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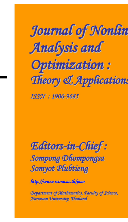

EXPANDING THE APPLICABILITY OF A TWO STEP NEWTON LAVRENTIEV METHOD FOR ILL-POSED PROBLEMS

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ABSTRACT. In [3] we presented a cubically convergent Two Step Directional Newton Method (TSDNM) for approximating a solution of an operator equation in a Hilbert space setting. George and Pareth in [13] use the analogous Two Step Newton Lavrentiev Method (TSNLM) to approximate a solution of an ill-posed equation. In the present paper we show how to expand the applicability of (TSNLM). In particular, we present a semilocal convergence analysis of (TSNLM) under: weaker hypotheses, weaker convergence criteria, tighter error estimates on the distances involved and an at least as precise information on the location of the solution.

KEYWORDS: Newton-Lavrentiev regularization method; ill-posed problem; Hilbert space; semilocal convergence.
AMS Subject Classification: 65J20 65J15 47J36.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution of the nonlinear ill-posed operator equation

$$F(x) = f, \tag{1.1}$$

where F is a Fréchet differentiable operator defined on an open and convex subset $D(F)$ of a real Hilbert space X and $f \in X$. Let $\langle .,. \rangle$ and $\|.\|$, stand, respectively for the inner product and the corresponding norm. Let also $U(x,r)$ and $\overline{U(x,r)}$, stand, respectively for the open and closed balls in X with center $x \in X$ and radius $r > 0$. Let $L(X)$ denote the space of bounded linear operators from X into X. We suppose that F is a monotone operator. That is

$$\langle F(x) - F(y), x - y \rangle \geq 0, \qquad \forall x, y \in D(F).$$

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Set $S = \{x : F(x) = f\}$. It is known that S is closed and convex provided that F is monotone and continuous (see, e.g., [28]). It follows that there exists a unique element of minimal norm. Denoted such an element by $\hat{x}$. Then we have $F(\hat{x}) = f$. Suppose that $F'(.)$ is a positive self adjoint operator. Then, $(F'(.) + \alpha I)^{-1} \in L(X)$ for each $\alpha > 0$. Here, we need $\sigma(F'(.)) \subseteq [0, \infty)$ and $\|(F'(.) + \alpha I)^{-1}\| \leq \frac{c}{\alpha}$ for some constant $c > 0$ and for any $\alpha > 0$. Such conditions are weaker than the self adjointness of $F'(.)$. In practice, only noisy data f^δ is available, such that $\|f - f^\delta\| \leq \delta$. Hence, the problem of computing $\hat{x}$ from equation $F(x) = f^\delta$ is ill-posed. Since a small perturbation in the data can cause large deviation in the solution. The Lavrentiev regularization method has been used to solve (1.1) by obtaining an approximation x_α^δ of equation

$$F(x) + \alpha(x - x_0) = f^\delta, \tag{1.2}$$

where $\alpha > 0$ is the regularization parameter and $x_0 \in D(F)$ is an initial point which is an approximation to $\hat{x}$ [11] ,[13], [15], [29, 30]. If $\alpha > 0$ is chosen appropriately then, it is known that $x_\alpha^\delta \to \hat{x}$ as $\alpha \to 0$ and $\delta \to 0$ [27], [30].

Many problems from computational sciences and other disciplines can be brought in a form similar to equation (1.1) using mathematical modelling [1], [4], [8], [20], [25], [26]. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

In [3], we introduced a third order Two Step Directional Newton Method (TSDNM) for approximating a zero x^* of a differentiable operator F on a Banach space setting. Motivated by (TSDNM) George and Pareth [13] used the analogous Two Step Newton Lavrentiev Method (TSNLM)

$$y_{n,\alpha}^\delta = x_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)^{-1}[F(x_{n,\alpha}^\delta) - f^\delta + \alpha(x_{n,\alpha}^\delta - x_0)] \tag{1.3}$$

and

$$x_{n+1,\alpha}^\delta = y_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)^{-1}[F(y_{n,\alpha}^\delta) - f^\delta + \alpha(y_{n,\alpha}^\delta - x_0)] \tag{1.4}$$

to generate a cubically convergent iteration $\{x_{n,\alpha}^\delta\}$ approximating x_α^δ, where $x_{0,\alpha}^\delta = x_0, \alpha > \alpha_0 > 0$ and $R_\alpha(x) = F'(x) + \alpha I$ [13].

Note that we have

$$\|R_\alpha(x)^{-1} F'(x)\| \leq 1, \quad \forall x \in D(F). \tag{1.5}$$

The semilocal convergence analysis was based on the following conditions which has been used extensively in the study of iterative procedures for solving ill-posed problems [11], [16], [29].

(C1) There exists a constant $L > 0$ such that for each $x, u \in D(F)$ and $v \in X$, there exists an element $P(x, u, v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)P(x, u, v), \qquad \|P(x, u, v)\| \leq L\|v\|\|x - u\|.$$

They used the additional restriction that

$$0 < L \leq 1. \tag{1.6}$$

In the present paper, we extend the convergence domain of (TSNLM) and also drop restrictive condition (1.6) under weaker sufficient semilocal convergence criteria.

Moreover, the upper bounds on the distances $\|x^\delta_{n+1,\alpha} - x^\delta_{n,\alpha}\|, \|x^\delta_{n,\alpha} - x^\delta_\alpha\|$ are tighter and the information on the location of the solution x^δ_α at least as precise (see Section 3).

There are cases when Lipschitz-type condition $(C1)$ is violated (see Section 4) but the following weaker central-Lipschitz-type condition is satisfied:

(C1)' Let $x_0 \in D(F)$ be fixed. There exists a constant $L_0 > 0, r > 0$ such that for each $x, u \in U(x_0, r) \cup U(\hat{x}, r) \subseteq D(F)$ and $v \in X$, there exists an element $\Phi(x, u, v) \in X$ such that

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq L_0\|v\|(\|x_0 - u\| + \|x - x_0\|).$$

We note that since $\|u - x\| \leq \|u - x_0\| + \|x - x_0\|$ condition $(C1)$ always implies $(C1)'$ with $L_0 = L$ and $\Phi = P$ but not necessarily vice versa. Note that, under $(C1)'$ we also have the following special case:

(C1)" Let $x_0 \in D(F)$ be fixed. There exists a constant $l_0 > 0$ such that for all $w_\theta = x_0 + \theta(\hat{x} - x_0) \in D(F)$ and $v \in X$, there exists an element $\Phi(x_0, w_\theta, v) \in X$ such that

$$[F'(x_0) - F'(w_\theta)]v = F'(x_0)\Phi(x_0, w_\theta, v), \qquad \|\Phi(x_0, w_\theta, v)\| \leq l_0\|v\|\|x_0 - w_\theta\|.$$

Note that $l_0 \leq L_0 \leq L$ hold in general and $\frac{L_0}{l_0}$ and $\frac{L}{L_0}$ can be arbitrarily large [1]-[5].

In section 2 we provide a semilocal convergence analysis for (TSNLM) using $(C1)'$ instead of $(C1)$. We shall refer to [3], [13] for some of the proofs omitted in this study.

2. Semilocal convergence of (TSNLM) under $(C1)'$

We present the semilocal convergence of (TSNLM) using $(C1)'$. We need to introduce some sequences and parameters:

$$e^\delta_{n,\alpha} := \|y^\delta_{n,\alpha} - x^\delta_{n,\alpha}\|, \qquad \forall n = 0, 1, \cdots, \tag{2.1}$$

for $\delta_0 < (17 - 12\sqrt{2})\alpha_0$ for some $\alpha_0 > 0$ and $\|x_0 - \hat{x}\| \leq \rho$,

$$\rho \leq \frac{\sqrt{1 + 2l_0(17 - 12\sqrt{2} - \frac{\delta_0}{\alpha_0})} - 1}{l_0} = \rho_0. \tag{2.2}$$

Let

$$b_\rho = \frac{l_0}{2}\rho^2 + \rho + \frac{\delta_0}{\alpha_0}, \tag{2.3}$$

$$r = \frac{1}{L_0}\frac{2b_\rho}{1 - b_\rho + \sqrt{(1 - b_\rho)^2 - 32b_\rho}}, \tag{2.4}$$

$$\gamma_\rho = \frac{L}{2}\rho^2 + \rho + \frac{\delta_0}{\alpha_0}, \tag{2.5}$$

and

$$p = 2L_0 r, q = 2p^2. \tag{2.6}$$

Note that r is well defined, since $\frac{p}{2} < 1, q \in (0, 1)$ and $b_\rho \in (0, 17 - 12\sqrt{2}]$. We also have that

$$b_\rho \leq \gamma_\rho \tag{2.7}$$

and

$$\frac{1 + L_0 r}{1 - 8L_0^2 r^2}b_\rho = \frac{1 + \frac{p}{2}}{1 - q}b_\rho = L_0 r. \tag{2.8}$$

Estimate (2.7) holds as strict inequality if $l_0 < L$. Parameter γ_ρ was used in [13]. In order for us to simplify the notation, let x_n, y_n and e_n, stand, respectively for $x^\delta_{n,\alpha}, y^\delta_{n,\alpha}$ and $e^\delta_{n,\alpha}$. If we simply use the needed $(C1)''$ instead of $(C1)$ we arrive at:

LEMMA 2.1. Suppose that $(C1)''$ holds and b_ρ is given by (2.3). Then, the following assertion holds

$$e_0 \le b_\rho < 17 - 12\sqrt{2} = 0.029437252\cdots.$$

Proof. Using (2.1), (2.2), (2.3) and $(C1)''$ we obtain in turn that

$$\begin{aligned}
e_0 = \|y_0 - x_0\| &= \|R_\alpha(x_0)^{-1}(F(x_0) - f^\delta)\| \\
&= \|R_\alpha(x_0)^{-1}[F(x_0) - F(\hat{x}) - F'(x_0)(x_0 - \hat{x}) \\
&\quad + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&= \|R_\alpha(x_0)^{-1}[\int_0^1 (F'(x_0 + t(\hat{x} - x_0)) - F'(x_0))dt(x_0 - \hat{x}) \\
&\quad + F'(x_0)(x_0 - \hat{x}) + F(\hat{x}) - f^\delta]\| \\
&\le \|\int_0^1 \Phi(x_0 + t(\hat{x} - x_0), x_0, x_0 - \hat{x})\| + \|x_0 - \hat{x}\| \\
&\quad + \|R_\alpha(x_0)^{-1}(F(\hat{x}) - f^\delta)\| \\
&\le \frac{l_0}{2}\|x_0 - \hat{x}\|^2 + \|x_0 - \hat{x}\| + \frac{1}{\alpha}\|F(\hat{x}) - f^\delta\| \\
&\le \frac{l_0}{2}\rho^2 + \rho + \frac{\delta}{\alpha} \\
&\le \frac{l_0}{2}\rho^2 + \rho + \frac{\delta_0}{\alpha_0} = b_\rho \le 17 - 12\sqrt{2}.
\end{aligned}$$

The proof of the Lemma is complete.

REMARK 2.2. If $l_0 = L$ Lemma 2.1 reduces to Lemma 11 in [13]. Otherwise, i.e., if $l_0 < L$ it constitutes an improvement according to (2.7).

With the notion introduced so far we can present the semilocal convergence analysis of (TSNLM) using the next three results.

THEOREM 2.3. Suppose that $(C1)'$ holds and $\delta \in (0, \delta_0]$. Then, the following assertions hold

(a) $\|x_n - y_n\| \le p\|y_{n-1} - x_{n-1}\| = pe_{n-1}$,
(b) $\|x_n - x_{n-1}\| \le (1 + \frac{p}{2})e_{n-1}$,
(c) $e_n \le qe_{n-1}$.

Proof. Using (1.3) we get that

$$\begin{aligned}
x_n - y_{n-1} &= y_{n-1} - x_{n-1} - R_\alpha(x_{n-1})^{-1}(F(y_{n-1}) - F(x_{n-1}) \\
&\quad + \alpha(y_{n-1} - x_{n-1})) \\
&= R_\alpha(x_{n-1})^{-1}[R_\alpha(x_{n-1})(y_{n-1} - x_{n-1}) \\
&\quad - (F(y_{n-1}) - F(x_{n-1})) - \alpha(y_{n-1} - x_{n-1})] \\
&= R_\alpha(x_{n-1})^{-1}\int_0^1 \{F'(x_{n-1}) - F'(x_{n-1} + t(y_{n-1} - x_{n-1}))\} \\
&\quad \times(y_{n-1} - x_{n-1})dt \\
&= R_\alpha(x_{n-1})^{-1}\int_0^1 \{F'(x_{n-1}) - F'(x_0) + F'(x_0) - F'(x_{n-1} + t(y_{n-1} - x_{n-1}))\} \\
&\quad \times(y_{n-1} - x_{n-1})dt.
\end{aligned}$$

In view of $(C1)'$ and (1.5) we have that

$$\|x_n - y_{n-1}\| \le \|\int_0^1 \Phi(x_{n-1}, x_0, y_{n-1} - x_{n-1})dt\|$$

$$
\begin{aligned}
&+\|\int_0^1 \Phi(x_0, x_{n-1}+t(y_{n-1}-x_{n-1}), y_{n-1}-x_{n-1})dt\| \\
\leq\ & L_0[\|x_{n-1}-x_0\| + \int_0^1 \|x_{n-1}-x_0+t(y_{n-1}-x_{n-1})\|dt]\|y_{n-1}-x_{n-1}\| \\
\leq\ & 2L_0 r\|y_{n-1}-x_{n-1}\| = p\|y_{n-1}-x_{n-1}\| = pe_{n-1}.
\end{aligned}
$$

This proves (a). Now (b) follows from (a) and the triangle inequality;

$$\|x_n - x_{n-1}\| \leq \|x_n - y_{n-1}\| + \|y_{n-1} - x_{n-1}\|.$$

To prove (c) we first use (1.3) to obtain in turn the identity

$$
\begin{aligned}
y_n - x_n &= x_n - y_{n-1} - R_\alpha(x_n)^{-1}(F(x_n) - f^\delta + \alpha(x_n - x_0)) \\
&\quad + R_\alpha(x_{n-1})^{-1}(F(y_{n-1}) - f^\delta + \alpha(y_{n-1} - x_0)) \\
&= x_n - y_{n-1} - R_\alpha(x_n)^{-1}(F(x_n) - F(y_{n-1}) + \alpha(x_n - y_{n-1})) \\
&\quad + [R_\alpha(x_{n-1})^{-1} - R_\alpha(x_n)^{-1}](F(y_{n-1}) - f^\delta + \alpha(y_{n-1} - x_0)) \\
&= R_\alpha(x_n)^{-1}[R_\alpha(x_n)(x_n - y_{n-1}) - (F(x_n) - F(y_{n-1})) \\
&\quad - \alpha(x_n - y_{n-1})] + [R_\alpha(x_{n-1})^{-1} - R_\alpha(x_n)^{-1}] \\
&\quad \times(F(y_{n-1}) - f^\delta + \alpha(y_{n-1} - x_0)). \qquad (2.9)
\end{aligned}
$$

Then, again by $(C1)'$ and (2.9) we obtain that

$$
\begin{aligned}
e_n &\leq \|R_\alpha(x_n)^{-1}\int_0^1 [F'(x_n) - F'(y_{n-1}+t(x_n-y_{n-1}))]dt(x_n-y_{n-1})\| \\
&\quad + \|R_\alpha(x_n)^{-1}(F'(x_n) - F'(x_{n-1}))R_\alpha(x_{n-1})^{-1}(F(y_{n-1}) - f^\delta \\
&\quad + \alpha(y_{n-1}-x_0))\| \\
&\leq \|R_\alpha(x_n)^{-1}\int_0^1 [F'(x_n) - F'(y_{n-1}+t(x_n-y_{n-1}))]dt(x_n-y_{n-1})\| \\
&\quad + \|R_\alpha(x_n)^{-1}(F'(x_n) - F'(x_{n-1}))(y_{n-1}-x_n)\| \\
&\leq L_0[\|x_n - x_0\| + \int_0^1 \|y_{n-1}-x_0+t(x_n-y_{n-1})\|dt]\|x_n-y_{n-1}\| \\
&\quad + L_0[\|x_n-x_0\| + \|x_{n-1}-x_0\|]\|x_n-y_{n-1}\| \\
&\leq 4L_0 r\|y_{n-1}-x_n\| = 4L_0 r(2L_0 r)e_{n-1} \\
&= qe_{n-1}.
\end{aligned}
$$

This completes the proof of the Theorem.

THEOREM 2.4. Under the hypotheses of Theorem 2.3 further suppose that

$$\rho < \rho_0 \text{ and } L_0 \leq 1. \qquad (2.10)$$

Moreover, suppose that

$$\overline{U(x_0, r)} \subseteq D(F). \qquad (2.11)$$

Then, $x_n, y_n \in U(x_0, r)$ for each $n = 0, 1, 2, \cdots$.

Proof. We note by (2.10) that we have

$$q \in (0, 1). \qquad (2.12)$$

Using Lemma 2.1, Theorem 2.3 and (2.11) we get that

$$\|x_1 - x_0\| \leq (1 + L_0 r)e_0 \leq (1 + L_0 r)b_\rho < r.$$

Hence, $x_1 \in U(x_0, r)$. Similarly, we obtain that

$$\|y_1 - x_0\| \quad \leq \quad \|y_1 - x_1\| + \|x_1 - x_0\| \qquad (2.13)$$

$$\begin{aligned} &\leq qe_0 + (1+\frac{p}{2})b_\rho && (2.14)\\ &\leq [q+1+\frac{p}{2}]b_\rho < L_0 r \leq r, && (2.15)\end{aligned}$$

which implies $y_1 \in U(x_0, r)$. Moreover, we have that

$$\begin{aligned} \|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\leq (1+\frac{p}{2})\|y_1 - x_1\| + (1+\frac{p}{2})b_\rho \\ &\leq (1+\frac{p}{2})qb_\rho + (1+\frac{p}{2})b_\rho \\ &= (1+q)(1+\frac{p}{2})b_\rho < L_0 r \leq r, \end{aligned}$$

which also implies $x_2 \in U(x_0, r)$. Furthermore, we obtain that

$$\begin{aligned} \|y_2 - x_0\| &\leq \|y_2 - x_2\| + \|x_2 - x_0\| \\ &\leq q\|y_1 - x_1\| + (1+q)(1+\frac{p}{2})b_\rho \\ &\leq q^2(1+\frac{p}{2})b_\rho + (1+q)(1+\frac{p}{2})b_\rho \\ &\leq (1+q+q^2)(1+\frac{p}{2})b_\rho < L_0 r \leq r. \end{aligned}$$

Hence, we proved that $y_2 \in U(x_0, r)$. Proceeding in an analogous way we prove that $x_n, y_n \in U(x_0, r)$. That completes the proof of the Theorem.

THEOREM 2.5. Suppose that the hypotheses of Theorem 2.4 hold. Then, sequence $\{x^\delta_{n,\alpha}\}$ remains in $U(x_0, r)$ for each $n = 0, 1, 2, \cdots$ and converges to a solution $x^\delta_\alpha \in \overline{U(x_0, r)}$ of equation (1.2). Moreover, the following estimates hold

$$\|x_n - x^\delta_\alpha\| \leq b_0 e^{-\gamma_0 n}, \qquad (2.16)$$

where $b_0 = (1+\frac{p}{2})b_\rho$ and $\gamma_0 = -\ln q > 0$.

Proof. Using (b) of Theorem 2.3 and (2.10) we get that

$$\|x_{n+m} - x_n\| \leq \sum_{i=0}^{m-1} \|x_{n+i+1} - x_{n+i}\|. \qquad (2.17)$$

But, we have

$$\|x_{n+i+1} - x_{n+i}\| \leq (1+\frac{p}{2})q^{n+i}e_0. \qquad (2.18)$$

In view of (2.18), inequality (2.17) gives that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq [1+q+q^2+\cdots+q^{m-1}]q^n(1+\frac{p}{2})e_0 \\ &\leq \frac{1-q^m}{1-q}(1+\frac{p}{2})q^n e_0. && (2.19)\end{aligned}$$

It follows from (2.19) that sequence $\{x_n\}$ is complete in a Hilbert space X and as such it converges to some $x^\delta_\alpha \in \overline{U(x_0, r)}$ (since $\overline{U(x_0, r)}$ is closed set). By letting $m \to \infty$ we obtain (2.16). Finally, to prove x^δ_α is a solution of (1.2), note that

$$\begin{aligned} \|F(x_n) - f^\delta + \alpha(x_n - x_0)\| &= \|R_\alpha(x_n)(x_n - y_n)\| \\ &\leq (\|F'(x_n)\| + \alpha)e_n \\ &\leq (\|F'(x_n)\| + \alpha)q^n b_\rho \to 0 \text{ as } n \to \infty. \end{aligned}$$

That completes the proof of the Theorem.

REMARK 2.6. (a) The convergence order of (TSNLM) is three [13] under $(C1)$. In Theorem 2.5 the error bounds are too pessimistic. That is why in practice we shall use the computational order of convergence (COC) (see eg. [4]) defined by

$$\varrho \approx \ln\left(\frac{\|x_{n+1} - x_\alpha^\delta\|}{\|x_n - x_\alpha^\delta\|}\right) / \ln\left(\frac{\|x_n - x_\alpha^\delta\|}{\|x_{n-1} - x_\alpha^\delta\|}\right).$$

The (COC) ϱ will then be close to 3 which is the order of convergence of (TSNLM).

(b) In the rest of this section we suppose that

$$\rho_0^* \leq r \tag{2.20}$$

which is possible for x_0 sufficiently close to $\hat{x}$.

Next, we present the results concerning error bounds under source conditions. We need a condition on the source function.

(C2) (George and Pareth [13]) There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(\hat{x})\|$ satisfying $lim_{\lambda \to 0}\varphi(\lambda) = 0$ and $v \in X$ with $\|v\| \leq 1$ such that

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))v$$

and

$$sup_{\lambda \geq 0}\frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq c_\varphi \varphi(\alpha), \qquad \forall \lambda \in (0, a].$$

REMARK 2.7. It can easily be seen that functions

$$\varphi(\lambda) = \lambda^\nu, \quad \lambda > 0$$

for $0 < \nu \leq 1$ and

$$\varphi(\lambda) = \begin{cases} (ln\frac{1}{\lambda})^{-\beta} & , \quad 0 < \lambda \leq e^{-(\beta+1)} \\ 0 & , \quad otherwise \end{cases}$$

for $\beta \geq 0$ satisfy $(C2)$ (cf. [23]).

PROPOSITION 2.8. (cf. [30], Proposition 3.1) Let $F : D(F) \subseteq X \to X$ be a monotone operator in X. Let x_α^δ be the unique solution of (1.2) and $x_\alpha := x_\alpha^0$. Then

$$\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\alpha}.$$

THEOREM 2.9. (cf. [29], Proposition 4.1 or [30], Theorem 3.3) Suppose that $(C1)', (C2)$ and hypotheses of Proposition 2.8 hold. Let $\hat{x} \in D(F)$ be a solution of (1.1). Then, the following assertion holds

$$\|x_\alpha - \hat{x}\| \leq (L_0 r + 1)\varphi(\alpha).$$

THEOREM 2.10. Suppose hypotheses of Theorem 2.5 and Theorem 2.9 hold. Then, the following assertion holds

$$\|x_n - \hat{x}\| \leq b_0 e^{-\gamma_0 n} + c_1(\varphi(\alpha) + \frac{\delta}{\alpha})$$

where $c_1 = \max\{1, (L_0 r + 1)\}$.

Let

$$\bar{c} := max\{b_0 + 1, (L_0 r + 1)\}, \tag{2.21}$$

and let

$$n_\delta := min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\alpha}\}. \tag{2.22}$$

THEOREM 2.11. Let $\bar{c}$ and n_δ be as in (2.21) and (2.22) respectively. Suppose that hypothese of Theorem 2.10 hold. Then, the following assertions hold

$$\|x_{n_\delta} - \hat{x}\| \leq \bar{c}(\varphi(\alpha) + \frac{\delta}{\alpha}). \tag{2.23}$$

Note that the error estimate $\varphi(\alpha) + \frac{\delta}{\alpha}$ in (2.23) is of optimal order if $\alpha := \alpha_\delta$ satisfies, $\varphi(\alpha_\delta)\alpha_\delta = \delta$.

Now using the function $\psi(\lambda) := \lambda\varphi^{-1}(\lambda), 0 < \lambda \leq a$ we have $\delta = \alpha_\delta\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, so that $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. In view of the above observations and (2.23) we have the following.

THEOREM 2.12. Let $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ for $0 < \lambda \leq a$, and the assumptions in Theorem 2.11 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ and let n_δ be as in (2.22). Then

$$\|x_{n_\delta} - \hat{x}\| = O(\psi^{-1}(\delta)).$$

In this section, we present a parameter choice rule based on the balancing principle studied in [22], [27]. In this method, the regularization parameter α is selected from some finite set

$$D_M(\alpha) := \{\alpha_i = \mu^i\alpha_0, i = 0, 1, \cdots, M\}$$

where $\mu > 1$, $\alpha_0 > 0$ and let

$$n_i := min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\alpha_i}\}.$$

Then for $i = 0, 1, \cdots, M$, we have

$$\|x^\delta_{n_i,\alpha_i} - x^\delta_{\alpha_i}\| \leq c\frac{\delta}{\alpha_i}, \qquad \forall i = 0, 1, \cdots M.$$

Let $x_i := x^\delta_{n_i,\alpha_i}$. The parameter choice strategy that we are going to consider in this paper, we select $\alpha = \alpha_i$ from $D_M(\alpha)$ and operate only with corresponding $x_i, \quad i = 0, 1, \cdots, M$. Proof of the following theorem is analogous to the proof of Theorem 3.1 in [29].

THEOREM 2.13. (cf. [29], Theorem 3.1) Assume that there exists $i \in \{0, 1, 2, \cdots, M\}$ such that $\varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}$. Suppose the hypotheses of Theorem 2.11 and Theorem 2.12 hold and let

$$l := max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\alpha_i}\} < M,$$

$$k := max\{i : \|x_i - x_j\| \leq 4\bar{c}\frac{\delta}{\alpha_j}, \qquad j = 0, 1, 2, \cdots, i\}.$$

Then $l \leq k$ and

$$\|\hat{x} - x_k\| \leq c\psi^{-1}(\delta)$$

where $c = 6\bar{c}\mu$.

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 2.13 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < \alpha_0$ and $\mu > 1$.
- Choose M big enough but not too large and $\alpha_i := \mu^i\alpha_0, i = 0, 1, 2, \cdots, M$.
- Choose $\rho \leq \rho_0^*$.

2.1. **Algorithm.**

1. Set $i = 0$.
2. Choose $n_i = min\{n : e^{-\gamma_0 n} \leq \frac{\delta}{\alpha_i}\}$.
3. Solve $x_i = x^{\delta}_{n_i,\alpha_i}$ by using the iteration (1.3).
4. If $\|x_i - x_j\| > 4\bar{c}\frac{\delta}{\alpha_j}, j < i$, then take $k = i - 1$ and return x_k.
5. Else set $i = i + 1$ and return to Step 2.

3. Semilocal convergence of (TSNLM) under (C1)

We present the semilocal convergence of (TSNLM) under $(C1)$. As in [3], [13] let us define function $g : (0, \infty) \to (0, \infty)$ by

$$g(t) = \frac{L^2}{8}(4 + 3Lt)t^2. \tag{3.1}$$

Note that if

$$t < \rho_1 = \frac{0.706442399}{L} \tag{3.2}$$

then

$$g(t) < 1. \tag{3.3}$$

Set

$$\rho^* = \min\{\rho_1, \rho_0\}. \tag{3.4}$$

Then, as in Section 2 (see also [3] and [13]) we were at the main semilocal convergence result for (TSNLM):

THEOREM 3.1. Suppose that $(C1), (C1)''$ hold and

$$\rho < \rho^* \tag{3.5}$$

where ρ^* is defined by (3.4) and $\overline{U(x_0, R)} \in D(F)$ with

$$R = (\frac{1}{1 - g(b_\rho)} + \frac{L}{2}\frac{b_\rho}{1 - g(b_\rho)^2})b_\rho. \tag{3.6}$$

Then, the following assertions hold

$$\|x_n - y_{n-1}\| \leq \frac{Le_{n-1}}{2}\|y_{n-1} - x_{n-1}\|,$$

$$\|x_n - x_{n-1}\| \leq (1 + \frac{Le_{n-1}}{2})\|y_{n-1} - x_{n-1}\|,$$

$$\|y_n - x_n\| \leq g(e_{n-1})\|y_{n-1} - x_{n-1}\|,$$

$$g(e_n) \leq g(b_\rho)^{3^n},$$

$$e_n \leq g(b_\rho)^{\frac{3^n-1}{2}} b_\rho.$$

Sequence $\{x_n\}$ generated by (1.3) remains in $U(x_0, R)$ for each $n = 0, 1, 2, \cdots$ and converges to a solution $x^\delta_\alpha \in \overline{U(x_0, R)}$ of equation (1.2). Moreover, the following assertion hold

$$\|x_n - x^\delta_\alpha\| \leq de^{-\gamma^{3^n}},$$

where $d = (\frac{1}{1-g(b_\rho)^3} + \frac{Lb_\rho}{2}\frac{1}{1-g(b_\rho)^3}g(b_\rho))b_\rho$ and $\gamma = -\ln g(b_\rho)$.

REMARK 3.2. Even if $l_0 = L$ Theorem 3.1 improves Theorem 3 in [13], since we do not assume $L \leq 1$. Otherwise, i.e., if $l_0 < L$, it constitutes a further improvement with advantages as stated in the introduction of this study since $b_\rho < \gamma_\rho$. Note that the results in [13] use γ_ρ instead of b_ρ. Therefore our constants c, γ are tighter if $l_0 < L$ or $L_0 < L$.

REMARK 3.3. In the rest of this section we suppose that

$$\rho^* \leq R \tag{3.7}$$

which is possible for x_0 sufficiently close to $\hat{x}$. The rest of the results of Section 2 hold in this setting if we replace $L_0, b's, \gamma_0, e^{-\gamma_0 n}, \rho_0^*$ respectively by $L, c's, \gamma, e^{-\gamma 3^n}, \rho^*$.

4. Examples

In this section we first consider the example considered in [29] for illustrating the algorithm considered in section 2. We apply the algorithm by choosing a sequence of finite dimensional subspace (V_n) of X with $dim V_n = n+1$. Precisely we choose V_n as the linear span of $\{v_1, v_2, \cdots, v_{n+1}\}$ where $v_i, i = 1, 2, \cdots, n+1$ are the linear splines in a uniform grid of $n+1$ points in $[0, 1]$.

EXAMPLE 4.1. (see [29], section 4.3) Let $F : D(F) \subseteq L^2(0,1) \longrightarrow L^2(0,1)$ defined by

$$F(u) := \int_0^1 k(t,s)u^3(s)ds,$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1 \\ (1-s)t, & 0 \leq t \leq s \leq 1 \end{cases}.$$

Then for all $x(t), y(t) : x(t) > y(t)$:

$$\langle F(x) - F(y), x - y \rangle = \int_0^1 \left[\int_0^1 k(t,s)(x^3 - y^3)(s)ds \right] (x-y)(t)dt \geq 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3\int_0^1 k(t,s)(u(s))^2 w(s)ds. \tag{4.1}$$

Note that for $u, v > 0$,

$$\begin{aligned} (F'(v) - F'(u))w &= 3\int_0^1 k(t,s)[(v(s))^2 - (u(s))^2]w(s)ds \\ &:= F'(u)\Phi(v,u,w), \end{aligned}$$

where $\Phi(v,u,w) = \frac{(v^2-u^2)w}{u^2}$.

Observe that

$$\begin{aligned} \Phi(v,u,w) &= \frac{(v^2-u^2)w}{u^2} \\ &= \frac{(u+v)(v-u)w}{u^2}. \end{aligned}$$

So condition $(C1)'$ satisfies with $L_0 \geq \|\frac{u+v}{u^2}\|$.

In our computation, we take $f(t) = \frac{6\cos(\pi t) + \cos^3(\pi t) + 14t - 7}{9\pi^2}$ and $f^\delta = f + \delta$. Then the exact solution

$$\hat{x}(t) = \cos(\pi t).$$

We use

$$x_0(t) = \cos(\pi t) + \frac{3(t\pi^2 - t^2\pi^2 + sin^2(\pi t))}{4\pi^2}$$

as our initial guess, so that the function $x_0 - \hat{x}$ satisfies the source condition

$$x_0 - \hat{x} = \varphi(F'(\hat{x}))\frac{1}{4}$$

where $\varphi(\lambda) = \lambda$.

Observe that while performing numerical computation on finite dimensional subspace (V_n) of X, one has to consider the operator $P_nF'(.)P_n$ instead of $F'(.)$, where P_n is the orthogonal projection on to V_n. Thus incurs an additional error $\|P_nF'(.)P_n - F'(.)\| = O(\|F'(.)(I - P_n)\|)$.

Let $\|F'(.)(I - P_n)\| \leq \varepsilon_n$. For the operator $F'(.)$ defined in (4.1), $\varepsilon_n = O(n^{-2})$ (cf. [14]). Thus we expect to obtain the rate of convergence $O((\delta + \varepsilon_n)^{\frac{1}{2}})$.

We choose $\alpha_0 = (1.1)(\delta + \varepsilon_n)$, $\mu = 1.1$. The results of the computation are presented in Table 1. The plots of the exact solution for ($n = 128$ to $n = 1024$) and the approximate solution obtained are given in Figures 1 and 2.

TABLE 1. Iterations and corresponding error estimates

n	k	n_k	$\delta + \varepsilon_n$	α_k	$\|x_k - \hat{x}\|$	$\frac{\|x_k - \hat{x}\|}{(\delta+\varepsilon_n)^{1/2}}$
8	2	2	0.0135	0.0180	0.3575	3.0782
16	2	2	0.0134	0.0178	0.2573	2.2247
32	2	2	0.0133	0.0178	0.1871	1.6196
64	2	2	0.0133	0.0177	0.1394	1.2073
128	2	2	0.0133	0.0177	0.1079	0.9344
256	2	2	0.0133	0.0177	0.0880	0.7619
512	2	2	0.0133	0.0177	0.0761	0.6588
1024	2	2	0.0133	0.0177	0.0694	0.6007

FIGURE 1. Curves of the exact and approximate solutions

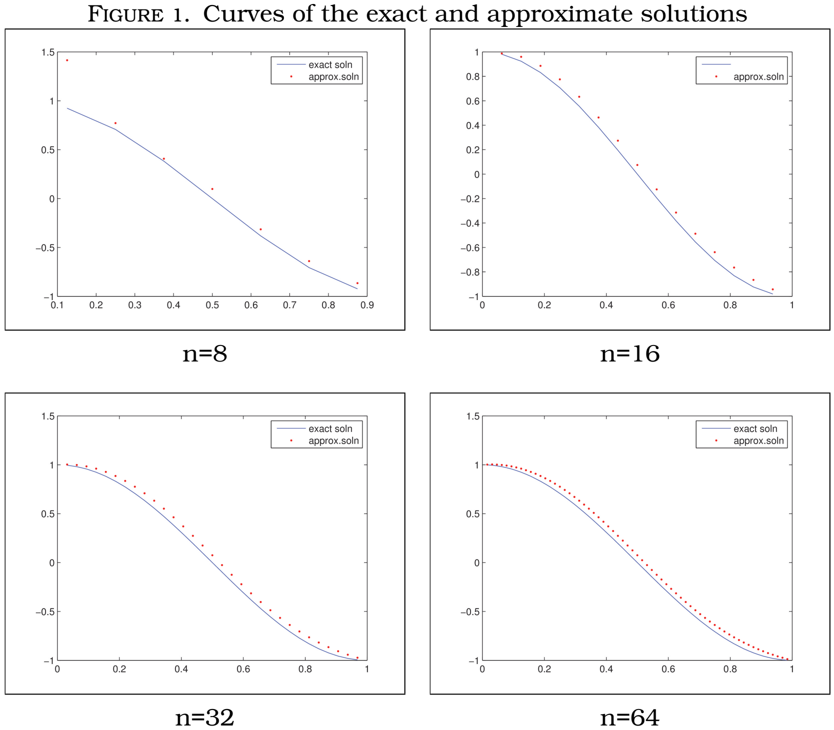
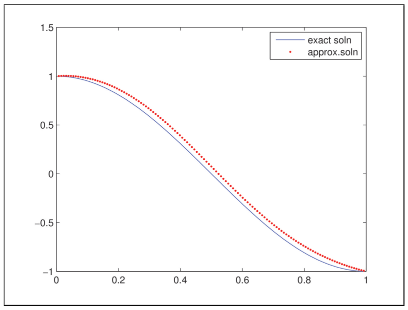
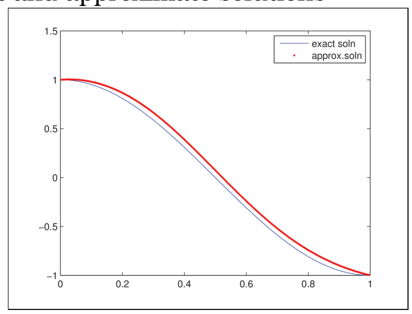


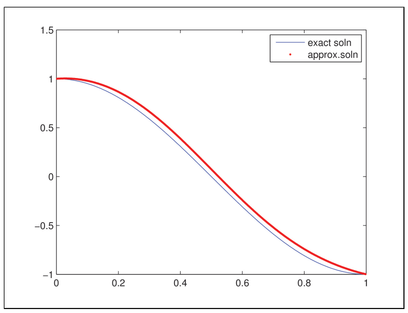
FIGURE 2. Curves of the exact and approximate solutions



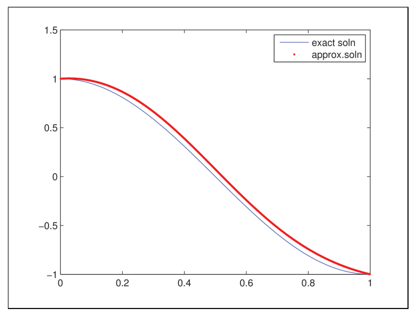
n=128



n=256



n=512



n=1024

Next we present two examples where $(C1)$ is not satisfied but $(C1)'$ is satisfied.

EXAMPLE 4.2. Let $X = Y = \mathbb{R}$, $D = [0, \infty), x_0 = 1$ and define function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1 x + c_2, \tag{4.2}$$

where c_1, c_2 are real parameters and $i > 2$ an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. However central Lipschitz condition $(C1)'$ holds for $L_0 = 1$.

Indeed, we have

$$\begin{aligned} \|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{x_0^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}} \\ &\leq L_0|x - x_0|. \end{aligned}$$

EXAMPLE 4.3. We consider the integral equations

$$u(s) = f(s) + \tau \int_a^b G(s,t) u(t)^{1+1/n} dt, \quad n \in \mathbb{N}. \tag{4.3}$$

Here, f is a given continuous function satifying $f(s) > 0, s \in [a, b], \tau$ is a real number, and the kernel G is continuous and positive in $[a, b] \times [a, b]$.

For example, when $G(s, t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \tau u^{1+1/n} \tag{4.4}$$

$$u(a) = f(a), u(b) = f(b). \tag{4.5}$$

These type of problems have been considered in [1], [2], [19].

Equation of the form (4.3) generalize equations of the form

$$u(s) = \int_a^b G(s,t)u(t)^n dt \tag{4.6}$$

studied in [1], [2], [19]. Instead of (4.3) we can try to solve the equation $F(u) = 0$ where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \Omega = \{u \in C[a,b] : u(s) \geq 0, s \in [a,b]\},$$

and

$$F(u)(s) = u(s) - f(s) - \tau \int_a^b G(s,t)u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm.

The derivative F' is given by

$$F'(u)v(s) = v(s) - \tau(1+\frac{1}{n}) \int_a^b G(s,t)u(t)^{1/n} v(t)dt, \quad v \in \Omega.$$

First of all, we notice that F' does not satisfy a Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1], G(s,t) = 1$ and $y(t) = 0$. Then $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\tau|(1+\frac{1}{n}) \int_a^b x(t)^{1/n} dt.$$

If F' were a Lipschitz function, then

$$\|F'(x) - F'(y)\| \leq L_1 \|x - y\|,$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n} dt \leq L_2 \max_{x \in [0,1]} x(s), \tag{4.7}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the functions

$$x_j(t) = \frac{t}{j}, \quad j \geq 1, \quad t \in [0,1].$$

If these are substituted into (4.7)

$$\frac{1}{j^{1/n}(1+1/n)} \leq \frac{L_2}{j} \Leftrightarrow j^{1-1/n} \leq L_2(1+1/n), \quad \forall j \geq 1.$$

This inequality is not true when $j \to \infty$.

Therefore, condition (4.7) is not satisfied in this case. However, condition $(C1)'$ holds. To show this, let $x_0(t) = f(t)$ and $\gamma = \min_{s\in[a,b]} f(s), \alpha > 0$ Then for $v \in \Omega$,

$$\begin{aligned}
\|[F'(x) - F'(x_0)]v\| &= |\tau|(1+\frac{1}{n}) \max_{s\in[a,b]} |\int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n})v(t)dt| \\
&\leq |\tau|(1+\frac{1}{n}) \max_{s\in[a,b]} G_n(s,t)
\end{aligned}$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}} \|v\|$.

Hence,

$$\begin{aligned}
\|[F'(x) - F'(x_0)]v\| &= \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b G(s,t)dt \|x - x_0\| \\
&\leq L_0 \|x - x_0\|,
\end{aligned}$$

where $L_0 = \frac{|\tau|(1+1/n)}{\gamma^{(n-1)/n}}N$ and $N = \max_{s\in[a,b]}\int_a^b G(s,t)dt$. Then condition $(C1)'$ holds for sufficiently small τ.

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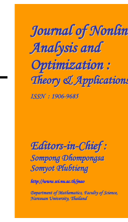

SUZUKI-TYPE FIXED POINT THEOREMS FOR TWO MAPS ON METRIC-TYPE SPACES

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ABSTRACT. In this paper, we generalize the Suzuki-type fixed point theorems in [N. Hussain, D. Dorić, Z. Kadelburg, and S. Radenović, *Suzuki-type fixed point results in metric type spaces*, Fixed Point Theory Appl **2012:126** (2012), 1 - 10] for two maps on metric-type spaces. Examples are given to validate the results.

KEYWORDS: Suzuki-type fixed point; metric-type space.
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1. INTRODUCTION AND PRELIMINARIES

In [2], Hussain, Dorić, Kadelburg and Radenović have proved the following theorems. These results are generalizations of Suzuki-type fixed point theorems in [8] and [9].

Theorem 1.1 ([2], Theorem 3). *Let (X, D, K) be a complete metric-type space, let $T : X \longrightarrow X$ be a map and let $\theta = \theta_K : [0, 1) \longrightarrow \left(\frac{1}{K+1}, 1\right]$ be defined by*

$$\theta(r) = \theta_K(r) = \begin{cases} 1 & \text{if } 0 \le r \le \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} < r \le b_K \\ \frac{1}{K+r} & \text{if } b_K < r < 1 \end{cases}$$

where $b_K = \frac{1-K+\sqrt{1+6K+K^2}}{4}$ is the positive solution of $\frac{1-r}{r^2} = \frac{1}{K+r}$, satisfying the following conditions

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(i) D *is continuous in each variable.*
(ii) *There exists* $r \in [0, 1)$ *such that for each* $x, y \in X$,

$$\theta(r)D(x, Tx) \leq D(x, y) \text{ implies } D(Tx, Ty) \leq \frac{r}{K} M(x, y) \tag{1.1}$$

where

$$M(x, y) = \max\Big\{D(x, y), D(x, Tx), D(y, Ty), \frac{1}{2K}\big[D(x, Ty) + D(y, Tx)\big]\Big\}.$$

Then we have

(i) T *has a unique fixed point* $z \in X$.
(ii) *For each* $x \in X$, *the sequence* $\{T^n x\}$ *converges to* z.
(iii) T *has the property* (P).

Theorem 1.2 ([2], Theorem 4)**.** *Let* (X, D, K) *be a metric-type space and let* $T : X \longrightarrow X$ *be a map satisfying the following conditions*

(i) X *is compact.*
(ii) D *is continuous.*
(iii) *For all* $x, y \in X$ *and* $x \neq y$,

$$\frac{1}{1 + K} D(x, Tx) < D(x, y) \text{ implies } D(Tx, Ty) < \frac{1}{K} D(x, y). \tag{1.2}$$

Then T *has a unique fixed point in* X.

In this paper, we extend the main results in [2] for two maps on metric-type spaces. Examples are given to validate the results.

First we recall some notions and lemmas which will be useful in what follows.

Definition 1.3 ([6], Definition 6)**.** Let X be a nonempty set, let $K \geq 1$ be a real number and let $D : X \times X \longrightarrow [0, \infty)$ satisfy the following properties

(i) $D(x, y) = 0$ if and only if $x = y$.
(ii) $D(x, y) = D(y, x)$ for all $x, y \in X$;
(iii) $D(x, z) \leq K\big[D(x, y) + D(y, z)\big]$ for all $x, y, z \in X$.

Then (X, D, K) is called a *metric-type space.*

Note that a metric-type space was introduced and studied under the name of a b*-metric space* by Czerwik in [1]. Moreover, in [5], Khamsi introduced another definition of a metric-type space with a bit difference, where the condition (3) in Definition 1.3 is replaced by

$D(x, z) \leq K\big[D(x, y_1) + \cdots + D(y_n, z)\big]$ for all $x, y_1, \cdots, y_n, z \in X$.

Definition 1.4 ([6], Definition 7)**.** Let (X, D, K) be a metric-type space.

(i) A sequence $\{x_n\}$ is called *convergent* to $x \in X$ if $\lim_{n\to\infty} D(x_n, x) = 0$.
(ii) A sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.
(iii) (X, D, K) is called *complete* if every Cauchy sequence is a convergent sequence.

Definition 1.5 ([3], page 2)**.** A map $T : X \to X$ is called to have the *property* (P) if $\mathcal{F}(T) = \mathcal{F}(T^n)$ for all $n \in \mathbb{N}$, where $\mathcal{F}(T) = \{x \in X : Tx = x\}$.

Definition 1.6 ([7], Definition 1.2)**.** Let (X, d) be a metric space and $T : X \longrightarrow X$ be a map. T is called *sequentially convergent* if $\{y_n\}$ is convergent provided $\{Ty_n\}$ is convergent.

Lemma 1.7 ([4], Lemma 3.1)**.** *Let $\{y_n\}$ be a sequence in a metric-type space (X, D, K) such that*

$$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n) \tag{1.3}$$

for some $\lambda \in [0, \frac{1}{K})$ and all $n \in \mathbb{N}$. Then $\{y_n\}$ is a Cauchy sequence in (X, D, K).

2. MAIN RESULTS

The following result is a sufficient condition for a map on a metric-type space having the property (P). If $K = 1$, this result becomes [3, Theorem 1.1].

Lemma 2.1. *Let (X, D, K) be a metric-type space and $T : X \longrightarrow X$ be a map such that*

$$D(Tx, T^2x) \leq \lambda D(x, Tx) \tag{2.1}$$

for some $0 \leq \lambda < 1$ and all $x \in X$. Then T has property (P).

Proof. If $u \in \mathcal{F}(T^n)$, that is, $T^n u = u$, then from (2.1) we have

$$D(u, Tu) = D(TT^{n-1}u, T^2T^{n-1}u) \leq \lambda D(T^{n-1}u, TT^{n-1}u) \leq \cdots \leq \lambda^n D(u, Tu).$$

Since $0 \leq \lambda^n < 1$, we get $D(u, Tu) = 0$, that is, $u \in \mathcal{F}(T)$.

If $u \in \mathcal{F}(T)$, that is $Tu = u$, then

$$D(u, T^n u) = D(u, T^{n-1}u) = \ldots = D(u, Tu) = 0.$$

Then $T^n u = u$, that is $u \in \mathcal{F}(T^n)$. This proves that T has property (P). □

The first main result of the paper is as follows.

Theorem 2.2. *Let (X, D, K) be a complete metric-type space, let $T, F : X \longrightarrow X$ be two maps and let $\theta = \theta_K : [0, 1) \longrightarrow \left(\frac{1}{K+1}, 1\right]$ be defined by*

$$\theta(r) = \theta_K(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2} \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} < r \leq b_K \\ \frac{1}{K+r} & \text{if } b_K < r < 1 \end{cases} \tag{2.2}$$

where $b_K = \frac{1 - K + \sqrt{1 + 6K + K^2}}{4}$ is the positive solution of $\frac{1-r}{r^2} = \frac{1}{K+r}$, satisfying the following conditions

(i) *D is continuous in each variable.*

(ii) *There exists $r \in [0, 1)$ such that for each $x, y \in X$*

$$\theta(r)D(Fx, FTx) \leq D(Fx, Fy) \text{ implies } D(FTx, FTy) \leq \frac{r}{K}M(x, y) \tag{2.3}$$

where

$$M(x, y) = \max\Big\{D(Fx, Fy), D(Fx, FTx), D(Fy, FTy), \frac{1}{2K}\big[D(Fx, FTy)+D(Fy, FTx)\big]\Big\}.$$

(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point $a \in X$.*

(ii) *For each $x \in X$, the sequence $\{FT^n x\}$ converges to Fa.*

(iii) *If $TF = FT$, then T has the property (P) and F, T have a unique common fixed point.*

Proof. (1). For each $x \in X$, since $\theta(r) \le 1$, we have $\theta(r)D(Fx, FTx) \le D(Fx, FTx)$. It follows from (2.3) that

$$\begin{aligned} & D(FTx, FT^2x) \qquad (2.4) \\ \le\ & \frac{r}{K}\max\Big\{D(Fx,FTx), D(Fx,FTx), D(FTx,FT^2x), \\ & \frac{1}{2K}\big[D(Fx,FT^2x)+D(FTx,FTx)\big]\Big\} \\ \le\ & \frac{r}{K}\max\Big\{D(Fx,FTx), D(FTx,FT^2x), \frac{1}{2K}K\big[D(Fx,FTx)+D(FTx,FT^2x)\big]\Big\} \\ =\ & \frac{r}{K}\max\Big\{D(Fx,FTx), D(FTx,FT^2x)\Big\}. \end{aligned}$$

We consider following two cases.

Case 1. $\max\Big\{D(Fx,FTx), D(FTx,FT^2x)\Big\} = D(FTx,FT^2x)$. Then (2.4) becomes $D(FTx,FT^2x) \le \frac{r}{K}D(FTx,FT^2x)$. Since $\frac{r}{K} < 1$, we have

$$D(FTx, FT^2x) = 0 \qquad (2.5)$$

that is $FTx = FT^2x$. Note that F is one-to-one, then $Tx = T^2x$. Therefore, $a = Tx$ is a fixed point of T.

Case 2. $\max\Big\{D(Fx,FTx), D(FTx,FT^2x)\Big\} = D(Fx,FTx)$. Then (2.4) becomes

$$D(FTx, FT^2x) \le \frac{r}{K}D(Fx, FTx). \qquad (2.6)$$

Put $x_{n+1} = Tx_n$ and $y_n = FTx_n$ for all $n \in \mathbb{N}$ where $x_0 = x$. We also have $x_n = T^n x$ and $y_n = Fx_{n+1}$. It follows from (2.6) that

$$D(y_n, y_{n+1}) = D(FTx_n, FT^2x_n) \le \frac{r}{K}D(Fx_n, FTx_n) = \frac{r}{K}D(y_{n-1}, y_n). \qquad (2.7)$$

Using Lemma 1.7, we conclude that $\{y_n\}$ is a Cauchy sequence in the compete metric-type space X. Then y_n converges to z for some $z \in X$. Since F is sequentially convergent, $\{x_n\}$ converges to some $a \in X$ and also from the continuity of F, $\{Fx_n\}$ converges to Fa. Note that $\{y_{n-1}\}$ converges to z, then

$$y_{n-1} = FTx_{n-1} = Fx_n \to Fa = z. \qquad (2.8)$$

Let us prove now that

$$D(FTx, z) \le \frac{r}{K}\max\Big\{D(Fx, z), D(Fx, FTx)\Big\} \qquad (2.9)$$

holds for each $x \neq a$. Indeed, since $Fx_n \to z$ and $FTx_n \to z$ and by the continuity of D, we have

$$D(Fx_n, FTx_n) \to 0 \text{ and } D(Fx_n, Fx) \to D(z, Fx) \neq 0. \qquad (2.10)$$

Then there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\theta(r)D(Fx_n, FTx_n) < D(Fx_n, Fx). \qquad (2.11)$$

From (2.3) and (2.11), we have for such n

$$\begin{aligned} D(FTx_n, FTx) \ \le\ & \frac{r}{K}\max\Big\{D(Fx_n,Fx), D(Fx_n,FTx_n), D(Fx,FTx), \qquad (2.12) \\ & \frac{1}{2K}\big[D(Fx_n,FTx)+D(Fx,FTx_n)\big]\Big\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in (2.12) and using (2.10) and the continuity of D, we get

$$\begin{aligned} & D(z, FTx) \\ \leq\ & \frac{r}{K}\max\Big\{D(z,Fx), D(Fx,FTx), \frac{1}{2K}\big(D(z,FTx)+D(Fx,z)\big)\Big\} \\ \leq\ & \frac{r}{K}\max\Big\{D(z,Fx), D(Fx,FTx), \frac{1}{2K}K\big(D(z,Fx)+D(Fx,FTx)\big)+\frac{1}{2K}D(Fx,z)\Big\} \\ \leq\ & \frac{r}{K}\max\Big\{D(z,Fx), D(Fx,FTx)\Big\}. \end{aligned}$$

Hence, we have (2.9) .

For each $n \geq 1$, put $x = T^{n-1}a$. Therefore,

$$D(FT^n a, FT^{n+1}a) \leq \frac{r}{K}D(FT^{n-1}a, FT^n a)$$

holds for each $n \in \mathbb{N}$ where $FT^0 a = z$. By induction, we have

$$D(FT^n a, FT^{n+1}a) \leq \frac{r^n}{K^n}D(z, FTa). \tag{2.13}$$

Now we will prove that

$$D(FT^n a, z) \leq D(FTa, z) \tag{2.14}$$

holds for all $n \geq 1$ by induction. For $n = 1$ this relation is obvious. Suppose that it holds for some n. If $FT^n a = z$, note that $z = Fa$ and F is one-to-one, then $T^n a = a$. It implies that $FT^{n+1}a = FTa$ and $D(FT^{n+1}a, z) = D(FTa, z)$. If $FT^n a \neq z$, then from (2.9), (2.13) and the induction hypothesis, we get

$$\begin{aligned} D(FT^{n+1}a, z) &\leq \frac{r}{K}\max\Big\{D(FT^n a, z), D(FT^n a, FT^{n+1}a)\Big\} \\ &\leq \frac{r}{K}\max\Big\{D(FTa, z), \frac{r^n}{K^n}D(z, FTa)\Big\} \\ &\leq \frac{r}{K}D(FTa, z) \end{aligned}$$

and that (2.14) is proved.

Now we will prove that a is a fixed point of T. Suppose to the contrary that $Ta \neq a$, that is, $FTa \neq Fa$ or equivalently,

$$FTa \neq z. \tag{2.15}$$

We consider following two subcases.

Subcase 2.1. $0 \leq r < b_K$. That implies $\theta(r) \leq \dfrac{1-r}{r^2}$.

We will prove

$$D(FT^n a, FTa) \leq \frac{r}{K}D(FTa, z) \tag{2.16}$$

holds for all $n \geq 1$ by induction. For $n = 1$, (2.16) obvious and for $n = 2$, (2.16) follows from (2.13). Suppose that (2.16) holds for some $n > 2$. Then we have

$$D(z, FTa) \leq K\big[D(z, FT^n a) + D(FT^n a, FTa)\big] \leq K\big[D(z, FT^n a) + \frac{r}{K}D(FTa, z)\big].$$

Hence

$$D(z, FTa) \leq \frac{K}{1-r}D(z, FT^n a). \tag{2.17}$$

Since $\theta(r) \le \frac{1-r}{r^2}$ and by using (2.8), (2.13) and (2.17), we get

$$\begin{aligned}\theta(r)D(FT^n a, FT^{n+1}a) &\le \frac{1-r}{r^2}D(FT^n a, FT^{n+1}a)\\ &\le \frac{1-r}{r^n}D(FT^n a, FT^{n+1}a)\\ &\le \frac{1-r}{K^n}D(z, FTa)\\ &\le \frac{1}{K^{n-1}}D(z, FT^n a)\\ &\le D(z, FT^n a)\\ &= D(Fa, FT^n a).\end{aligned}$$

Assumption (2.3) implies that

$$\begin{aligned}D(FTa, FT^{n+1}a) \le \frac{r}{K}\max\Big\{&D(Fa, FT^n a), D(Fa, FTa), D(FT^n a, FT^{n+1}a),\\ &\frac{1}{2K}\big(D(Fa, FT^{n+1}a) + D(FT^n a, FTa)\big)\Big\}.\end{aligned}$$

Using (2.13), (2.14) and the induction hypothesis, we obtain the last maximum is equal to $D(FTa, z)$. That is $D(FTa, FT^{n+1}a) \le \frac{r}{K}D(FTa, z)$ and (2.16) is proved by induction.

From (2.15), we have $FT^n a \neq z$ for each $n \in \mathbb{N}$. If $FT^n a = z$ for some $n \in \mathbb{N}$, then from (2.16) we get $D(z, FTa) = 0$. It is a contradiction with (2.15). So $FT^n a \neq z$ for each $n \in \mathbb{N}$. Hence, (2.9) and (2.13) imply that

$$\begin{aligned}D(FT^{n+1}a, z) &\le \frac{r}{K}\max\Big\{D(FT^n a, z), D(FT^n a, FT^{n+1}a)\Big\}\\ &\le \frac{r}{K}\max\Big\{D(FT^n a, z), \frac{r^n}{K^n}D(z, FTa)\Big\}.\end{aligned} \tag{2.18}$$

Since $D(FTa, z) \le K\big[D(FTa, FT^n a) + D(FT^n a, z)\big]$, it follows from (2.16) that

$$D(FT^n a, z) \ge \frac{1}{K}D(FTa, z) - D(FTa, FT^n a) \ge \frac{1-r}{K}D(FTa, z).$$

Note that there exists $n_1 \in \mathbb{N}$ such that $1 - r \ge r^n$ for all $n \ge n_1$ and $0 \le r \le b_K$. For $n \ge n_1$, we have

$$D(FT^n a, z) \ge \frac{r^n}{K}D(FTa, z) \ge \frac{r^n}{K^n}D(FTa, z).$$

Using (2.18), we have

$$0 \le D(FT^{n+1}a, z) \le \frac{r}{K}D(FT^n a, z) \le \cdots \le \Big(\frac{r}{K}\Big)^{n-n_1+1}D(FT^{n_1}a, z). \tag{2.19}$$

Taking the limit as $n \to \infty$ in (2.19), we get $FT^n a \to z$ and let again $n \to \infty$ in (2.16), we get $D(FTa, z) \le \frac{r}{K}D(FTa, z)$ that means $D(FTa, z) = 0$. Therefore, $FTa = z$. It is a contradiction with (2.15).

Subcase 2.2. $b_K \le r < 1$. That implies $\theta(r) = \frac{1}{K+r}$. We will prove there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\theta(r)D(Fx_{n_j+1}, FTx_{n_j+1}) = \theta(r)D(y_{n_j}, y_{n_j+1}) \le D(y_{n_j}, z) \tag{2.20}$$

holds for each $j \in \mathbb{N}$. If

$$\frac{1}{K+r}D(y_{n-1}, y_n) > D(y_{n-1}, z) \text{ and } \frac{1}{K+r}D(y_n, y_{n+1}) > D(y_n, z)$$

hold for some $n \in \mathbb{N}$, then (2.7) we have

$$\begin{aligned} D(y_{n-1}, y_n) &\leq K\big[D(y_{n-1}, z) + D(z, y_n)\big] \\ &< \frac{K}{K+r}\big[D(y_{n-1}, y_n) + D(y_n, y_{n+1})\big] \\ &\leq \frac{K}{K+r}\big[D(y_{n-1}, y_n) + \frac{r}{K}D(y_{n-1}, y_n)\big] \\ &= D(y_{n-1}, y_n). \end{aligned}$$

It is impossible. Hence

$$\theta(r)D(y_{n-1}, y_n) \leq D(y_{n-1}, z) \text{ or } \theta(r)D(y_n, y_{n+1}) \leq D(y_n, z)$$

holds for some $n \in \mathbb{N}$. In particular

$$\theta(r)D(y_{2n-1}, y_{2n}) \leq D(y_{2n-1}, z) \text{ or } \theta(r)D(y_{2n}, y_{2n+1}) \leq D(y_{2n}, z)$$

holds for all $n \in \mathbb{N}$. In other words there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ that satisfies (2.20) for each $j \in \mathbb{N}$. But the assumption (2.3) implies that

$$\begin{aligned} &D(FTx_{n_j+1}, FTa) \\ \leq\ &\frac{r}{K}.\max\Big\{D(Fx_{n_j+1}, Fa), D(Fx_{n_j+1}, FTx_{n_j+1}), D(Fa, FTa), \\ &\qquad \frac{r}{2K}\big[D(Fx_{n_j+1}, FTa) + D(Fa, FTx_{n_j+1})\big]\Big\}. \end{aligned} \tag{2.21}$$

Taking the limit as $j \to \infty$ in (2.21), we obtain

$$D(z, FTa) \leq \frac{r}{K}.D(Fa, FTa) = \frac{r}{K}D(z, FTa).$$

It implies $D(z, FTa) = 0$, that is $z = FTa$. It is a contradiction with (2.15).

From two above subcases, we get $Ta = a$, that is a is a fixed point of T.

Finally, we prove that a is a unique fixed point of T. Indeed, if a and b are two fixed points of T, then (2.9) implies that

$$D(Fa, Fb) = D(FTa, Fb) \leq \frac{r}{K}\max\Big\{D(Fa, Fb), D(Fa, FTa)\Big\} = \frac{r}{K}D(Fa, Fb).$$

Since $\frac{r}{K} < 1$, we have $D(Fa, Fb) = 0$, that is $Fa = Fb$. Also since F is one-to-one, we get $a = b$.

(2). It is a direct consequence of (2.8).

(3). From (2.5) and (2.6), we have

$$D(FTx, FT^2x) \leq \frac{r}{K}D(Fx, FTx). \tag{2.22}$$

Note that the property (P) follows from (2.22) and Lemma 2.1. We need only prove T and F have a unique common fixed point. Let a be the unique fixed point of T. Suppose to the contrary that $Fa \neq a$. Since F is one-to-one, $F^2a \neq Fa$. Then

$$\theta(r)D(Fa, FTa) = 0 < D(Fa, F^2a).$$

It follows from (2.3) that

$$D(FTa, FTFa) = D(FTa, F^2Ta) = D(Fa, F^2a) \leq \frac{r}{K} M(a, Fa)$$

where

$$\begin{aligned} & M(a, Fa) \\ = \;& \max\Big\{D(Fa, F^2a), D(Fa, FTa), D(F^2a, F^2Ta), \frac{1}{2K}\big[D(Fa, F^2Ta) + D(F^2a, FTa)\big]\Big\} \\ = \;& D(Fa, F^2a). \end{aligned}$$

Therefore,

$$D(Fa, F^2a) \leq \frac{r}{K} D(Fa, F^2a) < D(Fa, F^2a).$$

It is a contradiction. This proves that a is a unique common fixed point of T and F. □

Remark 2.3. By choosing F is the identity in Theorem 2.2, we get Theorem 1.1.

From Theorem 2.2, we get following corollaries.

Corollary 2.4. *Let (X, D, K) be a complete metric-type space, let $T, F : X \longrightarrow X$ be two maps and let $\theta = \theta_K : [0, 1) \longrightarrow \Big(\frac{1}{K+1}, 1\Big]$ be defined by (2.2) and satisfy the following conditions*

(i) *D is continuous in each variable.*
(ii) *There exists $r \in [0, 1)$ such that for each $x, y \in X$,*

$$\theta(r)D(Fx, FTx) \leq D(Fx, Fy) \text{ implies } D(FTx, FTy) \leq \frac{r}{K} D(Fx, Fy). \quad (2.23)$$

(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point $z \in X$.*
(ii) *For each $x \in X$, the sequence $\{FT^n x\}$ converges to Fz.*
(iii) *If $TF = FT$ then T has the property (P) and F, T have a unique common fixed point.*

Corollary 2.5. *Let (X, D, K) be a complete metric-type space, let $T, F : X \longrightarrow X$ be two maps and let $\theta = \theta_K : [0, 1) \longrightarrow \Big(\frac{1}{K+1}, 1\Big]$ be defined by (2.2) and satisfy the following conditions*

(i) *D is continuous in each variable.*
(ii) *There exists $r \in [0, 1)$ such that for each $x, y \in X$,*

$$\theta(r)D(Fx, FTx) \leq D(Fx, Fy)$$

$$\text{implies } D(FTx, FTy) \leq \frac{r}{K} \max\Big\{D(Fx, FTx), D(Fy, FTy)\Big\}. \quad (2.24)$$

(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point $z \in X$.*
(ii) *For each $x \in X$, the sequence $\{FT^n x\}$ converges to Fz.*
(iii) *If $TF = FT$ then T has the property (P) and F, T have a unique common fixed point.*

Corollary 2.6. *Let (X, D, K) be a complete metric-type space, let $T, F : X \longrightarrow X$ be two maps and let $\theta = \theta_K : [0, 1) \longrightarrow \left(\frac{1}{K+1}, 1\right]$ be defined by (2.2) and satisfy the following conditions*

(i) *D is continuous in each variable.*
(ii) *There exists $r \in [0, 1)$ such that for each $x, y \in X$,*

$$\theta(r)D(Fx, FTx) \leq D(Fx, Fy)$$

$$\text{implies } D(FTx, FTy) \leq \frac{r}{2K}\big[D(Fx, FTy) + D(Fy, FTx)\big]. \tag{2.25}$$

(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point $z \in X$.*
(ii) *For each $x \in X$, the sequence $\{FT^n x\}$ converges to Fz.*
(iii) *If $TF = FT$ then T has the property (P) and F, T have a unique common fixed point.*

Remark 2.7. Corollary 2.4 is a generalization of [2, Corollary 1], Corollary 2.5 is a generalization of [2, Corollary 2] and Corollary 2.6 is a generalization of [2, Corollary 3].

The second main result of the paper is as follows.

Theorem 2.8. *Let (X, D, K) be a metric-type space where D is continuous and let $T, F : X \longrightarrow X$ be two maps satisfying the conditions*

(i) *For all $x, y \in X$ and $x \neq y$,*

$$\frac{1}{1+K}D(Fx, FTx) < D(Fx, Fy) \text{ implies } D(FTx, FTy) < \frac{1}{K}D(Fx, Fy). \tag{2.26}$$

(ii) *$F(X)$ is compact.*
(iii) *F is one-to-one, continuous and sequentially convergent.*

Then we have

(i) *T has a unique fixed point in X.*
(ii) *If $TF = FT$ then F, T have a unique common fixed point.*

Proof. (1). First, denote $\beta = \inf\{D(Fx, FTx) : x \in X\}$ and choose a sequence $\{x_n\}$ in X such that $D(Fx_n, FTx_n) \to \beta$. Since $F(X)$ is compact, so there exist $Fv, Fw \in F(X)$ such that $Fx_n \to Fv$ and $FTx_n \to Fw$. Since F is continuous, one-to-one and sequentially convergent, we get $x_n \to v$ and $Tx_n \to w$. Note that the continuity of D implies

$$\lim D(Fx_n, Fw) = \lim D(Fv, Fw) = \lim D(Fx_n, FTx_n) = \beta.$$

We will prove $\beta = 0$. Suppose to the contrary that $\beta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\frac{2+K}{2+2K}\beta < D(Fx_n, Fw) \text{ and } D(Fx_n, FTx_n) < \frac{2+K}{2}\beta.$$

Then $\frac{1}{1+K}D(Fx_n, FTx_n) < D(Fx_n, Fw)$ and the assumption (2.26) implies that

$$D(FTx_n, FTw) < \frac{1}{K}D(Fx_n, Fw). \tag{2.27}$$

Taking the limit as $n \to \infty$ in (2.27), we obtain $D(Fw, FTw) \leq \frac{1}{K}\beta$.

If $K > 1$, then $D(Fw, FTw) < \beta$. It is impossible by the definition of β.

If $K = 1$, then $D(Fw, FTw) = \beta$ and

$$\frac{1}{1+K}D(Fw, FTw) < D(Fw, FTw).$$

It follows from (2.26) that

$$D(FTw, FT^2w) < \frac{1}{K}D(Fw, FTw) = \beta.$$

It is also impossible by the definition of β.

Hence, in all cases we obtain a contradiction and it follows that $\beta = 0$ and so $Fv = Fw$. Since F is one-to-one, we have $v = w$.

Now we prove that T has a fixed point. Suppose to the contrary that $Tz \neq z$ for all $z \in X$. Since F is one-to-one, we have $FTz \neq Fz$ for all $z \in X$. In particular, we get

$$0 < \frac{1}{1+K}D(Fx_n, FTx_n) < D(Fx_n, FTx_n).$$

It follows from (2.26) that $D(FTx_n, FT^2x_n) < \frac{1}{K}D(Fx_n, FTx_n)$. Therefore,

$$\begin{aligned} D(Fv, FT^2x_n) &\leq K\big[D(Fv, FTx_n) + D(FTx_n, FT^2x_n)\big] \\ &< KD(Fv, FTx_n) + D(Fx_n, FTx_n). \end{aligned} \tag{2.28}$$

Taking the limit as $n \to \infty$ in (2.28), we get $D(Fv, FT^2x_n) \to 0$, that is, $FT^2x_n \to Fv$.

Suppose that

$$\frac{1}{1+K}D(Fx_n, FTx_n) \geq D(Fx_n, Fv)$$

and

$$\frac{1}{1+K}D(FTx_n, FT^2x_n) \geq D(FTx_n, Fv)$$

both hold for some $n \in \mathbb{N}$. Then

$$\begin{aligned} D(Fx_n, FTx_n) &\leq K\big[D(Fx_n, Fv) + D(FTx_n, Fv)\big] \\ &\leq \frac{K}{1+K}\big[D(Fx_n, FTx_n) + D(FTx_n, FT^2x_n)\big] \\ &< \frac{K}{1+K}\big[D(Fx_n, FTx_n) + \frac{1}{K}.D(Fx_n, FTx_n)\big] \\ &= D(Fx_n, FTx_n). \end{aligned}$$

That is impossible. Thus, for each $n \in \mathbb{N}$, either

$$\frac{1}{1+K}D(Fx_n, FTx_n) < D(Fx_n, Fv)$$

or

$$\frac{1}{1+K}D(FTx_n, FT^2x_n) < D(FTx_n, Fv)$$

holds. It follows from (2.26) that, for each $n \in \mathbb{N}$, either

$$D(FTx_n, FTv) < \frac{1}{K}D(Fx_n, Fv) \tag{2.29}$$

or

$$D(FT^2x_n, FTv) < \frac{1}{K}D(FTx_n, Fv) \tag{2.30}$$

holds. If (2.29) holds only for finitely many $n \in \mathbb{N}$, then (2.32) holds for infinitely many $n \in \mathbb{N}$. Thus, there exists a sequence $\{n_k\}$ such that

$$D(FT^2x_{n_k}, FTv) < \frac{1}{K}D(FTx_{n_k}, Fv) \tag{2.31}$$

holds for each $k \in \mathbb{N}$. If (2.29) holds for infinitely many $n \in \mathbb{N}$, then there exists a sequence $\{n_j\}$ such that

$$D(FTx_{n_j}, FTv) < \frac{1}{K}D(Fx_{n_j}, Fv) \tag{2.32}$$

holds for each $j \in \mathbb{N}$.

In both cases, taking the limit as $k \to \infty$ in (2.31) or $j \to \infty$ in (2.32), we obtain $D(Fv, FTv) = 0$, that is, $Fv = FTv$. Since F is one-to-one, we get $v = Tv$. This is a contradiction with the assumption that T has no any fixed point.

Finally, we prove the uniqueness of the fixed point. Suppose to the contrary that y, z are two fixed points of T and $z \neq y$. Then $Fz = FTz$ and $Fy \neq Fz$. Therefore,

$$\frac{1}{1+K}D(Fz, FTz) < D(Fz, Fy)$$

and (2.26) implies that

$$D(FTz, FTy) < \frac{1}{K}D(Fz, Fy) = \frac{1}{K} \cdot D(FTz, FTy).$$

This is impossible since $K \geq 1$. Thus T has a unique fixed point in X.

(2). Let v be the unique fixed point of T. Suppose to the contrary that $Fv \neq v$. Since F is one-to-one, $F^2v \neq Fv$. Then

$$\frac{1}{1+K}D(Fv, FTv) = 0 < D(Fv, F^2v).$$

It follows from (2.26) that

$$D(FTv, FTFv) = D(FTv, F^2Tv) = D(Fv, F^2v) < \frac{1}{K}D(Fv, F^2v) \leq D(Fv, F^2v).$$

It is a contradiction. This proves that v is a unique common fixed point of T and F. □

The following example shows that Theorem 2.2 is a proper generalization of Theorem 1.1.

Example 2.9. Let $X = [0, +\infty)$, let D be the usual metric on $\mathbb{R}$, that is $K = 1$, and let T, F be defined by

$$Tx = \frac{x^2}{x+1}, Fx = \mathrm{e}^x - 1$$

for all $x \in X$. We have

$$\begin{aligned} D(Tx, T2x) &= \frac{x^2(2x+3)}{(2x+1)(x+1)} \\ D(x, 2x) &= x \\ D(x, Tx) &= \frac{x}{x+1} \\ D(2x, T2x) &= \frac{2x}{2x+1} \\ D(x, T2x) &= \left|\frac{2x^2 - x}{2x+1}\right| \end{aligned}$$

$$D(2x, Tx) \quad = \quad \frac{x^2+2x}{x+1}.$$

Let the condition (1.2) hold. Since

$$\theta(r).D(x,Tx) = \theta(r)\frac{x}{x+1} \leq \frac{x}{x+1} \leq x = D(x,2x)$$

for all $x \in X$, then

$$D(Tx, T2x) \leq rM(x,2x)$$

where

$$M(x,2x) = \max\Big\{x, \frac{x}{x+1}, \frac{2x}{2x+1}, \frac{1}{2}\Big(\Big|\frac{2x^2-x}{2x+1}\Big| + \frac{x^2+2x}{x+1}\Big)\Big\} \leq \frac{x^2+2x}{x+1}.$$

Then we have

$$\frac{x^2(2x+3)}{(2x+1)(x+1)} \leq r\frac{x^2+2x}{x+1}$$

that is

$$\frac{x(2x+3)}{(2x+1)(x+1)} \leq r\frac{x+2}{x+1}$$

for all $x \in X$. Taking the limit as $x \to +\infty$, we get $r \geq 1$. It is a contradiction. This proves that Theorem 1.1 is not applicable to T.

On the other hand, we have

$$\begin{aligned} D(FTx, FTy) &= \Big|e^{\frac{x^2}{x+1}} - e^{\frac{y^2}{y+1}}\Big| \\ D(Fx, Fy) &= |e^x - e^y|. \end{aligned}$$

We consider two following cases.

Case 1. $x \geq y$. Then $D(FTx, FTy) \leq \dfrac{1}{2}D(Fx, Fy)$ is equivalent to

$$2e^{\frac{x^2}{x+1}} - e^x \leq 2e^{\frac{y^2}{y+1}} - e^y.$$

Now we shall prove that $\varphi(x) = 2e^{\frac{x^2}{x+1}} - e^x$ is decreasing on $[0,+\infty)$. Indeed, we have

$$\varphi'(x) = e^x\Big(2\frac{x^2+2x}{(x+1)^2}e^{\frac{-x}{x+1}} - 1\Big).$$

Note that $\psi(x) = 2\dfrac{x^2+2x}{(x+1)^2}e^{\frac{-x}{x+1}} - 1$ satisfies $\psi'(x) = e^{\frac{-x}{x+1}}\dfrac{4-2x^2}{(x+1)^4}$. It implies that

$$\max_{[0,+\infty)} \psi(x) = \psi(\sqrt{2}) < 0.$$

Therefore, $\varphi'(x) < 0$ on $[0,+\infty)$. This proves that $\varphi(x)$ is decreasing. Then we have

$$D(FTx, FTy) < \frac{1}{2}D(Fx, Fy) \tag{2.33}$$

for all $x, y \in X$. This proves that (2.23) holds with $r = \dfrac{1}{2}$.

Case 2. $x < y$. Then $D(FTx, FTy) \leq \dfrac{1}{2}D(Fx, Fy)$ is equivalent to

$$2e^{\frac{y^2}{y+1}} - e^y \leq 2e^{\frac{x^2}{x+1}} - e^x.$$

As the same as Case 1, we also get that (2.23) holds with $r = \dfrac{1}{2}$.

By two above cases, we see that (2.23) holds with $r = \frac{1}{2}$. Note that other conditions in Corollary 2.4 are also satisfied, then Corollary 2.4 is applicable to T and F. We see that $x = 0$ is the unique fixed point of T.

The following example shows that Corollary 2.4 is a proper generalization of [2, Corollary 1].

Example 2.10. For X and F, T as in Example 2.9, we have

$$D(Tx, T2x) = \frac{x^2(2x+3)}{(2x+1)(x+1)}, D(x, 2x) = x.$$

If the condition in [2, Corollary 1] holds, then $\frac{x^2(2x+3)}{(2x+1)(x+1)} \leq r.x$, that is

$$\frac{x(2x+3)}{(2x+1)(x+1)} \leq r \tag{2.34}$$

for all $x \in X$. Taking the limit as $x \to +\infty$ in (2.34), we get $r \geq 1$. It is a contradiction. This proves that [2, Corollary 1] is not applicable to T. As in Example 2.9, Corollary 2.4 is applicable to F and T. Note that $x = 0$ is the unique fixed point of T.

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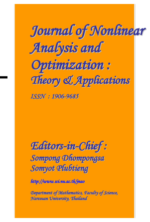

EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR A NONLINEAR NEUTRAL DIFFERENCE EQUATION WITH VARIABLE DELAY

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ABSTRACT. In this paper, we study the existence of positive periodic solutions of the nonlinear neutral difference equation with variable delay

$$x(n+1) = a(n)x(n) + \triangle g(n, x(n-\tau(n))) + f(n, x(n-\tau(n))).$$

The main tool employed here is the Krasnoselskii's hybrid fixed point theorem dealing with a sum of two mappings, one is a contraction and the other is completely continuous. The results obtained here generalize the work of Raffoul and Yankson [7].

KEYWORDS : Positive periodic solutions, nonlinear neutral difference equations, fixed point theorem.
AMS Subject Classification: 39A10, 39A12, 39A23.

1. INTRODUCTION

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential and difference equations, see the references in this article and references therein.

In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of delay difference equations. Motivated by the papers [1]-[5],[7],[8] and the references therein, we concentrate on the existence of positive periodic solutions for the nonlinear neutral difference equation with variable delay

$$x(n+1) = a(n)x(n) + \triangle g(n, x(n-\tau(n))) + f(n, x(n-\tau(n))), \tag{1.1}$$

where

$$g, f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R},$$

with $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Throughout this paper $\triangle$ denotes the forward difference operator $\triangle x(n) = x(n+1) - x(n)$ for any

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sequence $\{x(n), n \in \mathbb{Z}\}$. Also, we define the operator E by $Ex(n) = x(n+1)$. For more on the calculus of difference equations, we refer the reader to [6].

The purpose of this paper is to use Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions for equation (1.1). To apply Krasnoselaskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is completely continuous. In the case $g(n,x) = cx$, Raffoul and Yankson in [7] to show that (1.1) has a positive periodic solutions by using Krasnoselskii's fixed point theorem.

The organization of this paper is as follows. In Section 2, we present the inversion of difference equation (1.1) and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [9]. In Section 3, we present our main results on existence of positive periodic solutions of (1.1). The results presented in this paper generalize the main results in [7].

2. PRELIMINARIES

Let T be an integer such that $T \geq 1$. Define $P_T = \{\varphi \in C(\mathbb{Z}, \mathbb{R}) : \varphi(n+T) = \varphi(n)\}$ where $C(\mathbb{Z}, \mathbb{R})$ is the space of all real valued functions. Then $(P_T, \|.\|)$ is a Banach space with the maximum norm

$$\|x\| = \sup_{n \in [0, T-1] \cap \mathbb{Z}} |x(n)|.$$

Since we are searching for the existence of periodic solutions for equation (1.1), it is natural to assume that

$$a(n+T) = a(n), \ \tau(n+T) = \tau(n), \tag{2.1}$$

with τ being scalar sequence and $\tau(n) \geq \tau^* > 0$. Also, we assume

$$0 < a(n) < 1. \tag{2.2}$$

We also assume that the functions $g(n,x)$ and $f(n,x)$ are continuous in x and periodic in n with period T, that is,

$$g(n+T, x) = g(n,x), \ f(n+T, x) = f(n,x). \tag{2.3}$$

The following lemma is fundamental to our results.

Lemma 2.1. *Suppose (2.1)-(2.3) hold. If $x \in P_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= g(n, x(n-\tau(n))) \\ &\quad + \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, x(u-\tau(u))) - (1-a(u))\, g(u, x(u-\tau(u)))\right], \end{aligned} \tag{2.4}$$

where

$$G(n,u) = \frac{\prod_{s=u+1}^{n+T-1} a(s)}{1 - \prod_{s=n}^{n+T-1} a(s)}. \tag{2.5}$$

Proof. We consider two cases, $n \geq 1$ and $n \leq 0$. Let $x \in P_T$ be a solution of (1.1). For $n \geq 1$ equation (1.1) is equivalent to

$$\triangle\left[x(n) \prod_{s=0}^{n-1} a^{-1}(s)\right] = \left[\triangle g(n, x(n-\tau(n))) + f(n, x(n-\tau(n)))\right] \prod_{s=0}^{n} a^{-1}(s). \tag{2.6}$$

By summing (2.6) from n to $n+T-1$, we obtain

$$\sum_{u=n}^{n+T-1} \triangle \left[x(u) \prod_{s=0}^{u-1} a^{-1}(s) \right]$$
$$= \sum_{u=n}^{n+T-1} \left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u))) \right] \prod_{s=0}^{u} a^{-1}(s).$$

As a consequence, we arrive at

$$x(n+T) \prod_{s=0}^{n+T-1} a^{-1}(s) - x(n) \prod_{s=0}^{n-1} a^{-1}(s)$$
$$= \sum_{u=n}^{n+T-1} \left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u))) \right] \prod_{s=0}^{u} a^{-1}(s).$$

Since $x(n+T) = x(n)$, we obtain

$$x(n) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s) \right]$$
$$= \sum_{u=n}^{n+T-1} \left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u))) \right] \prod_{s=0}^{u} a^{-1}(s). \tag{2.7}$$

Rewrite

$$\sum_{u=n}^{n+T-1} \triangle g(u, x(u-\tau(u))) \prod_{s=0}^{u} a^{-1}(s)$$
$$= \sum_{u=n}^{n+T-1} E \left[\prod_{s=0}^{u-1} a^{-1}(s) \right] \triangle g(u, x(u-\tau(u))).$$

Performing a summation by parts on the on the above equation, we get

$$\sum_{u=n}^{n+T-1} \triangle g(u, x(u-\tau(u))) \prod_{s=0}^{u} a^{-1}(s)$$
$$= g(n, x(n-\tau(n))) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s) \right]$$
$$- \sum_{u=n}^{n+T-1} g(u, x(u-\tau(u))) \triangle \left[\prod_{s=0}^{u-1} a^{-1}(s) \right]$$
$$= g(n, x(n-\tau(n))) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s) \right]$$
$$- \sum_{u=n}^{n+T-1} g(u, x(u-\tau(u))) \left[1 - a(u) \right] \prod_{s=0}^{u} a^{-1}(s). \tag{2.8}$$

Substituting (2.8) into (2.7), we obtain

$$x(n) \left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s) \right]$$

$$= g\left(n, x\left(n-\tau\left(n\right)\right)\right)\left[\prod_{s=0}^{n+T-1} a^{-1}\left(s\right) - \prod_{s=0}^{n-1} a^{-1}\left(s\right)\right]$$
$$- \sum_{u=n}^{n+T-1} g\left(u, x\left(u-\tau\left(u\right)\right)\right)\left[1-a\left(u\right)\right]\prod_{s=0}^{u} a^{-1}\left(s\right)$$
$$+ \sum_{u=n}^{n+T-1} f\left(u, x\left(u-\tau\left(u\right)\right)\right)\prod_{s=0}^{u} a^{-1}\left(s\right).$$

Dividing both sides of the above equation by $\prod_{s=0}^{n+T-1} a^{-1}\left(s\right) - \prod_{s=0}^{n-1} a^{-1}\left(s\right)$, we obtain (2.4).

Now for $n \leq 0$, equation (1.1) is equivalent to

$$\triangle\left[x\left(n\right)\prod_{s=n}^{0} a^{-1}\left(s\right)\right] = \left[\triangle g\left(n, x\left(n-\tau\left(n\right)\right)\right) + f\left(n, x\left(n-\tau\left(n\right)\right)\right)\right]\prod_{s=n+1}^{0} a^{-1}\left(s\right).$$

Summing the above expression from n to $n+T-1$, we obtain (2.4) by a similar argument. This completes the proof. □

To simplify notation, we let

$$m = \min\left\{G\left(n,u\right) : n \geq 0,\ u \leq T\right\} = G\left(n,n\right) > 0, \tag{2.9}$$

and

$$M = \max\left\{G\left(n,u\right) : n \geq 0,\ u \leq T\right\} = G\left(n, n+T-1\right) = G\left(0, T-1\right) > 0. \tag{2.10}$$

It is easy to see that for all $n, u \in \mathbb{Z}$, we have

$$G\left(n+T, u+T\right) = G\left(n,u\right). \tag{2.11}$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1.1). For its proof we refer the reader to [9].

Theorem 2.1 (Krasnoselskii). *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $\left(\mathbb{B}, \|.\|\right)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*
(i) $x, y \in \mathbb{D}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,
(ii) $\mathcal{A}$ is completely continuous,
(iii) $\mathcal{B}$ is a contraction mapping.
Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

To apply Theorem 2.1, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $\left(\mathbb{B}, \|.\|\right) = \left(P_T, \|.\|\right)$ and $\mathbb{D} = \left\{\varphi \in \mathbb{B} : L \leq \varphi \leq K\right\}$, where L is non-negative constant and K is positive constant. We express equation (2.4) as

$$\varphi\left(n\right) = \left(\mathcal{B}\varphi\right)\left(n\right) + \left(\mathcal{A}\varphi\right)\left(n\right) := \left(H\varphi\right)\left(n\right),$$

where $\mathcal{A}, \mathcal{B} : \mathbb{D} \rightarrow \mathbb{B}$ are defined by

$$\left(\mathcal{A}\varphi\right)\left(n\right) = \sum_{u=n}^{n+T-1} G\left(n,u\right)\left[f\left(u, \varphi\left(u-\tau\left(u\right)\right)\right) - \left(1-a\left(u\right)\right)g\left(u, \varphi\left(u-\tau\left(u\right)\right)\right)\right], \tag{3.1}$$

and

$$\left(\mathcal{B}\varphi\right)\left(n\right) = g\left(n, \varphi\left(n-\tau\left(n\right)\right)\right). \tag{3.2}$$

In this section we obtain the existence of a positive periodic solution of (1.1) by considering the two cases; $g(n,x) \geq 0$ and $g(n,x) \leq 0$ for all $n \in \mathbb{Z}$, $x \in \mathbb{D}$. We assume that function $g(n,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|g(n,x) - g(n,y)| \leq k\|x-y\|, \text{ for all } n \in [0,T-1] \cap \mathbb{Z},\ x,y \in \mathbb{D}. \tag{3.3}$$

In the case $g(n,x) \geq 0$, we assume that there exist a non-negative constant k_1 and positive constant k_2 such that

$$k_1 x \leq g(n,x) \leq k_2 x, \text{ for all } n \in [0,T-1] \cap \mathbb{Z},\ x \in \mathbb{D}, \tag{3.4}$$

$$k_2 < 1, \tag{3.5}$$

and for all $n \in [0,T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$\frac{L(1-k_1)}{mT} \leq f(n,x) - [1-a(n)]\, g(n,x) \leq \frac{K(1-k_2)}{MT}, \tag{3.6}$$

where m and M are defined by (2.9) and (2.10), respectively.

Lemma 3.1. *Suppose that the conditions (2.1)-(2.3) and (3.4)-(3.6) hold. Then $\mathcal{A} : \mathbb{D} \to \mathbb{B}$ is completely continuous.*

Proof. We first show that $(\mathcal{A}\varphi)(n+T) = (\mathcal{A}\varphi)(n)$.
Let $\varphi \in \mathbb{D}$. Then using (3.1) we arrive at

$$\begin{aligned}&(\mathcal{A}\varphi)(n+T)\\ &= \sum_{u=n+T}^{n+2T-1} G(n+T,u)\left[f(u,\varphi(u-\tau(u))) - (1-a(u))\, g(u,\varphi(u-\tau(u)))\right].\end{aligned}$$

Let $j = u - T$, then

$$\begin{aligned}&(\mathcal{A}\varphi)(n+T)\\ &= \sum_{j=n}^{n+T-1} G(n+T,j+T)\left[f(j+T,\varphi(j+T-\tau(j+T)))\right.\\ &\left.-(1-a(j+T))\, g(j+T,\varphi(j+T-\tau(j+T)))\right]\\ &= \sum_{j=n}^{n+T-1} G(n,j)\left[f(j,\varphi(j-\tau(j))) - (1-a(j))\, g(j,\varphi(j-\tau(j)))\right]\\ &= (\mathcal{A}\varphi)(n),\end{aligned}$$

by (2.1), (2.3) and (2.11).

To see that $\mathcal{A}(\mathbb{D})$ is uniformly bounded, we let $n \in [0,T-1] \cap \mathbb{Z}$ and for $\varphi \in \mathbb{D}$, we have by (3.6) that

$$\begin{aligned}&|(\mathcal{A}\varphi)(n)|\\ &\leq \left|\sum_{u=n}^{n+T-1} G(n,u)\left[f(u,\varphi(u-\tau(u))) - (1-a(u))\, g(u,\varphi(u-\tau(u)))\right]\right|\\ &\leq MT\frac{K(1-k_2)}{MT} = K(1-k_2).\end{aligned}$$

From the estimation of $|(\mathcal{A}\varphi)(n)|$ it follows that

$$\|\mathcal{A}\varphi\| \leq K(1-k_2).$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded.

Next, we show that $\mathcal{A}$ maps bounded subsets into compact sets. As $\mathcal{A}(\mathbb{D})$ is uniformly bounded in $\mathbb{R}^T$, then $\mathcal{A}(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. Therefore $\mathcal{A}$ is completely continuous. This completes the proof. □

Lemma 3.2. *Suppose that (3.3) holds. If $\mathcal{B}$ is given by (3.2) with*

$$k < 1, \tag{3.7}$$

then $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction.

Proof. Let $\mathcal{B}$ be defined by (3.2). Obviously, $(\mathcal{B}\varphi)(n+T) = (\mathcal{B}\varphi)(n)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned} |(\mathcal{B}\varphi)(n) - (\mathcal{B}\psi)(n)| &\leq |g(n, \varphi(n-\tau(n))) - g(n, \psi(n-\tau(n)))| \\ &\leq k\|\varphi - \psi\|. \end{aligned}$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \leq k\|\varphi - \psi\|$. Thus $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction by (3.7). □

Theorem 3.1. *Suppose (2.1)-(2.3) and (3.3)-(3.7) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

Proof. By Lemma 3.1, the operator $\mathcal{A} : \mathbb{D} \to \mathbb{B}$ is completely continuous. Also, from Lemma 3.2, the operator $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned} &(\mathcal{B}\psi)(n) + (\mathcal{A}\varphi)(n) \\ &= g(n, \psi(n-\tau(n))) \\ &+ \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\ &\leq k_2 K + M \sum_{u=n}^{n+T-1} \left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\ &\leq k_2 K + MT\frac{K(1-k_2)}{MT} = K. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(\mathcal{B}\psi)(n) + (\mathcal{A}\varphi)(n) \\ &= g(n, \psi(n-\tau(n))) \\ &+ \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\ &\geq k_1 L + m \sum_{u=n}^{n+T-1} \left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\ &\geq k_1 L + mT\frac{L(1-k_1)}{mT} = L. \end{aligned}$$

This shows that $\mathcal{B}\psi + \mathcal{A}\varphi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}x + \mathcal{B}x$. By Lemma 2.1 this fixed point is a solution of (1.1) and the proof is complete. □

Remark 3.3. When $g(n,x) = cx$, Theorem 3.1 reduces to Theorem 3.2 of [7].

In the case $g(n,x) \leq 0$, we substitute conditions (3.4)-(3.6) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 x \leq g(n,x) \leq k_4 x, \text{ for all } n \in [0, T-1] \cap \mathbb{Z},\ x \in \mathbb{D}, \tag{3.8}$$

$$-k_3 < 1, \tag{3.9}$$

and for all $n \in [0, T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$\frac{L - k_3 K}{mT} \leq f(n,x) - [1 - a(n)]\, g(n,x) \leq \frac{K - k_4 L}{MT}. \tag{3.10}$$

Theorem 3.2. *Suppose (2.1)-(2.3), (3.3) and (3.7)-(3.10) hold. Then equation (1.1) has a positive T-periodic solution x in the subset $\mathbb{D}$.*

The proof follows along the lines of Theorem 3.1, and hence we omit it.

Remark 3.4. When $g(n,x) = cx$, Theorem 3.2 reduces to Theorem 3.3 of [7].

Example 3.5. Consider the following nonlinear neutral difference equation

$$x(n+1) = a(n)\, x(n) + \triangle g(n, x(n - \tau(n))) + f(n, x(n - \tau(n))), \tag{3.11}$$

where

$$T = 4,\ \tau(n) = 5,\ a(n) = \frac{1}{5},\ g(n,x) = 0.8 \sin(x),$$

and

$$f(n,x) = \frac{1}{1000} \frac{1}{x^2 + 0.03} + 0.64 \sin(x) + 0.024.$$

Then Equation (3.11) has a positive 4-periodic solution x satisfying $0.004 \leq x \leq \frac{\pi}{2}$. To see this, we have $L = 0.004$, $K = \frac{\pi}{2}$. A simple calculation yields

$$k = 0.8,\ m = \frac{5}{224},\ M = \frac{225}{224},\ k_1 = \frac{2}{\pi},\ k_2 = 0.8.$$

Define the set $\mathbb{D} = \left\{\varphi \in P_4 : 0.004 \leq \varphi(n) \leq \frac{\pi}{2},\ n \in [0,3] \cap \mathbb{Z}\right\}$. Then for $x \in \left[0.004, \frac{\pi}{2}\right]$ we have

$$\begin{aligned} f(n,x) - [1 - a(n)]\, g(n,x) &= \frac{1}{1000} \frac{1}{x^2 + 0.03} + 0.024 \\ &\leq 0.058 < 0.078 \simeq \frac{K(1 - k_2)}{MT}. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(n,x) - [1 - a(n)]\, g(n,x) &= \frac{1}{1000} \frac{1}{x^2 + 0.03} + 0.024 \\ &\geq 0.024 > 0.016 \simeq \frac{L(1 - k_1)}{mT}. \end{aligned}$$

By Theorems 3.1, Equation (3.11) has a positive 4-periodic solution x such that $0.004 \leq x \leq \frac{\pi}{2}$.

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GENERALIZATIONS OF THE KKMF PRINCIPLE HAVING COERCING FAMILIES

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ABSTRACT. In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano [1] obtained a generalization of Ky Fan's 1984 KKM theorem on the intersection of a family of closed sets on non-compact convex sets in a topological vector space. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. In 2011, Chebbi, Gourdel, and Hammami [5] introduced a generalized coercivity type condition for multimaps defined on topological spaces endowed with a generalized convex structure and extended Fan's KKM theorem. In this paper, we show that better forms of theorems in [1, 3–5] can be deduced from a KKM theorem on abstract convex spaces in Park's sense [13–17].

KEYWORDS: KKM theorem; Fan's 1961 KKM lemma; 1984 KKM theorem; Fan-Browder fixed point theorem; Abstract convex space; (partial) KKM principle; (partial) KKM space.
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1. INTRODUCTION

One of the earliest equivalent formulations of the Brouwer fixed point theorem of 1912 is the theorem of Knaster, Kuratowski, and Mazurkiewicz (the KKM theorem for short) of 1929 [10] on the intersection of a family of closed sets. Actually, the KKM theorem was concerned with a particular type of multimaps, later called KKM maps by Dugundji and Granas [6]. The KKM theory (first called by the author in 1992; see [12, 15]) is the study of applications of various equivalent formulations of the KKM theorem and their generalizations.

From 1961 Ky Fan showed that the KKM theorem provides the foundation for many of the modern essential results in diverse areas of mathematical sciences. Actually, a milestone on the history of the KKM theory was erected by Fan in 1961 [7]. His 1961 KKM Lemma (or the Fan-KKM theorem or the KKMF principle [1]) extended the KKM theorem to arbitrary topological vector spaces and was applied to

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various problems in his subsequent papers. Moreover, his lemma was extended in 1979 and 1984 [8, 9] to the 1984 KKM theorem with a new coercivity (or compactness) condition for noncompact convex sets with new applications; see [12, 15].

In 2005, Ben-El-Mechaiekh, Chebbi, and Florenzano [1] obtained a generalization of the 1984 KKM theorem for KKM maps admitting a coercing family, and gave several deep examples of the family related to an exceptional family, an escaping sequence, an attracting trajectory, and others. They also extended the Fan-Browder fixed point theorem to multimaps on non-compact convex sets. Their generalizations of the KKM theorem is applied by Chebbi to some minimax inequality and equilibria in [3] and to some quasi-variational inequalities in [4]. Moreover, Chebbi, Gourdel, and Hammami in 2011 [5] introduced a generalized coercivity type condition for multimaps defined on topological spaces endowed with a generalized convex structure and Fan's KKM lemma.

Since 2006, the present author initiated the KKM theory on abstract convex spaces and obtained very general forms of KKM type theorems in [18, 19]. Moreover, in a recent paper [21], we introduced several generalizations of the 1984 KKM theorem and some known direct applications in order to reveal the close relationship among such generalizations. As a continuation, in the present paper, we show that some better forms of the KKM theorem, the fixed point theorem, and other results in [1, 3–5] can be deduced from a KKM theorem on abstract convex spaces in the sense of the author [13–23].

In Section 2 of this paper, we introduce the recent concepts of abstract convex spaces and partial KKM spaces. We also introduce one of the recent versions of general KKM type theorems in our previous works [18–23]. Section 3 is devoted to generalize the coercivity type conditions in [1] and [5]. In Sections 4 and 5, we show that better forms of main theorems in these two papers [1, 5] can be deduced from a KKM theorem on abstract convex spaces in the sense of [13–23]. Finally, Section 6 deals with improvements of results of [3, 4].

2. ABSTRACT CONVEX SPACES

Since 2006 we have introduced the concepts of abstract convex spaces, KKM spaces, and partial KKM spaces; see our recent works [17–22] and the references therein.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if, for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Some remarks and examples on $\mathfrak{KC}$-maps and $\mathfrak{KO}$-maps can be seen in [13, 14]. In this paper, we need only the fact that any continuous function $s : E \longrightarrow Z$ belongs to $\mathfrak{KC}(E, D, Z)$.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Example 2.4. The following are typical examples of KKM spaces. Others can be seen in [15–17] and the references therein.

(1) A *convex space* $(X, D) = (X, D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\operatorname{co} D \subset X$, and each Γ_A is the convex hull of $A \in \langle D\rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde [11] for $X = D$.

(2) A *generalized convex space* or a G*-convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \longrightarrow \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$.

When $X = D$, a G-convex space is called an L-space; see [5].

(3) A *space having a family* $\{\phi_A\}_{A\in\langle D\rangle}$ or simply a ϕ_A*-space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \longrightarrow X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

A subset C of X is said to be ϕ_A*-convex* with respect to a subset $D' \subset D$ if for each $B \in \langle D'\rangle$, we have $\operatorname{Im} \phi_B := \phi_B(\Delta_{|B|-1}) \subset C$.

For a ϕ_A-space $(X, D; \{\phi_A\})$, the corresponding abstract convex space $(X, D; \Gamma)$ with $\Gamma_A := \phi_A(\Delta_n)$ for $A \in \langle D\rangle$ with $|A| = n + 1$ is a KKM space. This KKM space may not be G-convex.

We have the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Convex space $\Longrightarrow$ H-space
$\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

Consider the following related four conditions for a multimap $G : D \multimap Z$ from a set D into a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D}\overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D}\overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the definition of $\mathfrak{KC}$-maps, we have a whole intersection property of the Fan type under certain "coercivity" conditions. The following is given in [18, 19, 21–23]:

Theorem 2.5. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*
(1) *$\overline{G}$ is a KKM map w.r.t. F; and*
(2) *there exists a nonempty compact subset K of Z such that either*
(i) *$K \supset \bigcap\{\overline{G(y)} \mid y \in M\}$ for some $M \in \langle D\rangle$; or*
(ii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap\{\overline{G(y)} \mid y \in D'\}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap\{\overline{G(y)} \mid y \in D\} \neq \emptyset.$$

Furthermore,
(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*
(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

Remark 2.6. 1. Taking $\overline{K}$ instead of K, we may assume K is closed and the closure notations in (i) and (ii) can be erased.

2. In a recent work [23], we showed that a particular form of Theorem 2.5 for $F = 1_E$ unifies several important KKM type theorems appeared in history.

3. GENERALIZATIONS OF VARIOUS COERCING FAMILIES

In this section, we obtain generalizations of coercivity conditions considered in [1] and [5].

Let us begin with the following particular form of the condition (ii) in Theorem 2.5 with $sG : D \multimap Z$ instead of $G : D \multimap Z$:

(I) Let $(E, D; \Gamma)$ be an abstract convex space, $G : D \multimap E$ a multimap, Z a topological space, and $s : E \longrightarrow Z$ a continuous map such that

(C) there exists a nonempty compact subset K of Z such that, for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and

$$s(L_N) \cap \bigcap_{y\in D'} sG(y) \subset K.$$

Note that $s \in \mathfrak{KC}(E, D, Z)$.

Under the situation of (I), we have the following:

Proposition 3.1. *If $(E, D; \Gamma)$ is a partial KKM space, then so is the abstract convex space $(Z, D; s\Gamma)$, where $s\Gamma : \langle D\rangle \multimap Z$.*

Proof. Let $G' : D \multimap Z$ be a closed-valued KKM map, that is, for any $A \in \langle D\rangle$, $s\Gamma(A) \subset G'(A)$ or $\Gamma(A) \subset (s^{-1}G')(A)$. Then $s^{-1}G' : D \multimap E$ is a closed-valued KKM map on the partial KKM space $(E, D; \Gamma)$. Hence $\{s^{-1}G'(a)\}_{a\in D}$ has the finite intersection property and so does the family $\{G'(a)\}_{a\in D}$. □

Note that the proof of Proposition 3.1 also hods for open-valued KKM maps and for KKM spaces.

Proposition 3.2. *The set* $s(L_N)$ *is a compact* $s\Gamma$*-convex subset of* $(Z, D; s\Gamma)$.

Proof. Since L_N is compact and s is continuous, $s(L_N)$ is compact. Since L_N is Γ-convex relative to some $D' \subset D$ such that $N \subset D'$, for any $A \in \langle D' \rangle$, we have $\Gamma(A) \subset L_N$ and hence $s\Gamma(N) \subset s(L_N)$. Therefore, $s(L_N)$ is $s\Gamma$-convex relative to $D' \subset D$. □

In 2011, Chebbi et al. [5] introduced the notion of coercing family in L-spaces for a given map as follows:

(II) Let D be an arbitrary set in an L-space (E, Γ), Z a topological space, and $s : E \longrightarrow Z$ a continuous map. A family $\{(C_a, K)\}_{a \in E}$ is said to be L-coercing for a map $F : D \multimap Z$ with respect to s if

(i) K is a compact subset of Z;

(ii) for each $N \in \langle D \rangle$, there exists a compact L-convex set L_N in E containing N such that

$$x \in L_N \;\Rightarrow\; C_x \cap D \subset L_N \cap D;$$

(iii) $\{x \in E \mid s(x) \in \bigcap_{y \in C_x \cap Z} F(y)\} \subset s^{-1}(K)$.

Proposition 3.3. *Definition* (II) *implies* (I).

Proof. Under the situation of (II), note that $(E, D; \Gamma)$ is a G-convex space and hence a (partial) KKM space. Let $G := s^{-1}F : D \multimap E$ and, for any $N \in \langle D \rangle$, we have a compact Γ-convex subset L_N of E containing N. Choose an $x \in L_N$ and let $D' \equiv (C_x \cap D) \cup N \subset L_N \cap D$ by (ii) (and Remark 1 in [3]). Then L_N is Γ-convex relative to $D' \subset D$ containing N. Moreover, by (iii),

$$x \in \bigcap_{y \in D'} G(y) = \bigcap_{y \in C_x \cap D} s^{-1}F(y) \subset s^{-1}(K).$$

Hence

$$s(x) \in s(L_N) \cap \bigcap_{y \in D'} sG(y) \subset K.$$

Therefore (I) holds. □

Motivated by [1], we define the following:

(III) Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. We say that a map $G : D \multimap Z$ has a *coercing family* $\{(D_i, K_i)\}_{i \in I}$ if and only if

(1) for each $i \in I$, K_i is a compact subset of Z and $D_i \subset D$ such that, for each $N \in \langle D \rangle$, there exist a compact subset L_N^i of E that is Γ-convex relative to $D_i \cup N$;

(2) for each $i \in I$, there exists $k \in I$ with $\bigcap_{y \in D_k} F(y) \subset K_i$.

This definition improves the following coercivity in the sense of Ben-El-Mechaiekh, Chebbi, and Florenzano in [1];

(IV) [1] Consider a subset X of a Hausdorff topological vector space and a topological space Z. A family $\{(D_i, K_i)\}_{i \in I}$ of pairs of sets is said to be *coercing* for a map $F : X \multimap Z$ if and only if:

(i) for each $i \in I$, D_i is contained in a compact convex subset of X, and K_i is a compact subset of Z;

(ii) for each $i, j \in I$, there exists $k \in I$ such that $D_i \cup D_j \subset D_k$;

(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in D_k} F(x) \subset K_i$.

If I is a singleton, the family is called a *single* coercing family.

Remark 3.4. In [1], it is noted that the condition (iii) holds *if and only if* the "dual" map $\Phi : Z \multimap X$ of F, defined by $\Phi(z) = X \setminus F^-(z),\ z \in Z$ verifies
(iii)$'$ $\forall i \in I,\ \exists k \in I,\ \forall z \in Z \setminus K_i,\quad \Phi(z) \cap C_k \neq \emptyset.$

In [1], there are given several deep examples of condition (iii)$'$ related to an exceptional family, an escaping sequence, an attracting trajectory, and others.

Remark 3.5. In [5], it is shown that L-coercing families in (II) contain coercing families in the sense of (IV).

Here we show that (III) is equivalent to a particular case of the coercivity (I) for abstract convex spaces:

Proposition 3.6. *Let $(E, D; \Gamma)$ be an abstract convex space. A map $G : D \multimap E$ admits a coercing family in the sense of* (III) *if and only if the coercivity* (I) *with $E = Z$ and $s = 1_E$ holds.*

Proof. When I is a singleton, then the existence of a coercing family implies the coercivity (I) with $E = Z$ and $s = 1_E$.

Conversely, choose an arbitrary $i \in I$ and let $K := K_i$. For an $N \in \langle D \rangle$, let $D' := D_k \cup N$ with D_k in (III)(2). Since there exists a compact Γ-convex subset $L_N := L_N^k$ of E relative to D', by (III)(2) again, we have

$$L_N \cap \bigcap_{y \in D'} F(y) \subset L_N \cap \bigcap_{y \in D_k} G(y) \subset K.$$

Therefore, the coercivity condition (I) with $E = D$ and $s = 1_E$ holds. □

Note that all of (I)-(IV) are examples of the coercivity (ii) in Theorem 2.5.

4. GENERALIZATIONS OF THE KKM THEOREM

In this section, we show that better forms of KKM theorems in [1] and [5] can be deduced from the KKM theorem 2.5 on abstract convex spaces.

Theorem 4.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z an arbitrary topological space and $G : D \multimap Z$ a closed-valued multimap. Suppose that there exists a continuous map $s : E \longrightarrow Z$ such that:*
(1) *the multimap $R : D \multimap E$ defined by $R(y) := s^{-1}(G(y))$ is KKM;*
(2) *the coercivity condition* (I) *holds for R instead of G.*
Then we have $K \cap \bigcap_{y \in D} G(y) \neq \emptyset$.

Proof. We apply Theorem 2.5 with $F := s$.

(1) Since $s^{-1}G$ is a closed-valued KKM map by (1), $\Gamma_A \subset R(A) = s^{-1}G(A)$ and $s\Gamma_A \subset sR(A) = G(A)$ for all $A \in \langle D \rangle$. Therefore $\overline{G}$ is a KKM map w.r.t. s.

(2) Condition (2) implies (ii) in Theorem 2.5 with $F := s$ and $G := sR$.

Therefore, by the case (ii) of Theorem 2.5, we have

$$s(E) \cap K \cap \bigcap_{y \in D} sR(y) \neq \emptyset.$$

This implies the conclusion. □

The main theorem of [5] is the particular case of Theorem 4.1 under the assumption of (II) as follows:

Corollary 4.2. *[5] Let D be an arbitrary set in the L-space (E, Γ), Z an arbitrary topological space and $F : D \multimap Z$ a map with quasi-compactly closed values. Suppose that there exists a continuous function $s : E \longrightarrow Z$ such that:*

(1) *the map $R : D \multimap E$ defined by $R(y) = s^{-1}(F(y))$ is KKM;*

(2) *there exists an L-coercing family for F with respect to s as in* (II).

Then $K \cap \bigcap_{x \in D} F(x) \neq \emptyset$.

The main theorem of [1] is the particular case $s = 1_Y$ of Theorem 4.1 under the assumption of (IV) as follows:

Corollary 4.3. *[1] Let E be a Hausdorff topological vector space, Y a convex subset of E, X a non-empty subset of Y, and $F : X \multimap Y$ a KKM map with compactly closed (in Y) values. If F admits a coercing family in the sense of* (IV), *then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

Corollary 4.4. *From the above corollaries, we notice the following:*

1. The quasi-compactly closed sets are compactly closed sets in modern usage and can be replaced by mere closed sets by adopting compactly generated extension of the original topology.

2. Our proofs are based on Theorem 2.5 and different from that of [1] and [3].

3. In view of (III), condition (ii) in (IV) of a coercing family is redundant.

4. The existence of a coercing family (IV) is simply equivalent to that of a single coercing family.

5. In Corollaries 4.2 and 4.3, F can be transfer closed-valued or intersectionally closed-valued.

6. As was noted in [1], if the coercing family is single, then Corollary 4.3 reduces to the 1984 KKM theorem 4 of Fan [9] which in turn generalizes the KKMF principle.

5. GENERALIZATIONS OF THE FAN-BROWDER FIXED POINT THEOREMS

In this section, we show that better forms of the Fan-Browder fixed point theorems in [1] and [5] can be deduced from the KKM theorem 2.5 on abstract convex spaces.

From Theorem 2.5, we also obtain the following Fan-Browder type alternative:

Theorem 5.1. *Let $(E, D; \Gamma)$ be a partial KKM space, and $S : E \multimap D$, $T : E \multimap E$ maps satisfying*

(1) *$S^{-}(y)$ is open for each $y \in D$;*

(2) *$T(x) \supset \mathrm{co}_\Gamma S(x)$ for each $x \in E$.*

Suppose that there exists a nonempty compact subset K of E satisfying

(3) *for each $N \in \langle D \rangle$, there exist $D' \subset D$ containing N and a compact Γ-convex subset L_N of E relative to D' such that*

$$L_N \cap \bigcap \{E \setminus S^{-}(y) \mid y \in D'\} \subset K.$$

Then either (a) *S has a maximal element $x_0 \in K$, that is, $S(x_0) = \emptyset$; or* (b) *T has a fixed point $x_1 \in E$, that is, $x_1 \in T(x_1)$.*

Proof. Suppose T has no fixed point. Define a map $G : D \multimap E$ by

$$G(y) := E \setminus S^{-}(y) = \{x \in E \mid y \notin S(x)\}, \quad y \in D.$$

Then G is closed-valued. Moreover, G is a KKM map.

In fact, suppose on the contrary that there exists an $N \in \langle D \rangle$ such that $\Gamma_N \not\subset G(N)$; that is, there exists an $x \in \Gamma_N$ such that $x \notin G(y)$ for all $y \in N$. In other words, $N \in \langle D \setminus G^-(x) \rangle$ and

$$y \notin G^-(x) \Leftrightarrow x \notin G(y) = E \setminus S^-(y) \Leftrightarrow x \in S^-(y) \Leftrightarrow y \in S(x)$$

for all $y \in N$. Hence $N \subset S(x)$ and, by (2), we have $x \in \Gamma_N \subset T(x)$. This is a contradiction.

Note that (3) implies condition (ii) of Theorem 2.5 with $E = D$ and $s = 1_E \in \mathfrak{KC}(E, D, E)$ since $(E, D; \Gamma)$ is a partial KKM space. Therefore, by Theorem 2.5, we have $K \cap \bigcap_{y \in D} G(y) \neq \emptyset$.

Then we have an $x_0 \in K$ and $x_0 \in G(y) = E \setminus S^-(y)$ or $y \notin S(x_0)$ for all $y \in D$. Hence S has a maximal element $x_0 \in K$. □

From Theorem 5.1, we obtain the following fixed point result [1]:

Corollary 5.2. *[1] Let X be a non-empty convex subset of a Hausdorff topological vector space and let $\Phi : X \multimap X$ be a map with open fibers (in X) and non-empty values. If Φ admits a single coercing family in the sense of* (IV) *satisfying* (iii)$'$, *then the map* co(Φ) *has a fixed point.*

Proof. We will use Theorem 5.1 with $E = D = X$, $\Gamma = \text{co}$, $S = \Phi$, $T = \text{co}(\Phi)$. Since Φ has non-empty values, it does not have a maximal element. Now it suffices to show that (iii)$'$ implies condition (3) of Theorem 5.1.

Suppose K is a compact subset of X and C is contained in a compact convex subset L of X. Let $N \in \langle X \rangle$. Since X is a convex subset of a Hausdorff topological vector space, there exists a compact convex subset L_N of X containing $D' := L \cup N$. Note that $\Phi(x) \cap D' \supset \Phi(x) \cap C \neq \emptyset$ for all $x \in X \setminus K$ by (iii)$'$, that is,

$$x \in X \setminus K \Rightarrow \Phi(x) \cap D' \neq \emptyset \Rightarrow \exists y \in \Phi(x) \cap D' \Rightarrow x \in \Phi^-(y),\ \exists y \in D'.$$

Now we have

$$x \in L_N \cap \bigcap_{y \in D'} (X \setminus \Phi^-(y)) \Rightarrow x \in X \setminus \Phi^-(y),\ \forall y \in D' \Rightarrow x \notin \Phi^-(y),\ \forall y \in D'.$$

Therefore, $x \notin X \setminus K$ and hence $x \in K$. So condition (3) of Theorem 5.1 holds. □

Now we can obtain an equivalent variant of Theorem 5.1:

Corollary 5.3. *Let $(E, D; \Gamma)$ be a partial KKM space, Z an arbitrary topological space and $G : D \multimap Z$, $H : E \multimap Z$ multimaps. Suppose that there exists a continuous map $s : E \longrightarrow Z$ such that:*

(1) *$G(y)$ is open for each $y \in D$;*

(2) *$s^{-1}H(x) \supset \text{co}_\Gamma G^- s(x)$ for each $x \in E$.*

Suppose that there exists a nonempty compact subset K of E satisfying

(3) *for each $N \in \langle D \rangle$, there exist $D' \subset D$ containing N and a compact Γ-convex subset L_N of E relative to D' such that*

$$s(L_N) \cap \bigcap \{Z \setminus s^{-1}G(y) \mid y \in D'\} \subset K.$$

Then either (a) *$G^- s$ has a maximal element $x_0 \in E$, that is, $G^- s(x_0) = \emptyset$; or* (b) *$s^{-1}H$ has a fixed point $x_1 \in E$, that is, $s(x_1) \in H(x_1)$.*

Proof. In view of Propositions 3.1 and 3.2, $(Z, D; s\Gamma)$ is a partial KKM space and $s(L_N)$ is a compact $s\Gamma$-convex subset relative to D'. We apply Theorem **??** replacing $(E, D; \Gamma)$ by $(Z, D; s\Gamma)$, $S := G^- s$ and $T := s^{-1}H$. Then

(1) $S^- = s^{-1}G$ is open-valued since so is G and s is continuous.

(2) $T(x) = s^{-1}H(x) \supset \mathrm{co}_\Gamma G^- s(x) = \mathrm{co}_\Gamma S(x)$ for each $x \in E$.
(3) Condition (3) of Theorem 5.1 with $S^- = s^{-1}G$ holds.
Therefore, by Theorem 5.1, the conclusion follows. □

Remark 5.4. Note that we deduced Corollary 5.3 from Theorem 5.1. Conversely, Corollary 5.3 for $E = Z$ and $s = 1_E$ reduces to Theorem 5.1.

The following is Theorem 2 in [5]:

Corollary 5.5. *[5] Let $(X;\Gamma)$ be an L-space, Z an arbitrary topological space, $s : X \longrightarrow Z$ a continuous map and $S : X \multimap Z$ a multimap such that:*
(i) *for each $x \in X$, $S(x)$ is quasi-compactly open in Z;*
(ii) *for each $z \in Z$, $S^{-1}(z)$ is nonempty and L-convex;*
(iii) *there exists an L-coercing family $\{(C_x, K)\}_{x\in X}$ for the map $Q(x) = Z \setminus S(x)$ with respect to s.*
Then there exists $x_0 \in X$ such that $s(x_0) \in S(x_0)$. In particular, if s is the identity map, then S has a fixed point.

Proof. Put $E = D = X$ and $G = H = S$ in Corollary 5.3. □

As was noted in [1], Theorems in Sections 4 and 5 can be used to extend existing results on various equilibrium problems, solvability of complementarity problems, existence of zero on non-compact domains, and existence of equilibria for qualitative games and abstract economies.

6. COMMENTS ON GENERAL MINIMAX INEQUALITIES AND APPLICATIONS

It is well-known that any KKM type theorem can be reformulated equivalently to the Fan-Browder type fixed point theorems, matching theorems, minimax inequalities, and so on.
In this section, we indicate that results of Chebbi in [3, 4] can be improved following our preceding arguments.
The following is a KKM type minimax inequality given in Theorem 5.3 in [22]

Theorem 6.1. *[22] Let $(E, D;\Gamma)$ be a partial KKM space. Let $f : E \times D \longrightarrow \overline{\mathbb{R}}$ be an extended real-valued function and $\gamma \in \overline{\mathbb{R}}$ such that*
(1) *for each $y \in D$, $\{x \in E \mid f(x, y) \le \gamma\}$ is intersectionally closed [resp., transfer closed];*
(2) *for each $N \in \langle D\rangle$ and $x \in \Gamma_N$, $\min\{f(x, y) \mid y \in N\} \le \gamma$; and*
Suppose that there exists a nonempty compact subset K of E satisfying
(3) *for each $N \in \langle D\rangle$, there exist $D' \subset D$ containing N and a compact Γ-convex subset L_N of E relative to D' such that*

$$L_N \cap \bigcap_{y\in D'} \{x \in E \mid f(x, y) \le \gamma\} \subset K.$$

Then (a) *there exists a $\hat{x} \in E$ [resp., $\hat{x} \in K$] such that*

$$f(\hat{x}, y) \le \gamma \quad \text{for all } y \in D; \text{ and}$$

(b) *if $E = D$ and $\gamma = \sup_{x\in E} f(x, x)$, then we have the minimax inequality:*

$$\inf_{y\in E} \sup_{x\in E} f(x, y) \le \sup_{x\in E} f(x, x).$$

Corollary 6.2. *[3] Let X be a nonempty convex subset of a t.v.s. E, and $f : X \times X \longrightarrow \overline{\mathbb{R}}$ be a function satisfying*

(i) *f is l.s.c. in the first variable on each compact convex subsets of X;*
(ii) *for each $A \in \langle X \rangle$, $\sup_{x \in \mathrm{co}\, A} \min_{y \in A} f(x, y) \leq 0$; and*
(iii) *the coercivity condition* (IV) *with $X = Z$ and*

$$F(y) := \{x \in X \mid f(x, y) \leq 0\} \text{ for } y \in X.$$

Then there exists an $x_0 \in X$ such that $f(x_0, y) \leq 0$ for all $y \in X$.

Proof. Note that X can be regarded a convex space in the sense of Lassonde [11] and endowed the compactly generated extension of of its original topology. Then (i) becomes simply "f is l.s.c." and hence, condition (1) of Theorem 6.1 is satisfied. Moreover, it is clear that (ii) implies (2) of Theorem 6.1. Further, (iii) implies the coercivity condition (I) in Section 3 with $s = 1_E$ and $G(y) := \{x \in E \mid f(x, y) \leq \gamma\}$ for $y \in D$. Therefore, the conclusion of Corollary 5.2 follows from Theorem 6.1(a) with $\gamma = 0$. □

Corollary 6.2 is applied to some equilibrium problems in [3] and to some quasi-variational inequalities in [4]. Note that Corollary 6.2 can be improved by adopting more general conditions (I)–(III) with $s = 1_E$ and $Z = E$. Moreover, any interested reader can check that all results in [3] and [4] can be improved by applying Theorem 6.1 instead of Corollary 6.2.

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ON THE MEANS OF PROJECTIONS ON CAT(0) SPACES

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ABSTRACT. We improve a result on approximation a common element of two closed convex subsets of a complete CAT(0) space appeared as Theorem 4.1 in [2]. New practical iterative scheme is presented and conditions on two given sets are relaxed.

KEYWORDS: Projection; CAT(0) space.
AMS Subject Classification: 47H09 47H10

1. INTRODUCTION

von Neumann introduced the alternating projection method and proved the following strong convergence in Hilbert spaces [cf. 2]:

Theorem 1.1 (von Neumann)**.** *Let H be a Hilbert space and $A, B \subset H$ its closed subspaces. Assume $x_0 \in H$ is a starting point and $\{x_n\} \subset H$ the sequence generated by*

$$x_{2n-1} = P_A(x_{2n-2}), \quad x_{2n} = P_B(x_{2n-1}), \; n \in \mathbb{N}, \tag{1.1}$$

where P_A, P_B are projection mappings from H to A and B respectively. Then $\{x_n\}$ converges in norm to a point from $A \cap B$.

When "subspaces" are replaced by "convex subsets", we only have "weak convergence" for the alternating projections:

Theorem 1.2. [3] *Let H be a Hilbert space and $A, B \subset H$ closed convex sets with $A \cap B \neq \emptyset$. Assume $x_0 \in H$ is a starting point and $\{x_n\} \subset H$ the sequence generated by (1.1). Then $\{x_n\}$ weakly converges to a point from $A \cap B$.*

It took 39 years since 1965 until Hundal [7] in 2004 could provide a counter example:

Example 1.3. [7] *There exist a hyperplane $A \subset \ell_2$, a convex cone $B \subset \ell_2$ and a point $x_0 \in \ell_2$ such that the sequence generated by (1.1) from the starting point x_0 converges weakly to a point in $A \cap B$ but not in norm.*

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In 2011, Bačák, Searston, Sims [2] extend the result of Bregman for CAT(0) spaces.

Theorem 1.4. [2, Theorem 4.1] *Let X be a complete CAT(0) space and $A, B \subset X$ convex closed subsets such that $A \cap B \neq \emptyset$. Let $x_0 \in X$ be a starting point and $\{x_n\} \subset X$ be the sequence generated by (1.1). Then:*

(i) *$\{x_n\}$ weakly converges to a point $x \in A \cap B$.*
(ii) *If A and B are boundedly regular, then $x_n \longrightarrow x$.*
(iii) *If A and B are boundedly linearly regular, then $x_n \longrightarrow x$ linearly.*
(iv) *If A and B are linearly regular, then $x_n \longrightarrow x$ linearly with a rate independent of the starting point.*

It is the aim of this paper to present an iterative sequence which strongly converges to a common point of the sets A and B. We do not impose any requirements on A and B as stated in (ii).

2. PRELIMINARIES

Let (X, d) be a metric space. A *geodesic* joining $x \in X$ to $y \in X$ is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Obviously, c is an isometry and $d(x, y) = l$. We call the image of c a *geodesic segment* joining x and y. If it is unique this geodesic is denoted $[x, y]$. Write $c(\alpha 0 + (1-\alpha)l) = \alpha x \oplus (1-\alpha)y$ for $\alpha \in (0,1)$. We also write the midpoint $\frac{1}{2}x \oplus \frac{1}{2}y$ of a segment $[x, y]$ as $\frac{x \oplus y}{2}$. The space X is said to be a *geodesic space* if every two points of X are joined by a geodesic. It is said to be of *hyperbolic type* [6] if it satisfies the following inequality:

$$d(p, \alpha x \oplus (1-\alpha)y) \leq \alpha d(p, x) + (1-\alpha) d(p, y) \tag{2.1}$$

for all $p \in X$. Following [5], let $\{v_1, v_2, ..., v_n\} \subset X$ and $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset (0, 1)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and write, by induction,

$$\bigoplus_{i=1}^{n} \lambda_i v_i := (1-\lambda_n)\Big(\frac{\lambda_1}{1-\lambda_n} v_1 \oplus \frac{\lambda_2}{1-\lambda_n} v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1-\lambda_n} v_{n-1}\Big) \oplus \lambda_n v_n. \tag{2.2}$$

Note for an example that $\frac{1}{3}v_1 \oplus \frac{1}{3}v_2 \oplus \frac{1}{3}v_3$ and $\frac{1}{3}v_2 \oplus \frac{1}{3}v_1 \oplus \frac{1}{3}v_3$ are not necessary coincide. Under (2.1) we can see that

$$d\left(\bigoplus_{i=1}^{n} \lambda_i v_i, x\right) \leq \sum_{n=1}^{n} \lambda_i d(v_i, x) \tag{2.3}$$

for each $x \in X$.

A metric space X is said to be a *CAT(0) space* (cf.[4] p.163) if it is a geodesic space satisfying one of the following equivalent conditions.

(i) **(CN) inequality**: If $x_0, x_1 \in X$, then
$$d^2\left(y, \frac{x_0 \oplus x_1}{2}\right) \leq \frac{1}{2} d^2(y, x_0) + \frac{1}{2} d^2(y, x_1) - \frac{1}{4} d^2(x_0, x_1), \text{ for all } y \in X.$$

(ii) **Law of cosine**: If $a = d(p, q), b = d(p, r), c = d(q, r)$ and ξ is the Alexandrov angle at p between $[p, q]$ and $[p, r]$, then $c^2 \geq a^2 + b^2 - 2ab\cos\xi$.

Lemma 2.1. [4, Proposition 2.2] *Let X be a CAT(0) space. Then for each $p, q, r, s \in X$ and $\alpha \in [0, 1]$,*

$$d(\alpha p \oplus (1-\alpha)q, \alpha r \oplus (1-\alpha)s) \leq \alpha d(p, r) + (1-\alpha) d(q, s). \tag{2.4}$$

In particular, (2.1) holds in CAT(0) spaces.

Let C be a nonempty subset of X. We will denote the family of nonempty bounded closed subsets of C by $BC(C)$ and the family of nonempty compact subsets of C by $K(C)$. Let $H(\cdot,\cdot)$ be the *Hausdorff distance* on $BC(X)$, that is,

$$H(A,B) = \max\left\{\sup_{a\in A} dist(a,B), \sup_{b\in B} dist(b,A)\right\}, \quad A,B\in BC(X),$$

where $dist(a,B) = \inf\{d(a,b) : b\in B\}$ is the distance from the point a to the subset B.

A mapping $t : C \longrightarrow C$ and a multivalued mapping $T : C \longrightarrow BC(C)$ are said to be *nonexpansive* if for each $x, y \in C$,

$$d(tx,ty) \le d(x,y), \text{ and}$$
$$H(Tx,Ty) \le d(x,y),$$

respectively. If $tx = x$, we call x a fixed point of a single valued mapping t. And if $x\in Tx$, we call x a fixed point of a multivalued mapping T. We use the notation $Fix(S)$ to stand for the set of all fixed points of a mapping S. Thus $Fix(t)\cap Fix(T)$ is the set of common fixed points of t and T, i.e., $x \in Fix(t)\cap Fix(T)$ if and only if $x = tx \in Tx$.

Let $\{\lambda_n\}$ be a given sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n = 1$, let $\{v_n\}$ be a bounded sequence in X and let v_0 be an arbitrary point in X. Let $\lambda'_n = \sum_{i=n+1}^{\infty}\lambda_i$ and assume that $\sum_{i=n}^{\infty}\lambda'_i \longrightarrow 0$ as $n\longrightarrow\infty$. In [5] the element $\bigoplus_{n=1}^{\infty}\lambda_n v_n$ has been defined. Here is its description. Set

$$s_n := \lambda_1 v_1 \oplus \lambda_2 v_2 \oplus \cdots \oplus \lambda_n v_n \oplus \lambda'_n v_0.$$

Thus, by (2.2),

$$s_n = \Big(\sum_{i=1}^{n}\lambda_i\Big) w_n \oplus \lambda'_n v_0, \tag{2.5}$$

where $w_1 = v_1$ and for each $n \ge 2$,

$$w_n = \frac{\lambda_1}{\sum_{i=1}^{n}\lambda_i} v_1 \oplus \frac{\lambda_2}{\sum_{i=1}^{n}\lambda_i} v_2 \oplus \cdots \oplus \frac{\lambda_n}{\sum_{i=1}^{n}\lambda_i} v_n.$$

We know that $\{s_n\}$ is a Cauchy sequence. Thus $s_n \longrightarrow x$ as $n\longrightarrow\infty$ for some $x\in X$. Write

$$x = \bigoplus_{n=1}^{\infty}\lambda_n v_n.$$

By (2.5), $d(s_n,w_n) \le \lambda'_n d(w_n,v_0)$, it is seen that $\lim_{n\longrightarrow\infty} s_n = \lim_{n\longrightarrow\infty} w_n$. Thus the limit x is independent of the choice of v_0. Moreover, it had been shown in [5] that

(A): if y_0 and v_n belong to X, $d(v_n,y_0) = d(x,y_0)$ for all n where $x = \bigoplus_{n=1}^{\infty}\lambda_n v_n$, then $v_n = x$ for all n.

Lemma 2.2. [5, Lemma 3.8] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, let $\{t_n : n\in\mathbb{N}\}$ be a family of single-valued nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} Fix(t_n)$ is nonempty. Define $t : C\longrightarrow C$ by*

$$t(x) = \bigoplus_{n=1}^{\infty}\lambda_n t_n(x)$$

for all $x\in C$ where $\{\lambda_n\}\subset(0,1)$ with $\sum_{n=1}^{\infty}\lambda_n = 1$ and $\sum_{i=n}^{\infty}\lambda'_i \longrightarrow 0$ as $n\longrightarrow\infty$. Then t is nonexpansive and $Fix(t) = \bigcap_{n=1}^{\infty} Fix(t_n)$.

Theorem 2.3. [8, Lemma 2.2] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, let $t : C \to C$ be nonexpansive, fix $u \in C$, and for each $s \in (0, 1)$ let x_s be the point of $[u, t(x_s)]$ satisfying*

$$d(u, x_s) = sd(u, t(x_s)).$$

Then $Fix(t) \neq \emptyset$ if and only if $\{x_s\}$ remains bounded as $s \longrightarrow 1$. In this case, the following statements hold:

(1) *$\{x_s\}$ converges to the unique fixed point z of t which is nearest to u;*
(2) *$d^2(u, z) \leq \mu_n d^2(u, u_n)$ for all Banach limits μ and all bounded sequences $\{u_n\}$ with $d(u_n, t(u_n)) \longrightarrow 0$.*

We will follow the proof of the following theorem to prove our main result (Theorem 3.1).

Theorem 2.4. [5, Theorem 3.7] *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{t_n : C \longrightarrow C\}$ be a countable family of nonexpansive mappings and $T : C \longrightarrow K(C)$ be a nonexpansive mapping with $\bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T) \neq \emptyset$. Suppose that $T(p) = \{p\}$ for all $p \in \bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T)$. Let t and $\{\lambda_n\}$ be as in Lemma 2.2. Suppose that $u, z_1 \in C$ are arbitrarily chosen and $\{z_n\}$ is defined by*

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n)\left(\frac{1}{2}w_n(z_n) \oplus \frac{1}{2}y_n\right), \quad n \in \mathbb{N}, \tag{2.6}$$

such that $d(y_n, y_{n+1}) \leq d(z_n, z_{n+1})$ for all $n \in \mathbb{N}$, where $y_n \in T(z_n)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\longrightarrow\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Then $\{z_n\}$ converges to the unique point of $\bigcap_{n=1}^{\infty} Fix(t_n) \cap Fix(T)$ which is nearest to u.

In the course of the proof of Theorem 2.4, the following results play important role.

Lemma 2.5. [9, Proposition 2] *Let a be a real number and let $(a_1, a_2, ...) \in \ell^{\infty}$ be such that $\mu_n(a_n) \leq a$ for all Banach limits μ and $\limsup_n(a_{n+1} - a_n) \leq 0$. Then $\limsup_n a_n \leq a$.*

Lemma 2.6. [1, Lemma 2.3] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\eta_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \eta_n < \infty$, and $\{\gamma_n\}$ a sequence of real numbers with $\limsup_{n\longrightarrow\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \eta_n \text{ for all } n \in \mathbb{N}.$$

Then $\lim_{n\longrightarrow\infty} s_n = 0$.

3. MAIN RESULTS

We first consider a convergence result.

Theorem 3.1. *Let C be a closed convex subset of a complete CAT(0) space X, $t : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(t) \neq \emptyset$ and M a positive real number. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0, 1)$ satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in C$ be arbitrarily chosen and $\{z_n\}$ be defined by*

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n)u_n, \quad u_n \in C$$

such that

$$d(u_n, tz_n) \leq \varepsilon_n M \tag{3.1}$$

for all $n \in \mathbb{N}$. If $\{z_n\}$ is bounded, then the sequence $\{z_n\}$ converges to the unique point of $Fix(t)$ which is nearest to u.

Proof. We follow the proof of Theorem 2.4. By (3.1), we see that

$$\begin{array}{rcl} d(u_n, u_{n+1}) & \leq & d(u_n, tz_n) + d(tz_n, tz_{n+1}) + d(tz_{n+1}, u_{n+1}) \\ & \leq & d(z_n, z_{n+1}) + M(\varepsilon_n + \varepsilon_{n+1}). \end{array}$$

From the definition of z_n, we have

$$\begin{array}{rcl} d(z_{n+1}, z_n) & = & d(\alpha_n u \oplus (1-\alpha_n)u_n, \alpha_{n-1} u \oplus (1-\alpha_{n-1})u_{n-1}) \\ & \leq & d(\alpha_n u \oplus (1-\alpha_n)u_n, \alpha_n u \oplus (1-\alpha_n)u_{n-1}) \\ & & +d(\alpha_n u \oplus (1-\alpha_n)u_{n-1}, \alpha_{n-1} u \oplus (1-\alpha_{n-1})u_{n-1}) \\ & \leq & (1-\alpha_n)d(u_n, u_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, u_{n-1}) \\ & \leq & (1-\alpha_n)d(z_n, z_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, u_{n-1}) \\ & & +(1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1}). \end{array}$$

Putting in Lemma 2.6, $[s_n = d(z_n, z_{n-1}), \gamma_n = 0$ and $\eta_n = |\alpha_n - \alpha_{n-1}| d(u, u_{n-1}) + (1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1})]$ or $[s_n = d(z_n, z_{n-1}), \gamma_n = \left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| d(u, u_{n-1})$ and $\eta_n = (1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1})]$ according to $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n \longrightarrow \infty} (\alpha_n / \alpha_{n+1}) = 1$, respectively. Thus, using (C3) and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, we obtain

$$\lim_{n \longrightarrow \infty} d(z_{n+1}, z_n) = 0.$$

It follows from (C1) that

$$\begin{array}{rcl} d(z_n, u_n) & \leq & d(z_n, z_{n+1}) + d(z_{n+1}, u_n) \\ & = & d(z_n, z_{n+1}) + d(\alpha_n u \oplus (1-\alpha_n)u_n, u_n) \\ & \leq & d(z_n, z_{n+1}) + \alpha_n d(u, u_n) \longrightarrow 0. \end{array}$$

This implies

$$\begin{array}{rcl} d(u_n, tu_n) & \leq & d(u_n, tz_n) + d(tz_n, tu_n) \\ & \leq & \varepsilon_n M + d(z_n, u_n) \longrightarrow 0. \end{array}$$

Let $x_s \in [u, tx_s]$ satisfying $d(u, x_s) = s d(u, tx_s)$ for all $s \in (0,1)$. By Theorem 2.3, we have $z =: \lim_{s \longrightarrow 1} x_s$ which is the unique point of $Fix(t)$ nearest to u and $\mu_n(d^2(u,z) - d^2(u, u_n)) \leq 0$ for all Banach limits μ. Moreover, since $d(u_n, u_{n+1}) \leq d(z_n, z_{n+1}) + M(\varepsilon_n + \varepsilon_{n+1}) \longrightarrow 0$,

$$\limsup_{n \longrightarrow \infty} \left(d^2(u,z) - d^2(u, u_n)\right) - \left(d^2(u,z) - d^2(u, u_{n+1})\right) = 0.$$

Therefore Lemma 2.5 implies

$$\limsup_{n \to \infty} \left(d^2(u,z) - (1-\alpha_n)d^2(u, u_n)\right) = \limsup_{n \to \infty} \left(d^2(u,z) - d^2(u, u_n)\right) \leq 0.$$

Consider the following estimates:

$$\begin{array}{rcl} d^2(z_{n+1}, z) & = & d^2(\alpha_n u \oplus (1-\alpha_n)u_n, z) \\ & \leq & \alpha_n d^2(u,z) + (1-\alpha_n)d^2(u_n, z) - \alpha_n(1-\alpha_n)d^2(u, u_n) \\ & = & (1-\alpha_n)d^2(u_n, z) + \alpha_n \left(d^2(u,z) - (1-\alpha_n)d^2(u, u_n)\right) \\ & \leq & (1-\alpha_n)(d(u_n, tz_n) + d(tz_n, z))^2 + \alpha_n \left(d^2(u,z) - (1-\alpha_n)d^2(u, u_n)\right) \\ & \leq & (1-\alpha_n)(d^2(z_n, z) + 2\varepsilon_n M d(z_n, z) + \varepsilon_n^2 M^2) \\ & & +\alpha_n \left(d^2(u,z) - (1-\alpha_n)d^2(u, u_n)\right) \end{array}$$

$$\begin{aligned}
&= (1-\alpha_n)d^2(z_n,z)+\alpha_n\left(d^2(u,z)-(1-\alpha_n)d^2(u,u_n)\right)\\
&\quad +(1-\alpha_n)(2\varepsilon_n M d(z_n,z)+\varepsilon_n^2 M^2)\\
&\leq (1-\alpha_n)d^2(z_n,z)+\alpha_n\left(d^2(u,z)-(1-\alpha_n)d^2(u,u_n)\right)\\
&\quad +(1-\alpha_n)(2\varepsilon_n MN+\varepsilon_n^2 M^2),
\end{aligned}$$

where $N=\sup\{d(z_n,z):n\in\mathbb{N}\}$. We can now use Lemma 2.6 to conclude the proof. □

Here is our first main result.

Theorem 3.2. *Let X be a complete CAT(0) space and $\{A_i : i \in \mathbb{N}\}$ be a family of closed convex subsets of X such that $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n = 1$, $\sum_{i=n}^{\infty}\lambda_i' \longrightarrow 0$ as $n\longrightarrow\infty$ where $\lambda_i' = \sum_{j=i+1}^{\infty}\lambda_j$. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying $\sum_{n=1}^{\infty}\varepsilon_n<\infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in X$ be arbitrarily chosen and set*

$$r_n=\sup_{i\in\mathbb{N}}\{dist(z_n,A_i)\},\quad \beta_n\in\left(0,\frac{1}{2}\sqrt{4r_n^2+4\varepsilon_n^2}-r_n\right),$$

$$z_{n+1}=\alpha_n u\oplus(1-\alpha_n)u_n,\ \text{where}$$

$$u_n=\bigoplus_{i=1}^{\infty}\lambda_i u_n^{A_i},\quad u_n^{A_i}\in A_i\cap B(z_n:dist(z_n,A_i)+\beta_n^2)$$

for all $n\in\mathbb{N}$. Then the sequence $\{z_n\}$ converges to the unique point of $\bigcap_{i=1}^{\infty}A_i$ which is nearest to u.

Proof. For each $i\in\mathbb{N}$, let $p_i : X\longrightarrow A_i$ be the projection mapping. Using the law of cosine and the definition of β_n, we have

$$\begin{aligned}
d^2(u_n^{A_i},p_iz_n) &\leq d^2(z_n,u_n^{A_i})-d^2(z_n,p_iz_n)\\
&\leq (d(z_n,p_iz_n)+\beta_n)^2-d^2(z_n,p_iz_n)\\
&= 2\beta_n d(z_n,p_iz_n)+\beta_n^2\leq\beta_n(2r_n+\beta_n)\\
&< \left(\frac{1}{2}\sqrt{4r_n^2+4\varepsilon_n^2}-r_n\right)\left(\frac{1}{2}\sqrt{4r_n^2+4\varepsilon_n^2}+r_n\right)=\varepsilon_n^2.
\end{aligned}$$

Hence $d(u_n^{A_i},p_iz_n)<\varepsilon_n$ for all $n\in\mathbb{N}$. Let $p:X\longrightarrow X$ be defined by

$$px=\bigoplus_{i=1}^{\infty}\lambda_i p_i x$$

for each $x\in X$. From Lemma 2.2, p is nonexpansive and $Fix(p)=\bigcap_{i=1}^{\infty}Fix(p_i)=\bigcap_{i=1}^{\infty}A_i$. For each n, we can choose $m_n\in\mathbb{N}$ such that

$$d\left(\bigoplus_{i=1}^{\infty}\lambda_i u_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i}\right)+d\left(\bigoplus_{i=1}^{\infty}\lambda_i p_i z_n,\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_i z_n\right)<\varepsilon_n.$$

Thus

$$\begin{aligned}
d(u_n,pz_n) &\leq d\left(\bigoplus_{i=1}^{\infty}\lambda_i u_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i}\right)+d\left(\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_i z_n\right)\\
&\quad +d\left(\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_i z_n,\bigoplus_{i=1}^{\infty}\lambda_i p_i z_n\right)\\
&< \sum_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}d(u_n^{A_i},p_iz_n)+\varepsilon_n<2\varepsilon_n.
\end{aligned}$$

Let $q \in \bigcap_{i=1}^{\infty} A_i$. Then

$$\begin{aligned} d(z_{n+1},q) &= d(\alpha_n u \oplus (1-\alpha_n)u_n, q) \\ &\leq \alpha_n d(u,q) + (1-\alpha_n) d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, q\right) \\ &\leq \alpha_n d(u,q) + (1-\alpha_n) d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} u_n^{A_i}\right) \\ &\quad +(1-\alpha_n) d\left(\bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} u_n^{A_i}, q\right) \\ &\leq \alpha_n d(u,q) + (1-\alpha_n)\left(\varepsilon_n + \sum_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}(d(u_n^{A_i}, p_i z_n) + d(p_i z_n, q))\right) \\ &\leq \alpha_n d(u,q) + (1-\alpha_n) d(z_n, q) + 2(1-\alpha_n)\varepsilon_n \\ &\leq \max\{d(u,q), d(z_n,q)\} + 2(1-\alpha_n)\varepsilon_n. \end{aligned}$$

By induction we have

$$d(z_{n+1},q) \leq \max\{d(u,q), d(z_1,q)\} + 2\sum_{n=1}^{\infty}(1-\alpha_n)\varepsilon_n < \infty \text{ for all } n \in \mathbb{N}.$$

This implies the sequence $\{z_n\}$ is bounded. The result now follows from Theorem 3.1. □

When the domain is bounded, we have the following result where the sequence $\{z_n\}$ is computable.

Theorem 3.3. *Let X be a complete CAT(0) space and $\{A_i : i \in \mathbb{N}\}$ be a family of closed convex subsets of X such that $\bigcap_{i=1}^{\infty} A_i \neq \emptyset$ and $\bigcup_{i=1}^{\infty} A_i$ is bounded. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n = 1$, $\sum_{i=n}^{\infty}\lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$ where $\lambda_i' = \sum_{j=i+1}^{\infty}\lambda_j$. Let $\{\varepsilon_n\}$ be a sequence in $(0,\frac{1}{2})$ and $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in C$ be arbitrarily chosen. For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $\lambda_i' < \varepsilon_n$ for all $i \geq k_n$ and set*

$$r_n = \sup_{i\in\mathbb{N}}\{dist(z_n, A_i)\}, \quad \beta_n \in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right),$$

$$z_{n+1} = \alpha_n u \oplus (1-\alpha_n)u_n', \text{ where}$$

$$u_n' = \bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j} u_n^{A_i}, \quad u_n^{A_i} \in A_i \cap B(z_n : dist(z_n, A_i) + \beta_n^2).$$

Then the sequence $\{z_n\}$ converges to the unique point of $\bigcap_{i=1}^{\infty} A_i$ which is nearest to u.

Proof. Let p_i and p be as in the proof of Theorem 3.2. Thus we have

$$d(u_n^{A_i}, p_i z_n) < \varepsilon_n$$

for all $n \in \mathbb{N}$. For each n, we can choose $m_n > k_n$ such that

$$d\left(\bigoplus_{i=1}^{\infty} \lambda_i p_i z_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} p_i z_n\right) < \varepsilon_n.$$

Since $\lambda_i' < \varepsilon_n < \frac{1}{2}$, we have

$$d\left(\bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j} p_i z_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} p_i z_n\right)$$

$$\leq d\left(\bigoplus_{i=1}^{k_n}\frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}p_iz_n, \bigoplus_{i=1}^{k_n+1}\frac{\lambda_i}{\sum_{j=1}^{k_n+1}\lambda_j}p_iz_n\right)+\cdots+d\left(\bigoplus_{i=1}^{m_n-1}\frac{\lambda_i}{\sum_{j=1}^{m_n-1}\lambda_j}p_iz_n, \bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}p_iz_n\right)$$

$$\leq \frac{\lambda_{k_n+1}}{\sum_{j=1}^{k_n+1}\lambda_j}d\left(\bigoplus_{i=1}^{k_n}\frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}p_iz_n, p_{k_n+1}z_n\right)+\cdots+\frac{\lambda_{m_n}}{\sum_{j=1}^{m_n}\lambda_j}d\left(\bigoplus_{i=1}^{m_n-1}\frac{\lambda_i}{\sum_{j=1}^{m_n-1}\lambda_j}p_iz_n, p_{m_n}z_n\right)$$

$$\leq K\sum_{i=k_n+1}^{m_n}\frac{\lambda_i}{1-\lambda_i'} < 2K\sum_{i=k_n+1}^{m_n}\lambda_i < 2K\lambda'_{k_n+1} < 2K\varepsilon_n,$$

where $K = \sup_{n\in\mathbb{N}}\left\{\sup_{l\in\mathbb{N}}\left\{d\left(\bigoplus_{i=1}^{l}\frac{\lambda_i}{\sum_{j=1}^{l}\lambda_j}p_iz_n, p_{l+1}z_n\right)\right\}\right\} < \infty.$
Thus

$$d(u_n', pz_n) \leq \varepsilon_n(2K+2).$$

The result now follows from Theorem 3.1. □

As corollaries, with the same lines of proofs, the corresponding results hold for a finite family $\{t_i : i = 1, 2, ..., N\}$ of mappings.

Applications

Let X be a complete CAT(0) space. For a function $h : X \longrightarrow (-\infty, \infty]$, the $\alpha-$sublevel set is defined by

$$A_h^\alpha = \{x \in X : h(x) \leq \alpha\}.$$

Let $\{h_i : i \in \mathbb{N}\}$ be a family of lower semi-continuous and convex functions from X into $(-\infty, \infty]$. Bačák, Searston and Sims [2] introduced the method for approximating a minimizer of the functional $H : X \longrightarrow (-\infty, \infty]$, where $H = \sup_{i\in\mathbb{N}} h_i$ as the following:

Proposition 3.4. [2, Proposition 5.2] *Let X be a complete CAT(0) space and a mapping $F : X \longrightarrow (-\infty, \infty]$ be of the form $F = \max\{f, g\}$, where $f, g : X \longrightarrow (-\infty, \infty]$ are lower semi-continuous and convex functions. Let $\alpha > \inf_{x\in X} F(x) > -\infty$, and A_F^α be nonempty. Assume that f is both uniformly convex and uniformly continuous on bounded sets of X. Let $x_0 \in X$ be a starting point and $\{x_n\} \subset X$ be the sequence generated by*

$$x_{2n-1} = P_f(x_{2n-1}), \quad x_{2n} = P_g(x_{2n-1}),\ n \in \mathbb{N},$$

where P_f and P_g are projection mappings from X to A_f^α and A_g^α respectively. Then $\{x_n\}$ converges to $z \in A_F^\alpha$.

We now show Propositions providing the strong convergence of the sequence $\{z_n\}$ to an (approximative) minimizer of the functional H.

Proposition 3.5. *Let X be a complete CAT(0) space and a mapping $H : X \longrightarrow (-\infty, \infty]$ be of the form $H = \sup_{i\in\mathbb{N}} h_i$, where $h_i : X \longrightarrow (-\infty, \infty]$ are lower semi-continuous and convex functions for all $i \in \mathbb{N}$. Let $\alpha > \inf_{x\in X} H(x) > -\infty$. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^\infty \lambda_n = 1$, $\sum_{i=n}^\infty \lambda_i' \longrightarrow 0$ as $n \longrightarrow \infty$ where $\lambda_i' = \sum_{j=i+1}^\infty \lambda_j$. Let $\{\varepsilon_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$ satisfying $\sum_{n=1}^\infty \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, z_1 \in X$ are arbitrarily chosen and set*

$$r_n = \sup_{i\in\mathbb{N}}\{dist(z_n, A_{h_i}^\alpha)\}, \quad \beta_n \in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right),$$

$$z_{n+1} = \alpha_n u \oplus (1-\alpha_n)u_n,$$

where

$$u_n = \bigoplus_{i=1}^{\infty} \lambda_i u_n^i, \ \ u_n^i \in A_{h_i}^{\alpha} \cap B(z_n : dist(z_n, A_{f_i}^{\alpha}) + \beta_n^2)$$

for all $n \in \mathbb{N}$. *Then the sequence* $\{z_n\}$ *converges to the unique point of* A_H^{α} *which is nearest to* u.

Proof. Since $h_i : X \longrightarrow (-\infty, \infty]$ are lower semi-continuous and convex functions, $A_{h_i}^{\alpha}$ is closed and convex for all $i \in \mathbb{N}$. The result then follows from Theorem 3.2. □

Proposition 3.6. *Let* X *be a complete CAT(0) space and a mapping* $H : X \longrightarrow (-\infty, \infty]$ *be of the form* $H = \sup_{i \in \mathbb{N}} h_i$, *where* $h_i : X \longrightarrow (-\infty, \infty]$ *are lower semi-continuous and convex functions for all* $i \in \mathbb{N}$. *Let* $\alpha > \inf_{x \in X} H(x) > -\infty$. *Let* $\{\lambda_n\}$ *be a sequence in* $(0,1)$ *such that* $\sum_{n=1}^{\infty} \lambda_n = 1$, $\sum_{i=n}^{\infty} \lambda_i' \longrightarrow 0$ *as* $n \longrightarrow \infty$ *where* $\lambda_i' = \sum_{j=i+1}^{\infty} \lambda_j$. *Let* $\{\varepsilon_n\}$ *be a sequence in* $(0, \frac{1}{2})$ *and* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *satisfying* $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, *(C1), (C2) and (C3) respectively. Let* $u, z_1 \in C$ *be arbitrarily chosen. For each* $n \in \mathbb{N}$, *choose* $k_n \in \mathbb{N}$ *such that* $\lambda_i' < \varepsilon_n$ *for all* $i \geq k_n$ *and set*

$$r_n = \sup_{i \in \mathbb{N}} \{dist(z_n, A_{h_i}^{\alpha})\}, \quad \beta_n \in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right),$$

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) u_n',$$

where

$$u_n' = \bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n} \lambda_j} u_n^i, \ \ u_n^i \in A_{h_i}^{\alpha} \cap B(z_n : dist(z_n, A_{h_i}^{\alpha}) + \beta_n^2).$$

If $\{z_n\}$ *is bounded, then the sequence* $\{z_n\}$ *converges to the unique point of* A_H^{α} *which is nearest to* u.

Proof. Here we apply Theorem 3.3. □

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REFINEMENTS OF ε-DUALITY THEOREMS FOR A NONCONVEX PROBLEM WITH AN INFINITE NUMBER OF CONSTRAINTS

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ABSTRACT. Some remarks on approximate optimality conditions of a nonconvex optimization problem which has an infinite number of constraints are given. Results on ϵ-duality theorems of the problem are refined by using a mixed type dual problem of Wolfe and Mond-Weir type.

KEYWORDS: Almost ε-quasi solutions; ε-quasi solutions; ε-duality theorems.
AMS Subject Classification: 90C30 49N15 90C26 90C46.

1. INTRODUCTION

It was known that one of the first results dealing with approximate optimality solutions of a nonconvex programming problem was the paper "Necessary condition for ε-optimality" published on 1982 by P. Loridan [8]. A bit earlier, the such results can be found in the book of P.-J. Laurent [7] and in the paper of S.S Kutateladze [6]. Since the appearance of these results, there were many papers concerning in approximate necessary/sufficient optimality conditions of nonconvex problems such as [2, 4, 5, 9, 11, 12, 14, 15]. Besides concept of ε-solutions of an optimization problem which have global character, there were concepts of approximate solutions which have local one such as ε-quasi solutions, almost ε-quasi solutions. If the concept of global solutions is suitable for convex problems, the concept of local solutions is crucial for nonconvex problems.

Recently, in [12], some sufficient ε-optimality conditions and ε-duality theorems of a nonconvex optimization problem which has an infinite number of constraints have been established without assuming any constraint qualification condition. These results can be improved. Let us reconsider the problem:

$$\begin{array}{lll}(\mathrm{P}) & \text{Minimize} & f(x) \\ & \text{s.t} & f_t(x) \le 0, t \in T, \\ & & x \in C,\end{array}$$

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where $f, f_t : X \longrightarrow \mathbb{R}$, $t \in T$ are locally Lipschitz functions on a Banach space X, T is an arbitrary index set (not necessarily finite), C is a closed convex subset of X. In that paper, approximate optimality conditions are established based on a generalized Karush-Kuhn-Tucker condition up to ε and the properties of regularity or ε-semiconvexity applied for locally involved Lipschitz functions. Results on approximate duality theorems of Wolfe type are also presented.

The aim of this paper is to give some remarks on approximate sufficient conditions presented in [12]. Concretely, the approximate sufficient conditions can rebuilt with relaxed assumptions. Moreover, results on ϵ-duality theorems in the paper will be refined. Besides, relations between (P) and its dual problems via approximate dual theorems will be clarified. To improve results on ε-optimality conditions, we use the properties of ε-regularity and ε-semiconvexity for the locally involved Lipschitz functions instead of the properties of regularity and semiconvexity. To refine the results of ϵ-duality theorems given in the paper, we use a mixed type dual problem for (P). Then results on ϵ-duality theorems of Wolfe type and Mond-Weir type are derived. We also note that, the mixed type dual scheme was applied recently for a nonconvex multiobjective programming problem which has an infinite number of constraints [13].

The paper is organized as follows. The next section is devoted to preliminaries including basic concepts and definitions such as ϵ-semiconvex functions and locally approximate solutions. The main results are in the last two sections. In Section 3, some versions of approximate sufficient optimality conditions for (P) are given by using properties of ε-regularity or ε-semiconvexity applied for the functions involved of (P). In the last section, by formulating the dual problem of (P) in a mixed type, some new results on ϵ-duality theorems are proposed. Then some results on ϵ-duality theorems presented in [12] can be covered. Finally, evaluations between the approximate optimal values of primal-dual problems are studied.

2. PRELIMINARIES AND NOTATIONS

Throughout the paper, X is a Banach space, T is a compact topological space, C is a closed convex subset of X, and $f : X \longrightarrow \mathbb{R}$ is a locally Lipschitz function. We also assume that the constraint functions $f_t : X \longrightarrow \mathbb{R}$, $t \in T$, are locally Lipschitz with respect to x uniformly in t, i.e., that for each $x \in X$, there exists a neighbourhood U of x and a constant $K > 0$ such that

$$|f_t(z) - f_t(z')| \leq K \|z - z'\| \quad \forall\, z,\, z' \in U \text{ and } \forall\, t \in T.$$

Let $g : X \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The directional derivative of g at $z \in X$ in direction $d \in X$ is

$$g'(z; d) = \lim_{t \longrightarrow 0^+} \frac{g(z + td) - g(z)}{t}$$

if the limit exists.

The Clarke generalized directional derivative at $z \in X$ in direction $d \in X$ and the Clarke subdifferential of g at z are defined by

$$g^c(z; d) := \lim_{\substack{x \longrightarrow z \\ t \longrightarrow 0^+}} \sup \frac{g(x + td) - g(x)}{t},$$

$$\partial^c g(z) := \{v \in X^* \mid v(d) \leq g^c(z; d), \forall d \in X\},$$

respectively, where X^* is a dual space of X.

A locally Lipschitz function g is said to be quasidifferentiable (or regular in the sense of Clarke) at $z \in X$ if $g'(z;d)$ exists and

$$g^c(z;d) = g'(z;d), \forall d \in X.$$

For a closed subset D of X, the tangent cone to D is defined by

$$T_D(x) = \{v \in X \mid d_D^{\circ}(x;v) = 0\},$$

where d_D denotes the distance function to D, and the normal cone to D at x is defined by

$$N_D(x) = \{x^* \in X^* \mid \langle x^*, v\rangle \leq 0, \forall v \in T_D(x)\}.$$

If D is convex, the normal cone to D at x coincides with the one in the sense of convex analysis, i.e,

$$N_D(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq 0, \forall y \in D\}.$$

Definition 2.1. [12] Let C be a subset of X and let $\alpha \geq 0$. A locally Lipschitz function $g : X \longrightarrow \mathbb{R}$ is said to be α-semiconvex at $z \in C$ if g is regular at z, and the following condition is satisfied

$$g'(z; x - z) + \sqrt{\alpha}\|x - z\| \geq 0 \implies g(x) + \sqrt{\alpha}\|x - z\| \geq g(z),\ \forall x \in C. \tag{2.1}$$

The function g is said to be α-semiconvex on C if g is α-semiconvex at every $z \in C$.

As $\alpha = 0$, we obtain the definition of semiconvex function proposed in [10].

Lemma 2.2. [10] *If $g : X \longrightarrow \mathbb{R}$ is a semiconvex function on a convex set $C \subset X$, $z \in C, z + d \in C$ then $g(z + d) \leq g(z)$ implies that $g'(z;d) \leq 0$.*

Definition 2.3. [8] Let $\varepsilon \geq 0$. A locally Lipschitz function $g : X \longrightarrow \mathbb{R}$ is said to be ε-regular at $z \in X$, provided that

$$0 \leq g^c(z;d) - g'(z;d) \leq \sqrt{\varepsilon}\|d\|, \forall d \in X.$$

We use the following linear space:

$$\mathbb{R}^{(T)} := \{(\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

With $\lambda = (\lambda_t) \in \mathbb{R}^{(T)}$, the supporting set according to λ is

$$T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}.$$

Obviously, it is a finite subset of T. We also denote by $\mathbb{R}_+^{(T)}$ the non-negative cone of $\mathbb{R}^{(T)}$,

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t) \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0, t \in T\}.$$

It is easy to see that this cone is convex. For every $\lambda \in \mathbb{R}^{(T)}$, we define

$$\|\lambda\|_1 := \sum_{t\in T} |\lambda_t| = \sum_{t\in T(\lambda)} |\lambda_t|.$$

For $\alpha \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}^{(T)}$, $\lambda = (\lambda_t)_{t\in T}, \mu = (\mu_t)_{t\in T}$, we understand that

$$\begin{array}{lll} \lambda + \mu & := & (\lambda_t + \mu_t)_{t\in T}, \\ \alpha.\lambda & := & (\alpha\lambda_t)_{t\in T}. \end{array}$$

With $\lambda \in \mathbb{R}^{(T)}$ and $\{z_t\}_{t\in T} \subset Z$, Z being a real linear space, we define

$$\sum_{t\in T} \lambda_t z_t := \begin{cases} \sum_{t\in T(\lambda)} \lambda_t z_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

For $\lambda \in \mathbb{R}^{(T)}, f_t, t \in T$, and $\{Y_t\}_{t\in T}$, a family of non-empty subsets of X, we understand that

$$\sum_{t\in T} \lambda_t f_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t f_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset, \end{cases}$$

and

$$\sum_{t\in T} \lambda_t Y_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t Y_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

We denote by A the feasible set of (P). Let $\epsilon > 0$, the ε-feasible set of (P) is defined by

$$A_\varepsilon := \{x \in C \mid f_t(x) \leq \sqrt{\varepsilon}, \forall t \in T\}.$$

Definition 2.4. Let $\varepsilon \geq 0$. A point $z_\varepsilon \in X$ is said to be

(i) an almost ε-solution of (P) if

$$z_\varepsilon \in A_\varepsilon \text{ and } f(z_\varepsilon) \leq f(x) + \varepsilon, \forall x \in A;$$

(ii) an almost ε-quasisolution of (P) if

$$z_\varepsilon \in A_\varepsilon \text{ and } f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon}\, \|x - z_\varepsilon\|, \forall x \in A;$$

(iii) an almost regular ε-solution of (P) if z_ε is an almost ε-solution and is an almost ε-quasisolution of (P).

As $z_\varepsilon \in A$, we obtain the definitions of ε-solution, ε-quasisolution, and regular ε-solution of (P), respectively.

3. ε-OPTIMALITY CONDITIONS

To give some remarks and to improve results in [12], some theorems are recalled for the sake of convenience. Firstly, we need the following conditions:

$(\mathcal{A})$ (a1) X is separable, or
(a2) T is metrizable and $\partial^c f_t(x)$ is upper continuous (w^*) in t for each $x \in X$.

$(\mathcal{B})$ $\exists d \in T_C(z), f_t^c(z; d) < 0, \forall t \in I(z)$, where $z \in A$, $I(z) = \{t \in T \mid f_t(z) = 0\}$.

Theorem 3.1. [12] *Let $\varepsilon \geq 0$ and z be an ε-quasisolution for* (P). *If the conditions $(\mathcal{A})$ and $(\mathcal{B})$ are satisfied and the convex hull of $\{\cup\partial^c f_t(z), t \in I(z)\}$ is $weak^*$-closed then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that*

$$0 \in \partial^c f(z) + \sum_{t\in T} \lambda_t \partial^c f_t(z) + N_C(z) + \sqrt{\varepsilon} B^*, f_t(z) = 0, \forall t \in T(\lambda), \tag{3.1}$$

where B^* is a closed unit ball in X^*.

A pair (z, λ) satisfies (3.1) is called a Karush-Kuhn-Tucker (KKT) pair up to ε. From the theorem above, a generalized KKT condition up to ε was proposed as follows.

Definition 3.1. [12] Let $\varepsilon \geq 0$. A pair $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ is said to be satisfied generalized KKT condition up to ε corresponding to (P) if

$$\begin{cases} 0 \in \partial^c f(z_\varepsilon) + \sum\limits_{t\in T(\lambda)} \lambda_t \partial^c f_t(z_\varepsilon) + N(C, z_\varepsilon) + \sqrt{\varepsilon} B^* \\ f_t(z_\varepsilon) \geq 0, \forall t \in T(\lambda). \end{cases}$$

The pair (z_ε, λ) is called a generalized KKT pair up to ε. It is called strict if $f_t(z_\varepsilon) > 0$ for all $t \in T(\lambda)$, which is equivalent to $\lambda_t = 0$ if $f_t(z_\varepsilon) \leq 0$.

Then, a sufficient condition for a strict generalized KKT pair up to ε was given.

Theorem 3.2. [12] *Let $\varepsilon > 0$ and the condition $(\mathcal{A})$ be satisfied. For every $x \in A_\varepsilon$, let the strong closure of the subset $co\{\cup \partial^c f_t(x), t \in I(x)\}$ be $weak^*$-closed. Then there exists an almost regular ε-solution z for* (P) *and $\lambda \in \mathbb{R}_+^{(T)}$ such that (z, λ) is a strict generalized KKT pair up to ε.*

The such generalized KKT pair condition was used as a hypothesis to survey almost ε-quasisolutions of (P).

Theorem 3.3. [12] *For the problem* (P)*, assume that C is convex and that the functions f_t, $t \in T$, are convex. Let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. If f is ε-semiconvex at z_ε with respect to C, then*

$$\begin{aligned}&f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| \text{ for all } x \in C \text{ such that}\\ &f_t(x) \leq f_t(z_\varepsilon),\ \forall t \in T(\lambda).\end{aligned}$$

In particular, z_ε is an almost ε-quasisolution for (P).

By modifying the assumptions applied for the involved functions of (P), we extend the theorem above to the ones as follows. Firstly, assumptions posed on the involved functions of (P) are relaxed.

Theorem 3.4. *For the problem* (P)*, let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. Suppose that the function f is ε-regular at z_ε and the functions $f_t, t \in T$, are semiconvex on C. If the condition (2.1) of Definition 2.1 holds for f at $z = z_\varepsilon$ with $\alpha \geq 4\varepsilon$ then*

$$\begin{aligned}&f(z_\varepsilon) \leq f(x) + 2\sqrt{\varepsilon}\,\|x - z_\varepsilon\| \quad \text{for all } x \in C \text{ such that}\\ &f_t(x) \leq f_t(z_\varepsilon) \quad \text{for all } t \in T(\lambda).\end{aligned}$$

In particular, z_ε is an almost 4ε-quasisolution for (P).

Proof. Let $\varepsilon \geq 0$. Assume that $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$is a generalized KKT pair up to ε. If $T(\lambda) \neq \emptyset$, we obtain $u \in \partial^c f(z_\varepsilon), u_t \in \partial^c f_t(z_\varepsilon), \forall t \in T(\lambda), w \in N(C, z_\varepsilon), v \in B^*$ and $f_t(z_\varepsilon) \geq 0$ for all $t \in T(\lambda)$ such that

$$u(x - z_\varepsilon) + \sum_{t \in T(\lambda)} \lambda_t u_t(x - z_\varepsilon) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| = -w(x - z) \geq 0, \forall x \in C, \tag{3.2}$$

Note that $f_t, t \in T$, are semiconvex at z_ε. If $f_t(x) \leq f_t(z_\varepsilon)$ for all $t \in T(\lambda)$ then

$$u_t(x - z_\varepsilon) \leq f_t^c(z_\varepsilon; x - z_\varepsilon) = f_t'(z_\varepsilon; x - z_\varepsilon) \leq 0, \forall t \in T(\lambda), \forall x \in C.$$

Then, from (3.2), we obtain $u(x - z_\varepsilon) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| \geq 0$ for all $x \in C$. Since $u \in \partial^c f(z_\varepsilon)$ and f is ε-regular at z_ε, $f^c(z_\varepsilon; x - z_\varepsilon) \leq f'(z_\varepsilon; x - z_\varepsilon) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\|$. We get

$$f'(z_\varepsilon; x - z_\varepsilon) + \sqrt{4\varepsilon}\,\|x - z_\varepsilon\| \geq 0, \forall x \in C.$$

Since the condition (2.1) of Definition 2.1 holds for f at $z = z_\varepsilon$ with $\alpha \geq 4\varepsilon$, from the inequality above, we deduce the desired result. As $T(\lambda) = \emptyset$, we get

$$u(x - z_\varepsilon) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| = -w(x - z) \geq 0, \forall x \in C.$$

It is easy to see that the conclusion can be derived. □

Corollary 3.2. *For the problem* (P)*, let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. If $f_t, t \in T$, are semiconvex at z_ε and f is ε-semiconvex at z_ε then*

$$\begin{aligned}&f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| \quad \text{for all } x \in C \text{ such that}\\ &f_t(x) \leq f_t(z_\varepsilon) \quad \text{for all } t \in T(\lambda).\end{aligned}$$

In particular, z_ε is an almost ε-quasisolution for (P).

Proof. Arguing as in the proof of the theorem above with noticing that f is regular and $\alpha = \epsilon$, we can obtain the desired result. □

Remark 3.3. Since a convex function is a semiconvex function (see [8], [12]), we can see that Theorem 3.3 is a corollary of the one above.

Frequently, the Lagrangian function corresponding to (P) is formulated by

$$L(y, \lambda) = f(y) + \sum_{t \in T} \lambda_t f_t(y), \text{ for all } (y, \lambda) \in X \times \mathbb{R}_+^{(T)}.$$

It is obvious that, for every $\lambda \in \mathbb{R}_+^{(T)}$, the function $L(\cdot, \lambda)$ is locally Lipschitz on X. Note that if the functions f and $f_t, t \in T$, are semiconvex or ε- semiconvex at z_ε then $L(\cdot, \lambda)$ may not achieve the same property. We propose another version of theorem above.

Theorem 3.5. *For the problem* (P)*, let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. Assume that $f_t, t \in T$, are regular at z_ε and f is ε-regular at z_ε. If the condition (2.1) of Definition 2.1 holds for $L(\cdot, \lambda)$ at $z = z_\varepsilon$ with $\alpha \geq 4\varepsilon$ then*

$$\begin{aligned} &f(z_\varepsilon) \leq f(x) + 2\sqrt{\varepsilon}\,\|x - z_\varepsilon\| \quad \textit{for all } x \in C \textit{ such that} \\ &f_t(x) \leq f_t(z_\varepsilon) \quad \textit{for all } t \in T(\lambda). \end{aligned}$$

In particular, z_ε is an almost 4ε-quasisolution for (P).

Proof. Let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a generalized KKT pair up to ε. If $T(\lambda) = \emptyset$, the proof is similar to the case in the proof of Theorem 3.4. When $T(\lambda) \neq \emptyset$, we get $f_t(z_\varepsilon) \geq 0$, for all $t \in T(\lambda)$. Using an argument similar to the one of the proof of Theorem 3.4, we obtain $u \in \partial^c f(z_\varepsilon), u_t \in \partial^c f_t(z_\varepsilon), \forall t \in T(\lambda), w \in N(C, z_\varepsilon), v \in B^*$ such that

$$u(x - z_\varepsilon) + \sum_{t \in T(\lambda)} \lambda_t u_t(x - z_\varepsilon) + \sqrt{\varepsilon} v(x - z_\varepsilon) = -w(x - z_\varepsilon) \geq 0 \geq 0, \forall x \in C.$$

Since $f_t, t \in T$, are regular at z_ε and f is ε-regular at z_ε, we derive

$$f'(z_\varepsilon; x - z_\varepsilon) + \sum_{t \in T(\lambda)} \lambda_t f_t'(z_\varepsilon; x - z_\varepsilon) + \sqrt{4\varepsilon}\,\|x - z_\varepsilon\| \geq 0, \forall x \in C,$$

i.e.,

$$L'(\cdot, \lambda)(z_\varepsilon; x - z_\varepsilon) + \sqrt{4}\varepsilon\,\|x - z_\varepsilon\| \geq 0, \forall x \in C.$$

Since the condition (2.1) of Definition 2.1 holds for $L(\cdot, \lambda)$ at z_ε with $\mu \geq 4\varepsilon$, it follows

$$f(x) + \sum_{t \in T(\lambda)} \lambda_t f_t(x) + \sqrt{4\varepsilon}\,\|x - z_\varepsilon\| \geq f(z_\varepsilon) + \sum_{t \in T(\lambda)} \lambda_t f_t(z_\varepsilon), \forall x \in C.$$

On the other hand, we have $f_t(x) \leq f_t(z_\varepsilon)$, for all $t \in T(\lambda)$. Then,

$$f(x) + \sqrt{4\varepsilon}\,\|x - z_\varepsilon\| \geq f(z_\varepsilon), \forall x \in C.$$

Since $A \subset C$, it is easy to deduce that z_ε is an almost 4ε-quasisolution for (P). □

Corollary 3.4. *For the problem* (P)*, let $\varepsilon \geq 0$ and let $(z_\varepsilon, \lambda) \in A_\varepsilon \times \mathbb{R}_+^{(T)}$ be a KKT pair up to ε. If $f, f_t, t \in T$, are regular at z_ε and $L(\cdot, \lambda)$ is ε-semiconvex at z_ε then*

$$\begin{aligned} &f(z_\varepsilon) \leq f(x) + \sqrt{\varepsilon}\,\|x - z_\varepsilon\| \quad \textit{for all } x \in C \textit{ such that} \\ &f_t(x) \leq f_t(z_\varepsilon) \quad \textit{for all } t \in T(\lambda). \end{aligned}$$

In particular, z_ε is an almost ε-quasisolution for (P).

Proof. If $L(\cdot,\lambda)$ is ε-semiconvex then the condition (2.1) of Definition 2.1 holds for for $L(\cdot,\lambda)$ with $\alpha=\varepsilon$. On the other hand, if f is regular at z_ε then $u(x-z_\varepsilon)\le f^c(z_\varepsilon;x-z_\varepsilon)=f'(z_\varepsilon;x-z_\varepsilon), u\in\partial^c(z_\varepsilon)$. Using a similar argument as in the proof of theorem above, we can deduce the desired result. □

4. ε-DUALITY THEOREMS

In [12], the dual problem of (P) was formulated in Wolfe type and some results on ε-duality theorems was established. In this part, we are interested in a dual problem of (P) in a mixed type of Wolfe and Mond-Weir type. With this approach, we can cover some results established before. In addition, ε-duality theorems in Mond-Weir are also derived. Besides ε-duality theorems, our results attempt to evaluate the relations between the approximate optimal values of (P) and its dual problems.

Let us consider the mixed type dual problem of (P):

$$
\begin{array}{lll}
\text{(D)} & \text{Maximize} & L(x,\lambda):=f(y)+\sum_{t\in T}\lambda_t f_t(y)\\
 & \text{s.t} & 0\in\partial^c f(y)+\sum_{t\in T}(\lambda_t+\mu_t)\partial^c f_t(y)+N(C,x)+\sqrt{\varepsilon}B^*,\\
 & & \mu_t f_t(y)\ge 0, t\in T,\\
 & & (y,\lambda,\mu)\in C\times\mathbb{R}^{(T)}_+\times\mathbb{R}^{(T)}_+.
\end{array}
$$

Denote by F the feasible set of (D).

Based on the definition of ε-quasisolutions of the dual problem of (P) in Wolfe type presented in [12], we propose the definition of ε-quasisolutions of (D) as follows.

Definition 4.1. A point $(y_\varepsilon,\bar\lambda,\bar\mu)\in F$ is called an ε-quasisolution of (D) if

$$L(y_\varepsilon,\bar\lambda)\ge L(y,\lambda)-\sqrt{\varepsilon}\|y-y_\varepsilon\|-\sqrt{\varepsilon}\|\lambda-\bar\lambda\|_1,\forall(y,\lambda,\mu)\in F.$$

Theorem 4.1. *If $f,f_t,t\in T$, are regular on C and $L(\cdot,\zeta)$ is ε-semiconvex on C for every $\zeta\in\mathbb{R}^{(T)}_+$ then ε-weak duality between* (P) *and* (D) *holds, i.e.,*

$$f(x)+\sqrt{\varepsilon}\|x-y\|\ge L(y,\lambda),\forall x\in A,\forall(y,\lambda,\mu)\in F.$$

Proof. Let x and (y,λ,μ) be the feasible solutions of (P) and (D), respectively. We have

$$0\in\partial^c f(y)+\sum_{t\in T}(\lambda_t+\mu_t)\partial^c f_t(y)+N(C,y)+\sqrt{\varepsilon}B^*,\mu_t f_t(y)\ge 0,t\in T.$$

Using an argument as in the proofs of theorem above, we deduce that

$$L(x,\lambda+\mu)+\sqrt{\varepsilon}\|x-y\|\ge L(y,\lambda+\mu),\forall x\in C.$$

As $x\in A$, we get $f_t(x)\le 0$ for all $t\in T$. From this and $\mu_t f_t(y)\ge 0,t\in T$, the inequality above implies

$$f(x)+\sqrt{\varepsilon}\|x-y\|\ge L(y,\lambda).$$

□

Theorem 4.2. *Let $(z,\bar\lambda)\in A_\epsilon\times\mathbb{R}^{(T)}_+$ be a strict generalized KKT pair up to ε. If $f,f_t,t\in T$, are regular at z and $L(\cdot,\zeta)$ is ε-semiconvex at z for every $\zeta\in\mathbb{R}^{(T)}_+$ then $(z,\bar\lambda,0)$ is an ε-quasisolution of* (D).

Proof. Let $(y,\lambda,\mu)\in F$. Using an argument similar to the one in the proof the theorem above, we can deduce that

$$L(x,\lambda+\mu)+\sqrt{\varepsilon}\|x-y\|\ge L(y,\lambda+\mu)\ge L(y,\lambda),\forall x\in C.$$

Hence,

$$L(z, \lambda + \mu) \geq L(y, \lambda) - \sqrt{\varepsilon}\|z - y\|. \tag{4.1}$$

Note that

$$L(z, \bar{\lambda}) - L(z, \lambda + \mu) = \sum_{t \in T}(\bar{\lambda}_t - \lambda_t - \mu_t) f_t(z) \tag{4.2}$$

and $(z, \bar{\lambda})$ is a strict KKT pair up to ε. Hence, $\bar{\lambda}_t = 0$ if $f_t(z) \leq 0$.
So, if $f_t(z) \leq 0$ then we get

$$L(z, \bar{\lambda}) - L(z, \lambda + \mu) = -\sum_{t \in T}(\lambda_t + \mu_t) f_t(z) \geq 0. \tag{4.3}$$

If $0 < f_t(z) \leq \sqrt{\varepsilon}$ then

$$(\bar{\lambda}_t - \lambda_t - \mu_t) f_t(z) \geq -|\bar{\lambda}_t - \lambda_t - \mu_t| f_t(z) \geq -|\bar{\lambda}_t - \lambda_t| f_t(z).$$

Combining this, (4.2), and (4.3), we obtain $L(z, \bar{\lambda}) - L(z, \lambda + \mu) \geq -\sqrt{\varepsilon}\|\bar{\lambda} - \lambda\|_1$. This and (4.1) imply that

$$L(z, \bar{\lambda}) - L(y, \lambda) = L(z, \bar{\lambda}) - L(z, \lambda + \mu) + L(z, \lambda + \mu) - L(y, \lambda) \geq -\sqrt{\varepsilon}\|\bar{\lambda} - \lambda\|_1 - \sqrt{\varepsilon}\|z - y\|.$$

Furthermore, since $(z, \bar{\lambda}, 0) \in F$, the desired conclusion follows. □

When $\mu = 0$, Problem (D) becomes the dual problem of (P) in Wolfe type and corresponding theorems can be derived. As $\lambda = 0$ we obtain the dual problem of (P) in Mond-Weir type as follows.

$$\begin{array}{lll} (\mathrm{D_M}) & \text{Maximize} & f(y) \\ & \text{s.t} & 0 \in \partial^c f(y) + \sum_{t \in T} \mu_t \partial^c f_t(y) + N(C, x) + \sqrt{\varepsilon} B^*, \\ & & \mu_t f_t(y) \geq 0, t \in T, \\ & & (y, \mu) \in C \times \mathbb{R}_+^{(T)}. \end{array}$$

The feasible set of $(\mathrm{D_M})$ is denoted by F_M.

Definition 4.2. A point $(z, \bar{\mu}) \in F_M$ is called an ε-quasisolution of $(\mathrm{D_M})$ if

$$f(z) + \sqrt{\varepsilon}\|z - y\| \geq f(y), \forall (y, \mu) \in F_M.$$

Remark 4.3. When $\lambda = 0$, from Theorem 4.1, we get $f(x) + \sqrt{\varepsilon}\|x - y\| \geq f(y), x \in A, (y, 0, \mu) \in F$. Combining this and the problem $(\mathrm{D_M})$ we obtain an ε-weak duality theorem in Mond-Weir type. As $\mu = 0$, Theorem 4.1 becomes the ε-weak duality theorem presented in [12], and Theorem 4.2 reduces to Corollary 5.2 in [12].

Theorem 4.3. *Let $(z, \bar{\mu})$ be a KKT pair up to ϵ. Suppose that $f, f_t, t \in T$, are regular at z and $L(\cdot, \zeta)$ is ε-semiconvex at z for every $\zeta \in \mathbb{R}_+^{(T)}$. Then z is an ε-quasisolution of $(\mathrm{D_M})$.*

Proof. Let $(y, \mu) \in F_M$. By using an argument as in the proof of Theorem 4.1, we can deduce

$$L(x, \mu) + \sqrt{\epsilon}\|x - y\| \geq L(y, \mu) \geq f(y), \forall x \in C.$$

Since $(z, \bar{\mu})$ is a KKT pair up to ϵ, $(z, \bar{\mu})$ is a point of F_M. Furthermore, since $z \in A$, $f_t(z) \leq 0$ for all $t \in T$. Consequently, from the inequality above,

$$f(z) + \sqrt{\epsilon}\|z - y\| \geq L(x, \bar{\mu}) + \sqrt{\epsilon}\|z - y\| \geq L(y, \bar{\mu}) \geq f(y).$$

The desired result follows. □

Relations between (P) and its mixed type dual problem will be clarified some more by Theorem 4.4 and 4.5 below.

Theorem 4.4. *Suppose that $f, f_t, t \in T$, are regular at z_ε and $L(\cdot, \zeta)$ is ε-semiconvex at z_ε for every $\zeta \in \mathbb{R}_+^{(T)}$. Let $(z_\varepsilon, \bar{\lambda}, \bar{\mu})$ be a feasible point of* (D) *such that $\bar{\lambda}_t f_t(z_\varepsilon) \geq 0$ for all $t \in T$. If $z_\varepsilon \in A_\varepsilon$ then it is an almost ε-quasisolution for* (P).

Proof. Let $(z_\varepsilon, \bar{\lambda}, \bar{\mu})$ be a feasible point of (D). Using argument as above, we can deduce that

$$L(x, \bar{\lambda}, \bar{\mu}) + \sqrt{\epsilon}\|x - z\| \geq L(z_\varepsilon, \bar{\lambda}, \bar{\mu}) \geq f(z), \forall x \in C.$$

If $z_\varepsilon \in A_\varepsilon$ then for all $x \in A$ we obtain

$$f(x) + \sqrt{\epsilon}\|x - z\| \geq L(z_\varepsilon, \bar{\lambda}, \bar{\mu}) \geq f(z).$$

□

Remark 4.4. As $\mu = 0$, the theorem above becomes Proposition 5.2 in [12].

The following theorem is a small modification of the one above. The proof is omitted.

Theorem 4.5. *Suppose that $f_t, t \in T$, are semiconvex at z_ε and f is ε-semiconvex at z_ε. Let $(z_\varepsilon, \bar{\lambda}, \bar{\mu})$ be a feasible point of* (D) *such that $\bar{\lambda}_t f_t(z_\varepsilon) \geq 0$ for all $t \in T$. If $z_\varepsilon \in A_\varepsilon$ then it is an almost ε-quasisolution for* (P).

Remark 4.5. When $\mu = 0$, Theorems 4.4 and 4.5 are the ε-converse dual theorems in Wolfe type.

It is well known that, for an optimization problem, if strong duality between the problem and its dual problem appears then they have the same optimal value. In case approximate duality, it may need to know the error estimation between the optimal values of primal and dual problems. The next part is devoted to propose some results on error estimation between the value of (P) and the value of its dual problem at their ε-quasisolutions, respectively.

Theorem 4.6. *Given $\varepsilon > 0$, suppose that z_ε is an ε-quasisolution of* (P) *and there exist $\lambda^*, \mu^* \in \mathbb{R}_+^{(T)}$ such that $(z_\varepsilon, \lambda^* + \mu^*)$ is a KKT pair up to ε. Let $(y_\varepsilon, \bar{\lambda}, \bar{\mu})$ is an ε-quasisolution of* (D). *If $L(\cdot, \zeta)$ is ε-semiconvex on C for every $\zeta \in \mathbb{R}_+^{(T)}$ then*

$$-\sqrt{\varepsilon}\|\bar{\lambda} - \lambda^*\|_1 - \sqrt{\varepsilon}\|y_\varepsilon - z_\varepsilon\| \leq L(y_\varepsilon, \bar{\lambda}) - f(z_\varepsilon) \leq \sqrt{\varepsilon}\|y_\varepsilon - z_\varepsilon\| \tag{4.4}$$

Proof. Let $(y_\varepsilon, \bar{\lambda}, \bar{\mu})$ be an ε-quasisolution of (D). We get

$$L(y_\varepsilon, \bar{\lambda}) \geq L(y, \lambda) - \sqrt{\varepsilon}\|y - y_\varepsilon\| - \sqrt{\varepsilon}\|\lambda - \bar{\lambda}\|_1, \forall (y, \lambda, \mu) \in F. \tag{4.5}$$

Let z_ε be an ε-quasisolution of (P) and $(z_\varepsilon, \lambda^* + \mu^*)$ be a KKT pair up to ε. We obtain $f_t(z_\varepsilon) = 0$ for all $t \in T(\lambda^* + \mu^*)$. Note that $T(\lambda^*), T(\mu^*) \subset T(\lambda^* + \mu^*)$. It implies that $f_t(z_\varepsilon) = 0$ for all $t \in T(\mu^*) \cup T(\lambda^*)$. Hence, $\mu_t^* f_t(z_\varepsilon) = 0$ for all $t \in T$. This deduces that $(z_\varepsilon, \lambda^* + \mu^*)$ is also a feasible point of (D). From (4.5), we obtain

$$L(y_\varepsilon, \bar{\lambda}) \geq L(z_\varepsilon, \lambda^*) - \sqrt{\varepsilon}\|z_\varepsilon - y_\varepsilon\| - \sqrt{\varepsilon}\|\lambda^* - \bar{\lambda}\|_1.$$

Note that $f_t(z_\varepsilon) = 0$ for all $t \in T(\lambda^*)$. Hence, $L(z_\varepsilon, \lambda^*) = f(z_\varepsilon)$. So,

$$L(y_\varepsilon, \bar{\lambda}) \geq f(z_\varepsilon) - \sqrt{\varepsilon}\|z_\varepsilon - y_\varepsilon\| - \sqrt{\varepsilon}\|\lambda^* - \bar{\lambda}\|_1. \tag{4.6}$$

On the other hand, by applying Theorem 4.1 with $L(\cdot, \bar{\lambda})$ to be ε-semiconvex on C, we obtain

$$f(z_\varepsilon) + \sqrt{\varepsilon}\|z_\varepsilon - y_\varepsilon\| \geq L(y_\varepsilon, \bar{\lambda}).$$

This and (4.6) imply the conclusion. □

The following corollary can be obtained directly if the dual problem is formulated in Mond-Weir type.

Corollary 4.6. *Given $\varepsilon > 0$, suppose that z_ε is an ε-quasisolution of* (P) *and there exist $\mu^* \in \mathbb{R}_+^{(T)}$ such that (z_ε, μ^*) is a KKT pair up to ε. Let y_ε is an ε-quasisolution of the problem $(\mathrm{D_M})$. If f is ε-semiconvex on C then*

$$|f(y_\varepsilon) - f(z_\varepsilon)| \leq \sqrt{\varepsilon}\|y_\varepsilon - z_\varepsilon\|.$$

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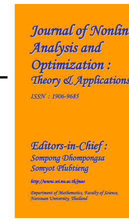

STRONG CONVERGENCE THEOREMS FOR COMMON FIXED POINT OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. The purpose of this paper is to give some strong convergence theorems for the problem of finding a common fixed point of a finite family of nonexpansive mappings in Banach spaces by the combination of the regularization method and the viscosity approximation method.

KEYWORDS: Accretive operators; weak sequential continuous mapping; and contraction mapping.
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1. INTRODUCTION

Numerous problems in mathematics and physical sciences can be recast in terms of a fixed point problem for nonexpansive mappings. For instance, if the nonexpansive mappings are projections onto some closed and convex sets, then the fixed point problem becomes the famous convex feasibility problem. Due to the practical importance of these problems, algorithms for finding fixed points of nonexpansive mappings continue to be flourishing topic of interest in fixed point theory.
The problem of finding a common fixed point of nonexpansive mappings has been investigated by many researchers: see, for instance, Bauschke [4], O' Hara et al. [23], Jung [16], Chang et al. [8], Ceng et al. [9], Chidume et al. [11], Kang et al. [18], N. Buong et al. [6] and others.

In 2000, Moudafi [22] proposed a viscosity approximation method which was considered by many authors [7, 10, 24, 27, 30, 32, 33] of selecting a particular fixed point of a given nonexpansive mapping in Hilbert spaces. If H is a Hilbert space, $T: C \to C$ is nonexpansive self-mapping on a nonempty closed convex C of H and $f: C \to C$ is a contraction mapping, then he proved the following results:

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(1) The sequence $\{x_n\}$ in C generated by the iterative scheme:

$$x_0 \in C,\ x_n = \frac{1}{1+\varepsilon_n}T(x_n) + \frac{\varepsilon_n}{1+\varepsilon_n}f(x_n),\ \forall n \geq 0,$$

converges strongly to the unique solution of the variational inequality

$$\overline{x} \in F(T) \text{ such that } \langle (I-f)(\overline{x}), \overline{x} - x\rangle \leq 0,\ \forall x \in F(T),$$

where $\{\varepsilon_n\}$ is a sequence of positive numbers tending to zero.

(2) With a initial $z_0 \in C$, define the sequence $\{z_n\}$ in C by

$$z_{n+1} = \frac{1}{1+\varepsilon_n}T(z_n) + \frac{\varepsilon_n}{1+\varepsilon_n}f(z_n),\ \forall n \geq 0.$$

Suppose that $\lim_{n\to\infty}\varepsilon_n = 0$ and $\sum_{n=1}^{\infty}\varepsilon_n = +\infty$, and $\lim_{n\to\infty}\left|\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}\right| = 0$. Then $\{z_n\}$ converges strongly to the unique solution of the variational inequality

$$\overline{x} \in F(T) \text{ such that } \langle (I-f)(\overline{x}), \overline{x} - x\rangle \leq 0,\ \forall x \in F(T).$$

On the other hand, the problem of finding a fixed point of a nonexpansive mapping is equivalent to the problem of finding a zero of the operator equation $0 \in A(x)$, involving the accretive mapping A.

One popular method of solving the problem of finding a zero of a maximal monotone operator is the proximal point algorithm, this algorithm is proposed by Rockafellar. In 1976, Rockafellar [25] proved the weak convergence of his algorithm, if the regularization sequence is bounded away from zero and if the sequence of the errors satisfies the suitable condition. In 1991, Güler [14] gave an example showing that Rockafellar's proximal point algorithm did not converge strongly in an infinite-dimensional Hilbert space. So, to have strong convergence, one has modify this algorithm. Recently, several authors proposed modifications of Rockafellar's proximal point algorithm to have strong convergence. Solodov and Svaiter [26] initiated such investigation followed by Kamimura and Takahashi [17] (in which the work of [26] is extended to the framework of uniformly convex and uniformly smooth Banach spaces). Lehdili and Moudafi [20] combined the technique of the proximal map and the Tikhonov regularization to introduce the prox-Tikhonov method. In 2006, Xu [31]; in 2009, Song and Yang [28] combined the regularization proximal point algorithm and a modification of iterative algorithms of Hapern's type [19] to obtain strong convergence theorems for the problem of finding a zero of maximal monotone operator in Hilbert space.

In 2011, by using the regularization proximal point algorithm of Xu [31], J. K. Kim and T. M. Tuyen [19] introduced an implicit iterative method in the form

$$r_n \sum_{i=1}^{N} A_i(x_{n+1}) + x_{n+1} = t_n u + (1-t_n)x_n,\ n \geq 0, \tag{1.1}$$

where $u, x_0 \in E$, and $A_i = I - T_i$ to find a common fixed point of a finite family of nonexpansive mappings $T_i:\ E \longrightarrow E,\ i = 1, 2, ..., N$ in Banach spaces. With this algorithm they are obtained the strong convergence of iterative $\{x_n\}$ generated by (1.1) to a common fixed point of T_i, when the sequences $\{r_n\}$ and $\{t_n\}$ are chosen suitable.

In this paper, we combine the regularization method and the viscosity approximation method, and use the tecnique of accretive operators to get convergence theorems for the problem of finding a common fixed point of a finite family of nonexpansive mappings in Banach spaces. And also, we consider the stability of algorithms and we give an application for the convex feasibility problem in Banach spaces.

2. PRELIMINARIES AND NOTATIONS

Let E be a Banach space with its dual space E^*. For the sake of simplicity, the norms of E and E^* are denoted by the same symbol $\|.\|$. We write $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. We use the symbols $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ and $\longrightarrow$ to denote the weak convergence, weak* convergence and strong convergence, respectively.

Definition 2.1. A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ the inequalities $\|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon$, imply there exists a $\delta = \delta(\varepsilon) \ge 0$ such that

$$\frac{\|x + y\|}{2} \le 1 - \delta.$$

The function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x + y\| : \ \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon\} \tag{2.1}$$

is called the modulus of convexity of the space E. The function $\delta_E(\varepsilon)$ defined on the interval $[0, 2]$ is continuous, increasing and $\delta_E(0) = 0$. The space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0,\ \forall\varepsilon \in (0, 2]$.
The function

$$\rho_E(\tau) = \sup\{2^{-1}\big(\|x + y\| + \|x - y\|\big) - 1 : \ \|x\| = 1,\ \|y\| = \tau\}, \tag{2.2}$$

is called the modulus of smoothness of the space E. The function $\rho_E(\tau)$ defined on the interval $[0, +\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$.

Definition 2.2. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0. \tag{2.3}$$

It is well known that every uniformly convex and uniformly smooth Banach space is reflexive. In what follows, we denote

$$h_E(\tau) := \frac{\rho_E(\tau)}{\tau}. \tag{2.4}$$

The function $h_E(\tau)$ is nondecreasing. In addition, we have the following estimate

$$h_E(K\tau) \le LKh_E(\tau),\ \forall K > 1,\ \tau > 0, \tag{2.5}$$

where L is the Figiel's constant [1, 2, 13], $1 < L < 1.7$.

Definition 2.3. A mapping j from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\} \tag{2.6}$$

is called the normalized duality mapping of E.

In any smooth Banach space $J(x) = 2^{-1}\text{grad}\|x\|^2$ and, if E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E^* is stricly convex or E is smooth, then J is single valued. Suppose that J be single valued, then J is said to be weakly sequentially continuous if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x,\ J(x_n) \overset{*}{\rightharpoonup} J(x)$. We denote the single valued normalized duality mapping by j.

Definition 2.4. An operator $A :\ D(A) \subseteq E \rightrightarrows E$ is called accretive, if for all $x, y \in D(A)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \ge 0,\ \forall u \in A(x),\ v \in A(y). \tag{2.7}$$

Definition 2.5. An operator $A: \ D(A) \subseteq E \rightrightarrows E$ is called $m-$accretive, if it is an accretive operator and the range $R(\lambda A + I) = E$ for all $\lambda > 0$.

If A is a $m-$accretive operator, then it is a demiclosed operator, i.e., if the sequence $\{x_n\} \subset D(A)$ satisfies $x_n \rightharpoonup x$ and $A(x_n) \longrightarrow f$, then $A(x) = f$ [2].

Definition 2.6. A mapping $T: \ C \longrightarrow E$ is said to be nonexpansive on a closed and convex subset C of Banach space E if

$$\|T(x) - T(y)\| \leq \|x - y\|, \ \forall x, y \in C. \tag{2.8}$$

If $T: \ C \longrightarrow E$ is a nonexpansive then $I - T$ is accretive operator.

Definition 2.7. A mapping $f: \ C \longrightarrow E$ is said to be contraction on a closed and convex subset C of Banach space E, if there exists $c \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq c\|x - y\| \ \forall x, y \in C. \tag{2.9}$$

Definition 2.8. Let G be a nonempty closed and convex subset of E. A mapping $Q_G: \ E \longrightarrow G$ is said to be

i) a retraction onto G if $Q_G^2 = Q_G$;

ii) a nonexpansive retraction, if it also satisfies the inequality

$$\|Q_G x - Q_G y\| \leq \|x - y\|, \ \forall x, y \in E; \tag{2.10}$$

iii) a sunny retraction, if for all $x \in E$ and for all $t \in [0, +\infty)$,

$$Q_G(Q_G x + t(x - Q_G x)) = Q_G x. \tag{2.11}$$

A closed and convex subset C of E is said to be a nonexpansive retract of E, if there exists a nonexpansive retraction from E onto C and is said to be a sunny nonexpansive retract of E, if there exists a sunny nonexpansive retraction from E onto C.

Definition 2.9. Let $C_1, \ C_2$ be convex subsets of E. The quantity

$$\beta(C_1, C_2) = \sup_{u \in C_1} \inf_{v \in C_2} \|u - v\| = \sup_{u \in C_1} d(u, C_2)$$

is said to be semideviation of the set C_1 from the set C_2. The function

$$\mathcal{H}(C_1, C_2) = \max\{\beta(C_1, C_2), \ \beta(C_2, C_1)\}$$

is said to be a Hausdorff distance between C_1 and C_2.

In what follows, we shall make use of the following lemmas:

Lemma 2.10. [3] *If E is a uniformly smooth Banach space, C_1 and C_2 are closed and convex subsets of E such that the Hausdorff $\mathcal{H}(C_1, C_2) \leq \delta$, and Q_{C_1} and Q_{C_2} are the sunny nonexpansive retractions onto the subsets C_1 and C_2, respectively, then*

$$\|Q_{C_1} x - Q_{C_2} x\|^2 \leq 16R(2r + d)h_E(\frac{16L\delta}{R}), \tag{2.12}$$

where L is Figiel's constant, $r = \|x\|, \ d = \max\{d_1, d_2\}$, and $R = 2(2r + d) + \delta$. Here $d_i = dist(\theta, C_i), \ i = 1, 2$, and θ is the origin of the space E.

Lemma 2.11. [1] *Let E be an uniformly convex and uniformly smooth Banach space. If $A = I - T$ with a nonexpansive mapping $T: \ D(A) \subseteq E \longrightarrow E$, then for all $x, y \in D(T)$, the domain of T,*

$$\langle Ax - Ay, j(x - y)\rangle \geq L^{-1}R^2\delta_E\left(\frac{\|Ax - Ay\|}{4R}\right), \tag{2.13}$$

where $\|x\| \leq R, \ \|y\| \leq R$ and $1 < L < 1.7$ is Figiel constant.

Lemma 2.12. [1] *In an uniformly smooth Banach space E, for all $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + c\rho_E(\|y\|), \tag{2.14}$$

where $c = 48\max(L, \|x\|, \|y\|)$.

Lemma 2.13. [12] *Let A be a continuous and accretive operator on the real Banach space E with $D(A) = E$. Then A is $m-$accretive.*

Lemma 2.14. [5, 29] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n,\ \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ for each $n \geq 0$ such that (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose either (a) $\sigma_n = o(\alpha_n)$, or (b) $\sum_{n=1}^{\infty} |\sigma_n| < \infty$, or (c) $\limsup \dfrac{\sigma_n}{\alpha_n} \leq 0$. Then $a_n \to 0$ as $n \to \infty$.

3. MAIN RESULTS

Firstly, we consider the following problem

$$\text{Finding an element } x^* \in S = \cap_{i=1}^{N} Fix(T_i), \tag{3.1}$$

where $Fix(T_i)$ is the set of fixed points of the nonexpansive mapping $T_i:\ E \longrightarrow E$, $i = 1, 2, ..., N$.

Let $x_0 \in E$ and let $f:\ E \longrightarrow E$ be contraction mapping on E with the contractive coefficient $k \in [0,1)$, we define the sequence $\{x_n\}$ as follow:

$$r_n \sum_{i=1}^{N} A_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1-t_n)x_n,\ n \geq 0, \tag{3.2}$$

where $\{r_n\}$ and $\{t_n\}$ are sequences of positive real numbers.

Remark 3.1. The algorithm (1.1) is a special case of the algorithm (3.2), when $f(x) = u$ for all $x \in E$.

Remark 3.2. In this paper, we use the symbol f to denote the contraction mapping on E with the contractive coefficient $k \in [0,1)$.

Theorem 3.3. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $T_i:\ E \longrightarrow E,\ i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0,1)$ satisfy*

- i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
- ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{x_n\}$ generated by (3.2) converges strongly to a common fixed point $q \in S$, which is unique solution of the following variational inequality

$$\langle (I-f)(q), j(q-p)\rangle \leq 0,\ \forall p \in S. \tag{3.3}$$

Proof. Firstly, we show that equation (3.2) defines a unique sequence $\{x_n\} \subset E$. Indeed, since the operator $\sum_{i=1}^{N} A_i$ is Lipschitz continuous and accretive on E, it is $m-$accretive (Lemma 2.13). Therefore equation (3.2) has a unique solution $x_{n+1} \in E$.

For every $x^* \in S$, we have

$$\langle \sum_{i=1}^{N} A_i(x_{n+1}), j(x_{n+1} - x^*)\rangle \geq 0,\ \forall n \geq 0. \tag{3.4}$$

Therefore,

$$\langle t_n f(x_n) + (1-t_n)x_n - x_{n+1}, j(x_{n+1} - x^*)\rangle \geq 0,\ \forall n \geq 0. \tag{3.5}$$

It gives the inequality

$$\|x_{n+1} - x^*\|^2 \leq [t_n\|f(x_n) - x^*\| + (1-t_n)\|x_n - x^*\|].\|x_{n+1} - x^*\|.$$

Consequently, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq t_n\|f(x^*) - x^*\| + t_n\|f(x_n) - f(x^*)\| + (1-t_n)\|x_n - x^*\|\\ &\leq t_n\|f(x^*) - x^*\| + [1 - t_n(1-k)]\|x_n - x^*\|\\ &\leq \max\big(\frac{\|f(x^*) - x^*\|}{1-k}, \|x_n - x^*\|\big)\\ &\vdots\\ &\leq \max\big(\frac{\|f(x^*) - x^*\|}{1-k}, \|x_0 - x^*\|\big),\ \forall n \geq 0.\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\} \subseteq \{x_n\}$, which converges weakly to a limit point $\overline{x} \in E$.

Suppose $\|x_n\| \leq R$ and $\|x^*\| \leq R$ with $R > 0$. By Lemma 2.11, we have

$$\begin{aligned}\delta_E\left(\frac{\|A_i(x_{n+1})\|}{4R}\right) &\leq \frac{L}{R^2}\langle A_i(x_{n+1}), j(x_{n+1} - x^*)\rangle\\ &\leq \frac{L}{R^2}\langle \sum_{k=1}^{N} A_k(x_{n+1}), j(x_{n+1} - x^*)\rangle\\ &\leq \frac{L}{R^2 r_n}\|t_n f(x_n) + (1-t_n)x_n - x_{n+1}\|.\|x_{n+1} - x^*\|\\ &\longrightarrow 0,\ n \longrightarrow \infty,\end{aligned}$$

for every $i = 1, 2, ..., N$.

Since modulus of convexity δ_E is continuous and E is the uniformly convex Banach space, $A_i(x_{n+1}) \longrightarrow 0$, $i = 1, 2, ..., N$. It is clear that $\overline{x} \in S$ from the demiclosedness of A_i.

Let q be unique solution of the variational inequality (3.3). Then, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)(q), j(q - x_n)\rangle &= \lim_{k\to\infty}\langle (I-f)(q), j(q - x_{n_k})\rangle\\ &= \langle (I-f)(q), j(q - \overline{x})\rangle \leq 0.\end{aligned} \tag{3.6}$$

Next, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \langle -r_n\sum_{i=1}^{N} A_i(x_{n+1}) + t_n f(x_n) + (1-t_n)x_n - q, j(x_{n+1} - Q_S u)\rangle\\ &= -r_n\langle \sum_{i=1}^{N} A_i(x_{n+1}), j(x_{n+1} - q)\rangle + \langle t_n f(x_n) + (1-t_n)x_n - q, j(x_{n+1} - q)\rangle\\ &\leq \frac{1}{2}[\|t_n f(x_n) + (1-t_n)x_n - q\|^2 + \|x_{n+1} - q\|^2].\end{aligned}$$

By Lemma 2.12 and the estimate above, we conclude that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \|t_nf(x_n)+(1-t_n)x_n-q\|^2\\ &= \|t_n(f(x_n)-f(q))+(1-t_n)(x_n-q)+t_n(f(q)-q)\|^2\\ &\le (1-t_n)^2\|x_n-q\|^2+2t_n(1-t_n)\langle (f(x_n)-f(q))+(f(q)-q), j(x_n-q)\rangle\\ &+c\rho_E(t_n\|f(x_n)-q\|)\\ &\le [(1-t_n)^2+2kt_n(1-t_n)]\|x_n-q\|^2+2t_n(1-t_n)\langle f(q)-q, j(x_n-q)\rangle\\ &+c\rho_E(t_n\|f(x_n)-q\|)\\ &\le \begin{cases}(1-t_n)\|x_n-q\|^2+2t_n(1-t_n)\langle f(q)-q, j(x_n-q)\rangle\\ +c\rho_E(t_n\|f(x_n)-q\|),\ \text{if } k\in[0,\dfrac{1}{2}],\\ [1-2(1-k)t_n]\|x_n-q\|^2+2t_n(1-t_n)\langle f(q)-q, j(x_n-q)\rangle\\ +c\rho_E(t_n\|f(x_n)-q\|),\ \text{if } k\in(\dfrac{1}{2},1).\end{cases}\end{aligned}$$

Consequently, we have

$$\|x_{n+1}-q\|^2 \le \begin{cases}(1-t_n)\|x_n-q\|^2+\sigma_n,\ \text{if } k\in[0,\dfrac{1}{2}],\\ [1-2(1-k)t_n]\|x_n-q\|^2+\sigma_n,\ \text{if } k\in(\dfrac{1}{2},1),\end{cases} \tag{3.7}$$

where

$$\sigma_n = t_n\big[2(1-t_n)\langle f(q)-q, j(x_n-q)\rangle + c\frac{\rho_E(t_n\|f(x_n)-q\|)}{t_n}\big].$$

Since E is the uniformly smooth Banach space, the property of function $\rho_E(t)$ and the boundedness of $\{f(x_n)\}$, we get that $\dfrac{\rho_E(t_n\|f(x_n)-q\|)}{t_n} \longrightarrow 0,\ n\longrightarrow\infty$. By (3.6), we obtain $\limsup_{n\to\infty}\dfrac{\sigma_n}{t_n}\le 0$. So, an application of Lemma 2.14 onto (3.7) yields the desired result. □

Theorem 3.4. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $T_i:\ E\longrightarrow E,\ i=1,2,...,N$ be nonexpansive mappings with $S=\cap_{i=1}^N Fix(T_i)\neq\emptyset$. If the sequences $\{r_n\}\subset(0,+\infty)$ and $\{t_n\}\subset(0,1)$ satisfy*

- i) $\lim_{n\to\infty}t_n=0$; $\sum_{n=0}^\infty t_n=+\infty,\ \sum_{n=0}^\infty|t_{n+1}-t_n|<+\infty$;
- ii) $\inf_n r_n = r>0,\ \sum_{n=0}^\infty\big|1-\dfrac{r_n}{r_{n+1}}\big|<+\infty$,

then the sequence $\{x_n\}$ generated by (3.2) converges strongly to a common fixed point q, which is unique solution of the variational inequality (3.3).

Proof. From the proof of Theorem 3.3, we obtain the sequence $\{x_n\}$ is bounded and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\overline{x}\in E$. Now, we show that $\overline{x}\in S$.

In (3.2), replacing n by $n+1$, we get

$$r_{n+1}\sum_{i=1}^N A_i(x_{n+2})+x_{n+2}=t_{n+1}f(x_{n+1})+(1-t_{n+1})x_{n+1}. \tag{3.8}$$

From (3.2) and (3.8) and by the accretiveness of $\sum_{i=1}^{N} A_i$, we have

$$\begin{aligned} r_{n+1}\langle x_{n+2}-x_{n+1}, j(x_{n+2}-x_{n+1})\rangle &- (r_{n+1}-r_n)\langle x_{n+2}, j(x_{n+2}-x_{n+1})\rangle \\ &\le \langle r_n[t_{n+1}f(x_{n+1})+(1-t_{n+1})x_{n+1}] \\ &- r_{n+1}[t_nf(x_n)+(1-t_n)x_n], j(x_{n+2}-x_{n+1})\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} r_{n+1}\|x_{n+2}-x_{n+1}\| &\le \|r_n[t_{n+1}f(x_{n+1})+(1-t_{n+1})x_{n+1}] - r_{n+1}[t_nf(x_n)+(1-t_n)x_n]\| \\ &+ |r_{n+1}-r_n|.\|x_{n+2}\| \\ &\le r_{n+1}(1-t_{n+1})\|x_{n+1}-x_n\| + (1-t_{n+1})|r_n-r_{n+1}|.\|x_{n+1}\| \\ &+ r_{n+1}|t_{n+1}-t_n|.\|x_n\| + |r_{n+1}-r_n|.\|x_{n+2}\| \\ &+ r_{n+1}t_{n+1}\|f(x_{n+1}-f(x_n))\| + t_{n+1}|r_{n+1}-r_n|.\|f(x_{n+1})\| \\ &+ r_{n+1}|t_{n+1}-t_n|.\|f(x_n)\| \\ &\le r_{n+1}[1-(1-k)t_{n+1}]\|x_{n+1}-x_n\| + (1-t_{n+1})|r_n-r_{n+1}|.\|x_{n+1}\| \\ &+ r_{n+1}|t_{n+1}-t_n|.\|x_n\| + |r_{n+1}-r_n|.\|x_{n+2}\| \\ &+ t_{n+1}|r_{n+1}-r_n|.\|f(x_{n+1})\| + r_{n+1}|t_{n+1}-t_n|.\|f(x_n)\|. \end{aligned}$$

By $\{t_n\} \subset (0,1)$ and $r_n > 0$ for all n, we deduce

$$\|x_{n+2}-x_{n+1}\| \le [1-(1-k)t_{n+1}]\|x_{n+1}-x_n\| + \left(2\big|t_{n+1}-t_n\big| + 3\big|1-\frac{r_n}{r_{n+1}}\big|\right)K, \quad (3.9)$$

where $K = \max\{\sup\|f(x_n)\|, \sup\|x_n\|\} < +\infty$. By Lemma 2.14, $\|x_{n+1}-x_n\| \longrightarrow 0$, as $n \longrightarrow \infty$.

Suppose $R > \max\{K, \|x^*\|\}$. By Lemma 2.11, we have

$$\begin{aligned} \delta_E\left(\frac{\|A_i(x_{n+1})\|}{4R}\right) &\le \frac{L}{R^2}\langle A_i(x_{n+1}), j(x_{n+1}-x^*)\rangle \\ &\le \frac{L}{R^2}\langle \sum_{k=1}^{N} A_k(x_{n+1}), j(x_{n+1}-x^*)\rangle \\ &\le \frac{L}{R^2 r_n}\|t_nf(x_n)+(1-t_n)x_n-x_{n+1}\|.\|x_{n+1}-x^*\| \\ &\le \frac{2L}{Rr}(2Rt_n + \|x_{n+1}-x_n\|) \\ &\longrightarrow 0,\ n \longrightarrow \infty, \end{aligned}$$

for every $i = 1, 2, ..., N$.

Since modulus of convexity δ_E is continuous and E is the uniformly convex Banach space, $A_i(x_{n+1}) \longrightarrow 0$, $i = 1, 2, ..., N$. It is clear that $\overline{x} \in S$ from the demiclosedness of A_i.

The rest of the proof follows the pattern of Theorem 3.3. ☐

Now, we will give a method to solve the following more general problem

$$\text{Finding an element } x^* \in S = \cap_{i=1}^{N} Fix(T_i), \quad (3.10)$$

where $T_i : C_i \longrightarrow C_i$, $i = 1, 2, ..., N$ is nonexpansive mapping and C_i is a closed, convex and nonexpansive retract of E.

Obviously, we have the following lemma:

Lemma 3.5. *Let E be a Banach space and let C be a closed, convex and retract of E. Let $T : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $Fix(T) = Fix(TQ_C)$, where $Q_C : E \longrightarrow C$ is a retraction of E.*

We consider the iterative sequence $\{x_n\}$ defined by

$$r_n \sum_{i=1}^{N} B_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1-t_n)x_n,\ x_0 \in E,\ \ n \geq 0, \tag{3.11}$$

where $B_i = I - T_i Q_{C_i},\ i = 1, 2, ..., N$ and $Q_{C_i} :\ E \longrightarrow C_i$ is a nonexpansive retraction from E onto $C_i,\ i = 1, 2, ..., N$.

Theorem 3.6. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed nonexpansive retract subset of E and let $T_i : C_i \longrightarrow C_i$, $i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{x_n\}$ generated by (3.11) converges strongly to a common fixed point $q \in S$, which is unique solution of the following variational inequality

$$\langle (I - f)(q), j(q - p)\rangle \leq 0,\ \forall p \in S. \tag{3.12}$$

Proof. By Lemma 3.5, we have $S = \cap_{i=1}^{N} Fix(T_i Q_{C_i})$ and apply Theorem 3.3 we obtain the proof of this theorem. □

Theorem 3.7. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed nonexpansive retract subset of E and let $T_i : C_i \longrightarrow C_i$, $i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^{N} Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty,\ \sum_{n=0}^{\infty} |t_{n+1} - t_n| < +\infty$;
ii) $\inf_n r_n = r > 0,\ \sum_{n=0}^{\infty} \left|1 - \dfrac{r_n}{r_{n+1}}\right| < +\infty$,

then the sequence $\{x_n\}$ generated by (3.11) converges strongly to a common fixed point $q \in S$, which is unique solution of the variational inequality (3.12).

Proof. By Lemma 3.5, we have $S = \cap_{i=1}^{N} Fix(T_i Q_{C_i})$ and apply Theorem 3.4 we obtain the proof of this theorem. □

Next, we study stability of regularization algorithm (3.11) in the case that each C_i is closed, convex and sunny nonexpansive retract of E with respect to pertubations of operators T_i and constraints C_i, $i = 1, 2, ..., N$ satisfying conditions:

(P1) Instead of C_i, there is a sequence of closed, convex and sunny nonexpansive retracts $C_i^n \subset E$, $n = 1, 2, 3, ...$ such that

$$\mathcal{H}(C_i^n, C_i) \leq \delta_n,\ i = 1, 2, ..., N,$$

where $\{\delta_n\}$ is a sequence of positive numbers.

(P2) On the each set C_i^n, there is a nonexpansive self-mapping $T_i^n : C_i^n \longrightarrow C_i^n$, $i = 1, 2, ..., N$ satisfying the conditions: if for all $t > 0$, there exists the increasing positive functions $g(t)$ and $\xi(t)$ such that $g(0) \geq 0,\ \xi(0) = 0$ and $x \in C_i$, $y \in C_i^m,\ \|x - y\| \leq \delta$, then

$$\|T_i x - T_i^m y\| \leq g(\max\{\|x\|, \|y\|\})\xi(\delta). \tag{3.13}$$

Remark 3.8. Note that, the conditions (P1) and (P2) are considered in [1] by Y. Alber.

We establish the convergence and stability of regularization method (3.11) in the form

$$r_n \sum_{i=1}^{N} B_i^n(z_{n+1}) + z_{n+1} = t_n f(x_n) + (1-t_n)z_n, \; z_0 \in E, \;\; n \geq 0, \tag{3.14}$$

where $B_i^n = I - T_i^n Q_{C_i^n}, \; i = 1,2,...,N, \; f : \; E \longrightarrow E$ is a contraction and $Q_{C_i^n} : E \longrightarrow C_i^n$ is a sunny nonexpansive retraction from E onto $C_i^n, \; i = 1,2,...,N$.

Theorem 3.9. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed sunny nonexpansive retract subset of E and let $T_i : \; C_i \longrightarrow C_i, \; i = 1,2,...,N$ be nonexpansive mappings with $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. If the conditions (P1) and (P2) are fulfilled, and the sequences $\{r_n\}, \; \{\delta_n\}$ and $\{t_n\}$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^\infty t_n = +\infty$;
ii) $\lim_{n\to\infty} r_n = +\infty$;
iii) $\sum_{n=0}^\infty r_n \xi(a\sqrt{h_E(\delta_n)}) < +\infty$ *or* $\lim_{n\to\infty} \dfrac{r_n \xi(a\sqrt{h_E(\delta_n)})}{t_n} = 0$ *for each* $a > 0$,

then the sequence $\{z_n\}$ generated by (3.14) converges strongly to a common fixed point $q \in S$, which is unique solution of the variational inequality (3.12).

Proof. For each n, $\sum_{i=1}^N B_i^n$ is $m-$accretive operator on E, so the equation (3.14) define unique element $z_{n+1} \in E$.
From the equation (3.11) and (3.14) we have

$$\begin{aligned} & r_n \langle \sum_{i=1}^{N} B_i^n(z_{n+1}) - B_i^n(x_{n+1}), j(z_{n+1} - x_{n+1}) \rangle \\ & \quad + r_n \langle \sum_{i=1}^{N} B_i^n(x_{n+1}) - B_i(x_{n+1}), j(z_{n+1} - x_{n+1}) \rangle + \|z_{n+1} - x_{n+1}\|^2 \\ & \quad = (1-t_n)\langle z_n - x_n, j(z_{n+1} - x_{n+1}) \rangle + t_n \langle f(z_n) - f(x_n), j(z_{n+1} - x_{n+1}) \rangle. \end{aligned} \tag{3.15}$$

By the accretiveness of $\sum_{i=1}^N B_i^n$ and the equation (3.15), we deduce

$$\|z_{n+1} - x_{n+1}\| \leq [1-(1-k)t_n]\|z_n - x_n\| + r_n \sum_{i=1}^{N} \|B_i^n(x_{n+1}) - B_i(x_{n+1})\|. \tag{3.16}$$

For each $i \in \{1,2,...,N\}$,

$$\|B_i^n(x_{n+1}) - B_i(x_{n+1})\| = \|T_i^n Q_{C_i^n} x_{n+1} - T_i Q_{C_i} x_{n+1}\|. \tag{3.17}$$

Since $\{x_n\}$ is bounded and $\mathcal{H}(C_i, C_i^n) \leq \delta_n$, there exist constants $K_{1i} > 0$ and $K_{2i} > 1$ such that inequalities

$$\|Q_{C_i^n} x_{n+1} - Q_{C_i} x_{n+1}\| \leq K_{1i}\sqrt{h_E(K_{2i}\delta_n)} \leq K_{1i}\sqrt{K_{2i}L}\sqrt{h_E(\delta_n)} \tag{3.18}$$

hold.
By the condition (P2),

$$\|T_i^n Q_{C_i^n} x_{n+1} - T_i Q_{C_i} x_{n+1}\| \leq g(M_i)\xi(K_{1i}\sqrt{K_{2i}L}\sqrt{h_E(\delta_n)}), \tag{3.19}$$

where $M_i = \max\{\sup \|Q_{C_i^n} x_{n+1}\|, \sup \|Q_{C_i} x_{n+1}\|\} < +\infty$.
From (3.16), (3.17) and (3.19), we obtain

$$\|z_{n+1} - x_{n+1}\| \leq [1 - (1-k)t_n]\|z_n - x_n\| + Ng(M)r_n\xi(\gamma_{12}\sqrt{h_E(\delta_n)}), \quad (3.20)$$

where $M = \max\{M_1, M_2, ..., M_N\} < +\infty$ and $\gamma_{12} = \max\limits_{i=1,2,...,N}\{K_{1i}\sqrt{K_{2i}L}\}$.

By the assumption and Lemma 2.14, we conclude that $\|z_n - x_n\| \longrightarrow 0$. In addition, by Theorem 3.6,

$$\|z_n - q\| \leq \|z_n - x_n\| + \|x_n - q\| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \quad (3.21)$$

which implies that z_n converges strongly to q. □

By a proof similar to the proof of Theorem 3.9 we have the following result:

Theorem 3.10. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed sunny nonexpansive retract subset of E and let $T_i : C_i \longrightarrow C_i$, $i = 1, 2, ..., N$ be nonexpansive mappings with $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. If the conditions (P1) and (P2) are fulfilled, and the sequences $\{r_n\}$, $\{\delta_n\}$ and $\{t_n\}$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^\infty t_n = +\infty$, $\sum_{n=0}^\infty |t_{n+1} - t_n| < +\infty$;
ii) $\inf\limits_n r_n = r > 0$, $\sum_{n=0}^\infty \left|1 - \frac{r_n}{r_{n+1}}\right| < +\infty$,
iii) $\sum_{n=0}^\infty r_n\xi(a\sqrt{h_E(\delta_n)}) < +\infty$ *or* $\lim_{n\to\infty} \frac{r_n\xi(a\sqrt{h_E(\delta_n)})}{t_n} = 0$ *for each* $a > 0$,

then the sequence $\{z_n\}$ generated by (3.14) converges strongly to a common fixed point $q \in S$, which is unique solution of the variational inequality (3.12).

Finally, in this section we give a method to solve the following problem:

$$\text{Finding an element } x^* \in S = \cap_{i=1}^N Fix(T_i), \quad (3.22)$$

where $T_i : C_i \longrightarrow E$, $i = 1, 2, ..., N$ is nonexpansive nonself-mapping and C_i is a closed, convex and sunny nonexpansive retract of E.

Lemma 3.11. [21] *Let C be a closed and convex subset of a strictly convex Banach space E and let $T : C \longrightarrow E$ be a nonexpansive mapping from C into E. Suppose that C is a sunny nonexpansive retract of E. If $Fix(T) \neq \emptyset$, then $Fix(T) = Fix(Q_C T)$, where Q_C is a sunny nonexpansive retraction from E onto C.*
We have the following results:

Theorem 3.12. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed sunny nonexpansive retract subset of E, let $T_i : C_i \longrightarrow E$, $i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^\infty t_n = +\infty$;
ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{u_n\}$ defined by

$$r_n \sum_{i=1}^N f_i(u_{n+1}) + u_{n+1} = t_n f(x_n) + (1 - t_n)u_n, \; u_0 \in E, \;\; n \geq 0, \quad (3.23)$$

converges strongly to a common fixed point $q \in S$, which is unique solution of the following variational inequality

$$\langle (I - f)(q), j(q - p)\rangle \leq 0, \; \forall p \in S, \quad (3.24)$$

where $f_i = I - Q_{C_i} T_i Q_{C_i},\ i = 1, 2, ..., N.$

Proof. By Lemma 3.5 and Lemma 3.11, $S = \cap_{i=1}^N Fix(T_i) = \cap_{i=1}^N Fix(f_i)$. Apply Theorem 3.3 we obtain the proof of this theorem. □

Theorem 3.13. *Suppose that E is a uniformly convex and uniformly smooth Banach space, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C_i be a convex closed sunny nonexpansive retract subset of E, let $T_i : C_i \longrightarrow E,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N Fix(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^\infty t_n = +\infty,\ \sum_{n=0}^\infty |t_{n+1} - t_n| < +\infty$;

ii) $\inf_n r_n = r > 0,\ \sum_{n=0}^\infty \left|1 - \dfrac{r_n}{r_{n+1}}\right| < +\infty$,

then the sequence $\{u_n\}$ defined by

$$r_n \sum_{i=1}^N f_i(u_{n+1}) + u_{n+1} = t_n f(x_n) + (1 - t_n) u_n,\ u_0 \in E,\ \ n \geq 0, \tag{3.25}$$

converges strongly to a common fixed point $q \in S$, which is unique solution of the variational inequality (3.24).

Proof. By Lemma 3.5 and Lemma 3.11, $S = \cap_{i=1}^N Fix(T_i) = \cap_{i=1}^N Fix(f_i)$. Apply Theorem 3.4 we obtain the proof of this theorem. □

4. An Application

Consider the following convex feasibility problem:

$$\text{Finding an element } x^* \in S = \cap_{i=1}^N S_i \neq \emptyset, \tag{4.1}$$

where $S_i,\ i = 1, 2, ..., N$ are closed, convex and nonexpansive retracts of a uniformly convex and uniformly smooth Banach space E.

In this section, we give an application of regularization algorithms (3.2) to find a solution of (4.1).
Let Q_{S_i} denote the nonexpansive retraction from E onto $S_i,\ i = 1, 2, ..., N$. It is clear that $F(Q_{S_i}) = S_i,\ i = 1, 2, ..., N$. Thus, the problem (4.1) is equivalent to the problem of finding a common fixed point of finite family of nonexpansive mappings $T_i = Q_{S_i},\ i = 1, 2, ..., N$.

By Theorem 3.3 and Theorem 3.4, we have the following results:

Theorem 4.1. *If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^\infty t_n = +\infty$;

ii) $\lim_{n\to\infty} r_n = +\infty$,

then the sequence $\{x_n\}$ defined by

$$r_n \sum_{i=1}^N A_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1 - t_n) x_n,\ u, x_0 \in E,\ n \geq 0 \tag{4.2}$$

converges strongly to a solution of (4.1), *where* $A_i = I - Q_{S_i},\ i = 1, 2, ..., N$

Theorem 4.2. *If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

i) $\lim_{n\to\infty} t_n = 0$; $\sum_{n=0}^\infty t_n = +\infty,\ \sum_{n=0}^\infty |t_{n+1} - t_n| < +\infty$;

ii) $\inf_n r_n = r > 0,\ \sum_{n=0}^\infty \left|1 - \dfrac{r_n}{r_{n+1}}\right| < +\infty$, *then the sequence* $\{x_n\}$ *defined by*

$$r_n \sum_{i=1}^{N} A_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1-t_n)x_n,\ u, x_0 \in E,\ n \geq 0 \tag{4.3}$$

converges strongly to a solution of (4.1), *where* $A_i = I - Q_{S_i},\ i = 1, 2, ..., N$

Now, we consider a special case of problem (4.1), it is the problem of finding a solution of a general system of linear equations.

Let S denote the set of solutions of the general system of linear equations

$$\sum_{j=1}^{k} a_{ij}x_j = b_i,\ i = 1, 2, ..., N, \tag{4.4}$$

and we suppose $S \neq \emptyset$, and $\sum_{j=1}^{k} a_{ij}^2 > 0,\ \forall i = 1, 2, ..., N.$

Let

$$S_i = \{(x_1, x_2, ..., x_k) \mid \sum_{j=1}^{k} a_{ij}x_j = b_i\},\ i = 1, 2, ..., N. \tag{4.5}$$

Then, S_i is a hyperplane in $\mathbb{R}^k$.

It is well - known that, the orthogonal projection P_i from $\mathbb{R}^k$ onto S_i is also the sunny nonexpansive retraction from $\mathbb{R}^k$ onto $S_i,\ i = 1, 2, ..., N$. Moreover,

$$P_i(x) = \left(x_l - a_{il} \frac{\sum\limits_{j=1}^{k} a_{ij}x_j - b_l}{\sum\limits_{j=1}^{n} a_{ij}^2} \right)_{l=1}^{k},\ i = 1, 2, ..., N, \tag{4.6}$$

for all $x = (x_1, ..., x_k) \in \mathbb{R}^k$.

We have a corollary of Theorem 4.1 and Theorem 4.2:

Corollary 4.3. *If the sequences* $\{r_n\} \subset (0, +\infty)$ *and* $\{t_n\} \subset (0, 1)$ *satisfy the conditions i) and ii) in Theorem 4.1 or the conditions i) and ii) in Theorem 4.2, then the sequence* $\{x_n\}$ *defined by*

$$r_n \sum_{i=1}^{N} B_i(x_{n+1}) + x_{n+1} = t_n f(x_n) + (1-t_n)x_n,\ u, x_0 \in E,\ n \geq 0 \tag{4.7}$$

converges strongly to a solution x^* *of system* (4.4), *where* $B_i = I - P_i,\ i = 1, 2, ..., N$.

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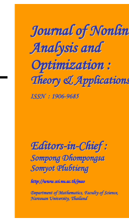

A UNIFYING SEMI-LOCAL ANALYSIS FOR ITERATIVE ALGORITHMS OF HIGH CONVERGENCE ORDER

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ABSTRACT. We present a unifying semi-local convergence analysis of two-step Newton-type methods for solving nonlinear equations in a Banach space setting. Convergence order of these methods is higher than two. Our analysis expands the applicability of these methods by providing weaker convergence criteria and a convergence analysis – which is tighter than earlier studies [1–4, 24–34] – is also presented. Numerical examples illustrating the developed theoretical results are also given.

KEYWORDS: Two-point Newton type methods; Banach space; majorizing sequence; semi-local convergence; recurrent functions.
AMS Subject Classification: 65B05 65J15 65N30 65N35 65H10 47H17 49M15.

1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where, $\mathcal{F}$ is a twice Fréchet differentiable operator defined on a convex subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. Numerous problems in science and engineering can be reduced to solving the above equation [18, 32]. Consequently, solving these equations is an important scientific field of research. In many situations, finding a closed form solution for the non-linear equation (1.1) is not possible. Therefore, iterative solution techniques are employed for solving these equations. The study about convergence analysis of iterative methods is usually divided into two categories : semi-local and local convergence analysis. The semilocal convergence analysis is based upon the information around an initial point to give criteria ensuring the convergence of the iterative procedure. While the

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local convergence analysis is based on the information around a solution to find estimates of the radii of convergence balls.

In the present paper, we study the semi-local convergence of the Two-step Newton-type method (**TSNTM**) defined by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1}\mathcal{T}_{\mathcal{F}}(x_n)\mathcal{F}(y_n) \end{aligned}\right\} \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $x_0 \in \mathbf{D}$ is an initial point, the operator $\mathcal{T}_{\mathcal{F}}(x) : \mathbf{D} \to \mathbf{Y}$ is given as

$$\mathcal{T}_{\mathcal{F}}(x) = \mathcal{I} + \mathcal{V}_{\mathcal{F}}(x) + \mathcal{V}_{\mathcal{F}}(x)^2\mathcal{G}_{\mathcal{F}}(x),$$

where the operator $\mathcal{V}_{\mathcal{F}}(x) : \mathbf{D} \to \mathbf{Y}$ is defined by

$$\mathcal{V}_{\mathcal{F}}(x) = \mathcal{F}'(x)^{-1}\mathcal{F}''(x)\mathcal{F}'(x)^{-1}\mathcal{F}(x)$$

and $\mathcal{G}_{\mathcal{F}} : \mathbf{D} \to \mathbf{L}(\mathbf{X}, \mathbf{X})$ is a given linear operator for each $x \in \mathbf{D}$. Some special cases of (**TSNTM**) are

Case $-$ 1. two-step Newton method of order three (**TSNM-O-3**) defined by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(y_n) \end{aligned}\right\} \tag{1.3}$$

for each $n = 0, 1, 2, \ldots,$

Case $-$ 2. Two-step Newton method of order four (**TSNM-O-4**) defined by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1}(\mathcal{I} + \mathcal{V}_{\mathcal{F}}(x_n))\mathcal{F}(y_n) \end{aligned}\right\} \tag{1.4}$$

for each $n = 0, 1, 2, \ldots,$

Case $-$ 3. Two-step Newton method of order five (**TSNM-O-5**) defined by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ x_{n+1} &= y_n - \mathcal{F}'(x_n)^{-1}\Big(\mathcal{I} + \mathcal{V}_{\mathcal{F}}(x_n) \\ &\qquad + \frac{\mathcal{V}_{\mathcal{F}}(x_n)^2}{2}\Big(\frac{5}{2}\mathcal{I} - \mathcal{V}_{\mathcal{F}'}(x_n)\Big)\Big)\mathcal{F}(y_n) \end{aligned}\right\} \tag{1.5}$$

for each $n = 0, 1, 2, \ldots.$

Many other choices of operator $\mathcal{T}_{\mathcal{F}}$ lead to other popular iterative methods such as Halley's-type or Chebyshev-type methods []. Concerning the order of convergence of such methods - in the case when $\mathbf{X} = \mathbf{Y} = \mathbb{R}$ - a theorem by Traub [33] states that for sufficiently smooth $\mathcal{G}_{\mathcal{F}}(x)$ (**TSNTM**) has order four.

The following set of conditions (**C**) have been used to perform semi-local convergence analysis of these method [1–29]

$\mathbf{C}_1$. there exists $x_0 \in \mathbf{D}$ such that $\mathcal{F}'(x_0)^{-1} \in \mathbf{L}(\mathbf{Y}, \mathbf{X})$,

$\mathbf{C}_2$. $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \le \eta$,

$\mathbf{C}_3$. $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(x)\right\| \le \mathcal{L}$ for each $x \in \mathbf{D}$ or $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(y))\right\| \le \mathcal{L}\left\|x - y\right\|$ for each $x, y \in \mathbf{D}$,

$\mathbf{C}_4$. $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}''(x) - \mathcal{F}''(y))\right\| \le \mathcal{M}\left\|x - y\right\|$ for each $x, y \in \mathbf{D}$,

$\mathbf{C}_5$. $\eta \le \dfrac{\mathcal{L}^2 + 4\mathcal{M} - \mathcal{L}\sqrt{L^2 + 2\mathcal{M}}}{3\mathcal{M}(\mathcal{L} + \sqrt{\mathcal{L}^2 + 2\mathcal{M}})}$,

$\mathbf{C}_6$. $\overline{U}(x_0, R_0) \subseteq \mathbf{D}$ where R_0 is the small positive root of

$$p(t) = \frac{\mathcal{M}}{6}t^2 + \frac{\mathcal{L}}{2}t - t + \eta.$$

However, simple numerical examples can be used to show that even though the condition $(\mathbf{C}_5)$ is not satisfied but still **(TSNTM)** converges to the solution $x^\star$. As an example, let $\mathbf{X} = \mathbf{Y} = \mathbb{R}$, $x_0 = 1$ and $\mathbf{D} = [\zeta, 2-\zeta]$ for $\zeta \in (0,1)$. Define function $\mathcal{F}$ on $\mathbf{D}$ by

$$\mathcal{F}(x) = x^5 - \zeta. \tag{1.6}$$

Then, through some simple calculations, the conditions **(C)** yield

$$\eta = \frac{(1-\zeta)}{5}, \quad \mathcal{L} = 4(2-\zeta)^3, \quad \mathcal{M} = 12(2-\zeta)^2.$$

Figure 1 plots the criterion $(\mathbf{C}_4)$ for the problem (1.6). The curve (defined by the right

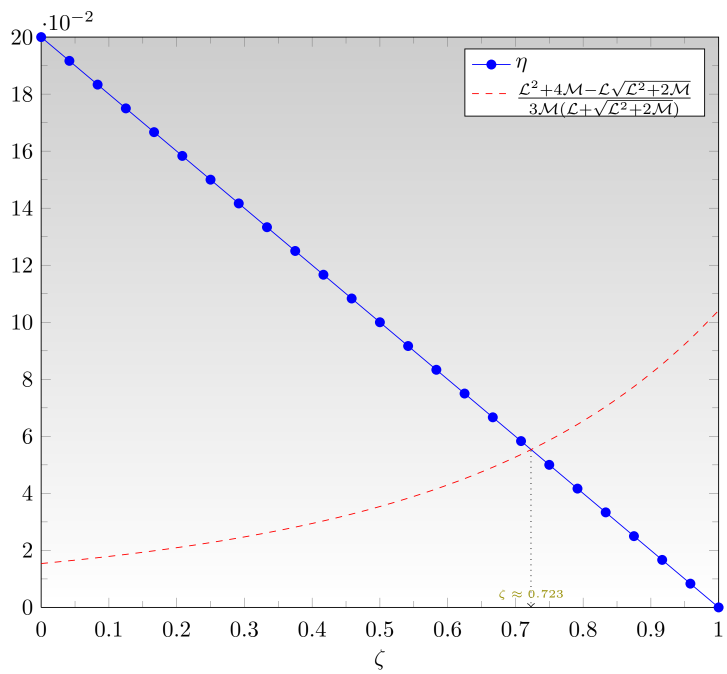

FIGURE 1. Convergence criterion $(\mathbf{C}_5)$ for (1.6).

hand side of the inequality $(\mathbf{C}_4)$) intersect the line η (see Figure 1) at $\zeta \approx 0.72$. We notice in the Figure 1 that for $\zeta < 0.72$ the criterion $(\mathbf{C}_4)$ is not satisfied. However, one may see that the method (1.2) is convergent. For additional examples, see the Section 4.

In this paper, we are concerned with expanding the applicability of **(TSNTM)** where the the condition $(\mathbf{C}_5)$ (or $(\mathbf{C}_6)$) fails. To achieve this, we introduce the center-Lipschitz conditions

$\mathbf{C}_7$. $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x_0))\right\| \leq \mathcal{L}_0 \left\|x - x_0\right\|$ for each $x \in \mathbf{D}$,
$\mathbf{C}_8$. $\left\|\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x)\mathcal{F}'(x_0)\right\| \leq b$ for each $x \in \mathbf{D}$,
$\mathbf{C}_9$. $\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x))\mathcal{F}'(x_0)\right\| \leq c$ for each $x \in \mathbf{D}$.

Here onwards, the conditions $(\mathbf{C}_1)$, $(\mathbf{C}_2)$, $(\mathbf{C}_3)$, $(\mathbf{C}_4)$, $(\mathbf{C}_7)$, $(\mathbf{C}_8)$ and $(\mathbf{C}_9)$ are referred as the **(H)** conditions.

Several techniques are usually considered to study the convergence of iterative methods, as we can see in the studies [1–33]. Among these, the most popular techniques are based on majorizing sequences. In the studies that lead to convergence condition (**C**$_5$), the condition ($\mathbf{C}_3$) was used to compute the upper bound

$$\left\|\mathcal{F}'(x_n)^{-1}\mathcal{F}'(x_0)\right\| \leq \frac{1}{1-\mathcal{L}\left\|x_n - x_0\right\|}. \tag{1.7}$$

Instead of using (**C**$_3$), we use the more precise and less expensive condition (**C**$_4$) which leads to

$$\left\|\mathcal{F}'(x_n)^{-1}\mathcal{F}'(x_0)\right\| \leq \frac{1}{1-\mathcal{L}_0\left\|x_n - x_0\right\|}. \tag{1.8}$$

Note that

$$\mathcal{L}_0 \leq \mathcal{L} \tag{1.9}$$

holds in general and $\mathcal{L}/\mathcal{L}_0$ can be arbitrarily large [23]. This change - in the study of semi-local convergence of method - leads to tighter error estimates on the distances $\|y_n - x_n\|$, $\|x_{n+1} - y_n\|$, $\|x_{n+1} - y_n\|$, $\|y_n - x^\star\|$, $\|x_n - x^\star\|$ and weaker convergence criteria.

The rest of the paper is organized as follows. Section 2 develop results on majorizing sequences for (**TSNTM**) (1.2), where as in the Section 3 we develop the semilocal convergence of the (**TSNTM**). Section 4 presents a Lemma about the special case Two-point Newton method. Finally, numerical examples are given in the concluding Section 5.

2. Majorizing sequences

Here, we find sufficient conditions for the convergence of scalar sequences that will be shown - in the next section - to be majorizing for (**TSNTM**). Let $\mathcal{L}_0 > 0$, $\mathcal{L} > 0$, $b \geq 0$, $c \geq 0$ and $\eta > 0$ be some positive constants. It is convenient for us to define functions γ, α and h_i for $i = 1, 2, 3$ by

$$\gamma(t) = \frac{b\mathcal{L}\,t}{2}, \quad \gamma = \gamma(\eta), \tag{2.1}$$

$$\alpha(t) = \frac{\left[\dfrac{\mathcal{L}\gamma(t)^2}{2} + \mathcal{L}\gamma(t) + \dfrac{c\mathcal{L}}{2}\right]t}{1 - \mathcal{L}_0(1+\gamma(t))t}, \quad \alpha = \alpha(\eta), \tag{2.2}$$

$$h_1(t) = [a(t) + \mathcal{L}_0(1+\gamma(t))]t - 1, \tag{2.3}$$

$$h_2(t) = \frac{b\mathcal{L}}{2}\alpha(t)t + \mathcal{L}_0\gamma(t)(1+\gamma(t))t - \gamma(t) \tag{2.4}$$

and

$$h_3(t) = a(t)t + \mathcal{L}_0(1+\gamma(t))(1+\alpha(t))t - 1 \tag{2.5}$$

where

$$a(t) = \frac{\mathcal{L}}{2}\gamma(t)^2 + \mathcal{L}\gamma(t) + \frac{c\mathcal{L}}{2}, \quad a = a(\eta).$$

Let the minimum positive zeros of the functions h_1, h_2 and h_3 be η_1, η_2 and η_3, respectively. Note that - by the choice of η_1 - $\alpha(t)$ is well defined on $(0, \eta_1)$ and $\alpha \in (0, 1)$. We set

$$\eta_0 = \min\{\eta_1, \eta_2, \eta_3\}. \tag{2.6}$$

Then, for all $t \in (0, \eta_0)$ we have

$$\alpha \in (0, 1) \tag{2.7}$$

$$h_1(t) < 0 \tag{2.8}$$

$$h_2(t) \leq 0 \tag{2.9}$$

and

$$h_3(t) \leq 0. \tag{2.10}$$

We can show the following result about the convergence of majorizing sequences.

Lemma 2.1. *Let the positive constants be* $\mathcal{L}_0 > 0$, $\mathcal{L} > 0$, $b \geq 0$, $c \geq 0$, $\mathcal{M} \geq 0$ *and* $\eta > 0$. *Furthermore suppose that*

$$\eta \begin{cases} \leq \eta_0 & \text{if} \quad \eta_0 \neq \eta_1, \\ < \eta_0 & \text{if} \quad \eta_0 = \eta_1. \end{cases} \tag{2.11}$$

Then, scalar sequence $\{t_n\}$ *generated by*

$$\begin{aligned} t_0 = 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{b\mathcal{L}(s_n - t_n)^2}{2(1 - \mathcal{L}_0 t_n)}, \\ s_{n+1} = t_{n+1} + \frac{\frac{\mathcal{L}}{2}(t_{n+1} - s_n)^2 + \mathcal{L}(s_n - t_n)(t_{n+1} - s_n) + \frac{c\mathcal{L}}{2}(s_n - t_n)^2}{1 - \mathcal{L}_0 t_{n+1}} \end{aligned} \tag{2.12}$$

is increasing, bounded from above by

$$t^{\star\star} = \Big(\frac{1+\gamma}{1-\alpha}\Big)\eta \tag{2.13}$$

and converges to its unique least upper bound $t^\star$ *which satisfies*

$$0 \leq t^\star \leq t^{\star\star}. \tag{2.14}$$

Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$0 \leq t_{n+1} - s_n \leq \gamma(s_n - t_n) \leq \gamma\alpha^n\eta \tag{2.15}$$

and

$$0 < s_{n+1} - t_{n+1} \leq \alpha(s_n - t_n) \leq \alpha^{n+1}\eta. \tag{2.16}$$

Proof. We use mathematical induction to prove (2.15) and (2.16). By (2.1), (2.2) and (2.12), estimates (2.15) and (2.16) hold for $n = 0$ since

$$t_1 - s_0 = \frac{b\mathcal{L}}{2}(s_0 - t_0)(s_0 - t_0) = \gamma(s_0 - t_0) \tag{2.17}$$

and

$$\begin{aligned} s_1 - t_1 &= \frac{\frac{\mathcal{L}}{2}(t_1 - s_0)^2 + \mathcal{L}(s_0 - t_0)(t_1 - s_0) + \frac{c\mathcal{L}}{2}(s_0 - t_0)^2}{1 - \mathcal{L}_0 t_1}, \\ &\leq \frac{\frac{\mathcal{L}}{2}\gamma^2(s_0 - t_0)^2 + \mathcal{L}\gamma(s_0 - t_0)^2 + \frac{c\mathcal{L}}{2}(s_0 - t_0)^2}{1 - \mathcal{L}_0(1+\gamma)\eta}, \\ &\leq \frac{a(s_0 - t_0)}{1 - \mathcal{L}_0(1+\gamma)\eta}(s_0 - t_0) = \alpha(s_0 - t_0). \end{aligned} \tag{2.18}$$

Let us assume that (2.15) and (2.16) hold for all $k \leq n$. Then, we have

$$t_{k+1} - s_k \leq \gamma(s_k - t_k) \leq \gamma\alpha^k\eta,$$
$$s_{k+1} - t_{k+1} \leq \alpha(s_k - t_k) \leq \alpha^{k+1}\eta$$

and

$$\begin{aligned} t_{k+1} &\leq s_k + \gamma\alpha^k\eta \leq t_k + \alpha^k\eta + \gamma\alpha^k\eta \\ &\leq t - k - 1 + \alpha^{k-1}\eta + \alpha^k\eta + \gamma\alpha^{k-1}\eta + \gamma\alpha^k\eta \end{aligned}$$

$$\begin{aligned}
&\leq \cdots \leq t_2 + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\leq s_1 + \gamma\alpha\eta + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\leq t_1 + \alpha\eta + \gamma\alpha\eta + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&\leq \eta + \gamma\eta + \alpha\eta + \gamma\alpha\eta + (\alpha^2\eta + \alpha^3\eta + \cdots + \alpha^k\eta) + (\gamma\alpha^2\eta + \cdots + \gamma\alpha^k\eta) \\
&= \frac{1-\alpha^{k+1}}{1-\alpha}(1+\gamma)\eta < \frac{1+\gamma}{1-\alpha}\eta = t^{\star\star}. \qquad (2.19)
\end{aligned}$$

Evidently, estimates (2.15) and (2.16) are true provided that

$$\frac{b\mathcal{L}(s_k - t_k)}{2(1-\mathcal{L}_0 t_k)} \leq \gamma \qquad (2.20)$$

and

$$\frac{a(s_k - t_k)}{(1-\mathcal{L}_0 t_{k+1})} \leq \alpha. \qquad (2.21)$$

The estimate (2.20) can be written as

$$\frac{b\mathcal{L}}{2}\alpha^k\eta + \gamma\mathcal{L}_0(1+\gamma)\frac{1-\alpha^k}{1-\alpha}\eta - \gamma \leq 0. \qquad (2.22)$$

Inequality (2.22) motivates us to define recurrent functions f_k on $[0, 1)$ for each $k = 1, 2, 3, \ldots$ by

$$f_k(t) = \frac{b\mathcal{L}}{2}t^k\eta + \gamma\mathcal{L}_0(1+\gamma)\frac{1-t^k}{1-t}\eta - \gamma. \qquad (2.23)$$

We need a relationship between two consecutive functions f_k. We have by (2.23) that

$$\begin{aligned}
f_{k+1}(t) &= f_k(t) + \frac{b\mathcal{L}}{2}t^{k+1}\eta - \frac{b\mathcal{L}}{2}t^k\eta + \gamma\mathcal{L}_0(t^k\eta - t^{k-1}\eta + \gamma t^k\eta - \gamma t^{k-1}\eta) \\
&= f_k(t)(t-1)\Big[\frac{b\mathcal{L}}{2}t + +\gamma\mathcal{L}_0(1+\gamma)\Big]t^{k-1}\eta. \qquad (2.24)
\end{aligned}$$

It follows from (2.24) that

$$f_{k+1}(t) \leq f_k(t) \leq \cdots \leq f_1(t). \qquad (2.25)$$

In view of (2.22) and (2.25) it suffices to show that

$$f_1(\alpha) \leq 0 \qquad (2.26)$$

which is true by the choice of η_2, (2.4) and (2.11). Similarly, estimate (2.21) can be written as

$$a\alpha^{k-1}\eta + \mathcal{L}_0(1+\gamma)\frac{1-\alpha^{k+1}}{1-\alpha}\eta - 1 \leq 0. \qquad (2.27)$$

Define recurrent functions g_k on $[0, 1)$ for each $k = 1, 2, \ldots$ by

$$g_k(t) = at^{k-1}\eta + \mathcal{L}_0(1+\gamma)\frac{1-t^{k+1}}{1-t}\eta - 1. \qquad (2.28)$$

Then, using (2.28) we get that

$$g_{k+1}(t) = g_k(t) + (t-1)\Big[a + \mathcal{L}_0(1+\gamma)(1+t)\Big]t^{k-1}\eta. \qquad (2.29)$$

It follows from (2.29) that

$$g_{k+1}(t) \leq g_k(t) \leq \cdots \leq g_1(t). \qquad (2.30)$$

We can show instead of (2.27) that

$$g_1(\alpha) \leq 0, \qquad (2.31)$$

which is true by the choice of η_3, (2.5) and (2.11). The induction for (2.15) and (2.16) is complete. Hence, sequence $\{t_n\}$ is increasing, bounded from above by $t^{\star\star}$ (given by (2.13)) and converges to its unique least upper bound $t^\star$. The proof of the Lemma is complete. □

We have the following useful and obvious extension of Lemma 2.1.

Lemma 2.2. *Suppose there exists $N \geq 0$ such that*

$$t_0 < s_0 < t_1 < \cdots < t_N < s_N < t_{N+1} < \frac{1}{\mathcal{L}_0}. \tag{2.32}$$

and

$$s_N - t_N \begin{cases} \leq \eta_0 & \text{if} \quad \eta_0 \neq \eta_1 \\ < \eta_0 & \text{if} \quad \eta_0 = \eta_1. \end{cases} \tag{2.33}$$

Then, the conclusions of the Lemma 2.1 hold for sequence $\{t_n\}$. Moreover, the following estimates hold for each $n = 0, 1, 2, 3, \ldots$

$$0 < t_{N+1+n} - s_{N+n} \leq \gamma_N (s_{N+n} - t_{N+n}) \tag{2.34}$$

and

$$0 < s_{N+1+n} - t_{N+1+n} \leq \alpha_N (s_{N+n} - t_{N+n}) \tag{2.35}$$

where $\gamma_N = \gamma(s_N - t_N)$, $\alpha_N = \alpha(s_N - t_N)$ and $t_N^{\star\star} = \dfrac{1+\gamma_N}{1-\alpha_N}(s_N - t_N)$.

Remark 2.3.

R1. Note that for $N = 0$, the Lemma 2.2 reduces to Lemma 2.1 with $\alpha_0 = \alpha$ and $\gamma_0 = \gamma$.

3. Semi-local convergence analysis

We need the following Ostrowski-type representation connecting $\mathcal{F}(x_{n+1})$ to the method [1–28].

Lemma 3.1. *Suppose that all iterates of the method* (**TSNTM**) (1.2) *are well defined. Then, the following identity holds for each $n = 0, 1, 2, \ldots$*

$$\mathcal{F}(x_{n+1}) = \int_0^1 \Big[\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)\Big](x_{n+1} - y_n) \mathrm{d}\theta + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n)$$
$$+ (\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x_n)) \int_0^1 [\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)](y_n - x_n) \mathrm{d}\theta. \tag{3.1}$$

Proof. We have – by the definition of the method (**TSNTM**) (1.2) – that

$$\begin{aligned} \mathcal{F}(y_n) &= \mathcal{F}(y_n) - \mathcal{F}(x_n) - \mathcal{F}'(x_n)(y_n - x_n) \\ &= \int_0^1 [\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)](y_n - x_n) \mathrm{d}\theta. \end{aligned} \tag{3.2}$$

Moreover, we get in turn that

$$\begin{aligned} \mathcal{F}(x_{n+1}) &= \mathcal{F}(x_{n+1}) - \mathcal{F}(y_n) - \mathcal{F}'(y_n)(x_{n+1} - y_n) + \mathcal{F}(y_n) + \mathcal{F}'(y_n)(x_{n+1} - y_n) \\ &= \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)](x_{n+1} - y_n) \mathrm{d}\theta \\ &\qquad + \mathcal{F}(y_n) + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n) + \mathcal{F}'(x_n)(x_{n+1} - y_n) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)](x_{n+1} - y_n)\mathrm{d}\theta \\
&\qquad + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n) + \mathcal{F}(y_n) + \mathcal{F}'(x_n)\mathcal{F}'(x_n)^{-1}\mathcal{T}_{\mathcal{F}}(x_n)\mathcal{F}(y_n) \\
&= \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)](x_{n+1} - y_n)\mathrm{d}\theta \\
&\qquad + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n) + (\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x_n))\mathcal{F}(y_n) \\
&= \int_0^1 [\mathcal{F}'(y_n + \theta(x_{n+1} - y_n)) - \mathcal{F}'(y_n)](x_{n+1} - y_n)\mathrm{d}\theta \\
&\qquad + (\mathcal{F}'(y_n) - \mathcal{F}'(x_n))(x_{n+1} - y_n) + (\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x_n)) \\
&\qquad\qquad \int_0^1 [\mathcal{F}'(x_n + \theta(y_n - x_n)) - \mathcal{F}'(x_n)](y_n - x_n)\mathrm{d}\theta.
\end{aligned}
$$

The proof of the Lemma is complete. □

We can show the main semi-local convergence result for the method (1.2) under the (**H**) conditions.

Theorem 3.2. *Suppose that the (**H**) conditions and the conditions of Lemma 2.1 hold. Moreover, suppose that*

$$\overline{U}(x_0, t^\star) \subseteq \mathbf{D}. \tag{3.3}$$

*Then, sequence $\{x_n\}$ generated by the (**TSNTM**) (1.2) is well defined, remain in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $\mathcal{F}(x) = 0$. Moreover, the following estimates hold*

$$\|y_n - x_n\| \leq s_n - t_n, \tag{3.4}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n, \tag{3.5}$$

$$\|x_n - x^\star\| \leq t^\star - t_n \tag{3.6}$$

and

$$\|y_n - x^\star\| \leq t^\star - s_n. \tag{3.7}$$

Furthermore, if there exists $R \geq t^\star$ such that

$$U(x_0, R) \subseteq \mathbf{D} \tag{3.8}$$

and

$$\frac{\mathcal{L}_0}{2}(t^\star + R) = 1 \tag{3.9}$$

then, the solution $x^\star$ is unique in $U(x_0, R)$.

Proof. We shall prove that (3.4) and (3.5) hold using mathematical induction. Using ($\mathbf{C}_2$), (1.2) and (2.12) , we get that

$$\|y_0 - x_0\| = \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| \leq \eta = s_0 - t_0 \leq t^\star.$$

That is (3.4) holds for $n = 0$ and $y_0 \in \overline{U}(x_0, t^\star)$ (by (2.13)). In view of (1.2), (2.12), ($\mathbf{C}_3$) and (3.2), we obtain that

$$
\begin{aligned}
\|x_1 - y_0\| &\leq \left\|\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x_0)\mathcal{F}'(x_0)\right\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_0)\right\| \\
&\leq \frac{b\mathcal{L}}{2}(s_0 - t_0)^2 = t_1 - s_0,
\end{aligned} \tag{3.10}
$$

which shows that (3.5) hold for $n = 0$. We also get that

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 \leq t^\star,$$

which implies that $x_1 \in \overline{U}(x_0, t^\star)$. Let us assume that (3.4), (3.5), $y_k \in \overline{U}(x_0, t^\star)$ and $x_{k+1} \in \overline{U}(x_0, t^\star)$ hold for all $k \leq n$. It follows from the proof of Lemma 2.1 and (**C**$_5$) that

$$\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x_{k+1}) - \mathcal{F}'(x_0))\right\| \leq \mathcal{L}_0 \|x_{k+1} - x_0\| \leq \mathcal{L}_0 t_{k+1} < 1. \tag{3.11}$$

Estimate (3.11) and the Banach Lemma on invertible operators [23] imply that

$$\mathcal{F}'(x_{k+1})^{-1} \in \mathcal{L}(\mathbf{Y}, \mathbf{X}),$$
$$\left\|\mathcal{F}'(x_{k+1})^{-1}\mathcal{F}'(x_0)\right\| \leq \frac{1}{1 - \mathcal{L}_0 \|x_{k+1} - x_0\|} \leq \frac{1}{1 - \mathcal{L}_0 t_{k+1}}. \tag{3.12}$$

Then, we have by (1.2), (**C**$_3$), (2.12) and (3.12) (for k replacing by $k+1$) and the induction hypotheses that

$$\begin{aligned}\|x_{k+1} - y_k\| &\leq \left\|\mathcal{F}'(x_k)^{-1}\mathcal{F}'(x_0)\right\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x_k)\mathcal{F}'(x_0)\right\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(y_k)\right\| \\ &\leq \frac{b\mathcal{L}}{2(1 - \mathcal{L}_0 t_k)}(s_k - t_k)^2 = t_{k+1} - s_k. \end{aligned} \tag{3.13}$$

Using (1.2), (**C**$_3$), (**C**$_4$), (2.12), (3.1), (3.12), (3.13) and the induction hypotheses we obtain in turn that

$$\begin{aligned}\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\right\| &\leq \left\|\int_0^1 \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(y_k + \theta(x_{k+1} - y_k)) - \mathcal{F}'(y_k)]\mathrm{d}\theta\right\| \|x_{k+1} - y_k\| \\ &\quad + \left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(y_k) - \mathcal{F}'(x_k))\right\| \|x_{k+1} - y_k\| + \left\|\mathcal{F}'(x_0)^{-1}(\mathcal{I} - \mathcal{T}_{\mathcal{F}}(x_k))\mathcal{F}'(x_0)\right\| \\ &\quad \left\|\int_0^1 \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x_k + \theta(y_k - x_k)) - \mathcal{F}'(x_k)]\mathrm{d}\theta\right\| \|y_k - x_k\| \\ &\leq \frac{\mathcal{L}}{2}\|x_{k+1} - y_k\|^2 + \mathcal{L}\|y_k - x_k\| \|x_{k+1} - y_k\| + \frac{c\mathcal{L}}{2}\|y_k - x_k\|^2 \\ &\leq \frac{\mathcal{L}}{2}(t_{k+1} - s_k)^2 + \mathcal{L}(s_k - t_k)(t_{k+1} - s_k) + \frac{c\mathcal{L}}{2}(s_k - t_k)^2. \end{aligned} \tag{3.14}$$

Then, by (1.2), (2.12), (3.13) and (3.14), we get that

$$\begin{aligned}\|y_{k+1} - x_{k+1}\| &\leq \|\mathcal{F}'(x_{k+1})\mathcal{F}'(x_0)\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k+1})\right\| \\ &\leq \frac{\frac{\mathcal{L}}{2}(t_{k+1} - s_k)^2 + \mathcal{L}(s_k - t_k)(t_{k+1} - s_k) + \frac{c\mathcal{L}}{2}(s_k - t_k)^2}{1 - \mathcal{L}_0 t_{k+1}} \\ &= s_{k+1} - t_{k+1}. \end{aligned} \tag{3.15}$$

We shall also have that

$$\|y_{k+1} - x_0\| \leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \leq s_{k+1} - t_{k+1} + t_{k+1} - t_0 = s_{k+1} \leq t^\star$$

and

$$\|x_{k+2} - x_0\| \leq \|x_{k+2} - y_{k+1}\| + \|y_{k+1} - x_0\| \leq t_{k+2} - s_{k+1} + s_{k+1} - t_0 = t_{k+2} \leq t^\star$$

Hence, y_{k+1} and x_{k+2} belongs to $\overline{U}(x_0, t^\star)$. It follows from (3.7), (3.8) and Lemma 2.1 that sequence $\{x_n\}$ is complete in a Banach space $\mathbf{X}$ and a such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). By letting $k \longrightarrow \infty$ in (3.14) we obtain $\mathcal{F}(x^\star) = 0$. Estimates (3.9) and (3.10) follows from (3.7) and (3.8) by using standard majorization techniques. Finally to the uniqueness part, $y^\star \in \overline{U}(x_0, R)$

be a solution of equation $\mathcal{F}(x)=0$. Let $Q=\int_0^1 \mathcal{F}'(x^\star+\theta(y^\star-x^\star))\mathrm{d}\theta$. Using (C_5), (3.11) and (3.12), we get that

$$\left\|\mathcal{F}'(x_0)^{-1}(Q-\mathcal{F}'(x_0))\right\| \leq \int_0^1 \left\|\mathcal{F}'(x_0)^{-1}\left[\int_0^1 [\mathcal{F}'(x^\star+\theta(y^\star-x^\star))-\mathcal{F}'(x_0)]\mathrm{d}\theta\right]\right\| \leq \frac{\mathcal{L}_0}{2}(t^\star+R)=1. \tag{3.16}$$

It follows from (3.16) and the Banach lemma on invertible operators that $Q^{-1}\in \mathbf{L}(\mathbf{Y},\mathbf{X})$. Then, using the identity

$$0=\mathcal{F}(y^\star)-\mathcal{F}(x^\star)=Q(y^\star-x^\star)$$

we deduce that $x^\star=y^\star$. The proof of the Theorem is complete. □

Remark 3.3.

R1. The limit point $t^\star$ can be replaced by $t^{\star\star}$ (given in closed from by (2.13)) in Theorem 3.2.

R2. The conclusions of Theorem 3.2 hold if hypotheses of Lemma 2.1 are replaced by those of Lemma 2.2.

R3. It follows from the **(H)** conditions that there exist b_0, c_0, $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ satisfying

$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x_0)\mathcal{F}'(x_0)\right\| \leq b_0, \tag{3.17}$$

$$\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x_1)-\mathcal{F}'(x_0))\right\| \leq \mathcal{L}_1\left\|x_1-x_0\right\|, \tag{3.18}$$

$$\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(y_0+\theta(x_1-y_0))-\mathcal{F}'(y_0)]\mathrm{d}\theta\right\| \leq \mathcal{L}_2\theta\left\|x_1-y_0\right\|, \tag{3.19}$$

$$\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(y_0)-\mathcal{F}'(x_0))\right\| \leq \mathcal{L}_2\left\|y_0-x_0\right\|, \tag{3.20}$$

$$\left\|\mathcal{F}'(x_0)^{-1}(\mathcal{I}-\mathcal{T}_{\mathcal{F}}(x_0))\mathcal{F}'(x_0)\right\| \leq c_0, \tag{3.21}$$

and

$$\left\|\int_0^1 \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x_0+\theta(y_0-x_0))-\mathcal{F}'(x_0)]\mathrm{d}\theta\right\| \leq \mathcal{L}_3\theta\left\|y_0-x_0\right\|, \tag{3.22}$$

where

$$y_0=x_0-\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)$$

and

$$x_1=x_0-\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)-\mathcal{F}'(x_0)^{-1}\mathcal{T}_{\mathcal{F}}(x_0)\mathcal{F}(x_0-\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)).$$

Note that

$$b_0\leq b,\quad c_0\leq c,\quad \mathcal{L}_1\leq\mathcal{L}_0,\quad \mathcal{L}_2\leq\mathcal{L}\quad\text{and}\quad \mathcal{L}_3\leq\mathcal{L} \tag{3.23}$$

and b/b_0, c/c_0, $\mathcal{L}_0/\mathcal{L}_1$, $\mathcal{L}/\mathcal{L}_2$, $\mathcal{L}/\mathcal{L}_3$ can be arbitrarily large [23].

We may notice that estimates (3.17) - (3.21) are not additional to the **(H)** conditions, since in practice the verifications of $(\mathbf{C}_2)$-$(\mathbf{C}_5)$ require the computation of b_0, c_0, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$. Note that finding these constants only involve computations at the initial data. We define

$$\begin{aligned} r_0 &= 0, \quad q_0 = \eta, \quad r_1 = q_0 + \frac{b_0 \mathcal{L}_3 (q_0 - r_0)^2}{2}, \\ q_1 &= r_1 + \frac{\frac{\mathcal{L}_2}{2}(r_1 - q_0)^2 + \mathcal{L}_2 (q_0 - r_0)(r_1 - q_0) + \frac{c_0 \mathcal{L}_3}{2}(q_0 - r_0)^2}{(1 - \mathcal{L}_1 r_1)} \\ r_{n+1} &= q_n + \frac{b \mathcal{L}(q_n - r_n)^2}{2(1 - \mathcal{L}_0 r_n)}, \\ q_{n+1} &= r_{n+1} + \frac{\frac{\mathcal{L}}{2}(r_{n+1} - q_n)^2 + \mathcal{L}(q_n - r_n)(r_{n+1} - q_n) + \frac{c\mathcal{L}}{2}(q_n - r_n)^2}{(1 - \mathcal{L}_0 r_{n+1})} \end{aligned} \tag{3.24}$$

Furthermore, according to the proof of Theorem 3.2, $\{r_n\}$ is a majorizing sequence for $\{x_n\}$ (see also (3.4) - (3.6)) and the tables in the next section. Note that the majorizing sequence $\{v_n\}$ - for the method (1.2) - is given by

$$\begin{aligned} v_0 &= 0, \quad v_{n+1} = u_n + \frac{b\mathcal{L}(u_n - v_n)^2}{2(1 - \mathcal{L} v_n)}, \\ u_{n+1} &= v_{n+1} + \frac{\frac{\mathcal{L}}{2}(v_{n+1} - u_n)^2 + \mathcal{L}(u_n - v_n)(v_{n+1} - u_n) + \frac{c\mathcal{L}}{2}(u_n - v_n)^2}{(1 - \mathcal{L} v_{n+1})}. \end{aligned} \tag{3.25}$$

A simple inductive argument shows that

$$q_n \le s_n \le u_n \tag{3.26}$$

$$r_n \le t_n \le v_n \tag{3.27}$$

$$r_{n+1} - q_n \le t_{n+1} - s_n \le v_{n+1} - u_n \tag{3.28}$$

$$q_{n+1} - r_{n+1} \le s_{n+1} - t_{n+1} \le u_{n+1} - v_{n+1} \tag{3.29}$$

and

$$r^\star = \lim_{n \longrightarrow \infty} r_n \le t^\star \le v^\star = \lim_{n \longrightarrow \infty} v_n. \tag{3.30}$$

Left hand side in the estimates (3.26) - (3.30) hold as strict inequalities if any of the inequalities in (3.23) is strict. Moreover, right hand side in the estimates (3.26) - (3.30) also hold as strict inequalities for $n > 1$ if $\mathcal{L}_0 < \mathcal{L}$. Furthermore, $\{r_n\}$, $\{t_n\}$ can replace $\{v_n\}$ in the convergence results in the literature under the sufficient convergence conditions given there [1–4] (see also ($\mathbf{C}_5$)).

Finally note that the conditions of Lemma 2.1 or Lemma 2.2 can be weaker than those in the literature. In practice we shall use $\{r_n\}$ or $\{t_n\}$ to estimate error bounds on the distances $\|x_{n+1} - y_n\|$, $\|y_n - x_n\|$, $\|x_n - x^\star\|$, $\|y_n - x^\star\|$ and we shall test if conditions of Lemma 2.1 or Lemma 2.2 or those in the literature hold.

4. Special case I : Two-point Newton method

Let $\mathcal{T}_{\mathcal{F}}(x) = \mathcal{I}$. Then, we can choose $b = 1$ and $c = 0$. In this case method (1.2) reduces to the two-point Newton method. In this case, Lemma 2.1 reduces to the following Lemma.

Lemma 4.1. *Let the positive constants be $\mathcal{L}_0 > 0$, $\mathcal{L} > 0$ and $\eta > 0$. Suppose that*

$$\eta \begin{cases} \le \eta_0 & \text{if} \quad \eta_0 \ne \eta_1 \\ < \eta_0 & \text{if} \quad \eta_0 = \eta_1. \end{cases} \tag{4.1}$$

Then, scalar sequence $\{t_n\}$ *generated by*

$$t_0 = 0, \quad s_0 = \eta, \quad t_{n+1} = s_n + \frac{\mathcal{L}(s_n - t_n)^2}{2(1 - \mathcal{L}_0 t_n)}$$
$$s_{n+1} = t_{n+1} + \frac{\frac{\mathcal{L}}{2}(t_{n+1} - s_n)^2 + \mathcal{L}(t_{n+1} - s_n)(s_n - t_n)}{1 - \mathcal{L}_0 t_{n+1}} \tag{4.2}$$

is increasing, bounded from above by

$$t^{\star\star} = \left(\frac{1+\gamma}{1-\alpha}\right)\eta \tag{4.3}$$

and converges to its unique least upper bound $t^\star$ *which satisfies*

$$0 \le t^\star \le t^{\star\star}. \tag{4.4}$$

Moreover, the following estimates hold for each $n = 0, 1, 2, \dots$

$$0 < t_{n+1} - s_n \le \gamma(s_n - t_n) \le \gamma\alpha^n\eta \tag{4.5}$$

and

$$0 < s_{n+1} - t_{n+1} \le \alpha(s_n - t_n) \le \alpha^{n+1}\eta. \tag{4.6}$$

5. Numerical examples

Example 5.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}$ be equipped with the max-norm, $x_0 = \omega$, $\mathbf{D} = [-2, 2]$. Let us define $\mathcal{F}$ on $\mathbf{D}$ by

$$\mathcal{F}(x) = x^3 - 1. \tag{5.1}$$

Here, $w \in \mathbf{D}$. Through some algebraic manipulations, for the conditions (**H**), we obtain

$$\eta = \frac{|\omega^3 - 1|}{3\omega^2}, \quad \mathcal{L} = \frac{4}{\omega^2}, \quad \mathcal{M} = \frac{2}{\omega^2}, \quad \mathcal{L}_0 = \frac{2 + |\omega|}{\omega^2}, \quad b = \frac{179}{144}, \quad c = \frac{35}{144}.$$

For $\omega = 1.21$, the convergence criterion ($\mathbf{C}_5$) yields

$$0.1756621815 \le 0.1731485558.$$

Thus the criterion ($\mathbf{C}_5$) does not hold. Even though the criterion ($\mathbf{C}_5$) is not satisfied. We can see that the method (1.2) converges. For example, let us choose $\mathcal{G}_{\mathcal{F}}(x) = -\mathcal{I}$ and which will result in a fourth order convergent iterative procedure. The performance of this method for (5.1) is reported in the table 2.

Now let us validate the hypotheses of Lemma 2.1 and 2.2. From (2.1) – (2.5), we obtain

$$\eta_1 = 0.2196968398, \quad \eta_2 = 0.1803308682, \quad \eta_3 = 0.1803308682$$

and from the formulation (2.6), we obtain

$$\eta_0 = \eta_2 = 0.1803308682.$$

We notice that the condition (2.11) - of Lemma 2.1 - holds. That is : $0.1756621815 < 0.1803308682$. For the sequence (2.12), we obtain the Table 1. From (2.13), we get

$$t^{\star\star} = 0.4114076922.$$

Comparing the $t^{\star\star}$ with the values in the Table 1, we notice that the inequality (2.14) holds. Furthermore, we notice in the Table 1 the hypothesis of Lemma 2.2 also hold. Since the conditions of Lemma 2.1 - and also that of Lemma 2.2 - holds thus the Theorem 3.2 is applicable. Comparing tables 1 and 2, we see that the estimates (3.4) – (3.7) hold. Comparing Tables 1 and 2, we notice that the estimates of Theorem 3.2 hold.

Example 5.2. In this example, we provide an application of our results to a special nonlinear Hammerstein integral equation of the second kind. Consider the integral equation

$$x(s) = 1 + \frac{4}{5}\int_0^1 G(s,t)x(t)^3\,dt, \quad s \in [0,1], \tag{5.2}$$

where, G is the Green kernel on $[0,1]\times[0,1]$ defined by

$$G(s,t) = \begin{cases} t(1-s), & t \le s; \\ s(1-t), & s \le t. \end{cases} \tag{5.3}$$

Let $\mathbf{X} = \mathbf{Y} = \mathcal{C}[0,1]$ and $\mathbf{D}$ be a suitable open convex subset of $\mathbf{X}_1 := \{x \in \mathbf{X} : x(s) > 0, s \in [0,1]\}$, which will be given below. Define $\mathcal{F} : \mathbf{D} \to \mathbf{Y}$ by

$$[\mathcal{F}(x)](s) = x(s) - 1 - \frac{4}{5}\int_0^1 G(s,t)x(t)^3\,dt, \quad s \in [0,1]. \tag{5.4}$$

The first and second derivatives of $\mathcal{F}$ are given by

$$[\mathcal{F}(x)'y](s) = y(s) - \frac{12}{5}\int_0^1 G(s,t)x(t)^2 y(t)\,dt, \quad s \in [0,1], \tag{5.5}$$

and

$$[\mathcal{F}(x)''yz](s) = \frac{24}{5}\int_0^1 G(s,t)x(t)y(t)z(t)\,dt, \quad s \in [0,1], \tag{5.6}$$

respectively. We use the max-norm. Let $x_0(s) = 1$ for all $s \in [0,1]$. Then, for any $y \in \mathbf{D}$, we have

$$[(I - \mathcal{F}'(x_0))(y)](s) = \frac{12}{5}\int_0^1 G(s,t)y(t)\,dt, \quad s \in [0,1], \tag{5.7}$$

which means

$$\|I - \mathcal{F}'(x_0)\| \le \frac{12}{5}\max_{s\in[0,1]}\int_0^1 G(s,t)\,dt = \frac{12}{5\times 8} = \frac{3}{10} < 1. \tag{5.8}$$

It follows from the Banach theorem that $\mathcal{F}'(x_0)^{-1}$ exists and

$$\|\mathcal{F}'(x_0)^{-1}\| \le \frac{1}{1-\dfrac{3}{10}} = \frac{10}{7}. \tag{5.9}$$

On the other hand, we have from (5.4) that

$$\|\mathcal{F}(x_0)\| = \frac{4}{5}\max_{s\in[0,1]}\int_0^1 G(s,t)\,dt = \frac{1}{10}.$$

Then, we get $\eta = 1/7$. Note that $\mathcal{F}''(x)$ is not bounded in $\mathbf{X}$ or its subset $\mathbf{X}_1$. Take into account that a solution $x^\star$ of equation (1.1) with $\mathcal{F}$ given by (5.3) must satisfy

$$\|x^\star\| - 1 - \frac{1}{10}\|x^\star\|^3 \le 0, \tag{5.10}$$

i.e., $\|x^\star\| \le \rho_1 = 1.153467305$ and $\|x^\star\| \ge \rho_2 = 2.423622140$, where ρ_1 and ρ_2 are the positive roots of the real equation $z - 1 - z^3/10 = 0$. Consequently, if we look for a solution such that $x^\star < \rho_1 \in \mathbf{X}_1$, we can consider $\mathbf{D} := \{x : x \in \mathbf{X}_1 \quad \text{and} \quad \|x\| <$

$r\}$, with $r \in (\rho_1, \rho_2)$, as a nonempty open convex subset of $\mathbf{X}$. For example, choose $r = 1.7$. Using (3.7) and (3.8), we have that for any $x, y, z \in \mathbf{D}$

$$\begin{aligned} \|[(\mathcal{F}'(x) - \mathcal{F}'(x_0))y]\,(s)\| &= \frac{12}{5}\left\|\int_0^1 G(s,t)(x(t)^2 - x_0(t)^2)y(t)\,dt\right\| \\ &\le \frac{12}{5}\int_0^1 G(s,t)\|x(t) - x_0(t)\|\,\|x(t) + x_0(t)\|y(t)\,dt \\ &\le \frac{12}{5}\int_0^1 G(s,t)\,(r+1)\|x(t) - x_0(t)\|y(t)\,dt, \quad s \in [0,1] \end{aligned} \tag{5.11}$$

and

$$\|(F''(x)yz)(s)\| = \frac{24}{5}\int_0^1 G(s,t)x(t)y(t)z(t)\,dt, \quad s \in [0,1]. \tag{5.12}$$

Then, we get

$$\|\mathcal{F}'(x) - \mathcal{F}'(x_0)\| \le \frac{12}{5}\frac{1}{8}(r+1)\|x - x_0\| = \frac{81}{100}\|x - x_0\|, \tag{5.13}$$

$$\|F''(x)\| \le \frac{24}{5} \times \frac{r}{8} = \frac{51}{50} \tag{5.14}$$

and

$$\|[[F''(x) - \mathcal{F}''(\overline{x})]\,yz]\,(s)\| = \frac{24}{5}\left\|\int_0^1 G(s,t)\,(x(t) - \overline{x}(t)))y(t)z(t)\right\|dt \tag{5.15}$$

$$\le \frac{24}{5}\frac{1}{8}\|x - \overline{x}\| = \frac{3}{5}\|x - \overline{x}\|. \tag{5.16}$$

Now we can choose constants as follows:

$$\mathcal{M} = \frac{6}{7}, \quad \mathcal{L} = \frac{51}{35}, \quad \mathcal{L}_0 = \frac{81}{70}, \quad b = \frac{22}{15}, \quad c = \frac{7}{15},$$
$$b_0 = \frac{11}{15}, \quad c_0 = \frac{2}{15}, \quad \mathcal{L}_1 = \frac{11}{70}, \quad \mathcal{L}_2 = \frac{16}{35}, \quad \mathcal{L}_2 = \frac{16}{35}, \quad \text{and} \quad \eta = \frac{1}{7}.$$

We can verify that the condition ($\mathbf{C}_5$) holds. From equations (2.1) – (2.6), we obtain

$$\eta_1 = 0.5292437221, \quad \eta_2 = 0.4285556173, \quad \eta_3 = 0.4285556173.$$

From the formulation (2.7), we get

$$\eta_0 = \eta_2 = 0.4285556173.$$

We may see that the hypothesis (2.11) of Lemma 2.1 holds. Now let us compare the sequences (2.12), (3.24) and (3.25), with (3.7). Comparison – among sequences (2.12), (3.24) and (3.25) – is reported in Table 3. In the Table 3, we observe that the sequence $\{q_n\}$ is finer than the sequence $\{s_n\}$ and $\{s_n\}$ is finer than than $\{u_n\}$ – which is also true by the estimates (3.26) and (3.29).

Concerning the uniqueness balls, let us denote the radii [1, 3, 4, 7, 9, 18–21] by γ_1 and γ_2, respectively. These are given as the smallest positive roots of the polynomials

$$p_1(t) = \mathcal{L}_0\,t - 1 \quad (\text{for} \quad t^\star = \mathbb{R}) \tag{5.17}$$

and

$$p_2(t) = \frac{\mathcal{M}}{6}t^3 + \frac{\mathcal{L}}{2}t^2 - t + \eta \tag{5.18}$$

respectively. Using the values of $\mathcal{L}_0$, $\mathcal{L}$, $\mathcal{M}$ and η we get

$$\gamma_1 = 0.8641975309, \quad \gamma_2 = 0.1517444889. \tag{5.19}$$

Note that $\overline{U}(x_0, r-1) \subseteq \mathbf{D}$, $\mathcal{L}_0 < \mathcal{L}$ and $\gamma_2 < \gamma_1$. Therefore, the new approach provides the largest uniqueness ball and since $r - 1 < \gamma_1$, we deduce that $x^\star$ is unique in $\overline{U}(x_0, r-1) = \overline{U}(1, 0.7) \subseteq \mathbf{D}$.

Example 5.3. We consider nonlinear Hammerstein integral equation

$$x(s) = 1 + \int_0^1 G(s,t)x(t)^2\,dt, \quad s, t \in [0,1] \tag{5.20}$$

where $s \in \mathcal{C}[0,1]$, and the kernel $G(s,t)$ is given as

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ (1-t)s, & s \leq t. \end{cases}$$

Hammerstein integral equations are associated with boundary value problems for differential equations [1]. For these equations higher order methods – utilizing information about the second derivatives – may be advantageous [1].

To solve the nonlinear integral equation (4.1), we divide the interval ($s, t \in [0,1]$) into $n-$points and approximate the integral part through an $n-$point Gauss-Legendre quadrature. Let these $n-$points be ξ_i with $i = 1, 2, \ldots, n$. Thus we obtain

$$x(\xi_j) = 1 + \int_0^1 G(\xi_j, t)x(t)^2\,dt \approx 1 + \sum_{i=1}^{n} \omega_i\, G(\xi_j, \xi_i)x(\xi_i)^2 \tag{5.21}$$

where the nodes ξ_i and weights w_i are given as

$$\xi_i = \frac{1}{2}z_i + \frac{1}{2}, \quad \omega_i = \frac{2}{(1-z_i^2)(\mathcal{P}_n'(z_i))^2}$$

where z_i (also known as $i-$th Gauss-node) are the $i-$th zeros of the normalized Legendre, i.e. $\mathcal{P}_n(1) = 1$, polynomial $\mathcal{P}_n(z)$

$$\mathcal{P}_n(z) = \frac{1}{2^n\, n!}\frac{d^n}{d\,x^n}[(x^2-1)^n].$$

From (5.21), we get the nonlinear-system $\mathcal{F}\colon \mathbb{R}^n \to \mathbb{R}^n$

$$\mathcal{F}(\mathbf{x}) \equiv \mathbf{x} - \mathbf{1} - \mathcal{A}\,\mathbf{v}_x = 0 \tag{5.22}$$

where

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}, \quad \mathbf{1} = [1, 1, \ldots, 1]^{\mathrm{T}}, \quad \mathcal{A} = [a_{i,j}]_{i,j=1}^{n}, \quad \mathbf{v}_x = [x_1^2, x_2^2, \ldots, x_n^2]^{\mathrm{T}}$$

where $a_{i,j} = \omega_i\, G(\xi_j, \xi_i)$. Moreover, $\mathcal{F}'(\mathbf{x}) = \mathbf{I} - 2\mathcal{A}\mathbf{D}(x)$ where $\mathbf{D}(\mathbf{x}) = \operatorname{diag}\{x_1, x_2, \ldots, x_n\}$ and $\mathcal{F}''(\mathbf{x}) = \mathcal{A}$. The discretized system of equations (5.22) satisfies the condition ($\mathbf{C}_5$) and it also satisfies the hypothesis – condition (2.11) – of Lemma 2.1.

To solve the nonlinear integral equation (4.1), we divide the interval through a $20-$point Gauss-Legendre quadrature rule which results in $20-$nonlinear equations with 20 unknowns. Solution is reported in the Table 4 when the residual is $\|x_{n+1} - x_n\|_{\mathrm{L}_2} \leq 1 \times 10^{-50}$. For a second derivative $\mathcal{F}''(\mathbf{x})$ of size $m \times m$ the computational cost of order is $\mathrm{O}\big(m^2\big)$ [1]. As a result, for sufficiently large systems the computational cost during each iteration of the four methods (**NM-O2**, **TSNM-O3**, **TSNM-O4**, **TSNM-O5**) is of the same order [1]. Therefore, the fifth order method **TSNM-O5** is the most computationally efficient for solving such systems.

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n	t_n	s_n	$s_n - t_n$	$t_{n+1} - s_n$	$t^\star - t_n$	$t^\star - s_n$
0	$0.00 \times 10^{+00}$	3.85×10^{-02}	3.85×10^{-02}	1.04×10^{-02}	6.39×10^{-02}	2.54×10^{-02}
1	4.89×10^{-02}	6.14×10^{-02}	1.25×10^{-02}	1.67×10^{-03}	1.49×10^{-02}	2.46×10^{-03}
2	6.31×10^{-02}	6.39×10^{-02}	7.79×10^{-04}	7.25×10^{-06}	7.86×10^{-04}	7.46×10^{-06}
3	6.39×10^{-02}	6.39×10^{-02}	2.05×10^{-07}	5.07×10^{-13}	2.05×10^{-07}	5.07×10^{-13}
4	6.39×10^{-02}	6.39×10^{-02}	3.77×10^{-18}	1.71×10^{-34}	3.77×10^{-18}	1.71×10^{-34}
5	6.39×10^{-02}	6.39×10^{-02}	2.34×10^{-50}	6.56×10^{-99}	2.34×10^{-50}	6.56×10^{-99}
6	6.39×10^{-02}	6.39×10^{-02}	5.55×10^{-147}	3.71×10^{-292}	5.55×10^{-147}	3.71×10^{-292}
7	6.39×10^{-02}	6.39×10^{-02}	7.47×10^{-437}	6.70×10^{-872}	7.47×10^{-437}	6.70×10^{-872}
8	6.39×10^{-02}	6.39×10^{-02}	1.81×10^{-1306}	$0.00 \times 10^{+00}$	1.81×10^{-1306}	$0.00 \times 10^{+00}$
9	6.39×10^{-02}	6.39×10^{-02}	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$

TABLE 1. Majorizing sequence (2.12) for (4.1).

n	$\|x_{n+1} - x_n\|$	$\|x_{n+1} - y_n\|$	$\|x_n - y_n\|$	$\|x_n - x^\star\|$	$\|y_n - x^\star\|$
0	4.00×10^{-02}	1.50×10^{-03}	3.85×10^{-02}	4.00×10^{-02}	1.52×10^{-03}
1	1.61×10^{-05}	2.58×10^{-10}	1.61×10^{-05}	1.61×10^{-05}	2.58×10^{-10}
2	5.35×10^{-19}	2.86×10^{-37}	5.35×10^{-19}	5.35×10^{-19}	2.86×10^{-37}
3	6.53×10^{-73}	4.27×10^{-145}	6.53×10^{-73}	6.53×10^{-73}	4.27×10^{-145}
4	1.46×10^{-288}	2.12×10^{-576}	1.46×10^{-288}	1.46×10^{-288}	2.12×10^{-576}
5	3.59×10^{-1151}	$0.00 \times 10^{+00}$	3.59×10^{-1151}	3.59×10^{-1151}	$0.00 \times 10^{+00}$
6	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$
7	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$
8	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$
9	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$	$0.00 \times 10^{+00}$

TABLE 2. Method (1.2) applied to $\mathcal{F}(x) = x^3 - 1$.

n	q_n	s_n	u_n	$r_{n+1} - q_n$	$t_{n+1} - s_n$	$v_{n+1} - u_n$
0	1.43×10^{-01}	1.43×10^{-01}	1.43×10^{-01}	3.42×10^{-03}	2.18×10^{-02}	2.18×10^{-02}
1	1.47×10^{-01}	1.76×10^{-01}	1.80×10^{-01}	9.69×10^{-07}	1.85×10^{-04}	3.40×10^{-04}
2	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	1.24×10^{-13}	5.28×10^{-09}	2.17×10^{-08}
3	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	2.00×10^{-27}	3.79×10^{-18}	6.91×10^{-17}
4	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	5.23×10^{-55}	1.96×10^{-36}	7.02×10^{-34}
5	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	3.56×10^{-110}	5.20×10^{-73}	7.23×10^{-68}
6	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	1.65×10^{-220}	3.68×10^{-146}	7.68×10^{-136}
7	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	3.57×10^{-441}	1.84×10^{-292}	8.66×10^{-272}
8	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	1.66×10^{-882}	4.62×10^{-585}	1.10×10^{-543}
9	1.47×10^{-01}	1.77×10^{-01}	1.81×10^{-01}	3.60×10^{-1765}	2.90×10^{-1170}	1.78×10^{-1087}

TABLE 3. Comparison among the sequences (2.12), (3.24) and (3.25). Estimates (3.26) – (3.30) hold.

n	$\|x_{n+1} - x_n\|_{L_2}$			
	NM-O2	**TSNM-O3**	**TSNM-O4**	**TSNM-O5**
1	9.869×10^{-2}	1.931×10^{-3}	1.074×10^{-4}	6.652×10^{-5}
2	4.275×10^{-4}	4.233×10^{-6}	2.139×10^{-16}	4.122×10^{-23}
3	3.957×10^{-8}	8.426×10^{-18}	4.275×10^{-63}	1.886×10^{-123}
4	1.931×10^{-16}	3.957×10^{-50}	$---$	$---$
5	2.224×10^{-33}	$---$	$---$	$---$
6	8.001×10^{-65}	$---$	$---$	$---$

TABLE 4. Errors for the Newton (**NM-O2**) and the methods (1.2) applied to (5.20).

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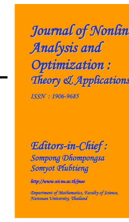

ON THE COMPUTATION OF FIXED POINTS FOR RANDOM OPERATOR EQUATIONS

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ABSTRACT. We approximate fixed points of random operator equation on a complete probability space using Newton's method. Error bounds on the distances involved and some applications are also provided in this study.

KEYWORDS : Newton's method; Complete probability space; Semilocal convergence; Fixed point; Random operator equation; Probabilistic contraction mapping principle.
AMS Subject Classification: 49M15 46S50 47S50

1. INTRODUCTION

Many problems in economics, linear programming and physics lead to random matrix equations [14]. Systems of random equations can also be found in the study of random difference and differential equations [10, 15, 18, 25]. Most methods to approximate solutions are iterative and the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [1–4, 6–9, 11, 13, 16–20, 22–24]. Consider the stochastic initial value problem (see for example [17]):

$$\begin{array}{l} dX(t) = \varphi(t, X(t))\, dB(t) + b(t, X(t))\, dt, \quad 0 \le t \le T \\ X(0) = \zeta, \end{array} \tag{1.1}$$

where, $\{X(t), B(t)\}$ is a family of stochastic processes satisfying some properties (see [17, Definition 2.1]). Eq. (1.1) is also known as Ito–type stochastic differential

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equation. We can solve problem (1.1) using the iterative scheme

$$\begin{aligned} X_0(t) &= \zeta \\ X_{n+1}(t) &= X(0) + \int_0^t \varphi(s, X_n(s))\, dB(s) + \int_0^t b(s, X_n(s))\, ds + \\ & \int_0^t \varphi_x(s, X_n(s))\,(X_{n+1}(s) - X_n(s))\, dB(s) + \\ & \int_0^t b_x(s, X_n(s))\,(X_{n+1}(s) - X_n(s))\, ds. \end{aligned} \tag{1.2}$$

Scheme (1.2) is exactly the Newton method for the stochastic problem (1.1). Note that we can write (1.1) in the following form

$$F(Z)(t) = Z(t) - Z(0) - \int_0^t \varphi(t, X(t))\, dB(t) - \int_0^t b(t, X(t))\, dt. \tag{1.3}$$

In this study we are concerned with the problem of approximating a locally unique solution $x_\star$ of the general random operator equation

$$F(x) = 0. \tag{1.4}$$

We use Newton's method to generate a sequence approximating a locally unique solution of a random operator equation on a complete probability space. A brief survey of some of the general algorithms approximating the solutions of random integral equations is presented in [12].

The paper is organized as follows: Section 2 contains the necessary background results and concepts from probabilistic functional analysis. In Section 3 we provide the semilocal convergence analysis of Newton's method which is faster than the modified Newton's method studied by Bharucha–Reid and Kannan in [13]. We also provide computable upper bounds on the distances involved.

2. PRELIMINARIES

In order for us to make the paper as self contained as possible, we need to introduce some basic concepts and results from probabilistic functional analysis. We refer the reader to [1, 11, 13, 18–20] for more material in this area.

Let $(\Omega, \mathcal{C}, m)$ be a probability measure space and let $(\mathcal{X}, \mathcal{B})$ be a measurable space, where $\mathcal{X}$ is a Banach space and $\mathcal{B}$ is the σ–algebra of all Borel subsets of $\mathcal{X}$. The set Ω is a nonempty abstract set, whose elements ω are called elementary events. $\mathcal{C}$ is a σ–algebra of subsets of Ω. That is, $\mathcal{C}$ is a nonempty class of subsets of Ω satisfying the conditions:

(1) $\Omega \in \mathcal{C}$;
(2) If $A_i \in \mathcal{C}$ $(i = 1, 2)$, then $A_1 - A_2 \in \mathcal{C}$;
(3) If $A_i \in \mathcal{C}$ $(i \geq 1)$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.

The elements of $\mathcal{C}$ are called events. We denote by m a probability measure on $\mathcal{C}$. That is, m is a set function, with domain $\mathcal{C}$, which is nonnegative, countably additive and such that $m(A) \in [0, 1]$ for all $A \in \mathcal{C}$, with $m(\Omega) = 1$. In this study, we assume that m is a complete probability measure. That is m is such that, if $A \in \mathcal{C}$, $m(A) = 0$ and $A_1 \subseteq A$ then $A_1 \in \mathcal{C}$.

A mapping $Q : \Omega \longrightarrow \mathcal{X}$ is said to be a random variable with values in $\mathcal{X}$, if the inverse image under the mapping Q of every Borel set B_0 belongs to $\mathcal{C}$. Let $Q_1(\omega)$

and $Q_2(\omega)$ be $\mathcal{X}$–valued random variables defined on the same probability space. $Q_1(\omega)$ and $Q_2(\omega)$ are said to be equivalent if for every $B_0 \in \mathcal{B}$, we have

$$m(\{\omega : Q_1(\omega) \in B_0\}\Delta\{\omega : Q_2(\omega) \in B_0\}) = 0.$$

If $\mathcal{X}$ is separable, then $m(\{\omega : Q_1(\omega) \neq Q_2(\omega)\}) = 0$.

$Q(\omega)$ is said to be a bounded random operator if there exists a nonnegative real–valued random variable $K(\omega)$ such that for all $x, y \in \mathcal{X}$,

$$\| Q(\omega)x - Q(\omega)y \| \leq K(\omega) \ \| x - y \|, \quad \text{almost surely (a.s.)}$$

A sequence of bounded linear random operators $L_n(\omega)$ is said to be strongly convergent to a bounded linear operator $L_0(\omega)$, if for any $x \in \mathcal{X}$

$$m(\{\omega : \lim_{n\to\infty} \| L_n(\omega)x - L_0(\omega)x \| = 0\}) = 1.$$

An operator equation

$$Q(\omega)x = y(\omega), \tag{2.1}$$

where, $y(\omega)$ is a given $\mathcal{X}$–valued random variable and $Q(\omega)$ is a given random operator on $\mathcal{X}$ is said to be a random operator equation; and for any $\mathcal{X}$–valued random variable $x_\star(\omega)$ satisfying

$$m(\{\omega : Q(\omega)x_\star(\omega) = y(\omega)\}) = 1 \tag{2.2}$$

is said to be a random solution of equation (2.1).

If

$$m(\{\omega : Q(\omega)x_\star(\omega) = x_\star(\omega)\}) = 1 \tag{2.3}$$

then $x_\star(\omega)$ is a random fixed point of random operator equation

$$Q(\omega)x(\omega) = x(\omega). \tag{2.4}$$

We also need the following results on random contraction mappings, fixed points and inverses of random operators. Let $Q(\omega) : \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ and let $\ell(\omega)$ be a nonnegative real–valued random variable, such that $m(\{\omega : \ell(\omega) < 1\}) = 1$. A random operator $Q(\omega)$ on $\mathcal{X}$ is said to be a random contraction operator if

$$m(\{\omega : \| Q(\omega)x - Q(\omega)y \| \leq \ell(\omega) \ \| x - y \|\}) = 1 \quad \textit{for all} \quad x, y \in \mathcal{X}. \tag{2.5}$$

We have the following two extentions of the Banach contraction mapping principle [4, 16] for random fixed point theorems due to Hanš [15] (see also [10, 11, 18]).

Theorem 2.1. *Let $Q(\omega) : \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ be a continuous random operator, where $\mathcal{X}$ is a separable Banach space.*

Let $\omega \in \Omega$, $x \in \mathcal{X}$ and define sequence $\{Q^n(\omega)\}$ by

$$\begin{aligned} &Q^1(\omega)x = Q(\omega)x \\ &Q^{n+1}(\omega)x = Q(\omega)(Q^n(\omega)x), \quad n \geq 1. \end{aligned} \tag{2.6}$$

If $Q(\omega)$ satisfies the condition

$$m\left(\bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{x\in\mathcal{X}}\bigcap_{y\in\mathcal{X}}\left\{\omega : \| Q^n(\omega)x - Q^n(\omega)y \| \leq \left(1 - \frac{1}{i}\right) \ \| x - y \| \right\}\right) = 1, \tag{2.7}$$

then, there exists an $\mathcal{X}$-valued random variable $x_\star(\omega)$, which satisfies (2.3). Moreover, sequence $\{x_n(\omega)\}$ given by

$$x_n(\omega) = Q(\omega)x_{n-1}(\omega), \quad (n \geq 1) \tag{2.8}$$

converges to $x_\star(\omega)$ a.e.

Furthemore, if $y_\star(\omega)$ is another $\mathcal{X}$-valued random variable, which satisfies (2.3), then $x_\star(\omega)$ and $y_\star(\omega)$ are equivalent.

We also have the following consequence of Theorem 2.1.

Proposition 2.2. *Let $Q(\omega)\,:\,\Omega\times\mathcal{X}\longrightarrow\mathcal{X}$ be a continuous random contraction operator, where $\mathcal{X}$ is a separable Banach space. Then, there exists an $\mathcal{X}$-valued random variable $x_\star(\omega)$, which is the unique fixed point of $Q(\omega)$. Moreover, sequence $\{x_n(\omega)\}$ generated by (2.7) converges to $x_\star$ a.s.*

Proof. Set

$$E=\{\omega\,:\,\ell(\omega)<1\}, \tag{2.9}$$

$$P=\{\omega\,:\,Q(\omega)x \text{ is continuous in } x\} \tag{2.10}$$

and

$$G(x,y)=\{\omega\,:\parallel Q(\omega)x-Q(\omega)y\parallel\le\ell(\omega)\ \parallel x-y\parallel\}. \tag{2.11}$$

The intersections in

$$\bigcap_{x\in\mathcal{X}}\bigcap_{y\in\mathcal{X}}\{G(x,y)\cap E\cup P\}$$

can be replaced by intersections over a countable dense set in $\mathcal{X}$, since $\mathcal{X}$ is separable. It follows that condition (2.7) holds with $n=1$. That completes the proof of Proposition 2.2. □

Let $\mathcal{L}(\mathcal{X})$ be the algebra of bounded linear operator on $\mathcal{X}$. Let $Q(\omega)$ be a random operator with values in $\mathcal{L}(\mathcal{X})$. Then, $Q^{-1}(\omega)$ is the random operator $\mathcal{L}(\mathcal{X})$ mapping $Q(\omega)x$ into x a.s.

$Q(\omega)$ is said to be invertible if $Q^{-1}(\omega)$ exists. If $Q(\omega)$ is an invertible random operator with values in $\mathcal{L}(\mathcal{X})$, then $Q^{-1}(\omega)$ is a random operator with values in $\mathcal{L}(\mathcal{X})$.

3. SEMILOCAL CONVERGENCE OF NEWTON'S METHOD

We need the notion of the Fréchet–derivative of a random operator.

Definition 3.1. *Let $Q(\omega)\,:\,\Omega\times\mathcal{X}\longrightarrow\mathcal{X}$ be a continuous random operator, where $\mathcal{X}$ is a separable Banach space. Assume*

$$\lim_{h\to0}\frac{Q(\omega)(x_0+h)-Q(\omega)x_0}{h} \tag{3.1}$$

exists. That is we assume that for every $\omega\in\Omega$, the operator $Q(\omega)\,:\,\mathcal{X}\longrightarrow\mathcal{X}$ is differentiable. The $\mathcal{X}$-valued element given by (3.1) and denoted by $Q'(\omega)x_0\,:\,\Omega\times\mathcal{X}\longrightarrow\mathcal{X}$ is the Fréchet-derivative at x_0 of $Q(\omega)$. That is we define:

$$Q'(\omega)x_0=\lim_{h\to0}\frac{Q(\omega)(x_0+h)-Q(\omega)x_0}{h}. \tag{3.2}$$

The randomness of $Q(\omega)$ implies that $Q'(\omega)x_0$ is random linear operator [13].

Note that the definition of $Q'(\omega)$ is not the same with the deterministic notion of the Fréchet–derivative, where, $Q'x_0\,:\,\mathcal{X}\longrightarrow\mathcal{L}(\mathcal{X})$ [2, 9]. Here, $Q'(\omega)\,:\,\mathcal{X}\longrightarrow\mathcal{X}$. As in [13, Example 3.2, p. 233], let us consider random integral operator on $\mathcal{X}=\mathcal{C}[a,b]$, equipped with the sup–norm:

$$Q(\omega)x=\int_a^b H'_v(t,\theta,x(\theta),\omega)\,d\theta, \tag{3.3}$$

where, $H(t,\theta,v,\omega)$ is measurable; $H(t,\theta,v,\omega)$ and $H'_v(t,\theta,u,\omega)$ are jointly continuous for $a\le t$, $\theta\le b$, $|v|\le R$, $R\ge0$.

Then $Q(\omega) : U(0,R) = \{x : \| x \| \leq R\} \longrightarrow \mathcal{X}$ is Fréchet–differentiable for all $\omega \in \Omega$. It follows from (3.3) that

$$Q'(\omega)x_0[q] = \int_a^b H'_v(t,\theta,x_0(\theta),\omega)\, q(\theta)\, d\theta. \tag{3.4}$$

Let $\mathcal{D}$ be an open subset of $\mathcal{X}$ and $Q(\omega) : \Omega \times \mathcal{D} \longrightarrow \mathcal{X}$ be a random nonlinear operator. Let $Q(\omega)$ be continuous Fréchet–differentiable a.s. Let $x(\omega) : \Omega \longrightarrow \mathcal{D}$ be a $\mathcal{X}$–fixed random variable, such that $(\mathcal{I} - Q'(\omega)\, x(\omega))^{-1} : \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ is defined and bounded. Clearly, $(\mathcal{I} - Q'(\omega)\, x(\omega))^{-1}$ is a random bounded linear operator, since $Q'(\omega)x$ is random [15, 21].

In order for us to simplify the notation, we denote

$$F = F(\omega) = \mathcal{I} - Q(\omega), \quad x = x(\omega)$$

$$F'(x) = F'(\omega)x(\omega) = \mathcal{I} - Q'(\omega)x(\omega)$$

and

$$F'(x)^{-1} = (F'(\omega)x(\omega))^{-1} = (\mathcal{I} - Q'(\omega)x(\omega))^{-1}.$$

With this notation, we shall use Newton's method (NM)

$$x_{n+1} = x_n - F'(x_n)^{-1}\, F(x_n), \quad (n \geq 0), \quad (x_0 \in \mathcal{D}) \tag{3.5}$$

to generate a sequence $\{x_n\}$ converging to a zero $x_\star = x_\star(\omega)$ of F.

Note also that $x_\star$ is a fixed point of $Q(\omega)$ satisfying (2.4). A semilocal convergence result was given in [13, p. 234] for the modified Newton method (MNM)

$$x_{n+1} = x_n - F'(x_0)^{-1}\, F(x_n), \quad (n \geq 0), \quad (x_0 \in \mathcal{D}). \tag{3.6}$$

Here, we are motivated by optimization considerations and the work of Bharucha–Reid, Kannan [13] on (MNM). We provide a semilocal convergence result for (NM), which a faster than (MNM).

Theorem 3.2. *Let $Q(\omega) : \Omega \times \mathcal{D} \longrightarrow \mathcal{X}$ be a continuous Fréchet-differentiable a.s. random nonlinear operator. Let $x = x(\omega) : \Omega \longrightarrow \mathcal{D}$ be a $\mathcal{X}$-valued random variable, such that $F'(x)^{-1} = (\mathcal{I} - Q'(\omega)x(\omega))^{-1} : \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ is defined and bounded. Let $x_0 = x_0(\omega) \in \mathcal{D}$ be a fixed $\mathcal{X}$-valued random variable. Then, there exists $U(x_0, r)$ for x_0 and $r > 0$, such that if*

$$\| F'(x)^{-1}\, F'(x_0) \| \leq a(\omega) = a \tag{3.7}$$

for all $x \in \mathcal{D}$ and for some real-valued random variable $a(\omega)$;

$$\| F'(x_0)^{-1}\, F(x_0) \| \leq r\,(1 - \ell); \tag{3.8}$$

$$a \,\| F'(x_0)^{-1}\, (F'(x) - F'(x_0)) \| \leq \ell(\omega) \quad for\ all\ x \in \mathcal{D}; \tag{3.9}$$

and

$$\overline{U}(x_0, r) \subseteq \mathcal{D}. \tag{3.10}$$

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (NM) (3.5) is well defined, remains in $\overline{U}(x_0, r)$ for all $n \geq 0$ and converges to a unique solution $x_\star = x_\star(\omega)$ of equation $F(x) = 0$ in $\overline{U}(x_0, r)$.

Moreover, the following error estimates hold for all $n \geq 1$:

$$\| x_{n+1} - x_n \| \leq \ell^n \;\| x_n - x_{n-1} \| \tag{3.11}$$

and

$$\| x_n - x_\star \| \leq \frac{\ell^n}{1 - \ell} \;\| x_1 - x_0 \| . \tag{3.12}$$

Proof. Using (3.5) for $n = 0$ and (3.8), we get x_1 is well defined and $x_1 \in \overline{U}(x_0, r)$. Let us assume $x_k \in \overline{U}(x_0, r)$ for all $k \leq n$. Then x_{k+1} is well defined by (3.5). We shall show $x_{k+1} \in \overline{U}(x_0, r)$. In view of (3.5), we obtain in turn the approximation

$$\begin{aligned}
&\| x_{k+1} - x_k \| = \| F'(x_k)^{-1} F(x_k) \| \\
&= \| F'(x_k)^{-1} (F(x_k) - F(x_{k-1}) + F(x_{k-1})) \| \\
&= \| F'(x_k)^{-1} (F(x_k) - F(x_{k-1}) - F'(x_k)(x_k - x_{k-1})) \| \\
&= \| F'(x_k)^{-1} (F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1})) + \\
&(F'(x_0) - F'(x_k))(x_k - x_{k-1}) \| \\
&= \Bigg\| (F'(x_k)^{-1} F'(x_0)) \Bigg(F'(x_0)^{-1} (F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1})) + \\
&F'(x_0)^{-1} (F'(x_0) - F'(x_k))(x_k - x_{k-1}) \Bigg) \Bigg\| .
\end{aligned} \tag{3.13}$$

Now, since $Q(\omega)$, $F = \mathcal{I} - Q(\omega)$ are continuously Fréchet–differentiable a.s., we denote by G the set of all $\omega \in \Omega$, such that if $x = x(\omega)$, $y = y(\omega)$ belong in $U(x_0, r)$, then by (3.2)

$$a \; \| F'(x_0)^{-1} (F(y) - F(x) - F'(x_0)(y - x)) \| \leq \epsilon \; \| y - x \|, \tag{3.14}$$

where, $\epsilon > 0$ is such that

$$a \, (\| F'(x_0)^{-1} (F(x) - F(x_0)) \| + \epsilon) < \ell. \tag{3.15}$$

In particular for $y = x_k$ and $x = x_{k-1}$, we have by (3.14) and (3.15)

$$a \; \| F'(x_0)^{-1} (F(x_k) - F(x_{k-1}) - F'(x_0)(x_k - x_{k-1})) \| \leq \epsilon \; \| x_k - x_{k-1} \|, \tag{3.16}$$

and

$$a \, (\| F'(x_0)^{-1} (F(x_k) - F(x_0)) \| + \epsilon) < \ell. \tag{3.17}$$

Then , using (3.14), (3.16) and (3.17), we get

$$\| x_{k+1} - x_k \| \leq \ell \; \| x_k - x_{k-1} \|, \tag{3.18}$$

so,

$$\| x_{k+1} - x_k \| \leq \| x_k - x_{k-1} \| \leq \cdots \leq \ell^k \; \| x_1 - x_0 \| \tag{3.19}$$

and by (3.8)

$$\begin{aligned}
\| x_{k+1} - x_0 \| &\leq \sum_{i=1}^{i=k+1} \| x_i - x_{i-1} \| \\
&\leq \sum_{i=0}^{i=k} \ell^k \; \| x_1 - x_0 \| \\
&= \frac{1 - \ell^{k+1}}{1 - \ell} \; \| x_1 - x_0 \| \leq \frac{\| x_1 - x_0 \|}{1 - \ell} \leq r,
\end{aligned} \tag{3.20}$$

which implies $x_{k+1} \in \overline{U}(x_0, r)$. It follows from (3.18) that $\{x_n\}$ is Cauchy in a complete space $\mathcal{X}$ and as such it converges to some $x_\star \in \overline{U}(x_0, r)$.

Moreover, define

$$E = \{\omega \, : \, \ell(\omega) < 1\}, \tag{3.21}$$

and

$$P = \{\omega \, : \, Q(\omega)x \text{ is continuous in } x\}. \tag{3.22}$$

Then, by Proposition 2.2, there exists an $\mathcal{X}$–valued random variable $x_\star$, which is the unique solution of equation $F(x) = 0$ in $\overline{U}(x_0, r)$. Then, sequence $\{x_n\}$ converges to $x_\star$ a.s.

Furthemore, we have for all $i \geq 0$:

$$\| x_{k+i} - x_k \| \leq \sum_{j=k+i-1}^{j=k} \| x_{j+1} - x_j \| \leq \frac{1-\ell^i}{1-\ell}\, \ell^k \, \| x_1 - x_0 \| . \tag{3.23}$$

By letting $i \longrightarrow \infty$, we get

$$\| x_\star - x_k \| \leq \frac{\ell^k}{1-\ell} \, \| x_1 - x_0 \| \leq r. \tag{3.24}$$

That completes the proof of Theorem 3.2. □

In the case of the (MNM) (3.6), we can let $a(\omega) = 1$ in Theorem 3.2, so that conditions (3.7) and (3.9) are satisfied. Hence, we arrive at the following Corollary of Theorem 3.2 for the semilocal convergence of (MNM). This result was also essentially (without all the details in the proof) given in [13, p. 234].

Corollary 3.3. *Let $Q(\omega) : \Omega \times \mathcal{D} \longrightarrow \mathcal{X}$ be a continuous Fréchet-differentiable a.s. random nonlinear operator. Let $x_0 = x_0(\omega) : \Omega \longrightarrow \mathcal{D}$ be a $\mathcal{X}$-valued random variable, such that $F'(x_0)^{-1} = (\mathcal{I} - Q'(\omega) x_0(\omega))^{-1} : \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ is defined and bounded. Let $\ell = \ell(\omega) : \Omega \longrightarrow (0,1)$ be any real valued random variable. Then, there exists $U(x_0, r)$, such that if*

$$\| F'(x_0)^{-1} F(x_0) \| \leq r\,(1-\ell) \quad \text{and} \quad \overline{U}(x_0, r) \subseteq \mathcal{D}.$$

Then, The conclusions of Theorem 3.2 hold for sequence $\{x_n\}$ $(n \geq 0)$ generated by (MNM) (3.6).

As in the deterministic case [4], [6]–[9], [16, 22], the error estimates (3.11) and (3.12) can be improved so that the usual quadratic convergence of (NM) can be attained. However additional hypothesis of "Lipschitz–type" are needed. We provide such a case here, but first we need the following result on majorizing sequence for (NM).

Lemma 3.4. *[5, 6, 8] Assume that there exist constants $L_0 \geq 0$, $L \geq 0$, with $L_0 \leq L$ and $\eta \geq 0$, such that:*

$$q_{AH} = \overline{L}\,\eta \leq \frac{1}{2}, \tag{3.25}$$

where,

$$\overline{L} = \frac{1}{8}\left(L + 4\,L_0 + \sqrt{L^2 + 8\,L_0\,L} \right). \tag{3.26}$$

The inequality in (3.25) is strict, if $L_0 = 0$.

Then, sequence $\{t_k\}$ ($k \geq 0$) given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_{k+1} = t_k + \frac{L\,(t_k - t_{k-1})^2}{2\,(1 - L_0\,t_k)} \qquad (k \geq 1), \tag{3.27}$$

is well defined, nondecreasing, bounded from above by $t^{\star\star}$, and converges to its unique least upper bound $t^\star \in [0, t^{\star\star}]$, where

$$t^{\star\star} = \frac{2\,\eta}{2 - \delta}, \tag{3.28}$$

$$1 \leq \delta = \frac{4\,L}{L + \sqrt{L^2 + 8\,L_0\,L}} < 2 \quad \text{for } L_0 \neq 0. \tag{3.29}$$

Moreover, the following estimates hold:

$$L_0\,t^\star \leq 1, \tag{3.30}$$

$$0 \leq t_{k+1} - t_k \leq \frac{\delta}{2}\,(t_k - t_{k-1}) \leq \cdots \leq \left(\frac{\delta}{2}\right)^k \eta, \quad (k \geq 1), \tag{3.31}$$

$$t_{k+1} - t_k \leq \left(\frac{\delta}{2}\right)^k (2\,q_{AH})^{2^k-1}\,\eta, \quad (k \geq 0), \tag{3.32}$$

$$0 \leq t^\star - t_k \leq \left(\frac{\delta}{2}\right)^k \frac{(2\,q_{AH})^{2^k-1}\,\eta}{1-(2\,q_{AH})^{2^k}}, \quad (2\,q_{AH} < 1), \quad (k \geq 0). \tag{3.33}$$

Then, as in Theorem 3.2, we can show the following semilocal result for the quadratic convergence of (NM).

Theorem 3.5. *Let $Q(\omega) \,:\, \Omega \times \mathcal{D} \longrightarrow \mathcal{X}$ be a continuous Fréchet-differentiable a.s. random nonlinear operator. Let $x = x(\omega) \,:\, \Omega \longrightarrow \mathcal{D}$ be a $\mathcal{X}$-valued random variable, such that $F'(x)^{-1} = (F'(\omega)x(\omega))^{-1} = (\mathcal{I} - Q'(\omega)x(\omega))^{-1} \,:\, \Omega \times \mathcal{X} \longrightarrow \mathcal{X}$ is defined and bounded. Let $x_0 = x_0(\omega) \,:\, \Omega \longrightarrow \mathcal{D}$ be a fixed $\mathcal{X}$-valued random variable.*

Assume:

$$\| \, F'(x_0)^{-1}\,F(x_0) \, \| \leq \eta(\omega) = \eta; \tag{3.34}$$

for any $x, y \in \mathcal{D}$, $\overline{L_0} = \overline{L_0}(\omega) = L_0(\omega) \,\, \| \, x - x_0 \, \| = L \,\, \| \, x - x_0 \, \| : \, \Omega \longrightarrow (0,1)$ and $\overline{L} = \overline{L}(\omega) = \dfrac{L(\omega)}{2} \,\, \| \, y - x \, \| = \dfrac{L}{2} \,\, \| \, y - x \, \| : \, \Omega \longrightarrow (0,1)$ are real-valued random variables;

$$\| \, F'(x_0)^{-1}\,(F'(x) - F'(x_0)) \, \| \leq L_0(\omega) \,\, \| \, x - x_0 \, \|; \tag{3.35}$$

$$\| \, F'(x)^{-1}\,F'(x_0) \, \| \leq \frac{1}{1 - L_0(\omega) \,\, \| \, x - x_0 \, \|}; \tag{3.36}$$

$$\| \, F'(x_0)^{-1}\,(F(y) - F(x) - F'(x)\,(y-x)) \, \| \leq \frac{L(\omega)}{2} \,\, \| \, y - x \, \|^2; \tag{3.37}$$

hypotheses of Lemma 3.4 hold with $\eta(\omega) = \eta$, $L_0(\omega) = L_0$ and $L(\omega) = L$;

and

$$\overline{U}(x_0, t^\star) \subseteq \mathcal{D}, \tag{3.38}$$

where $t^\star$ is given in Lemma 3.4.

Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by (NM) (3.5) is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$ and converges to a unique solution $x_\star = x_\star(\omega)$ of equation $F(x) = 0$ in $\overline{U}(x_0, t^\star)$.

Moreover, the following error estimates hold for all $n \geq 1$:

$$\| \, x_{n+1} - x_n \, \| \leq t_{n+1} - t_n \tag{3.39}$$

and

$$\| \, x_n - x_\star \, \| \leq t^\star - t_n. \tag{3.40}$$

Remark 3.6. *Note that in view of Lemma 3.4, the convergence order of sequence $\{x_n\}$ is quadratic for $q_{AH} < 1/2$ and linear if $q_{AH} = 1/2$.*

Proof. (of Theorem 3.5) We follow the proof of Theorem 3.2. But this time using (3.5), (3.27), (3.34)–(3.37) and the approximation

$$F'(x_0)^{-1}\,F(x_k) = F'(x_0)^{-1}\,(F(x_k) - F(x_{k-1}) - F'(x_{k-1})\,(x_k - x_{k-1})),$$

we get that

$$\begin{aligned}
&\| \, x_{k+1} - x_k \, \| \\
&\leq \| \, F'(x_k)^{-1}\,F'(x_0) \, \| \| \, F'(x_0)^{-1}\,(F(x_k) - F(x_{k-1}) - F'(x_{k-1})\,(x_k - x_{k-1})) \, \| \\
&\leq \frac{1}{1 - L_0 \,\, \| \, x_k - x_0 \, \|} \, \frac{L}{2} \,\, \| \, x_k - x_{k-1} \, \|^2 \leq \frac{L\,(t_k - t_{k-1})^2}{1 - L_0\,(1 - L_0\,t_k)^2} = t_{k+1} - t_k.
\end{aligned} \tag{3.41}$$

Hence, $\{t_k\}$ ($k \geq 0$) is a majorizing sequence for $\{x_k\}$. The rest follows as in Theorem 3.2 and the deterministic case [4], [6]–[9]. That completes the proof of Theorem 3.5. □

As an example, the results obtained here find applications in the solution of equations of the form

$$Q\,x = z(t,\omega),$$
$$(Q - \lambda\,\mathcal{I})\,x = z(t,\omega),$$
$$Q(\omega)\,x = z(t),$$
$$(Q(\omega) - \lambda\,\mathcal{I})\,x = z(t),$$
$$Q(\omega)\,x = z(t,\omega)$$

and

$$(Q(\omega) - \lambda\,\mathcal{I})\,x = z(t,\omega),$$

where, the imput is a random function. Then, "Lipschitz–type" estimates (3.35)–(3.37) can be realized using inversion theorems, which can be found , e.g. in [9], [13].

Applications and examples, including the solution of nonlinear Chandrasekhar–type integral equations appearing in radiative transfer are also find in [4], [6]–[9].

CONCLUSION

We provided new convergence results for (NM) and (MNM) for solving random operator equations. The sufficient convergence conditions are obtained using Lipschitz and center–Lipschitz conditions instead of the only Lipschitz condition used in [13]. Our results extend the applicability of this method studied in [13]. Some remarks and applications in the deterministic case are also provided in this study.

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EXISTENCE THEOREMS FOR GENERALIZED NASH EQUILIBRIUM PROBLEMS: AN ANALYSIS OF ASSUMPTIONS

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ABSTRACT. The generalized Nash equilibrium, where the feasible sets of the players depend on other players' action, becomes increasingly popular among academics and practitionners. In this paper, we provide a thorough study of theorems guaranteeing existence of generalized Nash equilibria and analyze the assumptions on practical parametric feasible sets.

KEYWORDS: Noncooperative games; Existence theorem; Nash equilibrium.
AMS Subject Classification: 91A10.

1. INTRODUCTION

In noncooperative game theory, solution concepts had been searched for years at the beginning of the XX$^{\text{th}}$ century, cf. [33]. Thankfully, the Nobel prize laureate, John F. Nash, proposed a unified solution concept for noncooperative games, latter called Nash equilibrium, with [26, 27]. Despite some critiscism, this solution concept is widely used among academics to model noncooperative behavior. Classical applications of Nash equilibrium include computer science, telecommunication, energy markets, and many others, see [14] for a recent survey. In this note, we focus on noncooperative games with infinite action space and one-period horizon. Let be N the number of players. The strategy set of player i is denoted by $X_i \subset \mathbb{R}^{n_i}$ and the payoff function by $\theta_i : X \to \mathbb{R}$ (to be maximized), where $X = X_1 \times \cdots \times X_N$. Player i's (pure) strategy is denoted by $x_i \in X_i$ while $x_{-i} \in X_{-i}$ denotes the other players' action, i.e. $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ and $X_{-i} = X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$. A game is thus described by $(N, X_i, \theta_i(.))$.

Definition 1.1. A Nash equilibrium is a strategy point $x^\star \in X$ such that no player has an incentive to deviate, i.e. for all $i \in \{1, \ldots, N\}$,

$$\forall x_i \in X_i,\ \theta_i(x_i, x^\star_{-i}) \leq \theta_i(x^\star_i, x^\star_{-i}). \tag{1.1}$$

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Originally, Nash introduced the equilibrium concept to finite games in [26, 27], i.e. X_i is a finite set. Therefore, he used the mixed strategy concept (i.e. a probability distribution over the pure strategies) and proved the existence of such an equilibrium in that context. We report here the existence theorem of [28] for infinite games.

Theorem 1.2 (Nash). *Let N agents be characterized by an action space X_i and an objective function θ_i. If $\forall i \in \{1, \dots, N\}$, X_i is nonempty, convex and compact; $\theta_i : X \to \mathbb{R}$ is continuous with $X = X_1 \times \cdots \times X_N$ and $\forall x_{-i} \in X_{-i}, x_i \to \theta_i(x_i, x_{-i})$ is concave on X_i, then there exists a Nash equilibrium.*

The concavity assumption of the objective function θ_i with respect to x_i is sometimes called player-concavity. When dealing with cost functions rather payoff functions, the concavity assumption has to be replaced by a convexity assumption. Most existence theorems of Nash equilibrium rely on a fixed-point argument, and this from the very beginning. Indeed when Nash introduced his equilibrium concept, a fixed-point theorem is used: the Kakutani theorem in [26] and the Brouwer theorem in [27].

Since the introduction of games $(N, X_i, \theta_i(.))$, many extensions have been proposed in the literature: discontinuous payoffs (e.g. [8]), non concave payoffs (e.g. [4]), topological action spaces (e.g. [24, 30]), constrained strategy sets (e.g. [9, 32]). In the following, we consider the latter extension dealing with games where each player has a range of actions which depends on the actions of other players. This new extension leads to the so-called generalized Nash equilibrium.

Let 2^{X_i} be the family of subsets of X_i. Let $C_i : X_{-i} \to 2^{X_i}$ be constraint correspondence of Player i, i.e. a function mapping a point in X_{-i} to a subset of X_i. Thus, $C_i(x_{-i})$ defines the ith player action space given other players' action x_{-i}. Typically, the constraint correspondence C_i is defined by a parametrized action space as $C_i(x_{-i}) = \{x_i \in X_i,\ g_i(x_i, x_{-i}) \geq 0\}$, where $g_i : X \to \mathbb{R}^{m_i}$ is a constraint function. When g_i does not depend on x_{-i}, we get back to standard game. A generalized game is described by $(N, X_i, C_i(.), \theta_i(.))$ and is also called an abstract economy in reference to Debreu's economic work [1, 9].

Definition 1.3. The generalized Nash equilibrium for a generalized game (N, X_i, C_i, θ_i) is defined as a point $x^\star$ solving for all $i \in \{1, \dots, N\}$,

$$x_i^\star \in \underset{x_i \in C_i(x_{-i}^\star)}{\arg\max}\ \theta_i(x_i, x_{-i}^\star). \tag{1.2}$$

In the present paper, we provide a self-contained survey of existence theorems for generalized Nash equilibrium. We also emphasize the use of fixed-point theorems in the proof of such theorems. A second purpose of this paper is to analyze the assumptions of those theorems on practical applications, and in particular the assumption on the constraint correspondence. Now, we set the outline of this paper. Section 2 gives the minimum required mathematical tools to study generalized Nash equilibria. Then, Section 3 presents the most recent existence theorems. Finally, Section 4 focuses on the analysis of assumptions when dealing with parametrized constrained sets.

2. PRELIMINARIES AND NOTATIONS

The mathematical tools needed are briefly summarized in this section, so that this paper is self-contained. For further details, we refer readers to the following books [2, 7, 12, 20, 29]. In the following, X and Y are two metric spaces.

2.1. **Quasiconcavity.** Refinements of concavity are first presented with characterizations based directly on the function f. Special characterizations when f is continuously differentiable or twice continuously differentiable exists but are omitted here, see [10] for a comprehensive study.

Definition 2.1. A function $f : X \to Y$ is concave (resp. convex) iff $\forall x, y \in X, \forall \lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \geq \text{(resp. } \leq\text{)}\ \lambda f(x) + (1-\lambda)f(y). \tag{2.1}$$

Strict convexity/concavity is obtained when Inequality (2.1) is strict.

Concavity of f can be also be defined in terms of the graph of f. Let hyp(f), epi(f) be the hypograph and the epigraph of f, defined as hyp$(f) = \{(x, y), y \geq f(x)\}$ and epi$(f) = \{(x, y), y \leq f(x)\}$. The concavity (resp. convexity) of a function f is equivalent to the convexity of hyp(f) (resp. epi(f)). So, it is immediate that hyp$(\min(f_1, f_2))$ = hyp$(f_1) \cap$ hyp(f_2): an intersection of two convex sets (resp. epi$(\max(f_1, f_2))$ = epi$(f_1) \cap$ epi(f_2)). The quasiconcavity is now introduced by relaxing Inequality (2.1).

Definition 2.2. A function $f : X \to Y$ is quasiconcave (resp. quasiconvex) iff $\forall x, y \in X, \forall \lambda \in]0, 1[$,

$$f(\lambda x+(1-\lambda)y) \geq \min(f(x), f(y)), \text{ resp. } f(\lambda x+(1-\lambda)y) \leq \max(f(x), f(y)). \tag{2.2}$$

Again, strict quasiconvexity/concavity is obtained when Inequality (2.2) is strict.

A univariate quasiconvex (resp. quasiconcave) function is either monotone or unimodal. Obviously convexity implies quasiconvexity. To better catch the meaning of quasi-concavity in contrast to concavity, we plot on Figure 1 examples of a concave function, a non-concave quasi-concave function and a non-quasiconcave function.

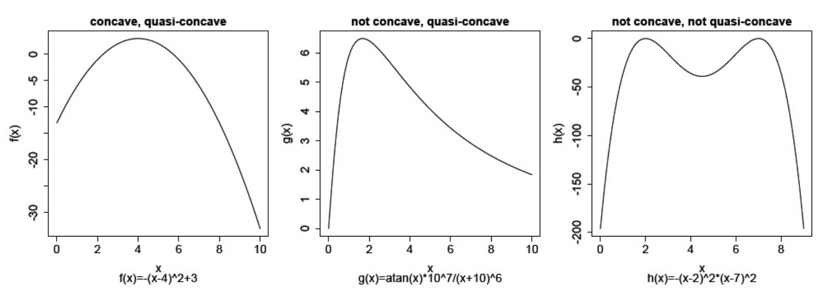

FIGURE 1. Examples and counter-examples of quasi-concavity

2.2. **Correspondences.** As unveiled in the introduction, new tools are used to refine the action strategy set from a compact set in Equation (1.1) to a player-dependent constrained set in Equation (1.2). Thus, correspondences, also called multi-valued functions, point-to-set maps or set-valued mappings, are introduced.

Definition 2.3. A correspondence $F : X \to 2^Y$ is an application such that $\forall x \in X$, $F(x)$ is a subset of Y. A correspondence is also denoted by $F : X \to \mathcal{P}(Y)$ or $F : X \rightrightarrows Y$. Given F, the domain is $dom(F) = \{x \in X, F(x) \neq \emptyset\}$, the range is $rg(F) = \bigcup_x F(x)$ and the graph is $Gr(F) = \{(x, y) \in X \times Y, y \in F(x)\}$.

Example 2.4. Typical examples of correspondences are the inverse of a function f, $x \to f^{-1}(x)$ (since $f^{-1}(x)$ might be empty, a singleton and a set); a constraint set $x \to \{x, f(x) \leq c\}$; the generalized gradient $x \to \partial f(x)$ or $F : x \to [-|x|, |x|]$.

Definition 2.5. For a correspondence $F : X \mapsto 2^Y$, the image of a subset B by F is defined as $F^{-1}(B) = \{x \in X, F(x) = B\}$. The exterior image (also called upper inverse) is $F^+(B) = \{x \in X, F(x) \subset B\}$ whereas the interior image (also called lower inverse) is $F^-(B) = \{x \in X, F(x) \cap B \neq \emptyset\}$.

A type of continuity for set-valued mappings has been introduced by Bouligand and Kuratowski in 1932: the lower and the upper semicontinuity (abbreviated l.s.c. and u.s.c.). In the literature, there are two concurrent definitions: the semicontinuity in the sense of Berge (e.g. [6, page 109]) and the semicontinuity in the sense of Hausdorff (e.g. [3, page 38-39]). These two definitions depend on the property of the set $F(x)$, yet, they are equivalent if F is compact-valued. In that case, the u.s.c./l.s.c. continuity can be defined on the exterior/interior images of F.

Definition 2.6. $F : X \to 2^Y$ is upper semicontinuous (u.s.c.) at x, iff $\forall y \in Y$, $F^+(\{y\})$ is an open set in X. $F : X \to 2^Y$ is lower semicontinuous (l.s.c.) at x, iff $\forall y \in Y$, $F^-(\{y\})$ is an open set in X.

When studying parametrized constrained sets, it is generally more convenient to work with characterizations by sequences. Therefore, the equivalent definition of the semicontinuity in terms of sequences are now given, see e.g. [19].

Definition 2.7. $F : X \to 2^Y$ is upper semicontinuous (u.s.c.) at x, iff for all sequence $(x_n)_n \in X, x_n \underset{n\to+\infty}{\longrightarrow} x, \forall y_n \in T(x_n)$ and $\forall y \in Y$, $y_n \underset{n\to+\infty}{\longrightarrow} y \Rightarrow y \in T(x)$.

Definition 2.8. $F : X \to 2^Y$ is lower semicontinuous (l.s.c.) at x, iff for all sequence $(x_n)_n \in X, x_n \underset{n\to+\infty}{\longrightarrow} x, \forall y \in T(x)$, there exists a sequence $(y_k)_k \in Y$, such that $y_k \underset{n\to+\infty}{\longrightarrow} y$ and $\forall k \in \mathbb{N}, y_k \in T(x_k)$.

Example 2.9. Let F defined by $F(x) = [-1, 1]$ if $x \neq 0$ or $\{0\}$ otherwise. F is l.s.c. in 0 but not u.s.c. Let G defined by $G(x) = \{0\}$ if $x \neq 0$ or $[-1, 1]$ otherwise. G is u.s.c. in 0 but not l.s.c. Let H defined by $H(x) = \{0\}$ if $x \neq 0$ or $\{-1, 1\}$ otherwise. H is neither u.s.c. nor l.s.c. in 0.

2.3. **Theorems for correspondences.** Thirdly, the necessary fixed-point theorems (for correspondences) are given, namely the Kakutani theorem [21] and the Begle theorem [5]. Let us first recall by the Brouwer theorem that a continuous function f from a finite-dimensional ball into itself admits a fixed-point. The Kakutani theorem is a valuable extention of the Brouwer's theorem to correspondences and is reported from [3]. The original theorem of [21] does not have any ambiguity about upper semicontinuity, since the author works in a finite-dimensional space with compact-valued mapping.

Theorem 2.10 (Kakutani). *Let K be a nonempty compact convex subset of a Banach space (e.g. $\mathbb{R}^n$) and $T : K \to 2^K$ a correspondence. If T is u.s.c. such that $\forall x \in K$, $T(x)$ is nonempty, closed and convex, then T admits a fixed-point theorem.*

The Begle theorem ([5]) is an extension of a fixed-point theorem for locally connected spaces by [11, 23], which in turn extends the Brouwer fixed-point theorem. To introduce this theorem, contractible polyhedrons have first to be defined: contractibility is in a sense related to convexity, see e.g. [22].

Definition 2.11. A geometric polyhedron is a finite union of convex hulls of finite-point sets.

Definition 2.12. A polyhedron is a subset S of $\mathbb{R}^n$ homeomorphic to a geometric polyhedron P, i.e. there exists a bijective function between S and P.

Definition 2.13. Contractible sets are nonempty sets deformable into a point by a continuous function (homotopy).

Example 2.14. Any star domain of Euclidean spaces is contractible whereas a finite-dimensional sphere is not. Any convex set of Euclidean spaces is contractible.

The Begle theorem reported here is the version from [9], originally contractible sets are replaced by absolute retracts in [5].

Theorem 2.15 (Begle)**.** *Let Z be a contractible polyhedron and $\phi : Z \to 2^Z$ be upper semicontinuous. If $\forall z \in Z, \phi(z)$ is contractible, then ϕ admits a fixed point.*

Finally, a last theorem needed is the Berge's maximum theorem, see e.g. [29, page 229] and [7, page 64].

Theorem 2.16 (Berge's maximum theorem)**.** *Let X, Y be two metric spaces, $f : X \times Y \to \mathbb{R}$ be an objection function and $F : X \to 2^Y$ a constraint correspondence. Assume that f is continuous, F is both l.s.c. and u.s.c.; and F is nonempty and compact valued. Then we have*

(i) *$\phi : x \to \max_{y \in F(x)} f(x, y)$ is a continuous function from X in $\mathbb{R}$.*

(ii) *$\Phi : x \to \arg\max_{y \in F(x)} f(x, y)$ is u.s.c. correspondence from X in 2^Y and compact-valued.*

As shown in the proof of Theorem 3.1, if in addition f is quasiconcave in y, then Φ is convex valued. Note that $\Phi(x)$ is sometimes written $\{y \in F(x), f(x, y) = \phi(x)\}$. The sequel demonstrates that the maximum theorem and the two fixed-point theorems 2.10 and 2.15 are the base recipes for showing the existence of a generalized Nash equilibrium.

3. STATE-OF-THE-ART EXISTENCE THEOREMS

Showing the existence of a generalized Nash equilibrium can be tackled in two different ways: either a direct approach based on fixed-point theorems or a reformulation based on quasi-variational inequalities. Proofs are given to emphasize how the maximum theorem is the link between optimization subproblem (1.2) and fixed-point theorems.

3.1. **The direct approach.**

Firstly, we investigate the direct approach. Theorem 3.1 was established by [1] in the context of abstract economy, so a simplified version by [20] is reported below. Some equivalent reformulations of Theorem 3.1 using a preference correspondence rather than a payoff function are also available in the following books: Theorem 19.8 in [7] and Theorem 3.7.1 in [12]. Note that [2] propose a different version, called the Arrow-Debreu-Nash theorem, where objective functions are player-concave rather than player-quasiconcave.

Theorem 3.1. *Let N players be characterized by an action space X_i, a constraint correspondence C_i and an objective function $\theta_i : X \to \mathbb{R}$. Assume for all players, we have*

(i) *X_i is nonempty, convex and compact subset of a Euclidean space,*

(ii) C_i *is both u.s.c. and l.s.c. in* X_{-i},
(iii) $\forall x_{-i} \in X_{-i}$, $C_i(x_{-i})$ *is nonempty, closed, convex,*
(iv) θ_i *is continuous on the graph* $Gr(C_i)$[1],
(v) $\forall x \in X$, $x_i \to \theta_i(x_i, x_{-i})$ *is quasiconcave on* $C_i(x_{-i})$,

Then there exists a generalized Nash equilibrium.

Proof. Since θ_i is continuous, C_i is both l.s.c. and u.s.c.; and C_i is nonempty and compact valued, the maximum theorem implies that the best response correspondence defined as

$$x_{-i} \overset{P_i}{\longmapsto} \underset{x_i \in C_i(x_{-i})}{\arg\max} \; \theta_i(x_i, x_{-i})$$

is u.s.c. and compact valued. Furthermore, as θ_i is player quasiconcave, P_i is convex valued. Let $z_i, y_i \in P_i(x_{-i})$. By definition of maximal points, $\forall x_i \in C_i(x_{-i})$, we have $\theta_i(y_i, x_{-i}) \geq \theta_i(x_i, x_{-i})$ and $\theta_i(z_i, x_{-i}) \geq \theta_i(x_i, x_{-i})$. Let $\lambda \in]0, 1[$. By the quasiconcaveness assumption, we get

$$\theta_i(\lambda y_i + (1-\lambda) z_i, x_{-i}) \geq \min\left(\theta_i(y_i, x_{-i}), \theta_i(z_i, x_{-i})\right) \geq \theta_i(x_i, x_{-i}).$$

Hence, $\lambda y_i + (1-\lambda) z_i \in P_i(x_{-i})$, i.e. $P_i(x_{-i})$ is a convext set. Furthermore, P_i is also nonempty valued since $C_i(x_{-i})$ is nonempty. Now, consider the Cartesian product of $P_i(x_{-i})$ to define Φ as

$$\begin{array}{rccc} \Phi: & X & \to & 2^{X_1} \times \cdots \times 2^{X_N} \\ & x & \to & P_1(x_{-1}) \times \cdots \times P_N(x_{-N}) \end{array}$$

where X is a subset of $\mathbb{R}^n$ with $n = \sum_i n_i$. This multiplayer best response is nonempty, convex and compact valued. In our finite-dimensional setting and with a finite Cartesian product, the upper semicontinuity of each component P_i implies the upper semicontinuity of Φ, see Prop 3.6 of [19]. Finally, the Kakutani theorem gives the existence result. □

The Debreu theorem ([9]) based on contractible sets is now given. Originally, the upper-semicontinuity is replaced by the closedness of the graph $Gr(C_i)$, but this is equivalent since contractible sets are closed and compact sets.

Theorem 3.2. *Let N agents be characterized by an action space X_i and $X = X_1 \times \cdots \times X_N$. Let a payoff function $\theta_i : X \to \mathbb{R}$ and a restricted action space $C_i(x_{-i})$ given other player actions x_{-i}. Each agent i maximizes its payoff on $C_i(x_{-i})$. If for all agents, we have*

(i) X_i *is a contractible polyhedron,*
(ii) $C_i : X_{-i} \to 2^{X_i}$ *is u.s.c.,*
(iii) θ_i *is continuous from* $Gr(C_i)$ *to* $\mathbb{R}$,
(iv) $\phi_i : x_{-i} \to \max\limits_{x_i \in C_i(x_{-i})} \theta_i(x_i, x_{-i})$ *is continuous,*
(v) $\forall x_{-i} \in X_{-i}$, *the best response set* $M_{x_{-i}} = \{x_i \in X_i(x_{-i}),\ \theta_i(x_i, x_{-i}) = \phi_i(x_{-i})\}$ *is contractible,*

Then there exists a generalized Nash equilibrium.

Proof. Let $G_i = Gr(C_i)$. Again, we work on the best response set, which is defined as

$$M_i = \{(x_i, x_{-i}) \in X_i \times X_{-i},\ x_i \in M_{x_{-i}}\} = \{(x_i, x_{-i}) \in G_i,\ \theta_i(x_i, x_{-i}) = \phi_i(x_{-i})\}.$$

[1] There is no need for θ_i to be continuous on the whole space X, since only feasible points (i.e. those in $Gr(C_i)$) matters in Equation (1.2).

This set is closed since the functions ϕ_i and θ_i are continuous and c_i is u.s.c.. Let Φ be the correspondence defined as $\Phi(x) = M_{x_{-1}} \times \cdots \times M_{x_{-N}}$. Using the Cartesian product, the graph of Φ is given by

$$Gr(\Phi) = \{(x, y) \in X \times X,\ y \in \Phi(x)\} = \bigcap_{i=1}^{N} \{(x, y) \in X \times X,\ (y_i, x_{-i}) \in M_i\},$$

a finite intersection of closed sets. As $Gr(\Phi)$ is closed, Φ is u.s.c.. Moreover for all x, $\Phi(x)$ is contractible as a finite Cartesian product of contractible sets. Applying the Begle fixed-point theorem completes the proof. □

3.2. **The QVI reformulation.** The generalized Nash equilibrium problem (1.2) can be reformulated in the quasi-variational inequality (QVI) framework. Variational inequality and QVI are first described and then the existence theorem is given.

Definition 3.3 (Variational Inequality)**.** Given a function $F : \mathbb{R}^n \to \mathbb{R}^n$ and a set $K \subset \mathbb{R}^n$ a Variational Inequality, denoted by $VI(K, F(.))$, is to find a vector $x^\star$ such that $x^\star \in K$ and

$$\forall y \in K, (y - x^\star)^T F(x^\star) \geq 0.$$

Definition 3.4 (Quasi-Variational Inequality)**.** Given a function $F : \mathbb{R}^n \to \mathbb{R}^n$ and a correspondence $K : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, a Quasi-Variational Inequality, denoted by $QVI(K(.), F(.))$, is to find a vector $x^\star$ such that $x^\star \in K(x^\star)$ and

$$\forall y \in K(x^\star),\ (y - x^\star)^T F(x^\star) \geq 0.$$

For examples of applications of variational inequality problems and links with optimization, see e.g. [15, 16]. Definition 1.3 can be reformulated as the Quasi-Variational Inequality problem $QVI(C(.), \Theta(.))$ with

$$C(x) = C_1(x_{-1}) \times \cdots \times C_N(x_{-N}) \text{ and } \Theta(x) = \begin{pmatrix} \nabla_{x_1}\theta_1(x) \\ \vdots \\ \nabla_{x_N}\theta_N(x) \end{pmatrix}, \tag{3.1}$$

see e.g. [18] and [14] for a proof. Note that this reformulation assumes the differentiability of objective function θ_i. An existence theorem based on the QVI approach developped in [17] is now given.

Theorem 3.5. *Let N players be characterized by an action space X_i, a constraint correspondence C_i and an objective function $\theta_i : X \to \mathbb{R}$. Assume for all players, θ_i is continuously differentiable on the graph $Gr(C_i)$. Let $C(x) = C_1(x_{-1}) \times \cdots \times C_N(x_{-N})$. Assume there exists a compact convex subset $T \subset \mathbb{R}^n$,*

(i) *$\forall x \in T$, $C(x)$ is nonempty, closed, convex subset of T,*
(ii) *C is both u.s.c. and l.s.c. in T,*

Then there exists a generalized Nash equilibrium.

Proof. Using Θ as given in Equation (3.1), the correspondence $F : T \to 2^T$ is defined as

$$F(x) = \arg\max_{z \in C(x)} -(z - x)^T \Theta(x).$$

Let $x^\star$ be a fixed-point of F. We have

$$\begin{aligned} x^\star \in F(x^\star) \quad &\Leftrightarrow \quad x^\star \in \arg\max_{z \in C(x^\star)} -(z - x^\star)^T \Theta(x^\star) \\ &\Leftrightarrow \quad \forall z \in C(x^\star), -(z - x^\star)^T \Theta(x^\star) \leq -(x^\star - x^\star)^T \Theta(x^\star) = 0 \\ &\Leftrightarrow \quad \forall z \in C(x^\star), (z - x^\star)^T \Theta(x^\star) \geq 0. \end{aligned}$$

Thus, the QVI reformulation of (1.2) turns out to be a fixed-point problem. By assumption, C is nonempty, compact valued and both u.s.c. and l.s.c.. The function $(z, y) \to -(z-y)^T\Theta(y)$ is continuous since θ_i is continuously differentiable. Therefore by the maximum theorem, the correspondence F is u.s.c. and compact valued. Furthermore, the function $z \to -(z-x)^T\Theta(x)$ is a linear function (hence convex). So, F is also convex-valued (by the maximum theorem), hence $F(x)$ is a contractible set for all $x \in T$. Applying the Begle fixed-point theorem completes the proof. □

A significative part of games are such that player strategies are required to satisfy a common coupling constraint (such games are called jointly convex games) see [14] and the references therein. In jointly convex games, the contraint correspondence simplifies to

$$\begin{array}{rll} C_i: & X_{-i} \to & 2^{X_i} \\ & x_{-i} \to & \{x_i \in X_i,\ (x_i, x_{-i}) \in K\}, \end{array} \tag{3.2}$$

where $K \subset X_1 \times \cdots \times X_N$ is a nonempty convex set. We are now interested in points solving the (classical) variational inequality problem $VI(K, \Theta(.))$ with K given in Equation (3.2) and Θ given in Equation (3.1). As expected, not all solutions of the generalized Nash equilibrium (1.2) (i.e. solutions of $QVI(C(.), \Theta(.))$) solves this variational inequality problem. Therefore, a special type of generalized Nash equilibrium has been introduced: a variational equilibrium also called a normalized equilibrium. A variational equilibrium has a special interpretation in terms of Lagrange multipliers of the corresponding KKT systems of the GNEP, see e.g. [13, 17].

Definition 3.6 (Variational equilibrium)**.** A strategy $\bar{x}$ is a variational equilibrium of a generalized game $(N, X_i, C_i(.), \theta_i(.))$ if $\bar{x}$ solves $VI(K, \Theta(.))$ with K given in Equation (3.2) and Θ given in Equation (3.1).

Theorem 3.7. *Let N players be characterized by an action space X_i, a constraint correspondence C_i and an objective function $\theta_i : X \to \mathbb{R}$. Assume for all players, θ_i is continuously differentiable on the graph $Gr(C_i)$ and there exists a nonempty convex compact set $K \subset \mathbb{R}^n$ such that $C_i(x_{-i}) = \{x_i \in X_i,\ (x_i, x_{-i}) \in K\}$, then there exists a variational equilibrium.*

Proof. Same proof as Theorem 3.5 with $C(x)$ replace by K which has the same properties. □

4. PARAMETRIZED CONSTRAINED SETS

This final section aims to provide criteria to guarantee the assumptions of previous theorems, as well as, proofs for such criteria. For this purpose, a parametrized constraint set is considered

$$\begin{array}{rll} C_i: & X_{-i} \to & 2^{X_i} \\ & x_{-i} \to & \{x_i \in X_i,\ g_i(x_i, x_{-i}) \geq 0\}, \end{array} \tag{4.1}$$

for $x_{-i} \in X_{-i}$, where $g_i : \mathbb{R}^n \to \mathbb{R}^{m_i}$ and $C_i(x_{-i}) = \emptyset$ for $x_{-i} \notin X_{-i}$. The jth component of constraint function g_i is denoted by g_{ij}.

A central assumption of Theorems 3.1, 3.2 and 3.5 is to require C_i to be both lower and upper semicontinuous. Yet, Theorem 3.2 requires C_i to be u.s.c. and ϕ to be continuous, by Berge's maximum theorem, a sufficient condition is that C_i is also l.s.c.. Other assumptions of these theorems are nonemptyness, convexity and closedness of $C_i(x_{-i})$. Theorem 3.2 also requires X_i and M_i to be contractible: a sufficient condition is X_i to be a convex and θ_i to be player-quasiconcave.

4.1. **The upper-semicontinuity.** [31] devote a full chapter on set-valued analysis. Despite they formulate the outer and inner semicontinuity through superior limit of sets, they work with the Berge's semicontinuity (respectively the lower and upper semicontinuity). Their Theorem 5.7 of [31] gives equivalent reformulations of upper semicontinuity using the graph properties, for which their Example 5.10 is a direct application. A small variant is given here by removing equality constraints.

Proposition 4.1. *Let $C_i : X_{-i} \to 2^{X_i}$ be the feasible set mapping defined in Equation (4.1). Assume $X_i \subset \mathbb{R}^{n_i}$ is closed and all components g_{ij}'s are continuous on $X_i \times X_{-i} \subset \mathbb{R}^n$, then the correspondence C_i is u.s.c. on X_{-i}.*

Proof. For all $j = \{1, \dots, m\}$, by the continuity of the jth component g_{ij}, the set of $x_i \in X_i$ such that $g_{ij}(x_i, x_{-i}) \geq 0\}$ is closed. So, $C_i(x_{-i})$ is a finite intersection of closed sets, thus a closed set. Let $(x_{i,n}, x_{-i,n})_n \to x$ and $y_{i,n} \in C_i(x_{-i,n})$, the closedness of $C_i(x_{-i})$ guarantees that $y_{i,n} \to y_i$ implies $y_i \in C_i(x_{-i})$. □

A weaker assumption on the constraint function g_i is given in Theorem 10 of [19]. [19] only assumes that each component g_{ij} is an upper semicontinuous function (i.e. the closedness of the hypograph g_{ij}).

4.2. **The lower-semicontinuity.** Generally, conditions on g_i in order that the correspondence is l.s.c. are harder to find. Nevertheless, [31, 19] provide conditions for it. An application of Theorem 5.9 of [31] to the constraint correspondence in Equation (4.1) is presented.

Proposition 4.2. *Let $C_i : X_{-i} \to 2^{X_i}$ be the feasible set mapping defined in Equation (4.1). Assume g_{ij}'s are continuous and concave in x_i for each x_{-i}. If there exists $\bar{x}$ such that $g_i(\bar{x}_i, \bar{x}_{-i}) > 0$ for all i, then C_i is l.s.c. at $\bar{x}_{-i}$ and in some neighborhood of $\bar{x}_{-i}$ (and also u.s.c.).*

Proof. For ease of notation, the subscript i is removed, x_i and x_{-i} are denoted by x and w, respectively. Let f be the function $f(x, w) = \min(g_1(x, w), \dots, g_m(x, w))$. which is continuous by the continuity of g_j. By the concavity with respect to x, f is also concave with respect to x. The upper level $\text{lev}_{\geq 0} f$ is the graph of C. By the continuity of f, the graph $Gr(f)$ is closed.

$$\begin{aligned}\text{lev}_{\geq 0} f &= \{(x, w), f(x, w) \geq 0\} = \{(x, w), \forall i = 1, \dots, m, g_i(x, w) \geq 0\} \\ &= \{(x, w), x \in C(w)\} = Gr(C).\end{aligned}$$

So C is u.s.c.. The level set of a convex function is also convex, so is $\text{lev}_{\geq 0} f(., w)$ with respect to x for all w. As f is continuous and $g(\bar{x}, \bar{w}) > 0$, there exists an open set O, such that

$$\forall w \in O, f(\bar{x}, w) > 0.$$

Since the upper level set $\text{lev}_{\geq 0} f(., w)$ for any w is convex and f is continuous, the interior $\text{int}C(w)$ is nonempty. This guarantees the lower semicontinuity of C at $\bar{w}$. Indeed, for all $\tilde{w} \in O$ and all $\tilde{x} \in \text{int}C(\tilde{w})$, by the continuity of f and the assumption at $(\bar{x}, \bar{w})$, there exists a neighborhood $W \subset O \times \text{int}C(\tilde{w})$ such that f is strictly positive on W, and also $W \subset \text{gph}(C)$. As $w \to \tilde{w}$, then certainly $\tilde{x}$ belongs to the inner limit of $C(w)$. This inner limit is a closed set, and so includes $\text{int}C(\tilde{w}) \supset \text{cl}(\text{int}C(\tilde{w}))$. Since $C(\tilde{w})$ is a closed convex set with noempty interior, $\text{cl}(\text{int}C(\tilde{w})) = C(\tilde{w})$. Hence, the inner limit of $C(w)$ contains $C(\tilde{w})$, i.e. C is l.s.c. at $\tilde{w}$ by Theorem 5.9 of [31]. □

The previous property is also given in Theorem 12 of [19] and proved using the sequence characterization of semicontinuity. Theorem 13 of [19] is reported here as it gives weaker conditions for the correspondence to be lower semicontinuous than Theorem 12.

Proposition 4.3. *Let $C_i : X_{-i} \to 2^{X_i}$ be the feasible set mapping as defined above. Let $\widetilde{C}_i$ be the correspondence $\widetilde{C}_i(x_{-i}) = \{x_i \in X_i, g_i(x_i, x_{-i}) > 0\}$. If each component g_{ij} is lower semicontinuous (i.e. closedness of the epigraph) on $\bar{x}_{-i} \times \widetilde{C}_i(\bar{x}_{-i})$ and $C_i(\bar{x}_{-i}) \subset$ cl$(\widetilde{C}_i(\bar{x}_{-i}))$, then C_i is lower semicontinuous at $\bar{x}_{-i}$.*

Proof. For ease of notation, the subscript i is removed, x_i and x_{-i} are denoted by x and w, respectively. If $\widetilde{C}(\bar{w}) = \emptyset$, then by assumption, $C(\bar{w}) = \emptyset$ and the conclusion is trivial. Otherwise, when $\widetilde{C}(\bar{w}) \neq \emptyset$, we choose $\bar{x} \in C(\bar{w})$ and $w_n \to \bar{w}$. Since $C(\bar{w}) \subset \text{cl}(\widetilde{C}(\bar{w}))$, there exists a sequence $(x_m)_m$ of elements in $\widetilde{C}(\bar{w})$ such that $x_m \to \bar{x}$. Construct the sequence n_m such that $n_0 = 0$ and $n_m = \max(n_{m-1} + 1, \arg\min_k(\forall l \geq k, g(w_l, x_m) > 0))$. The sequence is well defined by the lower semicontinuity of g. Furthermore, the sequence $(x_{n_m})_{m \geq 0}$ is such that $x_{n_m} \in C(w_{n_m})$ with $x_{n_m} \to \bar{x}$, $w_{n_m} \to \bar{w}$ and $\bar{x} \in C(\bar{w})$, i.e. C is l.s.c. at $\bar{w}$. □

Continuous selections introduced by [25] can be used to further relaxed assumptions on g_i. Originally, [25] works with topological spaces and uses the Berge's semicontinuity. Their Proposition 2.3 is reported below.

Proposition 4.4. *If $\phi : X \to 2^Y$ is l.s.c. and $\phi : X \to 2^Y$ such that for every $x \in X$, cl$(\phi(x)) =$ cl$(\psi(x))$, then ψ is also l.s.c..*

Property 4.4 has strong consequences on the lower semicontinuity of the correspondence C_i and justifies the [19]'s approach to use the correspondence $\widetilde{C}_i$ rather than C_i, since images have the same closure set. Therefore, the lower semicontinuity of each component g_{ij} suffices to get the lower semicontinuity of C_i. With the continuity of g_{ij}, it is even more straightforward to see that for all $x_i \in \widetilde{C}_i(x_{-i})$, there exists a sequence $(x_{-i,n})_n$ and $x_{i,n} \in \widetilde{C}_i(x_{-i,n})$ such that for all $n \geq n_0$, $g_i(x_{i,n}, x_{-i,n}) > 0$. Other types of conditions not based on strict inequalities are given in [19] .

4.3. **The nonemptyness, the closedness and the convexity.** Finally, we turn our attentions to other assumptions. Theorems 3.1, 3.2 and 3.5 also require C_i to be nonempty, convex and closed valued. The convexity assumption on $C_i(x_{-i})$ is satisfied when g_i is quasi-concave with respect to x_i. This is not immediate with the definition of quasi-concavity given in Section 2. But an equivalent definition for a function f to be quasiconcave is that all upper level sets $U_f(r) = \{x \in X, f(x) \geq r\}$ are convex for all r, see [10]. Thus, if g_i is quasiconcave, then $U_{g_i}(0)$ is convex. The nonemptyness assumption is the most challenging assumption. Except to have a strict inequality condition and the continuity of g_i's, it is hard to find general conditions. Finally, the closedness assumption is satisfied when g_i's are continuous.

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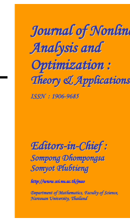

AN EMBEDDING THEOREM FOR A CLASS OF CONVEX SETS IN NONARCHIMEDEAN NORMED SPACES

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ABSTRACT. In this article we show that the class of all compact convex sets of a real nonarchimedean normed space can be embedded in a real nonarchimedean normed space.

KEYWORDS : Embedding theorem; Nonarchimedean normed space.
AMS Subject Classification: Primary: 46A55 , Secondary: 46S10

1. INTRODUCTION AND PRELIMINARIES

In [1], Rådström showed that the class of all compact convex sets of a real normed space can be embedded in a real normed space. In this article we give a nonarchimedean counterpart for this fact. We start by recalling a few essential concepts from [2].

Let K be a field. A nonarchimedean absolute value on K is a function $|\cdot| : K \to \mathbb{R}$ such that, for any $a, b \in K$ we have

1. $|a| \geq 0$,
2. $|a| = 0$ if and only if $a = 0$,
3. $|ab| = |a| \cdot |b|$,
4. $|a + b| \leq \max(|a|, |b|)$.

The field K is called nonarchimedean if it is equipped with a nonarchimedean absolute value such that the corresponding metric is complete.

Let X be a vector space over field K which is equipped with a nonarchimedean absolute value (nonarchimedean vector space, for short). A nonarchimedean norm $\|\cdot\|$ on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that

1. $\|x\| = 0$ implies that $x = 0$;
2. $\|ax\| = |a| \cdot \|x\|$, for any $a \in K$ and $x \in X$;
3. $\|x + y\| \leq \max(\|x\|, \|y\|)$, for any $x, y \in X$.

Moreover, a nonarchimedean vector space X equipped with a nonarchimedean norm is called a nonarchimedean normed space. Nonarchimedean normed spaces

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over the nonarchimedean field $\mathbb{R}$, will be called real nonarchimedean normed spaces.

Throughout this paper we assume that K is a nonarchimedean field and X is a nonarchimedean normed space over K. We set $\mathcal{B} =: \{a \in K : |a| \leq 1\}$.

A subset $A \subseteq X$ is called convex if either A is empty or is of the form $A = x + A_0$ for some vector $x \in X$ and some $\mathcal{B}$-submodule $A_0 \subseteq X$. A lattice L in X is an $\mathcal{B}$-submodule which satisfies the condition that for any vector $x \in X$ there is a nonzero scalar $a \in K$ such that $ax \in L$. For more basic facts see [2].

2. MAIN RESULTS

For nonempty convex subsets A and B of X and scalar $\lambda \in K$, let $A + B =: \{x + y : x \in A, y \in B\}$ and $\lambda A =: \{\lambda x : x \in A\}$. Addition and scalar multiplication satisfy $(A + B) + C = A + (B + C)$, $A + B = B + A$, and $\lambda(A + B) = \lambda A + \lambda B$.

Lemma 2.1. *Let A, B, and C be subsets of a nonarchimedean normed space X, where C is closed and B is nonempty convex and bounded. Then $A + B \subseteq C + B$ implies $A \subseteq C$.*

Proof. Since B is convex, there exist a $\mathcal{B}$-submodule B_0 and $b \in X$ such that $B = b + B_0$. By assumption we have $A + B_0 \subseteq C + B_0$. Let $a \in A \backslash C$. There is a lattice L such that $(a + L) \cap (C) = \emptyset$. Since L is $\mathcal{B}$-submodule of X, $(a + L) \cap (C + L) = \emptyset$. Boundedness of B implies that B_0 is bounded and so there is $\alpha \in K$ such that $B_0 \subseteq \alpha L$. If $|\alpha| \leq 1$, then $(a + B_0) \cap (C + B_0) = \emptyset$ which is a contradiction. If $|\alpha| > 1$, then $a = z + b$, for some $z \in C$ and $b \in B_0$. This implies that $(z + b + \alpha^{-1}B_0) \cap (C + \alpha^{-1}B_0) = \emptyset$, which is a contradiction since $(b + \alpha^{-1}B_0) \cap \alpha^{-1}B_0 \neq \emptyset$. □

Lemma 2.1 implies that:

Corollary 2.2. *Let A, B, and C be subsets of a nonarchimedean normed space X, where A and C are closed and B is nonempty convex and bounded. Then $A + B = C + B$ implies $A = C$.*

For subsets A and C of X, define

$$\mathfrak{h}(A, C) =: \inf\{\varepsilon > 0 : C \subseteq N_\varepsilon(A), A \subseteq N_\varepsilon(C)\},$$

where $N_\varepsilon(A) =: \{z \in X : d(z, A) < \varepsilon\}$ and $d(z, A)$ denotes distance of z from A. By convention $\inf \emptyset = \infty$. The extended real valued function $\mathfrak{h}$ has the following properties for each subset A, B, and C:

(i) $\mathfrak{h}(A, B) \geq 0$ and $\mathfrak{h}(A, A) = 0$;
(ii) $\mathfrak{h}(A, B) = \mathfrak{h}(B, A)$;
(iii) $\mathfrak{h}(A, B) \leq \max(\mathfrak{h}(A, C), \mathfrak{h}(C, B))$;
(iv) $\mathfrak{h}(A, B) = 0$ if and only if $\overline{A} = \overline{B}$, where $\overline{A}$ denotes the closure of A in X.

The Proof of Properties 1 and 2 are easy and we just give the proof of Properties 3 and 4. By contradiction, let $\mathfrak{h}(A, B) > \max(\mathfrak{h}(A, C), \mathfrak{h}(C, B))$ for some subsets A, B, C. Then there would be positive numbers λ_1 and λ_2 where $\lambda_1 < \mathfrak{h}(A, B)$, $\lambda_2 < \mathfrak{h}(A, B)$, $A \subseteq N_{\lambda_1}(C)$, $C \subseteq N_{\lambda_1}(A)$ and $C \subseteq N_{\lambda_2}(B)$, $B \subseteq N_{\lambda_2}(C)$. Therefore $B \subseteq N_\lambda(A)$ and $A \subseteq N_\lambda(B)$ where $\lambda = \max(\lambda_1, \lambda_2)$. This is a contradiction since $\lambda < \mathfrak{h}(A, B)$. To prove 4, let $\mathfrak{h}(A, B) = 0$ and $x \in \overline{A}$. For each $\gamma > 0$ there exists nonzero $\lambda > 0$ such that $\lambda \leq \gamma$ with $B \subseteq N_\lambda(A)$, $A \subseteq N_\lambda(B)$ and $N_\lambda(x) \cap A \neq \emptyset$. Since $A \subseteq N_\lambda(B)$ so $N_\lambda(x) \cap B \neq \emptyset$ and consequently $N_\gamma(x) \cap B \neq \emptyset$, that is $x \in \overline{B}$. By a similar way we have $\overline{B} \subseteq \overline{A}$. Conversely, if $\overline{A} = \overline{B}$ and $\mathfrak{h}(A, B) > 0$, then there exists $\lambda > 0$ such that either $B \nsubseteq N_\lambda(A)$ or $A \nsubseteq N_\lambda(B)$. If $x \in A \backslash N_\lambda(B)$, then $N_\lambda(x) \cap B = \emptyset$. That is to say x is not an element of $\overline{B}$, which is a contradiction.

Lemma 2.3. *If A and C are convex sets in a nonarchimedean normed space, then for each nonempty convex and bounded set B we have*

$$\mathfrak{h}(A, C) = \mathfrak{h}(A + B, C + B).$$

Proof. If $C \subseteq N_\lambda(A)$ and $A \subseteq N_\lambda(C)$, for some $\lambda \geq 0$, then $C + B \subseteq N_\lambda(A + B) = B + N_\lambda(A)$, $A + B \subseteq N_\lambda(C + B) = B + N_\lambda(C)$. Therefore $\mathfrak{h}(A + B, C + B) \leq \mathfrak{h}(A, C)$. The inverse inequality is obtained by Lemma 2.1. □

By part A of Theorem 1 in [1], if M is a commutative semigroup with the law of cancellation, then M can be embedded in a group N. Also, if G is a group in which M is embedded, then N is isomorphic to a subgroup of G containing M. Therefore, by Corollary 2.2, the semigroup of all nonempty compact convex subsets of a nonarchimedean normed space can be embedded in a minimal group N as a semigroup.

Hereafter let $\mathbb{R}$ be equipped with a nonarchimedean absolute value $|\cdot|$.

Theorem 2.4. *Let M be an additive commutative semigroup with the law of cancellation. If a multiplication by real scalars is defined on M which satisfies*

$$\lambda(A+B) = \lambda A + \lambda B, \quad (\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A, \quad \lambda_1(\lambda_2 A) = \lambda_1 \cdot \lambda_2 A, \quad 1A = A,$$

for every $A, B \in M$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$, then M can be embedded in a minimal real nonarchimedean vector space N.

Moreover, if a metric d is given on M with

$$d(A + C, B + C) = d(A, B), \quad d(\lambda A, \lambda B) = \lambda d(A, B),$$

for every $A, B \in M$ and $\lambda \in \mathbb{R}$ and the operations addition and scalar multiplication are continuous in the topology induced by d, then a nonarchimedean norm can be defined on N which makes it as a real nonarchimedean normed space.

Proof. Following to the proof of Theorem 1 in [1], consider the equivalence relation $\sim$ defined as $(A, B) \sim (C, D)$ if and only if $A + D = B + C$, for $A, B, C, D \in M$. By $[A, B]$ denote the equivalence class containing the pair (A, B). The set N shall consist of equivalence classes $[A, B]$, where A and B are elements of M. Addition and scalar multiplication in N are defined by $[A, B] + [C, D] = [A + C, B + D]$ and $\lambda[A, B] = [\lambda A, \lambda B]$ for $\lambda \in [0, +\infty)$, otherwise $\lambda[A, B] = [-\lambda B, -\lambda A]$. Obviously the given operations are well defined and N constitutes a nonarchimedean vector space. For some $B \in M$ define $f : M \to N$ by $f(A) = [A + B, B]$ for each $A \in M$. The mapping f is well defined and embeds M in the nonarchimedean vector space N. Clearly, for $\lambda \in \mathbb{R}$ and $A \in M$ the scalar product λA coincides with the one given on M.

Let d be a nonarchimedean metric on M satisfying the assumptions of theorem. Define d_0 on $N \times N$ as

$$d_0([A, B], [C, D]) = d(A + D, B + C).$$

Let $[A, B], [C, D] \in N$ and $d_0([A, B], [C, D]) = 0$. So $d(A + D, B + C) = 0$ which implies that $A + D = B + C$, that is $(A, B) \sim (C, D)$. Conversely, if $(A, B) \sim (C, D)$, then $d_0([A, B], [C, D]) = 0$. Obviously

$$d_0([A, B], [C, D]) = d_0([C, D], [A, B]).$$

Also

$$\begin{aligned} d_0([A,B],[C,D]) &= d(A+D,B+C) \\ &\leq \max(d(A+F+E+D,B+E+E+D),d(B+E+F \\ &\quad +C,B+E+E+D)) \\ &= \max(d(A+F,B+E),d(E+D,F+C)) \\ &= \max(d_0([A,B],[E,F]),d_0([E,F],[C,D])). \end{aligned}$$

Since nonarchimedean metric d_0 is invariant under translation, so the function $\|\cdot\| : N \to \mathbb{R}$, where $\|[A,B]-[C,D]\| =: d_0([A,B],[C,D])$ is a nonarchimedean norm on N. Therefore addition and scalar multiplication are continuous operations, and if $A,B \in M$, the distance between A and B equals $d(A,B)$. □

By Corollaries 2.2 and 2.3 and Theorem 2.4, we have the following.

Theorem 2.5. *Let M be a class of nonempty closed, bounded convex subsets of X which is closed under addition and scalar multiplication and equipped with a nonarchimedean metric. Then M can be isometrically embedded in a real nonarchimedean normed space N. In particular the operations addition and scalar multiplication of M are induced by the operations of N.*

Moreover, if H is a nonarchimedean normed space in which M is embedded in the above way, then H contains a subspace containing M and isometric to N.

It is worth mentioning that the class of all nonempty compact convex sets of a real nonarchimedean normed space satisfies Theorem 2.5.

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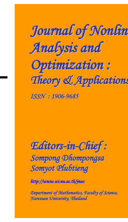

MINIMUM-NORM FIXED POINT OF A FINITE FAMILY OF $\lambda-$STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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ABSTRACT. Let K be a nonempty closed and convex subset of a real Hilbert space H and for each $1 \leq i \leq N$, let $T_i : K \to K$ be λ_i-strictly pseudocontractive mapping. Then for $\beta \in (0, 2\lambda]$, where $\lambda := \min\{\lambda_i : i = 1, 2, ..., N\}$, and each $t \in (0, 1)$, it is proved that, there exists a sequence $\{y_t\} \subset K$ satisfying $y_t = P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big]$, where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_N T_N$, for $\theta_1 + \theta_2 + ... + \theta_N = 1$, which converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. Moreover, we provide an explicit iteration process which converges strongly to a common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. Corresponding results, for a common minimum-norm solution of a finite family of $\alpha-$inverse strongly monotone mappings are also discussed. Our theorems improve several results in this direction.

KEYWORDS: Minimum-norm fixed point; nonexpansive mappings; $\lambda-$strict pseudocontractive mappings; monotone mappings.
AMS Subject Classification: 47H06 47H09 47H10 47J05 47J05.

1. INTRODUCTION

Let K be a nonempty subset of a real Hilbert space H and T be a self-mapping of K. The mapping T is called *Lipschitzian* if there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$, for all $x, y \in K$. If $L = 1$, then T is called *nonexpansive* and if $L \in [0, 1)$, T is called *contraction*. A mapping T with domain $D(T)$ and range $R(T)$ in H is called $\lambda-$*strictly pseudocontractive* in the terminology of Browder and Petryshyn [2] if for all $x, y \in D(T)$ there exists $\lambda > 0$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2 - \lambda||x - y - (Tx - Ty)||^2. \tag{1.1}$$

It is obvious that $\lambda-$strictly pseudocontractive mapping is Lipschitzian with $L = \frac{1+\lambda}{\lambda}$. Without loss of generality we may assume $\lambda \in (0, 1)$. If I denotes the identity operator, then (1.1) can be written in the form

$$\langle (I - T)x - (I - T)y, x - y\rangle \geq \lambda||(I - T)x - (I - T)y||^2. \tag{1.2}$$

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Moreover, one can show that (1.1) (and hence (1.2)) is equivalent to the inequality

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, k = (1 - 2\lambda) < 1.$$

It is easy to see that a class of nonexpansive mappings which includes a class of contraction mappings is contained in a class of $\lambda-$ strictly pseudocontractive mappings. However, the converse may not be true (see, [1, 8] for details).

Interest in $\lambda-$strictly pseudocontractive mappings stems mainly from their firm connection with the important class of nonlinear $\alpha-$*inverse strongly monotone* mappings where a mapping A with domain $D(A)$ and range in H is called $\alpha-$*inverse strongly monotone* if there exists $\alpha \in (0, 1)$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha ||Ax - Ay||^2, \text{ for every } x, y \in D(A).$$

Observe that A is $\alpha-$inverse strictly monotone if and only if $(I - A)$ is $\lambda-$strictly pseudocontractive, where $\lambda = \alpha$. It is known [9] that $\lambda-$strictly pseudo-contractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. Therefore, it is interesting to develop the algorithms for λ-strictly pseudo-contractive mappings. Consequently, considerable research efforts, especially within the past 20 years or so, have been devoted to iterative methods for approximating fixed points of T when T is nonexpansive or $\lambda-$strictly pseudocontractive (see for example [6, 7, 8, 12, 14, 15, 16, 17] and the references contained therein).

Recently, we notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. In an abstract way, we may formulate such problems as follows:

$$\text{find } x^* \in K \text{ such that } ||x^*|| := \min_{x \in K} ||x||, \tag{1.3}$$

that is, x^* is a metric projection of the origin onto K, $P_K 0$.

A typical example is the split feasibility problem which was introduced by Censor and Elfving [4]. The problem is formulated as finding:

$$x^* \in C \text{ and } Ax^* \in Q, \tag{1.4}$$

where C and Q are nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \longrightarrow H_2$ is a bounded linear operator. A split feasibility problem in finite dimensional Hilbert spaces was used for modeling inverse problems which arise in medical image constructions [3] and intensity-modulated radiation therapy [4].

Set

$$\min_{x \in C} \varphi(x) := \min_{x \in C} \frac{1}{2} ||Ax - P_Q Ax||^2. \tag{1.5}$$

It is clear that $\bar{x}$ is a solution to the split feasibility problem (1.4) if and only if $\bar{x}$ solves the minimization problem (1.5) with the minimum equal to 0. Now, assume that (1.4) is consistent (i.e., (1.4) has a solution) and let Γ denote the (closed convex) solution set of (1.4) (or equivalently, solution of (1.5)). Then, in this case, Γ has a unique element $\bar{x}$ if and only if it is a solution of the following equation:

$$\bar{x} \in C, \text{ such that } \bigtriangledown \varphi(\bar{x}) = A^*(I - P_Q)A\bar{x} = 0, \tag{1.6}$$

where A^* is the adjoint of A. Let $Tx := (I - \gamma A^*(I - P_Q)A)x$, for any $\gamma > 0$. Then problem (1.6) is equivalent to the fixed point problem equation

$$\bar{x} = T\bar{x} = (I - \gamma A^*(I - P_Q)A)\bar{x}. \tag{1.7}$$

Therefore, finding the solution of the split feasibility problem (1.4) is equivalent to finding the minimization problem (1.5) with the minimum equal to zero if and only it is the minimum-norm of fixed point of the mapping $x \mapsto Tx = (I - \gamma A^*(I - P_Q)A)x$.

Thus, we study the general case of finding the minimum-norm fixed point of $\lambda-$strict pseudocontractive mapping $T : K \longrightarrow K$; that is, we find $x^* \in K$ which satisfies

$$x^* \in F(T) \text{ such that } ||x^*|| = \min\{||x|| : x \in F(T)\}. \tag{1.8}$$

In connection with the iterative approximation of the minimum-norm fixed point of nonexpansive mapping T, Yang *et.al* [12] introduced an implicit scheme given by

$$y_t = \beta T y_t + (1 - \beta)P_K[(1 - t)y_t], t \in (0, 1).$$

They proved that, under appropriate conditions on t and β, the path $\{y_t\}$ converges strongly to the minimum-norm fixed point of T, in real Hilbert spaces. Furthermore, they showed that an explicit scheme given by

$$x_{n+1} = \beta T x_n + (1 - \beta)P_K[(1 - \alpha_n)x_n], n \geq 1,$$

under appropriate conditions on $\{\alpha_n\}$ and β, converges strongly to the minimum-norm fixed point of T.

More recently, Yao and Xu [13] have also introduced and proved that the implicit scheme given by

$$y_t = P_K[(1 - t)T y_t], t \in (0, 1),$$

under appropriate conditions on t, converge strongly to the minimum-norm fixed point of nonexpansive self-mapping T. In addition, they showed that an explicit scheme given by

$$x_{n+1} = P_K[(1 - t_n)T x_n], n \geq 1,$$

under appropriate conditions on $\{t_n\}$, converges strongly to the minimum-norm fixed point of T.

A natural question arises whether we can extend the results of Yang et.al [12] *and Yao and Xu* [13] *to a class of mappings more general than nonexpansive mappings or not?*

Let K be a closed convex subset of a real Hilbert space H and let $T_i : K \to K, i = 1, 2, ..., N$ be λ_i-strictly pseudocontractive mapping.

It is our purpose in this paper to prove that for $\beta \in (0, 1)$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset K$ satisfying $y_t = P_K\big[(1 - t)(\beta T y_t + (1 - \beta)y_t)\big]$, which converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. Moreover, we provide an explicit iteration process which converges strongly to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. Finally, we also give a numerical example which support our results. Our theorems improve several results in this direction.

2. PRELIMINARIES AND NOTATIONS

In what follows we shall make use of the following lammas.

Lemma 2.1. [10] *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_K x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0, \forall z \in K.$$

Lemma 2.2. [15] *Let K be a closed and convex subset of a real Hilbert space H. Let $T_i : K \longrightarrow K, i = 1, 2, ..., N$, be λ_i-strictly pseudocontractive mappings with $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_N T_N$, where $\theta_1 + \theta_2 + ... + \theta_N = 1$. Then T is λ-strictly pseudocontractive with $\lambda := \min\{\lambda_i : i = 1, 2, ..., N\}$ and $F(T) = \cap_{i=1}^{N} F(T_i)$.*

Lemma 2.3. [11] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\longrightarrow\infty} \delta_n \leq 0$. Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.4. [18] *Let H be a real Hilbert space, K be a closed convex subset of H and $T : K \longrightarrow K$ be a $\lambda-$strictly pseudo-contractive mapping, then*
(i) $F(T)$ is closed convex subset of K;
(ii) $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in K such that $x_n \rightharpoonup x$ and $T x_n - x_n \longrightarrow 0$, as $n \longrightarrow \infty$, then $x = T(x)$.

Lemma 2.5. [19] *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \longrightarrow K$ be $\lambda-$strictly pseudocontractive mapping and $T_\beta x := \beta T x + (1 - \beta)x$. Then for $\beta \in (0, 2\lambda]$, and $x, y \in K$ we have that*

$$||T_\beta x - T_\beta y|| \leq ||x - y||.$$

Lemma 2.6. *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle.$$

3. MAIN RESULTS

We now prove the following theorem.

Theorem 3.1. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $T_i : K \longrightarrow K$ be λ_i- strictly pseudocontractive mapping with $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Then, for $\beta \in (0, 2\lambda]$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset K$ satisfying the following condition:*

$$y_t = P_K\big[(1 - t)(\beta T y_t + (1 - \beta) y_t)\big], \tag{3.1}$$

and the net $\{y_t\}$ converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$, where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_N$, for $\theta_1 + \theta_2 + ... + \theta_n = 1$ and $\lambda := \min\{\lambda_i : i = 1, 2, ..., N\}$.

Proof. For $\beta \in (0, 2\lambda]$ and each $t \in (0, 1)$, let $T_t(y) := P_K\big[(1 - t)(\beta T y + (1 - \beta) y)\big]$. Then using nonexpansiveness of P_K, Lemma 2.2 and Lemma 2.5 we have for $x, y \in K$ that

$$||T_t x - T_t y||^2 = ||P_K\big[(1 - t)(\beta T x + (1 - \beta) x)\big] - P_K\big[(1 - t)(\beta T y + (1 - \beta) y)\big]||^2$$

$$\begin{aligned} &\leq (1-t)||\beta(Tx-Ty)+(1-\beta)(x-y)||^2 \\ &\leq (1-t)||x-y||^2, \end{aligned} \tag{3.2}$$

and hence

$$||T_t x - T_t y|| \leq \sqrt{(1-t)}||x-y||.$$

Thus, we get that T_t is a contraction mapping on K and hence T_t has a unique fixed point, y_t, in K. This means that the equation

$$y_t := P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big]. \tag{3.3}$$

has a unique solution for each $t \in (0,1)$. Furthermore, since $F(T) \neq \emptyset$ and $\beta \in (0, 2\lambda]$, for $y^* \in F(T)$, we have from (3.1), convexity of $||.||^2$ and Lemma 2.5 that

$$\begin{aligned} ||y_t - y^*||^2 &= ||P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big] - y^*||^2 \\ &\leq ||(1-t)[\beta(Ty_t - y^*) + (1-\beta)(y_t - y^*)] - ty^*||^2 \\ &\leq ||(1-t)||\beta(Ty_t - y^*) + (1-\beta)(y_t - y^*)||^2 + t||y^*||^2 \\ &\leq (1-t)||y_t - y^*||^2 + t||y^*||^2 \end{aligned}$$

which implies that

$$||y_t - y^*|| \leq ||y^*||.$$

Therefore, $\{y_t\}$ and hence $\{Ty_t\}$ are bounded.

Furthermore, from (3.3) and the fact that P_K is nonexpansive we get that

$$\begin{aligned} ||y_t - Ty_t|| &= ||P_K\big[(1-t)(\beta T y_t + (1-\beta)y_t)\big] - P_K T y_t|| \\ &\leq ||(1-t)(1-\beta)(y_t - Ty_t) - tTy_t|| \\ &\leq (1-t)(1-\beta)||Ty_t - y_t|| + t||Ty_t||, \end{aligned}$$

which implies that

$$||y_t - Ty_t|| \leq \frac{t}{1-(1-\beta)(1-t)}||Ty_t|| \longrightarrow 0, \text{ as } t \longrightarrow 0^+. \tag{3.4}$$

Now, let $z_t := (1-t)(\beta T y_t + (1-\beta)y_t)$. Then from (3.4) we get that

$$||z_t - y_t|| \leq (1-t)\beta||Ty_t - y_t|| + t||y_t|| \longrightarrow 0, \text{ as } t \longrightarrow 0^+. \tag{3.5}$$

In addition, since $\{z_t\}$ is bounded there exists a sequence $\{t_n\} \subset (0,1)$ such that $z_{t_n} \rightharpoonup z$. Thus, from (3.5) we get that

$$y_{t_n} \rightharpoonup z, \text{ as } n \longrightarrow \infty, \tag{3.6}$$

and hence from (3.4) and Lemma 2.4 we have that $z \in F(T)$. Furthermore, from (3.3), nonexpansiveness of P_K and Lemma 2.5 we get that

$$\begin{aligned} ||y_t - y^*||^2 &\leq ||z_t - y^*||^2 \\ &= \langle (1-t)(\beta T y_t + (1-\beta)y_t) - y^*, z_t - y^* \rangle \\ &= (1-t)\langle \beta(Ty_t - Ty^*) + (1-\beta)(y_t - y^*), z_t - y^* \rangle \\ &\quad + t\langle -y^*, z_t - y^* \rangle \\ &\leq (1-t)||y_t - y^*||.||z_t - y^*|| + t\langle -y^*, z_t - y^* \rangle. \\ &\leq (1-t)||z_t - y^*||^2 + t\langle -y^*, z_t - y^* \rangle. \end{aligned}$$

This implies that

$$||z_t - y^*||^2 \leq \langle y^*, y^* - z_t \rangle. \tag{3.7}$$

Since $z \in F(T)$, substituting z for y^* and t_n for t in (3.7) we get that

$$z_{t_n} \longrightarrow z. \tag{3.8}$$

Thus, from (3.7) and (3.8) we have that

$$||z - y^*||^2 \quad \leq \quad \langle y^*, y^* - z\rangle, \text{ as } n \longrightarrow \infty \ ,$$

which is equivalent to the inequality

$$\langle z, y^* - z\rangle \geq 0 \text{ and hence } z = P_F 0.$$

If there is another subsequence $\{z_{t_m}\}$ of $\{z_t\}$ such that $z_{t_m} \rightharpoonup z'$, similar argument gives that $z' = P_F 0$, which implies, by uniqueness of $P_F 0$, that $z' = z$. Therefore, the net $z_t \longrightarrow z = P_F 0$ and hence from (3.5) the net $y_t \longrightarrow z = P_F 0$, which is the minimum-norm fixed point of T. Therefore, from Lemma 2.2, $\{y_t\}$ converges strongly, as $t \longrightarrow 0$, to the common minimum-norm fixed point of $T_i, i = 1, 2, ..., N$. The proof is complete. □

If we assume $T_i's, i = 1, ..., N$ to be nonexpansive mappings we get the following corollary.

Corollary 3.1. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $T_i : K \longrightarrow K$ be a finite family of nonexpansive mapping with $\cap_{i=1}^N F(T_i) \neq \emptyset$. Then, for $\beta \in (0, 1)$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset K$ satisfying the following condition:*

$$y_t = P_K\big[(1 - t)(\beta T y_t + (1 - \beta)y_t)\big],$$

and the net $\{y_t\}$ converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$, where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_N$, for $\theta_1 + \theta_2 + ... + \theta_n = 1$.

If in Theorem 3.1, we consider $\{\beta_n\} \subset (a, 2\lambda]$, for some $a > 0$, and $\{t_n\} \subset (0, 1)$ such that $t_n \longrightarrow 0$, and $y_n := y_{t_n}$ the method of proof of Theorem 3.1 provides the following corollary.

Corollary 3.2. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $T_i : K \longrightarrow K$ be λ_i-strictly pseudocontractive mapping with $\cap_{i=1}^N F(T) \neq \emptyset$. Then, for $\beta_n \subset (a, 2\lambda]$, for some $a > 0$, and $\{t_n\} \subset (0, 1)$, there exists a sequence $\{y_{t_n}\} \subset K$ satisfying the following condition:*

$$y_{t_n} = P_K\big[(1 - t_n)(\beta_n T y_{t_n} + (1 - \beta_n)y_{t_n})\big], \tag{3.9}$$

and the net $\{y_{t_n}\}$ converges strongly, as $t_n \longrightarrow 0$, to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$, where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_N$, for $\theta_1 + \theta_2 + ... + \theta_n = 1$ and $\lambda := \min\{\lambda_i : i = 1, 2, ..., N\}$.

We now state and prove a convergence theorem for the common minimum-norm zero of finite family of $\alpha-$inverse strongly monotone mappings.

Theorem 3.2. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $A_i : K \longrightarrow H$ be α_i- inverse strongly monotone mapping satisfying $(I - A)(K) \subseteq K$ and $\cap_{i=1}^N N(A_i) \neq \emptyset$. Then, for $\beta \in (0, 2\alpha]$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset H$ satisfying the following condition:*

$$y_t = P_K\big[(1 - t)(\beta(I - A)y_t + (1 - \beta)y_t)\big], \tag{3.10}$$

and the net $\{y_t\}$ converges strongly, as $t \longrightarrow 0^+$, to the common minimum-norm zero of $\{A_i : i = 1, 2, ..., N\}$, where $A := \theta_1 A_1 + \theta_2 A_2 + ... + \theta_n A_N$, for $\theta_1 + \theta_2 + ... + \theta_n = 1$ and $\alpha := \min\{\alpha_i : i = 1, 2, ..., N\}$.

Proof. For $x \in K$, let $T_i(x) = (I - A_i)x$, for $i = 1, 2, ..., N$. Then, we get that each T_i is α_i- strictly pseudocontractive self- mapping with $\cap_{i=1}^N N(A_i) = \cap_{i=1} NF(T_i)$. Now, replacing A_i with $(I - T_i)$ we get that scheme (3.10) reduces to (3.1) and hence the conclusion follows from Theorem 3.1. □

Now, we prove strong convergence of an explicit scheme for a common minimum-norm fixed point of λ-strictly pseudocontractive mappings.

Theorem 3.3. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \le i \le N$, let $T_i : K \to K$ be λ_i-strictly pseudocontractive mapping with $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = P_K\big[(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n)\big], n \ge 1, \tag{3.11}$$

where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_n$, for $\theta_1 + \theta_2 + ... + \theta_N = 1$ and $\{\alpha_n\} \subset (0,1)$, $\beta_n \in [a, 2\lambda]$, for some $a > 0$ and $\lambda := \min\{\lambda_i, i = 1, 2, ..., N\}$ satisfying $\lim_{n\longrightarrow\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Then, $\{x_n\}$ converges strongly to the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$.

Proof. We note that by Lemma 2.2 we have that T is λ-strictly pseudocontractive and $F(T) = \cap_{n=1}^N F(T_i)$. Let $x^* \in F(T)$. Then from (3.11), using nonexpansiveness of P_K, convexity of $||.||^2$, $\beta_n \in [a, 2\lambda]$ and Lemma 2.5 we get that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||P_K\big[(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n)\big] - P_K x^*||^2 \\ &\le ||(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n) - x^*||^2 \\ &= ||(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n - x^*) - \alpha_n x^*||^2 \\ &\le (1-\alpha_n)||(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - Tx^*)||^2 + \alpha_n||x^*||^2 \\ &\le (1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x^*||^2. \end{aligned}$$

Thus, by induction we obtain that

$$||x_{n+1} - x^*||^2 \le \max\{||x_0 - x^*||^2, ||x^*||^2\}.$$

Consequently, $\{x_n\}$ and hence $\{Tx_n\}$ are bounded. Furthermore, from (3.11) we have that

$$\begin{aligned} ||x_{n+2} - x_{n+1}|| &= ||(1-\alpha_{n+1})(\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}) \\ &\quad -(1-\alpha_n)(\beta_n T x_n + (1-\beta_n)x_n)|| \\ &= ||(1-\alpha_{n+1})(\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}) \\ &\quad -(1-\alpha_n)(\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}) \\ &\quad +(1-\alpha_n)(\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}) \\ &\quad -(1-\alpha_n)(\beta_n T x_n + (1-\beta_n)x_n)|| \\ &\le |\alpha_{n+1} - \alpha_n| ||\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1}|| \\ &\quad +(1-\alpha_n)||\beta_{n+1} T x_{n+1} + (1-\beta_{n+1})x_{n+1} \\ &\quad -(1-\alpha_n)(\beta_n T x_n + (1-\beta_n)x_n)|| \\ &\le M|\alpha_{n+1} - \alpha_n| + (1-\alpha_n)||T_{\beta_{n+1}} x_{n+1} - T_{\beta_n} x_n||, \end{aligned} \tag{3.12}$$

for some $M > 0$, where $T_{\beta_n} x_n := \beta_n T x_n + (1-\beta_n)x_n$. Furthermore, we have that

$$\begin{aligned} ||T_{\beta_{n+1}} x_{n+1} - T_{\beta_n} x_n|| &\le ||T_{\beta_{n+1}} x_{n+1} - T_{\beta_{n+1}} x_n|| + ||T_{\beta_{n+1}} x_n - T_{\beta_n} x_n|| \\ &\le ||x_{n+1} - x_n|| + ||\beta_{n+1} T x_n + (1-\beta_{n+1})x_n \\ &\quad -(\beta_n T x_n + (1-\beta_n)x_n)|| \end{aligned}$$

$$\begin{aligned} &\leq ||x_{n+1} - x_n|| + |\beta_{n+1} - \beta_n|||x_n - Tx_n|| \\ &\leq ||x_{n+1} - x_n|| + M'|\beta_{n+1} - \beta_n|, \end{aligned} \tag{3.13}$$

for some $M' > 0$. Then putting (3.13) into (3.12) we obtain that

$$||x_{n+2} - x_{n+1}|| \leq (1 - \alpha_n)||x_{n+1} - x_n|| + |\beta_{n+1} - \beta_n|M'' + |\alpha_{n+1} - \alpha_n|M'',$$

for some $M'' > 0$, and hence using the assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$ and following the method in [6] we get that

$$\lim_{n \longrightarrow \infty} ||x_{n+1} - x_n|| = 0. \tag{3.14}$$

Let $y_n = \beta_n Tx_n + (1 - \beta_n)x_n$. Then we observe that $\{y_n\}$ is bounded and from (3.11) and the fact that $\alpha_n \longrightarrow 0$, as $n \longrightarrow 0$, we obtain that

$$\begin{aligned} ||x_{n+1} - y_n|| &= ||P_K\big[(1 - \alpha_n)(\beta_n Tx_n + (1 - \beta_n)x_n)\big] - P_K y_n|| \\ &\leq ||(1 - \alpha_n)(\beta_n Tx_n + (1 - \beta_n)x_n) - y_n|| \\ &\leq \alpha_n||y_n|| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \end{aligned} \tag{3.15}$$

Thus, from (3.14) and (3.15) we get that

$$||y_n - x_n|| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.16}$$

Consequently,

$$\lim_{n \longrightarrow \infty} ||x_n - Tx_n|| = \lim_{n \longrightarrow \infty} \frac{||x_n - y_n||}{\beta_n} = 0. \tag{3.17}$$

Now, let $z_n = (1 - \alpha_n)(\beta_n Tx_n + (1 - \beta_n)x_n)$ and $\tilde{x} := P_F 0$. Then we have that $\{z_n\}$ is bounded. Furthermore, (3.17) and the assumption that $\alpha_n \longrightarrow 0$, as $n \longrightarrow \infty$, give that

$$z_n - x_n = (1 - \alpha_n)[\beta_n(Tx_n - x_n) - \alpha_n x_n] \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{3.18}$$

Let $\{z_{n_k}\}$ be a subsequence of $\{z_n\}$ such that

$$\limsup_{n \longrightarrow \infty} \langle \tilde{x}, z_n - \tilde{x} \rangle = \lim_{k \longrightarrow \infty} \langle \tilde{x}, z_{n_k} - \tilde{x} \rangle,$$

and $z_{n_k} \rightharpoonup z$. Then, from (3.18) we have that $x_{n_k} \rightharpoonup z$. Thus, from (3.17) and Lemma 2.4 we get that $z \in F(T)$. Therefore, by Lemma 2.1 we obtain that

$$\limsup_{n \longrightarrow \infty} \langle \tilde{x}, z_n - \tilde{x} \rangle = \lim_{k \longrightarrow \infty} \langle \tilde{x}, z_{n_k} - \tilde{x} \rangle = \langle \tilde{x}, z - \tilde{x} \rangle \geq 0. \tag{3.19}$$

Now, we show that $x_{n+1} \longrightarrow \tilde{x}$, as $n \longrightarrow \infty$. But from (3.11), Lemma 2.5 and Lemma 2.6 we have that

$$\begin{aligned} ||x_{n+1} - \tilde{x}||^2 &= ||P_K\big[(1 - \alpha_n)((1 - \beta_n)x_n + \beta_n Tx_n)\big] - P_K\tilde{x}||^2 \\ &\leq ||(1 - \alpha_n)((1 - \beta_n)x_n + \beta_n Tx_n) - \tilde{x}||^2 \\ &= ||(1 - \alpha_n)((1 - \beta_n)(x_n - \tilde{x}) + \beta_n(Tx_n - T\tilde{x})) - \alpha_n\tilde{x}||^2 \\ &\leq (1 - \alpha_n)||((1 - \beta_n)(x_n - \tilde{x}) + \beta_n(Tx_n - T\tilde{x})||^2 + 2\alpha_n\langle -\tilde{x}, z_n - \tilde{x} \rangle \\ &\leq (1 - \alpha_n)||x_n - \tilde{x}||^2 + 2\alpha_n\langle -\tilde{x}, z_n - \tilde{x} \rangle. \end{aligned} \tag{3.20}$$

Therefore, this with (3.19) and Lemma 2.3 give that $x_n \longrightarrow \tilde{x}$, as $n \longrightarrow \infty$, which is the minimum-norm fixed point of T and hence, by Lemma 2.2, the common minimum-norm fixed point of $\{T_i : i = 1, 2, ..., N\}$. □

If we take $T_i's, i = 1, ..., N$ to be nonexpansive mappings we get the following corollary.

Corollary 3.3. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $1 \leq i \leq N$, let $T_i : K \to K$ be a finite family of nonexpansive mapping with $\cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in K$ by*

$$x_{n+1} = P_K\big[(1-\alpha_n)((1-\beta_n)x_n + \beta_n T x_n)\big], n \geq 1,$$

where $T := \theta_1 T_1 + \theta_2 T_2 + ... + \theta_n T_n$, for $\theta_1 + \theta_2 + ... + \theta_N = 1$ and $\{\alpha_n\} \subset (0,1)$, $\beta_n \in [a,1)$, for some $a > 0$ and satisfying $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Then, $\{x_n\}$ converges strongly to the common minimum-norm fixed point of $\{T_i, i = 1,2,...,N\}$.

The following corollary is implied from Theorem 3.3.

Theorem 3.4. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $A_i : K \to H$ be α_i-inverse strongly monotone mapping satisfying $(I-A)(K) \subseteq K$ and $\cap_{i=1}^{N} N(A_i) \neq \emptyset$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in H$ by*

$$x_{n+1} = P_K\big[(1-\alpha_n)((1-\beta_n)x_n + \beta_n (I-A) x_n)\big], n \geq 1, \tag{3.21}$$

where $A := \theta_1 A_1 + \theta_2 A_2 + ... + \theta_n A_n$, for $\theta_1 + \theta_2 + ... + \theta_N = 1$ and $\{\alpha_n\} \subset (0,1)$, $\beta_n \in [a, 2\alpha]$, for some $a > 0$ and $\alpha := \min\{\alpha_i, i = 1,2,...,N\}$ satisfying $\lim_{n \longrightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$. Then, $\{x_n\}$ converges strongly to the common minimum-norm zero of $\{A_i : i = 1,2,...,N\}$, as $n \to \infty$.

Proof. The method of proof of Theorem 3.2 and using Theorem 3.3 provide the required assertion.

□

4. Numerical example

Now, we give an example of a finite family of λ-strictly pseudocontractive mappings satisfying Theorem 3.3 and some numerical experiment results to explain the conclusion of Theorem 3.3 as follows:

Example 4.1. Let $H = \mathbb{R}$ with absolute value norm. Let $K = [0,1]$ and $T_1, T_2 : K \longrightarrow K$ be defined by

$$T_1 x := \begin{cases} x + (x - \frac{1}{2})^2, & x \in [0, \frac{1}{2}], \\ x, & x \in (\frac{1}{2}, 1], \end{cases} \tag{4.1}$$

and

$$T_2 x := \begin{cases} x, & x \in [0, \frac{2}{3}], \\ x - (x - \frac{2}{3})^2, & x \in (\frac{2}{3}, 1], \end{cases} \tag{4.2}$$

Then we first show that T_1 is λ-strictly pseudocontractive mapping with $\lambda = \frac{1}{2}$. If $x, y \in [0, \frac{1}{2}]$ then

$$\begin{aligned} \langle (I-T_1)x - (I-T_1)y, x-y \rangle &= \langle -(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2, x-y \rangle \\ &= \big[(x-\frac{1}{2})^2 - (y-\frac{1}{2})^2\big](y-x) \\ &= \big[(x-\frac{1}{2})^2 - (y-\frac{1}{2})^2\big]\big[(y-\frac{1}{2}) - (x-\frac{1}{2})\big] \end{aligned}$$

$$\begin{aligned}
&= \big[(x-\frac{1}{2})^2-(y-\frac{1}{2})^2\big]\frac{\big[(y-\frac{1}{2})^2-(x-\frac{1}{2})^2\big]}{(y-\frac{1}{2})+(x-\frac{1}{2})}\\
&= \big[(x-\frac{1}{2})^2-(y-\frac{1}{2})^2\big]\frac{\big[(x-\frac{1}{2})^2-(y-\frac{1}{2})^2\big]}{(\frac{1}{2}-x)+(\frac{1}{2}-y)}\\
&\geq \frac{1}{2}\big|(x-\frac{1}{2})^2-(y-\frac{1}{2})^2\big|^2\\
&= \frac{1}{2}\big|(I-T_1)x-(I-T_1)y\big|^2.
\end{aligned}$$

If $x\in[0,\frac{1}{2}]$ and $y\in(\frac{1}{2},1]$ we get that

$$\begin{aligned}
\langle (I-T_1)x-(I-T_1)y, x-y\rangle &= \langle -(x-\frac{1}{2})^2, x-y\rangle = (x-\frac{1}{2})^2(y-x)\\
&= (x-\frac{1}{2})^2\big[(y-\frac{1}{2})-(x-\frac{1}{2})\big]\\
&\geq (x-\frac{1}{2})^2(\frac{1}{2}-x)\\
&\geq (x-\frac{1}{2})^2\frac{(\frac{1}{2}-x)^2}{(\frac{1}{2}-x)} \geq \frac{1}{2}|(x-\frac{1}{2})^2|^2\\
&= \frac{1}{2}\big|(I-T_1)x-(I-T_1)y\big|^2.
\end{aligned}$$

If $x,y\in(\frac{1}{2},1]$ then we get that $|(I-T_1)x-(I-T_1)y|=0$ and hence

$$\langle (I-T_1)x-(I-T_1)y, x-y\rangle \geq \frac{1}{2}|(I-T_1)x-(I-T_1)y|^2.$$

Therefore, T_1 is $\lambda-$ strictly pseudocontractive mapping with $\lambda=\frac{1}{2}$ and $F(T_1)=[\frac{1}{2},1]$.

Similarly, we can show that T_2 is $\lambda-$ strictly pseudocontractive mapping with $\lambda=\frac{1}{2}$ and $F(T_2)=[0,\frac{2}{3}]$.

Thus, if $Tx:=\theta T_1x+(1-\theta)T_2x$, where $\theta=\frac{1}{2}$, then T is given by

$$Tx=\begin{cases} x+\frac{1}{2}(x-\frac{1}{2})^2, x\in[0,\frac{1}{2}],\\ x, \qquad\qquad x\in(\frac{1}{2},\frac{2}{3}],\\ x-\frac{1}{2}(x-\frac{2}{3})^2, x\in(\frac{2}{3},1],\end{cases} \tag{4.3}$$

which is $\lambda-$ strictly pseudocontractive mapping with $\lambda=\frac{1}{2}$ and $F(T)=[\frac{1}{2},\frac{2}{3}]=F(T_1)\cap F(T_2)$. Now, taking $\alpha_n=\frac{1}{n+1}$, $\beta_n=\frac{1}{100}+\frac{1}{n+1}$, scheme (3.11) reduces to

$$x_{n+1}=P_K\Big[(1-\frac{1}{n+1})\Big((1-(\frac{1}{100}+\frac{1}{n+1}))x_n+(\frac{1}{100}+\frac{1}{n+1})Tx_n\Big)\Big], n\geq 1. \tag{4.4}$$

Therefore, by Theorem 3.3, the sequence $\{x_n\}$ in (4.4) converges strongly to $\frac{1}{2}$, the common minimum norm fixed point of T_1 and T_2.

Next, we show the numerical experiment result tables using software Mathlab 7.5 for the iteration process of the sequence $\{x_n\}$ with initial point $x_1=0.2$ and $x_1=0.7$, respectively.

n	1	200	600	1000	1200	1400	1600	1800	2000
x_n	0.200	0.386	0.432	0.446	0.451	0.457	0.457	0.460	0.462

n	1	200	600	1000	1200	1400	1600	1800	2000
x_n	0.700	0.386	0.432	0.446	0.451	0.454	0.457	0.460	0.462

Remark 4.2. Theorem 3.1 improves Theorem 3.1 of Yang *et.al* [12] and Yao and Xu [13] to a more general class of finite family of $\lambda-$strictly pseudocontractive mappings. Moreover, Theorem 3.3 improves Theorem 3.2 of Yang *et.al* [12] and Yao and Xu [13] in the sense that our scheme provides a minimum-norm fixed point of finite family of $\lambda-$strict pseudocontractive mapping T.

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SOME FIXED POINT RESULTS FOR UNIFORMLY QUASI-LIPSCHITZIAN MAPPINGS IN CONVEX METRIC SPACES

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ABSTRACT. In this paper, an iteration process for approximating common fixed points of two uniformly quasi Lipschitzian mappings in convex metric spaces is defined. Without using "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n-1)<\infty$ associated with asymptotically (quasi-)nonexpansive mappings, some convergence theorems are also proved. The results presented generalize, improve and unify some recent results.

KEYWORDS: Uniformly quasi-Lipschitzian mappings; Common fixed points; Convex metric spaces.
AMS Subject Classification: 47H09 65J15.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers. We also denote by $F(T)$ the set of fixed points of T and by $F=F(T)\cap F(S)$ the set of common fixed points of two mappings T and S.

Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be asymptotically nonexpansive, if there exists a sequence $k_n\in[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ such that

$$d(T^nx,T^ny)\le k_nd(x,y),\quad \forall x,y\in X,\ n\in\mathbb{N}.$$

If $F(T)\neq\emptyset$, then T is said to be asymptotically quasi-nonexpansive, if there exists $k_n\in[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ such that

$$d(T^nx,p)\le k_nd(x,p),\quad \forall x\in X,\ p\in F(T),\ n\in\mathbb{N}.$$

T is said to be uniformly quasi-Lipschitzian, if there exists a constant $L>0$ (called Lipschitz constant) such that

$$d(T^nx,p)\le Ld(x,p),\quad \forall x\in X,\ p\in F(T),\ n\in\mathbb{N}.$$

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Remark 1.1. If $F(T) \neq \emptyset$, it follows from the above definitions that each asymptotically nonexpansive mapping must be an asymptotically quasi-nonexpansive, and each asymptotically quasi-nonexpansive mapping must be a uniformly quasi-Lipschitzian, where $L = \sup_{n\geq 0}\{k_n\} < \infty$. But the converse may not necessarily hold.

The approximation problems concerned with the fixed points of the asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been studied extensively by many authors in recent years. Takahashi [4] introduced the notion of a convex metric space and studied the fixed point theory for nonexpansive mappings in such a setting. A normed linear space is a special example of a convex metric space. But there are many examples of convex metric spaces which are not embedded in any normed linear space (see [4]). Later on, Tian [5] gave some sufficient and necessary conditions such that Ishikawa iteration process for an asymptotically quasi-nonexpansive mapping converges to a fixed point in a convex metric space. Liu et al. [3] and Wang and Liu [6] gave some sufficient and necessary conditions for Ishikawa iteration process with errors to approximate common fixed points of two uniformly quasi-Lipschitzian mappings in a convex metric space. Also, Chang et al. [1], Khan and Abbas [2], Yildirim and Khan [8] and other authors have studied fixed point theorems in convex metric spaces.

We recall the following which can be found in [5].

Let (X, d) be a metric space.

- A mapping $W : X^3 \times [0,1]^3 \to X$ is said to be a convex structure on X, if it satisfies the following condition: for any $(x, y, z; a, b, c) \in X^3 \times [0,1]^3$ with $a + b + c = 1$, and $u \in X$:
$$d(W(x, y, z; a, b, c), u) \leq ad(x, u) + bd(y, u) + cd(z, u).$$
- If (X, d) is a metric space with a convex structure W, then (X, d) is called a convex metric space.
- Let (X, d) be a convex metric space, a nonempty subset E of X is said to be convex, if $W(x, y, z; a, b, c) \in E$, $\forall (x, y, z) \in E^3$, $(a, b, c) \in [0,1]^3$ with $a + b + c = 1$.

Recently, Wang and Liu [6] considered the following iteration process for uniformly quasi-Lipschitzian mappings S and T in convex metric spaces:

$$\begin{aligned} x_{n+1} &= W(x_n, S^n y_n, u_n; a_n, b_n, c_n), \\ y_n &= W\left(x_n, T^n x_n, v_n; a_n^{'}, b_n^{'}, c_n^{'}\right) \end{aligned} \tag{1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}, \{c_n'\}$ are six sequences in $[0,1]$ with $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$, $n \in \mathbb{N}$ and $\{u_n\}, \{v_n\}$ are two sequences in X satisfying condition: For any nonnegative integers $n, m, 0 \leq n < m$, if $\delta(A_{nm}) > 0$, then

$$\max_{n\leq i,j\leq m}\{d(x, y) : x \in \{u_i, v_i\}, y \in \{x_j, y_j, Sy_j, Tx_j, u_j, v_j\}\} < \delta(A_{nm}),$$

where $A_{nm} = \{x_i, y_i, Sy_i, Tx_i, u_i, v_i : n \leq i \leq m\}$,

$$\delta(A_{nm}) = \sup_{x,y\in A_{nm}} d(x, y).$$

They also proved convergence of the iteration process (1.1) to a common fixed point of S and T.

Motivated by the above studies, we introduce, in this paper, an iteration process to approximate common fixed points for two uniformly quasi-Lipschitzian mappings as follows:

Let (X, d) be a convex metric space with a convex structure W. Let $S, T : X \to X$ be uniformly quasi-Lipschitzian mappings with respective Lipschitz constants $L_1 > 0$ and $L_2 > 0, \{a_n\}, \{b_n\}, \{c_n\}$ be three sequences in $[0, 1]$ with $a_n + b_n + c_n = 1$, $n \in \mathbb{N}$. For any given $x_0 \in X$, define a sequence $\{x_n\}$ as follows:

$$x_{n+1} = W(x_n, S^n x_n, T^n x_n; a_n, b_n, c_n). \tag{1.2}$$

While acknowledging the process (1.1) due to Wang and Liu, we underscore that our process

- is independent of (1.1) due to Wang and Liu : none reduces to the other.
- is one-step process as compared with the two-step process (1.1) and still able to compute common fixed points.
- being one-step process is simpler than (1.1).

Having introduced this process, we use it to prove some strong convergence results for quasi-Lipschitzian mappings. Moreover, as opposed to Wang and Liu [6], some convergence theorems are proved for asymptotically (quasi-)nonexpansive mappings without using "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ associated with such mappings.

In order to prove our main results, the following lemma will be needed:

Lemma 1.2. [9] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of non-negative numbers such that*

$$a_{n+1} \le (1 + b_n) a_n, \quad n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

2. MAIN RESULTS

In what follows, we take $L = \max\{L_1, L_2\}$ where $L_1 > 0$ and $L_2 > 0$ are Lipschitz constants of the quasi-Lipschitzian mappings S and T respectively.

Theorem 2.1. *Let (X, d) be a convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be uniformly quasi-Lipschitzian mappings. Let the sequence $\{x_n\}$ be as in (1.2) with the sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

If $F \neq \emptyset$, then:

(1) *for all $p \in F$ and for each $n \in \mathbb{N}$,*

$$d(x_{n+1}, p) \le (1 + L(1 - a_n)) d(x_n, p),$$

(2) *there exists a constant $M > 0$ such that, for all $n, m \in \mathbb{N}$ and for every $p \in F$,*

$$d(x_{n+m}, p) \le M d(x_n, p).$$

Proof. (1) For any $p \in F$, from (1.2), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(x_n, S^n x_n, T^n x_n; a_n, b_n, c_n), p) \\ &\le a_n d(x_n, p) + b_n d(S^n x_n, p) + c_n d(T^n x_n, p) \\ &\le a_n d(x_n, p) + b_n L_1 d(x_n, p) + c_n L_2 d(x_n, p) \\ &\le (a_n + b_n L + c_n L) d(x_n, p) \end{aligned}$$

$$\leq \quad (1 + L(1 - a_n))\, d(x_n, p). \tag{2.1}$$

This completes the proof of (1).

(2) It is well known that $1 + x \leq e^x$ for all $x \geq 0$. Using it for the inequality (2.1), we have

$$\begin{aligned} d(x_{n+m}, p) &\leq (1 + L(1 - a_{n+m-1}))\, d(x_{n+m-1}, p) \\ &\leq e^{L(1-a_{n+m-1})} d(x_{n+m-1}, p) \\ &\leq e^{L(1-a_{n+m-1})} \left[(1 + L(1 - a_{n+m-2}))\, d(x_{n+m-2}, p)\right] \\ &\leq e^{L[(1-a_{n+m-1})+(1-a_{n+m-2})]} d(x_{n+m-2}, p) \\ &\;\;\vdots \\ &\leq M d(x_n, p), \end{aligned} \tag{2.2}$$

where $M = e^{L \sum_{k=0}^{\infty}(1-a_k)}$. This completes the proof of (2). □

Now we give the main theorems of this paper. Our first theorem deals with uniformly quasi-Lipschitzian mappings.

Theorem 2.2. *Let (X, d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be uniformly quasi-Lipschitzian mappings and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty} (1 - a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.

Proof. The necessity is obvious. Thus, we will only prove the sufficiency. From Theorem 2.1, we have

$$d(x_{n+1}, F) \leq (1 + L(1 - a_n))\, d(x_n, F).$$

As $\sum_{n=0}^{\infty} (1 - a_n) < \infty$, therefore $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 1.2. But by hypothesis, $\liminf_{n\to\infty} d(x_n, F) = 0$, therefore we must have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F) = 0$, so for each $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, F) < \frac{\varepsilon}{M+1} \quad \forall n \geq n_1. \tag{2.3}$$

Thus, there exists $p_1 \in F$ such that

$$d(x_n, p_1) < \frac{\varepsilon}{M+1} \quad \forall n \geq n_1. \tag{2.4}$$

From (2.2) and (2.4), we obtain

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(x_n, p_1) \\ &\leq M d(x_n, p_1) + d(x_n, p_1) \\ &= (M+1)\, d(x_n, p_1) \\ &< (M+1)\left(\frac{\varepsilon}{M+1}\right) \\ &= \varepsilon, \end{aligned}$$

for all $n, m \geq n_1$. Hence $\{x_n\}$ is a Cauchy sequence in closed convex subset E of the complete metric space X, therefore, it must converge to a point of E. Suppose $\lim_{n\to\infty} x_n = p$; we prove that $p \in F$. To this end, we only need to prove that F is closed because

$$d(p, F) = \lim_{n\to\infty} d(x_n, F) = 0. \tag{2.5}$$

Let $p_n \in F$ be a sequence such that $\lim_{n\to\infty} p_n = p^*$. We show that $p^* \in F$. In fact,

$$\begin{aligned} d(Sp^*, p^*) &\leq d(p^*, p_n) + d(Sp^*, p_n) \\ &\leq d(p^*, p_n) + Ld(p^*, p_n) \\ &= (1 + L)\, d(p^*, p_n) \end{aligned}$$

yields that $d(Sp^*, p^*) = 0$. Similarly, $d(Tp^*, p^*) = 0$. Thus $p^* \in F$ and so F is closed. Thus by (2.5), $p \in F$. This completes the proof. □

In the following results concerned with asymptotically (quasi-)nonexpansive mappings, we do not need "the rate of convergence condition" $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ associated with such type of mappings.

Theorem 2.3. *Let (X, d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be asymptotically quasi-nonexpansive mappings with sequences $\{k_n\}$ and $\left\{k_n^{'}\right\}$ (without the conditions $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=0}^{\infty}\left(k_n^{'} - 1\right) < \infty$), and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. $\{k_n\}, \left\{k_n^{'}\right\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = \lim_{n\to\infty} k_n^{'} = 1$; therefore there exist $L_1 > 0$ and $L_2 > 0$ such that $L_1 = \sup_{n\geq 0}\{k_n\} < \infty$ and $L_2 = \sup_{n\geq 0}\left\{k_n^{'}\right\} < \infty$. In this case, S and T are uniformly quasi-Lipschitzian mappings with $L_1 > 0$ and $L_2 > 0$. Hence, Theorem 2.3 can be proven by Theorem 2.2. □

Theorem 2.4. *Let (X, d) be a complete convex metric space, E be a nonempty closed convex subset of X and $S, T : E \to E$ be asymptotically nonexpansive mappings with sequences $\{k_n\}$ and $\left\{k_n^{'}\right\}$ (without the conditions $\sum_{n=0}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=0}^{\infty}\left(k_n^{'} - 1\right) < \infty$), and $F \neq \emptyset$. Suppose that $\{x_n\}$ is the iteration process defined by (1.2), and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $[0, 1]$ satisfying*

$$a_n + b_n + c_n = 1 \text{ and } \sum_{n=0}^{\infty}(1 - a_n) < \infty.$$

Then $\{x_n\}$ converges to a fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Remark 2.1. All the results proved in this paper can also be proved for the iteration process with error terms. In this case our main iteration process (1.2) looks like

$$x_{n+1} = W(x_n, S^n x_n, T^n x_n, u_n; a_n, b_n, c_n, d_n), \tag{2.6}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n + d_n = 1$, $n \in \mathbb{N}$.

Remark 2.2. (i) From computational point of view, our iteration processes (1.2) and (2.6) are simpler than iteration processes of Chang et al. [1], Liu et al. [3], Wang and Liu [6].

(ii) Our results also generalize results of Yao et al. [7] to two uniformly quasi-Lipschitzian mappings in convex metric spaces.

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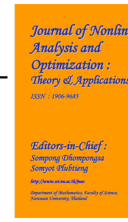

BANACH FIXED POINT THEOREM IN APPLICATION TO NONLINEAR ELLIPTIC SYSTEMS

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ABSTRACT. This paper is concerned with some nonlinear elliptic systems. Under suitable conditions on the nonlinearities f and g, we obtain weak solution in Sobolev space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ by applying the Banach fixed point theorem.

KEYWORDS: Weak solution; Nonlinear elliptic system; The Laplace operator; Banach fixed point theorem.
AMS Subject Classification: 35B30 35J60 35P15

1. INTRODUCTION

We study the nonlinear elliptic system of the form:

$$\begin{cases} -\Delta u - \operatorname{div}(h_1(|\nabla u|^2)\nabla u) = \lambda f(x,u,v) & \text{in } \Omega \\ -\Delta v - \operatorname{div}(h_2(|\nabla v|^2)\nabla v) = \lambda g(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth open set in $\mathbb{R}^N$, $(N \geq 3)$, λ is a real number, $-\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u, and $h_1, h_2 \in C(\mathbb{R}, \mathbb{R})$.

Theorems concerning the existence and properties of fixed point are known as fixed-point theorems. Such theorems are the most important tools for proving the existence and uniqueness of the solutions to various mathematical models(differential, integral and partial differential equations, and variational inequalities, etc.) representing phenomena arising in different fields, such as steady-state temperature distribution, chemical reactions, neutron transport theory, economic theories, epidemics and flow of fluids. They are also used to study the problems of optimal control related to this systems. For details, one can refer to [1, 3].

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In recent years, many publications have appeared concerning quasilinear elliptic systems which have been used in a great variety of applications, we refer the readers to [5, 6, 7, 8, 9] and the references therein. J. Zhang and Z. Zhang [11] used variational methods to obtain weak solution of semilinear elliptic system and quasilinear elliptic system.

Motivated by [4], in this paper, we will discuss problem (1.1). Under the suitable condition on the nonlinearities $f(x,u,v)$ and $g(x,u,v)$, using Banach fixed point theorem (see [10]), we show that system (1.1) has a unique weak solutions.

Throughout this paper for $(u,v) \in \mathbb{R}^2$, denote $|(u,v)|^2 = |u|^2 + |v|^2$. We assume that f and g are L^2 functions which are Lipschitz continuous with respect to the second variable, i.e., there exist constants $c_1, c_2 > 0$ such that for a.e. $x \in \Omega$ and for any $(u,v), (u_1,v_1) \in \mathbb{R}^2$

$$|f(x,u,v) - f(x,u_1,v_1)| \le c_1|(u,v) - (u_1,v_1)| \tag{1.2}$$

$$|g(x,u,v) - g(x,u_1,v_1)| \le c_2|(u,v) - (u_1,v_1)|. \tag{1.3}$$

We assume that h_1 and h_2 are the continuous and nondecreasing functions satisfying the following growth conditions:
There exist β_1, β_2 and $M_1, M_2 > 0$ such that

$$0 < h_1(t) \le \beta_1 \; 0 < h_2(t) \le \beta_2. \tag{1.4}$$

and

$$|h_1'(t)| \le \frac{M_1}{1+t} \; |h_2'(t)| \le \frac{M_2}{1+t} \tag{1.5}$$

for all $t \in R$.

Let λ_1 be the first eigenvalue of Dirichlet problem $-\Delta u = \lambda u$. The main result of this paper is as follows:

Definition 1.1. We say that $(u,v) \in H$ is a weak solution of (1.1) if

$$\int_\Omega [\nabla u \nabla \xi + \nabla v \nabla \eta + h_1(|\nabla u|^2)\nabla u \nabla \xi + h_2(|\nabla v|^2)\nabla v \nabla \eta] \, dx$$
$$-\lambda \int_\Omega [f(x,u,v)\xi + g(x,u,v)\eta] \, dx = 0$$

for all $(\xi, \eta) \in H$

Theorem 1.2. *Suppose that conditions $(2)-(5)$ hold. For any $\lambda \in (0, \frac{\lambda_1}{c_1+c_2})$ there exists a unique weak solution of (1.1).*

This paper is organized as follows. In section 2, we present some relevant lemmas. We reserve the section 3 for the proof of the main result.

2. Preliminary lemmas

Given a bounded smooth open set $\Omega \subset \mathbb{R}^N$. Let us consider the Hilbert space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ and $\langle \, , \, \rangle_{L^2}$ the inner product in $L^2(\Omega)$. The norm on H is given by

$$\|(u,v)\| = (\int_\Omega (|\nabla u|^2 + |\nabla v|^2) \, dx)^{\frac{1}{2}}$$

and the norm on $L^2(\Omega) \times L^2(\Omega)$ is given by

$$\|(u,v)\|_{L^2 \times L^2} = (\int_\Omega (|u|^2 + |v|^2) \, dx)^{\frac{1}{2}}.$$

First, we define the operator $a : H \times H \to \mathbb{R}$ by

$$a((u,v),(\xi,\eta)) \quad = \quad \int_\Omega \nabla u \nabla \xi dx + \int_\Omega \nabla v \nabla \eta dx + \int_\Omega h_1(|\nabla u|^2)\nabla u \nabla \xi dx$$

$$+\int_\Omega h_2(|\nabla v|^2)\nabla v\nabla\eta dx,$$

respectively $b_\lambda : H\times H\to\mathbb{R}$ by

$$b_\lambda((u,v),(\xi,\eta))=\lambda[\int_\Omega f(x,u,v)\xi dx+\int_\Omega g(x,u,v)\eta dx].$$

We point out certain properties of the operators a, respectively b_λ.

Lemma 2.1. The operators a and b_λ satisfy the following properties:

$(A1)$ for each $(u,v)\in H$, the application $(\xi,\eta)\mapsto a((u,v),(\xi,\eta))$ is linear and continuous.

$(A2)$ $a((u,v),\ (u,v)-(u_1,v_1))-a((u_1,v_1),\ (u,v)-(u_1,v_1))\geq\|(u,v)-(u_1,v_1)\|^2$ for all $(u,v),(u_1,v_1)\in H$.

$(A3)$ there exist $M>0$ such that

$$|\ a((u,v),(\xi,\eta))-a((u_1,v_1),(\xi,\eta))|\leq M\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|$$

for all $(u,v),(u_1,v_1),(\xi,\eta)\in H$.

$(B1)$ for each $(u,v)\in H$, the application $(\xi,\eta)\mapsto b_\lambda((u,v),(\xi,\eta))$ is linear and continuous.

$(B2)$ $b_\lambda((u,v),\ (u,v)-(u_1,v_1))-b_\lambda((u_1,v_1),\ (u,v)-(u_1,v_1))\leq\dfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v)-(u_1,v_1)\|^2$ for all $(u,v),(u_1,v_1)\in H$.

$(B3)$ there exist $N=N(\lambda)>0$ such that

$$|\ b_\lambda((u,v),(\xi,\eta))-b_\lambda((u_1,v_1),(\xi,\eta))|\leq N\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|$$

for all $(u,v),(u_1,v_1),(\xi,\eta)\in H$.

Proof. $(A1)$ We fix $(u,v)\in H$. It is clear that $(\xi,\eta)\mapsto a((u,v),\ (\xi,\eta))$ is linear. On the other hand, using Holder's inequality we have

$$\begin{aligned}|a((u,v),(\xi,\eta))| &= |\int_\Omega\nabla u\nabla\xi dx+\int_\Omega\nabla v\nabla\eta dx+\int_\Omega h_1(|\nabla u|^2)\nabla u\nabla\xi dx\\ &\quad+\int_\Omega h_2(|\nabla v|^2)\nabla v\nabla\eta dx|\\ &\leq (\beta_1+1)\|(u,v)\|\ \|(\xi,\eta)\|+(\beta_2+1)\|(u,v)\|\ \|(\xi,\eta)\|\\ &= (\beta_1+\beta_2+2)\|(u,v)\|\ \|(\xi,\eta)\|.\end{aligned}$$

It follows that $(\xi,\eta)\mapsto a((u,v),(\xi,\eta))$ is continuous.
$(A2)$ we have

$$\begin{aligned}&a((u,v),\ (u,v)-(u_1,v_1))-a((u_1,v_1),\ (u,v)-(u_1,v_1))\\ &=\|(u,v)-(u_1,v_1)\|^2+[\int_\Omega(h_1(|\nabla u|^2)|\nabla u|^2+h_1(|\nabla u_1|^2)|\nabla u_1|^2-h_1(|\nabla u|^2)\nabla u\nabla u_1\\ &-h_1(|\nabla u_1|^2)\nabla u\nabla u_1)\ dx]+[\int_\Omega(h_2(|\nabla v|^2)|\nabla v|^2+h_2(|\nabla v_1|^2)|\nabla v_1|^2-h_2(|\nabla v|^2)\nabla v\nabla v_1\\ &-h_2(|\nabla v_1|^2)\nabla v\nabla v_1)\ dx]\end{aligned}$$

on the other hand we have

$$h_1(|\nabla u|^2)|\nabla u|^2+h_1(|\nabla u_1|^2)|\nabla u_1|^2\geq h_1(|\nabla u|^2)\nabla u\nabla u_1+h_1(|\nabla u_1|^2)\nabla u\nabla u_1. \tag{2.1}$$

Indeed, it is sufficient to show that

$$h_1(|\nabla u|^2)|\nabla u|^2+h_1(|\nabla u_1|^2)|\nabla u_1|^2\geq|h_1(|\nabla u|^2)\nabla u\nabla u_1+h_1(|\nabla u_1|^2)\nabla u\nabla u_1|$$

or

$$h_1(|\nabla u|^2)|\nabla u|^2+h_1(|\nabla u_1|^2)|\nabla u_1|^2\geq h_1(|\nabla u|^2)|\nabla u.\nabla u_1|+h_1(|\nabla u_1|^2)|\nabla u.\nabla u_1|.$$

So we shall prove that

$$h_1(|\nabla u|^2)|\nabla u|^2 + h_1(|\nabla u_1|^2)|\nabla u_1|^2 \geq h_1(|\nabla u|^2)|\nabla u||\nabla u_1| + h_1(|\nabla u_1|^2)|\nabla u||\nabla u_1|$$

or

$$(h_1(|\nabla u|^2)|\nabla u| - h_1(|\nabla u_1|^2)|\nabla u_1|)\ (|\nabla u| - |\nabla u_1|) \geq 0 \tag{2.2}$$

we define the auxiliary function $\psi_1 : [0, \infty) \to R$ by $\psi_1(t) = h_1(t^2)t$. Obviously ψ_1 is increasing on $[0, \infty)$ which implies

$$[\psi(t_1) - \psi(t_2)]\ (t_1 - t_2) \geq 0\ \forall t_1, t_2 \in [0, \infty).$$

Taking $t_1 = |\nabla u|, t_2 = |\nabla u_1|$ the inequality (7) follows.

Similarly by auxiliary function $\psi_2 : [0, \infty) \to R$ by $\psi_2(t) = h_2(t^2)t$ we obtain

$$h_2(|\nabla v|^2)|\nabla v|^2 + h_2(|\nabla v_1|^2)|\nabla v_1|^2 - h_2(|\nabla v|^2)\nabla v \nabla v_1 - h_2(|\nabla v_1|^2)\nabla v \nabla v_1 \geq 0.$$

Hence

$$\begin{aligned} a((u,v),\ (u,v)-(u_1,v_1)) - a((u_1,v_1),\ (u,v)-(u_1,v_1)) \\ \geq \|(u,v)-(u_1,v_1)\|^2,\ \forall (u,v),(u_1,v_1) \in H. \end{aligned}$$

$(A3)$

$$\begin{aligned} &|\ a((u,v),(\xi,\eta)) - a((u_1,v_1),(\xi,\eta))| = |\int_\Omega (\nabla u - \nabla u_1)\nabla \xi dx + \int_\Omega (\nabla v - \nabla v_1)\nabla \eta dx \\ &+ \int_\Omega [h_1(|\nabla u|^2)\nabla u - h_1(|\nabla u_1|^2)\nabla u_1]\nabla \xi dx + \int_\Omega [h_2(|\nabla v|^2)\nabla v - h_2(|\nabla v_1|^2)\nabla v_1]\nabla \eta dx| \\ &\leq 2\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\| + \int_\Omega \Sigma_{j=1}^N\ |h_1(|\nabla u|^2)\frac{\partial u}{\partial x_j} - h_1(|\nabla u_1|^2)\frac{\partial u_1}{\partial x_j}|\ |\frac{\partial \xi}{\partial x_j}| \\ &+ \int_\Omega \Sigma_{j=1}^N\ |h_2(|\nabla v|^2)\frac{\partial v}{\partial x_j} - h_2(|\nabla v_1|^2)\frac{\partial v_1}{\partial x_j}|\ |\frac{\partial \eta}{\partial x_j}| \end{aligned}$$

on the other hand we have

$$|h_1(|\nabla u|^2)\frac{\partial u}{\partial x_j} - h_1(|\nabla u_1|^2)\frac{\partial u_1}{\partial x_j}| \leq (\beta_1 + M_1(N+1))|\nabla u - \nabla u_1|$$

and

$$|h_2(|\nabla v|^2)\frac{\partial v}{\partial x_j} - h_2(|\nabla v_1|^2)\frac{\partial v_1}{\partial x_j}| \leq (\beta_2 + M_2(N+1))|\nabla v - \nabla v_1|.$$

Indeed let $x \in \overline{\Omega}$ be fixed and $H_i : R^N \to R$ be defined by

$$H_i(\xi) = h_1(|\xi|^2)\xi_i\ \forall \xi \in R^N,\ \forall i \in \{1, 2, ...N\}.$$

Using the mean value theorem we deduce that

$$|H_i(\xi) - H_i(\eta)| \leq |\xi - \eta| \sup_{\theta \in [\xi,\eta]} |\nabla H_i(\theta)|$$

where $[\xi, \eta]$ is the line segment in R^N between the point ξ and η, i.e., $[\xi, \eta] = \{t\xi + (1-t)\eta :\ t \in [0,1]\ \}$. But

$$|\nabla H_i(\theta)| = (\Sigma_{j=1}^N\ (\frac{\partial H_i(\theta)}{\partial \theta_j})^2)^{\frac{1}{2}} \leq \Sigma_{j=1}^N\ |\frac{\partial H_i(\theta)}{\partial \theta_j}|. \tag{2.3}$$

For $j \neq i$

$$\frac{\partial H_i(\theta)}{\partial \theta_j} = 2\ h_1'(|\theta|^2)\ \theta_i\ \theta_j$$

and for $j = i$

$$\frac{\partial H_i(\theta)}{\partial \theta_i} = h_1(|\theta|^2) + 2\ h_1'(|\theta|^2)\ \theta_i^2.$$

Thus by (4), (5), (8), we find

$$\begin{aligned} |\nabla H_i(\theta)| \leq \Sigma_{j=1}^N\ |\frac{\partial H_i(\theta)}{\partial \theta_j}| \leq h_1(|\theta|^2) + 2\ |h_1'(|\theta|^2)|\ \Sigma_{j=1}^N|\theta_i\theta_j| \\ \leq h_1(|\theta|^2) + 2\ |h_1'(|\theta|^2)|\Sigma_{j=1}^N\ \frac{\theta_i^2 + \theta_j^2}{2} \end{aligned}$$

$$\leq \beta_1 + 2|h_1'(|\theta|^2)|\frac{N+1}{2}|\theta|^2$$
$$\leq \beta_1 + M_1(N+1).$$

Similarly for $G_i(\xi) = h_2(|\xi|^2)\xi_i$ we have $|\nabla G_i(\theta)| \leq \beta_2 + M_2(N+1)$.
Hence

$$|a((u,v),(\xi,\eta)) - a((u_1,v_1),\ (\xi,\eta))| \leq 2\|(u,v)-(u_1,v_1\|\ \|(\xi,\eta)\|$$
$$+(\beta_1 + M_1(N+1))\int_\Omega \Sigma_{j=1}^N\ |\nabla u - \nabla u_1|\ |\frac{\partial \xi}{\partial x_j}|dx$$
$$+(\beta_2 + M_2(N+1))\int_\Omega \Sigma_{j=1}^N\ |\nabla v - \nabla v_1|\ |\frac{\partial \eta}{\partial x_j}|dx$$
$$\leq 2\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\| + (\beta_1+M_1(N+1))N(\int_\Omega |\nabla u - \nabla u_1|^2 dx)^{\frac{1}{2}}\ (\int_\Omega |\nabla \xi|^2 dx)^{\frac{1}{2}}$$
$$+(\beta_2+M_2(N+1))N(\int_\Omega |\nabla v - \nabla v_1|^2 dx)^{\frac{1}{2}}\ (\int_\Omega |\nabla \eta|^2 dx)^{\frac{1}{2}} \leq M\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|$$

where $M = 2 + N(\beta_1 + \beta_2 + (N+1)(M_1+M_2))$.

$(B1)$ We fix $(u,v) \in H$. Obviously, the application $(\xi,\eta) \mapsto b_\lambda((u,v),\ (\xi,\eta))$ is linear. Using Holder's inequality, we have

$$|b_\lambda((u,v),\ (\xi,\eta))| = |\lambda \textstyle\int_\Omega [f(x,u,v)\xi + g(x,u,v)\eta]dx|$$
$$\leq \lambda(\textstyle\int_\Omega |f(x,u,v)|^2dx)^{\frac{1}{2}}(\int_\Omega |\xi|^2dx)^{\frac{1}{2}} + \lambda(\int_\Omega |g(x,u,v)|^2dx)^{\frac{1}{2}}(\int_\Omega |\eta|^2dx)^{\frac{1}{2}}$$
$$\leq K\|(\xi,\eta))\|,$$

where K is positive constant.

$(B2)$ By (2), (3) we have

$$b_\lambda((u,v),\ (u,v)-(u_1,v_1)) - b_\lambda((u_1,v_1),\ (u,v)-(u_1,v_1))$$
$$= \lambda \textstyle\int_\Omega [f(x,u,v) - f(x,u_1,v_1)](u-u_1)dx + \lambda\int_\Omega [g(x,u,v)-g(x,u_1,v_1)](v-v_1)dx$$
$$\leq \lambda c_1(\textstyle\int_\Omega |u-u_1|^2 + |v-v_1|^2)dx)^{\frac{1}{2}}(\int_\Omega |u-u_1|^2dx)^{\frac{1}{2}}$$
$$+\lambda c_2(\textstyle\int_\Omega |u-u_1|^2 + |v-v_1|^2)dx)^{\frac{1}{2}}(\int_\Omega |v-v_1|^2dx)^{\frac{1}{2}}$$
$$\leq \lambda c_1\|(u,v)-(u_1,v_1)\|^2_{L^2\times L^2} + \lambda c_2\|(u,v)-(u_1,v_1)\|^2_{L^2\times L^2}$$
$$= \lambda(c_1+c_2)\|(u,v)-(u_1,v_1)\|^2_{L^2\times L^2}$$
$$\leq \tfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v)-(u_1,v_1)\|^2.$$

$(B3)$ Using Holder's inequality and (2), (3) we obtain

$$|b_\lambda((u,v),(\xi,\eta)) - b_\lambda((u_1,v_1),(\xi,\eta))|$$
$$= |\lambda \textstyle\int_\Omega [f(x,u,v)-f(x,u_1,v_1)]\xi dx + \lambda\int_\Omega [g(x,u,v)-g(x,u_1,v_1)]\eta dx|$$
$$\leq \lambda c_1\|(u,v)-(u_1,v_1)\|_{L^2\times L^2}\ (\textstyle\int_\Omega |\xi|^2dx)^{\frac{1}{2}} + \lambda c_2\|(u,v)-(u_1,v_1)\|_{L^2\times L^2}\ (\int_\Omega |\eta|^2dx)^{\frac{1}{2}}$$
$$\leq \lambda(c_1+c_2)\|(u,v)-(u_1,v_1)\|_{L^2\times L^2}\|(\xi,\eta)\|_{L^2\times L^2}$$
$$\leq \tfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|$$
$$= N\|(u,v)-(u_1,v_1)\|\ \|(\xi,\eta)\|,$$

where $N = \frac{\lambda(c_1+c_2)}{\lambda_1}$.

3. Proof of main theorem

In this section we give the proof of theorem 1.2.

Proof of theorem 1.2. Let $\lambda \in (0, \frac{\lambda_1}{c_1+c_2})$ be arbitrary but fixed. By Lemma 2.1(A1) and the Riesz theorem (see e.g. Brezis [2], Theorem V.5) we deduce that for each $(u,v) \in H$ there exists a unique element denote by $A(u,v) \in H$ such that

$$a((u,v),(\xi,\eta)) = \langle A(u,v),(\xi,\eta)\rangle.$$

Thus we can define the operator $A : H \to H$. Using Lemma 2.1(A2) it follows that

$$\langle A(u,v) - A(u_1,v_1),\ (u,v) - (u_1,v_1)\rangle \geq \|(u,v) - (u_1,v_1)\|^2 \tag{3.1}$$

for all $(u,v),(u_1,v_1) \in H$ i.e., A is strongly monotone.
Lemma 2.1(A3) yields

$$|\langle A(u,v),(\xi,\eta)\rangle - \langle A(u_1,v_1),(\xi,\eta)\rangle| \leq M\|(u,v) - (u_1,v_1)\|\ \|(\xi,\eta)\|$$

for all $(u,v),(u_1,v_1),(\xi,\eta) \in H$. Hence,

$$\|A(u,v) - A(u_1,v_1)\| = \sup_{\|(\xi,\eta)\|\leq 1} |\langle A(u_1,v_1),(\xi,\eta)\rangle| \leq M\|(u,v) - (u_1,v_1)\| \tag{3.2}$$

i.e., A is Lipschitz continuous.

By Lemma 2.1$(B1)$ and the Riesz theorem we deduce that for each $(u,v) \in H$ there exists a unique element $B_\lambda(u,v) \in H$ such that

$$b_\lambda((u,v),(\xi,\eta)) = \langle B_\lambda(u,v),\ (\xi,\eta)\rangle,\ \forall(\xi,\eta) \in H.$$

Thus, we can also define the operator $B_\lambda : H \to H$ which satisfies

$$\langle B_\lambda(u,v),\ (u,v) - (u_1,v_1)\rangle - \langle B_\lambda(u,v),\ (u,v) - (u_1,v_1)\rangle \leq \frac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v) - (u_1,v_1)\|^2. \tag{3.3}$$

Using Lemma 2.1$(B3)$ we find

$$\begin{aligned} &\|B_\lambda(u,v) - B_\lambda(u_1,v_1)\| = \sup|\langle B_\lambda(u,v),(\xi,\eta)\rangle - \langle B_\lambda(u_1,v_1),(\xi,\eta)\rangle| \\ &= \sup_{\|w\|\leq 1} |b_\lambda((u,v),(\xi,\eta)) - b_\lambda((u_1,v_1),(\xi,\eta))| \leq N\|(u,v) - (u_1,v_1)\| \end{aligned} \tag{3.4}$$

we define the operator $T : H \to H$ by

$$T(u,v) = (u,v) - t(A(u,v) - B_\lambda(u,v))$$

where $t \in (0,\ \frac{2(1-\frac{\lambda(c_1+c_2)}{\lambda_1})}{(N+M)^2}\ )$. The relation (9)-(12) shows that for each $(u,v),(u_1,v_1) \in H$ we have

$$\begin{aligned} &\|T(u,v) - T(u_1,v_1)\|^2 = \langle T(u,v) - T(u_1,v_1),\ T(u,v) - T(u_1,v_1)\rangle \\ &= \langle (u,v) - t(A(u,v) - B_\lambda(u,v)) - (u_1,v_1) + t(A(u_1,v_1) - B_\lambda(u_1,v_1)), \\ &(u,v) - t(A(u,v) - B_\lambda(u,v)) - (u_1,v_1) + t(A(u_1,v_1) - B_\lambda(u_1,v_1))\rangle \\ &= \|(u,v) - (u_1,v_1)\|^2 - 2t\langle A(u,v) - A(u_1,v_1),\ (u,v) - (u_1,v_1)\rangle \\ &+2t\langle B_\lambda(u,v) - B_\lambda(u_1,v_1),\ (u,v) - (u_1,v_1)\rangle - 2t^2\langle A(u,v) - A(u_1,v_1), \\ &B_\lambda(u,v) - B_\lambda(u_1,v_1)\rangle + t^2\|A(u,v) - A(u_1,v_1)\|^2 + t^2\|B_\lambda(u,v) - B_\lambda(u_1,v_1)\|^2 \\ &\leq \|(u,v) - (u_1,v_1)\|^2 - 2t\|(u,v) - (u_1,v_1)\|^2 + 2t\tfrac{\lambda(c_1+c_2)}{\lambda_1}\|(u,v) - (u_1,v_1)\|^2 \\ &+2t^2(M\|(u,v) - (u_1,v_1)\|)\ (N\|(u,v) - (u_1,v_1)\|) + t^2M^2\|(u,v) - (u_1,v_1)\|^2 \\ &+t^2N^2\|(u,v) - (u_1,v_1)\|^2 \\ &= [1 - 2t(1 - \tfrac{\lambda(c_1+c_2)}{\lambda_1}) + M^2t^2 + N^2t^2 + 2NMt^2]\ \|(u,v) - (u_1,v_1)\|^2 \\ &= \beta\|(u,v) - (u_1,v_1)\|^2, \end{aligned}$$

where

$$\beta = 1 - 2(1 - \frac{\lambda(c_1+c_2)}{\lambda_1})t + (M+N)^2t^2 \geq 0.$$

If $t = 0$ or $t = \frac{2(1-\frac{\lambda(c_1+c_2)}{\lambda_1})}{(M+N)^2}$ then $\beta = 1$. This implies that $\sqrt{\beta} < 1$ for all $t \in (0,\ \frac{2(1-\frac{\lambda(c_1+c_2)}{\lambda_1})}{(M+N)^2})$. Hence

$$\|T(u,v) - T(u_1,v_1)\| \leq \sqrt{\beta}\|(u,v) - (u_1,v_1)\|,\ \forall\ (u,v),(u_1,v_1) \in H$$

i,e., T is $\sqrt{\beta}$- contractive with $\sqrt{\beta} < 1$. By Banach fixed point theorem (see Zeidler [10], section 1.6) it follows that there is a unique solution $(u,v) \in H$ of problem $T(u,v) = (u,v)$ i.e., the problem $A(u,v) = B_\lambda(u,v)$ has a unique solution $(u,v) \in H$. It follows that

$$\langle A(u,v),\ (\xi,\eta)\rangle = \langle B_\lambda(u,v),\ (\xi,\eta)\rangle,\ \forall\ (\xi,\eta) \in H$$

$$a((u,v),(\xi,\eta)) = b_\lambda((u,v),(\xi,\eta)).$$

Thus the proof of Theorem 1.2 is complete.

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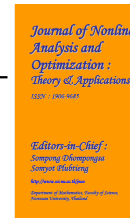

PPF DEPENDENT FIXED POINT THEOREMS FOR RATIONAL TYPE CONTRACTION MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we prove the existence of the PPF dependent fixed point theorems in the Razumikhin class for rational type contraction mappings where the domain and range of the mappings are not the same. We also use this result to prove the PPF dependent coincidence point theorems. Our results extend and generalize some results of Bernfeld *et al.* in [S. R. Bernfeld, V. Lakshmikatham and Y. M. Reddy, Fixed point theorems of operators with PPF dependence in Banach spaces, Applicable Anal. 6 (1977), 271–280.].

KEYWORDS: PPF fixed points; Razumikhin classes; Rational type contraction.
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1. INTRODUCTION

The applications of fixed point theory are very important in diverse disciplines of mathematics since they can be applied for solving various problems, for instance, equilibrium problems, variational problems, and optimization problems. The Banach's contraction mapping principle is one of the cornerstones in the development of fixed point theory. In particular, this principle is used to demonstrate the existence and uniqueness of a solution of differential equations, integral equations, functional equations, partial differential equations and others. Due to the importance, generalizations of Banach's contraction mapping principle have been investigated heavily by many mathematicians. One of the most interesting is extension of Banach's contraction mapping principle in case of non-self mappings.

In 1997, Bernfeld *et al.* [1] introduced the concept of PPF dependent fixed point or the fixed point with PPF dependence which is a one type of fixed point for mappings that have different domains and ranges. They also proved the existence of PPF dependent fixed point theorems in the Razumikhin class for Banach type contraction mappings. The PPF dependent fixed point theorems are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data

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and future consideration. Afterward, a number of papers appeared in which PPF dependent fixed point theorems have been discussed (see [3, 4, 5] and references therein).

On the other hand, Dass and Gupta [2] and Jaggi [6] were first to establish the existence of fixed point theorems using contractive conditions involving rational expressions. To the best of our knowledge, there is no discussion so far concerning the PPF dependent fixed point theorems via rational contractive conditions.

In this paper, we will introduce the rational type contraction non-self mapping and also establish the existence of PPF dependent fixed point theorems for such mapping in Razumikhin class. Furthermore, we apply this result to the existence of PPF dependence coincidence point theorems. Our results extend some result in [1].

2. PRELIMINARIES

Throughout this paper, let E denotes a Banach space with the norm $\|\cdot\|_E$, I denotes a closed interval $[a, b]$ in $\mathbb{R}$, and $E_0 = C(I, E)$ denotes the set of all continuous E-valued functions on I equips with the supremum norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E.$$

A point $\phi \in E_0$ is said to be *a PPF dependent fixed point* or *a fixed point with PPF dependence* of mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

For a fixed element $c \in I$, the Razumikhin or minimal class of functions in E_0 is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_E\}.$$

It is easy to see that if the function $\widetilde{\phi} \in E_0$ is a constant function, then $\widetilde{\phi} \in \mathcal{R}_c$.

The class $\mathcal{R}_c$ is algebraically closed with respect to difference if $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Similarly, $\mathcal{R}_c$ is topologically closed if it is closed with respect to the topology on E_0 generated by the norm $\|\cdot\|_{E_0}$.

The Razumikhin class play an important role in proving the existence of PPF fixed points with different domain and range of the mappings in abstract spaces.

Definition 2.1 (Bernfeld *et al.* [1]). The mapping $T : E_0 \to E$ is called Banach type contraction if there exists a real number $\alpha \in [0, 1)$ such that

$$\|T\phi - T\xi\|_E \le \alpha\|\phi - \xi\|_{E_0} \tag{2.1}$$

for all $\phi, \xi \in E_0$.

The following PPF dependent fixed point theorem is proved in Bernfeld *et al.* [1].

Theorem 2.2 (Bernfeld *et al.* [1]). *Let $T : E_0 \to E$ be a Banach type contraction. If $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

3. PPF DEPENDENT FIXED POINT THEOREMS

First of all, we introduce the definition of the rational type contraction mappings.

Definition 3.1. The mapping $T : E_0 \to E$ is called rational type contraction if there exist real numbers $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $c \in I$ such that

$$\|T\phi - T\xi\|_E \le \alpha\|\phi - \xi\|_{E_0} + \frac{\beta\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \tag{3.1}$$

for all $\phi, \xi \in E_0$.

It is easy to see that every Banach type contraction mapping is rational type contraction mapping, but the converse is necessarily not true.

Next, we prove PPF dependent fixed point theorems for rational type contraction mappings.

Theorem 3.2. *Let $T : E_0 \to E$ be a rational type contraction mapping. If $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathcal{R}_c$ defined by

$$T\phi_{n-1} = \phi_n(c) \tag{3.2}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

Proof. Let ϕ_0 be an arbitrary function in $\mathcal{R}_c \subseteq E_0$. Since $T\phi_0 \in E$, there exists $x_1 \in E$ such that $T\phi_0 = x_1$. Choose $\phi_1 \in \mathcal{R}_c$ such that

$$x_1 = \phi_1(c).$$

Since $\phi_1 \in \mathcal{R}_c \subseteq E_0$ and by hypothesis, we get $T\phi_1 \in E$. This implies that there exists $x_2 \in E$ such that $T\phi_1 = x_2$. Thus, we can choose $\phi_2 \in \mathcal{R}_c$ such that

$$x_2 = \phi_2(c).$$

By continuing this process, by induction, we can construct the sequence $\{\phi_n\}$ in $\mathcal{R}_c \subseteq E_0$ such that

$$T\phi_{n-1} = \phi_n(c)$$

for all $n \in \mathbb{N}$. Since $\mathcal{R}_c$ is algebraically closed with respect to difference, we have

$$\|\phi_{n-1} - \phi_n\|_{E_0} = \|\phi_{n-1}(c) - \phi_n(c)\|_E$$

for all $n \in \mathbb{N}$.

Next, we will show that $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c$.

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E \\
&= \|T\phi_{n-1} - T\phi_n\|_E \\
&\leq \alpha\|\phi_{n-1} - \phi_n\|_{E_0} \\
&\quad + \frac{\beta\|\phi_{n-1}(c) - T\phi_{n-1}\|_E\|\phi_n(c) - T\phi_n\|_E}{1 + \|T\phi_{n-1} - T\phi_n\|_E} \\
&= \alpha\|\phi_{n-1} - \phi_n\|_{E_0} \\
&\quad + \frac{\beta\|\phi_{n-1}(c) - \phi_n(c)\|_E\|\phi_n(c) - \phi_{n+1}(c)\|_E}{1 + \|\phi_n(c) - \phi_{n+1}(c)\|_E} \\
&\leq \alpha\|\phi_{n-1} - \phi_n\|_{E_0} + \beta\|\phi_{n-1}(c) - \phi_n(c)\|_E \\
&= \alpha\|\phi_{n-1} - \phi_n\|_{E_0} + \beta\|\phi_{n-1} - \phi_n\|_{E_0} \\
&= (\alpha + \beta)\|\phi_{n-1} - \phi_n\|_{E_0}.
\end{aligned}$$

Hence, by repeated application of the above relation yields

$$\|\phi_n - \phi_{n+1}\|_{E_0} \leq k^n\|\phi_0 - \phi_1\|_{E_0}$$

for all $n \in \mathbb{N}$, where $k = \alpha + \beta$.

For $m, n \in \mathbb{N}$ with $m > n$, we obtain that

$$\begin{aligned}
\|\phi_n - \phi_m\|_{E_0} &\leq \|\phi_n - \phi_{n+1}\|_{E_0} + \|\phi_{n+1} - \phi_{n+2}\|_{E_0} \\
&\quad + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
&\leq (k^n + k^{n+1} + \cdots + k^{m-1})\|\phi_0 - \phi_1\|_{E_0} \\
&\leq \frac{k^n}{1-k}\|\phi_0 - \phi_1\|_{E_0}.
\end{aligned}$$

This implies that the sequence $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{R}_c \subseteq E_0$. By the completeness of E_0, we get $\{\phi_n\}$ converges to a limit point $\phi^* \in E_0$, that is, $\lim_{n\to\infty} \phi_n = \phi^*$. Since $\mathcal{R}_c$ is topologically closed, we have $\phi^* \in \mathcal{R}_c$.

Now we prove that ϕ^* is a PPF dependent fixed point of T. From the assumption of rational type contraction of T, we get

$$\|T\phi^* - \phi^*(c)\|_E \leq \|T\phi^* - \phi_n(c)\|_E + \|\phi_n(c) - \phi^*(c)\|_E$$

$$
\begin{aligned}
&= \|T\phi^* - T\phi_{n-1}\|_E + \|\phi_n - \phi^*\|_{E_0} \\
&\leq \alpha\|\phi^* - \phi_{n-1}\|_{E_0} \\
&\quad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1}(c) - T\phi_{n-1}\|_E}{1 + \|T\phi^* - T\phi_{n-1}\|_E} \\
&\quad + \|\phi_n - \phi^*\|_{E_0} \\
&= \alpha\|\phi^* - \phi_{n-1}\|_{E_0} \\
&\quad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E\|\phi_{n-1}(c) - \phi_n(c)\|_E}{1 + \|T\phi^* - \phi_n(c)\|_E} \\
&\quad + \|\phi_n - \phi^*\|_{E_0}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in the above inequality, we have

$$\|T\phi^* - \phi^*(c)\|_E = 0$$

and so

$$T\phi^* = \phi^*(c).$$

This implies that ϕ^* is a PPF dependent fixed point of T in $\mathcal{R}_c$.

Finally, we prove the uniqueness of PPF dependent fixed point of T in $\mathcal{R}_c$. Let ϕ^* and ξ^* be two PPF dependent fixed points of T in $\mathcal{R}_c$. Therefore,

$$
\begin{aligned}
\|\phi^* - \xi^*\|_{E_0} &= \|\phi^*(c) - \xi^*(c)\|_E \\
&= \|T\phi^* - T\xi^*\|_E \\
&\leq \alpha\|\phi^* - \xi^*\|_{E_0} \\
&\quad + \frac{\beta\|\phi^*(c) - T\phi^*\|_E\|\xi^*(c) - T\xi^*\|_E}{1 + \|T\phi^* - T\xi^*\|_E} \\
&= \alpha\|\phi^* - \xi^*\|_{E_0}.
\end{aligned}
$$

Since $\alpha < 1$, we have $\|\phi^* - \xi^*\|_{E_0} = 0$ and hence $\phi^* = \xi^*$. Therefore, T has a unique PPF dependent fixed point in $\mathcal{R}_c$. This completes the proof. □

Remark 3.3. If the Razumikhin class $\mathcal{R}_c$ is not topologically closed, then the limit of the sequence $\{\phi_n\}$ in Theorem 3.2 may be outside of $\mathcal{R}_c$. Therefore, a PPF dependent fixed point of T may not be unique.

By applying Theorem 3.2, we obtain the following corollaries.

Corollary 3.4. *Let $T : E_0 \to E$ and there exists a real number $\alpha \in [0, 1)$ such that*

$$\|T\phi - T\xi\|_E \leq \alpha\|\phi - \xi\|_{E_0} \tag{3.3}$$

for all $\phi, \xi \in E_0$.

If there exists $c \in I$ such that $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathcal{R}_c$.

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathcal{R}_c$ defined by

$$T\phi_{n-1} = \phi_n(c) \tag{3.4}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

Corollary 3.5. *Let $T : E_0 \to E$ and there exists a real number $\beta \in [0, 1)$ and $c \in I$ such that*

$$\|T\phi - T\xi\|_E \leq \frac{\beta\|\phi(c) - T\phi\|_E\|\xi(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \tag{3.5}$$

for all $\phi, \xi \in E_0$.

If $\mathcal{R}_c$ is topologically closed and algebraically closed with respect to difference, then T has a unique PPF dependent fixed point in $\mathcal{R}_c$.

Moreover, for a fixed $\phi_0 \in \mathcal{R}_c$, if a sequence $\{\phi_n\}$ of iterates of T in $\mathcal{R}_c$ defined by

$$T\phi_{n-1} = \phi_n(c) \tag{3.6}$$

for all $n \in \mathbb{N}$, then $\{\phi_n\}$ converges to a PPF dependent fixed point of T in $\mathcal{R}_c$.

4. PPF DEPENDENT COINCIDENCE POINT THEOREMS

Definition 4.1. Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. A point $\phi \in E_0$ is said to be a PPF dependent coincidence point or a coincidence point with PPF dependence of T and S if $T\phi = (S\phi)(c)$ for some $c \in I$.

Next, we introduce the condition of the rational type contraction for a pair of two mappings.

Definition 4.2. Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. The ordered pair (T, S) is said to satisfy the condition of rational type contraction if there exist real numbers $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $c \in I$ such that

$$\|T\phi - T\xi\|_E \leq \alpha\|S\phi - S\xi\|_{E_0} + \frac{\beta\|(S\phi)(c) - T\phi\|_E\|(S\xi)(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \tag{4.1}$$

for all $\phi, \xi \in E_0$.

It easy to see that if (T, S) satisfie the condition of rational type contraction and S is identity mapping, then T is a rational type contraction mapping.

Now, we apply Theorem 3.2 to the PPF dependent coincidence point theorem.

Theorem 4.3. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. Suppose that the following conditions hold:*

(i): *(T, S) satisfies the condition of rational type contraction;*

(ii): *$S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.*

If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point.

Proof. Consider the mapping $S : E_0 \to E_0$. We obtain that there exists $F_0 \subseteq E_0$ such that $S(F_0) = S(E_0)$ and $S|_{F_0}$ is one-to-one. Since

$$T(F_0) \subseteq T(E_0) \subseteq E,$$

we can define a mapping $\mathcal{A} : S(F_0) \to E$ by

$$\mathcal{A}(S\phi) = T\phi \tag{4.2}$$

for all $\phi \in F_0$. Since $S|_{F_0}$ is one-to-one, then $\mathcal{A}$ is well-defined.

From (4.2) and the condition of rational type contraction of (T, S), we have

$$\|\mathcal{A}(S\phi) - \mathcal{A}(S\xi)\|_E \leq \alpha\|S\phi - S\xi\|_{E_0} + \frac{\beta\|(S\phi)(c) - \mathcal{A}(S\phi)\|_E\|(S\xi)(c) - \mathcal{A}(S\xi)\|_E}{1 + \|\mathcal{A}(S\phi) - \mathcal{A}(S\xi)\|_E}$$

for all $S\phi, S\xi \in S(E_0)$. This shows that $\mathcal{A}$ is a rational type contraction mapping.

Now, we use Theorem 3.2 with a mapping $\mathcal{A}$, then there exists a unique PPF dependent fixed point $\varphi \in S(F_0)$ of $\mathcal{A}$, that is, $\mathcal{A}\varphi = \varphi(c)$. Since $\varphi \in S(F_0)$, we can find $\omega \in F_0$ such that $\varphi = S\omega$. Therefore, we get

$$T\omega = \mathcal{A}(S\omega) = \mathcal{A}\varphi = \varphi(c) = (S\omega)(c).$$

This implies that ω is a PPF dependent coincidence point of T and S. This completes the proof. □

By applying Theorem 4.3, we obtain the following corollaries.

Corollary 4.4. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. Suppose that the following conditions hold:*

(i): *there exists a real number $\alpha \in [0, 1)$ such that*

$$\|T\phi - T\xi\|_E \leq \alpha\|S\phi - S\xi\|_{E_0} \tag{4.3}$$

for all $\phi, \xi \in E_0$;

(ii): *there exists $c \in I$ such that $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.*

If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point in $\mathcal{R}_c$.

Corollary 4.5. *Let $T : E_0 \to E$ and $S : E_0 \to E_0$ be two given mappings. Suppose that the following conditions hold:*

(i): *there exists a real number $\beta \in [0, 1)$ and $c \in I$ such that*

$$\|T\phi - T\xi\|_E \leq \frac{\beta \|(S\phi)(c) - T\phi\|_E \|(S\xi)(c) - T\xi\|_E}{1 + \|T\phi - T\xi\|_E} \tag{4.4}$$

for all $\phi, \xi \in E_0$;

(ii): $S(\mathcal{R}_c) \subseteq \mathcal{R}_c$.

If $S(\mathcal{R}_c)$ is topologically closed and algebraically closed with respect to difference, then T and S have a PPF dependent coincidence point in $\mathcal{R}_c$.

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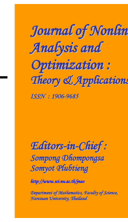

GENERALIZED MIXED GENERAL VECTOR VARIATIONAL-LIKE INEQUALITIES IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT. In this work, we consider and study new kinds of generalized mixed general vector variational-like inequalities in real topological vector spaces. We use the Ferro minimax theorem to discuss the existence of weak and strong solutions for the generalized mixed general vector variational-like inequality problems.

KEYWORDS: Generalized mixed general vector variational-like inequality; Weak solution; Strong solution; Ferro minimax theorem; Topological vector space.
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1. INTRODUCTION

Variational inequalities were introduced and considered by Stampacchia [1] in the early sixties. It has been shown that a wide class of linear and nonlinear problem arising in various branches of mathematical and engineering sciences can be studied within the unified and general framework of variational inequalities. Variational inequalities have been generalized and extended in several directions by using novel techniques. In 1980, Giannessi [2] initiated the vector variational inequality in finite dimensional Euclidean space. Since then, Chen *et al.* [3], Lee *et al.* [4, 5], Khan and Salahuddin [6], Yang [7], Ding [8], Ding and Tarafdar [9, 10], Peng [11], Usman and Khan [12], and Irfan and Ahmad [13] have investigated vector variational inequalities in abstract spaces.

The variational-like inequality also known as the pre-variational inequalities is one of the generalized form of variational inequalities. The variational-like inequalities and generalized variational-like inequalities are powerful tools for studying nonconvex optimization problems and nonconvex and nondifferentiable optimization problems respectively, see [2, 14].

In 1988, Noor [15] introduced and studied general variational inequality in Hilbert spaces which can be used to study both odd-order and even-order free and moving boundary value problems. Since then, many authors have further studied various generalizations of general variational inequalities in Hilbert spaces and Banach spaces respectively. For example, see [16–22].

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It is well known that the variational inequality theory and equilibrium problems have wide applications in finance, economics, transportation, optimization and operation research, and the solution sets for variational inequalities are of considerable interest [23–26].

Let X, Y be arbitrary real Hausdorff topological vector spaces. Let 2^Y denotes the family of all nonempty subsets of Y and $L(X,Y)$ the space of all continuous linear mappings from X to Y. Let K be a nonempty set of $X, C : K \to 2^Y$ a set valued mapping such that for each $x \in K, C(x)$ is proper closed convex pointed cone with apex at the origin and $int\ C(x) \neq \emptyset$. The mappings $g : K \to K, A : K \times L(X,Y) \to L(X,Y), T : K \to 2^{L(X,Y)}, h : K \times K \longrightarrow Y$ and $\eta : K \times K \longrightarrow K$ are given. For each $x \in K$, we define the relations $\leq_{C(x)}$ and $\not\leq_{C(x)}$ as follows:

(i) $z \leq_{C(x)} y \Leftrightarrow y - z \in C(x)$,
(ii) $z \not\leq_{C(x)} y \Leftrightarrow y - z \notin C(x)$.

Similarly we can define the relations $\leq_{intC(x)}$ and $\not\leq_{int\ C(x)}$ if we replace the set $C(x)$ by $intC(x)$. If the mapping $C(x)$ is constant, then we write $C(x)$ as C.

Inspired and motivated by recent works of authors see [27–30, 32–34], in this paper, we consider the following generalized mixed general vector variational-like inequality problem (GMGVVLIP): find $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{int\ C(\bar{x})} 0. \tag{1.1}$$

Such solution $\bar{x}$ is called a weak solution of the GMGVVLIP (1.1). If $\bar{s}$ does not depend on y, then GMGVVLIP (1.1) reduces to the following problem: find $\bar{x} \in K$ and $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{int\ C(\bar{x})} 0,\ \forall\ y \in K. \tag{1.2}$$

Such solution $\bar{x}$ is called a strong solution of the GMGVVLIP (1.2).

If $Y = \mathcal{R}$ and $C(x) = (-\infty, 0]$, then $X^* = L(X, \mathcal{R})$ and $T : K \longrightarrow 2^{X^*}$ is a mapping from K into 2^{X^*} and the GMGVVLIP (1.1) reduces to the following generalized mixed general variational-like inequality problem: find $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \geq 0. \tag{1.3}$$

We remark that if $\bar{s}$ does not depend on y, then the solution of problem (1.3) is called strong solution.

Definition 1.1 Let Ω be a vector space, Σ a topological vector space, K a nonempty convex subset of $\Omega, C : K \to 2^\Sigma$ a set-valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. For any $x \in K, \psi : K \to \Sigma$ is said to be

(i) $C(x)$-convex iff
$\psi(tx_1 + (1-t)x_2) \leq_{C(x)} t\psi(x_1) + (1-t)\psi(x_2)$ for every $x_1, x_2 \in K$ and $t \in [0,1]$,
(ii) properly quasi $C(x)$-convex iff we have either
$\psi(tx_1 + (1-t)x_2) \leq_{C(x)} \psi(x_1)$ or
$\psi(tx_1 + (1-t)x_2) \leq_{C(x)} \psi(x_2)$ for every $x_1, x_2 \in K$ and $t \in [0,1]$.

Definition 1.2 Let Ω be a vector space, Σ a topological vector space, K a nonempty convex subset of $\Omega, C : K \to 2^\Sigma$ a set-valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. Further, let A be a nonempty subset of Σ, then for any fixed $x \in K$,

(i) a point $z \in A$ is called a minimal point A with respect to the cone $C(x)$ iff $A \cap (z - C(x)) = \{z\}; Min^{C(x)}A$ is the set of all minimal points of A with respect to the cone $C(x)$;
(ii) a point $z \in A$ is called a maximal point A with respect to the cone $C(x)$ iff $A \cap (z + C(x)) = \{z\}; Max^{C(x)}A$ is the set of all maximal points of A with respect to the cone $C(x)$;
(iii) a point $z \in A$ is called a weakly minimal point of A with respect to the cone $C(x)$ iff $A \cap (z - intC(x)) = \emptyset; Min_w^{C(x)}A$ is the set of all weakly minimal points of A with respect to the cone $C(x)$;

(iv) a point $z \in A$ is called a weakly maximal point of A with respect to the cone $C(x)$ iff $A \cap (z + intC(x)) = \emptyset$; $Max_w^{C(x)} A$ is the set of all weakly maximal points of A with respect to the cone $C(x)$.

Definition 1.3 Let X, Y be real topological vector spaces. The set valued mapping $T : X \to 2^Y$ is a closed mapping iff the following holds:
the net $(x_\alpha) \to x_0, (y_\alpha) \to y_0,\ y_\alpha \in T(x_\alpha) \Rightarrow y_0 \in T(x_0)$.

Lemma 1.1[35] *Let K be a nonempty subset of a Hausdorff topological vector space X. Let $G : K \to 2^X$ be a KKM mapping such that for any $y \in K, G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$. Then there exists $x^* \in K$ such that $x^* \in G(y)$ for all $y \in K$.*

Lemma 1.2[9] *Let X and Y be Hausdorff topological vector spaces and $L(X, Y)$ be the topological vector space under the σ-topology. Then, the bilinear mapping*

$$\langle \cdot, \cdot \rangle : L(X, Y) \times X \longrightarrow Y$$

is continuous on $L(X, Y) \times X$, where $\langle l, x \rangle$ denotes the evaluation the linear operator $l \in L(X, Y)$ at $x \in X$.
Lemma 1.3[33] *Let X, Y, Z be the real topological vector spaces, let K and C be two nonempty subsets of X and Y respectively. Let $F : K \times C \to 2^Z,\ T : K \to 2^Y$ be the set valued mappings. If both F and T are upper semicontinuous with nonempty compact values, then the multivalued mapping $G : K \to 2^Z$ defined by*

$$G(x) = \bigcup_{y \in T(x)} F(x, y) = F(x, T(x))$$

is upper semicontinuous with nonempty compact values.

2. EXISTENCE OF WEAK SOLUTIONS FOR THE GMGVVLIP (1.1)

Theorem 2.1 *Let X, Y be the real Hausdorff topological vector spaces, K a nonempty closed convex subset of $X, C : K \to 2^Y$ a set-valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and int $C(x) \neq \emptyset$. Given the mappings $A : K \times L(X, Y) \to L(X, Y), h : K \times K \to Y, \eta : K \times K \longrightarrow K, g : K \to K, T : K \to 2^{L(X,Y)}$ and $\upsilon : K \times K \to Y$, suppose that*

(i) *$0 \leq_{C(x)} \upsilon(x, x)$ for all $x \in K$;*
(ii) *for each $x \in K$, there is an $s \in T(x)$ such that for all $y \in K$*

$$\upsilon(x, y) - \langle A(x, s), \eta(y, g(x)) \rangle + h(g(x), y) \leq_{C(x)} 0;$$

(iii) *for each $x \in K$, the set $\{y \in K : 0 \not\leq_{C(x)} \upsilon(x, y)\}$ is convex;*
(iv) *there is a nonempty compact convex subset D of K such that for every $x \in K \backslash D$, there is a $y \in D$ such that for all $s \in T(x)$*

$$\langle A(x, s), \eta(y, g(x)) \rangle + h(g(x), y) \leq_{intC(x)} 0;$$

(v) *for each $y \in K$, the set*

$$\{x \in K : \langle A(x, s), \eta(y, g(x)) \rangle + h(g(x), y) \leq_{intC(x)} 0,\ \forall\ s \in T(x)\}$$

is open in K.

Then there exists $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x})) \rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

That is, $\bar{x} \in K$ is a solution of the problem (1.1).

Proof Define a set-valued mapping $\Omega : K \to 2^D$ by

$$\Omega(y) = \{x \in D : \exists\ s \in T(x) \text{ such that } \langle A(x, s), \eta(y, g(x)) \rangle + h(g(x), y) \not\leq_{intC(x)} 0\},$$

for all $y \in K$. From condition (v), we know that for each $y \in K$, the set $\Omega(y)$ is closed in K and hence it is compact in D because of the compactness of D. Next we claim that the family $\{\Omega(y) : y \in K\}$ has the finite intersection property, then whole intersection $\bigcap_{y \in K} \Omega(y)$ is nonempty and any element in the intersection $\bigcap_{y \in K} \Omega(y)$ is a solution of (1.1). For any given nonempty finite subset N of K, let $D_N = conv\{D \cup N\}$, the convex hull of $D \cup N$. Then D_N is a compact convex subset of K. Define the set-valued mappings $S, R : D_N \to 2^{D_N}$ respectively by

$$S(y) = \{x \in D_N : \exists\, s \in T(x) \text{ such that } \langle A(x,s), \eta(y, g(x))\rangle + h(g(x), y) \nleq_{intC(x)} 0\},$$
$$R(y) = \{x \in D_N : 0 \leq_{C(x)} \upsilon(x,y)\}, \text{ for each } y \in D_N.$$

From the conditions (i) and (ii), we have

$$0 \leq_{C(y)} \upsilon(y,y) \text{ for all } y \in D_N, \tag{2.1}$$

and for each $y \in K$ there is an $s \in T(y)$ such that

$$\upsilon(y,y) - \langle A(y,s), \eta(y, g(y))\rangle + h(g(y), y) \leq_{C(y)} 0.$$

Hence

$$0 \leq_{C(y)} \langle A(y,s), \eta(y, g(y))\rangle + h(g(y), y)$$

and then $y \in S(y)$ for all $y \in D_N$. We can easily see that S has closed value in D_N. Since for each $y \in D_N, \Omega(y) = S(y) \cap D$. If we prove that whole intersection of the family $\{S(y) : y \in D_N\}$ is nonempty, we can deduce that the family $\{\Omega(y) : y \in K\}$ has the finite intersection property because $N \subset D_N$ and due to the condition (iv). In order to deduce the conclusion of our theorem we can apply Lemma 1.1 if we claim that S is a KKM-mapping. Indeed if S is not a KKM- mapping neither is R since $R(y) \subset S(y)$ for each $y \in D_N$, then there is a nonempty finite subset M of D_N such that

$$conv\ M \not\subset \bigcup_{u \in M} R(u).$$

Thus there is an element $\bar{u} \in conv\ M \subset D_N$ such that $\bar{u} \notin R(u)$ for all $u \in M$, that is

$$0 \nleq_{C(\bar{u})} \upsilon(\bar{u}, u), \text{ for all } u \in M.$$

By (iii) we have

$$\bar{u} \in convM \subset \{y \in K : 0 \nleq_{C(\bar{u})} \upsilon(\bar{u}, y)\}$$

and hence

$$0 \nleq_{C(\bar{u})} \upsilon(\bar{u}, \bar{u}),$$

which contradicts (2.1). Hence R is a KKM-mapping and so is S. Therefore there exists $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \nleq_{intC(\bar{x})} 0.$$

. That is, $\bar{x} \in K$ is a solutions of the problem (1.1). This completes the proof.

If we further assume that $h : K \times K \longrightarrow Y$ is continuous on $K, \eta : K \times K \longrightarrow K$ is also continuous, let the mappings $A : K \times L(X,Y) \to L(X,Y), g : K \to K$ be continuous and $T : K \to 2^{L(X,Y)}$ be upper semicontinuous with nonempty compact values. Then, by using Lemma 1.2 and Lemma 1.3, we can prove that the condition (v) of Theorem 2.1 is satisfied. Hence we have the following result.

Theorem 2.2 *Let X, Y be real Hausdorff topological vector spaces, K a nonempty closed convex subset of $X, C : K \to Y$ a set valued mapping such that for each $x \in K, C(x)$ is a proper closed convex pointed cone with apex at the origin and $intC(x) \neq \emptyset$. Let the mappings $A : K \times L(X,Y) \to L(X,Y), h : K \times K \to Y, \eta : K \times K \to K$ and $g : K \to K$ be continuous. Let $T : K \to 2^{L(X,Y)}$ be the upper semicontinuous with nonempty compact values and $\upsilon : K \times K \to Y$. Suppose that*

(i) *the conditions (i)-(iv) of Theorem 2.1 hold;*

(ii) *the mapping* $W : K \longrightarrow 2^Y$ *defined by* $W(x) = Y \setminus (-intC(x)),\ \forall\, x \in K$ *is upper semicontinuous.*

Then there exists $\bar{x} \in K$ *such that for each* $y \in K$*, there exists* $\bar{s} \in T(\bar{x})$ *satisfying*

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

That is, $\bar{x} \in K$ *is a solution of the problem (1.1).*

Proof For each fixed $y \in K$, define the mappings $F : L(X, Y) \times K \longrightarrow Y$ and $G : K \longrightarrow 2^Y$ by

$$F(s, x) = \langle s, \eta(y, g(x))\rangle + h(g(x), y) \text{ and}$$
$$G(x) = \bigcup_{s \in T(x)} F(s, x).$$

It follows from the continuity of the mapping A, h, η, g and Lemma 1.2 that the mapping F is continuous. Since T is upper semicontinuous with nonempty compact values, it follows from Lemma 1.3 that G is also upper semicontinuous on K with nonempty compact values. We claim that for each $y \in Y$, the set $M = \{x \in K : G(x) \not\subseteq (-intC(x))\}$ is closed in K. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in M and $x_\lambda \longrightarrow x_0$. Then we have $x_\lambda \in K$, $x_0 \in K$ and $G(x_\lambda) \not\subseteq (-intC(x))$ for each $\lambda \in \Lambda$. Hence, for each $\lambda \in \Lambda$, there exists $u_\lambda \in G(x_\lambda)$ such that $u_\lambda \notin (-intC(x))$ and hence $u_\lambda \in K \setminus (-intC(x))$. Noting that the set $L = \{x_\lambda\}_{\lambda\in\Lambda} \bigcup \{x_0\}$ is compact and G upper semicontinuous with compact values we have $G(L)$ is compact. Since $\{u_\lambda\}_{\lambda\in\Lambda} \subseteq G(L)$, without any loss of generality, we can assume $u_\lambda \longrightarrow u_o$. By the upper semicontinuous of the mappings G and W, we have $u_o \in G(x_o)$ and so $u_o \notin (-intC(x_o))$. Hence $G(x_o) \not\subseteq (-intC(x_o))$, $x_0 \in M$ and M is a closed set. It follows that the set

$$\{x \in K : \langle A(x, s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{intC(x)} 0,\ \forall s \in T(x)\}$$
$$= \{x \in K : G(x) \subseteq -intC(x)\} = K \setminus M$$

is open in K. Then all the conditions of Theorem 2.1 hold. By Theorem 2.1, there exists $\bar{x} \in K$ such that for each $y \in K$, there exists $\bar{s} \in T(\bar{x})$ satisfying

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

That is, $\bar{x} \in K$ is a solutions of the problem (1.1).

Theorem 2.3 *Let* $X, Y, K, C, A, h, \eta, g, T$ *be the same as in Theorem 2.1. Assume that for each* $x \in K, h$ *is* $C(x)$*-convex in* K *such that*

(i) *for each* $x \in K$*,* h *is* $C(x)$*-convex in the second argument;*

(ii) η *is affine at first argument;*

(iii) *for each* $x \in K$ *there is an* $s \in T(x)$ *such that*

$$\langle A(x, s), \eta(x, g(x))\rangle + h(g(x), x) \not\leq_{intC(x)} 0;$$

(iv) *there is a nonempty compact convex subset* D *of* K *such that for every* $x \in K \backslash D$ *there is* $y \in D$ *such that*

$$\langle A(x, s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{intC(x)} 0,\ \forall\ s \in T(x);$$

(v) *for each* $y \in K$*, the set*

$$\{x \in K : \langle A(x, s), \eta(y, g(x))\rangle + h(g(x), y) \leq_{intC(x)} 0,\ \forall\ s \in T(x)\}$$

is open in K*.*

Then, there exists $\bar{x} \in K$ *such that for each* $y \in K$*, there exists* $\bar{s} \in T(\bar{x})$ *satisfying*

$$\langle A(\bar{x}, \bar{s}), \eta(y, g(\bar{x}))\rangle + h(g(\bar{x}), y) \not\leq_{intC(\bar{x})} 0.$$

That is, $\bar{x} \in K$ *is a solutions of the problem (1.1).*

Proof For any given nonempty finite subset N of K let $D_N = conv(D \cup N)$. Then D_N is a nonempty compact convex subset of K. Define $\Omega : K \longrightarrow 2^D$ and $S : D_N \to 2^{D_N}$ as in the proof of Theorem 2.1. We note that for each $x \in D_N, S(x)$ is nonempty and closed since

$x \in S(x)$ by conditions (iii) and (v). For each $y \in K, \Omega(y)$ is nonempty and compact in D. Next we claim that S is a KKM-mapping. Indeed if not there is a nonempty finite subset M of D_N such that

$$conv\ M \not\subset \bigcup_{x \in M} S(x).$$

Then there is an $x^* \in convM \subset D_N$ such that

$$\langle A(x^*, s), \eta(x, g(x^*))\rangle + h(g(x^*), x) \leq_{intC(x^*)} 0,\ \forall\ x \in M,\ s \in T(x^*).$$

Since η is affine in the first argument and h is $C(x^*)$-convex in the second variable, the mapping

$$x \rightarrow \langle A(x^*, s), \eta(x, g(x^*))\rangle + h(g(x^*), x), \forall s \in T(x^*)$$

is also $C(x^*)$-convex on D_N. Hence we can deduce that

$$\langle A(x^*, s), \eta(x^*, g(x^*))\rangle + h(g(x^*), x^*) \leq_{intC(x^*)} 0, \text{ for all } s \in T(x^*).$$

This contradict the condition (iii). Therefore S is a KKM-mapping by Lemma 1.4, we have

$$\bigcap_{x \in D_N} S(x) \neq \emptyset.$$

Note that for any $u \in \bigcap_{x \in D_N} S(x)$, we have $u \in D$ by condition (iv). Hence, we have

$$\bigcap_{y \in N} \Omega(y) = \bigcap_{y \in N} (S(y) \cap D) \neq \emptyset$$

for each nonempty finite subset N of K. Therefore the whole intersection $\bigcap_{y \in K} \Omega(y)$ is nonempty. Let $\bar{x} \in \bigcap_{y \in K} \Omega(y)$. Then $(\bar{x}, \bar{s})$ is a solution of problem (1.1).

3. EXISTENCE OF STRONG SOLUTIONS FOR THE GMGVVLIP (1.2)

Theorem 3.1 *Let X be a real Banach space, Y, K, C, η, A, h, g and υ be the same as in Theorem 2.2. Under the assumptions of Theorem 2.2, we have a weak solution $\bar{x}$ of the GMGVVLIP (1.1) with $\bar{s} \in T(\bar{x})$. In addition, if K is compact, $x \rightarrow Y \backslash \{-intC(x)\}$ a closed mapping on $K, T(\bar{x})$ is convex, h is $C(\bar{x})$-convex in the second argument and continuous on K, the mappings $A : K \times L(X, Y) \rightarrow L(X, Y), g : K \rightarrow K$ are continuous, $\eta : K \times K \longrightarrow K$ is continuous and affine in the first argument, $T : K \rightarrow 2^{L(X,Y)}$ is upper semicontinuous with nonempty compact values and the mapping $s \rightarrow -\langle A(x, s), \eta(x, g(\bar{x}))\rangle$ is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $x \in K$. Assume that*

$$(L^*)\ Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset$$

$$Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} + C(\bar{x}), \forall s \in T(\bar{x}).$$

Assume also that

(*i*) *for any fixed $x \in K$, if*

$$\delta \in Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\}$$

and δ can not be compared with

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)$$

which does not equal to δ, then

$$\delta \not\leq_{intC(\bar{x})} 0,$$

(ii) if

$$Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y \backslash (-intC(\bar{x})),$$

there exists an $s \in T(\bar{x})$ such that

$$Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y \backslash (-intC(\bar{x})).$$

Then $\bar{x}$ is a strong solution of the GMGVVLIP (1.2), that is there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x) \nleq_{intC(\bar{x})} 0, \ \forall\, x \in K.$$

Furthermore, the set of all strong solutions of problem (1.2) is compact.

Proof Since η is affine in the first argument and h is $C(\bar{x})$-convex in the second argument on K, the mapping

$$x \to \langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)$$

is also $C(\bar{x})$-convex on K. Since the mapping

$$s \to -\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle$$

is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $\bar{x} \in K$, it follows that the mapping

$$s \to -\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x), \text{ for each } x \in K$$

is also properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $\bar{x} \in K$. From Theorem 2.1, we know that $\bar{x} \in K$ such that (1.1) holds for all $x \in K$ and for some $\bar{s} \in T(\bar{x})$. Then

$$\forall \gamma \in Min^{C(\bar{x})} \bigcup_{x \in K} Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\},$$

by (i), we have

$$\gamma \nleq_{intC(\bar{x})} 0.$$

From condition (L^*), the convexity of $T(\bar{x})$, and the Ferro Minimax Theorem [27] we have, for every

$$\alpha \in \text{Max}^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} \text{Min}_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\}, \ \alpha \nleq_{intC(\bar{x})} 0.$$

This implies that

$$\text{Max}^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} \text{Min}_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y \backslash (-intC(\bar{x})).$$

From (ii) there is an $\bar{s} \in T(\bar{x})$ such that

$$\text{Min}_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y \backslash (-intC(\bar{x})).$$

Hence we know that

$$\forall \rho \in \bigcup_{x \in K} \{\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\},$$

therefore

$$\rho \nleq_{int\ C(\bar{x})} 0.$$

Hence there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x) \nleq_{intC(\bar{x})} 0, \text{ for all } x \in K,$$

such that $\bar{x}$ is a strong solution of the GMGVVLIP (1.2).
Finally to see that the solution set of the GMGVVLIP (1.2) is compact, it is sufficient to show that the solution set is closed due to the coercivity condition (iv) of Theorem 2.1. To this end

let Γ denote the solution set of the GMGVVLIP (1.2). Suppose that a net $\{x_\lambda\} \subset \Gamma$ which converges to p. Fix any $y \in K$, for each λ, there is $s_\lambda \in T(x_\lambda)$ such that

$$\langle A(x_\lambda, s_\lambda), \eta(y, g(x_\lambda))\rangle + h(g(x_\lambda), y) \not\leq_{intC(x_\lambda)} 0. \tag{3.1}$$

Since T is upper semicontinuous with nonempty compact values and the set $\{x_\lambda\} \cup \{p\}$ is compact, it follows that $T(\{x_n\} \cup \{p\})$ is compact. Therefore, without loss of generality, we may assume that the sequence $\{s_\lambda\}$ converges to some s. Then $s \in T(p)$ and

$$h(g(x_\lambda), y) - \langle A(x_\lambda, s_\lambda), \eta(y, g(x_\lambda))\rangle \notin intC(x_\lambda).$$

This implies that

$$h(g(x_\lambda), y) - \langle A(x_\lambda, s_\lambda), \eta(y, g(x_\lambda))\rangle \in Y\backslash(-intC(x_\lambda)).$$

By the continuity of A, η, g and h and Lemma 1.2, we have

$$h(g(p), y) - \langle A(x, s), \eta(y, g(p))\rangle$$

$$= \lim_{x \to \infty} h(g(x_n), y) - \langle A(x_n, s_n), \eta(y, g(x_n))\rangle \in Y\backslash(-intC(p)).$$

Then we obtain

$$\langle A(x, s), \eta(y, g(p))\rangle + h(g(p), y) \not\leq_{intC(p)} 0.$$

Hence $p \in \Gamma$ and Γ is closed.

Theorem 3.2 *Let X be a real Banach space, let Y, K, C, A, h, g, η and T be as in Theorem 2.2. Under the assumption of Theorem 2.2, we have a weak solution $\bar{x}$ of the problem (1.1) with $\bar{s} \in T(\bar{x})$. In addition, if $T(\bar{x})$ is convex, h is $C(\bar{x})$-convex with respect to the first variable, $x \to Y\backslash(-intC(x))$ a closed mapping on K and the mappings $A : K \times L(X,Y) \to L(X,Y), g : K \to K, h : K \times K \to Y$ are continuous. Suppose that $T : K \to 2^{L(X,Y)}$ is upper semi continuous with nonempty compact values and the mapping $s \to -\langle A(x, s), \eta(x, g(\bar{x}))\rangle$ is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $x \in K$. Assume for any nonempty compact subset M of K :*

$$(L^*)\ Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset$$

$$Min_w^{C(\bar{x})} \bigcup_{x \in K} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} + C(\bar{x}), \forall s \in T(\bar{x}).$$

Assume also that

(i) *for any fixed $x \in M$, if*

$$\delta \in Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\}$$

and δ can not be compared with

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x),$$

which is not equal to δ, then

$$\delta \not\leq_{int\ C(\bar{x})} 0,$$

(ii) *if*

$$Max^{C(\bar{x})} \bigcup_{s \in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{\bar{x} \in M} \{\langle A(\bar{x}, s), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\} \subset Y\backslash(-intC(\bar{x})).$$

Then $(\bar{x}, \bar{s})$ is a strong solution of the problem (1.2), that is there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x}, \bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x) \not\leq_{intC(\bar{x})} 0, \text{ for all } x \in K.$$

Furthermore, the set of all strong solutions of the problem (1.2) is compact.

proof Let $\bar{B}(0,r) = \{x \in X : \|x\| \leq r\}$ for each $r > 0$, then the set $K_r = \bar{B}(0,r) \cap K$ is compact in X. If $K_r \neq \emptyset$ and we replace K by K_r, in Theorem 3.1, all the conditions of Theorem 3.1 hold. Hence by Theorem 3.1 there exists $\bar{s} \in T(\bar{x})$ such that

$$\langle A(\bar{x},\bar{s}), \eta(z, g(\bar{x}))\rangle + h(g(\bar{x}), z) \nleq_{int\ C(\bar{x})} 0, \text{ for all } z \in K_r. \tag{3.2}$$

Let us choose $r > \|g(\bar{x})\|$. Since g is continuous and convex for any $x \in K$, choose $t \in (0,1)$ small enough such that $(1-t)\bar{x} + tx \in K_r$. Putting $z = (1-t)\bar{x} + tx$ in (3.3), we have

$$\langle A(\bar{x},\bar{s}), \eta((1-t)\bar{x} + tx, g(\bar{x}))\rangle + h(g(\bar{x}), (1-t)\bar{x} + tx) \nleq_{intC(\bar{x})} 0.$$

We note that

$$\begin{aligned} & t\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), (1-t)\bar{x} + tx) \\ \leq_{C(\bar{x})} & \ t\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + (1-t)h(g(\bar{x}), \bar{x}) + th(g(\bar{x}), x) \\ = & \ t\{\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x)\}, \end{aligned}$$

which implies that

$$\langle A(\bar{x},\bar{s}), \eta(x, g(\bar{x}))\rangle + h(g(\bar{x}), x) \nleq_{intC(\bar{x})} 0, \text{ for all } x \in K.$$

This completes the proof.

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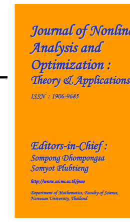

MINIMAX PROGRAMMING WITH (G,α)-INVEXTY

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ABSTRACT. In this paper, we deal with the minimax programming (P) under the differentiable (G,α)-invexity which was proposed in [J. Nonlinear Anal. Optim. 2(2): 305-315]. With the help of auxiliary programming problem (G-P), some new Kuhn-Tucker necessary conditions, namely for G-Kuhn-Tucker necessary conditions, is presented for the minimax programming (P). Also G-Karush-Kuhn-Tucker sufficient conditions under (G,α)-invexity assumptions are obtained for the minimax programming (P). Making use of these optimality conditions, we construct a dual problem (DI) for (P) and establish weak, strong and strict converse duality theorems between problems (P) and (DI).

Keywords: (G,α)-invexity; minimax programming, optimal solution, G-Kuhn-Tucker necessary optimality conditions
AMS Subject Classification: 90C29, 90C46.

1. INTRODUCTION

Convexity plays a central role in many aspects of mathematical programming including analysis of stability, sufficient optimality conditions and duality. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1, 2, 10]. One of these concepts, invexity, was introduced by Hanson in [7]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush-Kuhn-Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [6] introduced the concept of pre-invex functions which is a special case of invexity.

Recently, Antczak extended further invexity to G-invexity [3] for scalar differentiable functions and introduced new necessary optimality conditions for differentiable mathematical programming problem. Antczak also applied the introduced G-invexity notion to develop sufficient optimality conditions and new duality results for differentiable mathematical programming problems. Furthermore, in the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions. In [4], Antczak defined

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vector G-invex (G-incave) functions with respect to η, and applied this vector G-invexity to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. He also established the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable vector optimization problems under the Kuhn-Tucker constraint qualification [4]. With this vector G-invexity concept, Antczak proved new duality results for nonlinear differentiable multiobjective programming problems [5]. A number of new vector duality problems such as G-Mond–Weir, G-Wolfe and G-mixed dual vector problems to the primal one were also defined in [5].

Motivated by [4, 5, 9], we [12] presented the vector (G, α)-invexity concept. In this sequel, we deal with nonlinear minimax programming problems with the vector (G, α)-invexity, and the nonlinear minimax programming problem is presented as follows.

$$\text{(P)} \qquad \min \sup_{y\in Y} \phi(x, y)$$
$$\text{subject to} \quad g_j(x) \leq 0,\ j \in M = \{1, \cdots, m\}$$

where Y is a compact subset of $\mathbb{R}^p$, $\phi(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g_j(\cdot) : \mathbb{R}^n \to \mathbb{R}$ $(j \in M)$. Let E be the set of feasible solutions of problem (P); in other words, $E = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in M\}$. For convenience, let us define the following sets for every $x \in E$.

$$J(x) = \{j \in M \mid\ g_j(x) = 0\},\ Y(x) = \left\{ y \in Y | \varphi(x, y) = \sup_{z\in Y} \varphi(x, z) \right\}.$$

The rest of the paper is organized as follows. In Section 2, we present concepts regarding to vector (G, α)-invexity. In Section 3, we present G-Karush-Kuhn-Tucker sufficient and necessary optimality conditions for the minimax fractional mathematical programming problems. When the sufficient conditions are utilized, dual problem is formulated and duality results are presented in Section 4.

2. Vector (G, α)-invex functions

In this section, we provide some definitions and some results that we shall use in the sequel. The following convenience for equalities and inequalities will be used throughout the paper. For any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T$, we define:

$$x > y \text{ if and only if } x_i > y_i, \text{ for } i = 1, 2, \cdots, n;$$
$$x \geqq y \text{ if and only if } x_i \geq y_i, \text{ for } i = 1, 2, \cdots, n;$$
$$x \geqslant y \text{ if and only if } x_i \geq y_i, \text{ for } i = 1, 2, \cdots, n, \text{ but } x \neq y;$$
$$x \not> y \text{ is the negation of } x > y.$$

We say that a vector $z \in \mathbb{R}^n$ is negative if $z \leqq 0$ and strictly negative if $z < 0$.

Let $g = (g_1, \cdots, g_m) : X \to \mathbb{R}^m$ be a vector-valued differentiable function defined on a nonempty set $X \subset \mathbb{R}^n$; let $I_{g_i}(x)$ be the range of g_i, that is, the image of X under g_i for each $i \in M$. Further, suppose that $G_g = (G_{g_1}, \cdots, G_{g_m}) : \mathbb{R} \to \mathbb{R}^m$ be a vector-valued function such that $G_{g_i} : I_{g_i}(X) \to \mathbb{R}$ is strictly increasing on $I_{g_i}(X)$ for each $i \in M$. The following Definition 2.1 is taken from [12]

Definition 2.1. Let $g = (g_1, \cdots, g_m) : X \to \mathbb{R}^m$ be a vector-valued differentiable function defined on a nonempty open set $X \subset \mathbb{R}^n$; let $I_{g_i}(x)$ be the range of g_i for each $i \in M$. If there exist a differentiable vector-valued function $G_g = (G_{g_1}, \cdots, G_{g_m}) : R \to \mathbb{R}^m$ such that any its component $G_{g_i} : I_{g_i}(X) \to \mathbb{R}$ is strictly increasing on $I_{g_i}(X)$, a vector-valued function $\eta : X \times X \to \mathbb{R}^n$ and real functions $\alpha_i : X \times X \to \mathbb{R}_+ (i \in M)$ such that, for all $x \in X$ $(x \neq u)$,

$$G_{g_i}(g_i(x)) - G_{g_i}(g_i(u)) \geq \alpha_i(x, u)G'_{g_i}(g_i(u))\nabla g_i(u)\eta(x, u), i \in M, \qquad (2.1)$$

then g is said to be a (strictly) vector (G_g, α)-invex function at u on X (with respect to η) (or shortly, (G_g, α)-invex function at u on X), where $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_m)^T$. If (2.1) is satisfied for each $u \in X$, then g is vector (G_g, α)-invex on X with respect to η.

Remark 2.2. In order to define an analogous class of (strictly) vector (G_g, α)-incave functions with respect to η, the direction of the inequality in the definition of these functions should be changed to the opposite one.

We note that the (G_g, α)-invex function is a generalization of α-invex and G_g-invex function.

For convenience, we need the following nonlinear fractional programming problem (G-P).

$$(G\text{-P})\quad \min \sup_{y\in Y} G_\phi(\phi(x,y))$$
$$s.t.\ G_g g(x) := (G_{g_1}(g_1(x)), G_{g_2}(g_2(x)), \cdots, G_{g_m}(g_m(x))) \leqq G_g(0),$$

where $G_g(0) := (G_{g_1}(0), G_{g_2}(0), \cdots, G_{g_m}(0))$. We denote by $X_{G\text{-}P} = \{x \in \mathbb{R}^n \mid G_g g(x) \leq G_g(0)\}$, $J'(\bar{x}) := \{j \in M : G_{g_j} g_j(\bar{x}) = G_{g_j}(0)\}$. If function G_{g_j} is strictly increasing on $I_{g_j}(X)$ for each $j \in M$, then $X_P = X_{G\text{-}P}$ and $J(\bar{x}) = J'(\bar{x})$. So, we represent the set of all feasible solutions and the set of constraint active indices for either (CVP) or (G-CVP) by the notations E and $J(\bar{x})$, respectively.

Theorem 2.3. *Let G_ϕ be a strictly increasing function defined on $I_\phi(X, Y)$, G_{g_j} be a strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$, and $0 \in I_{g_j}(X)$, $j \in M$. Then $\bar{x}$ is an optimal solution for (P) if and only if $\bar{x}$ is also an optimal solution for $(G\text{-}P)$.*

Proof "if" part, we prove that if $\bar{x}$ is an optimal solution for (G-P), then $\bar{x}$ is an optimal solution for (P). On the contrary, let $\bar{x}$ be an optimal solution for (G-P) but not an optimal solution for (P). Define

$$f(x) := \sup_{y\in Y} \phi(x,y),$$

Then there exists $x_0 \in E$ such that

$$f(x_0) < f(\bar{x}).$$

This means that

$$\phi(x_0, y) < \phi(\bar{x}, z), \forall\, y \in Y(x_0), \forall\, z \in Y(\bar{x}).$$

Note that the strictly monotonicity of G_ϕ, we have

$$G_\phi(\phi(x_0, y)) < G_\phi(\phi(\bar{x}, z)), \forall\, y \in Y(x_0), \forall\, z \in Y(\bar{x}).$$

This contradicts to the assumption that $\bar{x}$ is an optimal solution for (G-P).

The proof of "only if" part is similar to "if " part, we omitted it.

3. Optimality conditions in minimax programming

In [4], Antczak introduced the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable multiobjective programming problem. Under G-invexity assumptions, he considered also G-Karush-Kuhn-Tucker sufficient optimality conditions for this kind of multiobjective programming problem. Here, we firstly present some G-Kuhn-Tucker necessary optimality conditions for differentiable minimax programming problem through an auxiliary programming problem. After that, we give some sufficient optimality conditions under (G, α)-invexity. We shall use the following Theorem 3.1 proved by Schmitendrof in [13].

Theorem 3.1. *Let x^* be an optimal solution to the minimax problem (P). Moreover, the vectors $\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Then there exist positive integer q^* and vectors $y_i \in Y(x^*)$ together with scalars $\lambda_i^* \geqslant 0$ $(i = 1, \ldots, q^*)$ and $\mu_j^* \geqslant 0$ $(j \in M)$ such that*

$$\sum_{i=1}^{q^*} \lambda_i^* \nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) = 0,$$
$$\mu_j^* g_j(x^*) = 0, j \in M,$$
$$\sum_{i=1}^{q^*} \lambda_i^* = 1.$$

Furthermore, if α is the number of nonzero λ_i^*, and β is the number of nonzero μ_j^*, then

$$1 \leq \alpha + \beta \leq n + 1.$$

Theorem 3.2 (G-Karush-Kuhn-Tucker necessary optimality conditions)**.** *Let x^* be an optimal solution to the minimax problem (P). Suppose that G_ϕ is a differentiable and strictly increasing function defined on $I_\phi(X,Y)$, G_{g_j} is a differentiable and strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$. Moreover, the vectors $G'_{g_j}(g_j(x^*))\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Then there exist positive integer q^* $(1 \leq q^* \leq n+1)$ and vectors $y_i \in Y(x^*)$ together with scalars $\lambda_i^* > 0$ $(i = 1, \ldots, q^*)$ and $\mu_j^* \geq 0$ $(j \in M)$ such that*

$$\sum_{i=1}^{q^*} \lambda_i^* G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* G'_{g_j}(g_j(x^*))\nabla g_j(x^*) = 0, \tag{3.1}$$

$$\mu_j^*(G_{g_j}(g_j(x^*)) - G_{g_j}(0)) = 0, j \in M, \tag{3.2}$$

$$\sum_{i=1}^{q^*} \lambda_i^* = 1. \tag{3.3}$$

Proof Since x^* is an optimal solution to the minimax problem (P), we can choose $y_i \in Y(x^*), i = 1, \cdots, q^*$ such that they satisfy Theorem 3.1. For each y_i, we consider the programming problem (P$_{y_i}$) as follows.

$$(\mathrm{P}_{y_i}) \qquad \min \ \ \phi(x, y_i)$$
$$\text{subject to} \ \ g_j(x) \leq 0, \ j \in M = \{1, \cdots, m\}.$$

It is evident that x^* is an optimal solution to (P$_{y_i}$). Using similar arguments as in the proof of Theorem 2.3, we can prove that x^* is an optimal solution to (G-P$_{y_i}$)

$$(G\text{-}\mathrm{P}_{y_i}) \quad \min \quad G_\phi(\phi(x, y_i))$$
$$s.t. \ G_g g(x) := (G_{g_1}(g_1(x)), G_{g_2}(g_2(x)), \cdots, G_{g_m}(g_m(x))) \leqq G_g(0).$$

Thus, there exist $\lambda_i > 0$, $\mu_{ji} \geq 0$ $(j \in M)$ such that

$$\lambda_i \nabla_x (G_\phi(\phi(x^*, y_i))) + \sum_{j=1}^{m} \mu_{ji} \nabla \left(G_{g_j}(g_j(x^*))\right) = 0, \tag{3.4}$$

$$\mu_{ji} \left(G_{g_j}(g_j(x^*)) - G_{g_j}(0)\right) = 0, j \in M.$$

Note that

$$\nabla_x (G_\phi(\phi(x^*, y_i))) = G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i),$$
$$\nabla \left(G_{g_j}(g_j(x^*))\right) = G'_{g_j}(g_j(x^*))\nabla g_j(x^*), j \in M,$$

One obtains from (3.4) that

$$\lambda_i G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_{ji} G'_{g_j}(g_j(x^*))\nabla g_j(x^*) = 0, \tag{3.5}$$

Therefore, one obtains from (3.5) that

$$\sum_{i=1}^{q^*} \lambda_i G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \left(\sum_{i=1}^{q^*} \mu_{ji}\right) \nabla \left(G_{g_j}(g_j(x^*))\right) = 0,$$

or

$$\sum_{i=1}^{q^*} \frac{\lambda_i}{\sum_{j=1}^{q^*} \lambda_j} G'_\phi(\phi(x^*, y_i))\nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \left(\frac{\sum_{i=1}^{q^*} \mu_{ji}}{\sum_{i=1}^{q^*} \lambda_i}\right) \nabla \left(G_{g_j}(g_j(x^*))\right) = 0.$$

Let $\lambda_i^* = \frac{\lambda_i}{\sum_{j=1}^{q^*} \lambda_j}$ and $\mu_j^* = \frac{\sum_{i=1}^{q^*} \mu_{ji}}{\sum_{i=1}^{q^*} \lambda_i}$ in the above equation. Then we can deduce the required results.

Theorem 3.3 (G-Karush-Kuhn-Tucker necessary optimality conditions)**.** *Let x^* be an optimal solution to the minimax problem (P). Suppose that G_ϕ is a differentiable and strictly increasing function defined on $I_\phi(X, Y)$, G_{g_j} is a differentiable and strictly increasing function defined on $I_{g_j}(X)$ such that $G'_{g_j}(g_j(x^*)) > 0$ for each $j \in M$. Moreover, the vectors $\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Then there exist positive integer q^* $(1 \le q^* \le n+1)$ and vectors $y_i \in Y(x^*)$ together with scalars $\lambda_i^* > 0$ $(i = 1, \dots, q^*)$ and $\mu_j^* \ge 0$ $(j \in M)$ such that*

$$\sum_{i=1}^{q^*} \lambda_i^* G'_\phi(\phi(x^*, y_i)) \nabla_x \phi(x^*, y_i) + \sum_{j=1}^{m} \mu_j^* G'_{g_j}(g_j(x^*)) \nabla g_j(x^*) = 0,$$
$$\mu_j^* G_{g_j}(g_j(x^*)) = G_{g_j}(0), j \in M,$$
$$\sum_{i=1}^{q^*} \lambda_i^* = 1.$$

Proof Since $G'_{g_j}(g_j(x^*)) > 0$ for each $j \in M$, and the vectors $\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Then we can deduce that the vectors $G'_{g_j}(g_j(x^*))\nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. Now, from Theorem 3.2, we obtain the required results.

Next, we establish the sufficient optimality conditions for the minimax programming problems (P). In the following theorem, we assume that functions constituting the considered nonlinear optimization problem (P) are (G, α)-invex, and we prove that a feasible point $\bar{x}$, at which the G-Karush-Kuhn-Tucker necessary optimality conditions are fulfilled, is an optimal solution.

Theorem 3.4. *Let x^* be a feasible point for (P), G_ϕ be a differentiable and strictly increasing function defined on $I_\phi(X, Y)$, G_{g_j} be a differentiable and strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$. Suppose that G-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at x^*. Further, assume that $\phi(\cdot, y_i)$ is (G_ϕ, α_i)-invex with respect to η at x^* on X for $i = 1, \cdots, q^*$, g is vector (G_g, β)-invex with respect to the same function η at x^* on X. Then x^* is an optimal solution to (P).*

Proof Suppose, contrary to the result, that x^* is not an optimal solution for (P). Hence, there exists $x_0 \in X$ such that

$$\sup_{y \in Y} \phi(x_0, y) < \phi(x^*, y_i), i = 1, \cdots, q^*.$$

Thus,

$$\phi(x_0, y_i) < \phi(x^*, y_i), i = 1, \cdots, q^*.$$

Since G_ϕ is strictly increasing on $I_\phi(X, Y)$, then

$$G_\phi(\phi(x_0, y_i)) < G_\phi(\phi(x^*, y_i)), i = 1, \cdots, q^*. \tag{3.6}$$

By the generalized invexity assumptions of $\phi(\cdot, y_i)$ and g_j, we have

$$G_\phi(\phi(x_0, y_i)) - G_\phi(\phi(x^*, y_i)) \ge \alpha_i(x_0, x^*) G'_\phi(\phi(x^*, y_i)) \nabla_x \phi(x^*, y_i) \eta(x_0, x^*), i = 1, \cdots, q^*, \tag{3.7}$$

$$G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(x^*)) \ge \beta_j(x_0, x^*) G'_{g_j}(g_j(x^*)) \triangledown g_j(x^*) \eta(x_0, x^*), j \in M \tag{3.8}$$

Multiplying (3.7) and (3.8) by λ_i^* and μ_j^* for $i = 1, \cdots, q^*$ and $j \in M$, respectively, we get

$$\left(\sum_{i=1}^{q^*} \bar{\lambda}_i G'_\phi(\phi(x^*, y_i)) \triangledown \phi(x^*, y_i) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(x^*)) \triangledown g_j(x^*) \right) \eta(x_0, x^*) < 0$$

which contradicts the G-Karush-Kuhn-Tucker necessary optimality condition (3.1). Hence, x^* is an optimal solution for (P), and the proof is complete.

4. Duality theorems

Making use of the optimality conditions of the preceding section, we present dual problem (DI) to the minimax problem (P), and establish weak, strong and strict converse duality theorems. For convenience, we use the following notation.

$$K(x) \quad = \{(q,\, \lambda,\, \bar{y}) \in \mathrm{N} \times \mathbb{R}^q_+ \times \mathbb{R}^{mq} | 1 \leqslant q \leqslant n+1, \lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{R}^q_+ \text{ with } \sum_{i=1}^q \lambda_i = 1,\ \bar{y} = (y_1,\, \ldots,\, y_q) \text{ with } y_i \in Y(x), i = 1,\, \ldots,\, q\}.$$

$H_1(q,\, \lambda,\, \bar{y})$ denotes the set of all triplets $(z, \mu, \nu) \in R^n \times R^n_+ \times R_+$ satisfying

$$\sum_{i=1}^{q} \lambda_i G'_\phi(\phi(z, y_i)) \nabla_z \phi(z, y_i) + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(z)) \nabla g_j(z) = 0, \tag{4.1}$$

$$\phi(z, y_i) \geq \nu, i = 1,\, 2,\, \cdots,\, q, \tag{4.2}$$

$$\mu_j g_j(z) \geq 0, j \in M, \tag{4.3}$$

$$y_i \in Y(z),\ (q,\, \lambda,\, \bar{y}) \in K(z).$$

Our dual problem (DI) can be stated as follows.

$$\text{(DI)} \quad \max_{(q,\, \lambda,\, \bar{y}) \in K(z)} \ \sup_{(z, \mu, \nu) \in H_1(q,\, \lambda,\, \bar{y})} \nu$$

Note that if $H_1(q,\, \lambda,\, \bar{y})$ is empty for some triplet $(q,\, \lambda,\, \bar{y}) \in K(z)$, then $\sup_{(z,\mu,\nu) \in H_1(q,\, \lambda,\, \bar{y})} \nu = -\infty$.

Theorem 4.1 (Weak duality). *Let x and $(z, \mu, \nu, q, \lambda, \bar{y})$ be (P)-feasible and (DI)-feasible, respectively; let G_ϕ be a differentiable and strictly increasing function defined on $I_\phi(X, Y)$, and G_{g_j} be a differentiable and strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$. Suppose that $\phi(\cdot, y_i)$ is (G_ϕ, α_i)-invex at z for $i = 1, \ldots, q$, g_j is (G_{g_j}, β_j)-invex at z for $j \in M$. Then*

$$\sup_{y \in Y} \phi(x,\, y) \geqslant \nu.$$

Proof Suppose to the contrary that $\sup_{y \in Y} \phi(x,\, y) < \nu$. Therefore, we obtain

$$\phi(x, y) < \nu \leq \phi(z, y_i),\ \forall\, y \in Y.$$

Thus

$$\phi(x, y_i) < \phi(z, y_i),\ i = 1, \ldots, q.$$

Note that

$$g_j(x) \leq 0,\ \mu_j g_j(z) \geq 0,\ \mu_j \geq 0,\ j \in M.$$

By the increase of G_ϕ and G_{g_j}, we obtain

$$\sum_{i=1}^{q} \lambda_i \frac{G_\phi(\phi(x, y_i)) - G_\phi(\phi(z, y_i))}{\alpha(x, z)} + \sum_{j=1}^{m} \mu_j \frac{G_{g_j}(g_j(x)) - G_{g_j}(g_j(z))}{\beta_j(x, z)} < 0. \tag{4.4}$$

Similar to the proof of Theorem 3.4, by (4.4) and the generalized invexity assumptions of $\phi(\cdot, y_i)$ and g_j for $i = 1, \cdots, q$ and $j \in M$, we have

$$\left(\sum_{i=1}^{q} \lambda_i G'_\phi(\phi(z, y_i)) \nabla_z \phi(z, y_i) + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(z)) \nabla g_j(z) \right) \eta(x, z) < 0.$$

Thus, we have a contradiction to (4.1). So $\sup_{y \in Y} \phi(x,\, y) \geqslant \nu$.

Theorem 4.2 (Strong duality). *Let x^* be an optimal solution of (P). Suppose that G_ϕ is a strictly increasing differentiable function defined on $I_\phi(X, Y)$, G_{g_j} is a strictly increasing differentiable function defined on $I_{g_j}(X)$ for each $j \in M$. Moreover, the vectors $G'_{g_j}(g_j(x^*)) \nabla g_j(x^*)$, $j \in J(x^*)$, are linearly independent. If the hypothesis of Theorem 4.1 holds for all (DI)-feasible points $(z,\, \mu,\, \nu,\, q,\, \lambda,\, \bar{y})$, then there exists $(q^*,\, \lambda^*,\, \bar{y}^*) \in K$, $(x^*,\, \mu^*,\, \nu^*) \in H_1(q^*,\, \lambda^*,\, \bar{y}^*)$ such that $(q^*,\, \lambda^*,\, \bar{y}^*,\, x^*,\, \mu^*,\, \nu^*)$ is a (DI) optimal solution, and the two problems (P) and (DI) have the same optimal values.*

Proof By Theorem 3.2, there exists $\nu^* = \phi(x^*,\, y_i^*)$ $(i = 1, \cdots, q^*)$, satisfying the requirements specified in the theorem, such that $(q^*,\, \lambda^*,\, \bar{y}^*,\, x^*,\, \mu^*,\, \nu^*)$ is a (DI) feasible solution and $\nu^* = \phi(x^*,\, y_i^*)$, then the optimality of this feasible solution for (DI) follows from Theorem 4.1.

Theorem 4.3 (Strict converse duality). *Let $\bar{x}$ and $(z,\, \mu,\, \nu,\, q,\, \lambda,\, \bar{y})$ be optimal solutions of (P) and (DI), respectively. Suppose that G_ϕ is a differentiable and strictly increasing function defined on $I_\phi(X, Y)$, G_{g_j} is a differentiable and strictly increasing function defined on $I_{g_j}(X)$ for each $j \in M$. Suppose that $\phi(\cdot, y_i)$ is (G_ϕ, α_i)-invex at z for $i = 1, \ldots, q$, g_j is (G_{g_j}, β_j)-invex at z for $j \in M$. Then $\bar{x} = z$; that is, z is a (P)-optimal solution and $\sup_{y\in Y} \phi(\bar{x},\, y) = \nu$.*

Proof Suppose to the contrary that $\bar{x} \neq z$. Using similar arguments as in the proof of Theorem 3.4, we have

$$0 = \left(\sum_{i=1}^{q} \lambda_i G'_\phi(\phi(z, y_i)) \nabla_z \phi(z, y_i) + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(z)) \nabla g_j(z) \right) \eta(\bar{x}, z)$$
$$< \sum_{i=1}^{q} \lambda_i \frac{G_\phi(\phi(\bar{x}, y_i)) - G_\phi(\phi(z, y_i))}{\alpha_i(\bar{x}, z)} + \sum_{j=1}^{m} \mu_j \frac{G_{g_j}(g_j(\bar{x})) - G_{g_j}(g_j(z))}{\beta_j(\bar{x}, z)}$$

and

$$\sum_{j=1}^{m} \mu_j \frac{G_{g_j}(g_j(\bar{x})) - G_{g_j}(g_j(z))}{\beta_j(\bar{x}, z)} \leq 0.$$

Therefore,

$$\sum_{i=1}^{q} \lambda_i \frac{G_\phi(\phi(\bar{x}, y_i)) - G_\phi(\phi(z, y_i))}{\alpha_i(\bar{x}, z)} > 0.$$

From the above inequality, we can conclude that there exists a certain $i_0 \in \{1, \ldots, q\}$, such that

$$G_\phi(\phi(\bar{x}, y_i)) - G_\phi(\phi(z, y_i)) > 0.$$

It follows that

$$\sup_{y\in Y} \phi(\bar{x},\, y) \geq \phi(\bar{x},\, y_{i_0}) > \phi(z, y_{i_0}) > \nu. \tag{4.5}$$

On the other hand, we know from Theorem 4.1 that

$$\sup_{y\in Y} \phi(\bar{x},\, y) = \nu.$$

This contradicts to (4.5).

5. Conclusion

This paper deals with the minimax programming under (G_f, α)-invexity assumptions which was introduced in [12]. Note that this invexity unifies the G-invexity and α-invexity presented in [4] and [9], respectively. By constructing auxiliary mathematical programmings (G-P), we have discussed the relations between programming problems (G-P) and (P). We have proved G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable minimax programming problem (P). We pointed out that our statement of the so-called G-Kuhn-Tucker necessary optimality conditions established in this paper is more general than the classical Kuhn-Tucker necessary optimality conditions found in the literature. Also, we have proved the sufficiency of the introduced G-Karush-Kuhn-Tucker (G-Kuhn-Tucker) necessary optimality conditions for minimax programming problem (P) involving (G, α)-invexity. Making use of the optimality conditions presented in Section 3, we present dual problem (DI) to the minimax problem (P), and establish weak, strong and strict converse duality theorems.

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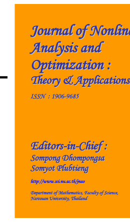

A NOTE ON ULAM-HYERS STABILITY OF A FIXED POINT EQUATION VIA GENERALIZED PICARD OPERATORS

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ABSTRACT. In this note, we introduce new classes operators, which is a generalization of Picard operators, and obtain some Ulam-Hyers stability results for the operators which extend results in [5]. As application, an existence and uniqueness result for an integral equation is given.

KEYWORDS: Ulam-Hyers stability; Generalized Ulam-Hyers stability; Fixed point; Weakly Picard operator.
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1. INTRODUCTION

Let (X, d) be a metric space, Y be a nonempty subset of X and $f : Y \longrightarrow X$ be an operator. The set of fixed points of f will be denoted by $Fix(f) := \{x \in X | x = f(x)\}$. We will denote by $\tilde{B}(x_0, r)$ the closed ball centered in $x_0 \in X$ with radius $r > 0$, i.e., $\tilde{B}(x_0, r) = \{x \in X | d(x_0, x) \le r\}$. Following [3] we present the basic notions of weakly Picard operators.
$I(f) := \{Z \subset Y | f(Z) \subset Z, Z \neq \emptyset\}$ - the set of all invariant subsets of f;
$(MI)_f := \bigcup_{Z \in I(f)} Z$ - the maximal invariant subset of f;
$(AB)_f(x^*) := \{x \in Y | f^n(x)$ is defined for all $n \in \mathbb{N}$ and $f^n(x) \xrightarrow{d} x^* \in Fix(f)\}$- the attraction basin of $x^* \in Fix(f)$ with respect to f;
$(AB)_f := \bigcup_{x^* \in Fix(f)} (AB)_f(x^*)$- the attraction basin of f.

Definition 1.1. ([2]) An operator $f : Y \longrightarrow X$ is nonself weakly Picard operator if $Fix(f) \neq \emptyset$ and $(MI)_f = (AB)_f$. If $Fix(f) = \{x^*\}$, then a nonself weakly Picard operator is said to be nonself Picard operator.

Definition 1.2. ([2]) For each nonself weakly Picard operator $f : Y \longrightarrow X$ we define the operator $f^\infty : (AB)_f \longrightarrow Fix(f) \subset (AB)_f$, by $f^\infty(x) = \lim_{n \longrightarrow \infty} f^n(x)$.

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Definition 1.3. ([2]) Let $f : Y \longrightarrow X$ be a nonself weakly Picard operator and $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing function which is continuous at 0 and $\psi(0) = 0$. The operator f is nonself ψ-weakly Picard operator if

$$d(x, f^\infty(x)) \le \psi(d(x, f(x))), \text{ for all } x \in (MI)_f.$$

In the case that $\psi(t) := ct$ (for some $c > 0$), for each $t \in \mathbb{R}_+$, we say that f is c-weakly Picard operator.

For some examples of nonself weakly Picard operators and ψ-weakly Picard operators, see [2].

If $f : Y \longrightarrow X$ is an operator, let us consider the fixed point equation

$$x = f(x),\ x \in Y \tag{1.1}$$

and the inequation

$$d(y, f(y)) \le \varepsilon. \tag{1.2}$$

Definition 1.4. ([5]) The equation (1.1) is called generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in (AB)_f$ of (1.2) there exists a solution x^* of the fixed point equation (1.1) such that

$$d(y^*, x^*) \le \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, the equation (1.1) is said to be Ulam-Hyers stable.

In 2009, Rus [5] proved the following result:

Theorem 1.5. *Let (X, d) be a metric space, Y be a nonempty subset of X and $f : Y \longrightarrow X$ be a ψ-weakly Picard operator. Then, the fixed point equation (1.1) is generalized Ulam-Hyers stable. In particular, if f is c-weakly Picard operator, then the equation (1.1) is Ulam-Hyers stable.*

This paper is organized as follows: In Section 2, we extend Theorem 1.5 to wider classes of operators. Examples of such opertors are given. Then, in Section 3, an application to an integral equation is also given.

2. MAIN RESULTS

Let (X, d) be a metric space, Y be a nonempty subset of X and $f : Y \longrightarrow X$ be an operator. For a sequence $S = \{s_n\}$ of selfmaps on X, we define the following notions:

$C(S)_f(x^*) = \{x \in X | s_n(x)$ is defined for all $n \in \mathbb{N}$ and $s_n(x) \xrightarrow{d} x^* \in Fix(f)\}$-the convergence set of S at x^*;

$C(S)_f = \bigcup_{x^* \in Fix(f)} C(S)_f(x^*)$-the convergence set of S.

We will denote the composition $f_n \circ f_{n-1} \circ ... \circ f_j$ simply by $\prod_{i=j}^n f_i = f_n \circ f_{n-1} \circ ... \circ f_j$. In particular, $\prod_{i=1}^n f$ is simply the n-th iterate f^n of f. We now introduce new classes of operators.

Definition 2.1. Let S be a sequence of selfmaps on X. An operator $f : Y \longrightarrow X$ is nonself weakly convergence operator with respect to S (nonself WCO wrpt S) if $Fix(f) \ne \emptyset$ and $(MI)_f = C(S)_f$. If $Fix(f) = \{x^*\}$, then a nonself WCO wrpt S is said to be nonself convergence operator with respect to S (nonself CO wrpt S).

It is obvious that if f is a Picard operator, then it is a CO wrpt $S = \{f^n\}$. The converse is not true, as the following example shows:

Example 2.2. Put $X = [\frac{1}{2}, 2]$ and define a mapping $f : X \longrightarrow X$ by $f(x) = \frac{1}{x}$ for $x \in X$. Then f is a CO wrpt $S = \{((1-\lambda)I + \lambda fI)^n\}$ for some $\lambda \in (0,1)$ but it is not a Picard operator.

Proof. It is easy to see that $Fix(f) = \{1\}$. Let $S = \{((1-\lambda)I + \lambda fI)^n\}$, where I denotes the identity map with $\lambda \in (0,1)$. By Example 4.3 in [1], we get that $(MI)_f = C(S)_f = X$. Therefore, f is a CO wrpt $S = \{f^n\}$. We know that $(MI)_f = X \neq \{1\} = (AB)_f$, so f is not a Picard operator. □

Similarity, if f is a weakly Picard operator, then it is a WCO wrpt $S = \{f^n\}$. The converse is not true.

Example 2.3. Let $X = [0,1]$ and $f : X \longrightarrow X$ be given by $f(x) = x$, for all $x \in (0,1)$ and $f(0) = 1$ and $f(1) = 0$. Then f is a WCO wrpt $S = \{((1-\lambda)I + \lambda fI)^n\}$ for some $\lambda \in (0,1)$ but it is not a weakly Picard operator.

Proof. Let $S = \{((1-\lambda)I + \lambda fI)^n\}$, with $\lambda \in (0,1)$. It is easy to see that $Fix(f) = (0,1)$ and $(MI)_f = C(S)_f = X$. Hence, f is a WCO wrpt S. Since $\{f^n(0)\}, \{f^n(1)\}$ do not converge and $(MI)_f = X \neq (0,1) = (AB)_f$, f is not a weakly Picard operator. □

Definition 2.4. Let $S = \{s_n\}$ be a sequence of selfmaps on X and $f : Y \longrightarrow X$ be a nonself WCO wrpt S. We define the operator $r : C(S)_f \longrightarrow Fix(f) \subset C(S)_f$, by $r(x) = \lim_{n\longrightarrow\infty} s_n(x) \in Fix(f)$.

Definition 2.5. Let $S = \{s_n\}$ be a sequence of selfmaps on X, $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ an increasing function which is continuous at 0 and $\psi(0) = 0$. An operator $f : Y \longrightarrow X$ is said to be a nonself ψ-weakly convergence operator with respect to S (nonself ψ-WCO wrpt S) if it is a nonself WCO wrpt S and

$$d(x, r(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in (MI)_f.$$

In the case that $\psi(t) := ct$ (for some $c > 0$), for each $t \in \mathbb{R}_+$, we say that f is a nonself c-WCO wrpt S.

For sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$, if $S = \{\prod_{i=1}^n g_i\}$ is a sequence such that

$$g_i = (1-\alpha_i)I + \alpha_i f[(1-\beta_i)I + \beta f]$$

for each $i \in \mathbb{N}$, a nonself ψ-WCO wrpt S is called nonself ψ-weakly Ishikawa type operator associated to sequences $\{\alpha_n\}$ and $\{\beta_n\}$. When $\{\beta_n\} = \{0\}$, a nonself ψ-WCO wrpt S is called nonself ψ-weakly Mann type operator associated to sequences $\{\alpha_n\}$. A nonself ψ-weakly Ishikawa type operator associated to constant sequence is called nonself ψ-weakly Krasnoselskij type operator.

It is easy to see that if $f : Y \longrightarrow X$ is a ψ-weakly Picard operator, then it is a ψ-WCO wrpt $S = \{f^n\}$. The following example shows that the converse is not true.

Example 2.6. For X and f as in Example 2.2, we obtain f is a ψ-WCO wrpt $S = \{((1-\lambda)I + \lambda fI)^n\}$ for some $\lambda \in (0,1)$ but it is not a ψ-weakly Picard operator.

Proof. From Example 2.2, f is CO wrpt $S = \{((1-\lambda)I + \lambda fI)^n\}$ for some $\lambda \in (0,1)$. Consider

$$d(x, r(x)) = |x-1| \leq |x - \frac{1}{x}| = d(x, f(x)) \leq \psi(d(x, f(x))),$$

where $\psi(t) = at + 1$, $a \geq 1$. Since f is not a Picard operator, it is not a ψ-weakly Picard operator. □

Definition 2.7. If $f : Y \longrightarrow X$ is an operator and $S = \{s_n\}$ be a sequence of selfmaps on X. The equation (1.1) is called generalized Ulam-Hyers stable with respect to S if there exists $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in C(S)_f$ of (1.2) there exists a solution x^* of the fixed point equation (1.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, the equation (1.1) is said to be Ulam-Hyers stable with respect to S.

An Ulam-Hyers stability result is the following:

Theorem 2.8. *Let (X, d) be a metric space, Y be a nonempty subset of X, $f : Y \longrightarrow X$ be a ψ-WCO wrpt S and $S = \{s_n\}$ be a sequence of selfmaps on X. Then, the equation (1.1) is generalized Ulam-Hyers stable with respect to S. In particular, if f is c-WCO wrpt S, then the equation (1.1) is Ulam-Hyers stable with respect to S.*

Proof. Let $\varepsilon > 0$ and $y^* \in C(S)_f$ such that $d(y^*, f(y^*)) \leq \varepsilon$. Since f is ψ-WCO wrpt S, we get

$$d(x, r(x)) \leq \psi(d(x, f(x))),\ x \in (MI)_f.$$

From $(MI)_f = C(S)_f$, we take $x^* := r(y^*)$. Thus, $d(y^*, x^*) \leq \psi(\varepsilon)$. □

The proof presented here based on a standard proof in [5](see [3]). However, we obtain a result for larger classes of operators and the following results are immediate:

Corollary 2.9. *([3]) Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $f : \tilde{B}(x_0, r) \longrightarrow X$ be an α-contraction, such that $d(x_0, f(x_0)) \leq (1 - \alpha)r$. Then, the fixed point equation (1.1) is Ulam-Hyers stable with respect to $S = \{f^n\}$.*

Corollary 2.10. *([3]) Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $f : \tilde{B}(x_0, r) \longrightarrow X$ be an φ-contraction, such that $d(x_0, f(x_0)) \leq r - \varphi(r)$. Suppose also that the function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and onto. Then, the fixed point equation (1.1) is generalized Ulam-Hyers stable with respect to $S = \{f^n\}$.*

We will present some consequences of Theorem 2.8. We need first some definitions and theorems.

Definition 2.11. ([1],[4]) A mapping $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a comparison function if it is increasing and $\varphi^k(t) \longrightarrow 0$ as $k \longrightarrow +\infty$.
As a consequence, we also have $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$ and φ is continuous at 0.

Definition 2.12. ([1]) Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. An operator $f : H \longrightarrow H$ is said to be

(i) generalized pseudo-contraction if there exists a constant $M > 0$ such that

$$\langle f(x) - f(y), x - y \rangle \leq M \cdot \|x - y\|^2,\ x, y \in H;$$

(ii) Lipschitzian if there exists $L > 0$ such that

$$\|f(x) - f(y)\| \leq L \cdot \|x - y\|,\ x, y \in H.$$

Theorem 2.13. ([1]) *Let K be a nonempty closed convex subset of a real Hilbert space and $f : K \longrightarrow K$ a generalized pseudocontractive and Lipschitzian operator with the corresponding constants M and L fulfilling the conditions*

$$0 < M < 1 \text{ and } M \leq L.$$

Then

(i) *f has an unique fixed point p;*
(ii) *for each x_0 in K, the Kranoselskij iteration $\{x_n\}_{n=0}^{\infty}$, given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda f(x_n),\ n = 0, 1, 2, ...$$

converges to p, for all $\lambda \in (0,1)$ satisfying

$$0 < \lambda < 2(1-M)/(1-2M+L^2);$$

(iii) *Both a priori*

$$\|x_n - p\| \leq \frac{\theta^n}{1-\theta} \cdot \|x_1 - x_0\|,\ n = 1, 2, ...$$

and a posteriori

$$\|x_n - p\| \leq \frac{\theta}{1-\theta} \cdot \|x_n - x_{n-1}\|,\ n = 1, 2, ...$$

estimates hold, with

$$\theta = ((1-\lambda)^2 + 2\lambda(1-\lambda)M + \lambda^2 L^2)^{1/2}.$$

Using the previous Theorem, we can prove the following.

Theorem 2.14. *Let K be a nonempty closed convex subset of a real Hilbert space and $f : K \longrightarrow K$ a generalized pseudocontractive and Lipschitzian operator with the corresponding constants M and L fulfilling the conditions*

$$0 < M < 1 \text{ and } M \leq L.$$

Then, the fixed point equation (1.1) is Ulam-Hyers stable with respect to $S = \{((1-\lambda)I + \lambda f I)^n\}$ where $\lambda \in (0,1)$ satisfying $0 < \lambda < 2(1-M)/(1-2M+L^2)$.

Proof. Let $S = \{g^n\}$ such that

$$g = (1-\lambda)I + \lambda f I$$

where $\lambda \in (0,1)$ satisfying $0 < \lambda < 2(1-M)/(1-2M+L^2)$. By Theorem 2.13, $Fix(f) = \{p\}$, $(MI)_f = C(S)_f = K$ and for each $x \in K$,

$$\|x - p\| \leq \frac{\lambda}{1-\theta} \cdot \|x - f(x)\|,$$

where $\theta = ((1-\lambda)^2 + 2\lambda(1-\lambda)M + \lambda^2 L^2)^{1/2}$. Then f is a c-weakly Krasnoselskij type operator with $c := \dfrac{\lambda}{1-\theta} > 0$. Hence, by Theorem 2.8, the fixed point equation (1.1) is Ulam-Hyers stable with respect to S. □

3. Application

Consider the integral equation

$$x(t) = \int_a^b K(t,s,x(s))ds + g(t), \ t \in [a,b]. \tag{3.1}$$

Theorem 3.1. *Assume*

(i) $K : [a,b] \times [a,b] \times \mathbb{R}^n$ *and* $g : [a,b] \longrightarrow \mathbb{R}^n$ *are continuous;*

(ii) K *is Lipschitzian with respect to the third variable, i.e., there exists* $L > 0$ *such that*

$$|K(t,s,u) - K(t,s,v)| \le L|u-v|, \text{ for each } t,s \in [a,b], u,v \in \mathbb{R}^n;$$

(iii) $\int_a^b K(t,s,u) - K(t,s,v)ds \le R(u-v)$, *for each* $t \in [a,b], u,v \in \mathbb{R}^n$ *where* $0 < R < 1$ *and* $R \le L(b-a)$.

Then the following conclusions hold;

(a) *the integral equation (3.1) has a unique solution* x^* *in* $L_2([a,b],\mathbb{R}^n)$,

(b) *there exists a sequence* S *of selfmaps on* X *such that the integral equation (3.1) is Ulam-Hyers stable with respect to* S.

Proof. Let $X := L_2([a,b],\mathbb{R}^n)$ with norm $\|x\| := (\int_a^b x^2(t)dt)^{1/2}$ and inner product $\langle x,y\rangle = \int_a^b x(t)y(t)dt$ for $x,y \in X$. Define $T : X \longrightarrow X$ by

$$Tx(t) := \int_a^b K(t,s,x(s))ds + g(t), \ t \in [a,b].$$

For $x,y \in X$

$$|Tx(t) - Ty(t)| \le \int_a^b |K(t,s,x(s)) - K(t,s,y(s))|ds \le L\int_a^b |x(s)-y(s)|ds.$$

Thus

$$|Tx(t)-Ty(t)|^2 \le L^2(\int_a^b |x(s)-y(s)|ds)^2 \le L^2 \cdot \int_a^b |x(s)-y(s)|^2 ds \cdot \int_a^b 1ds = L^2(b-a)\|x-y\|^2.$$

We have

$$\begin{aligned} \|Tx(t) - Ty(t)\|^2 &= \int_a^b |Tx(t)-Ty(t)|^2 dt \le \int_a^b L^2(b-a)\|x-y\|^2 dt \\ &= L^2(b-a)^2\|x-y\|^2. \end{aligned}$$

Therefore T is Lipschitzian operator, i.e.,

$$\|Tx - Ty\| \le L(b-a)\|x-y\|.$$

Consider

$$\begin{aligned} \langle Tx(t) - Ty(t), x(t)-y(t)\rangle &= \langle \int_a^b K(t,s,x(s)) - K(t,s,y(s))ds, x(t)-y(t)\rangle \\ &= \int_a^b \int_a^b K(t,s,x(s)) - K(t,s,y(s))ds \cdot (x(t)-y(t))dt \\ &\le R\int_a^b (x(t)-y(t))^2 dt = R\|x-y\|^2. \end{aligned}$$

Hence we obtain T is a generalized pseudocontractive and Lipschitzian operator. The conclusion follows from Theorem 2.14. □

Remark 3.2. Note that the operator f in Example 2.2 is a generalized pseudocontractive and Lipschitzian operator with the corresponding constants $M > 0$ and $L = 4$ but it fails to be Picard operator. This means that the operator T in Theroem 3.1 does not satisfy condition in Theorem 1.5.

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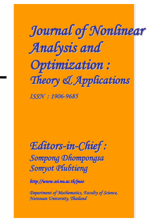

WEAK CONVERGENCE OF FIXED POINT ITERATIONS IN METRIC SPACES

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ABSTRACT. The concept of convergence in normed spaces is extended to metric spaces; and weak convergence of fixed point iterations of contractions on metric spaces is obtained in this article.

KEYWORDS: Directed set; Fixed point iteration; Semi metric.
AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

It is true in a non zero normed space $(X, \|\cdot\|)$ that $\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\} = \sup\{\sup\{|f(x)| : f \in F\} : F$ is a finite nonempty subset of the set $\{g \in X^* : \|g\| = 1\}\}$. Here the collection of all finite nonempty subsets of $\{g \in X^* : \|g\| = 1\}$ is a directed set under the inclusion relation. This article is to consider metrics of the type $d(x, y) = \sup\{d_i(x, y) : i \in I\}$ on a nonempty set X, when each d_i is a semi metric (i.e., $d_i(x, y) = 0$ need not imply $x = y$; following the book [1], p.100) on X, for every i in a directed set $(I, \leq)$; and when $d_i \leq d_j$ whenever $i \leq j$. Convergence of a fixed point iteration through each d_i is considered as weak convergence. For some results in connection with weak convergence for fixed point results in nonlinear functional analysis see [2, 3, 5, 6].

The following two results (see [4]) are fundamental results which are applied to obtain extensions for weak convergence. Many other generalized results can also be applied to obtain results on weak convergence. If d_i is a semi metric on a nonempty set X, then X is said to be d_i-complete or (X, d_i) is said to be complete, if for a sequence $(x_n)_{n=1}^{\infty}$ in X such that $d_i(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$, there is a point x in X such that $d_i(x_n, x) \longrightarrow 0$, as $n \longrightarrow \infty$.

Theorem 1.1. *Suppose d_i is a semi metric on a nonempty set X such that (X, d_i) is complete. Let $T : X \longrightarrow X$ be a given function such that $d_i(T^2(x), T(x)) \leq k d_i(T(x), x), \forall x \in X$, for some $k \in (0, 1)$. Fix $x_0 \in X$ and define $x_1, x_2, \ldots,$ by $x_{n+1} = T(x_n), \forall n = 0, 1, 2, \ldots.$ Then there is a point x^* in X such that*

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$d_i(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$ and $d_i(T(x^*), x^*) = 0$. Moreover, if d_i is a metric, then the fixed point of T is unique.

Theorem 1.2. *Suppose (X, d) is a nonempty compact metric space. Let $T : X \longrightarrow X$ be a function such that $d(T(x), T(y)) < d(x, y)$ whenever $d(x, y) \neq 0$. Then T has a unique fixed point x^*. Moreover, if $x_0 \in X$ is fixed and $x_1, x_2, \ldots$ are defined by $x_{n+1} = T(x_n), \forall n = 0, 1, 2, \ldots$, then $d(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$.*

2. MAIN RESULTS

Let X be a nonempty metric space with a metric d. Suppose $(d_i)_{i\in I}$ is a family of semi metrics on X such that $d(x, y) = \sup_{i\in I} d_i(x, y), \forall x, y \in X$. Suppose further that $(I, \leq)$ is a directed set such that $d_i(x, y) \leq d_j(x, y), \forall x, y \in X$, whenever $i \leq j$ in I. These things are assumed in the following two results. The next theorem 2.1 assumes that one more condition is satisfied.

Consider a nonempty set of the form $A_i = \{y \in X : d_i(x_i, y) = 0\}$, for some $x_i \in X$. If a set of this form A_i is called an i-zero set, and if there is a collection $(A_i)_{i\in I}$ of i-zero sets such that $A_i \supseteq A_j$ whenever $i \leq j$ in I, then it is assumed in the next theorem 2.1 that $\bigcap_{i\in I} A_i \neq \emptyset$.

Theorem 2.1. *Let $(k_i)_{i\in I}$ be a given family of numbers in the open interval $(0, 1)$. Let $T : X \longrightarrow X$ be a mapping such that $d_i(T^2(x), T(x)) \leq k_i d_i(T(x), x)$, $\forall x \in X, \forall i \in I$. Suppose further that each (X, d_i) is complete, for every $i \in I$. Then there is a unique fixed point x^* of T in X. Moreover, if $x_0 \in X$ is fixed and $x_1, x_2, \ldots$ are defined by $x_{n+1} = T(x_n), \forall n = 0, 1, 2, \ldots$, then $d_i(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$, for each $i \in I$.*

Proof. Fix $x_0 \in X$, and define $x_1, x_2, \ldots$ in X by $x_{n+1} = T(x_n), \forall n = 0, 1, 2, \ldots$. Then, by theorem 1.1, for each $i \in I$, there is a point x_i^* in X such that $d_i(x_n, x_i^*) \longrightarrow 0$ as $n \longrightarrow \infty$ and $d_i(T(x_i^*), x_i^*) = 0$.

Write $A_i = \{x \in X : d_i(x, x_i^*) = 0\}$, an i-zero set, for every $i \in I$. For $i \leq j$ in I, if $x \in A_j$, then

$$\begin{aligned} 0 &\leq d_i(x, x_i^*) \\ &\leq d_i(x, x_j^*) + d_i(x_j^*, x_n) + d_i(x_n, x_i^*) \\ &\leq d_j(x, x_j^*) + d_j(x_j^*, x_n) + d_i(x_n, x_i^*) \\ &= d_j(x_j^*, x_n) + d_i(x_n, x_i^*); \end{aligned}$$

and the right hand side tends to zero as n tends to infinity. Thus $A_j \subseteq A_i$, whenever $i \leq j$ in I. So, by assumption, $\bigcap_{i\in I} A_i \neq \emptyset$. Suppose $x^* \in \bigcap_{i\in I} A_i$. Since $0 \leq d_i(x^*, x_n) \leq d_i(x^*, x_i^*) + d_i(x_i^*, x_n) = d_i(x_i^*, x_n)$, then $d_i(x^*, x_n) \longrightarrow 0$ as $n \longrightarrow \infty$, for every $i \in I$. Also, $0 \leq d_i(T(x^*), x^*) \leq d_i(T(x^*), T(x_i^*)) + d_i(T(x_i^*), x_i^*) + d_i(x_i^*, x^*) \leq k_i d_i(x^*, x_i^*) + 0 + 0 = 0, \forall i \in I$, imply that $T(x^*) = x^*$. Moreover, if $y^* = T(y^*)$ for some $y^* \in X$, then $0 \leq d_i(x^*, y^*) = d_i(T(x^*), T(y^*)) \leq k_i d_i(x^*, y^*), \forall i \in I$, imply that $x^* = y^*$. This proves the theorem. □

Note that the assumption made before the statement of the theorem 2.1 is not necessary in the previous theorem, if X is a compact metric space.

Lemma 2.2. *Suppose (X, d) is compact. Let $T : X \longrightarrow X$ be a mapping such that $d_i(T(x), T(y)) < d_i(x, y)$ whenever $d_i(x, y) \neq 0$, with $x, y \in X$ and $i \in I$. Then T has a unique fixed point x^* in X. Moreover, if $x_0 \in X$, and if $x_{n+1} = T(x_n)$, for $n = 0, 1, 2, \ldots$, then $d_i(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$, for each $i \in I$.*

Proof. Fix $x_0 \in X$ and define $x_1, x_2, \dots$ in X by $x_{n+1} = T(x_n)$, for $n = 0, 1, 2, \dots$. To each $i \in I$ and to each $x \in X$, let $[x]_i = \{y \in X : d_i(x, y) = 0\}$. Then for given $x, y \in X$, either $[x]_i = [y]_i$ or $[x]_i \cap [y]_i = \emptyset$, for any $i \in I$. Define $\tilde{d}_i([x]_i, [y]_i) = d_i(x, y), \forall x, y \in X$ and define $\tilde{X}_i = \{[x]_i : x \in X\}$, for any $i \in I$. Then $(\tilde{X}_i, \tilde{d}_i)$ is a compact metric space, for any $i \in I$. Define $T_i : (\tilde{X}_i, \tilde{d}_i) \longrightarrow (\tilde{X}_i, \tilde{d}_i)$ by $T_i([x]_i) = T(x), \forall x \in X$, for any $i \in I$. Then $\tilde{d}_i(T_i([x]_i), T_i([y]_i)) < \tilde{d}_i([x]_i, [y]_i)$ whenever $\tilde{d}_i([x]_i, [y]_i) \neq 0$. Then, by theorem 1.2, for each $i \in I$, there is a point x_i^* in X such that $d_i(T(x_i^*), x_i^*) = 0$, $[x_i^*]_i$ is the unique fixed point of T_i, and $d_i(x_n, x_i^*) \longrightarrow 0$ as $n \longrightarrow \infty$. Consider a subnet of $(x_i^*)_{i \in I}$ that converges to some x^* in (X, d). Then $d_i(T(x^*), x^*) \leq d_i(T(x^*), T(x_i^*)) + d_i(T(x_i^*), x_i^*) + d_i(x_i^*, x^*) \leq 2d_i(x^*, x_i^*) \leq 2d(x^*, x_i^*), \forall i \in I$, imply that $T(x^*) = x^*$. If $y^* = T(y^*)$ for some $y^* \in X$, then $d_i(x^*, y^*) = d_i(T(x^*), T(y^*)) < d_i(x^*, y^*)$, whenever $d_i(x^*, y^*) \neq 0$, for any $i \in I$. This proves the uniqueness of the fixed point of T. Moreover, $d_i(x^*, x_i^*) \leq d_i(T(x^*), T(x_i^*)) + d_i(T(x_i^*), x_i^*) < d_i(x^*, x_i^*)$ whenever $d_i(x^*, x_i^*) \neq 0$. This proves that $d_i(x^*, x_i^*) = 0, \forall i \in I$. So, for every $i \in I, d_i(x_n, x^*) \longrightarrow 0$ as $n \longrightarrow \infty$. This completes the proof. □

Example 2.3. Let X be the collection of all bounded continuous real valued functions defined on the real line R. This is a complete metric space under the metric d defined by $d(f, g) = \sup_{x \in R} |f(x) - g(x)|, \forall f, g \in X$. To each $i = 1, 2, \dots$, define $B_i = (-\infty, -1 - \frac{1}{4^i}] \cup [-1 + \frac{1}{4^i}, 1 - \frac{1}{4^i}] \cup [1 + \frac{1}{4^i}, \infty)$, and define $d_i(f, g) = \sup\{|f(x) - g(x)| : x \in B_i\}, \forall f, g \in X$. Then define $T : X \longrightarrow X$ by

$$(T(f))(x) = \begin{cases} \frac{f(x)}{x} & \text{for } |x| \geq 1 \\ x f(x) & \text{for } |x| \leq 1. \end{cases}$$

Note that $d(f, g) = \sup_{i \in I} d_i(f, g), \forall f, g \in X$, with $I = \{1, 2, \dots\}$, which is a directed set under the usual ordering relation. It can be verified that X, d_i, d, and I satisfy the conditions of the theorem 2.1 with $k_n = \max\left\{\frac{1}{1 + \frac{1}{4^n}}, 1 - \frac{1}{4^n}\right\}$. Here the zero function is the unique fixed point.

This example 2.3 also reveals that the fixed point iteration may not converge strongly with respect to d. But this is not the case when (X, d) is compact. Now the proof of the theorem 2.2 is to be analyzed. The uniqueness part of the proof implies that the net $(x_i^*)_{i \in I}$ converges to x^*. If a subsequence $(z_m)_{m=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ converges to some z^* in (X, d), then $d(z_m, z^*) \longrightarrow 0$ as $m \longrightarrow \infty$, and hence $d_i(z_m, z^*) \longrightarrow 0$ as $m \longrightarrow \infty, \forall i \in I$; whereas $d_i(z_m, x^*) \longrightarrow 0$ as $m \longrightarrow \infty, \forall i \in I$. Thus $d_i(x^*, z^*) = 0, \forall i \in I$ and hence $d(x^*, z^*) = 0$. Thus $x^* = z^*$. So, every subsequence of $(x_n)_{n=1}^{\infty}$ should converge to x^* in the compact metric space (X, d).

Theorem 2.4. *Under the assumptions of lemma 2.2, and for the sequence $(x_n)_{n=1}^{\infty}$ given in lemma 2.2, the following strong conclusion holds:*

$$d(x_n, x^*) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

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LAVRENTIEV REGULARIZATION OF NONLINEAR ILL-POSED EQUATIONS UNDER GENERAL SOURCE CONDITION

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ABSTRACT. Analogues to the procedure adopted by Scherzer et.al (1993) for choosing the regularization parameter in Tikhonov regularization of nonlinear ill-posed equations of the form $F(x) = y$, Tautenhahn (2002) considered an a posteriori parameter choice strategy for Lavrentiev regularization in the case of monotone F, and derived order optimal error estimates under Hölder type source conditions. In this paper, we derive order optimal error estimates under a general source condition so that the results are applicable for both mildly and exponentially ill-posed problems. Results in this paper generalize results of Tautenhahn (2002) and also extend results of Nair and Tautenhahn (2004) to the nonlinear case.

KEYWORDS: Lavrentiev regularization; Inverse problems; Ill-posed problems; Discrepancy principle; Monotone operator.
AMS Subject Classification: 65F22 65J15 65J20 65J22 65M30 65M32.

1. INTRODUCTION

In this paper we are interested in finding a stable approximate solution for an ill-posed equation
$$F(x) = y, \tag{1.1}$$
where $F : D(F) \subset X \to X$ is a nonlinear operator and X is a Hilbert space. We shall denote the inner product and the corresponding norm X by $\langle .,. \rangle$ and $\|.\|$ respectively.

We assume that (1.1) has a solution, say $x^\dagger$ and for $\delta \geq 0$, $y^\delta \in X$ is an available noisy data with
$$\|y - y^\delta\| \leq \delta. \tag{1.2}$$
We also assume that the operator F possesses a Fréchet derivative in a neighbourhood of $x^\dagger$, i.e, there exists $r > 0$ such that the Fréchet derivative $F'(x)$ exists for every $x \in B_r(x^\dagger) := \{u \in X : \|u - x^\dagger\| < r\}$.

Since (1.1) is ill-posed, the solution $x^\dagger$ need not depend continuously on the data. So, in order to obtain stable approximate solutions, it is required to regularize (1.1).

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Tikhonov regularization is one of the widely used regularization methods which has been extensively studied in the literature (cf. [2], [3], [6], [7], [9], [13]). In this method, the regularized solution is obtained by minimizing the Tikhonov functional

$$J_{\alpha,\delta}(x) := \|F(x) - y^{\delta}\|^2 + \alpha\|x - \bar{x}\|^2, \quad x \in D(F), \tag{1.3}$$

for each $\alpha > 0$, where $\bar{x} \in D(F)$ is a known initial approximation of $x^{\dagger}$. As the given operator is Fréchet differentiable, a minimum for the functional $J_{\alpha,\delta}$ in (1.3), if exists, is a solution of the associated *Euler -Lagrange equation*

$$F'(x)^*(F(x) - y^{\delta}) + \alpha(x - \bar{x}) = 0, \tag{1.4}$$

where $F'(x)^*$ is the adjoint of operator $F'(x)$, the Fréchet derivative of F at x.

Now, suppose that the given operator F is monotone, i.e.,

$$\langle F(x_2) - F(x_1), x_2 - x_1\rangle \geq 0 \qquad \forall\, x_1, x_2 \in D(F). \tag{1.5}$$

Then to get a regularized solution for (1.1), one can use an equation simpler than (1.4), namely,

$$F(x) + \alpha(x - \bar{x}) = y^{\delta}. \tag{1.6}$$

This method, in which the regularized solution is obtained by solving the equation (1.6), is known as Lavrentiev regularization. The existence and uniqueness of the solution of (1.6) can be asserted from the proof of Theorem 11.2 in [1] by making use of the hemicontinuity and the monotonicity of F. Note that the equation (1.6) does not involve Fréchet derivatives of F at any point. However, for deriving the error estimates, we shall make use of an equivalent form of (1.6), namely,

$$x_{\alpha}^{\delta} = \bar{x} + (A_{\alpha,\delta} + \alpha I)^{-1}[y^{\delta} - F(x_{\alpha}^{\delta}) + A_{\alpha}^{\delta}(x_{\alpha,\delta} - \bar{x})], \tag{1.7}$$

where $A_{\alpha,\delta} := F'(x_{\alpha}^{\delta})$.

After getting a regularized solution by solving (1.6) for each $\alpha > 0$, the next important aspect is to choose the regularization parameter $\alpha := \alpha(\delta)$ such that $x_{\alpha}^{\delta} \to x^{\dagger}$ as $\delta \to 0$. This choice may be a priori or a posteriori. Due to the practical applicability, a posteriori parameter strategy gains importance over a priori one. One such procedure is proposed by Scherzer et.al (cf. [9]) for Tikhonov regularization. For Lavrentiev regularization (1.7), Tautenhahn (cf. [11]) considered an analogous a posteriori strategy in which α is required to satisfy the equation

$$\|R_{\alpha,\delta}[F(x_{\alpha}^{\delta}) - y^{\delta}]\| = c\delta, \tag{1.8}$$

where $R_{\alpha,\delta} = \alpha(F'(x_{\alpha}^{\delta}) + \alpha I)^{-1}$ and $c > 0$ is an appropriate constant, and derived an order optimal error estimate under the assumption that the solution satisfies a *Hölder type source condition*. It is to be mentioned that Hölder type source conditions, though considered in the literature are suitable for mildly ill-posed problems, they are not applicable for many of the severely ill-posed cases where a *logarithmic type source condition* is sometimes more suitable (See [4], [5]).

In [8], Lavrentiev regularization for linear ill-posed problem is considered under a general source condition and optimal error estimates are obtained under the discrepancy principle of the form (1.8). Such general source conditions are useful for mildly and severely ill-posed problems, in particular for both Hölder type and logarithmic type source conditions. It is the purpose of this paper to extend the above analysis to the case of nonlinear ill-posed problems so that the result can be applied to a wider class of problems.

We note that for deriving the error estimates, Tautenhahn [11] used the assumption that there exists a constant $k_0 > 0$ such that for every $x \in D(F)$ and $v \in X$, there exists an element $k(x, x^\dagger, v) \in X$ satisfying

$$(F'(x) - F'(x^\dagger))v = F'(x^\dagger)k(x, x^\dagger, v), \qquad \|k(x, x^\dagger, v)\| \leq k_0\|v\|. \tag{1.9}$$

However, for deriving an estimate for the error $\|x_\alpha - x^\dagger\|$, using the notation $M_\alpha := \int_o^1 F'(x^\dagger + t(x_\alpha - x^\dagger))dt$, the following relation has been used:

$$\|(M_\alpha + \ I)^{-1}(F'(x^\dagger) - M_\alpha)u\| \leq k_0\|(M_\alpha + \ I)^{-1}M_\alpha\| \, \|u\|.$$

(cf. [11], step following (3.7) in the proof of Theorem 3.3). The above relation does not seem to follow from the above assumption (1.9). What follows from the assumption (1.9) is the relation

$$\|(M_\alpha + \ I)^{-1}(F'(x^\dagger) - M_\alpha)u\| \leq k_0\|(M_\alpha + \ I)^{-1}F'(x^\dagger)\| \, \|u\|.$$

It is also the purpose of this paper to fill the above apparent gap in the analysis in [11] by using the following alternate assumption on the nonlinearity of F, which has been used in [6], [7], [13] for Tikhonov regularization, so as to suit for analysis under a general source condition as well.

Assumption 1.1. *There exists a constant $k_0 > 0$ such that for every $x, u \in B_r(x^\dagger)$ and $v \in X$, there exists an element $g(x, u, v) \in X$ satisfying*

$$(F'(x) - F'(u))v = F'(u)g(x, u, v), \qquad \|g(x, u, v)\| \leq k_0\|v\|\|x - u\|.$$

It is shown in [9] that some parameter identification problems and nonlinear Hammerstein operator equation does satisfy Assumption 1.1.

2. ERROR ESTIMATE FOR LAVRENTIEV REGULARIZATION

Recall that, in Lavrentiev regularization, regularized solution x_α^δ is obtained by solving the nonlinear equation (1.6). As we have already spelt out in the last section, we shall assume that F is Fréchet differentiable in a neighbourhood $B_r(x^\dagger)$, where $x^\dagger$ is a solution of (1.1), and F is also a monotone operator so that for every $x \in B_r(x^\dagger)$, $F'(x)$ is a positive self adjoint operator and $F'(x) + \alpha I$ has continuous inverse for every $\alpha > 0$.

We derive bounds for the term $\|x_\alpha^\delta - x^\dagger\|$ under the following source condition:

Assumption 2.1. *There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \longrightarrow (0, \infty)$ with $a \geq \|F'(x^\dagger)\|$ satisfying*

(i) $\lim_{\lambda \longrightarrow 0} \varphi(\lambda) = 0,$

(ii) *there exists $c_\varphi > 0$ such that*

$$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{(\lambda + \alpha)} \leq c_\varphi \varphi(\alpha) \quad \forall \, \alpha \in \ (0, a],$$

(iii) *there exist $\rho > 0$ and $v \in X$ with $\|v\| \leq \rho$ such that*

$$\bar{x} - x^\dagger = \varphi(F'(x^\dagger)\,)v.$$

Assumption 2.1, known as a general source condition, is similar to the one considered in [8] for linear case. It can be seen easily that it includes both the well known source conditions namely, the Hölder type source condition, that is, with $\varphi(\lambda) = \lambda^\nu$, $0 < \lambda \leq 1$, and the logarithmic source condition, that is, with $\varphi(\lambda) = [\log(1/\lambda)]^{-\nu}$, $\nu > 0$.

2.1. **General Error estimate.** We find out error bound for $\|x_\alpha^\delta - x_\alpha\|$ and $\|x_\alpha - x^\dagger\|$ so that a bound for $\|x_\alpha^\delta - x^\dagger\|$ is obtained by triangle inequality. First we quote a result from [11] for the error bound for $\|x_\alpha^\delta - x_\alpha\|$.

Theorem 2.1. *Let (1.5) hold and x_α be the solution of the (1.6) with y in place of y^δ. Then*

(i) $\|x_\alpha^\delta - x_\alpha\| \le \delta/\alpha$,
(ii) $\|x_\alpha - x^\dagger\| \le \|\bar{x} - x^\dagger\|$,
(iii) $\|F(x_\alpha^\delta) - F(x_\alpha)\| \le \delta$.

Remark 2.2. From the relation $\|x_\alpha^\delta - x^\dagger\| \le \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x^\dagger\|$ and Theorem 2.1 we obtain that

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \|\bar{x} - x^\dagger\|.$$

Therefore, we see that equation (1.7) is meaningful if r in $B_r(x^\dagger)$ satisfies the relation

$$r > \frac{\delta}{\alpha} + \|\bar{x} - x^\dagger\|.$$

Our next result deals with error bound for $\|x_\alpha - x^\dagger\|$. We shall denote

$$A := F'(x^\dagger), \qquad A_\alpha := F'(x_\alpha).$$

Theorem 2.2. *Let the Assumption 1.1, 2.1 and (1.5) hold, and let $k_0\|\bar{x} - x^\dagger\| < 2$. Then*

$$\|x_\alpha - x^\dagger\| \le \tilde{c}_\varphi \varphi(\alpha)\rho, \tag{2.1}$$

where $\tilde{c}_\varphi = c_\varphi(1 + k_0\|\bar{x} - x^\dagger\|)/(1 - k_0\|\bar{x} - x^\dagger\|/2)$.

Proof. Denote $A_\alpha := F'(x_\alpha)$ and $A := F'(x^\dagger)$. From (1.6), we have

$$x_\alpha = \bar{x} + (A_\alpha + \alpha I)^{-1}[y - F(x_\alpha) + A_\alpha(x_\alpha - \bar{x})].$$

We observe that

$$\begin{aligned}
x_\alpha - x^\dagger &= \bar{x} - x^\dagger + (A_\alpha + \alpha I)^{-1}[y - F(x_\alpha) + A_\alpha(x_\alpha - \bar{x})] \\
&= \bar{x} - x^\dagger + (A_\alpha + \alpha I)^{-1}[y - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger + x^\dagger - \bar{x})] \\
&= \alpha(A_\alpha + \alpha I)^{-1}(\bar{x} - x^\dagger) + (A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)] \\
&= \alpha(A_\alpha + \alpha I)^{-1}(\bar{x} - x^\dagger) + (A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)] \\
&= \alpha(A + \alpha I)^{-1}(\bar{x} - x^\dagger) + \alpha((A_\alpha + \alpha I)^{-1} - (A + \alpha I)^{-1})(\bar{x} - x^\dagger) \\
&\quad + (A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)] \\
&= v_\alpha + (A_\alpha + \alpha I)^{-1}(A - A_\alpha)v_\alpha \\
&\quad + (A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)]
\end{aligned}$$

where $v_\alpha := \alpha(A + \alpha I)^{-1}(\bar{x} - x^\dagger)$. From Assumption 2.1, we have

$$\|v_\alpha\| = \|\alpha(A + \alpha I)^{-1}[\varphi(A)]v\| \le c_\varphi \rho \varphi(\alpha). \tag{2.2}$$

Thus,

$$\|x_\alpha - x^\dagger\| \le c_\varphi \rho \varphi(\alpha) + a_\alpha + b_\alpha, \tag{2.3}$$

$$\begin{aligned}
a_\alpha &:= \|(A_\alpha + \alpha I)^{-1}(A - A_\alpha)v_\alpha\| \\
b_\alpha &:= \|(A_\alpha + \alpha I)^{-1}[F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger)]\|.
\end{aligned}$$

Now, let us find estimates for the quantities a_α and b_α. By Assumption 1.1, we have $(A - A_\alpha)v_\alpha = A_\alpha g(x^\dagger, x_\alpha, v_\alpha)$, with

$$\|g(x^\dagger, x_\alpha, v_\alpha)\| \le k_0\|x_\alpha - x^\dagger\| c_\varphi \rho\varphi(\alpha).$$

Thus,

$$\begin{aligned} a_\alpha := \|(A_\alpha + \alpha I)^{-1}(A - A_\alpha)v_\alpha\| &\le \|(A_\alpha + \alpha I)^{-1} A_\alpha g(x^\dagger, x_\alpha, v_\alpha)\| \\ &\le k_0\|x^\dagger - x_\alpha\| c_\varphi \rho\varphi(\alpha). \end{aligned}$$

For obtaining a bound for b_α, we first observe from fundamental theorem of calculus and the Assumption 1.1 that

$$\begin{aligned} F(x^\dagger) - F(x_\alpha) + A_\alpha(x_\alpha - x^\dagger) &= \int_0^1 [F'(x_\alpha + t(x^\dagger - x_\alpha)) - A_\alpha](x^\dagger - x_\alpha)dt \\ &= A_\alpha \int_0^1 g(x_\alpha + t(x^\dagger - x_\alpha), x_\alpha, x^\dagger - x_\alpha)dt, \end{aligned} \tag{2.4}$$

where

$$\|g(x_\alpha + t(x^\dagger - x_\alpha), x_\alpha, x^\dagger - x_\alpha)\| \le k_0\|x_\alpha - x^\dagger\|^2 t.$$

Using (2.4), we get

$$b_\alpha := \|(A_\alpha + \alpha I)^{-1} A_\alpha \int_0^1 g(x_\alpha + t(x^\dagger - x_\alpha), x_\alpha, x^\dagger - x_\alpha)dt\| \le \frac{k_0}{2}\|x_\alpha - x^\dagger\|^2.$$

Hence, we get

$$\|x_\alpha - x^\dagger\| \le c_\varphi\varphi(\alpha)\rho + c_\varphi k_0\|x_\alpha - x^\dagger\|\varphi(\alpha)\rho + \frac{k_0\|x_\alpha - x^\dagger\|^2}{2}.$$

Using $\|x_\alpha - x^\dagger\| \le \|\bar{x} - x^\dagger\|$, we get

$$\|x_\alpha - x^\dagger\| \le c_\varphi\varphi(\alpha)\rho + c_\varphi k_0\|\bar{x} - x^\dagger\|\varphi(\alpha)\rho + \frac{k_0\|\bar{x} - x^\dagger\|\|x_\alpha - x^\dagger\|}{2}.$$

Hence,

$$\|x_\alpha - x^\dagger\| \le c_\varphi \left(\frac{1 + k_0\|\bar{x} - x^\dagger\|}{1 - k_0\|\bar{x} - x^\dagger\|/2}\right)\varphi(\alpha)\rho.$$

□

Combining Theorem 2.1 and Theorem 2.2 we obtain a bound for $\|x_\alpha^\delta - x^\dagger\|$ as in the following theorem.

Theorem 2.3. *Under the assumptions of Theorem 2.2,*

$$\|x_\alpha^\delta - x^\dagger\| \le \hat{c}_\varphi\left(\frac{\delta}{\alpha} + \rho\varphi(\alpha)\right),$$

where $\hat{c}_\varphi := max\{\tilde{c}_\varphi, 1\}$.

2.2. **A priori parameter choice.** We note that

$$\frac{\delta}{\alpha} = \rho\varphi(\alpha) \iff \frac{\delta}{\rho} = \psi(\varphi(\alpha)),$$

where $\psi : (0, \varphi(a)] \longrightarrow (0, a\varphi(a)]$ is defined as

$$\psi(\lambda) := \lambda\varphi^{-1}(\lambda),$$

for $\lambda \in (0, \varphi(a)]$. Our next theorem gives error bound for $\|x_\alpha^\delta - x^\dagger\|$ under an a-priori parameter choice.

Theorem 2.4. *Let the assumptions of Theorem 2.2 be satisfied. If the regularization parameter is chosen as* $\alpha = \varphi^{-1}\psi^{-1}(\delta/\rho)$ *with* ψ *defined by* $\psi(\lambda) := \lambda\varphi^{-1}(\lambda)$ *for* $\lambda \in (0, \varphi(a)]$, *then*

$$\|x_\alpha^\delta - x^\dagger\| \le \hat{c}_\varphi \rho\, \psi^{-1}(\delta/\rho), \tag{2.5}$$

where $\hat{c}_\varphi := max\{\tilde{c}_\varphi, 1\}$.

3. ERROR ESTIMATE UNDER AN A POSTERIORI CHOICE OF PARAMETER

Throughout this section we assume that the regularization parameter is chosen according to the discrepancy principle (1.8). The following lemma, proved in [11] ensures the existence of the regularization parameter α for which (1.8) holds.

Lemma 3.1 ([11], Proposition 4.1)**.** *Let the monotonicity property (1.5) be satisfied and* $\|F(\bar{x}) - y^\delta\| \ge c\delta$ *with* $c > 1$. *Then there exists an* $\alpha \ge \beta_0 := (c-1)\delta/\|\bar{x} - x^\dagger\|$ *satisfying (1.8).*

As in the last section, we use the notations $A := F'(x^\dagger)$ and $A_\alpha := F'(x_\alpha)$. We shall also use the notations

$$R_\alpha := \alpha(A_\alpha + \alpha I)^{-1}, \qquad R_\alpha^\delta := \alpha(A_\alpha^\delta + \alpha I)^{-1}.$$

For obtaining the main result of this section, we first prove some lemmas.

Lemma 3.2. *Let the Assumption 1.1 and assumptions in Lemma 3.1 hold and* $\alpha := \alpha(\delta)$ *is chosen according to (1.8). Then*

$$\|\alpha(A + \alpha I)^{-1}(F(x_\alpha) - y)\| \ge \frac{(c-2)\delta}{1 + k_1}, \tag{3.1}$$

where $k_1 = k_0 c\|\bar{x} - x^\dagger\|/(c-1)$.

Proof. By (1.8), we have

$$\|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha^\delta) - y^\delta)\| = c\delta.$$

Now consider

$$\begin{aligned} &|c\delta - \alpha\|(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\|| \\ &= \quad |\|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha^\delta) - y^\delta)\| - \|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\|| \\ &\le \quad \|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha^\delta) - y^\delta - (F(x_\alpha) - y))\| \\ &\le \quad \|F(x_\alpha^\delta) - F(x_\alpha)\| + \|y^\delta - y\|. \end{aligned}$$

Using (1.2) and Theorem 2.1 we get

$$|c\delta - \alpha\|(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\|| \le 2\delta$$

which gives

$$(c-2)\delta \le \|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\| \le (c+2)\delta. \tag{3.2}$$

Now let

$$\begin{aligned} a &= \|\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\| \\ b &= \|\alpha(A + \alpha I)^{-1}(F(x_\alpha) - y)\|. \end{aligned}$$

Then

$$\begin{aligned} a &\le b + \|\alpha((A_\alpha^\delta + \alpha I)^{-1} - (A + \alpha I)^{-1})(F(x_\alpha) - y)\| \\ &\le b + \|(A_\alpha^\delta + \alpha I)^{-1}(A - A_\alpha^\delta)\alpha(A + \alpha I)^{-1}(F(x_\alpha) - y))\| \\ &\le b + \|(A_\alpha^\delta + \alpha I)^{-1}A_\alpha^\delta g(x^\dagger, x_\alpha^\delta, \alpha(A + \alpha I)^{-1}(F(x_\alpha) - y))\| \\ &\le b + k_0\|x_\alpha^\delta - x^\dagger\|\|\alpha(A + \alpha I)^{-1}(F(x_\alpha) - y)\| \\ &\le (1 + k_1)b \end{aligned}$$

where $k_1 = k_0 c\|\bar{x} - x^\dagger\|/(c-1)$. Now (3.2) gives

$$(c-2)\delta \leq (1+k_1)b$$

which in turn implies

$$\|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - y)\| \geq \frac{(c-2)\delta}{1+k_1}. \tag{3.3}$$

□

Lemma 3.3. *Let assumptions of Theorem 2.2 and Lemma 3.1 hold. Then*

(i) $\|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - y)\| \leq \mu\alpha\varphi(\alpha)\rho$,

(ii) $\alpha \geq \varphi^{-1}\psi^{-1}(\xi\delta/\rho)$,

where $\mu = \tilde{c}_\varphi \left(1 + k_0\|\bar{x} - x^\dagger\|/2\right)$ *with* $\tilde{c}_\varphi$ *as in Theorem 2.2 and* $\xi = (c-2)/(1+k_1)\mu$.

Proof. By Assumption 1.1, we know that for every $x, z \in B_r(x^\dagger)$ and $u \in X$,

$$F'(x)u = F'(z)[u + F'(z)g(x,z,u)], \qquad \|g(x,z,u)\| \leq k_0\|u\|\|x-z\|.$$

Hence,

$$\begin{aligned}
F(x_\alpha) - F(x^\dagger) &= \int_0^1 F'(x^\dagger + t(x_\alpha - x^\dagger))(x_\alpha - x^\dagger)dt \\
&= A(x_\alpha - x^\dagger) + \int_0^1 Ag(x^\dagger + t(x_\alpha - x^\dagger), x^\dagger, x_\alpha - x^\dagger)dt \\
&= A\left((x_\alpha - x^\dagger) + \int_0^1 g(x^\dagger + t(x_\alpha - x^\dagger), x^\dagger, x_\alpha - x^\dagger)\right)dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - F(x^\dagger))\| \\
= & \|\alpha(A+\alpha I)^{-1}A\left((x_\alpha - x^\dagger) + \int_0^1 g(x^\dagger + t(x_\alpha - x^\dagger), x^\dagger, x_\alpha - x^\dagger)\right)dt\| \\
\leq & \alpha\|(A+\alpha I)^{-1}A\|\left(\|x_\alpha - x^\dagger\| + \int_0^1 \|g(x^\dagger + t(x_\alpha - x^\dagger), x^\dagger, x_\alpha - x^\dagger)dt\|\right) \\
\leq & \alpha\left(\|x_\alpha - x^\dagger\| + \frac{k_0\|x_\alpha - x^\dagger\|^2}{2}\right) \\
\leq & \alpha\|x_\alpha - x^\dagger\|\left(1 + \frac{k_0\|x_\alpha - x^\dagger\|}{2}\right).
\end{aligned}$$

Using Theorems 2.1 and 2.2, we have

$$\|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - F(x^\dagger))\| \leq \tilde{c}_\varphi\left(1 + \frac{k_0\|\bar{x} - x^\dagger\|}{2}\right)\alpha\varphi(\alpha)\rho,$$

where $\tilde{c}_\varphi = c_\varphi(1 + k_0\|\bar{x} - x^\dagger\|)/(1 - k_0\|\bar{x} - x^\dagger\|/2)$. Thus,

$$\|\alpha(A+\alpha I)^{-1}(F(x_\alpha) - F(x^\dagger))\| \leq \mu\alpha\varphi(\alpha)\rho, \tag{3.4}$$

where $\mu = \tilde{c}_\varphi\left(1 + k_0\|\bar{x} - x^\dagger\|/2\right)$. In view of the relation (3.1), we get

$$\frac{(c-2)\delta}{1+k_1} \leq \mu\alpha\varphi(\alpha)\rho$$

which implies, using the definition of ψ,

$$\psi(\varphi(\alpha)) = \alpha\varphi(\alpha) \geq \frac{(c-2)\delta}{(1+k_1)\mu\rho} = \frac{\xi\delta}{\rho}.$$

Thus,

$$\alpha \geq \varphi^{-1}\psi^{-1}\left(\xi\delta/\rho\right).$$

This completes the proof. □

Lemma 3.4. *Let Assumption 1.1 be satisfied and* $k_0\|\bar{x} - x^\dagger\| < 1$. *Then for all* $0 < \alpha_0 \leq \alpha$

$$\|x_\alpha - x_{\alpha_0}\| \leq \frac{\|R_\alpha(F(x_\alpha) - y)\|}{(1 - k_0\|\bar{x} - x^\dagger\|)\alpha_0}. \tag{3.5}$$

Proof. From (1.6) we know that

$$(F(x_\alpha) - y) + \alpha(x_\alpha - \bar{x}) = 0 \tag{3.6}$$

$$(F(x_{\alpha_0}) - y) + \alpha_0(x_{\alpha_0} - \bar{x}) = 0. \tag{3.7}$$

Hence,

$$\alpha_0(x_\alpha - x_{\alpha_0}) = (\alpha - \alpha_0)(\bar{x} - x_\alpha) + \alpha_0(\bar{x} - x_{\alpha_0}) - \alpha(\bar{x} - x_\alpha).$$

Using (3.6) and (3.7), we get

$$\alpha_0(x_\alpha - x_{\alpha_0}) = \frac{\alpha - \alpha_0}{\alpha}(F(x_\alpha) - y) + F(x_{\alpha_0}) - F(x_\alpha).$$

Adding $A_\alpha(x_\alpha - x_{\alpha_0})$ on both sides of the above equation, we get

$$(A_\alpha + \alpha_0 I)(x_\alpha - x_{\alpha_0}) = \frac{\alpha - \alpha_0}{\alpha}(F(x_\alpha) - y) + (F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0}))$$

which implies

$$x_\alpha - x_{\alpha_0} = \frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y) + (A_\alpha + \alpha_0 I)^{-1}(F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0})).$$

We first observe from fundamental theorem of calculus and the Assumption 1.1 that

$$\begin{aligned} F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0}) &= \int_0^1 [F'(x_\alpha + t(x_{\alpha_0} - x_\alpha)) - A_\alpha](x_{\alpha_0} - x_\alpha)dt \\ &= A_\alpha \int_0^1 g(x_\alpha + t(x_{\alpha_0} - x_\alpha), x_\alpha, x_{\alpha_0} - x_\alpha) \\ &\quad dt. \end{aligned} \tag{3.8}$$

Thus,

$$\|x_\alpha - x_{\alpha_0}\| \leq \|\frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| + c_\alpha,$$

where

$$c_\alpha = \|(A_\alpha + \alpha_0 I)^{-1}A_\alpha \int_0^1 g(x_\alpha + t(x_{\alpha_0} - x_\alpha), x_\alpha, x_{\alpha_0} - x_\alpha)dt\|.$$

Using the estimate $\|(A_\alpha + \alpha_0 I)^{-1}A_\alpha\| \leq 1$ and again Assumption 1.1, we have

$$\begin{aligned} c_\alpha &\leq \int_0^1 \|g(x_\alpha + t(x_{\alpha_0} - x_\alpha), x_\alpha, x_{\alpha_0} - x_\alpha)\|dt \\ &\leq \int_0^1 k_0\|x_{\alpha_0} - x_\alpha\|^2 t\, dt \\ &\leq \frac{k_0\|x_{\alpha_0} - x_\alpha\|^2}{2} \end{aligned}$$

Since $\|x_\alpha - x^\dagger\| \leq \|\bar{x} - x^\dagger\|$ and

$$\|x_{\alpha_0} - x_\alpha\| \leq \|x_{\alpha_0} - x^\dagger\| + \|x^\dagger - x_\alpha\| \leq 2\|\bar{x} - x^\dagger\|,$$

we have

$$c_\alpha \leq k_0\|\bar{x} - x^\dagger\|\|x_{\alpha_0} - x_\alpha\|.$$

Thus,

$$\|x_\alpha - x_{\alpha_0}\| \leq \|\frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| + k_0\|\bar{x} - x^\dagger\|\|x_{\alpha_0} - x_\alpha\|.$$

We observe that

$$\begin{aligned} \|\frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| &\leq \|(A_\alpha + \alpha_0)^{-1}(F(x_\alpha) - y)\| \\ &= \frac{1}{\alpha_0}\|\alpha_0(A_\alpha + \alpha_0 I)^{-1}R_\alpha^{-1}R_\alpha(F(x_\alpha) - y)\| \\ &\leq \|\alpha_0(A_\alpha + \alpha_0 I)^{-1}R_\alpha^{-1}\|\frac{\|R_\alpha(F(x_\alpha) - y)\|}{\alpha_0}. \end{aligned}$$

Note that

$$\|\alpha_0(A_\alpha + \alpha_0 I)^{-1}R_\alpha^{-1}\| \leq \sup_{\lambda \geq 0} \frac{\alpha_0(\lambda + \alpha)}{\alpha(\lambda + \alpha_0)}.$$

But

$$\frac{\alpha_0(\lambda + \alpha)}{\alpha(\lambda + \alpha_0)} \leq \frac{\alpha\lambda + \alpha_0\alpha}{\alpha(\lambda + \alpha_0 I)} = 1.$$

Thus, we have

$$\|x_\alpha - x_{\alpha_0}\| \leq \frac{\|R_\alpha(F(x_\alpha) - y)\|}{\alpha_0} + k_0\|\bar{x} - x^\dagger\|\|x_{\alpha_0} - x_\alpha\|$$

so that using the assumption $k_0\|\bar{x} - x^\dagger\| < 1$, we obtain

$$\|x_\alpha - x_{\alpha_0}\| \leq \frac{\|R_\alpha(F(x_\alpha) - y)\|}{\alpha_0(1 - k_0\|\bar{x} - x^\dagger\|)}.$$

□

Next lemma gives a bound for $\|R_\alpha(F(x_\alpha) - y)\|$.

Lemma 3.5. *Let assumptions of Lemma 3.3 hold. Then*

$$\|R_\alpha(F(x_\alpha) - y)\| \leq \beta\delta,$$

where $\beta = (c+2)[1 + k_0\|\bar{x} - x^\dagger\|/(c-1)]$.

Proof. Let $a = \|R_\alpha(F(x_\alpha) - y)\|$ and $b = \|R_\alpha^\delta(F(x_\alpha) - y)\|$. We note that

$$\begin{aligned} a &\leq \|R_\alpha^\delta(F(x_\alpha) - y)\| + \|(R_\alpha - R_\alpha^\delta)[F(x_\alpha) - y]\| \\ &= b + \|(A_\alpha + \alpha I)^{-1}(A_\alpha^\delta - A_\alpha)\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - F(x^\dagger))\| \\ &= b + k_0\|x_\alpha^\delta - x_\alpha\|\|R_\alpha^\delta(F(x_\alpha) - y)\|. \end{aligned}$$

Using $\|x_\alpha^\delta - x_\alpha\| \leq \delta/\alpha$, we get

$$a \leq (1 + k_0\delta/\alpha)b$$

and using Lemma 3.1, we have

$$a \leq \left(1 + \frac{k_0\|\bar{x} - x^\dagger\|}{c-1}\right)b. \tag{3.9}$$

We observe that

$$\begin{aligned} b &= \|R_\alpha^\delta(F(x_\alpha) - y)\| \\ &= \|R_a^\delta(F(x_\alpha) - y^\delta + y^\delta - y)\| \\ &\leq \|R_\alpha^\delta(F(x_\alpha) - y^\delta)\| + \|y^\delta - y\|. \end{aligned}$$

From (1.2), (1.8), we get

$$\|R_\alpha^\delta(F(x_\alpha) - y)\| \le (c+2)\delta. \tag{3.10}$$

Using (3.9) and (3.10), we obtain

$$a = \|R_\alpha(F(x_\alpha) - y)\| \le (c+2)\left(1 + \frac{k_0\|\bar{x} - x^\dagger\|}{c-1}\right)\delta.$$

□

We now give our main result.

Theorem 3.1. *Let assumptions of Lemma 3.3 hold. If, in addition, $k_0\|\bar{x} - x^\dagger\| \le 1$, then*

$$\|x_\alpha^\delta - x^\dagger\| \le \kappa_\varphi \psi^{-1}\left(\frac{\eta\delta}{\rho}\right)\rho, \tag{3.11}$$

where

$$\kappa_\varphi = ((1 - k_0\|\bar{x} - x^\dagger\|)^{-1} + \tilde{c}_\varphi + 1/\xi)$$

and $\eta = max\{\beta, \xi\}$ with $\tilde{c}_\varphi, \xi, \beta$, as in Theorem 2.2, Lemma 3.3 and 3.5 respectively.

Proof. Let $\Phi(\lambda) := \varphi^{-1}\psi^{-1}(\lambda)$ and $\alpha_0 := \Phi(\beta\delta/\rho)$ with $\beta = (c+2)[1 + k_0\|\bar{x} - x^\dagger\|/(c-1)]$.

First consider the case when $\alpha(\delta) \le \alpha_0$. We have

$$\|x_\alpha^\delta - x^\dagger\| \le \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x^\dagger\|.$$

Using Theorem 2.1 and 2.2, we have

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \tilde{c}_\varphi\varphi(\alpha)\rho,$$

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \tilde{c}_\varphi\varphi(\alpha_0)\rho. \tag{3.12}$$

Next assume that $\alpha(\delta) \ge \alpha_0$. In this case, using Lemma 3.4, Theorem 2.1 and 2.2 in

$$\|x_\alpha^\delta - x^\dagger\| \le \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x_{\alpha_0}\| + \|x_{\alpha_0} - x^\dagger\|,$$

we get

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \frac{\|R_\alpha(F(x_\alpha) - y)\|}{(1 - k_0\|\bar{x} - x^\dagger\|)\alpha_0} + \tilde{c}_\varphi\varphi(\alpha_0)\rho. \tag{3.13}$$

Since the error bound in (3.12) is smaller than the error bound in (3.13), the error bound for the latter case will be the error bound for the $\|x_\alpha^\delta - x^\dagger\|$, for any $\alpha \in (0, a]$.

Using Lemma 3.5, we get

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\alpha} + \frac{\beta\delta}{(1 - k_0\|\bar{x} - x^\dagger\|)\alpha_0} + \tilde{c}_\varphi\varphi(\alpha_0)\rho.$$

Using Lemma 3.3 in the first term of right hand side and using the value of α_0 in the second and last term, we obtain

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\delta}{\Phi(\frac{\xi\delta}{\rho})} + \frac{\beta\delta}{(1 - k_0\|\bar{x} - x^\dagger\|)\Phi\left(\frac{\beta\delta}{\rho}\right)} + \tilde{c}_\varphi\psi^{-1}\left(\frac{\beta\delta}{\rho}\right)\rho. \tag{3.14}$$

But, since $\varphi^{-1}(\lambda) = \frac{1}{\lambda}\psi(\lambda)$, we have $\Phi(\lambda) = \varphi^{-1}\psi^{-1}(\lambda) = \lambda/\psi^{-1}(\lambda)$ so that

$$\|x_\alpha^\delta - x^\dagger\| \le \frac{\psi^{-1}(\xi\delta/\rho)\rho}{\xi} + \frac{\psi^{-1}(\beta\delta/\rho)\rho}{(1 - k_0\|\bar{x} - x^\dagger\|)} + \tilde{c}_\varphi\psi^{-1}\left(\frac{\beta\delta}{\rho}\right)\rho.$$

Hence,

$$\|x_\alpha^\delta - x^\dagger\| \le \left(\frac{1}{\xi} + \frac{1}{(1 - k_0\|\bar{x} - x^\dagger\|)} + \tilde{c}_\varphi\right)\psi^{-1}\left(\frac{\eta\delta}{\rho}\right)\rho.$$

where $\eta = max\{\beta, \xi\}$. Thus,

$$\|x_\alpha^\delta - x^\dagger\| \le \kappa_\varphi \psi^{-1}\left(\frac{\eta\delta}{\rho}\right)\rho$$

with $\kappa_\varphi := (1 - k_0\|\bar{x} - x^\dagger\|)^{-1} + \tilde{c}_\varphi + 1/\xi$. □

4. EXAMPLE

Here we give an example, taken from [10], of a nonlinear ill-posed operator equation for the purpose of illustration of the Assumption 1.1 which is a modified form of a condition considered by Tautenhan [11].

For $x \in L^2(0,1)$, let

$$[F(x)](t) := \int_0^1 k(s,t)x^3(s)ds, \quad t \in (0,1),$$

where

$$k(t,s) = \begin{cases}(1-t)s, & 0 \le s \le t \le 1\\(1-s)t, & 0 \le t \le s \le 1.\end{cases}$$

Then for all $x, y \in L^2(0,1)$ with $x(t) > y(t)$ for $t \in (0,1)$, we have

$$\langle F(x) - F(y),\, x - y\rangle = \int_0^1 \Big(\int_0^1 k(t,s)(x^3 - y^3)(s)ds\Big)(x-y)(t)dt \ge 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$[F'(x)v](t) = 3\int_0^1 k(t,s)x^2(s)v(s)ds, \quad x, v \in L^2(0,1), \quad t \in (0,1).$$

Now, let us restrict the domain of F to

$$D(F) := \{x \in L^2(0,1) : x > c \text{ a.e. }\}$$

for some constant $c > 0$. Then for $u \in D(F)$ and $v, w \in L^2(0,1)$, we have

$$[(F'(x) - F'(u))v](t) = 3\int_0^1 k(t,s)u^2(s)\frac{[x^2(s) - u^2(s)]}{u^2(s)}v(s)ds, \quad t \in (0,1).$$

Thus, for $x, u \in D(F)$ and $v \in L^2(0,1)$,

$$[F'(x) - F'(u)]v = F'(u)g(x,u,v),$$

where

$$g(x,u,v)(s) = \frac{[x^2(s) - u^2(s)]}{u^2(s)}v(s) = \frac{[x(s)+u(s)]\,[x(s)-u(s)]}{u^2(s)}v(s).$$

Observe that

$$\|g(v,u,w)\|_2 \le \frac{1}{c^2}\|x+u\|_2\|x-u\|_2\|v\|_2.$$

So Assumption 1.1 is satisfied if $k_0 > 0$ is taken such that

$$\frac{\|x+u\|_2}{c^2} \le k_0 \quad \text{for all} \quad x, y \in B_r(x^\dagger).$$

If we take

$$y(t) = \frac{6\sin(\pi t) + \sin^3(\pi t)}{9\pi^2}, \quad t \in (0,1),$$

then the exact solution is $x^\dagger(t) = \sin(\pi t)$, $t \in (0,1)$. If we use

$$x_0(t) = \sin(\pi t) + \frac{3(t\pi^2 - t^2\pi^2 + \sin^2(\pi t))}{4\pi^2}$$

as the initial guess, then $x_0 - \hat{x} = \varphi(F'(x^\dagger))\frac{1}{4}$ with $\varphi(\lambda) = \lambda$.

5. CONCLUDING REMARKS

Laverntiev regularization of nonlinear ill-posed operator equation $F(x) = y$ is considered when F is monotone and Fréchet differentiable in the neighbourhood of a solution $x^\dagger$. Order optimal error estimates are derived under a general source condition by choosing the regularization parameter a priori and a posteriori manners. The results of this paper generalize the results in [11], [12], fills an apparent gap in the analysis in [11] by using an alternate assumption on the nonlinearity of F, and extend the results in [8] to nonlinear case.

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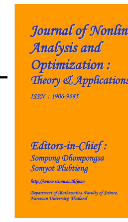

EXISTENCE OF NONLINEAR NEUTRAL IMPULSIVE INTEGRODIFFERENTIAL EVOLUTION EQUATIONS OF SOBOLEV TYPE WITH TIME VARYING DELAYS

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ABSTRACT. In this paper, we prove the existence of solutions for nonlinear neutral impulsive evolution integrodifferential equations of Sobolev type with time varying delays. The results are obtained by using semigroup theory and the Monch's fixed point theorem. An application of the same problem is discussed. An example is provided to illustrate the theory.

KEYWORDS: Existence; Neutral differential equation; Impulsive differential equation; Measure of noncompactness; Fixed point theorem.
AMS Subject Classification: 34A37, 47D06, 47H10, 74H20, 34K40.

1. INTRODUCTION

A large class of scientific and engineering problems is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations which can be described as differential equations in infinite dimensional spaces using semigroups. In general functional differential equations or evolution equations serve as an abstract formulations of many partial differential equations which arise in problems connected with heat-flow in materials with memory, viscoelasticity and many other physical phenomena. Using the method of semigroups, various solutions of nonlinear and semilinear evolution equations have been discussed by Pazy [27] and the nonlocal problem for the same equations has been first studied by Byszewskii [11–13]. Because it is demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. Such problems with nonlocal conditions have been extensively studied in literature [1, 5, 6, 31]. Balachandran et al. [8] studied the nonlocal Cauchy problem for delay integrodifferential equations of Sobolev type in Banach spaces. Bahuguna and

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Shukla [9] established the approximation of solutions to nonlinear Sobolev type evolution equations. Showalter [30] established the existence of solutions of semi-linear evolution equations of Sobolev type in Banach spaces. This type of equations arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics, and shear in second-order uids. For more details, we refer the reader to [10, 21, 22].

A delay differential equation is a special type of functional differential equation. Delay differential equations are similar to ordinary differential equation, but their evolution involves past values of the state variable. Time delay is inherently the character of most dynamical systems to some extent. Particularly the delays in many engineering systems such as power systems are often time-varying and sometimes vary violently with time. Time delays are frequently encountered in various engineering systems such as aircraft, long transmission lines in pneumatic models and chemical or process control systems. These delays may be the source of instability and lead to serious deterioration in the performance of closed loop systems. Theory of neutral differential equations has been studied by several authors in Banach spaces [15, 16, 18–20].

Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is in the form of impulses. The theory of impulsive differential equations has become an active area of investigation due to their applications in the field such as mechanics, electrical engineering, medicine biology and so on. However, one may easily visualize that abrupt changes such as shock, harvesting and disasters may occur in nature. These phenomena are short time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is in the form of impulses. The theory of impulsive differential equation [23, 26, 29] is much richer than the corresponding theory of differential equations without impulsive effects. The impulsive condition

$$\Delta u(t_i) = u(t_i^+) - u(t_i^-) = I_i(u(t_i^-)),\ i = 1,\ 2, \ldots, m,$$

is a combination of traditional initial value problems and short-term perturbations whose duration is negligible in comparison with the duration of the process. Lin and Liu [24] discussed the iterative methods for the solution of impulsive functional differential systems.

Measures of noncompactness are a very useful tool in many branches of mathematics. They are used in the fixed point theory, linear operators theory, theory of differential and integral equations and others [3]. There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\sigma(X)$ of a bounded set X in a metric space is defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of sets of diameter smaller than r. The Hausdorff measure of noncompactness $\chi(X)$ defined as infimum of numbers $r > 0$ such that X can be covered with a finite number of balls of radii smaller than r. There exist many formula on $\chi(X)$ in various spaces [3, 4]. The notion of a measure of weak compactness was introduced by De Blasi [14] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations. Several authors have studied the measures of noncompactness in Banach spaces [2, 4].

Motivated by the above literature, the goal of this paper is to use the fixed point theorem to obtain the existence of mild solution of sobolev type nonlinear neutral impulsive integrodifferential evolution equations with time varying delays.

2. PRELIMINARIES

In this section, we recall some definitions, notations and results that we need in the sequel. Throughout this paper, $(X, \|\cdot\|)$ is a Banach space and $A(t)$ generates the evolution operator in X. Also $A(t),\ t \in I$ is closed linear operator defined on a common domain $\mathcal{D} := D(A(t))$, which is dense in X.

The purpose of this paper is to prove the existence of mild solutions for a nonlinear impulsive neutral delay integrodifferential equation of Sobolev type with nonlocal conditions of the form

$$\begin{aligned} & \frac{d}{dt}\left[Bx(t) + e(t, x(\sigma_1(t)))\right] + A(t)x(t) \\ & \quad = f(t, x(\sigma_2(t))) + \int_0^t k(t,s)h(s, x(\sigma_3(s)))ds,\ t \in I,\ t \neq t_k, \quad (2.1) \end{aligned}$$

$$x(0) + g(x) = x_0, \quad (2.2)$$

$$\Delta x(t_k) = I_k(x_{t_k}), \quad k = 1, 2, \ldots, m \quad (2.3)$$

where $A(t), B$ are two closed operators such that $-A(t)\ B^{-1}$ generates the strongly continuous semigroup of bounded linear operators $U(t,s)$ in a Banach space X and $I = [0, a]$. The nonlinear operators $f : [0,a] \times X \to X, k : [0,a] \times [0,a] \to \mathcal{R}$, $h : [0,a] \times X \to X$, $e : [0,a] \times X \to X$, $g : \mathcal{PC}([0,a], X) \to D(B)$ and the delay $\sigma_i(t) \leq t, i = 1,2,3$ are given appropriate functions; $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, for $0 = t_0 < t_1 ... < t_k < t_{k+1} = a$.

Let $(X, \|.\|)$ be a real Banach space. $\{A(t) : t \in \mathbb{R}\}$ is a family of closed linear operators defined on a common domain $\mathcal{D}$ which is dense in X and we assume that the linear non-autonomous system

$$\begin{aligned} u'(t) &= A(t)u(t), \quad s \leq t \leq a, \\ u(s) &= x \in X, \quad (2.4) \end{aligned}$$

has associated evolution family of operators $\{U(t,s) : 0 \leq s \leq t \leq a\}$. In the next definition, $\mathcal{L}(X)$ is a space of bounded linear operator from X into X endowed with the uniform convergence topology.

Definition 2.1. A family of operators $\{U(t,s) : 0 \leq s \leq t \leq a\} \subset \mathcal{L}(X)$ is called a evolution family of operators for (2.4), if the following properties hold:

(i) $U(t,s)U(s,\tau) = U(t,\tau)$ and $U(t,t)x = x$, for every $s \leq \tau \leq t$ and all $x \in X$;
(ii) For each $x \in X$, the functions for $(t,s) \longrightarrow U(t,s)x$ is continuous and $U(t,s) \in \mathcal{L}(X)$ for every $t \geq s$ and
(iii) For $0 \leq s \leq t \leq a$, the function $t \longrightarrow U(t,s)$, for $(s,t] \in \mathcal{L}(X)$, is differentiable with $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$.

We denote by $\mathcal{PC}([0,a], X)$ the space of $X-$ valued continuous functions on $[0,a]$ with the norm $\|x\| = \sup\{\|x(t)\|, t \in [0,a]\}$ and by $\mathcal{L}^1([0,a], X)$ the space of $X-$valued Bochner integrable functions on $[0,a]$ with the norm

$$\|f\|_{\mathcal{L}^1} = \int_0^a \|f(t)\|dt.$$

Let us recall the following definition.

Definition 2.2. A continuous solution $x(t)$ of the integral equation

$$\begin{aligned} x(t) &= B^{-1}U(t,0)B[x_0 - g(x)] + B^{-1}U(t,0)e(0, x(0)) \\ & \quad -B^{-1}e(t, x(\sigma_1(t)) + \int_0^t U(t,s)A(s)B^{-1}e(s, x(\sigma_1(s))ds \end{aligned}$$

$$+\int_0^t U(t,s)B^{-1}\left[f(s,x(\sigma_2(s)))+\int_0^s k(s,\tau)h(\tau,x(\sigma_3(\tau)))d\tau\right]ds$$
$$+\sum_{0<t_i<t} B^{-1}U(t,t_k)I_kx(t_k) \tag{2.5}$$

is said to be a mild solution of problem $(2.1)-(2.3)$ on $[0,a]$.

To prove our main theorem we assume certain conditions on the operators $A(t)$ and B. Let X and Y be Banach spaces with norm $|\cdot|$ and $\|\cdot\|$ respectively. The operators $A(t):D(A(t))\subset X\longrightarrow Y$ and $B:D(B)\subset X\longrightarrow Y$ satisfy the following hypothesis:

- **(A1)** $A(t)$ and B are closed linear operators;
- **(A2)** $D(B)\subset D(A(t))$ and B is bijective;
- **(A3)** $B^{-1}:Y\to D(B)$ is continuous.

The hypothesis $(\mathbf{A1})-(\mathbf{A3})$ and the closed graph theorem imply the boundedness of the linear operator $A(t)B^{-1}:X\to X$ and $-A(t)B^{-1}$ generates a uniformly continuous evolution operators $U(t,s), t\geq 0$, of bounded linear operators on Banach space X.

Next, we introduce the Hausdorff's measure of noncompactness $\psi(\cdot)$ defined on each bounded subset E of Banach space Y by

$$\psi(B)=\inf\{\epsilon>0;\ B\ has\ a\ finite\ \epsilon-\ net\ in\ Y\}.$$

Lemma 2.3. *[3] Let Y be a real Banach space and $C,E\subseteq Y$ be bounded, with the following properties:*

- (i) *C is pre-compact if and only if $\psi_Y(B)=0$.*
- (ii) *$\psi_Y(C)=\psi_Y(\bar{C})=\psi_Y(conC)$, where $\bar{C}$ and $conC$ mean the closure and convex hull of C respectively.*
- (iii) *$\psi_Y(C)\leq\psi_Y(E)$, where $C\subseteq E$.*
- (iv) *$\psi_Y(C+E)\leq\psi_Y(C)+\psi_Y(E)$, where $C+E=\{x+y:\ x\in C,\ y\in E\}$.*
- (v) *$\psi_Y(C\cup E)\leq\max\{\psi_Y(C),\ \psi_Y(E)\}$.*
- (vi) *$\psi_Y(\lambda C)\leq|\lambda|\psi_Y(C)$, for any $\lambda\in\mathcal{R}$.*
- (vii) *If the map $\mathcal{F}:D(\mathcal{F})\subseteq Y\to Z$ is Lipschitz continuous with constant r,then $\psi_Z(\mathcal{F}B)\leq r\psi_Y(B)$, for any bounded subset $B\subseteq D(\mathcal{F})$, where Z be a Banach space.*

Before we prove the existence results, we need the following Lemmas.

Lemma 2.4. *[3] If $\mathbb{W}\subseteq\mathcal{PC}([0,a],X)$ is bounded, then $\psi(\mathbb{W}(t))\leq\psi_c(\mathbb{W})$ for all $t\in[0,a]$, where $\mathbb{W}(t)=\{u(t);u\in\mathbb{W}\}\subseteq X$. Furthermore if $\mathbb{W}$ is equicontinuous on $[0,a]$, then $\psi(\mathbb{W}(t))$ is continuous on $[0,a]$ and $\psi_c(\mathbb{W})=\sup\{\psi(\mathbb{W}(t)),\ t\in[0,a]\}$.*

Lemma 2.5. *[17, 25] If $\{u_n\}_{n=1}^{\infty}\subset\mathcal{L}^1([0,a],X)$ is uniformly integrable, then the function $\psi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and*

$$\psi\Big\{\Big(\int_0^t u_n(s)ds\Big)_{n=1}^{\infty}\Big\}\leq 2\int_0^t\psi(\{u_n(s)\}_{n=1}^{\infty})ds. \tag{2.6}$$

The following fixed point theorem, a nonlinear alternative of Monch type, plays a key role in our existence of mild solutions for nonlocal Cauchy problem $(2.1)-(2.3)$.

Theorem 2.6. *Let Y be a Banach space, U an open subset of Y and $0\in U$. Suppose that $F:\overline{U}\to Y$ is a continuous map which satisfies Monch's condition (that is, if $D\subseteq\overline{U}$ is countable and $D\subseteq\overline{co}(0\cup F(D))$, then $\overline{D}$ is compact) and assume that*

$$x\neq\lambda F(x),\quad for\ \ x\in\partial U\ \ and\ \ \lambda\in(0,1) \tag{2.7}$$

holds. Then F has a fixed point in $\overline{U}$.

3. MAIN RESULTS

In this section, we study the existence of mild solutions of neutral impulsive evolution integrodifferential equations of Sobolev type.

To prove our existence results, we assume the following hypotheses:

(M1) $A(t)$ generates a family of evolution operator $U(t,s)$, when $t > s > 0$, of C_0- semigroups on X and there exists a constant $M > 0$ such that

$$\|U(t,s)\| \leq M, \quad \text{for } 0 \leq s \leq t \leq a.$$

(M2) (i) The nonlinear function $e : [0,a]\times X \to X$, for a.e $t \in [0,a]$, the function $e(\cdot,x)$ is continuous and for all $x \in X$, the function $e(\cdot,x) : [0,a] \to X$ is measurable, for all $x \in X$.

(ii) There exist functions $\phi_0, \phi_1, \phi_2 \in \mathcal{L}^1([0,a],\mathcal{R}^+)$ and nondecreasing continuous functions $\Omega_e, \Omega_{Ae} : \mathcal{R}^+ \to \mathcal{R}^+$ such that for every $x \in X$, we have

$$\begin{aligned} \|e(t,x)\| &\leq \phi_1(t)\ \Omega_e\|x\|,\ a.e.\ t \in [0,a] \\ \|A(t)e(t,x)\| &\leq \phi_2(t)\ \Omega_{Ae}\|x\|,\ a.e.\ t \in [0,a] \\ \|e(0,x)\| &\leq \phi_0,\ \ a.e.\ t \in [0,a]. \end{aligned}$$

(iii) There exist functions $\gamma_0, \gamma_e, \gamma_{Ae} \in \mathcal{L}^1([0,a],\mathcal{R}^+)$ such that for every bounded $D \subset X$, we have

$$\begin{aligned} \psi(e(t,D)) &\leq \gamma_e(t)\psi(D),\ \ a.e.\ t \in [0,a] \\ \psi(A(t)e(t,D)) &\leq \gamma_{Ae}(t)\psi(D),\ \ a.e.\ t \in [0,a] \\ \psi(e(0,D)) &\leq \gamma_0,\ \ a.e.\ t \in [0,a]. \end{aligned}$$

(M3) (i) The nonlinear function $f : [0,a]\times X \to X$, for a.e $t \in [0,a]$, the function $f(\cdot,x)$ is continuous and for all $x \in X$, the function $f(\cdot,x) : [0,a] \to X$ is measurable for all $x \in X$.

(ii) There exists a function, $\phi_3 \in \mathcal{L}^1([0,a],\mathcal{R}^+)$ and a nondecreasing continuous function $\Omega_f : \mathcal{R}^+ \to \mathcal{R}^+$ such that for every $x \in X$, we have

$$\|f(t,x)\| \leq \phi_3(t)\Omega_f\|x\|,\ \ a.e\ t \in [0,a].$$

(iii) There exists a function, $\gamma_f \in \mathcal{L}^1([0,a],\mathcal{R}^+)$ such that for every bounded $D \subset X$, we have

$$\psi(f(t,D)) \leq \gamma_f(t)\psi(D), \quad a.e\ t \in [0,a].$$

(M4) (i) The nonlinear function $h : [0,a]\times X \to X$, for a.e $t \in [0,a]$, the function $h(\cdot,x)$ is continuous and for all $x \in X$, the function $f(\cdot,x) : [0,a] \to X$ is strongly measurable, for all $x \in X$.

(ii) There exists a function, $\phi_4 \in \mathcal{L}^1([0,a],\mathcal{R}^+)$ and a nondecreasing continuous function $\Omega_h : \mathcal{R}^+ \to \mathcal{R}^+$ such that for every $x \in X$, we have

$$\|h(t,x)\| \leq \phi_4(t)\Omega_h\|x\|,\ \ a.e\ t \in [0,a].$$

(iii) There exists a function, $\gamma_h \in \mathcal{L}^1([0,a],\mathcal{R}^+)$ such that for every bounded $D \subset X$, we have

$$\psi(h(t,D)) \leq \gamma_h(t)\psi(D),\ \ a.e\ t \in [0,a].$$

(M5) The function $k : [0, a] \times [0, a] \to R$ is measurable function such that there exist a constant K such that

$$\|k(t, s)\| \le K, \text{ for } s, t \in I.$$

(M6) (i) $I_k : X \longrightarrow X$ is continuous. There exists a nondecreasing continuous function $\Omega_I : \mathcal{R}^+ \to \mathcal{R}^+$ such that for every $x \in X$, we have

$$\|I_k(x(t_k))\| \le \Omega_I \|x\|, \text{ where } k = 1, 2, 3, \ldots, m.$$

(ii) There exists a function, $\gamma_I \in \mathcal{L}^1([0, a], \mathcal{R}^+)$ such that, for every bounded $D \subset X$, we have

$$\psi(I_k(D)) \le \gamma_I(t)\psi(D), \quad k = 1, 2 \ldots, m.$$

(M7) The function $g : \mathcal{PC}([0, a], X) \to D(B)$ is continuous compact map such that $\|g(x)\| \le c\|x\| + d$, for all $x \in \mathcal{PC}([0, a], X)$, for some positive constants c and d.

Now, we give the existence results for $(2.1) - (2.3)$.

Theorem 3.1. *Assume that the conditions $(M1) - (M7)$ are satisfied. Then, for every $x_0 \in D(B)$ the impulsive nonlocal problem $(2.1) - (2.3)$ has at least one mild solution $[0, a]$ provided that there exists a constant $\mathcal{N} > 0$ with*

$$\frac{(1 - \alpha\beta M c)\mathcal{N}}{\alpha\beta M(d + \|x_0\|) + \alpha\phi_1\Omega_e(\mathcal{N}) + \alpha M[\phi_0 + \phi_2\Omega_{Ae}(\mathcal{N}) + \phi_3\Omega_f(\mathcal{N}) + K\phi_4\Omega_h(\mathcal{N}) + \Omega_I(\mathcal{N})]} > 1 \tag{3.1}$$

and that

$$2\alpha[\|\gamma_e\| + M\{\|\gamma_0\| + \|\gamma_{Ae}\| + \|\gamma_f\| + K\|\gamma_h\| + \|\gamma_I\|\}] < 1. \tag{3.2}$$

Proof. We consider the operator $\mathcal{F} : \mathcal{PC}([0, a], X) \to \mathcal{PC}([0, a], X)$ defined by

$$(\mathcal{F}x)(t) = (\mathcal{F}_1 x)(t) + (\mathcal{F}_2 x)(t) \tag{3.3}$$

with

$$(\mathcal{F}_1 x)(t) = B^{-1}U(t, 0)B[x_0 - g(x)] \tag{3.4}$$

$$\begin{aligned} (\mathcal{F}_2 x)(t) = {} & B^{-1}U(t, 0)e(0, x(0)) - B^{-1}e(t, x(\sigma_1(t)) \\ & + \int_0^t U(t, s)A(s)B^{-1}e(s, x(\sigma_1(s))ds \\ & + \int_0^t U(t, s)B^{-1}\left[f(s, x(\sigma_2(s))) + \int_0^s k(s, \tau)h(\tau, x(\sigma_3(\tau)))d\tau\right]ds \\ & + \sum_{0 < t_i < t} B^{-1}U(t, t_k)I_k x(t_k), \quad \text{for all } t \in [0, a]. \end{aligned} \tag{3.5}$$

It is easy to see that the fixed point of $\mathcal{F}$ is the mild solutions of impulsive nonlocal problem $(2.1) - (2.3)$. Subsequently, we will prove that $\mathcal{F}$ has a fixed point by using Theorem 2.6.

First, we claim that the operator $\mathcal{F}$ is continuous on $\mathcal{PC}([0, a], X)$. For this purpose, we assume that $x_n \to x$ in $\mathcal{PC}([0, a], X)$. Then by $(M2 - (ii))$ we get that

$$e(t, x_n(\sigma_1(t))) \to e(t, x(\sigma_1(t))), \quad a.e.\ t \in [0, a]$$

$$A(s)e(s, x_n(\sigma_1(s))) \to A(s)e(s, x(\sigma_1(s))), \quad a.e.\ \ s \in [0, a].$$

By the same reason $(M3 - (ii))$ and $(M4 - (ii))$ we get

$$f(s, x_n(\sigma_2(s))) \to f(s, x(\sigma_2(s))), \quad a.e.\ s \in [0, a]$$

$$h(\tau, x_n(\sigma_2(\tau))) \to h(\tau, x(\sigma_2(\tau))), \quad a.e.\ \tau \in [0,a].$$

Since $(M4-(ii)),(M5)$ hold, by the dominated convergence theorem, for every $s \in [0,a]$ we have

$$\int_0^s k(s,\tau)h(\tau, x_n(\sigma_3(\tau)))d\tau \to \int_0^s k(s,\tau)h(\tau, x(\sigma_3(\tau)))d\tau, \quad (n \to +\infty).$$

Thus

$$\begin{aligned}
\|\mathcal{F}x_n - \mathcal{F}x\| \;\leq\;& \alpha\beta M\|g(x_n)-g(x)\| + \alpha\|e(t,x_n(\sigma_1(t))) - e(t,x(\sigma_1(t)))\| \\
&+\alpha M\int_0^t \|A(s)e(s,x_n(\sigma_1(s))) - A(s)e(s,x(\sigma_1(s)))\|ds \\
&+\alpha M\int_0^t \|f(s,x_n(\sigma_2(s))) - f(s,x(\sigma_2(s)))\|ds \\
&+\alpha M\int_0^t\int_0^s \|k(s,\tau)h(\tau,x_n(\sigma_3(\tau)))d\tau - k(s,\tau)h(\tau,x(\sigma_3(\tau)))d\tau\|ds \\
&+\alpha M\sum_{0<t_i<t}\|I_kx_n(t_k) - I_kx(t_k)\| \\
\to\;& 0 \qquad as \quad n \to \infty. \qquad (3.6)
\end{aligned}$$

That is $\mathcal{F}$ is continuous.

Next, we claim that the Monch's condition holds.
Suppose that $D \subseteq B_r$ is countable and $D \subseteq \overline{co}(0 \cup \mathcal{F}(D))$, we show that $\psi(D) = 0$, where B_r is the open ball of the radius r centered at the zero in $\mathcal{PC}([0,a],X)$. Without loss of generality, we may suppose that $D = \{x_n\}_{n=1}^{+\infty}$. By using the condition $(M1)-(M7)$, we can easily verify that $\{\mathcal{F}x_n\}_{n=1}^{+\infty}$ is equicontinuous. So, $D \subseteq \overline{co}(0 \cup \mathcal{F}(D))$ is also equicontinuous.

Now, from the Lemma 2.3, 2.4, 2.5 and the continuity of $B^{-1}U(t,0)B$, it follows that

$$\begin{aligned}
\psi(\{\mathcal{F}x_n\}_{n=1}^{+\infty}) \;\leq\;& \sup_{t\in[0,a]}\psi(\{B^{-1}U(t,0)Bg(x_n)\}_{n=1}^{+\infty}) + \psi(\{B^{-1}U(t,0)e(0,x(0))\}) \\
&+\psi\Big(\{B^{-1}e(t,x_n(\sigma_1(t)))\}_{n=1}^{+\infty}\Big) \\
&+\psi\Big(\{\int_0^t U(t,s)A(s)B^{-1}e(s,x_n(\sigma_1(s)))ds\}_{n=1}^{+\infty}\Big) \\
&+\psi\Big(\{\int_0^t U(t,s)B^{-1}f(s,x_n(\sigma_2(s)))ds\}_{n=1}^{+\infty}\Big) \\
&+\psi\Big(\{\int_0^t\int_0^s U(t,s)B^{-1}k(s,\tau)h(\tau,x_n(\sigma_3(\tau)))d\tau ds\}_{n=1}^{+\infty}\Big) \\
&+\psi\Big(\{\sum_{0<t_i<t}B^{-1}U(t,t_k)I_kx_n(t_k)\}_{n=1}^{+\infty}\Big) \\
\leq\;& 2\alpha M\gamma_0 + 2\alpha\gamma_e(t)\psi(\{x_n(\sigma_1(t))\}_{n=1}^{+\infty}) \\
&+2\alpha M\int_0^t \gamma_{Ae}(s)\psi(\{x_n(\sigma_1(s))\}_{n=1}^{+\infty})ds \\
&+2\alpha M\int_0^t \gamma_f(s)\psi(\{x_n(\sigma_1(s))\}_{n=1}^{+\infty})ds \\
&+2\alpha MK\int_0^t\int_0^s \gamma_h(s)\psi(\{x_n(\sigma_3(\tau))\}_{n=1}^{+\infty})d\tau ds
\end{aligned}$$

$$+2\alpha M\gamma_I\psi(\{x_n(t_k)\}_{n=1}^{+\infty})$$
$$\leq 2\alpha\Big[\|\gamma_e\| + M\{\|\gamma_0\|+\|\gamma_{Ae}\|+\|\gamma_f\|+K\|\gamma_h\|+\|\gamma_I\|\}\Big]\psi(\{x_n\}_{n=1}^{+\infty}).$$

Thus, we get that

$$\begin{aligned}\psi(D) &\leq \psi(\overline{co}(0\cup\mathcal{F}(D)))\\ &= \psi(\mathcal{F}(D))\\ &\leq 2\alpha\Big[\|\gamma_e\| + M\{\|\gamma_0\| + \|\gamma_{Ae}\| + \|\gamma_f\| + K\|\gamma_h\| + \|\gamma_I\|\}\Big]\psi(D)\end{aligned}$$

which implies that $\psi(D) = 0$, since the condition (3.2) holds.

Now let $\lambda \in (0,1)$ and $x = \lambda\mathcal{F}(x)$. Then, for $t \in [0,a]$

$$\begin{aligned}x(t) &= \lambda B^{-1}U(t,0)BE[x_0 - g(x)] + \lambda B^{-1}U(t,0)e(0,x(0))\\ &\quad -\lambda B^{-1}e(t,x(\sigma_1(t)) + \lambda\int_0^t U(t,s)A(s)B^{-1}e(s,x(\sigma_1(s))ds\\ &\quad +\lambda\int_0^t U(t,s)B^{-1}\left[f(s,x(\sigma_2(s)) + \int_0^s k(s,\tau)h(\tau,x(\sigma_3(\tau)))d\tau\right]ds\\ &\quad +\lambda\sum_{0<t_i<t} B^{-1}U(t,t_k)I_kx(t_k)\end{aligned}$$

and one has

$$\begin{aligned}\|x(t)\| &\leq \alpha\beta M(\|x_0\| + c\|x\| + d) + \alpha M\phi_0 + \alpha\phi_1(t)\Omega_e\|x\|\\ &\quad +\alpha M\int_0^a \phi_2(s)\Omega_{Ae}\|x\|ds + \alpha M\int_0^a \phi_3(s)\Omega_f\|x\|ds\\ &\quad +\alpha M\int_0^a K\phi_4(s)\Omega_h\|x\|ds + \alpha M\Omega_I\|x\|\\ &\leq \alpha\beta M(\|x_0\| + c\|x\| + d) + \alpha\|\phi_1\|_{\mathcal{L}^1}\Omega_e\|x\|\\ &\quad +\alpha M\Big[\|\phi_0\|+\|\phi_2\|_{\mathcal{L}^1}\Omega_{Ae}\|x\|+\|\phi_3\|_{\mathcal{L}^1}\Omega_f\|x\|+K\|\phi_4\|_{\mathcal{L}^1}\Omega_h\|x\|+\Omega_I\|x\|\Big].\end{aligned}$$

Consequently,

$$\frac{(1-\alpha\beta Mc)\|x\|}{\alpha\beta M(d+\|x_0\|) + \alpha\phi_1\Omega_e\|x\| + \alpha M[\phi_0 + \phi_2\Omega_{Ae}\|x\| + \phi_3\Omega_f\|x\| + K\phi_4\Omega_h\|x\| + \Omega_I\|x\|]}.$$

Then by (3.1) there exists $\mathcal{N}$ such that $\|x\| \neq \mathcal{N}$. Set

$$\mathcal{U} = \{x \in \mathcal{PC}([0,a],X) : \|x\| < \mathcal{N}\}.$$

From the choice of $\mathcal{U}$ there is no $x \in \partial U$ such that $x = \lambda\mathcal{F}(x)$, for some $\lambda \in (0,1)$. Thus we get a fixed point of $\mathcal{F}$ in $\overline{\mathcal{U}}$ due to Theorem 2.6, which is a mild solution to $(2.1)-(2.3)$. The proof is completed. □

Now, we will give the existence for $(2.1)-(2.3)$ when the nonlocal item g has no compactness. Assume the following holds:

- (M8) The function $g : \mathcal{PC}([0,a],X) \to D(B)$ is Lipschitz continuous with constant $\mathcal{L}$.

Theorem 3.2. *Assume that the conditions $(M1)-(M6)$ and $(M8)$ are satisfied. Then for every $x_0 \in D(B)$the impulsive nonlocal problem $(2.1)-(2.3)$ has at least one mild solution $[0,a]$ provided that there exists a constant $\mathcal{N} > 0$ with*

$$\frac{(1-\alpha\beta M\mathcal{L})\mathcal{N}}{\alpha\beta M(\|g(0)\|+\|x_0\|)+\alpha\phi_1\Omega_e(\mathcal{N})+\alpha M[\phi_0+\phi_2\Omega_{Ae}(\mathcal{N})+\phi_3\Omega_f(\mathcal{N})+K\phi_4\Omega_h(\mathcal{N})+\Omega_I(\mathcal{N})]} > 1 \quad (3.7)$$

and that

$$\alpha\beta M\mathcal{L} + 2\alpha\Big[\|\gamma_e\| + M\{\|\gamma_0\| + \|\gamma_{Ae}\| + \|\gamma_f\| + K\|\gamma_h\| + \|\gamma_I\|\}\Big] < 1. \tag{3.8}$$

Proof. On account of Theorem 3.1, we can prove that operator $\mathcal{F}$ defined by (3.3) is continuous on $\mathcal{PC}([0,a],X)$.

We prove that $\mathcal{F}$ satisfies the Monch's condition holds.

For this purpose, Let $D \subseteq B_r$ is countable and $D \subseteq \overline{co}(0 \cup \mathcal{F}(D))$, we show that $\psi(D) = 0$. Without loss of generality, we may suppose that $D = \{x_n\}_{n=1}^{+\infty}$. By using the condition $(\mathbf{A1}) - (\mathbf{A3})$, we can easily verify that $\{\mathcal{F}_2 x_n\}_{n=1}^{+\infty}$ is equicontinuous. Moreover, $\mathcal{F}_1 : D \to \mathcal{PC}([0,a],X)$ is Lipschitz continuous with constant $\alpha\beta M\mathcal{L}$ due to the condition $(M8)$. In fact, for $x, y \in D$, we have

$$\begin{aligned}\|R_1x - R_1y\| &= \sup_{t\in[0,a]} \|B^{-1}U(t,0)Bg(x) - B^{-1}U(t,0)g(y)\| \\ &\le \alpha\beta M\|g(x) - g(y)\| \\ &\le \alpha\beta M\mathcal{L}\|x - y\|.\end{aligned}$$

So, from $(M2-(iii)), (M3-(iii)), (M4-(iii)), (M8)$ and Lemma 2.3, 2.4, 2.5 it follows that

$$\psi(\{\mathcal{F}x_n\}_{n=1}^{+\infty}) \le \psi(\{\mathcal{F}_1x_n\}_{n=1}^{+\infty}) + \psi(\{\mathcal{F}_2x_n\}_{n=1}^{+\infty})$$

$$\begin{aligned}&\psi(\{\mathcal{F}x_n\}_{n=1}^{+\infty}) \\ &\le \alpha\beta M\mathcal{L}\psi(\{x_n\}_{n=1}^{+\infty}) + \sup_{t\in[0,a]} \psi(\{B^{-1}U(t,0)e(0,x(0))\}) \\ &\quad +\psi\Big(\{B^{-1}e(t,x_n(\sigma_1(t)))\}_{n=1}^{+\infty}\Big) \\ &\quad +\psi\Big(\{\int_0^t U(t,s)A(s)B^{-1}e(s,x_n(\sigma_1(s)))ds\}_{n=1}^{+\infty}\Big) \\ &\quad +\psi\Big(\{\int_0^t U(t,s)B^{-1}f(s,x_n(\sigma_2(s)))ds\}_{n=1}^{+\infty}\Big) \\ &\quad +\psi\Big(\{\int_0^t\int_0^s U(t,s)B^{-1}k(s,\tau)h(\tau,x_n(\sigma_3(\tau)))d\tau ds\}_{n=1}^{+\infty}\Big) \\ &\quad +\psi\Big(\{\sum_{0<t_i<t} B^{-1}U(t,t_k)I_kx_n(t_k)\}_{n=1}^{+\infty}\Big) \\ &\le \alpha\beta M\mathcal{L}\psi(\{x_n\}_{n=1}^{+\infty}) \\ &\quad +2\alpha\Big[\|\gamma_e\| + M\{\|\gamma_0\| + \|\gamma_{Ae}\| + \|\gamma_f\| + K\|\gamma_h\| + \|\gamma_I\|\}\Big]\psi(\{x_n\}_{n=1}^{+\infty}) \\ &\le \Big\{\alpha\beta M\mathcal{L}+2\alpha[\|\gamma_e\|+M\{\|\gamma_0\|+\|\gamma_{Ae}\|+\|\gamma_f\|+K\|\gamma_h\|+\|\gamma_I\|\}]\Big\}\psi(\{x_n\}_{n=1}^{+\infty}).\end{aligned}$$

Thus, we get that

$$\begin{aligned}\psi(D) &\le \psi(\overline{co}(0 \cup \mathcal{F}(D))) \\ &= \psi(\mathcal{F}(D)) \\ &\le \alpha\Big\{\beta M\mathcal{L} + 2[\|\gamma_e\| + M\{\|\gamma_0\| + \|\gamma_{Ae}\| + \|\gamma_f\| + K\|\gamma_h\| + \|\gamma_I\|\}]\Big\}\psi(D)\end{aligned}$$

which implies that $\psi(D) = 0$, since the condition (3.8) holds.

Now with analogous arguments as in the proof of theorem 3.1, we can get an open ball U by the condition of (3.7), and there is no $x \in \partial\mathcal{U}$ such that $x = \lambda\mathcal{F}(x)$ for some $\lambda \in (0,1)$. Thus we get a fixed point of $\mathcal{F}$ in $\overline{\mathcal{U}}$ due to Theorem 2.3, which is a mild solution to $(2.1) - (2.3)$. The proof is completed. □

4. APPLICATION

The notion of controllability is of great importance in mathematical control theory. Many fundamental problems of control theory such as pole-assignment, stabilizability and optimal control may be solved under the assumption that the system is controllable. It means that it is possible to steer any initial state of the system to any final state in some finite time using an admissible control. During the last few decades, several authors have discussed the existence, uniqueness, and asymptotic behavior of the solution of these systems. Apart from these, the study of controllability and observability properties of a system in control theory is certainly, at present, one of the most active interdisciplinary areas of research. Control theory arises in most modern applications. On the other hand, control theory has remained a discipline where many mathematical ideas and methods have fused to produce a new body of important mathematics. In control theory, one of the most important qualitative aspects of a dynamical system is controllability. As far as the controllability problems associated with finite-dimensional systems modelled by ODEs are concerned, this theory has been extensively studied during the last decades. In the finite-dimensional context, a system is controllable if and only if the algebraic Kalman rank condition is satisfied. According to this property, when a system is controllable for some time, it is controllable for all the time. But this is no longer true in the context of infinite-dimensional systems modelled by PDEs.

As an application of Theorem 3.1 we shall consider the system $(2.1)-(2.3)$ with a control parameter such as

$$\frac{d}{dt}\left[Bx(t)+e(t,x(\sigma_1(t)))\right]+A(t)x(t) = f(t,x(\sigma_2(t)))+Cu(t)+\int_0^t k(t,s)h(s,x(\sigma_3(s)))ds,\ t\neq t_k, \quad (4.1)$$

$$x(0)+g(x) = x_0 \quad (4.2)$$

$$\Delta x(t_k) = I_k(x_{t_k}), \quad k=1,2,\ldots,m \quad (4.3)$$

where A,B,f,g,e,h and I_k are as before and C is a bounded linear operator from a Banach space U into X and $u\in\mathcal{L}^2([0,a],U)$. The mild solution of $(4.1)-(4.3)$ is given by

$$\begin{aligned} x(t) &= B^{-1}U(t,0)B[x_0-g(x)]+B^{-1}U(t,0)e(0,x(0)) \\ &\quad -B^{-1}e(t,x(\sigma_1(t))+\int_0^t U(t,s)A(s)B^{-1}e(s,x(\sigma_1(s))ds \\ &\quad +\int_0^t U(t,s)B^{-1}\left[f(s,x(\sigma_2(s)))+Cu(s)+\int_0^s k(s,\tau)h(\tau,x(\sigma_3(\tau)))d\tau\right]ds \\ &\quad +\sum_{0<t_i<t} B^{-1}U(t,t_k)I_k x(t_k). \end{aligned} \quad (4.4)$$

Definition 5.1 ([7, 28]) System $(4.1)-(4.3)$ is said to be controllable on the interval $[0,a]$, if for every $x_0,\ x_1\in X$, there exists a control $u\in\mathcal{L}^2(I,U)$ such that the mild solution $u(\cdot)$ of $(4.1)-(4.3)$ satisfies $x(b)=x_1$.

To study the controllability result we need the following additional condition:

$(M8)$ The linear operator $W:\mathcal{L}^2(I,U)\to X$, defined by

$$Wu=\int_0^a B^{-1}U(a,s)Cu(s)ds$$

induces an inverse operator W^{-1} defined an $\mathcal{L}^2(I,V)/kerW$ and there exists a positive constant $\mathcal{M}_1 > 0$ such that $\|CW^{-1}\| \leq \mathcal{M}_1$.

Theorem 4.1. *If the assumptions* $(M1)-(M8)$ *are satisfied, then the system* $(4.1)-(4.3)$ *is controllable on* I.

Proof. Using the assumption $(M8)$, for an arbitrary function $u(\cdot)$, define the control

$$
\begin{aligned}
u(t) \;=\; & W^{-1}\Big[x_1 - B^{-1}U(t,0)B[x_0 - g(x)] + B^{-1}U(t,0)e(0,x(0)) \\
& -B^{-1}e(t,x(\sigma_1(t))) + \int_0^t U(t,s)A(s)B^{-1}e(s,x(\sigma_1(s))ds \\
& + \int_0^t U(t,s)B^{-1}\left[f(s,x(\sigma_2(s)))+Cu(s) + \int_0^s k(s,\tau)h(\tau,x(\sigma_3(\tau)))d\tau\right] ds \\
& + \sum_{0<t_i<t} B^{-1}U(t,t_k)I_k x(t_k)\Big](t).
\end{aligned}
$$

We shall show that when using this control, the operator $\mathcal{H} : Z \to Z$ defined by

$$
\begin{aligned}
&(\mathcal{H}u)(t) \\
&= B^{-1}U(t,0)B[x_0 - g(x)] + B^{-1}U(t,0)e(0,x(0)) \\
&\quad -B^{-1}e(t,x(\sigma_1(t))) + \int_0^t U(t,s)A(s)B^{-1}e(s,x(\sigma_1(s))ds \\
&\quad + \int_0^t U(t,s)B^{-1}\left[f(s,x(\sigma_2(s))) + \int_0^s k(s,\tau)h(\tau,x(\sigma_3(\tau)))d\tau\right] ds \\
&\quad + \int_0^t U(t,s)B^{-1}CW^{-1}\Big[x_1 - B^{-1}U(t,0)B[x_0 - g(x)] + B^{-1}U(t,0)e(0,x(0)) \\
&\quad -B^{-1}e(t,x(\sigma_1(t))) + \int_0^a U(t,s)A(s)B^{-1}e(s,x(\sigma_1(s))ds \\
&\quad + \int_0^t U(t,s)B^{-1}\left[f(s,x(\sigma_2(s))) + Cu(s) + \int_0^s k(s,\tau)h(\tau,x(\sigma_3(\tau)))d\tau\right] ds \\
&\quad + \sum_{0<t_i<t} B^{-1}U(t,t_k)I_k x(t_k)\Big](t) + \sum_{0<t_i<t} B^{-1}S(t-t_k)I_k x(t_k)
\end{aligned}
$$

has a fixed point. This fixed point is, then a solution of $(4.1)-(4.3)$. Clearly, $(\mathcal{H}u)(a) = x(a) = x_1$, which means that the control u steers the system $(4.1)-(4.3)$ from the initial state x_0 to the final state x_1 at time a, provided we can obtain a fixed point of the nonlinear operator $\mathcal{H}$. The remaining part of the proof is similar to Theorem 3.1 and hence, it is omitted. □

5. EXAMPLE

Consider the partial integrodifferential equation of neutral type

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big[z(t,x) - z_{xx}(t,x) + \int_{-\infty}^t b(s-t)b_1(s,z(\sin s,x))ds\Big] \\
&= -a(t,x)\frac{\partial^2}{\partial y^2}z(t,x) + b_2(t,z(\sin t,x)) \\
&\quad + \frac{\sin z(t,x)}{(1+t)(1+t^2)}\int_0^t k(t,s)e^{-z(\sin s,x)}ds, \\
&\qquad 0 \leq x \leq \pi, \quad t \in I \qquad (5.1)
\end{aligned}
$$

$$z(t,0) = z(t,\pi) = 0, \quad t \geq 0 \tag{5.2}$$

$$z(0,x) + \sum_{i=1}^{m} e_i \phi_{t_i}(s,x) = z_0(x) \in \mathcal{PC}, \quad 0 \leq x \leq \pi \tag{5.3}$$

$$\Delta z|_{t=t_i} = I_i(z(x)) = (\gamma_i(z(x)) + t_i)^{-1}, \ z \in X, \ 1 \leq i \leq m, \tag{5.4}$$

where $a(t,x)$ is continuous on $0 \leq y \leq \pi, \ 0 \leq t \leq a$ and the constant e_i, γ_i are small and $b_1(t,s)$ is continuous such that $\|k(t,s)\| \leq K_B$. Let us take $X = U = L^2[0,\pi]$ endowed with the usual norm $|\cdot|_{L^2}$. Put $x(t) = z(t,x)$ is continuous norm $\|\cdot\|_{\mathcal{L}_2}$ and let

$$\begin{aligned}
e(t,\psi)(x) &= \int_0^{\pi} b_1(s-t)\psi(s,x)ds; \\
f(t,\psi)(x) &= b_2(t,\psi(\sin t, x)); \\
h(s,\psi)(x)ds &= \frac{\sin\psi(t,x)}{(1+t)(1+t^2)} \int_0^t e^{-\psi(\sin s, x)} ds; \\
I_i(\psi)(x) &= (\gamma_i|\psi(x)| + t_i)^{-1}; \\
g(\psi)(x) &= \sum_{i=1}^{m} e_i \phi_{t_i}(s,x).
\end{aligned}$$

Define the operator $A : \mathcal{D}(A) \subset X \longrightarrow X$ and $B : \mathcal{D}(B) \subset X \longrightarrow X$ by

$$Az = -z_{xx}, \quad Bz = z - z_{xx},$$

where each domain $\mathcal{D}(A)$ and $\mathcal{D}(B)$ is given by

$$\{z \in X : z, z_x \text{ are absolutely continuous}, \ z_{xx} \in X, z(0) = z(\pi) = 0\}.$$

Then A and B can be written, respectively, as

$$\begin{aligned}
Az &= \sum_{n=1}^{\infty} n^2 \langle z, z_n \rangle z_n, \ z \in \mathcal{D}(A) \\
Bz &= \sum_{n=1}^{\infty} (1+n^2) \langle z, z_n \rangle z_n, \ z \in \mathcal{D}(B),
\end{aligned}$$

where $z_n(x) = \sqrt{2/\pi}\sin(nx)$, $n = 1, 2, \ldots,$ is the orthogonal set of vectors of A. Furthermore for $z \in X$, we have

$$\begin{aligned}
B^{-1}z &= \sum_{n=1}^{\infty} \frac{1}{1+n^2} \langle z, z_n \rangle z_n; \\
-AB^{-1}z &= \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2} \langle z, z_n \rangle z_n; \\
S(t)z &= \sum_{n=1}^{\infty} \exp\left(\frac{-n^2 t}{1+n^2}\right) \langle z, z_n \rangle z_n.
\end{aligned}$$

It is easy to see that AB^{-1} generates a strongly continuous semigroup $S(t)$ on Y and $S(t)$ is compact such that $|S(t)| \leq e^{-t}$ for each $t > 0$. Now we define the operator $A(t) : \mathcal{D}(A) \subset X \longrightarrow X$ by $A(t)z = a(t,x)Az(x)$. By assuming that $x \longrightarrow a(t,x)$ is continuous in t and there exist $\rho > 0$ such that $a(t,x) \leq -\rho$ for all $t \in I, \ x \in [0,\pi]$, it follows that the system

$$\begin{aligned}
z'(t) &= A(t)z(t), \quad t \geq s, \\
z(s) &= x \in X,
\end{aligned}$$

generates an evolution system $U(t,s)$ as $U(t,s)z = T(t-s)\exp(\int_s^t a(\tau,x)d\tau)z$, for $z \in X$ and $\|U(t,s)\| \leq e^{-(1+\rho)(t-s)}$, for every $t \geq s$.

With this choice of $A(t),\ e,\ f,\ h,\ g$ and I_i, we see that $(5.1)-(5.4)$ can be written in the abstract formulation of $(2.1)-(2.3)$. So all the conditions of the Theorem 3.1 are satisfied. Hence the equation $(5.1)-(5.4)$ has a mild solution.

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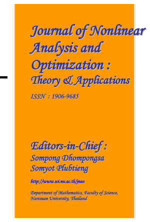

RECENT FIXED POINT THEOREMS FOR T-CONTRACTIVE MAPPINGS AND T-WEAK (ALMOST) CONTRACTIONS IN METRIC AND CONE METRIC SPACES ARE NOT REAL GENERALIZATIONS

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ABSTRACT. The purpose of this research article is to show that recent fixed point theorems obtained in metric and cone metric spaces for T-contractive mappings and TW-contractions are equivalent to previously existing theorems in the literature; hence are redundant. We also show that Proposition 2.5 of [4] is invalid

KEYWORDS: Fixed point theorem; T-contraction; T-contractive mapping; T-Kannan mapping; TCaterjea mapping; $T-$Zamfirescu mapping; $T-$weak contraction; Cone metric space.
AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 2007 Huang and Zhang [10] introduced the notion of cone metric spaces, replacing the set R of real numbers by an ordered real Banach space E as the codomain of a metric and they defined several notions related to sequences in cone metric spaces and proved their properties. They also generalized the famous Banach contraction principle and Kannan's fixed point theorem to such spaces. Subsequently, many authors studied fixed point theory of various kinds of self mappings defined on cone metric spaces.

Many authors [17–24] have noticed that fixed point results in cone metric spaces can be obtained from the existing results in the usual metric space setting. For instance Du [17] obtained the equivalence between three fixed point theorems and their metric space versions. I.D. Arandelovic and D.J keckic [24] also proved that a large number of generalizations of fixed point results to topological vector space valued cone metric spaces (and hence to cone metric spaces) are not real generalizations but a complicated way to formulate a result that is a special case of an old one. But none of these authors proved that all fixed point results that are

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provable in cone metric spaces are reducible to (or obtainable from) corresponding fixed point results in metric spaces.

In [5] Berinde introduced the notion of weak contraction which he renamed later as almost contraction in [6] and proved that such mappings have a fixed point. But the fixed point may not be unique. The class of almost contractions includes Kannan mappings, Chaterjea mappings, Zamfirescu mappings and some quasi-contractions as special cases.

In [4], A. Beiranvand, S. Moradi, M.Omid and H. Pazadeh introduced the notion of T-Banach contraction and extended the Banach contraction principle [3] to such contraction types. In the same paper, they introduced the notion of T-contractive mapping and extended one of Edelstein fixed point theorems ([8], Remark 3.1) to T-contractive maps. In [12], S.Moradi introduced T-Kannan mappings and extended Kannan's fixed point theorem [11] to such maps. All these extensions were done in the setting of metric spaces.

In [13–15], R. Morales and E. Rojas studied T-contractive mappings, T-Kannan contractions, T-Chaterjea contractions, T-Zamfirescu contractions and T-weak (almost) contractions in cone metric spaces.

In [9], Haghi et al showed that some recent fixed point theorems which are supposed to be generalizations of previously existing theorems are not real generalizations. Also, Aydi et al [2] showed that the fixed point theorem for TB-contractions which was obtained by Beiranvand et al [4] is equivalent to the Banach contraction principle.

In this paper we show that the fixed point theorem recently obtained for T-contractive mappings in metric spaces ([4], Theorem 2.9) is equivalent to one of Edelstein's fixed point theorems ([8], Remark 3.1). Secondly, we show that the fixed point theorem for T-weak contractions in cone metric spaces ([15], Theorem 3.4) is equivalent to the cone metric space version of Berinde's fixed point theorem for almost contractions ([1], Theorem 2.1 and [6], Theorem 1). As a corollary of the later, we show that the fixed point theorems obtained in [13–15] for T-Kannan, T-Chaterjea and T-Zamfirescu mappings are not real generalizations. We also provide a counter example to disprove Proposition 2.5 of [4].

2. PRELIMINARIES AND NOTATIONS

Definition 2.1. ([10]) Let E be a real Banach space and $P \subseteq E$. The subset P of E is called a cone if:

(i) P is closed, nonempty and nontrivial (i.e., $P \neq \{0\}$);
(ii) ax + by ∈ P for all $x, y \in P$ and nonnegative real numbers a and b;
(iii) P $\bigcap$ (-P) = {0}.

A cone P of a real Banach space E induces a partial ordering on E as follows. Define $x \preceq y$ if and only if $y - x \in P$ for every $x, y \in E$. Then $\preceq$ is a partial ordering on E.

Notation: For x, y ∈ E, we write $x \prec y$ if $x \preceq y$ and $x \neq y$. Likewise, we write $x \ll y$ if $y - x \in intP$, where int P denotes the interior of P.

Definition 2.2. ([10]): Let E be a real Banach space and $P \subseteq E$ be a cone. The cone P is called normal if there is a number $K > 1$ such that for all $x, y \in E, 0 \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number K satisfying the above inequality is called the normal constant of P. The cone P is called a solid cone if $intP \neq \phi$.

Throughout this paper let E denote a real Banach space and P denote a solid cone of E. Moreover, let $\preceq$ represents the partial ordering on E induced by P.

Definition 2.3. ([10]) Let X be a nonempty set. Suppose a mapping $d : X \times X \to E$ satisfies

(d1) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) d(x, y) = d(y, x) and
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric for X and the pair (X, d) is called a cone metric space.

Definition 2.4. ([10]) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}, n = 1, 2, \cdots$ be a sequence in X. Then we say

(i) $\{x_n\}$ converges to x if for every $c \in intP$ there exists a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.
(ii) $\{x_n\}$ is a Cauchy sequence if for every $c \in intP$ there exists a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(iii) (X, d) is a complete cone metric space if every Cauchy sequence in (X, d) is convergent in (X, d).

Definition 2.5. [4, 12–15] Let (X, d) be a cone metric space and T, S be two self maps of X. The mapping S is said to be:

(i) T-Banach contraction (TB- contraction) if there exists $k \in [0, 1)$

$$d(TSx, TSy) \preceq kd(Tx, Ty). \tag{2.1}$$

(ii) T- Contractive mapping if

$$d(TSx, TSy) \prec d(Tx, Ty) \forall x, y \in X with\ x \neq y. \tag{2.2}$$

(iii) T-Kannan contraction (TK- contraction) if there exists $b \in [0, 1/2)$

$$d(TSx, TSy) \preceq b[d(Tx, TSx) + d(Ty, TSy)]. \tag{2.3}$$

(iv) T- Chaterjea contraction (TC -contraction) if there exists $c \in [0, 1/2)$

$$d(TSx, TSy) \preceq c[d(Tx, TSy) + d(Ty, TSx)] \forall x, y \in X. \tag{2.4}$$

(v) T- Zamfirescu contraction (TZ- contraction) if there are real numbers a, b and c with $0 \leq a < 1, 0 \leq b, c < \frac{1}{2}$ such that for all $x, y \in X$ at least one of the following conditions hold:

$$\begin{aligned} (TZ1) &: d(TSx, TSy) \preceq ad(Tx, Ty) \\ (TZ2) &: d(TSx, TSy) \preceq b[d(Tx, TSx) + d(Ty, TSy)] \\ (TZ3) &: d(TSx, TSy) \preceq c[d(Tx, TSy) + d(Ty, TSx)] \end{aligned} \tag{2.5}$$

(vi) T-Weak contraction(TW-contraction) if there exists real numbers $a \in (0, 1)$ and $b \geq 0$ such that

$$d(TSx, TSy) \preceq ad(Tx, Ty) + bd(Ty, TSx) \forall x, y \in X. \tag{2.6}$$

.

Proposition 2.6. *([15]) Let (X, d) be a cone metric space and T, S be two self maps of X.*

(i) *If S is a TB-contraction, then S is a T-weak contraction.*
(ii) *If S is a TK-contraction, then S is a T-weak contraction.*

(iii) *If S is a TC-contraction, then S is a T-weak contraction.*
(iv) *If S is a TZ-contraction, then S is a T-weak contraction.*

Definition 2.7. ([13]) Let (X, d) be a cone metric space, P a normal cone with normal constant K and $T : X \to X$. Then T is said to be

(i) continuous if for every sequence (x_n) in X and x ∈ X, $lim_{n\longrightarrow\infty} x_n = x$ implies that
$$lim_{n\longrightarrow\infty} Tx_n = Tx;$$
(ii) sequentially convergent if the following holds: For every sequence (y_n) in X, if T(y_n) is convergent, so is (y_n).
(iii) subsequentially convergent if we have, for every sequence (y_n) in X, if $T(y_n)$ is convergent, then (y_n) has a convergent subsequence.

Theorem 2.8. *([8]) Let (X, d) be a compact metric space and S be a self mapping of X satisfying the condition $d(Sx, Sy) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then S has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

A Cone metric space version of Theorem 2.8 is given in [10]. On the other hand, Beiranvand et al [4] extended Theorem 2.8 to T-contractive mappings as follows.

Theorem 2.9. *([4]) Let (X, d) be a compact metric space and S, T be self mappings of X such that T is injective, continuous and S is a T-contractive mapping. Then S has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

Morales et al [13] extended Theorem 2.9 to cone metric spaces as follows.

Theorem 2.10. *([13])Let (X, d) be a compact cone metric space, P be a normal cone with normal constant K and $T, S : X \to X$ functions such that T is injective, continuous and S is T- contractive mapping. Then,*

(i) *S has a unique fixed point;*
(ii) *For any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to the fixed point of S.*

Berinde proved the following theorem.

Theorem 2.11. *([5]) Let (X, d) be a complete metric space and S be a weak (almost) contraction. Then,*

(i) *S has a fixed point;*
(ii) *For any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to a fixed point of S. Further if, for some $\theta \in (0, 1)$ and $L_1 \in [0, \infty)$, S satisfies $d(TSx, TSy) \preceq \theta d(Tx, Ty) + L_1 d(Tx, TSy)$ for all x, y∈ X, then S has a unique fixed point.*

A Cone metric space version of Theorem 2.11 is given in [1]. It is stated as follows.

Theorem 2.12. *([1]) Let (X, d) be a complete cone metric space and the mapping $T : X \to X$ a weak contraction (i.e., there exists a constant $a \in (0, 1)$ and some $b \geq 0$ such that $d(Tx, Ty) \preceq ad(x, y) + bd(y, Tx)$ for all $x, y \in X$. Then T has a fixed point in X.*

Morales et al proved the following result, which can be thought as a common generalization of Theorem 2.11 and Theorem 2.12.

Theorem 2.13. *([15]) Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K and $T, S : X \to X$ functions such that T is injective, continuous and S is a continuous T-weak contraction. Then,*

(i) *If T is subsequentially convergent, then S has a fixed point;*
(ii) *If T is sequentially convergent, then the sequence of iterates $\{S^n x_0\}$ converges to a fixed point of S for any $x_0 \in X$.*

3. MAIN RESULTS

We start with a disproof of proposition 2.5 of [4], which is stated as follows.

Proposition 3.1. *([4]) If (X, d) is a compact metric space, then every function $T : X \to X$ is subsequentially convergent and every continuous function $T : X \to X$ is sequentially convergent.*

Disproof: If X has at least two elements, then the proposition is invalid; in view of the following simple example. Let (X, d) be a compact metric space and T be a constant map on X (i.e., there exists $z \in X$ such that Tx = z $\forall x \in X$). Clearly, T is continuous. Consider the alternate sequence

$$y_n := \begin{cases} x & if\ n = 0, 2, 4, \cdots \\ y & if\ n = 1, 3, 5, \cdots ; \end{cases}$$

where x, y $\in X$ and $x \neq y$. Since (Ty_n) is a constant sequence, so it is convergent. However, (y_n) is not convergent. Thus, Proposition 2.5 of [4] is invalid. □

Theorem 3.2. *Theorem 2.8 is equivalent to Theorem 2.9.*

Proof:

Part I (Theorem 2.9 ⇒ Theorem 2.8):

If T is the identity mapping on X, then Theorem 2.9 reduces to Theorem 2.8.

Part II (Theorem 2.8 ⇒ Theorem 2.9):

Define $\delta(x, y) := d(Tx, Ty) \forall x, y \in X$. Then δ is a metric on X. We now show that (X, δ) is a compact metric space. Let (x_n) be a sequence in X. Since (X, d) is compact, so there exist a subsequence (x_{n_i}) of (x_n) and an element y of X such that x_{n_i} converges to y (with respect to d), i.e. $d(x_{n_i}, y) \to 0$ as $i \to \infty$. Since T is continuous (w.r.t d), so $d(Tx_{n_i}, Ty) \to 0$. This implies that $\delta(x_{n_i}, y) \to 0$. Hence x_{n_i} converges to y with respect to δ, too. Therefore (X, δ) is compact. Furthermore, condition (2.2) reduces to $\delta(Sx, Sy) < \delta(x, y)$ for all $x, y \in X$ with $x \neq y$. By Theorem 2.8, S has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point. □

A similar argument may be used to establish Theorem 3.3 below.

Theorem 3.3. *Theorem 2.10 is equivalent to the cone metric space version of Theorem 2.8.*

Theorem 3.4. *Theorem 2.13 is equivalent to Theorem 2.12.*

Proof:

Part I (Theorem 2.13 ⇒ Theorem 2.12):

If T is the identity mapping on X, then Theorem 2.13 reduces to Theorem 2.12.

Part II (Theorem 2.12 ⇒ Theorem 2.13):

Define $\delta(x, y) := d(Tx, Ty)$ for all $x, y \in X$. Then δ is a cone metric on X. We now show that (X, δ) is a complete cone metric space. Let $\{x_n\}$ be a Cauchy sequence in (X, δ). From the definition of δ, this implies that $\{Tx_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $y \in X$ such that $d(Tx_n, y) \to 0$ as n $\to \infty$. But T is sequentially convergent, and then there exists $x \in X$, such that $d(x_n, x) \to 0$ as $n \to \infty$. Since T is continuous, this implies that $d(Tx_n, Tx) \to 0$ as $n \to \infty$, that is, $\delta(x_n, x) \to 0$ as $n \to \infty$. This proves that (X, δ) is complete. Furthermore, condition (2.6) reduces to $\delta(Sx, Sy) \preceq ad(x, y) + b\delta(x, Sy)$ for all $x, y \in X$. By Theorem 2.12, S has a unique fixed point. Also for any $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point. □

Corollary 3.5.

(i) *Theorem 1 in [10] is equivalent the cone metric space version of Theorem 2.6 in [4].*
(ii) *Theorem 3 in [10] is equivalent to Theorem 3.1 in [14].*
(iii) *Theorem 4 in [10] is equivalent to Theorem 3.5 in [14].*
(iv) *Theorem 3.2 in [15] is equivalent to the cone metric space version of Zamfirescu fixed point theorem.*

Proof Since TB-contractions, TK-contractions, TC-contractions and TZ-contractions are TW-contractions, so corollary 3.5 follows easily from Theorem 3.4. □

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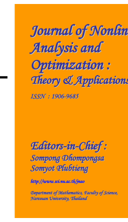

GENERALIZED MINIMAX FRACTIONAL PROGRAMMING PROBLEMS WITH GENERALIZED NONSMOOTH $(F, \alpha, \rho, d, \theta)$-UNIVEX FUNCTIONS

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ABSTRACT. The aim of this paper is to establish the sufficient optimality conditions for a class of nondifferentiable multiobjective generalized minimax fractional programming problems involving $(F, \alpha, \rho, d, \theta)$-univex functions. Subsequently, we apply the optimality condition to formulate a dual model and prove weak, strong and strict converse duality theorems.

KEYWORDS: Generalized minimax fractional programming; $(F, \alpha, \rho, d, \theta)$-univexity; Sufficient optimality conditions; Duality.
AMS Subject Classification: 90C26 90C29 90C32 90C46.

1. INTRODUCTION

Fractional programming is a nonlinear programming method that has knows increasing exposure in the last few decades. Interest of this subject was generated by the fact that various optimization problems from engineering and economics consider the minimization of a ratio between physical and/or economical functions, for example cost/time, cost/volume, cost/profit, or other quantities that measure the efficiency of a system. For example, the productivity of industrial systems, defined as ratio between the realized services in a system within a given period of time and the utilized resources, is used as one of the best indicators of the quality of their operation. See Stancu-Minasian's book [21] which contains the state-of-the art theory and practice developments.

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Consider the following multiobjective generalized fractional programming problem [7]:

$$\text{(GFPP)} \qquad \begin{cases} \min E(x) = \big(E_1(x), E_2(x), ..., E_p(x)\big)^T, \\ \text{subject to} \\ g(x) = \big(g_1(x), g_2(x), ..., g_r(x)\big)^T \leq 0, \\ x \in X, \end{cases}$$

where $E_i(x) = \max_{y \in Y} \dfrac{f_i(x,y) + \Phi_i(x)}{h_i(x,y) - \Psi_i(x)}, i = 1, 2, ..., p.$
In addition, X is a closed convex subset of R^n and Y is a compact subset of R^m, $f_i(x,y) : X \times Y \to R$, $h_i(x,y) : X \times Y \to R$, $g : R^n \to R^r$, $\nabla_x f_i(x,y)$ and $-\nabla_x h_i(x,y)$ exist and are continuous with respect to (x,y) for $i = 1, 2, ..., p$, $f_i(x,y)$ and $-h_i(x,y)$ are upper semicontinuous functions with respect to y on Y for $i = 1, 2, ..., p$, g is a locally Lipschitz function on X, $\Phi_i(x), \Psi_i(x) : R^n \to R$ are convex functions on X for $i = 1, 2, ..., p$, $f_i(x,y) + \Phi_i(x) \geq 0, h_i(x,y) - \Psi_i(x) > 0$, $\forall (x,y) \in R^n \times Y, i = 1, 2, ..., p$.

Minimax fractional programming problems have been widely reviewed by many authors and several approaches for sufficient optimality conditions and duality theorems have been studied under different kinds of generalized convexity, see for example [1, 2, 5, 8, 11, 15, 16, 19, 20, 23], and the references therein.

Liang et al. [17] introduced the concept of (F, α, ρ, d)-convexity and obtained some corresponding optimality conditions and duality results for the single objective fractional problem. Also, Liang et al. [18] extended their results to multiobjective fractional programs. Ahmad and Husain [1, 2] obtained sufficient optimality conditions and duality theorems for a class of nondifferentiable minimax fractional programming problems under generalized (F, α, ρ, d)-convexity assumptions. Later on, Ahmad [3] extended the work Ahmad and Husain [1, 2] to establish second order duality results for the nondifferentiable minimax fractional programming problem under the assumptions of generalized second order (F, α, ρ, d)-convexity.

On the other hand, Bector et al. [4] defined a new class of function called univex functions in nonlinear programming, which were further generalized by several researcher, and obtained optimality and duality results for a nonlinear multiobjective programming problem. Jayswal [11] focus his study on a nondifferentiable minimax fractional programming problem and established sufficient optimality conditions and duality theorems under the assumption of generalized α-univexity. Gupta et al. [9] obtained duality results for two types of second-order dual models of a nondifferentiable minimax fractional programming problem involving second-order α-univex functions.

Recently, Zheng and Cheng [23] given the concept of generalized (F, ρ, θ)-d-univexity in the setting of Clarke's derivative and derived Kuhn-Tucker type sufficient optimality conditions and duality theorems for a nondifferentiable minimax fractional problem with inequality constraints and its three different types of dual problems.

The notion of (V, ρ)-invexity for vector-valued functions was introduced by Kuk et al. [14], which is generalization of the V-invex function given in [13]. Very recently, Tong and Zheng [22] introduced the concept of generalized $(F, \alpha, \rho, \theta)$-$d$-$V$-univex functions involving locally Lipschitz functions and established some alternatives theorems and saddle point necessary optimality conditions for properly efficient solutions of vector optimization problems.

Gao and Rong [7] established Karush-Kuhn-Tucker type necessary conditions for the generalized fractional programming problem (GFPP). Moreover, they also formulated two kinds of dual models for (GFPP) and obtained sufficient optimality conditions and duality theorems under the assumptions of generalized $(F, \alpha, \rho, \theta)$-$V$-convexity.

In this paper, inspired from the work of Ahmad and Husain [1, 2], Gao and Rong [7], Tong and Zheng [22] and Zheng and Cheng [23], we established sufficient optimality conditions and duality theorems for generalized minimax fractional programming problem (GFPP) involving $(F, \alpha, \rho, d, \theta)$-univex functions.

The paper is organized as follow. Some definition and notations are given in Section 2. In Section 3, we derive sufficient optimality conditions for nondifferentiable minimax fractional programming problems under the assumption of generalized $(F, \alpha, \rho, d, \theta)$-univexity. After utilized the optimality condition, a dual problem is formulated and duality results are presented in Section 4. Concluding remarks are presented in Section 5.

2. PRELIMINARIES AND NOTATIONS

Let R^n be the n-dimensional Euclidean space and R^n_+ its non-negative orthant. For $x, y \in R^n$, we let $x \leq y \Leftrightarrow y - x \in R^n_+$; $x < y \Leftrightarrow y - x \in R^n_+ \setminus \{0\}$.

Let $S = \{x \in X : g(x) \leq 0\}$ be the set of all feasible solutions to (GFPP). For each $x \in S$, we define

$$I(x) = \{j : g_j(x) = 0, j = 1, 2, ..., r\},$$

$$Y_i(x) = \left\{ y \in Y : \frac{f_i(x, y) + \Phi_i(x)}{h_i(x, y) - \Psi_i(x)} = \max_{z \in Y} \frac{f_i(x, z) + \Phi_i(x)}{h_i(x, z) - \Psi_i(x)} \right\}, i = 1, 2, ..., p,$$

$$K(x) = \Big\{ (s, \hat{t}, \hat{y}) \in N \times R^{s \times p} \times R^{p \times m \times s} : 1 \leq s \leq n + 1, \hat{t} = (t^1, t^2, ..., t^p),$$
$$t^i = (t^i_1, t^i_2, ..., t^i_s)^T \geq 0, \sum_{i=1}^{s} t^i_l = 1, \hat{y} = (y^1, y^2, ..., y^p)^T,$$
$$y^i = (y^i_1, y^i_2, ..., y^i_s), y^i_l \in Y_i(x), l = 1, 2, ..., s, i = 1, 2, ..., p \Big\}.$$

Definition 2.1. A feasible point $\bar{x}$ is said to be an efficient solution of the multiobjective generalized fractional programming problem (GFPP) if there exists no other feasible x such that

$$E_i(x) \leq E_i(\bar{x}), \text{ for all } i \in P = \{1, 2, ..., p\},$$
$$E_k(x) < E_k(\bar{x}), \text{ for at least one } k \neq i.$$

Definition 2.2. [6] The function $f : X \to R$ is said to be locally Lipschitz on X if for each bounded subset B of X, there exists a constant K such that

$$|f(y) - f(x)| \leq K \|y - x\|, \text{ for all points } y \text{ and } x \text{ of } B,$$

where $\|.\|$ denotes the Euclidean norm.

For the function f Lipschitzian on X , Clarke defined the *generalized directional derivative* of f at a point $x \in X$ in the direction $\nu \in R^n$ by

$$f^0(x; \nu) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda\nu) - f(y)}{\lambda}.$$

Also, he defined the subdifferential (or generalized gradient) of the function f at a point x by the unique, nonempty, convex and compact set

$$\partial f(x) = \left\{\xi \in R^n | f^0(x;\nu) \geq \xi^T \nu, \forall \nu \in R^n\right\}.$$

The elements of $\partial f(x)$ are called subgradients.
It then follows that

$$f^0(x;\nu) = \max\left\{\xi^T \nu | \xi \in \partial f(x)\right\}, \text{ for any } x \text{ and } \nu.$$

We remark that when the function f is smooth (continuously differentiable), $\partial f(x)$ is the singleton set $\{\nabla f(x)\}$ and when f is convex, $\partial f(x)$ coincides with the subdifferential of convex functions.

Definition 2.3. A functional $F : X \times X \times R^n \to R$ is said to be sublinear in its third argument, if for $\forall x, \bar{x} \in X$

(i) $F(x,\bar{x};a_1 + a_2) \leq F(x,\bar{x};a_1) + F(x,\bar{x};a_2), \qquad \forall a_1, a_2 \in R^n,$
(ii) $F(x,\bar{x};\alpha a) = \alpha F(x,\bar{x};a), \qquad \forall \alpha \in R_+, a \in R^n.$

By (ii), it is clear that $F(x,\bar{x};0) = 0$.

To impose the convexity assumptions in the above problem (GFPP), we propose the following definition. Let $f : X \to R$ be a locally Lipschitz and $F : X \times X \times R^n \to R$ be a sublinear functional. Also let $\alpha : X \times X \to R_+ \setminus \{0\}$, $b : X \times X \to R_+$, $\theta : X \times X \to R_+$ such that $x \neq y \Rightarrow \theta(x,y) \neq 0$, $d : R \to R$ with the property that $d(0) = 0$, $\phi : R \to R$ and ρ is a real number.

Definition 2.4. The function f is said to be (F,α,ρ,d,θ)-univex at $y \in X$ with respect to b and ϕ, if the inequality

$$b(x,y)\phi[f(x) - f(y)] \geq F(x,y;\alpha(x,y)\xi) + \rho d^2(\theta(x,y)),$$

holds, for each $x \in X$ and $\xi \in \partial f(y)$.

The function f is said to be (F,α,ρ,d,θ)-univex on X with respect to b and ϕ if it is (F,α,ρ,d,θ)-univex at any point $y \in X$ with respect to the same b and ϕ. In particular, f is said to be strongly (F,α,ρ,d,θ)-univex or (F,α)-univex if $\rho > 0$ or $\rho = 0$, respectively.

It has been revealed in [22] by means of an example that the above class of functions is an extension of F-convex function [10] or η-invex function [12].
Let C be a nonempty subset of X and $d_c(.) : X \to R$ its distance function,

$$d_c(x) = \inf\left\{\, \|x - c\| : c \in C\right\}.$$

Throughout the paper, we assume that the sublinear functional F satisfies the following condition D.

Condition D: Let sublinear functional $F : X \times X \times R^n \to R$ satisfy the following relation for some

$$K > 0, K\partial d_x(\bar{x}) \subset \left\{\epsilon \in R^n : F(x,\bar{x};\epsilon) \leq 0, \forall x \in X\right\}.$$

The following result from [7] is needed in the sequel.

Theorem 2.5. *Let $\bar{x}$ be an efficient solution for (GFPP). If (GFPP) satisfies Calmness Constraints Qualification [6] at $\bar{x}$, in other words, for every $i \in \{1,2,...,p\}$, the following problem*

$$(P)_i \qquad \qquad Min\ E_i(x)$$

subject to

$$E_k(x) - E_k(\bar{x}) \leq 0, k \neq i, x \in S = \{x \in X : g(x) \leq 0\},$$

satisfies Calmness Constraints Qualification at $\bar{x}$, then there exist

$$(s, t, y) \in K(\bar{x}), \lambda \in R^p, \bar{u} \in R^r_+, \bar{e} \in R^p_+,$$

and $K > 0$ such that

$$0 \in \sum_{i=1}^{p} \lambda_i \left\{ \sum_{l=1}^{s} t_l^i \left(\nabla_x f_i(\bar{x}, y_l^i) - \bar{e}_i \nabla_x h_i(\bar{x}, y_l^i) \right) + \partial \Phi_i(\bar{x}) + \bar{e}_i \partial \Psi_i(\bar{x}) \right\}$$

$$+ \sum_{j=1}^{r} \bar{u}_j \partial g_j(\bar{x}) + K \partial d_x(\bar{x}), \tag{2.1}$$

$$f_i(\bar{x}, y_l^i) - \bar{e}_i h_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) + \bar{e}_i \Psi_i(\bar{x}) = 0, i = 1, 2, ..., p, l = 1, 2, ..., s, \tag{2.2}$$

$$\sum_{j=1}^{r} \bar{u}_j g_j(\bar{x}) = 0, \tag{2.3}$$

$$\sum_{l=1}^{s} t_l^i = 1, t_l^i \geq 0, i = 1, 2, ..., p, l = 1, 2, ..., s, \tag{2.4}$$

$$\sum_{i=1}^{p} \lambda_i = 1, \lambda_i > 0, i = 1, 2, ..., p. \tag{2.5}$$

Throughout the paper we denote

$$H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(., y_l^i) + \Phi_i(.) - e_i h_i(., y_l^i) + e_i \Psi_i(.) \right\}.$$

3. SUFFICIENT OPTIMALITY CONDITION

In this section, we shall establish a sufficient optimality condition involving generalized convexity assumptions discussed in the previous section.

Theorem 3.1 (Sufficient optimality conditions). *Let $\bar{x}$ be a feasible solution to (GFPP). Assume that there exist $(s, t, y) \in K(\bar{x}), \lambda \in R^p, u \in R^r_+, e \in R^p_+$, and $K > 0$ satisfying the relations (2.1)-(2.5). Assume also that $H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{ f_i(., y_l^i) + \Phi_i(.)$ $-e_i(h_i(., y_l^i) - \Psi_i(.)) \}$ is $(F, \alpha_1, \rho_1, d, \theta)$-univex at $\bar{x}$ with respect to b_0 and ϕ_0 with $b_0 > 0, V < 0 \Rightarrow \phi_0(V) < 0$ and $\sum_{j=1}^{r} u_j g_j(.)$ is $(F, \alpha_2, \rho_2, d, \theta)$-univex at $\bar{x}$ with respect to b_1 and ϕ_1 with $b_1 \geq 0$, $V \leq 0 \Rightarrow \phi_1(V) \leq 0$. Furthermore, assume*

$$\frac{\rho_1}{\alpha_1(x, \bar{x})} + \frac{\rho_2}{\alpha_2(x, \bar{x})} \geq 0. \tag{3.1}$$

Then $\bar{x}$ is an efficient solution to (GFPP).

Proof. Suppose to the contrary that $\bar{x}$ is not an efficient solution of (GFPP). Then there exists $x \in S$ such that

$$E_i(x) \leq E_i(\bar{x}) = e_i, \text{ for all } i \in P,$$

$$E_k(x) < E_k(\bar{x}) = e_k, \text{ for at least one } k \neq i.$$

Since $\lambda > 0, \sum_{i=1}^{p} \lambda_i = 1$, we have

$$\sum_{i=1}^{p} \lambda_i \left\{ \max_{y \in Y} \{f_i(x, y) + \Phi_i(x) - e_i(h_i(x, y) - \Psi_i(x))\} \right\} < 0.$$

The above inequality together with (2.2), (2.4) and $y_l^i \in Y_i(\bar{x}), l = 1, 2, ..., s, i = 1, 2, ..., p$, yield

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(x, y_l^i) + \Phi_i(x) - e_i h_i(x, y_l^i) + e_i \Psi_i(x) \right\} < 0$$

$$= \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i h_i(\bar{x}, y_l^i) + e_i \Psi_i(\bar{x}) \right\}.$$

That is,

$$H(x) - H(\bar{x}) < 0.$$

Since $b_0(x, \bar{x}) > 0$ and $V < 0 \Rightarrow \phi_0(V) < 0$, we get

$$b_0(x, \bar{x})\phi_0(H(x) - H(\bar{x})) < 0.$$

From $(F, \alpha_1, \rho_1, d, \theta)$-univexity of $H(.)$ at $\bar{x}$, we obtain

$$0 > b_0(x, \bar{x})\phi_0(H(x) - H(\bar{x}))$$
$$\geq F\left(x, \bar{x}; \alpha_1(x, \bar{x}) \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i(\xi_l^i + e_i \eta_l^i)\right) + \rho_1 d^2(\theta(x, \bar{x})),$$
$$\forall \xi_l^i \in \nabla_x f_i(\bar{x}, y_l^i) + \partial\Phi_i(\bar{x}), \forall \eta_l^i \in -\nabla_x h_i(\bar{x}, y_l^i) + \partial\Psi_i(\bar{x}),$$
$$l = 1, 2, ..., s, i = 1, 2, ..., p.$$

Since $\alpha_1(x, \bar{x}) > 0$, by the sublinearity of F, we obtain

$$F\left(x, \bar{x}; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i(\xi_l^i + e_i \eta_l^i)\right) + \frac{\rho_1 d^2(\theta(x, \bar{x}))}{\alpha_1(x, \bar{x})} < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(\bar{x}, y_l^i) + \partial\Phi_i(\bar{x}), \forall \eta_l^i \in -\nabla_x h_i(\bar{x}, y_l^i) + \partial\Psi_i(\bar{x}),$$
$$l = 1, 2, ..., s, i = 1, 2, ..., p. \quad (3.2)$$

By the feasibility of x and from (2.3), we have

$$\sum_{j=1}^{r} u_j(g_j(x) - g_j(\bar{x})) \leq 0.$$

Since $b_1(x, \bar{x}) \geq 0$ and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, from the above inequality, we get

$$b_1(x, \bar{x})\phi_1\left(\sum_{j=1}^{r} u_j(g_j(x) - g_j(\bar{x}))\right) \leq 0.$$

From $(F, \alpha_2, \rho_2, d, \theta)$-univexity of $\sum_{j=1}^{r} u_j g_j(.)$ at $\bar{x}$, we obtain

$$0 \geq b_1(x, \bar{x})\phi_1\left(\sum_{j=1}^{r} u_j(g_j(x) - g_j(\bar{x}))\right) \geq F\left(x, \bar{x}; \alpha_2(x, \bar{x}) \sum_{j=1}^{r} u_j \gamma_j\right)$$
$$+\rho_2 d^2(\theta(x, \bar{x})), \quad \forall \gamma_j \in \partial g_j(\bar{x}), j = 1, 2, ..., r.$$

Since $\alpha_2(x,\bar{x}) > 0$, by the sublinearity of F, we obtain

$$F\left(x,\bar{x};\sum_{j=1}^{r}u_j\gamma_j\right)+\frac{\rho_2 d^2(\theta(x,\bar{x}))}{\alpha_2(x,\bar{x})}\leq 0, \forall\gamma_j\in\partial g_j(\bar{x}), j=1,2,...,r. \tag{3.3}$$

On adding (3.2), (3.3) and with the sublinear functional F satisfying condition D, we get

$$F\left(x,\bar{x};\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i(\xi_l^i+e_i\eta_l^i)+\sum_{j=1}^{r}u_j\gamma_j+K\eta\right)$$
$$+\left(\frac{\rho_1}{\alpha_1(x,\bar{x})}+\frac{\rho_2}{\alpha_2(x,\bar{x})}\right)d^2(\theta(x,\bar{x}))<0,$$
$$\forall\xi_l^i\in\nabla_x f_i(\bar{x},y_l^i)+\partial\Phi_i(\bar{x}), \forall\eta_l^i\in-\nabla_x h_i(\bar{x},y_l^i)+\partial\Psi_i(\bar{x}),$$
$$\forall\gamma_j\in\partial g_j(\bar{x}), \forall\eta\in\partial d_x(\bar{x}), l=1,2,...,s, i=1,2,...,p, j=1,2,...,r.$$

By the assumption $\dfrac{\rho_1}{\alpha_1(x,\bar{x})}+\dfrac{\rho_2}{\alpha_2(x,\bar{x})}\geq 0$, we have

$$F\left(x,\bar{x};\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i(\xi_l^i+e_i\eta_l^i)+\sum_{j=1}^{r}u_j\gamma_j+K\eta\right)<0,$$
$$\forall\xi_l^i\in\nabla_x f_i(\bar{x},y_l^i)+\partial\Phi_i(\bar{x}), \forall\eta_l^i\in-\nabla_x h_i(\bar{x},y_l^i)+\partial\Psi_i(\bar{x}),$$
$$\forall\gamma_j\in\partial g_j(\bar{x}), \forall\eta\in\partial d_x(\bar{x}), l=1,2,...,s, i=1,2,...,p, j=1,2,...,r,$$

which contradicts (2.1). This completes the proof. □

Corollary 3.2. *Let $\bar{x}$ be a feasible solution to (GFPP). Assume that there exist $(s,t,y)\in K(\bar{x}), \lambda\in R^p, u\in R_+^r, e\in R_+^p$, and $K>0$ satisfying the relations (2.1)-(2.5). Assume also that $H(.)=\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i\left\{f_i(.,y_l^i)+\Phi_i(.)\ -e_i(h_i(.,y_l^i)-\Psi_i\right.$ $\left.(.))\right\}$ is strongly $(F,\alpha_1,\rho_1,d,\theta)$-univex at $\bar{x}$ with respect to b_0 and ϕ_0 with $b_0>0, V<0\Rightarrow\phi_0(V)<0$ and $\sum_{j=1}^{r}u_jg_j(.)$ is strongly $(F,\alpha_2,\rho_2,d,\theta)$-univex at $\bar{x}$ with respect to b_1 and ϕ_1 with $b_1\geq 0$, $V\leq 0\Rightarrow\phi_1(V)\leq 0$. Then $\bar{x}$ is an efficient solution to (GFPP).*

Proof. Under the assumptions of this corollary, we know that inequality (3.1) holds. Therefore, $\bar{x}$ is an efficient solution to (GFPP). □

4. DUALITY MODEL

In this section, we consider the following dual for (GFPP) and establish weak, strong and strict converse duality results.

(GFMD) $$\max_{(s,t,y)\in K(z)}\ \sup_{(z,\lambda,u,e,K)\in H_1(s,t,y)}\quad e=(e_1,e_2,...,e_p)^T,$$

subject to

$$0\in\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i\left\{\nabla_x f_i(z,y_l^i)+\partial\Phi_i(z)-e_i\left(\nabla_x h_i(z,y_l^i)-\partial\Psi_i(z)\right)\right\}$$

$$+\sum_{j=1}^{r} u_j \partial g_j(z) + K\partial d_x(z), \tag{4.1}$$

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(z, y_l^i) + \Phi_i(z) - e_i \left(h_i(z, y_l^i) - \Psi_i(z) \right) \right\} \geq 0, \tag{4.2}$$

$$\sum_{j=1}^{r} u_j g_j(z) \geq 0, \tag{4.3}$$

$$\sum_{i=1}^{p} \lambda_i = 1, \lambda > 0, e \geq 0, u \geq 0, K > 0,$$

where $H_1(s,t,y) = \{(z, \lambda, u, e, K) \in R^n \times R^p \times R^r \times R^p \times R\}$.

Theorem 4.1 (Weak duality). *Let x and $(z, \lambda, u, e, K, s, t, y)$ be the feasible solution to (GFPP) and (GFMD), respectively. Suppose that $H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{ f_i(., y_l^i) + \Phi_i(.)$ $-e_i(h_i(., y_l^i) - \Psi_i(.))\}$ is $(F, \alpha_1, \rho_1, d, \theta)$-univex at z with respect to b_0 and ϕ_0 with $b_0 > 0, V < 0 \Rightarrow \phi_0(V) < 0$ and $\sum_{j=1}^{r} u_j g_j(.)$ is $(F, \alpha_2, \rho_2, d, \theta)$-univex at z with respect to b_1 and ϕ_1 with $b_1 \geq 0$, $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, and*

$$\frac{\rho_1}{\alpha_1(x,z)} + \frac{\rho_2}{\alpha_2(x,z)} \geq 0. \tag{4.4}$$

Then the following can not hold:

$$E_i(x) \leq e_i, \text{ for } i = 1, 2, ..., p,$$

and

$$E_k(x) < e_k, \text{ for at least one } k \in \{1, 2, ..., p\}.$$

Proof. Suppose to the contrary that $E(x) < e$, then we have

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(x, y_l^i) + \Phi_i(x) - e_i h_i(x, y_l^i) + e_i \Psi_i(x) \right\} < 0.$$

The above inequality together with (4.2) and $y_l^i \in Y_i(x)$ for $l = 1, 2, ..., s, i = 1, 2, ..., p$, yield

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(x, y_l^i) + \Phi_i(x) - e_i h_i(x, y_l^i) + e_i \Psi_i(x) \right\} < 0$$

$$\leq \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{ f_i(z, y_l^i) + \Phi_i(z) - e_i h_i(z, y_l^i) + e_i \Psi_i(z) \right\}.$$

That is,

$$H(x) - H(z) < 0.$$

Since $b_0(x,z) > 0$ and $V < 0 \Rightarrow \phi_0(V) < 0$, we get

$$b_0(x,z)\phi_0(H(x) - H(z)) < 0.$$

From $(F, \alpha_1, \rho_1, d, \theta)$-univexity of $H(.)$ at z, we obtain

$$0 > b_0(x,z)\phi_0(H(x) - H(z))$$

$$\geq F\left(x, z; \alpha_1(x,z) \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i (\xi_l^i + e_i \eta_l^i) \right) + \rho_1 d^2(\theta(x,z)),$$

$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z),$$
$$l = 1, 2, ..., s, i = 1, 2, ..., p.$$

Since $\alpha_1(x, z) > 0$, by the sublinearity of F, we obtain

$$F\left(x, z; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i(\xi_l^i + e_i \eta_l^i)\right) + \frac{\rho_1 d^2(\theta(x, z))}{\alpha_1(x, z)} < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z),$$
$$l = 1, 2, ..., s, i = 1, 2, ..., p. \quad (4.5)$$

Utilizing the feasibility of x and (4.3), we have

$$\sum_{j=1}^{r} u_j(g_j(x) - g_j(z)) \le 0.$$

Since $b_1(x, z) \ge 0$ and $V \le 0 \Rightarrow \phi_1(V) \le 0$, we get

$$b_1(x, z)\phi_1\left(\sum_{j=1}^{r} u_j(g_j(x) - g_j(z))\right) \le 0.$$

From $(F, \alpha_2, \rho_2, d, \theta)$-univexity of $\sum_{j=1}^{r} u_j g_j(.)$ at z, we obtain

$$0 \ge b_1(x, z)\phi_1\left(\sum_{j=1}^{r} u_j(g_j(x) - g_j(z))\right) \ge F\left(x, z; \alpha_2(x, z)\sum_{j=1}^{r} u_j\gamma_j\right)$$
$$+\rho_2 d^2(\theta(x, z)), \quad \forall \gamma_j \in \partial g_j(z), j = 1, 2, ..., r.$$

Since $\alpha_2(x, z) > 0$, by the sublinearity of F, we obtain

$$F\left(x, z; \sum_{j=1}^{r} u_j\gamma_j\right) + \frac{\rho_2 d^2(\theta(x, z))}{\alpha_2(x, z)} \le 0, \forall \gamma_j \in \partial g_j(z), j = 1, 2, ..., r. \quad (4.6)$$

On adding (4.5), (4.6) and with the sublinear functional F satisfying condition D, we get

$$F\left(x, z; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i(\xi_l^i + e_i \eta_l^i) + \sum_{j=1}^{r} u_j\gamma_j + K\eta\right)$$
$$+\left(\frac{\rho_1}{\alpha_1(x, z)} + \frac{\rho_2}{\alpha_2(x, z)}\right) d^2(\theta(x, z)) < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z),$$
$$\forall \gamma_j \in \partial g_j(z), \forall \eta \in \partial d_x(z), l = 1, 2, ..., s, i = 1, 2, ..., p, j = 1, 2, ..., r.$$

By the assumption $\dfrac{\rho_1}{\alpha_1(x, z)} + \dfrac{\rho_2}{\alpha_2(x, z)} \ge 0$, we have

$$F\left(x, z; \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i(\xi_l^i + e_i \eta_l^i) + \sum_{j=1}^{r} u_j\gamma_j + K\eta\right) < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z),$$
$$\forall \gamma_j \in \partial g_j(z), \forall \eta \in \partial d_x(z), l = 1, 2, ..., s, i = 1, 2, ..., p, j = 1, 2, ..., r,$$

which contradicts the relation (4.1). Therefore the proof is completed. □

Corollary 4.2. *Let x and $(z, \lambda, u, e, K, s, t, y)$ be the feasible solution to (GFPP) and (GFMD), respectively. Suppose that $H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(., y_l^i) + \Phi_i(.) - e_i(h_i(., y_l^i) - \Psi_i(.))\}$ is strongly $(F, \alpha_1, \rho_1, d, \theta)$-univex at z with respect to b_0 and ϕ_0 with $b_0 > 0, V < 0 \Rightarrow \phi_0(V) < 0$ and $\sum_{j=1}^{r} u_j g_j(.)$ is strongly $(F, \alpha_2, \rho_2, d, \theta)$-univex at z with respect to b_1 and ϕ_1 with $b_1 \geq 0$, $V \leq 0 \Rightarrow \phi_1(V) \leq 0$. Then the following can not hold:*

$$E_i(x) \leq e_i, \text{ for } i = 1, 2, ..., p,$$

and

$$E_k(x) < e_k, \text{ for at least one } k \in \{1, 2, ..., p\}.$$

Proof. Under the assumptions of this corollary, we know that inequality (4.4) holds. So, we get the corollary from Theorem 4.1. □

Theorem 4.3 (Strong duality)**.** *Assume that $\bar{x}$ is efficient solution to (GFPP) and let (GFPP) satisfies Calmness Constraints Qualification [6] at $\bar{x}$. Then, there exist $\bar{\lambda} \in R^p, \bar{u} \in R^r, \bar{e} \in R^p_+, (\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$, and $\bar{K} > 0$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{e}, \bar{K}, \bar{s}, \bar{t}, \bar{y})$ is feasible solution to (GFMD). Further, if the hypothesis of weak duality theorem 4.1 holds for all feasible $(z, \lambda, u, e, K, s, t, y)$ to (GFMD), then $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{e}, \bar{K}, \bar{s}, \bar{t}, \bar{y})$ is an efficient solution to (GFMD) and the two objectives have the same optimal values.*

Proof. By Theorem 2.5, there exist $(\bar{s}, \bar{t}, \bar{y}) \in K(\bar{x})$ and $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{e}, \bar{K}) \in H_1(\bar{s}, \bar{t}, \bar{y})$ such that $(\bar{x}, \bar{\lambda}, \bar{u}, \bar{e}, \bar{K}, \bar{s}, \bar{t}, \bar{y})$ is feasible for (GFMD). Since (GFPP) and (GFMD) have the same objective values, the optimality of this feasible solution follows from weak duality Theorem 4.1. □

Theorem 4.4 (Strict converse duality)**.** *Let $\bar{x}$ and $(z, \lambda, u, e, K, s, t, y)$ be the feasible solution to (GFPP) and (GFMD), respectively. Suppose that $H(.) = \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(., y_l^i) + \Phi_i(.) - e_i(h_i(., y_l^i) - \Psi_i(.))\}$ is $(F, \alpha_1, \rho_1, d, \theta)$-univex at z with respect to b_0 and ϕ_0 with $b_0 > 0, V < 0 \Rightarrow \phi_0(V) < 0$ and $\sum_{j=1}^{r} u_j g_j(.)$ is $(F, \alpha_2, \rho_2, d, \theta)$-univex at z with respect to b_1 and ϕ_1 with $b_1 \geq 0$, $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, and let the inequalities*

(a) $\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i(h_i(\bar{x}, y_l^i) - \Psi_i(\bar{x}))\} < 0,$

(b) $\dfrac{\rho_1}{\alpha_1(\bar{x}, z)} + \dfrac{\rho_2}{\alpha_2(\bar{x}, z)} \geq 0,$

hold. Then, $\bar{x} = z$; that is, z is optimal to (GFPP).

Proof. Suppose to the contrary that $\bar{x} \neq z$. By the feasibility of $\bar{x}$ and $(z, \lambda, u, e, K, s, t, y)$ to (GFPP) and (GFMD), respectively and the hypothesis (a), we have

$$\sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \{f_i(\bar{x}, y_l^i) + \Phi_i(\bar{x}) - e_i(h_i(\bar{x}, y_l^i) - \Psi_i(\bar{x}))\} < 0$$

$$\leq \sum_{i=1}^{p} \lambda_i \sum_{l=1}^{s} t_l^i \left\{f_i(z, y_l^i) + \Phi_i(z) - e_i \left(h_i(z, y_l^i) - \Psi_i(z)\right)\right\},$$

and

$$\sum_{j=1}^{r} u_j(g_j(\bar{x}) - g_j(z)) \leq 0.$$

That is,

$$H(\bar{x}) - H(z) < 0,$$

and

$$\sum_{j=1}^{r} u_j(g_j(\bar{x}) - g_j(z)) \leq 0.$$

Since $b_0(\bar{x}, z) > 0$, $b_1(\bar{x}, z) \geq 0$, $V < 0 \Rightarrow \phi_0(V) < 0$, and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, we get

$$b_0(\bar{x}, z)\phi_0(H(\bar{x}) - H(z)) < 0,$$

and

$$b_1(\bar{x}, z)\phi_1\left(\sum_{j=1}^{r} u_j(g_j(\bar{x}) - g_j(z))\right) \leq 0.$$

From $(F, \alpha_1, \rho_1, d, \theta)$-univexity of $H(.)$ and $(F, \alpha_2, \rho_2, d, \theta)$-univexity of $\sum_{j=1}^{r} u_j g_j(.)$ at z, we have

$$\begin{aligned} 0 &> b_0(\bar{x}, z)\phi_0(H(\bar{x}) - H(z)) \\ &\geq F\left(\bar{x}, z; \alpha_1(\bar{x}, z)\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i(\xi_l^i + e_i\eta_l^i)\right) + \rho_1 d^2(\theta(\bar{x}, z)), \\ &\qquad \forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z), \\ &\qquad l = 1, 2, ..., s, i = 1, 2, ..., p, \end{aligned}$$

and

$$0 \geq b_1(\bar{x}, z)\phi_1\left(\sum_{j=1}^{r} u_j(g_j(\bar{x}) - g_j(z))\right) \geq F\left(\bar{x}, z; \alpha_2(\bar{x}, z)\sum_{j=1}^{r} u_j\gamma_j\right)$$
$$+\rho_2 d^2(\theta(\bar{x}, z)), \quad \forall \gamma_j \in \partial g_j(z), j = 1, 2, ..., r.$$

Since $\alpha_1(\bar{x}, z) > 0$, $\alpha_2(\bar{x}, z) > 0$, by the sublinearity of F, above inequalities imply

$$F\left(\bar{x}, z; \sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i(\xi_l^i + e_i\eta_l^i)\right) + \frac{\rho_1 d^2(\theta(\bar{x}, z))}{\alpha_1(\bar{x}, z)} < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z),$$
$$l = 1, 2, ..., s, i = 1, 2, ..., p, \tag{4.7}$$

and

$$F\left(\bar{x}, z; \sum_{j=1}^{r} u_j\gamma_j\right) + \frac{\rho_2 d^2(\theta(\bar{x}, z))}{\alpha_2(\bar{x}, z)} \leq 0, \forall \gamma_j \in \partial g_j(z), j = 1, 2, ..., r. \tag{4.8}$$

On adding (4.7), (4.8) and with the sublinear functional F satisfying condition D , we get

$$F\left(\bar{x}, z; \sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i(\xi_l^i + e_i\eta_l^i) + \sum_{j=1}^{r} u_j\gamma_j + K\eta\right)$$
$$+\left(\frac{\rho_1}{\alpha_1(\bar{x}, z)} + \frac{\rho_2}{\alpha_2(\bar{x}, z)}\right) d^2(\theta(\bar{x}, z)) < 0,$$
$$\forall \xi_l^i \in \nabla_x f_i(z, y_l^i) + \partial\Phi_i(z), \forall \eta_l^i \in -\nabla_x h_i(z, y_l^i) + \partial\Psi_i(z),$$
$$\forall \gamma_j \in \partial g_j(z), \forall \eta \in \partial d_x(z), l = 1, 2, ..., s, i = 1, 2, ..., p, j = 1, 2, ..., r.$$

By the assumption $\dfrac{\rho_1}{\alpha_1(\bar{x},z)}+\dfrac{\rho_2}{\alpha_2(\bar{x},z)}\geq 0$, we have

$$F\left(\bar{x},z;\sum_{i=1}^{p}\lambda_i\sum_{l=1}^{s}t_l^i(\xi_l^i+e_i\eta_l^i)+\sum_{j=1}^{r}u_j\gamma_j+K\eta\right)<0,$$

$$\forall\xi_l^i\in\nabla_x f_i(z,y_l^i)+\partial\Phi_i(z),\forall\eta_l^i\in-\nabla_x h_i(z,y_l^i)+\partial\Psi_i(z),$$

$$\forall\gamma_j\in\partial g_j(z),\forall\eta\in\partial d_x(z),l=1,2,...,s,i=1,2,...,p,j=1,2,...,r,$$

which contradicts the relation (4.1). Therefore the proof is completed. □

5. CONCLUDING REMARK

This paper addressed the sufficient optimality conditions for generalized minimax fractional programming problems involving generalized (F,α,ρ,d,θ)-univex function. For the class of problems, we formulated a dual model and proved weak, strong and strict converse duality theorems. The question arises whether the second and higher order dual and duality theorems for the considered problems hold. This would be the task of our forthcoming works.

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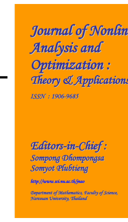

FIXED POINT THEOREMS FOR SOME GENERALIZED NONEXPANSIVE MAPPINGS IN $CAT(0)$ SPACES

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ABSTRACT. In this paper, at first we introduce C_α condition, which is weaker than α-nonexpansivity and present some fixed point theorems for mappings satisfying this condition, in $CAT(0)$ spaces. Our results extend and improve some results in [6]. In the sequal, we introduce fundamentaly nonexpansive mapping which generalizes the Suzuki's generalized nonexpansive mapping and consequently we give some fixed point results for this kind of mappings.

KEYWORDS: $CAT(0)$ spaces; α-nonexpansive mappings; fixed point; Condition C; Opial property.
AMS Subject Classification: 47H09, 47H10.

1. INTRODUCTION

Fixed point theory for nonexpansive and related mappings has played a fundamental role in many aspects of functional analysis for many years. In this paper, we apply generalized nonexpansive definitions which are strong enough to generate a fixed point but do not force the map to be continuous in spite of this fact that in most of the fixed point theorems in this field either continuity is explicitly assumed or, the nonexpansive definitions themselves imply continuity. In 2008, Suzuki [13] introduced condition C as below:
Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy condition (C) (or Suzuki's generalized nonexpansive) if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \quad implies \quad ||Tx - Ty|| \leq ||x - y||,$$

for all $x, y \in C$.

Proposition 1.1. *Every nonexpansive mapping satisfies condition (C), but the inverse is not true (see* [13] *example1).*

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As can be seen, this condition does not force the map to be continuous. Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x, c(l) = y$ and $d(c(t), c(\acute{t})) = |t - \acute{t}|$ for all $t, \acute{t} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x to y for each $x, y \in X$. Write $c(\alpha 0 + (1-\alpha)l) = \alpha x \oplus (1-\alpha)y$ for $\alpha \in (0, 1)$. The space X is said to be of hyperbolic type [8] if it satisfies

$$d(p, \alpha x \oplus (1-\alpha)y) \leq \alpha d(p, x) + (1-\alpha)d(p, y) \ \ \forall p \in X. \tag{1.1}$$

Let $v_1, v_2, ..., v_n \subset X$ and $\lambda_1, \lambda_2, ..., \lambda_n \subset (0, 1)$ with $\sum_{i=1}^{n} \lambda_i = 1$. We write, by induction,

$$\bigoplus_{i=1}^{n} \lambda_i \nu_i := (1-\lambda_n)(\frac{\lambda_1}{1-\lambda_n}\nu_1 \oplus \frac{\lambda_2}{1-\lambda_n}\nu_2 \oplus ... \oplus \frac{\lambda_{n-1}}{1-\lambda_n}\nu_{n-1})\lambda_n \nu_n. \tag{1.2}$$

The definition of $\oplus$ in (3.3) is an ordered one in the sense that it depends on the order of points $v_1, ..., v_n$. Under (3.2) we can see that

$$d(\bigoplus_{i=1}^{n} \lambda_i \nu_i, x) \leq \sum_{i=1}^{n} \lambda_i d(\nu_i, x) \tag{1.3}$$

for each $x \in X$. A subset $Y \subseteq X$ is said to convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane E^2 such that $d_{E^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a $CAT(0)$ space [1] if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$: $d(x, y) \leq d_{E^2}(\bar{x}, \bar{y})$.

Lemma 1.2 ([1], see Proposition 2.2). *Let X be a $CAT(0)$ space. Then for each $p, q, r, s \in X$ and $\alpha \in [0, 1]$,*

$$d(\alpha p \oplus (1-\alpha)q, \alpha r \oplus (1-\alpha)s) \leq \alpha d(p, r) + (1-\alpha)d(q, s).$$

In particular, (3.2) holds in $CAT(0)$ spaces. Let X be a complete $CAT(0)$ space and (x_n) be a bounded sequence in X. For $x \in X$ set:

$$r(x, (x_n)) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r((x_n))$ of (x_n) is given by

$$r((x_n)) = \inf\{r(x, (x_n)) : x \in X\},$$

and the asymptotic center $A((x_n))$ of (x_n) is the set:

$$A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}.$$

It is known that in a $CAT(0)$ space, $A((x_n))$ consists of exactly one point [4], and distance function in $CAT(0)$ spaces, is convex (see page 159 of [1]). Also every $CAT(0)$ space has the *Opial* property, i.e. if (x_n) is a sequence in K and $\Delta - \lim x_n = x$, then for each $y(\neq x) \in K$ we have

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y)$$

Definition 1.3. (see [11], Definition 3.1) A sequence (x_n) in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of (u_n) for every sequence (u_n) of (x_n). In this case, we write $\Delta - \lim_n x_n = x$ and call x the $\Delta - \lim$ of (x_n).

We also need the following theorem which is presented in [12] (see Corollary 2.8).

Theorem 1.1. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ satisfies condition (C) then $F(T)$ (the set of fixed points of T) is nonempty, closed and convex.*

2. GENERALIZED α-NONEXPANSIVE MAPPINGS

Recently, in 2010, the authors in [6] proved some fixed point theorems for α-nonexpansive mappings introduced by Goebel and Pineda [9] as follows :
A mapping T on a nonempty closed convex subset C of a Banach space X is said to be α-nonexpansive if for given multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ satisfies $\alpha_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^n \alpha_i = 1$ we have

$$\sum_{i=1}^n \alpha_i ||T^i x - T^i y|| \leq ||x - y||, \quad \forall x, y \in C.$$

The above definition generalizes the nonexpansive one. Now, we are going to generalize α-nonexpansivity by Suzuki's method:

Definition 2.1. Let C be a nonempty closed convex subset of a Banach space X. For a given multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ satisfies $\alpha_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^n \alpha_i = 1$, $p \in \{1, 2, ..., n\}$, a mapping $T : C \longrightarrow C$ is said to satisfy condition C_α if

$$\frac{1}{2}||x - \sum_{i=1}^p \alpha_i T^i x|| \leq ||x - y|| \quad \textit{implies} \quad \sum_{i=1}^p \alpha_i ||T^i x - T^i y|| \leq ||x - y||, \qquad (2.1)$$

for all $x, y \in C$.

In the case $p = n$, it is easy to show every α-nonexpansive mapping satisfies condition C_α, but the converse is not necessarily true.

Example 2.2. Define a mapping T on $[0, \infty]$ by $Tx = [\frac{x}{3}]$. Then for $\alpha = (\frac{1}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10}, \frac{2}{5})$ and $x = 3k, y = 3k - p$ for $0 < p < 1$ (for example let $x = 729$ and $y = 728.5$, therefore $Tx = 243, T^2 x = 81, T^3 x = 27, T^4 x = 9, T^5 x = 3$ and $Ty = 242, T^2 y = 80, T^3 y = 26, T^4 y = 8, T^5 y = 2$) we have

$$\sum_{i=1}^5 \alpha_i d(T^i x, T^i y) \nleq d(x, y)$$

thus T is not α-nonexpansive, but T satisfies condition C_α.

For technical reason we always assume that the first coefficient α_1 is nonzero. If T satisfies condition C_α then

$$\frac{1}{2}||x - \sum_{i=1}^p \alpha_i T^i x|| \leq ||x - y||$$

implies

$$\sum_{i=1}^{p} \alpha_i ||T^i x - T^i y|| \leq ||x - y||,$$

on the other hand

$$||\sum_{i=1}^{p} \alpha_i T^i x - \sum_{i=1}^{p} \alpha_i T^i y|| \leq \sum_{i=1}^{p} \alpha_i ||T^i x - T^i y||.$$

So if we set $T_{\alpha_p} x = \sum_{i=1}^{p} \alpha_i T^i x$ for all $x \in C$ then it follows that the mapping T_{α_p} satisfies condition C. However, we can't imply that if T_{α_p} satisfies condition C then T satisfies condition C_α because it is much weaker.

3. Fixed point theorems

In this section, we prove some fixed point theorems for mapping satisfying condition C_α in a $CAT(0)$ space. First, we mentioned the definition of condition C_α in $CAT(0)$ spaces as follow:

Definition 3.1. Let C be a nonempty bounded, closed and convex subset of a $CAT(0)$ space X. For a given multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ satisfies $\alpha_i \geq 0, i = 1, 2, ..., n$ and $\sum_{i=1}^{n} \alpha_i = 1$, $p \in \{1, 2, ..., n\}$, a mapping $T : C \longrightarrow C$ is said to satisfy condition C_α if

$$\frac{1}{2} d(x, \bigoplus_{i=1}^{p} \alpha_i T^i x) \leq d(x, y) \quad implies \quad \sum_{i=1}^{p} \alpha_i d(T^i x, T^i y) \leq d(x, y), \tag{3.1}$$

for all $x, y \in C$.

Theorem 3.1. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ satisfies condition C_α and for all $n \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be such that $\alpha_i \geq 0, i = 2, ..., n, \alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $\sum_{i=1}^{n} \alpha_i = 1$, then $F(T) = F(T_{\alpha_p})$ for all $p \in \{1, ..., n\}$.*

Proof. It is clear that $F(T) \subset F(T_{\alpha_p})$. Next, we show that $F(T_{\alpha_p}) \subset F(T)$. Since T satisfies condition C_α, for $x \in F(T_{\alpha_p})$ and for all $k \in \{1, 2, ..., m\}$ we have

$$0 = \frac{1}{2} d(x, \bigoplus_{i=1}^{p} \alpha_i T^i x) \leq d(x, T^k x),$$

let $x \neq Tx$, then for all $m \in \{1, 2, ..., n\}$ we can write

$$\begin{aligned}
d(T^m x, Tx) \leq & \quad \tfrac{1}{\alpha_1} d(T^{m-1} x, x) \\
\leq & \quad \tfrac{1}{\alpha_1} (d(T^{m-1} x, Tx) + d(Tx, x)) \\
\leq & \quad \tfrac{1}{\alpha_1^2} d(T^{m-2} x, x) + \tfrac{1}{\alpha_1} d(Tx, x) \\
\leq & \quad \tfrac{1}{\alpha_1^2} (d(T^{m-2} x, Tx) + d(Tx, x)) + \tfrac{1}{\alpha_1} d(Tx, x) \\
& \quad . \\
& \quad . \\
& \quad . \\
\leq & \quad (\tfrac{1}{\alpha_1^{m-1}} + ... + \tfrac{1}{\alpha_1^2} + \tfrac{1}{\alpha_1}) d(Tx, x).
\end{aligned}$$

So one can write

$$\begin{aligned} d(x,Tx) = & \ d(T_{\alpha_p}x,Tx) \\ = & \ d(\bigoplus_{i=1}^{p}\alpha_i T^i x, Tx) \\ \leq & \ \alpha_2 d(T^2x,Tx)+\alpha_3 d(T^3x,Tx)+...+\alpha_p d(T^px,Tx) \\ \leq & \ \frac{\alpha_2}{\alpha_1}d(Tx,x)+(\frac{\alpha_3}{\alpha_1^2}+\frac{\alpha_3}{\alpha_1})d(Tx,x)+...+(\frac{\alpha_p}{\alpha_1^{p-1}}+...+\frac{\alpha_p}{\alpha_1^2}+\frac{\alpha_p}{\alpha_1})d(Tx,x) \\ = & \ (\frac{\alpha_2+\alpha_3+...+\alpha_p}{\alpha_1}+\frac{\alpha_3+...+\alpha_p}{\alpha_1^2}+...+\frac{\alpha_p}{\alpha_1^{p-1}})d(Tx,x) \\ \leq & \ (\frac{1-\alpha_1}{\alpha_1}+\frac{1-\alpha_1}{\alpha_1^2}+...+\frac{1-\alpha_1}{\alpha_1^{p-1}})d(x,Tx) \\ = & \ \frac{1-\alpha_1^{p-1}}{\alpha_1^{p-1}}d(x,Tx). \end{aligned}$$

Since $\alpha_1 > \frac{1}{\sqrt[n-1]{2}} \geq \frac{1}{\sqrt[p-1]{2}}$ this implies that $\frac{1-\alpha_1^{p-1}}{\alpha_1^{p-1}} < 1$ which lead to a contradiction, therefore $x = Tx$ and this complete the proof. □

Corollary 3.2. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ satisfies condition C_α and for all $n \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be such that $\alpha_i \geq 0$ for $i = 2, ..., n, \alpha_1 > \frac{1}{\sqrt[n-1]{2}}$ and $\sum_{i=1}^{n}\alpha_i = 1$ then $F(T)$ is nonempty closed and convex.*

Proof. Since T_{α_p} satisfies condition C, it follows by Theorem 1.1 and Theorem 3.1 that $F(T)$ is nonempty closed and convex. □

Therefore the existence problem of a fixed point of mapping $T : K \longrightarrow K$ satisfying condition C_α can be directly obtained by the existence of a fixed point of mapping T_α which satisfies condition C. Next, we show that the approximate fixed point sequences for these two mappings are the same.

Theorem 3.2. *Let $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be as in Theorem 3.1. Let K be a bounded closed convex subset of a complete $CAT(0)$ space X and $T : K \to K$ satisfies condition C_α. Suppose (x_m) be a bounded sequence in K and $\alpha_1 d(T^n x_m, T^{n+1}x_m) \leq d(T^n x_m, T^{n-1}x_m)$. Then $d(x_m, Tx_m) \longrightarrow 0$ if and only if $d(x_m, T_{\alpha_p}x_m) \longrightarrow 0$ as $m \longrightarrow \infty$.*

Proof. Let $d(x_m, Tx_m) \longrightarrow 0$. Since $\alpha_1 d(T^n x_m, T^{n+1}x_m) \leq d(T^n x_m, T^{n-1}x_m)$ one can write

$$\begin{aligned} d(T^k x_m, x_m) \leq & \ d(T^k x_m, T^{k-1}x_m)+...+d(T^2x_m,Tx_m)+d(Tx_m,x_m) \\ \leq & \ (\frac{1}{\alpha_1^{k-1}}+...+\frac{1}{\alpha_1}+1)d(Tx_m,x_m). \end{aligned}$$

So $d(T^k x_m, x_m) \longrightarrow 0$ as $m \longrightarrow 0$ for all $k \in \{1, 2, ..., n\}$. Thus by the above equation

$$\begin{aligned} d(T_{\alpha_p}x_m, x_m) = & \ d(\bigoplus_{i=1}^{p}\alpha_i T^i x_m, x_m) \\ \leq & \ \sum_{i=1}^{p}\alpha_i d(T^i x_m, x_m) \longrightarrow 0, \quad as \quad m \longrightarrow \infty. \end{aligned}$$

Again we can write

$$\begin{aligned} d(T^k x_m, Tx_m) \leq & \ d(T^k x_m, T^{k-1}x_m)+...+d(T^2x_m,Tx_m) \\ \leq & \ (\frac{1}{\alpha_1^{k-1}}+...+\frac{1}{\alpha_1})d(Tx_m,x_m). \end{aligned}$$

Now, conversely, assume that $d(x_m, T_{\alpha_p}x_m) \longrightarrow 0$ as $m \longrightarrow \infty$. Since

$$\begin{aligned} d(x_m, Tx_m) &\leq d(x_m, T_{\alpha_p}x_m) + d(T_{\alpha_p}x_m, Tx_m) \\ &= d(x_m, T_{\alpha_p}x_m) + d(\bigoplus_{i=1}^{p} \alpha_i T^i x_m, Tx_m) \\ &\leq d(x_m, T_{\alpha_p}x_m) + \alpha_2 d(T^2x_m, Tx_m) + ... + \alpha_p d(T^p x_m, Tx_m) \\ &\leq d(x_m, T_{\alpha_p}x_m) + \frac{\alpha_2}{\alpha_1} d(Tx_m, x_m) + ... + (\frac{\alpha_p}{\alpha_1^{p-1}} + ... + \frac{\alpha_p}{\alpha_1}) d(Tx_m, x_m) \\ &= d(x_m, T_{\alpha_p}x_m) + (\frac{\alpha_2 + ... + \alpha_p}{\alpha_1} + ... + \frac{\alpha_p}{\alpha_1^{p-1}}) d(Tx_m, x_m) \\ &\leq d(x_m, T_{\alpha_p}x_m) + (\frac{1-\alpha_1}{\alpha_1} + \frac{1-\alpha_1}{\alpha_1^2} + ... + \frac{1-\alpha_1}{\alpha_1^{p-1}}) d(x_m, Tx_m) \\ &= d(x_m, T_{\alpha_p}x_m) + \frac{1-\alpha_1^{p-1}}{\alpha_1^{p-1}} d(x_m, Tx_m), \end{aligned}$$

and $\beta_p = \frac{1-\alpha_1^{p-1}}{\alpha_1^{p-1}} < 1$, hence

$$(1 - \beta_p) d(x_m, Tx_m) \leq d(x_m, T_{\alpha_p}x_m).$$

Which implies that $d(x_m, Tx_m) \longrightarrow 0$ as $m \longrightarrow \infty$. □

Remark 3.3. Note that if K is a bounded closed convex subset of a strictly convex Banach space and $T : K \longrightarrow K$ satisfies condition C,then $F(T)$ is closed and convex [13]. Hence if we use this, instead of Theorem 1.1, then we can write all the above results in the setting where Chakkrid Klin-eam and Suthep Suantai [6] worked in and generalize all their mentioned results.

4. FUNDAMENTALLY NONEXPANSIVE MAPPINGS

In this section, we want to generalize Suzuki's generalized nonexpansive mappings in another manner as follow:

Definition 4.1. Let X be a $CAT(0)$ space and K be a bounded closed convex subset of X. A mapping $T : K \longrightarrow K$ is said to be fundamentally nonexpansive if

$$d(T^2x, Ty) \leq d(Tx, y),$$

for all $x, y \in K$.

Proposition 4.2. *Every mapping which satisfies condition (C) is fundamentally nonexpansive, but the inverse is not true.*

Proof. By taking $\acute{x} = Tx, \acute{y} = y$, we see that every nonexpansive mapping is fundamentally nonexpansive. So by Lemma 3.4 part (iii) in [13] the desired result is obtained. □

Example 4.3. Define a mapping T on $[0, 2]$ by

$$T(x) = \begin{cases} 0 & if x \neq 2, \\ 1 & if x = 2. \end{cases}$$

By taking $x = 2, y = 1.5$ we have

$$\frac{1}{2} d(T(2), 2) \leq d(2, 1.5)$$

but

$$d(T(2), T(1.5)) \nleq d(2, 1.5).$$

Therefore T is fundamentally nonexpansive, but T is not nonexpansive or even satisfies condition (C).

Theorem 4.1. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Let $T : K \to K$ be fundamentally nonexpansive and $F(T) \neq \emptyset$, then $F(T)$ is Δ-closed and convex set.*

Proof. Suppose (x_n) is a sequence in $F(T)$ which Δ-converges to some $y \in K$. We want to show $y \in F(T)$. In order to prove this, one can write

$$d(x_n, Ty) = d(T^2 x_n, Ty) \leq d(Tx_n, y) = d(x_n, y)$$

therefore

$$\limsup_n d(x_n, Ty) \leq \limsup_n d(x_n, y).$$

By the uniqueness of asymptotic center, we obtain $Ty = y$.

$F(T)$ is convex: let $x, z \in F(T)$, then we have:

$$d(x, Ty) = d(T^2 x, Ty) \leq d(Tx, y) = d(x, y),$$

and

$$d(z, Ty) = d(T^2 z, Ty) \leq d(Tz, y) = d(z, y).$$

For $y \in [x, z]$, we have $d(x, y) + d(y, z) = d(x, z)$

$$d(x, z) \leq d(x, Ty) + d(Ty, z) \leq d(x, y) + d(y, z) = d(x, z).$$

Therefore $d(x, Ty) = d(x, y)$ and $d(Ty, z) = d(y, z)$, because if $d(x, Ty) < d(x, y)$ or $d(Ty, z) < d(y, z)$, then we obtain the contradiction $d(x, z) < d(x, z)$, therefore $Ty \in [x, z]$ and $Ty = y$, which means $[x, z] \subset F(T)$. □

Lemma 4.4. [7] *Let (z_n) and (w_n) be bounded sequences in K and $\lambda \in (0, 1)$. Suppose that $z_{n+1} = \lambda w_n + (1 - \lambda) z_n$ and $d(w_{n+1}, w_n) \leq d(z_{n+1}, z_n)$ for all $n \in N$. Then $\limsup_n d(w_n, z_n) = 0$.*

Lemma 4.5. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Let $T : K \to K$ be fundamentally nonexpansive, then always there exists an approximate fixed point sequence for T.*

Proof. Define a sequence (x_n) in K by $x_1 \in K$ and

$$x_{n+1} = \alpha T x_n \oplus (1 - \alpha) x_n$$

for $n \in \mathbb{N}$, where α is a real number belonging to $[0, 1]$. Then we have

$$d(Tx_{n+1}, Tx_n) = \alpha d(T^2 x_n, Tx_n) \leq \alpha d(Tx_n, x_n) = d(x_{n+1}, x_n).$$

for $n \in \mathbb{N}$, hence

$$d(Tx_{n+1}, Tx_n) \leq d(x_{n+1}, x_n).$$

So by Lemma 4.4,

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0$$

holds. □

Lemma 4.6. [5] *Let (x_n) be a bounded sequence in K, then the asymptotic center of (x_n) is in K.*

Theorem 4.2. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. Let $T : K \to K$ be fundamentally nonexpansive, then $F(T)$ is nonempty.*

Proof. By Lemma 4.6, the asymptotic center of any bounded sequence is in K, particularly, the asymptotic center of approximate fixed point sequence for T is in K. Let $A((x_n)) = \{y\}$, we want to show that y is a fixed point of T. In order to prove this, one can write

$$d(x_n, Ty) = d(T^2 x_n, Ty) \leq d(Tx_n, y) = d(x_n, y)$$

therefore

$$\limsup_n d(x_n, Ty) \leq \limsup_n d(x_n, y).$$

By the uniqueness of the asymptotic center $Ty = y$. □

Corollary 4.7. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ is fundamentally nonexpansive, then $F(T)$ is nonempty, Δ-closed and convex.*

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