ \sum_{i=1}^{k}\left(C_{i0} + C_{i1}W_{i1} + C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ K_i > 1,\ 2 \le n_i \le N_i \\ and\ n_i\ are\ integers; i = 1, 2, \ldots, k \end{array}\right\} \tag{5.1}$$

Using (3.5), the equation (5.1) can be written as

$$\left.\begin{array}{l} Minimize\ Z = \left[\frac{V(\bar{y}_{1st}^*)}{\bar{Y}_1^2}, \frac{V(\bar{y}_{2st}^*)}{\bar{Y}_2^2}, \ldots, \frac{V(\bar{y}_{pst}^*)}{\bar{Y}_p^2}\right] \\ Subject\ to\ \sum_{i=1}^{k}\left(C_{i0} + C_{i1}W_{i1} + C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ K_i > 1,\ 2 \le n_i \le N_i \\ and\ n_i\ are\ integers; i = 1, 2, \ldots, k \end{array}\right\} \tag{5.2}$$

where $V(\bar{y}_{1st}^*),\ j = 1, 2, \ldots, p$ is defined in (2.8).
The equation (5.2) can be written as

$$\left.\begin{array}{l} Minimize\ Z = [Z_1, Z_2, \ldots, Z_p] \\ Subject\ to\ \sum_{i=1}^{k}\left(C_{i0} + C_{i1}W_{i1} + C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ K_i > 1,\ 2 \le n_i \le N_i \\ and\ n_i\ are\ integers; i = 1, 2, \ldots, k \end{array}\right\} \tag{5.3}$$

where $Z_j = \frac{V(\bar{y}_{jst}^*)}{\bar{Y}_j^2} = \bar{Y}_j^{-2}\{\sum_{i=1}^{k}\frac{p_i^2 S_{ji}^2}{n_i} + \sum_{i=1}^{k}\left(\frac{k_i-1}{n_i}\right)W_{i2}p_i^2 S_{ji2}^2\},\ j = 1, 2, \ldots, p.$
or
$Z_j = \sum_{i=1}^{k}\frac{p_i^2 S_{ji}^2 \bar{Y}_j^{-2}}{n_i} + \sum_{i=1}^{k}\left(\frac{k_i-1}{n_i}\right)W_{i2}p_i^2 S_{ji2}^2 \bar{Y}_j^{-2},\ j = 1, 2, \ldots, p$. Let Z_j^* be the optimum value of Z_j obtained by solving the following Nonlinear Programming Problem(NLPP)

$$\left.\begin{array}{l} Minimize\ Z_j,\ j = 1, 2, \ldots, p \\ Subject\ to\ \sum_{i=1}^{k}\left(C_{i0} + C_{i1}W_{i1} + C_{i2}\frac{W_{i2}}{K_i}\right)n_i \le C_0 \\ K_i > 1,\ 2 \le n_i \le N_i \\ and\ n_i\ are\ integers; i = 1, 2, \ldots, k \end{array}\right\} \tag{5.4}$$

Further let

$$\tilde{Z}_j = \tilde{Z}_j(n_1, n_2, \ldots, n_i, \ldots, n_k) = \sum_{i=1}^{k}\frac{p_i^2 S_{ji}^2 \bar{Y}_j^{-2}}{n_i} + \sum_{i=1}^{k}\left(\frac{k_i - 1}{n_i}\right)W_{i2}p_i^2 S_{ji2}^2 \bar{Y}_j^{-2} \tag{5.5}$$

Denote the variance under the compromise allocation, where n_i and $K_i;\ i = 1, 2, \dots, k$ are to be worked out.
Obviously $\tilde{Z}_j \geq Z_j^*$ and $\tilde{Z}_j - Z_j^* \geq 0;\ j = 1, 2, \dots, p$ will give the increase in the variance due to not using the individual optimum allocation for j^{th} characteristic.
Now consider the following goal:
"Find n_i and K_i such that the increase in the value of the variance for each characteristic due to the use of compromise allocation, n_i instead of individual optimum allocation, should not greater than $x_j(j = 1, 2, , p)$". Where $x_j \geq 0(j = 1, 2, , p)$ are the unknown goal variables.
To achieve these goals n_i must satisfy

$$\tilde{Z}_j - Z_j^* \leq x_j;\ j = 1, 2, \dots, p \tag{5.6}$$

or

$$\tilde{Z}_j - x_j \leq Z_j^*;\ j = 1, 2, \dots, p$$

or

$$\sum_{i=1}^{k} \frac{p_i^2 S_{ji}^2 \bar{Y}_j^{-2}}{n_i} + \sum_{i=1}^{k} \left(\frac{k_i - 1}{n_i}\right) W_{i2} p_i^2 S_{ji2}^2 \bar{Y}_j^{-2} - x_j \leq Z_j^*;\ j = 1, 2, \dots, p \tag{5.7}$$

The value $\sum_{j=1}^{p} x_j$ will give us the total increase in variances by not using the individual optimum allocations.
This suggests the following Goal Programming Problem (GPP) to solve.

$$\left.\begin{array}{c} Minimize\ \sum_{j=1}^{p} x_j \\ \\ Subject\ to \\ \\ \sum_{i=1}^{k} \frac{p_i^2 S_{ji}^2 \bar{Y}_j^{-2}}{n_i} + \sum_{i=1}^{k} \left(\frac{k_i-1}{n_i}\right) W_{i2} p_i^2 S_{ji2}^2 \bar{Y}_j^{-2} - x_j \leq Z_j^*;\ j = 1, 2, \dots, p \\ \\ \sum_{i=1}^{k} \left(C_{i0} + C_{i1} W_{i1} + C_{i2} \frac{W_{i2}}{K_i}\right) n_i \leq C_0 \\ \\ K_i > 1,\ 2 \leq n_i \leq N_i \\ \\ and\ n_i\ are\ integers; i = 1, 2, \dots, k \end{array}\right\} \tag{5.8}$$

6. NUMERICAL ILLUSTRATION

The following numerical example is presented to illustrate the practical use and the computational details of working out the optimum sampling fractions among non-respondents $1/K_i$ (where K_i are defined in 4.6) and proposed allocation defined by (4.9). Consider a population of size N=3850 divided into four strata. Let the population means of the two characteristics defined on each unit of the population are assumed to be known as $\bar{Y}_1 = 24.73$ and $\bar{Y}_2 = 31.53$. It is also assumed that the relative values of variances of the non-respondent and respondents, that is, $S_{ji2}^2/S_{ji}^2 = 0.25$ for $j = 1, 2$ and $i = 1, 2, \dots, 4$. Further, let the total amount available for the survey is $C_0 = 5000$ units. Table 5.1 shows the relevant information.
Substituting the values from Table 5.1 in (3.8), we get

$$a_1^2 = 16.05 \qquad a_2^2 = 17.89$$

TABLE 1. Data for four strata and two characteristics

i	N_i	p_i	S_{1i}^2	S_{2i}^2	W_{i1}	W_{i2}	C_{i0}	C_{i1}	C_{i2}
1	1214	0.32	4817.72	8121.15	0.70	0.30	1	2	3
2	822	0.21	6251.26	7613.52	0.80	0.20	1	3	4
3	1028	0.27	3066.16	1456.40	0.75	0.25	1	4	5
4	786	0.20	6207.25	6977.72	0.72	0.28	1	5	6

$$a_3^2 = 6.48 \qquad a_4^2 = 17.18$$

$$b_1^2 = 4.02 \qquad b_2^2 = 4.47$$

$$b_3^2 = 1.62 \qquad b_4^2 = 4.29$$

We redefine $A_i = a_i^2 - W_{i2}b_i^2$ and $B_i = C_{i0} + C_{i1}W_{i1},\ i = 1, 2, \ldots, 4$

$$A_1 = 14.85 \qquad A_2 = 16.99$$

$$A_3 = 6.08 \qquad A_4 = 15.97$$

$$B_1 = 2.4 \qquad B_2 = 3.4$$

$$B_3 = 4.0 \qquad B_2 = 4.6$$

From column (11) &(12) of the Table 5.2, Sampling fractions among the non-respondents are $\frac{1}{K_1} = 0.46, \frac{1}{K_2} = 0.47, \frac{1}{K_3} = 0.46$ and $\frac{1}{K_4} = 0.45$ and the rounded off compromise allocations using the proposed method are obtained as: $n_1 = 528, n_2 = 311, n_3 = 221$ and $n_4 = 247$ with the value of the objective function as $Z = .0113$.

Solution by using Goal Programming:

Using the equation (5.4) the non linear programming problem for the first characteristic is

$$\left.\begin{array}{c} Minimize\ Z_1 = \frac{.8066}{n_1} + \frac{.4507}{n_2} + \frac{.3655}{n_3} + \frac{.4059}{n_4} \\ + \frac{(K_1-1)}{n_1} * 0.06 + \frac{(K_2-1)}{n_2} * 0.02 + \frac{(K_3-1)}{n_3} * 0.02 + \frac{(K_4-1)}{n_4} * 0.02 \\ Subject\ to \\ (2.4 + \frac{0.9}{K_1})n_1 + (3.4 + \frac{0.8}{K_2})n_2 + (4 + \frac{1.25}{K_3})n_3 + (4.6 + \frac{1.68}{K_4})n_4 \leq 5000 \\ 2 \leq n_1 \leq 1214;\ 2 \leq n_2 \leq 822 \\ 2 \leq n_3 \leq 1028;\ 2 \leq n_4 \leq 786 \\ K_i > 1,\ and\ n_i\ are\ integers; i = 1, 2, \ldots, 4 \end{array}\right\}$$

The optimal solution provided by LINGO is
$n_{1,1}^* = 482, n_{1,2}^* = 307, n_{1,3}^* = 253, n_{1,4}^* = 247, K_1 = 2.15, K_2 = 2.12, K_3 = 2.17, K_4 = 2.20$ with the value of the objective function as $Z_1 = 0.0067$. Where $n_{1,i}^*$ denote the optimum allocation for the first characteristic.

Similarly using (5.4) the non linear programming problem for the second characteristic is

$$\left.\begin{aligned}
&Minimize\ Z_2 = \frac{.8365}{n_1} + \frac{.3377}{n_2} + \frac{.1068}{n_3} + \frac{.2807}{n_4} \\
&+\frac{(K_1-1)}{n_1} * 0.06 + \frac{(K_2-1)}{n_2} * 0.02 + \frac{(K_3-1)}{n_3} * 0.07 + \frac{(K_4-1)}{n_4} * 0.02 \\
&Subject\ to \\
&(2.4 + \frac{0.9}{K_1})n_1 + (3.4 + \frac{0.8}{K_2})n_2 + (4 + \frac{1.25}{K_3})n_3 + (4.6 + \frac{1.68}{K_4})n_4 \leq 5000 \\
&2 \leq n_1 \leq 1214;\ 2 \leq n_2 \leq 822 \\
&2 \leq n_3 \leq 1028;\ 2 \leq n_4 \leq 786 \\
&K_i > 1,\ and\ n_i\ are\ integers; i = 1, 2, \ldots, 4
\end{aligned}\right\}$$

The solution of NLPP provided by LINGO is given as
$n^*_{2,1} = 594, n^*_{2,2} = 320, n^*_{2,3} = 149, n^*_{2,4} = 249, K_1 = 2.16, K_2 = 2.11, K_3 = 1.0, K_4 = 2.21$ with the value of the objective function as $Z_2 = 0.0046$. Where $n^*_{2,i}$ denote the optimum allocation for the second characteristic.
After finding the optimum values of $Z^*_j; j = 1, 2$ the GPP (5.8) takes the form

$$\left.\begin{aligned}
&Minimize\ x_1 + x_2 \\
&Subject\ to \\
&\frac{.8066}{n_1} + \frac{.4507}{n_2} + \frac{.3655}{n_3} + \frac{.4059}{n_4} \\
&+\frac{(K_1-1)}{n_1} * 0.06 + \frac{(K_2-1)}{n_2} * 0.02 + \frac{(K_3-1)}{n_3} * 0.02 + \frac{(K_4-1)}{n_4} * 0.02 - x_1 \leq 0.0067 \\
&\frac{.8365}{n_1} + \frac{.3377}{n_2} + \frac{.1068}{n_3} + \frac{.2807}{n_4} \\
&+\frac{(K_1-1)}{n_1} * 0.06 + \frac{(K_2-1)}{n_2} * 0.02 + \frac{(K_3-1)}{n_3} * 0.07 + \frac{(K_4-1)}{n_4} * 0.02 - x_2 \leq 0.0046 \\
&(2.4 + \frac{0.9}{K_1})n_1 + (3.4 + \frac{0.8}{K_2})n_2 + (4 + \frac{1.25}{K_3})n_3 + (4.6 + \frac{1.68}{K_4})n_4 \leq 5000 \\
&2 \leq n_1 \leq 1214;\ 2 \leq n_2 \leq 822 \\
&2 \leq n_3 \leq 1028;\ 2 \leq n_4 \leq 786 \\
&x_j \geq 0;\ j = 1, 2;\ K_i > 1,\ and\ n_i\ are\ integers; i = 1, 2, \ldots, 4
\end{aligned}\right\}$$

Using the LINGO, the optimum compromise allocation is found to be
$n^*_1 = 524, n^*_2 = 309, n^*_3 = 204, n^*_4 = 246, K_1 = 2.15, K_2 = 2.11, K_3 = 1.16, K_4 = 2.21$. Sampling fractions among the non-respondents are $\frac{1}{K_1} = 0.46, \frac{1}{K_2} = 0.47, \frac{1}{K_3} = 0.86$ and $\frac{1}{K_4} = 0.45$. The values of the objective function as $Z = 0.0115$.

7. CONCLUSION

In this paper, we have formulated the problem of non-response in multivariate stratified sample surveys as a problem of multi-objective mathematical programming problem (MOMPP). The formulated MOMPP has been solved by using two different optimization techniques namely goal programming and Lagrange Multiplier's Technique (LMT). The compromise allocations obtained by LMT gives the minimum coefficients of variation as compared to the goal programming technique

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TABLE 2. Calculation for proposed allocation n_i and optimum sampling fractions among Non-respondents $1/K_i$n(where K_i is defined in **??**)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
i	A_i	B_i	$A_i * B_i$	$\sqrt{(4)}$	A_i/B_i	$\sqrt{(6)}$	$\sqrt{C_{i2}/b_i^2}$	$p_iC_0\sqrt{(6)}$	$p_i\{\sqrt{(5)} + W_{i2}b_i\sqrt{C_{i2}}\}$	$K_i = (7) * (8)$	$n_i = (9)/\sum(10)$
1	14.85	2.4	35.64	5.97	6.19	2.49	0.86	3980.06	2.24	2.15	527.92
2	16.99	3.4	57.78	7.60	4.99	2.23	0.94	2347.40	1.77	2.11	311.36
3	6.07	4.0	24.30	4.93	1.52	1.23	1.76	1663.68	1.52	2.16	220.67
4	15.97	4.6	73.48	8.57	3.47	1.86	1.18	1863.44	1.99	2.20	247.17

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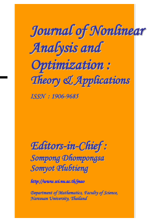

A METHOD FOR APPROXIMATING SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS BY USING FUZZY TRANSFORMS

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ABSTRACT. Fuzzy transforms developed by Irina Perfilieva is a novel, mathematically well founded soft computing tool with many applications. These techniques are based on mainly two transforms, direct fuzzy transform and inverse fuzzy transform. However a lot of works has still to be done on it for e.g. the extension of fuzzy transform and its applications in ordinary differential equations. In this paper we develop an approximating model based on fuzzy transform and apply the model for numerical solution of ordinary differential equations.

KEYWORDS : Fuzzy transforms; Fuzzy partitions; Ordinary Differential Equations.
AMS Subject Classification:

1. INTRODUCTION

In classical Mathematics, various types of transforms are introduced (e.g. Laplace transform, Fourier transform, wavelet transform etc.) by various researchers. In 2001 Irina Perfilieva introduced fuzzy transform in her paper [2]. Latter on fuzzy transform is applied in to various fields, like image processing, data mining etc in the papers [5, 10]. The fuzzy transform provides a relation between the space of continuous functions defined on a bounded domain of real line R and R^n.Similarly inverse fuzzy transform identified each vector of R^n with a continuous map. The central idea of the fuzzy transform is to partition the domain of the function by fuzzy sets.

Definition1.1([5]): Let $[a, b]$ be an interval of real numbers and $x_1, x_2, \ldots x_n$ be fixed nodes within $[a, b]$ such that $x_1 = a, x_n = b$ and $n \geq 2$.We say that fuzzy sets $A_1, A_2, \ldots A_n$, identified with their membership functions $A_1(x), A_2(x), \ldots A_n(x)$ and defined on $[a, b]$ form a fuzzy partition of $[a, b]$ if they fulfill the following conditions for$i = 1, 2, \ldots n$.

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1. $A_i : [a,b] \to [0,1]$, for $i = 1,2,\ldots,n$.

   $A_i(x_i) = 1$, for $i = 1,2,\ldots,n$
2. $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$
3. $A_i(x)$is continuous.
4. $A_i(x)$is monotonically increasing on $x_{i-1}, x_i$and monotonically decreasing on x_i, x_{i+1}.
5. $\sum_{i=1}^{n} A_i(x) = 1$, for all $x \in [a,b]$.
6. $A_i(x_i - x) = A_i(x_i + x)$,$\forall x \in [0,h], i = 2,3,\ldots,n-1, n > 2$.
7. $A_{i+1}(x) = A_i(x-h)$,$\forall x \in [a+h,b]$,for $i = 2,3,\ldots,n-2, n > 2$

Where h is the uniform distance between two nodes.
Let us remark that the shape of basic functions is specified by a set of nodes $x_1, x_2, \ldots x_n$ and the properties 1-7. The shape of basic functions is not predetermined and therefore it can be chosen additionally according to further requirements.
Figure 1 shows a fuzzy partition of the interval $[-4,4]$, with triangular membership functions.
The following expression gives the formal representation of such triangular membership functions.

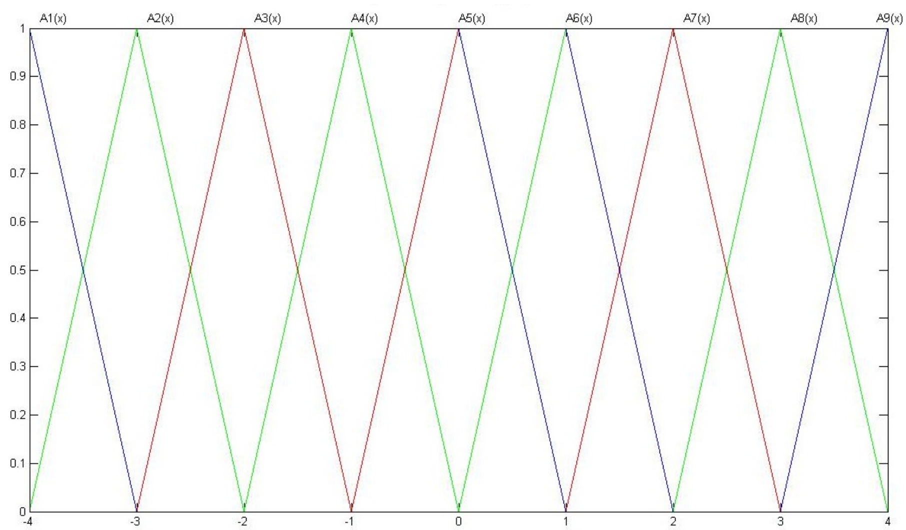

FIGURE 1. An example fuzzy partition of [-4,4].

$$A_1(x) = \begin{cases} -3 - x & \text{if} x \in [x_1, x_2] \\ 0 & \text{otherwise} \end{cases}$$

and for $i = 2,3,,n-1$.

$$A_i(x) = \begin{cases} x - x_{i-1} & \text{if} x \in [x_{i-1}, x_i] \\ 1 - x + x_i & \text{if} x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

$$A_n(x) = \begin{cases} x - x_{n-1} & \text{if} x \in [x_{n-1}, x_n] \\ 0 & \text{otherwise} \end{cases}$$

Lemma 1.1. [6]: *Let a uniform fuzzy partition of* $[a, b]$ *is given by basic functions* $A_1, A_2, \cdots, A_n, n \geq 3$, *then*

$$\int_{x_1}^{x_2} A_1(x)dx = \int_{x_{n-1}}^{x_n} A_n(x)dx = \frac{h}{2} \tag{1.1}$$

and for $i = 2, 3, \cdots, n-1$

$$\int_{x_{i-1}}^{x_{i+1}} A_i(x)dx = h \tag{1.2}$$

The above lemma shows that the integral of the basic functions does not depend upon the particular shape of the basic functions.

2. FUZZY TRANSFORM

In this section we first give the definition of fuzzy transform given by Irina Perfilieva in 2006.

Definition2.1([6]): Let $f(x)$ be a continuous function on $[a, b]$ and $A_1(x), A_2(x), \cdots, A_n(x)$ be basis functions determining a uniform fuzzy partition of $[a, b]$.Then the n-tuple of real numbers $[F_1, F_2, \cdots, F_n]$ such that

$$F_i = \frac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx}, i = 1, 2, \cdots, n \tag{2.1}$$

will be called the F- transform of f w.r.t. the given basis functions. Real's F_i are called components of the F-transform.

Lemma 2.1. [5]: *Let f be any continuous function defined on* $[a, b]$, *but function f is twice continuously differentiable in* (a, b) *and let* $A_1(x), A_2(x), \cdots, A_n(x)$ *be basis functions determining a uniform fuzzy partition of* $[a, b]$ *.Then for each* $i = 1, 2, \cdots, n$

$$F_i = f(x_i) + O(h^2)$$

Proof. Perfilieva (2004). □

Now a question arises in the minds that can we get back the original function by its fuzzy transform. The answer is we can reconstruct an approximate function to the original function. For that purpose Perfilieva define inverse fuzzy transform.

Definition2.2[6]: Let $A_1, A_2, \cdots, A_n$ be basic functions which form a uniform fuzzy partition of $[a, b]$ and f be a function from $c([a, b])$. Let $F_n[f] = [F_1, F_2, \cdots, F_n]$ be the fuzzy transform of f with respect
to $A_1, A_2, \cdots, A_n$. Then the function defined by

$$f_{F,n}(x) = \sum_{i=1}^{n} F_i.A_i(x)$$

is called the inverse fuzzy transform of f with respect to $A_1, A_2, \cdots, A_n$.

The following theorem shows that the inverse fuzzy transform can approximate the original continuous function f with a very small precision.

Theorem 2.2. [6]:*Let f be a continuous functions defined on $[a, b]$. Then for any $\epsilon > 0$ there exist n_ϵ and a uniform fuzzy partition $A_1, A_2, \cdots, A_n$ of [a, b] such that for all $x \in [a, b]$*

$$|f(x) - f_{F,n_\epsilon}| \le \epsilon$$

3. APPROXIMATE SOLUTIONS OF SECOND ORDER O.D.E. BY USING FUZZY TRANSFORM

In this section, we show that how the fuzzy transform can be used for solution of second order ordinary differential equation. Consider the following differential equation:

$$y''(x) = f(x, y) \tag{3.1}$$
$$y'(x_1) = c, y(x_1) = d.$$

Here we show that this differential equation can be solved by using fuzzy transform.

For solving the above equation we need a uniform fuzzy partition of the domain. Let $a = x_1 < x_2 < \cdots < x_n = b$ be fixed nodes within the domain and consider the fuzzy partition, $A_1, A_2, \cdots, A_n$ defined on these domain. Here we also assume that all the node points are equidistant, i.e. $x_i - x_{i-1} = h$ (say).

Now we approximate $y\prime(x)$ and $y''(x)$ by the following formula

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h) \tag{3.2}$$

$$y''(x) = \frac{y(x-h) - 2y(x) + y(x+h)}{h^2} + O(h^2) \tag{3.3}$$

Denote

$$y_1(x) = y(x + h)$$

as a new function and

$$y_2(x) = y(x - h)$$

as another new function. Now if we apply the F-transform on both sides of equation (3.3), then by using these new functions and by linearity of F-transform, we obtain the relation between the fuzzy transform components of y, y_1, y_2 and y'' as follows

$$F_n[y''] = \frac{F_n[y_1] - 2F_n[y] + F_n[y_2]}{h^2}$$

where, $F_n[y''] = [Y_2'', Y_3'', \cdots, Y_{n-1}'']$, $F_n[y_1] = [Y1_2, Y1_3, \cdots, Y1_{n-1}]$, $F_n[y_2] = [Y2_2, Y2_3, \cdots, Y2_{n-1}]$ and $F_n[y] = [Y_2, Y_3, \cdots, Y_{n-1}]$ are the fuzzy transform components of y'', y_1, y_2 and y respectively. Note that these vectors are two components shorter since y_1 may not be defined on $[x_{n-1}, x_n]$ and y_2 may not be defined on $[x_1, x_2]$.

Now by using the definition of fuzzy transform it can be easily proved that, $Y1_k = Y_{k+1}$ and $Y2_k = Y_{k-1}$, for $k = 2, 3, \cdots, n-1$. Indeed,for values of $k = 2, 3, \cdots, n-2$,

$$Y1_k = \frac{1}{h}\int_{x_{k-1}}^{x_{k+1}} y(x+h)A_k(x)dx = \frac{1}{h}\int_{x_k}^{x_{k+2}} y(t)A_{k+1}(t)dt$$

For, $k = n-1$ the proof is similar.The proof of $Y2_k = Y_{k-1}$,for $k = 2, 3, \cdots, n-1$ is analogous. Therefore, we can write the components of the F-transform of y'' via components of the F-transform of y. So, we can write the equation (3.1) component wise as

$$Y_k'' = \frac{Y_{k+1} - 2Y_k + Y_{k-1}}{h^2}, fork = 2, 3, \cdots, n-1.$$

Now for $k = 2, 3, \cdots, n-1$, we introduce the following system of linear equations.

$$\begin{aligned} Y_2'' &= \frac{Y_3 - 2Y_2 + Y_1}{h^2} \\ Y_3'' &= \frac{Y_4 - 2Y_3 + Y_2}{h^2} \\ &\vdots \\ Y_{n-1}'' &= \frac{Y_n - 2Y_{n-1} + Y_{n-2}}{h^2} \end{aligned}$$

The above system can be written in matrix form as

$$[Y_2'', Y_3'', \cdots, Y_{n-1}'']^T = D[Y_1, Y_2, \cdots, Y_n]^T \tag{3.4}$$

where, D is the $(n-2) \times n$ matrix, given by

$$\mathsf{D} = \frac{1}{h^2} \times \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \end{pmatrix}$$

Now, by using equation (3.4) and equation (3.1), we can write

$$D[Y_1, Y_2, \cdots, Y_n]^T = [F_2, F_3, \cdots, F_{n-1}]^T$$

where, $F_2, F_3, \cdots, F_{n-1}$ are the corresponding fuzzy transform components of $f(x, y)$. Now we use the initial conditions and make the matrix D as $n \times n$ matrix. The initial conditions are given as,

$$y(x_1) = c, \Rightarrow Y_1 = c \text{ (by using lemma2.1)}$$

and

$$y'(x_1) = d, \Rightarrow y(x_2) - y(x_1) = dh \Rightarrow Y2 - Y1 = dh, \text{ (by using lemma2.1)}$$

By using the above initial conditions, we make the matrix D as square matrix of order $n \times n$ and also write the system of linear equations as

$$D^c[Y_1, Y_2, \cdots, Y_n]^T = [\frac{c}{h^2}, \frac{d}{h}, F_2, \cdots, F_{n-1}]^T \tag{3.5}$$

where,

$$\mathsf{D^c} = \frac{1}{h^2} \times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & -2 & 1 \end{pmatrix}$$

Now, we solve the system of linear equation (3.5) by using any numerical techniques. Note here that the solution must exist since D^c is a invertible matrix.

Note3.1: Here for solving both the systems of linear equations we need the value of F_i, which is the fuzzy transform components of the function $f(x, y)$, with respect to x, therefore F_i will be dependent on y also. For overcoming these difficulties we approximate F_i as

$$F_i' = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x, Y_k) A_k(x) dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx}$$

That means here we have assumed the value of the function y as constant for the interval (x_{k-1}, x_{k+1}). Here we omit the proof that

$$F_i - F_i' = O(h^2).$$

Now after evaluating all the fuzzy transform components of y, we will find an approximation of the function y by using inverse fuzzy transform. For illustration of the above method we give the following example.

Example3.1: Consider the following initial value problem

$$y''(x) = 2x + y, -4 \leq x \leq 4$$
$$y(-4) = 8, y'(-4) = -2$$

For solving the above system we consider the fuzzy partition given in figure above. Now here the system of linear equations (3.5) can be written as

$$D^c[Y_1, Y_2, \cdots, Y_n]^T = [8, -2, F_2, \cdots, F_{n-1}]^T$$

Where D^c is as given above with $h = 1$.Now after solving this system of linear equations we find

$Y1 = 8$, $Y2 = 6$, $Y3 = 4 + F_2$, $Y4 = 2 + 2F_2 + F_3$, $Y5 = 3F_2 + 2F_3 + F_4$
$Y6 = -2 + 4F_2 + 3F_3 + 2F_4 + F_5$, $Y7 = -4 + 5F_2 + 4F_3 + 3_F4 + 2F_5 + F_6$
$Y8 = -6 + 6F_2 + 5F_3 + 4F_4 + 3F_5 + 2F_6 + F_7$ and
$Y9 = -8 + 7F_2 + 6F_3 + 5F_4 + 4F_5 + 3F_6 + 2F_7 + F_8$

Now, for finding

$$Y3, Y4, Y5, Y6, Y7, Y8 \text{ and } Y9$$

, we need the value of

$$F_2, F_3, F_4, F_5, F_6, F_7 \text{ and } F_8$$

, which are the 2nd ,3rd and 4th, 5th ,6th, 7th and 8th fuzzy transform components of the function $f(x, y)$. Now by using the note (3.1), we evaluate $F_2, F_3, F_4, F_5, F_6, F_7$ and F_8 by using software "MATLAB" and then subsequently we find $Y1 = 8, Y2 = 6, Y3 = 4, Y4 = 2, Y5 = 0, Y6 = -2, Y7 = -4, Y8 = -6$ and $Y9 = -8$. Now by using these fuzzy transform components of y we approximate y by using inverse fuzzy transform, which graph is shown below. Since the exact solution is $y = -2x$. Therefore from the graph given in Figure 2, we can say that in this case our approximate solution coincides with the exact solution.

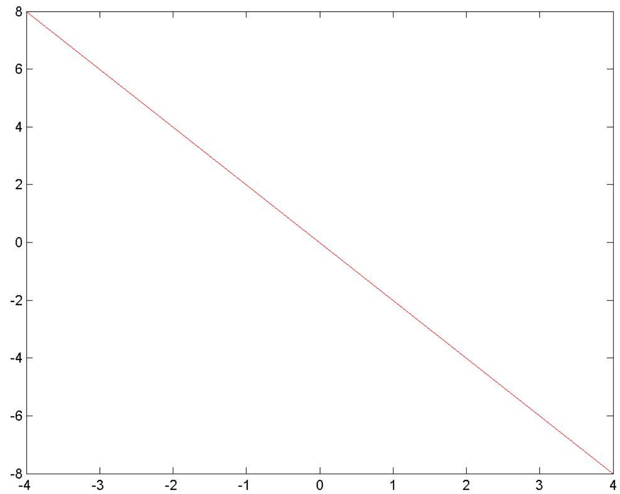

FIGURE 2. The approximation graph.

4. CONCLUSION

We have introduced a new numerical technique by using fuzzy transform, which enables us to construct various approximating models depending on the choice of basic functions. With this new technique (fuzzy transform) we solved second order initial value ordinary differential equations.

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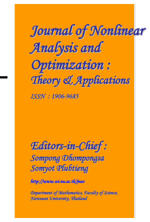

CONTROLLABILITY OF STOCHASTIC IMPULSIVE NEUTRAL INTEGRODIFFERENTIAL SYSTEMS WITH INFINITE DELAY

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ABSTRACT. The paper is concerned with the controllability of stochastic impulsive neutral integrodifferential systems with infinite delay in an abstract space. Sufficient conditions for controllability are obtained by means of semigroup theory and Banach contraction principle. An example is provided to illustrate the theory.

KEYWORDS: Stochastic impulsive neutral integrodifferential system; Infinite delay; Mild solution; Banach fixed point theorem.
AMS Subject Classification: 93B05, 34A37.

1. INTRODUCTION

Controllability is one of the fundamental concept in mathematical control theory and plays an important role in both deterministic and stochastic control systems. It is illustrious that controllability of deterministic systems are widely used in many fields of science and technology. The deterministic models often fluctuate due to noise, which is random or atleast appears to be so. Therefore, we must move from deterministic problems to stochastic problems. Many systems in physics and biology exhibit impulsive dynamical behavior due to sudden jumps at certain instants during the dynamical process. Differential equations involving impulsive effects occur in many applications: pharmacokinetics, the radiation of electromagnetic waves, population dynamics [8], the abrupt increase of glycerol in fed-batch culture, bio-technology, nanoelectronics, etc., and for basic theory refer [16, 26]. The theory of impulsive integrodifferential equations with their applications in fields such as mechanics, electrical engineering, medicine, ecology and so on have become an active areas of investigation since the theory provides a natural framework for mathematical modeling of many physical phenomena. Moreover, impulsive control which based on the theory of impulsive integrodifferential equations has gained renewed interest recently for its promising applications towards controlling systems exhibiting chaotic behavior (see [29]). As the generalization of classic impulsive integrodifferential equations, impulsive neutral stochastic functional integrodifferential equations have attracted the researchers great interest.

Byszewski [9] introduced the nonlocal initial conditions into the initial value problems and argued that the corresponding models more accurately describe the phenomena since

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more information was taken into account at the onset of the experiment, thereby reducing the ill effects incurred by a single (possibly erroneous) initial measurement.

The controllability of nonlinear deterministic systems represented by evolution equations in abstract spaces have been studied by several authors [2, 3, 5]. Balachandran and Anandhi [1] and Balachandran et al. [7] discussed the controllability of neutral functional integrodifferential systems in abstract spaces. Li et al. [18] studied the controllability of impulsive functional differential systems in Banach spaces. Chang [10] and Park et al. [22] investigated the controllability of impulsive functional differential systems and impulsive neutral integrodifferential systems with infinte delay in Banach spaces. The controllability of nonlinear stochastic systems in finite and infinite-dimensional spaces have been studied by many authors [15, 19]. Balachandran and Karthikeyan [4] and Balachandran et al. [6] derived the sufficient conditions for controllability of stochastic integrodifferential systems in finite dimensional spaces. Park et al. [23] discussed the controllability of neutral stochastic functional integrodifferential infinite delay systems in abstract space. Sakthivel et al. [25] investigated the controllability of non-linear impulsive stochastic systems. Subalakshmi and Balachandran [27, 28] studied the controllability of semilinear stochastic functional integrodiffferential systems and approximate controllability of nonlinear stochastic impulsive integrodifferential systems in Hilbert spaces. Based on [21], Hu and Ren [14] proved the existence results for impulsive neutral stochastic functional integrodifferential equations with infinite delays in the phase space $\mathcal{B}_h$. Motivated by these literatures, in this paper we study the controllability of stochastic impulsive neutral integrodifferential system with infinite delay in the phase space $\mathcal{B}_h$ which is an untreated topic sofar.

Consider the following class of stochastic impulsive neutral integrodifferential equation with infinite delay and nonlocal conditions

$$
\begin{aligned}
d\Big[x(t) - g\Big(t, x_t, \int_0^t c(t,s,x_s)ds\Big)\Big] &= \Big[Ax(t) + f\Big(t, x_t, \int_0^t e(t,s,x_s)ds\Big) + Bu(t)\Big]dt \\
&\quad + \int_{-\infty}^t \sigma(t,s,x_s)dw(s),\ t \in J := [0,a],\ t \neq \tau_k, \\
\triangle x(\tau_k) &= x(\tau_k^+) - x(\tau_k^-) = I_k(x(\tau_k^-)), \quad k = 1,2,\cdots,m, \\
x(s) + \big(h_1(x_{t_1}, x_{t_2}, x_{t_3}, \cdots, x_{t_p})\big)(s) &= \phi(s) \in \mathcal{L}_2(\Omega, \mathcal{B}_h),\ for\ a.e.\ s \in J_0 := (-\infty, 0] \quad (1.1)
\end{aligned}
$$

where $0 < t_1 < t_2 < t_3 < \cdots < t_p \leq a \quad (p \in N)$. Here, the state variable $x(\cdot)$ takes values in a real separable Hilbert space H with innerproduct $(\cdot,\cdot)$ and norm $|\cdot|$ and A is the infinitesimal generator of a strongly continuous semigroup of a bounded linear operator $\mathcal{S}(t),\ t \geq 0$ in H. The control function $u(\cdot)$ takes values in $L^2(J,U)$ of admissible control functions for a separable Hilbert space U and B is a bounded linear operator from U into H. Let K be another separable Hilbert space with innerproduct $(\cdot,\cdot)_K$ and the norm $|\cdot|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$. We employ the same notation $|\cdot|$ for the norm $\mathcal{L}(K,H)$, where $\mathcal{L}(K,H)$ denotes the space of all bounded linear operators from K into H. Further, $c : D \times \mathcal{B}_h \longrightarrow H, e : D \times \mathcal{B}_h \longrightarrow H, g : J \times \mathcal{B}_h \times H \longrightarrow H, f : J \times \mathcal{B}_h \times H \longrightarrow H, \sigma : D \times \mathcal{B}_h \longrightarrow \mathcal{L}_Q(K,H)$ and $h_1 : \mathcal{B}_h^p \longrightarrow \mathcal{B}_h$ are measurable mappings in H-norm, $\mathcal{L}_Q(K,H)$-norm and $\mathcal{B}_h$-norm respectively. Here $\mathcal{L}_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H which will be defined in Section 2 and $D = \{(t,s) \in J \times J : s \leq t\}$. Here, $I_k \in C(H,H)\ (k = 1,2,\cdots,m)$ are bounded functions. Furthermore, the fixed times τ_k satisfies $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = a$, $x(\tau_k^+)$ and $x(\tau_k^-)$ denote the right and left limits of $x(t)$ at $t = \tau_k$. And $\triangle x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ represents the jump in the state x at time τ_k, where I_k determines the size of the jump. The histories $x_t : \Omega \longrightarrow \mathcal{B}_h$, $t \geq 0$, which are defined by setting $x_t = \{x(t+s), s \in (-\infty, 0]\}$, belong to the phase space $\mathcal{B}_h$, which will be defined in Section 2. The initial data $\phi = \{\phi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variables independent of $\{w(t) : t \geq 0\}$ with finite second moment.

2. PRELIMINARIES

Throughout the paper $(H, |\cdot|)$ and $(K, |\cdot|_K)$ denote real separable Hilbert spaces.

Let $(\Omega, \mathcal{F}, P; \mathbf{F})$ $\{\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}\}$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t)\colon\ \Omega \longrightarrow H$ and the collection of random variables $S = \{x(t,\omega) : \Omega \longrightarrow H \backslash t \in J\}$ is called a stochastic process. Generally, we just write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \longrightarrow H$ in the space of S. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\omega_i(t)e_i$, where $\{\omega_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_a = \mathcal{F}$. Let $\Psi \in \mathcal{L}(K, H)$ and define

$$|\Psi|_Q^2 = Tr(\Psi Q \Psi^*) = \sum_{i=1}^{\infty} |\sqrt{\lambda_i}\Psi e_i|^2.$$

If $|\Psi|_Q < \infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(K, H)$ denote the space of all Q-Hilbert-Schmidt operators $\Psi : K \longrightarrow H$. The completion $\mathcal{L}_Q(K, H)$ of $\mathcal{L}(K, H)$ with respect to the topology induced by the norm $|\cdot|_Q$ where $|\Psi|_Q^2 = (\Psi, \Psi)$ is a Hilbert space with the above norm topology. For more details refer to Da Prato [11].

Now, we present the abstract phase space $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \longrightarrow (0, \infty)$ is a continuous function with $l = \int_{-\infty}^{0} h(t)dt < \infty$. For any $b > 0$, define

$$\mathcal{B}_h = \Big\{\Psi : (-\infty, 0] \longrightarrow H : (E|\Psi(\theta)|^2)^{1/2} \text{ is a bounded and measurable function on } [-b, 0] \text{ and } \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (E|\Psi(\theta)|^2)^{1/2} ds < \infty\Big\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\Psi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} (E|\Psi(\theta)|^2)^{1/2} ds, \quad \forall \ \Psi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space [13].

Let A be the infinitesimal generator of an analytic semigroup $\mathcal{S}(t)$ in H. Then, $(A - \alpha I)$ is invertible and generates a bounded analytic semigroup for $\alpha > 0$ large enough. Therefore, we can assume that the semigroup $\mathcal{S}(t)$ is bounded and the generator A is invertible. We suppose that $0 \in \rho(A)$, which is the resolvent set of A. It follows that $(-A)^{\alpha},\ 0 < \alpha \leq 1$ can be defined as a closed linear invertible operator with its domain $D(-A)^{\alpha}$ being dense in H. We denote by H_{α} the Banach space $D(-A)^{\alpha}$ endowed with the norm $\|x\|_{\alpha} = \|(-A)^{\alpha}x\|$, which is equivalent to the graph norm of $(-A)^{\alpha}$. Furthermore, we have $H_{\beta} \subset H_{\alpha},\ 0 < \alpha < \beta$ and the embedding is continuous. For semigroup theory literature we refer [24].

Lemma 2.1. *The following two properties hold:*

(i) *If $0 < \beta < \alpha \leq 1$, then $H_{\alpha} \subset H_{\beta}$ and the embedding is compact whenever the resolvent operator of A is compact.*

(ii) *For every $0 < \alpha \leq 1$, there exists C_{α} such that*

$$\|(-A)^{\alpha}\mathcal{S}(t)\| \leq \frac{C_{\alpha}}{t^{\alpha}}, \quad t > 0. \tag{2.1}$$

Let $J_1 = (-\infty, a]$. Now, we define the mild solution of (1.1) as in [14].

Definition 2.2. A stochastic process $x : J_1 \times \Omega \longrightarrow H$ is called a mild solution of (1.1) if

(a) $x(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t \geq 0$;

(b) $\triangle x(\tau_k) = x(\tau_k^+) - x(\tau_k^-) = I_k(x(\tau_k^-)),\ k = 1, 2, \cdots, m;$

(c) $x(t) \in H$ and for every $0 \leq s < t$, the function $A\mathcal{S}(t-s)g(s, x_s, \int_0^s c(s,\eta,x_\eta)d\eta)$ is integrable such that the following integral equation is satisfied

$$\begin{aligned} x(t) \;=\; & \mathcal{S}(t)\big[\phi(0) - \big(h_1(x_{t_1}, x_{t_2}, x_{t_3}, \cdots, x_{t_p})\big)(0) - g(0, x_0, 0)\big] + g\Big(t, x_t, \int_0^t c(t,s,x_s)ds\Big) \\ & + \int_0^t A\mathcal{S}(t-s)g\Big(s, x_s, \int_0^s c(s,\eta,x_\eta)d\eta\Big)ds + \int_0^t \mathcal{S}(t-s)Bu(s)ds \\ & + \int_0^t \mathcal{S}(t-s)f\Big(s, x_s, \int_0^s e(s,\eta,x_\eta)d\eta\Big)ds + \int_0^t \mathcal{S}(t-s)\Big(\int_{-\infty}^s \sigma\big(s,\eta,x_\eta)dw(\eta)\Big)ds \\ & + \sum_{0<\tau_k<t} \mathcal{S}(t-\tau_k)I_k(x(\tau_k^-)), \quad \text{for a.e. } t \in J; \end{aligned} \tag{2.2}$$

(d) $x_0(\cdot) + \big(h_1(x_{t_1}, x_{t_2}, x_{t_3}, \cdots, x_{t_p})\big)(0) = \phi \in \mathcal{L}_2(\Omega, \mathcal{B}_h)$ on J_0 satisfies $\|\phi\|^2_{\mathcal{B}_h} < \infty$.

Definition 2.3. The system (1.1) is said to be controllable on the interval $[0,a]$, if for every initial function $\phi \in \mathcal{L}_2(\Omega, \mathcal{B}_h)$ and $x_1 \in H$, there exists a control $u \in L^2(J,U)$ such that the solution $x(\cdot)$ of (1.1) satisfies $x(a) = x_1$.

In order to establish our result we assume the following hypotheses:

(H1) A is the infinitesimal generator of an analytic semigroup $\mathcal{S}(t)$ in H, $0 \in \rho(A)$ and there exists a positive constant M such that

$$\|\mathcal{S}(t)\|^2 \leq M \text{ for all } t \geq 0.$$

(H2) The linear operator $W : L^2(J,U) \longrightarrow H$ defined by

$$Wu = \int_0^a \mathcal{S}(a-s)Bu(s)ds$$

is invertible with inverse operator W^{-1} taking values in $L^2(J,U) \setminus kerW$ and there exist positive constants M_b and M_W such that

$$\|B\|^2 \leq M_b \; and \; \|W^{-1}\|^2 \leq M_W.$$

(H3) There exists a constant $M_c > 0$ such that for all $x, y \in \mathcal{B}_h, (t,s) \in D$,

$$E\Big|\int_0^t \Big[c(t,s,x) - c(t,s,y)\Big]ds\Big|^2 \leq M_c\|x-y\|^2_{\mathcal{B}_h}.$$

(H4) There exist constants $0 < \beta < 1$ and M_g such that g is H_β -valued, $(-A)^\beta g$ is continuous and for $t \in J,\; x_1, x_2 \in \mathcal{B}_h,\; y_1, y_2 \in H$ such that

$$E\big|(-A)^\beta g(t,x_1,y_1) - (-A)^\beta g(t,x_2,y_2)\big|^2 \leq M_g\Big[\|x_1 - x_2\|^2_{\mathcal{B}_h} + E|y_1 - y_2|^2\Big].$$

(H5) The function $\sigma : D \times \mathcal{B}_h \longrightarrow \mathcal{L}_Q(K,H)$ is continuous and there exist positive constants $M_\sigma, \tilde{M}_\sigma$ for all $(t,s) \in D$ and $x, y \in \mathcal{B}_h$ such that

$$E|\sigma(t,s,x) - \sigma(t,s,y)|^2_Q \leq M_\sigma\|x-y\|^2_{\mathcal{B}_h},$$

$$E|\sigma(t,s,x)|^2_Q \leq \tilde{M}_\sigma.$$

(H6) For each $\phi \in \mathcal{B}_h$,

$$\mathcal{H}(t) = \lim_{b \longrightarrow \infty} \int_{-b}^0 \sigma(t,s,\phi)dw(s)$$

exists and is continuous. Further, there exists a positive constant $M_{\mathcal{H}}$ such that

$$|\mathcal{H}(s)|^2_Q \leq M_{\mathcal{H}}.$$

(H7) For each $(t,s) \in D$, $x, y \in \mathcal{B}_h$, the function $e : D \times \mathcal{B}_h \longrightarrow H$ is continuous and there exist constants $M_e,\; \tilde{M}_e$ such that

$$E\Big|\int_0^t \Big[e(t,s,x) - e(t,s,y)\Big]ds\Big|^2 \leq M_e\|x-y\|^2_{\mathcal{B}_h}$$

and $\tilde{M}_e = \sup_{(t,s)\in D}\big(|\int_0^t e(t,s,0)ds|^2\big)$.

(H8) The function $f : J\times\mathcal{B}_h\times H \longrightarrow H$ is continuous and there exist positive constants $M_f, \tilde{M}_f$ for $t\in J, x_1, x_2\in\mathcal{B}_h,\ y_1, y_2\in H$ such that

$$E\big|f(t,x_1,y_1)-f(t,x_2,y_2)\big|^2 \le M_f\Big[\|x_1-x_2\|^2_{\mathcal{B}_h}+E|y_1-y_2|^2\Big]$$

and $\tilde{M}_f=\sup_{t\in J}|f(t,0,0)|^2$.

(H9) The function $h_1 : \mathcal{B}_h^p \longrightarrow \mathcal{B}_h$ is continuous and there exist positive constants $M_h,\ \tilde{M}_h$ for $x,y\in\mathcal{B}_h,\ s\in(-\infty,0]$ such that

$$E\|\big(h_1(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_p})\big)(s)-\big(h_1(y_{t_1},y_{t_2},y_{t_3},\cdots,y_{t_p})\big)(s)\|^2\le M_h\|x-y\|^2_{\mathcal{B}_h},$$

$$E\|\big(h_1(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_p})\big)(s)\|^2\le\tilde{M}_h.$$

(H10) $I_k\in C(H,H)$ and there exist positive constants $\beta_k,\ \tilde{\beta}_k$ such that for all $x,y\in H$,

$$E|I_k(x)-I_k(y)|^2\le\beta_k\|x-y\|^2_{\mathcal{B}_h},\quad k=1,2,\cdots m,$$

$$E|I_k(x)|^2\le\tilde{\beta}_k\quad k=1,2,\cdots m.$$

(H11) There exists a constant $\nu>0$ such that

$$\begin{aligned}\nu &= 12l^2\Big\{\big(1+6MM_bM_Wa^2\big)\Big[\Big(M_0^2+\frac{(C_{1-\beta}a^\beta)^2}{2\beta-1}\Big)M_g(1+2M_c)+2a^2MM_\sigma\\&\quad+Ma^2M_f(1+M_e)+mM\sum_{k=1}^m\beta_k\Big]+6M^2M_bM_Wa^2M_h\Big\}<1,\\ \tilde{M} &= 7\Big\{(1+9MM_bM_Wa^2)\Big[2MM_0^2\big[M_g\|\widehat{\phi}\|^2_{B_h}+c_2\big]+2Ma^2[2M_f\tilde{M}_e+\tilde{M}_f]\\&\quad+2\Big(M_0^2+\frac{\big(C_{1-\beta}a^\beta\big)^2}{2\beta-1}\Big)[2M_gc_1+c_2]+2Ma^2[M_{\mathcal{H}}+Tr(Q)\tilde{M}_\sigma]\\&\quad+mM\sum_{k=1}^m\tilde{\beta}_k\Big]+9MM_bM_Wa^2(|x_1|^2+ME|\phi(0)|^2+M\tilde{M}_h)\Big\},\text{ where}\\ M_0 &= \|(-A)^{-\beta}\|, c_1=\sup_{(t,s)\in D}\Big|\int_0^t c(t,s,0)ds\Big|^2\text{ and }c_2=\sup_{t\in J}|(-A)^\beta g(t,0,0)|^2.\end{aligned}$$

We now consider the space

$$\begin{aligned}\mathcal{B}_a=\Big\{&x:J_1\longrightarrow H, x_k\in C(J_k,H)\text{ and there exist }x(\tau_k^-)\text{ and }x(\tau_k^+)\text{ with}\\&x(\tau_k)=x(\tau_k^-),\ x_0+\big(h_1(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_p})\big)(0)=\phi\in\mathcal{B}_h, k=0,1,2,\cdots,m\Big\},\end{aligned}$$

where x_k is the restriction of x to $J_k=(\tau_k,\tau_{k+1}]$ and $C(J_k,H)$ denotes the space of all continuous H-valued stochastic processes $\{\xi(t):t\in J_k\}, k=0,1,2,\cdots,m$. Set $\|\cdot\|_a$ be a seminorm in $\mathcal{B}_a$ defined by

$$\|x\|_a=\|x_0\|_{\mathcal{B}_h}+\sup_{0\le s\le a}\Big(E|x(s)|^2\Big)^{1/2},\ x\in\mathcal{B}_a.$$

We give a useful lemma appeared in [17].

Lemma 2.4. *Assume that $x\in\mathcal{B}_a$, then for $t\in J, x_t\in\mathcal{B}_h$. Moreover,*

$$l(E|x(t)|^2)^{1/2}\le\|x_t\|_{\mathcal{B}_h}\le l\sup_{0\le s\le t}(E|x(s)|^2)^{1/2}+\|x_0\|_{\mathcal{B}_h},$$

where $l=\int_{-\infty}^0 h(s)ds<\infty$.

3. CONTROLLABILITY RESULT

Theorem 3.1. *If the conditions* $(H1)-(H11)$ *are satisfied then the system* (1.1) *is controllable on* J *provided that*

$$7l^2\Big\{\big(1+9MM_bM_Wa^2\big)\Big[8\Big(M_0^2+\frac{(C_{1-\beta}a^{\beta})^2}{2\beta-1}\Big)M_g(1+2M_c)+8Ma^2M_f(1+2M_e)\Big]\Big\}<1 \quad (3.1)$$

Proof: Using the hypothesis $(H2)$ for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned} u_x^a(t) &= W^{-1}\Big\{x_1-\mathcal{S}(a)\big[\phi(0)-\big(h_1(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_p})\big)(0)-g(0,x_0,0)\big] \\ &\quad -g\Big(a,x_a,\int_0^a c(a,s,x_s)ds\Big)-\int_0^a A\mathcal{S}(a-s)g\Big(s,x_s,\int_0^s c(s,\eta,x_\eta)d\eta\Big)ds \\ &\quad -\int_0^a \mathcal{S}(a-s)f\Big(s,x_s,\int_0^s e(s,\eta,x_\eta)d\eta\Big)ds-\int_0^a \mathcal{S}(a-s)\Big[\mathcal{H}(s)+\int_0^s \sigma(s,\eta,x_\eta)dw(\eta)\Big]ds \\ &\quad -\sum_{0<\tau_k<a}\mathcal{S}(a-\tau_k)I_k(x(\tau_k^-))\Big\}(t). \end{aligned} \quad (3.2)$$

Consider the mapping $\Phi:\mathcal{B}_a\longrightarrow\mathcal{B}_a$ defined by

$$(\Phi x)(t)=\begin{cases} \phi(t)-\big(h_1(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_p})\big)(t), & t\in J_0, \\ \mathcal{S}(t)\big[\phi(0)-\big(h_1(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_p})\big)(0)-g(0,x_0,0)\big] \\ +g\Big(t,x_t,\int_0^t c(t,s,x_s)ds\Big)+\int_0^t A\mathcal{S}(t-s)g\Big(s,x_s,\int_0^s c(s,\eta,x_\eta)d\eta\Big)ds \\ +\int_0^t \mathcal{S}(t-s)Bu_x^a(s)ds+\int_0^t \mathcal{S}(t-s)f\Big(s,x_s,\int_0^s e(s,\eta,x_\eta)d\eta\Big)ds \\ +\int_0^t \mathcal{S}(t-s)\Big[\mathcal{H}(s)+\int_0^s \sigma(s,\eta,x_\eta)dw(\eta)\Big]ds+\sum_{0<\tau_k<t}\mathcal{S}(t-\tau_k)I_k(x(\tau_k^-)), \\ & \text{for a.e. } t\in J. \end{cases} \quad (3.3)$$

We shall show that the operator Φ has a fixed point, which is then a solution of system (1.1). Clearly, $(\Phi x)(a)=x_1$. For

$$\begin{aligned} E\Big|\int_0^t A\mathcal{S}(t-s)g\Big(s,x_s,\int_0^s c(s,\eta,x_\eta)d\eta\Big)ds\Big|^2 &\le E\Big|\int_0^t(-A)^{1-\beta}\mathcal{S}(t-s)(-A)^{\beta}g\Big(s,x_s,\int_0^s c(s,\eta,x_\eta)d\eta\Big)ds\Big|^2 \\ &\le \int_0^t\frac{2C_{1-\beta}^2}{(t-s)^{2(1-\beta)}}\Big[M_g(1+2M_c)\|x_s\|_{\mathcal{B}_h}^2+2M_gc_1+c_2\Big]ds, \end{aligned}$$

then from the Bochner theorem [20], it follows that $A\mathcal{S}(t-s)g\Big(s,x_s,\int_0^s c(s,\eta,x_\eta)d\eta\Big)$ is integrable on J.
For $\phi\in\mathcal{B}_h$, define

$$\widehat{\phi}(t)=\begin{cases} \phi(t)-\big(h_1(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_p})\big)(t), & t\in J_0, \\ \mathcal{S}(t)\Big[\phi(0)-\big(h_1(x_{t_1},x_{t_2},x_{t_3},\cdots,x_{t_p})\big)(0)\Big], & t\in J, \end{cases} \quad (3.4)$$

then $\widehat{\phi}(t)\in\mathcal{B}_a$. Set

$$x(t)=z(t)+\widehat{\phi}(t),\quad t\in J_1.$$

It is clear that x satisfies (2.2) if and only if z satisfies $z_0=0$ and

$$\begin{aligned} z(t) &= -\mathcal{S}(t)g(0,\widehat{\phi}_0,0)+g\Big(t,z_t+\widehat{\phi}_t,\int_0^t c(t,s,z_s+\widehat{\phi}_s)ds\Big) \\ &\quad +\int_0^t A\mathcal{S}(t-s)g\Big(s,z_s+\widehat{\phi}_s,\int_0^s c(s,\eta,z_\eta+\widehat{\phi}_\eta)d\eta\Big)ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \mathcal{S}(t-s)f\Big(s, z_s+\widehat{\phi}_s, \int_0^s e(s,\eta,z_\eta+\widehat{\phi}_\eta)d\eta\Big)ds \\
&+\int_0^t \mathcal{S}(t-s)\Big(\mathcal{H}(s)+\int_0^s \sigma\big(s,\eta,z_\eta+\widehat{\phi}_\eta)dw(\eta)\Big)ds \\
&+\int_0^t \mathcal{S}(t-s)Bu^a_{z+\widehat{\phi}}(s)ds+\sum_{0<\tau_k<t}\mathcal{S}(t-\tau_k)I_k(z(\tau_k^-)+\widehat{\phi}(\tau_k^-)), \quad t\in J,
\end{aligned}
$$

where,

$$
\begin{aligned}
u^a_{z+\widehat{\phi}}(t) \ = W^{-1}\Big\{ & x_1-\mathcal{S}(a)\big[\phi(0)-\big(h_1((z+\widehat{\phi})_{t_1},(z+\widehat{\phi})_{t_2},\cdots,(z+\widehat{\phi})_{t_p})\big)(0) \\
& -g(0,z_0+\widehat{\phi}_0,0)\big]-g\Big(a,z_a+\widehat{\phi}_a,\int_0^a c(a,s,z_s+\widehat{\phi}_s)ds\Big) \\
& -\int_0^a A\mathcal{S}(a-s)g\Big(s,z_s+\widehat{\phi}_s,\int_0^s c(s,\eta,z_\eta+\widehat{\phi}_\eta)d\eta\Big)ds \\
& -\int_0^a \mathcal{S}(a-s)f\Big(s,z_s+\widehat{\phi}_s,\int_0^s e(s,\eta,z_\eta+\widehat{\phi}_\eta)d\eta\Big)ds \\
& -\int_0^a \mathcal{S}(a-s)\Big[\mathcal{H}(s)+\int_0^s \sigma\big(s,\eta,z_\eta+\widehat{\phi}_\eta)dw(\eta)\Big]ds \\
& -\sum_{0<\tau_k<a}\mathcal{S}(a-\tau_k)I_k(z(\tau_k^-)+\widehat{\phi}(\tau_k^-))\Big\}(t).
\end{aligned}
$$

Let $\mathcal{B}_a^0=\{y\in\mathcal{B}_a : y_0=0\in\mathcal{B}_h\}$. For any $y\in\mathcal{B}_a^0$, we have

$$
\|y\|_a=\|y_0\|_{\mathcal{B}_h}+\sup_{0\le s\le a}\Big(E|y(s)|^2\Big)^{1/2}=\sup_{0\le s\le a}\Big(E|y(s)|^2\Big)^{1/2}
$$

and thus $(\mathcal{B}_a^0,\|\cdot\|_a)$ is a Banach space. Set

$$
\mathcal{B}_q=\{y\in\mathcal{B}_a^0:\ y(0)=0,\ \|y\|_a^2\le q\}\ \text{for some}\ q\ge 0,
$$

then $\mathcal{B}_q\subseteq\mathcal{B}_a^0$ is a bounded closed convex set, and for $z\in\mathcal{B}_q$, we have

$$
\begin{aligned}
\|z_t+\widehat{\phi}_t\|^2_{\mathcal{B}_h} &\le 2\big(\|z_t\|^2_{\mathcal{B}_h}+\|\widehat{\phi}_t\|^2_{\mathcal{B}_h}\big) \\
&\le 4\big(l^2\sup_{0\le s\le t}E|z(s)|^2+\|z_0\|^2_{\mathcal{B}_h}+l^2\sup_{0\le s\le t}E|\widehat{\phi}(t)|^2+\|\widehat{\phi}_0\|^2_{B_h}\big) \\
&\le 4l^2\big(q+2M(E|\phi(0)|^2+M_h)\big)+4\|\widehat{\phi}\|^2_{\mathcal{B}_h} \\
&:= q\prime. \qquad (3.5)
\end{aligned}
$$

Let the operator $\widehat{\Phi}:\mathcal{B}_a^0\longrightarrow\mathcal{B}_a^0$ defined by

$$
(\widehat{\Phi}z)(t)=\begin{cases}
0, & t\in J_0,\\
-\mathcal{S}(t)g(0,\widehat{\phi}_0,0)+g\Big(t,z_t+\widehat{\phi}_t,\int_0^t c(t,s,z_s+\widehat{\phi}_s)ds\Big) & \\
+\int_0^t A\mathcal{S}(t-s)g\Big(s,z_s+\widehat{\phi}_s,\int_0^s c(s,\eta,z_\eta+\widehat{\phi}_\eta)d\eta\Big)ds & \\
+\int_0^t \mathcal{S}(t-s)f\Big(s,z_s+\widehat{\phi}_s,\int_0^s e(s,\eta,z_\eta+\widehat{\phi}_\eta)d\eta\Big)ds & \\
+\int_0^t \mathcal{S}(t-s)\Big(\mathcal{H}(s)+\int_0^s \sigma\big(s,\eta,z_\eta+\widehat{\phi}_\eta)dw(\eta)\Big)ds & \\
+\int_0^t \mathcal{S}(t-s)Bu^a_{z+\widehat{\phi}}(s)ds+\sum\limits_{0<\tau_k<t}\mathcal{S}(t-\tau_k)I_k(z(\tau_k^-)+\widehat{\phi}(\tau_k^-)),\ t\in J. &
\end{cases} \qquad (3.6)
$$

Obviously, the operator Φ has a fixed point which is equivalent to prove that $\widehat{\Phi}$ has a fixed point. Since all the functions involved in the operator are continuous therefore $\widehat{\Phi}$ is continuous.

From our assumptions we have

$$\begin{aligned} E|u^a_{z+\widehat{\phi}}|^2 &\leq 9M_W\Big\{|x_1|^2 + ME|\phi(0)|^2 + M\tilde{M}_h + 2MM_0^2\Big[M_g\|\widehat{\phi}\|^2_{B_h} + c_2\Big] \\ &\quad +2\Big(M_0^2 + \frac{\big(C_{1-\beta}a^\beta\big)^2}{2\beta-1}\Big)\Big[M_g(1+2M_c)q\prime + 2M_gc_1 + c_2\Big] \\ &\quad +2Ma^2[M_f(1+2M_e)q' + 2M_f\tilde{M}_e + \tilde{M}_f] + 2Ma^2[M_{\mathcal{H}} + Tr(Q)\tilde{M}_\sigma] \\ &\quad +mM\sum_{k=1}^m \tilde{\beta}_k\Big\} := \mathcal{G} \text{ and} \\ E|u^a_{z+\widehat{\phi}} - u^a_{w+\widehat{\phi}}|^2 &\leq 6M_W\Big\{MM_h + \Big(M_0^2 + \frac{\big(C_{1-\beta}a^\beta\big)^2}{2\beta-1}\Big)M_g(1+M_c) + Ma^2M_f(1+M_e) \\ &\quad +2a^2MM_\sigma + mM\sum_{k=1}^m \beta_k\Big\}\|z_t - w_t\|^2_{\mathcal{B}_h}. \end{aligned}$$

Step 1: $\widehat{\Phi}(B_q) \subseteq B_q$ for some $q > 0$.

We claim that there exists a positive integer q such that $\widehat{\Phi}(B_q) \subseteq B_q$. If it is not true, then for each positive number q, there exists a function $z^q(\cdot) \in B_q$, but $\widehat{\Phi}(z^q) \notin B_q$, i.e. $|(\widehat{\Phi}z^q)(t)|^2 > q$ for some $t \in J$. However, on the other hand from $(H1) - (H11)$ we have

$$\begin{aligned} q &< E|(\widehat{\Phi}z^q)(t)|^2 \\ &\leq 7\Big\{2MM_0^2\Big[M_g\|\widehat{\phi}\|^2_{B_h} + c_2\Big] + 2\Big(M_0^2 + \frac{\big(C_{1-\beta}a^\beta\big)^2}{2\beta-1}\Big)\Big[M_g(1+2M_c)q\prime + 2M_gc_1 + c_2\Big] \\ &\quad +MM_ba^2\mathcal{G} + 2Ma^2[M_f(1+2M_e)q' + 2M_f\tilde{M}_e + \tilde{M}_f] + 2Ma^2[M_{\mathcal{H}} + Tr(Q)\tilde{M}_\sigma] \\ &\quad +mM\sum_{k=1}^m \tilde{\beta}_k\Big\} \\ &\leq 7\Big\{(1 + 9MM_bM_Wa^2)\Big[2MM_0^2[M_g\|\widehat{\phi}\|^2_{B_h} + c_2] + 2\Big(M_0^2 + \frac{\big(C_{1-\beta}a^\beta\big)^2}{2\beta-1}\Big)[M_g(1+2M_c)q\prime \\ &\quad +2M_gc_1 + c_2\big] + 2Ma^2[M_f(1+2M_e)q' + 2M_f\tilde{M}_e + \tilde{M}_f] + 2Ma^2[M_{\mathcal{H}} + Tr(Q)\tilde{M}_\sigma] \\ &\quad +mM\sum_{k=1}^m \tilde{\beta}_k\Big] + 9MM_bM_Wa^2\Big(|x_1|^2 + ME|\phi(0)|^2 + M\tilde{M}_h\Big)\Big\} \end{aligned}$$

$$\begin{aligned} q &\leq \tilde{M} + 7\Big\{(1 + 9MM_bM_Wa^2)\Big[\Big(M_0^2 + \frac{\big(C_{1-\beta}a^\beta\big)^2}{2\beta-1}\Big)\Big[2M_g(1+2M_c)q\prime\Big] \\ &\quad +2Ma^2[M_f(1+2M_e)q']\Big]\Big\}, \end{aligned}$$

where $\tilde{M}$ is independent of q. Dividing both sides by q and noting that

$$q' = 4l^2\Big(q + 2M(E|\phi(0)|^2 + M_h)\Big) + 4\|\widehat{\phi}\|^2_{\mathcal{B}_h} \longrightarrow \infty \text{ as } q \longrightarrow \infty$$

and thus we have

$$7l^2\Big\{\Big(1+9MM_bM_Wa^2\Big)\Big[8\Big(M_0^2 + \frac{(C_{1-\beta}a^\beta)^2}{2\beta-1}\Big)M_g(1+2M_c) + 8Ma^2M_f(1+2M_e)\Big]\Big\} \geq 1.$$

This contradicts (3.1). Hence $\widehat{\Phi}(B_q) \subseteq B_q$, for some positive number q.

Step 2: $\widehat{\Phi} : \mathcal{B}_a^0 \longrightarrow \mathcal{B}_a^0$ is a contraction mapping.

Let $z, w \in \mathcal{B}_a^0$ then we have

$$
\begin{aligned}
E\Big|\widehat{\Phi}z(t)-\widehat{\Phi}w(t)\Big|^2 \le\ & E\Big|g\Big(t, z_t+\widehat{\phi}_t, \int_0^t c(t,s,z_s+\widehat{\phi}_s)ds\Big)-g\Big(t, w_t+\widehat{\phi}_t, \int_0^t c(t,s,w_s+\widehat{\phi}_s)ds\Big)\Big|^2 \\
&+E\Big|\int_0^t \mathcal{S}(t-s)B\big(u^a_{z+\widehat{\phi}}(s)-u^a_{w+\widehat{\phi}}(s)\big)ds\Big|^2 \\
&+E\Big|\int_0^t A\mathcal{S}(t-s)\Big(g\Big(s, z_s+\widehat{\phi}_s, \int_0^s c(s,\eta,z_\eta+\widehat{\phi}_\eta)d\eta\Big) \\
&\qquad -g\Big(s, w_s+\widehat{\phi}_s, \int_0^s c(s,\eta,w_\eta+\widehat{\phi}_\eta)d\eta\Big)\Big)ds\Big|^2 \\
&+E\Big|\int_0^t \mathcal{S}(t-s)\Big(f\Big(s, z_s+\widehat{\phi}_s, \int_0^s e(s,\eta,z_\eta+\widehat{\phi}_\eta)d\eta\Big) \\
&\qquad -f\Big(s, w_s+\widehat{\phi}_s, \int_0^s e(s,\eta,w_\eta+\widehat{\phi}_\eta)d\eta\Big)\Big)ds\Big|^2 \\
&+E\Big|\int_0^t \mathcal{S}(t-s)\Big(\int_0^s \sigma\big(s,\eta,z_\eta+\widehat{\phi}_\eta\big)dw(\eta)-\int_0^s \sigma\big(s,\eta,w_\eta+\widehat{\phi}_\eta\big)dw(\eta)\Big)ds\Big|^2 \\
&+E\Big|\sum_{0<\tau_k<t}\mathcal{S}(t-\tau_k)\Big(I_k(z(\tau_k^-)+\widehat{\phi}(\tau_k^-))-I_k(w(\tau_k^-)+\widehat{\phi}(\tau_k^-))\Big)\Big|^2 \\
\le\ & 6\Big\{(1+6MM_bM_Wa^2)\Big[\Big(M_0^2+\frac{(C_{1-\beta}a^\beta)^2}{2\beta-1}\Big)M_g(1+2M_c)+2a^2MM_\sigma \\
&+Ma^2M_f(1+M_e)+mM\sum_{k=1}^m\beta_k\Big]+6M^2M_bM_Wa^2M_h\Big\}\|z_t-w_t\|^2_{\mathcal{B}_h} \\
\le\ & 12\Big\{(1+6MM_bM_Wa^2)\Big[\Big(M_0^2+\frac{(C_{1-\beta}a^\beta)^2}{2\beta-1}\Big)M_g(1+2M_c)+2a^2MM_\sigma \\
&+Ma^2M_f(1+M_e)+mM\sum_{k=1}^m\beta_k\Big]+6M^2M_bM_Wa^2M_h\Big\} \\
&\times\Big[l^2\sup_{0\le s\le t}E|z(s)-w(s)|^2+\|z_0-w_0\|^2_{\mathcal{B}_h}\Big] \\
\le\ & \nu\sup_{0\le s\le a}E|z(s)-w(s)|^2,
\end{aligned}
$$

since $\nu<1$ by $(H11)$ and we have used the fact that $z_0=0, w_0=0$. Taking the supremum over t, we get

$$\|\widehat{\Phi}z-\widehat{\Phi}w\|_a^2 \le \nu\|z-w\|_a^2,$$

and so $\widehat{\Phi}$ is a contraction. Hence by Banach fixed point theorem there exists a unique fixed point $x\in\mathcal{B}_a$ such that $(\Phi x)(t)=x(t)$. This fixed point is then the solution of the system (1.1) and clearly, $x(a)=(\Phi x)(a)=x_1$ which implies that the system (1.1) is controllable on J. □

4. EXAMPLE

Consider the following partial neutral integrodifferential equation of the form

$$d\Big[v(t,y) \quad + \quad \int_{-\infty}^t r_1(t,y,s-t)G_1(v(s,y))ds+\int_0^t\int_{-\infty}^s \mu_1(s-\xi)G_2(v(\xi,y))d\xi ds\Big]$$

$$
\begin{aligned}
&= \Big[\frac{\partial^2}{\partial y^2}v(t,y)+\int_{-\infty}^{t} r_2(t,y,s-t)F_1(v(s,y))ds+\int_0^t\int_{-\infty}^{s}\mu_2(s-\xi)F_2(v(\xi,y))d\xi ds \\
&\quad +c(y)u(t)\Big]dt+\int_{-\infty}^{t}\mu_3(s-t)v(s,y)d\beta(s), y\in[0,\pi], t\in J=[0,a], t\neq\tau_k, \\
v(t,0) &= v(t,\pi)=0,\ t\geq 0, \\
v(t,y) &+ \sum_{i=0}^{p}\int_0^{\pi}k_i(y,\eta)v(t_i,\eta)d\eta=\phi(t,y),\ t\in(-\infty,0], y\in[0,\pi], \\
\triangle v(\tau_i)(y) &= \int_{-\infty}^{\tau_i}q_i(\tau_i-s)v(s,y)ds,\ y\in[0,\pi]. \qquad (4.1)
\end{aligned}
$$

where $0<\tau_1<\tau_2<\cdots<\tau_n<a$ are prefixed numbers, $\phi\in\mathcal{B}_h,\ p\in N$ and $0<t_0<t_1<t_2<\cdots<t_p<a$. We take $H=K=U=L^2([0,\pi])$ with the norm $|\cdot|_{L^2}$. Define the operator $A:H\longrightarrow H$ by $A\omega=\omega''$ with domain

$$D(A)=\{\omega(\cdot)\in H:\omega,\omega' \text{ are absolutely continuous and } \omega''\in H,\omega(0)=\omega(\pi)=0\}.$$

Then

$$A\omega=\sum_{n=1}^{\infty}n^2(\omega,\omega_n)\omega_n,\ \omega\in D(A),$$

where $\omega_n(s)=\sqrt{\frac{2}{\pi}}\sin ns,\ n=1,2,\cdots$ is the orthogonal set of eigenvectors of A. It is well known that the A is the infinitesimal generator of an analytic semigroup $\mathcal{S}(t),\ t\geq 0$ in H and is given by

$$\mathcal{S}(t)\omega=\sum_{n=1}^{\infty}e^{-n^2t}(\omega,\omega_n)\omega_n,\ \omega\in H.$$

By [12], $(-A)^{-3/4}\omega=\sum_{n=1}^{\infty}\frac{1}{(\sqrt{n})^3}(\omega,\omega_n)\omega_n$, for every $\omega\in H$ and $\|(-A)^{-3/4}\|$ is bounded. Let $\|(-A)^{-3/4}\|=\lambda$. The operator $(-A)^{3/4}\omega$ is given by

$$(-A)^{3/4}\omega=\sum_{n=1}^{\infty}(\sqrt{n})^3(\omega,\omega_n)\omega_n$$

on the space $D((-A)^{3/4})=\{\omega\in H:\sum_{n=1}^{\infty}(\sqrt{n})^3(\omega,\omega_n)\omega_n\in H\}$. Since the semigroup $\mathcal{S}(t)$ is analytic there exists a constant $M>0$ such that $\|\mathcal{S}(t)\|^2\leq M$ and satisfies $(H1)$.

We assume the following conditions hold:

(a) $B:L^2([0,\pi])\longrightarrow H$ is a bounded linear operator defined by

$$Bu(y)=c(y)u,\ 0\leq y\leq\pi,\ u\in L^2(J,U).$$

(b) The linear operator $W:L^2(J,U)\longrightarrow H$ is defined by

$$Wu=\int_0^a\mathcal{S}(a-s)c(y)u(s)ds$$

has an inverse operator W^{-1} defined on $L^2(J,U)\setminus kerW$ and satisfies condition $(H2)$.

(c) $\beta(t)$ denotes an one-dimensional standard Brownian motion.

(d) The function $k_i:[0,\pi]\times[0,\pi]\longrightarrow R$ are C^2-functions, for each $i=1,2,\cdots,p$.

(e) $q_i : R \longrightarrow R$ are continuous functions and $\beta_i = \int_{-\infty}^{0} h(s)q_i^2(s)ds < \infty$ for $i = 1, 2, \cdots, m$

Let $h(s) = e^{4s}, s < 0$, then $l = \int_{-\infty}^{0} h(s)ds = 1/4$ and define

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta\in[s,0]} |\varphi(\theta)|_{L^2} ds.$$

Hence for $(t, \varphi) \in [0, a] \times \mathcal{B}_h$, where $\varphi(\theta)(y) = \varphi(\theta, y),\ (\theta, y) \in (-\infty, 0] \times [0, \pi]$. Set

$$v(t)(y) = v(t, y), \quad g(t, \varphi, N_1\varphi)(y) = \int_{-\infty}^{0} r_1(t, y, \theta)G_1(\varphi(\theta)(y))d\theta + N_1\varphi(y),$$

$$f(t, \varphi, N_2\varphi)(y) = \int_{-\infty}^{0} r_2(t, y, \theta)F_1(\varphi(\theta)(y))d\theta + N_2\varphi(y)$$

$$\text{and} \ \ \sigma(t, s, \varphi)(y) = \int_{-\infty}^{0} \mu_3(\theta)\varphi(\theta)(y)d\theta,$$

where

$$N_1\varphi(y) = \int_0^t \int_{-\infty}^{0} \mu_1(s-\theta)G_2(\varphi(\theta)(y))d\theta ds,\ N_2\varphi(y) = \int_0^t \int_{-\infty}^{0} \mu_2(s-\theta)F_2(\varphi(\theta)(y))d\theta ds.$$

Then the above equation can be written in the abstract form as the system (1.1). Moreover

$$g([0, a] \times \mathcal{B}_h \times L^2) \subseteq D((-A)^{3/4})$$

and

$$|(-A)^{3/4}g(t, \varphi_1, u_1)(y) - (-A)^{3/4}g(t, \varphi_2, u_2)(y)|^2 \leq M_g\left[\|\varphi_1 - \varphi_2\|^2_{\mathcal{B}_h} + |u_1 - u_2|^2\right]$$

for some constant $M_g > 0$ depending on r_1, μ_1 and G_1, G_2 and $|u_1 - u_2|^2 = |N_1\varphi_1 - N_1\varphi_2|^2 \leq M_c\|\varphi_1 - \varphi_2\|^2_{\mathcal{B}_h}$ for $M_c > 0$. Further, other assumptions $(H5) - (H11)$ are satisfied such that $\frac{7}{2}\big\{\big(1 + 9MM_bM_Wa^2\big)\big[(\lambda^2 + 2C^2_{1/4}(\sqrt{a})^3)M_g(1 + 2M_c) + Ma^2M_f(1 + 2M_e)\big]\big\} < 1$ and it is possible to choose q_i, k_i in such a way that the constant $\nu < 1$. Hence by Theorem (3.1) the system (4.1) is controllable on J. □

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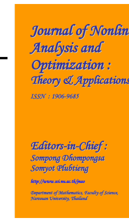

ON PARANORMED I-CONVERGENT SEQUENCE SPACES OF INTERVAL NUMBERS

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ABSTRACT. In this article we introduce and study the paranormed I-convergent sequence spaces $\mathcal{C}^I(\bar{\mathcal{A}}, p)$, $\mathcal{C}_0^I(\bar{\mathcal{A}}, p)$, $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$ and $\mathcal{M}_{\mathcal{C}_\circ}^I(\bar{\mathcal{A}}, p)$ on the sequence of interval numbers with the help of a bounded sequence $p = (p_k)$ of strictly positive real numbers. We study some topological and algebraic properties and some inclusion relations on these spaces.

KEYWORDS: Interval numbers; Ideal; Filter; I-convergent sequence; Solid and monotone space; Banach space.

AMS Subject Classification: 46A40

1. INTRODUCTION AND PRELIMINARIES

It is an admitted fact that the real and complex numbers are playing a vital role in the world of mathematics. Many mathematical structures have been constructed with the help of these numbers. In recent years, since 1965 fuzzy numbers and interval numbers also managed their place in the world of mathematics and credited into account some alike structures . Interval arithmetic was first suggested by P.S.Dwyer [5] in 1951. Further development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by R.E.Moore

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[13] in 1959 and Moore and Yang [14] and others and have developed applications to differential equations.

Recently, Chiao [4] introduced sequences of interval numbers and defined usual convergence of sequences of interval numbers. Şengönül and Eryılmaz [19] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete.

Here after, we give the notation and definitions that will be used in the paper.

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus, we can investigate some properties of interval numbers for instance, arithmetic properties or analysis properties. Let us denote the set of all real valued closed intervals by $I\mathbb{R}$. Any element of $I\mathbb{R}$ is called a closed interval and it is denoted by $\bar{A} = [x_l, x_r]$. An interval number is closed subset of real numbers [4]. The algebraic operations for interval numbers can be found in [19].

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by

$$d(\bar{A}_1, \bar{A}_2) = \max\{\mid x_{1_l} - x_{2_l} \mid, \mid x_{1_r} - x_{2_r} \mid\}, (\text{see})[14, 19] \tag{1.1}$$

where x_l and x_r be first and last points of $\bar{A}$, respectively.

In a special case, $\bar{A}_1 = [a, a]$, $\bar{A}_2 = [b, b]$, we obtain the usual metric of $\mathbb{R}$ with

$$d(\bar{A}_1, \bar{A}_2) = \mid a - b \mid .$$

Let us define transformation f from $\mathbb{N}$ to $I\mathbb{R}$ by $k \rightarrow f(k) = \bar{A}$, $\bar{\mathcal{A}} = (\bar{A}_k)$. The function f is called sequence of interval numbers, where $\bar{A}_k$ is the k^{th} term of the sequence $(\bar{A}_k)$.

Let us denote the set of sequences of interval numbers with real terms by

$$\omega(\bar{\mathcal{A}}) = \{\bar{\mathcal{A}} = (\bar{A}_k) : \bar{A}_k \in I\mathbb{R}\}. \tag{1.2}$$

The algebraic properties of $\omega(\bar{\mathcal{A}})$ can be found in [4, 19].

The following definitions were given by Şengönül and Eryılmaz in [19].

A sequence $\bar{\mathcal{A}} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ of interval numbers is said to be convergent to an interval number $\bar{A}_0 = [x_{0_l}, x_{0_r}]$ if for each $\epsilon > 0$, there exists a positive integer n_0 such that $d(\bar{A}_k, \bar{A}_0) < \epsilon$, for all $k \geq n_0$ and we denote it as $\lim_k \bar{A}_k = \bar{A}_0$.

Thus, $\lim_k \bar{A}_k = \bar{A}_0 \Leftrightarrow \lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$ and it is said to be Cauchy sequence of interval numbers if for each $\epsilon > 0$, there exists a positive integer k_0

such that $d(\bar{A}_k, \bar{A}_m) < \epsilon$, whenever $k,\ m \geq k_0$.

Let us denote the space of all convergent, null and bounded sequences of interval numbers by $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$, respectively. The sets $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$ are complete metric spaces with the metric

$$\widehat{d}(\bar{A}_k, \bar{B}_k) = \sup_k \max\{|x_{kl} - y_{kl}|, |x_{rl} - y_{rl}|\}. \tag{1.3}$$

If we take $\bar{B}_k = \bar{O}$ in (3) then, the metric $\widehat{d}$ reduces to (see,[19])

$$\widehat{d}(\bar{A}_k, \bar{O}) = \sup_k \max\{|x_{kl}|, |x_{rl}|\}. \tag{1.4}$$

In this paper, we assume that a norm $||\bar{A}_k||$ of the sequence of interval numbers $(\bar{A}_k)$ is the distance from $(\bar{A}_k)$ to $\bar{O}$ and satisfies the following properties:
$\forall \bar{A}_k, \bar{B}_k \in \lambda(\bar{\mathcal{A}})$ and $\forall \alpha \in \mathbb{R}$
(N_1). $\forall \bar{A}_k \in \lambda(\bar{\mathcal{A}}) - \{\bar{O}\}, ||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} > 0$;
(N_2). $||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} = 0 \Leftrightarrow \bar{A}_k = \bar{O}$;
(N_3). $||\bar{A}_k + \bar{B}_k||_{\lambda(\bar{\mathcal{A}})} \leq ||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})} + ||\bar{B}_k||_{\lambda(\bar{\mathcal{A}})}$
(N_4). $||\alpha \bar{A}_k||_{\lambda(\bar{\mathcal{A}})} = |\alpha|||\bar{A}_k||_{\lambda(\bar{\mathcal{A}})}$, where $\lambda(\bar{\mathcal{A}})$ is a subset of $\omega(\bar{\mathcal{A}})$.

Let $\bar{\mathcal{A}} = (\bar{A}_k) = ([x_{k_l}, x_{k_r}])$ be the element of $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ or $\ell_\infty(\bar{\mathcal{A}})$. Then, with respect to the above discussion the classes of sequences $\mathcal{C}(\bar{\mathcal{A}})$, $\mathcal{C}_\circ(\bar{\mathcal{A}})$ and $\ell_\infty(\bar{\mathcal{A}})$ are normed interval spaces normed by

$$\| \bar{\mathcal{A}} \| = \sup_k \max\{| x_{k_l} |, | x_{k_r} |\} \text{ (see[19])}. \tag{1.5}$$

Throughout, $\bar{O} = [0, 0]$ and $\bar{I} = [1, 1]$ represent zero and identity interval numbers according to addition and multiplication, respectively.

As a generalisation of usual convergence for the sequences of real or complex numbers, the concept of statistical convergence was first introduced by Fast [6] and also independently by Buck [3] and Schoenberg [18]. Later on, it was further investigated from a sequence space point of view and linked with the Summability Theory by Fridy [7], Šalát [15], Tripathy [20] and many others. The notion of statistical convergence has been extended to interval numbers by Esi as follows in [1, 2].

Let us suppose that $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$. If, for every $\epsilon > 0$,

$$\lim_k \frac{1}{k} |\{n \in \mathbb{N} : \| \bar{A}_k - \bar{A}_0 \| \geq \epsilon,\ n \leq k\}| = 0 \tag{1.6}$$

then the sequence $\bar{\mathcal{A}} = (\bar{A}_k)$ is said to be statistically convergent to an interval number $\bar{A}_0$, where vertical lines denote the cardinality of the enclosed set.

That is, if $\delta(A(\epsilon)) = 0$, where $A(\epsilon) = \{k \in \mathbb{N} :\| \bar{A}_k - \bar{A}_0 \| \geq \epsilon\}$.

The notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt and Wilczyński [11, 12]. Later on, it was studied by Šalát, Tripathy and Ziman [16, 17], Esi and Hazarika [1], Tripathy and Hazarika [21], Khan *et al* [8, 9, 10] and many others.

Theorem 1.1. *Let suppose that I be an ideal.*

Then, a sequence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) \subset \omega(\bar{\mathcal{A}})$
(i) is said to be I-convergent to an interval number$\bar{A}_0$ if for every $\epsilon > 0$, the set $\{k \in N :\| \bar{A}_k - \bar{A}_0 \| \geq \epsilon\} \in I$.

In this case, we write $I - \lim \bar{A}_k = \bar{A}_0$. If $\bar{A}_0 = \bar{O}$ then the sequence $\bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}})$ is said to be I-null. In this case, we write $I - \lim \bar{A}_k = \bar{O}$.
(ii) is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in N :\| \bar{A}_k - \bar{A_m} \| \geq \epsilon\} \in I$.
(iii) is said to be I-bounded if there exists some $M > 0$ such that $\{k \in N :\| \bar{A}_k \| \geq M\} \in I$.

Let us denote the classes of I-convergent, I-null, bounded I-convergent and bounded I-null sequences of interval numbers with $\mathcal{C}^I(\bar{\mathcal{A}})$, $\mathcal{C}^I_\circ(\bar{\mathcal{A}})$, $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}})$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}})$, respectively.
We know that for each ideal I, there is a filter $\pounds(I)$ corresponding to I, i.e $\pounds(I)$ = $\{K \subseteq N : K^c \in I\}$, where $K^c = N \setminus K$.

Theorem 1.2. *A sequence space $\lambda(\bar{\mathcal{A}})$ of interval numbers*
(iv) is said to be solid(normal) if $(\alpha_k \bar{A}_k) \in \lambda(\bar{\mathcal{A}})$ whenever $(\bar{A}_k) \in \lambda(\bar{\mathcal{A}})$ and for any sequence (α_k) of scalars with $| \alpha_k | \leq 1$, for all $k \in \mathbb{N}$,
(v) is said to be symmetric if $(\bar{A}_{\pi(k)}) \in \lambda(\bar{\mathcal{A}})$ whenever $\bar{A}_k \in \lambda(\bar{\mathcal{A}})$, where π is a permutation on $\mathbb{N}$,
(vi) is said to be sequence algebra if $(\bar{A}_k) * (\bar{B}_k) = (\bar{A}_k.\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$ whenever $(\bar{A}_k)$, $(\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$,
(vii) is said to be convergence free if $(\bar{B}_k) \in \lambda(\bar{\mathcal{A}})$ whenever $(\bar{A}_k) \in \lambda(\bar{\mathcal{A}})$ and $\bar{A}_k = \bar{O}$ implies $\bar{B}_k = \bar{O}$, for all k.

Theorem 1.3. *Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$. The K-step space of the $\lambda(\bar{\mathcal{A}})$ is a sequence space $\mu_K^{\lambda(\bar{\mathcal{A}})} = \{(\bar{A}_{k_n}) \in \omega(\bar{\mathcal{A}}) : (\bar{A}_k) \in \lambda(\bar{\mathcal{A}})\}$.*

Theorem 1.4. *A canonical pre-image of a sequence* $(\bar{A}_{k_n}) \in \mu_K^{\lambda(\bar{\mathcal{A}})}$ *is a sequence*

$(\bar{B}_k) \in \omega(\bar{\mathcal{A}})$ *defined by*

$$\bar{B}_k = \begin{cases} \bar{A}_k, & \text{if } k \in K, \\ \bar{O}, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space $\mu_K^{\lambda(\bar{A})}$ *is a set of canonical preimages of all elements in* $\mu_K^{\lambda(\bar{A})}$, *i.e.* $\bar{\mathcal{B}}$ *is in the canonical preimage of* $\mu_K^{\lambda(\bar{A})}$ *iff* $\bar{\mathcal{B}}$ *is the canonical preimage of some* $\bar{\mathcal{A}} \in \mu_K^{\lambda(\bar{A})}$.

Theorem 1.5. *A sequence space* $\lambda(\bar{\mathcal{A}})$ *is said to be monotone if it contains the canonical preimages of its step space.*

Now, we give some important Lemmas:

Lemma 1.6. *Every solid space is monotone.*

Lemma 1.7. *Let* $K \in \pounds(I)$ *and* $M \subseteq N$. *If* $M \notin I$, *then* $M \cap K \notin I$, *where* $\pounds(I) \subseteq 2^N$ *filter on* N.

Lemma 1.8. *If* $I \subseteq 2^N$ *and* $M \subseteq N$. *If* $M \notin I$, *then* $M \cap N \notin I$.

2. MAIN RESULTS

Let us give an instrumental and important definition for this paper:

Theorem 2.1. *Let* $\bar{X}$ *be a set of interval numbers. A function* $g : \bar{X} \longrightarrow \mathbb{R}$ *is called paranorm on* $\bar{X}$, *if for all* $A,\ B \in \bar{X}$,
(P_1) $g(A) = 0 \; if \; A = \bar{0}$,
(P_2) $g(-A) = g(A)$,
(P_3) $g(A + B) \leq g(A) + g(B)$,
(P_4) *If* (λ_n) *is a sequence of scalars with* $\lambda_n \to \lambda$ $(n \to \infty)$ *and* $(A_n), A_0 \in \bar{X}$ *with* $A_n \to A_0$ $(n \to \infty)$ *in the sense that* $g(A_n - A_0) \to 0$ $(n \to \infty)$, *then* $g(\lambda_n A_n - \lambda A_0) \to 0$ $(n \to \infty)$.

In this article, we introduce and study the following paranormed classes of sequences of interval numbers

$$\mathcal{C}^I(\bar{\mathcal{A}}, p) = \left\{ \bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \left\{ k \in \mathbb{N} : \left(\parallel \bar{A}_k - \bar{A} \parallel \right)^{p_k} \geq \epsilon \right\} \in I, \right\}, \quad (2.1)$$

$$\mathcal{C}_0^I(\bar{\mathcal{A}}, p) = \left\{ \bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \left\{ k \in \mathbb{N} : \left(\parallel \bar{A}_k \parallel \right)^{p_k} \geq \epsilon \right\} \in I \right\}, \quad (2.2)$$

$$\ell_\infty(\bar{\mathcal{A}}, p) = \left\{ \bar{\mathcal{A}} = (\bar{A}_k) \in \ell_\infty(\bar{\mathcal{A}}) : \sup_k (\parallel \bar{A}_k \parallel)^{p_k} < \infty \right\}, \quad (2.3)$$

where $p = (p_k)$ is a bounded sequence of strictly positive real numbers.

We also denote

$$\mathcal{M}_{\mathcal{C}}^{I}(\bar{\mathcal{A}}, p) = \ell_\infty(\bar{\mathcal{A}}, p) \cap \mathcal{C}^I(\bar{\mathcal{A}}, p) \text{ and } \mathcal{M}_{\mathcal{C}_\circ}^{I}(\bar{\mathcal{A}}, p) = \ell_\infty(\bar{\mathcal{A}}, p) \cap \mathcal{C}_0^I(\bar{\mathcal{A}}, p).$$

Theorem 2.2. *The classes of sequences $\mathcal{M}_{\mathcal{C}}^{I}(\bar{\mathcal{A}}, p)$ and $\mathcal{M}_{\mathcal{C}_\circ}^{I}(\bar{\mathcal{A}}, p)$ are paranormed spaces paranormed by*

$$g(\bar{\mathcal{A}}) = \sup_k \parallel \bar{A}_k \parallel^{\frac{p_k}{M}}, \text{ where } M = \max\{1, \sup_k p_k\}.$$

Proof. Let $\bar{\mathcal{A}} = (\bar{A}_k),\ \bar{\mathcal{B}} = (\bar{B}_k) \in \mathcal{M}_{\mathcal{C}}^{I}(\bar{\mathcal{A}}, p)$.
(P_1) It is Clear that $g(\bar{\mathcal{A}}) = 0$ if $\bar{\mathcal{A}} = \bar{\theta}$.
(P_2) $g(\bar{\mathcal{A}}) = g(-\bar{\mathcal{A}})$ is obvious.
(P_3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality, we have

$$\begin{aligned} g(\bar{\mathcal{A}} + \bar{\mathcal{B}}) = g(\bar{A}_k + \bar{B}_k) &= \sup_k \parallel \bar{A}_k + \bar{B}_k) \parallel^{\frac{p_k}{M}} \\ &\leq \sup_k \parallel \bar{A}_k \parallel^{\frac{p_k}{M}} + \sup_k \parallel \bar{B}_k \parallel^{\frac{p_k}{M}} \\ &= g(\bar{A}_k) + g(\bar{B}_k) = g(\bar{\mathcal{A}}) + g(\bar{\mathcal{B}}) \end{aligned}$$

Therefore, $g(\bar{\mathcal{A}} + \bar{\mathcal{B}}) \leq g(\bar{\mathcal{A}}) + g(\bar{\mathcal{B}})$.
(P_4) Let (λ_k) be a sequence of scalars with $(\lambda_k) \to \lambda$ $(k \to \infty)$ and $(\bar{A}_k),\ \bar{A}_0 \in \mathcal{M}_{\mathcal{C}}^{I}(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k \to \bar{A}_0$ $(k \to \infty)$, in the sense that $g(\bar{A}_k - \bar{A}_0) \to 0$ $(k \to \infty)$. Then, since the inequality

$$g(\bar{A}_k) \leq g(\bar{A}_k - \bar{A}_0) + g(\bar{A}_0)$$

holds by subadditivity of g, the sequence $\{g(\bar{A}_k)\}$ is bounded.
Therefore,

$$\begin{aligned} g\big[(\lambda_k \bar{A}_k - \lambda \bar{A}_0)\big] &= g\big[(\lambda_k \bar{A}_k - \lambda \bar{A}_k + \lambda \bar{A}_k - \lambda \bar{A}_0)\big] \\ &= g\big[(\lambda_k - \lambda)\bar{A}_k + \lambda(\bar{A}_k - \bar{A}_0)\big] \\ &\leq g\big[(\lambda_k - \lambda)\bar{A}_k\big] + g\big[\lambda(\bar{A}_k - \bar{A}_0)\big] \\ &\leq \mid (\lambda_k - \lambda) \mid^{\frac{p_k}{M}} g(\bar{A}_k) + \mid \lambda \mid^{\frac{p_k}{M}} g(\bar{A}_k - \bar{A}_0) \to 0 \end{aligned}$$

as $(k \to \infty)$. That is to say that scalar multiplication is continuous.
Hence $\mathcal{M}_{\mathcal{C}}^{I}(\bar{\mathcal{A}}, p)$ is a paranormed space. For $\mathcal{M}_{\mathcal{C}_\circ}^{I}(\bar{\mathcal{A}}, p)$, the proof is similar. □

Theorem 2.3. *The set $\mathcal{M}_{\mathcal{C}}^{I}(\bar{\mathcal{A}}, p)$ is closed subspace of $\ell_\infty(\bar{\mathcal{A}}, p)$.*

Proof. Let $(\bar{A}_k^{(n)})$ be a Cauchy sequence in $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k^{(n)} \to \bar{A}$. We show that $\bar{A} \in \mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$. Since $(\bar{A}_k^{(n)}) \in \mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p)$ there exists $\bar{A}_n$ such that

$$\{k \in \mathbb{N} : \| \bar{A}_k^{(n)} - \bar{A}_n \|^{p_k} \geq \epsilon\} \in I.$$

We need to show that

(1) $(\bar{A}_n)$ converges to $\bar{A}_0$.

(2) If $U = \{k \in \mathbb{N} : \big(\| \bar{A}_k - \bar{A}_0 \| \big)^{p_k} < \epsilon\}$, then $U^c \in I$.

(1) Since $(\bar{A}_k^{(n)})$ is Cauchy sequence in $\mathcal{M}_{\mathcal{C}}^I(\bar{\mathcal{A}}, p) \Rightarrow$ for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sup_k \big(\| \bar{A}_k^{(n)} - \bar{A}_k^{(q)} \| \big)^{\frac{p_k}{M}} < \frac{\epsilon}{3}$, for all $n,\ q \geq k_0$. For $\epsilon > 0$, we have

$$B_{nq} = \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(n)} - \bar{A}_k^{(q)} \| \big)^{p_k} < (\frac{\epsilon}{3})^M\},$$

$$B_q = \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q)} - \bar{A}_q \| \big)^{p_k} < (\frac{\epsilon}{3})^M\},$$

$$B_n = \left\{k \in \mathbb{N} : \big(\| \bar{A}_k^{(n)} - \bar{A}_n \| \big)^{p_k} < (\frac{\epsilon}{3})^M\right\}.$$

Then, B_{nq}^c, B_q^c and $B_n^c \in I$. Let $B^c = B_{nq}^c \cup B_q^c \cup B_n^c$, where $B = \{k \in \mathbb{N} : \big(\| \bar{A}_q - \bar{A}_n \| \big)^{p_k} < \epsilon\}$. Then $B^c \in I$.

If we choose a integer $k_0 \in B^c$ then for each $n,\ q \geq k_0$, we have

$$\begin{aligned} &\big\{k \in \mathbb{N} : \big(\| \bar{A}_q - \bar{A}_n \| \big)^{p_k} < \epsilon\big\} \\ &\supseteq \Bigg[\{k \in \mathbb{N} : \big(\| \bar{A}_q - \bar{A}_k^{(q)} \| \big)^{p_k} < (\frac{\epsilon}{3})^M\} \\ &\cap \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q)} - \bar{A}_k^{(n)} \| \big)^{p_k} < (\frac{\epsilon}{3})^M\} \\ &\cap \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(n)} - \bar{A}_n \| \big)^{p_k} < (\frac{\epsilon}{3})^M\}\Bigg] \end{aligned} \tag{2.4}$$

Then $(\bar{A}_n)$ is a Cauchy sequence of interval numbers, so there exists some interval number $\bar{A}_0$ such that $\bar{A}_n \to \bar{A}_0$ as $n \to \infty$.

(2) Let $0 < \delta < 1$ be given. Then, we show that if

$$U = \{k \in \mathbb{N} : \big(\| \bar{A}_k - \bar{A}_0 \| \big)^{p_k} < \delta\},$$

then $U^c \in I$. Since $(\bar{A}_k^{(n)}) \to \bar{A}$ then, there exists $q_0 \in \mathbb{N}$ such that

$$P = \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q_0)} - \bar{A}_k \| \big)^{p_k} < (\frac{\delta}{3D})^M\} \tag{2.5}$$

implies $P^c \in I$. The number q_0 can be chosen that together with (11), we have

$$Q = \{k \in \mathbb{N} : \big(\| \bar{A}_{q_0} - \bar{A}_0 \| \big)^{p_k} < (\frac{\delta}{3D})^M\}$$

such that $Q^c \in I$. Since $\{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \| \big)^{p_k} \geq \delta\} \in I$ we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where

$$S = \{k \in \mathbb{N} : \big(\| \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \| \big)^{p_k} < (\frac{\delta}{3D})^M\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where $U = \{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A}_0 \parallel \big)^{p_k} < \delta\}$. Therefore, for each $k \in U^c$, we have

$$\begin{aligned} &\{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A}_0 \parallel \big)^{p_k} < \delta\} \\ &\supseteq [\{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A}_k^{(q_0)} \parallel \big)^{p_k} < (\frac{\delta}{3})^M\} \\ &\cap \{k \in \mathbb{N} : \big(\parallel \bar{A}_k^{(q_0)} - \bar{A}_{q_0} \parallel \big)^{p_k} < (\frac{\delta}{3})^M\} \\ &\cap \{k \in \mathbb{N} : \big(\parallel \bar{A}_{q_0} - \bar{A}_0 \parallel \big)^{p_k} < (\frac{\delta}{3})^M\}] \end{aligned} \tag{2.6}$$

Then, the result follows from (12). □

Since the inclusions $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p) \subset \ell_\infty(\bar{\mathcal{A}}, p)$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p) \subset \ell_\infty(\bar{\mathcal{A}}, p)$ are strict so in view of Theorem (2.3) we have the following result.

Theorem 2.4. *The spaces $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ are nowhere dense subsets of $\ell_\infty(\bar{\mathcal{A}}, p)$.*

Theorem 2.5. *The spaces $\mathcal{C}^I_0(\bar{\mathcal{A}}, p)$ and $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ are both solid and monotone.*

Proof. We shall prove the result for $\mathcal{C}^I_0(\bar{\mathcal{A}}, p)$. For $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$, the result follows similarly.

For, let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I_0(\bar{\mathcal{A}}, p)$ and (α_k) be a sequence of scalars with $\mid \alpha_k \mid \leq 1$, for all $k \in \mathbb{N}$.

Since $\mid \alpha_k \mid^{p_k} \leq \max\{1, \mid \alpha_k \mid^G\} \leq 1$, for all $k \in \mathbb{N}$, where $G = \sup_k p_k$

we have

$$\big(\parallel \alpha_k \bar{A}_k \parallel \big)^{p_k} \leq \big(\parallel \bar{A}_k \parallel \big)^{p_k}, \text{ for all k} \in \mathbb{N}.$$

which further implies that

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\} \supseteq \{k \in \mathbb{N} : \big(\parallel \alpha_k \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\}.$$

But

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\} \in I$$

Therefore,

$$\{k \in \mathbb{N} : \big(\parallel \alpha_k \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\} \in I.$$

Thus, $\alpha_k(\bar{A}_k) \in \mathcal{C}^I_0(\bar{\mathcal{A}}, p)$.
Therefore, the space $\mathcal{C}^I_0(\bar{\mathcal{A}}, p)$ is solid and hence by Lemma (1.6), it is monotone. □

Here, we will give a definition that will be used in the following theorem.

Theorem 2.6. *A non-trivial ideal $I \subseteq 2^N$ is called admissible if $\{\{x\} : x \in N\} \subseteq I$.*

Theorem 2.7. *Let $G = \sup_k p_k < \infty$ and I be an admissible ideal. Then, the following are equivalent:*
(1) $(\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, p)$;
(2) there exists $(\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k = \bar{B}_k$, for a.a.k.r.I.
(3) there exists $(\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, p)$ and $\bar{C}_k \in \mathcal{C}_0^I(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k = \bar{B}_k + \bar{C}_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : \big(\parallel \bar{B}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \in I$
(4) there exists a subset $K = \{k_1 < k_2 < k_3 < k_4 ...\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n\to\infty} \big(\parallel \bar{A}_{k_n} - \bar{A} \parallel \big)^{p_{k_n}} = 0$.

Proof. (1) implies (2).
Let $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, p)$. Then, there exists interval number $\bar{A}$ such that the set

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : \big(\parallel \bar{A}_k - \bar{A} \parallel)^{p_k} \geq t^{-1}\} \in I$$

Define a sequence $(\bar{B}_k)$ as

$$\bar{B}_k = \bar{A}_k, \quad \text{for all k} \leq m_1.$$

For $m_t < k \leq m_{t+1},\ t \in \mathbb{N}$

$$\bar{B}_k = \begin{cases} \bar{A}_k, \text{ if } \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} < t^{-1}, \\ \qquad \bar{A}, otherwise. \end{cases}$$

Then $\bar{B}_k \in \mathcal{C}(\bar{\mathcal{A}}, p)$ and from the inclusion

$$\{k \leq m_t : \bar{A}_k \neq \bar{B}_k\} \subseteq \{k \leq m_t : f\big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \in I.$$

we get $\bar{A}_k = \bar{B}_k$ for a.a.k.r.I.
(2) implies (3). For $\bar{\mathcal{A}} = (\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, p)$, then, there exists $(\bar{B}_k) \in \mathcal{C}(\bar{\mathcal{A}}, p)$ such that $\bar{A}_k = \bar{B}_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : \bar{A}_k \neq \bar{B}_k\}$, then $K \in I$. Define $\bar{C}_k$ as follows:

$$\bar{C}_k = \begin{cases} \bar{A}_k - \bar{B}_k, \text{if } k \in K, \\ O, \text{ if } k \notin K. \end{cases}$$

Then $\bar{C}_k \in \mathcal{C}_0^I(\bar{\mathcal{A}}, p)$ and $\bar{B}_k \in \mathcal{C}(\bar{\mathcal{A}}, p)$.
(3) implies (4). Suppose (3) holds. Let $\epsilon > 0$ be given. Let

$$P_1 = \{k \in \mathbb{N} : \big(\parallel \bar{C}_k \parallel)^{p_k} \geq \epsilon\} \in I.$$

and

$$K = P_1^c = \{k_1 < k_2 < k_3 < k_4 ...\} \in \pounds(I).$$

Then we have

$$\lim_{k\to\infty} \big(\parallel \bar{A}_{k_n} - \bar{A} \parallel \big)^{p_{k_n}} = 0.$$

(4) implies (1). Let $K = \{k_1 < k_2 < k_3 < k_4...\} \in \pounds(I)$ and

$$\lim_{k\to\infty} \big(\parallel \bar{A}_{k_n} - \bar{A} \parallel)^{p_{k_n}} = 0.$$

Then for any $\epsilon > 0$, and Lemma (1.7), we have

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\} \subseteq K^c \cup \{k \in K : \big(\parallel \bar{A}_k - \bar{A} \parallel \big)^{p_k} \geq \epsilon\}.$$

Thus, $(\bar{A}_k) \in \mathcal{C}^I(\bar{\mathcal{A}}, p)$ □

Theorem 2.8. *Let* (p_k) *and* (q_k) *be two sequences of positive real numbers. Then* $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p) \supseteq \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q)$ *if and only if* $\lim_{k\in K}\inf \frac{p_k}{q_k} > 0$, *where* $K \subseteq \mathbb{N}$ *such that* $K \in \pounds(I)$.

Proof. Let $\lim_{k\in K}\inf \frac{p_k}{q_k} > 0$ and $(\bar{A}_k) \in \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q)$. Then, there exists $\beta > 0$ such that $p_k > \beta q_k$ for sufficiently large $k \in K$.
Since $(\bar{A}_k) \in \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q)$. For a given $\epsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{q_k} \geq \epsilon\} \in I.$$

Let $G_0 = K^c \cup B_0$. Then $G_0 \in I$. Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{p_k} \geq \epsilon\} \subseteq \{k \in \mathbb{N} : \big(\parallel \bar{A}_k \parallel \big)^{\beta q_k} \geq \epsilon\} \in I.$$

Therefore, $(\bar{A}_k) \in \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$. The converse part of the result follows obviously. □

Theorem 2.9. *Let* (p_k) *and* (q_k) *be two sequences of positive real numbers. Then* $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q) \supseteq \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ *if and only if* $\lim_{k\in K}\inf \frac{q_k}{p_k} > 0$, *where* $K \subseteq \mathbb{N}$ *such that* $K \in \pounds(I)$.

Proof. The proof follows similarly as that of the proof Theorem (2.8). □

Theorem 2.10. *Let* (p_k) *and* (q_k) *be two sequences of positive real numbers. Then* $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, q) = \mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ *if and only if* $\lim_{k\in K}\inf \frac{p_k}{q_k} > 0$ *and* $\lim_{k\in K}\inf \frac{q_k}{p_k} > 0$, *where* $K \subseteq \mathbb{N}$ *such that* $K \in \pounds(I)$.

Proof. On combining Theorem (2.8) and (2.9), we get the desired result. □

Theorem 2.11. *The spaces* $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ *and* $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$ *are not seperable.*

Proof. By a counter example we prove the result for the space $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$
For $\mathcal{M}^I_{\mathcal{C}_\circ}(\bar{\mathcal{A}}, p)$, the result follows similarly.

Counter Example.

Let M be an infinite subset of increasing natural numbers such that $M \in I$.
Let

$$p_k = \begin{cases} 1, \text{if k} \in M, \\ 2, \text{otherwise}. \end{cases}$$

Let $P_0 = \{(\bar{A}_k) : \bar{A}_k = [0,0]$ or $[1,1]$, for $k \in M$ and $\bar{A}_k = [0,0],$ otherwise$\}$.
Since M is infinite, so P_0 is uncountable. Consider the class of open balls $\mathcal{B}_1 = \{B(\bar{Z}, \frac{1}{2}) : \bar{Z} \in P_0\}$. Let $\mathcal{C}_1$ be an open cover of $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ containing $\mathcal{B}_1$. Since $\mathcal{B}_1$ is uncountable, so $\mathcal{C}_1$ cannot be reduced to a countable subcover for $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$. Thus $\mathcal{M}^I_{\mathcal{C}}(\bar{\mathcal{A}}, p)$ is not seperable. Hence the result. □

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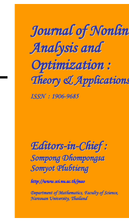

A COMMON FIXED POINT THEOREM VIA FAMILY OF R-WEAKLY COMMUTING MAPS

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ABSTRACT. In the present paper we prove a unique common fixed point theorem for a family of R-weakly commuting maps in non-Archimedean Menger PM-spaces without using the notion of continuity. Our result generalizes and extends the result of Khan and Sumitra [5] and few others, also suggest a path to a new inequality containing rational, product and minimum of some terms under implicit relation.

KEYWORDS : Non-Archimedean Menger PM-spaces; R-weakly commuting maps; Common fixed point.
AMS Subject Classification:$47H10, 54H25$.

1. INTRODUCTION

Non-Archimedean probabilistic metric space and some topological preliminaries on them were first studied by Istratescu and Babescu [9] and Istratescu and Crivat [10]. Some fixed point theorems for mappings on non-Archimedean Menger spaces have been proved by Istratescu ([11],[12]) as a result of the generalizations of some of the results of Sehgal and Bharucha-Reid [13] and Sherwood [2]. Achari [3] studied the fixed points of quasi-contraction type mappings in non-Archimedean PM-spaces and generalized the results of Istratescu [11]. In 1994, Pant [6] introduced the concept of R-weakly commuting maps in metric spaces. Later on Cho et al. [14] generalised this this idea and gave the concept of R-weakly commuting maps of type A_g. Vasuki [7] proved some common fixed point theorem for R-weakly commuting maps in fuzzy metric spaces. Recently Khan and Sumitra [5] introduced the concept of R-weakly commuting maps in non-Archimedean menger PM-spaces and proved a common fixed point theorem for three point wise

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R-weakly commuting mappings in complete non-Archimedean Menger PM-spaces. In the present paper we prove a unique common fixed point theorem for a family of R-weakly commuting maps in non-Archimedean Menger PM-spaces without using the notion of continuity. our result generalizes and extends the result of Khan and Sumitra [5] and other, also suggest a path to a new inequality containing rational, product and minimum of some terms under implicit relation.

2. PRELIMINARIES

Definition 2.1. [10],[11] Let X be any non-empty set and D be the set of all left continuous distribution functions. An ordered pair (X, F) is said to be non-Archimedean probabilistic metric space (N.A. PM-space) if F is a mapping from $X \times X$into D satisfying the following conditions, where the value of Fat $(x, y) \in X \times X$ is represented by $F_{x,y}$ or $F\,(x, y)$ for all $x, y \in X$ such that
(i) $F(x, y; t) = 1$ for all $t > 0$ if and only if $x = y$;
(ii) $F(x, y; t) = F(y, x; t)$;
(iii) $F(x, y; 0) = 0$;
(iv) if $F(x, y; t_1) = F(y, z; t_2) = 1$ then $F(x, z; \max\{t_1, t_2\}) = 1$ for all $x, y, z \in X$.

Definition 2.2. [4] A t-norm is a function $\Delta : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ which is associative, commutative, non-decreasing in each coordinate and $\Delta\,(a, 1) = a$ for all $a \in [0, 1]$.

Definition 2.3. [1],[9] A non-Archimedean Menger PM-space is an ordered triplet (X, F, Δ) where Δ is a t-norm and (X, F) is a N.A. PM-space satisfying the following condition: $F(x, z; \max\{t_1, t_2\}) \geq \Delta(F(x, y; t_1),\ F(y, z; t_2))$ for all $x, y, z \in X, t_1, t_2 \geq 0$. For details of topological preliminaries on non-Archimedean Menger PM-spaces, we refer to Cho et al.[15].

Definition 2.4. [8],[15] An N.A. Menger PM-space (X, F, Δ) is said to be of type $(C)_g$ if there exists a $g \in \Omega$ such that
$g(F(x, z; t)) \leq g(F(x, y; t)) + g(F(y, z; t))$ for all $x, y, z \in X, t \geq 0$, where $\Omega = \{g | g : [0, 1] \longrightarrow [0, \infty)$ is continuous, strictly decreasing with $g(1) = 0$ and $g(0) < \infty$.

Definition 2.5. [8],[15] A N.A. Menger PM-space (X, F, Δ) is said to be of type $(D)_g$ if there exists a, $g \in \Omega$ such that $g(\Delta(t_1, t_2)) \leq g(t_1) + g(t_2)$ for all $t_1, t_2 \in [0, 1]$.

Remark 2.6. [8],[15] (i) If N.A. Menger PM-space is of type $(D)_g$then $(X, F, \Delta)$is of type $(C)_g$.
(ii) If (X, F, Δ) is N.A. Menger PM-space and $\Delta \geq \Delta(r, s) = \max(r + s - 1, 1)$ then (X, F, Δ) is of type $(D)_g$ for $g \in \Omega$ and $g(t) = 1 - t$.

Throughout this paper (X, F, Δ) is a complete N.A. Menger PM-space with a continuous strictly increasing t-norm Δ. Let $\emptyset : [0, \infty) \longrightarrow [0, \infty)$ be a function satisfying the condition (Φ); (Φ) (ϕ) is upper semi-continuous from the right and $\phi(t) < t$ for $t > 0$.

Definition 2.7. [8],[15] A sequence $\{x_n\}$ in the N.A. Menger PM-space (X, F, Δ) converges to x if and only if for each $\epsilon > 0, \lambda > 0$ there exists $M(\epsilon, \lambda)$ such that $g(F(x_n, x; \epsilon)) < g(1 - \lambda)$ for all $n > M$.

Definition 2.8. [15] A sequence x_n in the N.A. Menger PM-space is a Cauchy sequence if and only if for each$\epsilon > 0, \lambda > 0$ there exists $M(\epsilon, \lambda)$ such that $g(F(x_n, x_{n+p}; \epsilon)) < g(1 - \lambda)$ for all $n > M$ and $p \geq 1$.

Example 2.9. [15] Let X be any set with at least two elements. If we define $F(x,x;t) = 1$ for all $x \in X, t > 0$ and

$$F\left(x,y;t\right) = \begin{cases} 0\; if\; t \leq 1 \\ 1\; if\; t > 1 \end{cases}$$

where $x, y \in X, x = y$, then (X, F, Δ) is the N.A. Menger PM-space with $\Delta(a,b) = \min(a,b) or (a.b)$.

Proof: condition (i),(ii) and (iii) are trivial. Let us go for condition (iv) Suppose that $F(x,y;t_1) = 1 = F(y,z;t_2), x \neq y, y \neq z$ then $t_1, t_2 > 1$ implies $\max(t_1,t_2) > 1 \Rightarrow F(x,z;\max(t_1,t_2) = 1, x \neq z$. Also, Menger inequality $F(x,z;\max(t_1,t_2)) \geq \triangle(F(x,y;t_1), \triangle F(x,y;t_1))$ is obvious.Thus $(X,F;\triangle)$ is an N.A. Menger space.

Example 2.10. [15] Let $X = R$ be the set of real numbers equipped with metric defined as $d(x,y) = |x-y|$ Set

$$F\left(x,\ y;\ t\right) = \frac{t}{t + d\left(x,y\right)}$$

Then (X, F, Δ) is a N.A. Menger PM-space with Δ as continuous t-norm Satisfying $\Delta(r,s) = min(r,s) or (r,s)$.

Lemma 2.11. [15] *If a function $\phi : [0,\infty) \longrightarrow [0,\infty)$ satisfies the condition (Φ)then we*
(i) For all $t \geq 0, \lim_{\mathrm{n}\longrightarrow\infty} \phi^{\mathrm{n}}(t) = 0$, where $\phi^{\mathrm{n}}\left(t\right)$ is the nth iteration of $\phi(t)$.
(ii) If $\{t_n\}$ is a non decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)\ n = 1,2,\ldots$ then $\lim_{\mathrm{n}\longrightarrow\infty} t_n = 0$. In particular, if $t \leq \phi(t)$, for each $t \geq 0$, than $t = 0$.

Lemma 2.12. [15] *Let $\{y_n\}$ be a sequence in X such that $lim_{n\longrightarrow\infty} F(y_n, y_{n+1}; t) = 1$ for each $t > 0$. If the sequence $\{y_n\}$ is not a Cauchy sequence in X, then there exists $\in_0 > 0, t_0 > 0$, and two sequences $\{m_i\}$ and $\{n_i\}$ of positive inte-gers such that*
(i) $m_i \geq n_{i+\ 1}$ and $n_i \longrightarrow \infty$ as $i \longrightarrow \infty$
(ii) $F(y_{mi}, y_{ni};\ t_0\ ) < 1 - \in_0$ and $F\left(y_{m_i-1}, y_{ni};\ t_0\ \right) \geq 1 - \in_0, i = 1,2,\ldots$

Definition 2.13. [5] Two maps A and S of a Non-Archimedean Menger PM space (X, F, Δ) into itself are said to be R-weakly commuting if there exists some $R > 0$ such that $g(F(ASx, SAx;t) \leq g(F(ASx, SAx; t/R)$ for every $x \in X, t > 0$.

Theorem 2.14. *Let $(X,\ F, *)$ be a complete fuzzy metric space and let f and g be R-weakly commuting self mappings of X satisfying the condition: $M\left(fx, fy, t\right) \geq r.M(gx, gy, t)$ where $r : [0,1] \longrightarrow [0,1]$ is a continuous function such that $r\left(t\right) > t$ for each $0 \leq t < 1$ and $r\left(1\right) = 1$ and the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\{x_n\} \longrightarrow x, \{y_n\} \longrightarrow y$ implies $M(x_n,\, y_n, t) \longrightarrow M(x,\, y, t)$. If the range of g contains the range of f and either f or g is continuous, then f and g have a unique common fixed point.*

3. MAIN RESULTS

Theorem 3.1. *Let S and T be a complete N. A. Menger PM-space $(X, F, \triangle)$. Let $\{R_n\}_{n=1}^{\infty}$ be a family of self mappings satisfying:*

(i) *$R_i(X) \subseteq T(X), R_j(X) \subseteq S(X)$ and the pair $\{R_i, S\}$ and $\{R_j, T\}$ are point wise R-weakly commuting;*

(ii)

$$g(F(R_ix,R_jy;t)) \leq \phi[\max\{g(F(Sx,Ty;t)),g(F(Sx,R_ix;t)),g(F(Ty,R_jy;t)),$$
$$\frac{1}{2}(g(F(Sx,R_jy;t))+g(F(Ty,R_ix;t))),$$
$$\min\{g(F(R_jy,Ty;t)),g(F(R_ix,Sx;t))\},$$
$$\sqrt{g(F(Ty,R_jy;t)).g(F(Ty,R_ix;t))},$$
$$\frac{g(F(Sx,R_jy;t)).g(F(Ty,R_jy;t))}{g(F(Sx,Ty;t))}\}]$$

for every $x,\ y \in X,\ i = 2n-1,\ j = 2n,\ (n \in N)$ *and* $i \neq j$, *where* ϕ *satisfies the condition* (Φ). *Then* $\{R_n\}$,S *and* T *have a unique common fixed point in* X.

Proof:- Since $R_i(X) \subseteq T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $R_1(x_0) = Tx_1$.
Since $R_j(X) \subseteq S(X)$, for this x_1 we can choose a point $x_2 \in X$ such that $R_2(x_1) = Sx_2$ and so on. Inductively, We can define a sequence $\{y_n\}$in X,

$$y_{2n} = R_{2n+1}(x_{2n}) = T_{x_{2n+1}}, y_{2n-1} = R_{2n}(x_{2n-1}) = Sx_{2n} \qquad n = 1,2,3... \tag{3.1}$$

Let $M_n = g(F(R_{2n+1}(x_n), R_{2n}(x_{n-1});t)) = g(f(y_n,y_{n-1};t))$ $n = 1,2,3,...$ then

$$M_{2n} = g(F(R_{2n+1}(x_{2n}),R_{2n}(x_{2n-1});t))$$
$$\leq \phi[max\{g(F(Sx_{2n},Tx_{2n-1};t)),g(F(Sx_{2n},R_{2n+1}(x_{2n});t)),$$
$$g(F(Tx_{2n-1},R_{2n}(x_{2n-1});t)),$$
$$\frac{1}{2}(g(F(Sx_{2n},R_{2n}(x_{2n-1});t))+g(F(Tx_{2n-1},R_{2n+1}(x_{2n});t))),$$
$$\min\{g(F(R_{2n}(x_{2n-1}),Tx_{2n-1};t)),g(F(R_{2n+1}(x_{2n}),Sx_{2n};t))\},$$
$$\sqrt{g(F(Tx_{2n-1},R_{2n}(x_{2n-1});t)).g(F(Tx_{2n-1},R_{2n+1}(x_{2n});t))},$$
$$\frac{g(F(R_{2n}(x_{2n-1}),Sx_{2n};t)).g(F(Tx_{2n-1},R_{2n}(x_{2n-1});t))}{g(F(Sx_{2n},Tx_{2n-1};t)))}\}]$$

$$M_{2n} = \phi[max\{g(F(y_{2n-1},y_{2n-2};t)),g(F(y_{2n-1},y_{2n};t)),g(F(y_{2n-2},y_{2n-1};t)),$$
$$\frac{1}{2}(g(F(y_{2n-1},y_{2n-1};t))+g(F(y_{2n-2},y_{2n};t))),$$
$$\min\{g(F(y_{2n-1},y_{2n-2};t)),g(F(y_{2n},y_{2n-1};t))\},$$
$$\sqrt{g(F(y_{2n-1},y_{2n-2};t)).g(F(y_{2n-1},y_{2n-2};t))},$$
$$\frac{g(F(y_{2n-1},y_{2n-1};t)).g(F(y_{2n-1},y_{2n-2};t))}{g(F(y_{2n-1},y_{2n-2};t))}\}]$$

i.e.

$$M_{2n} \leq \phi[Max\{M_{2n-1},M_{2n},M_{2n-1},\frac{1}{2}(M_{2n-1}+M_{2n}),\min\{M_{2n-1},M_{2n}\},$$
$$\sqrt{(M_{2n-1})^2},g(1)\}] \tag{3.2}$$

Case I : If $M_{2n} > M_{2n-1}$then by (3.2) $M_{2n} \leq \phi(M_{2n})$Which is contradiction.
Case II : If $M_{2n-1} > M_{2n}$ then by (3.2) gives $M_{2n} \leq \phi M_{2n-1}$
Then by lemma (2.12) we get $lim_{n\longrightarrow\infty}M_{2n} = 0$ i.e.
$lim_{n\longrightarrow\infty}g(F(Rx_{2n+1}(x_{2n}),R_{2n}(x_{2n-1});t)) = 0$ or
$lim_{n\longrightarrow\infty}g(F(y_{2n},y_{2n-1};t)) = 0$ Similarly, we can show that
$\lim_{n\longrightarrow\infty}g(F(R_{2n}(x_{2n+1}),R_{2n+1}(x_{2n+2});t)) = 0$ or

$\lim_{n\longrightarrow\infty} g(F(y_{2n+1}, y_{2n+2}; t)) = 0$ Thus we have
$\lim_{n\longrightarrow\infty} g(F(R_{2n+1}(x_n), R_{2n}(x_{2n+1}); t)) = 0$ For all $t > 0$ or

$$\lim_{n\longrightarrow\infty} g(F(y_n, y_{n+1}; t)) = 0 \tag{3.3}$$

Before preceding the proof of the theorem, we first prove the following claim
Claim: Let $\{R_n\}_{n=1}^{\infty}$,S and $T : X \longrightarrow X$ be maps satisfying equations(3.1), (3.2), (3.3) and $\{y_n\}$ defined by (3.1) such that

$$\lim_{n\longrightarrow\infty} g(F(y_n, y_{n+1}; t)) = 0 \tag{3.4}$$

for all n, is a Cauchy sequence.
Proof of the Claim :- Since $g \in \Omega$, it follows that $\lim_{n\longrightarrow\infty} F(y_n, y_{n+1}) = 1$ for each $t > 0$ iff $\lim_{n\longrightarrow\infty} g(f(y_n, y_{n+1}; t)) = 1$ for each $t > 0$. By Lemma (2.12) If $\{y_n\}$ is not a Cauchy sequence In X, there exists $\in_0 > 0, t_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that:
 (a) $m_i > n_i + 1$ and $n_i \longrightarrow \infty$ as $i \longrightarrow \infty$
 (b) $g(F(y_{m_i}, y_{n_i}; t_0) > g(1- \in_0)$ and

$$g(F(y_{m_i-1}, y_{n_i}; t_0) \leq (1- \in_0), i = 1, 2, ...$$

Since $g(t) = 1 - t$, we have

$$g(1- \in_0) < g(f(y_{m_i}, y_{n_i}; t_0))$$

$$\begin{aligned} g(1- \in_0) &\leq g(F(y_{m_i}, y_{m_i-1}; t_0)) + g(F(y_{m_i-1}, y_{n_i}; t_0)) \\ g(1- \in_0) &\leq g(F(y_{m_i}, y_{m_i-1}; t_o)) + g(1- \in_0) \end{aligned} \tag{3.5}$$

As $i \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} g(F(y_{m_i}, y_{n_i}; t_0)) = g(1- \in_0) \tag{3.6}$$

on the other hand , we have

$$\begin{aligned} g(1- \in_0) &< g(f(y_{m_i}, y_{n_i}; t_0)) \\ g(1- \in_0) &\leq g(F(y_{n_i}, y_{n_i+1}; t_0)) + g(F(y_{m_i}, y_{n_i+1}; t_0)) \end{aligned} \tag{3.7}$$

Now consider $g(F(y_{m_i}, y_{n_i+1}; t_0)$ in (3.7) and assume that both m_i and n_i are even. Then by (ii) of Theorem 3.1, we have

$$\begin{aligned} g(F(y_{m_i}, y_{n_i+1}; t_0)) \quad = \quad & g(F(R_{2n+1}(x_{mi}), R_{2n}(x_{n_i+1}); t_0)) \\ & \leq \phi[max\{g(F(Sx_{m_i}, Tx_{n_i+1}; t_0)), g(F(Sx_{m_i}, R_{2n+1}(x_{m_i}); t_0)), \\ & g(F(Tx_{n_i+1}, R_{2n}(x_{n_i+1}); t_0)), \\ & \frac{1}{2}(g(F(Sx_{m_i}, R_{2n}(x_{n_i+1}); t_0)) + g(F(Tx_{n_i+1}, R_{2n+1}(x_{m_i}); t_0))), \\ & \min\{g(F(R_{2n}(x_{n_i+1}), Tx_{n_i+1}; t_0)), g(F(R_{2n+1}(x_{m_i}), Sx_{m_i}; t_0))\}, \\ & \sqrt{g(FR_{2n}(x_{n_i+1}), Tx_{n_i+1}; t_0)).g(F(R_{2n}(x_{n_i+1}), Tx_{n_i+1}; t_0))}, \\ & \frac{g(F(R_{2n}(x_{n_i+1}), Sx_{m_i}; t_0)).g(F(R_{2n}(x_{n_i+1}), Tx_{n_i+1}; t_0))}{g(F(Sx_{m_i}, Tx_{n_i+1}; t_0))}\}] \end{aligned}$$

$$\begin{aligned} \leq \quad & \phi[\max\{g(F(y_{m_i-1}, y_{n_i}; t_0)), g(F(y_{m_i-1}, y_{m_i}; t_o)), \\ & g(f(y_{n_i}, y_{n_i+1}; t_0)), \\ & \frac{1}{2}(g(F(y_{m_i-1}, y_{n_i+1}; t_0)) + g(F(y_{n_i}, y_{m_i}; t_0)))\}, \\ & \min\{g(F(y_{n_i+1}, y_{n_i}; t_0)), g(F(y_{m_i-1}, y_{m_i}; t_o))), \\ & \sqrt{g(F(y_{n_i+1}, y_{n_i}; t_0)), g(F(y_{n_i}, y_{n_i+1}; t_o))}, \end{aligned}$$

$$\frac{g(F(y_{n_i+1}, y_{m_i-1}; t_0)), g(F(y_{n_i+1}, y_{n_i}; t_o))}{g(f(y_{n_i}, y_{m_i-1}; t_o))}\}]$$

Which on letting $n \longrightarrow \infty$, reduces to

$$g(1-\in_0) \le \phi[max\{g(1-\in_0), 0, 0, g(1-\in_0), \min\{0,0\}, 0, 0\}]$$
$$g(1-\in_0) \le \phi g(1-\in_0)$$

which is contradiction. Hence the sequence $\{y_n\}$ defined by (3.1) is a cauchy sequence, which concludes the proof of the claim.

Since X is complete, then the sequence $\{y_n\}$ converges to a point z in X and so the Sub-sequences $\lim_{n\longrightarrow\infty} R_{2n+1}(x_{2n})$, $\lim_{n\longrightarrow\infty} R_{2n}(x_{2n+1})$, $\lim_{n\longrightarrow\infty} Sx_{2n}$ and $\lim_{x\longrightarrow\infty} Tx_{2n+1}$ of seq. $\{y_n\}$ also Converge to the limit z.
Since the pair (R_i, S) are R-weakly commuting, So by definition (2.13)

$$g(F(R_iSx_{2n+1}, SR_ix_{2n+1}; t)) \le g(F(R_ix_{2n+1}, Sx_{2n+1}; t/R))$$

which gives

$$\lim_{n\longrightarrow\infty} R_iSx_{2n+1} = \lim_{n\longrightarrow\infty} SR_ix_{2n+1} = Sz \text{ as } S \text{ is continuous}$$

Implies $\lim_{n\longrightarrow\infty} R_iSx_{2n+1} = Sz$ and $\lim_{n\longrightarrow\infty} SR_ix_{2n+1} = Sz$.
Now we claim that $Sz = z$. Contrary suppose contrary that $Sz \neq z$ then by (ii) of Theorem (3.1)

$$\begin{aligned} g(F(R_iSx_{2n+1}, R_jx_{2n}; t)) \le& \phi[max\{g(F(SSx_{2n+1}, Tx_{2n}; t)), \\ & g(F(SSx_{2n+1}, R_ix_{2n+1}; t)), g(F(Tx_{2n}, R_jx_{2n}; t)), \\ & \frac{1}{2}(g(F(SSx_{2n+1}; R_jx_{2n}; t)) + g(F(Tx_{2n}, R_iSx_{2n+1}; t))), \\ & \min\{g(F(R_jx_{2n}, Tx_{2n}; t)), g(F(R_iSx_{2n+1}, SSx_{2n+1}; t))\}, \\ & \sqrt{g(F(R_jx_{2n}, Tx_{2n}; t)).g(F(Tx_{2n}, R_jx_{2n+1}; t))}, \\ & \frac{g(F(R_jx_{2n}, SSx_{2n+1}; t)).g(F(Tx_{2n}, R_jx_{2n}; t))}{g(F(SSx_{2n+1}, Tx_{2n}; t))}\}] \end{aligned}$$

Which on letting limit $n \longrightarrow \infty$

$$\begin{aligned} g(F(Sz, z; t)) \le& \phi[max\{g(F(Sz, z; t)), g(F(Sz, z; t))g(F(z, z; t)), \\ & \frac{1}{2}(g(F(Sz, z; t)) + g(F(z, Sz; t))), \\ & \min\{g(F(z, z; t)), g(F(Sz, Sz; t))\}, \\ & \sqrt{g(F(z, z; t)).g(F(z, z; t))}, \\ & \frac{g(F(z, Sz; t))g(F(z, z; t))}{g(F(Sz, z; t))}\}] \\ =& \phi[max\{g(F(Sz, z; t)), g(F(Sz, z; t)), g(1), g(F(Sz, z, t)), \\ & g(1), g(1), g(1)\}] \\ g(F(Sz, z; t)) \le& \phi(g(F(Sz, z; t))) < g(F(Sz, z; t)) \end{aligned}$$

Thus z is a fixed point of S Similarly we can show that z is a fixed point of R_i

Again pair (R_j, T) is R-weakly commuting so by definition of (2.13)

$$g(F(R_jTx_{2n+1}, TR_jx_{2n+1}; t)) \le g(F(R_jx_{2n+1}, Tx_{2n+1}; t/R))$$

which gives

$$\lim_{n\longrightarrow\infty} R_jTx_{2n+1} = \lim_{n\longrightarrow\infty} TR_jx_{2n+1} = Tz \text{ as } T \text{ is continuous}$$

Implies $\lim_{n\longrightarrow\infty} R_jTx_{2n+1} = Tz$ and $\lim_{n\longrightarrow\infty} TR_jx_{2n+1} = Tz$.
We have to show that $Tz = z$, to do this contrary suppose that $Tz \neq z$ then by (ii) of Theorem (3.1)

$$\begin{aligned} g(R_iz, R_jTx_{2n}; t) \leq \phi[max\{& g(F(Sz, TTx_{2n}; t)), g(F(Sz, R_iz; t)),\\ & g(F(TTx_{2n}, R_j(Tx_{2n}); t)),\\ & \frac{1}{2}(g(F(Sz, R_jTx_{2n}; t)) + g(F(TTx_{2n}, R_iz; t)))\\ & \min\{g(F(R_j(Tx_{2n}, TTx_{2n}; t))), g(F(R_iz, Sz; t))\}\\ & \sqrt{g(F(R_j(Tx_{2n}), TTx_{2n}; t)).g(F(R_j(Tx_{2n}), TTx_{2n}; t))},\\ & \frac{g(F(R_j(Tx_{2n}, Sz; t))).g(F(R_j(Tx_{2n}), TTx_{2n}; t))}{g(F(Sz, , TTx_{2n}; t))}\}] \end{aligned}$$

which on letting limit $n \longrightarrow \infty$

$$\begin{aligned} g(F(Rz, Tz; t)) \leq & \phi[max\{g(F(z, Tz; t)), g(F(z, z; t)), g(F(Tz, Tz; t)),\\ & \frac{1}{2}g(F(z, Tz; t)) + g(F(Tz, z; t)),\\ & \min\{g(F(Tz, Tz; t)), g(F(z, z; t))\},\\ & \sqrt{(g(F(Tz, Tz; t)))^2},\\ & \frac{g(F(Tz, z; t)).g(F(Tz, Tz; t))}{g(F(z, Tz; t))}\}] \end{aligned}$$

i.e. $g(F(z, Tz; t) \leq \phi(g(F(z, Tz; t)) < g(F(z, Tz; t))$.
Which is a contradiction, Thus z is a fixed point of T. Similarly we can show that z is a fixed point of R_j. Hence $R_iz = R_jz = Sz = Tz = z$. Thus z is a common fixed point of R_i,R_j,S & T. The uniqueness of the common fixed point follows from inequality (ii) of Theorem (3.1).

In the paper Khan & Sumitra [5], obtained a common fixed point theorem in 2 non Archimedean Menger spaces for R-weakly commuting maps inspired by this result we motivate to prove more generalized version in the setting of non-Archimedean Menger PM spaces.

Corollary 3.2. *Let R_1, R_2, S and T be four continuous self maps of a complete N.A Menger PM spaces (X, F, Δ) ,satisfying*
(i) $R_1(X) \subseteq T(X), R_2(X) \subseteq S(X)$, and $\{R_1, S\}$ and $\{R_2, S\}$ are R-weakly Commuting
(ii)

$$\begin{aligned} g(F(R_1(x), R_2(y); t)) \leq & \phi[max\{g(F(Sx, Ty; t)), g(F(Sx, R_1x; t)), g(F(Ty, R_2y; t))\\ & \frac{1}{2}(g(F(Sx, R_2y; t)) + g(F(Ty, R_1x; t))),\\ & \min\{g(F(R_2y, Ty, t)), g(F(R_1x, Sx, t))\},\\ & \sqrt{g(F(R_2y, Sy; t)).g(F(R_2y, Ty; t))},\\ & \frac{g(F(R_2y, Sx; t)).g(F(R_2y, Ty; t))}{g(F(Sx, Ty; t))}\}] \end{aligned}$$

for every $x, y \in X$, where ϕ satisfies the condition (Φ) . Then R_1, R_2, S and T have a unique common fixed point in X.

Corollary 3.3. *Let R, S, T be three continuous self maps of a complete N.A Menger PM- spaces (X, F, Δ) satisfies;*
(i) $R(x) \subseteq S(x) \cap T(x)$, and pair $\{R, S\}$ and $\{R, T\}$ are R-weakly commuting
(ii)

$$\begin{aligned} g(F(Rx, Ry; t)) \leq & \phi[max\{g(F(Sx, Ty; t)), g(F(Sx, Rx; t)), g(F(Ty, Ry; t)), \\ & \frac{1}{2}(g(F(Sx, Ry; t)) + g(F(Ty, Rx; t))), \\ & \min\{g(F(Ry, Ty, t)), g(F(Rx, Sx, t))\}, \\ & \sqrt{g(F(Ry, Sy; t)).g(F(Ry, Ty; t))}, \\ & \frac{g(F(Ry, Sx; t)).g(F(Ry, Ty; t))}{g(F(Sx, Ty; t))}\}] \end{aligned}$$

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then R, S and T have a unique common fixed point in X.

Corollary 3.4. *Let R, S be two continuous self maps of a complete N.A Menger PM space (X, F, Δ) satisfying;*
(i) $R(X) \subseteq S(X)$ and the pair $\{R, S\}$ is R-weakly commuting
(ii)

$$\begin{aligned} g(F(Rx, Ry; t)) \leq & \phi[man\{g(F(Sx, Sy; t)), g(F(Sx, Rx; t)), g(F(Sy, Ry; T)), \\ & \frac{1}{2}(g(F(Sx, Ry; t)) + g(F(Sy, Rx; t))), \\ & \min\{g(F(Ry, Sy, t)), g(F(Rx, Sx, t))\}, \\ & \sqrt{g(F(Ry, Sy; t)).g(F(Ry, Sy; t))}, \\ & \frac{g(F(Ry, Sx; t)).g(F(Ry, Sy; t))}{g(F(Sx, Sy; t))}\}] \end{aligned}$$

for every $x, y \in X$, Where ϕ satisfies the condition (Φ). Then $R$and S have a unique common fixed point in X.

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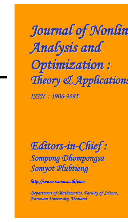

A NONLINEAR INTEGER PROGRAMMING FORMULATION FOR THE AIRLIFT LOADING PROBLEM WITH INSUFFICIENT AIRCRAFT

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ABSTRACT. The Airlift Loading Problem with Insufficient Aircraft (ALPIA) is frequently faced by members of the United States Department of Defense when conducting airlift missions. The ALPIA is a combination of knapsack, assignment, and packing problems; items are *selected* for shipment based on a utility measure then *assigned* to pallets which will be loaded into an aircraft in a specific pallet position. These pallets are then *packed* in a manner to optimize both the pallet and aircraft characteristics, such as item utility, aircraft and pallet utilization, pallet center of gravity, aircraft center of balance, etc. Since not all items have the same destination, it is necessary to perform the packing in an intelligent fashion to ensure ease of unpacking at a destination. This paper formulates the ALPIA as an integer programming problem which allows items to be stably packed onto pallets with any specified orientation (i.e. accounting for “this side up” constraints). Rather than addressing the knapsack, assignment and packing problems separately in a hierarchical manner, this formulation simultaneously accounts for each of these problems.

KEYWORDS : Nonlinear Programming, Integer Programming, Knapsack, Bin-Packing.
AMS Subject Classification : 90C30

1. INTRODUCTION AND LITERATURE REVIEW

The airlift loading problem (ALP) was first defined by Chocolaad [2] as a knapsack problem and redefined by Roesener, et al. [8] as a bin-packing problem. This airlift process involves: (1) *packing* cargo items onto pallets, (2) *partitioning* the set of packed pallets into aircraft loads, (3) *selecting* a set of aircraft from a pool of aircraft, and (4) *placing* the cargo in the best available positions within the aircraft. There are very strict differences between the ALP and other packing problems. In addition to the normal spatial packing constraints, factors such as weight, center of balance, and temporal restrictions on cargo loading availability and cargo delivery requirements must be considered while solving the ALP [8].

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The two common problems used to model the ALP are knapsack and bin-packing problems. The single knapsack problem was proven to be a combinatorial NP-Hard problem by Karp [5]. It involves the selection of items from an available set of n items each with weight w_i and utility u_i, i =1,2,. . . ,n, to be packed in a container with a total weight capacity of b (i.e. a single constraint). The overall goal is to maximize the combined utility of the items placed into the container. The multidimensional knapsack problem extends the single dimensional knapsack problem by allowing more than one constraint in the problem.

The bin packing problem has also been proven to be a combinatorial NP-Hard problem [5]. It is defined as the placement of objects into a given number of bins of limited capacity in a way that minimizes the number of bins required. There are different types of bin packing problems named according to descriptions of the items to be packed. In this research effort, only two-dimensional (2D) and three-dimensional (3D) bin packing problems of orthogonal items are considered due to their relationship to the ALP. Numerous exact, heuristic, and meta-heuristic solution methods for 2D bin packing problems exist [3, 6]. In 2006, Harwig et al. [4] used tabu search to solve 2D bin packing problems using two dimensional packing, achieving excellent results on a well-known problem set. Nance, et al. [7] and Roesener, et al. [8] defined and solved special cases of aircraft loading problems as 2D bin packing problems. The 3D bin packing problem is an extension of the 2D bin packing problem in which an additional dimension is added to the problem. Pallet packing and container loading are common applications of 3D bin packing problems for which solution methods exist.

2. PROBLEM DESCRIPTION

This research focuses on a problem called the Aircraft Loading Problem with Insufficient Aircraft (ALPIA). In general, the ALPIA includes *selecting* cargo items to be transported, *packing* the items onto pallets, *partitioning* the pallets into aircraft sized loads, and *assigning* the pallets to specific positions within the available aircraft. These sub-problems are described in more detail below.

1. *Selecting Cargo Items*: For regularly scheduled missions, the cargo items to be transported could surpass the amount of available space within the airlift aircraft. Thus, some of the items will remain at the aerial port of embarkation (APOE). In order to carry the maximum amount of cargo items and ensure that the most important items are transported, a special evaluation or utility for each cargo item is needed. The goal of this sub-problem involves maximization of *both* the number of items and the utility associated with those items.
2. *Packing Items onto Pallets*: For some airlift missions (including deployments), the deploying command packs the pallets prior to the aircraft arrival. In other airlift missions, members of an aerial-port squadron are responsible for pallet packing. Unless the pallet is properly balanced (i.e. the pallet center of gravity (CG) is approximately in the geometric center of the pallet), the safety of the ground handlers and aircrew could be in jeopardy. This sub-problem considers how to best pack selected items onto a pallet while ensuring safety requirements (proper CG position, heavier items on lower levels, etc.).
3. *Partitioning Packed Pallets*: For multi-aircraft/multi-destination missions, pallets that have the same destinations should be partitioned and assigned

to the same aircraft. This sub-problem does not occur in single aircraft operations.

4. *Assigning Pallets to Aircraft*: This sub-problem involves assigning the pallets to specific pallet stations inside the aircraft while reducing the number of aircraft required. For this sub-problem, the aircraft's center of balance (CB), or the point within the aircraft where the cargo load is balanced, and any regional constraints associated with a specific pallet position in an aircraft (i.e. height and weight of packed pallet) must still be satisfied.

Additionally, each of the ALPIA sub-problems is restricted to the same aircraft constraints which affect the ALP. These constraints were defined by Roesener, et al. [9]; they are modified for this problem formulation and presented in subsequent sections.

1. *Aircraft CB*
2. *Operational Allowable Cabin Load (ACL)*
3. *Pallet Position Restrictions*
4. *Available Space for Loading*
5. *Route of Flight and Item Destination*
6. *Pallet CG*

Although the sub-problems of the ALPIA can be viewed separately, they are not independent. Rather, changes in one sub-problem can have a dramatic impact on the feasibility and/or optimality of another sub-problem. With this in mind, it is best to formulate and solve the ALPIA in its entirety, rather than sequentially solving the sub-problems.

3. ALPIA FORMULATION

To enhance understanding, initially only constraints for a single pallet will be presented. These constraints with an associated objective function were previously detailed by Roesener, et al. [9]. After the single pallet constraints are detailed, multi-pallet, multi-destination and multi-aircraft packing formulations will be presented, respectively. Lastly, the objective function will be explained.

3.1. **ALPIA Set Notation and Variable Description.** Before a valid formulation can be presented, the set notation used in the formulation must be adequately explained. Additionally, the variables associated with these sets must also be defined.

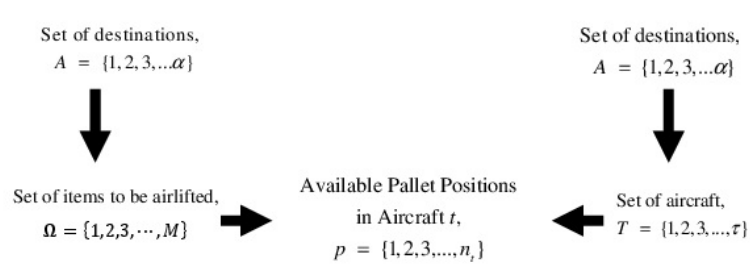

FIGURE 1. ALPIA Sets and their Relationships

There are three major data sets associated with the ALPIA: *Destinations*, *Cargo Items*, and *Available Aircraft*. The aircraft set has a subset, *Available Pallet Positions*, which varies among aircraft of different types. The overall goal of this research is to efficiently and feasibly place the items on the available pallets. The sets and their relationships are shown in Figure 1.

The constants and decision variables used in this formulation will be explained in the context of these sets. Some of the input data for the ALPIA are values that vary as a function of the decision variables. Although these values are not constant throughout the formulation, their values are not allowed to vary arbitrarily. Thus, they will be explained in the same section as the constants (i.e., maximum, minimum and optimal CB values). The constants and functions of decision variables are:

a. *Destination Set* (A)*:* This is the set of destinations for cargo items and aircraft.
   - Item ($a \in A$): This index refers to a destination, where a is a positive integer value (i.e., $a \in \{1,2,\ldots,\alpha = |A|\}$).

b. *Item Set* (Ω)*:* This is the set of cargo items that may be loaded. The following defines the parameters (i.e., utility, weight, and dimensions) associated with each item. Figure 2 presents a visual illustration of an element of this set.

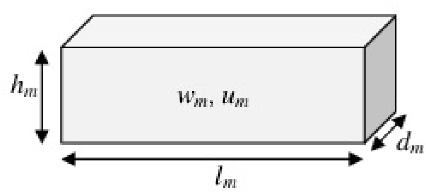

FIGURE 2. Physical Characteristics of Item m ($m \in \Omega$)

   - Item ($m \in \Omega$): This is an index which refers to the items to be transported, where m is a positive integer value (i.e., $m \in \{1,2,\ldots,M = |\Omega|\}$).
   - Destination of the item (I_{ma}, $m \in \Omega$, $a \in A$): This is a binary constant; $I_{ma} = 1$ if item $m \in \Omega$ has destination $a \in A$. Otherwise, $I_{ma} = 0$.
   - Utility of the item (u_m): This is a positive, integer-valued constant. It is the assigned utility of item m. This value accounts for the priority of the item and the usefulness that can be currently derived from transporting the item.
   - Weight (w_m): This is a real-valued constant. It is the actual weight of item m (in pounds).
   - Dimensions, (d_m, l_m, h_m): In this context, d_m is the depth, l_m is the length, and h_m is the height of item m (in inches). They are positive, real-valued constants.

   The CG of item $m \in \Omega$ (not of a packed pallet) is assumed to be in the geometric center of the item.

c. *Aircraft Set* (T)*:* This is the set of aircraft that are available for loading.
   - Available Aircraft ($t \in T$): This designates the index for an available aircraft; it is a positive integer value.

- Available Aircraft Routes ($R_t = \{a_1, \ldots, a_\alpha\}$, $t \in T, a \in A$): This is the set of ordered destinations that gives the route of aircraft t.
- Cargo Load (Ψ_t): This is the total weight of the packed items that can be placed on aircraft t.
- CB Limits for aircraft t ($CB_{t(max)}$, $CB_{t(min)}$, $CB_{t(ideal)}$): These are the maximum, minimum, and ideal values of the CB for aircraft t, respectively. These values are predetermined for each aircraft type, and they depend upon the total cargo weight assigned to the aircraft. The maximum and minimum CB values are constraints which cannot be violated without the aircraft departing from safe flight. The ideal CB, however, denotes the target CB value; it is the CB location (for a given cargo load) at which the aircraft exhibits the best fuel consumption rate. Each of these CB limits is a real-valued constant that is a function of the decision variables.
- Number of Pallet Positions, (n_t, $t \in T$): This is a positive, integer valued constant. It is the total number of the pallet positions inside the t^{th} aircraft.

d. *Pallet Positions Set* (P_t, $t \in T$)*:* This is the set of available pallet positions within aircraft t.

- Pallet Positions ($p \in P_t$, $t \in T$)*:* This is the index for the pallet positions for each aircraft in the aircraft set. Note that $|P_t| = n_t$, and that this is directly dependent upon the type and route of aircraft t.
- Assigned Arrival Point (I_{pa}, $p \in P_t, t \in T, a \in A$): This is a binary constant. A value of 1 indicates that the pallet p of aircraft t is assigned to the destination a.
- Fuselage station (b_p, $p \in P_t, t \in T$): This is a real-valued constant that refers to the distance from the aircraft's reference datum point to the center of pallet position p in aircraft t.
- Pallet Position Restrictions (W_p, H_p, D_p , L_p, $p \in P_t, t \in T$): These are constants that represent the weight, height, depth and length (respectively) restrictions for a packed pallet located in pallet position p in aircraft t.

In addition to the parameters, several decision variables which require detailed explanation are used in the formulation. Pixel based packing (i.e., the packing bins and items are partitioned into uniform unit pixels) was first used in a nonlinear 3D bin packing formulation by Ballew [1]; however, the advantages for implementing this type of approach was not adequately addressed by Ballew. In this research, ALPIA is formulated as a multi-constraint bipartite maximal matching problem. The objective is matching the maximum number of item unit pixels to the pallet pixels, which is equivalent to occupying the maximum amount of available space on the pallet.

As seen in Figure 3, items are placed on the pallets on specific grids. A similar idea was proposed by Ballew [1].

a. Coordinates inside the pallet: (i, j, k) denote the coordinates of a cubic grid made up of small "pixels" or grid cubes of unit volume. The volume of each grid cube depends upon the units of the cargo items to be packed. For example, if all items are measured in inches, then a 1 in^3 pixel would logically be used as a grid cube; if the items are measured in centimeters, then a 1 cm^3 pixel would be used. A smaller grid cube volume allows for higher fidelity in the packing procedure, but requires more computation

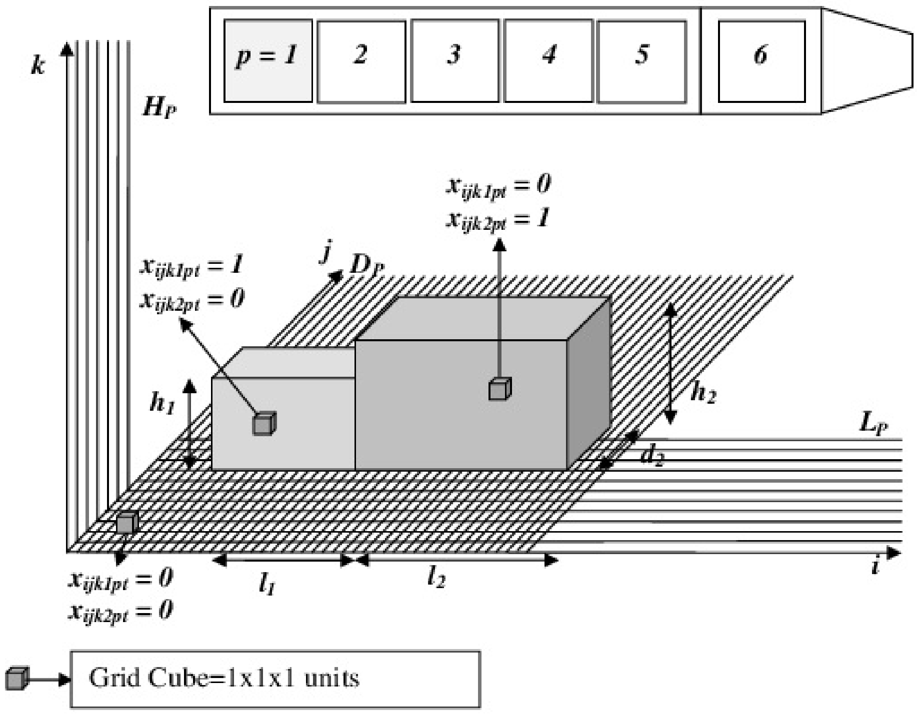

FIGURE 3. Required variables for packing an item m ($m \in \Omega$) on a pallet p ($p \in P_t$) based on grids or pixels

time. For this research, all items are measured in inches; thus, a grid cube volume of 1 in^3 is used.

b. Occupation of pixels (x_{ijkmpt}): This is a binary decision variable. A value of 1 indicates that the grid cube on the (*i*,*j*,*k*) coordinates of the pallet occupying the p^{th} pallet position in aircraft t is occupied by item m.
c. Item-Pallet Relation (X_{mpt}): This is a binary decision variable. A value of 1 indicates that item m is packed on the pallet occupying pallet position p in aircraft t. When considering a given pallet p in aircraft t, the decision variable is denoted by X_{mpt}.
d. Item-Pallet Orientation (y_{mptsz}): This is a binary decision variable. A value of 1 indicates that the z^{th} (z = 1,2,3) dimension of item m is parallel to the s^{th}(s = 1,2,3) dimension of the pallet occupying the p^{th} pallet position in aircraft t. It will be used to determine the orientation of a packed item with reference to the pallet. This variable allows for different orientations of items as well as ensuring any item with a "This side up" constraint is properly packed. Figure 4 provides a visual description of these decision variables.

Now that the sets, constants and variables have been adequately explained, the actual formulation for the ALPIA can be presented. First, the formulation of a single pallet, which was previously detailed by Roesener, et al. [9], will be explained. The single pallet formulation will then be expanded to encompass multiple pallets with a single destination on a single aircraft. This formulation will be further expanded

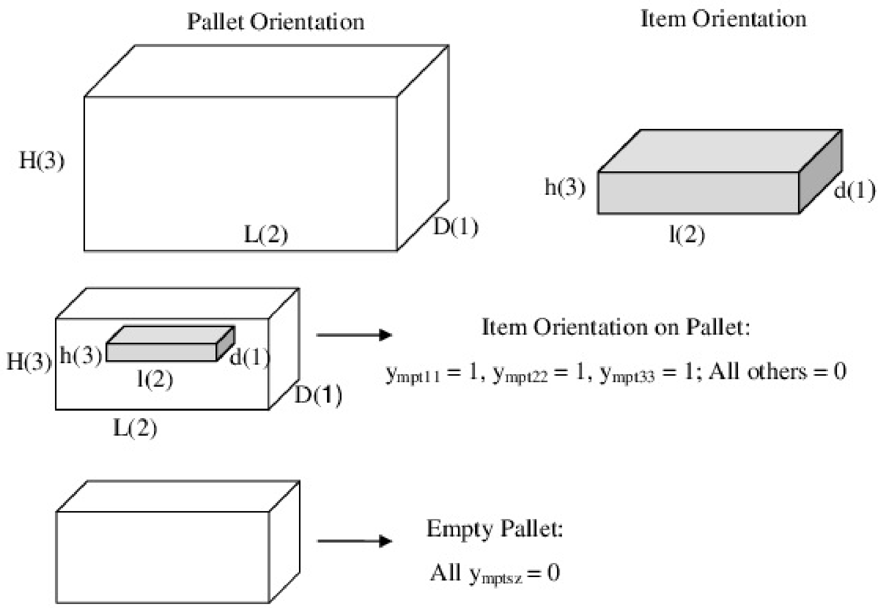

FIGURE 4. Pallet Orientation Variables

to allow for multiple destinations for items and pallets that are on a single aircraft. Finally, the formulation of a problem with multiple aircraft which have multiple destinations will be presented.

3.2. **Formulation of the Single Aircraft, Single Destination, Single Pallet Packing Problem.** For a single pallet formulation, the indices for the aircraft t and the pallet occupying the p^{th} pallet position will have a constant value. These subscripts will therefore be omitted in the mathematical presentation. The constraints necessary for this formulation are:

a. Overlap Constraint [1]: Multiple items cannot simultaneously occupy the same grid cube.

$$\sum_{m=1}^{M} x_{ijkm} \leq 1 \;\; (\forall\, i, j, k) \tag{3.1}$$

b. Stability Constraint [1]: Each occupied grid cube requires support; in other words, it must be placed upon another *occupied* grid cube or on the surface of the pallet.

$$\sum_{m=1}^{M} x_{ij(k+1)m} - \sum_{m=1}^{M} x_{ijkm} \leq 0 \;\; (\forall\, i, j, k) \tag{3.2}$$

c. Dimensional Constraints: The total number of grid cubes occupied by items along each depth, length and height dimension cannot exceed the pallet's limitation for depth (D), length (L) and height (H), respectively.

$$\sum_{i=1}^{D}\sum_{m=1}^{M} x_{ijkm} \leq D \ \ (\forall\ j,k) \tag{3.3}$$

$$\sum_{j=1}^{L}\sum_{m=1}^{M} x_{ijkm} \leq L \ \ (\forall\ i,k) \tag{3.4}$$

$$\sum_{k=1}^{H}\sum_{m=1}^{M} x_{ijkm} \leq H \ \ (\forall\ i,j) \tag{3.5}$$

d. Volume Constraint: The total volume of packed items on a pallet can be at most the allowable volume for that pallet, which is *D·L·H*. In other words, the total number of grid cubes occupied by all packed items cannot exceed the total number of available grid cubes.

$$\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}\sum_{m=1}^{M} x_{ijkm} \leq D \cdot L \cdot H \tag{3.6}$$

e. Weight Constraint: Total weight of packed items on a pallet cannot exceed the structural weight limitations (W) for the actual pallet or the pallet position within the aircraft (W represents the smallest of these weight limitations).

$$\sum_{m=1}^{M} w_m \cdot X_m \leq W \tag{3.7}$$

Figure 5 shows the 2-dimensional placement (i.e. the “footprint”) of two different items on a theoretical pallet or dimensions ten units by eight units. Observe that for each column and each row the total occupied pixels are different from each other. This is one of the reasons for using a grid-based formulation.

f. CG Constraints: A set of constraints are required to ensure the Pallet CG in the lateral (CG_{length}) and longitudinal (CG_{depth}) direction from the center of the pallet does not exceed a given amount. The ideal center of balance is in the center of the pallet, which is determined by $L/2$ and $D/2$ for the lateral and longitudinal dimensions, respectively. These values along with the CG for the vertical dimension must be calculated as a “soft” constraint which will negatively impact the objective function if it is violated (but not cause the problem to become infeasible).

$$\left[\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(i \cdot x_{ijkm})}{d_m \cdot l_m \cdot h_m} - \frac{D \cdot X_m}{2}\right) \cdot w_m - CG_{depth}\sum_{m=1}^{M} w_m \cdot X_m\right] \leq 0 \tag{3.8}$$

$$\left[-\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(i \cdot x_{ijkm})}{d_m \cdot l_m \cdot h_m} - \frac{D \cdot X_m}{2}\right) \cdot w_m + CG_{depth}\sum_{m=1}^{M} w_m \cdot X_m\right] \leq 0 \tag{3.9}$$

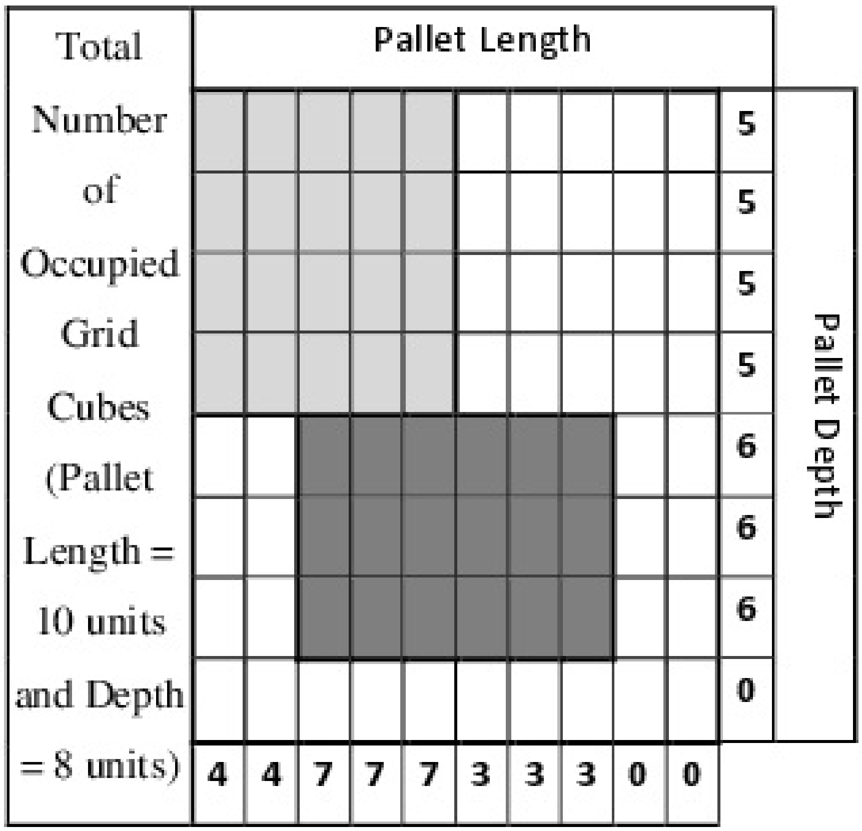

FIGURE 5. Example of different items ($m \in \Omega$) occupying grids on the pallet p ($p \in P_t$)

$$\left[\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(i \cdot x_{ijkm})}{d_m \cdot l_m \cdot h_m} - \frac{L \cdot X_m}{2}\right) \cdot w_m - CG_{length}\sum_{m=1}^{M} w_m \cdot X_m\right] \leq 0 \tag{3.10}$$

$$\left[-\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(i \cdot x_{ijkm})}{d_m \cdot l_m \cdot h_m} - \frac{L \cdot X_m}{2}\right) \cdot w_m + CG_{length}\sum_{m=1}^{M} w_m \cdot X_m\right] \leq 0 \tag{3.11}$$

g. Vertical CG Constraint: The pallet CG in the vertical dimension should be in the lower half of the pallet to prevent tipping. This constraint ensures that heavier items are placed under lighter items without imposing unnecessary constraints on the problem. The smaller values imply the vertical CG is below half the height ($H/2$) of the pallet and the majority of the heavier items are closer to the pallet surface.

$$\left[\sum_{m=1}^{M}\left(\frac{\sum_{i=1}^{D}\sum_{j=1}^{L}\sum_{k=1}^{H}(k\cdot x_{ijkm})}{d_m\cdot l_m\cdot h_m}-\frac{H\cdot X_m}{2}\right)\cdot w_m - \sum_{m=1}^{M} w_m\cdot X_m\right]\leq\frac{H}{2} \tag{3.12}$$

h. Item Integrity: A set of constraints are necessary to ensure that all occupied cube grids associated with a single item are contiguous. In other words, cargo items cannot be divided into small pieces and placed on the pallet. These constraints also ensure that the item's dimensions and volume coincide with the actual values for the item while allowing the item to be packed in different orientations. This non-linear formulation ensures the item integrity while allowing packing in different orientations.

$$\left[y_{m11}\left(d_m\sum_{i=1}^{D-d_m}\prod_{s=0}^{d_m-1}x_{(i+s)jkm}\right)+y_{m12}\left(l_m\sum_{i=1}^{D-l_m}\prod_{s=0}^{l_m-1}x_{(i+s)jkm}\right) +y_{m13}\left(h_m\sum_{i=1}^{D-h_m}\prod_{s=0}^{h_m-1}x_{(i+s)jkm}\right)\right]=\left(\sum_{i=1}^{D}x_{ijkm}\right)(\forall\ j,k,m) \tag{3.13}$$

$$\left[y_{m21}\left(d_m\sum_{j=1}^{L-d_m}\prod_{s=0}^{d_m-1}x_{i(j+s)km}\right)+y_{m22}\left(l_m\sum_{j=1}^{L-l_m}\prod_{s=0}^{l_m-1}x_{i(j+s)km}\right) +y_{m23}\left(h_m\sum_{j=1}^{L-h_m}\prod_{s=0}^{h_m-1}x_{i(j+s)km}\right)\right]=\left(\sum_{j=1}^{L}x_{ijkm}\right)(\forall\ i,k,m) \tag{3.14}$$

$$\left[y_{m31}\left(d_m\sum_{j=1}^{H-d_m}\prod_{s=0}^{d_m-1}x_{ij(k+s)m}\right)+y_{m32}\left(l_m\sum_{j=1}^{H-l_m}\prod_{s=0}^{l_m-1}x_{ij(k+s)m}\right) +y_{m33}\left(h_m\sum_{j=1}^{H-h_m}\prod_{s=0}^{h_m-1}x_{ij(k+s)m}\right)\right]=\left(\sum_{j=1}^{L}x_{ijkm}\right)(\forall\ i,j,m) \tag{3.15}$$

i. Item Orientation Constraint: The last set of constraints only applies to items which have a "this side up" constraint. These constraints force the height dimension of the item to be used in the vertical orientation.

$$y_{m11}+y_{m21}+y_{m31}\leq 1\ \ (\forall\ m) \tag{3.16}$$

$$y_{m12}+y_{m22}+y_{m32}\leq 1\ \ (\forall\ m) \tag{3.17}$$

$$y_{m13}+y_{m23}+y_{m33}\leq 1\ \ (\forall\ m) \tag{3.18}$$

$$y_{m11}+y_{m12}+y_{m13}\leq 1\ \ (\forall\ m) \tag{3.19}$$

$$y_{m21} + y_{m22} + y_{m23} \leq 1 \ \ (\forall\ m) \tag{3.20}$$

$$y_{m31} + y_{m32} + y_{m33} \leq 1 \ \ (\forall\ m) \tag{3.21}$$

$$y_{m33} = X_m \tag{3.22}$$

3.3. **Formulation of the Single Aircraft, Single Destination, Multi-Pallet Packing Problem.** For a multi-pallet ALPIA formulation in a single aircraft with a single destination, the previously defined constraints are used with an additional index ($p \in P$) to account for multiple pallet positions. The aircraft index will still remain constant and will therefore be omitted. Additional constraints are necessary to ensure that none of the items are placed on multiple pallets and to distinguish between pallet positions.

$$\sum_{p=1}^{n} X_{mp} \leq 1 \ \ (\forall\ m) \tag{3.23}$$

Another important constraint for flight safety is the aircraft CB. A constraint that assures that the aircraft CB is within the acceptable range is given by:

$$CB_{min} \leq \left(\frac{\sum_{p=1}^{n} \left(b_p \cdot \sum_{m=1}^{M} \left(w_m \cdot X_{mp} \right) \right)}{\sum_{m=1}^{M} w_m \cdot X_{mp}} \right) \leq CB_{max} \tag{3.24}$$

The last constraint for aircraft is the total ACL. The total weight of the cargo load cannot exceed the allowable cabin load for the aircraft.

$$\sum_{p=1}^{n} \sum_{m=1}^{M} \left(w_m \cdot X_{mp} \right) \leq \psi \tag{3.25}$$

3.4. **Formulation of the Single Aircraft, Multi-Destination, Multi-Pallet Packing Problem.** In the multi-destination, single-aircraft instance of the ALPIA problem, an additional constraint is added to the problem. The aircraft subscript is still not required in this formulation. The additional constraint is:

$$I_{ma} \cdot I_{pa} = X_{mp} \ \ (\forall\ a, m, p) \tag{3.26}$$

This constraint ensures that none of the items are allowed to be packed on a pallet that has a different destination than the item.

3.5. **Formulation of Multi-Aircraft, Multi-Destination, Multi-Pallet Packing Problem with Insufficient Aircraft.** When the formulations used in the previous subproblems are augmented with the index accounting for multiple aircraft ($t \in T$) and combined, the following formulation is derived. The first constraints are for packing the pallet:

$$\sum_{m=1}^{M} x_{ijkmpt} \leq 1 \ \ (\forall\ i, j, k, p, t) \quad \rightarrow Avoid\ Item\ Overlap \tag{3.27}$$

$$\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} \sum_{m=1}^{M} x_{ijkmpt} \leq D_{pt} \cdot L_{pt} \cdot H_{pt} \, (\forall\ p, t) \quad \rightarrow Pallet\ Volume \tag{3.28}$$

$$\sum_{i=1}^{D_{pt}} \sum_{m=1}^{M} x_{ijkmpt} \leq D_{pt} \, (\forall \; j, k, p, t) \quad \rightarrow Pallet \; Depth \tag{3.29}$$

$$\sum_{j=1}^{L_{pt}} \sum_{m=1}^{M} x_{ijkmpt} \leq L_{pt} \, (\forall \; i, k, p, t) \quad \rightarrow Pallet \; Length \tag{3.30}$$

$$\sum_{k=1}^{H_{pt}} \sum_{m=1}^{M} x_{ijkmpt} \leq H_{pt} \, (\forall \; i, j, p, t) \quad \rightarrow Pallet \; Height \tag{3.31}$$

$$\sum_{m=1}^{M} X_{mpt} \cdot w_m \leq W_{pt} \;\; (\forall \; p, t) \quad \rightarrow Pallet \; Weight \tag{3.32}$$

$$\sum_{m=1}^{M} x_{ij(k+1)mpt} - \sum_{m=1}^{M} x_{ijkmpt} \leq 0 \quad (\forall \; i, j, k, p, t) \quad \rightarrow Packing \; Stability \tag{3.33}$$

$$\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} x_{ijkmpt} \leq d_m \cdot l_m \cdot h_m \cdot X_{mpt} \, (\forall \; m, p, t) \quad \rightarrow Linking \; Constraint \tag{3.34}$$

$$I_{ma} \cdot I_{pa} = X_{mpt} \, (\forall \; a, m, p, t) \quad \rightarrow Destination \; Constraint \tag{3.35}$$

The Pallet CG constraints are given by (depth, length, and height, respectively):

$$\begin{aligned} &\left| \sum_{m=1}^{m} \left[\left(\frac{\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} (i \cdot x_{ijkmpt})}{d_m \cdot l_m \cdot h_m} - \frac{D_{pt} \cdot X_{mpt}}{2} \right) \cdot w_m \right] \right. \\ &\quad \left. - \sum_{m=1}^{M} X_{mpt} \cdot w_m \right| \leq CG_{depth,pt} \end{aligned} \tag{3.36}$$

$$\begin{aligned} &\left| \sum_{m=1}^{m} \left[\left(\frac{\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} (j \cdot x_{ijkmpt})}{d_m \cdot l_m \cdot h_m} - \frac{L_{pt} \cdot X_{mpt}}{2} \right) \cdot w_m \right] \right. \\ &\quad \left. - \sum_{m=1}^{M} X_{mpt} \cdot w_m \right| \leq CG_{length,pt} \end{aligned} \tag{3.37}$$

$$\begin{aligned} &\left(\sum_{m=1}^{m} \left[\left(\frac{\sum_{i=1}^{D_{pt}} \sum_{j=1}^{L_{pt}} \sum_{k=1}^{H_{pt}} (k \cdot x_{ijkmpt})}{d_m \cdot l_m \cdot h_m} - \frac{H_{pt} \cdot X_{mpt}}{2} \right) \cdot w_m \right] \right. \\ &\quad \left. - \sum_{m=1}^{M} X_{mpt} \cdot w_m \right) \leq \frac{H_{pt}}{2} \end{aligned} \tag{3.38}$$

As previously mentioned in the single pallet packing process, pixels must be kept contiguous by using another constraint set. This formulation requires the same type of constraints with the additional indices for multiple pallets and aircraft.

$$\left[y_{m11pt}\left(d_m \sum_{i=1}^{D-d_m}\prod_{s=0}^{d_m-1} x_{(i+s)jkmpt}\right) + y_{m12pt}\left(l_m \sum_{i=1}^{D-l_m}\prod_{s=0}^{l_m-1} x_{(i+s)jkmpt}\right)\right.$$
$$\left.+y_{m13pt}\left(h_m \sum_{i=1}^{D-h_m}\prod_{s=0}^{h_m-1} x_{(i+s)jkmpt}\right)\right] = \left(\sum_{i=1}^{D} x_{ijkmpt}\right) (\forall\ j,k,m,p,t) \tag{3.39}$$

$$\left[y_{m21pt}\left(d_m \sum_{j=1}^{L-d_m}\prod_{s=0}^{d_m-1} x_{i(j+s)kmpt}\right) + y_{m22pt}\left(l_m \sum_{j=1}^{L-l_m}\prod_{s=0}^{l_m-1} x_{i(j+s)kmpt}\right)\right.$$
$$\left.+y_{m23pt}\left(h_m \sum_{j=1}^{L-h_m}\prod_{s=0}^{h_m-1} x_{i(j+s)kmpt}\right)\right] = \left(\sum_{j=1}^{L} x_{ijkmpt}\right) (\forall\ i,k,m,p,t) \tag{3.40}$$

$$\left[y_{m31pt}\left(d_m \sum_{j=1}^{H-d_m}\prod_{s=0}^{d_m-1} x_{ij(k+s)mpt}\right) + y_{m32pt}\left(l_m \sum_{j=1}^{H-l_m}\prod_{s=0}^{l_m-1} x_{ij(k+s)mpt}\right)\right.$$
$$\left.+y_{m33pt}\left(h_m \sum_{j=1}^{H-h_m}\prod_{s=0}^{h_m-1} x_{ij(k+s)mpt}\right)\right] = \left(\sum_{j=1}^{L} x_{ijkmpt}\right) (\forall\ i,j,m,p,t) \tag{3.41}$$

These constraints make the problem distinctly non-linear. Furthermore, they are very detailed and (possibly) difficult to understand. The following constraints could replace the pixel contiguity constraints; however, both sets of constraints will result in a non-linear formulation of the problem.

$$max\left(\sum_{s=i}^{i+d_m} x_{(s)jkmpt}\right) - d_m y_{m11pt} = 0 \qquad (\forall\ j,k,m,p,t) \tag{3.42}$$

$$max\left(\sum_{s=j}^{j+d_m} x_{i(s)kmpt}\right) - d_m y_{m21pt} = 0 \qquad (\forall\ i,k,m,p,t) \tag{3.43}$$

$$max\left(\sum_{s=k}^{k+d_m} x_{ij(s)mpt}\right) - d_m y_{m31pt} = 0 \qquad (\forall\ i,j,m,p,t) \tag{3.44}$$

$$max\left(\sum_{s=i}^{i+l_m} x_{(s)jkmpt}\right) - l_m y_{m11pt} = 0 \qquad (\forall\ j,k,m,p,t) \tag{3.45}$$

$$max\left(\sum_{s=j}^{j+l_m} x_{i(s)kmpt}\right) - l_m y_{m21pt} = 0 \qquad (\forall\ i,k,m,p,t) \tag{3.46}$$

$$max\left(\sum_{s=k}^{k+l_m} x_{ij(s)mpt}\right) - l_m y_{m31pt} = 0 \qquad (\forall\ i,j,m,p,t) \tag{3.47}$$

$$max\left(\sum_{s=i}^{i+h_m} x_{(s)jkmpt}\right) - h_m y_{m11pt} = 0 \qquad (\forall\ j,k,m,p,t) \tag{3.48}$$

$$max\left(\sum_{s=j}^{j+h_m} x_{i(s)kmpt}\right) - h_m y_{m21pt} = 0 \qquad (\forall\ i,k,m,p,t) \tag{3.49}$$

$$max\left(\sum_{s=k}^{k+h_m} x_{ij(s)mpt}\right) - h_m y_{m31pt} = 0 \qquad (\forall\ i,j,m,p,t) \tag{3.50}$$

To account for different orientations and still ensure feasible solutions, the following set of constraints is required.

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m11pt} + y_{m21pt} + y_{m31pt}) \leq 1 \qquad (\forall\ m) \tag{3.51}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m12pt} + y_{m22pt} + y_{m32pt}) \leq 1 \qquad (\forall\ m) \tag{3.52}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m13pt} + y_{m23pt} + y_{m33pt}) \leq 1 \qquad (\forall\ m) \tag{3.53}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m11pt} + y_{m12pt} + y_{m13pt}) \leq 1 \qquad (\forall\ m) \tag{3.54}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m21pt} + y_{m22pt} + y_{m23pt}) \leq 1 \qquad (\forall\ m) \tag{3.55}$$

$$\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(y_{m31pt} + y_{m32pt} + y_{m33pt}) \leq 1 \qquad (\forall\ m) \tag{3.56}$$

$$y_{m33pt} = X_{mpt} \tag{3.57}$$

Finally, the constraints related to the available aircraft are given by:

$$\sum_{p=1}^{n} X_{mpt} \leq 1 \ \ (\forall\ m,t) \tag{3.58}$$

$$CB_{min,t} \leq \left(\frac{\sum_{p=1}^{n}\left(b_{pt}\cdot\sum_{m=1}^{M}(w_m\cdot X_{mpt})\right)}{\sum_{m=1}^{M} w_m\cdot X_{mpt}}\right) \leq CB_{max,t} \quad (\forall\ t) \tag{3.59}$$

$$\sum_{p=1}^{n}\sum_{m=1}^{M}(w_m\cdot X_{mpt}) \leq \psi_t \quad (\forall\ t) \tag{3.60}$$

3.6. **Objective Function.** Unlike bin, pallet or container packing problems which have an objective of minimizing the number of bins required to pack all items, the ALPIA primary objective is to maximize the total utility of the packed items while also maximizing the total volume and weight of the transported items. Of course, leaving any available space on the pallet empty may cause the overall problem to require an additional bin (possibly more). Thus, without exceeding the weight or volume limitations of pallets, a goal of the ALPIA is to pack them efficiently; this aids the primary objective by ensuring pallet space is available for as many items as possible.

The ALPIA objective function is also similar to that of a knapsack problem. In both types of problems, the number of "bins" or "knapsacks" is limited. In knapsack problems, the objective involves maximizing the total utility of the selected items; however, for the ALPIA, simply maximizing the number of the packed, high priority items while only focusing on utility may lead to inefficient use of available capacity (weight and space).

In the following section, two different objective functions are introduced. The first function is a weighted sum of sub-objectives. The sub-objectives of the ALPIA are:

a. *Maximizing the utility of the packed items and reducing the number of unpacked high utility items*: Both packed and unpacked items are relevant in the ALPIA. Failing to pack items of high priority is as undesirable as neglecting to utilize the available aircraft capacity. Thus, the first objective function, f_1, is given by:

$$f_1 = \left[\frac{\left(\sum_{m=1}^{M} u_m\right) - \left(\sum_{m=1}^{M} u_m \cdot X_m\right)}{\left(\sum_{m=1}^{M} u_m\right)} \right] \cdot 100 \tag{3.61}$$

This objective function is a percentage of the utility of the packed items. Unfortunately, minimizing this value may lead to packing the highest priority items whenever there is sufficient space (regardless of whether some other item is better suited for the space).

b. *Maximum Aircraft Capacity Usage*: ALPIA involves packing the aircraft efficiently by placing properly packed pallets within the aircraft. Placing packed pallets with the available volume maximized may not result in the best possible solution; weight and volume maximization should also be included. Additional portions of the objective function which account for these considerations are labeled f_2 and f_3 and are given by:

$$f_2 = \left[\frac{min\left[\left(\sum_{t=1}^{\tau} \sum_{p=1}^{n_t} W_p\right), \left(\sum_{t=1}^{\tau} \psi_t\right)\right] - \left(\sum_{m=1}^{M} w_m \cdot X_m\right)}{min\left[\left(\sum_{t=1}^{\tau} \sum_{p=1}^{p_t} W_p\right), \left(\sum_{t=1}^{\tau} \psi_t\right)\right]} \right] \cdot 100 \tag{3.62}$$

$$f_3 = \left[\frac{\left(\sum_{t=1}^{\tau} \sum_{p=1}^{n_t} L_{pt} \cdot D_{pt} \cdot H_{pt}\right) - \left(\sum_{m=1}^{M} w_m \cdot X_m\right)}{\left(\sum_{t=1}^{\tau} \sum_{p=1}^{n_t} L_{pt} \cdot D_{pt} \cdot H_{pt}\right)} \right] \cdot 100 \tag{3.63}$$

The overall goal of these two portions of the objective function is to minimize the unused capacity by minimizing the unused weight and volume pallet capacity, respectively. These functions make the objective function non-linear. The weight limitation of the aircraft depends on the relationship between the ACL and the total weight capacity of the pallet positions on the same aircraft.

c. *Balanced Aircraft:* An unbalanced aircraft with pallets whose volume is maximized will be an infeasible solution to the ALPIA; packing the pallets with respect to each other is important. CB feasibility is assured with the constraints; thus this does not require inclusion in the objective function.

d. *Efficiently Packed Pallets*: Packing balanced and stable pallets is more important than completely maximizing their volume. CG feasibility and stable packing of the items are assured by the constraints. As a result, this does not require inclusion in the objective function.

After defining the sub-objectives, the overall objective is a weighted summation of the sub-objective functions and is given by:

$$min\left[(\lambda_1 \cdot f_1) + (\lambda_2 \cdot f_2) + (\lambda_3 \cdot f_3)\right] = min \sum_{i=1}^{3} (\lambda_i \cdot f_i) \tag{3.64}$$

where f_1, f_2, and f_3 are as previously defined, and λ_1, λ_2, and λ_3 are penalty weights.

Despite the similarities with knapsack and bin packing problems, the ALPIA has additional aspects which require consideration. These considerations may result in a non-convex solution space of ALPIA. In addition to the drawbacks of a non-linear function, possible non-convex portions of the solution set of ALPIA may not be obtained by minimizing convex combinations of the objectives.

Thus, another objective function for the ALPIA may be given as:

$$max\left[\sum_{m=1}^{M} u_m^{\lambda} \cdot l_m \cdot d_m \cdot h_m \cdot w_m \cdot X_m\right] \tag{3.65}$$

This objective function attempts to simultaneously maximize the total utility, total volume and total height without using penalty multipliers. The use of the superscript ensures the importance of the priority aspect of the items. Lower values may be used in routine missions for higher aircraft utilization; higher values may be used in deployment (less frequent) missions for value based aircraft utilization.

4. COMPLEXITY OF ALPIA

Clearly, ALPIA is an NP-Hard optimization problem since the 0-1 Knapsack problem is a special case of sub-problem 1 (i.e., *Selecting Cargo Items*). Although the polynomial transformation is not presented in this research, the 0-1 Knapsack problem can be reduced to the ALPIA in polynomial time. Karp [5] previously proved the 0-1 Knapsack problems to be NP-Hard.

5. SUMMARY

In this research effort, the Airlift Loading Problem with Insufficient Aircraft (ALPIA) was introduced and explained in detail. Similarities and differences between the ALPIA and knapsack, bin-packing and multi-constraint bipartite maximal matching problems were also presented. For the first time, a formulation considering all the constraints of "packing an aircraft" and an objective function that achieves the ALPIA objective is presented. Except for the contiguity constraint and the objective function, this is an integer-linear formulation.

The number of variables associated with the constraints presented in the ALPIA formulation is very large; there are

$$\left(\sum_{t=1}^{\tau}\sum_{p=1}^{n_t}(L_{pt}\cdot D_{pt}\cdot H_{pt})\right)+7 \qquad (t\in T, p\in P_T) \tag{5.1}$$

variables required for each item. Couple this with the fact that the ALPIA is an NP-Hard problem, and classical optimization methods are insufficient to solve the ALPIA in a reasonable amount of computational time and effort. Therefore, heuristics or other algorithmic techniques can be applied to this problem to provide a high quality solution in a reasonable amount of time.

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