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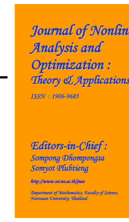

SOME DOUBLE SEQUENCE SPACES IN n-NORMED SPACES USING IDEAL CONVERGENCE AND A SEQUENCE OF ORLICZ FUNCTIONS

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ABSTRACT. In the present paper we introduce some double sequence spaces using ideal convergence and a sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ in n-normed spaces and examine some topological properties of the resulting sequence spaces.

KEYWORDS: Paranorm space; I-convergence; Difference sequence spaces; Orlicz function; Musielak-Orlicz function; n-normed spaces.
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1. INTRODUCTION AND PRELIMINARIES

The initial works on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [13], Moricz [20], Moricz and Rhoades [21], Tripathy ([38, 39]), Başarır and Sonalcan [2] and many others. Hardy [13] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [41] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [25] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [23] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals

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of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Recently Başar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Now, recently Raj and Sharma [33] have introduced double sequence spaces of entire functions. By the convergence of a double sequence we mean the convergence in the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [28]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.

The concept of 2-normed spaces was initially developed by Gähler [8] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [22]. Since then, many others have studied this concept and obtained various results, see Gunawan ([10, 11]) and Gunawan and Mashadi [12] and references therein. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \cdots, \cdot\|$ on X^n satisfying the following four conditions:

(i) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(ii) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutation;
(iii) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \; \|x_1, x_2, \cdots, x_n\|$ for any $\alpha \in \mathbb{K}$, and
(iv) $\|x + x', x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|x', x_2, \cdots, x_n\|$

is called a n-norm on X, and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $\|x_1, x_2, \cdots, x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \cdots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, \|\cdot, \cdots, \cdot\|)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \cdots, x_{n-1}, a_i\| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{\substack{k\to\infty \\ p\to\infty}} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

The notion of difference sequence spaces was introduced by Kızmaz [14], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and

$c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let r be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\},$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \begin{pmatrix} r \\ v \end{pmatrix} x_{k+v}.$$

Taking $r = 1$, we get the spaces which were introduced and studied by Kızmaz [14]. An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [17] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Let $\lambda = (\lambda_r)$ be a non-decreasing sequence of positive numbers tending to infinity and $\lambda_{r+1} \leq \lambda_r + 1,\ \ \lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k \in I_r} x_k,\ \ I_r = [r - \lambda_r + 1, r].$$

A single sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L if $t_r(x) \to L$ as $r \to \infty$ see [16]. If $\lambda_r = r$, then the (V, λ)-summability is reduced to $(C, 1)$-summability see ([36, 37]).

The double sequence $\lambda_2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \leq \lambda_{m,n} + 1,\ \ \lambda_{m,n+1} \leq \lambda_{m,n} + 1,$$

$$\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1},\ \ \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \Big\{(k, l) : m - \lambda_{m,n} + 1 \leq k \leq m,\ \ n - \lambda_{m,n} + 1 \leq l \leq n\Big\}.$$

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(i) $p(x) \geq 0$ for all $x \in X$,
(ii) $p(-x) = p(x)$ for all $x \in X$,
(iii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
(iv) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [40], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [18, 19, 24, 27, 29–31, 34]) and reference therein.
A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.
The notion of ideal convergence was introduced first by P. Kostyrko [15] as a generalization of statistical convergence which was further studied in topological spaces (see [5]). More applications of ideals can be seen in ([5, 6]).
Recently a lot of activities have started to study sumability, sequence spaces and related topics in these non linear spaces (see [9, 35]). In particular Sahiner [35] combined these two concepts and investigated ideal sumability in these spaces and introduced certain sequence spaces using 2-norm. Raj and Sharma [32] have introduced some sequence spaces of ideal convergence in 2-normed spaces.
We continue in this direction and by using a sequence of Orlicz functions, generalized sequences and also ideals we introduce I-convergence of generalized sequences with respect to a sequence of Orlicz functions in n-normed spaces.

Let $(X, \|\cdot, \cdots, \cdot\|)$ be a normed space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called statistically convergent to $x \in X$ if the set $A(\epsilon) = \Big\{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\Big\}$ has natural density zero for each $\epsilon > 0$.
A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if

(i) $\phi \in \mathcal{I}$;
(ii) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$;
(iii) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ (see [9]).

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\Big\}$ belongs to $\mathcal{I}$ (see [15]).
Let $\Lambda = (\lambda_{m,n})$ be non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \geq \lambda_n + 1$, $\lambda_1 = 0$ and let I be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $(X, \|\cdot, \cdots, \cdot\|)$ is a n-normed space. Let $p = (p_{k,l})$ be a bounded sequence of positive real numbers. By $S''(n - X)$ we denote the space of all sequences defined over $(X, \|\cdot, \cdots, \cdot\|)$. Now we define the following sequence spaces in this paper :

$$(W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) =$$

$$\Big\{x \in S''(n-X) : \forall\ \epsilon > 0 \ \Big\{m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \geq \epsilon\Big\} \in I$$

$$\text{for some } \ L, \rho > 0 \ \text{ and } \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$(W_0^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)=$

$$\Big\{x\in S''(n-X):\forall\ \epsilon>0\ \Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\epsilon\Big\}\in I$$

$$\text{for some}\ \ \rho>0\ \ \text{and}\ \ z_1,\cdots,z_{n-1}\in X\Big\},$$

$(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)=$

$$\Big\{x\in S''(n-X):\exists\ K>0\ \ \text{such that}\ \ \sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq K\ \ \text{for some}\ \ \rho>0\ \ \text{and}\ \ z_1,\cdots,z_{n-1}\in X\Big\},$$

$(W_\infty^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)=$

$$\Big\{x\in S''(n-X):\exists\ K>0\ \ \text{such that}\ \ \Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\geq K\Big\}\in I\ \ \text{for some}\ \ \rho>0\ \ \text{and}\ \ z_1,\cdots,z_{n-1}\in X\Big\}.$$

The following inequality will be used throughout the paper. If $0\leq p_{k,l}\leq\sup p_{k,l}=H$, $D=\max(1,2^{H-1})$ then

$$|a_{k,l}+b_{k,l}|^{p_{k,l}}\leq D\{|a_{k,l}|^{p_{k,l}}+|b_{k,l}|^{p_{k,l}}\}\tag{1.1}$$

for all k,l and $a_{k,l},b_{k,l}\in\mathbb{C}$. Also $|a|^{p_{k,l}}\leq\max(1,|a|^H)$ for all $a\in\mathbb{C}$.
The main aim of this paper is to study some topological properties and some inclusion relation between above defined sequence spaces.

2. MAIN RESULTS

Theorem 2.1. *Let $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions, $p=(p_{k,l})$ be a bounded sequence of positive real numbers and I be an admissible ideal of $\mathbb{N}$. Then $(W^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$, $(W_0^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$, $(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ and $(W_\infty^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ are linear spaces.*

Proof. Let $x=(x_{k,l}),y=(y_{k,l})\in(W^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ and $\alpha,\beta\in\mathbb{R}$. So

$$\Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\epsilon\Big\}\in I$$

$$\text{for some}\ \ L,\ \rho_1>0,\ \text{and}\ \ z_1,\cdots,z_{n-1}\in X\Big\}$$

and

$$\Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}-L}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\epsilon\Big\}\in I$$

$$\text{for some}\ \ L,\ \rho_2>0\ \text{and}\ \ z_1,\cdots,z_{n-1}\in X\Big\}.$$

Since $\|\cdot,\cdots,\cdot\|$ is a n-norm and $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions the following inequality holds:

$$\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r(\alpha x_{k,l}+\beta y_{k,l})-L}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq\quad D\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$+ \quad D\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}-L}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$
$$\leq \quad DF\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$
$$+ \quad DF\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}-L}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}},$$

where $F=\max\Big[1,\Big(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H,\Big(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H\Big]$. From the above inequality, we get $\Big\{m,n\in\mathbb{N}:\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r(\alpha x_{k,l}+\beta y_{k,l})-L}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq \epsilon\Big\}$

$$\subseteq \quad \Big\{m,n\in\mathbb{N}:DF\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\frac{\epsilon}{2}\Big\}$$
$$\cup \quad \Big\{m,n\in\mathbb{N}:DF\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}-L}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\geq\frac{\epsilon}{2}\Big\}.$$

Two sets on the right hand side belong to I and this completes the proof. Similarly, we can prove that $(W_0^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$, $(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ and $(W_\infty^I)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ are linear spaces. □

Theorem 2.2. *Let $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions and $p=(p_{k,l})$ be a bounded sequence of positive real numbers. For any fixed $m,n\in\mathbb{N}$, $(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$ is a paranormed space with the paranorm defined by*

$$g_{m,n}(x)=\inf\Big\{\rho^{\frac{p_{m,n}}{H}}:\rho>0 \quad \textit{is such that} \quad \sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$
$$\leq 1,\forall z_1,\cdots,z_{n-1}\in X\Big\}.$$

Proof. It is clear that $g_{m,n}(x)=g_{m,n}(-x)$. Since $M_{k,l}(0)=0$, we get $\inf\{\rho^{\frac{p_{m,n}}{H}}\}=0$ for $x=0$ therefore, $g_{m,n}(0)=0$. Let us take $x=(x_{k,l})$ and $y=(y_{k,l})$ in $(W_\infty)_2\Big(\lambda,\mathcal{M},\Delta^r,p,\|\cdot,\cdots,\cdot\|\Big)$. Let

$$B(x)=\Big\{\rho>0:\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\leq 1,\forall z_1,\cdots,z_{n-1}\in X\Big\},$$

$$B(y)=\Big\{\rho>0:\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\leq 1,\forall z_1,\cdots,z_{n-1}\in X\Big\}.$$

Let $\rho_1\in B(x)$ and $\rho_2\in B(y)$. Then if $\rho=\rho_1+\rho_2$, we have

$$\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}M_{k,l}\Big(\|\frac{\Delta^r(x_{k,l}+y_{k,l})}{\rho},z_1,\cdots,z_{n-1}\|\Big)$$
$$\leq \quad \frac{\rho_1}{\rho_1+\rho_2}\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)$$

$$+ \quad \frac{\rho_2}{\rho_1+\rho_2}\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r y_{k,l}}{\rho_2}, z_1,\cdots,z_{n-1}\|\Big).$$

Thus $\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r (x_{k,l}+y_{k,l})}{\rho_1+\rho_2}, z_1,\cdots,z_{n-1}\|\Big)^{p_{k,l}} \leq 1$ and

$$\begin{aligned} g_{m,n}(x+y) &\leq \inf\Big\{(\rho_1+\rho_2)^{\frac{p_{m,n}}{H}} : \rho_1\in B(x),\ \rho_2\in B(y)\Big\} \\ &\leq \inf\Big\{\rho_1^{\frac{p_{m,n}}{H}} : \rho_1\in B(x)\Big\} + \inf\Big\{\rho_2^{\frac{p_{m,n}}{H}} : \rho_2\in B(y)\Big\} \\ &= g_{m,n}(x)+g_{m,n}(y). \end{aligned}$$

Let $\sigma^s\to\sigma$ where $\sigma,\sigma^s\in\mathbb{C}$ and let $g_{m,n}(x^s_{k,l}-x)\to 0$ as $s\to\infty$.
We have to show that $g_{m,n}(\sigma^s x^s_{k,l}-\sigma x)\to 0$ as $s\to\infty$. Let

$$B(x^s)=\Big\{\rho_s>0:\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x^s_{k,l}}{\rho_s}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\leq 1, \forall z_1,\cdots,z_{n-1}\in X\Big\},$$

$$B(x^s-x)=\Big\{\rho'_s>0:\sup_{k,l}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big[M_{k,l}\Big(\|\frac{\Delta^r x^s_{k,l}-x_{k,l}}{\rho'_s}, z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}\leq 1,$$

$$\forall z_1,\cdots,z_{n-1}\in X\Big\}.$$

If $\rho_s\in B(x^s)$ and $\rho'_s\in B(x^s-x)$ then we observe that

$$\begin{aligned} &\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} M_{k,l}\|\frac{\Delta^r(\sigma^s x^s_{k,l}-\sigma x_{k,l})}{\rho_s|\sigma^s-\sigma|+\rho'_s|\sigma|}, z_1,\cdots,z_{n-1}\| \\ &\leq \frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r(\sigma^s x^s_{k,l}-\sigma x^s_{k,l})}{\rho_s|\sigma^s-\sigma|+\rho'_s|\sigma|}, z_1,\cdots,z_{n-1}\| \\ &+ \|\frac{\Delta^r(\sigma x^s_{k,l}-\sigma x_{k,l})}{\rho_s|\sigma^s-\sigma|+\rho'_s|\sigma|}, z_1,\cdots,z_{n-1}\|\Big) \\ &\leq \frac{|\sigma^s-\sigma|\rho_s}{\rho_s|\sigma^s-\sigma|+\rho'_s|\sigma|}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r(x^s_{k,l})}{\rho_s}, z_1,\cdots,z_{n-1}\|\Big) \\ &+ \frac{|\sigma|\rho'_s}{\rho_s|\sigma^s-\sigma|+\rho'_s|\sigma|}\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}} M_{k,l}\Big(\|\frac{\Delta^r(x^s_{k,l}-x_{k,l})}{\rho'_s}, z_1,\cdots,z_{n-1}\|\Big). \end{aligned}$$

From the above inequality, it follows that

$$\frac{1}{\lambda_{m,n}}\sum_{k,l\in I_{m,n}}\Big(M_{k,l}\Big(\|\frac{\Delta^r(\sigma^s x^s_{k,l}-\sigma x_{k,l})}{\rho_s|\sigma^s-\sigma|+\rho'_s|\sigma|}, z_1,\cdots,z_{n-1}\|\Big)\Big)^{p_{k,l}}\leq 1$$

and consequently,

$$\begin{aligned} g_{m,n}(\sigma^s x^s-\sigma x) &\leq \inf\Big\{\Big(\rho_s|\sigma^s-\sigma|+\rho'_s|\sigma|\Big)^{\frac{p_{m,n}}{H}} : \rho_s\in B(x^s), \rho'_s\in B(x^s-x)\Big\} \\ &\leq (|\sigma^s-\sigma|)^{\frac{p_{m,n}}{H}}\inf\Big\{\rho^{\frac{p_{m,n}}{H}} : \rho_s\in B(x^s)\Big\} \\ &+ (|\sigma|)^{\frac{p_{m,n}}{H}}\inf\Big\{(\rho'_s)^{\frac{p_{m,n}}{H}} : \rho'_s\in B(x^s-x)\Big\} \\ &\longrightarrow 0 \quad \text{as} \quad m\longrightarrow\infty. \end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions which satisfies Δ_2-condition. Then $(W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subset (W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subset (W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ and the inclusions are strict.*

Proof. The inclusion $(W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subset (W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ is obvious. We have only show that $(W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subset (W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$. Let $(x_{k,l}) \in (W^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$. Then

$$\begin{aligned}
&\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{2\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&= \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l} + L - L}{2\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&\leq \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l} - L}{2\rho}, z_1, \cdots, z_{n-1}\| + \|\frac{L}{2\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&\leq DG\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\
&+ DG\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}},
\end{aligned}$$

where $G = \max\big\{1, \left(\frac{1}{2}\right)^H\big\}$. Thus from Δ_2-condition, we have $x \in (W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ and this completes the proof of the theorem. □

Theorem 2.4. *Let $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ are sequences of Orlicz functions. Then we have*
(i) $(W_0^I)_2\Big(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subseteq (W_0^I)_2\Big(\lambda, \mathcal{M} \circ \mathcal{M}', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ provided $(p_{k,l})$ is such that $H_0 = \inf p_{k,l} > 0$.
(ii) $(W_0^I)_2\Big(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \cap (W_0^I)_2\Big(\lambda, \mathcal{M}'', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \subseteq (W_0^I)_2\Big(\lambda, \mathcal{M}' + \mathcal{M}'', \Delta^r, p, \|\cdot, \cdots, \cdot\|)$.

Proof. (i) For given $\epsilon > 0$, first choose $\epsilon_0 > 0$ such that $\max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \epsilon$. Now using the continuity of $M_{k,l}$. Choose $0 < \delta < 1$ such that $0 < t < \delta$, this implies that $M_{k,l}(t) < \epsilon_0$. Let $(x_{k,l}) \in (W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$. Now from the definition

$$B(\delta) = \Big\{m, n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \geq \delta^H\Big\} \in I.$$

Thus if $m, n \notin B(\delta)$ then

$$\begin{aligned}
&\frac{1}{\lambda_{m,n}} \sum_{k,l \in I_{m,n}} \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \delta^H \\
&\Longrightarrow \sum_{k,l \in I_{m,n}} \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \lambda_{m,n}\delta^H \\
&\Longrightarrow \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \delta^H \text{ for all } k, l \in I_{m,n} \\
&\Longrightarrow \Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big] < \delta \text{ for all } k, l \in I_{m,n}.
\end{aligned}$$

Hence from above using the continuity of $\mathcal{M} = (M_{k,l})$ we must have

$$M_{k,l}\Big(M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big) < \epsilon_0 \ \forall \ \ k,l \in I_{m,n}$$

which consequently implies that

$$\begin{aligned}\sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big)\Big]^{p_{k,l}} &< \lambda_{m,n}\max\{\epsilon_0^H, \epsilon_0^{H_0}\} \\ &< \lambda_{m,n}\epsilon \\ \Longrightarrow \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big)\Big]^{p_{k,l}} &< \epsilon.\end{aligned}$$

This shows that

$$\Big\{m,n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big)\Big]^{p_{k,l}} \geq \epsilon\Big\} \subset B(\delta)$$

and so belongs to I. This proves the result.

(ii) Let $(x_{k,l}) \in (W_0^I)_2\Big(\lambda, \mathcal{M}', \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big) \cap (W_0^I)_2\Big(\lambda, \mathcal{M}''_{k,l}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$. Then the fact

$$\begin{aligned}&\frac{1}{\lambda_{m,n}}\Big[(M'_{k,l} + M''_{k,l})\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\ &\qquad \leq D\frac{1}{\lambda_{m,n}}\Big[M'_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \\ &\qquad + D\frac{1}{\lambda_{m,n}}\Big[M''_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\end{aligned}$$

gives the result. □

Theorem 2.5. *The sequence spaces* $(W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *and* $(W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *are solid.*

Proof. Let $(x_k) \in (W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$, let $(\alpha_{k,l})$ be a sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$. Then we have

$$\Big\{m,n \in \mathbb{N} : \frac{1}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{(\Delta^r \alpha_{k,l} x_{k,l})}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}}\Big\} \subset$$

$$\Big\{m,n \in \mathbb{N} : \frac{C}{\lambda_{m,n}} \sum_{k,l\in I_{m,n}} \Big[M_{k,l}\Big(\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} \geq \epsilon\Big\} \in I,$$

where $C = \max\{1, |\alpha_{k,l}|^H\}$. Hence $(\alpha_{k,l}x_{k,l}) \in (W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ for all sequences of scalars $\alpha_{k,l}$ with $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$ whenever $(x_{k,l}) \in (W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$.

Similarly, we can prove that $(W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ is also solid. □

Theorem 2.6. *The sequence spaces* $(W_0^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *and* $(W_\infty^I)_2\Big(\lambda, \mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|\Big)$ *are monotone.*

Proof. It is easy to prove so we omit the details. □

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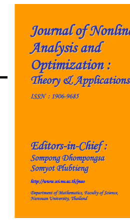

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COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF SELFMAPS SATISFYING GENERALIZED WEAKLY CONTRACTIVE CONDITION

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ABSTRACT. In this paper, we prove a common fixed point theorem for two pairs of selfmaps satisfying certain generalized weakly contractive condition. Also, we prove the same for two pairs of such selfmaps in which one pair is compatible, reciprocally continuous and the other pair is weakly compatible. Some existing results are drawn as corollaries from the main results of this paper. Examples are given in support of the main results of the paper.

KEYWORDS : Common fixed point; Generalized weakly contractive condition; Complete metric space, Compatible maps; Weakly compatible maps and reciprocally continuous maps
AMS Subject Classification: 47H10 54H25

1. INTRODUCTION

In 1977, Rhoades [10] compared various definitions of contractive mappings on a complete metric space which were used to generalize Banach contraction mapping principle. After 20 years, in 1997, weakly contractive maps were introduced by Alber and Guerre-Delabriere [1] in Hilbert spaces which generalize contraction maps, and established a fixed point theorem in Hilbert space setting. Rhoades [11] extended this idea to Banach spaces and proved the existence of fixed points of weakly contractive selfmaps in Banach space setting. Weakly contractive maps have been considered in several works by different researchers namely Alber, Guerre-Delabrier [1], Babu, Nageswara Rao and Alemayehu [2], Babu and Alemayehu [3], Choudhury, Konar, Rhoades and Metiya [4], Doric [5], Dutta and Choudhury [6] and Rhoades [11] and some references cited in these papers in order to establish the existence of fixed points.

Throughout this paper we denote
$\Phi = \{\phi : [0,\infty) \to [0,\infty) \mid \phi \text{ is lower semicontinuous and } \phi(t) = 0 \Leftrightarrow t = 0\}.$
$\Psi = \{\psi : [0,\infty) \to [0,\infty) \mid \psi \text{ is continuous, nondecreasing and } \psi(t) = 0 \Leftrightarrow t = 0\},$

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Definition 1.1. (Rhoades [11]) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be weakly contractive if there exists $\psi \in \Psi$ such that

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y)) \text{ for all } x, y \in X.$$

Theorem 1.1. (*Rhoades* [11] *Let* (X, d) *be a complete metric space and* T *be a weakly contractive mapping. Then* T *has a unique fixed point in X.*

Definition 1.2. (Choudhury, Konar, Rhoades and Metiya [4]) Let (X, d) be a metric space and T be a selfmap of X. T is a generalized weakly contractive map if there exist maps $\psi \in \Psi$ and $\phi : [0, \infty) \to [0, \infty)$ satisfying ϕ is continuous and$\phi(\mathrm{t}) = 0 \iff \mathrm{t} = 0$ such that
$d(Tx, Ty) \le \psi(M(x, y)) - \phi(\max\{d(x, y),\ d(y, Ty)\})$ for all $x, y \in X$,
where

$$M(x, y) = \max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}.$$

Definition 1.3. (Jungck [7]) Let f and g be selfmaps of a metric space (X, d). The pair (f, g) is said be a *compatible pair* on X, if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$.

Definition 1.4. (Jungck and Rhoades [8]) Let f and g be selfmaps of a metric space (X, d). The pair (f, g) is said be *weakly compatible* if they commute at their coincidence point, *i.e.*, $fgx = gfx$ whenever $gx = fx$, $x \in X$.

Here we note that every compatible pair is weakly compatible pair of maps but its converse need not be true [7].

Definition 1.5. (Pant [9]) Let f and g be selfmaps of a metric space (X, d). Then f and g are said to be *reciprocally continuous* if $\lim_{n\to\infty} fgx_n = ft$ and
$\lim_{n\to\infty} gfx_n = gt$ whenever $\{x_n\}$ is a sequence in X such that
$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$.

Clearly if f and g are continuous then they are reciprocally continuous, but its converse need not be true (Pant [9]).

Theorem 1.2. (*Choudhury, Konar, Rhoades and Metiya* [4]) *Let* (X, d) *be a complete metric space and* T *a generalized weakly contractive mapping of* X. *Then T has a unique fixed point.*

Theorem 1.3. (*Choudhury, Konar, Rhoades and Metiya* [4]) *Let* (X, d) *be a complete metric space. Let* f *and* g *be selfmaps of* X. *Suppose that there exist maps* $\psi \in \Psi$ *and* $\phi : [0, \infty) \to [0, \infty)$ satisfying ϕ is continuous and

$\phi(t) = 0$ if and only $\mathrm{t} = 0$ *such that*
$d(fx, gy) \le \psi(M(x, y)) - \phi(m(x, y)\})$ *for all* $x, y \in X$, *where*

$M(x, y) = \max\{d(x, y),\ d(x, fx),\ d(y, gy),\ \frac{1}{2}[d(x, gy) + d(y, fx)]\}$
and
$m(x, y) = \max\{d(x, y),\ d(x, fx),\ d(y, gy)\}$.
Then f *and* g *have a unique common fixed point. Moreover, any fixed point of* f *is a fixed point of* g *and conversely.*

Definition 1.6. (Babu, Nageswara Rao and Alemayehu [2]) Let f and g be two selfmaps of a metric space (X, d). The pair (f, g) *is said to be a generalized weakly contractive pair* if there exists a function $\phi \in \Phi$ such that

$d(fx, gy) \leq M(x, y) - \phi(M(x, y))$ for all x, y in X,
where
$M(x, y) = \max\{d(x, y),\ d(x, fx),\ d(y, gy),\ \frac{1}{2}[d(x, gy) + d(y, fx)]\}$.

Definition 1.7. (Babu, Nageswara Rao and Alemayehu [2]) Let f, g, S and T be selfmaps of a metric space (X, d). We say that the pair (f, g) *is* (S, T) *generalized weakly contractive* if there exists a function $\phi \in \Phi$ such that
$d(fx, gy) \leq M(x, y) - \phi(M(x, y))$ for all x, y in X,
where
$M(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$.

Theorem 1.4. *(Babu, Nageswara Rao and Alemayehu [2]) Let f, g, S and T be selfmaps of a complete metric space (X, d) such that $fX \subseteq TX$ and $gX \subseteq SX$ and (f, g) is (S, T) generalized weakly contractive pair. If one of the ranges fX, gX, SX and TX is closed, then f, g, S and T have a unique common fixed point in X.*

Theorem 1.5. *(Babu, Nageswara Rao and Alemayehu [2]) Let f, g, S and T be selfmaps of a complete metric space (X, d) such that $fX \subseteq TX$ and $gX \subseteq SX$ and (f, g) is (S, T) generalized weakly contractive pair. Further assume that either*

(i) (f, S) is reciprocally continuous and compatible pair of maps and (g, T) a pair of weakly compatible maps

or

(ii) (g, T) is reciprocally continuous and compatible pair of maps and (f, S) a pair of weakly compatible maps.
Then f, g, S and T have a unique common fixed point in X.

Motivated by the works of Doric [5], Dutta and Choudhury [6], Choudhury, Konar, Rhoades and Metiya [4] we extend the concept of (ψ, ϕ)- weakly contractive maps to four maps.

Definition 1.8. Let f, g, S and T be four selfmaps of a metric space (X, d). If there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that
$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \phi(m(x, y))$ for all x, y in X $\quad (A)$
where
$M(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$
and
$m(x, y) = \max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty)\}$

then we say that f, g, S and T satisfy *generalized* (ψ, ϕ)- *weakly contractive condition*.

In this paper, we prove a common fixed point theorem for two pairs of selfmaps satisfying generalized (ψ, ϕ)- weakly contractive condition. Also, we prove the same for two pairs of such selfmaps in which one pair is compatible, reciprocally continuous and the other pair is weakly compatible. Some existing results are drawn as corollaries from the main results of this paper. Examples are given in support of the main results of the paper.

2. A COMMON FIXED POINT OF TWO PAIRS OF WEAKLY CONTRACTIVE MAPS

Let f, g, S and T be selfmaps of a metric space (X, d) satisfying $fX \subseteq TX$ and $gX \subseteq SX$. Let $x_0 \in X$. Since $fX \subseteq TX$, we can choose $x_1 \in X$ such that

$fx_0 = Tx_1 = y_0$ (say).
Since $gX \subseteq SX$, corresponding to $x_1 \in X$ we can choose $x_2 \in X$ such that
$gx_1 = Sx_2 = y_1$ (say).
Continuing the same process we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = Sx_{2n+2},\ n = 0, 1, 2, \dots . \qquad (B)$$

The following proposition is useful in our subsequent discussion.

Proposition 2.1. *Let f, g, S and T be selfmaps of a metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ succh that f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition. Assume also that (f, S) and (g, T) are weakly compatible.*
Then $F(f, S) \neq \emptyset$ if and only if $F(g, T) \neq \emptyset$, where
$F(f, S)$={$x \in X : f(x) = S(x) = x$} and
$F(g, T)$={$x \in X : g(x) = T(x) = x$}.
In this case f, g, S and T have a unique common fixed point.

Proof. Assume that $F(f, S) \neq \emptyset$. Let $z \in F(f, S)$ and so

$$z = fz = Sz. \qquad (2.1)$$

Now, we show that $z \in F(g, T)$.
Since $fX \subseteq TX$ there exists $w \in X$ such that

$$fz = Tw. \qquad (2.2)$$

Then, from (2.1) and (2.2), we get

$$fz = Tw = Sz = z. \qquad (2.3)$$

Next we show that $gw = z$.
Now from (A) we have

$$\psi(d(z, gw)) = \psi(d(fz, gw)) \leq \psi(M(z, w)) - \phi(m(z, w)) \qquad (2.4)$$

where

$$\begin{aligned} M(z, w) &= max\{d(Sz, Tw),\ d(fz, Sz),\ d(gw, Tw), \frac{1}{2}[d(Sz, gw) + d(fz, Tw)]\} \\ &= max\{0,\ 0,\ d(gw, z),\ \frac{1}{2}d(z, gw)\} \\ &= d(z, gw). \end{aligned} \qquad (2.5)$$

and

$$\begin{aligned} m(z, w) &= max\{d(Sz, Tw),\ d(fz, Sz),\ d(gw, Tw)\} \\ &= max\{0,\ 0,\ d(gw, z)\} = d(z, gw). \end{aligned} \qquad (2.6)$$

On using (2.5) and (2.6) in (2.4), we have

$$\psi(d(z, gw)) \leq \psi(d(z, gw)) - \phi(d(z, gw))$$

which implies that $\phi(d(z, gw)) = 0$. Hence

$$z = gw. \qquad (2.7)$$

From (2.3) and (2.7) it follows that

$$gw = Tw = z. \qquad (2.8)$$

Since g and T are weakly compatible, we have by (2.8)

$$gz = gTw = Tgw = Tz.$$

Hence

$$gz = Tz. \tag{2.9}$$

Now, we show that $gz = z$.

From (A) we have

$$\psi(d(z, gz)) = \psi(d(fz, gz)) \leq \psi(M(z, z)) - \phi(m(z, z)) \tag{2.10}$$

where

$$\begin{aligned} M(z,z) &= \{d(Sz,Tz),\ d(fz,Sz),\ d(gz,Tz),\ \frac{1}{2}[d(Sz,gz)+d(fz,Tz)]\} \\ &= max\{d(z,gz),\ 0,\ 0,\ \frac{1}{2}[d(z,gz)+d(z,gz)]\} \\ &= d(z,gz). \end{aligned} \tag{2.11}$$

Also, it is easy to see that

$$m(z, z) = d(z, gz). \tag{2.12}$$

Therefore, on using (2.11) and (2.12) in (2.10), we have

$$\psi(d(z, gz)) \leq \psi(d(z, gz)) - \phi(d(z, gz))$$

which implies that

$$\phi(d(z, gz)) = 0$$

i.e.,

$$z = gz. \tag{2.13}$$

Hence, from (2.9) and (2.13), we have $z = gz = Tz$.

Therefore

$$F(g, T) \neq \emptyset \tag{2.14}$$

Hence, from (2.1) and (2.14), we have

$$F(f, S) \subseteq F(g, T). \tag{2.15}$$

Conversely assume that $F(g, T) \neq \emptyset$.

Let $z \in F(g, T)$, then

$$gz = Tz = z. \tag{2.16}$$

Now, we show that $z \in F(f, S)$. Since $gX \subseteq SX$, there exists $u \in X$ such that

$$gz = Su. \tag{2.17}$$

Then, by (2.16) and (2.17), we have

$$gz = Su = Tz = z. \tag{2.18}$$

Next we show that $fu = z$.

From (A) we have

$$\psi(d(fu, z)) = \psi(d(fu, gz)) \leq \psi(M(u, z)) - \phi(m(u, z)) \tag{2.19}$$

where

$$\begin{aligned} M(u,z) &= max\{d(Su,Tz),\ d(fu,Su),\ d(gz,Tz),\ \frac{1}{2}[d(Su,gz)+d(fu,Tz)]\} \\ &= max\{0,\ d(fu,z),\ 0,\ \frac{1}{2}d(fu,z)\} \end{aligned}$$

$$= \quad d(fu, z). \tag{2.20}$$

Also we have

$$m(u, z) = d(fu, z). \tag{2.21}$$

Now on using (2.20) and (2.21) in (2.19), we have

$$\psi(d(fu, z)) \leq \psi(d(fu, z)) - \phi(d(fu, z))$$

which implies that $\phi(d(fu, z)) = 0$. Hence

$$fu = z.$$

Therefore from (2.18), it follows that

$$fu = Su = z.$$

Since f and S are weakly compatible we have

$$fz = fSu = Sfu = Sz,$$

so that

$$fz = Sz. \tag{2.22}$$

Now, we show that $fz = z$.

From (A) we have

$$\psi(d(fz, z)) = \psi(d(fz, gz)) \leq \psi(M(z, z)) - \phi(m(z, z)) \tag{2.23}$$

where

$$\begin{aligned} M(z, z) &= max\{d(Sz, Tz),\ d(fz, Sz),\ d(gz, Tz), \frac{1}{2}[d(Sz, gz) + d(fz, Tz)]\} \\ &= d(fz, z). \end{aligned} \tag{2.24}$$

Also, it is easy to see that

$$m(z, z) = d(fz, z). \tag{2.25}$$

On using (2.25) and (2.24) in (2.23), we have

$$\psi(d(fz, z)) \leq \psi(d(fz, z)) - \phi(d(fz, z))$$

which implies that $\phi(d(fz, z)) = 0$ so that

$$fz = z. \tag{2.26}$$

Hence from (2.22) and (2.26) we have $fz = Sz = z$. Therefore

$$F(f, S) \neq \emptyset. \tag{2.27}$$

Thus from (2.16) and (2.27) we get

$$F(g, T) \subseteq F(f, S). \tag{2.28}$$

Therefore from (2.15) and (2.28) we have $F(f, S) = F(g, T)$. □

Proposition 2.2. *Let f, g, S and T be selfmaps of a metric space (X, d) satisfying $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition. Then for each $x_0 \in X$ the sequence $\{y_n\}$ defined by (B) is Cauchy in X.*

Proof. First we suppose that $y_n = y_{n+1}$ for some n.

If $n = 2m$ then

$$y_{2m} = y_{2m+1}.$$

Now, we have

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) & \\ &= max\{d(Sx_{2m+2}, Tx_{2m+1}),\ d(fx_{2m+2}, Sx_{2m+2}),\ d(gx_{2m+1}, Tx_{2m+1}), \\ &\quad \frac{1}{2}[d(Sx_{2m+2}, gx_{2m+1}) + d(fx_{2m+2}, Tx_{2m+1})]\} \\ &= max\{d(y_{2m+1}, y_{2m}),\ d(y_{2m+2}, y_{2m+1}),\ d(y_{2m+1}, y_{2m}), \\ &\quad \frac{1}{2}[d(y_{2m+1}, y_{2m+1}) + (d(y_{2m+2}, y_{2m})]\} \\ &= max\{0,\ d(y_{2m+2}, y_{2m+1}),\ 0\ , \frac{1}{2}[0 + (d(y_{2m+2}, y_{2m})]\} \\ &= max\{d(y_{2m+2}, y_{2m+1}),\ \frac{1}{2}d(y_{2m+2}, y_{2m})\} \\ &\leq max\{d(y_{2m+2}, y_{2m+1}),\ \frac{1}{2}[d(y_{2m+2}, y_{2m+1}) + d(y_{2m+1}, y_{2m})]\} \\ &= max\{d(y_{2m+2}, y_{2m+1}), \frac{1}{2}d(y_{2m+2}, y_{2m+1})\} \\ &= d(y_{2m+2}, y_{2m+1}), \end{aligned}$$

but we have

$$d(y_{2m+2}, y_{2m+1}) \leq M(x_{2m+2}, x_{2m+1}).$$

Hence we have

$$M(x_{2m+2}, x_{2m+1}) = d(y_{2m+2}, y_{2m+1}). \tag{2.29}$$

Also, we have

$$\begin{aligned} m(x, y) &= max\{d(Sx_{2m+2}, Tx_{2m+1}), d(fx_{2m+2}, Sx_{2m+2}), d(gx_{2m+1}, Tx_{2m+1})\} \\ &= max\{d(y_{2m+1}, y_{2m}),\ d(y_{2m+2}, y_{2m+1}),\ d(y_{2m+1}, y_{2m})\} \\ &= max\{0,\ d(y_{2m+2}, y_{2m+1}),\ 0)\} \\ &= d(y_{2m+2}, y_{2m+1}). \end{aligned} \tag{2.30}$$

Now, from (A) we have

$$\begin{aligned} \psi(d(y_{2m+2}, y_{2m+1})) &= \psi(d(fx_{2m+2}, gx_{2m+1})) \\ &\leq \psi(M(x_{2m+2}, x_{2m+1}) - \phi(m(x_{2m+2}.x_{2m+1})) \end{aligned} \tag{2.31}$$

On using (2.29) and (2.30) in (2.31) we get

$$\psi(d(y_{2m+2}, y_{2m+1})) \leq \psi(d(y_{2m+2}, y_{2m+1}) - \phi(d(y_{2m+2}, y_{2m+1})),$$

which implies that $\phi(d(y_{2m+2}, y_{2m+1})) \leq 0$.

Hence $d(y_{2m+2}, y_{2m+1}) = 0$, i.e.,

$$y_{2m+2} = y_{2m+1}. \tag{2.32}$$

In a similar way it is easy to see that

$$y_{2m+3} = y_{2m+2}. \tag{2.33}$$

Hence, from (2.32) and (2.33), we have

$$y_{n+1} = y_{n+2}.$$

Now by applying mathematical induction it follows that

$$y_n = y_{n+k},$$

for all $k \geq 0$. Therefore, $\{y_m\}$ is a constant sequence for $m \geq n$ and hence it is a Cauchy sequence in X.

Now, we suppose that

$$y_n \neq y_{n+1}, \text{ for all } n. \tag{2.34}$$

Then from (A) we have

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(M(x_{2n+2}, x_{2n+1})) - \phi(m(x_{2n+2}, x_{2n+1})) \tag{2.35}$$

where

$$\begin{aligned} M(x_{2n+2}, x_{2n+1}) &= max\{d(Sx_{2n+2},\ Tx_{2n+1}),\ d(fx_{2n+2}, Sx_{2n+2}),\ d(gx_{2n+1}, Tx_{2n+1}) \\ &\quad \frac{1}{2}[d(Sx_{2n+2}, gx_{2n+1}) + d(fx_{2n+2}, Tx_{2n+1})]\} \\ &= max\{d(y_{2n+1}, y_{2n}),\ d(y_{2n+2}, y_{2n+1}),\ d(y_{2n+1}, y_{2n}), \\ &\quad \frac{1}{2}[d(y_{2n+1}, y_{2n+1}) + d(y_{2n+2}, y_{2n})]\} \\ &= max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \frac{1}{2}d(y_{2n+2}, y_{2n})\} \\ &\leq max\{d(y_{2n+1}, y_{2n}),\ d(y_{2n+2}, y_{2n+1}),\ \frac{1}{2}[d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n})]\} \\ &\leq max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), max\{d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n})\}\} \\ &= max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\}. \end{aligned} \tag{2.36}$$

Also we have

$$m(x_{2n+2}, x_{2n+1}) = max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\}. \tag{2.37}$$

Hence from (2.36) and (2.37) we get

$$M(x_{2n+2}, x_{2n+1}) = m(x_{2n+2}, x_{2n+1})$$

If

$$max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\} = d(y_{2n+2}, y_{2n+1}) \tag{2.38}$$

then using (2.38) in (2.35) we get

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(d(y_{2n+2}, y_{2n+1})) - \phi(d(y_{2n+2}, y_{2n+1}))$$

which implies that

$$\phi(d(y_{2n+2}, y_{2n+1})) \leq 0.$$

It follows that $y_{2n+2} = y_{2n+1}$, which is a contradiction with (2.34).

Therefore

$$max\{d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1})\} = d(y_{2n+1}, y_{2n})$$

and

$$\begin{aligned} \psi(d(y_{2n+2}, y_{2n+1})) &\leq \psi(d(y_{2n+1}, y_{2n})) - \phi(d(y_{2n+1}, y_{2n})) \\ &< \psi(d(y_{2n+1}, y_{2n})). \end{aligned}$$

Since ψ is nondecreasing we have

$$d(y_{2n+2}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n}). \tag{2.39}$$

With a similar argument it follows that

$$d(y_{2n+3}, y_{2n+2}) \leq d(y_{2n+2}, y_{2n+1}). \tag{2.40}$$

Therefore, from (2.39) and (2.40) we have

$$d(y_{n+2}, y_{n+1}) \leq d(y_{n+1}, y_n), \text{ for } n = 0, 1, 2.3, \ldots.$$

Hence the sequence $\{d(y_{n+1}, y_n)\}$ is a nonincreasing sequence of nonnegative real numbers and hence it converges to some real number δ (say), $\delta \geq 0$.

Now, we show that $\delta = 0$. If possible, suppose that

$$\delta > 0. \tag{2.41}$$

Since $M(x_{2n+2}, x_{2n+1}) = m(x_{2n+2}, x_{2n+1}) = d(y_{2n+1}, y_{2n})$ and from (2.35), we have

$$\psi(d(y_{2n+2}, y_{2n+1})) \leq \psi(d(y_{2n+1}, y_{2n})) - \phi(d(y_{2n+1}, y_{2n})). \tag{2.42}$$

On taking upper limit as $n \to \infty$, using the continuity of ψ and lower semicontinuity of ϕ in (2.42) we get $\psi(\delta) \leq \psi(\delta) - \phi(\delta)$, a contradiction.

Therefore

$$\delta = 0.$$

Next, we show that the sequence $\{y_n\}$ is a Cauchy sequence in X. It suffices to show that $\{y_{2n}\}$ is a Cauchy sequence in X.

If possible, suppose that $\{y_{2n}\}$ is not a Cauchy sequence.

Then there exist $\epsilon > 0$ and sequences of even positive integers $\{2m_k\}$, $\{2n_k\}$ with $2m_k > 2n_k > k$ such that

$$d(y_{2m_k}, y_{2n_k}) \geq \epsilon. \tag{2.43}$$

Let $2m_k$ be the least positive integer exceeding $2n_k$ and satisfying (2.43). Then it follows that

$d(y_{2m_k}, y_{2n_k}) \geq \epsilon$ and

$$d(y_{2m_k-2}, y_{2n_k}) < \epsilon. \tag{2.44}$$

We now prove

(i) $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon$, (ii) $\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) = \epsilon$,

(iii) $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k-1}) = \epsilon$, (iv) $\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k-1}) = \epsilon$.

Since the proof in each case is similar we prove (i).

Now from (2.43) we have

$$\epsilon \leq d(y_{2m_k}, y_{2n_k})$$

which implies that

$$\epsilon \leq \limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}).$$

By using the triangle inequality and (2.44) we have

$$\begin{aligned} d(y_{2m_k}, y_{2n_k}) &\leq d(y_{2m_k}, y_{2n_k-2}) + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}) \\ &< \epsilon + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}). \end{aligned}$$

Therefore we have

$$\limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}) \leq \epsilon.$$

It follows that

$$\epsilon \leq \limsup_{k\to\infty} \; d(y_{2m_k}, y_{2n_k}) \leq \epsilon$$

so that

$$\limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon. \tag{2.45}$$

On the other hand, we have

$$\epsilon \leq \liminf_{k\to\infty} \; d(y_{2m_k}, y_{2n_k}) \leq \limsup_{k\to\infty} \; d(y_{2m_k}, y_{2n_k}) = \epsilon$$

so that

$$\liminf_{k\to\infty} \; d(y_{2m_k}, y_{2n_k}) = \epsilon. \tag{2.46}$$

Hence, from (2.45) and (2.46), we have

$$\epsilon = \liminf_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \limsup_{k\to\infty} d(y_{2m_k}, y_{2n_k}).$$

Therefore $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k})$ exists and $\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \epsilon$.

Now we have

$$\begin{aligned} M(x_{2n_k}, x_{2m_k+1}) &= max\{d(Sx_{2n_k}, Tx_{2m_k+1}),\ d(fx_{2n_k}, Sx_{2n_k}),\ d(gx_{2m_k+1}, Tx_{2m_k+1}), \\ &\quad \frac{1}{2}[d(Sx_{2n_k}, gx_{2m_k+1}) + d(fx_{2n_K}, Tx_{2m_k+1})]\} \\ &= max\{d(y_{2n_k-1}, y_{2m_k}),\ d(y_{2n_k}, y_{2n_k-1}),\ d(y_{2m_k+1}, y_{2m_k}) \\ &\quad \frac{1}{2}[d(y_{2n_k-1}, y_{2m_k+1}) + d(y_{2n_k}, y_{2m_k})]\} \end{aligned}$$

On taking limits as $k \to \infty$ we get

$$\lim_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = max\{\epsilon, 0, 0, \epsilon\} = \epsilon. \tag{2.47}$$

In a similarly way it is easy to see that

$$\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k+1}) = \epsilon. \tag{2.48}$$

Now putting $x = x_{2n_k}$ and $y = x_{2m_k+1}$ in (A) we obtain

$$\begin{aligned} \psi(d(y_{2n_k}, y_{2m_k+1})) &= \psi(d(fx_{2n_k}, gx_{2m_k+1})) \\ &\leq \psi(M(x_{2n_k}, x_{2m_k+1})) - \phi(m(x_{2n_k}, x_{2m_k+1})). \end{aligned}$$

On taking upper limit as $k \to \infty$ using (2.47), (2.48), the continuity of ψ and lower semicontinuity of ϕ in the last inequality we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence so that $\{y_n\}$ is a Cauchy sequence. □

Theorem 2.1. *Let f, g, S and T be selfmaps of a complete metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition. If the pairs (f, S) and (g, T) are weakly compatible and one of the ranges fX, gX, SX and TX is closed, then for each $x_0 \in X$ the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim_{n\to\infty} y_n = z$ (say) and z is a unique common fixed point of f, g, S and T.*

Proof. By Proposition 2.2 the sequence $\{y_n\}$ is Cauchy in X. Since X is complete there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Thus

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \tag{2.49}$$

Case (i): Suppose that SX is closed. Then z is in SX and hence there exists $u \in X$ such that

$$Su = z. \tag{2.50}$$

Now, we show that $fu = z$. Now we have

$$\begin{aligned} M(u, x_{2n+1}) &= max\{d(Su, Tx_{2n+1}),\ d(fu, Su),\ d(gx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2}[d(Su, gx_{2n+1}) + d(fu, Tx_{2n+1})]\} \end{aligned}$$

and on taking limits as $n \to \infty$ we have

$$\lim_{n\to\infty} M(u, x_{2n+1}) = d(fu, z) \tag{2.51}$$

Similarly it is easy to see that

$$\lim_{n\to\infty} m(u, x_{2n+1}) = d(fu, z). \tag{2.52}$$

Using (A), we have

$$\psi(d(fu, gx_{2n+1})) \le \psi(M(u, x_{2n+1})) - \phi(m(u, x_{2n+1})). \tag{2.53}$$

On taking upper limit as $n \to \infty$ and using (2.51), (2.52), the continuity of ψ and lower semicontinuity of ϕ in (2.53), we get

$$\psi(d(fu, z)) \le \psi(d(fu, z)) - \phi(d(fu, z))).$$

Hence it follows that $\phi(d(fu, z))) \le 0$. Therefore

$$fu = z.$$

Hence from (2.50), we get

$$Su = fu = z.$$

Since f and S are weakly compatible we have $fz = fSu = Sfu = Sz$. Therefore

$$fz = Sz. \tag{2.54}$$

Now, we show $fz = z$. We have

$$\begin{aligned} M(z, x_{2n+1}) &= \max\{d(Sz, Tx_{2n+1}), d(fz, Sz), d(gx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2}[d(Sz, gx_{2n+1}) + d(fz, Tx_{2n+1})]\} \end{aligned}$$

and on taking limits as $n \to \infty$ we have

$$\lim_{n\to\infty} M(z, x_{2n+1}) = d(fz, z). \tag{2.55}$$

Also we have

$$\lim_{n\to\infty} m(z, x_{2n+1}) = d(fz, z). \tag{2.56}$$

Now, from (A) we have

$$\psi(d(fz, gx_{2n+1})) \le \psi(M(z, x_{2n+1})) - \phi(m(u, x_{2n+1})). \tag{2.57}$$

On taking upper limit as $n \to \infty$ using (2.55), (2.56), the continuity of ψ and lower semicontinuity of ϕ in (2.57) we get

$$\psi(d(fz, z)) \leq \psi(d(fz, z)) - \phi(d(fz, z))$$

so that $\phi(d(fz, z)) \leq 0$.

Hence

$$fz = z.$$

Therefore from (2.54) we have $z = fz = Sz$. By Proposition 2.1, $F(g, T) \neq \emptyset$ with z in $F(g, T)$. Hence $z = fz = gz = Sz = Tz$.

Case (ii): Suppose that gX is closed.

In this case, $z \in gX \subseteq SX$ which implies that $z \in SX$ and hence the proof follows as in case (i).

For the cases TX is closed and fX is closed we follow the arguments similar to the cases of SX is closed and gX is closed respectively. □

By choosing ψ as the identity map on $[0, \infty)$ in Theorem 2.3 we get the following corollary.

Corollary 2.3. *Let f, g, S and T be selfmaps of a complete metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there is $\phi \in \Phi$ such that*
$d(fx, gy) \leq M(x, y) - \phi(m(x, y))$ *for all* $x, y \in X$
where
$M(x, y) = max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty),\ \frac{1}{2}[d(Sx, gy) + d(fx, Ty)]\}$
and
$m(x, y) = max\{d(Sx, Ty),\ d(fx, Sx),\ d(gy, Ty)\}$.
If the pairs (f, S) and (g, T) are weakly compatible and one of the ranges fX, gX, SX and TX is closed, then for each x_0 in X the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim_{n\to\infty} y_n = z$ (say) and z is a unique common fixed point of f, g, S and T.

Remark 2.4. Theorem 1.3 of (Choudhury, Konar, Rhoades and Metiya [4]) follows as a corollary to Theorem 2.1 by choosing $S = T = I_X$ (I_X, the identity mapping on X).

Remark 2.5. Theorem 1.4 (Babu, Nageswara Rao and Alemayehu[2]) follows as a corollary to Corollary 2.3 by choosing $\phi \in \Phi$ nondecreasing.

Now we give an example in support of Theorem 2.1.

Example 2.6. Let $X = [0, 1]$ with the usual metric and let f, g, S and T be self maps on X defined by

$$gx = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1, \end{cases} \qquad fx = \begin{cases} \frac{1}{2}, & 0 \leq x < 1 \\ 1, & x = 1, \end{cases}$$

$$Sx = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \text{ and } \frac{3}{4} \leq \text{x} \leq 1, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ 1, & \frac{1}{2} < x < \frac{3}{4}. \end{cases} \qquad Tx = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ \frac{1}{20}, & \frac{1}{2} < x \leq 1. \end{cases}$$

Define $\psi, \phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^2$, $t \geq 0$ and $\phi(t) = \frac{1}{20}t^2$, $0 \leq t \leq \frac{19}{20}$ and $\phi(t) = \frac{1}{4}t$, $t > \frac{19}{20}$ then $\psi \in \Psi$ and $\phi \in \Phi$ and the maps f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition so that f, g, S and T satisfy all the hypotheses of Theorem 2.1 and f, g, S and T have a unique common fixed point $\frac{1}{2}$.

3. A COMMON FIXED POINT THEOREM WITH RECIPROCAL CONTINUITY

Theorem 3.1. *Let f, g, S and T be selfmaps of a complete metric space (X, d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that f, g, S and T satisfy generalized (ψ, ϕ)- weakly contractive condition.*
Assume that either

(i) (f, S) *is reciprocally continuous and compatible pair of maps and* (g, T)

a pair of weakly compatible maps

or

(ii) (g, T) *is reciprocally continuous and compatible pair of maps and* (f, S)

a pair of weakly compatible maps.
Then for each x_0 in X the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim\limits_{n\to\infty} y_n = z$ (say), and z is a unique common fixed point of f, g, S and T.

Proof. By Proposition 2.2 the sequence $\{y_n\}$ is Cauchy in X. Since X is complete there exist $z \in X$ such that $\lim\limits_{n\to\infty} y_n = z$.
Thus $\lim\limits_{n\to\infty} y_{2n} = \lim\limits_{n\to\infty} fx_{2n} = \lim\limits_{n\to\infty} Tx_{2n+1} = z$
and $\lim\limits_{n\to\infty} y_{2n+1} = \lim\limits_{n\to\infty} gx_{2n+1} = \lim\limits_{n\to\infty} Sx_{2n+2} = z$.

Suppose (i) holds.
Since (f, S) is reciprocally continuous it follows that
$\lim\limits_{n\to\infty} fSx_{2n+2} = fz$ and $\lim\limits_{n\to\infty} Sfx_{2n+2} = Sz$.
Since (f, S) is a compatible pair, we have $\lim\limits_{n\to\infty} d(fSx_{2n+2}, Sfx_{2n+2}) = 0$. Hence we have $fz = Sz$. Since $fX \subseteq TX$ there exists $u \in X$ such that

$$fz = Tu.$$

Thus we have

$$fz = Tu = Sz. \tag{3.1}$$

Now, we show that $fz = gu$. Using (A) we have

$$\psi(d(fz, gu))) \leq \psi(M(z, u)) - \phi(m(z, u)), \tag{3.2}$$

where

$$\begin{aligned} M(z,u) &= \max\{d(Sz,Tu),\ d(fz,Sz),\ d(gu,Tu),\ \frac{1}{2}[d(Sz,gu)+d(fz,Tu)]\} \\ &= \max\{0,\ 0,\ d(gu,fz),\ \frac{1}{2}[d(fz,gu)]\} = d(fz,gu). \end{aligned} \tag{3.3}$$

Also it follows that

$$m(z, u) = d(fz, gu). \tag{3.4}$$

Therefore by using (3.3) and (3.4) in (3.2) we get

$$\psi(d(fz, gu)) \leq \psi(d(fz, gu)) - \phi(d(fz, gu)).$$

Hence it follows that $\phi(d(fz, gu)) \leq 0$. Therefore

$$fz = gu. \tag{3.5}$$

Therefore from (3.1) we have $fz = Sz = gu = Tu$. Since every compatible pair is weakly compatible, the pair (f, S) is weakly compatible. Hence from $fz = Sz$ we get that

$$ffz = Sfz. \tag{3.6}$$

Next, we show that $ffz = fz$. By using (A), we have

$$\begin{aligned}\psi(d(ffz, fz)) &= \psi(d(ffz, gu)) \\ &\leq \psi(M(fz, u)) - \phi(m(fz, u))\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}M(fz, u) &= \max\{d(Sfz, Tu),\ d(ffz, Sfz),\ d(gu, Tu), \\ &\quad \frac{1}{2}[d(Sfz, gu) + d(ffz, Tu)]\} \\ &= \max\{d(ffz, fz),\ 0,\ 0,\ \frac{1}{2}[d(ffz, fz) + d(ffz, fz)]\} \\ &= d(ffz, fz).\end{aligned} \tag{3.8}$$

Also we have

$$m(fz, u) = d(ffz, fz). \tag{3.9}$$

On using (3.8) and (3.9) in (3.7) we have

$$\psi(d(ffz, fz)) \leq \psi(d(ffz, fz)) - \phi(d(ffz, fz))$$

which implies that $\phi(d(ffz, fz)) \leq 0$. Hence

$$ffz = fz. \tag{3.10}$$

Therefore, from (3.6) and (3.10), we have

$$ffz = Sfz = fz. \tag{3.11}$$

Hence fz is a common fixed point of f and S. Since (g, T) is weakly compatible and $gu = Tu$ we have

$$gTu = Tgu. \tag{3.12}$$

Therefore, from (3.5)and (3.12),we have

$$gfz = Tfz. \tag{3.13}$$

Now, we show that $gfz = fz$.
By using (A) we have

$$\psi(d(fz, gfz)) \leq \psi(M(z, fz)) - \phi(m(z, fz)) \tag{3.14}$$

where

$$\begin{aligned}M(z, fz) &= \max\{d(Sz, Tfz),\ d(fz, Sz),\ d(gfz, Tfz) \\ &\quad \frac{1}{2}[d(Sz, gfz) + d(fz, Tfz)]\} \\ &= \max\{d(fz, gfz),\ 0,\ 0,\ \frac{1}{2}[d(fz, gfz) + d(fz, gfz)]\} \\ &= d(fz, gfz)\end{aligned} \tag{3.15}$$

Also we have

$$m(fz, u) = d(fz, gfz). \tag{3.16}$$

Now using (3.15) and (3.16) in (3.14) we have

$$\psi(d(fz,gfz)) \leq \psi(d(fz,gfz)) - \phi(d(fz,gfz))$$

which implies that $\phi(d(fz,gfz)) \leq 0$. Hence

$$fz = gfz. \tag{3.17}$$

Therefore

$$fz = gfz = Tfz. \tag{3.18}$$

From (3.11) and (3.18) we have

$$ffz = gfz = Sfz = Tfz = fz. \tag{3.19}$$

Hence fz is a common fixed point of f, g , S and T.
Finally we show that $fz = z$.
From (A), we have

$$\psi(d(fz,gx_{2n+1})) \leq \psi(M(z,x_{2n+1}) - \phi(m(z,x_{2n+1}) \tag{3.20}$$

where

$$\begin{aligned} M(z,x_{2n+1}) &= \max\{d(Sz,Tx_{2n+1}),\ d(fz,Sz),\ d(gx_{2n+1},Tx_{2n+1}) \\ &\quad \frac{1}{2}[d(Sz,gx_{2n+1}) + d(fz,Tx_{2n+1})]\}. \end{aligned}$$

On letting $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} M(z,x_{2n+1}) &= \max\{d(fz,z),\ 0,\ 0, \frac{1}{2}[d(fz,z) + d(fz,z)]\} \\ &= d(fz,z). \end{aligned} \tag{3.21}$$

Also we have

$$\lim_{n\to\infty} m(z,x_{2n+1}) = d(fz,z). \tag{3.22}$$

Now, on taking limits as $n \to \infty$, using (3.21), (3.22) the continuity of ψ and lower semicontinuity ϕ in (3.20) we get

$\psi(d(fz,z)) \leq \psi(d(fz,z)) - \phi(d(fz,z))$ which implies that $\phi(d(fz,z)) \leq 0$. Hence

$$fz = z. \tag{3.23}$$

Therefore from (3.19) and (3.23) we have $z = fz = gz = Sz = Tz$.

The proof of case (ii) is similar and hence is omitted. □

By choosing ψ as the identity map on $[0,\infty)$ in Theorem 3.1 we get the following corollary.

Corollary 3.1. *Let f, g, S and T be selfmaps of a complete metric space (X,d) which satisfy $fX \subseteq TX$ and $gX \subseteq SX$. Assume that there is $\phi \in \Phi$ such that*
$d(fx,gy) \leq M(x,y) - \phi(m(x,y))$ for all $x,y \in X$ where
$M(x,y) =$ max$\{d(Sx,Ty),\ d(fx,Sx),\ d(gy,Ty),\ \frac{1}{2}[d(Sx,gy) + d(fx,Ty)]\}$
and
$m(x,y) =$ max$\{d(Sx,Ty),\ d(fx,Sx),\ d(gy,Ty)\}$.
Assume that either

(i) (f,S) is reciprocally continuous and compatible pair of maps and (g,T) a pair of weakly compatible maps

or

(ii) (g,T) is reciprocally continuous and compatible pair of maps and (f,S) a pair of weakly compatible maps.
Then for each x_0 in X the sequence $\{y_n\}$ defined by (B) is Cauchy in X and $\lim_{n\to\infty} y_n = z\ (say)$ and z is a unique common fixed point of f, g, S and T.

Remark 3.2. Theorem 1.5 (Babu, Nageswara Rao and Alemayehu [2]) follows as a corollary to Corollary 3.1 by choosing $\phi \in \Phi$ nondecreasing.

Now, we give an example in support of Theorem 3.1.

Example 3.3. Let $X = [0,1]$ with the usual metric and let f, g, S and T be selfmaps on X defined by

$$fx = \begin{cases} 0, & x = 0 \\ \frac{2}{3}, & 0 < x < 1 \\ \frac{3}{4}, & x = 1, \end{cases} \qquad gx = \begin{cases} \frac{1}{3}, & x = 0 \\ \frac{2}{3}, & 0 < x < 1 \\ \frac{5}{12}, & x = 1, \end{cases}$$

$$Sx = \begin{cases} 1, & x = 0 \\ \frac{1}{3}, & 0 < x < \frac{2}{3} \\ \frac{4}{3} - x, & \frac{2}{3} \le x < 1 \\ 0, & x = 1, \end{cases} \qquad Tx = \begin{cases} 1, & x = 0 \\ 1 - \frac{1}{2}x, & 0 < x < 1 \\ 0, & x = 1. \end{cases}$$

Define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t^2$, $t \ge 0$ and $\phi(t) = \frac{1}{2}t^2$, if $0 \le t \le \frac{2}{3}$ and $\phi(t) = \frac{1}{3}t$, if $t > \frac{2}{3}$, then $\psi \in \Psi$ and $\phi \in \Phi$. Here we observe that (f,S) is reciprocally continuous, (f,S) is a compatible pair and (g,T) is a weakly compatible pair of maps and the maps f, g, S and T satisfy generalized (ψ,ϕ)-weakly contractive condition so that f, g, S and T satisfy all the hypotheses of Theorem 3.1 and f, g, S and T have a unique common fixed point $\frac{2}{3}$.

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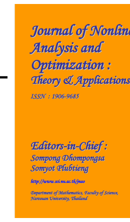

FIXED POINT THEOREMS FOR KANNAN TYPE CYCLIC WEAKLY CONTRACTIONS

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ABSTRACT. In this article, we introduce the notion of Kannan Type cyclic weakly contraction and derive the existence of fixed point theorems in the setup of complete metric spaces. Our main theorems extend and improve some fixed point theorems in the literature. Examples are given to support the usability of the results.

KEYWORDS : Fixed point theory; Cyclic φ-contraction; Kannan type mapping.
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1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is one of the cornerstone in the development of nonlinear functional analysis. Besides mathematics, fixed point theory has been used effectively in many other discipline such as economics, chemistry, biology, computer science, engineering, and others. In particular, Banach's contraction mapping principle [2] has a significant role in fixed point theory and hence in nonlinear functional analysis.

Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a self-map. If there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \le kd(x, y)$$

for all $x, y \in X$, then T has a unique fixed point.

Banach fixed point theorem not only guarantee the existence and uniqueness of a fixed point but also show how to get it. All things considered, Banach's contraction mapping principle differ from the origin and antecedents results. We also notice that a self mapping T, in Banach fixed point theorem, is necessarily continuous. Due to its importance, fixed point theory draw interest of many researcher (see, e.g.,[19, 11]).

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In 1968, Kannan [9] prove one of the interesting generalization of the Banach Contraction Principle.

Definition 1.1. [9, 10] A self-mapping $T : X \longrightarrow X$, on a metric space (X, d) is said to be Kannan type mapping if there exists $0 < k < 1$ such that, for all $x, y \in X$, the following inequality holds:

$$d(Tx, Ty) \leq \frac{k}{2}[d(x, Tx) + d(y, Ty)]].$$

In Kannan fixed point theorem, a self mapping T need not to be continuous. This is the most important gains of the Kannan's result.

On the other hand, Kirk et al.[17] in 2003 introduce the following notion of cyclic representation and characterize the Banach Contraction Principle in the context of cyclic mapping.

Definition 1.2. [17] Let X be a non-empty set and $T : X \longrightarrow X$ an operator. By definition, $X = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to T if

(a) X_i, $i = 1, \ldots, m$ are non-empty sets,
(b) $T(X_1) \subset X_2, \cdots, T(X_{m-1}) \subset X_m, T(X_m) \subset X_1$.

After the distinguished notion and related fixed point result of Kirk et al.[17], a number of fixed point theorems are reported in the literature, for operators T defined on a complete metric space X with a cyclic representation of X with respect to T (see, e.g., [5] - [23]).

Very recently, Karapınar [11] characterize the notion of the cyclic weak φ-contraction and prove fixed point theorems for such types contractions in the context of cyclic mapping.

Definition 1.3. (See [11]) Let (X, d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ closed nonempty subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. An operator $T : Y \longrightarrow Y$ is called a cyclic weak φ-contraction if

(1) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T;
(2) there exists a continuous, non-decreasing function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$ where $A_{m+1} = A_1$.

Let $\mathbf{F}$ denote all the continuous functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$.

Theorem 1.4. *(See [11]) Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty subsets of X and $X = \bigcup_{i=1}^{m} A_i$. Suppose that T is a cyclic weak φ-contraction with $\varphi \in \mathbf{F}$. Then, T has a fixed point $z \in \bigcap_{i=1}^{n} A_i$.*

The aim of this paper is to use the concept of cyclic contraction and Kannan type mapping and introduce the notions of Kannan type cyclic weakly contractions and then derive fixed point theorems on it in the setup of complete metric spaces. Our results generalize fixed point theorems [11, 17] in the sense of metric spaces.

2. MAIN RESULTS

In this section, we introduce the notion of Kannan type cyclic weakly contraction in metric space. Before this, we introduce the following class of functions: Let $\mathbf{F_1}$ denote all the continuous functions $\psi : [0, \infty)^2 \longrightarrow [0, \infty)$ satisfying $\psi(x, y) = 0$ if and only if $x = y = 0$.

We introduce the notion of Kannan type cyclic weakly contraction in metric space, in the following way.

Definition 2.1. Let (X, d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. An operator $T : Y \longrightarrow Y$ is called a Kannan type cyclic weakly contraction if

(1) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;

(2)

$$d(Tx, Ty) \quad \leq \quad \tfrac{1}{2}[d(x, Tx) + d(y, Ty)] - \psi(d(x, Tx), d(x, Ty)),$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$ where $A_{m+1} = A_1$ and

$$\psi(d(x, Tx), d(y, Ty)) \in \mathbf{F_1}.$$

We state the main result of this section as follows:

Theorem 2.2. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that T is a Kannan type cyclic weakly contraction. Then, T has a fixed point $z \in \bigcap_{i=1}^{n} A_i$.*

Proof. Take $x_0 \in X$. We construct a sequence in the following way:

$$x_{n+1} = Tx_n, \text{ for all } n = 0, 1, 2, \cdots.$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then, the existence of the fixed point is proved. Indeed, we have $Tx_{n_0} = x_{n_0+1} = x_{n_0}$. On the occasion of that we assume $x_{n+1} \neq x_n$ for any $n = 0, 1, 2, \cdots$. Regarding that $X = \bigcup_{i=1}^{m} A_i$, for any $n > 0$ there exists $i_n \in \{1, 2, \cdots, m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since T is a Kannan type cyclic weakly contraction, we have

(2.1)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \frac{1}{2}[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] - \psi(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)) \\ &= \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] - \psi(d(x_{n-1}, x_n), d(x_n, x_{n+1})) \quad (2.2) \\ &\leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

Consequently,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \text{ for any } n \in \mathbb{N}.$$

By virtue of the fact that we conclude $\{d(x_n, x_{n+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Therefore, there exists $\gamma \geq 0$ such that

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = \gamma. \tag{2.3}$$

Letting $n \longrightarrow \infty$ in (2.2) we derive that

$$\gamma \leq \lim_{n \longrightarrow \infty} \frac{1}{2}[2\gamma] \leq \gamma,$$

or

$$\lim_{n \longrightarrow \infty} d(x_{n-1}, x_{n+1}) = 2\gamma. \tag{2.4}$$

Setting $n \longrightarrow \infty$ in (2.2) and by using (2.3)and (2.4) we get

$$\begin{aligned} \gamma &\leq \frac{1}{2}(2\gamma) - \liminf_{n \longrightarrow \infty} \psi(d(x_{n-1}, x_n), d(x_n, x_{n+1})) \\ &\leq \gamma - \psi(\gamma, \gamma) \end{aligned}$$

or, $\psi(\gamma,\gamma) \leq 0$ by the continuity of ψ. This is a contradiction unless $\gamma = 0$, that is,

$$\lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

We shall show that the sequence $\{x_n\}$ is a Cauchy sequence. For this goal, we prove the following claim first:

(C) For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q \equiv 1(m)$, then $d(x_r, x_q) < \varepsilon$.

Assume the contrary of (C). Thus, there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$ satisfying

$$d(x_{q_n}, x_{r_n}) \geq \varepsilon. \tag{2.6}$$

Now, we take $n > 2m$. Then, corresponding to $q_n \geq n$ use can choose r_n in such a way that it is the smallest integer with $r_n > q_n$ satisfying $r_n - q_n \equiv 1(m)$ and $d(x_{q_n}, x_{r_n}) \geq \varepsilon$. Therefore, $d(x_{q_n}, x_{r_{n-m}}) \leq \varepsilon$. By using the triangular inequality

$$\begin{aligned} \varepsilon &\leq d(x_{q_n}, x_{r_n}) \leq d(x_{q_n}, x_{r_{n-m}}) + \sum_{i=1}^{m} d(x_{r_{n-i}}, x_{r_{n-i+1}}) \\ &< \varepsilon + \sum_{i=1}^{m} d(x_{r_{n-i}}, x_{r_{n-i+1}}). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the last inequality and taking into account that $\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = 0$, we find

$$\lim_{n \longrightarrow \infty} d(x_{q_n}, x_{r_n}) = \varepsilon. \tag{2.7}$$

Again, by the triangular inequality

$$\begin{aligned} \varepsilon &\leq d(x_{q_n}, x_{r_n}) \\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{r_{n+1}}) + d(x_{r_{n+1}}, x_{r_n}) \\ &\leq d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_{n+1}}, x_{q_n}) + d(x_{q_n}, x_{r_n}) + d(x_{r_n}, x_{r_{n+1}}) + d(x_{r_{n+1}}, x_{r_n}) \\ &= 2d(x_{q_n}, x_{q_{n+1}}) + d(x_{q_n}, x_{r_n}) + 2d(x_{r_n}, x_{r_{n+1}}). \end{aligned} \tag{2.8}$$

Taking (2.5) and (2.7) into account, we obtain

$$\lim_{n \longrightarrow \infty} d(x_{q_{n+1}}, x_{r_{n+1}}) = \varepsilon \tag{2.9}$$

as $n \longrightarrow \infty$ in (2.7).

Due to the fact that x_{q_n} and x_{r_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, using the fact that T is a Kannan type cyclic weakly contraction, we have

$$\begin{aligned} d(x_{q_{n+1}}, x_{r_{n+1}}) &= d(Tx_{q_n}, Tx_{r_n}) \\ &\leq \frac{1}{2}[d(x_{q_n}, Tx_{q_n}) + d(x_{r_n}, Tx_{r_n})] - \psi(d(x_{q_n}, Tx_{q_n}), d(x_{r_n}, Tx_{r_n})) \\ &\leq \frac{1}{2}[d(x_{q_n}, x_{q_{n+1}}) + d(x_{r_n}, x_{r_{n+1}})] - \psi(d(x_{q_n}, x_{q_{n+1}}), d(x_{r_n}, x_{r_{n+1}})). \end{aligned}$$

Regarding (2.5) and (2.7) and the continuity of ψ, letting $n \longrightarrow \infty$ in the last inequality, we conclude that $\varepsilon = 0$. This is a contradiction and hence (C) is proved.

By the help of (C), we shall show $\{x_n\}$ is a Cauchy sequence in Y. Fix $\varepsilon > 0$. By (C), we find $n_0 \in \mathbb{N}$ such that if $r, q \geq n_0$ with $r - q \equiv 1(m)$

$$d(x_r, x_q) \leq \frac{\varepsilon}{2}. \tag{2.10}$$

Since $\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\varepsilon}{2m} \tag{2.11}$$

for any $n \geq n_1$. Assume that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \cdots, m\}$ such that $s-r \equiv k(m)$. Hence, $s-r+\varphi \equiv 1(m)$ for $\varphi = m-k+1$. So, we have

$$d(x_r, x_s) \leq d(x_r, x_{s+j}) + d(x_{s+j}, x_{s+j-1}) + \cdots + d(x_{s+1}, x_s).$$

By (2.10) and (2.11) and from the last inequality, we get

$$d(x_r, x_s) \leq \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \leq \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon$$

By reason of the fact that (x_n) is a Cauchy sequence in Y. Since Y is closed in X, then Y is also complete and there exists $x \in Y$ such that $\lim_{n\longrightarrow\infty} x_n = x$. In what follows, we prove that x is a fixed point of T. In fact, since $\lim_{n\longrightarrow\infty} x_n = x$ and, as $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T, the sequence (x_n) has infinite terms in each A_i for $i \in \{1, 2, \cdots, m\}$. Suppose that $x \in A_i$, $Tx \in A_{i+1}$ and we take a subsequence x_{n_k} of (x_n) with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by the above-mentioned comment). By using the contractive condition, we can obtain

$$\begin{aligned} d(x_{n_{k+1}}, Tx) &= d(Tx_{n_k}, Tx) \\ &\leq \frac{1}{2}[d(x_{n_k}, Tx_{n_k}) + d(x, Tx)] - \psi(d(x_{n_k}, Tx_{n_k}), d(x, Tx)) \\ &= \frac{1}{2}[d(x_{n_k}, x_{n_{k+1}}) + d(x, Tx)] - \psi(d(x_{n_k}, x_{n_{k+1}}), d(x, Tx)). \end{aligned}$$

Setting $n \longrightarrow \infty$ and using $x_{n_k} \longrightarrow x$, continuity of ψ, we have

$$d(x, Tx) \leq \frac{1}{2} d(x, Tx) - \psi(0, d(x, Tx)) \leq \frac{1}{2} d(x, Tx)$$

which is a contradiction unless $d(x, Tx) = 0$.Consequently, x is a fixed point of T.

We shall prove that x is a unique fixed point of T. Suppose, to the contrary that, there exists $z \in X$ with $x \neq z$ and $Tz = z$. By using the contractive condition we obtain

$$\begin{aligned} d(x, z) = d(Tx, Tz) &\leq \frac{1}{2}[d(x, Tx) + d(z, Tz)] \\ &\quad -\psi(d(x, Tx), d(z, Tz)) = 0. \end{aligned}$$

which is a contradiction. □

Corollary 2.3. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T : Y \longrightarrow Y$ be an operator such that*

(1) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;
(ii) there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \tag{2.12}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$ where $A_{m+1} = A_1$. Then, T has a fixed point $z \in \bigcap_{i=1}^{n} A_i$.

Proof. Let $\alpha \in [0, \frac{1}{2})$. Here, it suffices to take the function $\psi : [0, +\infty)^2 \longrightarrow [0, +\infty)$ as $\psi(a, b) = (\frac{1}{2} - \alpha)(a + b)$. It is clear that ψ satisfies the conditions:

(i) $\psi(a, b) = 0$ if and only if $a = b = 0$, and
(ii) $\psi(x, y) = (\frac{1}{2} - \alpha)(x + y) = \psi(x + y, 0)$.

Hence, we apply Theorem 2.2 and get the desired result. □

The following corollary gives us a fixed point theorem with a contractive condition of integral type for cyclic contractions.

Corollary 2.4. *Let (X, d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T : Y \longrightarrow Y$ be an operator such that*

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;
(ii) there exists $\alpha \in [0, \frac{1}{2})$ such that

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leq \alpha \int_0^{d(x,Tx)+d(y,Ty)} \rho(t)dt$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \cdots, m$ where $A_{m+1} = A_1$, and $\rho : [0, \infty) \longrightarrow [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt$ for $\varepsilon > 0$. Then T has unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Proof. It is easily proved that the function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ given by $\varphi(t) = \int_0^t \rho(s)ds$ satisfies that $\varphi \in \mathbf{F_1}$. Therefore, Corollary 2.3 is obtained from Theorem 2.2, taking as φ the above-defined function and as ψ the function $\psi(x, y) = (\frac{1}{2} - \alpha)(x + y) = \varphi(x + y, 0)$. □

If in Corollary 2.4, we take $A_i = X$ for $i = 1, 2, \cdots, m$, we obtain the following result.

Corollary 2.5. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping such that for any $x, y \in X$,*

$$\int_0^{d(Tx,Ty)} \rho(t)dt \leq \alpha \int_0^{d(x,Tx)+d(y,Ty)} \rho(t)dt$$

where $\rho : [0, \infty) \longrightarrow [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)dt$ for $\varepsilon > 0$ and the constant $\alpha \in [0, \frac{1}{2})$. Then T has unique fixed point.

If in Theorem 2.2 we put $A_i = X$ for $i = 1, 2, \cdots, m$ we get the generalized result of [9, 10].

Corollary 2.6. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Tx) + d(y, Ty)] - \psi(d(x, Tx) + d(y, Ty)),$$

where $\psi \in \mathbf{F_1}$. Then T has unique fixed point.

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ON THE GENERALIZED VARIATIONAL-LIKE INEQUALITIES PROBLEMS FOR MULTIVALUED MAPPINGS

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ABSTRACT. In this paper, two classes of generalized variational-like inequalities problems for multivalued mappings are introduced and then by using KKM technique and Kakutani-Fan-Glicksberg fixed point theorem the solvability of them are investigated when the mappings are relaxed $\eta-\alpha$-monotone. One can consider this paper the topological vector space version of reference [15]..

KEYWORDS : Generalized multivalued variational like inequalities; KKM-mappings, η-hemicontinuity; η-Coercivity; relaxed η-α-monotone; Relaxed η-α-semimonotone mappings
AMS Subject Classification: 47H05 49J40

1. INTRODUCTION

The existence of solutions for variational inequality problems, complementarity problems, equilibrium problems and others is mainly dependent on the monotonicity of a map (see [1, 2, 3, 4, 6, 8, 10, 14, 19]). Recently, many authors, see [7, 8, 9, 10, 11] considered the quasimonotonicity in dealing with variational inequality problems. Verma [17, 18] studied and established some existence theorems for a solution of a class of nonlinear variational inequality problems with p-monotone and p-Lipschitz mappings in the setting of reflexive Banach spaces.

Inspired and motivated by several authors[1, 3, 4, 7, 9, 13, 20], we introduce two new concepts of relaxed η-α-semimonotonicity as well as two classes of variational-like inequalities with relaxed η-α-monotone mappings and relaxed η-α-semimonotone mappings. Using KKM-technique, we obtain the existence of a solution for variational-like inequalities problems with relaxed η-α-monotone mappings in the setting of reflexive Banach spaces. We also present the solvability of variational-like inequalities problems with η-α-semimonotone mappings for an arbitrary Banach space by applying of Kakutani-Fan fixed point theorem [5, 20].

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2. VARIATIONAL-LIKE INEQUALITIES WITH RELAXED η-α-MONOTONE MAPPINGS

Throughout this paper, unless otherwise specified, we always let E be a Hausdorff topological vector space with dual space $E^\star$, K a nonempty closed convex subset of E, T a multivalued mapping from K to $E^\star$, and $\eta : K \times K \longrightarrow K$ and $\alpha : E \longrightarrow \Re$(the real numbers) are mappings. Furthermore, we assume that $\alpha(0) = 0$ and $\lim_{t\longrightarrow 0^+} \frac{\alpha(tz)}{t} = 0$, for all $z \in K$. This means that α the directional derivative at θ(zero of E) at every direction $z \in K$ is zero. For examples of these mappings, one can consider all α which has the property $\alpha(tz) = t^p\alpha(z)$ for all $t \geq 0$, $p > 1$ and $z \in E$. We note that if we take $E = \Re$ then it is easy to see that the directional derivative of the mapping $\alpha(x) = |x|$ at θ in each direction $z \in E$ is zero but it dose not satisfy $\alpha(tz) = t^p\alpha(z)$ for all $t \geq 0$, $p > 1$ and $z \in E$.

Definition 2.1. A multivalued mapping $T : K \longrightarrow 2^{E^\star}$ ($2^{E^\star}$ denotes the set of all subsets of $E^\star$) is said to be relaxed η-α-monotone if there exist mappings $\eta : K \times K \longrightarrow K$ and $\alpha : E \longrightarrow \Re$ such that the following inequality holds,

$$\langle u - v,\ \eta(x, y)\rangle\ \geq\ \alpha(x - y),\ \text{for all } x, y \in K,\ u \in T(x),\ \text{and } v \in T(y). \tag{2.1}$$

Remark:

(i) If $\eta(x, y) = x - y$, for all $x, y \in K$ then (2.1) becomes

$$\langle u - v,\ x - y\rangle\ \geq\ \alpha(x - y),\ \text{for all } u \in T(x), \text{and } v \in T(y), \tag{2.1a}$$

and T is called relaxed α-monotone.

(ii) If $T : K \longrightarrow E^\star$ is a single valued mapping then (2.1) becomes

$$\langle Tx - Ty,\ \eta(x, y)\rangle\ \geq\ \alpha(x - y),\ \text{for all } x, y \in K, \tag{2.1b}$$

and T is called relaxed η-α-monotone mapping (see [14]).

(iii) If $\eta(x, y) = x - y$, for all $x, y \in K$ and $\alpha(z) = k\|z\|^p$, where p and k are positive constants, then $(2.1b)$ reduces to

$$\langle Tx - Ty,\ x - y\rangle\ \geq\ K\|x - y\|^p,\ \text{for all } x, y \in K,$$

and T is called p-monotone (see [12, 20]).

Definition 2.1 Let X and Y be two topological spaces. A set-valued mapping $G : X \longrightarrow 2^Y$ is called:

(i) **upper semi-continuous** (u.s.c.) at $x \in X$ if for each open set V containing $G(x)$, there is an open set U containing x such that for each $t \in U,\ G(t) \subseteq V;\ G$ is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(iii) **lower semi-continuous** (l.s.c.) at $x \in X$ if for each open set V with $G(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U,\ G(t) \cap V \neq \emptyset;\ G$ is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(vi) **continuous** if G is both lower semi-continuous and upper semi-continuous.

Proposition 2.1 ([16]) Let X and Y be two topological spaces. A set-valued mapping $T : X \longrightarrow 2^Y$ is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$ which converges to x there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \longrightarrow y$.

Definition 2.2 Let $T : K \longrightarrow 2^{E^\star}$ and $\eta : K \times K \longrightarrow K$ be the two mappings. We say that T is lower η-hemicontinuous whenever, for any $x, y \in K$, the mapping

$f : [0, 1] \longrightarrow 2^{(-\infty,+\infty)}$ defined by,

$$f(t) = \langle T(x + t(y - x)), \ \eta(y, x)\rangle$$

is lower semicontinuous at 0.

Remark that this definition is weaker than the corresponding definition given in [3].

Definition 2.3 ([6]) A mapping $F : K \longrightarrow 2^E$ is said to be a KKM-mapping, if for any $\{x_1, x_2, \dots, x_n\} \subset K$, co$\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$, where $2^E\backslash\{\emptyset\}$ denotes the family of all nonempty subsets of E.

Lemma 2.1([6]) Let K be a nonempty subset of a topological vector space X and $F : K \to 2^X$ a KKM mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x\in B} F(x)$ is compact. Then

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

Theorem 2.1. Let $T : K \longrightarrow 2^{E^\star}$ be lower η-hemicontinuous and relaxed η-α-monotone mapping. Let $f : K \times K \longrightarrow R \cup \{+\infty\}$ be a proper function (that is $f \neq +\infty$) and $\eta : K \times K \longrightarrow E$ be a mapping. Assume that

(i) $\eta(x, x) = 0$, for all $x \in K$,
(ii) for any fixed $x \in K$ and $u \in Ty$, the mapping $y \longrightarrow \langle u, \eta(y, x)\rangle$ is convex,
(iii) for any fixed $x \in K$, the mapping $y \longrightarrow f(y, x)$ is convex.

Then the following two variational-like inequality problems are equivalent (that is, their solution sets are equal):

(i) Find $x \in K$ such that

$$\langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ 0, \text{ for all } y \in K \text{ and } u \in T(x). \tag{2.2}$$

(ii) Find $x \in K$ such that

$$\langle v, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ \alpha(y - x), \text{ for all } y \in K \text{ and } v \in T(y). \tag{2.3}$$

Proof. Let $x \in K$ be a solution of (2.2). Since T is relaxed η-α-monotone, we have

$$\langle v, \eta(y, x)\rangle + f(y, x) - f(x, x) \ \geq \ \langle u, \eta(y, x)\rangle + \alpha(y - x) + f(y, x) - f(x, x)$$

for all $y \in K$, $v \in T(y)$. Then $x \in K$ is a solution of (2.3).

For vice versa, let $x \in K$ be a solution of (2.3). Assume that y is an arbitrary element of K and $u \in T(x)$. Since x is a solution of (2.3) then $f(x, x) < \infty$. Letting

$$y_t = (1 - t)x + ty, \ t \in [0, 1],$$

(note K is a convex set) then $y_t \in K$. Moreover y_t approaches to x when t converges to zero and so by Proposition 2.1 (note $u \in T(x)$ and T is lower $\eta-$ hemicontinuous) there is $v_t \in T(y_t)$, such that

$$\langle v_t, \eta(y, x)\rangle \longrightarrow \langle u, \ \eta(y, x)\rangle \text{ if } t \longrightarrow 0 \quad (*)$$

and hence (note that x is a solution of (2.3))

$$\langle v_t, \eta(y_t, x)\rangle + f(y_t, x) - f(x, x) \ \geq \ \alpha(y_t - x) \ = \ \alpha(t(y - x)). \tag{2.4}$$

By condition (iii) we get

$$f(y_t,x)-f(x,x) \;=\; f((1-t)x+ty,x)-f(x,x) \;\leq\; t(f(y,x)-f(x,x)) \qquad (2.5)$$

and also conditions (ii) and (i) imply that

$$\begin{aligned}\langle v_t,\eta(y_t,x)\rangle \;&=\; \langle v_t,\eta((1-t)x+ty,x)\rangle \\ &\leq\; (1-t)\langle v_t,\eta(x,x)\rangle + t\langle v_t,\eta(y,x)\rangle \\ &=\; t\langle v_t,\eta(y,x)\rangle\;. \end{aligned} \qquad (2.6)$$

It follows from (2.4)-(2.6) , for $t\in]0,1]$, that,

$$\langle v_t,\eta(y,x)\rangle + f(y,x)-f(x,x) \;\geq\; \frac{\alpha(t(y-x))}{t} = \frac{\alpha(t(y-x))-\alpha(\theta)}{t}, \qquad (2.7)$$

for all $y\in K$ and $v_t\in T(y_t)$. Now the result follows by letting $t\longrightarrow 0$ in (2.7), using (*), and the fact that α has nonnegative directional derivative at zero in each direction. That is

$$\langle u,\eta(y,x)\rangle + f(y,x)-f(x,x) \;\geq\; 0, \text{ for all } y\in K \text{ and } u\in T(x).$$

Hence $x\in K$ is a solution of (2.2). This completes the proof.

We need the following theorem in the sequel.

Theorem 2.2. Let K be a nonempty closed convex subset of a topological vector space E and $E^\star$ the dual space of E. Let $T:K\longrightarrow 2^{E^\star}\backslash\{\emptyset\}, f:K\times K\longrightarrow R\cup\{+\infty\}$ and $\eta:K\times K\longrightarrow E$ be three mappings such that,

- (i) $\eta(x,y)+\eta(y,x)=0$, for all $x\in K$,
- (ii) for any fixed $y\in K$, the mapping $x\longrightarrow\langle Tx,\eta(y,x)\rangle+f(y,x)-f(x,x)$ is lower semi-continuous,
- (iii) for any fixed $y\in K$, the mappings $x\longrightarrow\eta(x,y)$ and $x\longrightarrow f(x,y)$ are concave and convex, respectively,
- (iv) $\langle u_i-u_j,\eta(a_i,a_j)\rangle\geq 0$, for each finite subset $A=\{a_1,a_2,...,a_n\}$ of K, $y\in coA$ and $u_i\in T(y)$,
- (v) there exist a compact convex subset D of K and a compact subset B of K such that

$$\forall x\in K\backslash B\ \exists z\in D:\ \langle u,\,\eta(z,x)\rangle + f(z,x)-f(x,x) \;<\; 0, \text{ for some } u\in T(z).$$

Then the solution set of problem (2.2) is nonempty and compact.

Proof. Define set-valued mapping, $F:K\longrightarrow 2^E$ as follows:

$$F(y) \;=\; \{x\in K\,:\,\forall\, u\in T(x),\ \langle u,\,\eta(y,x)\rangle + f(y,x)-f(x,x) \;\geq\; 0\}.$$

We claim that F is a KKM mapping. If F is not a KKM-mapping, then there exist subset $\{y_1,y_2,\dots,y_n\}\subset K$ and $t_i>0$, $i=1,2,\dots,n$, such that $\sum_{i=1}^n t_i=1$,

$$z \;=\; \sum_{i=1}^n t_i\, y_i \;\notin\; \bigcup_{i=1}^n F(y_i),$$

and hence there exist $u_i\in T(y)$, for $i=1,2,...,n$ such that

$$\langle u_i,\,\eta(y_i,z)\rangle + f(y_i,z)-f(z,z)<0, \text{ for } i=1,2,\dots,n,$$

and so

$$\sum_{i=1}^n t_i\langle u_i,\,\eta(y_i,z)\rangle + \sum_{i=1}^n t_i f(y_i,z)-f(z,z)<0,$$

and by (iii) (f is convex in the first variable) we have

$$\sum_{i=1}^{n} t_i\langle u_i,\ \eta(y_i, z) < 0,$$

and by (i) (note $\eta(y_i, z) = -\eta(z, y_i)$ and $z = \sum_{j=1}^{n} t_j y_j$) we get

$$-\sum_{i=1}^{n} t_i\langle u_i,\ \eta(z, y_i) < 0,$$

and it follows from (iii) and (i) that

$$-\sum_{j=1}^{n}\sum_{i=1}^{n} t_i t_j\langle u_i,\ \eta(y_j, y_i) < 0,$$

and so by (i) (note $\eta(y_i, y_i) = 0, \eta(y_i, y_j) = -\eta(y_j, y_i)$) we get

$$\sum_{i<j} t_i t_j\langle u_i - u_j,\ \eta(y_i, y_j)\rangle\ < 0,$$

and so $\langle u_i - u_j,\ \eta(y_i, y_j)\rangle\ < 0$, for some $i < j$,which is contradicted (by (iv)). This implies that F is a KKM-mapping. We claim that $F(y)$ is closed for all $y \in K$. Indeed, let $\{x_\alpha\}$ be a net in $F(y)$ which converges to $x \in K$. We have to show that $x \in F(y)$. To see this let $v \in T(x)$ be an arbitrary element. By (ii) through Proposition 2.1 there is net $\{v_\alpha\}$ in E^* with $v_\alpha \in T(x_\alpha)$ such that

$$\langle v_\alpha, \eta(y, x_\alpha)\rangle + f(y, x_\alpha) - f(x_\alpha, x_\alpha) \longrightarrow \langle v, \eta(y, x)\rangle + f(y, x) - f(x, x) \quad (I)$$

and since $x_\alpha \in F(y)$ we deduce from (I) that

$$\langle v, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0,$$

and hence $x \in F(y)$. Also it follows from (v) that $\cap_{z\in D}F(z) \subseteq B$, and so F satisfies all the assumptions of Lemma 2.1 and then there exists $\overline{x} \in \bigcap_{y\in K} F(y)$. This means that $\overline{x}$ is a solution of problem 2.2. Furthermore the solution set of problem 2.2 equals to the intersection $\bigcap_{y\in K} F(y)$ which by using (v) is a subset of the compact set B and, note $\bigcap_{y\in K} F(y)$ is closed, so it is compact. This completes the proof of theorem. □

Remark. (i) It is clear that one can omit condition (v) in Theorem 2.2 when the set K is compact.

(ii) In [7], the authors, instead of condition (v) in Theorem 2.2, considered the following condition for a reflexive Banach space, which consists of finding $x_0 \in K$ such that,

$$\frac{\langle u - u_0,\ \eta(x, x_0)\rangle - f(x_0, x) + f(x, x)}{\|\eta(x, x_0)\|} \longrightarrow +\infty\ , \qquad (II)$$

whenever $\|x\| \longrightarrow \infty$, for all $u \in T(x)$, $u_0 \in T(x_0)$.
They (II) called η-coercive. It is clear that (II) is a special case of condition (v) in Theorem 2.2. Because for each positive real number M there is another positive number N such that

$$\|x\| > N \Rightarrow \frac{\langle u - u_0,\ \eta(x, x_0)\rangle - f(x_0, x) + f(x, x)}{\|\eta(x, x_0)\|} > M. \qquad (III)$$

Now we can take $B = \{x : \|x\| \leq N\}$ and $D = \{x_0\}$ which are weakly compact (note E is a reflexive Banach space) and convex. Moreover by condition (i) of Theorem 2.2 $\eta(x, x_0) = -\eta(x_0, x)$ and by multiplying the relation (III) by -1 we get condition (v) in Theorem 2.2.

An special case of (II) has been given in [19] as follows ,

$$\frac{\langle u - u_0,\ \eta(x, x_0)\rangle + f(x) - f(x_0)}{\|\eta(x, x_0)\|} \longrightarrow +\infty\ ,$$

whenever $\|x\| \longrightarrow \infty$, for all $u \in T(x)$, $u_0 \in T(x_0)$.

By combining Theorems 2.1 and 2.2 one can deduce the next result.

Theorem 2.3. Let K be a nonempty closed convex subset of a topological vector space E and $E^\star$ the dual space of E. Let $T : K \longrightarrow 2^{E^\star}\setminus\{\emptyset\}$ be lower η-hemicontinuous and relaxed η-α-monotone and the conditions (i)-(v) of Theorem 2.2 and condition (ii) of Theorem 2.1 hold. Then the solution sets of problems (2.2) and (2.3) are equal and a nonempty compact subset of K.

We note that if T is a single valued mapping and f is a zero map, then the Theorems 2.1 and 2.2 are equivalent to the problems considered and studied by Bai et al [1].

3. VARIATIONAL-LIKE INEQUALITIES WITH RELAXED η-α-SEMIMONOTONE MAPPINGS

Throughout this section, let E be an arbitrary locally convex topological vector space(briefly, locally convex space) with its dual E^* and K a nonempty closed convex subset of E.

Definition 3.1. Let $A : K \times K \longrightarrow 2^{E^*}$, $\eta : K \times K \longrightarrow E$ and $\alpha : E \longrightarrow \Re$ be three mappings. The mapping A is called relaxed $\eta - \alpha$-semimonotone if the mapping $y \longrightarrow A(w, y)$ is relaxed $\eta - \alpha-$monotone, for each $w \in K$. In this section we consider the following problem of finding $x \in K$ such that

$$\langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0, \text{for all } y \in K \text{ and} u \in A(x, y). \quad (3.1)$$

where $f : K \times K \longrightarrow \Re$.

In order to prove our existence theorem we need the following result.

Theorem 3.1 (Kakutani-Fan-Glicksberg)([5]). Let X be a locally convex Hausdorff space, $D \subseteq X$ a nonempty, convex compact subset. Let $T : D \longrightarrow 2^D$ be upper semicontinuous with nonempty, closed convex $T(x)$, for all $x \in D$. Then T has a fixed point in D.

Theorem 3.2. Let E be a locally convex Hausdorff space, $K \subseteq E$ a nonempty closed convex set, $A : K \times K \longrightarrow 2^E$ a relaxed $\eta - \alpha$-semimonotone mapping, $f : KtimesK \longrightarrow \Re\bigcup\{+\infty\}$ a proper convex and weakly lower semicontinuous functional, and $\eta : K \times K \longrightarrow E$ a mapping. If for all $w \in K$, the mapping $y \in A(w, y)$ satisfies all the assumptions of Theorem 2.2 and the mapping, for all $w \in K$, $x \longrightarrow \langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0$, for all $y \in K$ and $u \in A(w, y)$, is convex and upper semicontinuous, then problem (3.1) has a solution. Moreover the solution set of problem (3.1) is compact and convex.

Proof. By Theorem 2.2, for each $w \in coB$, the set

$$G(w) = \{x \in coB : \langle u, \eta(y, x)\rangle + f(y, x) - f(x, x) \geq 0, \text{ for all } y \in K \text{ and } u \in A(w, y)\}$$

is nonempty convex and compact subset of $B \subset K$. Now the mapping $G : coB \longrightarrow 2^{coB}$ defined by $w \longrightarrow G(w)$ fulfils all the conditions of Theorem 3.1 and hence there is $x \in coB \subset K$ such that $x \in G(x)$ and so x is a solution of problem 3.1 and so the solution set of the problem 3.1 is nonempty. It is clear that the solution set of problem (3.1) is equal to the intersection

$$\bigcap_{w \in K} G(w) \subseteq \bigcap_{x \in coB} G(w) \subset D$$

and since $G(w)$, for all $w \in K$ is closed and D is compact then the solution set problem (3.1) is compact and the convexity of the solution set is obvious from the assumptions. This completes the proof. $\square$

Remark 3.1. If A is a single valued mapping and f is a zero map, then problem (3.1) is equivalent to the problem (3.1) considered and studied by Bai et al [1]. Note that Theorems 2.2 and 3.1 are topological vector space version of Theorems 2.1 and 2.6, respectively, in [3].

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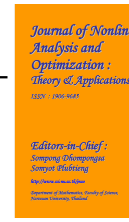

SOME COMMON FIXED POINT RESULTS FOR GENERALIZED WEAK CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. Some common fixed point results satisfying a generalized weak contractive condition in the framework of partially ordered metric spaces are obtained. The proved results generalize and extend some known results in the literature.

KEYWORDS : Common fixed point; f-Weakly contractive mappings; (μ, ψ)-Generalized f-weakly contractive mappings; Weakly compatible mappings
AMS Subject Classification: 41A50 47H10 54H25

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is an old and rich branch of analysis and has a large number of applications. Fixed point problems involving different contractive type inequalities have been studied by many authors (see [1]-[20] and references cited therein). The main aim of this work is to prove some common fixed point theorems for (μ, ψ)-generalized f-weakly contractive mappings in partially ordered metric spaces.

The Banach contraction mapping is one of the pivotal results of analysis. It is very popular tool for solving existence problems in many different fields of mathematics. There are a lot of generalizations of the Banach contraction principle in the literature. Ran and Reurings [18] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodŕiguez-López [17] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions.

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Alber and Guerre-Delabriere [1] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [20] proved the fixed point theorem which is one of the generalizations of Banach's Contraction Mapping Principle, because the weakly contractions contains contractions as a special case and he also showed that some results of [1] are true for any Banach space. In fact, weakly contractive mappings are closely related to the mappings of Boyd and Wong [3] and of Reich types [19]. Fixed point problems involving weak contractions and mappings satisfying weak contractive type inequalities have been studied by many authors (see [1], [7]-[15], [20] and references cited therein).

First, we recall some basic definitions and related results.

A map $T : X \longrightarrow X$ is called a *weakly contractive mapping* (see [1], [13], [20]) if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)) \tag{1.1}$$

where $\psi : [0, \infty) \longrightarrow [0, \infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim \psi(x) = \infty$.

If we take $\psi(x) = kx$, $0 < k < 1$, then a weakly contractive mapping is called a contraction.

A map $T : X \longrightarrow X$ is called a f*-weakly contractive mapping* (see [14]) if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(fx, fy) - \psi(d(fx, fy)) \tag{1.2}$$

where $f : X \longrightarrow X$ is a self-mapping, $\psi : [0, \infty) \longrightarrow [0, \infty)$ is continuous and nondecreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim \psi(x) = \infty$.

If we take $\psi(x) = (1 - k)x$, $0 < k < 1$, then a f-weakly contractive mapping is called a f-contraction. Further, if $f =$ identity mapping and $\psi(x) = (1 - k)x$, $0 < k < 1$, then a f-weakly contractive mapping is called a contraction.

A map $T : X \longrightarrow X$ is called a *generalized* f*-weakly contractive mapping* (see [7]) if for each $x, y \in X$,

$$d(Tx, Ty) \leq \frac{1}{2}[d(fx, Ty) + d(fy, Tx)] - \psi(d(fx, Ty), d(fy, Tx)) \tag{1.3}$$

where $f : X \longrightarrow X$ is a self-mapping, $\psi : [0, \infty)^2 \longrightarrow [0, \infty)$ is a continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

If $f =$identity mapping, then a generalized f-weakly contractive mapping is a generalized weakly contractive mapping (see [13]).

Khan et al. [16] initiated the use of a control function that alters distance between two points in a metric space, which they called an altering distance function.

A function $\mu : [0, \infty) \longrightarrow [0, \infty)$ is called an *altering distance function* if the following properties are satisfied:

(i) μ is monotone increasing and continuous;

(ii) $\mu(t) = 0$ if and only if $t = 0$.

A map $T : X \longrightarrow X$ is called a (μ, ψ)*-generalized* f*-weakly contractive mapping* (see [8]) if for each $x, y \in X$,

$$\mu(d(Tx, Ty)) \leq \mu(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]) - \psi(d(fx, Ty), d(fy, Tx)) \tag{1.4}$$

where $f : X \longrightarrow X$ is a self-mapping, $\mu : [0, \infty) \longrightarrow [0, \infty)$ is an altering distance function and $\psi : [0, \infty)^2 \longrightarrow [0, \infty)$ is a lower semi-continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

If $f =$identity mapping, then a (μ, ψ)-generalized f-weakly contractive mapping is a (μ, ψ)*-generalized weakly contractive mapping.*

Let M be a nonempty subset of a metric space (X, d), a point $x \in M$ is a *common fixed (coincidence) point* of f and T if $x = fx = Tx$ $(fx = Tx)$. The set of fixed points (respectively, coincidence points) of f and T is denoted by $F(f, T)$ (respectively, $C(f, T)$). The mappings $T, f : M \to M$ are called *commuting* if $Tfx = fTx$ for all $x \in M$; *compatible* if $\lim d(Tfx_n, fTx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim fx_n = t$ for some t in M; *weakly compatible* if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$.

Suppose $(X, \leq)$ is a partially ordered set and $T, f : X \longrightarrow X$. A mapping T is said to be *monotone f-nondecreasing* if for all $x, y \in X$,

$$fx \leq fy \text{ implies } Tx \leq Ty. \tag{1.5}$$

If f=identity mapping, then T is a *monotone nondecreasing*.

A subset W of a partially ordered set X is said to be *well ordered* if every two elements of W are comparable.

2. MAIN RESULTS

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T and f are self mappings on X, $T(X) \subseteq f(X)$, T is a monotone f-nondecreasing mapping and*

$$\mu(d(Tx, Ty)) \leq \mu(\frac{1}{2}[d(fx, Ty) + d(fy, Tx)]) - \psi(d(fx, Ty), d(fy, Tx)) \tag{2.1}$$

for all $x, y \in X$ for which $f(x) \geq f(y)$ where $\mu : [0, \infty) \longrightarrow [0, \infty)$ is an altering distance function and $\psi : [0, \infty)^2 \longrightarrow [0, \infty)$ is a lower semi-continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

If $\{f(x_n)\} \subset X$ is a nondecreasing sequence with $f(x_n) \longrightarrow f(z)$ in $f(X)$, then $f(x_n) \leq f(z)$, and $f(z) \leq f(f(z))$ for every n.

Also suppose that $f(X)$ is closed. If there exists an $x_0 \in X$ with $f(x_0) \leq T(x_0)$, then T and f have a coincidence point.

Further, if T and f are weakly compatible, then T and f have a common fixed point. Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point.

Proof. Let $x_0 \in X$ such that $f(x_0) \leq T(x_0)$. Since $T(X) \subseteq f(X)$, we can choose $x_1 \in X$ so that $fx_1 = Tx_0$. Since $Tx_1 \in f(X)$, there exists $x_2 \in X$ such that $fx_2 = Tx_1$. By induction, we construct a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$, for every $n \geq 0$.

Since $f(x_0) \leq T(x_0)$, $T(x_0) = f(x_1)$, $f(x_0) \leq f(x_1)$, T is monotone f-nondecreasing mapping, $T(x_0) \leq T(x_1)$. Similarly $f(x_1) \leq f(x_2)$, $T(x_1) \leq T(x_2)$, $f(x_2) \leq f(x_3)$. Continuing, we obtain

$$T(x_0) \leq T(x_1) \leq T(x_2) \leq \ldots \leq T(x_n) \leq T(x_{n+1}) \leq \ldots.$$

We suppose that $d(T(x_n), T(x_{n+1})) > 0$ for all n. If not then $T(x_{n+1}) = T(x_n)$ for some n, $T(x_{n+1}) = f(x_{n+1})$, i.e. T and f have a coincidence point x_{n+1}, and so we have the result.

Consider

$$\begin{aligned}
\mu(d(Tx_{n+1}, Tx_n)) &\leq \mu(\frac{1}{2}[d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})]) - \psi(d(fx_{n+1}, Tx_n), d(fx_n, Tx_{n+1})) \\
&= \mu(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1})) - \psi(0, d(Tx_{n-1}, Tx_{n+1})) \qquad (*) \\
&\leq \mu(\frac{1}{2}d(Tx_{n-1}, Tx_{n+1}))
\end{aligned}$$

$$\leq \quad \mu(\frac{1}{2}[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})])$$

Since μ is a non-decreasing function, for all $n = 1, 2 \ldots$, we have $d(Tx_{n+1}, Tx_n) \leq d(Tx_n, Tx_{n-1})$. Thus $\{d(Tx_{n+1}, Tx_n)\}$ is a monotone decreasing sequence of non-negative real numbers and hence is convergent. Hence there exists $r \geq 0$ such that $d(Tx_{n+1}, Tx_n) \longrightarrow r$.

From inequality $(*)$, we have

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq \frac{1}{2} d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \frac{1}{2}[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \end{aligned}$$

letting $n \longrightarrow \infty$, we have

$$r \leq \lim \frac{1}{2} d(Tx_{n-1}, Tx_{n+1}) \leq \frac{1}{2} r + \frac{1}{2} r,$$

i.e. $\lim d(Tx_{n-1}, Tx_{n+1}) = 2r$. Using the continuity of μ and lower semi-continuity of ψ, and inequality $(*)$, we have $\mu(r) \leq \mu(r) - \psi(0, 2r)$, and consequently, $\psi(0, 2r) \leq 0$. Thus $r = 0$. Hence

$$d(Tx_{n+1}, Tx_n) \longrightarrow 0.$$

Now, we show that $\{Tx_n\}$ is a Cauchy sequence. If otherwise, then there exists $\epsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k) > m(k) > k$ such that for every k, $d(Tx_{m(k)}, Tx_{n(k)}) \geq \epsilon$, $d(Tx_{m(k)}, Tx_{n(k)-1}) < \epsilon$. So, we have

$$\begin{aligned} \epsilon &\leq d(Tx_{m(k)}, Tx_{n(k)}) \\ &\leq d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)}) \\ &< \epsilon + d(Tx_{n(k)-1}, Tx_{n(k)}). \end{aligned}$$

Letting $k \longrightarrow \infty$ and using $d(Tx_{n+1}, Tx_n) \longrightarrow 0$, we have

$$\lim d(Tx_{m(k)}, Tx_{n(k)}) = \epsilon = \lim d(Tx_{m(k)}, Tx_{n(k)-1}). \tag{2.2}$$

Again,

$$d(Tx_{m(k)}, Tx_{n(k)-1}) \leq d(Tx_{m(k)}, Tx_{m(k)-1}) + d(Tx_{m(k)-1}, Tx_{n(k)}) + d(Tx_{n(k)}, Tx_{n(k)-1}),$$

and

$$d(Tx_{m(k)-1}, Tx_{n(k)}) \leq d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{m(k)}, Tx_{n(k)}).$$

Letting $k \longrightarrow \infty$ in the above two inequalities and using (2.2) we get,

$$\lim d(Tx_{m(k)-1}, Tx_{n(k)}) = \epsilon.$$

Also, we have

$$\begin{aligned} \mu(\epsilon) &\leq \mu(d(Tx_{m(k)}, Tx_{n(k)})) \\ &\leq \mu(\frac{1}{2}[d(fx_{m(k)}, Tx_{n(k)}) + d(fx_{n(k)}, Tx_{m(k)})]) - \\ &\quad \psi(d(fx_{m(k)}, Tx_{n(k)}), d(fx_{n(k)}, Tx_{m(k)})) \\ &= \mu(\frac{1}{2}[d(Tx_{m(k)-1}, Tx_{n(k)}) + d(Tx_{n(k)-1}, Tx_{m(k)})]) - \\ &\quad \psi(d(Tx_{m(k)-1}, Tx_{n(k)}), d(Tx_{n(k)-1}, Tx_{m(k)})). \end{aligned}$$

Taking $k \longrightarrow \infty$, and using the continuity of μ and lower semi-continuity of ψ, we have $\mu(\epsilon) \leq \mu(\frac{1}{2}[\epsilon + \epsilon]) - \psi(\epsilon, \epsilon)$ and consequently $\psi(\epsilon, \epsilon) \leq 0$, which is contradiction since $\epsilon > 0$. Thus $\{Tx_n\}$ is a Cauchy sequence. As $f(X)$ is closed and $fx_n = Tx_{n-1}$, $\{fx_n\}$ is also a Cauchy sequence, there is some $z \in X$ such that $\lim fx_{n+1} = \lim Tx_n = fz$. Since $\{f(x_n)\}$ is a nondecreasing sequence and $\lim fx_{n+1} = fz$, $f(x_n) \leq f(z)$, and $f(z) \leq f(f(z))$ for every n. Consider

$$\begin{aligned}\mu(d(Tz, fx_{n+1})) &= \mu(d(Tz, Tx_n)) \\ &\leq \mu(\frac{1}{2}[d(fz, Tx_n) + d(fx_n, Tz)]) - \psi(d(fz, Tx_n), d(fx_n, Tz)),\end{aligned}$$

letting $n \longrightarrow \infty$, we have

$$\mu(d(Tz, fz)) \leq \mu(\frac{1}{2}d(fz, Tz)) - \psi(0, d(fz, Tz))$$

This implies that $d(Tz, fz) = 0$, i.e. $Tz = fz$ and z is a coincidence point of T and f.

Now suppose that T and f are weakly compatible. Let $w = T(z) = f(z)$. Then $T(w) = T(f(z)) = f(T(z)) = f(w)$ and $f(z) \leq f(f(z)) = f(w)$. Consider

$$\begin{aligned}\mu(d(T(z), T(w))) &\leq \mu(\frac{1}{2}[d(fz, Tw) + d(fw, Tz)]) - \psi(d(fz, Tw), d(fw, Tz)) \\ &= \mu(\frac{1}{2}[d(Tz, Tw) + d(Tw, Tz)]) - \psi(d(Tz, Tw), d(Tw, Tz)) \\ &= \mu(d(Tw, Tz)) - \psi(d(Tz, Tw), d(Tw, Tz)).\end{aligned}$$

This implies that $d(Tz, Tw) = 0$, by the property of ψ. Therefore, $T(w) = f(w) = w$.

Now suppose that the set of common fixed points of T and f is well ordered. We claim that common fixed points of T and f is unique. Assume on contrary that, $Tu = fu = u$ and $Tv = fv = v$ but $u \neq v$. Consider

$$\begin{aligned}\mu(d(u, v)) &= \mu(d(Tu, Tv)) \\ &\leq \mu(\frac{1}{2}[d(fu, Tv) + d(fv, Tu)]) - \psi(d(fu, Tv), d(fv, Tu)) \\ &= \mu(\frac{1}{2}[d(u, v) + d(v, u)]) - \psi(d(u, v), d(v, u)) \\ &= \mu(d(u, v)) - \psi(d(u, v), d(u, v)).\end{aligned}$$

This implies that $d(u, v) = 0$, by the property of ψ. Hence $u = v$. Conversely, if T and f have only one common fixed point then the set of common fixed point of f and T being singleton is well ordered. □

If f =identity mapping, then we have the following result.

Corollary 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T is a self mapping on X, T is a monotone nondecreasing mapping and*

$$\mu(d(Tx, Ty)) \leq \mu(\frac{1}{2}[d(x, Ty) + d(y, Tx)]) - \psi(d(x, Ty), d(y, Tx)) \tag{2.3}$$

for all $x, y \in X$ for which $x \geq y$ where $\mu : [0, \infty) \longrightarrow [0, \infty)$ is an altering distance function and $\psi : [0, \infty)^2 \longrightarrow [0, \infty)$ is a lower semi-continuous mapping such that $\psi(x, y) = 0$ if and only if $x = y = 0$.

Also suppose that either

(i) $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \longrightarrow z$ in X, then $x_n \leq z$, for every n; or

(ii) T is continuous.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point.

Moreover, for arbitrary two points $x, y \in X$, there exists $w \in X$ such that w is comparable with both x and y. Then the fixed point of T is unique.

Proof. If (i) holds, then taking f =identity mapping in Theorem 2.1 we get the result.

If (ii) holds then proceeding as in Theorem 2.1 with f =identity mapping, we can prove that $\{Tx_n\}$ is a cauchy sequence, $z = \lim x_{n+1} = \lim T(x_n) = T(\lim x_n) = T(z)$ and hence T has a fixed point.

Let u and v be two fixed points of T such that $u \neq v$. Now, consider the following two cases:

(a) If u and v are comparable. Consider

$$\begin{aligned} \mu(d(u,v)) &= \mu(d(Tu,Tv)) \\ &\leq \mu(\frac{1}{2}[d(u,Tv)+d(v,Tu)]) - \psi(d(u,Tv),d(v,Tu)) \\ &\leq \mu(\frac{1}{2}[d(u,v)+d(v,u)]) - \psi(d(u,v),d(v,u)) \\ &= \mu(d(u,v)) - \psi(d(u,v),d(u,v)). \end{aligned}$$

This implies that $d(u,v)=0$, by the property of ψ. Hence $u=v$.

(b) If u and v are not comparable. Choose an element $w \in X$ comparable with both of them. Then also $u = T^n u$ is comparable with $T^n w$ for each n. Consider

$$\begin{aligned} \mu(d(u,T^n w)) &= \mu(d(T^n u, T^n w)) \\ &= \mu(d(TT^{n-1}u, TT^{n-1}w)) \\ &\leq \mu(\frac{1}{2}[d(T^{n-1}u,T^n w)+d(T^{n-1}w,T^n u)]) - \psi(d(T^{n-1}u,T^n w),d(T^{n-1}w,T^n u)) \\ &= \mu(\frac{1}{2}[d(u,T^n w)+d(T^{n-1}w,u)]) - \psi(d(u,T^n w),d(T^{n-1}w,u)) \qquad (**) \\ &\leq \mu(\frac{1}{2}[d(u,T^n w)+d(T^{n-1}w,u)]) \end{aligned}$$

and hence we get $d(u,T^n w) \leq d(u,T^{n-1}w)$. This proves that the nonnegative decreasing sequence $\{d(u,T^n w)\}$ is convergent. If $\lim_{n\longrightarrow\infty}\{d(u,T^n w)\} = r$, then, letting $n \longrightarrow \infty$ in $(**)$ and from the continuity of μ and lower semi-continuity of ψ we obtain $\mu(r) \leq \mu(r) - \psi(r,r) \leq \mu(r)$. This gives $\psi(r,r)=0$ and by our assumption about ψ, $r=0$. Consequently, $\lim_{n\longrightarrow\infty} d(u,T^n w)=0$. Analogously, it can be proved that $\lim_{n\longrightarrow\infty} d(v,T^n w)=0$. Since the limit is unique, we have $u=v$.

□

If $\mu(t)=t$, then we have the following result.

Corollary 2.3. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that T is a self mapping on X, T is a monotone nondecreasing mapping and*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Ty)+d(y,Tx)] - \psi(d(x,Ty),d(y,Tx)) \qquad (2.4)$$

for all $x,y \in X$ for which $x \geq y$ where $\mu : [0,\infty) \longrightarrow [0,\infty)$ is an altering distance function and $\psi : [0,\infty)^2 \longrightarrow [0,\infty)$ is a lower semi-continuous mapping such that $\psi(x,y)=0$ if and only if $x=y=0$.

Also suppose that either

(i) $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \longrightarrow z$ in X, then $x_n \leq z$, for every n; or

(ii) T is continuous.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point.

Moreover, for arbitrary two points $x, y \in X$, there exists $w \in X$ such that w is comparable with both x and y. Then the fixed point of T is unique.

If $\psi(x,y) = (\frac{1}{2} - k)(x+y)$, $0 < k < \frac{1}{2}$, we have the following result.

Corollary 2.4. [7] *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T is a nondecreasing self-mapping of X and T satisfies*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \text{ for } x \geq y, \tag{2.5}$$

where $0 < k < \frac{1}{2}$, for all $x, y \in X$. Also suppose either

(i) if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \longrightarrow z$ in X, then $x_n \leq z$ for every n.

or

(ii) T is continuous.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point.

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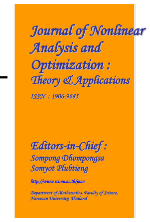

COINCIDENCE POINT THEOREMS IN HIGHER DIMENSION FOR NONLINEAR CONTRACTIONS

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ABSTRACT. In this manuscript, we introduce the concept of a coincidence point of N-order of $F : X^N \to X$ and $g : X \to X$ where $N \geq 2$ and X is an ordered set endowed with a metric d. We prove some coincidence point theorems of such mappings involving nonlinear contractions. The presented results are generalizations of the recent fixed point theorems due to Berzig and Samet [M. Berzig and B. Samet, An extension of coupled fixed point's concept in higher dimension and applications, Comput. Math. Appl. 63 (2012) 1319–1334]. Also, this work is an extension of M. Borcut [M. Borcut, Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput. 218 (2012) 7339–7346].

KEYWORDS : Coincidence point; Nonlinear contractions

1. INTRODUCTION

Banach fixed point theorem and its applications are well known. Many authors have extended this theorem, introducing more general contractive conditions, which imply the existence of a fixed point. Recently, there have been so many exciting developments in the field of existence of fixed point in partially ordered sets. The first result in this direction was given by Turinici [28], where he extended the Banach contraction principle in partially ordered sets. Ran and Reurings [25] presented some applications of Turinici's theorem to matrix equations. Subsequently, many other results in ordered sets have been obtained, see [1]-[4],[11],[12], [17]-[19], [21]-[24].

In [15], Bhaskar and Lakshmikantham introduced the concept of a coupled fixed point of a mapping $F : X\times X \to X$ and studied the problems of the uniqueness of a coupled fixed point in partially ordered metric spaces and applied their theorems to

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problems of the existence and uniqueness of solution for a periodic boundary value problem. In [20], Lakshmikantham and Ćirić introduced the concept of coupled coincidence point for mappings $F : X \times X \to X$ and $g : X \to X$, and proved some coupled coincidence point theorems for nonlinear contraction in partially ordered metric spaces.

We consider the following definitions and results which shall be required in the sequel.

Definition 1.1 ([20]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \quad \text{implies} \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \quad \text{implies} \quad F(x, y_1) \succeq F(x, y_2).$$

Definition 1.2 ([20]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = gx \ \text{ and } \ F(y, x) = gy.$$

Definition 1.3 ([20]). We say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if

$$gF(x, y) = F(gx, gy).$$

Lakshmikantham and Ćirić [20] obtained the following result.

Theorem 1.1 ([20]). *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ for each $t > 0$ and also suppose $F : X \times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \leq \phi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right)$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gv \preceq gy$. Assume that $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either F is continuous or X has the following properties:

(i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
(ii) *if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point.

Many generalizations and extensions of Theorem 1.1 exist in the literature, see [5],[6],[26],[27]. Recently, Berinde and Borcut [13] introduced the concept of tripled fixed point and established fixed point results for mappings having a monotone property and satisfying a contractive condition in ordered metric spaces. Later, Borcut [16] (see also [7]) established a tripled coincidence point theorem for a pair of mappings $F : X \times X \times X \to X$ and $g : X \to X$ satisfying a nonlinear contractive condition in ordered metric spaces. For other tripled fixed point results, see [8, 9, 10].

Definition 1.4 ([16]). Let $(X, \preceq)$ be a partially ordered set, and g a self map on X. The mapping $F : X \times X \times X \to X$ is said to has mixed g-monotone property if for any $x, y, z \in X$

$$x_1,\ x_2 \in X, \quad gx_1 \preceq gx_2 \Longrightarrow F(x_1, y, z) \preceq F(x_2, y, z),$$
$$y_1,\ y_2 \in X, \quad gy_1 \preceq gy_2 \Longrightarrow F(x, y_1, z) \succeq F(x, y_2, z),$$
$$z_1,\ z_2 \in X, \quad gz_1 \preceq gz_2 \Longrightarrow F(x, y, z_1) \preceq F(x, y, z_2).$$

Definition 1.5 ([16]). Let X be a non-empty set. Given $F : X \times X \times X \to X$ and $g : X \to X$. An element (x, y, z) is called a tripled coincidence point of F and g if

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \text{ and } F(z, y, x) = gz.$$

Definition 1.6 ([16]). Let X be a non-empty set. Let $F : X \times X \times X \to X$ and $g : X \to X$ are such that

$$g(F(x, y, z)) = F(gx, gy, gz)$$

whenever $x, y, z \in X$, then F and g are said to be commutative.

Consider also a class of function useful later.

Definition 1.7 (See ([20])). We denote by Θ the set of functions $\theta : [0, \infty) \longrightarrow [0, \infty)$ satisfying

(a) θ is non-decreasing,
(b) $\theta^{-1}(\{0\}) = \{0\}$,
(c) $\theta(t) < t$ for all $t > 0$,
(d) $\lim\limits_{r \longrightarrow t^+} \theta(r) < t$ for all $t > 0$.

Borcut [16] proved the following result.

Theorem 1.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F\colon X \times X \times X \to X$ and $g\colon X \to X$ are such that F has the mixed g-monotone property. Assume there is a function $\phi \in \Theta$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}),$$

for any $x, y, z, u, v, w \in X$ for which $gx \succeq gu$, $gv \succeq gy$ and $gz \succeq gw$. Assume that $F(X \times X \times X) \subseteq g(X)$, g is continuous and commutes with F. Also suppose either F is continuous or X has the following properties:

(i) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
(ii) *if a non-increasing sequence $x_n \to x$, then $x \preceq x_n$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0, z_0), \quad gy_0 \succeq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \preceq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = gx, \quad F(y, x, y) = gy \quad \text{and} \quad F(z, y, x) = gz,$$

that is, F and g have a tripled coincidence point.

Throughout this paper, we will use the following notations:

$$\underbrace{X \times X \cdots X \times X}_{N \text{ terms}} = X^N \quad \text{where } N \text{ is a positive integer,}$$

and

$$(x_{\varphi(p)}, x_{\varphi(p+1)}, \ldots, x_{\varphi(p+q)}) := x[\varphi(p : p + q)].$$

Now, following the concept of m-mixed monotone property introduced very recently by Berzig and Samet [14], we introduce:

Definition 1.8. Let $(X, \preceq)$ be an ordered set, N,m are positive integers, $0 \leq m \leq N$, $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two given mappings. We say that F has the m-g-mixed monotone property if $F(x_1, \ldots, x_m, x_{m+1}, \ldots, x_N)$ is monotone g-non-decreasing for the range of components from 1 to m and is monotone g-non-increasing for the range of components from $m + 1$ to N, that is,

$$\underline{x}_i, \overline{x}_i \in X, \quad g(\underline{x}_i) \preceq g(\overline{x}_i) \quad \text{implies} \quad F(x_1, \ldots, \underline{x}_i, \ldots, x_N) \preceq F(x_1, \ldots, \overline{x}_i, \ldots, x_N), \text{ for } i = 1, \ldots, m,$$

and

$$\underline{x}_i, \overline{x}_i \in X, \quad g(\underline{x}_i) \preceq g(\overline{x}_i) \quad \text{implies} \quad F(x_1, \ldots, \underline{x}_i, \ldots, x_N) \succeq F(x_1, \ldots, \overline{x}_i, \ldots, x_N), \text{ for } i = m+1, \ldots, N,$$

for all $(x_1, \ldots, x_N) \in X^N$.

Also, we introduce the concept of a coincidence point of N-order of $F : X^N \to X$ and $g : X \to X$ as follows:

Definition 1.9. Let $(X, \preceq)$ be an ordered set, N,m are positive integers, $0 \leq m \leq N$, $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two given mappings such that F has the m-g-mixed monotone property. An element $U = (x_1, x_2, \ldots, x_N) \longrightarrow X^N$ is called a coincidence point of N-order of F and g if there exist $2N$ maps $\varphi_1, \ldots, \varphi_m : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$, $\psi_1, \ldots, \psi_m : \{m + 1, \ldots, N\} \longrightarrow \{m + 1, \ldots, N\}$, $\varphi_{m+1}, \ldots, \varphi_N : \{1, \ldots, m\} \longrightarrow \{m + 1, \ldots, N\}$, and $\psi_{m+1}, \ldots, \psi_N : \{m + 1, \ldots, N\} \longrightarrow \{1, \ldots, m\}$ such that

$$gx_i = F(x[\varphi_i(1 : m)], x[\psi_i(m + 1 : N)]), \text{ for } i = 1, \ldots, N. \tag{1.1}$$

Definition 1.10. Let X be a non-empty set. Let $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two given mappings. We say F and g are commutative if

$$g(F(x_1, x_2, \ldots, x_N)) = F(gx_1, gx_2, \ldots, gx_N)$$

for all $(x_1, x_2, \ldots, x_N) \in X^N$.

In this paper, we establish some coincidence point theorems of N-order for $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ satisfying a contractive condition in complete ordered metric spaces. The presented results extend and generalize many results in literature.

2. MAIN RESULTS

Let (X, d) be a metric space and N be a positive integer, $N \geq 1$. We endow the product set X^N with the metric $\overline{d} : X^N \longrightarrow [0, \infty)$, given by

$$\overline{d}((u_1, u_2, \ldots, u_N), (v_1, v_2, \ldots, v_N)) = \max_{1 \leq i \leq N} d(u_i, v_i), \tag{2.1}$$

which also will be denoted by d.

Our first result is the following.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. For N, m positive integers, $0 \leq m \leq N$, let $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings such that F has the m-g-mixed monotone property. Suppose that there exists $\theta \in \Theta$ such that*

$$d(F(U), F(V)) \leq \theta \left(\max_{1 \leq i \leq N} d(gx_i, gy_i) \right), \tag{2.2}$$

for all $U = (x_1, \ldots, x_N), V = (y_1, \ldots, y_N) \in X^N$ *such that*

$$gx_i \preceq gy_i, \text{ for } i = 1, \ldots, m \quad \text{and} \quad gx_i \succeq gy_i, \text{ for } i = m+1, \ldots, N.$$

Suppose $F(X^N) \subseteq g(X)$, g *is continuous and commute with* F *and suppose either*

(a) F *is continues or*

(b) X *has the following property*

 (i) *if non-decreasing sequence* $\{x_n\} \longrightarrow x$, *then* $x_n \preceq x$ *for all* n,

 (ii) *if non-increasing sequence* $\{y_n\} \longrightarrow x$, *then* $y_n \succeq x$ *for all* n.

If there exists $U^{(0)} = (x_1^{(0)}, \ldots, x_N^{(0)}) \in X^N$ *such that*

$$\begin{aligned} gx_i^{(0)} &\preceq F(x^{(0)}[\varphi_i(1:m)], x^{(0)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, m, \\ gx_i^{(0)} &\succeq F(x^{(0)}[\varphi_i(1:m)], x^{(0)}[\psi_i(m+1:N)]), \text{ for } i = m+1, \ldots, N, \end{aligned} \tag{2.3}$$

where $\varphi_1, \ldots, \varphi_m : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$,$\psi_1, \ldots, \psi_m : \{m+1, \ldots, N\} \longrightarrow \{m+1, \ldots, N\}$, $\varphi_{m+1}, \ldots, \varphi_N : \{1, \ldots, m\} \longrightarrow \{m+1, \ldots, N\}$, *and* $\psi_{m+1}, \ldots, \psi_N : \{m+1, \ldots, N\} \longrightarrow \{1, \ldots, m\}$, *then there exists* $(x_1, x_2, \ldots, x_N) \longrightarrow X^N$ *satisfying* (1.1), *that is,* F *and* g *have a coincidence point of* N*-order.*

Proof. Let $U^{(0)} = (x_1^{(0)}, \ldots, x_N^{(0)}) \in X^N$ satisfying (2.3). Since $F(X^N) \subseteq g(X)$, we can choose $U^{(1)} = (x_1^{(1)}, \ldots, x_N^{(1)})$ such that

$$gx_i^{(1)} = F(x^{(0)}[\varphi_i(1:m)], x^{(0)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N.$$

Again from $F(X^N) \subseteq g(X)$, we can choose $U^{(2)} = (x_1^{(2)}, \ldots, x_N^{(2)})$ such that

$$gx_i^{(2)} = F(x^{(1)}[\varphi_i(1:m)], x^{(1)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N.$$

Continuing this process we can construct sequences $\{U^{(n)}\} = \{(x_1^{(n)}, \ldots, x_N^{(n)})\}$ in X^N such that

$$gx_i^{(n+1)} = F(x^{(n)}[\varphi_i(1:m)], x^{(n)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N. \tag{2.4}$$

Since F has the m-g-mixed monotone property, we have

$$\begin{aligned} gx_i^{(0)} \preceq gx_i^{(1)} \preceq gx_i^{(2)}, \text{ for } i = 1, \ldots, m. \\ gx_i^{(0)} \succeq gx_i^{(1)} \succeq gx_i^{(2)}, \text{ for } i = m+1, \ldots, N. \end{aligned}$$

Continuing this process, we can construct N sequences $\{gx_1^{(n)}\}, \ldots, \{gx_N^{(n)}\}$ in X such that

$$\begin{aligned} gx_i^{(n)} &= F(x^{(n-1)}[\varphi_i(1:m)], x^{(n-1)}[\psi_i(m+1:N)]) \preceq gx_i^{(n+1)} \\ &= F(x^{(n)}[\varphi_i(1:m)], x^{(n)}[\psi_i(m+1:N)]), i = 1, \ldots, m \\ gx_i^{(n)} &= F(x^{(n-1)}[\varphi_i(1:m)], x^{(n-1)}[\psi_i(m+1:N)]) \succeq gx_i^{(n+1)} \\ &= F(x^{(n)}[\varphi_i(1:m)], x^{(n)}[\psi_i(m+1:N)]), i = m+1, \ldots, N. \end{aligned}$$

Assume $(gx_1^{(n+1)}, \ldots, gx_N^{(n+1)}) \neq (gx_1^{(n)}, \ldots, gx_N^{(n)})$ for all $n \geq 0$, that is, $(x_1^{(n)}, \ldots, x_N^{(n)})$ is not a coincidence point of N-order of F and g. For $n \geq 0$, let

$$t_n = \max_{1 \leq i \leq N} d(gx_i^{(n)}, gx_i^{(n+1)}).$$

By assumption, $t_n > 0$ for all $n \geq 0$. We shall prove that $\{t_n\}$ is a decreasing sequence. Since

$$gx_i^{(n)} \preceq gx_i^{(n+1)}, \text{ for } i = 1, \ldots, m, \quad \text{and} \quad gx_i^{(n)} \succeq gx_i^{(n+1)}, \text{ for } i = m+1, \ldots, N. \tag{2.5}$$

we have

$$d(gx_1^{(n)}, gx_1^{(n+1)}) = d(F(x^{(n-1)}[\varphi_1(1:m)], x^{(n-1)}[\psi_1(m+1:N)]), F(x^{(n)}[\varphi_1(1:m)], x^{(n)}[\psi_1(m+1:N)]))$$
$$\leq \theta\left(\max_{\substack{1\leq i\leq m\\ m+1\leq j\leq N}}\left\{d(gx^{(n-1)}_{\varphi_1(i)}, gx^{(n)}_{\varphi_1(i)}), d(gx^{(n-1)}_{\psi_1(j)}, gx^{(n)}_{\psi_1(j)})\right\}\right).$$

Since θ is non-decreasing and $\max_{\substack{1\leq i\leq m\\ m+1\leq j\leq N}}\left\{d(gx^{(n-1)}_{\varphi_1(i)}, gx^{(n)}_{\varphi_1(i)}), d(gx^{(n-1)}_{\psi_1(j)}, gx^{(n)}_{\psi_1(j)})\right\} \leq$ t_{n-1}, then

$$d(gx_1^{(n)}, gx_1^{(n+1)}) \leq \theta(t_{n-1}). \tag{2.6}$$

Similarly, we obtain

$$d(gx_i^{(n)}, gx_i^{(n+1)}) \leq \theta(t_{n-1}), \quad i = 2, \dots, N. \tag{2.7}$$

Using (2.6) and (2.7) we obtain

$$0 < t_n = \max_{1\leq i\leq N} d(gx_i^{(n)}, gx_i^{(n+1)}) \leq \theta(t_{n-1}) < t_{n-1} \text{ ; since } \theta(t) < t; \text{ for all } t > 0.$$

Thus, a sequence $\{t_n\}$ is monotone decreasing. Therefore, there is some $t_+ > 0$ such that $\lim_{n\longrightarrow\infty} t_n = t_+$. We show that $t = 0$.

Suppose, on the contrary, that $t > 0$. Then, taking the limit as $n \longrightarrow \infty$ of both sides of $t_n \leq \theta(t_n)$ where $\theta \in \Theta$, we obtain

$$t = \lim_{n\longrightarrow\infty} t_n \leq \lim_{n\longrightarrow\infty} \theta(t_{n-1}) < \lim_{n\longrightarrow\infty} t_{n-1} = t$$

which is a contradiction. Thus $t = 0$, that is,

$$\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} \max_{1\leq i\leq N} d(gx_i^{(n)}, gx_i^{(n+1)}) = 0. \tag{2.8}$$

Now we prove that $\left\{gx_i^{(n)}\right\}, i = 1, \dots, N$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\left\{gx_i^{(n)}\right\}, i = 1, \dots, N$ is not a Cauchy sequence. Then there exist an $\varepsilon > 0$ and two subsequences of integers $\{\sigma(k)\}$, $\{\mu(k)\}$, $\mu(k) > \sigma(k) \geq k$ with

$$r_k = \max_{1\leq i\leq N} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) \geq \varepsilon. \tag{2.9}$$

We may also assume

$$\max_{1\leq i\leq N} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k)-1)}) < \varepsilon \tag{2.10}$$

by choosing $\sigma(k)$ to be the smallest number exceeding $\sigma(k)$ for which $r_k \geq \varepsilon$. By (2.10) and the triangle inequality, we get for $i = 1, \dots, N$

$$\begin{aligned} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) &\leq d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k)-1)}) + d(gx_i^{(\mu(k)-1)}, gx_i^{(\mu(k))}) \\ &< \varepsilon + d(gx_i^{(\mu(k)-1)}, gx_i^{(\mu(k))}). \end{aligned}$$

Letting $k \to \infty$ in above inequality and using (2.8), we have

$$\lim_{k\to\infty} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) \leq \varepsilon, \quad i = 1, \dots, N. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) &\leq d(gx_i^{(\sigma(k))}, gx_i^{(\sigma(k)-1)}) + d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)}) \\ &+ d(gx_i^{(\mu(k)-1)}, gx_i^{(\mu(k))}) \\ &\leq d(gx_i^{(\sigma(k))}, gx_i^{(\sigma(k)-1)}) + d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k))}) \end{aligned}$$

$$
\begin{aligned}
&+ \quad d(gx_i^{(\mu(k))}, gx_i^{(\mu(k)-1)}) + d(gx_i^{(\mu(k)-1)}, gx_i^{(\mu(k))}) \\
&< \quad d(gx_i^{(\sigma(k))}, gx_i^{(\sigma(k)-1)}) + \varepsilon + 2d(gx_i^{(\mu(k))}, gx_i^{(\mu(k)-1)}).
\end{aligned}
$$

Letting again $k \to \infty$ in above inequality and using (2.8), we have

$$
\lim_{k\to\infty} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) \le \lim_{k\to\infty} d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)}) \le \varepsilon, \quad i = 1, \dots, N. \tag{2.12}
$$

By (2.9), (2.11) and (2.12), we may get

$$
\lim_{k\to\infty} r_k = \lim_{k\to\infty} \max_{1\le i\le N} d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)}) = \varepsilon. \tag{2.13}
$$

From (2.5), we have

$$
gx_i^{(\sigma(k))} \preceq gx_i^{(\mu(k))}, \text{ for } i = 1, \dots, m, \quad \text{and} \quad gx_i^{(\sigma(k))} \succeq gx_i^{(\mu(k))}, \text{ for } i = m+1, \dots, N.
$$

Now using this, (2.2), (2.4) and monotonicity of θ, we get

$$
\begin{aligned}
d(gx_1^{(\sigma(k))}, gx_1^{(\mu(k))}) \quad &= \quad d(F(x^{(\sigma(k)-1)}[\varphi_1(1:m)], x^{(\sigma(k)-1)}[\psi_1(m+1:N)]), \\
&\quad\quad F(x^{(\mu(k)-1)}[\varphi_1(1:m(n))], x^{(\mu(k)-1)}[\psi_1(m+1:N)])) \\
&\le \quad \theta\left(\max_{1\le i\le N} d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)})\right).
\end{aligned}
$$

As a consequence, similarly we have

$$
r_k = \max_{1\le i\le N} d(gx_i^{(\sigma(k))}, gx_i^{(\mu(k))}) \le \theta\left(\max_{1\le i\le N} d(gx_i^{(\sigma(k)-1)}, gx_i^{(\mu(k)-1)})\right), \text{ for } i = 2, \dots, N. \tag{2.14}
$$

Letting $k \longrightarrow \infty$ and using (2.13), we get

$$
\varepsilon \le \theta(\varepsilon) < \varepsilon, \tag{2.15}
$$

which is a contradiction. Therefore, we proved that $\left\{gx_i^{(n)}\right\}, i = 1, \dots, N$ are Cauchy sequences. Since X is complete, there exist $U = (x_1, \dots, x_N) \in X$ such that,

$$
\lim_{n\longrightarrow\infty} gx_i^{(n)} = x_i, \quad i = 1, \dots, N. \tag{2.16}
$$

Thus, by continuity of g, we get

$$
\lim_{n\longrightarrow\infty} g(gx_i^{(n)}) = gx_i, \quad i = 1, \dots, N. \tag{2.17}
$$

From, (2.4) and commutativity of F and g, we have,

$$
g(gx_i^{(n+1)}) = g(F(x^{(n)}[\varphi_i(1:m)], x^{(n)}[\psi_i(m+1:N)])) \tag{2.18}
$$

$$
= \quad F(gx_{\varphi_i(1)}^{(n)}, \dots, gx_{\varphi_i(m)}^{(n)}, gx_{\psi_i(m+1)}^{(n)}, \dots, gx_{\psi_i(N)}^{(n)}) \tag{2.19}
$$

for $i = 1, \dots, N$.

Suppose now that (a) holds. We take $n \longrightarrow \infty$ and using the continuity of F, we get

$$
\begin{aligned}
gx_i = \lim_{n\longrightarrow\infty} g(gx_i^{(n+1)}) &= \lim_{n\longrightarrow\infty} F(gx_{\varphi_i(1)}^{(n)}, \dots, gx_{\varphi_i(m)}^{(n)}, gx_{\psi_i(m+1)}^{(n)}, \dots, gx_{\psi_i(N)}^{(n)}) \\
&= F(\lim_{n\longrightarrow\infty} gx_{\varphi_i(1)}^{(n)}, \dots, \lim_{n\longrightarrow\infty} gx_{\varphi_i(m)}^{(n)}, \lim_{n\longrightarrow\infty} gx_{\psi_i(m+1)}^{(n)}, \dots, \lim_{n\longrightarrow\infty} gx_{\psi_i(N)}^{(n)}) \\
&= F(x_{\varphi_i(1)}, \dots, x_{\varphi_i(m)}, x_{\psi_i(m+1)}, \dots, x_{\psi_i(N)}) \\
&= F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]), \quad \text{for} \quad i = 1, \dots, N.
\end{aligned}
$$

Thus, we proved that F and g have a coincidence point of N-order.

Suppose now that (b) holds. Since $\left\{g(x_i^{(n)})\right\}$ is monotone non-decreasing for $i = 1,\dots,m$ and non-increasing for $i = m+1,\dots,N$, and $gx_i^{(n)} \longrightarrow x_i$, $i = 1,\dots,N$, from (b) for all n, we have two cases. The first case is $gx_{\varphi(i)}^{(n)} \preceq x_{\varphi(i)}$, for $i = 1,\dots,m$ and $gx_{\psi(i)}^{(n)} \succeq x_{\psi(i)}$, for $i = m+1,\dots,N$. The second case is $gx_{\varphi(i)}^{(n)} \succeq x_{\varphi(i)}$, for $i = 1,\dots,m$ and $gx_{\psi(i)}^{(n)} \preceq x_{\psi(i)}$ for $i = m+1,\dots,N$. For both cases and by the triangle inequality, the monotonicity of θ, (2.2) and (2.18), we get

$$\begin{aligned}
& d(gx_i, F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)])) \\
& \leq d(gx_i, g(gx_i^{(n+1)})) + d(g(gx_i^{(n+1)}), F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)])) \\
& \leq d(gx_i, g(gx_i^{(n+1)})) + d(F(gx_{\varphi_i(1)}^{(n)},\dots,gx_{\varphi_i(m)}^{(n)}, gx_{\psi_i(m+1)}^{(n)},\dots,gx_{\psi_i(N)}^{(n)}), \\
& F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)])) \\
& \leq d(gx_i, g(gx_i^{(n+1)})) + \theta\left(\max_{\substack{1\leq i\leq m\\ m+1\leq j\leq N}} \left\{d(g(gx_{\varphi(i)}^{(n)}), gx_{\varphi(i)}^{(n)}), d(g(gx_{\psi(j)}^{(n)}), gx_{\psi(j)}^{(n)})\right\}\right) \\
& \leq d(gx_i, g(gx_i^{(n+1)})) + \theta(\max_{1\leq i\leq N} d(g(gx_i^{(n)}), gx_i^{(n)})).
\end{aligned}$$

So letting $n \longrightarrow \infty$ yields $d(gx_i, F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)])) \leq 0$. Hence $gx_i = F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)])$ for $i = 1,\dots,N$. Then, we proved that F and g have a coincidence point of N-order. This completes the proof of Theorem 2.1. □

Proposition 2.1. *Theorem 5 in* [16] *is a particular case of Theorem 2.1.*

Proof. Let $F : X \times X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings satisfying the hypotheses of Theorem 5 in [16]. For all $x_1, x_2, x_3 \in X$, define the mapping $G : X \times X \times X \longrightarrow X$ by

$$G(x_1, x_2, x_3) = F(x_1, x_3, x_2).$$

Since F has the g-mixed monotone property, the mapping G has the 2-g-mixed monotone property with $N = 3$. F is continuous, thus also G is continuous . Now, for all $X_1, X_2, X_3 \in X$ and $Y_1, Y_2, Y_3 \in X$ with $gX_1 \preceq gY_1, gX_2 \preceq gY_2$ and $gX_3 \succeq gY_3$, we have

$$\begin{aligned}
& d(G(X_1, X_2, X_3), G(Y_1, Y_2, Y_3)) = d(F(Y_1, Y_3, Y_2), F(X_1, X_3, X_2)) \\
& \leq \theta(\max\{d(X_1, Y_1); d(X_2, Y_2); d(X_3, Y_3)\}).
\end{aligned}$$

Moreover, from the hypotheses of Theorem 5 in [16], we know that there exist $x_1^{(0)}, x_2^{(0)}, x_3^{(0)} \in X$ such that $gx_1^{(0)} \preceq F(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$, $gx_2^{(0)} \succeq F(x_2^{(0)}, x_1^{(0)}, x_2^{(0)})$ and $gx_3^{(0)} \preceq F(x_3^{(0)}, x_2^{(0)}, x_1^{(0)})$. Denote $X_1^{(0)} = x_3^{(0)}, X_2^{(0)} = x_1^{(0)}$ and $X_3^{(0)} = x_2^{(0)}$, we have

$$gX_1^{(0)} \preceq F(X_1^{(0)}, X_3^{(0)}, X_2^{(0)}), gX_2^{(0)} \preceq F(X_2^{(0)}, X_3^{(0)}, X_1^{(0)}) \quad \text{and} \quad gX_3^{(0)} \succeq F(X_3^{(0)}, X_2^{(0)}, X_3^{(0)}).$$

This implies that

$$gX_1^{(0)} \preceq G(X_1^{(0)}, X_2^{(0)}, X_3^{(0)}), gX_2^{(0)} \preceq G(X_2^{(0)}, X_1^{(0)}, X_3^{(0)}) \quad \text{and} \quad gX_3^{(0)} \succeq G(X_3^{(0)}, X_3^{(0)}, X_2^{(0)}).$$

Now, all the required hypotheses of Theorem 2.1 are satisfied with $N = 3$, $m = 2$, $\varphi_1(1) = 1$, $\varphi_1(2) = 2$, $\varphi_2(1) = 2$, $\varphi_2(2) = 1$ and $\psi_3(3) = 2$. Applying Theorem 2.1, we get that there exist $X_1, X_2, X_3 \in X$ such that

$$gX_1 = G(X_1, X_2, X_3), gX_2 = G(X_2, X_1, X_3) \quad \text{and} \quad gX_3 = G(X_3, X_3, X_2),$$

that is,

$$gX_1 = F(X_1, X_3, X_2), gX_2 = F(X_2, X_3, X_1) \quad \text{and} \quad gX_3 = F(X_3, X_2, X_3).$$

This implies that $(u_1, u_2, u_3) = (X_2, X_3, X_1)$ is a tripled coincidence point of F and g. □

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. For N, m positive integers, $0 \leq m \leq N$, let $F : X^N \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings such that F has the m-g-mixed monotone property. Suppose that there exists $k \in [0, 1)$ such that*

$$d(F(U), F(V)) \leq k \max_{1 \leq i \leq N} d(gx_i, gy_i), \tag{2.20}$$

for all $U = (x_1, \ldots, x_N), V = (y_1, \ldots, y_N) \longrightarrow X^N$ such that

$$gx_i \preceq gy_i, \text{ for } i = 1, \ldots, m \quad \text{and} \quad gx_i \succeq gy_i, \text{ for } i = m+1, \ldots, N.$$

Suppose $F(X^N) \subseteq g(X)$, g is continuous and commute with F and suppose either

(a) *F is continues or*

(b) *X has the following property*

 (i) *if non-decreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$ for all n.*

 (ii) *if non-increasing sequence $\{y_n\} \longrightarrow x$, then $y_n \succeq x$ for all n.*

If there exists $U^{(0)} = (x_1^{(0)}, \ldots, x_N^{(0)}) \in X^N$ such that

$$\begin{aligned} gx_i^{(0)} &\preceq F(x^{(0)}[\varphi_i(1:m)], x^{(0)}[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, m, \\ gx_i^{(0)} &\succeq F(x^{(0)}[\varphi_i(1:m)], x^{(0)}[\psi_i(m+1:N)]), \text{ for } i = m+1, \ldots, N. \end{aligned} \tag{2.21}$$

where $\varphi_1, \ldots, \varphi_m : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$,$\psi_1, \ldots, \psi_m : \{m+1, \ldots, N\} \longrightarrow \{m+1, \ldots, N\}$, $\varphi_{m+1}, \ldots, \varphi_N : \{1, \ldots, m\} \longrightarrow \{m+1, \ldots, N\}$, and $\psi_{m+1}, \ldots, \psi_N : \{m+1, \ldots, N\} \longrightarrow \{1, \ldots, m\}$, then there exists $(x_1, x_2, \ldots, x_N) \longrightarrow X^N$ satisfying (1.1).

Proof. Following the proof of Theorem 2.1, for $\theta(t) = kt$ with $k \in [0, 1)$, then there exists $(x_1, x_2, \ldots, x_N) \in X^N$ satisfying (1.1). □

Remark 2.3. Corollary 2.2 generalizes Theorem 3.1 and Theorem 3.2 of Berzig and Samet [14] (when taking $\delta_i = k$ for each $i = 1, \ldots, N$).

Theorem 2.2. *By adding to the hypotheses of Theorem 2.1 the condition: for every $U = (x_1, \ldots, x_N), V = (y_1, \ldots, y_N) \in X^N$, there exists a $W = (z_1, \ldots, z_N) \in X^N$ such that $(F(z[\varphi_1(1:m)], z[\psi_1(m+1:N)]), \ldots, F(z[\varphi_N(1:m)], z[\psi_N(m+1:N)]))$ is comparable to $(gx_1, \ldots, gx_N)$ and $(gy_1, \ldots, gy_N)$. Then F and g have a unique N-order coincidence point.*

Proof. If $U = (x_1, \ldots, x_N)$ and $V = (y_1, \ldots, y_N) \in X^N$ are two N-order coincidence points of F and g, then we show that

$$d((gx_1, \ldots, gx_N), (gy_1, \ldots, gy_N)) = 0.$$

Since U and V are two N-order coincidence points, we have

$$gx_i = F(x[\varphi_i(1:m)], x[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N,$$

and

$$gy_i = F(y[\varphi_i(1:m)], y[\psi_i(m+1:N)]), \text{ for } i = 1, \ldots, N.$$

By assumption, there is $W = (z_1, \ldots, z_N) \in X^N$ such that $(F(z[\varphi_1(1:m)], z[\psi_1(m+1:N)]), \ldots, F(z[\varphi_N(1:m)], z[\psi_N(m+1:N)]))$ is comparable to $(gx_1, \ldots, gx_N)$

and $(gy_1,\dots,gy_N)$. Set $z_i^{(0)}=z_i$, for $i=1,\dots,N$, we can choose $W^{(1)}=(z_1^{(1)},\dots,z_N^{(1)})$ such that

$$gz_i^{(1)}=F(z^{(0)}[\varphi_i(1:m)],z^{(0)}[\psi_i(m+1:N)]),\ \text{for } i=1,\dots,N.$$

For $n>1$, continuing this process we can construct the sequences $\left\{gz_i^{(n)}\right\}, i=1,\dots,N$ such that

$$gz_i^{(n+1)}=F(z^{(n)}[\varphi_i(1:m)],z^{(n)}[\psi_i(m+1:N)]),\ \text{for } i=1,\dots,N.$$

Further, set $x_i^{(0)}=x_i$, for $i=1,\dots,N$ and $y_i^{(0)}=y_i$, for $i=1,\dots,N$, on the same way, define the sequences $\left\{gx_i^{(n)}\right\}, i=1,\dots,N$ and $\left\{gy_i^{(n)}\right\}, i=1,\dots,N$.

Then it is easy to show that, for all $n\geq 1$,

$$gx_i^{(n)}=F(x[\varphi_i(1:m)],x[\psi_i(m+1:N)]),\ \text{for } i=1,\dots,N,$$

and

$$gy_i^{(n)}=F(y[\varphi_i(1:m)],y[\psi_i(m+1:N)]),\ \text{for } i=1,\dots,N.$$

Since

$$F(x[\varphi_i(1:m)],x[\psi_i(m+1:N)])=gx_i^{(1)}=gx_i,\ \text{for } i=1,\dots,N,$$

and

$$F(z[\varphi_i(1:m)],z[\psi_i(m+1:N)])=gz_i^{(1)},\ \text{for } i=1,\dots,N,$$

are comparable, then $gx_i\preceq gz_i^{(1)}$ for $i=1,\dots,m$ and $gx_i\succeq gz_i^{(1)}$ for $i=m+1,\dots,N$. It is easy to show that gx_i for $i=1,\dots,N$ are also comparable to $gz_i^{(n)}$ for $i=1,\dots,N$, that is, $gx_i\preceq gz_i^{(n)}$ for $i=1,\dots,m$ and $gx_i\succeq gz_i^{(n)}$ for $i=m+1,\dots,N$ for all $n\geq 1$. Thus, from (2.2) and using the proof of Theorem 2.1 we have

$$\begin{aligned}
d(gx_i,gz_i^{(n+1)}) &= d(F(x[\varphi_i(1:m)],x[\psi_k(m+1:N)]),F(z^{(n)}[\varphi_i(1:m)],z^{(n)}[\psi_i(m+1:N)]))\\
&\leq \theta\left(\max_{\substack{1\leq k\leq m\\ m+1\leq j\leq N}}\left\{d(gx_{\varphi_i(k)},gz^{(n)}_{\varphi_i(k)}),d(gx_{\psi_i(j)},gz^{(n)}_{\psi_i(j)})\right\}\right)\\
&\leq \theta\left(\max_{1\leq i\leq N}\left\{d(gx_{(i)},gz^{(n)}_{(i)})\right\}\right),\quad \text{for } i=1,\dots,N.
\end{aligned}$$

Thus, we obtain

$$\max_{1\leq i\leq N} d(gx_i,gz_i^{(n+1)})\leq\theta\left(\max_{1\leq i\leq N}\left\{d(gx_i,gz_i^{(n)})\right\}\right)\leq\cdots\leq\theta^n\left(\max_{1\leq i\leq N}\left\{d(gx_i,gz_i^{(1)})\right\}\right). \tag{2.22}$$

But, it is known that the fact that $\theta\in\Theta$ implies

$$\lim_{n\to\infty}\theta^n(t)=0\quad\text{for all } t>0.$$

By letting $n\longrightarrow\infty$ in (2.22), we obtain

$$\lim_{n\longrightarrow\infty} d(gx_i,gz_i^{(n+1)})=0,\quad\text{for } i=1,\dots,N. \tag{2.23}$$

Similarly, one can prove that

$$\lim_{n\longrightarrow\infty} d(gy_i,gz_i^{(n+1)})=0,\quad\text{for } i=1,\dots,N. \tag{2.24}$$

By the triangle inequality, (2.23) and (2.24) we have

$$d(gx_i,gy_i)\leq d(gx_i,gz_i^{(n+1)})+d(gz_i^{(n+1)},gy_i)\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

which ends the proof. □

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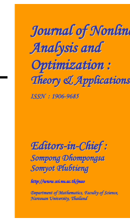

AN ITERATIVE ALGORITHM FOR A SYSTEM OF GENERALIZED IMPLICIT NONCONVEX VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. In this paper, we consider a new system of generalized implicit nonconvex variational inequality problems in the setting of two different Hilbert spaces. Using projection method, we establish the equivalence between the system of generalized implicit nonconvex variational inequality problems and a system of nonconvex variational inequality inclusions. Using this equivalence formulation, we suggest an iterative algorithm and show that the sequences generated by this iterative algorithm converge strongly to a solution of the system of generalized implicit nonconvex variational inequality problems. The results presented in this paper can be viewed as an improvement and refinement of previously known results for nonconvex (convex) variational inequality problems.

KEYWORDS : System of generalized implicit nonconvex variational inequality problems; uniformly prox-regular set; projection method; iterative algorithm; strongly monotone mapping; relaxed-Lipschitz continuous mapping.
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1. INTRODUCTION

In 1985, Pang [1] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities and discussed the convergence of method of decomposition for system of variational inequalities. Later, it was noticed that variational inequality over product sets and the system of variational inequalities both are equivalent, see for applications [1, 2, 3, 4]. Since then many authors, see for example [3, 4, 5, 6, 7, 8]

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studied the existence theory of various classes of system of variational inequalities by exploiting fixed-point theorems and minimax theorems. Recently, a number of iterative algorithms based on projection method and its variant forms have been developed for solving various systems of variational inequalities, see for instance [9, 10, 11, 12].

It is well known that the projection method and its variant forms based on projection operator over convex set are important tools for studying of existence and iterative approximation of solutions of various classes (systems) of variational inequality problems in the convexity settings, but these may not be applicable in general, when the sets are nonconvex. To overcome the difficulties that rise from the nonconvexity of underlying sets, the properties of projection operators over uniformly prox-regular sets are used.

In recent years, Bounkhel *et al.* [13], Moudafi [14], Wen [15], Kazmi *et al.* [16], Noor [17, 18, 19], and the relevant references cited therein], Alimohammady *et al.* [20], Balooee *et al.* [21] suggested and analyzed iterative algorithms for solving some classes (systems) of nonconvex variational inequality problems in the setting of uniformly prox-regular sets.

On the other hand, to the best of our knowledge, the study of iterative algorithms for solving the systems of variational inequality problems considered in [9, 11] in nonconvex setting has not been done so far.

Motivated and inspired by research going on in this area, we introduce a system of generalized implicit nonconvex variational inequality problems (in short, SGINVIP) defined on the uniformly prox-regular sets in different two Hilbert spaces. SGINVIP is different from those considered in [13, 14, 15, 16, 17, 18, 19, 20, 21] and includes the new and known systems of nonconvex (convex) variational inequality problems as special cases. Using the properties of projection operator over uniformly prox-regular sets, we suggest an iterative algorithm for finding the approximate solution of SGINVIP. Further, we prove that SGINVIP has a solution and the approximate solution obtained by iterative algorithm converges strongly to the solution of SGINVIP. The method presented in this paper extend, unify and improves the methods presented in [13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

2. PRELIMINARIES

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty closed set in H, not necessarily convex.

First, we recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis, see [23, 24, 25, 26].

Definition 2.1. The *proximal normal cone* of K at $u \in K$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K(u + \alpha\xi)\},$$

where $\alpha > 0$ is a constant and P_K is projection operator of H onto K, that is,

$$P_K(u) = \{u^* \in K : d_K(u) = \|u - u^*\|\},$$

where $d_K(u)$ is the usual distance function to the subset K, that is,

$$d_K(u) = \inf_{v\in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempty closed subset of H. Then $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha > 0$ such that

$$\langle \xi, v - u\rangle \ \leq \alpha\|v - u\|^2, \ \forall v \in K.$$

Definition 2.2. The *Clarke normal cone*, denoted by $N_K^C(u)$, is defined as

$$N_K^C(u) = \bar{\text{co}}[N_K^P(u)],$$

where $\bar{\text{co}}A$ means the closure of the convex hull of A.

Poliquin *et al.* [24] and Clarke *et al.* [25] have introduced and studied a class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important role in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have

Definition 2.3. For a given $r \in (0, \infty]$, a subset K of H is said to be *normalized uniformly r-prox-regular* if and only if every nonzero proximal normal to K can be realized by any r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in \ N_K^P(u)$ with $\|\xi\| = 1$, one has

$$\langle \xi, \ v - u\rangle \ \leq \ \frac{1}{2r} \ \|v - u\|^2, \ \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [23, 25]. It is clear that if $r = \infty$, then uniformly r-prox-regularity of K reduces to its convexity.

It is known that if K is a uniformly r-prox-regular set, the proximal normal cone $N_K^P(u)$ is closed as a set-valued mapping. Thus, we have $N_K^C(u) = N_K^P(u)$.

Now, let us state the following proposition which summarizes some important consequences of the uniformly prox-regularities:

Proposition 2.1. Let $r > 0$ and let K_r be a nonempty closed and uniformly r-prox-regular subset of H. Set $U_r = \{x \in H : d(x, K_r) < r\}$.

(i) For all $x \in U_r, \ P_{K_r}(x) \neq \emptyset$;

(ii) For all $r' \in (0, r), \ P_{K_r}$ is Lipschitz continuous with constant $\dfrac{r}{r - r'}$ on $U_{r'} = \{x \in H : d(x, K_r) < r'\}$.

3. System of generalized implicit nonconvex variational inequality problems

Throughout the rest part of the paper, we assume that, for each $i \in \{1, 2\}$, H_i is a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|_i$ and $\langle\cdot,\cdot\rangle_i$, respectively, and K_{i,r_i} is uniformly r_i-prox-regular subset of H_i.

For $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, assume that $A_i, \ C_i : H_i \longrightarrow H_j, \ B_i : H_j \longrightarrow H_i, \ N_i : H_j \times H_i \times H_j \longrightarrow H_i, \ g_i : H_i \longrightarrow H_i$ are single-valued mappings. For any constant $\rho_i > 0$ $(i = 1, 2)$, we consider the system of generalized implicit nonconvex variational inequality problems (SGINVIP): Find $(x_1, x_2) \in H_1 \times H_2$ such that $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ and

$$\langle \rho_i N_i(A_i x_i, B_i x_j, C_i x_i) + \rho_i x_i, y_i - g_i(x_i)\rangle_i + \frac{1}{2r_i}\|y_i - g_i(x_i)\|_i^2 \geq 0, \ \forall y_i \in K_{i,r_i}. \tag{3.1}$$

Some special cases of SGINVIP (3.1)

Case 1. For each $i \in \{1, 2\}$, if $g_i = I_i$, the identity operator, $N_1(A_1x_1, B_1x_2, C_1x_1) = G_1(x_1, x_2) - x_1, \ N_2(A_2x_2, B_2x_1, C_2x_2) = G_2(x_1, x_2) - x_2$ for all $x_i \in H_i$, where $G_i : H_1 \times H_2 \longrightarrow H_i$ is a nonlinear mapping then SGINVIP (3.1) reduces to the system of problems of finding $(x_1, x_2) \in K_{1,r_1} \times K_{2,r_2}$ such that

$$\langle \rho_i G_i(x_1, x_2), y_i - x_i\rangle_i + \frac{1}{2r_i}\|y_i - x_i\|_i^2 \geq 0, \ \forall y_i \in K_{i,r_i}, \tag{3.2}$$

which appears to be new.

Case 2. In Case 1, if $H_1 = H_2, K_{2,r_2} = K_{1,r_1}$ then SGINVIP (3.1) reduces to the nonconvex variational inequality problem of finding $x \in K_{1,r_1}$ such that

$$\langle \rho_1 G_1(x, x), y - x\rangle_1 + \frac{1}{2r_1}\|y - x\|_1^2 \geq 0, \ \forall y \in K_{1,r_1},$$

which appears to be new.

Case 3. In Case 1, for each $i \in \{1, 2\}$, if $r_i = \infty$, i.e., $K_{i,r_i} = K_i$, the convex set in H_i, then SGINVIP (3.1) reduces to the system of variational inequality problems of finding $(x_1, x_2) \in K_1 \times K_2$ such that

$$\langle G_i(x_1, x_2), y_i - x_i\rangle_i \geq 0, \ \forall y_i \in K_i \tag{3.3}$$

which has been studied by Ansari *et al.* [5] and Verma [9].

The following definitions are needed in the proof of main result.

Definition 3.1. A nonlinear mapping $g_1 : H_1 \longrightarrow H_1$ is said to be k_1-*strongly monotone* if there exists a constant $k_1 > 0$ such that

$$\langle g_1(x_1) - g_1(y_1), x_1 - y_1\rangle_1 \ \geq \ k_1\|x_1 - y_1\|_1^2, \ \forall x_1, y_1 \in H_1.$$

Definition 3.2. Let $N_1 : H_2 \times H_1 \times H_2 \longrightarrow H_1, \ A_1, C_1 : H_1 \longrightarrow H_2, \ B_1 : H_2 \longrightarrow H_1$ be nonlinear mappings. Then N_1 is said to be

(i) δ_1-*strongly monotone* with respect to A_1 in the first argument if there exists a constant $\delta_1 > 0$ such that

$$\langle N_1(A_1u, x_1, x_2) - N_1(A_1v, x_1, x_2), u - v\rangle_1 \ \geq \ \delta_1\|u - v\|_1^2,$$
$$\forall u, v, x_1 \in H_1, \ x_2 \in H_2;$$

(ii) σ_1-*relaxed Lipschitz continuous* with respect to C_1 in the third argument if there exists a constant $\sigma_1 > 0$ such that

$$\langle N_1(x_2, x_1, C_1u) - N_1(x_2, x_1, C_1v), u - v\rangle_1 \leq -\sigma_1\|u - v\|_1^2,$$
$$\forall u, v, x_1 \in H_1, \ x_2 \in H_2;$$

(iii) $L_{(N_1,1)}$-*Lipschitz continuous* in the first argument if there exists a constant $L_{(N_1,1)} > 0$ such that

$$\|N_1(u, x_1, x_2) - N_1(v, x_1, x_2)\|_1 \leq L_{(N_1,1)}\|u - v\|_1,$$
$$\forall x_1 \in H_1, \ u, v, x_2 \in H_2.$$

Similarly, we can define the Lipschitz continuity of N_1 in the second and third arguments. First, we prove the following technical lemmas.

Lemma 3.1. SGINVIP (3.1) is equivalent to the following system of generalized implicit nonconvex variational inclusions: Find $(x_1, x_2) \in H_1 \times H_2$ such that $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ and

$$\mathbf{0}_i \in x_i + N_i(A_i x_i, B_i x_j, C_i x_i) + \rho_i^{-1} N^P_{K_{i,r_i}}(g_i(x_i)), \tag{3.4}$$

for $i = 1, 2$, where $N^P_{K_{i,r_i}}(u)$ denotes the proximal normal cone of K_{i,r_i} at u in the sense of nonconvex analysis (See Definition 2.1), and $\mathbf{0}_i$ is the zero vector of H_i.

Proof. Let $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ be a solution of SGINVIP (3.1). If $N_i(A_i x_i, B_i x_j, C_i x_i) + x_i = \mathbf{0}_i$, then evidently the inclusuion (3.4) follows. If $N_i(A_i x_i, B_i x_j, C_i x_i) + x_i \neq \mathbf{0}_i$, then from (3.1) and Lemma 2.1, we get the inclusion (3.4). Conversely, let $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ be a solution of system (3.4) then it follows from Definition 2.3 that $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SGINVIP (3.1).

Lemma 3.2. $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SGINVIP (3.1) if and only if $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ satisfies the system of relations

$$g_i(x_i) = P_{K_{i,r_i}}[g_i(x_i) - \rho_i(x_i + N_i(A_i x_i, B_i x_j, C_i x_i))], \tag{3.5}$$

for $i = 1, 2$, where $P_{K_{i,r_i}}$ is the projection operator of H_i onto the uniformly r_i-prox-regular set K_{i,r_i}.

Proof. The result follows immediately from Lemma 3.1 and from the fact that $P_{K_{i,r_i}} = (I_i + N^P_{K_{i,r_i}})^{-1}$.

We can rewrite the equations (3.5) as follows:

$$g_i(x_i) = P_{K_{i,r_i}}(w_i), \ w_i = g_i(x_i) - \rho_i(x_i + N_i(A_i x_i, B_i x_j, C_i x_i)). \tag{3.6}$$

The alternative formulation (3.6) enables us to suggest the following iterative algorithm for solving SGINVIP (3.1).

Iterative algorithm 3.1. For given $(w_1^0, w_2^0) \in H_1 \times H_2$, compute the iterative sequences $\{w_1^n\}, \{w_2^n\}, \{x_1^n\}$ and $\{x_2^n\}$ defined by the iterative schemes:

$$g_i(x_i^n) = P_{K_{i,r_i}}(w_i^n), \tag{3.7}$$

$$w_i^{n+1} = (1 - \alpha^n) w_i^n + \alpha^n [g_i(x_i^n) - \rho_i(x_i^n + N_i(A_i x_i^n, B_i x_j^n, C_i x_i^n))], \tag{3.8}$$

for all $n = 0, 1, 2, ...$, and for each $i \in \{1, 2\}$ with $j \in \{1, 2\} \setminus \{i\}$, where $\alpha^n \in (0, 1)$ for $n > 0$ and $\alpha^0 = 1$ and $\sum_{n=1}^{\infty} \alpha^n = \infty$ and $\rho_1, \rho_2 > 0$ are constants.

In Case I, Iterative algorithm 3.1 reduces to the following iterative algorithm for solving the system (3.2).

Iterative algorithm 3.2. For given $(w_1^0, w_2^0) \in H_1 \times H_2$, compute the iterative sequences $\{w_1^n\}, \{w_2^n\}, \{x_1^n\}$ and $\{x_2^n\}$ defined by the iterative schemes:

$$x_i^n = P_{K_{i,r_i}}(w_i^n),$$

$$w_i^{n+1} = (1 - \alpha^n) w_i^n + \alpha^n [x_i^n - \rho_i G_i(x_1^n, x_2^n)], \tag{3.9}$$

for all $n = 0, 1, 2, ...$ and for each $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, where $\alpha^n \in (0, 1)$ for $n > 0$ and $\alpha^0 = 1$ and $\sum_{n=1}^{\infty} \alpha^n = \infty$ and $\rho_1, \rho_2 > 0$ are constants.

Now, we prove the existence and iterative approximation of solutions for SGINVIP (3.1).

Theorem 3.1. For each $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, let the projection operator $P_{K_{i,r_i}}$ be $(\frac{r_i}{r_i - r_i'})$-Lipschitz continuous; let $A_i,\ C_i : H_i \longrightarrow H_j$ and $B_i : H_j \longrightarrow H_i$ be L_{A_i}-Lipschitz continuous, L_{C_i}-Lipschitz continuous and L_{B_i}-Lipschitz continuous, respectively. Let $g_i : H_i \longrightarrow H_i$ be k_i-strongly monotone and continuous; let $N_i : H_j \times H_i \times H_j \longrightarrow H_i$ be δ_i-strongly monotone with respect to A_i in the first argument, τ_i-relaxed Lipschitz continuous with respect to C_i in the third argument, and $L_{(N_i,p)}$-Lipschitz continuous in the p^{th} argument, where $p = 1, 2, 3$. If the constant ρ_i satisfy the following conditions:

$$M_i - \triangle_i < \rho_i < \min\{M_i + \triangle_i, \Psi_i\}, \tag{3.10}$$

where

$$\begin{aligned}
M_i &= \frac{b_i k_i - a_i e_i}{b_i(1 - e_i^2)};\ \triangle_i = \frac{\sqrt{(b_i k_i - a_i e_i)^2 - b_i^2(1 - e_i^2)(1 - a_i^2)}}{b_i^2(1 - e_i^2)};\\
\Psi_i &< \frac{1}{b_j d_i};\ a_i = \frac{1}{\mu_i} - \phi_i;\ \phi_i = b_i \rho_j d_j;\\
b_i k_i &> a_i e_i + b_i\sqrt{(1 - e_i^2)(1 - a_i^2)};\ d_i = L_{(N_i,2)} L_{B_i};\\
\mu_i &= \frac{r_i}{r_i - r_i'};\ b_i = \frac{1}{\sqrt{2k_i + 3}};\\
e_i &= \sqrt{(1 - 2\delta_i + L_{(N_i,1)}^2 L_{A_i}^2)} + \sqrt{(1 - 2\sigma_i + L_{(N_i,3)}^2 L_{C_i}^2)};\\
&\frac{\rho_i^2 - 2\rho_i k_i}{2k_i + 3} \in [-1, 0);\ r_i' \in (0, r_i);\ r_i \in (0, \infty].
\end{aligned}$$

Then the sequences $\{x_i^n\}$ and $\{w_i^n\}$ generated by Iterative algorithm 3.1 converge strongly to x_i and w_i, respectively, where (x_1, x_2) with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SGINVIP (3.1).

Proof. From Iterative algorithm 3.1, we have

$$\begin{aligned}
\|w_i^{n+1} - w_i^n\|_i \leq\ & (1 - \alpha^n)\|w_i^{n+1} - w_i^n\|_i + \alpha^n \|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho_i(x_i^{n+1} - x_i^n)\|_i\\
& + \alpha_n \rho_i \|N_i(A_i x_i^{n+1}, B_i x_j^{n+1}, C_i x_i^{n+1}) - N_i(A_i x_i^n, B_i x_j^n, C_i x_i^n)\|_i\\
\leq\ & (1 - \alpha_n)\|w_i^{n+1} - w_i^n\|_i + \alpha^n \|g_i(x_i^{n+1}) - g_i(x_i^n)) - \rho_i(x_i^{n+1} - x_i^n)\|_i\\
& + \alpha^n \rho_i \big[\|[N_i(A_i x_i^{n+1}, B_i x_j^{n+1}, C_i x_i^{n+1}) - N_i(A_i x_i^n, B_i x_j^{n+1}, C_i x_i^{n+1})\\
& - (x_i^{n+1} - x_i^n)\|_i\\
& + \|N_i(A_i x_i^n, B_i x_j^{n+1}, C_i x_i^{n+1}) - N_i(A_i x_i^n, B_i x_j^{n+1}, C_i x_i^n) + (x_i^{n+1} - x_i^n)\|_i\\
& + \|N_i(A_i x_i^n, B_i x_j^{n+1}, C_i x_i^n)] - N_i(A_i x_i^n, B_i x_j^n, C_i x_i^n)\|_i\big].
\end{aligned} \tag{3.11}$$

Since A_i, B_i, C_i are L_{A_i}-, L_{B_i}-, L_{C_i}-Lipschitz continuous, N_i is δ_i-strongly monotone with respect to A_i in the first argument, σ_i-relaxed Lipschitz continuous with respect to C_i, and is $L_{(N_i,1)}$-, $L_{(N_i,2)}$-, $L_{(N_i,3)}$-Lipschitz continuous in the first, second and third arguments, respectively, one can obtain:

$$\|N_i(A_ix_i^{n+1}, B_ix_j^{n+1}, C_ix_i^{n+1}) - N_i(A_ix_i^n, B_ix_j^{n+1}, C_ix_i^{n+1}) - (x_i^{n+1} - x_i^n)\|_i$$
$$\leq \sqrt{(1 - 2\delta_i + L_{(N_i,1)}^2 L_{A_i}}\; \|x_i^{n+1} - x_i^n\|_i, \tag{3.12}$$
$$\|N_i(A_ix_i^n, B_ix_j^{n+1}, C_ix_i^{n+1}) - N_i(A_ix_i^n, B_ix_j^{n+1}, C_ix_i^n) + (x_i^{n+1} - x_i^n)\|_i$$
$$\leq \sqrt{(1 - 2\sigma_i + L_{(N_i,3)}^2 L_{C_i}}\; \|x_i^{n+1} - x_i^n\|_i, \tag{3.13}$$

and

$$\|N_i(A_ix_i^n, B_ix_j^{n+1}, C_ix_i^n)] - N_i(A_ix_i^n, B_ix_j^n, C_ix_i^n)\|_i \leq L_{(N_i,2)} L_{B_i} \|x_i^{n+1} - x_i^n\|_i. \tag{3.14}$$

Since g_i is k_i-strongly monotone and $P_{K_{i,r_i}}$ be $(\frac{r_i}{r_i - r_i'})$-Lipschitz continuous, then using (3.7), we have

$$\begin{aligned}\|x_i^{n+1} - x_i^n\|_i^2 &= \|x_i^{n+1} - x_i^n - (g_i(x_i^{n+1}) - g_i(x_i^n)) + (g_i(x_i^{n+1}) - g_i(x_i^n))\|_i^2 \\ &\leq \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 \\ &\quad -2\langle g_i(x_i^{n+1}) - g_i(x_i^n) + x_i^{n+1} - x_i^n, x_i^{n+1} - x_i^n\rangle_i \\ &= \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 - 2\langle g_i(x_i^{n+1}) - g_i(x_i^n), x_i^{n+1} - x_i^n\rangle_i \\ &\quad -2\langle x_i^{n+1} - x_i^n, x_i^{n+1} - x_i^n\rangle_i \\ &\leq \left(\frac{r_i}{r_i - r_i'}\right)^2 \|w_i^{n+1} - w_i^n\|_i^2 - (2k_i + 2)\|x_i^{n+1} - x_i^n\|_i^2,\end{aligned}$$

or

$$\|x_i^{n+1} - x_i^n\|_i \leq \left(\frac{\mu_i}{\sqrt{2k_i + 3}}\right) \|w_i^{n+1} - w_i^n\|_i, \tag{3.15}$$

where $\mu_i = \left(\dfrac{r_i}{r_i - r_i'}\right)$.

Next, we estimate

$$\begin{aligned}&\|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho(x_i^{n+1} - x_i^n)\|_i^2 \\ \leq\;& \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 \\ &-2\langle g_i(x_i^{n+1}) - g_i(x_i^n), x_i^{n+1} - x_i^n\rangle_i + \rho_i^2\|x_i^{n+1} - x_i^n\|_i^2 \qquad (3.16) \\ \leq\;& \mu_i^2\|w_i^{n+1} - w_i^n\|_i^2 + (\rho_i^2 - 2\rho_i k_i)\|x_i^{n+1} - x_i^n\|_i^2.\end{aligned}$$

From (3.15) and (3.16), we have

$$\|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho_i(x_i^{n+1} - x_i^n)\|_i \leq \mu_i\sqrt{1 + \frac{\rho_i^2 - 2\rho_i k_i}{2k_i + 3}}\; \|w_i^{n+1} - w_i^n\|_i. \tag{3.17}$$

Further, from (3.11)-(3.15) and (3.17), we have

$$\begin{aligned}\|w_i^{n+2} - w_i^{n+1}\|_i \leq (1 - \alpha^n)\|w_i^{n+1} - w_i^n\|_i + \alpha^n\mu_i\sqrt{1 + \frac{\rho_i^2 - 2\rho_i k_i}{2k_i + 3}}\|w_i^{n+1} - w_i^n\|_i \\ +\alpha^n\rho_i\left[\frac{\mu_i}{\sqrt{2k_i + 3}}\left\{\sqrt{(1 - 2\delta_i + L_{(N_i,1)}^2 L_{A_i}} + \sqrt{1 - 2\sigma_i + L_{(N_i,3)}^2 L_{C_i}}\right\}\right.\end{aligned}$$

$$\times\|w_i^{n+1} - w_i^n\|_i + \frac{\mu_j}{\sqrt{2k_j+3}}(L_{(N_i,2)}L_{B_i})\Bigg] \|w_j^{n+1} - w_j^n\|_j. \tag{3.18}$$

Define $\|.\|_\star$ on $H_1 \times H_2$ by $\|(y_1, y_2)\|_\star = \sum_{i=1}^{2} \|y_i\|_i$ for any $(y_1, y_2) \in H_1 \times H_2$. We note that $H_1 \times H_2$ is a Hilbert space with induced norm $\|.\|_\star$. It follows from (3.18) that

$$\|(w_1^{n+2}, w_2^{n+2}) - (w_1^{n+1}, w_2^{n+1})\|_\star = \sum_{i=1}^{2} \|w_i^{n+2} - w_i^{n+1}\|_i,$$

$$\leq [1 - \alpha^n(1-\theta)]\|(w_1^{n+1}, w_2^{n+1}) - (w_1^n, w_2^n)\|_\star, \tag{3.19}$$

where $\theta = \max\{\theta_1, \theta_2\};\ \theta_i = \mu_i[p_i + b_i(\rho_i e_i + \rho_j d_j)]$;

$$p_i = \sqrt{1 + \frac{\rho_i^2 - 2\rho_i k_i}{2k_i + 3}};\ b_i = \frac{1}{\sqrt{2k_i+3}};$$
$$d_i = L_{(N_i,2)}L_{A_i};\ e_i = \sqrt{(1 - 2\delta_i + L^2_{(N_i,1)}L_{A_i}} + \sqrt{(1 - 2\sigma_i + L^2_{(N_i,3)}L_{C_i}}.$$

From conditions (3.10), we have $0 < \theta < 1$, and hence, using the similar lines of proof of Theorem 4.3 [22], there exists an integer $n^0 > 0$ and a number $\alpha \in (0, 1)$ such that $(1 - \alpha^n(1-\theta)) \leq (1 - \alpha(1-\theta))$ for all $n > n^0$. Therefore, from (3.19), we have

$$\|(w_1^{n+1}, w_2^{n+1}) - (w_1^n, w_2^n)\|_\star \leq (1 - \alpha(1-\theta))^{n-n^0}\|(w_1^{n^0+1}, w_2^{n^0+1}) - (w_1^{n^0}, w_2^{n^0})\|_\star.$$

Hence for any $m \geq n \geq n^0$, it follows that

$$\|(w_1^m, w_2^m) - (w_1^n, w_2^n)\|_\star \leq \sum_{i=n}^{m-1} \|(w_1^{i+1}, w_2^{i+1}) - (w_1^{n^0}, w_2^{n^0})\|_\star$$

$$\leq \sum_{i=1}^{m-1}(1 - \alpha(1-\theta))^{i-n^0}\|(w_1^{n^0+1}, w_2^{n^0+1}) - (w_1^{n^0}, w_2^{n^0})\|_\star. \tag{3.20}$$

Since $0 < (1-\alpha(1-\theta)) < 1$, it follows from (3.20) that $\|(w_1^m, w_2^m) - (w_1^n, w_2^n)\|_\star \leq \sum_{i=n}^{m-1} \|w_i^m - w_i^n\| \longrightarrow 0$ as $n \longrightarrow \infty$, and hence for each $i \in \{1, 2\}$, $\{w_i^n\}$ is a Cauchy sequence in H_i. Assume $w_i^n \longrightarrow w_i$ in H_i as $n \longrightarrow \infty$. We observe from (3.15) that $\{x_i^n\}$ is a Cauchy sequence and hence assume that $x_i^n \longrightarrow x_i$ in H_i as $n \longrightarrow \infty$.

Further, from the continuity of $N_i, A_i, B_i, C_i, g_i, P_{K_{i,r_i}}$ and Iterative algorithm 3.1, we observe that

$$g_i(x_i) = P_{K_{i,r_i}}[g_i(x_i) - \rho_i(x_i + N_i(A_i x_i, B_i x_j, C_i x_i))].$$

Hence it follows from Lemma 3.2 that $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SGINVIP (3.1). This completes the proof.

Remark 3.1.

(i) The method presented in this paper unifies the methods considered in [13, 14, 15, 16, 17, 18, 19, 20, 21] to the system of nonconvex variational inequality problems defined on the product of two different Hilbert spaces.

(ii) The method presented in this paper improves the methods considered in [19, 20, 21] in the sense that the continuity of g is required instead of the Lipschitz continuity.

(iii) One needs further research effort to extend the method presented for solving the system of nonconvex variational inequality problems involving set-valued mappings.

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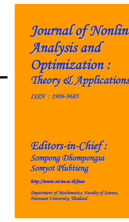

NEW CONVERGENCE ANALYSIS FOR COUNTABLE FAMILY OF RELATIVELY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we construct a new iterative scheme by hybrid methods and prove strong convergence theorem for approximation of a common fixed point of a countable family of relatively quasi-nonexpansive mappings in a uniformly smooth and strictly convex real Banach space with Kadec-Klee property using the properties of generalized f-projection operator. Our results extend many known recent results in the literature.

KEYWORDS : Relatively quasi-nonexpansive mappings; Generalized f-projection operator; Hybrid method; Banach spaces
AMS Subject Classification: 47H06 47H09 47J05 47J25

1. INTRODUCTION

Let E be a real Banach space with dual E^* and C be nonempty, closed and convex subset of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}.$$

The following properties of J are well known (The reader can consult [1-3] for more details): If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E; $J(x) \neq \emptyset$, $x \in E$; if E is reflexive, then J is a mapping from E onto E^* and if E is smooth, then J is single valued. Throughout this paper, we denote by ϕ, the functional on $E \times E$ defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, J(y)\rangle + ||y||^2, \ \forall x, y \in E. \tag{1.1}$$

From (1.1), we have $(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \ \forall x, y \in E$.
Let T be a mapping from C into E. A point $x \in C$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence

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$\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. We say that a mapping T is ***relatively nonexpansive*** (see, for example, [4-8]) if the following conditions are satisfied: $F(T) \neq \emptyset$; $\phi(p, Tx) \leq \phi(p, x),\ \forall x \in C,\ p \in F(T)$ and $F(T) = \widehat{F}(T)$. If T satisfies $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x),\ \forall x \in C,\ p \in F(T)$, then T is said to be ***relatively quasi-nonexpansive***. It is easy to see that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive mappings. Many authors have studied the methods of approximating the fixed points of relatively quasi-nonexpansive mappings (see, for example, [9-11] the references contained therein). Clearly, in Hilbert space H, relatively quasi-nonexpansive mappings and quasi-nonexpansive mappings are the same, for $\phi(x, y) = ||x - y||^2,\ \forall x, y \in H$ and this implies that $\phi(p, Tx) \leq \phi(p, x) \Leftrightarrow ||Tx - p|| \leq ||x - p||,\ \forall x \in C,\ p \in F(T)$. The examples of relatively quasi-nonexpansive mappings are given in [10].

We next give an example of a mapping that is relatively quasi-nonexpansive but not relatively nonexpansive.

Example 1.1. Let $E = \ell^2$ and

$$\begin{cases} x_0 = (1, 0, 0, 0, \ldots) \\ x_1 = (1, 1, 0, 0, \ldots) \\ x_2 = (1, 0, 1, 0, 0, \ldots) \\ x_3 = (1, 0, 0, 1, 0, 0, \ldots) \\ \ldots \\ x_n = (1, 0, 0, 0, \ldots, 0, 1, 0, 0, \ldots) \\ \ldots \end{cases}$$

Clearly, $\{x_n\}$ converges weakly to x_0. Define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \frac{n}{n+1} x_n, \text{ if } x = x_n (\exists n \geq 1), \\ -x, \text{ if } x \neq x_n (\forall n \geq 1). \end{cases}$$

We can see that $F(T) = \{0\} \neq \emptyset$ and

$$||Tx - 0|| = ||Tx|| \leq ||x|| = ||x - 0||,\ \forall x \in E.$$

Furthermore, since ℓ^2 is a Hilbert space, we obtain

$$\phi(Tx, 0) = ||Tx - 0||^2 = ||Tx||^2 \leq ||x||^2 = ||x - 0||^2 = \phi(x, 0),\ \forall x \in E.$$

It then follows that T is a relatively quasi-nonexpansive mapping. We next show that T is not a relatively nonexpansive mapping. Since $\{x_n\}$ converges weakly to x_0, then there exists $M > 0$ such that $||x_n|| \leq M,\ \forall n \geq 1$. We observe that

$$||Tx_n - x_n|| = \left|\left|\frac{n}{n+1} x_n - x_n\right|\right| = \frac{1}{n+1} ||x_n|| \leq \frac{1}{n+1} M \to 0,\ n \to \infty,$$

but $x_0 \notin F(T)$. Thus, $F(T) \neq \widehat{F}(T)$ even though $F(T) \neq \emptyset$ and $||Tx_n - x_n|| \to 0,\ n \to \infty$. Hence, T is not a relatively nonexpansive mapping.

The above Example 1.1 shows that the class of relatively nonexpansive mappings is properly contained in the class of relatively quasi-nonexpansive mappings.
In [7], Matsushita and Takahashi introduced a hybrid iterative scheme for approximation of fixed points of relatively nonexpansive mapping in a uniformly convex

real Banach space which is also uniformly smooth: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{w \in C : \phi(w, y_n) \le \phi(w, x_n)\}, \\ W_n = \{w \in C : \langle x_n - w, Jx_0 - Jx_n \rangle, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \ n \ge 0. \end{cases}$$

They proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{F(T)}x_0$, where $F(T) \neq \emptyset$.

In [12], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings: $x_0 \in C$,

$$\begin{cases} z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n) \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n) \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n(||x_0||^2 + 2\langle w, Jx_n - Jx_0 \rangle)\} \\ Q_n = \{z \in C : \langle x_n - z, Jx_n - Jx_0 \rangle \le 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are sequences in $(0, 1)$ satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ and T and S are relatively nonexpansive mappings and J is the single-valued duality mapping on uniformly smooth and uniformly convex Banach E. They proved under the appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (1.2) converges strongly to a common fixed point of T and S in a uniformly smooth and uniformly convex Banach space.

Recently, Li *et al.* [13] introduced the following hybrid iterative scheme for approximation of fixed points of a relatively nonexpansive mapping using the properties of generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C, \ C_0 = C$

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \le G(w, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \ n \ge 0, \end{cases}$$

They proved a strong convergence theorem for finding an element in the fixed points set of T in a uniformly smooth real Banach space which is also uniformly convex.

Motivated by the above mentioned results and the on-going research, it is our purpose in this paper to prove strong convergence theorem for a countable family of relatively quasi-nonexpansive mappings in a uniformly smooth and strictly convex real Banach space with the Kadec-Klee property using the properties of generalized f-projection operator. Our results extend the results of Matsushita and Takahashi [7], Plubtieng and Ungchittrakool [12], Li *et al.* [13] and many other recent known results in the literature.

2. PRELIMINARIES

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [14], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) := arg\min_{y \in C} \phi(y, x), \ \forall x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [3, 14-17]). If

E is a Hilbert space, then Π_C is the metric projection of H onto C. Next, we recall the concept of generalized f-projector operator, together with its properties. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = ||\xi||^2 - 2\langle \xi, \varphi\rangle + ||\varphi||^2 + 2\rho f(\xi), \tag{2.1}$$

where $\xi \in C,\ \varphi \in E^*,\ \rho$ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definitions of G and f, it is easy to see the following properties:
(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;
(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.

Definition 2.1. (Wu and Huang [18]) Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f \varphi = \Big\{u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi)\Big\},\ \forall \varphi \in E^*.$$

Recall that J is a single valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution in (2.1) gives

$$G(\xi, Jx) = ||\xi||^2 - 2\langle \xi, Jx\rangle + ||x||^2 + 2\rho f(\xi). \tag{2.2}$$

Definition 2.2. Let E be a real Banach space and C a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f x = \Big\{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\Big\},\ \forall x \in E.$$

Obviously, the definition of T is a relatively-quasi nonexpansive mapping is equivalent to: $F(T) \neq \emptyset$ and $G(p, JTx) \leq G(p, Jx),\ \forall x \in C,\ p \in F(T)$.

Lemma 2.3. *(Li et al. [13]) Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous convex functional. Then there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^*\rangle + \alpha,\ \forall x \in E.$$

Lemma 2.4. *(Li et al. [13]) Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Then the following statements hold:*
(i) $\Pi_C^f x$ is a nonempty closed and convex subset of C for all $x \in E$;
(ii) for all $x \in E,\ \hat{x} \in \Pi_C^f x$ if and only if

$$\langle \hat{x} - y, Jx - J\hat{x} + \rho f(y) - \rho f(x) \geq 0,\ \forall y \in C;$$

(iii) if E is strictly convex, then $\Pi_C^f x$ is a single valued mapping.

Lemma 2.5. *(Li et al. [13]) Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Let $x \in E$ and $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx),\ \forall y \in C.$$

We recall that a Banach space E has *Kadec-Klee property* if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $x_n \to x$ as $n \to \infty$. We note that every uniformly convex Banach space has the Kadec-Klee property. For more details on Kadec-Klee property, the reader is referred to [2, 16].

Lemma 2.6. *(Kim et al. [19]) Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which also has Kadec-Klee property. Let T be a closed relatively-quasi nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.7. *(Kim et al. [19]) Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R},\ g(0) = 0$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\Big|\Big|\sum_{n=1}^{\infty} \lambda_n x_n\Big|\Big|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

For the rest of this paper, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to p shall be denoted by $x_n \to p$ as $n \to \infty$, $\{x_n\}_{n=0}^{\infty}$ converges weakly to p shall be denoted by $x_n \rightharpoonup p$.

Lemma 2.8. *(Li et al. [13]) Let E be a Banach space and $y \in E$. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \rightharpoonup x \in \mathrm{int}(D(f))$ and $\lim_{n\to\infty} G(x_n, Jy) = G(x, Jy)$, then $\lim_{n\to\infty} ||x_n|| = ||x||$.*

3. MAIN RESULTS

Theorem 3.1. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_i\}_{i=1}^{\infty}$ is an infinite family of closed relatively-quasi nonexpansive mappings of C into itself such that $F := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset \mathrm{int}(D(f))$ and suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C,\ C_0 = C$,*

$$\begin{cases} y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J T_i x_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0,\ n \geq 0, \end{cases} \tag{3.1}$$

where J is the duality mapping on E. Suppose $\{\alpha_{ni}\}_{n=1}^{\infty}$ for each $i = 0, 1, 2, ...$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0,\ i = 1, 2, 3, ...,\ \sum_{i=0}^{\infty} \alpha_{ni} = 1$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F^f x_0$.

Proof. We first show that $C_n,\ \forall n \geq 0$ is closed and convex. It is obvious that $C_0 = C$ is closed and convex. Thus, we only need to show that C_n is closed and convex for each $n \geq 1$. Since $G(z, Jy_n) \leq G(z, Jx_n)$ is equivalent to

$$2\Big(\langle z, Jx_n\rangle - \langle z, Jy_n\rangle\Big) \leq ||x_n||^2 - ||y_n||^2.$$

This implies that C_n is closed and convex $\forall n \geq 0$. This shows that $\Pi_{C_{n+1}}^f x_0$ is well defined for all $n \geq 0$.

We now show that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. Since $f : E \to \mathbb{R}$ is a convex and lower semi-continuous, applying Lemma 2.3, we see that there exists $u^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, u^*\rangle + \alpha, \ \forall y \in E.$$

It follows that

$$\begin{aligned} G(x_n, Jx_0) &= ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho f(x_n) \\ &\geq ||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho\langle x_n, u^*\rangle + 2\rho\alpha \\ &= ||x_n||^2 - 2\langle x_n, Jx_0 - \rho u^*\rangle + ||x_0||^2 + 2\rho\alpha \\ &\geq ||x_n||^2 - 2||x_n||||Jx_0 - \rho u^*|| + ||x_0||^2 + 2\rho\alpha \\ &= (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha. \quad (3.2) \end{aligned}$$

Since $x_n = \Pi^f_{C_n} x_0$, it follows from (3.2) that

$$G(x^*, Jx_0) \geq G(x_n, Jx_0) \geq (||x_n|| - ||Jx_0 - \rho u^*||)^2 + ||x_0||^2 - ||Jx_0 - \rho u^*||^2 + 2\rho\alpha$$

for each $x^* \in C_n$. This implies that $\{x_n\}_{n=0}^\infty$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^\infty$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_n$. It then follows Lemma 2.5 that

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0). \quad (3.3)$$

It is obvious that

$$\phi(x_{n+1}, x_n) \geq (||x_{n+1}|| - ||x_n||)^2 \geq 0,$$

and so $\{G(x_n, Jx_0)\}_{n=0}^\infty$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^\infty$ exists.

We next show that $F \subset C_n, \ \forall n \geq 0$. For $n = 0$, we have $F \subset C = C_0$. Let $x^* \in F$. Since E is uniformly smooth, we know that E^* is uniformly convex. Then from Lemma 2.7, we have for any positive integer $j > 0$ that

$$\begin{aligned} G(x^*, Jy_n) &= G(x^*, (\alpha_{n0}Jx_n + \sum_{i=1}^\infty \alpha_{ni}JT_ix_n)) \\ &= ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^\infty \alpha_{ni}\langle x^*, JT_ix_n\rangle + ||\alpha_{n0}Jx_n + \sum_{i=1}^\infty \alpha_{ni}JT_ix_n||^2 + 2\rho f(x^*) \\ &\leq ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^\infty \alpha_{ni}\langle x^*, JT_ix_n\rangle + \alpha_{n0}||Jx_n||^2 + \sum_{i=1}^\infty \alpha_{ni}||JT_ix_n||^2 \\ &\quad -\alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) + 2\rho f(x^*) \\ &= ||x^*||^2 - 2\alpha_{n0}\langle x^*, Jx_n\rangle - 2\sum_{i=1}^\infty \alpha_{ni}\langle x^*, JT_ix_n\rangle + \alpha_{n0}||Jx_n||^2 + \sum_{i=1}^\infty \alpha_{ni}||JT_ix_n||^2 \\ &\quad -\alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) + 2\rho f(x^*) \\ &\leq G(x^*, Jx_n) - \alpha_{n0}\alpha_{nj}g(||Jx_n - JT_jx_n||) \quad (3.4) \\ &\leq G(x^*, Jx_n). \end{aligned}$$

So, $x^* \in C_n$. This implies that $F \subset C_n, \ \forall n \geq 0$.

Now since $\{x_n\}_{n=0}^\infty$ is bounded in C and E is reflexive, we may assume that $x_n \rightharpoonup p$ and since C_n is closed and convex for each $n \geq 0$, it is easy to see that $p \in C_n$ for each $n \geq 0$. Again since $x_n = \Pi^f_{C_n} x_0$, from the definition of $\Pi^f_{C_n}$, we obtain

$$G(x_n, Jx_0) \leq G(p, Jx_0),\ \forall n \geq 0.$$

Since

$$\begin{aligned}\liminf_{n\to\infty} G(x_n, Jx_0) &= \liminf_{n\to\infty} \Big\{||x_n||^2 - 2\langle x_n, Jx_0\rangle + ||x_0||^2 + 2\rho f(x_n)\Big\}\\ &\geq ||p||^2 - 2\langle p, Jx_0\rangle + ||x_0||^2 + 2\rho f(p) = G(p, Jx_0)\end{aligned}$$

then, we obtain

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(p, Jx_0).$$

This implies that $\lim_{n\to\infty} G(x_n, Jx_0) = G(p, Jx_0)$. By Lemma 2.8, we obtain $\lim_{n\to\infty} ||x_n|| = ||p||$. In view of Kadec-Klee property of E, we have that $\lim_{n\to\infty} x_n = p$.

We next show that $p \in \cap_{i=1}^{\infty} F(T_i)$. By the fact that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi^f_{C_{n+1}} x_0 \in C_n$, we obtain

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n).$$

Now, (3.3) implies that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0). \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.5), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0 = \lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

It then yields that $\lim_{n\to\infty} (||x_{n+1}|| - ||y_n||) = 0$. Since $\lim_{n\to\infty} ||x_{n+1}|| = ||p||$, we have

$$\lim_{n\to\infty} ||y_n|| = ||p|| \text{ and } \lim_{n\to\infty} ||Jy_n|| = ||Jp||.$$

This implies that $\{||Jy_n||\}_{n=0}^{\infty}$ is bounded in E^*. Since E is reflexive, and so E^* is reflexive, we can then assume that $Jy_n \rightharpoonup f_0 \in E^*$. In view of reflexivity of E, we see that $J(E) = E^*$. Hence, there exists $x \in E$ such that $Jx = f_0$. Since

$$\begin{aligned}\phi(x_{n+1}, y_n) &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Jy_n\rangle + ||y_n||^2\\ &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Jy_n\rangle + ||Jy_n||^2.\end{aligned} \tag{3.6}$$

Taking the limit inferior of both sides of (3.6) and in view of weak lower semicontinuity of $||.||$, we have

$$\begin{aligned}0 &\geq ||p||^2 - 2\langle p, f_0\rangle + ||f_0||^2 = ||p||^2 - 2\langle p, Jx\rangle + ||Jx||^2\\ &= ||p||^2 - 2\langle p, Jx\rangle + ||x||^2 = \phi(p, x),\end{aligned}$$

that is, $p = x$. This implies that $f_0 = Jp$ and so $Jy_n \rightharpoonup Jp$. It follows from $\lim_{n\to\infty} ||Jy_n|| = ||Jp||$ and Kadec-Klee property of E^* that $Jy_n \to Jp$. Note that $J^{-1} : E^* \to E$ is hemi-continuous, it yields that $y_n \rightharpoonup p$. It then follows from $\lim_{n\to\infty} ||y_n|| = ||p||$ and Kadec-Klee property of E that $\lim_{n\to\infty} y_n = p$. Hence,

$$\lim_{n\to\infty} ||x_n - y_n|| = 0 = \lim_{n\to\infty} ||Jx_n - Jy_n|| = 0.$$

It then follows from (3.4) that

$$\alpha_{n0}\alpha_{nj} g(||Jx_n - JT_j x_n||) \leq G(x^*, Jx_n) - G(x^*, Jy_n).$$

From $\lim_{n\to\infty} ||x_n - y_n|| = 0$ and $\lim_{n\to\infty} ||Jx_n - Jy_n|| = 0$, we can easily show that

$$G(x^*, Jx_n) - G(x^*, Jy_n) \to 0,\ n \to \infty.$$

Using the condition $\liminf_{n\to\infty}\alpha_{n0}\alpha_{nj} > 0$, we have for any $j \geq 1$ that

$$\lim_{n\to\infty} g(||Jx_n - JT_jx_n||) = 0.$$

By property of g, we have $\lim_{n\to\infty}||Jx_n - JT_jx_n|| = 0,\ j \geq 1$. Since $x_n \to p$ and J is uniformly continuous, we have $Jx_n \to Jp$. Now, from $\lim_{n\to\infty}||Jx_n - JT_jx_n|| = 0$, we obtain $\lim_{n\to\infty} JT_jx_n = Jp$. Furthermore, since J^{-1} is hemi-continuous, it follows that $T_jx_n \rightharpoonup p$. On the other hand,

$$\Big|||T_jx_n|| - ||p||\Big| = \Big|||JT_jx_n|| - ||Jp||\Big| \leq ||JT_jx_n - Jp|| \to 0, \text{ as } n \to \infty.$$

By $T_jx_n \rightharpoonup p$, $\lim_{n\to\infty}||T_jx_n|| = ||p||$ and Kadec-Klee property of E, we obtain that $T_jx_n \to p$, as $n \to \infty$, $j \geq 1$. Hence, we have

$$\lim_{n\to\infty}||x_n - T_jx_n|| = 0,\ j \geq 1. \tag{3.7}$$

Since $T_i,\ i \geq 1$ is closed and $x_n \to p$, we have $p \in F = \cap_{i=1}^{\infty}F(T_i)$.

Finally, we show that $p = \Pi_F^f x_0$. Since $F = \cap_{i=1}^{\infty}F(T_i)$ is a closed and convex set, from Lemma 2.4, we know that $\Pi_F^f x_0$ is single valued and denote $w = \Pi_F^f x_0$. Since $x_n = \Pi_{C_n}^f x_0$ and $w \in F \subset C_n$, we have

$$G(x_n, Jx_0) \leq G(w, Jx_0),\ \forall n \geq 0.$$

We know that $G(\xi, J\varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed. This implies that

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(w, Jx_0).$$

From the definition of $\Pi_F^f x_0$ and $p \in F$, we see that $p = w$. This completes the proof. □

Take $f(x) = 0$ for all $x \in E$ in Theorem 3.1, then $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C^f x_0 = \Pi_C x_0$. Then we obtain the following corollary.

Corollary 3.1. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_i\}_{i=1}^{\infty}$ is an infinite family of closed relatively-quasi nonexpansive mappings of C into itself such that $F := \cap_{i=1}^{\infty}F(T_i) \neq \emptyset$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$, $C_0 = C$,*

$$\begin{cases} y_n = J^{-1}(\alpha_{n0}Jx_n + \sum_{i=1}^{\infty}\alpha_{ni}JT_ix_n), \\ C_{n+1} = \{w \in C_n : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0,\ n \geq 0, \end{cases} \tag{3.8}$$

where J is the duality mapping on E. Suppose $\{\alpha_{ni}\}_{n=1}^{\infty}$ for each $i = 0, 1, 2, ...$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty}\alpha_{n0}\alpha_{ni} > 0$, $i = 1, 2, 3, ...$, $\sum_{i=0}^{\infty}\alpha_{ni} = 1$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Corollary 3.2. *Let E be a uniformly smooth and strictly convex real Banach space which also has Kadec-Klee property. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_i\}_{i=1}^{N}$ is a finite family of closed relatively-quasi nonexpansive mappings of C into itself such that $F := \cap_{i=1}^{N}F(T_i) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a convex* *and lower semicontinuous mapping with* $C \subset \operatorname{int}(D(f))$ *and suppose* $\{x_n\}_{n=0}^{\infty}$ *is iteratively generated by* $x_0 \in C,\ C_0 = C,$

$$\begin{cases} y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{N} \alpha_{ni} J T_i x_n), \\ C_{n+1} = \{w \in C_n : G(w, J y_n) \leq G(w, J x_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0,\ n \geq 0, \end{cases} \tag{3.9}$$

where J *is the duality mapping on* E. *Suppose* $\{\alpha_{ni}\}_{n=1}^{\infty}$ *for each* $i = 0, 1, 2, ..., N$ *is a sequence in* $(0, 1)$ *such that* $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0,\ i = 1, 2, 3, ..., N,\ \sum_{i=0}^{N} \alpha_{ni} = 1.$ *Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi^f_F x_0$.

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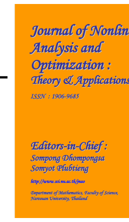

CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF GENERALIZED I-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we consider strong convergence of implicit iteration process to common fixed point for generalized I-asymptotically nonexpansive mappings. The main results extend to finite family of generalized I-asymptotically nonexpansive mappings in a Banach space.

KEYWORDS : I-asymptotically nonexpansive; Implicit iteration process; Common fixed point; Convergence theorem.
AMS Subject Classification: 47H09 47H10.

1. INTRODUCTION

Let K be a nonempty subset of a real normed linear space X and $T : K \to K$ be a mapping. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

We introduce the following definitions and statements which will be used in our main results(see references therein):

A mapping $T : K \longrightarrow K$ is called *nonexpansive* provided

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$ and $n \geq 1$. T is called *asymptotically nonexpansive* mapping if there exist a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim\limits_{n \longrightarrow \infty} \lambda_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda_n)\|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

The class of asymptotically nonexpansive maps which an important generalization of the class nonexpansive maps was introduced by Goebel and Kirk [4]. They proved that every asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space has a fixed point.

T is called *quasi-nonexpansive* mapping provided

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$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in K$ and $p \in F(T)$ and $n \geq 1$.

T is called *asymptotically quasi-nonexpansive* mapping if there exist a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim\limits_{n \longrightarrow \infty} \lambda_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda_n)\|x - p\|$$

for all $x \in K$ and $p \in F(T)$ and $n \geq 1$.

Remark 1.1. From above definitions, it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive.

We introduce the following definitions and statements which will be used in our main results(see [9]-[11]).

Let us recall some notions.

Let $T, I : K \longrightarrow K$. Then T is called *I-nonexpansive* on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\|$$

for all $x, y \in K$.

T is called *I- asymptotically nonexpansive* on K if there exists a sequence $\{\lambda'_n\} \subset [0, \infty)$ with $\lim\limits_{n \longrightarrow \infty} \lambda'_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda'_n)\|I^n x - I^n y\|$$

for all $x, y \in K$ and $n \geq 1$.

T is called *I- asymptotically quasi-nonexpansive* on K if there exists a sequence $\{\lambda'_n\} \subset [0, \infty)$ with $\lim\limits_{n \longrightarrow \infty} \lambda'_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda'_n)\|I^n x - p\|$$

for all $x \in K$ and $p \in F(T) \cap F(I)$ and $n = 1, 2, ...$

Remark 1.2. From the above definitions it follows that if $F(T) \cap F(I)$ is nonempty, a I-nonexpansive mapping must be I-quasi-nonexpansive, and linear I-quasi-nonexpansive mappings are I-nonexpansive mappings. But it is easily seen that there exist nonlinear continuous I-quasi-nonexpansive mappings which are not I-nonexpansive.

Now, we give the definition of the generalized asymptotically quasi-nonexpansive mapping as follows:

Definition 1.3. [7] Let X be a real normed linear space and K a nonempty subset of X. A mapping $T : K \longrightarrow K$ is called *generalized asymptotically quasi-nonexpansive* mapping if $F(T) \neq \emptyset$ and there exist sequences of real numbers $\{u_n\}$, $\{\varphi_n\}$ with $\lim\limits_{n \longrightarrow \infty} u_n = 0 = \lim\limits_{n \longrightarrow \infty} \varphi_n$ such that

$$\|T^n x - p\| \leq \|x - p\| + u_n\|x - p\| + \varphi_n$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

If, in Definition 1.3, $\varphi_n = 0$ for all $n \geq 1$ then T becomes asymptotically quasi-nonexpansive mapping and hence the class of generalized asymptotically quasi-nonexpansive mappings includes the class of asymptotically quasi-nonexpansive mappings.

Now we give generalized I-asymptotically quasi-nonexpansive mappings as follows:

Definition 1.4. Let X be a real normed linear space and K a nonempty subset of X. A mapping $T : K \longrightarrow K$ is called *generalized I-asymptotically quasi-nonexpansive* mapping if $F(T) \cap F(I) \neq \emptyset$ and there exist sequences of real numbers $\{u'_n\}$, $\{\varphi'_n\}$ with $\lim_{n \longrightarrow \infty} u'_n = 0 = \lim_{n \longrightarrow \infty} \varphi'_n$ such that

$$\|T^n x - p\| \leq \|I^n x - p\| + u'_n \|I^n x - p\| + \varphi'_n$$

for all $x \in K$, $p \in F(T) \cap F(I)$ and $n \geq 1$.

Also, if, in Definition 1.4, $\varphi'_n = 0$ for all $n \geq 1$ then T becomes I-asymptotically quasi-nonexpansive mapping and hence the class of generalized I-asymptotically quasi-nonexpansive mappings includes the class of I-asymptotically quasi-nonexpansive mappings.

Recently, concerning the convergence problems of an implicit(or non-implicit) iterative process to a common fixed point for finite family of asymptotically nonexpansive mappings(or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been obtained by a number of authors (see, the references therein).

Xu and Ori [13], in 2001, introduced an implicit iteration process for a finite family of nonexpansive mappings. Let K be a nonempty closed convex subset of $\mathcal{H}$ Hilbert space. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of K such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, the set of common fixed points of T_i, $i = 1, ..., N$. An implicit iteration process for finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ are defined as follows, with $\{\alpha_n\} \subset (0,1)$, and an initial point $x_0 \in K$, the sequence $\{x_n\}_{n \geq 1}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1 \\
x_2 &= \alpha_2 x_1 + (1-\alpha_2) T_2 x_2 \\
&\cdot \\
&\cdot \\
&\cdot \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N \\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_{N+1} x_{N+1} \\
&\cdot \\
&\cdot \\
&\cdot
\end{aligned}$$

The process is expressed in the following form

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n, \quad n \geq 1 \tag{1.1}$$

where $T_n = T_{n(modN)}$.

Xu and Ori [13] proved the weak convergence of the sequence $\{x_n\}$ defined implicity by (1.1) to a common fixed point of the finite family of nonexpansive mappings defined in Hilbert space. Zhou and Chang [14], in 2002, studied the weak and strong convergence of implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. Liu [5], in 2002, and Chidume - Shahzad [2], in 2005, proved the strong convergence of an implicit iteration process to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. Sun [8], in 2003, extended an implicit iteration process for a finite family of nonexpansive mappings due to Xu and Ori [13] to the case of asymptotically quasi-nonexpansive mappings in a setting of Banach spaces. Chang and et al.[1], in 2003, studied the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of asymptotically nonexpansive mappings in Banach spaces. Guo and Cho [3], in 2008, studied the weak and strong convergence of implicit iteration process with errors to a common fixed point for a finite family of nonexpansive mappings in Banach spaces. Shahzad and Zegeye[7], in 2007, studied the strong convergence of implicit iteration process to a common fixed point for a finite family of generalized asymptotically quasi-nonexpansive mappings in Banach spaces. Recently, in [11], the weak convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [9] the weak and strong convergence of implicit iteration process to a common fixed point of a finite family of I-asymptotically nonexpansive mappings were studied. More recently, Temir [10], studied the weak and strong convergence of the explicit iterative process of generalized I-asymptotically quasi-nonexpansive mappings to common fixed point in Banach space.

In this paper, we consider the following implicit iterative process with new type of conception which combines notions such as generalized asymptotically nonexpansive mapping and generalized I-asymptotically nonexpansive mapping. Let K be a nonempty subset of X Banach space.

Let $\{T_i\}_{i=1}^N$ be finite family of generalized I_i- asymptotically nonexpansive self-mappings and $\{I_i\}_{i=1}^N$ be finite family of generalized asymptotically nonexpansive self-mappings on K. $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. Then, an initial point $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$
\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1[\beta_1 x_1 + (1-\beta_1)I_1 x_1],\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2[\beta_2 x_2 + (1-\beta_2)I_2 x_2],\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N[\beta_N x_N + (1-\beta_N)I_N x_N],\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2[\beta_{N+1} x_{N+1} + (1-\beta_{N+1})I_1^2 x_{N+1}],\\
&\vdots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2[\beta_{2N} x_{2N} + (1-\beta_{2N})I_N^2 x_{2N}],\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3[\beta_{2N+1} x_{2N+1} + (1-\beta_{2N+1})I_1^3 x_{2N+1}]\\
&\vdots
\end{aligned}
$$

Let $x_0 \in K$ be any given point, the implicitly iterative sequence x_n generated by (1.2) should be written in the following compact form:

$$
\begin{cases}
y_n = \beta_n x_n + (1-\beta_n) I_{i(n)}^{k(n)} x_n; \\
x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} y_n,
\end{cases}
\tag{1.2}
$$

$\forall n \geq 1$, where $n = (k(n)-1)N + i(n), i(n) \in \{1,2,...,N\}, k(n) \geq 1$.

If we take $\{I_i\}_{i=1}^N$ identity mappings and $\forall n \geq 1, \beta_n = 0$ then the compact form induces (1.1) implicit iteration process defined in Xu and Ori [13].

The aim of this paper is to prove the strong convergence of implicit iterative sequence $\{x_n\}_{n\geq 1}$ defined by (1.2) to common fixed point for finite family of generalized I_i-asymptotically nonexpansive mappings in Banach space. We consider also $\{I_i\}_{i=1}^N$ be finite family of generalized asymptotically nonexpansive self-mappings of K subset of Banach space. Our results will thus improve and generalize corresponding results of [13], [7], [10] and [9].

2. PRELIMINARIES AND NOTATIONS

In order to prove the main results of this paper, we need the following statements:

Lemma 2.1. [12] *Let $\{a_n\}$, $\{b_n\}$ and $\{\kappa_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: $\forall n \geq 1$, $a_{n+1} \leq (1+\kappa_n)a_n + b_n$, where $\sum\limits_{n=0}^{\infty} \kappa_n < \infty$ and $\sum\limits_{n=0}^{\infty} b_n < \infty$. Then $\lim\limits_{n\longrightarrow\infty} a_n$ exists.*

Lemma 2.2. [6] *Let K be a nonempty closed bounded convex subset of a uniformly convex Banach space X and $\{\alpha_n\}$ a sequence $[\delta, 1-\delta]$, for some $\delta \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in K such that*

$$\limsup_{n\longrightarrow\infty} \|x_n\| \leq d,$$

$$\limsup_{n\longrightarrow\infty} \|y_n\| \leq d$$

and

$$\limsup_{n\longrightarrow\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = d$$

holds for some $d \geq 0$.Then

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0.$$

Let X be a uniformly convex Banach space and K a nonempty, closed and subset of X. Let $\{T_i : i \in \{1,...,N\}\}$ be N generalized I_i-asymptotically nonexpansive self-mappings of K with sequences of real numbers $\{\theta_{in}\}$, $\{\varphi_{in}\} \subset [0,\infty)$ and $\theta_{in},\ \ \varphi_{in} \to 0$ as $n \to \infty$ such that $\|T_i^k x - T_i^k y\| \leq (1+\theta_{in})\|I_i^k x - I_i^k y\| + \varphi_{in}$ for all $x, y \in K$ and $n \geq 1$ and $\{I_i : i \in \{1,...,N\}\}$ be N generalized asymptotically nonexpansive mappings of K with $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0,\infty)$ and $\tau_{in},\ \ \psi_{in} \to 0$ as $n \to \infty$ such that $\|I_i^k x - I_i^k y\| \leq \tau_{in}\|x-y\| + \psi_{in}$ for all $x, y \in K$, for each $i = 1,...N$ and $n \geq 1$.

Letting $\nu_n = \max\{\theta_{in}, \tau_{in}\}$ for all $i \in \{1,...,N\}$, $\nu_n \subset [0,\infty)$, with $\lim\limits_{k\longrightarrow\infty} \nu_n = 0$, $\sum\limits_{n=1}^{\infty} \nu_n < \infty$, also $\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ for all $i \in \{1,...,N\}$, $\phi_n \subset [0,\infty)$, with $\lim\limits_{n\longrightarrow\infty} \phi_n = 0$, $\sum\limits_{n=1}^{\infty} \phi_n < \infty$. Then there exist nonnegative real sequences $\{\nu_n\}$ and $\{\phi_n\}$ with $\nu_n,\ \ \phi_n \to 0$ as $n \to \infty$ such that

$$\|T_i^k x - T_i^k y\| \leq (1+\theta_{in})\|I_i^k x - I_i^k y\| + \varphi_{in} \leq (1+\nu_n)^2\|x-y\| + (2+\nu_n)\phi_n,$$

$$\|I_i^k x - I_i^k y\| \leq (1+\tau_{in})\|x-y\| + \psi_{in} \leq (1+\nu_n)\|x-y\| + \phi_n$$

for all $x, y \in K$,for each $i = 1,...N$ and $n \geq 1$.

Let denote the distance of x to set $F \subset K$, i.e., $d(x, F) = \inf\{\|x-p\| : p \in F\}$. A mapping $T : K \to K$ is said to *semi-compact* if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \longrightarrow \infty$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$

such that $\{x_{n_i}\} \to p \in K$.

The mappings $T, I : K \to K$ are said to satisfying condition *(A)* if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ such that $\frac{1}{2}(\|x - Tx\| + \|x - Ix\|) \geq f(d(x,F))$ for all $x \in K$, where $d(x,F) = \inf\{\|x - p\| : p \in F = F(T) \cap F(I)\}$.

A family $\{T_i : i \in \{1,...,N\}\}$ be N generalized I_i-asymptotically nonexpansive self-mappings of K and $\{I_i : i \in \{1,...,N\}\}$ be N generalized asymptotically nonexpansive mappings on K with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$ are said to satisfy condition *(B)* on K if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$, for all $r \in [0,\infty)$ and all $x \in K$ such that $\max_{1\leq \ell \leq N}\{\frac{1}{2}(\|x - T_\ell x\| + \|x - I_\ell x\|)\} \geq f(d(x,F))$ for at least one T_ℓ and I_ℓ, $\ell = \{1,...,N\}$.

3. STRONG CONVERGENCE OF IMPLICIT ITERATION FOR GENERALIZED I-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in \{1,...,N\}\}$ be N generalized I_i-asymptotically nonexpansive self-mappings of K with sequences of real numbers $\{\theta_{in}\}, \{\varphi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \theta_{in} < \infty$, $\sum_{n=1}^{\infty} \varphi_{in} < \infty$ and $\{I_i : i \in \{1,...,N\}\}$ be N generalized asymptotically nonexpansive mappings of K with $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \tau_{in} < \infty$ and $\sum_{n=1}^{\infty} \psi_{in} < \infty$. Letting $\nu_n = \max\{\theta_{in}, \tau_{in}\}$ for all $i \in \{1,...,N\}$, $\nu_n \subset [0,\infty)$, with $\lim_{k\longrightarrow\infty} \nu_n = 0$, $\sum_{n=1}^{\infty} \nu_n < \infty$, also $\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ for all $i \in \{1,...,N\}$, $\phi_n \subset [0,\infty)$, with $\lim_{n\longrightarrow\infty} \phi_n = 0$, $\sum_{n=1}^{\infty} \phi_n < \infty$. Let $z \in K$ be fixed and $\alpha, \beta \in (0,1)$.

Define $W : K \to K$

$$W(x) = \alpha z + (1-\alpha) T_{i(n)}^{k(n)}[\beta x + (1-\beta) I_{i(n)}^{k(n)} x]. \tag{3.1}$$

Then

$$\begin{aligned}
\|W(x) - W(y)\| &= \|\alpha z + (1-\alpha) T_{i(n)}^{k(n)}[\beta x + (1-\beta) I_{i(n)}^{k(n)} x] \\
&\quad -[\alpha z + (1-\alpha) T_{i(n)}^{k(n)}[\beta y + (1-\beta) I_{i(n)}^{k(n)} y]\| \\
&\leq (1-\alpha)\Big[(1+\nu_n)^2 \|\beta(x-y) \\
&\quad +(1-\beta)(I_{i(n)}^{k(n)} x - I_{i(n)}^{k(n)} y)\| + (2+\nu_n)\phi_n\Big] \\
&\leq (1-\alpha)\Big[(1+\nu_n)^2 \beta \|x-y\| \\
&\quad +(1+\nu_n)^2(1-\beta)\|I_{i(n)}^{k(n)} x - I_{i(n)}^{k(n)} y\| + (2+\nu_n)\phi_n\Big] \\
&\leq (1-\alpha)\Big[(1+\nu_n)^2 \beta \|x-y\| + (1+\nu_n)^3(1-\beta)\|x-y\| \\
&\quad +(1+\nu_n)^2(1-\beta)\phi_n + (2+\nu_n)\phi_n\Big]
\end{aligned}$$

$$\begin{aligned} &\le (1-\alpha)\Big[(1+\nu_n)^3\|x-y\| + (1+\nu_n)^2(1-\beta)\phi_n + (2+\nu_n)\phi_n\Big] \\ &\to (1-\alpha)\|x-y\| \ \ (n\to\infty). \end{aligned}$$

Thus, there exists a positive integer N_0 such that $\|W(x)-W(y)\| \le (1-\alpha)\|x-y\|$ for all $n \ge N_0$. Since $1-\alpha < 1$, then W is a contraction. By Banach contraction mapping principal, there exists a unique fixed point in K satisfiying the equation (3.1). This implies the implicit iterative process (1.2) is well defined.

Lemma 3.1. *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in \{1,...,N\}\}$ be N generalized I_i-asymptotically nonexpansive self-mappings of K with sequences of real numbers $\{\theta_{in}\}$, $\{\varphi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}\theta_{in} < \infty$, $\sum_{n=1}^{\infty}\varphi_{in} < \infty$ and $\{I_i : i \in \{1,...,N\}\}$ be N generalized asymptotically nonexpansive mappings of K with $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}\tau_{in} < \infty$ and $\sum_{n=1}^{\infty}\psi_{in} < \infty$. Let be $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)\cap F(I_i) \neq \emptyset$.*

(1) *$\nu_n = \max\{\theta_{in}, \tau_{in}\}$ for all $i \in \{1,...,N\}$, $\nu_n \subset [0,\infty)$, with $\lim_{n\longrightarrow\infty}\nu_n = 0$, also $\sum_{n=1}^{\infty}\nu_n < \infty$,*

(2) *$\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ for all $i \in \{1,...,N\}$, $\phi_n \subset [0,\infty)$, with $\lim_{n\longrightarrow\infty}\phi_n = 0$, also $\sum_{n=1}^{\infty}\phi_n < \infty$,*

(3) *$\{\alpha_n\}$ and $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$.*

Then the implicitly iterative sequence $\{x_n\}$ is generated by (1.2) converges to a common fixed point in $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)\cap F(I_i) \neq \emptyset$ if and only if

$$\liminf_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0.$$

Proof. The necessity is obvious and so it is omitted.

Now, we prove the sufficiency. From (1.2), we have for any $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i)\cap F(I_i) \neq \emptyset$,

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)}y_n - p\| \\ &\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\|T_{i(n)}^{k(n)}y_n - p\| \\ &\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\theta_{in})\|I_{i(n)}^{k(n)}y_n - p\| + \varphi_{in}\Big) \\ &\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)\|I_{i(n)}^{k(n)}y_n - p\| + \phi_n\Big) \\ &\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)(1+\tau_{in})\|y_n - p\| + (1+\tau_{in})\psi_{in}\Big) \\ &\quad +(1-\alpha_n)\phi_n \\ &\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)^2\|y_n - p\| + (1+\nu_n)\psi_{in}\Big) \\ &\quad +(1-\alpha_n)\phi_n \\ &\le \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)^2\|y_n - p\| + (1+\nu_n)\phi_n\Big) \\ &\quad +(1-\alpha_n)\phi_n \end{aligned}$$

$$\leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)^2\|y_n-p\| + (2+\nu_n)\phi_n\Big)$$

which implies that

$$\|x_n-p\| \leq \alpha_n\|x_{n-1}-p\| + (1-\alpha_n)\Big((1+\nu_n)^2\|y_n-p\| + (2+\nu_n)\phi_n\Big). \tag{3.2}$$

$$\begin{aligned}
\|y_n-p\| &= \|\beta_n x_n + (1-\beta_n)I_{i(n)}^{k(n)}x_n - p\| \\
&\leq \beta_n\|x_n-p\| + (1-\beta_n)\|I_{i(n)}^{k(n)}x_n - p\| \\
&\leq \beta_n\|x_n-p\| + (1-\beta_n)\Big((1+\tau_{in})\|x_n-p\| + \psi_{in}\Big) \\
&\leq \beta_n\|x_n-p\| + (1-\beta_n)\Big((1+\nu_n)\|x_n-p\| + \phi_n\Big) \\
&\leq (1+\nu_n)\|x_n-p\| + (1-\beta_n)\phi_n. \qquad (3.3)
\end{aligned}$$

Substituting (3.3) into (3.2), we obtain

$$\begin{aligned}
\|x_n-p\| \leq\ & \alpha_n\|x_{n-1}-p\| \\
&+(1-\alpha_n)\Big((1+\nu_n)^3\|x_n-p\| + (1-\beta_n)(1+\nu_n)^2\phi_n + (2+\nu_n)\phi_n\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
1-[(1-\alpha_n)(1+\nu_n)^3]\|x_n-p\| \leq\ & \alpha_n\|x_{n-1}-p\| \\
&+(1-\alpha_n)\Big((1-\beta_n)(1+\nu_n)^2\phi_n + (2+\nu_n)\phi_n\Big).
\end{aligned}$$

Then we get

$$\begin{aligned}
\|x_n-p\| &\leq \frac{\alpha_n}{1-[(1-\alpha_n)(1+\nu_n)^3]}\|x_{n-1}-p\| \\
&\quad +(1-\alpha_n)\Big((1-\beta_n)(1+\nu_n)^3\phi_n + (2+\nu_n)\phi_n\Big) \\
&= \Big[1+\frac{(1-\alpha_n)[(1+\nu_n)^3-1]}{1-[(1-\alpha_n)(1+\nu_n)^3]}\Big]\|x_{n-1}-p\| \\
&\quad +\frac{(1-\alpha_n)\Big((1-\beta_n)(1+\nu_n)^2\phi_n + (2+\nu_n)\phi_n\Big)}{1-[(1-\alpha_n)(1+\nu_n)^3]}.
\end{aligned} \tag{3.4}$$

We assume that $(1+\nu_n) \leq \sqrt[3]{(1+\frac{\delta}{2(1-\delta)})}$ for some $n \geq n_0$ and $\lambda < \frac{1}{\delta}$. Then we can write $1-[(1-\alpha_n)(1+\nu_n)^3] \geq \frac{\delta}{2}$, $\forall n \geq 1$. Then (3.4) becomes

$$\begin{aligned}
\|x_n-p\| &\leq \Big[1+\frac{2(1-\delta)[(\lambda^2+\lambda+1)(1+\nu_n-1)]}{\delta}\Big]\|x_{n-1}-p\| \\
&+ 2\frac{(1-\delta)\Big((1-\delta)(1+\frac{\delta}{2(1-\delta)})\phi_n + (2+\nu_n)\phi_n\Big)}{\delta} \\
&\leq (1+\kappa_{ik})\|x_{n-1}-p\| + 2\frac{(1-\delta)(\phi_n+(2+\nu_n)\phi_n)}{\delta},
\end{aligned} \tag{3.5}$$

where $\kappa_n = \Big[\frac{2(1-\delta)[(\lambda^2+\lambda+1)(\nu_n)]}{\delta}\Big]$ and $\Psi_{in} = 2\frac{(1-\delta)(3+\nu_n)\phi_n}{\delta}$. Moreover, from the condition (1) and (2), since $\sum_{n=1}^{\infty}\nu_n < \infty$ and $\sum_{n=1}^{\infty}\phi_n < \infty$, it follows that $\sum_{n=1}^{\infty}\kappa_n < \infty$

and $\sum_{n=1}^{\infty} \Psi_{in} < \infty$. Thus we obtain

$$\|x_n - p\| \leq (1 + \kappa_n)\|x_{n-1} - p\| + \Psi_{in}. \tag{3.6}$$

By Lemma 2.1, $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists for each $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$. By assumption $\liminf_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0$, we obtain

$$\lim_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. Notice that $1 + z \leq \exp(z)$ for all $z > 0$. From (3.6), for any $p \in \mathcal{F}$, we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq \exp\Big(\sum_{j=n}^{n+m-1} \kappa_j\Big)\|x_n - p\| + \exp\Big(\sum_{j=n}^{n+m-1} \kappa_j\Big)\Big(\sum_{j=n}^{n+m-1} \Psi_j\Big) \\ &\leq M\|x_n - p\| + M\Big(\sum_{j=1}^{\infty} \Psi_j\Big) \end{aligned}$$

for all natural numbers m, n, where $M = \exp\{\sum_{j=1}^{\infty} \kappa_j\} < +\infty$. Since $\lim_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0$, for any given $\epsilon > 0$, there exists a positive integer N_0 such that for all $n \geq N_0$, $d(x_n, \mathcal{F}) < \frac{\epsilon}{4M}$ and $\sum_{n=N_0}^{\infty} \Psi_n < \frac{\epsilon}{4M}$. There exists $p_1 \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$ such that $\|x_{N_0} - p_1\| < \frac{\epsilon}{4M}$.
Hence, for all $n \geq N_0$ and $m \geq 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\leq M\|x_{N_0} - p_1\| + M\Big(\sum_{n=N_0}^{\infty} \Psi_n\Big) + M\|x_{N_0} - p_1\| + M\Big(\sum_{n=N_0}^{\infty} \Psi_n\Big) \\ &\leq 2M\Big(\|x_{N_0} - p_1\| + \Big(\sum_{n=N_0}^{\infty} \Psi_n\Big)\Big) \\ &\leq 2M\Big(\frac{\epsilon}{4M} + \frac{\epsilon}{4M}\Big) = \epsilon \end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence in K.
Thus, the completeness of X implies that $\{x_n\}$ is convergent. Assume that $\{x_n\}$ converges to a point p.

Then $p \in K$, because K is closed subset of X. The set $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$ is closed. $\lim_{n\longrightarrow\infty} d(x_n, \mathcal{F}) = 0$ gives that $d(p, \mathcal{F}) = 0$.

Thus $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$. This completes the proof.

□

Lemma 3.2. *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in \{1, ..., N\}\}$ be N generalized I_i-asymptotically nonexpansive self-mappings of K with sequences of real numbers $\{\theta_{in}\}$, $\{\varphi_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_{in} < \infty$, $\sum_{n=1}^{\infty} \varphi_{in} < \infty$ and $\{I_i : i \in \{1, ..., N\}\}$ be N generalized* *asymptotically nonexpansive mappings of K with sequences $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \tau_{in} < \infty$ and $\sum_{n=1}^{\infty} \psi_{in} < \infty$. Let be $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$.*

(1) *$\nu_n = \max\{\theta_{in}, \tau_{in}\}$ for all $i \in \{1,...,N\}$, $\nu_n \subset [0,\infty)$, with $\lim_{n\longrightarrow\infty} \nu_n = 0$, also $\sum_{n=1}^{\infty} \nu_n < \infty$,*

(2) *$\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ for all $i \in \{1,...,N\}$, $\phi_n \subset [0,\infty)$, with $\lim_{n\longrightarrow\infty} \phi_n = 0$, also $\sum_{n=1}^{\infty} \phi_n < \infty$,*

(3) *$\{\alpha_n\}$ and $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$.*

Suppose that for any given $x \in K$, the sequence $\{x_n\}$ is generated by (1.2). Then

$$\lim_{n\longrightarrow\infty} \|T_\ell x_n - x_n\| = \lim_{n\longrightarrow\infty} \|I_\ell x_n - x_n\| = 0, \forall \ell = 1,2,...,N.$$

Proof. By Lemma 3.1, we can assume that $\lim_{n\longrightarrow\infty} \|x_n - p\| = d$ for all $p \in \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i)$.

Taking $\limsup_{n\longrightarrow\infty}$ on both sides in (3.3) inequality,

$$\limsup_{n\longrightarrow\infty} \|y_n - p\| \leq d. \tag{3.7}$$

Since $\{I_\ell : \ell \in \{1,...,N\}\}$ is N generalized asymptotically nonexpansive self-mappings of K, we can get that,

$$\|T_{i(n)}^{k(n)} y_n - p\| \leq (1+\nu_n)\|I_{i(n)}^{k(n)} y_n - p\| + \phi_n \leq (1+\nu_n)^2\|y_n - p\| + (2+\nu_n)\phi_n,$$

which on taking $\limsup_{n\longrightarrow\infty}$ and using (3.7) gives

$$\limsup_{n\longrightarrow\infty} \|T_{i(n)}^{k(n)} y_n - p\| \leq d.$$

Further,

$$\lim_{n\longrightarrow\infty} \|x_n - p\| = d$$

means that

$$\lim_{n\longrightarrow\infty} \|\alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} y_n - p\| = d,$$

$$\lim_{n\longrightarrow\infty} \|\alpha_n (x_{n-1} - p) + (1-\alpha_n)(T_{i(n)}^{k(n)} y_n - p)\| = d.$$

It follows from Lemma 2.2

$$\lim_{n\longrightarrow\infty} \|T_{i(n)}^{k(n)} y_n - x_{n-1}\| = 0. \tag{3.8}$$

Moreover,

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|(1-\alpha_n)[T_{i(n)}^{k(n)} y_n - x_{n-1}]\| \\ &\leq (1-\alpha_n)\|T_{i(n)}^{k(n)} y_n - x_{n-1}\|. \end{aligned}$$

Thus , from (3.8) we have

$$\lim_{n\longrightarrow\infty} \|x_n - x_{n-1}\| = 0 \tag{3.9}$$

and

$$\lim_{n\longrightarrow\infty} \|x_n - x_{n+j}\| = 0, \forall j = 1,...,N. \tag{3.10}$$

Now,

$$\begin{aligned}\|x_{n-1}-p\| &\leq \|x_{n-1}-T_{i(n)}^{k(n)}y_n\|+\|T_{i(n)}^{k(n)}y_n-p\| \\ &\leq \|x_{n-1}-T_{i(n)}^{k(n)}y_n\|+(1+\nu_n)^2\|y_n-p\|+(2+\nu_n)\phi_n\end{aligned}$$

which on taking $\lim_{n\longrightarrow\infty}$ implies

$$\begin{aligned}d=\lim_{n\longrightarrow\infty}\|x_{n-1}-p\| &\leq \limsup_{n\longrightarrow\infty}(\|x_{n-1}-T_{i(n)}^{k(n)}y_n\|+(1+\nu_n)^2\|y_n-p\| \\ &\quad +(2+\nu_n)\phi_n) \\ &\leq \limsup_{n\longrightarrow\infty}\|y_n-p\|\leq d.\end{aligned}$$

Then we have

$$\limsup_{n\longrightarrow\infty}\|y_n-p\|=d.$$

Next,

$$\|I_{i(n)}^{k(n)}x_n-p\| \quad\leq\quad (1+\nu_n)\|x_n-p\|+\phi_n.$$

Taking $\lim_{n\longrightarrow\infty}$ on both sides in the above inequality, we have

$$\lim_{n\longrightarrow\infty}\|I_{i(n)}^{k(n)}x_n-p\|\leq\lim_{n\longrightarrow\infty}\|x_n-p\|=d.$$

Further,

$$\lim_{n\longrightarrow\infty}\|\beta_n(x_n-p)+(1-\beta_n)(I_{i(n)}^{k(n)}x_n-p)\|=\lim_{n\longrightarrow\infty}\|y_n-p\|=d.$$

By Lemma 2.2, we have

$$\lim_{n\longrightarrow\infty}\|I_{i(n)}^{k(n)}x_n-x_n\|=0. \tag{3.11}$$

We have also,

$$\begin{aligned}\|T_{i(n)}^{k(n)}x_n-x_n\| &\leq \|T_{i(n)}^{k(n)}x_n-T_{i(n)}^{k(n)}y_n\|+\|T_{i(n)}^{k(n)}y_n-x_n\| \\ &\leq (1+\nu_n)^2\|x_n-y_n\|+\|T_{i(n)}^{k(n)}y_n-x_n\|+(2+\nu_n)\phi_n \\ &= (1+\nu_n)^2\|x_n-[\beta_nx_n+(1-\beta_n)I_{i(n)}^{k(n)}x_n]\| \\ &\quad +\|T_{i(n)}^{k(n)}y_n-x_n\|+(2+\nu_n)\phi_n \\ &= (1+\nu_n)^2\|(1-\beta_n)(x_n-I_{i(n)}^{k(n)}x_n)\|+\|T_{i(n)}^{k(n)}y_n-x_n\| \\ &\quad +(2+\nu_n)\phi_n \\ &= (1+\nu_n)^2(1-\beta_n)\|I_{i(n)}^{k(n)}x_n-x_n\| \\ &\quad +\|T_{i(n)}^{k(n)}y_n-x_{n-1}+x_{n-1}-x_n\|+(2+\nu_n)\phi_n \\ &\leq (1+\nu_n)^2(1-\beta_n)\|I_{i(n)}^{k(n)}x_n-x_n\|+\|T_{i(n)}^{k(n)}y_n-x_{n-1}\| \\ &\quad +\|x_n-x_{n-1}\|+(2+\nu_n)\phi_n.\end{aligned}$$

Taking $\lim_{n\longrightarrow\infty}$ on both sides in the above inequality, from (3.8), (3.9) and (3.11), we obtain

$$\lim_{n\longrightarrow\infty}\|T_{i(n)}^{k(n)}x_n - x_n\| = 0. \tag{3.12}$$

Now we prove that

$$\lim_{n\longrightarrow\infty}\|T_\ell x_n - x_n\| = \lim_{n\longrightarrow\infty}\|I_\ell x_n - x_n\| = 0, \forall \ell = 1, 2, ..., N$$

holds. In fact, since for each $n > N$,$n = (n-N)(modN)$ and $n = (k(n)-1)N+i(n)$, hence $n - N = ((k(n) - 1) - 1)N + i(n) = (k(n - N) - 1)N + i(n - N)$, that is, $k(n - N) = k(n) - 1$ and $i(n - N) = i(n)$.
From (3.11),

$$\begin{aligned}
\|x_n - I_n x_n\| &\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + \|I_{i(n)}^{k(n)}x_n - I_n x_n\| \\
&\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + (1+\nu_n)\|I_{i(n)}^{k(n)-1}x_n - x_n\| + \phi_n \\
&\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + (1+\nu_n)(\|I_{i(n)}^{k(n)-1}x_n - I_{i(n-N)}^{k(n)-1}x_{n-N}\| \\
&\quad + \|I_{i(n-N)}^{k(n)-1}x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|) + \phi_n \\
&\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + (1+\nu_n)^2\|x_n - x_{n-N}\| \\
&\quad + (1+\nu_n)\|I_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\| + (1+\nu_n)\|x_{n-N} - x_n\| \\
&\quad + (2+\nu_n)\phi_n \\
&\leq \|x_n - I_{i(n)}^{k(n)}x_n\| + (1+\nu_n)^2\|x_n - x_{n-N}\| \\
&\quad + (1+\nu_n)\|I_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\| + (2+\nu_n)\phi_n \to 0 \ \ (n\to\infty).
\end{aligned}$$

This implies that

$$\lim_{n\longrightarrow\infty}\|I_n x_n - x_n\| = 0. \tag{3.13}$$

Then we also have from (3.11) and (3.12)

$$\begin{aligned}
\|x_n - T_n x_n\| &\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + \|T_{i(n)}^{k(n)}x_n - T_n x_n\| \\
&\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + (1+\nu_n)\|I_{i(n)}^{k(n)}x_n - I_n x_n\| + \phi_n \\
&\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + (1+\nu_n)(\|I_{i(n)}^{k(n)-1}x_n - I_{i(n-N)}^{k(n)-1}x_{n-N}\| \\
&\quad + \|I_{i(n-N)}^{k(n)-1}x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|) + \phi_n \\
&\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + (1+\nu_n)^2\|x_n - x_{n-N}\| \\
&\quad + (1+\nu_n)\|I_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\| + (1+\nu_n)\|x_{n-N} - x_n\| \\
&\quad + (2+\nu_n)\phi_n \\
&\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + \Big[(1+\nu_n)^2 + (1+\nu_n)\Big]\|x_n - x_{n-N}\| \\
&\quad + (1+\nu_n)\|I_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\| + (2+\nu_n)\phi_n \to 0 \ \ (n\to\infty).
\end{aligned}$$

This implies that

$$\lim_{n\longrightarrow\infty}\|T_n x_n - x_n\| = 0. \tag{3.14}$$

Now for all $\ell = \{1, ..., N\}$.

$\|x_n - T_{n+\ell}x_n\| \leq \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - T_{n+\ell}x_{n+\ell}\| + \|T_{n+\ell}x_{n+\ell} - T_{n+\ell}x_n\|$.

Taking $\lim_{n\longrightarrow\infty}$ on both sides in the above inequality, then we get

$$\lim_{n\longrightarrow\infty} \|x_n - T_{n+\ell}x_n\| = 0$$

for all $\ell = \{1, ..., N\}$.
Thus, we have

$$\lim_{n\longrightarrow\infty} \|x_n - T_\ell x_n\| = 0. \tag{3.15}$$

$\|x_n - I_{n+\ell}x_n\| \leq \|x_n - x_{n+\ell}\| + \|x_{n+\ell} - I_{n+\ell}x_{n+\ell}\| + \|I_{n+\ell}x_{n+\ell} - I_{n+\ell}x_n\|$.

Taking $\lim_{n\longrightarrow\infty}$ on both sides in the above inequality, then we get

$$\lim_{n\longrightarrow\infty} \|x_n - I_{n+\ell}x_n\| = 0$$

for all $\ell = \{1, ..., N\}$.
Thus , we have

$$\lim_{n\longrightarrow\infty} \|x_n - I_\ell x_n\| = 0. \tag{3.16}$$

Then the proof is completed. □

Theorem 3.3. *Let X be a uniformly convex Banach space, K be a nonempty closed convex subset of X, $\{T_i : i \in \{1, ..., N\}\}$ be N generalized I_i-asymptotically nonexpansive self-mappings of K with sequences of real numbers $\{\theta_{in}\}$, $\{\varphi_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \theta_{in} < \infty$, $\sum_{n=1}^{\infty} \varphi_{in} < \infty$ and $\{I_i : i \in \{1, ..., N\}\}$ be N generalized asymptotically nonexpansive mappings of K with sequences $\{\tau_{in}\}$, $\{\psi_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \tau_{in} < \infty$ and $\sum_{n=1}^{\infty} \psi_{in} < \infty$.*

1. $\nu_n = \max\{\theta_{in}, \tau_{in}\}$ *for all* $i \in \{1, ..., N\}$, $\nu_n \subset [0, \infty)$, *with* $\lim_{n\longrightarrow\infty} \nu_n = 0$, *also* $\sum_{n=1}^{\infty} \nu_n < \infty$,
2. $\phi_n = \max\{\varphi_{in}, \psi_{in}\}$ *for all* $i \in \{1, ..., N\}$, $\phi_n \subset [0, \infty)$, *with* $\lim_{n\longrightarrow\infty} \phi_n = 0$, *also* $\sum_{n=1}^{\infty} \phi_n < \infty$,
3. $\{\alpha_n\}$ *and* $\{\beta_n\} \subset [\delta, 1 - \delta]$ *for some* $\delta \in (0, 1)$.

Let be $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \cap F(I_i) \neq \emptyset$. Suppose that one of the mappings $\{T_i : i \in \{1, ..., N\}\}$ and one of the mappings $\{I_i : i \in \{1, ..., N\}\}$ are semi-compact or satisfy condition (B). Then the implicit iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to a common fixed point of $\{T_i : i \in \{1, ..., N\}\}$ and $\{I_i : i \in \{1, ..., N\}\}$.

Proof. Without loss of generality, we can assume that $\{T_1\}$ and $\{I_1\}$ are semi-compact or satisfy condition *(B)*. It follows from (3.15) and (3.16) in Lemma 3.2 $\lim_{n\longrightarrow\infty} \|x_n - T_1x_n\| = 0 = \lim_{n\longrightarrow\infty} \|x_n - I_1x_n\| = 0$ By semi-compactness of $\{T_1\}$ and

$\{I_1\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \to p \in K$ strongly as $j \longrightarrow \infty$. From (3.15) and (3.16) in Lemma 3.2

$$\lim_{n \longrightarrow \infty} \|x_n - T_\ell x_n\| = \|p - T_\ell p\|$$

for all $\ell \in \{1, ..., N\}$, and

$$\lim_{n \longrightarrow \infty} \|x_n - I_\ell x_n\| = \|p - I_\ell p\|.$$

for all $\ell \in \{1, ..., N\}$. This implies that $p \in \mathcal{F}$. Since $\liminf_{n \longrightarrow \infty} d(x_n, \mathcal{F}) = 0$, Lemma 3.1 guarantees that $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$. If $\{T_1\}$ and $\{I_1\}$ satisfy condition *(B)*, then we have $\liminf_{n \longrightarrow \infty} d(x_n, \mathcal{F}) = 0$. From Lemma 3.1, we have that $\{x_n\}$ converges to a common fixed point in $\mathcal{F}$. This completes the proof. □

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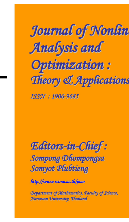

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A STUDY OF NON-ATOMIC MEASURES AND INTEGRALS ON EFFECT ALGEBRAS

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ABSTRACT. The present paper deals with the study of superior variation m^+, inferior variation m^- and total variation $|m|$ of an extended real-valued function m defined on an effect algebra L. Various properties in the context of functions m^+, m^- and $|m|$ are also established. Using the notion of an atom of a real-valued function, we have proved Intermediate value theorem for a non-atomic function m defined on a D-lattice L under suitable conditions. Finally, the notion of the integral for a bounded, real valued function with respect to a measure on effect algebras with Reisz decomposition property is introduced and studied.

KEYWORDS : Effect algebra; Superior variation; Inferior variation; Total variation; Jordan type decomposition theorem; m-Atoms; Non-atomic measure; Intermediate value theorem; Reisz decomposition property; μ-Integrable
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1. INTRODUCTION

If a quantum mechanical system $\mathcal{F}$ is represented in the usual way by a Hilbert space $\mathcal{H}$, then a self adjoint operator A on $\mathcal{H}$ such that $0 \leq A \leq I$ corresponds to an effect for $\mathcal{F}$ [19, 20, 29]. Effects are of significance in representing unsharp measurements or observations on the system $\mathcal{F}$ [4], and effect valued measures play an important role in stochastic quantum mechanics [1, 30]. As a consequence, there have been a number of recent efforts to establish appropriate axioms for logics, algebras, or posets suggested by or based on effects [13, 14]

In 1992, Kôpka defined D-posets of fuzzy sets in [13], which is closed under the formations of differences of fuzzy sets, while studying the axiomatical systems of fuzzy sets. A generalization of such structures of fuzzy sets to an abstract partially ordered set, where the basic operation is the difference, yields a very general and, at the same time, a very simple structure called a D-poset. A common generalization of orthomodular lattices and MV-algebras is termed as lattice ordered effect algebras introduced by Bennett and Foulis [4, 5] in 1994, while working on quantum

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mechanical systems. Such structures are being frequently used because of their wide range of applications in quantum physics, mathematical economics and fuzzy theory. For a list of nice examples of effect algebras, we refer to [8] and for some of its properties we refer also [5] and [6]. The equivalence of D-posets and effect algebras is proved by Foulis and Bennett [4] and independently by Pulmannová [25].

The decomposability of a vector measure was first studied by Rickart in 1943 [26], where he established a Lebesgue decomposition theorem for the class of "strongly bounded" additive measures. This result was later re-established (although it was not realized at that time) by Uhl. Jr. [30], who also presented a Yosida-Hewitt decomposition theorem for "strongly bounded" measures. Several Jordan type decomposition theorems are exhibited by Diestal and Faires in [7]. Afterwards, Faires and Morrison [9] exposed conditions on a vector valued measures that ensure vector valued Jordan type decomposition theorem to hold. A Jordan type decomposition theorem for vector measures, defined on an algebra of of sets, with values in an order complete Banach lattice is proved by Schmidt [27]. Upto slight modification, the result of [28]. extends to the case where domain of the vector measure is a ring of sets. It is also possible to give a common approach to vector measures on a Boolean ring and linear operators on a vector lattice. A first step in this direction was done in [27], where real-valued case was studied. The method presented there is based on a common abstraction of Boolean rings and lattice ordered groups. This approach can be refined and fitted to the vector valued case, and it then yields results of [7, 11] on Jordan decomposition without appeal to regularity of representing linear operators. The notion of non-atomic measures and their properties are studied by [15, 16, 17, 18, 21] and the references therein.

Aumann [2] introduced the concept of integral of a set-valued function which have many applications in mathematical economics, theory of control and many other fields. Different approaches has been use to extend and generalize the Integral theory. In the field of set-valued integrals, another approach was done by many authors using the Choquet integral or the Sugeno fuzzy integral (see [11, 12, 21, 24] and the references therein). Gould [11] investigated an integral of a real function with respect to an additive measure taking values in a Banach space X.

The objective of the present paper is study the notion of the non-atomic measures and integrals on effect algebras. The notion of non-atomic measures is used to establish an Intermediate value theorem on effect algebras. Moreover, the notion of integrals is introduced and studied with some of the basic natural properties of the integrals on effect algebras.

The paper is organized as follows: Section 2 contains prerequisites and basic results on an effect algebra L. The notions of superior variation m^+, inferior variation m^- and total variation $|m|$ [17] of an extended real-valued function m defined on L are studied elaborately in Section 3, followed by various properties in the context of functions m^+, m^- and $|m|$. Using the notion of an atom of a real-valued measure m [15, 17], we have proved the equivalence of the following: (i) m^+ and m^- are non-atomic, (ii) $|m|$ is non-atomic, (iii) m is non-atomic. In Section 4, we have proved Intermediate value theorem for a locally bounded real-valued σ-additive, non-atomic function m defined on a σ-continuous, σ-complete D-lattice L. Section 5 is concentrated on the objective to introduce the notion of the integral for a bounded, real valued function with respect to a measure on effect algebras with Reisz decomposition property (see [12, 22]).

2. PRELIMINARIES AND BASIC RESULTS

First of all, we shall give some preliminaries and basic results from effect algebras, which can be found in [3, 4, 5, 6, 22].

An *effect algebra* $(L; \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for $a, b, c \in L$:

(1) $a \oplus b = b \oplus a$, if $a \oplus b$ is defined;

(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, if one side is defined;

(3) for every $a \in L$, there exists a unique $b \in L$ such that $a \oplus b = 1$ (we put $a^{\perp} = b$);

(4) if $a \oplus 1$ is defined, then $a = 0$.

For brevity, we denote an effect algbera $(L; \oplus, 0, 1)$ by L. In an effect algebra L, a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a. We say that two elements $a, b \in L$ are *orthogonal*, and we write $a \perp b$, if $a \oplus b$ exists. If $a \oplus b = 1$, then b is called *orthocomplement* of a and write $b = a^{\perp}$. It is obvious that $1^{\perp} = 0$, $(a^{\perp})^{\perp} = a$, $a \perp 0$ and $a \oplus 0 = a$, for all $a \in L$. Also, for $a, b \in L$, we define $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. It may be proved that $\leq$ is a partial ordering on L and $0 \leq a \leq 1$; $a \leq b \Leftrightarrow b^{\perp} \leq a^{\perp}$, and $a \leq b^{\perp} \Leftrightarrow a \perp b$ for $a, b \in L$. If $a \leq b$, the element $c \in L$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b^{\perp})^{\perp}$ (we put $c = b \ominus a$).

In a natural way, the sum of more than two elements is obtained: If $a_1, a_2 \dots, a_n \in L$, we inductively define $a_1 \oplus a_2 \oplus \dots \oplus a_n = (a_1 \oplus \dots \oplus a_{n-1}) \oplus a_n$, provided that the right hand side exists. The definition is independent on permutations of the elements. We say that a finite subset $\{a_1, a_2, \dots, a_n\}$ of L is *orthogonal* if $a_1 \oplus a_2 \oplus \dots \oplus a_n$ exists. For a sequence $\{a_n\}$, we say that it is *orthogonal* if, for every n, $\bigoplus_{i \leqslant n} a_i$ exists. If, moreover, $\sup_n \bigoplus_{i \leqslant n} a_i$ exists, the *sum* $\bigoplus_{n \in \mathbb{N}} a_n$ of an orthogonal sequence $\{a_n\}$ in L is defined as $\sup \bigoplus_{i \leqslant n} a_i$; we denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}$ the set of all real numbers. An effect algebra L is called a *σ-complete effect algebra*, if every orthogonal sequence in L has its sum. If $(L, \leq)$ is a lattice, we say that effect algebra is a *lattice ordered effect algebra* (or a *D-lattice*). The notion of σ-continuity of a D-lattice is, as usual, expressed in terms of monotone sequences: we write $a_n \uparrow a$ (respectively, $a_n \downarrow a$) whenever $\{a_n\}$ is an increasing sequence in L and $a = \sup_n a_n$ (respectively, $\{a_n\}$ is a decreasing sequence and $a = \inf_n a_n$). The lattice $(L, \leq)$ is said to be *σ-continuous* if $a_n \uparrow a$ implies $a_n \wedge b \uparrow a \wedge b$ (or equivalently, $a_n \downarrow a$ implies $a_n \vee b \downarrow a \vee b$) for all $b \in L$.

2.1. We say that an effect algebra L satisfies the *Riesz decomposition property*, *RDP* for short, if for all $a, b_1, b_2 \in L$ $a \leq b_1 \oplus b_2$ implies that there exists two elements $a_1, a_2 \in L$ with $a_1 \leq b_1$ and $a_2 \leq b_2$ such that $a = a_1 \oplus a_2$. L has RDP if and only if, for $x_1, x_2, y_1, y_2 \in L$ such that $x_1 \oplus x_2 = y_1 \oplus y_2$ implies there exists four elements $c_{11}, c_{12}, c_{21}, c_{22} \in L$ such that $x_1 = c_{11} \oplus c_{12}$, $x_2 = c_{21} \oplus c_{22}$, $y_1 = c_{11} \oplus c_{21}$ and $y_2 = c_{12} \oplus c_{22}$.

2.2. A finite sequence $\mathcal{A} = \{a_i\}_{i=1}^{n}$ of nonzero elements of an effect algebra L is called a *partition* of unity of L if $a_1 \oplus a_2 \oplus \cdots \oplus a_n = 1$. A partition $\mathcal{B} = \{b_j\}_{j=1}^{m}$ is called a *refinement* of the partition $\mathcal{A}$ denoted by $\mathcal{A} \prec \mathcal{B}$, if for any element a_i ($i = 1, 2, \dots, n$) there is a subset $\alpha_i \subseteq \{1, 2, \dots, m\}$ such that $a_i = \bigoplus_{j \in \alpha_i} b_j$ and $\bigcup_{i=1}^{m} \alpha_i = \{1, 2, \dots, m\}$ and $\alpha_i \cap \alpha_k = \phi$ for $i \neq k$.

Let $\mathcal{A} = \{a_i\}_{i=1}^{n}$ and $\mathcal{B} = \{b_j\}_{j=1}^{m}$ be two partitions of unity in an effect algebra L with RDP. Due to RDP, there is a *Riesz refinement* (*or joint refinement*) $\mathcal{C} = \{c_{ij} :$

$1 \leq i \leq n, 1 \leq j \leq m\}$ of $\{a_i\}_{i=1}^n$ and $\{b_j\}_{j=1}^m$ such that, for all $1 \leq i \leq n$ and all $1 \leq j \leq m$. We have $a_i = c_{i1} \oplus \cdots \oplus c_{im}$, $b_j = c_{1j} \oplus \cdots \oplus c_{nj}$. In this case $\mathcal{C}$ is a partition of unity of L, such that $\mathcal{A} \prec \mathcal{C}$ and $\mathcal{B} \prec \mathcal{C}$. Also the family $(\mathfrak{P}, \prec)$ is a directed set, where $\mathfrak{P}$ is the set of all partitions of unity of L, $\prec$ is the order relation on $\mathfrak{P}$.

2.3. A function $\mu : L \to \mathbb{R}$ is said to be a *measure* if $\mu(a \oplus b) = \mu(a) + \mu(b)$, for every $a \oplus b \in L$. If μ is measure then $\mu(0) = 0$. If range of μ is $[0, \infty)$, then μ is monotone. A measure $\mu : L \to \mathbb{R}$ is said to be of *finite variation* if $\sup\{|\mu(a)| : a \in L\} < \infty$.

Let us recall the following results:

2.4. Assume that a, b, c are elements of an effect algebra L.

(i) If $a \leqslant b$, then $b = a \oplus (b \ominus a)$.

(ii) If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.

(iii) If $a \leqslant b \leqslant c$, then $(b \ominus a) \leq (c \ominus a)$.

(v) If $a \leq b \leq c$, then $a \oplus (c \ominus b) = c \ominus (b \ominus a)$ and $(c \ominus b) \oplus (b \ominus a) = (c \ominus a)$.

(vi) If $a \leq b \leq c$, then $(c \ominus b) \leq (c \ominus a)$ and $(c \ominus a) \ominus (c \ominus b) = (b \ominus a)$.

(vii) If $a \leq b \leq c$, then $(b \ominus a) \leq (c \ominus a)$ and $(c \ominus a) \ominus (b \ominus a) = (c \ominus b)$.

(viii) If $a \leq b \leq c$, then $a \perp (c \ominus b)$ and $a \oplus (c \ominus b) = c \ominus (b \ominus a)$.

(ix) If $a \leq b' \leq c'$, then $a \oplus (b \ominus c) = (a \oplus b) \ominus c$.

(x) If $a \perp b$ and $(a \oplus b) \leq c$, then $c \ominus (a \oplus b) = (c \ominus a) \ominus b = (c \ominus b) \ominus a$.

2.5. Let L be a σ-complete effect algebra. If $\{a_n\}$ is an increasing (respectively, decreasing) sequence, then $\sup_n a_n$ (respectively, $\inf_n a_n$) exists.

2.6. A function m defined on an effect algebra L with values in $\mathbb{R}$ is called a *measure* on L, if $a, b \in L$, $a \perp b$ implies $m(a \oplus b) = m(a) + m(b)$. It is clear that m is a measure if and only if $b \leq a$ implies $m(a) = m(b) + m(a \ominus b)$. Obviously, if m is a measure, then: (i) $m(0) = 0$; (ii) if $\beta \neq 0$ is a finite number, then βm is also a measure. We say that m is *σ-additive*, if for every orthogonal sequence $\{a_n\}$ in L such that $\bigoplus_n a_n$ exists, $m(\bigoplus_n a_n) = \sum_{n=1}^{\infty} m(a_n)$.

2.7. A function m defined on a D-lattice L with values in $\mathbb{R}$, is called *modular*, if $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$ for $a, b \in L$.

2.8. A function m defined on an effect algebra L with values in $\mathbb{R}$, is called *locally bounded* if, for any $a \in L$, $\sup\{m(b) : b \leqslant a, b \in L\}$ exists.

3. NON-ATOMIC MEASURES ON EFFECT ALGEBRAS

Definition 3.1. [17] Let m be an extended real-valued function defined on an effect algebra L, that is, $m : L \to [-\infty, \infty]$, with $m(0) = 0$. Then for $a \in L$,

(i) *superior variation* of m is defined by

$$m^+(a) = \sup\{m(b) : b \leqslant a, b \in L\};$$

(ii) *inferior variation* of m is defined by

$$\begin{aligned} m^-(a) &= -\inf\{m(b) : b \leqslant a, b \in L\} \\ &= \sup\{-m(b) : b \leqslant a, b \in L\}; \end{aligned}$$

(iii) *total variation* of m is defined by

$$|m| = m^+ + m^-.$$

Remark 3.2. *(i)* $0 \leq m^+(a) \leq \infty$, $0 \leq m^-(a) \leq \infty$, $0 \leq |m|(a) \leq \infty$, $a \in L$;

(ii) $m^+(0) = 0 = m^-(0)$, $|m|(0) = 0$;

(iii) $m^- = (-m)^+$, $m^+ = (-m)^-$;

(iv) $-m^-(a) \leqslant m(a) \leqslant m^+(a)$, $|m(a)| \leq |m|(a)$, $a \in L$.

Theorem 3.3. *If m is a locally bounded real-valued measure defined on an effect algebra L, then m can be written as*

$$m = m^+ - m^-.$$

Proof Let $\varepsilon > 0$. Let $a \in L$. Then there exists $b \in L$ such that $b \leqslant a$ and

$$m^+(a) - \varepsilon < m(b). \tag{3.1}$$

Since $a \ominus b \leq a$, we have

$$-m(a \ominus b) \leq m^-(a). \tag{3.2}$$

From (3.1) and (3.2), we get

$$m^+(a) - \varepsilon - m^-(a) < m(b) + m(a \ominus b),$$

which yields that

$$m^+(a) - m^-(a) - \varepsilon < m(a). \tag{3.3}$$

Letting $\varepsilon \to 0$, we obtain

$$m^+(a) - m^-(a) \leq m(a). \tag{3.4}$$

Further, since (3.4) is true for any m, with the aid of Remark 3.2(iii), we have

$$m^+(a) - m^-(a) \geq m(a). \tag{3.5}$$

Thus (3.4) and (3.5) yields that

$$m(a) = m^+(a) - m^-(a),$$

or $m = m^+ - m^-$. □

Theorem 3.4. *Let m be a real-valued modular measure defined on a D-lattice L. Then m^+ and m^- are measures (and hence $|m|$ is also a measure).*

Proof Firstly, let us consider about m^+. We have proved in [16], that for $a, b \in L$ with $a \perp b$,

$$m^+(a \oplus b) \leq m^+(a) + m^+(b). \tag{3.6}$$

By Definition 3.1 (i), (ii), there are sequences $\{a_n\}$ and $\{b_n\}$ of elements from L such that $a_n \leq a$, $b_n \leq b$ with

$$m(a_n) \to m^+(a), \;\; m(b_n) \to m^+(b). \tag{3.7}$$

Obviously, $a_n \perp b_n$ for each n. Therefore, from $m(a_n \oplus b_n) = m(a_n) + m(b_n)$, we have $m(a_n \oplus b_n) \to m^+(a) + m^+(b)$.

Further, $a_n \oplus b_n \leq a \oplus b$ yields that

$$m^+(a \oplus b) \geq m^+(a) + m^+(b). \tag{3.8}$$

From (3.6) and (3.8), we get

$$m^+(a \oplus b) = m^+(a) + m^+(b),$$

that is, m^+ is a measure.

By similar argument, we can show that m^- is a measure. From Definition 3.1(iii), $|m|$ is a measure.

Definition 3.5. An extended real-valued function m defined on an effect algebra L is called *continuous from below* (respectively, *continuous from above*), if $a, a_n \in L$, $a_n \uparrow a$, $n \in \mathbb{N} \Rightarrow m(a) = \lim_{n\longrightarrow\infty} m(a_n)$ (respectively, if $a, a_n \in L$, $a_n \downarrow a$, $n \in \mathbb{N}$ and $m(a_1) < \infty \Rightarrow m(a) = \lim_{n\longrightarrow\infty} m(a_n)$).

Proposition 3.6. *Let $m : L \to \mathbb{R}$ be a measure. Then the following assertions are equivalent:*

(i) m is σ-additive.

(ii) m is continuous from below.

(iii) m is continuous from above.

(iv) $a_n \downarrow 0$ implies $\lim_{n\to\infty} m(a_n) = 0$.

Theorem 3.7. *If m is a locally bounded real-valued σ-additive function defined on a σ-continuous D-lattice L, then m^+, m^- and $|m|$ are also σ-additive.*

Proof Let $a_n \uparrow a,\ a, a_n \in L$. Then $m^+(a_n) \leq m^+(a)$, for every n. Thus the increasing sequence $\{m^+(a_n)\}$ converges to a limit l, say, where $l \leq m^+(a)$.

For any element $b \in L$, $b \leq a$,

$$m(b \wedge a_n) \leq m^+(b \wedge a_n) \leq m^+(a_n);$$

also, from the σ-additivity of m,

$$m(b \wedge a_n) \to m(b).$$

Hence,

$$m(b) \leq l.$$

As $b \in L$ is arbitrary, we get

$$m^+(a) \leq l.$$

It follows that $m^+(a) = l$, that is, $m^+(a_n) \to m^+(a)$.

Further, since m^+ is a measure, in view of Proposition 3.6, m^+ is σ-additive.

The σ-additivity of m^- and $|m|$ are obvious.

Theorem 3.8. *If m is a locally bounded real-valued measure defined on an effect algebra L, then m can be written as*

$$m = m^+ - m^-.$$

Further, if m is a real-valued modular measure defined on a lattice effect algebra L, then the decomposed parts m^+ and m^- are measures on L (and hence $|m|$ is also a measure on L). Moreover, if m is a locally bounded real-valued σ-additive function defined on a σ-continuous D-lattice L, then the decomposed parts m^+, m^-, and $|m|$ are also σ-additive.

4. INTERMEDIATE VALUE THEOREM

Let m be a real-valued function defined on an effect algebra L. Firstly, we shall recall the notion of an atom of a measure m defined on an effect algebra L, which has been studied in [15, 17].

Definition 4.1. An element $a \in L$ with $m(a) \neq 0$ is called an *atom of* m (or an m-*atom*), if for $a, b \in L$ with $b \leq a$,

(i) $m(b) = 0$ (that is, $a =_m 0$) or

(ii) $m(a) = m(b)$.

In case there are no atoms of m in L, m is called *non-atomic* on L.

Theorem 4.2. *Let m be a locally bounded real-valued measure defined on an effect algebra L. Then the following conditions are equivalent:*
(i) m^+ *and* m^- *are non-atomic.*
(ii) $|m|$ *is non-atomic.*
(iii) m *is non-atomic.*

Proof $(i) \Rightarrow (ii)$: Let $a \in L$ be a $|m|$-atom. Let $b \leq a$, $b \in L$ with $m^+(b) \neq 0$. Obviously, $|m|(b) \neq 0$ and hence $|m|(a) = |m|(b)$, which yields that $a \in L$ is an m^+-atom.

$(ii) \Rightarrow (iii)$: See proof of Theorem 5.5 [17].

$(iii) \Rightarrow (i)$: Let $a \in L$ is an m^+-atom. Let $b \leq a$, $b \in L$ with $m(b) \neq 0$. Obviously, $m^+(b) \neq 0$ and hence $m^+(a) = m^+(b)$, which yields that

$$m(a) \leq m(b). \tag{4.1}$$

From (4.1) and Theorem 3.8, we have

$$m^+(a) - m^-(a) \leq m(b). \tag{4.2}$$

Replacing m by $-m$ in (4.2), we get

$$m(a) \geq m(b). \tag{4.3}$$

From (4.1) and (4.3), $a \in L$ is an m-atom.

Theorem 4.3. *Let m be a $[0, \infty)$-valued σ-additive function defined on a σ-complete effect algebra L. Then m is non-atomic on L if and only if for a given element $a \in L$ with $m(a) > 0$ and $\varepsilon > 0$, there exists $b \in L$, $b \leq a$, such that $0 < m(b) < \varepsilon$.*

Proof The *if* part: Obvious.

The *only if* part: Suppose the contrary and choose an element $a \in L$ with $m(a) > 0$ and $t_0 > 0$, for which $m(b) \geqslant t_0$ holds if $b \leqslant a$, $b \in L$ and $m(b) > 0$. Define

$$t_1 = \inf\{m(b) : b \in L, b \leqslant a, m(b) > 0\}.$$

Then obviously $0 < t_0 \leqslant t_1$. Take $a_1 \leqslant a$, $a_1 \in L$ with $t_1 \leqslant m(a_1) < t_1 + 1$ and setting

$$t_2 = \inf\{m(b) : b \in L, b \leqslant a_1, m(b) > 0\}.$$

Choose $a_2 \leqslant a_1$ with $t_2 \leqslant m(a_2) < t_2 + \frac{1}{2}$. Continuing the process in the same manner, we obtain sequences $\{t_n\}$ and $\{a_n\}$ such that $t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant m(a)$ and $a \geqslant a_1 \geqslant a_2 \geqslant \ldots$ with

$$t_n \leqslant m(a_n) < t_n + \frac{1}{2^n},$$

for all n. Using 2.2, put $a_0 = \bigwedge_{n=1}^{\infty} a_n$. Clearly, in view of Proposition 3.6, we have $m(a_0) = \lim_{n \longrightarrow \infty} m(a_n) = \lim_{n \longrightarrow \infty} t_n > 0$. Let $b \leqslant a_0$ with $m(b) > 0$. Then $\mu(a_0) \geqslant \mu(b) \geqslant t_n$, for any n and hence $\mu(b) = \mu(a_0)$. This gives that $a_0 \in L$ is an atom of m, a contradiction.

Theorem 4.4. *Let m be a $[0, \infty)$-valued σ-additive function defined on a σ-complete effect algebra L. If m is non-atomic on L, then m takes every value between 0 and $m(1)$.*

Proof Let $0 < t < m(1)$. According to Theorem 4.3, there are elements $c \in L$ such that $0 < m(c) < t$. Let

$$s_1 = \sup\{m(c) : c \in L, m(c) \leqslant t\}.$$

(Obviously $0 < s_1 \leqslant t$). Then there exists an element $c_1 \in L$ such that $\frac{s_1}{2} < m(c_1) \leqslant s_1$. Let

$$s_2 = \sup\{m(c) : c \in L, c_1 \leqslant c, m(c) \leqslant t\}.$$

Then there exists an element $c_2 \in L$ such that $c_2 \geqslant c_1$ and $s_2 - \frac{s_1}{2^2} < m(c_2) \leqslant s_2$. Continue this construction inductively to obtain

$$s_n = \sup\{m(c) : c \in L, c_{n-1} \leqslant c, m(c) \leqslant t\},$$

and then there exists $c_n \geqslant c_{n-1}$, $c_n \in L$ such that

$$s_n - \frac{s_1}{2^n} < m(c_n) \leqslant s_n.$$

It is clear that $\{s_n\}$ is a decreasing sequence and $\{c_n\}$ is an increasing sequence of elements in L such that $d = \bigvee_{n=1}^{\infty} c_n \in L$ (using 2.5) and therefore, in view of Proposition 3.6, we get $\lim_{n\longrightarrow\infty} s_n = \lim_{n\longrightarrow\infty} \mu(c_n) = \mu(d)$. Therefore $\mu(d) = \lim_{n\longrightarrow\infty} s_n = s$ (let). Clearly $s \leqslant t$. Now we claim that $s = t$. For, otherwise, let us suppose that $s < t$. Since $0 < t < \mu(1)$, we get $\mu(1 \ominus d) > 0$, $d \in L$ and therefore, by Theorem 4.3, we obtain an element b of L such that $b \leqslant (1 \ominus d)$ and $s < \mu(d \oplus b) < t$. But then $d \oplus b \geqslant c_{n-1}$, for all $n > 1$, which yields that $\mu(d \oplus b) \leqslant s_n$, for all n. This will further imply that $\mu(d \oplus b) \leqslant s$, a contradiction. Thus $\mu(d) = t$ as required.

Theorem 4.5. (*Intermediate value theorem*). *Let m be a locally bounded real-valued σ-additive function defined on a σ-continuous, σ-complete D-lattice L. If m is non-atomic on L, then m takes every value between $-m^-(1)$ and $m^+(1)$.*

Proof Follows from Theorem 3.7, Theorem 4.2 and Theorem 4.4.

5. GOULD TYPE INTEGRAL ON AN EFFECT ALGEBRA

In this section, we introduce and study the notion of Gould type integral with respect to a measure defined on an effect algebra. The Gould type integral was intensively studied for different types of set (multi)functions: vector valued measures [11], measures [12] monotone set multifunctions (called fuzzy multimeasures) [24]. From now onwards, let μ be a measure defined on an effect algebra L with RDP, which is not identically zero and let $f : L \to \mathbb{R}$ be a real valued bounded function.

Definition 5.1. [12] For a measure μ on L, define

$$\overline{\mu}(a) = \sup\{\sum_{i=1}^{n} \mid \mu(a_i) \mid\},$$

for every $a \in L$, where the supremum is extended over all finite partitions $\{a_i\}_{i=1}^n$ of a. We define $\widetilde{\mu}$ as

$$\widetilde{\mu}(a) = \inf\{\overline{\mu}(b) : a \leq b,\ b \in L\},$$

for every $a \in L$. It may be observed that $\overline{\mu}$ is a finitely additive monotone function on L and $\widetilde{\mu}(a) = \overline{\mu}(a)$, for every $a \in L$.
A property (M) is said to be *μ-almost every where* (μ-$a.e.$ in brief), if the property (M) is valid otherwise the set $\{a \in L : \widetilde{\mu}(a) = 0\}$. We shall assume $\widetilde{\mu}(1) < \infty$.

Definition 5.2. [12] Define $osc(f, a) = \sup_{x,y \leq a} \mid f(x) - f(y) \mid$, where $a \in L$. We observe that:

(1) $a \leq b \Rightarrow osc(f, a) \leq osc(f, b)$, for $a, b \in L$.
(2) $osc(f + g, a) \leq osc(f, a) + osc(g, a)$, for $a \in L$.
(3) $osc(\alpha f, a) =\mid \alpha \mid osc(f, a)$, for $a \in L$ and $\alpha \in \mathbb{R}$.

The function f is said to be *$\widetilde{\mu}$-measurable* on L if for every $\varepsilon > 0$ there exists a partition $\mathcal{A}_\varepsilon = \{a_i\}_{i=0}^n$ of unity of L such that

(1) $\widetilde{\mu}(a_0) < \varepsilon$,

(2) $\sup_{x,y\leq a_i} \mid f(x) - f(y) \mid = osc(f, a_i) < \varepsilon$, for every $i = 1, 2, \ldots, n$.

Such a partition $\mathcal{A}_\varepsilon$ is called an ε-*partition* of unity of L.
It is easy to see that if f and g are $\widetilde{\mu}$-measurable on L, then $f + g$ is $\widetilde{\mu}$-measurable, αf is $\widetilde{\mu}$-measurable for every $\alpha \in \mathbb{R}$ and $f + c$ is $\widetilde{\mu}$-measurable for every constant real number c.

Definition 5.3. [12] Let $\sigma(\mathcal{A}, t_i, \mu) \equiv \sigma(\mathcal{A}) = \sum_{i=1}^{n} f(t_i)\mu(a_i)$ for every partition $\mathcal{A}$ = $\{a_i\}_{i=1}^n$ of unity of L and $t_i \leq a_i, t_i \in L$, $i = 1, 2, \ldots, n$. The function f is said to be μ-*integrable* on L if there exists a $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a partition $\mathcal{A}_\varepsilon$ of unity of L so that for every partition $\mathcal{A}$ = $\{a_i\}_{i=1}^n \in \mathfrak{P}$ with $\mathcal{A}_\varepsilon \prec \mathcal{A}$ and every choice of $t_i \leq a_i, t_i \in L$, $i = 1, 2, \ldots, n$ we have $d(\sigma(\mathcal{A}), x) < \varepsilon$ (here d is the usual metric on $\mathbb{R}$). In this case I is called the integral of f in L and is denoted by $\int_L f d\mu$. That is, the net $\{\sigma(\mathcal{A})\}_{\mathcal{A}\in(\mathfrak{P},\prec)}$ is convergent in $(\mathbb{R}, d)$. Obviously, if $\mathcal{A}_\varepsilon$ exists, the integral is unique.

Example 5.4. Let $M_k = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k}{k}\}$, $k \geq 1$, be a finite MV-algebra. Then $s(\frac{1}{k}) = \frac{1}{k}$ is the measure on M_k and $\mathcal{A}$ = $\{\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}\}$ is the finest refinement of unity in M_k. Let $f : M_k \to \mathbb{R}$ is a bounded real valued function defined as $f(x) = c$, for every $x \in L$, where c is a real constant. Then $\sum_n f(\frac{1}{k})s(\frac{1}{k}) = c$ and hence $\int_L f d\mu = c$

Theorem 5.5. *If μ is nonnegative and $f = 0$ μ-almost everywhere, then f is $\widetilde{\mu}$-measurable, μ-integrable on L and $\int_L f d\mu = 0$.*

Proof Since f is a bounded function on L, there exists $M > 0$ so that $\mid f(a) \mid\leq M$ for every $a \in L$. Let $A = \{a \in L : f(a) \neq 0\}$. Then $\widetilde{\mu}(a) = 0$ for every $a \in A$, which implies that, for every $\varepsilon > 0$ there exists $b_\varepsilon \in L$, so that $a \leq b_\varepsilon$ and $\overline{\mu}(b_\varepsilon) < \frac{\varepsilon}{M}$. Let us consider $\mathcal{A}_\varepsilon = \{b_\varepsilon, b_\varepsilon^\perp\}$ which is a partition of unity of L. Since $\widetilde{\mu}(b_\varepsilon) = \overline{\mu}(b_\varepsilon) < \frac{\varepsilon}{M}$ and $\text{osc}(f, b_\varepsilon^\perp) = \sup_{x,y\leq b_\varepsilon} \mid f(x) - f(y) \mid= 0$, we get that f is $\widetilde{\mu}$- measurable.
Let us take an arbitrary partition $\mathcal{B}$ = $\{b_j\}_{i=1}^m$ of unity of L, such that $\mathcal{A}_\varepsilon \prec \mathcal{B}$. Let $t_i \leq b_i, t_i \in L$, $i = 1, 2, \ldots, m$, be chosen arbitrary. We may suppose that $b_1 \oplus b_2 \oplus \ldots, \oplus b_k = b_\varepsilon$ and $b_{k+1} \oplus b_{k+2} \oplus \cdots \oplus b_m = b_\varepsilon^\perp$. Then

$$\begin{aligned}\sigma(\mathcal{A}) =\mid \sum_{i=1}^{m} f(t_i)\mu(b_i) \mid &\leq \sum_{i=1}^{k} \mid f(t_i)\mu(b_i) \mid + \sum_{i=k+1}^{m} \mid f(t_i)\mu(b_i) \mid \\ &= \sum_{i=1}^{k} \mid f(t_i)\mu(b_i) \mid \\ &\leq M \cdot \sum_{i=1}^{k} \mid \mu(b_i) \mid \\ &\leq M \cdot \overline{\mu}(b_\varepsilon) \leq M \cdot \frac{\varepsilon}{M} = \varepsilon.\end{aligned}$$

Hence f is μ-integrable on L and $\int_L f d\mu = 0$.

Theorem 5.6. *If f is μ-integrable on L and $\alpha \in \mathbb{R}$, then*
(1) αf is μ-integrable on L and

$$\int_L \alpha f d\mu = \alpha \int_L f d\mu,$$

(2) f is $\alpha\mu$-integrable on L and

$$\int_L f d(\alpha\mu) = \alpha \int_L f d\mu.$$

Proof The case $\alpha = 0$ is trivial. Let $\alpha \neq 0$. Because f is μ-integrable on L, for every $\varepsilon > 0$ there exists a partition $\mathcal{A}_\varepsilon \in \mathfrak{P}$, so that for every partition $\mathcal{A} = \{a_i\}_{i=1}^n \in \mathfrak{P}$ with $\mathcal{A}_\varepsilon \prec \mathcal{A}$ and for every $t_i \leq a_i, t_i \in L$, $i = 1, 2, \ldots, n$ we have

$$d(\sum_{i=1}^{n} f(t_i)\mu(a_i), \int_L f d\mu) < \frac{\varepsilon}{\mid \alpha \mid}.$$

Then

$$d(\sum_{i=1}^{n} \alpha f(t_i)\mu(a_i), \alpha \int_L f d\mu) =\mid \alpha \mid \cdot d(\sum_{i=1}^{n} f(t_i)\mu(a_i), \int_L f d\mu),$$
$$<\mid \alpha \mid \cdot \frac{\varepsilon}{\mid \alpha \mid} = \varepsilon,$$

that is, αf is μ-integrable on L and $\int_L \alpha f d\mu = \alpha \int_L f d\mu$.

(2) The function $\mu : L \rightarrow \mathbb{R}$ defined by $(\alpha\mu)(a) = \alpha.\mu(a)$, for every $a \in L$ is a measure of finite variation and hence the theorem.

Theorem 5.7. *Let $f, g : L \rightarrow \mathbb{R}$ are two bounded μ-integrable functions on L. Then $f + g$ is also μ-integrable on L and*

$$\int_L (f+g) d\mu = \int_L f d\mu + \int_L g d\mu.$$

Proof Since f is μ-integrable, then for every $\varepsilon > 0$, there exists $\mathcal{A}_1 \in \mathfrak{P}$ so that for every $\mathcal{B}_1 \in \mathfrak{P}$, $\mathcal{B}_1 = \{a_i\}_{i=1}^n$ with $\mathcal{A}_1 \prec \mathcal{B}_1$ and for every $t_i \leq a_i, t_i \in L$, $i = 1, 2, \ldots, n$, we have

$$d(\sum_{i=1}^{n} f(t_i)\mu(a_i), \int_L f d\mu) < \frac{\varepsilon}{2}.$$

Similarly, g is also μ-integrable so there is a partition $\mathcal{A}_2 \in \mathfrak{P}$ such that for every $\mathcal{B}_2 \in \mathfrak{P}$, $\mathcal{B}_2 = \{b_j\}_{j=1}^m$ with $\mathcal{A}_2 \prec \mathcal{B}_2$ and every $s_j \leq b_j, s_j \in L$, $j = 1, 2, \ldots, m$, we have

$$d(\sum_{j=1}^{m} f(s_j)\mu(b_j), \int_L g d\mu) < \frac{\varepsilon}{2}.$$

Let $\mathcal{A}_0$ be a joint refinement of $\mathcal{A}_1$ and $\mathcal{A}_2$. Then for any partition $\mathcal{C} = \{c_k\}_{k=1}^p \in \mathfrak{P}$ with $\mathcal{A}_0 \prec \mathcal{C}$, we get that for every $k = 1, 2, \ldots, p$, there exists $i_k = 1, 2, \ldots, n$ and $j_k = 1, 2, \ldots, m$, so that $c_k \leq a_{i_k}, c_k \in L$ and $c_k \leq b_{j_k}, c_k \in L$. We have to prove that $d(\sum_{k=1}^p (f+g)(r_k)\mu(c_k), \int_L f d\mu + \int_L g d\mu) < \varepsilon$, for every $r_k \leq c_k, r_k \in L$, $k = 1, 2, \ldots, p$.

Now for every $r_k \leq c_k, r_k \in L$, $k = 1, 2, \ldots, p$ and $\mathcal{A}_1 \prec \mathcal{C}$, $\mathcal{A}_2 \prec \mathcal{C}$, we observe that

$$d(\sum_{k=1}^{p} (f+g)(r_k)\mu(c_k), \int_L f d\mu + \int_L g d\mu) = d(\sum_{k=1}^{p} f(r_k)\mu(c_k)$$
$$+ \sum_{k=1}^{p} g(r_k)\mu(c_k), \int_L f d\mu + \int_L g d\mu)$$
$$\leq d(\sum_{k=1}^{p} f(r_k)\mu(c_k), \int_L f d\mu)$$
$$+ d(\sum_{k=1}^{p} g(r_k)\mu(c_k), \int_L g d\mu)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $f + g$ is μ-integrable and thus the theorem is proved.

Corollary 5.8. *If $\mu : L \to [0, \infty)$ and $f = g$, μ-almost every where f and g are two μ-integrable bounded functions on L, then*

$$\int_L f d\mu = \int_L g d\mu.$$

Proof Using Theorem 5.5, 5.6 and 5.7, we have the above corollary.

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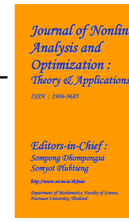

POSITIVE SOLUTIONS AND EIGENVALUES FOR SECOND ORDER INTEGRAL BOUNDARY VALUE PROBLEMS WITH SIGN-CHANGING NONLINEARITIES

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ABSTRACT. We investigate eigenvalue intervals for the existence of positive solutions for the second order integral boundary value problem

$$\begin{cases} u''(t) + \lambda f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \int_0^1 u(s)d\alpha(s), \quad u(1) = \int_0^1 u(s)d\beta(s) \end{cases}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ is sign-changing.

KEYWORDS : Positive solution; Sign-changing nonlinearities; Existence.
AMS Subject Classification: 34B10 34B18.

1. INTRODUCTION

In this paper, we are concerned with determining values of λ (eigenvalues), for which exist positive solutions of nonlinear second-order integral boundary-value problem

$$\begin{cases} u''(t) + \lambda f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = \int_0^1 u(s)d\alpha(s), \quad u(1) = \int_0^1 u(s)d\beta(s), \end{cases} \tag{1.1}$$

where $f \in C(\mathbb{R}, \mathbb{R})$; α and β are right continuous on $[0, 1)$, left continuous at $t = 1$, and nondecreasing on [0,1], with $\alpha(0) = \beta(0) = 0$; $\int_0^1 u(s)d\alpha(s)$ and $\int_0^1 u(s)d\beta(s)$ denote the Riemann-Stieltjes integrals of u with respect to α and β, respectively. We shall concentrate on the case when the nonlinearity $f(u)$ is allowed to change sign, which is of particular mathematical interest.

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The motivation for the present work stems from many recent investigations. In fact, boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. Concerning the existence of positive solutions for eigenvalue problem, we refer the reader to [4, 5, 7, 8]. For the special case $\lambda = 1$, (1.1) have been the subject matter of many recent publications on singular boundary value problems, for this we refer the reader to the papers by Karakostas and Tsamatos [10, 11], Yuhua Li and Fuyi Li [12], Webb and Infante [16, 17], Yang [18, 19, 20] and Zhang and Sun [22] and the references therein. For more information about the general theory of integral equations and their relation with boundary value problems we refer to the book of Corduneanu [6] and Agarwal and O'Regan [3]

Positive solutions are usually the ones of interest and because of the difficulties associated with proving the existence of such solutions using the techniques of nonlinear functional analysis, most of the recent work assumes nonnegativity of $f(u)$ in order to generate positive operators using the positivity of Green's function. To the authors's knowledge, there are few papers that have considered the existence of positive solutions for local, nonlocal and, particularly, integral boundary value problems involving sign-changing nonlinearities (see[1, 2, 13, 14, 15]).

Recently, using the a priori estimate method and the Leray-Schauder fixed point theorem, Yang [18] studied (1.1) with $\lambda = 1$ under the assumptions:

(H_1) $f(x) > 0, x \in (-\infty, 0]$ and there is $p > 0$ such that $f(x) < 0, x \in (p, +\infty)$.

(H_2) $\kappa_1 > 0, \kappa_4 > 0, \kappa = \kappa_1\kappa_4 - \kappa_2\kappa_3 > 0$, where

$$\kappa_1 = 1 - \int_0^1 (1-t)d\alpha(t), \quad \kappa_2 = \int_0^1 td\alpha(t),$$

$$\kappa_3 = \int_0^1 (1-t)d\beta(t), \quad \kappa_4 = 1 - \int_0^1 td\beta(t).$$

Motivated by [18], the purpose of this paper is to give sufficient conditions on $f(u)$ to determine ranges of λ for which positive solution exists. We make the following assumptions

(H_1') $f : \mathbb{R} \to \mathbb{R}$ is continuous and there is $p > 0$ such that $f(x) > 0,\ x \in (0, p)$ and $f(p) = 0$.

(H_2') $\int_0^1 d\alpha(t) < 1, \ \int_0^1 d\beta(t) < 1$.

Remark 1.1. It is obvious that the condition imposed on f in this paper is weaker than that of [18]. But the condition about $\alpha(t), \beta(t)$ is stronger than that of [18].

2. PRELIMINARIES

Let X be the Banach space $C[0, 1]$ with $\|u\| = \sup\limits_{t\in[0,1]} |u(t)|$. Define a set $K \subset X$ by

$$K = \{u \in X : u(t) \geq t(1-t)\|u\|, \ t \in [0, 1]\}.$$

It can be easily verified that K is indeed a cone in X. For any $r > 0$, defined Ω_r by $\Omega_r = \{u \in K : \ \|u\| < r\}$.

To study (1.1), consider the map $T : X \to X$ defined by

$$(Tu)(t) = \int_0^1 k(t,s)u(s)ds + \kappa^{-1}(1-t,t)\begin{pmatrix} \kappa_4 & \kappa_2 \\ \kappa_3 & \kappa_1 \end{pmatrix}\begin{pmatrix} \int_0^1 d\alpha(t)\int_0^1 k(t,s)u(s)ds \\ \int_0^1 d\beta(t)\int_0^1 k(t,s)u(s)ds \end{pmatrix}$$

and $F : X \to X$ defined by

$$(F(u))(t) = f(u(t)),$$

where $k(t,s)$ is given by

$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Then if (H_1) and (H_2) (or (H_1') and (H_2')) hold, by [18, 21], u is a solution of (1.1) if and only if $u \in C^2[0,1]$ is a solution of the equation

$$(I - \lambda TF)u = 0, \text{ that is, a fixed point of } \lambda TF,$$

where by standard arguments, $TF : X \to X$ is a compact map.

For the function $k(t,s)$, it is easy to know that

$$t(1-t)s(1-s) \le k(t,s) \le s(1-s), \quad t,s \in [0,1]. \tag{2.1}$$

Lemma 2.1. *$T(K) \subset K$ and the map $T : K \to K$ is completely continuous.*

Proof. The inequality (2.1) and the definition of T imply that $T(K) \subset K$. The complete continuity of the integral operator T is well known. □

The following lemma is needed in our argument.

Lemma 2.2. [9] *Let X be a Banach space and K a cone in X. Assume that $T : \Omega_r \to K$ is completely continuous such that $Tu \ne u$ for $u \in \partial\Omega_r$.*

(i) If $\|Tu\| \ge \|u\|$ for $u \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 0$.

(ii) If $\|Tu\| \le \|u\|$ for$u \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 1$.

3. MAIN RESULTS

Our first result is the following theorem in which we shall prove existence of at least a solution of the BVP(1.1).

Theorem 3.1. *If (H_1') and (H_2') hold, then for $\lambda > 0$, (1.1) has at least a solution $u \in C^2[0,1]$.*

Proof. We define a auxiliary functions $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(u) = \begin{cases} f(0), & u \le 0, \\ f(u), & 0 < u < p, \\ 0, & u \ge p. \end{cases}$$

Thus $\widetilde{f}$ is continuous and bounded on $\mathbb{R}$ so there exists $M > 0$ such that

$$|\widetilde{f}| \le M, \text{ on } \mathbb{R}.$$

Consider the modified problem

$$\begin{cases} u''(t) + \lambda\widetilde{f}(u(t)) = 0, & t \in (0,1), \\ u(0) = \int_0^1 u(s)d\alpha(s), & u(1) = \int_0^1 u(s)d\beta(s). \end{cases} \tag{3.1}$$

This is equivalent to the integral equation

$$\begin{aligned} u(t) &= \lambda \int_0^1 k(t,s)\widetilde{f}(u(s))ds \\ &\quad + \frac{\lambda}{\kappa}(1-t,t)\begin{pmatrix} \kappa_4 & \kappa_2 \\ \kappa_3 & \kappa_1 \end{pmatrix}\begin{pmatrix} \int_0^1 d\alpha(t)\int_0^1 k(t,s)\widetilde{f}(u(s))ds \\ \int_0^1 d\beta(t)\int_0^1 k(t,s)\widetilde{f}(u(s))ds. \end{pmatrix} \end{aligned} \tag{3.2}$$

We write (3.2) as an operator equation

$$(I - T_\lambda)u = 0,$$

where $T_\lambda = \lambda T\widetilde{F}$, and $\widetilde{F} : X \to X$ defined by

$$(\widetilde{F}(u))(t) = \widetilde{f}(u(t))$$

is bounded. Notice that $|\widetilde{f}| \leq M$. Thus for some constant $N > 0$, independent of λ, u, we have

$$\|T_\lambda u\|_{C[0,1]} \leq \lambda MN.$$

It follows from the Schauder fixed point theorem that T_λ has a fixed point $u_\lambda \in C[0,1]$. This together with the equation in (3.1) implies that $u_\lambda \in C^2[0,1]$.

Since results to be proved in this Theorem are true for any positive parameter λ. So in the rest of proof, we write u_λ as u for simplicity.

To finish the proof from the definition of $\widetilde{f}$, it suffices to show that any solution u of (3.1) satisfies

$$0 \leq u(t) \leq p, \quad t \in (0,1),$$

i.e., any solution u of (3.1) in fact is a solution of (1.1). Now we claim that any solution u of (3.1) satisfies $0 \leq u(t) \leq p,\ t \in (0,1)$. Firstly, we show that $u \geq 0,\ t \in [0,1]$. In fact, the nonnegativity of $\widetilde{f}$ ensures that

$$u''(t) \leq 0,\ t \in [0,1]. \tag{3.3}$$

So, in order to get the desired results, we only need to prove that $u(0) \geq 0$ and $u(1) \geq 0$. (3.3) implies

$$u(t) \geq (1-t)u(0) + tu(1), \quad t \in [0,1].$$

Therefore,

$$u(0) = \int_0^1 u(t)d\alpha(t) \geq u(0)\int_0^1 (1-t)d\alpha(t) + u(1)\int_0^1 td\alpha(t)$$

and

$$u(1) = \int_0^1 u(t)d\beta(t) \geq u(0)\int_0^1 (1-t)d\beta(t) + u(1)\int_0^1 td\beta(t).$$

The last two inequalities can be written as

$$\begin{pmatrix} \kappa_1 & -\kappa_2 \\ -\kappa_3 & \kappa_4 \end{pmatrix}\begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then by Remark 1.1, we have

$$\kappa_1 > 0,\ \kappa_2 \geq 0,\ \kappa_3 \geq 0,\ \kappa_4 > 0,\ \kappa > 0.$$

Thus

$$\begin{pmatrix} u(0) \\ u(1) \end{pmatrix} \geq \frac{1}{\kappa}\begin{pmatrix} \kappa_4 & \kappa_3 \\ \kappa_2 & \kappa_1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Next, we show that $u(t) \le p,\ t \in [0,1]$. If not, then there exists a $t_0 \in [0,1]$ such that $u(t_0) = \max_{t\in[0,1]} u(t) > p$.

If $t_0 = 0$, then $u(0) = \int_0^1 u(t)d\alpha(t) \le u(0)\int_0^1 d\alpha(t)$, from (H_2'), we have the contradiction $u(0) \le 0$. Similarly, the condition (H_2') gives $t_0 \ne 1$.

If $t_0 \in (0,1)$, then we have $u'(t_0) = 0$. We consider the following two cases: case (i), $u(t) \ge p,\ \ t \in [0,1]$; case (ii), there exists $t_1 \in [0,1]$ such that $u(t_1) < p$. In the second case , without loss of generality, we assume that $t_1 \in [0, t_0)$ and $u(t) > p,\ \ t \in (t_1, t_0]$.

For case (i), we have $u''(t) = -\lambda\widetilde{f}(u(t)) = -\lambda f(p) = 0,\ t \in (0,1)$. Thus, u' is constant on $[0,1]$. Since $u'(t_0) = 0$, it follows that $u'(t) = 0$ for $t \in [0,1]$. Consequently, $u(t) \equiv u(t_0) > p$ on $[0,1]$. On the other hand, we have $u(0) = \int_0^1 u(t)d\alpha(t) \le u(0)\int_0^1 d\alpha(t)$, from (H_2'), we obtain

$$u(t) \equiv u(0) \le 0.$$

Which is a contradiction.

For case (ii), we have $u''(t) = -\lambda\widetilde{f}(u(t)) = -\lambda f(p) = 0,\ t \in (t_1, t_0]$. This together with $u'(t_0) = 0$ implies that $u(t_1) = u(t_0) > p$, contradicting $u(t_1) < p$. □

When $f(0) > 0$, the existence results in Theorem 3.1 can be improved in

Theorem 3.2. *If* (H_1') *and* (H_2') *hold with* $f(0) > 0$*, then for* $\lambda > 0$*, (1.1) has at least one positive solution* $u \in C^2[0,1]$.

Remark 3.3. If $f(0) = 0$, then Theorem 3.2 may be false, as the following counterexample shows.

$$\begin{cases} -u''(t) = \lambda u(t)(1-u(t)), & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.4}$$

Suppose u is a positive solution of (3.4). Multiplying the equation in (3.4) by $\sin \pi t$ and integrating on $[0,1]$, we have

$$\begin{aligned} \pi^2 \int_0^1 u(t)\sin\pi t dt &= -\int_0^1 u''(t)\sin\pi t dt \\ &= \lambda\int_0^1 u(t)(1-u(t))\sin\pi t dt < \lambda\int_0^1 u(t)\sin\pi t dt. \end{aligned}$$

Thus, for every $\lambda \le \pi^2$, Theorem 3.2 is not true. However, Theorem 3.2 remain true for sufficiently large λ.

Theorem 3.4. *If* (H_1') *and* (H_2') *hold with* $f(0) = 0$. *Then exists a* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$*, (1.1) has at least one positive solution* $u \in C^2[0,1]$.

Proof. Using Theorem 3.1, we can show that any solution of the modified problem (3.1) satisfies

$$0 \le u(t) \le p, \quad t \in [0,1]$$

and hence is a solution of (1.1). So it suffices to show that the compact operator T_λ has a nonzero fixed point in $K\backslash\{\theta\}$.

Take $r \in (0,p)$. Let $\lambda_0 = \dfrac{512r}{11m}$, where $m = \min_{v\in[\frac{r}{4},r]} f(v) > 0$. Notice that for any $u \in K$ we have that $u(t) \ge t(1-t)\|u\|$ for all $t \in [0,1]$. In particular, we have

$u(t) \geq \frac{3}{16}\|u\|$ for all $t \in [\frac{1}{4}, \frac{3}{4}]$. Let $u \in \partial\Omega_r$, then $f(u(t)) \geq m$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. Hence for $\lambda > \lambda_0$, we have

$$\begin{aligned}\|T_\lambda u\| &\geq \lambda \min_{t\in[\frac{1}{4},\frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} k(t,s)f(u(s))ds \geq \lambda \min_{t\in[\frac{1}{4},\frac{3}{4}]} \left(t(1-t)\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)f(u(s))ds \right) \\ &\geq \frac{3\lambda}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)f(u(s))ds \\ &\geq \frac{11}{512}\lambda m > r = \|u\|, \quad u \in \partial\Omega_r.\end{aligned}$$

On the other hand, for each fixed $\lambda > \lambda_0$ since $\widetilde{f}(u)$ is bounded, there is an $R > r$ such that

$$\|T_\lambda u\| < R = \|u\|, \quad u \in \partial\Omega_R.$$

It follows from Lemma 2.2 that

$$i(T_\lambda, \Omega_r, K) = 0, \text{ while } i(T_\lambda, \Omega_R, K) = 1,$$

and hence,

$$i(T_\lambda, \Omega_R \backslash \overline{\Omega_r}, K) = 1.$$

Thus, T_λ has a fixed point u in $\Omega_R \backslash \overline{\Omega_r}$. Theorem 3.1 implies that the fixed point u is a solution of (1.1) such that

$$0 < r < \|u\| \leq p.$$

Consequently, (1.1) has at least a positive solution u for each $\lambda > \lambda_0$. □

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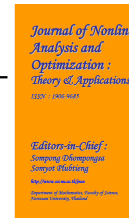

SOME SUBORDINATION RESULTS ASSOCIATED WITH GENERALIZED RUSCHEWEYH DERIVATIVES

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ABSTRACT. In this paper, we consider an unified class of functions of complex order associated with generalized Ruscheweyh derivative. We obtain a necessary and sufficient condition for functions to be in this class.

KEYWORDS : Starlike functions of complex order; Convex functions of complex order; Subordination; generalized Ruscheweyh derivative.
AMS Subject Classification: 30C45.

1. INTRODUCTION

Let A be the class of all analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U.

A function $f \in A$ is subordinate to an univalent function $g \in A$, written $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. Let Ω be the family of analytic functions $w(z)$ in the unit disk U satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in U$. Note that $f(z) \prec g(z)$ if there is a function $w(z) \in \Omega$ such that $f(z) = g(w(z))$.

Let $\phi(z)$ be an analytic function with positive real part on U and $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disc U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Ma and Minda [3] introduced and studied the class $S^*(\phi)$, consists of functions $f \in S$ for which

$$\tfrac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in U).$$

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Recently, Ravichandran et al.[5] defined classes related to the class of starlike functions of complex order as follows:

Definition 1.1. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \tfrac{1}{b}(\tfrac{zf'(z)}{f(z)} - 1) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in A$ satisfying

$$1 + \tfrac{1}{b}\tfrac{zf''(z)}{f'(z)} \prec \phi(z).$$

Following the work of Ma and Minda [3], Shanmugam and Sivasubramanian [7] obtained Fekete-Szegö inequality for the more general class $M_\alpha(\phi)$, defined by

$$\tfrac{\alpha z^2 f''(z) + zf'(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec \phi(z),$$

where $\phi(z)$ satisfies the conditions mentioned in Definition 1.1. Kamali et al.[2] introduced and studied a new class of functions $f \in T$ for which

$$Re(\tfrac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)}) > \beta,\ 0 \leq \alpha < 1, 0 \leq \beta < 1.$$

Shanmugum et. al.[8] remarked that the class of functions T is the familiar class of functions introduced and studied by Silverman [10]. In a later investigation, this particular class introduced by Kamali and Akbulut was generalized by Shanmugum et al. [9]. Shanmugum et. al.[8] introduced a more general class of complex order $M[b, \alpha](\phi)$ defined as follows:

Definition 1.2. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $M[b, \alpha](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \tfrac{1}{b}(\tfrac{\alpha z^3 f'''(z) + (1+2\alpha)z^2 f''(z) + zf'(z)}{\alpha z^2 f''(z) + zf'(z)} - 1) \prec \phi(z),\ 0 \leq \alpha < 1.$$

Clearly,

$$M[b, 0](\phi) \equiv C_b(\phi).$$

where the class $C_b(\phi)$ consists of functions $f \in A$ satisfying

$$1 + \tfrac{1}{b}\tfrac{zf''(z)}{f'(z)} \prec \phi(z).$$

In the present paper we shall need a recent generalization of the Ruscheweyh derivative which was introduced in [1].

Let $f \in A$, $\lambda \geq 0$ and $m \in \mathbb{R}$, $m > -1$, then we consider

$$D_\lambda^m f(z) = \tfrac{z}{(1-z)^{m+1}} * D_\lambda f(z), \quad z \in U,$$

where $D_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z), \quad z \in U.$

If $f \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U$ we obtain the power series expansion of the form

$$D_\lambda^m f(z) = z + \textstyle\sum_{n=2}^{\infty}[1 + (n-1)\lambda]\tfrac{(m+1)_{n-1}}{(1)_{n-1}} a_n z^n, \quad z \in U,$$

where $(a)_n$ is the Pochhammar symbol, given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{for } n = 0 \\ a(a+1)(a+2)...(a+n-1) & \text{for } n \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}$$

In the case $m \in \mathbb{N} = \{1, 2, 3, ...\}$, we have

$$D_\lambda^m f(z) = \frac{z(z^{m-1}D_\lambda f(z))^{(m)}}{m!}, \quad z \in U,$$

and for $\lambda = 0$ we obtain the mth Ruscheweyh derivative introduced in [6], $D_0^m = D^m$,

Since

$$\begin{aligned}\frac{(m+1)_{n-1}}{(1)_{n-1}} &= \frac{\Gamma(m+1+n-1)\Gamma(1)}{\Gamma(m+1)\Gamma(1+n-1)} \\ &= \frac{\Gamma(m+n)}{\Gamma(m+1)\Gamma(n)} \\ &= \frac{(m+n-1)!}{m!(n-1)!}.\end{aligned}$$

And

$$\begin{aligned}\sigma(m,n) &= \binom{m+n-1}{m} \\ &= \frac{(m+n-1)!}{m!(n-1)!}.\end{aligned}$$

So, we get

$$\sigma(m,n) = \frac{(m+1)_{n-1}}{(1)_{n-1}}$$

Hence

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty}[1+(n-1)\lambda]\sigma(m,n)a_n z^n, \quad z \in U,$$

So in this paper we introduce a more general class of complex order $H[b,\alpha,m,\lambda](\phi)$ which we define below.

Definition 1.3. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disc U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $H[b,\alpha,m,\lambda](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1 + \frac{1}{b}\left(\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\right) \prec \phi(z),$$
$$0 \leq \alpha < 1, \lambda \geq o, m > -1.$$

Clearly,

$$H[b,\alpha,m,\lambda](\phi) \equiv M[b,\alpha](\phi)$$

where $m = \lambda = 0$, [8].

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $H[b,\alpha,m,\lambda](\phi)$ which we have defined. The motivation of this paper is to generalize the results obtained by Shanmugum et. al.[8].

2. PRELIMINARIES AND NOTATIONS

In order to prove our main results, we need the following lemmas.

Lemma 2.1. [5] *Let ϕ be a convex function defined on U, $\phi(0) = 1$. Define $F(z)$ by*

$$F(z) = z\ exp\left(\int_0^z \frac{\phi(x)-1}{x}dx\right). \tag{2.1}$$

Let $q(z) = 1 + c_1 z + ...$ be analytic in U. Then

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z), \tag{2.2}$$

if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$\frac{q(tz)}{q(sz)} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.3}$$

Lemma 2.2 ([4], Corrolary 3.4h.1,p.135)**.** *Let $q(z)$ be univalent in U and let $\varphi(z)$ be analytic in a domain containing $q(U)$. If $\frac{zq'(z)}{\varphi(q(z))}$ is starlike, then*

$$zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z)),$$

implies that $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

3. MAIN RESULTS

3.1. **Subordination Results.** Applying Lemma 2.1 we have

Theorem 3.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b, \alpha, m, \lambda](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s[\alpha z^2 (D_\lambda^m f)''(tz) + z(D_\lambda^m f)'(tz)]}{t[\alpha z^2 (D_\lambda^m f)''(sz) + z(D_\lambda^m f)'(sz)]}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.1}$$

Proof. Define the function $p(z)$ by

$$p(z) = \left(\frac{\alpha z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{z}\right)^{\frac{1}{b}}. \tag{3.2}$$

By taking logarithmic derivative of $p(z)$ given by (3.2), we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left\{\frac{\alpha z^3 (D_\lambda^m f)'''(z) + (1+2\alpha) z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\right\}. \tag{3.3}$$

Now, by definition 1.3

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}\left\{\frac{\alpha z^3 (D_\lambda^m f)'''(z) + (1+2\alpha) z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2 (D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\right\} \prec \phi(z), \quad (0 \leq \alpha < 1). \tag{3.4}$$

Then applying Lemma 2.1 we get the result.
This completes the proof of Theorem 3.1. □

Putting $\lambda = 0$ in Theorem 3.1. Then we have the Ruscheweyh derivative and we get the following new result:

Corollary 3.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b, \alpha, m,](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s[\alpha z^2 (D^m f)''(tz) + z(D^m f)'(tz)]}{t[\alpha z^2 (D^m f)''(sz) + z(D^m f)'(sz)]}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.5}$$

And putting $m = \lambda = 0$ in Theorem 3.1 gives Theorem 2.1 [8]. Then we have.

Corollary 3.2. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b, \alpha,](\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{s[\alpha z^2 f''(tz) + zf'(tz)]}{t[\alpha z^2 f''(sz) + zf'(sz)]}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.6}$$

And putting $\alpha = m = \lambda = 0$ in Theorem 3.1 gives a result in Definition 1.1 [5]. Then we have

Corollary 3.3. *Let $\phi(z)$ and $F(z)$ be as in Lemma 2.1. The function $f \in H[b](\phi)$ if and only if for all $|s| \le 1$ and $|t| \le 1$, we have*

$$(\frac{sf'(tz)}{tf'(sz)})^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)} \tag{3.7}$$

Theorem 3.2. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b,\alpha,m,\lambda](\phi)$. Then*

$$\frac{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b. \tag{3.8}$$

Proof. Define the functions $p(z)$ and $q(z)$ by

$$p(z) = (\tfrac{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{z})^{\frac{1}{b}},\ q(z) = \tfrac{F(z)}{z}$$

Then a computation yields

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\{\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)} - 1\}, \tag{3.9}$$

now, by Definition 1.3 we have

$$1+\frac{zp'(z)}{p(z)} = 1+\frac{1}{b}\{\frac{\alpha z^3(D_\lambda^m f)'''(z) + (1+2\alpha)z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z) + z(D_\lambda^m f)'(z)}-1\} \prec \phi(z), \tag{3.10}$$

and

$$\tfrac{zq'(z)}{q(z)} = \tfrac{zF'(z)}{F(z)} - 1 = \phi(z) - 1.$$

Since $f \in H[b,\alpha,m,\lambda](\phi)$, we have

$$\tfrac{zp'(z)}{p(z)} = \tfrac{1}{b}\{\tfrac{\alpha z^3(D_\lambda^m f)'''(z)=(1+2\alpha)z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)} - 1\} \prec \phi(z) - 1 = \tfrac{zq'(z)}{q(z)}.$$

so

$$\tfrac{zp'(z)}{p(z)} \prec \tfrac{zq'(z)}{q(z)}.$$

Now in Lemma 2.2 putting $\varphi(p(z)) = \frac{1}{p(z)}$ and $\varphi(q(z)) = \frac{1}{q(z))}$ we get that
$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}$ implies that $p(z) \prec q(z)$
and $(p(z))^b \prec (q(z))^b$
Hence
$\frac{\alpha z^2(D_\lambda^m f)''(z)+z(D_\lambda^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b$.
This completes the proof of Theorem 3.2. □
□

Putting $\lambda = 0$ in Theorem 3.2. Then we have the Ruscheweyh derivative and we get the following new result.

Corollary 3.4. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b,\alpha,m](\phi)$. Then*

$$\frac{\alpha z^2(D^m f)''(z) + z(D^m f)'(z)}{z} \prec (\frac{F(z)}{z})^b. \tag{3.11}$$

And putting $m = \lambda = 0$ in Theorem 3.2 gives Theorem 2.3 [8]. Then we have.

Corollary 3.5. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b,\alpha](\phi)$. Then*

$$\frac{\alpha z^2 f''(z)+zf'(z)}{z} \prec (\frac{F(z)}{z})^b$$

Also putting $\alpha = m = \lambda = 0$ in Theorem 3.2 we have the following new result.

Corollary 3.6. *Let ϕ be starlike with respect to 1 and $F(z)$ is given by (2.1) be starlike. If $f \in H[b](\phi)$. Then*

$$f'(z) \prec (\frac{F(z)}{z})^b$$

3.2. **Coefficients Estimates.** This section is about the class β-convex functions involving complex order defined as follows.

Definition 3.7. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $H[b,\beta,m,\lambda](\phi)$ consists of all analytic functions $f \in A$ satisfying

$$1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\} \prec \phi(z),\ 0 \le \beta \le 1, \lambda \ge o, m > -1.$$

We note that, for $m = \lambda = 0$ we get $H[b,\beta,m,\lambda](\phi) \equiv M_{\beta,b}(\phi)$, [8].

To prove our main result of this section, we need the following:

Lemma 3.8. [5] *If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is a function with positive real part. Then*

$$|c_2 - \mu c_1^2| \le 2max\{1, |2\mu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2},\quad p(z) = \frac{1+z}{1-z}.$$

Our main result is the following

Theorem 3.3. *Let $0 \le \beta \le 1$. Further let $\phi(z) = 1 + B_1 z + B_2 z^2 + ..., z \in U$, where $B_n' s$ are real with $B_1 > 0$ and $B_2 \ge 0$. If $f(z)$ given by (1.1) belongs to $H[b,\beta,m,\lambda](\phi)$. Then*

$$|a_3 - \mu a_2^2| \le$$
$$\frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)} max\{1, |\frac{B_2}{B_1} + \frac{bB_1}{(1+\beta)^2}(1+3\beta-\mu\frac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})c_1^2|\}.$$

Proof. If $f(z) \in H[b,\beta,m,\lambda](\phi)$, then there is a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\} = \phi(w(z)). \tag{3.12}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots. \tag{3.13}$$

Since $w(z)$ is a Schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1$.
Define the function $p(z)$ by

$$\begin{aligned} p(z) &= 1+\frac{1}{b}\{(1-\beta)(\frac{z(D_\lambda^m f)'(z)}{(D_\lambda^m f)(z)})+\beta(1+\frac{z(D_\lambda^m f)''(z)}{(D_\lambda^m f)'(z)})-1\} \\ &= 1 + b_1 z + b_2 z^2 + \ldots . \end{aligned} \tag{3.14}$$

From (3.13) we get

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}[c_1 z + (c_2 - \frac{c_1^2}{2})z^2 + (c_3 - \frac{c_1^3}{4} - c_1 c_2)z^3 + ...] \ . \tag{3.15}$$

Since

$$\phi(z) = 1 + B_1 z + B_2 z^2 + ..., z \in U,$$

so, we get

$$\phi(\frac{p_1(z) - 1}{p_1(z) + 1}) = 1 + \frac{1}{2}B_1 c_1 z + [\frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{B_2 c_1^2}{4}]z^2 + ... \ . \tag{3.16}$$

Using (3.12),(3.14) and (3.15) we have

$$p(z) = \phi(\frac{p_1(z) - 1}{p_1(z) + 1}), \tag{3.17}$$

hence

$$1 + b_1 z + b_2 z^2 + ... = 1 + \frac{1}{2}B_1 c_1 z + [\frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{B_2 c_1^2}{4}]z^2 + ... \ . \tag{3.18}$$

Equating the coefficients in (3.18) we get

$$b_1 = \frac{1}{2}B_1 c_1, \tag{3.19}$$

$$b_2 = \frac{1}{2}(B_1(c_2 - \frac{1}{2}c_1^2)) + \frac{1}{4}B_2 c_1^2. \tag{3.20}$$

For $f(z)$ in (1.1) we obtain from (3.14) that

$$\begin{aligned} 1 + \frac{1}{b}\{(m+1)(1+\lambda)(1+\beta)a_2 z + [(m+1)(m+2)(1+2\lambda)(1+2\beta)a_3 - \\ (m+1)^2(1+\lambda)^2(1+3\lambda)a_2^2]z^2 + ...\} = 1 + b_1 z + b_2 z^2 + ... \end{aligned} \tag{3.21}$$

Equating the coefficients in (3.21) we get

$$a_2 = \frac{b b_1}{(m+1)(1+\lambda)(1+\beta)}, \tag{3.22}$$

$$a_3 = \frac{b b_2 + (m+1)^2(1+\lambda)^2(1+3\beta)a_2^2}{(m+1)(m+2)(1+2\lambda)(1+2\beta)}. \tag{3.23}$$

By applying (3.19) and (3.20) in (3.22) and (3.23) respectively we obtain

$$a_2 = \frac{b B_1 c_1}{2(m+1)(1+\lambda)(1+\beta)}, \tag{3.24}$$

$$\begin{aligned} a3 = \frac{b B_1 c_2}{2(m+1)(m+2)(1+2\lambda)(1+2\beta)} + \\ \frac{c_1^2}{4(m+1)(m+2)(1+2\lambda)}[\frac{1+3\beta}{(1+\beta)^2}b^2 B_1^2 - b(B_1 - B_2)]. \end{aligned} \tag{3.25}$$

Now, we have

$$a_3 - \mu a_2^2 = \frac{b B_1}{2(m+1)(m+2)(1+2\lambda)(1+2\beta)}[c_2 - \nu c_1^2], \tag{3.26}$$

where

$$\nu = \tfrac{1}{2}[1 - \tfrac{B_2}{B_1} - \tfrac{b B_1}{(1+\beta)^2}(1 + 3\beta - \mu\tfrac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})].$$

Then, applying lemma 3.8 on (3.26) we obtain

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)}|c_2 - \nu c_1^2| \leq \frac{B_1|b|}{2(m+1)(m+2)(1+2\beta)(1+2\lambda)} 2max\{1, |2\nu - 1|\},$$

for

$$\nu = \frac{1}{2}[1 - \frac{B_2}{B_1} - \frac{bB_1}{(1+\beta)^2}(1 + 3\beta - \mu\frac{(m+2)(1+2\lambda)(1+2\beta)}{(m+1)(1+\lambda)^2})].$$

This completes the proof of Theorem 3.3. □

□

Putting $m = \lambda = 0$ in Theorem 3.3 gives Theorem 3.3 in [8]. Then we have

Corollary 3.9. *Let $0 \leq \beta \leq 1$. Further let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ..., z \in U$, where $B_n' s$ are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1.1) belongs to $H[b, \beta](\phi)$.Then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(1+2\beta)} max\{1, |\frac{B_2}{B_1} + (1 - 2\mu + \beta(3 - 4\mu))\frac{bB_1}{(1+\beta)^2}|\}.$$

Putting $m = \lambda = \beta = 0$ in Theorem 3.3, gives a result obtained in [5]. Then we have

Corollary 3.10. *Let $0 \leq \beta \leq 1$. Further let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ..., z \in U$, where $B_n' s$ are real with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1.1) belongs to $H[b](\phi)$. Then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2} max\{1, |\frac{B_2}{B_1} + (1 - 2\mu)bB_1|\}.$$

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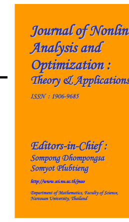

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A GENESIS OF GENERAL KKM THEOREMS FOR ABSTRACT CONVEX SPACES: REVISITED

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ABSTRACT. In our previous work [4], we obtained three general KKM type theorems A, B, and C for abstract convex spaces. In this paper, we show that these three theorems are mutually equivalent. Actually, by adopting a method of making new abstract convex spaces from old, we give a direct proof of Theorem C from Theorem B.

KEYWORDS : Abstract convex space; (partial) KKM principle; (partial) KKM space.
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1. INTRODUCTION

The celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929 is concerned with certain types of multimaps later called the KKM maps. The KKM theory, first named by the author, is the study of applications of equivalent formulations or generalizations of the KKM theorem. Actually the KKM theorem has several hundred generalizations in the literature.

Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify many known results in the theory; see [1, 3].

In our previous work [2], we reviewed some known facts on abstract convex spaces and obtained three general KKM type theorems which are equivalent or can be extended to Theorems A, B, and C in [4], resp. Each of them contains a large number of previously known particular forms which are generalizations, imitations, or modifications of the original KKM theorem due to many other authors. In [4], we recalled some historically important previous particular versions of our KKM type theorems in order to give a short history on each of them. Moreover, further remarks on related works were given in [5, 6].

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Note that the proofs of Theorems B and C in [4] are based on Theorem A, which is an abstract form of the original KKM Theorem. Some other proofs of Theorem C are also given in [5].

This paper is a continuation of [4]. In this paper, by adopting a method of making new abstract convex spaces from old, we give a direct proof of Theorem C from Theorem B. Consequently, Theorems A, B, and C are mutually equivalent.

2. ABSTRACT CONVEX SPACES

For the concepts of abstract convex spaces and KKM spaces, the reader may consult with the references in [3–6].

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all } \; A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

We had the following diagram for triples $(E, D; \Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Partial KKM space $\Longrightarrow$ Abstract convex space.

3. GENERAL KKM THEOREMS A, B, AND C

In [4], we gave standard forms of the KKM type theorems as follows:

Theorem A. *Let $(E, D; \Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E, D, E)$ [resp., $1_E \in \mathfrak{KO}(E, D, E)$], and $G : D \multimap E$ a multimap satisfying*

(1) *G has closed [resp., open] values; and*
(2) *$\Gamma_N \subset G(N)$ for any $N \in \langle D\rangle$ (that is, G is a KKM map).*

Then $\{G(y)\}_{y\in D}$ *has the finite intersection property.*

Further, if

(3) $\bigcap_{y\in M}\overline{G(y)}$ *is compact for some* $M \in \langle D\rangle$,

then we have

$$\bigcap_{y\in D}\overline{G(y)} \neq \emptyset.$$

Recall that Theorem A is a simple consequence of the definitions of the partial KKM principle or the KKM principle.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D}\overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D}\overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D}\overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the partial KKM principle we have a whole intersection property of the Fan type as follows:

Theorem B. *Let* $(E, D; \Gamma)$ *be a partial KKM space* [*that is,* $1_E \in \mathfrak{KC}(E, D, E)$] *and* $G : D \multimap E$ *a map such that*

(1) $\overline{G}$ *is a KKM map* [*that is,* $\Gamma_A \subset \overline{G}(A)$ *for all* $A \in \langle D\rangle$]; *and*

(2) *there exists a nonempty compact subset* K *of* E *such that either*

(i) $\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ *for some* $M \in \langle D\rangle$; *or*

(ii) *for each* $N \in \langle D\rangle$, *there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{y\in D'}\overline{G(y)} \subset K.$$

Then we have $K \cap \bigcap_{y\in D}\overline{G(y)} \neq \emptyset$.

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$;

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

Recall that conditions (i) and (ii) in Theorem B are usually called the *compactness conditions* or the *coercivity conditions*, and (ii) has numerous variations or particular forms appeared in a very large number of literature. Note that Theorem B can be easily deduced from the compact case of Theorem A; see [4, 5].

Theorem B can be extended for $F \in \mathfrak{KC}(E, D, Z)$ instead of $1_E \in \mathfrak{KC}(E, D, E)$) as the following in [4, 5]:

Theorem C. *Let* $(E, D; \Gamma)$ *be an abstract convex space,* Z *a topological space,* $F \in \mathfrak{KC}(E, D, Z)$, *and* $G : D \multimap Z$ *a map such that*

(1) $\overline{G}$ *is a KKM map w.r.t.* F; *and*

(2) *there exists a nonempty compact subset* K *of* Z *such that either*

(i) $\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ *for some* $M \in \langle D\rangle$; *or*

(ii) *for each* $N \in \langle D\rangle$, *there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$, $\overline{F(L_N)}$ *is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y\in D'}\overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$*; and*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

Remark 3.1. 1. Taking $\overline{K}$ instead of K, the closure notation in (i) and (ii) can be erased.

2. It is clear from Theorem C ⇒ Theorem B ⇒ Theorem A(closed case). In our previous proofs of Theorems B and C are based on Theorem A.

4. A WAY OF MAKING NEW ABSTRACT CONVEX SPACES FROM OLD

In this section we deduce Theorem C from Theorem B, based on a method of making new abstract convex spaces from old.

Consider the situation in Theorem C:

Definition 4.1. Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, and $F : E \multimap Z$ a map. Let $\Lambda_A := F(\Gamma_A)$ for each $A \in \langle D \rangle$. Then $(Z, D; \Lambda)$ is called the *abstract convex space induced by F*.

Let $Y \subset Z$ and $D' \subset D$ such that $\Lambda_B \subset Y$ for each $B \in \langle D' \rangle$. Then Y is called a Λ-*convex subset* of Z relative to D', and $(Y, D'; \Lambda')$ a *subspace* of $(Z, D; \Lambda)$ whenever $\Lambda' = \Lambda|_{\langle D' \rangle}$.

Proposition 4.2. *A KKM map* $G : D \multimap Z$ *on an abstract convex space* $(E, D; \Gamma)$ *with respect to* $F : D \multimap Z$ *is simply a KKM map on the corresponding abstract convex space* $(Z, D; \Lambda)$ *induced by* F.

Proof. Simply note that $\Lambda_A := F(\Gamma_A) \subset G(A)$ for each $A \in \langle D \rangle$. □

Proposition 4.3. *For an abstract convex space* $(E, D; \Gamma)$*, the corresponding abstract convex space* $(Z, D; \Lambda)$ *induced by* $F : D \multimap Z$ *is a partial KKM space if and only if* $F \in \mathfrak{KC}(E, D, Z)$.

The abstract convex space $(Z, D; \Lambda)$ *induced by* $F : D \multimap Z$ *is a KKM space if and only if* $F \in \mathfrak{KC}(E, D, Z) \cap \mathfrak{KO}(E, D, Z)$.

Proof. $(Z, D; \Lambda)$ is a partial KKM space

$\Longleftrightarrow$ For every closed-valued KKM map $G : D \multimap Z$ (that is, $\Lambda_A = F(\Gamma_A) \subset G(A)$ for each $A \in \langle D \rangle$), it has the finite intersection property of map-values.

$\Longleftrightarrow$ For every closed-valued KKM map $G : D \multimap Z$ with respect F, it has the finite intersection property of map-values.

$\Longleftrightarrow$ $F \in \mathfrak{KC}(E, D, Z)$.

Similarly, for open-valued KKM map $G : D \multimap Z$, it has the finite intersection property of map-values $\Longleftrightarrow$ $F \in \mathfrak{KO}(E, D, Z)$. □

The following is our main result in this paper:

Proposition 4.4. *Theorems B and C are equivalent.*

Proof. By putting $E = Z$ and $F = 1_E$, Theorem C reduces to Theorem B. Now we show that Theorem C follows from Theorem B as follows.

Let $(Z, D; \Lambda)$ be the abstract convex space induced by F with $\Lambda_A := F(\Gamma_A) \subset G(A)$ for each $A \in \langle D \rangle$. Then

(1) $\overline{G}$ is a KKM map on $(Z, D; \Lambda)$.

(2) Condition (i) implies Theorem B(i) with $\overline{F(E)} \cap K$ instead of K.

In fact, since $F(\Gamma_A) \subset \overline{G}(A)$ for each $A \in \langle D \rangle$, we have $F(\Gamma_{\{y\}}) \subset \overline{G}(y)$ for each $y \in D$ and so

$$K \supset \bigcap_{y \in M} \overline{G}(y) \supset \bigcap_{y \in M} F(\Gamma_{\{y\}}) \cap \bigcap_{y \in M} \overline{G}(y).$$

Hence

$$\overline{F(E)} \cap K \supset \bigcap \{\overline{G(y)} \mid y \in M\} \text{ for some } M \in \langle D \rangle.$$

(3) Condition (ii) implies Theorem B(ii) with $\overline{F(E)} \cap K$ instead of K and with a compact Λ-convex subset $\overline{F(L_N)}$ of Z instead of L_N.

In fact, this can be shown by the following two facts:

(a) $A \in \langle D' \rangle$ implies $\Gamma_A \subset L_N$ and hence $\Lambda_A = F(\Gamma_A) \subset F(L_N) \subset \overline{F(L_N)}$. So $\overline{F(L_N)}$ is compact and Λ-convex.

(b) $K \supset \overline{F(L_N)} \cap \bigcap \{\overline{G(y)} \mid y \in D'\}$ by assumption and clearly we have $\overline{F(E)} \supset \overline{F(L_N)} \cap \bigcap \{\overline{G(y)} \mid y \in D'\}$. Hence

$$\overline{F(E)} \cap K \supset \overline{F(L_N)} \cap \bigcap \{\overline{G(y)} \mid y \in D'\}.$$

Consequently, replacing $(E, D; \Gamma),\ K,\ L_N$ in Theorem B by $(Z, D; \Lambda),\ \overline{F(E)} \cap K,\ \overline{F(L_N)}$, respectively, all of the requirements of Theorem B are satisfied. Therefore we have the conclusion

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

The statements (α) and (β) are routine. □

5. COMMENTS ON RELATED WORKS

In this section, we give some comments on our previous versions of generalized KKM theorems A, B, and C appeared in our previous works:

(1) Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify many known results in the theory on convex spaces, H-spaces, G-convex spaces, and others; see [1].

(2) In 2008 and 2010 [1, 3], we established the basis of the KKM theory and gave particular forms of Theorems A and B. Note that [3] contains some incorrectly stated statements such as (VI), Theorem 4, (XVI), and (XVII). These can be corrected easily.

(3) In 2009 [2], we reviewed some known facts on abstract convex spaces and obtained three general KKM type theorems which are equivalent or can be extended to Theorems A, B, and C in this paper, resp. Each of them contains a large number of previously known particular forms which are generalizations, imitations, or modifications of the original KKM theorem due to many other authors.

(4) In 2011 [4], we established the basic KKM theorems A, B, and C, and recalled some historically important previous particular versions of these theorems in order to give a short history on each of them. Moreover, further comments on related works are given.

(5) In 2011 [5], we deduced Theorems B and C from Theorem A and added a new proof of Theorem C. Some corrections of the coercivity conditions appeared in previous versions of such KKM type theorems were given.

(6) In 2012 [6], we give several generalizations of the 1984 KKM theorem of Ky Fan and some known applications in order to recover the close relationship among them. Theorems A, B, and C are stated as the final ones in the evolution of the KKM theorem from 1929. It is stated that, as far as the author knows, Theorem C contains several hundred generalizations of the KKM theorem appeared in the existing literature; see the references therein.

(7) Further study on our new method of making abstract convex spaces from old will appear.

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COUPLED COMMON FIXED POINT THEOREMS OF CIRIC TYPE $g-$WEAK CONTRACTIONS WITH CLR_g PROPERTY

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In this paper we define Ciric type $g-$weak contractions in the context of coupled fixed points and prove the existence of coupled common fixed points for a pair of w-compatible maps using CLR_g property. Further, we consider a pair of maps satisfying a new class of implicit relation with CLR_g property and prove the existence of coupled common fixed points. The results of Long, Rhoades and Rajovic [15] and our results are independent. Examples are provided to illustrate this phenomenon.. ...

KEYWORDS : Coupled fixed point, coupled coincidence point, coupled common fixed point, $w-$compatible maps, property (E. A), CLR_g property, implicit relation..
AMS Subject Classification: 47H10, 54H25

1. INTRODUCTION

In 2006, Bhaskar and Lakshmikantham [9] established a coupled contraction principle and proved the existence of coupled fixed points in partially ordered compete metric spaces. In 2009, Lakshmikantham and Ciric [14] introduced the concept of commuting maps, coupled coincidence points and coupled common fixed points and established coupled coincidence, coupled common fixed point theorems in partially ordered complete metric spaces. In 2010, Abbas, Khan, Radenovic [3] introduced the concept of $w-$compatible maps in the context of coupled fixed points in cone metric spaces. Recently Long, Rhoades and Rajovic [15] established coupled coincidence point theorems in complete metric spaces and cone metric spaces too. Some works in this line of research in different spaces are [3, 8, 10, 13, 22, 23, 24, 25].

Throughout this paper, $\mathbb{N}$ denotes the set of all natural numbers, R is the set of all real numbers and $R_+ = [0, \infty)$.

In the following definitions, we suppose that X is a non-empty set.

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Definition 1.1. [9] An element (x, y) in $X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.2. [14] An element (x, y) in $X \times X$ is called a *coupled coincidence point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$.

Definition 1.3. [14] An element (x, y) in $X \times X$ is called a *coupled common fixed point* of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.4. [14] The mappings $F : X \times X \to X$ and $g : X \to X$ are called *commutative* if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

Definition 1.5. [3] The mappings $F : X \times X \to X$ and $g : X \to X$ are called *$w-$compatible* if $gF(x, y) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

We denote $\Phi_1 = \big\{\varphi/\varphi : R_+ \to R_+$ satisfying φ is non-decreasing and $\lim_{n\longrightarrow\infty} \varphi^n(t) = 0$ for $t > 0\big\}$.

Long, Rhoades and Rajovic [15] proved the following theorem in complete metric spaces.

Theorem 1.1. [15] *Let (X, d) be a complete metric space. Assume that $F : X \times X \to X$, $g : X \to X$ are two mappings satisfying*
(H_1) : *there exists $\varphi \in \Phi_1$ such that*

$$d\big(F(x, y), F(u, v)\big) \le \varphi\big(M_F^g(x, y, u, v)\big) \text{ for all } x, y, u, v \in X; \tag{1.1}$$

where

$$M_F^g(x, y, u, v) = max\{d(gx, gu),\ d(gy, gv),\ d(gx, F(x, y)),\ d(gu, F(u, v)), d(gy, F(y, x)),\ d(gv, F(v, u)),\ \tfrac{d(gx,F(u,v))+d(gu,F(x,y))}{2},\ \tfrac{d(gy,F(v,u))+d(gv,F(y,x))}{2}\},$$

(H_2) : $F(X \times X) \subseteq g(X)$ *and $g(X)$ is a closed subset of X.*

Then (i) F and g have a coupled coincidence point in X and
(ii) F and g have a unique common fixed point whenever F and g are $w-$compatible.

Popa [16] introduced *implicit relations* and established the existence of fixed points and common fixed points in metric spaces. The importance of using an implicit relation in proving fixed point theorems is that it includes many known contractive conditions so that the known results follow as corollaries. Some works on this line of research are [4, 5, 6, 7, 17].

In 2002, Amari and Moutawakil [1] introduced the notion of property ($E.\ A$) and proved the existence of common fixed points for a pair of self maps. Many researchers [2, 11, 18] worked in this direction.

In 2011, Sintunavarat and Kumam [20] introduced a new property called *common limit in the range of* g (CLR_g) in both metric and fuzzy metric spaces and proved common fixed point theorems in fuzzy metric specs. CLR_g property never requires the closedness of the range space of g for the existence of fixed points. For more details and works on CLR_g property we refer [19, 20, 21].

Recently Jain, Tas, Sanjay Kumar and Gupta [13] extended the notation of property ($E.\ A$) and CLR_g property to the context of coupled fixed points in metric spaces and fuzzy metric spaces and proved the coupled fixed point results in fuzzy metric spaces.

Definition 1.6. [13] Let (X, d) be a metric space. Two mappings $F : X \times X \to X$ and $g : X \to X$ are said to satisfy *property* ($E.\ A$) if there exist

two sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$\lim_{n\longrightarrow\infty} F(x_n, y_n) = \lim_{n\longrightarrow\infty} g(x_n) = t_1$ and $\lim_{n\longrightarrow\infty} F(y_n, x_n) = \lim_{n\longrightarrow\infty} g(y_n) = t_2$
for some $t_1, t_2 \in X$.

Definition 1.7. [13] Let (X, d) be a metric space. Two mappings $F : X \times X \to X$ and $g : X \to X$ are said to satisfy *common limit in the range of* g (CLR_g) *property* if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$\lim_{n\longrightarrow\infty} F(x_n, y_n) = \lim_{n\longrightarrow\infty} g(x_n) = gt_1$ and $\lim_{n\longrightarrow\infty} F(y_n, x_n) = \lim_{n\longrightarrow\infty} g(y_n) = gt_2$
for some $t_1, t_2 \in X$.

Remark 1.8. If F and g satisfy 'property ($E.A$) with range of g is closed' then F and g satisfy 'CLR_g property'. But its converse is not true due to the following example.

Example 1.9. Let $X = (-4, 4)$. We define $F : X \times X \to X$ and $g : X \to X$ by

$$F(x, y) = \frac{x - y}{4},\ x, y \in X$$

and

$$gx = \frac{x}{2},\ x \in X.$$

Here $g(X) = (-2, 2)$ is not a closed set. Now we choose two sequences $\{x_n\}$ and $\{y_n\}$ in X by
$x_n = -2 - \frac{1}{n}$ and $y_n = 2 + \frac{1}{n},\ n = 1, 2, 3\ldots$. Hence

$$\lim_{n\longrightarrow\infty} F(x_n, y_n) = \lim_{n\longrightarrow\infty} g(x_n) = -1 = g(-2)$$

and

$$\lim_{n\longrightarrow\infty} F(y_n, x_n) = \lim_{n\longrightarrow\infty} g(y_n) = 1 = g(2).$$

Thus the pair (F, g) satisfy CLR_g property.

Hence CLR_g property is more general than property ($E.A$) with $g(X)$ is closed.

In this paper, we prove a coupled common fixed point theorem for Ciric type $g-$weak contractions by using CLR_g property. Further, we consider a pair of maps satisfying a new class of implicit relation with CLR_g property and prove the existence of coupled common fixed points.

In the following, we define
$\Phi = \{\varphi/\varphi : R_+ \to R_+$ satisfying φ is continuous and $\varphi(t) = 0$ if and only if $t = 0\}$.
Here we note that the classes of functions Φ_1 and Φ are independent, in the sense that neither Φ_1 is contained in Φ nor Φ is contained in Φ_1. We illustrate it in the following examples.

Example 1.10. $\varphi = [0, +\infty) \to [0, +\infty)$ defined by $\varphi(t) = \begin{cases} t^2 & if\ t \in [0, 1] \\ \frac{1}{t} & if\ t \in (1, \infty). \end{cases}$

Clearly $\varphi \in \Phi$, but φ is not an increasing function. Hence φ does not belong to Φ_1.

Example 1.11. $\varphi = [0, +\infty) \to [0, +\infty)$ defined by $\varphi(t) = \begin{cases} \frac{t^2}{12} & if\ t \in [0, 1] \\ \frac{t}{10} & if\ t \in (1, \infty). \end{cases}$

Clearly $\varphi \in \Phi_1$, but φ is not a continuous function. Hence φ does not belong to Φ.

2. PRELIMINARIES

We define Ciric type $g-$weak contractions and a class of implicit relation in the context of coupled fixed points.

Definition 2.1. Let (X, d) be a metric space. Let $F : X \times X \to X,\ g : X \to X$ be two maps of a metric space X. We say that F is a *Ciric type g-weak contraction map* if there exists $\varphi \in \Phi$ such that

$$d(F(x, y), F(u, v)) \leq M(x, y, u, v) - \varphi(M(x, y, u, v)) \text{ for all } x, y, u, v \in X; \qquad (2.1)$$

where

$$M(x, y, u, v) = max\{d(gx, gu),\ d(gy, gv),\ d(gx, F(x, y)),\ d(gy, F(y, x)), d(gu, F(u, v)),$$
$$d(gv, F(v, u)),\ d(gx, F(u, v)), d(gy, F(v, u)),\ d(gu, F(x, y)),\ d(gv, F(y, x))\}.$$

Remark 2.2. Suppose that F and g satisfy the inequality (1.1) with $\varphi \in \Phi_1$. If φ is continuous then F is a Ciric type $g-$ weak contraction. But its converse need not be true (Example 2.3).
For, we assume that (1.1) holds.

$$\begin{aligned} i.e.,\ d(F(x, y), F(u, v)) &\leq \varphi(M_F^g(x, y, u, v)) \\ &\leq M(x, y, u, v) - (I - \varphi)M(x, y, u, v) \\ &= M(x, y, u, v) - \phi_\varphi(M(x, y, u, v)), \end{aligned}$$

where $\phi_\varphi = I - \varphi$ and it is clear that $\phi_\varphi(t) = 0$ if and only if $t = 0$.

Example 2.3. Let $X = [-1, 1)$ with the usual metric. We define $F : X \times X \to X$ and $g : X \to X$ by $F(x, y) = \begin{cases} \frac{1}{4} & if\ x \geq y \\ -\frac{1}{4} & if\ x < y; \end{cases}$ and $gx = \begin{cases} \frac{1}{2} & if\ x \neq 0 \\ 0 & if\ x = 0. \end{cases}$

We define $\varphi(t) = \frac{1}{8}t,\ t \geq 0$. Clearly $\varphi \in \Phi$ and F is a Ciric type $g-$weak contraction.

But for $x = 1,\ y = u = 0$ and $v = 1$, we have
$d(F(x, y), F(u, v) = \frac{1}{2} \not\leq \varphi(max\{\frac{1}{2},\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{2},\ \frac{1}{2}\}) = \varphi(\frac{1}{2})$ for any $\varphi \in \Phi$,
since $\varphi(t) < t$ for $t > 0$.
Hence the inequality (1.1) fails to hold.

Definition 2.4. Let Λ be the set of all continuous functions $T : R_+^{11} \to R$ satisfying the following conditions:
(T_1) : there exists a mapping $f : R_+ \to R_+,\ f(t) < t$ for $t > 0$ such that
$T(u, 0, 0, 0, 0, v_1, v_2, v_1, v_2, 0, 0) \leq 0$ for $u > 0$ or
$T(u, v_1, v_2, 0, 0, 0, 0, v_1, v_2) \leq 0$ for $u > 0$
implies that $u \leq f(max\{v_1, v_2\})$.

(T_2) : $T(u, 0, 0, u, u, 0, 0, 0, 0, 0, u) > 0$ for $u > 0$.

Example 2.5. $T(t_1, t_2, ..., t_{11}) = t_1 - k\ max\{t_2, t_3\}$, where $k \in [0, 1)$.
Let $T(u, v_1, v_2, 0, 0, 0, 0, v_1, v_2, v_1, v_2) = u - k\ max\{v_1, v_2\} \leq 0$
$i.e.,\ u \leq k\ max\{v_1, v_2\}$.
Thus $u \leq f(max\{v_1, v_2\})$ with $f(t) = kt$. Hence T_1 satisfied.
Also $T(u, 0, 0, u, u, 0, 0, 0, 0, 0, u) = u > 0$ for $u > 0$. Thus $T \in \Lambda$.

Example 2.6. $T(t_1,t_2,...,t_{11}) = t_1 - \varphi(max\{t_2,t_3,t_4,t_5,t_6,t_7,\frac{t_8+t_9}{2},\frac{t_{10}+t_{11}}{2}\})$
with $\varphi(t) < t$ for $t > 0,\ \varphi(t) = 0$ if and only if $t = 0$ and φ is continuous.
Let $u > 0$ and $T(u,v_1,v_2,0,0,0,0,v_1,v_2,v_1,v_2) = u - \varphi\big(max\{v_1,v_2\}\big) \leq 0$.
Hence $u \leq f(max\{v_1,v_2\})$ with $f = \varphi$.
Also $T(u,0,0,u,u,0,0,0,0,0,u) = u > 0$ for $u > 0$. Thus $T \in \Lambda$.

Example 2.7. $T(t_1,t_2,...,t_{11}) = t_1 - \alpha\frac{t_8t_9+t_{10}t_{11}}{1+t_2+t_3+t_4+t_5+t_6+t_7}$ where $0 \leq \alpha < 1$.
Let $u > 0,\ T(u,0,0,0,0,v_1,v_2,v_1,v_2,0,0) = u - \alpha\frac{v_1v_2}{1+v_1+v_2} \leq 0$.
$i.e.,\ u \leq \alpha\frac{v_1v_2}{1+v_1+v_2} \leq \alpha\ max\{v_1,v_2\}$. Hence $u \leq f(max\{v_1,v_2\})$ with $f(t) = \alpha$ for all $t \geq 0$. Also $T(u,0,0,u,u,0,0,0,0,0,u) = u > 0$ for $u > 0$. Thus $T \in \Lambda$.

Example 2.8. $T(t_1,t_2,...,t_{11}) = t_1 - (a_1t_2 + a_2t_3 + \cdots + a_{10}t_{11})$
where $\sum_{i=0}^{10} a_i < 1$. Let $u > 0$,
$T(u,0,0,0,0,v_1,v_2,v_1,v_2,0,0) = u - [(a_6+a_8)v_1 + (a_7+a_9)v_2] \leq 0$.
$i.e.,\ u \leq (a_6+a_8)max\{v_1,v_2\} + (a_7+a_9)max\{v_1,v_2\}$
$= (a_6+a_7+a_8+a_9)max\{v_1,v_2\}$.
Thus $u \leq f(max\{v_1,v_2\})$ with $f(t) = (a_6+a_7+a_8+a_9)t$.
Also $T(u,0,0,u,u,0,0,0,0,0,u) = u - (a_3+a_4+a_{11})u > 0$ for $u > 0$.
Hence $T \in \Lambda$.

3. MAIN RESULTS

The following is the main result of this section.

Theorem 3.1. *Let $(X,\ d)$ be a metric space and $F : X \times X \to X,\ g : X \to X$ be two maps, the pair (F,g) satisfy CLR_g property and F is a Ciric type $g-$weak contraction map then F and g have a coupled coincidence point. Further, F and g have a unique coupled common fixed point provided F and g are w-compatible.*

Proof. Since F and g satisfy CLR_g property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\longrightarrow\infty} F(x_n,y_n) = \lim_{n\longrightarrow\infty} g(x_n) = gx$ and $\lim_{n\longrightarrow\infty} F(y_n,x_n) = \lim_{n\longrightarrow\infty} g(y_n) = gy$ for some $x,y \in X$.
Now, we prove that $gx = F(x,y)$ and $gy = F(y,x)$.
Assume that $d(gx,F(x,y)) > 0$ or $d(gy,F(y,x)) > 0$.
Now, we consider

$$\begin{aligned} d(gx,F(x,y)) &\leq d(gx,F(x_n,y_n)) + d\big(F(x_n,y_n),F(x,y)\big) \\ &\leq d(gx,F(x_n,y_n)) + M(x_n,y_n,x,y) - \varphi(M(x_n,y_n,x,y)) \qquad (3.2) \end{aligned}$$

where

$$\begin{aligned} M(x_n,y_n,x,y) = max\{&d(gx_n,gx),\ d(gy_n,gy),\ d(gx_n,F(x_n,y_n)),d(gy_n,F(y_n,x_n)),\\ &d(gx,F(x,y)),d(gy,F(y,x)),d(gx_n,F(x,y)),\ d(gy_n,F(y,x)),\\ &d(gx,F(x_n,y_n)),d(gy,F(y_n,x_n))\}. \end{aligned}$$

On taking limits as $n \to \infty$, in $M(x_n,y_n,x,y)$, we get
$\lim_{n\longrightarrow\infty} M(x_n,y_n,x,y) = max\{d(gx,F(x,y)),\ d(gy,F(y,x))\}$.
Now, on taking limits as $n \to \infty$ in (3.2), we get
$d(gx,F(x,y)) \leq max\{d(gx,F(x,y)),\ d(gy,F(y,x))\}$

$$-\varphi(max\{d(gx, F(x,y)),\ d(gy, F(y,x))\})$$
$$< max\{d(gx, F(x,y)),\ d(gy, F(y,x))\}. \quad (3.3)$$

Similarly we get,

$$d(gy, F(y,x)) < max\{d(gx, F(x,y)),\ d(gy, F(y,x))\}. \quad (3.4)$$

Hence from (3.3) and (3.4) we get

$max\{d(gx, F(x,y)),\ d(gy, F(y,x))\} < max\{d(gx, F(x,y)),\ d(gy, F(y,x))\}$,
a contradiction.

Hence $gx = F(x,y)$ and $gy = F(y,x)$.

Thus (x,y) is a coupled coincidence point of F and g.

Let (x,y) and (x^*,y^*) be two coupled coincidence points of F and g.

Now, we prove that $gx = gx^*$ and $gy = gy^*$.

We assume that $d(gx, gx^*) > 0$ or $d(gy, gy^*) > 0$.
Now, we consider

$$\begin{aligned} d(gx, gx^*) &= d(F(x,y), F(x^*,y^*)) \\ &\leq M(x,y,x^*,y^*) - \varphi(M(x,y,x^*,y^*)) \\ &< M(x,y,x^*,y^*) \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} M(x,y,x^*,y^*) = max\{&d(gx, gx^*),\ d(gy, gy^*),\ d(gx, F(x,y)),\ d(gy, F(y,x)), \\ &d(gx^*, F(x^*,y^*)),\ d(gy^*, F(y^*,x^*)),\ d(gx, F(x^*,y^*)), \\ &d(gy, F(y^*,x^*)),\ d(gx^*, F(x,y)),\ d(gy^*, F(y,x))\} \\ = max\{&d(gx, gx^*),\ d(gy, gy^*)\}. \end{aligned}$$

Similarly we get

$$d(gy, gy^*) < max\{d(gx, gx^*),\ d(gy, gy^*)\}. \quad (3.6)$$

Henc, from (3.5) and (3.6), we get

$max\{d(gx, gx^*),\ d(gy, gy^*)\} < max\{d(gx, gx^*),\ d(gy, gy^*)\}$,
a contradiction.

Hence $gx = gx^*$ and $gy = gy^*$. (3.7)

Now, we prove that $gx = gy^*$ and $gy = gx^*$.

We assume that $d(gx, gy^*) > 0$ or $d(gy, gx^*) > 0$.
Consider

$$\begin{aligned} d(gx, gy^*) &= d(F(x,y), F(y^*,x^*)) \\ &\leq M(x,y,y^*,x^*) - \varphi(M(x,y,y^*,x^*)) \\ &< M(x,y,y^*,x^*) \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} M(x,y,y^*,x^*) = max\{&d(gx, gy^*),\ d(gy, gx^*),\ d(gx, F(x,y)),\ d(gy, F(y,x)), \\ &d(gy^*, F(y^*,x^*)),\ d(gx^*, F(x^*,y^*)),\ d(gx, F(y^*,x^*)), \\ &d(gy, F(x^*,y^*)),\ d(gy^*, F(x,y)),\ d(gx^*, F(y,x))\} \\ = max\{&d(gx, gy^*),\ d(gy, gx^*)\}. \end{aligned}$$

Similarly we get

$$d(gy, gx^*) < max\{d(gx, gy^*),\ d(gy, gx^*)\}. \quad (3.9)$$

Hence, from (3.8) and (3.9), we get

$max\{d(gx, gy^*),\ d(gy, gx^*)\} < max\{d(gx, gy^*),\ d(gy, gx^*)\}$,
a contradiction. Hence

$gx = gy^*$ and $gy = gx^*$. (3.10)
Thus, from (3.7) and (3.10), we get

$$gx = gx^* = gy = gy^*. \quad (3.11)$$

Let (x, y) be a coupled coincidence point of F and g, hence $gx = F(x, y)$ and $gy = F(y, x)$. Let us take $u = gx$ and $v = gy$. Since F and g are $w-$compatible, we have

$$gu = ggx = gF(x, y) = F(gx, gy) = F(u, v)$$

and

$$gv = ggy = gF(y, x) = F(gy, gx) = F(v, u).$$

Hence (u, v) is a coupled coincidence point, hence from (3.7) we have
$gu = gx$ and $gv = gy$. Thus
$u = gx = gu = F(u, v)$ and $v = gy = gv = F(v, u)$. (3.12)

Hence (u, v) is a coupled common fixed point.
And from (3.11) we have $u = v$.
Let (u_1, v_1) be another coupled common fixed point of F and g.
i.e., $u_1 = gu_1 = F(u_1, v_1)$ and $v_1 = gv_1 = F(v_1, u_1)$ (3.13)
From (3.11), (3.12) and (3.13), we get
$u_1 = gu_1 = gu = u$ and $v_1 = gv_1 = gv = v$.
Hence coupled common fixed point is unique. □

Corollary 3.1. *Let $(X,\ d)$ be a metric space and $F : X \times X \to X,\ g : X \to X$ be two maps, the pair (F, g) satisfy property $(E.A)$, $g(X)$ is closed and F is a Ciric type $g-$weak contraction map then F and g have a coupled coincidence point. Further, F and g have a unique coupled common fixed point provided F and g are w-compatible.*

Proof. Since the pair (F, g) satisfies property $(E.A)$ and $g(X)$ is closed, by Remark 1.8 we have F and g satisfy CLR_g property and hence by Theorem 3.1 the conclusion of this corollary follows. □

Example 3.2. Let $X = [0, 1)$ with the usual metric.
We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x-y}{3} & if\ x, y \in [0, \frac{1}{3})\ with\ x \geq y \\ \frac{1}{2} & if\ x, y \in [\frac{1}{3}, 1)\ with\ x \geq y \\ 0 & otherwise; \end{cases}$$

and $g : X \to X$ defined by $gx = \begin{cases} x & if\ x \in [0, \frac{1}{3}) \\ \frac{9}{10} & if\ x \in [\frac{1}{3}, 1). \end{cases}$

Now, we choose the sequences $\{x_n\}$ and $\{y_n\}$ in X by $x_n = \frac{1}{n+3}$ and $y_n = \frac{1}{3n+1}$, $n = 1, 2, 3, ...$, then

$\lim\limits_{n\longrightarrow\infty} F(x_n, y_n) = \lim\limits_{n\longrightarrow\infty} g(x_n) = 0 = g0$ and

$\lim\limits_{n\longrightarrow\infty} F(y_n, x_n) = \lim\limits_{n\longrightarrow\infty} g(y_n) = 0 = g0.$

Hence the pair (F, g) satisfy CLR_g property.

We define $\varphi : R_+ \to R_+$ by $\varphi(t) = \frac{1}{8}t,\ t \geq 0$; here we observe that $t - \varphi(t)$ is an increasing function.

Now, we consider the following cases to check the inequality (2.1).

First we consider the case $x, y, u, v \in [0, \frac{1}{3})$.
Now, we have the following four subcases.

Subcase (*i*) : $x \geq y$ and $u \geq v$.
Now

$$d(F(x,y),\ F(u,v)) \leq \begin{cases} \frac{1}{3}[(x-u)+(v-y)] & if\ x \geq u,\ v \geq y \\ \frac{1}{3}[(x-u)+(y-v)] & if\ x \geq u,\ v < y \\ \frac{1}{3}[(u-x)+(v-y)] & if\ x < u,\ v \geq y \\ \frac{1}{3}[(u-x)+(y-v)] & if\ x < u,\ v < y. \end{cases} \tag{3.14}$$

and

$$M(x,y,u,v) = max\{|x-u|,\ |y-v|,\ \tfrac{2x+y}{3},\ y,\ \tfrac{2u+v}{3},\ v,\ |\tfrac{3x-u+v}{3}|,\ y,\ |\tfrac{3u-x+y}{3}|,\ v\}.$$

$$Hence\ (3.14) \leq \begin{cases} \frac{7}{8}[\frac{3x-u+v}{3}] & whenever\ (x \geq u, v \geq y)\ or\ (x \geq u, v < y) \\ \frac{7}{8}[\frac{3u-x+y}{3}] & whenever\ (x < u, v \geq y)\ or\ (x < u, v < y) \end{cases}$$
$$= M(x,y,u,v) - \varphi(M(x,y,u,v)).$$

Subcase (*ii*) : $x \geq y,\ u < v$.
In this subcase, we have

$$d(F(x,y),\ F(u,v)) = \frac{1}{3}(x-y) \leq \begin{cases} \frac{7}{8}x & if\ max\{x,y,u,v\} = x \\ \frac{7}{8}v & if\ max\{x,y,u,v\} = v \end{cases}$$
$$= M(x,y,u,v) - \varphi(M(x,y,u,v))$$

where

$$M(x,y,u,v) = max\{|x-u|,\ |y-v|,\ \tfrac{2x+y}{3}, y, u, \tfrac{2v+u}{3}, x,\ |y - \tfrac{v-u}{3}|,\ |u - \tfrac{x-y}{3}|,\ v\}.$$

Subcase (*iii*) : $x < y,\ u \geq v$.
By symmetry in the inequality (2.1), it is clear that the inequality (2.1) holds as in Sub case (*ii*).

Subcase (*iv*) : $x < y,\ u < v$. Inequality (2.1) holds trivially.

In the following cases, *i.e.*,
(*i*) $x, y, u, v \in [\frac{1}{3}, 1)$ or $u, v \in [0, \frac{1}{3})$ and $x, y \in [\frac{1}{3}, 1)$ with $x \geq y,\ u < v$;
(*ii*) $u \in [0, \frac{1}{3})$ and $x, y, v \in [\frac{1}{3}, 1)$ with $x \geq y$;
(*iii*) $v \in [0, \frac{1}{3})$ and $x, y, u \in [\frac{1}{3}, 1)$ with $x \geq y$.
In these cases, we have
$d\big(F(x,y),\ F(u,v)\big) = \frac{1}{2} \leq \frac{7}{8}\frac{9}{10} = M(x,y,u,v) - \varphi(M(x,y,u,v))$,
where $M(x,y,u,v) = \frac{9}{10}$.

Now we consider the following cases:
(*i*) $x, y \in [0, \frac{1}{3})$ and $u, v \in [\frac{1}{3}, 1)$ with $x \geq y,\ u < v$;
(*ii*) $x, y, u \in [0, \frac{1}{3})$ and $v \in [\frac{1}{3}, 1)$ with $x \geq y$;
(*iii*) $x, y, v \in [0, \frac{1}{3})$ and $u \in [\frac{1}{3}, 1)$ with $x \geq y$.
In these cases, we have
$d\big(F(x,y),\ F(u,v)\big) = \frac{1}{3}(x-y) \leq \frac{7}{8}\frac{9}{10} = M(x,y,u,v) - \varphi(M(x,y,u,v))$,
where $M(x,y,u,v) = \frac{9}{10}$.

Also, we consider the following case :
$x, y \in [0, \frac{1}{3})$ and $u, v \in [\frac{1}{3}, 1)$ with $x \geq y,\ u \geq v$ then
$d\big(F(x,y),\ F(u,v)\big) = \frac{3-2(x-y)}{6} \leq \frac{7}{8}\frac{9}{10} = M(x,y,u,v) - \varphi(M(x,y,u,v))$,
where $M(x,y,u,v) = \frac{9}{10}$.

Further, we have the following cases :
(i) $x < y,\ u < v;\ x, y, u, v \in X$;
(ii) $u,\ v$ are in different intervals and $x,\ y$ are in different intervals;
(iii) $x,\ y$ are in different intervals with $u < v$;
(iv) $u,\ v$ are in different intervals with $x < y$.
In these cases, we have $d\big(F(x,y),\ F(u,v)\big) = 0$.

Since the inequality (2.1) is symmetric, the other cases $i.e.,\ x$ is replaced by u and y is replaced by v also hold.
Now at $(x,y) = (0,0)$ we have $gx = F(x,y),\ gy = F(y,x)$ and $gF(x,y) = F(gx,gy)$. Hence F and g satisfy all the hypotheses of Theorem 3.1 and $(0,0)$ is a coupled common fixed point. In fact $(0,0)$ is unique.

Remark 3.3. In Theorem 3.1, we considered Ciric type $g-$ weak contraction which is more general than the inequality (1.1) and relaxed the condition $F(X \times X) \subseteq g(X)$ but imposed a condition namely φ *is continuous on* R_+. Thus Theorem 3.1 is a partial generalization of Theorem 1.1.

Theorem 3.2. *Let* (X,d) *be a metric space,* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings such that*

(i) F *and* g *satisfy* CLR_g *property,*
(ii) *there exists* $T \in \Lambda$ *such that*

$$\begin{aligned}&Td(F(x,y),F(u,v)),\ d(gx,gu), d(gy,gv), d(gx,F(x,y)), d(gy,F(y,x)),\\ &\quad d(gu,F(u,v)), d(gv,F(v,u)), d(gx,F(u,v)),\ d(gy,F(v,u)),\\ &\quad d(gu,F(x,y)),\ d(gv,F(y,x))) \leq 0 \textit{ for all } x,y,u,v \in X. \qquad (3.15)\end{aligned}$$

Then (a) *the pair* (F,g) *has a coupled fixed point and*

(b) *the pair* (F,g) *has a unique coupled common fixed point provided it is* $w-$ *compatible.*

Proof. By (i), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \longrightarrow \infty} F(x_n, y_n) = \lim_{n \longrightarrow \infty} g(x_n) = gx \text{ and}$$

$$\lim_{n \longrightarrow \infty} F(y_n, x_n) = \lim_{n \longrightarrow \infty} g(y_n) = gy \quad \text{for some } x, y \in X.$$

Now, we prove that $gx = F(x,y)$ and $gy = F(y,x)$.

We assume that $d(gx, F(x,y)) > 0$ or $d(gy, F(y,x)) > 0$.
Now, we consider

$$\begin{aligned}&T(d(F(x_n,y_n),F(x,y)),\ d(gx_n,gx),\ d(gy_n,gy),\ d(gx_n,F(x_n,y_n)),\\ &\quad d(gy_n,F(y_n,x_n)),\ d(gx,F(x,y)), d(gy,F(y,x)),\ d(gx_n,F(x,y)),\\ &\quad d(gy_n,F(y,x)),\ d(gx,F(x_n,y_n)),\ d(gy,F(y_n,x_n))) \leq 0.\end{aligned}$$

On taking limits as $n \to \infty$, we get

$$\begin{aligned}&T\big(d(gx,F(x,y)),0,0,0,0,\ d(gx,F(x,y)),\ d(gy,F(y,x)),\ d(gx,F(x,y)),\\ &\qquad d(gy,F(y,x)),0,0\big) \leq 0.\end{aligned}$$

Hence from condition (T_1) of Definition 2.4 we get

$$d(gx, F(x,y)) \leq f(max\{d(gx, F(x,y)),\ d(gy, F(y,x))\}). \qquad (3.16)$$

Again we consider

$$T\big(d(F(y_n,x_n), F(y,x)),\ d(gy_n, gy),\ d(gx_n, gx),\ d(gy_n, F(y_n,x_n)),$$
$$d(gx_n, F(x_n,y_n)),\ d(gy, F(y,x)), d(gx, F(x,y)),\ d(gy_n, F(y,x)),$$
$$d(gx_n, F(x,y)),\ d(gy, F(y_n,x_n)),\ d(gx, F(x_n,y_n))) \leq 0.$$

On taking limits as $n \to \infty$, we get

$$T\big(d(gy, F(y,x)), 0, 0, 0, 0,\ d(gy, F(y,x)),\ d(gx, F(x,y)),\ d(gy, F(y,x)),$$
$$d(gx, F(x,y)), 0, 0\big) \leq 0.$$

Hence from condition (T_1) of Definition 2.4 we get

$$d(gy, F(y,x)) \leq f\big(max\{d(gx, F(x,y)),\ d(gy, F(y,x))\}\big). \qquad (3.17)$$

From (3.16) and (3.17) we get

$$max\{d(gx, F(x,y)), d(gy, F(y,x))\} \leq f(max\{d(gx, F(x,y)), d(gy, F(y,x))\}),$$
$$< max\{d(gx, F(x,y)), d(gy, F(y,x))\},$$

a contradiction.
Hence $gx = F(x,y)$ and $gy = F(y,x)$.

Thus (x,y) is a coupled fixed point of F and g.
Let (x,y) and (x^*,y^*) be two coupled coincidence points of F and g.
Now, we prove that $gx = gx^*$ and $gy = gy^*$.

We assume that $d(gx, gx^*) > 0$ and $d(gy, gy^*) > 0$.
Now, we consider

$$T(d(F(x,y), F(x^*,y^*)),\ d(gx, gx^*),\ d(gy, gy^*),\ d(gx, F(x,y)),\ d(gy, F(y,x))$$
$$d(gx^*, F(x^*,y^*)),\ d(gy^*, F(y^*,x^*)),\ d(gx, F(x^*,y^*)),\ d(gy, F(y^*,x^*))$$
$$d(gx^*, F(x,y)),\ d(gy^*, F(y,x))) \leq 0. \text{ Hence}$$

$$T\big(d(gx, gx^*),\ d(gx, gx^*),\ d(gy, gy^*), 0, 0, 0, 0,\ d(gx, gx^*),\ d(gy, gy^*),$$
$$d(gx^*, gx),\ d(gy, gy^*),\ d(gx^*, gx), d(gy^*, gy)\big) \leq 0.$$

Hence from condition (T_1) of Definition 2.4 we get

$$d(gx, gx^*) \leq f\big(max\{d(gx, gx^*),\ d(gy, gy^*)\}\big). \qquad (3.18)$$

Similarly it follows that

$$d(gy, gy^*) \leq f\big(max\{d(gx, gx^*),\ d(gy, gy^*)\}\big). \qquad (3.19)$$

From (3.18) and (3.19) we have

$$max\{d(gx, gx^*),\ d(gy, gy^*)\} \leq f\big(max\{d(gx, gx^*),\ d(gy, gy^*)\}\big)$$
$$< max\{d(gx, gx^*),\ d(gy, gy^*)\},$$

a contradiction.
Hence $gx = gx^*$ and $gy = gy^*$.
Now we prove that $gx = gy^*$ and $gy = gx^*$.

We assume that either $d(gx, gy^*) > 0$ or $d(gy, gx^*) > 0$.
Now, we consider

$$T(d(F(x,y), F(y^*,x^*)),\ d(gx, gy^*),\ d(gy, gx^*),\ d(gx, F(x,y)),\ d(gy, F(y,x)),$$
$$d(gy^*, F(y^*,x^*)),\ d(gx^*, F(x^*,y^*)),\ d(gx, F(y^*,x^*)),\ d(gy, F(x^*,y^*))$$

$d(gy^*, F(x,y)),\ d(gx^*, F(y,x))) \leq 0.$

$$T(d(gx, gy^*),\ d(gx, gy^*),\ d(gy, gx^*), 0, 0, 0, 0,\ d(gx, gy^*), d(gy, gx^*),$$
$$d(gy^*, gx),\ d(gx^*, gy)) \leq 0.$$ Hence

$$T(d(gx, gy^*),\ d(gx, gx^*),\ d(gy, gy^*), 0, 0, 0, 0,\ d(gx, gx^*),\ d(gy, gy^*),$$
$$d(gx^*, gx),\ d(gy, gy^*),\ d(gx^*, gx), d(gy^*, gy)) \leq 0.$$

Hence from condition (T_1) of Definition 2.4, we get

$$d(gx, gy^*) \leq f\big(max\{d(gx, gy^*),\ d(gy, gx^*)\}\big). \tag{3.20}$$

Similarly it follows that

$$d(gy, gx^*) \leq f\big(max\{d(gx, gy^*),\ d(gy, gx^*)\}\big). \tag{3.21}$$

From (3.20) and (3.21) we have

$$max\{d(gx, gy^*),\ d(gy, gx^*)\} \leq f(max\{d(gx, gy^*),\ d(gy, gx^*)\})$$
$$< max\{d(gx, gy^*),\ d(gy, gx^*)\},$$

a contradiction.

Hence $gx = gy^*$ and $gy = gx^*$.

Thus $gx = gx^* = gy = gy^*$. (3.22)

Let (x, y) be a coupled coincidence point and take $u = gx,\ v = gy$.

Hence $u = gx = F(x, y)$ and $v = gy = F(y, x)$.

Since the pair (F, g) is $w-$compatible, we have

$gu = ggx = gF(x, y) = F(gx, gy) = F(u, v)$

and

$gv = ggy = gF(y, x) = F(gy, gx) = F(v, u)$

so that (u, v) is a coupled coincidence point. Hence $gu = gx$ and $gy = gv$.

Thus $u = gx = gu = F(u, v)$ and $v = gy = gv = F(v, u)$.

Hence (u, v) is a coupled common fixed point.

Moreover from (3.22) we get $u = v$.

Now we suppose that (u_1, v_1) be another coupled common fixed point

$i.e.,\ u_1 = gu_1 = F(u_1, v_1)$ and $v_1 = gv_1 = F(v_1, u_1)$.

From (3.22) we get

$u_1 = gu_1 = gu = u$ and $v_1 = gv_1 = gv = v$.

Hence coupled fixed point is unique. □

Corollary 3.4. *Let (X, d) be a metric space, $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

(i) *F and g satisfy property (E.A),*

(ii) *g(X) is a closed subset of X,*

(iii) *there exists $T \in \Lambda$ such that*

$$T(d(F(x,y), F(u,v)),\ d(gx, gu), d(gy, gv), d(gx, F(x,y)), d(gy, F(y,x)), d(gu, F(u,v),$$
$$d(gv, F(v,u)), d(gx, F(u,v)),\ d(gy, F(v,u)), d(gu, F(x,y)),\ d(gv, F(y,x))) \leq 0$$

for all $x, y \in X$,

then (a) the pair (F, g) has a coupled fixed point

(b) the pair (F, g) has a unique coupled common fixed point provided it is $w-$ compatible.

Example 3.5. Let $X = [0, 1)$ with the usual metric. We define $T : R_+^{11} \to R$ by $T(t_1, t_1, ..., t_{11}) = t_1 - h\ max\{t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$, where $h = \frac{2}{3}$.

Clearly $T \in \Lambda$. Now, we define $F : X \times X \to X$ and $g : X \to X$ by

$$F(x,y) = \begin{cases} \frac{2x-y}{3} & if\ x \geq y \\ 0 & if\ x < y; \end{cases} \qquad gx = \begin{cases} x & if\ x \in [0, \frac{1}{5}) \\ \frac{9}{10} & if\ x \in [\frac{1}{5}, 1). \end{cases}$$

Now, we choose the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$x_n = \frac{1}{n+5}$ and $y_n = \frac{1}{2(n+2)}$, $n = 1, 2, 3...$, then

$$\lim_{n\longrightarrow\infty} F(x_n, y_n) = \lim_{n\longrightarrow\infty} g(x_n) = 0 = g0$$

and

$$\lim_{n\longrightarrow\infty} F(y_n, x_n) = \lim_{n\longrightarrow\infty} g(y_n) = 0 = g0 .$$

We define $\varphi : R_+ \to R_+$ by $\varphi(t) = \frac{t}{9}$, $t \geq 0$; here we observe that $t - \varphi(t)$ is an increasing function. And at $(x, y) = (0, 0)$ we have $gF(x, y) = F(gx, gy)$. Here we note that F and g satisfy the inequality (3.15). Hence F and g satisfy all the hypotheses of Theorem 3.2. and $(0, 0)$ is a coupled common fixed point. Moreover $(0, 0)$ is unique.

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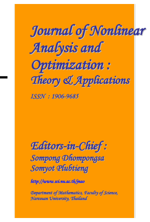

FIXED POINT THEOREMS FOR NONLINEAR CONTRACTIONS IN ORDERED PARTIAL METRIC SPACES

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ABSTRACT. In this paper we develope the weak contraction mapping principle in the context of partial metric spaces which are generalizations of metric spaces meant for the study of denotational semantics of programming languages. We consider certain control conditions for this purpose and accomplish the task in partial metric spaces. Additionally, a partial order is defined on this space. An illustrative example is given. The method we use in this paper is a combination of analytic and order theoretic methodologies.

KEYWORDS : Partially ordered set; Partial metric; Non-decreasing mapping; Weak contraction; Control functions; Fixed point.
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1. INTRODUCTION

Fixed points play an important role in computer science especially for justifications of induction and recursive definitions. In 1994 Matthews [23, 24] introduced the conception of partial metric spaces as generalizations of metric spaces where self distance may be non-zero. The motivation for such a generalization comes from the study of denotational semantics of programming languages in computer science [38] where it was felt that a metric approach to this study is not possible unless the definition of the metric is suitably modified. Our interest is to prove fixed point results in this space. Fixed points have important roles to play in computer science, especially in semantics [37]. The study of fixed points in partial metric spaces was initiated in [23] where Matthews established a contraction mapping theorem in partial metric spaces. Other fixed point results followed this work. Some instances of these works are in [1, 4-6, 21, 22, 31, 33, 35, 40].

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In metric spaces we find a lot of efforts to generalize the Banachs contraction mapping principle as, for instances, in [7, 9, 25, 39]. Particularly, Alber and Guerre-Delabriere in [3] introduced the concept of weak contraction in Hilbert spaces. Rhoades in [34] has shown that the result which Alber et al. proved in [3] is also valid in complete metric spaces. A weak contraction is intermediate to contraction mapping and a nonexpansive mapping. The weak contraction principle established by Rhoades in metric spaces as mentioned above has been generalized in a number of ways. Dutta and Choudhury [15] has proved a generalization employing a method different from that used by Rhoades. Another approach of generalisation was initiated by Eslamian and Abkar [17] and was further adopted by Choudhury and Kundu [14].
A separate methodology was applied to this problem by Popescu [32] and proved that some of the control conditions used by Doric [16] are not required. There are several other fixed point results of weakly contractive mappings and their generalizations. Some instances of these works are noted in [10, 12-13, 16, 28, 29, 41].

In recent years fixed point theory has experienced a rapid development in partially ordered metric spaces. References [2, 8, 11, 19, 27, 30] are some instances of these works. Particularly, Harjani et. al have established a generalized weak contraction principle in partially ordered metric spaces [20].

The purpose of this paper is to weaken the contractive conditions in partial metric spaces having a partial ordering defined on them. We have shown that the weak contractions necessarily have fixed points in partially ordered partial metric spaces. using the notion of weak control conditions two fixed point theorems in ordered partial metric spaces in view of Popescu [32] conditions has been proved. Here our effort is to show that a parallel development is also possible in partial metric spaces with a partial order. Our approach is a blending of analytic and order theoretic methods. We have given an illustrative examples.

The following are some essential concepts for our discussion in this paper.

Definition 1.1. [23] Let X be a nonempty set and let $p : X \times X \to \mathrm{R}^+$ be such that the following are satisfied, for all $x, y, z \in X$

$(P1)$ $x = y \Longleftrightarrow p(x,x) = p(y,y) = p(x,y)$
$(P2)$ $p(x,x) \leq p(x,y)$
$(P3)$ $p(x,y) = p(y,x)$
$(P4)$ $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then the pair (X, p) is called a partial metric space and p is called a partial metric on X.
It is clear that, if $p(x,y) = 0$, then from $(p1)$ and $(p2)$ $x = y$. But if $x = y$, $p(x,y)$ may not be 0. If p be a partial metric on X, then the function $d_p : X \times X \to R^+$ defined as

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

satisfies the conditions of an usual metric on X [23]. Each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open $p-$balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$ [23].

The concepts of convergence, Cauchy sequence, completeness and continuity in partial metric space is given in the following definition.

Definition 1.2. [23] Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in the partial metric space (X, p) converges to the limit x if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(2) A sequence $\{x_n\}$ in the partial metric space (X, p) is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and is finite.

(3) A partial metric space (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that
$p(x, x) = \lim_{m,n\to\infty} p(x_m, x_n)$.

(4) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subseteq B_p(fx_0, \varepsilon)$.

The following implication follows from the above definition.

If a function $f : X \to X$ where (X, p) is a partial metric space is continuous then $fx_n \to fx$ whenever $x_n \to x$ as $n \to \infty$.

Lemma 1.3. *[23] Let (X, p) be a partial metric space.*

(1) A sequence $\{x_n\}$ is a Cauchy sequence in the partial metric space (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).

(2) A partial metric space (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover, $\lim_{n\to\infty} d_p(x, x_n) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{m,n\to\infty} p(x_m, x_n)$.

Definition 1.4. A function $f : R \to R$ is said to be monotone non-decreasing (or monotone increasing) if $x \geq y$ implies $f(x) \geq f(y)$.

The following are examples of a partial metric spaces.

Example 1.5. [22] Let X =[0,1] and $p : X \times X \to \mathrm{R}^+$ be defined as $p(x, y) = \max\{x, y\}$. Then, (X, p) is a partial metric space and it is also complete.

We construct the following example of a partial metric spaces.

Example 1.6. Let $X = \{0, 1, 2, 3, 4, ...\}$. We define $p : X \times X \to \mathrm{R}^+$ as

$$p(x, y) = \begin{cases} x + y + 2, \text{if } x \neq y, \\ 1, \text{if } x = y. \end{cases}$$

Then p is a partial metric on X.

The properties $(P1)$, $(P2)$ and $(P3)$ are directly verified by inspection. We prove $(P4)$ in the following. Let $a, b, c \in X$. If $a \neq c$ then

i) $p(a, c) = a + c + 2 < a + b + 2 + b + c + 2 - 1 = p(a, b) + p(b, c) - p(b, b)$ (if $b \neq a$ and $b \neq c$).

ii) $p(a, c) = a + c + 2 < 1 + a + c + 2 = p(a, b) + p(b, c) - p(b, b)$ (if $b = a$ and $b \neq c$).

If $a = c$ then $p(a, c) = 1 \leq p(a, b) + p(b, c) - 1 = p(a, b) + p(b, c) - p(b, b)$. Thus (P4) is satisfied.

In view of (1.1) the function $d_p : X \times X \to R^+$ defined as

$$d_p(x, y) = \begin{cases} 2x + 2y + 2, \text{if } x \neq y, \\ 0, \text{if } x = y. \end{cases}$$

It is a metric on X.

2. MAIN RESULTS

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and let there be a partial metric p on X such that (X, p) is a complete partial metric space. Let $f : X \to X$ be a continuous and non-decreasing mapping such that*

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \beta(M(x, y)) \text{ whenever } x, y \in X \text{ and } x \preceq y, \quad (2.1)$$

with

$$M(x, y) = \max\{p(x, y), p(Tx, x), p(y, Ty), \frac{1}{2}[p(y, Tx) + p(x, Ty)]\} \quad (2.2)$$

where

i) $\psi : [0, \infty) \to [0, \infty)$ is a monotone non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$,

ii) $\beta : [0, \infty) \to [0, \infty)$ is a function satisfying $\beta(0) = 0$, $\liminf_{n\to\infty} \beta(a_n) > 0$ whenever $\lim_{n\to\infty} a_n = a > 0$,

iii) $\beta(t) > \psi(t) - \psi(t^-)$ for all $t > 0$, where $\psi(t^-)$ is the left limit of ψ at t.

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Starting with $x_0 \in X$, and following the same steps as in theorem 2.1, we obtain a sequence $\{x_n\}$ in X defined as

$$fx_n = x_{n+1} \text{ for all } n \geq 0, \quad (2.3)$$

for which

$$x_0 \preceq fx_0 = x_1 \preceq fx_1 = x_2 \preceq fx_2 \preceq .... \preceq fx_{n-1} = x_n \preceq fx_n = x_{n+1} \preceq ..... \quad (2.4)$$

If $x_n = x_{n+1}$, then f has a fixed point. Therefore we assume that

$$x_n \neq x_{n+1}, \text{ for all } n \geq 0.$$

Then it follows from the definition of p that

$$p(x_n, x_{n+1}) \neq 0 \text{ for all } n \geq 0. \quad (2.5)$$

Let, if possible, for some n

$$p(x_{n-1}, x_n) < p(x_n, x_{n+1}). \quad (2.6)$$

By triangular inequality of partial metric space,

$$\begin{aligned} \frac{1}{2}(p(x_{n-1}, x_{n+1}) + p(x_n, x_n)) &\leq \frac{1}{2}(p(x_{n-1}, x_n) + p(x_n, x_{n+1})) \\ &\leq \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} \end{aligned}$$

Now,

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2}[p(x_n, x_n) + p(x_{n-1}, x_{n+1})]\} \\ &= p(x_n, x_{n+1}) \quad \text{[by (2.6)]} \end{aligned}$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.1), using (2.2), (2.3), (2.4) and the monotone property of ψ, for all $n \geq 0$, we have

$$\begin{aligned}\psi(p(x_n, x_{n+1})) &= \psi(p(fx_{n-1}, fx_n)) \\ &\leq \psi(M(x_{n-1}, x_n)) - \beta(M(x_{n-1}, x_n)) \\ .&\leq \psi(p(x_n, x_{n+1})) - \beta(p(x_n, x_{n+1})) \end{aligned} \tag{2.7}$$

A consequence of the properties of β given in condition (ii) of the theorem is that $\beta(a) > 0$ for $a > 0$. Then from (2.5), $\beta(p(x_n, x_{n+1})) > 0$. With this, (2.7) leads to a contradiction. Therefore, for all $n \geq 1$,

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n).$$

Thus the sequence $\{p(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers and consequently there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{2.8}$$

Suppose that $r > 0$. If there exists n such that $p(x_n, x_{n+1}) = r$, then, by (2.7) we have $\psi(r) \leq \psi(r) - \beta(r)$. Since $\beta(r) > 0$, this is a contradiction. So $p(x_n, x_{n+1}) > r$, for all $n \geq 0$. Then taking limit infimum as $n \to \infty$ in (2.7), using (2.8) and the fact that $\{p(x_n, x_{n+1})\}$ is monotone decreasing, we have

$$\psi(r^+) \leq \psi(r^+) - \lim_{n\to\infty} \inf \beta(p(x_n, x_{n+1})).$$

By virtue of condition (ii), $\lim_{n\to\infty} \inf \beta(p(x_n, x_{n+1})) > 0$. So the above inequality leads to a contradiction. Hence

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.9}$$

It follows by (P1) and (P2) of definition 1.1 that

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{2.10}$$

Since from (1.1), $d_p(x, y) \leq 2p(x, y)$ for all $x, y \in X$, for all $n \geq 0$, from (2.9) it follows that

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{2.11}$$

Next we show that $\{x_n\}$ is a Cauchy sequence in (X, d_p). If not, then there exists some $\varepsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all $k \geq 0$,

$$n(k) > m(k) > k,$$
$$d_p(x_{m(k)}, x_{n(k)}) \geq \varepsilon \tag{2.12}$$

and

$$d_p(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.13}$$

Now, for all $k \geq 0$, we have

$$\begin{aligned}\varepsilon \leq d_p(x_{m(k)}, x_{n(k)}) &\leq d_p(x_{m(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{n(k)}) \\ &< \varepsilon + d_p(x_{n(k)-1}, x_{n(k)}) \qquad \text{(by(2.13))}.\end{aligned}$$

Taking $k \to \infty$ in the above inequality, using (2.11), we obtain

$$\lim_{k\to\infty} d_p(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{2.14}$$

Also, for all $k \geq 0$, we have

$d_p(x_{m(k)-1}, x_{n(k)-1}) \leq d_p(x_{m(k)-1}, x_{m(k)}) + d_p(x_{m(k)}, x_{n(k)}) + d_p(x_{n(k)}, x_{n(k)-1})$
and $d_p(x_{m(k)}, x_{n(k)}) \leq d_p(x_{m(k)}, x_{m(k)-1}) + d_p(x_{m(k)-1}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{n(k)})$.

Taking limit as $k \to \infty$ in the above two inequalities, using (2.11) and (2.14) we obtain

$$\lim_{k\to\infty} d_p(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.15}$$

For k=1,2,3...

$$d_p(x_{n(k)-1}, x_{m(k)}) \leq d_p(x_{n(k)-1}, x_{n(k)}) + d_p(x_{n(k)}, x_{m(k)}) \tag{2.16}$$

and

$$d_p(x_{n(k)}, x_{m(k)}) \leq d_p(x_{n(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{m(k)}). \tag{2.17}$$

Making $k \to \infty$ in (2.16) and (2.17) respectively, and using (2.14) and (2.11) we have

$$\lim_{k\to\infty} d_p(x_{n(k)-1}, x_{m(k)}) = \varepsilon. \tag{2.18}$$

Again for k=1,2,3...

$$d_p(x_{n(k)}, x_{m(k)-1}) \leq d_p(x_{n(k)}, x_{m(k)}) + d_p(x_{m(k)}, x_{m(k)-1})$$

$$and \;\; d_p(x_{n(k)}, x_{m(k)}) \leq d_p(x_{n(k)}, x_{m(k)-1}) + d_p(x_{m(k)-1}, x_{m(k)}).$$

Making $k \to \infty$ in the above two inequalities and using (2.14) and (2.11) we obtain

$$\lim_{k\to\infty} d_p(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{2.19}$$

Since for all $x, y \in X$, $d_p(x,y) \leq 2p(x,y) - p(x,x) - p(y,y)$ by using, (2.14), (2.15), (2.18) and (2.19) we get

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}, \tag{2.20}$$

$$\lim_{k\to\infty} p(x_{n(k)+1}, x_{m(k)}) = \frac{\varepsilon}{2}, \tag{2.21}$$

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)+1}) = \frac{\varepsilon}{2} \tag{2.22}$$

and

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)-1}) = \frac{\varepsilon}{2}. \tag{2.23}$$

Next we show that for sufficiently large k, $p(x_{m(k)}, x_{n(k)}) \leq \frac{\varepsilon}{2}$.

If not, then there exists a subsequence $\{k(i)\}$ of $\mathbb{N}$ such that for all $i > 0$,

$$\frac{\varepsilon}{2} < p(x_{m(k(i))}, x_{n(k(i))})). \tag{2.24}$$

In view of (2.4), substituting $x = x_{m(k(i))-1}$ and $y = x_{n(k(i))-1}$ in (2.1), for all $i > 0$, we have

$$\begin{aligned} & \psi(p(x_{m(k(i))}, x_{n(k(i))})) \\ = \;& \psi(p(fx_{m(k(i))-1}, fx_{n(k(i))-1})) \\ \leq \;& \psi(M(x_{m(k(i))-1}, x_{n(k(i))-1})) - \beta(M(x_{m(k(i))-1}, x_{n(k(i))-1})). \end{aligned} \tag{2.25}$$

$$\begin{aligned} & M(x_{m(k(i))-1}, x_{n(k(i))-1}) \\ & = \max\{p(x_{m(k(i))-1}, x_{n(k(i))-1}), p(x_{m(k(i))-1}, x_{m(k(i))}), p(x_{n(k(i))-1}, x_{n(k(i))}), \\ & \qquad \tfrac{1}{2}(p(x_{m(k(i))-1}, x_{n(k(i))}) + p(x_{n(k(i))-1}, x_{m(k(i))}))\} \end{aligned}$$

Taking limit as $i \to \infty$ in (2.25), using (2.20)- (2.23) in the above inequality, taking into account the inequality (2.24) and the monotone property of ψ, we obtain

$$\psi(\frac{\varepsilon}{2}^+) \leq \psi(\frac{\varepsilon}{2}^+) - \liminf_{i\to\infty} \beta(M(x_{m(k(i))-1}, x_{n(k(i))-1})).$$

But by a property of β, the last inequality implies that

$$\liminf_{i\to\infty} \beta(M(x_{m(k(i))-1}, x_{n(k(i))-1})) > 0.$$

Then the above inequality gives a contradiction. Thus for sufficiently large k,

$$p(x_{m(k)}, x_{n(k)}) \leq \frac{\varepsilon}{2}. \tag{2.26}$$

Again from (1.1) we have

$$d_p(x_{m(k)}, x_{n(k)}) = 2p(x_{m(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}).$$

Taking $k \to \infty$ and using (2.14) and (2.10), we have $p(x_{m(k)}, x_{n(k)}) \geq \frac{\varepsilon}{2}$. Then the above observation along with (2.26) implies that, there exists a positive integer k_1 such that for all $k \geq k_1$,

$$p(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}. \tag{2.27}$$

Substituting $x = x_{m(k)}$ and $y = y_{n(k)}$ in (2.1), (2.2), using (2.3), (2.4), we obtain

$$\begin{aligned} \psi(p(x_{m(k)+1}, x_{n(k)+1})) &= \psi(p(fx_{m(k)}, fx_{n(k)})) \\ &\leq \psi(M(x_{m(k)}, x_{n(k)})) - \beta(M(x_{m(k)}, x_{n(k)})) \end{aligned} \tag{2.28}$$

where,

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}) &= \max\{p(x_{m(k)}, x_{n(k)}), p(x_{m(k)}, x_{m(k)+1}), p(x_{n(k)}, x_{n(k)+1}), \\ &\quad \frac{1}{2}(p(x_{m(k)}, x_{n(k)+1}) + p(x_{m(k)+1}, x_{n(k)}))\} \end{aligned} \tag{2.29}$$

Then, for all $k \geq k_1$, $M(x_{m(k)}, x_{n(k)}) = \frac{\varepsilon}{2}$ and (2.28) becomes

$$\psi(p(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(\frac{\varepsilon}{2}) - \beta(\frac{\varepsilon}{2}) < \psi(\frac{\varepsilon}{2}). \tag{2.30}$$

Thus, by (2.30), using the monotone property of ψ, for all $k \geq k_1$, we have

$$p(x_{m(k)+1}, x_{n(k)+1}) < \frac{\varepsilon}{2}. \tag{2.31}$$

Taking the limit as $k \to \infty$ in (2.28), using (2.29) and (2.31), we obtain $\psi(\frac{\varepsilon}{2}^{-}) \leq \psi(\frac{\varepsilon}{2}) - \beta(\frac{\varepsilon}{2})$, which contradicts condition (iii).

Therefore the sequence $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, by lemma 1.3, (X, d_p) is complete and consequently the sequence $\{x_n\}$ is convergent to z in X, that is,

$$\lim_{n\to\infty} x_n = z. \tag{2.32}$$

Thus, by lemma 1.3,

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.33}$$

Again by (1.1), for all $m, n \geq 0$

$$d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_n, x_n) - p(x_m, x_m).$$

Taking limit $m, n \to \infty$, using (2.10) and the fact that $\{x_n\}$ is a Cauchy sequence in (X, d_p), we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Then, from (2.33), it follows that

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{2.34}$$

Next we prove that $fz = z$. Suppose that

$$p(z, fz) > 0. \tag{2.35}$$

By virtue of (2.32), the continuity of f implies that $fx_n \to fz$ as $n \to \infty$. Then, by lemma 1.3, we have

$$p(fz, fz) = \lim_{n\to\infty} p(fx_n, fz) = \lim_{n\to\infty} p(x_{n+1}, fz). \tag{2.36}$$

Now,

$$\begin{aligned} p(z, fz) &\leq p(z, x_{n+1}) + p(x_{n+1}, fz) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_{n+1}) + p(x_{n+1}, fz). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, using (2.32), (2.34) and (2.36), we obtain

$$\begin{aligned} p(z, fz) &\leq \lim_{n\to\infty} p(z, x_{n+1}) + \lim_{n\to\infty} p(x_{n+1}, fz) \\ &= p(fz, fz). \end{aligned}$$

$M(z, z) = \max\{p(z, z), p(z, fz), p(z, fz), \frac{1}{2}(p(z, fz) + p(z, fz))\} = p(z, fz)$ [by (2.34)]

Using the last inequality and the monotone property of ψ, from (2.36) we obtain,

$$\begin{aligned} \psi(p(z, fz)) &\leq \psi(p(fz, fz)) \leq \psi(M(z, z)) - \beta(M(z, z)) \text{ (by (2.1) and (2.4)).} \\ &\leq \psi(p(z, fz)) - \beta(p(z, fz)) \end{aligned} \tag{2.37}$$

In view of (i), (ii) and (2.35) we obtain $p(z, fz) = 0$. Then from (P1) and (P2) of the definition 1.1, it follows that $z = fz$. □

Our next theorem is obtained by replacing the continuity of f by an ordered theoretic condition.

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is a complete partial metric space. We assume that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then*

$$x_n \preceq z \text{ for all } n \geq 0. \tag{2.38}$$

Let $f : X \to X$ be a non-decreasing mapping such that

$$\psi(p(fx, fy)) \leq \psi(M(x, y)) - \beta(M(x, y)) \text{ for all } x, y \in X \text{ and } x \prec y \ (x \neq y), \tag{2.39}$$

where ψ and β satisfies all the condition of theorem 2.3. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Following the steps identically as in the proof of the theorem 2.1 we obtain (2.32) and (2.34). In view of (2.4) we claim that $\{x_n\}$ is a non-decreasing sequence converges to z in (X, p) such that for all $n \geq 1$, $x_n \preceq z$. If $x_n = z$, for some n, then, from (2.29) and (2.54), it follows that $x_n = x_{n+1}$, in which case we have a fixed point. So we assume that $x_n \neq z$ for all $n \geq 0$. From (2.34) it is observed that $p(z, z) = 0$. Suppose $\varepsilon = p(z, fz) > 0$.

Thus for each k_0 there exists $k_0 \in \mathbb{N}$ such that for $n \geq k_0$,

$$p(x_{n+1}, z) < \frac{\varepsilon}{2}, \quad p(x_n, z) < \frac{\varepsilon}{2} \text{ and in view of (2.9) } p(x_{n+1}, x_n) < \frac{\varepsilon}{2}.$$

$$\begin{aligned} p(z, fz) &\leq p(z, x_{n+1}) + p(x_{n+1}, fz) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_{n+1}) + p(fx_n, fz) \end{aligned}$$

Taking $n \to \infty$, and using (2.34), we have

$$p(z, fz) \leq \lim_{n\to\infty} p(fx_n, fz) \tag{2.40}$$

Since $x_n \preceq z$, putting $x = x_n$ and $y = z$ in (2.39), using (2.40), and the property of ψ, we get

$$\begin{aligned} \psi(p(z, fz)) &\leq \lim_{n\to\infty} \psi(p(fx_n, fz)) \\ &\leq \lim_{n\to\infty} \psi(M(x_n, z)) - \lim_{n\to\infty} \beta(M(x_n, z)) \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} \varepsilon &= p(z, fz) \leq M(x_n, z) \\ &= \max\{p(x_n, z), p(z, fz), p(fx_n, x_n), \frac{1}{2}(p(x_n, fz) + p(z, fx_n))\} \\ &= \max\{p(x_n, z), p(x_{n+1}, x_n), p(fz, z), \frac{1}{2}(p(x_n, z) + p(z, fz) + p(z, x_{n+1}))\} \\ &\leq \max\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \varepsilon, \varepsilon\} = \varepsilon. \end{aligned}$$

Then by (2.41),

$$\psi(\varepsilon^-) \leq \psi(\varepsilon) - \beta(\varepsilon) \tag{2.42}$$

In view of the properties of (i)-(iii) we arrive at a contradiction, unless $p(fz, z) = 0$. Since $p(z, z) = 0$ and $p(z, fz) = 0$, from (P1) and (P2) of definition 1.1, it follows that $z = fz$. □

Remark 2.3. Under the assumption when partial metric is a metric we have the result of Popescu [32].

Example 2.4. Let X = [0, 1]∪{2, 3, 4, ...}. $p(x, y) = max\{x, y\}$ for $x, y \in X$. We define a partial order as follows

1) $0 \preceq x$ for all $x \in [0, 1]$ and $0 \preceq 2$, $1 \preceq 3$.

2) for all $x, y \in \{2, 3, 4, ...\}$ $x \preceq y$ iff $x \leq y$ and $(y - x)$ is divisible by 2.

That is we have the following two chains $0 \leq 2 \leq 4...$ and $0 \leq 1 \leq 3...$ Then "$\preceq$" satisfies all the conditions of a partially ordered set.

Also, (X, p) is a complete partial metric space.
Let $\psi : [0, \infty) \to [0, \infty)$ be defined as:

$$\psi(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ t^2, & \text{if } t > 1. \end{cases}$$

and

$\beta : [0, \infty) \to [0, \infty)$ be defined as:

$$\beta(t) = \begin{cases} \frac{t^2}{2} & \text{if } 0 \leq t \leq 1 \\ 2t - 1, & \text{if } t > 1. \end{cases}$$

Let $f : X \to X$ be defined as:

$$\psi(t) = \begin{cases} x - \frac{x^2}{2} & \text{if } 0 \leq x \leq 1 \\ x - 1, & \text{if } x \in \{2, 3, 4, ...\}. \end{cases}$$

without loss of generality, we assume that $x > y$.

$p(x,y) = \max\{x,\, y\} = x,\, p(x,fx) = \max\{x,\, fx\} = x,$
$p(y,fy) = \max\{y,\, fy\} = y.$

$$p(y,fx) = \begin{cases} \max\{y, x - \frac{x^2}{2}\}, & \text{if } \ 0 \le x \le 1 \\ \max\{y, x-1\}, & \text{if } \ x \in \{2,3,4,...\}. \end{cases}$$

or

$$p(x,fy) = \begin{cases} \max\{x, y - \frac{y^2}{2}\} = x, & \text{if } \ 0 \le y \le 1 \\ \max\{x, y-1\} = x, & \text{if } \ y \in \{2,3,4,...\}. \end{cases}$$

Then

$$M(x,y) = \max\{p(x,y), p(fx,x), p(y,fy), \tfrac{1}{2}(p(y,fx) + p(x,fy))\} = x$$

Therefore, we discuss the following cases.

Case-1: $x, y \in [0,1]$. Then

$\psi(p(fx,fy)) = \psi\left(\max\left(x - \frac{x^2}{2}, y - \frac{y^2}{2}\right)\right)$

$= \psi\left(x - \frac{x^2}{2}\right)$ [since $x + y < 2$]

$= x - \frac{x^2}{2}$

$= \psi\left(M\left(x,y\right)\right) - \beta\left(M\left(x,y\right)\right).$

Case-2: $x \in \{3,4,...\}$ and $y \in [0,1]$. Then

$\psi(p(fx,fy)) = \psi\left(\max\left(x-1, y - \frac{y^2}{2}\right)\right)$

$= \psi\left(x-1\right)$

$= \left(x-1\right)^2 = x^2 - 2x + 1$

$= \psi\left(M\left(x,y\right)\right) - \beta\left(M\left(x,y\right)\right).$

Case-3: $x = 2$ and $y \in \left[0,1\right], fx = 1$. Then

$\psi(p(fx,fy)) = \psi\left(\max\left(1, y - \frac{y^2}{2}\right)\right)$

$= \psi\left(1\right)$ =1

$= \psi\left(2\right) - \beta\left(2\right)$

$= \psi\left(M\left(x,y\right)\right) - \beta\left(M\left(x,y\right)\right).$

Hence the required conditions of theorem 2.1 are satisfied and it is seen that "0" is the fixed point of f in X.

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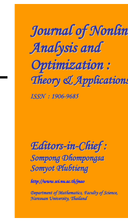

Retraction notice

RETRACTION NOTICE TO: "THE STABILITY OF GAUSS MODEL HAVING ONE-PREY AND TWO-PREDATORS" [JNAO. VOL.2 NO.2 (2011) 355-361]

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The article was found to contain errors and has been retracted at the request of the Editor-in-Chief and Authors. The Publisher apologizes for any inconvenience this may cause.

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