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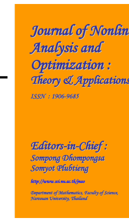

THE EIGENVALUE PROBLEMS FOR DIFFERENTIAL PENCILS ON THE HALF LINE

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ABSTRACT. In this paper, we study the solution of the boundary value problem for second-order differential operator on the half-line having jump point in an interior point. Using of the fundamental system of solutions, we investigate the asymptotic distribution of eigenvalues.

KEYWORDS : Asymptotic form; Jump point; Eigenvalues.
AMS Subject Classification: 34B07 34D05 34L20.

1. INTRODUCTION

We consider BVP(L) with the differential equation

$$y''(x) + (\rho^2 + i\rho q_1(x) + q_0(x))y(x) = 0, \quad x \geq 0, \tag{1.1}$$

on half-line and the boundary condition

$$U(y) := y'(0) + (\beta_1\rho + \beta_0)y(0) = 0, \tag{1.2}$$

and the jump condition

$$y(T-0) = a_1 y(T+0), \quad y'(T-0) = a_1^{-1} y'(T+0), \tag{1.3}$$

in an interior point $T > 0$. Here $a_1 > 0$, and the functions $q_j(x)$, $j = 0, 1$, are complex-valued, $q_1(x)$ is absolutely continuous and $(1+x)q_j^{(l)} \in L(0,\infty)$ for $0 \leq l \leq j \leq 1$.

Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics (see [2]). The boundary value problems on the finite interval with turning points and without discontinuities have been studied in [3]. Also the boundary value problem on the half-line with turning points but without discontinuities has been studied in [4]. Indefinite differential equations with discontinuity

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produce essential qualitative modification in the investigation of the inverse problem. For classical Sturm-Liouville operators with discontinuities inverse problems on the half line have been considered in [5]. In this paper we study discontinuous BVP(L) on the half-line for indefinite pencil (1.1).

The presence of discontinues inverse problems helps to study the blowup behavior of solutions. In section 2, we determine the asymptotic form of the solutions of (1.1) and using these asymptotic estimates, derive characteristic function and give eigenvalues.

2. MAIN RESULTS

Let $y(x)$ and $z(x)$ be continuously differentiable functions on $[0,T]$ and $[T,\infty)$. If $y(x)$ and $z(x)$ satisfy the jump condition (1.3), then

$$< y, z >_{|x=T-0} = < y, z >_{|x=T+0},$$

where $< y, z >= yz' - y'z$, and is called the Wronskian of the functions $y(x)$ and $z(x)$. Denote $\Pi\pm := \{\rho : \pm Im\rho > 0\}$ and $\Pi_0 := \{\rho : Im\rho = 0\}$. By the well-known method (see [4, 5]), we get that for $x \geq T$, $\rho \in \Pi\pm$, there exists a solution $e(x,\rho)$ of Eq.(1.1) (which is called the Jost-type solution) with the following properties :

1. For each fixed $x \geq T$, the functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are holomorphic for $\rho \in \Pi_+$ and $\rho \in \Pi_-$(i.e., they are piecewise holomorphic).

2. The functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are continuous for $x \geq T$, $\rho \in \overline{\Pi}_+$ and $\rho \in \overline{\Pi}_-$ (we differ the sides of the cut Π_0). In the other words, for real ρ, there exist the finite limits

$$e_{\pm}^{(\nu)}(x,\rho) = \lim_{z \longrightarrow \rho,\ z \in \Pi_{\pm}} e^{(\nu)}(x,z).$$

Moreover, the functions $e^{(\nu)}(x,\rho)$, $\nu = 0,1$, are continuously differentiable with respect to $\rho \in \overline{\Pi}_+ \setminus \{0\}$ and $\rho \in \overline{\Pi}_- \setminus \{0\}$.

3. For $x \longrightarrow \infty$, $\rho \in \overline{\Pi}_{\pm} \setminus \{0\}$, $\nu = 0,1$,

$$e^{(v)}(x,\rho) = (\pm i\rho)^v exp(\pm(i\rho x - Q(x)))(1 + o(1)), \tag{2.1}$$

where

$$Q(x) = \frac{1}{2}\int_0^x q_1(t)dt. \tag{2.2}$$

4. For $|\rho| \longrightarrow \infty, \rho \in \overline{\Pi}_{\pm}$, $\nu = 0,1$, uniformly in $x \geq T$,

$$e^{(\nu)}(x,\rho) = (\pm i\rho)^{\nu} exp(\pm(i\rho x - Q(x)))[1], \tag{2.3}$$

where $[1] := 1 + O(\rho^{-1})$. Denote

$$\Delta(\rho) := U(e(x,\rho)). \tag{2.4}$$

The function $\Delta(\rho)$ is called the characteristic function for BVP(L). The function $\Delta(\rho)$ is holomorphic in Π_+ and Π_-, and for real ρ, there exist the finite limits

$$\Delta\pm(\rho) = \lim_{z \longrightarrow \rho,\ z \in \Pi_{\pm}} \Delta(z).$$

Moreover, the function $\Delta(\rho)$ is continuously differentiable for $\rho \in \overline{\Pi}_{\pm} \setminus \{0\}$.

Theorem 2.1. For $|\rho| \longrightarrow \infty, \rho \in \overline{\Pi}_{\pm}$, the following asymptotical formula holds:

$$\Delta(\rho) = \frac{a_1\rho}{2} exp(\pm(i\rho T - Q(T)))((\beta_1 + i)exp(-i\rho T + Q(T))[1] + (\beta_1 - i)exp(i\rho T - Q(T))[1]).$$

Proof. Denote $\Pi^1_\pm := \{\rho : \pm Re\rho > 0\}$. Let $\{y_k(x,\rho)\}_{k=1,2}$ be the Birkhoff-type smooth fundamental system of solutions of Eq.(1.1) with the asymptotic forms

$$y_k^{(m)}(x,\rho) = ((-1)^{k-1}i\rho)^m exp((-1)^{k-1}(i\rho x - Q(x)))[1], \tag{2.5}$$

for $|\rho| \longrightarrow \infty, \rho \in \Pi^1_\pm$, m=0,1 (see[4, 5]). Then

$$e^{(m)}(x,\rho) = h_1(\rho)y_1^{(m)}(x,\rho) + h_2(\rho)y_2^{(m)}(x,\rho), \quad x \in [0,T]. \tag{2.6}$$

Using of solution (2.3) and the jump condition (1.3), we calculate coefficients $h_1(\rho)$ and $h_2(\rho)$ of the forms

$$h_1(\rho) = \frac{a_1\rho - a_1^{-1}i}{2\rho} exp(-i\rho T + Q(T))exp(\pm(i\rho T - Q(T)))[1],$$
$$h_2(\rho) = \frac{a_1\rho + a_1^{-1}i}{2\rho} exp(i\rho T - Q(T))exp(\pm(i\rho T - Q(T)))[1].$$

Substituting the results into (2.6), we obtain

$$e^{(m)}(x,\rho) = \frac{\rho a_1 i}{2} exp(\pm(i\rho T - Q(T)))(exp(-i\rho T + Q(T))exp(i\rho x - Q(x))$$
$$+(-1)^m exp(i\rho T - Q(T))exp(-i\rho x + Q(x))), \qquad x \in [0,T].$$

Together with (1.2) and (2.4), this yields the characteristic function $\Delta(\rho)$. □

Theorem 2.2. i_1) For sufficiently large k, the function $\Delta(\rho)$ has zeros of the form:

$$\rho_k = \frac{1}{T}(k\pi + \frac{-Q(T)i}{2} + \kappa_1 i) + O(k^{-1}), \tag{2.7}$$

where

$$\kappa_1 = \frac{1}{2} ln \frac{i - \beta_1}{i + \beta_1}.$$

i_2) The zeros of BVP(L) are simple, that is, $\Delta_1(\rho_k) = \frac{d\Delta(\rho)}{2\rho d\rho} \mid_{\rho=\rho_k} \neq 0$.

Proof. Using Theorem 2.1 and Rouche's theorem [1], we obtain the zeros of the function $\Delta(\rho)$ of the form (2.7). The relations

$$\begin{cases} e''(x,\rho) + (i\rho q_1(x) + q_0(x))e(x,\rho) = -\rho^2 e(x,\rho), \\ \varphi''(x,\rho_k) + (i\rho_k q_1(x) + q_0(x))\varphi(x,\rho_k) = -\rho_k^2\varphi(x,\rho_k), \end{cases}$$

result that

$$\frac{d}{dx} < e(x,\rho), \varphi(x,\rho_k) > + iq_1(x)e(x,\rho)\varphi(x,\rho_k)(\rho - \rho_k) = -(\rho^2 - \rho_k^2)e(x,\rho)\varphi(x,\rho_k).$$

Hence

$$-(\rho^2 - \rho_k^2)\int_0^\infty e(t,\rho)\varphi(t,\rho_k)dt = < e(x,\rho), \varphi(x,\rho_k) > (|_0^T + |_T^\infty)$$
$$+(\rho - \rho_k)\int_0^\infty iq_1(x)e(x,\rho)\varphi(x,\rho_k).$$

Since the Wronskian is continuous function, we have

$$\int_0^\infty e(t,\rho_k)\varphi(t,\rho_k)dt = -\Delta_1(\rho_k).$$

Using properties of the eigenfunctions i.e., $e(x,\rho_k) = \beta_k\varphi(x,\rho_k)$, $\beta_k \neq 0$ (see [4]), we arrive at $\beta_k\int_0^\infty \varphi^2(t,\rho_k)dt = \Delta_1(\rho_k) \neq 0$. □

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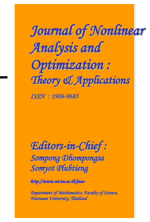

FIXED POINT THEOREMS IN SYMMETRIC $2-$CONE BANACH SPACE $\left(l_p, \|.,.\|_p^c\right)$

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ABSTRACT. The present article is concerned with l_p sequence spaces in point of their symmetric $2-$cone norm structure. Further, fixed point theorem for these spaces are proved.

KEYWORDS : Cone metric space; Cone normed space; $2-$Cone normed space; Fixed point theorems.
AMS Subject Classification: 47H10, 46A45, 57N17

1. INTRODUCTION

In [9] cone metric spaces were introduced by means of a partial ordering " $<$ " on a Banach space $(E, ||.||)$ via a cone P, where some fixed point theorems were proved to generalize the corresponding ones in metric spaces. In [10] Rezapour et al. proved that there were no normal cones with normal constant $M < 1$ and for each $k > 1$ there are cones with normal constant $M > k$. Abdeljawad et al. generalized the Banach spaces $\mathbb{R}^m$, l^∞ and $C[a, b]$ by defining $m-$Euclidean cone normed spaces E^m, E^∞ and the space $C_E(S)$ of continuous functions in cones [1].

It is well known that any metric space is paracompact. As a generalization of metric spaces, cone metric spaces play very important role in fixed point theory, computer science and some other research areas as well as in general topology.

Recently some interesting developments have occured in $2-$normed spaces, sequence spaces, and related topics in these nonlinear spaces (see [11],[6]).

In the followings we recall some preliminary notions which will be needed subsequently.

Definition 1.1. Let E be a real Banach space and P a subset of E. Then P is called cone if

(i) P is closed, non-empty, and $P \neq \{0\}$;
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b;
(iii) $P \cap (-P) = \{0\}$.

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For given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$, $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in int\ P$, where $int\ P$ denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } ||x|| \leq M||y||.$$

The least positive number satisfying the above is called the normal constant of P [10].

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} ||x_n - x|| = 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is normal cone.

In the following it is supposed that E is a Banach space, P is a cone in E with int $P \neq \varnothing$ and $\leq$ is partial ordering with respect to P.

Definition 1.2. A cone normed space is an ordered pair $(X, ||.||_c)$ where X is a vector space over R and $||.||_c : X \to (E, P, ||.||)$ is a function satisfying:

$(c1)$ $0 < ||x||_c$, for all $x \in X$;
$(c2)$ $||x||_c = 0$ if and only if $x = 0$;
$(c3)$ $||\alpha x||_c = |\alpha|\, ||x||_c$, for each $x \in X$ and $\alpha \in \mathbb{R}$;
$(c4)$ $||x + y||_c \leq ||x||_c + ||y||_c$, $x, y \in X$.

It is easy to see that each cone normed space is cone metric space. Namely, the cone metric is defined by $d(x, y) = ||x - y||_c$.

According to above definition a sequence $\{x_n\}$ of a cone normed space $(X, ||.||_c)$ over $(E, P, ||.||)$ is said to be convergent, if there exists $x \in X$ such that for all $c >> 0$, $c \in E$, there exists n_0 such that $||x - x_n||_c << c$ for all $n \geq n_0$. Also, we say that $\{x_n\}$ is Cauchy if for each $c >> 0$, there exists n_0 such that $||x_m - x_n||_c << 0$ for all $m, n \geq n_0$ [1].

Gahler introduced the concepts of $2-$metric spaces and linear $2-$normed spaces and their topological structures [2]. Many works can be found on the scientific literature related to $2-$normed spaces. The definition of a finite dimensional real $2-$normed space is given as the following:

Definition 1.3. [3] Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A $2-$norm on X is a function $\|., .\| : X \times X \to R$ which satisfies

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x, y\| = \|y, x\|$;
(iii) $\|\alpha x, y\| = |\alpha|\, \|x, y\|$, $\alpha \in \mathbb{R}$;
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|., .\|)$ is then called a $2-$normed space.

Definition 1.4. [8] Let X and Y be real linear spaces. Denote by D a non-empty subset $X \times Y$ such that for every $x \in X$, $y \in Y$ the sets

$$D_x = \{y \in Y; (x, y) \in D\} \text{ and } D^y = \{x \in X; (x, y) \in D\}$$

are linear subspaces of the space Y and X, respectively.

A function $\|., .\| : D \to [0; \infty)$ will be called a generalized $2-$norm on D if it satisfies the following conditions:

$(N1)$ $\|x, \alpha y\| = |\alpha| \, \|x, y\| = \|\alpha x, y\|$ for any real number α and all $(x, y) \in D$;
$(N2)$ $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for $x \in X$, $y, z \in Y$ such that$(x, y), (x, z) \in D$;
$(N3)$ $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for $x, y \in X$, $z \in Y$ such that$(x, z), (y, z) \in D$.

The set D is called a $2-$normed set.

In particular, if $D = X \times Y$, the function $\|.,.\|$ will be called a generalized $2-$norm on $X \times Y$ and the pair $(X \times Y, \|.,.\|)$ a generalized $2-$normed space. Moreover, if $X = Y$, then the generalized $2-$normed space will be denoted by $(X, \|.,.\|)$.

Assume that the generalized $2-$norm satifies, in addition, the symmetry condition. Then the symmetric $2-$norm can be defined as follows:

Definition 1.5. [8] Let X be a real linear space. Denote by χ a non-empty subset $X \times X$ with the property $\chi = \chi^{-1}$ and such that the set $\chi^y = \{x \in X; (x, y) \in \chi\}$ is a linear subspace of X, for all $y \in X$.

A function $\|.,.\| : \chi \to [0; \infty)$ satisfying the following conditions:
$(S1)$ $\|x, y\| = \|y, x\|$ for all $(x, y) \in \chi$;
$(S2)$ $\|x, \alpha y\| = |\alpha| \, \|x, y\|$ for any real number α and all $(x, y) \in \chi$;
$(S3)$ $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for $x, y, z \in X$ such that$(x, y), (x, z) \in \chi$;

will be called a generalized symmetric $2-$norm on χ. The set χ is called a symmetric $2-$normed set. In particular, if $\chi = X \times X$, the function $\|.,.\|$ will be called a generalized symmetric $2-$norm on X and the pair $(X, \|.,.\|)$ a generalized symmetric $2-$normed space.

Gunawan and Mashadi introduced the concepts of $n-$normed spaces and their topological structures [4]. Then Lewandowska defined generalized $2-$normed spaces and generalized symmetric $2-$normed spaces. In [5] Gunawan studied the space l_p, $1 \leq p \leq \infty$, its natural $n-$norm and proved a fixed point theorem for l_p as an $n-$normed space.

In this article, we introduce generalized symmetric $2-$cone normed space and a generalized symmetric $2-$cone Banach space and prove the fixed point theorem for some generalized symmetric $2-$cone Banach spaces.

In the main part of the article the results expressing under what conditions a self-mapping T of generalized symmetric $2-$cone Banach space $\left(l_p, \|.,.\|_p^c\right)$ has a unique fixed point are also given.

2. MAIN RESULTS

In the following we give the definition of $2-$cone normed space.

Definition 2.1. Let X be linear space over $\mathbb{R}$ with dimension greater than or equal to 2, E be Banach space with the norm $\|.\|$and $P \subset E$ be a cone. If the function

$$\|.,.\|_c : X \times X \to (E, P, \|.\|)$$

satisfies the following axioms:

(i) $\|x, y\|_c = 0 \Leftrightarrow x \; ve \; y$ are linearly dependent;
(ii) $\|x, y\|_c = \|y, x\|_c$;
(iii) $||\alpha x, y|| = |\alpha| \, ||x, y||_c$;
(iv) $\|x, y + z\|_c \leq \|x, y\|_c + \|x, z\|_c$;
then $(X, \|.,.\|_c)$ is called a $2-$cone normed space.

Example 2.2. Let $X = \mathbb{R}^n$, $E = \mathbb{R}^2$ and $P = \{(x_1, x_2) \in \mathbb{R}^2 : x_i \geqslant 0, \; i = 1, 2\}$. Then the function $\|.,.\|_c : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow (E, P, \|.\|)$ defined by

$$\|x_1, x_2\|_c = (A, A)$$

where

$$A = abs\left(\left|\begin{array}{cccc} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right|\right),$$

abs stands for absolute value and $|.|$ stands for determinant of a matrix, is a generalized symmetric $2-$cone norm and $(X, \|.,.\|_c)$ is a generalized symmetric $2-$cone normed space.

If we fix $\{u_1, u_2, ..., u_d\}$ to be a basis for X, we can give the following lemma.

Lemma 2.3. *Let $(X, \|.,.\|_c)$ be a $2-$cone normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if for each $c \in E$ with $c >> \theta$ (θ is zero element of E) there exists an $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x, u_i\|_c << c$ for every $i = 1, 2, ..., d$.*

Proof. We prove the necessity since the sufficiency is clear. But, in this case there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x, u_i\|_c << \frac{c}{d \max_i |\alpha_i|}$ for every $i = 1, 2, ..., d$. Since $\{u_1, u_2, ..., u_d\}$ is a basis for X, every y can be written of the form $y = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_d u_d$ for some $\alpha_1, \alpha_2, ..., \alpha_d$ in $\mathbb{R}$. Hence,

$$\begin{aligned} \|x_n - x, y\|_c &\leq |\alpha_1| \|x_n - x, u_1\|_c + ... + |\alpha_d| \|x_n - x, u_d\|_c \\ &\ll c. \end{aligned}$$

□

This gives us the following.

Lemma 2.4. *Let $(X, \|.,.\|_c)$ be a $2-$cone normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if $\lim_{n\to\infty} \max \|x_n - x, u_i\|_c = \theta$.*

Now we are ready to define a cone norm with respect to the basis $\{u_1, u_2, ..., u_d\}$ on X. Really, the function $\|.\|_{\frac{\infty}{c}} : X \to (E, P, \|.\|)$ defined by

$$\|.\|_{\frac{\infty}{c}} := \{\max \|x, u_i\|_c : i = 1, 2, ..., d\}$$

is a cone norm on X.

Note that if we choose another basis $\{v_1, v_2, ..., v_d\}$ then resulting $\|.\|_{\frac{\infty}{c}}$ will be equivalent to the one defined with respect to the basis $\{u_1, u_2, ..., u_d\}$.

Lemma 2.5. *Let $(X, \|.,.\|_c)$ be a $2-$cone normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if for each $c \in E$ with $c >> \theta$ (θ is zero element of E) there exists an $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x\|_{\frac{\infty}{c}} << c$.*

Definition 2.6. Let $\|.\|_{\frac{\infty}{c}} : X \to (E, P, \|.\|)$ and $r \in E$ with $r >> \theta$. Then the set

$$B_{\{u_1, u_2, ..., u_d\}}(x; r) = \left\{y : \|y - x\|_{\frac{\infty}{c}} << r\right\}$$

is called (open) ball centered at x, with radius r.

Then we have the following:

Lemma 2.7. *Let $(X, \|.,.\|_c)$ be a $2-$cone normed space. Then a sequence $\{x_n\}$ converges to x in X if and only if for each $r \in E$ with $r >> \theta$ (θ is zero element of E) there exists an $N = N(r) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x\|_{\frac{\infty}{c}} \in B_{\{u_1, u_2, ..., u_d\}}(x; r)$.*

Theorem 2.1. *Any $2-$cone normed space X is a cone normed space and its topology agrees with the norm generated by $\|.\|_{\frac{\infty}{c}}$.*

Now we introduce the notions of $2-$cone norm of the sequence space l_p, $1 \leq p \leq \infty$, consisting of all sequences $x = (x_k)$ such that $\sum_k |x_k|^p < \infty$ and prove some fixed point theorems.

Recall from [5] that the functions

$$\|.,.\|_p := \left[\frac{1}{2} \sum_k \sum_l \left| \det \begin{pmatrix} x_k & x_l \\ x_k & x_l \end{pmatrix} \right|^p \right]^{\frac{1}{p}}$$

and

$$\|.,.\|_\infty := \sup_k \sup_l \left| \det \begin{pmatrix} x_k & x_l \\ x_k & x_l \end{pmatrix} \right|$$

define a $2-$norm on l_p for $1 \leq p < \infty$ and for $p = \infty$ respectively. Then we have the following:

If $X = l_p$, $E = \mathbb{R}^n$ and $P = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_i \geqslant 0,\ i = 1, ..., n\}$. Then each of the functions $\|.,.\|_p^c : l_p \times l_p \longrightarrow (E, P, \|.\|)$ and $\|.,.\|_\infty^c : l_p \times l_p \longrightarrow (E, P, \|.\|)$ defined by

$$\|.,.\|_p^c := (\alpha_1 A, ..., \alpha_n A)$$

and

$$\|.,.\|_\infty^c := (\alpha_1 B, ..., \alpha_n B)$$

defines a $2-$norm on l_p for $1 \leq p < \infty$ and for $p = \infty$ respectively, where

$$A := \left[\frac{1}{2} \sum_k \sum_l \left| \det \begin{pmatrix} x_k & x_l \\ x_k & x_l \end{pmatrix} \right|^p \right]^{\frac{1}{p}},$$

$$B := \sup_k \sup_l \left| \det \begin{pmatrix} x_k & x_l \\ x_k & x_l \end{pmatrix} \right|$$

and $\alpha_i \geq 0$, $i = 1, 2, ..., n$.

Remember from [5] that for any $2-$ normed space X with dimension ≥ 2 an arbitrary linearly independent set $\{a_1, a_2\}$ can be choosen in X and with respect to $\{a_1, a_2\}$ a norm $\|.\|_p$ on X can be defined by

$$\|x\|_p^* := [\|x, a_1\|^p + \|x, a_2\|^p]^{\frac{1}{p}}$$

for $1 \leq p < \infty$ or

$$\|x\|_\infty^* := \sup [\|x, a_1\|, \|x, a_2\|]$$

for $p = \infty$. For instance for l_p we can choose $a_1 = (1, 0, 0, ...)$ and $a_2 = (0, 1, 0, ...)$.

The above facts allow us to define $2-$cone norms on l_p by

$$\|x\|_{p_c}^* := \left(\alpha_1 [\|x, a_1\|^p + \|x, a_2\|^p]^{\frac{1}{p}}, ..., \alpha_n [\|x, a_1\|^p + \|x, a_2\|^p]^{\frac{1}{p}} \right)$$

and by

$$\|x\|_{\infty_c}^* := (\alpha_1 \sup [\|x, a_1\|, \|x, a_2\|], ..., \alpha_n \sup [\|x, a_1\|, \|x, a_2\|])$$

where $\alpha_i \geq 0$, $i = 1, 2, ..., n$, for $1 \leq p < \infty$ and for $p = \infty$ respectively. Remember also that

$$\|x\|_p \leq \|x\|_p^* \leq 2^{\frac{1}{p}} \|x\|_p$$

for all $x \in l_p$, where $\|.\|_p$ is the usual norm on l_p. In particular one has $\|x\|_\infty^* = \|x\|_\infty$. Hence, if we take $\alpha_i = 1$ for all $i = 1, 2, ..., n$ in the above corollary we have $2-$cone norms $\|.,.\|_p^c := (A, ..., A)$ and $\|.,.\|_\infty^c := (B, ..., B)$ of l_p for $1 \leq p < \infty$ and for $p = \infty$ respectively. Thus we have

$$\left(\|x\|_p, ..., \|x\|_p \right) \leq \|x\|_{p_c}^* \leq \left(2^{\frac{1}{p}} \|x\|_p, ..., 2^{\frac{1}{p}} \|x\|_p \right)$$

where $\|x\|_P = \left(\|x\|_p, ..., \|x\|_p\right)$ is usual $p-$norm-like cone norm on $\left(l_p, \|.,.\|_p^c\right)$.

In order to show that $(l_p, \|.,.\|_c)$ is complete we need the following.

Lemma 2.8. *If a sequence in l_p is convergent in the usual norm $\|.\|_p$ then it is convergent in $2-$cone norm $\|.,.\|_p^c$. Similarly, if a sequence in l_p is Cauchy with respect to $\|.\|_p$ then it is Cauchy with respect to $\|.,.\|_p^c$.*

Theorem 2.2. *$\left(l_p, \|.,.\|_p^c\right)$ is a $2-$cone Banach space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $\left(l_p, \|.,.\|_p^c\right)$. Then for each $c \in E$ with $c >> \theta$ there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x_m, y\|_p^c << c$ for all y in l_p, if and only if for each $c \in E$ with $c >> \theta$ there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_n - x_m\|_{\substack{p \\ c}}^* << c$. This proves that $\{x_n\}$ is a Cauchy sequence in $2-$cone normed space $\left(l_p, \|.,.\|_p^c\right)$ if and only if $\{x_n\}$ is a Cauchy sequence in $\left(l_p, \|.\|_{\substack{p \\ c}}^*\right)$. □

Theorem 2.3. *Let T be a self-mapping of l_p such that*

$$\|Tx - Ty, z\|_p^c \leq K \|x - y, z\|_p^c$$

for all x, y, z in l_p, where $K \in (0, 1)$ is a constant. Then T has a unit fixed point in $\left(l_p, \|.,.\|_p^c\right)$.

Proof. Clearly T satisfies

$$\|Tx - Ty\|_{\substack{p \\ c}}^* \leq K \|x - y\|_{\substack{p \\ c}}^*$$

for all x, y, z in l_p. Since $\left(l_p, \|.\|_{\substack{p \\ c}}^*\right)$ is a cone Banach space T must have a unique fixed point. □

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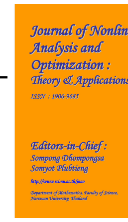

COAPPROXIMATION IN PROBABILISTIC 2-NORMED SPACES

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ABSTRACT. In this article, we studied the best coapproximation in probabilistic 2-normed spaces. We defined the best coapproximation on these spaces and generalized some definitions such as set of best coapproximation, P_b-coproximinal set and P_b-coapproximately compact and orthogonality relative to any set and proved some theorems about them.

KEYWORDS : Probabilistic 2-normed spaces; P_b-best coapproximation; P_b-coproximinal; P_b-coChebyshev.
AMS Subject Classification: 54E70, 46S50

1. INTRODUCTION

In [5], K. Menger introduced the notion of probabilistic metric spaces. The idea of K. Menger was to use distribution function in stead of non negative real numbers as values of the metric. The concept of probabilistic normed spaces (briefly, PN-spaces) was introduced by A. N. Sertnev in 1963, [6].
In [7],[4] the authors have introduced the concept of p-best approximation in probabilistic normed and 2-normed spaces. The main aim of this paper is to investigate another kind of best approximation that called best coapproximation in probabilistic 2-normed spaces. In the sequel after an introduction to probabilistic 2-normed spaces, we define the concept of best coapproximation in probabilistic 2-normed space and generalized some definitions such as set of best coapproximation, coproximinal set and coapproximatively compact set.
Chang et al. [1] defined some notions as follows:

A distance distribution function (briefly, $d.d.f.$), is a function F defined from extended interval $[0, +\infty]$ into the unit interval $I = [0, 1]$, that, is non decreasing and left continuous on $(0, +\infty)$ such that $F(0) = 0$ and $F(+\infty) = 1$. The family of all $d.d.f{'}s$ will be denoted by Δ^+ and we denote

$$D^+ = \{F \in \Delta^+ \mid lim_{t\to\infty} F(t) = 1\}.$$

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By setting $F \leq G$ when ever $F(t) \leq G(t)$, for all $t \in \mathbb{R}^+$, one introduces a natural ordering in D^+. If $a \in \mathbb{R}^+$ then H will be an element of D^+, defined by $H(t) = 0$ if $t \leq 0$ and $H(t) = 1$ if $t > 0$. It is obvious that $H \geq F$ if $t > 0$ for all $F \in D^+$.
A t-norm T is a two place function $T : I \times I \longrightarrow I$ which is associative, commutative, non decreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$.
Let T be a t-norm and T^* is the function given by

$$T^*(x, y) = 1 - T(1 - x, 1 - y)$$

for all $x, y \in I$. Then T^* is the t-conorm of T.
A triangle function is a mapping $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ which is associative, commutative, non decreasing and for which H is the identity, that is, $\tau(H, F) = F$, for every $F \in D^+$.

Definition 1.1. Let V be a linear space of dimension greater than 1 over filed $\mathbb{R}$ of real numbers, τ a triangle function, and let $\mathcal{F}$ be a mapping from $V \times V$ into D^+ satisfying the following conditions:
a) $F_{x,y} = H$ if and only if x and y are linearly dependent vectors.
b) $F_{x,y} \neq H$ if and only if x and y are linearly independent vectors.
c) $F_{x,y} = F_{y,x}$, for all $x, y \in V$.
d) $F_{\alpha x,y} = F_{x,y}(\frac{t}{|\alpha|})$, for every $t > 0$, $\alpha \neq 0$, $\alpha \in \mathbb{R}$ and $x, y \in V$.
e) $F_{x+y,z} \geq \tau(F_{x,z}, F_{y,z})$ for all $x, y, z \in V$.
Then $\mathcal{F}$ is called a probabilistic 2-norm on V and $(V, \mathcal{F}, \tau)$is called a probabilistic 2-normed space(briefly P2N- Space), and $\mathcal{F}$ is a strong probabilistic 2-norm if $b \in V$ and $t > 0$, $x \longrightarrow F_{x,b}(t)$ is a continuous map on V.
If the triangle inequality (e)is formulated under a t-norm T:
(f) $F_{x+y,z}(t_1 + t_2) \geq T(F_{x,z}(t_1), F_{y,z}(t_2)), for\ all\ x, y, z \in V,\ t_1, t_2 \in \mathbb{R}^+$,
then the triple $(V, \mathcal{F}, T)$ is called a Menger probabilistic 2-normed space.
If T is a left continuous t-norm and τ_T is the associated triangle function, then the inequalities (e)and(f)are equivalent.

Remark 1.2. It is easy to check that every 2-normed space $(V, \|.,.\|)$ can be made a probabilistic 2-normed space, in a natural way, by setting $F_{x,y} = H(t - \|x, y\|)$, for every $x, y \in V$, $t \in \mathbb{R}^+$ and $T = Min$.

Definition 1.3. Let $G \in \Delta^+$ be different from H, let $(V, \|.,.\|)$ be a 2-normed space. Define be a mapping from $\mathcal{F} : V \times V \to \Delta^+$, by $F_{x,y} = H$, if x and y are linearly dependent and

$$F_{x,y}(t) := G(\frac{t}{\|x, y\|})\ (t > 0)$$

when x and y are linearly independent. The pair $(V, \mathcal{F})$ is called the simple space generated by $(V, \|.,.\|)$ and G.

Let $(V, \|.,.\|)$ be a 2-normed space. Define for each $b \in V$, $\tau(F, G)(x) = F(x).G(x)$ for every $F, G \in \Delta^+$ and $F^{\|.,.\|}_{x,b}(t) = \frac{t}{(t+\|x,b\|)}$ for every $x \in V$, then $F^{\|.,.\|}$ is a $P - 2\ norm$ which is called the standard $P - 2\ norm$ induced by $\|.,.\|$.
I. Golet in [3] proved that if $(V, \mathcal{F}, \tau)$ is a probabilistic 2-normed space and $\mathcal{A}$ is the family of all finite and non-empty subsets of the linear space V. For every $A \in \mathcal{A}$, $\varepsilon > 0$ and $\lambda \in (0, 1), (V, \mathcal{F}, \tau)$ is a Hausdorff topological space in the topology τ induced by the family of (ε, λ)-neighborhoods of x_0 vector:

$$\nu_{x_0} = \{N_{x_0}(\varepsilon, \lambda, A) : \varepsilon > 0\ ,\ \lambda \in (0, 1)\ , A \in \mathcal{A}\}$$

Where

$$N_{x_0}(\varepsilon, \lambda, A) = \{x \in V : F_{x_0 - x,a}(\varepsilon) > 1 - \lambda\ , a \in A\}$$

Under a continuous triangle function τ such that $\tau \geq \tau_{T_m}$, where $T_m(a,b) = \max\{a+b-1,0\}$.

2. P_b-BEST COAPPROXIMATION IN PROBABILISTIC 2-NORMED SPACE

Definition 2.1. Let A be a nonempty subset of a P2N-space $(V, \mathcal{F})$. For $t > 0$ and $b \in V$, an element $a_0 \in A$ is called a P_b-best coapproximation to $x \in V$ from A if for every $a \in A$,

$$F_{a_0-a,b}(t) \geq F_{x-a,b}(t).$$

The set of all such elements a_0 that called a P_b-best coapproximation to $x \in V$, is denoted by $R^t_{A,b}(x)$, i.e.,

$$R^t_{A,b}(x) = \{a_0 \in A : F_{a_0-a,b}(t) \geq F_{x-a,b}(t) \text{ for all } a \in A,\ t > 0\}.$$

Putting

$$\breve{A}_b = \{x \in V : F_{a,b}(t) \geq F_{a-x,b}(t) \text{ for all } a \in A,\ t > 0\} = (R^t_{A,b})^{-1}(\{0\}),$$

it is clear $a_0 \in R^t_{A,b}(x)$ if and only if $x - a_0 \in \breve{A}_b$.

Definition 2.2. Let $(V, \mathcal{F})$ be a P2N-space. For $t > 0$ and $b \in V$, the nonempty subset $A \subset V$ is called P_b-coproximinal set if $R^t_{A,b}(x)$ is non-void for every $x \in V$ and A is called P_b-coChebyshev set if for every $x \in V$ the set $R^t_{A,b}(x)$ contains exactly one element.

Remark 2.3. Let A be a nonempty subset of a P2N-space $(V, \mathcal{F})$, and $\{x_n\}$ be a sequence of V.

(i)Then the sequence $\{x_n\}$ is said to be P_b-convergent to $x \in V$ and denoted by $x_n \xrightarrow{P_b} x$, if $\lim_{n\to\infty} F_{x_n-x,b}(t) = 1$, for all $x \in V$ and $t > 0$.

(ii)The set A is closed if and only if, whenever $\{a_n\}$ is a sequence of points in A converging to $x \in V$, then x is also in A.

Theorem 2.4. *Let A be a nonempty subset of a P2N-space $(V, \mathcal{F})$. Then for $t > 0$:*

(i) $R^t_{A+y,b}(x+y) = R^t_{A,b}(x) + y$, for every $x, y \in V$.

(ii) $R^{|\alpha|t}_{\alpha A,b}(\alpha x) = \alpha R^t_{A,b}(x)$, for every $x \in V$ and any scaler $\alpha \in \mathbb{R}\backslash\{0\}$.

(iii) A is P_b-coproxminal (respectively P_b-coChebyshev) if and only if $A + y$ is P_b-coproxminal (respectively P_b-coChebyshev) for every $y \in V$.

Proof. (i) For any $x, y \in V$, $t > 0$ and $b \in V$, let $a_0 \in R^t_{A+y,b}(x+y)$ if and only if, $F_{a_0-(a+y),b}(t) \geq F_{x+y-(a+y),b}(t)$ for all $(a+y) \in A+y$ if and only if, $F_{(a_0-y)-a,b}(t) \geq F_{x-a,b}(t)$ for all $a \in A$ if and only if, $(a_0 - y) \in R^t_{A,b}(x)$ i.e., $a_0 \in R^t_{A,b}(x) + y$.

(ii) Let $a_0 \in R^{|\alpha|t}_{\alpha A,b}(\alpha x)$, for any $x \in V$, $t > 0$ and $\alpha \in \mathbb{R}\backslash\{0\}$ if and only, $F_{a_0-\alpha a,b}(\mid \alpha \mid t) \geq F_{\alpha x-\alpha a,b}(\mid \alpha \mid t)$ for all $a \in A$ if and only, $F_{\frac{1}{\alpha}a_0-a,b}(t) \geq F_{x-a,b}(t)$ if and only, $\frac{1}{\alpha}a_0 \in R^t_{A,b}(x)$ if and only, $a_0 \in \alpha R^t_{A,b}(x)$. Therefore, $R^{|\alpha|t}_{\alpha A,b}(\alpha x) = \alpha R^t_{A,b}(x)$.

(iii) Is an immediate consequence of (i). □

Theorem 2.5. *Let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space and A be a convex subset of V. Then for $t > 0$, $b \in V$ and $x \in V$, $R^t_{A,b}(x)$ is a convex subset of A (for $R^t_{A,b}(x) \neq \emptyset$).*

Proof. Let $a_1, a_2 \in R^t_{A,b}(x)$, $t > 0$, $b \in V$ and $x \in V$, then

$$F_{a-a_1,b}(t) \geq F_{x-a,b}(t) \ and \ F_{a-a_2,b}(t) \geq F_{x-a,b}(t) \text{ for all } a \in A.$$

For $\lambda \in (0,1)$:

$$F_{a-(\lambda a_1+(1-\lambda)a_2),b}(t) = F_{\lambda a-\lambda a_1+a-\lambda a-a_2+\lambda a_2,b}(t)$$

$$
\begin{aligned}
&= F_{\lambda(a-a_1)+(1-\lambda)(a-a_2),b}(t)\\
&\geq \tau\big(F_{a-a_1,b}(\frac{\lambda t}{\lambda}), F_{a-a_2,b}(\frac{(1-\lambda)t}{(1-\lambda)})\big)\\
&\geq \tau\big(F_{x-a,b}(t), F_{x-a}(t)\big) = F_{x-a,b}(t),
\end{aligned}
$$

so for each $\lambda \in (0,1)$, we have

$$F_{a-(\lambda a_1+(1-\lambda)a_2),b}(t) \geq F_{x-a,b}(t),$$

then $\lambda a_1 + (1-\lambda)a_2 \in R^t_{A,b}(x)$.
Hence $R^t_{A,b}(x)$ is a convex. □

Theorem 2.6. *Let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space and A be a subset of V and $b \in V$. If $a_0 \in R^t_{A,b}(x)$ and $(1-\lambda)x + \lambda a_0 \in A$, for $x \in V$ and every scaler $\lambda \neq 0$, then $(1-\lambda)x + \lambda a_0 \in R^t_{A,b}(x)$.*

Proof. Let $a_0 \in R^t_{A,b}(x)$, $t > 0$, $b \in V$ and $x \in V$, then $F_{a-a_0,b}(t) \geq F_{x-a,b}(t)$ for all $a \in A$.
Then for, for $\lambda \neq 0$:

$$
\begin{aligned}
F_{a-[(1-\lambda)x+\lambda a_0],b}(t) &= F_{a-(1-\lambda)x-\lambda a+\lambda a-\lambda a_0,b}(t)\\
&= F_{(1-\lambda)a-(1-\lambda)x+\lambda(a-a_0),b}(t)\\
&= F_{(1-\lambda)(a-x)+\lambda(a-a_0),b}(t)\\
&\geq \tau\big(F_{a-x,b}(\frac{(1-\lambda)t}{(1-\lambda)}), F_{a-a_0,b}(\frac{\lambda t}{\lambda})\big)\\
&\geq \tau\big(F_{a-x,b}(t), F_{x-a,b}(t)\big) = F_{x-a,b}(t),
\end{aligned}
$$

for all $a \in A$, thus $(1-\lambda)x + \lambda a_0 \in R^t_{A,b}(x)$. □

Example 2.7. Let $V = \mathbb{R}^2$. Define $\mathcal{F} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathcal{D}^+$ as

$$F_{(x_1,x_2),(y_1,y_2)}(t) = \big(exp(|x_1y_2 - x_2y_1|/t)\big)^{-1}.$$

Then $(V, \mathcal{F}, \tau)$ is a Menger P2N-space where $\tau(F(t), G(t)) = F(t).G(t)$ for every F and G in $\mathcal{D}^+$. Let $A = \{(x_1,x_2) \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 1,\ 0 \leq x_2 \leq |x_1|\}$ and $x = (0,3), b = (0,2)$. Then for every $t > 0$, $(1,1), (-1,1) \in R^t_{A,b}(0,3)$.

Theorem 2.8. *For $t > 0$ and $b \in V$, let A be a P_b-coproximinal subspace of a P2N-space $(V, \mathcal{F})$. Then*

(1) if $\breve{A}_b$ is a compact set then $R^t_{A,b}(x)$ is compact, for every $x \in V$.

(2) if $\breve{A}_b$ is a close set then $R^t_{A,b}(x)$ is close, for every $x \in V$.

Proof. (1) Suppose $x \in V$ and $\{a_n\}$ is a sequence in $R^t_{A,b}(x)$. Since $x - a_n \in \breve{A}_b$ and $\breve{A}_b$ is a compact set, there is a subsequence $\{x - a_{n_k}\}$ that convergence to $u_0 \in \breve{A}_b$. Since $x - u_0 = a_0$, therefore $a_0 \in R^t_{A,b}(x)$.
(2) It is clear. □

The following lemma shows that the P_b-best coapproximation in probabilistic 2-normed spaces is a generalization of best coapproximation in 2-normed spaces.

Lemma 2.9. *Let $(V, \|.,.\|)$ be a 2-normed space and $F^{\|.,.\|}$ be the induced probabilistic 2-norm. Then for $b \in V$, $y_0 \in A$ is a best coapproximation to $x \in V$ in the 2-normed linear space if and only if y_0 is a P_b-best coapproximation to x in the induced probabilistic 2-normed linear space $(V, \mathcal{F}^{\|.,.\|}, \tau)$,* *Proof.* For $b \in V$, since y_0 is a best coapproximation to $x \in V$, we have $\{\|y-y_0,b\| \leq \|x-y,b\| \ ; \ \forall y \in A\}$ if and only if $\{\frac{t}{t+\|y-y_0,b\|} \geq \frac{t}{t+\|x-y,b\|} \ ; \ \forall y \in A\}$ if and only if $\{F^{\|.,.\|}_{y-y_0,b}(t) \geq F^{\|.,.\|}_{x-y,b}(t) \ ; \ \forall y \in A\}$ if and only if $y_0 \in R^t_{A,b}(x)$. □

Definition 2.10. For $t > 0$ and $b \in V$, let $(V, \mathfrak{F}, \tau)$ be a Menger P2N-space and A be a subset of V. An element $x \in V$ is said to be b-orthogonal to an element $y \in V$, and we denote $x \perp^b y$, if $F_{x+\lambda y,b}(t) \leq F_{x,b}(t)$ for all scalar $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $t > 0$.

Also, An element $x \in V$ is said to be $b-$orthogonal to A, and we denote $x \perp^b A$, if $x \perp^b y$, for all $y \in A$.

Theorem 2.11. *For $t > 0$ and $b \in V$, let $(V, \mathfrak{F}, \tau)$ be a Menger P2N-space and A be a subset of V. Then for $x \in V$, $y_0 \in R^t_{A,b}(x)$ if and only if $A \perp^b x - y_0$.*

Proof. Suppose $x \in V$ and $A \perp^b x - y_0$. Then $F_{a+\lambda(x-y_0),b}(t) \leq F_{a,b}(t)$ for all $a \in A$ and all scalar $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $t > 0$, if and only if $F_{x-y_0+\lambda^{-1}a,b}(\frac{t}{|\lambda|}) \leq F_{\lambda^{-1}a,b}(\frac{t}{|\lambda|})$, if and only if $F_{x-y_0+a',b}(\frac{t}{|\lambda|}) \leq F_{a',b}(\frac{t}{|\lambda|})$ and for $a^{'} = \lambda^{-1}a$ and $y_0 - a^{'} = a^{''}$, if and only if $F_{x-a'',b}(\frac{t}{|\lambda|}) \leq F_{y_0-a'',b}(\frac{t}{|\lambda|})$ for every $\lambda \neq 0$, if and only if $F_{a-y_0,b}(t) \geq F_{x-a,b}(t)$ for all $a \in A$ and for each $t > 0$, if and only if $y_0 \in R^t_{A,b}(x)$. □

Remark 2.12. For $t > 0$ and $b \in V$, let $(V, \mathfrak{F}, \tau)$ be a Menger P2N-space and A be a subset of V.

$$(R^t_{A,b})^{-1}(\{0\}) = \{x \in V : F_{a,b}(t) \geq F_{a-x,b}(t) \text{ for all } a \in A, \ t > 0\} = \{x \in V : A \perp^b x\},$$

$$\breve{A}_b = \{x \in V : A \perp^b x\}.$$

Theorem 2.13. *Let A be subspace of a Menger P2N-space $(V, \mathfrak{F}, \tau)$, then $\breve{A}_b \bigcap A = \{0\}$.*

Proof. Let $a \in \breve{A}_b \bigcap A$, we show that $a = 0$. To see this, we have $a \in \breve{A}_b$, then $A \perp^b a$ and $a \in A$, this implies that $h \perp^b a$ for all $h \in A$.
Therefore, $F_{h+\lambda a,b}(t) \leq F_{h,b}(t)$ for all $h \in A, \ t > 0$, and all scaler λ.
Now, if we choose $\lambda = -\frac{1}{3}$ and $h = a$, then $F_{a-\frac{1}{3}a,b}(t) \leq F_{a,b}(t)$, and so, $F_{\frac{2}{3}a,b}(t) = F_{a,b}(\frac{3}{2}t) \leq F_{a,b}(t)$, and hence, $a = 0$, i.e., $\breve{A}_b \bigcap A \subseteq \{0\}$. But $\{0\} \subseteq \breve{A}_b \bigcap A$, together, we get $\breve{A}_b \bigcap A = \{0\}$. □

Theorem 2.14. *For $t > 0$ and $b \in V$, let A be a P_b-coproximinal subspace of a Menger P2N space $(V, \mathfrak{F}, \tau)$. If $\breve{A}_b$ is a convex set, then A is P_b- coChebyshev, for every $x \in V$.*

Proof. Suppose $t > 0$, $x \in V$ and $a_1, a_2 \in R^t_{A,b}(x)$; then $x - a_1, x - a_2 \in \breve{A}_b$. Put $\breve{a}_1 = x - a_1$ and $\breve{a}_2 = x - a_2$ and let us have $x = a_1 + \breve{a}_1 = a_2 + \breve{a}_2$. Since $\frac{1}{2}(\breve{a}_1 - \breve{a}_2) \in \breve{A}_b$, it follows that $a_1 - a_2 \in \breve{A}_b \cap A = \{0\}$; then $a_1 = a_2$. □

Definition 2.15. For $t > 0$ and $b \in V$, let $(V, \mathfrak{F}, \tau)$ be a Menger P2N-space, A and H be subsets of V. Define: $R^t_{A,b}(H) = \bigcup_{h \in H} R^t_{A,b}(h)$.

Theorem 2.16. *For $t > 0$ and $b \in V$, let $(V, \mathfrak{F}, \tau)$ be a Menger P2N-space, A and $A^{'}$ be subspaces of V, such that $A \subseteq A^{'}$, and let $x \in V$. Then:*

$$R^t_{A,b}(R^t_{A',b}(x)) \subseteq R^t_{A,b}(x).$$

Proof. Suppose $a_0 \in R^t_{A,b}(R^t_{A',b}(x))$, then $a_0 \in R^t_{A,b}(a_0')$ for $a_0' \in R^t_{A',b}(x)$, so $A' \perp^b (x - a_0')$, and $A \perp^b (a_0' - a_0)$. Thus, $F_{a'+\lambda(x-a_0'),b}(t) \le F_{a',b}(t)$ for all $\lambda \in \mathbb{R}$ and $a' \in A'$, and $F_{a+\lambda(a_0'-a_0),b}(t) \le F_{a,b}(t)$ for all $\lambda \in \mathbb{R}$ and $a \in A$. Now since, $a + \lambda(a_0' - a_0) \in A'$ for $\lambda \in \mathbb{R}$ and $a \in A \subset A'$, therefore,

$$F_{a+\lambda(x-a_0),b}(t) = F_{a+\lambda(a_0'-a_0)+\lambda(x-a_0'),b}(t) \le F_{a+\lambda(a_0'-a_0),b}(t) \le F_{a,b}(t),$$

since $F_{a+\lambda(x-a_0),b}(t) \le F_{a,b}(t)$, so, $a \perp^b (x - a_0)$ for all $a \in A$, then $A \perp^b (x - a_0)$, i.e., $a_0 \in R^t_{A,b}(x)$. Hence $R^t_{A,b}(R^t_{A',b}(x)) \subseteq R^t_{A,b}(x)$. □

Corollary 2.17. *For $t > 0$ and $b \in V$, let $(V, \mathcal{F}, \tau)$ be a Menger P2N-space, and A be subspace of V. Then $R^t_{A,b}(x) = A \bigcap (x - \breve{A}_b)$.*

Proof. Let $a_0 \in A \bigcap (x - \breve{A}_b)$, if and only if $a_0 \in A$, and $a_0 \in (x - \breve{A}_b)$, if and only if $a_0 \in A$, and $a_0 = x - \breve{a}$, where $\breve{a} \in \breve{A}_b$, if and only if $a_0 \in A$, and $\breve{a} = x - a_0 \in \breve{A}$, if and only if $a_0 \in R^t_{A,b}(x)$. Therefore, $R^t_{A,b}(x) = A \bigcap (x - \breve{A}_b)$. □

Theorem 2.18. *Let A be subspace of a Menger P2N-space $(V, \mathcal{F}, \tau)$, then*
(1) A is a P_b-coproximinal subspace if and only if $V = A + \breve{A}_b$.
(2) A is a P_b-coChebyshev subspace if and only if $V = A \oplus \breve{A}_b$.

Proof. (1)(⇒)Let $t > 0$, assume that A is P_b-coproximinal, and let $x \in V$ and $a_0 \in R^t_{A,b}(x)$. Then, $x - a_0 \in \breve{A}_b$. Now, $x = a_0 + (x - a_0) \in A + \breve{A}_b$. Hence, $V = A + \breve{A}_b$.

(⇐) Let $t > 0$, $V = A + \breve{A}_b = \{a + y : \ a \in A, \ y \in \breve{A}_b\}$, and $x \in V$. Then $x = a_0 + y$, where $a_0 \in A$, $y \in \breve{A}_b$. Since $y \in \breve{A}_b = R^{-t}_{A,b}(0)$, then $0 \in R^t_{A,b}(y)$. Since $x = a_0 + y$, then $y = x - a_0$, so $R^t_{A,b}(y) = R^t_{A,b}(x - a_0)$, this implies that $0 \in R^t_{A,b}(y) = R^t_{A,b}(x - a_0)$.
Then $F_{0-(x-a_0),b}(t) \ge F_{a-(x-a_0),b}(t)$, so $F_{a_0-x,b}(t) \ge F_{(a+a_0)-x,b}(t)$ where $(a+a_0) \in A$; hence $a_0 \in R^t_{A,b}(x)$. Therefore A is P_b-coproximinal.

(2)(⇒) Suppose that A is P_b-coChebyshev subspace and $x \in V$, $x = a_1 + \breve{a}_1 = a_2 + \breve{a}_2$, where $a_1, a_2 \in A$ and $\breve{a}_1, \breve{a}_2 \in \breve{A}_b$.
We show that $a_1 = a_2$, and $\breve{a}_1 = \breve{a}_2$, since $x = a_1 + \breve{a}_1 = a_2 + \breve{a}_2$, then $x - a_1 = \breve{a}_1, x - a_2 = \breve{a}_2 \in \breve{A}$, this implies that $a_1, a_2 \in R^t_{A,b}(x)$.
Therefore, $a_1 = a_2$ because A is P_b-coChebyshev, it follows that $\breve{a}_1 = \breve{a}_2$. Thus $V = A \oplus \breve{A}_b$.

(⇐) Let $V = A \oplus \breve{A}_b$ and suppose for $x \in V$, there exist $a_1, a_2 \in R^t_{A,b}(x)$. We show $a_1 = a_2$.
Since $a_1, a_2 \in R^t_{A,b}(x)$, then $x - a_1, x - a_2 \in \breve{A}_b$ and therefore, $x = a_1 + \breve{a}_1 = a_2 + \breve{a}_2$, where $\breve{a}_1 = x - a_1$ *and* $\breve{a}_2 = x - a_2$. Since $V = A \oplus \breve{A}$, then $a_1 = a_2$ and $\breve{a}_1 = \breve{a}_2$. Hence A is P_b-coChebyshev. □

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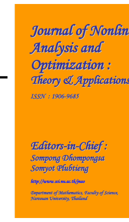

SOME NOTES ON (α, β)-GENERALIZED HYBRID MAPPINGS

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ABSTRACT. In 1973, Bruck generalized the notion nonexpansive mappings by introducing firmly nonexpansive mappings. Kohsaka and Takahashi introduced nonspreading mappings in 2008 and Takahashi introduced hybrid mappings in 2010. It is worth noting that each nonexpansive mapping is a 1-hybrid mapping and each nonspreading mapping is a 0-hybrid mapping. Thus, the notion of λ-hybrid mappings is a generalization of the notions of firmly nonexpansive mappings and nonspreading mappings. In 2011, Takahashi introduced generalized hybrid mappings and Aoyama and Kohsaka defined α-nonexpansive mappings on Banach spaces. Kocourek, Takahashi and Yao gave the notions of $(\alpha, \alpha - 1)$-generalized hybrid mappings and (α, β, γ)-super hybrid mappings. In this paper, we discuss (α, β)-generalized hybrid mappings. By using and combining ideas of some recent papers, we generalize the notion of α-nonexpansivity to (α, β)-nonexpansivity and give some results on the subject.

KEYWORDS : Ideal; Filter; Sequence of moduli; Lipschitz function; I-convergence field; I-convergent; Monotone; Solid spaces

1. INTRODUCTION

Let $(H, < ., . >)$ be a Hilbert space and C a nonempty subset of H. In 1973, Bruck generalized the notion nonexpansive mappings by introducing firmly nonexpansive mappings ([4]). We say that $T : C \longrightarrow H$ is a firmly nonexpansive mapping whenever $\|Tx - Ty\| \leq \|r(x - y) + (1 - r)(Tx - Ty)\|$ for all $r > 0$ and $x, y \in C$. A mapping $T : C \longrightarrow H$ is said to be quasi-nonexpansive whenever $F(T)$ is a nonempty set and $\|Tx - z\| \leq \|x - z\|$ for all $x \in C$ and $z \in F(T)$. In 2008, Kohsaka and Takahashi introduced nonspreading mappings ([8]). In 2010, Kurokawa and Takahashi proved some weak and strong convergence theorems for nonspreading mappings in Hilbert spaces ([9]). Later, Aoyama and Kohsaka generalized some of their results in 2011 ([2]). On the other hand, Aoyama, Iemoto, Kohsaka and Takahashi proved some fixed point results about λ-hybrid

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mappings ($\lambda \in \mathbb{R}$) ([1]). A mapping $T : C \longrightarrow H$ is said to be λ-hybrid if $2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\lambda Re < x - Tx, y - Ty >$ or equivalently $\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda) Re < x - Tx, y - Ty >$ for all $x, y \in C$. In fact, each nonexpansive mapping is a 1-hybrid mapping ([1]) and each nonspreading mapping is a 0-hybrid mapping ([2]). Also, T is $\frac{1}{2}$-hybrid if and only if T is a hybrid mapping in the sense of [13] (see for example, [2]). Let $\kappa \in [0, 1)$. A mapping $T : C \longrightarrow H$ is said to be κ-strictly pseudononspreading if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2Re < x - Tx, y - Ty > + \kappa\|x - Tx - (y - Ty)\|^2$$

for all $x, y \in C$ ([11]). Let $0 \leq \kappa \leq \beta < 1$ and T be a κ-strictly pseudononspreading mapping. Then, $T_\beta = \beta I + (1 - \beta)T$ is a $\frac{-\beta}{1-\beta}$-hybrid mapping ([2] and [11]). Recently, Takahashi introduced generalized hybrid mappings and proved some weak convergence theorems for generalized hybrid mappings in Banach spaces ([14]). In 2010, Klin-eam and Suantai, by using a multiindex $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ satisfying $\alpha_i \geq 0$ ($i = i, 2, \cdots, n$) and $\sum_{i=1}^{n} \alpha_i = 1$, introduced α-nonexpansive mappings and proved some fixed point results for the mappings ([6]). In 2011, Aoyama and Kohsaka introduced α-nonexpansive mappings on Banach spaces in a different form and provided some fixed point theorems for α-nonexpansive mappings ([3]). Let E be a Banach space, C a nonempty subset of E and α a real number such That $\alpha < 1$. A mapping $T : C \longrightarrow E$ is said to be α-nonexpansive if $\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2$ for all $x, y \in C$. Aoyama and Kohsaka proved that for $\lambda < 2$, T is a λ-hybrid mapping if and only if T is a $\frac{1-\alpha}{2-\alpha}$-nonexpansive mapping (see Proposition 2.2 in [3]). Let l^∞ be the Banach space of bounded real sequences with the supremum norm. It is known that there exists a bounded linear functional μ on l^∞ such that $\mu(\{t_n\}) \geq 0$ for all $\{t_n\} \in l^\infty$ with $t_n \geq 0$ ($n \geq 1$), $\mu(\{t_n\}) = 1$ for all $\{t_n\} \in l^\infty$ with $t_n = 1$ ($n \geq 1$) and $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\} \in l^\infty$. The functional μ is called Banach limit and the value of μ at $\{t_n\} \in l^\infty$ is denoted by $\mu_n t_n$ ([3] and [12]). In this paper, we give some results on (α, β)-generalized hybrid mapping. Also, by using and combining ideas of [1], [2], [3], [13] and [14], we generalize the notion of α-nonexpansivity to (α, β)-nonexpansivity and give some results about the subject. Finally, we appeal the following result which has been proved in [3].

Lemma 1.1. *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and T a selfmap on C such that $\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^n x - y\|^2$ for all $y \in C$. Then T has a fixed point.*

2. MAIN RESULTS

Now, we are ready to state and prove our main results. Our first result is another version of Theorem 3.1 in [7].

Theorem 2.1. *Let C be a nonempty, closed and convex subset of a Hilbert space H and T a selfmap on C such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\gamma Re < x - Ty, y - Tx > + k\|x - Tx - (y - Ty)\|^2$$

for all $x, y \in C$, where $\gamma + 2k < 0$. Then, T has a fixed point in C if and only if $\{T^n z\}$ is a bounded sequence for some $z \in C$.

Proof. Let $z \in F(T)$. Then $\{T^n z\} = \{z\}$ and so $\{T^n z\}$ is bounded. Now, suppose that there exists $z \in C$ such that $\{T^n z\}$ is bounded. Then, for each $x, y \in C$ we have

$$\| Tx - Ty \|^2 \leq \| x - y \|^2 + \gamma(\| x - Ty \|^2 + \| y - Tx \|^2)$$

$$+k\| x-y \|^2 + k\| Tx-Ty \|^2 + 2kRe\langle x-y, Ty-Tx\rangle.$$

Hence,

$$\| Tx-Ty \|^2 \leq \| x-y \|^2 + \gamma\| x-Ty \|^2 + \gamma\| y-Tx \|^2$$
$$+k\| x-y \|^2 + k\| Tx-Ty \|^2 + k\| x-y \|^2 + k\| Tx-Ty \|^2$$

for all $x, y \in C$. Let μ be the Banach limit. Since μ is a positive linear functional on l^∞, for each $y \in C$ and $n \geq 0$ we have

$$\mu_n\| T^{n+1}z-Ty \|^2 \leq \mu_n\| T^n z-y \|^2 + \gamma\mu_n\| T^n z-Ty \|^2 + \gamma\mu_n\| y-T^{n+1}z \|^2$$
$$+k\mu_n\| T^n x-y \|^2 + k\mu_n\| T^{n+1}z-Ty \|^2 + k\mu_n\| T^n z-y \|^2 + k\mu_n\| T^{n+1}z-Ty \|^2.$$

Thus, by using the property of μ we obtain

$$\mu_n\| T^n z-Ty \|^2 \leq \mu_n\| T^n z-y \|^2 + \gamma\mu_n\| T^n z-Ty \|^2 + \gamma\mu_n\| y-T^n z \|^2$$
$$+k\mu_n\| T^n z-y \|^2 + k\mu_n\| T^n z-Ty \|^2 + k\mu_n\| T^n z-y \|^2 + k\mu_n\| T^n z-Ty \|^2.$$

Hence, $(1-\gamma-2k)\mu_n \| T^n z-Ty \|^2 \leq (1+\gamma+2k)\mu_n \| T^n z-y \|$ and so

$$\mu_n\| T^n z-Ty \|^2 \leq \frac{1+\gamma+2k}{1-\gamma-2k}\mu_n \| T^n z-y \| \leq \mu_n \| T^n z-y \| .$$

Now by using Lemma 1.1, T has a fixed point. □

The following result is another version of Lemma 5.1 in [7].

Theorem 2.2. *Let C be a nonempty subset of a Hilbert space H and T a selfmap on C such that*

$$\| Tx-Ty \|^2 \leq \| x-y \|^2 + 2\gamma Re\langle x-Tx, y-Ty\rangle + k\| (I-T)x-(I-T)y \|^2$$

for all $x, y \in C$, where γ and k are real fixed numbers with $k < 1$. If $\{x_n\}$ converges weakly to z and $\{x_n - Tx_n\}$ tends to 0, then $I-T$ is demiclosed and $z \in F(T)$.

Proof. Suppose that $\{x_n\}$ converges weakly to z and $\{x_n - Tx_n\}$ tends to 0. Then, for each n we have

$$\| Tx_n-Tz \|^2 \leq \| x_n-z \|^2 + 2\gamma Re\langle x_n-Tx_n, z-Tz\rangle + k\| x_n-Tx_n+Tz-z \|^2$$

and so

$$\| Tx_n-x_n \|^2 + \| x_n-Tz \|^2 + 2Re\langle Tx_n-x_n, x_n-Tz\rangle = \| Tx_n-x_n+x_n-Tz \|^2$$
$$\leq \| x_n-z \|^2 + 2\gamma Re\langle x_n-Tx_n, z-Tz\rangle + k\| x_n-Tx_n+Tz-z \|^2.$$

Thus, we obtain

$$\mu(\{\| Tx_n-x_n \|^2\}) + \mu(\{\| x_n-Tz \|^2\}) + 2\mu(\{Re\langle Tx_n-x_n, x_n-Tz\rangle\})$$
$$\leq \mu(\{\| x_n-z \|^2\}) + 2\gamma\mu(\{Re\langle x_n-Tx_n, z-Tz\rangle\})$$
$$+k\mu(\{\| x_n-Tx_n+Tz-z \|^2\}).$$

Since μ is the Banach limit, $\{x_n\}$ converges weakly to z and $\{x_n - Tx_n\}$ tends to 0, we get

$$\mu_n\| x_n-Tz \|^2 \leq \mu_n\| x_n-z \|^2 + k\mu_n\| Tz-z \|^2$$

holds for all n. But, for each n we have

$$\mu_n\| x_n-z \|^2 + \mu_n\| z-Tz \|^2 + 2\mu_n Re\langle x_n-z, z-Tz\rangle$$
$$= \mu_n\| x_n-z+z-Tz \|^2 \leq \mu_n\| x_n-z \|^2 + k\mu_n\| Tz-z \|^2.$$

Since $\{x_n\}$ converges weakly to z, we obtain $(1-k)\mu_n\| Tz-z \|^2 \leq 0$ for all n. Hence, $\| z-Tz \|^2 \leq 0$ and so $Tz = z$. This implies that $I-T$ is demiclosed. □

The following result is a generalization of Lemma 2.7 in [2].

Theorem 2.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H and $\{x_n\}$ a sequence in C. Suppose that $T : C \to H$ and $T' : C \to H$ are two mappings and $\{\xi_n\}$ and $\{\xi'_n\}$ are two sequences of real numbers. Define the sequence $\{z_n\}$ in C by $z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$. Suppose that z is a weak cluster point of $\{z_n\}$,*

$$\xi_n + \xi'_n \leq \| x_n - z \|^2 - \| x_{n+1} - Tz \|^2 + \| x_n - z \|^2 - \| x_{n+1} - T'z \|^2$$

holds for all n, $\frac{1}{n}\sum_{k=1}^{n} \xi_k \longrightarrow 0$ and $\frac{1}{n}\sum_{k=1}^{n} \xi'_k \longrightarrow 0$. Then z is a common fixed point of T and T'.

Proof. First, note that

$$\xi_k + \xi'_k \leq \| x_k - z \|^2 - \| x_{k+1} - Tz \|^2 + \| x_k - z \|^2 - \| x_{k+1} - T'z \|^2$$

$$= \| x_k - Tz + Tz - z \|^2 - \| x_{k+1} - Tz \|^2 + \| x_k - T'z + T'z - z \|^2 - \| x_{k+1} - T'z \|^2$$

$$= \| x_k - Tz \|^2 - \| x_{k+1} - Tz \|^2 + 2Re\langle x_k - Tz, Tz - z\rangle + \| Tz - z \|^2$$

$$+ \| x_k - T'z \|^2 - \| x_{k+1} - T'z \|^2 + 2Re\langle x_k - T'z, T'z - z\rangle + \| T'z - z \|^2$$

holds for all k. By summing these inequalities from $k = 1$ to n and dividing by n, we obtain

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k + \frac{1}{n}\sum_{k=1}^{n} \xi'_k \leq \frac{1}{n}(\| x_1 - Tz \|^2 - \| x_{n+1} - Tz \|^2)$$

$$+2Re\langle \frac{1}{n}\sum_{k=1}^{n} x_k - Tz, Tz - z\rangle + \| Tz - z \|^2 + \frac{1}{n}(\| x_1 - T'z \|^2 - \| x_{n+1} - T'z \|^2)$$

$$+2Re\langle \frac{1}{n}\sum_{k=1}^{n} x_k - T'z, T'z - z\rangle + \| T'z - z \|^2$$

$$\leq \frac{1}{n}\| x_1 - Tz \|^2 + 2Re\langle z_n - Tz, Tz - z\rangle + \| Tz - z \|^2$$

$$+ \frac{1}{n}\| x_1 - T'z \|^2 + 2Re\langle z_n - T'z, T'z - z\rangle + \| T'z - z \|^2$$

for all n. Since z is a weak cluster point of $\{z_n\}$, there is a subsequence $\{z_{ni}\}$of $\{z_n\}$ such that $z_{ni} \longrightarrow z$. By replacing n by n_i, we get

$$\frac{1}{n_i}\sum_{k=1}^{n_i} \xi_k + \frac{1}{n_i}\sum_{k=1}^{n_i} \xi'_k \leq \frac{1}{n_i}\| x_1 - Tz \|^2 + 2Re\langle z_{ni} - Tz, Tz - z\rangle$$

$$+ \| Tz - z \|^2 + \frac{1}{n_i}\| x_1 - T'z \|^2 + 2Re\langle z_{ni} - T'z, T'z - z\rangle + \| T'z - z \|^2.$$

since $\frac{1}{n_i}\sum_{k=1}^{n_i} \xi_k \longrightarrow 0$, $\frac{1}{n_i}\sum_{k=1}^{n_i} \xi'_k \longrightarrow 0$ and $z_{ni} \longrightarrow z$, we obtain

$$0 \leq 2Re\langle z - Tz, Tz - z\rangle + \| Tz - z \|^2 + 2Re\langle z - T'z, T'z - z\rangle + \| T'z - z \|^2$$

$$= -\| Tz - z \|^2 - \| T'z - z \|^2.$$

Hence, $Tz = z$ and $T'z = z$. ☐

In 2011, Kocourek, Takahashi and Yao provided the notion of (α,β)-generalized hybrid mappings. Let C be a nonempty subset of a Hilbert space H and $\alpha,\beta\in\mathbb{R}$. We say that $T:C\longrightarrow C$ is a (α,β)-generalized hybrid mapping whenever

$$\alpha\| Tx-Ty\|^2+(1-\alpha)\| x-Ty\|^2\leq\beta\| Tx-y\|^2+(1-\beta)\| x-y\|^2$$

for all $x,y\in C$ ([7] and [14]). Note that, each (α,β)-generalized hybrid mapping is a nonexpansive mapping for $\alpha=1$ and $\beta=0$, a nonspreading mapping for $\alpha=2$ and $\beta=1$ and a hybrid mapping for $\alpha=\frac{3}{2}$ and $\beta=\frac{1}{2}$. Also, each (α,β)-generalized hybrid mapping is a quasi-nonexpansive mapping ([7]).

The following example shows that the conditions of Theorem 2.2 hold while a similar result is not true for (α,β)-generalized hybrid mappings.

Example 2.4. Consider $C=\{(1,0,0),(0,1,0),(0,0,0)\}$ in Euclidean metric space $\mathbb{R}^3$ and define the selfmap T on C by $T(1,0,0)=(1,0,0)$, $T(0,0,0)=(0,1,0)$ and $T(0,1,0)=(0,0,0)$. Then, T satisfies the conditions of Theorem 2.2 while T is not a (α,β)-generalized hybrid mapping, because by setting $x=(1,0,0)$ and $y=(0,0,0)$ we get a contradiction.

The following example shows that there is a $(2,1)$-generalized hybrid mapping which is not a nonexpansive mapping. One can find its main idea in [10].

Example 2.5. Let H be a Hilbert space. Consider the sets $E=\{x\in H:\| x\|\leq 1\}$, $D=\{x\in H:\| x\|\leq 2\}$ and $C=\{x\in H:\| x\|\leq 3\}$. Define the selfmap S on C by

$$Sx=\begin{cases} 0 & x\in D\\ P_E(x) & x\in C\backslash D\end{cases}$$

where P_E is the metric projection on E. It is easy to see that S is not a nonexpansive mapping while it is a $(2,1)$-generalized hybrid mapping.

The proof of the following result is straightforward (note that, $(\frac{\alpha-1}{\alpha})<1$).

Proposition 2.6. *Let C be a nonempty subset of a Hilbert space H, $\alpha>0$ and T a selfmap on C. Then, T is a $(\alpha,\alpha-1)$-generalized hybrid mapping if and only if T is a $\frac{\alpha-1}{\alpha}$-nonexpansive mapping if and only if T is a $(2-\alpha)$-hybrid mapping.*

The following example shows that there are discontinuous (α,β)-generalized hybrid mappings. Main idea of this example provided by Aoyama and Kohsaka in [3].

Example 2.7. Let E be a Banach space and $S,T:E\longrightarrow E$ two firmly nonexpansive mappings such that $S(E)$ and $T(E)$ are contained by rB_E for some $r>0$. Let α and δ be real numbers such that $1<\alpha\leq 2$ and $\delta\geq(1+\frac{2}{\sqrt{\frac{\alpha-1}{\alpha}}})r$. Define the map $U:E\longrightarrow E$ by

$$Ux=\begin{cases} Sx & x\in\delta B_E\\ Tx & otherwise\end{cases}$$

Then, U is a discontinuous $(\alpha,\alpha-1)$-generalized hybrid mapping.

Also, Kocourek, Takahashi and Yao provided the notion of (α,β,γ)-super hybrid mappings. Let C be a nonempty, closed and convex subset of a Hilbert space H and $\alpha,\beta,\gamma\in\mathbb{R}$ with $\gamma\geq 0$. We say that $T:C\longrightarrow C$ is a (α,β,γ)-super hybrid mapping whenever

$$\alpha\| Tx-Ty\|^2+(1-\alpha+\gamma)\| x-Ty\|^2\leq(\beta+(\beta-\alpha)\gamma)\| Tx-y\|^2$$
$$+(1-\beta-(\beta-\alpha-1)\gamma)\| x-y\|^2+(\alpha-\beta)\gamma\| x-Tx\|^2+\gamma\| y-Ty\|^2$$

for all $x, y \in C$ ([7]). Note that, each $(\alpha, \beta, 0)$-super hybrid mapping is a (α, β)-generalized hybrid mapping. By using this idea, we are going to generalize the notion of α-nonexpansivity in the following form. Let E be a Banach space, C a nonempty subset of E and α and β two real numbers with $\beta > \frac{-1}{2}$. A mapping $T : C \longrightarrow E$ is said to be (α, β)-nonexpansive if

$$(1-\alpha) \parallel Tx - Ty \parallel^2 + \alpha \parallel T^2x - T^2y \parallel^2 \leq (\frac{1}{2} - (\alpha+\beta)) \parallel Tx - y \parallel^2$$

$$+(\frac{1}{2} - (\alpha+\beta)) \parallel x - Ty \parallel^2 + \alpha \parallel T^2x - Ty \parallel^2 + \alpha \parallel Tx - T^2y \parallel^2 + 2\beta \parallel x - y \parallel^2$$

for all $x, y \in C$. If $\alpha = 0$ and $\alpha' = \frac{1}{2} - \beta$, then the notion of (α, β)-nonexpansivity reduces to the notion of α'-nonexpansivity. It is easy to see that each (α, β)-nonexpansive mapping is a quasi-nonexpansive mapping. Finally, note that by using a similar proof in Theorem 2.1, we can prove the following result.

Theorem 2.8. *Let C be a nonempty, closed and convex subset of a Hilbert space H, α and β two real numbers with $\beta > \frac{-1}{2}$ and $\alpha \leq 0$ and T a (α, β)-nonexpansive selfmap on C. Then, T has a fixed point in C if and only if $\{T^n z\}$ is a bounded sequence for some $z \in C$.*

The proof of the following result is straightforward (note that, $\frac{\frac{1}{2}-\beta}{1-\alpha} < 1$).

Proposition 2.9. *Let C be a nonempty subset of a normed space E, α and β two real numbers with $\alpha < 1$ and $\alpha - \beta < \frac{-1}{2}$ and T a (α, β)-nonexpansive selfmap on C such that T^2 is the identity map. Then T is a $\frac{\frac{1}{2}-\beta}{1-\alpha}$-nonexpansive mapping.*

Theorem 2.10. *Let C be a nonempty, closed and convex subset of a strictly convex Banach space E and T a (α, β)-nonexpansive selfmap on C. Then $F(T)$ is a closed and convex subset of E.*

Proof. If $F(T)$ is empty, then it is clear that $F(T)$ is closed and convex. Let $F(T) \neq \emptyset$. Since T is quasi-nonexpansive, by using a result of Itoh and Takahashi ([5]), we get that $F(T)$ is a closed and convex subset of E. □

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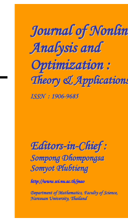

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ON A HALF-DISCRETE REVERSE MULHOLLAND'S INEQUALITY

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ABSTRACT. By using the way of weight functions and the technique of real analysis, a half-discrete reverse Mulholland's Inequality with a best constant factor is given. The extension with multi-parameters and the equivalent forms are also considered.

KEYWORDS : Mulholland's inequality; Weight function; Equivalent form; Reverse.
AMS Subject Classification: 26D15.

1. INTRODUCTION

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(\geq 0) \in L^p(0,\infty), g(\geq 0) \in L^q(0,\infty), ||f||_p = \{\int_0^\infty f^p(x)dx\}^{\frac{1}{p}} > 0$, $||g||_q > 0$, we have the following Hardy-Hilbert's integral inequality (cf. [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1.1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, ||a||_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0, ||b||_q > 0$, then we still have the following discrete Hardy-Hilbert's inequality with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} ||a||_p ||b||_q. \tag{1.2}$$

Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [10], [13], [18]). Also we have the following Mulholland's inequality with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [3], [15]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{\sum_{m=2}^\infty m^{p-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=2}^\infty n^{q-1} b_n^q\right\}^{\frac{1}{q}}. \tag{1.3}$$

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In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [11] gave an extension of (1.1) (for $p = q = 2$). Refinement the results of [11], Yang [14] gave some best extensions of (1.1) and (1.2) as follows: If $\lambda, \lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1 - 1}dt \in R_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f(\geq 0) \in L_{p,\phi}(0,\infty) = \{f|||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty\}, g(\geq 0) \in L_{q,\psi}(0,\infty), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.4}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover if $k_\lambda(x,y)$ is also finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing for $x > 0(y > 0)$, then for $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a|||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}, b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}, ||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.5}$$

where the constant factor $k(\lambda_1)$ is the best possible. For $\lambda = 1, k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (1.4) reduces to (1.1), and (1.5) reduces to (1.2). Some other results including the reverse Hilbert-type inequalities are provided by [19]-[9].

On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [3]. But they did not prove that the the constant factors in the inequalities are the best possible. And Yang [12] gave a result by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang [17] gave a half-discrete Hilbert's inequality and [16] gave the following half-discrete reverse Hilbert-type inequality with the best constant factor 4:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \min\{x,n\}a_n dx > 4\left\{\int_0^\infty (1-\theta_1(x))x^{\frac{3p}{2}-1}f^p(x)dx\right\}^{\frac{1}{p}} \times \left\{\sum_{n=1}^\infty n^{\frac{3q}{2}-1}a_n^q\right\}^{\frac{1}{q}} (\theta_1(x) \in (0,1)). \tag{1.6}$$

In this paper, by using the way of weight functions and the technique of real analysis, a half-discrete reverse Mulholland's inequality with a best constant factor is given as follows:

$$\int_1^\infty f(x) \sum_{n=2}^\infty \frac{a_n}{\ln xn} dx > \pi \left\{\int_1^\infty \frac{1-\theta_1(x)}{(\ln x)^{1-\frac{p}{2}}} x^{p-1}f^p(x)dx\right\}^{\frac{1}{p}} \times \left\{\sum_{n=2}^\infty \frac{n^{q-1}}{(\ln n)^{1-\frac{q}{2}}} a_n^q\right\}^{\frac{1}{q}} (\theta_1(x) \in (0,1)). \tag{1.7}$$

A best extension of (1.7) with multi-parameters, some equivalent forms are considered.

2. SOME LEMMAS

Lemma 2.1. *If $\lambda, \lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda$, setting weight functions $\omega(n)$ and $\varpi(x)$ as follows:*

$$\omega(n) \quad : \quad = (\ln n)^{\lambda_2} \int_1^\infty \frac{1}{x(\ln xn)^\lambda} (\ln x)^{\lambda_1 - 1} dx, n \in \mathbf{N}\backslash\{1\}, \tag{2.1}$$

$$\varpi(x) \quad : \quad = (\ln x)^{\lambda_1} \sum_{n=2}^{\infty} \frac{1}{n(\ln xn)^{\lambda}} (\ln n)^{\lambda_2 - 1}, x \in (1, \infty), \tag{2.2}$$

then we have

$$B(\lambda_1, \lambda_2)(1 - \theta_\lambda(x)) < \varpi(x) < \omega(n) = B(\lambda_1, \lambda_2), \tag{2.3}$$

where $\theta_\lambda(x) := \frac{1}{B(\lambda_1,\lambda_2)} \int_0^{\frac{1}{\ln x}} \frac{t^{\lambda_2-1}}{(1+t)^\lambda} dt \in (0,1)$ *and* $\theta_\lambda(x) = O(\frac{1}{(\ln x)^{\lambda_2}})(x \in (1,\infty))$.

Proof. Setting $t = \frac{\ln x}{\ln n}$ in (2.1), by calculation, we have

$$\omega(n) = \int_0^\infty \frac{1}{(1+t)^\lambda} t^{\lambda_1 - 1} dt = B(\lambda_1, \lambda_2).$$

Since for fixed $x > 1$,

$$h(x,y) := \frac{1}{y(\ln xy)^\lambda} (\ln y)^{\lambda_2 - 1} = \frac{1}{y(\ln x + \ln y)^\lambda (\ln y)^{1-\lambda_2}}$$

is strictly decreasing for $y \in (1, \infty)$, then we find

$$\varpi(x) < (\ln x)^{\lambda_1} \int_1^\infty \frac{1}{y(\ln xy)^\lambda} (\ln y)^{\lambda_2 - 1} dy$$

$$\overset{t=(\ln y)/(\ln x)}{=} \int_0^\infty \frac{t^{\lambda_2 - 1}}{(1+t)^\lambda} dt = B(\lambda_2, \lambda_1) = B(\lambda_1, \lambda_2),$$

$$\begin{aligned} \varpi(x) &> (\ln x)^{\lambda_1} \int_e^\infty \frac{(\ln y)^{\lambda_2 - 1}}{y(\ln xy)^\lambda} dy = \int_{\frac{1}{\ln x}}^\infty \frac{t^{\lambda_2-1}}{(1+t)^\lambda} dt \\ &= B(\lambda_1, \lambda_2)(1 - \theta_\lambda(x)) > 0, \end{aligned}$$

$$\begin{aligned} 0 &< \theta_\lambda(x) = \frac{1}{B(\lambda_1, \lambda_2)} \int_0^{\frac{1}{\ln x}} \frac{t^{\lambda_2 - 1}}{(1+t)^\lambda} dt \\ &< \frac{1}{B(\lambda_1, \lambda_2)} \int_0^{\frac{1}{\ln x}} t^{\lambda_2 - 1} dt = \frac{1}{\lambda_2 B(\lambda_1, \lambda_2)} \frac{1}{(\ln x)^{\lambda_2}}, \end{aligned}$$

and then (2.3) is valid. The lemma is proved. □

Lemma 2.2. *Let the assumptions of Lemma 2.1 be fulfilled and additionally,* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, n \in \mathbf{N}\backslash\{1\}, f(x)$ *is a non-negative measurable function in* $(1, \infty)$. *Then we have the following inequalities (Note: in this paper, if* $a_n = 0$, *then we think* $a_n^q = 0$ $(q < 0)$*):*

$$\begin{aligned} J \quad &: \quad = \left\{ \sum_{n=2}^\infty \frac{(\ln n)^{p\lambda_2 - 1}}{n} \left[\int_1^\infty \frac{f(x)}{(\ln xn)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} \\ &\geq [B(\lambda_1, \lambda_2)]^{\frac{1}{q}} \left\{ \int_1^\infty \varpi(x) x^{p-1} (\ln x)^{p(1-\lambda_1)-1} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} L_1 \quad &: \quad = \left\{ \int_1^\infty \frac{(\ln x)^{q\lambda_1 - 1}}{x[\varpi(x)]^{q-1}} \left[\sum_{n=2}^\infty \frac{a_n}{(\ln xn)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} \\ &\geq \left\{ B(\lambda_1, \lambda_2) \sum_{n=2}^\infty n^{q-1} (\ln n)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{2.5}$$

Proof. (i) By the reverse Hölder's inequality with weight (cf. [7]) and (2.3), it follows

$$\left[\int_1^\infty \frac{f(x)dx}{(\ln xn)^\lambda}\right]^p = \left\{\int_1^\infty \frac{1}{(\ln xn)^\lambda}\left[\frac{(\ln x)^{(1-\lambda_1)/q}}{(\ln n)^{(1-\lambda_2)/p}}\frac{x^{1/q}}{n^{1/p}}f(x)\right]\right.$$
$$\left.\times\left[\frac{(\ln n)^{(1-\lambda_2)/p}}{(\ln x)^{(1-\lambda_1)/q}}\frac{n^{1/p}}{x^{1/q}}\right]dx\right\}^p$$
$$\geq \int_1^\infty \frac{1}{(\ln xn)^\lambda}\frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln n)^{1-\lambda_2}}f^p(x)dx$$
$$\times\left\{\int_1^\infty \frac{1}{(\ln xn)^\lambda}\frac{n^{q-1}(\ln n)^{(1-\lambda_2)(q-1)}}{x(\ln x)^{1-\lambda_1}}dx\right\}^{p-1}$$
$$= \left\{\omega(n)\frac{(\ln n)^{q(1-\lambda_2)-1}}{n^{1-q}}\right\}^{p-1}\int_1^\infty \frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda(\ln n)^{1-\lambda_2}}f^p(x)dx$$
$$= \frac{[B(\lambda_1,\lambda_2)]^{p-1}n}{(\ln n)^{p\lambda_2-1}}\int_1^\infty \frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda(\ln n)^{1-\lambda_2}}f^p(x)dx.$$

Then by Lebesgue term by term integration theorem (cf. [6]), we have

$$J \geq [B(\lambda_1,\lambda_2)]^{\frac{1}{q}}\left\{\sum_{n=2}^\infty\int_1^\infty \frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda(\ln n)^{1-\lambda_2}}f^p(x)dx\right\}^{\frac{1}{p}}$$
$$= [B(\lambda_1,\lambda_2)]^{\frac{1}{q}}\left\{\int_1^\infty\sum_{n=2}^\infty \frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln xn)^\lambda(\ln n)^{1-\lambda_2}}f^p(x)dx\right\}^{\frac{1}{p}}$$
$$= [B(\lambda_1,\lambda_2)]^{\frac{1}{q}}\left\{\int_1^\infty \varpi(x)x^{p-1}(\ln x)^{p(1-\lambda_1)-1}f^p(x)dx\right\}^{\frac{1}{p}},$$

and (2.4) follows. Still by the reverse Hölder's inequality with weight ($q < 0$), we have

$$\left[\sum_{n=2}^\infty \frac{a_n}{(\ln xn)^\lambda}\right]^q = \left\{\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}\left[\frac{(\ln x)^{(1-\lambda_1)/q}}{(\ln n)^{(1-\lambda_2)/p}}\frac{x^{1/q}}{n^{1/p}}\right]\right.$$
$$\left.\times\left[\frac{(\ln n)^{(1-\lambda_2)/p}}{(\ln x)^{(1-\lambda_1)/q}}\frac{n^{1/p}}{x^{1/q}}a_n\right]\right\}^q$$
$$\leq \left\{\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}\frac{x^{p-1}(\ln x)^{(1-\lambda_1)(p-1)}}{n(\ln n)^{1-\lambda_2}}\right\}^{q-1}$$
$$\times\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}\frac{n^{q-1}(\ln n)^{(1-\lambda_2)(q-1)}}{x(\ln x)^{1-\lambda_1}}a_n^q$$
$$= \frac{x[\varpi(x)]^{q-1}}{(\ln x)^{q\lambda_1-1}}\sum_{n=2}^\infty \frac{(\ln x)^{\lambda_1-1}}{x(\ln xn)^\lambda}n^{q-1}(\ln n)^{(q-1)(1-\lambda_2)}a_n^q.$$

Then by Lebesgue term by term integration theorem, we have

$$L_1 \geq \left\{\int_1^\infty\sum_{n=2}^\infty \frac{(\ln x)^{\lambda_1-1}}{x(\ln xn)^\lambda}n^{q-1}(\ln n)^{(q-1)(1-\lambda_2)}a_n^q dx\right\}^{\frac{1}{q}}$$
$$= \left\{\sum_{n=2}^\infty\left[(\ln n)^{\lambda_2}\int_1^\infty \frac{(\ln x)^{\lambda_1-1}}{x(\ln xn)^\lambda}dx\right]n^{q-1}(\ln n)^{q(1-\lambda_2)-1}a_n^q\right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=2}^{\infty} \omega(n) n^{q-1} (\ln n)^{q(1-\lambda_2)-1} a_n^q \right\}^{\frac{1}{q}},$$

and then in view of (2.3), inequality (2.5) follows. □

3. MAIN RESULTS

In the following, for $0 < p < 1, q < 0$, we still use the normal expressions of $||f||_{p,\Phi}$ and $||a||_{q,\Psi}$. Setting $\Phi(x) := (1 - \theta_\lambda(x))x^{p-1}(\ln x)^{p(1-\lambda_1)-1} (x \in (1, \infty))$, $\Psi(n) := n^{q-1}(\ln n)^{q(1-\lambda_2)-1} (n \in \mathbf{N} \backslash \{1\})$, we have

$$[\Phi(x)]^{1-q} = \frac{(\ln x)^{q\lambda_1 - 1}}{x(1-\theta_\lambda(x))^{q-1}}, [\Psi(n)]^{1-p} = \frac{(\ln n)^{p\lambda_2 - 1}}{n}$$

and

Theorem 3.1. *If $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1 > 0, 0 < \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda, f(x), a_n \geq 0$, $f \in L_{p,\Phi}(1, \infty), a = \{a_n\}_{n=2}^{\infty} \in l_{q,\Psi}, ||f||_{p,\Phi} > 0$ and $||a||_{q,\Psi} > 0$, then we have the following equivalent inequalities:*

$$\begin{aligned} I &:= \sum_{n=2}^{\infty} a_n \int_1^{\infty} \frac{f(x)}{(\ln xn)^\lambda} dx \\ &= \int_1^{\infty} f(x) \sum_{n=2}^{\infty} \frac{a_n}{(\ln xn)^\lambda} dx > B(\lambda_1, \lambda_2) ||f||_{p,\Phi} ||a||_{q,\Psi}, \end{aligned} \tag{3.1}$$

$$J = \left\{ \sum_{n=2}^{\infty} \frac{(\ln n)^{p\lambda_2 - 1}}{n} \left[\int_1^{\infty} \frac{f(x)}{(\ln xn)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} > B(\lambda_1, \lambda_2) ||f||_{p,\Phi}, \tag{3.2}$$

$$L := \left\{ \int_1^{\infty} \frac{(\ln x)^{q\lambda_1 - 1}}{x(1-\theta_\lambda(x))^{q-1}} \left[\sum_{n=2}^{\infty} \frac{a_n}{(\ln xn)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} > B(\lambda_1, \lambda_2) ||a||_{q,\Psi}, \tag{3.3}$$

where the constant factor $B(\lambda_1, \lambda_2)$ in the above inequalities is the best possible.

Proof. By Lebesgue term by term integration theorem, there are two expressions for I in (3.1). In view of (2.4), for $\varpi(x) > B(\lambda_1, \lambda_2)(1 - \theta_\lambda(x))$, we have (3.2). By the reverse Hölder's inequality, we have

$$I = \sum_{n=2}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_1^{\infty} \frac{1}{(\ln xn)^\lambda} f(x) dx \right] [\Psi^{\frac{1}{q}}(n) a_n] \geq J ||a||_{q,\Psi}. \tag{3.4}$$

Then by (3.2), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$a_n := [\Psi(n)]^{1-p} \left[\int_1^{\infty} \frac{1}{(\ln xn)^\lambda} f(x) dx \right]^{p-1}, n \in \mathbf{N} \backslash \{1\},$$

then $J^{p-1} = ||a||_{q,\Psi}$. By (2.4), we find $J > 0$. If $J = \infty$, then (3.2) is naturally valid; if $J < \infty$, then by (3.1), we have

$$\begin{aligned} ||a||_{q,\Psi}^q &= J^p = I > B(\lambda_1, \lambda_2) ||f||_{p,\Phi} ||a||_{q,\Psi}, \\ ||a||_{q,\Psi}^{q-1} &= J > B(\lambda_1, \lambda_2) ||f||_{p,\Phi}, \end{aligned}$$

and we have (3.2), which is equivalent to (3.1).

In view of (2.5), for $[\varpi(x)]^{1-q} > [B(\lambda_1,\lambda_2)(1-\theta_\lambda(x))]^{1-q}$, we have (3.3). By the reverse Hölder's inequality, we find

$$I = \int_1^\infty [\Phi^{\frac{1}{p}}(x)f(x)]\left[\Phi^{\frac{-1}{p}}(x)\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}a_n\right]dx \geq ||f||_{p,\Phi}L. \tag{3.5}$$

Then by (3.3), we have (3.1). On the other-hand, assuming that (3.1) is valid, setting

$$f(x) := [\Phi(x)]^{1-q}\left[\sum_{n=2}^\infty \frac{1}{(\ln xn)^\lambda}a_n\right]^{q-1}, x \in (1,\infty),$$

then $L^{q-1} = ||f||_{p,\Phi}$. By (2.5), we find $L > 0$. If $L = \infty$, then (3.3) is naturally valid; if $L < \infty$, then by (3.1), we have

$$\begin{aligned} ||f||_{p,\Phi}^p &= L^q = I > B(\lambda_1,\lambda_2)||f||_{p,\Phi}||a||_{q,\Psi}, \\ ||f||_{p,\Phi}^{p-1} &= L > B(\lambda_1,\lambda_2)||a||_{q,\Psi}, \end{aligned}$$

and we have (3.3) which is equivalent to (3.1). Hence inequalities (3.1), (3.2) and (3.3) are equivalent.

For $0 < \varepsilon < p\lambda_1$, setting $\widetilde{f}(x) = 0, x \in (1,e); \widetilde{f}(x) = \frac{1}{x}(\ln x)^{\lambda_1-\frac{\varepsilon}{p}-1}, x \in [e,\infty)$, and $\widetilde{a}_n = \frac{1}{n}(\ln n)^{\lambda_2-\frac{\varepsilon}{q}-1}, n \in \mathbf{N}\backslash\{1\}$, if there exists a positive number $k(\geq B(\lambda_1,\lambda_2))$, such that (3.1) is valid as we replace $B(\lambda_1,\lambda_2)$ by k, then in particular, it follows

$$\begin{aligned} \widetilde{I} \quad : &= \sum_{n=2}^\infty \int_1^\infty \frac{1}{(\ln xn)^\lambda}\widetilde{a}_n\widetilde{f}(x)dx > k||\widetilde{f}||_{p,\Phi}||\widetilde{a}||_{q,\Psi} \\ &= k\left\{\int_e^\infty (1-O(\frac{1}{(\ln x)^{\lambda_2}}))\frac{dx}{x(\ln x)^{\varepsilon+1}}\right\}^{\frac{1}{p}} \\ &\quad \times\left\{\frac{1}{2(\ln 2)^{\varepsilon+1}} + \sum_{n=3}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\ &> k(\frac{1}{\varepsilon}-O(1))^{\frac{1}{p}}\left\{\frac{1}{2(\ln 2)^{\varepsilon+1}} + \int_2^\infty \frac{1}{y(\ln y)^{\varepsilon+1}}dy\right\}^{\frac{1}{q}} \\ &= \frac{k}{\varepsilon}(1-\varepsilon O(1))^{\frac{1}{p}}\left\{\frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}} + \frac{1}{(\ln 2)^\varepsilon}\right\}^{\frac{1}{q}}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \widetilde{I} &= \sum_{n=2}^\infty (\ln n)^{\lambda_2-\frac{\varepsilon}{q}-1}\frac{1}{n}\int_e^\infty \frac{1}{x(\ln xn)^\lambda}(\ln x)^{\lambda_1-\frac{\varepsilon}{p}-1}dx \\ &\overset{t=(\ln x)/(\ln n)}{=} \sum_{n=2}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}}\int_{1/\ln n}^\infty \frac{1}{(t+1)^\lambda}t^{\lambda_1-\frac{\varepsilon}{p}-1}dt \\ &\leq \int_0^\infty \frac{1}{(t+1)^\lambda}t^{\lambda_1-\frac{\varepsilon}{p}-1}dt\left[\frac{1}{2(\ln 2)^{\varepsilon+1}} + \sum_{n=3}^\infty \frac{1}{n(\ln n)^{\varepsilon+1}}\right] \\ &\leq B(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p})\left[\frac{1}{2(\ln 2)^{\varepsilon+1}} + \int_2^\infty \frac{dy}{y(\ln y)^{\varepsilon+1}}\right] \\ &= \frac{1}{\varepsilon}B(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p})\left[\frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}} + \frac{1}{(\ln 2)^\varepsilon}\right]. \end{aligned} \tag{3.7}$$

Hence by (3.6) and (3.7), it follows

$$\begin{aligned} & B(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}) \left[\frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}} + \frac{1}{(\ln 2)^{\varepsilon}} \right] \\ > \quad & k(1-\varepsilon O(1))^{\frac{1}{p}} \left\{ \frac{\varepsilon}{2(\ln 2)^{\varepsilon+1}} + \frac{1}{(\ln 2)^{\varepsilon}} \right\}^{\frac{1}{q}}, \end{aligned} \tag{3.8}$$

and $B(\lambda_1, \lambda_2) \geq k(\varepsilon \to 0^+)$. Hence $k = B(\lambda_1, \lambda_2)$ is the best value of (3.1).

We conform that the constant factor $B(\lambda_1, \lambda_2)$ in (3.2) ((3.3)) is the best possible. Otherwise we can come to a contradiction by (3.4) ((3.5)) that the constant factor in (3.1) is not the best possible. □

Remark 3.1. For $\lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2}$ in (3.1), (3.2) and (3.3), we have (1.7) and the following equivalent inequalities:

$$\left\{ \sum_{n=2}^{\infty} \frac{(\ln n)^{\frac{p}{2}-1}}{n} \left[\int_1^{\infty} \frac{f(x)dx}{\ln xn} \right]^p \right\}^{\frac{1}{p}} > \pi \left\{ \int_1^{\infty} \frac{(1-\theta_1(x))x^{p-1}}{(\ln x)^{1-\frac{p}{2}}} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{3.9}$$

$$\left\{ \int_1^{\infty} \frac{(\ln x)^{\frac{q}{2}-1}}{x(1-\theta_1(x))^{q-1}} \left[\sum_{n=2}^{\infty} \frac{a_n}{\ln xn} \right]^q dx \right\}^{\frac{1}{q}} > \pi \left\{ \sum_{n=2}^{\infty} \frac{n^{q-1}}{(\ln n)^{1-\frac{q}{2}}} a_n^q \right\}^{\frac{1}{q}}. \tag{3.10}$$

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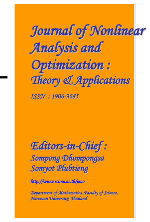

ON SOME I-CONVERGENT SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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ABSTRACT. In this article we introduce the sequence spaces $c_0^I(F)$, $c^I(F)$ and $l_\infty^I(F)$ for the sequence of modulii $F = (f_k)$ and study some of the properties of these spaces. The results here in proved are analogus to those by Vakeel.A.Khan and Khalid Ebadullah [Theory and Applications of Mathematics and Computer Science,1(2)(2011): 22-30].

KEYWORDS : Ideal; Filter; Sequence of moduli; Lipschitz function; I-convergence field; I-convergent; Monotone; Solid spaces

1. INTRODUCTION

Throughout the article $\mathbb{N}, \mathbb{R}, \mathbb{C}$and ω denotes the set of natural,real,complex numbers and the class of all sequences respectively.

The notion of the statistical convergence was introduced by H.Fast [5].Later on it was studied by J.A.Fridy[6, 7] from the sequence space point of view and linked it with the summability theory.

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko,Šalát and Wilezyński [18]. Later on it was studied by Šalát[26], Tripathy and Ziman [32] and Demirci[3], Das, Kostyrko, Wilczynski, and Malik [2], Mursaleen and Alotaibi [23], Mursaleen, Mohiuddine, and Edely [24], Mursaleen, and Mohiuddine [25], Mursaleen and Mohiuddine [26], Sahiner, Gurdal, Saltan and Gunawan [33] and Kumar [19]. Here we give some preliminaries about the notion of I-convergence.

Let X be a non empty set. Then a family of sets I$\subseteq 2^X$(power set of X)is said to be an ideal if I is additive i.e, A,B∈I ⇒A∪ B∈I and hereditary i.e., A∈I, B⊆A⇒B∈I.

A non-empty family of sets $\pounds(I) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin \pounds$(I),for A,B∈ $\pounds$(I) we have A∩B∈ $\pounds$(I) and for each A∈ $\pounds$(I)and A⊆B implies B∈ $\pounds$(I).

An Ideal I$\subseteq 2^X$ is called non-trivial if I$\neq 2^X$.

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A non-trivial ideal I$\subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\}$ $\subseteq$I.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal J$\neq$I containing I as a subset.

For each ideal I, there is a filter $\pounds$(I) corresponding to I.
i.e., $\pounds$(I) = $\{K \subseteq N : K^c \in I\}$,where K^c = N-K.

The idea of modulus was structured in 1953 by Nakano.(See[27]). A function f : [0,∞)$\longrightarrow$[0,∞) is called a modulus if

(1)f(t) = 0 if and only if t = 0,

(2) f(t+u)$\leq$ f(t)+ f(u) for all t,u$\geq$0,

(3) f is increasing, and

(4) f is continuous from the right at zero.

Ruckle [28, 29, 30] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}.$$

This space is an FK space ,and Ruckle[28] proved that that the intersecton of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$.Thus Ruckle[29] proved that,for any modulus f,

$$X(f) \subset l_1 \text{ and } X(f)^\alpha = l_\infty$$

Where

$$X(f)^\alpha = \{y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty\}.$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.\text{(See[29])}.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by B. Gramsch [10]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by D.J.H Garling[8, 9], G.Köthe[17] and W.H.Ruckle[28, 29, 30].

After then E.Kolk gave an extension of $X(f)$ by cosidering a sequence of modulii $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.\text{(See[15, 16])}.$$

Definition 1.1. A sequence space E is said to be solid or normal if $(x_k) \in$ E implies $(\alpha_k x_k) \in$ E for all sequence of scalars (α_k) with $|\alpha_k| < 1$ for all k $\in$ N.

Definition 1.2. A sequence space E is said to be monotone if it contains the cannonical preimages of all its stepspaces.

Definition 1.3. A sequence space E is said to be covergencefree if $(y_k)$$\in$ E whenever $(x_k)$$\in$ E and $x_k = 0$ implies $y_k = 0$.

Definition 1.4. A sequence space E is said to be a sequencealgebra if $(x_k y_k) \in$ E whenever $(x_k) \in$ E, $(y_k) \in$ E.

Definition 1.5. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in$ E whenever $(x_k) \in$ E where π(k) is a permutation on $\mathbb{N}$.

Definition 1.6. A sequence $(x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in N : |x_k - L| \geq \epsilon\} \in$ I. In this case we write $I - \lim x_k = L$.
The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) \in \omega : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C}\}.$$

Definition 1.7. A sequence $(x_k) \in \omega$ is said to be I-null if L = 0 .In this case we write $I - \lim x_k$ = 0.

Definition 1.8. A sequence $(x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m=m(ϵ) such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in$ I.

Definition 1.9. A sequence $(x_k) \in \omega$ is said to be I-bounded if there exists M >0 such that $\{k \in N : |x_k| > M\} \in I$.

Definition 1.10. Take for I the class I_f of all finite subsets of $\mathbb{N}$.Then I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X.(see[18]).

Definition 1.11. For I= I_δ the class of all $A \subset \mathbb{N}$ with $\delta(A) = 0$ respectively. I_δ is a non-trivial admissible ideal,I_δ-convergence is said to be logarithmic statistical covergence.(see[18]).

Definition 1.12. A map $\hbar$ defined on a domain $D \subset X$ i.e., $\hbar : D \subset X \to \boldsymbol{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where $K$is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$(see[32]).

Definition 1.13. A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{there exists } I - \lim x \in \boldsymbol{R}\}.$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$(See[31]).
Define a function $\hbar : F(I) \to \boldsymbol{R}$ such that $\hbar(x) = I - \lim x$, for all $x \in F(I)$, then the function $\hbar : F(I) \to \boldsymbol{R}$ is a Lipschitz function.(see [1, 4, 11, 12, 13, 14, 20, 21, 22, 26, 31, 34, 35]).

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and m_0^I represent the bounded , I-convergent, I-null, bounded I-convergent and bounded I-null sequence spaces respectively.

In this article we introduce the following classes of sequence spaces.

$$c^I(F) = \{(x_k) \in \omega : I - \lim f_k(|x_k|) = L \text{ for some L}\} \in I$$
$$c_0^I(F) = \{(x_k) \in \omega : I - \lim f_k(|x_k|) = 0\} \in I$$
$$l_\infty^I(F) = \{(x_k) \in \omega : \sup_k f_k(|x_k|) < \infty\} \in I$$

We also denote by

$$m^I(F) = c^I(F) \cap l_\infty(F)$$

and

$$m_0^I(F) = c_0^I(F) \cap l_\infty(F)$$

The following Lemmas will be used for establishing some results of this article.

Lemma 1.14. *Let E be a sequence space.If E is solid then E is monotone.*

Lemma 1.15. *Let K∈ £(I) and M⊆N. If M∉I, then M∩N ∉I.*

Lemma 1.16. *If I ⊂ 2^N and M⊆N. If M ∉I, then M∩N ∉I.*

2. MAIN RESULTS

Theorem 2.1. *For any sequence of moduli $F = (f_k)$,the classes of sequences $c^I(F), c_0^I(F), m^I(F)$ and $m_0^I(F)$ are linear spaces.*

Proof: We shall prove the result for the space $c^I(F)$.
The proof for the other spaces will follow similarly.
Let $(x_k), (y_k) \in c^I(F)$ and let α, β be the scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some} L_1 \in c;$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some} L_2 \in c.$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in N : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \quad (1)$$

$$A_2 = \{k \in N : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \quad (2)$$

Since f_k is a modulus function, we have

$$f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|)$$

$$\leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|).$$

Now, by (1) and (2), $\{k \in$ N: $f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$.
Therefore $(\alpha x_k + \beta y_k) \in c^I(F)$.
Hence $c^I(F)$ is a linear space.

Theorem 2.2. *A sequence $x = (x_k) \in m^I(F)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F). \quad (3)$$

Proof: Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m^I(F). \text{ For all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$.Then we have

$$|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$.
Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$.

Conversly, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$.
That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m^I(F)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I(F) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I(F)$ as well as $C_{\frac{\epsilon}{2}} \in m^I(F)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I(F)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi,$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m^I(F),$$

that is

$$diamJ \leq diamJ_\epsilon,$$

where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I(F)$ for (k=1,2,3,4,......).
Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$.

Result 2.3. The spaces $c_0^I(F)$ and $m_0^I(F)$ are solid and monotone .

Proof:We shall prove the result for $c_0^I(F)$. Let $x_k \in c_0^I$(F).Then

$$I - \lim_k f_k(|x_k|) = 0. \tag{4}$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all k $\in$ N. Then the result follows from (4) and the following inequality
$f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k|x_k|)$ for all k $\in$ N.
That the space c_0^I(F) is monotone follows from the Lemma 1.14.
For $m_0^I(F)$ the result can be proved similarly.

Result 2.4. The spaces $c^I(F)$ and $m^I(F)$ are neither solid nor monotone in general .

Proof:Here we give a counter example.
Let I = I_δ and f(x) = x^2 for all x$\in$ $[0, \infty)$. Consider the K-step space $X_K(f)$ of X defined as follows,

Let $(x_k) \in X$and let $(y_k) \in X_K$ be such that

$$(y_k) = \begin{cases} (x_k), \text{if k is even}, \\ 0, otherwise. \end{cases}$$

Consider the sequence (x_k) defined by $(x_k) = 1$ for all k $\in$ N.
Then $(x_k) \in c^I(F)$ but its K-stepspace preimage does not belong to $c^I(F)$. Thus $c^I(F)$ is not monotone.Hence $c^I(F)$ is not solid.

Result 2.5. The spaces $c^I(F)$ and $c_0^I(F)$ are sequence algebras.

Proof: We prove that $c_0^I(F)$ is a sequence algebra.
Let $(x_k), (y_k) \in c_0^I(F)$. Then

$$I - \lim f_k(|x_k|) = 0$$

and

$$I - \lim f_k(|y_k|) = 0.$$

Then we have

$$I - \lim f_k(|(x_k.y_k)|) = 0.$$

Thus $(x_k.y_k) \in c_0^I(F)$ is a sequence algebra.
For the space $c^I(F)$, the result can be proved similarly.

Result 2.6. The spaces $c^I(F)$ and $c_0^I(F)$ are not convergence free in general.

Proof:Here we give a counter example.
Let $I = I_f$ and $f(x) = x^3$ for all $x \in [0, \infty)$. Consider the sequence (x_k) and (y_k) defined by

$$x_k = \frac{1}{k} \quad and\ y_k = k \text{ for all } k \in \mathrm{N}.$$

Then $(x_k) \in c^I(F)$ and $c_0^I(F)$, but $(y_k) \notin c^I(F)$ and $c_0^I(F)$.
Hence the spaces $c^I(F)$ and $c_0^I(F)$ are not convergence free.

Result 2.7. If I is not maximal and $I \neq I_f$,then the spaces $c^I(F)$ and $c_0^I(F)$ are not symmetric.

Proof: Let $A \in I$ be infinite and $f(x) = x$ for all x$\in [0, \infty)$.
If

$$x_k = \begin{cases} 1, \text{for } k \in A, \\ 0, otherwise. \end{cases}$$

Then by lemma 1.16 $x_k \in c_0^I(F) \subset c^I(F)$.
Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$.Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), \text{for } k \in K, \\ \psi(k), otherwise, \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(k)} \notin c^I(F)$ and $x_{\pi(k)} \notin c_0^I(F)$.
Hence $c_0^I(F)$ and $c^I(F)$are not symmetric.

Theorem 2.3. *Let $F = (f_k)$ be the sequence of modulii. Then $c_0^I(F) \subset c^I(F) \subset l_\infty^I(F)$ and the inclusions are proper.*

Proof: Let $x_k \in c^I(F)$. Then there exists L $\in$ C such that

$$I - \lim f_k(|x_k - L|) = 0.$$

We have $f_k(|x_k|) \leq \frac{1}{2} f_k(|x_k - L|) + f_k \frac{1}{2}(|L|)$.
Taking the supremum over k on both sides we get $x_k \in l_\infty(F)$.
The inclusion $c_0^I(F) \subset c^I(F)$ is obvious.

Theorem 2.4. *The function $\hbar : m^I(F) \to \mathbb{R}$ is the Lipschitz function,where $m^I(F) = c^I(F) \cap l_\infty(F)$, and hence uniformly cotinuous.*

Proof:Let $x, y \in m^I(F)$,$x \neq y$.Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq ||x - y||\} \in I,$$
$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq ||x - y||\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < ||x - y||\} \in m^I(F),$$
$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < ||x - y||\} \in m^I(F).$$

Hence also $B = B_x \cap B_y \in m^I(F)$, so that $B \neq \phi$.
Now taking k in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3||x - y||.$$

Thus $\hbar$ is a Lipschitz function. For $m_0^I(F)$ the result can be proved similarly.

Result 2.10. If $x, y \in m^I(F)$,then $(x.y) \in m^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof: For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m^I(F),$$
$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m^I(F).$$

Now,

$$\begin{aligned} |x_k y_k - \hbar(x)\hbar(y)| &= |x_k y_k - x_k \hbar(y) + x_k \hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x_k||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)|. \qquad (8) \end{aligned}$$

As $m^I(F) \subseteq l_\infty(F)$,there exists an $M \in \boldsymbol{R}$ such that $|x_k| < M$ and $|\hbar(y)| < M$.
Using (8) we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon.$$

For all $k \in B_x \cap B_y \in m^I(F)$. Hence $(x.y) \in m^I(F)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.
For $m_0^I(F)$ the result can be proved similarly.

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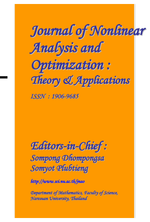

FIXED POINT THEOREMS IN NON-ARCHIMEDEAN MENGER PM-SPACES

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ABSTRACT. Recently, Sintunavarat and Kumam [Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. vol. 2011, Article ID 637958, 14 pages, 2011] defined the notion of $(CLRg)$ property which is more general than property (E.A). In the present paper, we prove a common fixed point theorem for a pair of weakly compatible mappings in Non-Archimedean Menger probabilistic metric spaces by using $(CLRg)$ property. As an application to our main result, we present a common fixed point theorem for two finite families of self mappings. Our results improve and extend several known results existing in the literature.

KEYWORDS : t-Norm; Non-Archimedean Menger probabilistic metric space; Weakly compatible mappings, Property (E.A); $(CLRg)$ property; Fixed point.
AMS Subject Classification: 47H10, 54H25

1. INTRODUCTION

In 1974, Istrătescu and Crivăt [10] introduced the concept of Non-Archimedean probabilistic metric spaces (shortly, N.A. PM-spaces) (see [9, 11]). Some fixed point theorems on N.A. Menger PM-spaces have been established by Istrătescu [7, 8] as a generalization of the results of Sehgal and Bharucha-Reid [20]. Further, Hadžić [5] studied the results of Istrătescu [7, 8].

In 1987, Singh and Pant [26] introduced the notion of weakly commuting mappings in N.A. Menger PM-spaces and proved some common fixed point theorems. Afterwards, Dimri and Pant [4] studied the application of N.A. Menger PM-spaces to product spaces. Jungck and Rhoades [12, 13] weakened the notion of compatible mappings by introducing the notion of weak compatibility and proved fixed point theorems without any requirement of continuity of the involved mappings. Many mathematicians proved common fixed point theorems in N.A. Menger PM-spaces

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using different contractive conditions (see [2, 3, 4, 14, 15, 16, 21, 22, 24, 25, 27]). In 2002, Aamri and El Moutawakil [1] defined the notion of property (E.A) which contained the class of non-compatible mappings. It is observed that property (E.A) requires the completeness (or closedness) of the underlying space (or subspaces) for the existence of the fixed points. Recently, Sintunavarat and Kumam [28] defined the notion of "common limit in the range property" with respect to mapping g (briefly, $(CLRg)$ property) in fuzzy metric spaces. They showed that $(CLRg)$ property never requires the closedness of the subspace (also see [29]).

The aim of this paper is to prove a common fixed point theorem for a pair of weakly compatible mappings in N.A. Menger PM-spaces employing $(CLRg)$ property. We give an example to support the useability of our main result. As an application to our main result, we present a common fixed point theorem for two finite families of self mappings.

2. PRELIMINARIES

Definition 2.1. [19] A triangular norm (shortly, t-norm) $\mathcal{T}$ is a binary operation on the unit interval [0,1] such that for all $a, b, c, d \in [0, 1]$ and the following conditions are satisfied:

(i) $\mathcal{T}(a, 1) = a$;
(ii) $\mathcal{T}(a, b) = \mathcal{T}(b, a)$;
(iii) $\mathcal{T}(a, b) \leq \mathcal{T}(c, d)$, whenever $a \leq c$ and $b \leq d$;
(iv) $\mathcal{T}(a, \mathcal{T}(b, c)) = \mathcal{T}(\mathcal{T}(a, b), c)$.

Definition 2.2. [19] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is said to be a distribution function if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$. We shall denote $\Im$ by the set of all distribution functions.

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by $F_{x,y}$ for all $x, y \in X$.

Definition 2.3. [8, 10] The ordered pair $(X, \mathcal{F})$ is said to be non-Archimedean probabilistic metric space (shortly N.A. PM-space) if X is a non-empty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, t_1, t_2 > 0$,

(i) $F_{x,y}(t) = 1 \Leftrightarrow x = y$;
(ii) $F_{x,y}(t) = F_{y,x}(t)$;
(iii) $F_{x,y}(0) = 0$;
(iv) If $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$ then $F_{x,z}(\max\{t_1, t_2\}) = 1$.

The ordered triplet $(X, \mathcal{F}, \mathcal{T})$ is called a N.A. Menger PM-space if $(X, \mathcal{F})$ is a N.A. PM-space, $\mathcal{T}$ is a t-norm and the following inequality holds:

$$F_{x,z}(\max\{t_1, t_2\}) \geq \mathcal{T}\left(F_{x,y}(t_1), F_{y,z}(t_2)\right),$$

for all $x, y, z \in X$ and $t_1, t_2 > 0$.

Example 2.4. Let X be any set with at least two elements. If we define $F_{x,x}(t) = 1$ for all $x \in X, t > 0$ and

$$F_{x,y}(t) = \begin{cases} 0, & \text{if } t \leq 1; \\ 1, & \text{if } t > 1, \end{cases}$$

where $x, y \in X, x \neq y$, then $(X, \mathcal{F}, \mathcal{T})$ is a N.A. Menger PM-space with $\mathcal{T}(a, b) = \min\{a, b\}$ or (ab) for all $a, b \in [0, 1]$.

Example 2.5. Let $X = \mathbb{R}$ be the set of real numbers equipped with metric defined by $d(x, y) = |x - y|$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

Then $(X, \mathcal{F}, \mathcal{T})$ is a N.A. Menger PM-space with $\mathcal{T}$ as continuous t-norm satisfying $\mathcal{T}(a, b) = \min\{a, b\}$ or (ab) for all $a, b \in [0, 1]$.

Definition 2.6. [3] A N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$ is said to be of type $(C)_{\mathfrak{g}}$ if there exists a $\mathfrak{g} \in \Omega$ such that

$$\mathfrak{g}(F_{x,z}(t)) \leq \mathfrak{g}(F_{x,y}(t)) + \mathfrak{g}(F_{y,z}(t)),$$

for all $x, y, z \in X, t \geq 0$, where $\Omega = \{\mathfrak{g} \mid \mathfrak{g} : [0, 1] \to [0, \infty)$ is continuous, strictly decreasing with $\mathfrak{g}(1) = 0$ and $\mathfrak{g}(0) < \infty\}$.

Definition 2.7. [3] A N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$ is said to be of type $(D)_{\mathfrak{g}}$ if there exists a $\mathfrak{g} \in \Omega$ such that

$$\mathfrak{g}(\mathcal{T}(t_1, t_2)) \leq \mathfrak{g}(t_1) + \mathfrak{g}(t_2),$$

for all $t_1, t_2 \in [0, 1]$.

Remark 2.8. [3]

(i) If a N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$ is of type $(D)_{\mathfrak{g}}$ then it is of type $(C)_{\mathfrak{g}}$.
(ii) If a N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$ is of type $(D)_{\mathfrak{g}}$, then it is metrizable, where the metric d on X is defined by

$$d(x, y) = \int_0^1 \mathfrak{g}(F_{x,y}(t))dt,$$

for all $x, y \in X$.

Throughout this paper $(X, \mathcal{F}, \mathcal{T})$ is a N.A. Menger PM-space with a continuous strictly increasing t-norm $\mathcal{T}$.

Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the condition (Φ): ϕ is upper semi-continuous from the right and $\phi(t) < t$ for $t > 0$.

Lemma 2.9. [3] *If a function $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition (Φ) then we have:*

(i) *for all $t \geq 0, \lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is the n^{th} iteration of $\phi(t)$.*
(ii) *If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$ where $n = 1, 2, \ldots$ then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$, for each $t \geq 0$ then $t = 0$.*

Definition 2.10. A pair (f, g) of self mappings of a N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$ is said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z,$$

for some $z \in X$.

Definition 2.11. [18] A pair (f, g) of self mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $fz = gz$ for some $z \in X$, then $fgz = gfz$.

It is known that a pair (f, g) of compatible mappings is weakly compatible but converse is not true in general.

Remark 2.12. It is noticed that the concepts of weak compatibility and property (E.A) are independent to each other (see [17, Example 2.2]).

Inspired by Sintunavarat and Kumam [28], we define the "common limit in the range property" with respect to mapping g in N.A. Menger PM-space as follows:

Definition 2.13. A pair (f, g) of self mappings of a N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$ is said to satisfy the $(CLRg)$ property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$.

Now, we show examples of self mappings f and g which are satisfying the $(CLRg)$ property.

Example 2.14. Let $(X, \mathcal{F}, \mathcal{T})$ be a N.A. Menger PM-space, where $X = [1, \infty)$ and metric d is defined as condition (2) of Remark 2.8. Define self mappings f and g on X by $f(x) = x + 2$ and $g(x) = 3x$ for all $x \in X$. Let a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n\in\mathbb{N}}$ in X, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 3 = g(1) \in X,$$

which shows that f and g satisfy the $(CLRg)$ property.

Example 2.15. Let $(X, \mathcal{F}, \mathcal{T})$ be a N.A. Menger PM-space, where $X = [0, \infty)$ and metric d is defined as condition (2) of Remark 2.8. Define self mappings f and g on X by $f(x) = \frac{x}{2}$ and $g(x) = \frac{2x}{3}$ for all $x \in X$. Let a sequence $\{x_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ in X. Since

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g(0) \in X,$$

therefore f and g satisfy the $(CLRg)$ property.

Definition 2.16. [6] Two families of self mappings $\{f_i\}$ and $\{g_j\}$ are said to be commuting pairwise if:

(i) $f_i f_j = f_j f_i$, $i, j \in \{1, 2, \ldots, m\}$,
(ii) $g_k g_l = g_l g_k$, $k, l \in \{1, 2, \ldots, n\}$,
(iii) $f_i g_k = g_k f_i$, $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, n\}$.

3. RESULTS

Theorem 3.1. *Let $(X, \mathcal{F}, \mathcal{T})$ be a N.A. Menger PM-space and the pair (f, g) of self mappings is weakly compatible such that*

$$\mathfrak{g}(F_{fx,fy}(t)) \le \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{gx,gy}(t)), \mathfrak{g}(F_{fx,gx}(t)), \mathfrak{g}(F_{fy,gy}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{gx,fy}(t)) + \mathfrak{g}(F_{fx,gy}(t))\right)\end{array}\right\}\right), \tag{3.1}$$

holds for all $x, y \in X$, $t > 0$, where $\mathfrak{g} \in \Omega$ and ϕ satisfies the condition (Φ). If f and g satisfy the $(CLRg)$ property, then f and g have a unique common fixed point.

Proof. Since the pair (f, g) satisfies the $(CLRg)$ property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu,$$

for some $u \in X$. We assert that $fu = gu$. On using inequality (3.1) with $x = x_n$, $y = u$, we get

$$\mathfrak{g}(F_{fx_n,fu}(t)) \le \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{gx_n,gu}(t)), \mathfrak{g}(F_{fx_n,gx_n}(t)), \mathfrak{g}(F_{fu,gu}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{gx_n,fu}(t)) + \mathfrak{g}(F_{fx_n,gu}(t))\right)\end{array}\right\}\right),$$

passing to limit as $n \to \infty$, we have

$$\mathfrak{g}(F_{gu,fu}(t)) \quad\le\quad \phi\left(\max\left\{\begin{array}{c}\mathfrak{g}(F_{gu,gu}(t)), \mathfrak{g}(F_{gu,gu}(t)), \mathfrak{g}(F_{fu,gu}(t)),\\ \frac{1}{2}\left(\mathfrak{g}(F_{gu,fu}(t)) + \mathfrak{g}(F_{gu,gu}(t))\right)\end{array}\right\}\right)$$

$$\begin{aligned}
&= \phi\left(\max\left\{\ \mathfrak{g}(1), \mathfrak{g}(1), \mathfrak{g}(F_{fu,gu}(t)), \tfrac{1}{2}\left(\mathfrak{g}(F_{gu,fu}(t)) + \mathfrak{g}(1)\right)\ \right\}\right) \\
&= \phi\left(\max\left\{0, 0, \mathfrak{g}(F_{fu,gu}(t)), \frac{1}{2}\left(\mathfrak{g}(F_{gu,fu}(t))\right)\right\}\right) \\
&= \phi(\mathfrak{g}(F_{fu,gu}(t))),
\end{aligned}$$

for all $t > 0$, which implies that $\mathfrak{g}(F_{fu,gu}(t)) = 0$. By Lemma 2.9, we get $fu = gu$.

Next, we let $z = fu = gu$. Since the pair (f, g) is weakly compatible, $fgu = gfu$ which implies that $fz = fgu = gfu = gz$. Now we show that $z = fz$. On using inequality (3.1) with $x = z$, $y = u$, we get

$$\mathfrak{g}(F_{fz,fu}(t)) \leq \phi\left(\max\left\{\begin{array}{c} \mathfrak{g}(F_{gz,gu}(t)), \mathfrak{g}(F_{fz,gz}(t)), \mathfrak{g}(F_{fu,gu}(t)), \\ \frac{1}{2}\left(\mathfrak{g}(F_{gz,fu}(t)) + \mathfrak{g}(F_{fz,gu}(t))\right) \end{array}\right\}\right),$$

and so

$$\begin{aligned}
\mathfrak{g}(F_{fz,z}(t)) &\leq \phi\left(\max\left\{\begin{array}{c} \mathfrak{g}(F_{fz,z}(t)), \mathfrak{g}(F_{fz,fz}(t)), \mathfrak{g}(F_{z,z}(t)), \\ \frac{1}{2}\left(\mathfrak{g}(F_{fz,z}(t)) + \mathfrak{g}(F_{fz,z}(t))\right) \end{array}\right\}\right) \\
&= \phi\left(\max\left\{\mathfrak{g}(F_{fz,z}(t)), \mathfrak{g}(1), \mathfrak{g}(1), \frac{1}{2}\left(\mathfrak{g}(F_{fz,z}(t)) + \mathfrak{g}(F_{fz,z}(t))\right)\right\}\right) \\
&= \phi\left(\max\left\{\mathfrak{g}(F_{fz,z}(t)), 0, 0, \mathfrak{g}(F_{fz,z}(t))\right\}\right) \\
&= \phi(\mathfrak{g}(F_{fz,z}(t))),
\end{aligned}$$

for all $t > 0$, which implies that $\mathfrak{g}(F_{fz,z}(t)) = 0$. By Lemma 2.9, we have $fz = z$ and so $z = fz = gz$. Therefore z is a common fixed point of f and g.

Uniqueness: Let $w(\neq z)$ be another common fixed point of f and g. On using inequality (3.1) with $x = z$, $y = w$, we have

$$\mathfrak{g}(F_{fz,fw}(t)) \leq \phi\left(\max\left\{\begin{array}{c} \mathfrak{g}(F_{gz,gw}(t)), \mathfrak{g}(F_{fz,gz}(t)), \mathfrak{g}(F_{fw,gw}(t)), \\ \frac{1}{2}\left(\mathfrak{g}(F_{gz,fw}(t)) + \mathfrak{g}(F_{fz,gw}(t))\right) \end{array}\right\}\right),$$

or

$$\begin{aligned}
\mathfrak{g}(F_{z,w}(t)) &\leq \phi\left(\max\left\{\begin{array}{c} \mathfrak{g}(F_{z,w}(t)), \mathfrak{g}(F_{z,z}(t)), \mathfrak{g}(F_{w,w}(t)), \\ \frac{1}{2}\left(\mathfrak{g}(F_{z,w}(t)) + \mathfrak{g}(F_{z,w}(t))\right) \end{array}\right\}\right) \\
&= \phi\left(\max\left\{\mathfrak{g}(F_{z,w}(t)), \mathfrak{g}(1), \mathfrak{g}(1), \frac{1}{2}\left(\mathfrak{g}(F_{z,w}(t)) + \mathfrak{g}(F_{z,w}(t))\right)\right\}\right) \\
&= \phi\left(\max\left\{\mathfrak{g}(F_{z,w}(t)), 0, 0, \mathfrak{g}(F_{z,w}(t))\right\}\right) \\
&= \phi(\mathfrak{g}(F_{z,w}(t))),
\end{aligned}$$

for all $t > 0$, which implies that $\mathfrak{g}(F_{z,w}(t)) = 0$. By Lemma 2.9, we get $z = w$. Therefore f and g have a unique a common fixed point. □

Remark 3.2. From the result, it is asserted that $(CLRg)$ property never requires any condition on closedness of the subspace, continuity of one or more mappings and containment of ranges of the involved mappings.

Our next theorem is proved for a pair of weakly compatible mappings in N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$ using property (E.A).

Theorem 3.3. *Let $(X, \mathcal{F}, \mathcal{T})$ be a N.A. Menger PM-space and the pair of self mappings (f, g) is weakly compatible satisfying inequality (3.1) of Theorem 3.1. If f and g satisfy the property (E.A) and the range of g is a closed subspace of X, then f and g have a unique common fixed point.*

Proof. Since the pair (f, g) satisfies the (E.A) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z,$$

for some $z \in X$. It follows from $g(X)$ being a closed subspace of X that there exists $u \in X$ such that $z = gu$. Therefore f and g satisfy the $(CLRg)$ property. It follows from Theorem 3.1 that there exists a unique common fixed point of f and g. □

Remark 3.4. Theorem 3.1 improves the results of Cho et al. [3], Singh et al. [24, Theorem 3.1, Corollary 3.3], Singh et al. [23, Theorem 3.1, Theorem 3.2], Singh et al. [25, Theorem 3.1, Corollary 3.1] and Singh and Dimri [22, Corollary 3.1] without any requirement of completeness (or closedness) of the underlying space (or subspaces), continuity of the mappings and containment of ranges of the involved mappings. Theorem 3.1 also generalize the results of Rao and Ramudu [18, Theorem 14].

The following example illustrates Theorem 3.1.

Example 3.5. Let $(X, \mathcal{F}, \mathcal{T})$ be a N.A. Menger PM-space, where $X = [1, 15)$ and metric d is defined as condition (2) of Remark 2.8. Define the self mappings f and g by

$$f(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 8, & \text{if } x \in (1, 3]. \end{cases} \quad g(x) = \begin{cases} 1, & \text{if } x = 1; \\ 7, & \text{if } x \in (1, 3]; \\ \frac{x+1}{4}, & \text{if } x \in (3, 15). \end{cases}$$

Taking $\{x_n\} = \left\{3 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$ or $\{x_n\} = \{1\}$, it is clear that the pair (f, g) satisfies the $(CLRg)$ property.

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1 = g(1) \in X.$$

It is noted that $f(X) = \{1, 8\} \not\subseteq [1, 4) \cup \{7\} = g(X)$. Thus, all the conditions of Theorem 3.1 are satisfied and 1 is a unique common fixed point of the pair (f, g). Also, all the involved mappings are even discontinuous at their unique common fixed point 1. Here, it is pointed out that $g(X)$ is not a closed subspace of X.

Now we utilize the notion of commuting pairwise and extend Theorem 3.1 to two finite families of self mappings in N.A. Menger PM-space.

Corollary 3.6. *Let $\{f_1, f_2, \ldots, f_p\}$ and $\{g_1, g_2, \ldots, g_q\}$ be two finite families of self mappings of a N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$ such that $f = f_1 f_2 \ldots f_p$ and $g = g_1 g_2 \ldots g_q$ which also satisfy inequality (3.1) of Theorem 3.1. Suppose that the pair (f, g) satisfies the $(CLRg)$ property.*

Moreover, if the family $\{f_i\}_{i=1}^p$ commutes pairwise with the family $\{g_i\}_{j=1}^q$, then (for all $i \in \{1, 2, \ldots, p\}$ and $j \in \{1, 2, \ldots, q\}$) f_i and g_j have a unique common fixed point.

Remark 3.7. Corollary 3.6 improves and extends the result of Singh and Dimri [22, Theorem 3.1].

By setting $f_1 = f_2 = \ldots = f_p = f$ and $g_1 = g_2 = \ldots = g_q = g$ in Corollary 3.6, we deduce the following:

Corollary 3.8. *Let f and g be self mappings of a N.A. Menger PM-space $(X, \mathcal{F}, \mathcal{T})$. Suppose that the pair (f^p, g^q) satisfies the $(CLRg)$ property such that*

$$\mathfrak{g}(F_{f^p x, f^p y}(t)) \leq \phi\left(\max\left\{\begin{array}{c} \mathfrak{g}(F_{g^q x, g^q y}(t)), \mathfrak{g}(F_{f^p x, g^q x}(t)), \mathfrak{g}(F_{f^p y, g^p y}(t)), \\ \frac{1}{2}\left(\mathfrak{g}(F_{g^q x, f^p y}(t)) + \mathfrak{g}(F_{f^p x, g^q y}(t))\right) \end{array}\right\}\right), \quad (3.2)$$

holds for all $x, y \in X$, $t > 0$, $\mathfrak{g} \in \Omega$ where ϕ satisfies the condition (Φ) and p, q are fixed positive integers. Then f and g have a unique common fixed point provided $fg = gf$.

Remark 3.9. The conclusion of Theorem 3.1 remains true if we replace inequality (3.1) by one of the following:

$$\mathfrak{g}(F_{fx,fy}(t)) \leq \phi\left(\max\left\{\mathfrak{g}(F_{gx,gy}(t)), \mathfrak{g}(F_{fx,gx}(t)), \mathfrak{g}(F_{fy,gy}(t)), \mathfrak{g}(F_{gx,fy}(t))\right\}\right), \tag{3.3}$$

for all $x, y \in X$, $t > 0$, where $\mathfrak{g} \in \Omega$ and ϕ satisfies the condition (Φ).
And

$$\mathfrak{g}(F_{fx,fy}(t)) \leq \phi\left(\max\left\{\mathfrak{g}(F_{gx,gy}(t)), \mathfrak{g}(F_{fx,gx}(t)), \mathfrak{g}(F_{fy,gy}(t))\right\}\right), \tag{3.4}$$

for all $x, y \in X$, $t > 0$, where $\mathfrak{g} \in \Omega$ and ϕ satisfies the condition (Φ).

Remark 3.10. Notice that results similar to Corollary 3.6 and Corollary 3.8 can also be outlined in respect of Remark 3.9 but we omit the details with a view to avoid any repetition.

Remark 3.11. The results (in view of Remark 3.9) improve the results of Khan and Sumitra [15, Theorem 2, Corollary 1], Singh et al. [23, Corollary 3.3, Corollary 3.4] and Singh et al. [21, Theorem 3.1].

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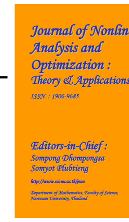

COMMON FIXED POINT THEOREM IN INTUITIONISTIC FUZZY METRIC SPACE UNDER $(S-B)$ PROPERTY

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ABSTRACT. The main purpose of this paper is to give common fixed point theorem in intuitionistic fuzzy metric space under strict contractive conditions for mappings satisfying $(S-B)$ property.

KEYWORDS : Intuitionistic fuzzy metric space; Common fixed point; Compatible maps; Weakly compatible maps; $(S-B)$ property.
AMS Subject Classification: 47H10 54H25

1. INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh [37] in 1965. Since, then to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Atanassov [5] Introduced and studied the concept of intuitionistic fuzzy sets. Intuitionistic fuzzy sets as a generalization of fuzzy sets can be useful in situations when description of a problem by a (fuzzy) linguistic variable, given in terms of a membership function only, seems too rough. Coker [7] introduced the concept of intuitionistic fuzzy topological spaces. Alaca et al. [3] proved the well- known fixed point theorems of Banach [6] in the setting of intuitionisitc fuzzy metric spaces. Later on, Turkoglu et al. [35] Proved Jungcks [12] common fixed point theorem in the setting of intuitionisitc fuzzy metric space. Turkoglu et al. [35] further formulated the notions of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of pants theorem [20]. Gregori et al. [10], Saadati and Park [26] studied the concept of intuitionistic fuzzy metric space and its applications. Recently, many authors have also studied the fixed point theory in fuzzy and intuitionistic fuzzy metric space (See [9, 11, 23, 24, 31, 32, 33, 34, 36].

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The study of common fixed points of non compatible mappings is also very interesting. Work along these lines has recently been initiated by pant [21, 22]. Sharma and Bamboria [30] defined a property $(S-B)$ for self maps and obtained some common fixed point theorems for such mappings under strict contractive conditions. The class of $(S-B)$ maps contains the class of non compatible maps. Kamran [15] obtained some coincidence and fixed point theorems for hybrid strict contractions.

Definition 1.1. [27] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-norm if $*$ is satisfying the following condition;
(i) * is commutative and associative;
(ii) * is continuous
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 1.2. [27] A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is continuous t-conorm if $\diamond$ is satisfying the following conditions;
(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous
(iii) $a \diamond 0 = a$ for all $a \in [0,1]$;
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 1.3. A 5 tuple$(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, $\diamond$ is a continuous t-conorm X^2 and $M, N,$ are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions;
(i) $M(x,y,t) + N(x,y,t) \leq 1$ for all $x, y \leq X$ and $t > 0$;
(ii) $M(x,y,0) = 0$ for all $x, y \in X$;
(iii) $M(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(iv) $M(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t > 0$;
(v) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ for all $x, y, z \in X$ and $s, t > 0$;
(vi) for all $x, y \in X, M(x,y,\cdot)$ is left continuous;
(vii) $\lim_{t \longrightarrow \infty}$ for all $x, y \in X$ and $t > 0$;
(viii) $N(x,y,0) = 1$ for all $x, y \in X$;
(ix) $N(x,y,t) = 0$ for all $x, y \in X$ and $t > 0$ iff $x = y$;
(x) $N(x,y,t) = N(y,x,t)$ for all $x, y \in X$ and $t > 0$;
(xi) $N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$ for all $x, y, z \in X$ and $s, t > 0$;
(xii) for all $x, y \in X, N(x,y,\cdot) : [0,\infty) \longrightarrow [0,1]$ is right continuous;
(xiii) $\lim_{t \longrightarrow \infty} N(x,y,t) = 0$ for all x, y in X.
(M, N) is called an intuitionistic fuzzy metric on X.
The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of nonnearness between x and y with respect to t respectively.

Remark 1.4. An intuitionistic fuzzy metric spaces with continous t-norm * and continous t-conorm $\diamond$ defined by $a * a \geq a$ and $(1-a) \diamond (1-a) \leq (1-a)$ for all $a \in [0,1]$. Then for all $x, y \in X, M(x,y,*)$ is non- decreasing and $N(x,y,\diamond)$ is nonincreasing.

Alaca, Turkoglu and Yildiz [3] introduced the following notions;

Definition 1.5. Let $(X, M, N*, \diamond)$ be an intuitionistic fuzzy metric space. Then (a) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$, $lim_{n \longrightarrow \infty} M(x_{n+p}, x_n, t) = 1, lim_{n \longrightarrow \infty} N(x_{n+p}, x_n, t) = 0$. (b) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if , for all $t > 0$, $lim_{n \longrightarrow \infty} M(x_n, x, t) = 1, lim_{n \longrightarrow \infty} N(x_n, x, t) = 0$;

Since * and $\diamond$ are continuous, the limit is uniquely determined from (v) and (xi) of definition 1.3, respectively. An intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 1.6. [32] A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be compatible if $\lim_{n\longrightarrow\infty} M(fgx_n, gfx_n, t) = 1$ and $\lim_{n\longrightarrow\infty} N(fgx_n, gfx_n, t) = 0$ for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = z$ for some $z \in X$.

Definition 1.7. A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be non-compatible $\lim_{n\longrightarrow\infty} M(fgx_n, gfx_n, t) \neq 1$ or non existent and $\lim_{n\longrightarrow\infty} N(fgx_n, gfx_n, t) \neq 0$ or non existent for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = z$ for some $z \in X$.

In 1998, Jungck and Rhodes [14] introduced the concept of weakly compatible maps as follows;

Definition 1.8. Two self maps f and g are said to be weakly compatible if they commute at coincidence points.

Aamri and Moutawakil [1] generalized the concept of non compatibility in metric spaces by defining the notion of E.A. Property and proved common fixed point theorems using this property.

Sharma and Bamboria [30] defined the $(S - B)$ property and proved common fixed point theorems in fuzzy metric spaces using this property.

Definition 1.9. A pair of self mappings (S, T) of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to satisfy $(S - B)$ property if there exists a sequence $\{x_n\}$ is X such that

$$\lim_{n\longrightarrow\infty} M(Sx_n, Tx_n, t) = 1, \lim_{n\longrightarrow\infty} N(Sx_n, Tx_n, t) = 0$$

Example 1.10. Let $X = [0, \infty)$ consider $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, where M and N are two fuzzy sets defined by $M(x, y, t) = t/[t + d(x, y)]$ and $N(x, y, t) = d(x, y)/[t + d(x, y)]$ where d is usual metric. Define $T, S : X \longrightarrow [0, \infty)$ by $Tx = x/5$ and $Sx = 2x/5$ for all x in X. Consider $x_n = 1/n$. Now, $\lim_{n\longrightarrow\infty} M(Sxn, Txn, t) = 1$ and $\lim_{n\longrightarrow\infty} N(Sxn, Txn, t) = 0$. Therefore S and T satisfy property $(S - B)$.

Lemma 1.11. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0, 1), M(x, y, kt).M(x, y, t)$ and $N(x, y, kt) \leq N(x, y, t)$ then $x = y$.*

Lemma 1.12. [2] *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in X. If there exist is a number $k \in (0, 1)$ such that;*

$$M(y_{n+2}, y_{n+1}, kt).M(y_{n+1}, y_n, t)$$

and

$$N(y_{n+2}, y_{n+1}, kt).N(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, \ldots$ Then $\{y_n\}$ is a Cauchy sequence in X.

2. MAIN RESULTS

Theorem 2.1. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space with $a*a \geq a$ and $(1-a) \diamond (1-a) \leq (1-a)$ for all $a \in [0,1)$, let A, B, S, and T be self mappings of X into itself such that,*

(1.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(1.2) (A, S) or (B, T) satisfies the property $(S-B)$.

(1.3) there exists a number $k \in (0,1)$ such that

$$[1 + PM(Sx, Ty, kt)] * M(Ax, By, kt)$$
$$\geq \acute{\phi}[PM(Ax, Sx, kt) * M(By, Ty, kt) + M(Ax, Ty, kt) * M(By, Sx, kt) + M(Sx, Ty, t)]$$
$$*M(Ax, Sx, t) * M(By, Ty, t) * M(By, Sx, t) * M(Ax, Ty, (2-\alpha)t)$$

and

$$[1 + PN(Sx, Ty, kt)] \diamond N(Ax, By, kt)$$
$$\leq \psi[PN(Ax, Sx, kt) \diamond N(By, Ty, kt) + N(Ax, Ty, kt) \diamond N(By, Sx, kt) + N(Sx, Ty, t)$$
$$\diamond(N(Ax, Sx, t) \diamond N(By, Ty, t) \diamond N(By, Sx, t) \diamond N(Ax, Ty, (2-a)t)]$$

for all $x, y \in X, P \geq 0$, $\alpha \in (0,2)$ and $t > 0$. Where $p\acute{h}i, \psi : [0,1] \longrightarrow [0,1]$ is continuous function such that $\acute{\phi}(S) > S$ and $\psi(S) < S$ for each $0 < S < 1$ with $M(x, y, t) > 0$.

(1.4) the pairs (A, S) and (B, T) are weakly compatible,

(1.5) one of A(X), B(X), S(X) or T(X) is a closed subset of X. Then A, B, S and T have a unique common fixed point in X.

Proof *Suppose that (B, T) satisfies the $(S-B)$ property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\longrightarrow\infty} Bx_n = \lim_{n\longrightarrow\infty} Tx_n = z$ for some $z \in X$. Since $B(X) \subset S(X)$ there exists a sequence $\{y_n\} \in X$ such that $\lim_{n\longrightarrow\infty} Bx_n = \lim_{n\longrightarrow\infty} Sy_n = z$. Now we shall show that $\lim_{n\longrightarrow\infty} Ay_n = z$. From (1.3) for $f\alpha = 1-q, q \in (0,1)$ we have;*

$$[1 + PM(Sy_n, Tx_n, kt)] * M(Ay_n, Bx_n, kt)$$
$$\acute{\phi}[PM(Ay_n, Sy_n, kt) * M(Bx_n, Tx_n, kt) + M(Ay_n, Tx_n, kt) * M(Bx_n, Sy_n, kt)$$
$$+M(Sy_n, Tx_n, t) * M(Ay_n, Sy_n, t) * M(Bx_n, Tx_n, t) * M(Bx_n, Sy_n, t)$$
$$*M(Ay_n, Tx_n(2-\alpha)t)]$$

and

$$[1 + PN(Sy_n, Tx_n, kt)] * N(Ay_n, Bx_n, kt)$$
$$\leq \psi[PN(Ay_n, Sy_n, kt) \diamond N(Bx_n, Tx_n, kt) + N(Ay_n, Tx_n, kt) \diamond N(Bx_n, Sy_n, kt)$$
$$+N(Sy_n, Tx_n, t) \diamond N(Ay_n, Sy_n, t) \diamond N(Bx_n, Tx_n, t) \diamond N(Bx_n, Sy_n, t)$$
$$\diamond N(Ay_n, Tx_n, (2-\alpha)t)]$$

$$M(Ay_n, Bx_n, kt) + P[M(Sy_n, Tx_n, kt) * M(Ay_n, Bx_n, kt)]$$
$$\geq \acute{\phi}[PM(Ay_n, Sy_n, kt) * M(Bx_n, Tx_n, kt) + M(Ay_n, Tx_n, kt) * M(Bx_n, Sy_n, kt)$$
$$+M(Sy_n, Tx_n, t) * M(Ay_n, Sy_n, t) * M(Bx_n, Tx_n, t)$$
$$*M(Bx_n, Sy_n, t) * M(Ay_n, Tx_n, (1+q)t)]$$

and

$$N(Ay_n, Bx_n, kt) + P[N(Sy_n, Tx_n, kt) \diamond N(Ay_n, Bx_n, kt)]$$
$$\leq \psi[PN(Ay_n, Sy_n, kt) \diamond N(Bx_n, Tx_n, kt) + N(Ay_n, Tx_n, kt) \diamond N(Bx + n, Sy_n, kt)$$
$$+N(Sy_n, Tx_n, t) \diamond N(Ay_n, Sy_n, t)$$

$$\diamond N(Bx_n, Tx_n, t) \diamond N(Bx_n, Sy_n, t) \diamond N(Ay_n, Tx_n, (1+q)t)]$$

$$M(Ay_n, Bx_n, kt) + P[M(Bx_n, Tx_n, kt) * M(Ay_n, Bx_n, kt)]$$
$$\geq \phi[PM(Ay_n, Bx_n, kt) * M(Bx_n, Tx_n, kt) + M(Ay_n, Tx_n, kt) * M(Bx_n, Bx_n, kt)$$
$$+M(Bx_n, Tx_n, t) * M(Ay_n, Bx_n, t) * M(Bx_n, Tx_n, t)$$
$$*M(Bx_n, Bx_n, t) * M(Ay_n, Bx_n, t) * M(Bx_n, Tx_n, qt)]$$

and

$$N(Ay)n, Bx_n, kt) + P[N(Bx_n, Tx_n, kt) \diamond N(Ay_n, Bx_n, kt)]$$
$$\leq \psi[PN(Ay_n, Bx_n, kt) \diamond N(Bx_n, Tx_n, kt) + N(Ay_n, Tx_n, kt) \diamond N(Bx_n, Bx_n, kt)$$
$$+N(Bx_n, Tx_n, t) \diamond N(Ay_n, Bx_n, t) \diamond N(Bx_n, Tx_n, t) \diamond N(Bx_n, Bx_n, t)$$
$$\diamond N(Ay_n, Bx_n, t) \diamond N(Bx_n, Tx_n, qt)]$$

Thus it follows that,

$$M(Ay_n, Bx_n, kt) \geq \phi[M(Bx_n, Tx_n, t) * M(Ay_n, Bx_n, t) * M(Bx_n, Tx_n, qt)]$$

and

$$N(Ay_n, Bx_n, kt) \leq \psi[N(Bx_n, Tx_n, t) \diamond N(Ay_n, Bx_n, t)) \diamond N(Bx_n, Tx_n, qt)]$$

*Since the t-norm * and t-conorm $\diamond$ is continuous and $M(x, y, \cdot)$ and $N(x, y, \cdot)$ is continuous, letting $q \longrightarrow 1$ we have,*

$$M(Ay_n, Bx_n, kt) \geq \phi[M(Bx_n, Tx_n, t) * M(Ay_n, Bx_n, t)]$$

and

$$N(Ay_n, Bx_n, kt) \leq \psi[N(Bx_n, Tx_n, t) \diamond N(Ay_n, Bx_n, t)]$$

It follows that

$$\begin{aligned} \lim_{n \longrightarrow \infty} M(Ay_n, Bx_n, kt) &\geq \phi[\lim_{n \longrightarrow \infty} M(Ay_n, Bx_n, t), M(\lim_{n \longrightarrow \infty} Ay_n, z, kt)] \\ &> M(\lim_{n \longrightarrow \infty} Ay_n, z, t) \end{aligned}$$

and

$$\begin{aligned} \lim_{n \longrightarrow \infty} N(Ay_n, Bx_n, kt) &\leq \psi[\lim_{n \longrightarrow \infty} N(Ay_n, Bx_n, t), N(\lim_{n \longrightarrow \infty} Ay_n, z, kt)] \\ &< N(\lim_{n \longrightarrow \infty} Ay_n, z, t) \end{aligned}$$

and we deduce that $\lim_{n \longrightarrow \infty} Ay_n = z$. Suppose $S(X)$ is a closed subset of X. Then $z = Su$ for some $u \in X$. Subsequently we have,

$$\lim_{n \longrightarrow \infty} Ay_n = \lim_{n \longrightarrow \infty} Bx_n = \lim_{n \longrightarrow \infty} Tx_n = \lim_{n \longrightarrow \infty} Sy_n = Su.$$

By (1.3) with $f\alpha = 1$, we have;

$$[1 + PM(Su, Tx_n, kt)] * M(Au, Bx_n, kt)$$
$$\geq \phi[PM(Au, Su, kt) * M(Bx_n, Tx_n, kt) + M(Bx_n, Su, t) * M(Au, Tx_n, t)$$
$$+M(Su, Tx_n, t) * M(Au, Su, t) * M(Bx_n, Tx_n, t) * M(Bx_n, Su, t) * M(Au, Tx_n, t)]$$

and

$$[1 + PN(Su, Tx_n, kt)] \diamond N(Au, Bx_n, kt)$$
$$\leq \psi[PN(Au, Su, kt) \diamond N(Bx_n, Tx_n, kt) + N(Bx_n, Su, t) \diamond N(Au, Tx_n, t) +$$
$$N(Su, Tx_n, t) \diamond N(Au, Su, t) \diamond N(Bx_n, Tx_n, t) \diamond N(Bx_n, Su, t) \diamond N(Au, Tx_n, t)]$$

$$M(Au, Bx_n, kt) + P[M(Su, Tx_n, kt) * M(Au, Bx_n, kt)]$$
$$\geq \phi[P[M(Au, Su, kt) * M(Bx_n, Tx_n, kt) + M(Au, Tx_n, kt) * M(Bx_n, Su, kt)]$$

$$+M(Su, Tx_n, kt) * M(Au, Su, t) * M(Bx_n, Tx_n, t) * M(Bx_n, Su, t) * M(Au, Tx_n, t)]$$

and

$$N(Au, Bx_n, kt) + P[N(Su, Tx_n, kt)) \diamond N(Au, Bx_n, kt)]$$
$$\leq \psi[PN(Au, Su, kt) \diamond N(Bx_n, Tx_n, kt) + N(Au, Tx_n, kt) \diamond N(Bx_n, Su, kt)$$
$$+[N(Su, Tx_n, kt) \diamond N(Au, Su, t) \diamond N(Bx_n, Tx_n, t) \diamond N(Bx_n, Su, t)) \diamond N(Au, Tx_n, t)]$$

Taking the $\lim_{n \longrightarrow \infty}$ *we have;*

$$M(Au, Su, kt) \geq \phi[PM(Au, Su, kt) * M(Su, Su, kt) + M(Su, Su, t) *$$
$$M(Au, Su, t) * M(Su, Su, t) * M(Su, Su, t) * M(Au, Su, t)]$$

and

$$N(Au, Su, kt) \leq \psi[PN(Au, Su, kt) \diamond N(Su, Su, kt) + N(Su, Su, t) \diamond$$
$$N(Au, Su, t) \Diamond N(Su, Su, t) \Diamond N(Su, Su, t) \Diamond N(Au, Su, t)]$$

This gives

$$M(Au, Su, kt) > M(Au, Su, t)$$

and

$$N(Au, Su, kt) < N(Au, Su, t)$$

Therefore by lemma 1, we have $Au = Su$. *I.e.* A *and* S *have a coincidence point. The weak compatibility of* A *and* S *implies that* $ASu = SAu$ *and then*

$$AAu = ASu = SAu = SSu.$$

On the other hand, since $A(X) \subset T(X)Y$, *there exists a point* $v \in X$ *such that* $Au = Tv$. *We claim that* $Tv = Bv$ *using (1.3) with* $\alpha = 1$, *we have,*

$$[1 + PM(Su, Tv, kt)] * M(Au, Bv, kt)$$
$$\geq \acute{\phi}[PM(Au, Su, kt) * M(Bv, Tv, kt) + M(Au, Tv, kt) * M(Bv, Su, kt) +$$
$$M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) * M(Bv, Su, t) * M(Au, Tv, t)]$$

and

$$[1 + PN(Su, Tv, kt)] \Diamond N(Au, Bv, kt)$$
$$\leq \Psi[PN(Au, Su, kt) \Diamond N(Bv, Tv, kt) + N(Au, Tv, kt) \Diamond N(Bv, Su, kt) + N(Su, Tv, t) \Diamond$$
$$N(Au, Su, t) \Diamond N(Bv, Tv, t) \Diamond N(Bv, Su, t) \Diamond N(Au, Tv, t)]$$

$$M(Au, Bv, kt) + P[M(Su, Tv, kt) * M(Au, Bv, kt)]$$
$$\geq \acute{\phi}[PM(Au, Su, kt) * M(Bv, Tv, kt) + M(Au, Tv, kt) * M(Bv, Su, kt) +$$
$$M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) * M(Bv, Su, t) * M(Au, Tv, t)]$$

And

$$N(Au, Bv, kt) + P[N(Su, Tv, kt) N(Au, Bv, kt)]$$
$$\leq \Psi[PN(Au, Su, kt) \Diamond N(Bv, Tv, kt) + N(Au, Tv, kt).N(Bv, Su, kt) +$$
$$N(Su, Tv, t).N(Au, Su, t) \Diamond N(Bv, Tv, t) \Diamond N(Bv, Su, t) \Diamond N(Au, Tv, t)]$$

Thus it follows that, $M(Au, Bv, kt) > M(Au, Bv, t)$ *and*

$$N(Au, Bv, kt) < N(Au, Bv, t)$$

$$M(AAu, Bv, kt) + P[M(SAu, Tv, kt) * M(AAu, Bv, kt)]$$
$$\geq \acute{\phi}[PM(AAu, SAu, kt) * M(Bv, Tv, kt) + M(AAu, Tv, kt) * M(Bv, SAu, kt)$$
$$+M(SAu, Tv, t) * M(AAu, SAu, t) * M(Bv, Tv, t) * M(Bv, SAu, t) * M(AAu, Tv, t)]$$

and

$$N(AAu, Bv, kt) + P[N(SAu, Tv, kt)\diamondsuit N(AAu, Bv, kt)]$$
$$\leq \Psi[PN(AAu, SAu, kt)\diamondsuit N(Bv, Tv, kt) + N(AAu, Tv, kt)\diamondsuit N(Bv, SAu, kt)$$
$$+N(SAu, Tv, t)\diamondsuit N(AAu, SAu, t)\diamondsuit N(Bv, Tv, t)\diamondsuit N(Bv, SAu, t)\diamondsuit N(AAu, Tv, t)]$$

$$M(AAu, Au, kt) + P[M(AAu, Au, kt) * M(AAu, Au, kt)]$$
$$\geq \acute{\phi}[PM(AAu, AAu, kt) * M(Au, Au, kt) + M(AAu, Au, kt) * M(Au, AAu, kt)$$
$$+M(AAu, Au, t) * M(AAu, AAu, t) * M(Au, Au, t)$$
$$*M(Au, Au, t) * M(AAu, Au, t)]$$

and

$$N(AAu, Au, kt) + P[N(AAu, Au, kt)\diamondsuit N(AAu, Au, kt)]$$
$$\leq \Psi[PN(AAu, AAu, kt)\diamondsuit N(Au, Au, kt) + N(AAu, Au, kt)\diamondsuit N(Au, AAu, kt)$$
$$+N(AAu, Au, t)\diamondsuit N(AAu, AAu, t)\diamondsuit N(Au, Au, t)$$
$$\diamondsuit N(Au, AAu, t)\diamondsuit N(AAu, Au, t)]$$

Thus it follows that

$$M(AAu, Au, kt) > M(AAu, Au, t)$$

and

$$N(AAu, Au, kt) < N(AAu, Au, t)$$

Therefore by lemma 1, we have $Au = AAu = SAu$. *I.e.* Au *is a common fixed point of* A *and* S. *Similarly, we prove that* Bv *is a common fixed point of* B *and* T. *Since* $Au = Bv$, *we conclude that* Au *is a common fixed point of* A, B, S *and* T. *If* $Au = Bu = Su = Tu = u$ *and* $Av = Bv = Sv = Tv = v$, *then by (1.3) with* $\alpha = 1$, *we have;*

$$[1 + PM(Su, Tv, kt)] * (Au, Bv, kt)$$
$$\geq \acute{\phi}[PM(Au, Su, kt) * M(Bv, Tv, kt) + M(Au, Tv, kt) * M(Bv, Su, kt) +$$
$$M(Su, Tv, t) * M(Au, Su, t) * M(Bv, Tv, t) * M(Bv, Su, t) * M(Au, Tv, t)]$$

$$[1 + PN(Su, Tv, kt)]\diamondsuit(Au, Bv, kt)$$
$$\leq \Psi[PN(Au, Su, kt)\diamondsuit N(Bv, Tv, kt) + N(Au, Tv, kt)\diamondsuit N(Bv, Su, kt) +$$
$$N(Su, Tv, t).N(Au, Su, t)\diamondsuit N(Bv, Tv, t)\diamondsuit N(Bv, Su, t)\diamondsuit N(Au, Tv, t)]$$

$$M(u, v, kt) + P[M(u, v, kt) * M(u, v, kt)]$$
$$\geq \acute{\phi}[PM(u, v, kt) * M(v, v, kt) + M(u, v, kt) * M(v, u, kt) + M(u, v, t) *$$
$$M(u, u, t) * M(v, v, t) * M(v, u, t) * M(u, v, t)]$$

and

$$N(u, v, kt) + P[N(u, v, kt).N(u, v, kt)]$$
$$\leq \Psi[PN(u, u, kt)\diamondsuit N(v, v, kt) + N(u, v, kt)\diamondsuit N(v, u, kt) + N(u, v, t)$$
$$\diamondsuit N(u, u, t)\diamondsuit N(v, v, t)\diamondsuit N(v, u, t)\diamondsuit N(u, v, t)]$$
$$M(u, v, kt) > M(u, v, t)$$

and

$$N(u, v, kt) < N(u, v, t)$$

By lemma 1, we have $u = v$. *Hence the common fixed point is a unique. This completes the proof of the theorem.*

Corollary 2.2. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space with $a*a \geq a$ and $(1-a)\diamondsuit(1-a) \leq (1-a)$ for all $a \in (0,1)$, let A, B, S, and T be self mappings of X into itself such that;*

(1.1) $A(X) \subset T(X)$ and $B(X) \subset S(X)$

(1.2) (A, S) or (B, T) satisfies the property $(S-B)$.

(1.3) there exists a number $k \in (0,1)$ such that

$$M(Ax, By, kt) \geq \acute{\phi}[M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(By, Sx, t) * M(Ax, Ty, (2-\alpha)t)]$$

and

$$N(Ax, By, kt) \leq \Psi[N(Sx, Ty, t)\diamondsuit N(Ax, Sx, t)\diamondsuit N(By, Ty, t)\diamondsuit N(By, Sx, t)\diamondsuit N(Ax, Ty, (2-\alpha)t)]$$

for all $x, y \in X$, $P \geq 0, \alpha \in (0,2)$ and $t > 0$. Where $\acute{\phi}, \Psi : [0,1] \to [0,1]$ is continuous function such that $\acute{\phi}(S) > S$ and $f\Psi(S) < S$ for each $0 < S < 1$ with $M(x, y, t) > 0$.

(1.4) the pairs $\{A, S\}$ and (B, T) are weakly compatible;

(1.5) one of $A(X), B(X), S(X)$ or $T(X)$ is a closed subset of X. Then A, B, S, and T have a unique common fixed point in X.

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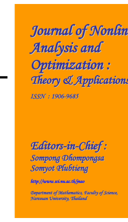

PICARD AND ADOMIAN DECOMPOSITION METHODS FOR A COUPLED SYSTEM OF QUADRATIC INTEGRAL EQUATIONS OF FRACTIONAL ORDER

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ABSTRACT. The comparison between the classical method of successive approximations (Picard) method and Adomian decomposition method was studied in many papers for example ([14] and [37]).
In this paper we are concerning with two analytical methods; the classical method of successive approximations (Picard)[18] and Adomian decomposition methods ([1]-[6], [16] and [17]) for a coupled system of quadratic integral equations of fractional order. Also, the existence and uniqueness of the solution and the convergence will be discussed for each method and some examples will be studied.

KEYWORDS : Coupled systems; Quadratic integral equation; Picard method; Adomian method; Continuous unique solution; Fractional-order integration; Convergence analysis; Error analysis.
AMS Subject Classification: 39B82 44B20 46C05

1. INTRODUCTION

Quadratic integral equations (QIEs) are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The quadratic integral equations can be very often encountered in many applications.
The quadratic integral equations have been studied in several papers and monographs (see for examples [8]-[12] and [20]-[26]).
The authors [27] proved the existence and the uniqueness of continuous solution for the quadratic integral equation

$$x(t) = a(t) \; + \; g(t, x(t)) \int_0^t f(s, x(s))\, ds$$

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by using the principle of contraction mapping and comparing the two analytical methods; the classical method of successive approximations (Picard)[18] which consists the construction of a sequence of functions such that the limit of this sequence of functions in the sense of uniform convergence is the solution of the quadratic integral equation, and Adomian decomposition method which gives the solution as a series see([1]-[6], [16] and [17]). Also, from the results of the examples the authors deduced that Picard method gives more accurate solution than ADM.

Systems occur in various problems of applied nature, for instance, see ([27]-[15], [29]-[31]). Recently, Su [36] discussed a two-point boundary value problem for a coupled system of fractional differential equations. Gafiychuk et al. [31] analyzed the solutions of coupled nonlinear fractional reaction-diffusion equations.
The coupled systems have been studied in many papers; see [27], [36] and [37].

This paper deals with the coupled system of quadratic integral equations of fractional order

$$\begin{aligned} x(t) &= a_1(t) + g_1(t,y(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s,y(s))\,ds,\ t\in[0,1], \\ y(t) &= a_2(t) + g_2(t,x(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s,x(s))\,ds,\ t\in[0,1], \end{aligned} \tag{1.1}$$

where $\alpha,\ \beta > 0$.
and comparing the results obtained from the two methods; Picard and Adomian decomposition methods. Also, some examples will be studied.

Now, the definition of the fractional-order integral operator is given by:

Definition 1.1. Let β be a positive real number, the fractional-order integral of order β of the function f is defined on the interval $[a,b]$ by (see [32], [33], [34] and [35])

$$I_a^\beta f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\,ds,$$

and when $a=0$, we have $I^\beta f(t) = I_0^\beta f(t)$.

For further properties of fractional-order integral operator (see [32]-[35]) for example.

2. METHOD OF SUCCESSIVE APPROXIMATIONS (PICARD METHOD)

Now, the coupled system (1.1) will be investigated under the assumptions:

(i) $a_i : I \to R_+ = [0,+\infty),\ i=1,2$ is continuous on I where $I=[0,1]$;

(ii) $f_i,\ g_i : I\times D \subset R_+ \to R_+,\ i=1,2$ are continuous and there exist positive constants M_i and $N_i,\ i=1,2$ such that $|g_i(t,x)| \le M_i$ and $|f_i(t,x)| \le N_i$ on D;

(iii) $f_i,\ g_i,\ i=1,2$ satisfy Lipschitz condition with Lipschitz constants L_i and K_i such that,

$$\begin{aligned} |g_i(t,x)-g_i(t,y)| &\le L_i|x-y|, \\ |f_i(t,x)-f_i(t,y)| &\le K_i|x-y|. \end{aligned}$$

Let $C = C(I)$ be the space of all real valued functions which are continuous on I. Define the operators T_1, T_2 by

$$T_1 y(t) = a_1(t) + g_1(t, y(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds, \ t \in I$$

$$T_2 x(t) = a_2(t) + g_2(t, x(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s))\, ds, \ t \in I,$$

where $\alpha, \ \beta > 0$.

Then the coupled (1.1) may be written as:

$$x(t) = T_1 y(t)$$

$$y(t) = T_2 x(t).$$

Define the operator T by

$$T(x, y)(t) = (T_1 y(t), \ T_2 x(t)).$$

Theorem 2.1. *Let the assumptions (i)-(iii) be satisfied. If* $(M_1 K_1 + N_1 L_1)(M_2 K_2 + N_2 L_2) < 1$, *then the coupled system of quadratic integral equations of fractional order (1.1) has a unique positive solution* $(x, y) \in C \times C$.

Proof. It is clear that the operators $T_1, \ T_2$ map C into C.

Applying Picard method to the coupled system of quadratic integral equation (1.1), the solution is constructed by the sequences

$$\begin{aligned} x_n(t) &= a_1(t) + g_1(t, y_{n-1}(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y_{n-1}(s))\, ds, \ n = 1, 2, \ldots, \\ x_0(t) &= a_1(t) \\ y_n(t) &= a_2(t) + g_2(t, x_{n-1}(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x_{n-1}(s))\, ds, \ n = 1, 2, \ldots, \\ y_0(t) &= a_2(t). \end{aligned} \tag{2.1}$$

All the functions $x_n(t)$ and $y_n(t)$ are continuous functions. Also, $x_n(t)$ and $y_n(t)$ can be written as a sum of successive differences:

$$x_n = x_0 + \sum_{j=1}^{n} (x_j - x_{j-1}),$$

$$y_n = y_0 + \sum_{j=1}^{n} (y_j - y_{j-1}).$$

This means that convergence of the two sequences $\{x_n\}$ and $\{y_n\}$ is equivalent to convergence of the two infinite series $\sum (x_j - x_{j-1})$, $\sum (y_j - y_{j-1})$ and the solution will be

$u(t) = (x(t), y(t))$, where

$$x(t) = \lim_{n \to \infty} x_n(t),$$

$$y(t) = \lim_{n \to \infty} y_n(t),$$

i.e. if the two infinite series $\sum (x_j - x_{j-1})$, $\sum (y_j - y_{j-1})$ converge, then the two sequence $\{x_n(t)\}$, $\{y_n(t)\}$ will converge to $x(t)$ and $y(t)$ respectively. To prove the uniform convergence of $\{x_n(t)\}$ and $\{y_n(t)\}$ we shall consider the two associated series

$$\sum_{n=1}^{\infty}[x_n(t)-x_{n-1}(t)],$$
$$\sum_{n=1}^{\infty}[y_n(t)-y_{n-1}(t)].$$

From (2.1) for $n=1$, we get

$$x_1(t)-x_0(t)=g_2(t,y_0(t))\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s,y_0(s))\ ds$$
$$y_1(t)-y_0(t)=g_1(t,x_0(t))\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_1(s,x_0(s))\ ds$$

and

$$|\ x_1(t)-x_0(t)\ |\le M_2N_2\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ ds\le M_2N_2\ \frac{t^{\beta}}{\Gamma(\beta+1)}.$$

Also,

$$|\ y_1(t)-y_0(t)\ |\le M_1N_1\ \frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{2.2}$$

Now, we shall obtain an estimate for $x_n(t)-x_{n-1}(t),\ n\ge 2$

$$\begin{aligned}
x_n(t)-x_{n-1}(t)\ &\le\ g_2(t,y_{n-1}(t))\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s,y_{n-1}(s))\ ds\\
&-\ g_2(t,y_{n-2}(t))\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s,y_{n-2}(s))\ ds\\
&+\ g_2(t,y_{n-1}(t))\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s,y_{n-2}(s))\ ds\\
&-\ g_2(t,y_{n-1}(t))\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s,y_{n-2}(s))\ ds\\
&\le\ g_2(t,y_{n-1}(t))\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ [f_2(s,y_{n-1}(s))-f_2(s,y_{n-2}(s))]\ ds\\
&+\ [\ g_2(t,y_{n-1}(t))-g_2(t,y_{n-2}(t))\ ]\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_2(s,y_{n-2}(s))\ ds,
\end{aligned}$$

using assumptions (ii) and (iii), we get

$$|x_n(t)-x_{n-1}(t)|\le M_2K_2\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ |y_{n-1}(s)-y_{n-2}(s)|\ ds$$
$$+\ N_2L_2|y_{n-1}(t)-y_{n-2}(t)|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ ds.$$

Putting $n=2,$ then using (2.2) we get

$$\begin{aligned}
|x_2(t)-x_1(t)|\ &\le\ M_2K_2\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ |y_1(s)-y_0(s)\ |\ ds\\
&+\ N_2\ L_2|\ y_1(t)-y_0(t)\ |\ \frac{t^{\beta}}{\Gamma(\beta+1)}\\
|x_2(t)-x_1(t)|\ &\le\ M_2M_1N_1K_2\frac{t^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+\alpha+1)}+M_1N_1N_2L_2\frac{t^{\alpha+\beta}}{\Gamma(\alpha+1).\Gamma(\beta+1)}\\
&\le\ M_1N_1(M_2K_2+N_2L_2)\ t^{\alpha+\beta}.
\end{aligned}$$

By the same way we can prove that:

$$|y_2(t) - y_1(t)| \leq M_2N_2(M_1K_1 + N_1L_1)\ t^{\alpha+\beta}$$

using the above estimate we get

$$\begin{aligned} |x_3(t) - x_2(t)| &\leq M_2K_2 \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ |y_2(s) - y_1(s)|\ ds \\ &+ N_2L_2\ |y_2(t) - y_1(t)| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ ds \\ &\leq M_2N_2(M_1K_1 + N_1L_1)(M_2K_2 + N_2L_2)\ t^{2\alpha+\beta}. \end{aligned}$$

by a similar way as done before we have the following:

$$|y_3(t) - y_2(t)| \leq M_2N_2(M_1K_1 + N_1L_1)(M_2K_2 + N_2L_2)\ t^{\alpha+2\beta}$$
$$|x_4(t) - x_3(t)| \leq M_2N_2(M_1K_1 + N_1L_1)^2(M_2K_2 + N_2L_2)\ t^{2\alpha+2\beta}$$
$$|y_4(t) - y_3(t)| \leq M_1N_1(M_1K_1 + N_1L_1)^2(M_2K_2 + N_2L_2)\ t^{2\alpha+2\beta}$$
$$|x_5(t) - x_4(t)| \leq M_1N_1(M_1K_1 + N_1L_1)^2(M_2K_2 + N_2L_2)^2\ t^{3\alpha+2\beta}$$

Repeating this technique, we obtain the general estimate for the terms of the series:

$$|x_n(t) - x_{n-1}(t)| \leq \begin{cases} M_2N_2(M_1K_1 + N_1L_1)^{\frac{n}{2}}(M_2K_2 + N_2L_2)^{\frac{n}{2}-1} & \text{for } n \text{ even} \\ \\ M_1N_1(M_1K_1 + N_1L_1)^{\frac{n-1}{2}}(M_2K_2 + N_2L_2)^{\frac{n-1}{2}} & \text{for } n \text{ odd} \end{cases}$$

and

$$|y_n(t) - y_{n-1}(t)| \leq \begin{cases} M_1N_1(M_1K_1 + N_1L_1)^{\frac{n}{2}}(M_2K_2 + N_2L_2)^{\frac{n}{2}-1} & \text{for } n \text{ even} \\ \\ M_2N_2(M_1K_1 + N_1L_1)^{\frac{n-1}{2}}(M_2K_2 + N_2L_2)^{\frac{n-1}{2}} & \text{for } n \text{ odd} \end{cases}$$

Since $(M_1K_1 + N_1L_1)(M_2K_2 + N_2L_2) < 1$, then the uniform convergence of

$$\sum_{n=1}^{\infty} [x_n(t) - x_{n-1}(t)]$$

and

$$\sum_{n=1}^{\infty} [y_n(t) - y_{n-1}(t)]$$

is proved and so the sequences $\{x_n(t)\}$ and $\{y_n(t)\}$ are uniformly convergent. Since $f_i(t,x)$ and $g_i(t,x)$ are continuous in the second argument then

$$\begin{aligned} x(t) &= a_1(t) + \lim_{n\to\infty} g_1(t, y_{n-1}(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_1(s, y_{n-1}(s))\ ds \\ &= a_1(t) + g_1(t, y(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f_1(s, y(s))\ ds. \end{aligned}$$

and

$$\begin{aligned} y(t) &= a_2(t) + \lim_{n\to\infty} g_2(t, x_{n-1}(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s, x_{n-1}(s))\ ds \\ &= a_2(t) + g_2(t, x(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f_2(s, x(s))\ ds. \end{aligned}$$

Therefore, the sequence $\{u_n(t)\}$ which is defined by $u_n(t) = (x_n(t),\ y_n(t))$ is uniformly convergent. Thus, the existence of a solution is proved.

To prove the uniqueness, let $\tilde{u}(t) = (\tilde{x}, \tilde{y})(t)$ be a continuous solution of (1.1). Then

$$\begin{aligned}\tilde{x}(t) &= a_1(t) + g_1(t, \tilde{y}(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, \tilde{y}(s))\, ds, \ t \in [0,1], \\ \tilde{y}(t) &= a_2(t) + g_1(t, \tilde{x}(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, \tilde{x}(s))\, ds, \ t \in [0,1],\end{aligned}$$

and

$$\begin{aligned}|\tilde{u}(t) - u_n(t)| &= |(\tilde{x}(t),\ \tilde{y}(t)) - (x_n(t),\ y_n(t))| \\ &= |(\tilde{x}(t) - x_n(t),\ \tilde{y}(t) - y_n(t))| \\ &\leq \sup_{t \in I} |(\tilde{x}(t) - x_n(t),\ \tilde{y}(t) - y_n(t))| \\ &\leq ||(\tilde{x}(t) - x_n(t),\ \tilde{y}(t) - y_n(t))|| \\ &\leq ||\tilde{x}(t) - x_n(t)|| + ||\tilde{y}(t)) - y_n(t)||,\end{aligned}$$

by a simple calculations we get

$$\lim_{n \to \infty} x_n(t) = x(t) = \tilde{x}(t),$$

$$\lim_{n \to \infty} y_n(t) = y(t) = \tilde{y}(t).$$

Therefore

$$\lim_{n \to \infty} u_n(t) = u(t) = \tilde{u}(t).$$

Which completes the proof. □

Corollary 2.1. *Let the assumptions of Theorem 2.1 be satisfied. If $\alpha,\ \beta \to 1$, then the coupled system of quadratic integral equation*

$$\begin{aligned}x(t) &= a_1(t) + g_1(t, y(t)) \int_0^t f_1(s, y(s))\, ds \\ y(t) &= a_1(t) + g_2(t, x(t)) \int_0^t f_2(s, x(s))\, ds\end{aligned}$$

has a unique continuous solution.

3. ADOMIAN DECOMPOSITION METHOD (ADM)

The Adomian decomposition method (ADM) is a non-numerical method for solving a wide variety of functional equations and usually gets the solution in a series form.

Since the beginning of the 1980s, Adomian ([1]-[6] and [16]-[17]) has presented and developed a so-called decomposition method for solving algebraic, differential, integro- differential, differential-delay, and partial differential equations. The solution is found as an infinite series which converges rapidly to accurate solutions. The method has many advantages over the classical techniques, mainly, it makes unnecessary the linearization, perturbation and other restrictive methods and assumptions which may change the problem being solved, sometimes seriously. In recent decades, there has been a great deal of interest in the Adomian decomposition method. The method was successfully applied to a large amount of applications in applied sciences. For more details about the method and its application, see ([1]-[6], [37] and [16]-[17]).

In this section, we shall study Adomian decomposition method (ADM) for the coupled system (1.1).
The solution algorithm of the coupled system (1.1) using ADM is,

$$x_0(t) = a_1(t), \; x_i(t) = A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} B_{i-1}(s)\, ds, \; i \geq 1, \tag{3.1}$$

$$y_0(t) = a_2(t), \; y_i(t) = C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} D_{i-1}(s)\, ds, \; i \geq 1, \tag{3.2}$$

where A_i, B_i, C_i and D_i are Adomian polynomials of the nonlinear terms

$g_1(t, y(t))$, $f_1(s, y(s))$, $g_2(t, x(t))$ and $f_2(s, x(s))$ respectively, which take the forms

$$A_i = \frac{1}{i!} \left[\frac{d^i}{d\lambda^i} g_1 \left(t, \sum_{k=0}^{\infty} \lambda^k y_k \right) \right]_{\lambda=0},$$

$$B_i = \frac{1}{i!} \left[\frac{d^i}{d\lambda^i} f_1 \left(s, \sum_{k=0}^{\infty} \lambda^k y_k \right) \right]_{\lambda=0},$$

$$C_i = \frac{1}{i!} \left[\frac{d^i}{d\lambda^i} g_2 \left(t, \sum_{k=0}^{\infty} \lambda^k x_k \right) \right]_{\lambda=0},$$

$$D_i = \frac{1}{i!} \left[\frac{d^i}{d\lambda^i} f_2 \left(s, \sum_{k=0}^{\infty} \lambda^k x_k \right) \right]_{\lambda=0}.$$

Finally, the solution of the coupled system (1.1) will be

$$x(t) = \sum_{i=0}^{\infty} x_i(t) \text{ and } y(t) = \sum_{i=0}^{\infty} y_i(t) \tag{3.3}$$

4. CONVERGENCE ANALYSIS

4.1. **Existence and Uniqueness theorem.**

Theorem 4.1. *Let* $a_1(t), a_2(t) \in C(I)$. *If* $0 < R < 1$ *then the coupled system (1.1) has a unique solution* $X \in C^2(I)$, *where* $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $R = \max\{r_1, r_2\}$, $r_1 = \frac{1}{\Gamma(\beta+1)} [L_2 N_2 + K_2 M_2], r_2 = \frac{1}{\Gamma(\alpha+1)} [L_1 N_1 + K_1 M_1]$.

Proof. The system (1.1):

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_2(t, x(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s))\, ds \\ g_1(t, y(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds \end{pmatrix}$$

can be written as,

$$X = G + DM,$$

where,

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, G = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, M = \begin{pmatrix} g_2(t, x(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, x(s))\, ds \\ g_1(t, y(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, y(s))\, ds \end{pmatrix}$$

The mapping $F : E \to E$ is defined as,

$$FX = G + DM,$$

Let $X, U \in E$, then

$$FU = G + DN,$$

where,

$$U = \begin{pmatrix} u \\ v \end{pmatrix},\ N = \begin{pmatrix} g_2\left(t, u(t)\right) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u\left(s\right)\right) ds \\ g_1\left(t, v(t)\right) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v\left(s\right)\right) ds \end{pmatrix}.$$

so,

$$\begin{aligned}
\|FX - FU\| &= \|D\| \, \|M - N\| \\
&= \left\| \begin{pmatrix} g_2\left(t, x(t)\right) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, x\left(s\right)\right) ds - g_2\left(t, u(t)\right) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u\left(s\right)\right) ds \\ \\ g_1\left(t, y(t)\right) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, y\left(s\right)\right) ds - g_1\left(t, v(t)\right) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v\left(s\right)\right) ds \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} g_2\left(t, x(t)\right) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, x\left(s\right)\right) ds - g_2\left(t, u(t)\right) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, x\left(s\right)\right) ds \\ +g_2\left(t, u(t)\right) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, x\left(s\right)\right) ds - g_2\left(t, u(t)\right) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, u\left(s\right)\right) ds \\ \\ g_1\left(t, y(t)\right) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, y\left(s\right)\right) ds - g_1\left(t, v(t)\right) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, y\left(s\right)\right) ds \\ +g_1\left(t, v(t)\right) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, y\left(s\right)\right) ds - g_1\left(t, v(t)\right) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, v\left(s\right)\right) ds \end{pmatrix} \right\|
\end{aligned}$$

$$= \left\| \begin{pmatrix} \left[g_2\left(t, x(t)\right) - g_2\left(t, u(t)\right)\right] \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2\left(s, x\left(s\right)\right) ds \\ +g_2\left(t, u(t)\right) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left[f_2\left(s, x\left(s\right)\right) - f_2\left(s, u\left(s\right)\right)\right] ds \\ \\ \left[g_1\left(t, y(t)\right) - g_1\left(t, v(t)\right)\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1\left(s, y\left(s\right)\right) ds \\ +g_1\left(t, v(t)\right) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[f_1\left(s, y\left(s\right)\right) - f_1\left(s, v\left(s\right)\right)\right] ds \end{pmatrix} \right\|$$

$$= \begin{pmatrix} \frac{L_2}{\Gamma(\beta)} \max\limits_{t \in I} |x(t) - u(t)| \int_0^t (t-s)^{\beta-1} \left|f_2\left(s, x\left(s\right)\right)\right| ds \\ + \frac{K_2}{\Gamma(\beta)} \max\limits_{t \in I} \left|g_2\left(t, u(t)\right)\right| |x(t) - u(t)| \int_0^t (t-s)^{\beta-1} ds \\ \\ \frac{L_1}{\Gamma(\alpha)} \max\limits_{t \in I} |y(t) - v(t)| \int_0^t (t-s)^{\alpha-1} \left|f_1\left(s, y\left(s\right)\right)\right| ds \\ + \frac{K_1}{\Gamma(\alpha)} \max\limits_{t \in I} \left|g_1\left(t, v(t)\right)\right| |y(t) - v(t)| \int_0^t (t-s)^{\alpha-1} ds \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\beta\Gamma(\beta)} \left[L_2 N_2 + K_2 M_2\right] \|x(t) - u(t)\| \\ \\ \frac{1}{\alpha\Gamma(\alpha)} \left[L_1 N_1 + K_1 M_1\right] \|y(t) - v(t)\| \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\Gamma(\beta+1)} \left[L_2 N_2 + K_2 M_2\right] \|x(t) - u(t)\| \\ \\ \frac{1}{\Gamma(\alpha+1)} \left[L_1 N_1 + K_1 M_1\right] \|y(t) - v(t)\| \end{pmatrix}$$

$$= \begin{pmatrix} r_1 \|x(t) - u(t)\| \\ \\ r_2 \|y(t) - v(t)\| \end{pmatrix}$$

where

$$r_1 = \frac{1}{\Gamma(\beta+1)} [L_2 N_2 + K_2 M_2], r_2 = \frac{1}{\Gamma(\alpha+1)} [L_1 N_1 + K_1 M_1]$$

which implies that

$$\|FX - FU\| \leq R \|X - U\|$$

where,

$$R = \max\{r_1, r_2\},$$

under the condition $0 < R < 1$, the mapping F is contraction and hence there exists a unique solution $X \in C^2(I)$ of the system (1.1) and this completes the proof. □

4.2. **Proof of convergence.**

Theorem 4.2. *Let the solution of the system (1.1) be exist. If $|x_{j1}(t)| < c$ where c is a positive constant then the series solution (3.3) of the system (1.1) using ADM converge.*

Proof. Define the two sequences $\{S_{1p}\}$ and $\{S_{2p}\}$ such that, $S_{1p} = \sum_{i=0}^{p} x_i(t)$ and $S_{2p} = \sum_{i=0}^{p} y_i(t)$ are the sequences of partial sums from the series solutions $\sum_{i=0}^{\infty} x_i(t)$ and $\sum_{i=0}^{\infty} y_i(t)$. Now,

$$\begin{aligned} g_1(t, y(t)) &= \sum_{i=0}^{\infty} A_i, \ f_1(s, y(s)) = \sum_{i=0}^{\infty} B_i, \\ g_2(t, x(t)) &= \sum_{i=0}^{\infty} C_i, \ f_2(s, x(s)) = \sum_{i=0}^{\infty} D_i, \end{aligned}$$

Let S_{jp} and S_{jq} ($j = 1, 2$), be arbitrary partial sums with $p > q$. We are going to prove that $\{S_{jp}\}$ are Cauchy sequences in this Banach space E.

$$\|S_{jp} - S_{jq}\| = \left\| \begin{pmatrix} \sum_{i=0}^{p} x_i - \sum_{i=0}^{q} x_i \\ \\ \sum_{i=0}^{p} y_i - \sum_{i=0}^{q} y_i \end{pmatrix} \right\|$$

$$\leq \|D\| \left\| \begin{pmatrix} \sum_{i=0}^{p} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{p} D_{i-1}(s)\, ds - \sum_{i=0}^{q} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{q} D_{i-1}(s)\, ds \\ \sum_{i=0}^{p} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{p} B_{i-1}(s)\, ds - \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{q} B_{i-1}(s)\, ds \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} \sum_{i=0}^{p} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{p} D_{i-1}(s)\, ds - \sum_{i=0}^{q} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{p} D_{i-1}(s)\, ds \\ + \sum_{i=0}^{q} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{p} D_{i-1}(s)\, ds - \sum_{i=0}^{q} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{q} D_{i-1}(s)\, ds \\ \\ \sum_{i=0}^{p} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{p} B_{i-1}(s)\, ds - \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{p} B_{i-1}(s)\, ds \\ + \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{p} B_{i-1}(s)\, ds - \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{q} B_{i-1}(s)\, ds \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} \left[\sum_{i=0}^{p} C_{i-1}(t) - \sum_{i=0}^{q} C_{i-1}(t)\right] \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{p} D_{i-1}(s)\, ds \\ + \sum_{i=0}^{q} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left[\sum_{i=0}^{p} D_{i-1}(s) - \sum_{i=0}^{q} D_{i-1}(s)\right] ds \\ \\ \left[\sum_{i=0}^{p} A_{i-1}(t) - \sum_{i=0}^{q} A_{i-1}(t)\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{p} B_{i-1}(s)\, ds \\ + \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\sum_{i=0}^{p} B_{i-1}(s) - \sum_{i=0}^{q} B_{i-1}(s)\right] ds \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} \left[\sum_{i=q+1}^{p} C_{i-1}(t)\right] \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{p} D_{i-1}(s)\, ds \\ + \sum_{i=0}^{q} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left[\sum_{i=q+1}^{p} D_{i-1}(s)\right] ds \\ \\ \left[\sum_{i=q+1}^{p} A_{i-1}(t)\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{p} B_{i-1}(s)\, ds \\ + \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\sum_{i=q+1}^{p} B_{i-1}(s)\right] ds \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} \left[\sum_{i=q}^{p-1} C_{i-1}(t)\right] \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sum_{i=0}^{p} D_{i-1}(s)\, ds \\ + \sum_{i=0}^{q} C_{i-1}(t) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left[\sum_{i=q}^{p-1} D_{i-1}(s)\right] ds \\ \\ \left[\sum_{i=q}^{p-1} A_{i-1}(t)\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{p} B_{i-1}(s)\, ds \\ + \sum_{i=0}^{q} A_{i-1}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[\sum_{i=q}^{p-1} B_{i-1}(s)\right] ds \end{pmatrix} \right\|$$

$$\leq \left\| \begin{pmatrix} \left[g_2(t,S_{1(p-1)}) - g_2(t,S_{1(q-1)})\right] \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left[f_2(t,S_{1p})\right] ds \\ +g_2(t,S_{1q}) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left[f_2(t,S_{1(p-)1}) - f_2(t,S_{1(q-1)})\right] ds \\ \\ \left[g_1(t,S_{2(p-1)}) - g_1(t,S_{2(q-1)})\right] \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[f_1(t,S_{2p})\right] ds \\ +g_1(t,S_{2q}) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[f_1(t,S_{2(p-1)}) - f_1(t,S_{2(q-1)})\right] ds \end{pmatrix} \right\|$$

$$\leq \begin{pmatrix} \frac{1}{\Gamma(\beta+1)} \left[L_2N_2 + M_2K_2\right] \left\|S_{1(p-1)} - S_{1(q-1)}\right\| \\ \\ \frac{1}{\Gamma(\alpha+1)} \left[L_1N_1 + M_1K_1\right] \left\|S_{2(p-1)} - S_{2(q-1)}\right\| \end{pmatrix}$$

$$\leq R\left\|S_{j(p-1)} - S_{j(q-1)}\right\|$$

Let $p = q + 1$ then,

$$\left\|S_{j(q+1)} - S_{jq}\right\| \leq R\left\|S_{jq} - S_{j(q-1)}\right\| \leq R^2\left\|S_{j(q-1)} - S_{j(q-2)}\right\| \leq \cdots \leq R^q\left\|S_{j1} - S_{j0}\right\|$$

From the triangle inequality we have,

$$\begin{aligned} \left\|S_{jp} - S_{jq}\right\| &\leq \left\|S_{j(q+1)} - S_{jq}\right\| + \left\|S_{j(q+2)} - S_{j(q+1)}\right\| + \cdots + \left\|S_{jp} - S_{j(p-1)}\right\| \\ &\leq \left[R^q + R^{q+1} + \cdots + R^{p-1}\right]\left\|S_{j1} - S_{j0}\right\| \\ &\leq R^q\left[1 + R + \cdots + R^{p-q-1}\right]\left\|S_{j1} - S_{j0}\right\| \\ &\leq R^q\left[\frac{1-R^{p-q}}{1-R}\right]\left\|x_{j1}\right\| \end{aligned}$$

where $\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. Now $0 < R < 1$, and $p > q$ implies that $(1 - R^{p-q}) \leq 1$. Consequently,

$$\begin{aligned} \left\|S_{jp} - S_{jq}\right\| &\leq \frac{R^q}{1-R}\left\|x_{j1}\right\| \\ &\leq \frac{R^q}{1-R}\max_{t\in I}\left|x_{j1}(t)\right| \end{aligned}$$

but, if $|x_{j1}(t)| < c$ then $\|S_{jp} - S_{jq}\| \to 0$ as $q \to \infty$ and hence, $\{S_{jp}\}$ are Cauchy sequences in this Banach space so, the series $\sum_{i=0}^{\infty} x_i(t)$ and $\sum_{i=0}^{\infty} y_i(t)$ converge and this completes the proof. □

5. NUMERICAL EXAMPLES

Example 1 Consider the following nonlinear FCSQIEs,

$$x(t) = \left(t^2 - \frac{t^{11/2}}{35\sqrt{\pi}}\right) + y^2(t)\int_0^t \frac{(t-s)^{-1/2}}{\Gamma(1/2)} y^3(s)\,ds, \tag{5.1}$$

$$y(t) = \left(\frac{t}{2} - \frac{1048576t^{39/2}}{22309287\sqrt{\pi}}\right) + x^4(t)\int_0^t \frac{(t-s)^{1/2}}{\Gamma(3/2)} x^5(s)\,ds,$$

and has the exact solution $x(t) = t^2, y(t) = \frac{t}{2}$.

Applying ADM to system (5.1), we get

$$x_0(t) = \left(t^2 - \frac{t^{11/2}}{35\sqrt{\pi}}\right),\ x_i(t) = A_{i-1}(t)\int_0^t \frac{(t-s)^{-1/2}}{\Gamma(1/2)} B_{i-1}(s)\,ds,\ i \geq 1,$$

$$y_0(t) = \left(\frac{t}{2} - \frac{1048576t^{39/2}}{22309287\sqrt{\pi}}\right),\ y_i(t) = C_{i-1}(t)\int_0^t \frac{(t-s)^{1/2}}{\Gamma(3/2)} D_{i-1}(s)\,ds,\ i \geq 1,$$

where A_i, B_i, C_i, and D_i are Adomian polynomials of the nonlinear terms y^2, y^3, x^4 and x^5 respectively and the solution will be,

$$x(t) = \sum_{i=0}^{q} x_i(t),\ y(t) = \sum_{i=0}^{q} y_i(t)$$

Table 1 shows the absolute error of ADM solution $(q = 2)$, while table 2 shows the absolute error of Picard solution $(q = 2)$.

Table 1: Absolute Error

t	$\lvert x_{exact} - x_{ADM}\rvert$	$\lvert y_{exact} - y_{ADM}\rvert$
0.1	6.61744×10^{-24}	3.13306×10^{-26}
0.2	4.65868×10^{-20}	2.62749×10^{-19}
0.3	7.80598×10^{-16}	2.94604×10^{-15}
0.4	7.77967×10^{-13}	2.19741×10^{-12}
0.5	1.64741×10^{-10}	3.70791×10^{-10}
0.6	1.30963×10^{-8}	2.44052×10^{-8}
0.7	5.29431×10^{-7}	8.37533×10^{-7}
0.8	0.0000130294	0.0000178146
0.9	0.000217161	0.000262602
1	0.00249546	0.00290512

Table 2: Absolute Error

t	$\lvert x_{exact} - x_{Picard}\rvert$	$\lvert y_{exact} - y_{Picard}\rvert$
0.1	6.61744×10^{-24}	3.76158×10^{-37}
0.2	0	3.9443×10^{-31}
0.3	1.31798×10^{-18}	7.37112×10^{-26}
0.4	3.61304×10^{-15}	1.11707×10^{-20}
0.5	1.66655×10^{-12}	1.17449×10^{-16}
0.6	2.49761×10^{-10}	2.26918×10^{-13}
0.7	1.72135×10^{-8}	1.36183×10^{-10}
0.8	6.71089×10^{-7}	3.47021×10^{-8}
0.9	0.0000169013	4.56449×10^{-6}
1	0.000299432	0.000340497

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ON SOME PROPERTIES OF p-WAVELET PACKETS VIA THE WALSH-FOURIER TRANSFORM

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ABSTRACT. A novel method for the construction of orthogonal p-wavelet packets on a positive half-line $\mathbb{R}^+$ was given by the author in [*Construction of wavelet packets on p-adic field,* Int. J. Wavelets Multiresolut. Inf. Process., 7(5) (2009), pp. 553-565]. In this paper, we investigate their properties by means of the Walsh-Fourier transform. Three orthogonal formulas regarding these p-wavelet packets are derived.

KEYWORDS : p-Multiresolution analysis; p-Wavelet packets; Riesz basis; Walsh functions; Walsh-Fourier transform.
AMS Subject Classification: 42C40 42C15 42C10

1. INTRODUCTION

In the early nineties a general scheme for the construction of wavelets was defined. This scheme is based on the notion of multiresolution analysis (MRA) introduced by Mallat [13]. Immediately specialists started to implement new wavelet systems and in recent years, the concept MRA of $\mathbb{R}^n$ has been extended to many different setups, for example, Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups [5], Lang [11] and [12] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ by following the procedure of Daubechies [6] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Later on, Farkov [7] extended the results of Lang [11] and [12] on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact Abelian group G which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. The construction of dyadic compactly supported wavelets for $L^2(\mathbb{R}^+)$ have been given by Protasov and Farkov in [14] where the latter author has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ arising from scaling filters with p^n many terms in [8].

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It is well-known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman *et al.* [4] constructed univariate orthogonal wavelet packets. The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application. The standard construction is to start from a multiresolution analysis (MRA) and generate the library using the associated quadrature mirror filters (QMFs). The internal structure of the MRA and the speed of the decomposition schemes make this an efficient adaptive method for simultaneous time and frequency analysis of signals. The concept of the wavelet packet was subsequently generalized to $\mathbb{R}^d$ by taking tensor products, whereas Shen [18] formulated non-tensor product wavelets in $L^2(\mathbb{R}^s)$. Other notable generalizations are the non-orthogonal version of wavelet packets [2], biorthogonal wavelet packets [3], vector-valued wavelet packets [1] and higher dimensional wavelet packets with arbitrary dilation matrix [10].

Recently, Shah [16] has constructed p-wavelet packets associated with the p-MRA on the positive half-line $\mathbb{R}^+$. He proved lemmas on the so-called splitting trick and several theorems concerning the Walsh-Fourier transform of the p-wavelet packets and the construction of p-wavelet packets to show that their translates form an orthonormal basis of $L^2(\mathbb{R}^+)$. Very recently, Shah and Debnath [17], have constructed the corresponding p-wavelet frame packets on the positive half-line $\mathbb{R}^+$ by using the Walsh-Fourier transform. As one of a series of works on positive half-line $\mathbb{R}^+$, the objective of this paper is to investigate certain properties of orthogonal p-wavelet packets on the positive half-line $\mathbb{R}^+$ by virtue of the Walsh-Fourier transform.

In order to make the paper self-contained, we state some basic preliminaries, notations and definitions including the Walsh-Fourier transform, Walsh functions and p-MRA in Section 2. In Section 3, we study certain properties of orthogonal p-wavelet packets on a half-line $\mathbb{R}^+$.

2. PRELIMINARIES AND p-WAVELET PACKETS

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, ...\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Set $\Omega_0 = \{0, 1, 2, ..., p-1\}$ and $\Omega = \Omega_0 - \{0\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x](\mathrm{mod}\, p), \; x_{-j} = [p^{1-j} x](\mathrm{mod}\, p). \tag{2.1}$$

We consider on $\mathbb{R}^+$ the addition defined as follows: if $z = x \oplus y$, then

$$z = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j}$$

with $\zeta_j = x_j + y_j (\mathrm{mod}\ p)$ $(j \in \mathbb{Z} \setminus \{0\})$, where $\zeta_j \in \Omega_0$ and x_j, y_j are calculated by (2.1). Moreover, we note that $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

For $x \in [0, 1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p); \\ \varepsilon_p^{\ell}, & \text{if } x \in [\ell p^{-1}, (\ell+1)p^{-1}),\ \ell \in \Omega, \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is denoted by the equality $r_0(x+1) = r_0(x),\ x \in \mathbb{R}^+$. Then, the generalized Walsh functions $\{w_m(x) : m \in \mathbb{Z}^+\}$ are defined by

$$w_0(x) \equiv 1,\ w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j\ ,\ \mu_j \in \Omega_0,\ \mu_k \neq 0$.

For $x, w \in \mathbb{R}^+$, let

$$\chi(x, w) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j w_{-j} + x_{-j} w_j)\right), \tag{2.2}$$

where x_j, w_j are given by (2.1). Note that $\chi(x, m/p^{n-1}) = \chi(x/p^{n-1}, m) = w_m(x/p^{n-1})$ for all $x \in [0, p^{n-1}),\ m \in \mathbb{Z}^+$.

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x)\overline{\chi(x, \xi)}\, dx, \tag{2.3}$$

where $\chi(x, \xi)$ is given by (2.2). The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see [15]). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and

$$\|\hat{f}\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}.$$

If $x, y, \xi \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x \oplus y, \xi) = \chi(x, \xi)\, \chi(y, \xi). \tag{2.4}$$

It is shown by Golubov *et al.* [9] that both the systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(., \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$.

In the following subsection, we introduce the notion of p-multiresolution analysis on $\mathbb{R}^+$ and give the definition of orthogonal wavelets of space $L^2(\mathbb{R}^+)$.

Definition 2.1. *A p-multiresolution analysis of $L^2(\mathbb{R}^+)$ is a nested sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ such that*

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^+)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(iii) $f \in V_j$ if and only if $f(p.) \in V_{j+1}$ for all $j \in \mathbb{Z}$,

(iv) there exists a function φ in V_0, called the scaling function, such that $\{\varphi(. \ominus k) : k \in \mathbb{Z}^+\}$ forms an orthonormal basis for V_0.

Since $\varphi(x) \in V_0 \subset V_1$, by Definition 2.1, there exists a finitely supported sequence $\{a_k\}_{k\in\mathbb{Z}^+} \in l^2(\mathbb{Z}^+)$ such that

$$\varphi(x) = \sum_{k\in\mathbb{Z}^+} a_k \, \varphi(px \ominus k). \tag{2.5}$$

The Walsh-Fourier transform of (2.5) is given by

$$\hat{\varphi}\,(\xi) = m_0(p^{-1}\xi)\, \hat{\varphi}(p^{-1}\xi), \tag{2.6}$$

where $m_0(\xi) = \sum_{k\in\mathbb{Z}^+} a_k \,\overline{\chi(k,\xi)}$, is a Walsh polynomial called the *symbol* of $\varphi(x)$.

Let $W_j, j \in \mathbb{Z}$ be the direct complementary subspaces of V_j in V_{j+1}. Assume that there exist a set of $p-1$ functions $\{\psi_1, \psi_2, ..., \psi_{p-1}\}$ in $L^2(\mathbb{R}^+)$ such that their translates and dilations form an orthonormal basis of W_j, i.e.,

$$W_j = \overline{\text{span}}\left\{\psi_\ell(p^j. \ominus k) : k \in \mathbb{Z}^+, \ell \in \Omega\right\}, \; j \in \mathbb{Z}. \tag{2.7}$$

Since $\psi_\ell(x) \in W_0 \subset V_1, \ell \in \Omega$, there exists a sequence $\left\{a_k^\ell\right\}_{k\in\mathbb{Z}^+}$ in $\Upsilon^2(\mathbb{Z}^+)$ such that

$$\psi_\ell\,(x) = \sum_{k\in\mathbb{Z}^+} a_k^\ell \,\varphi(px \ominus k), \;\; \ell \in \Omega. \tag{2.8}$$

Implementing the Walsh- Fourier transform for both sides of (2.8) gives

$$\hat{\psi}_\ell\,(\xi p) = m_\ell(\xi)\, \hat{\varphi}(\xi), \tag{2.9}$$

where

$$m_\ell(\xi) = \sum_{k\in\mathbb{Z}^+} a_k^\ell \,\overline{\chi(k,\xi)}. \tag{2.10}$$

Moreover, we say that $\psi_\ell, \in L^2(\mathbb{R}^+), \ell \in \Omega$ are orthogonal wavelets associated with the orthogonal scaling function $\varphi(x)$, if $\{\psi_\ell(x \ominus k) : k \in \mathbb{Z}^+, \ell \in \Omega\}$ is a basis of W_0 and

$$\langle \varphi(.), \varphi(. \ominus k)\rangle = \delta_{0,k}, \;\; k \in \mathbb{Z}^+, \tag{2.11}$$

$$\langle \varphi(.), \psi_\ell(. \ominus k)\rangle = 0, \;\; \ell \in \Omega, k \in \mathbb{Z}^+, \tag{2.12}$$

$$\langle \psi_\ell(.), \psi_{\ell'}(. \ominus k)\rangle = \delta_{\ell,\ell'}\delta_{0,k}, \;\; \ell, \ell' \in \Omega, k \in \mathbb{Z}^+. \tag{2.13}$$

We now introduce the definition of p-wavelet packets (as defined in [11]) associated with the scaling function $\varphi(x)$.

Definition 2.2. *Let the orthonormal scaling function $\varphi(x)$ and $\psi_\ell(x), \ell \in \Omega$ satisfy refinement equation (2.5) and wavelet equation (2.8), respectively. Then, for all $n \in \mathbb{Z}^+$, define the functions $\omega_n(x)$ recursively by*

$$\omega_n(x) = \omega_{pr+s}(x) = \sum_{k\in\mathbb{Z}^+} p\, a_k^s \,\omega_r(px \ominus k), \;\; s \in \Omega_0 \tag{2.14}$$

where $r \in \mathbb{Z}^+$ is the unique element such that $n = pr + s, s \in \Omega_0$ holds.

Applying the Walsh-Fourier transform for the both sides of (2.14) yields,

$$\hat{\omega}_{pr+s}(\xi) = m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi),\ s \in \Omega_0. \tag{2.15}$$

When $r = 0$ and $s \in \Omega$, we obtain

$$\hat{\omega}_s(p\xi) = m_s(\xi)\hat{\omega}_0(\xi)$$

which shows that $\omega_0(x) = \varphi(x)$ and $\omega_s(x) = \psi_\ell(x)$.

Lemma 2.3[8]. *Let $\omega_0(x) \in L^2(\mathbb{R}^+)$. The system $\{\omega_0(.\ominus k) : k \in \mathbb{Z}^+\}$ is orthogonal in $L^2(\mathbb{R}^+)$ if and only if*

$$\sum_{k\in\mathbb{Z}^+} \hat{\omega}_0(\xi \oplus k)\overline{\hat{\omega}_0(\xi \oplus k)} = 1 \text{ for a.e. } \xi \in \mathbb{R}^+. \tag{2.16}$$

3. THE PROPERTIES OF ORTHOGONAL p-WAVELET PACKETS

In this section, we investigate the orthogonality property of the p-wavelet packets on $\mathbb{R}^+$ by virtue of the Walsh-Fourier transform.

Lemma 3.1. *Let $\{\omega_n : n \in \mathbb{Z}^+\}$ be the p-wavelet packets associated with the p-MRA $\{V_j\}_{j\in\mathbb{Z}}$. Then, we have*

$$\sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi \oplus \ell)\big)\overline{m_s\big(p^{-1}(\xi \oplus \ell)\big)} = \delta_{r,s},\ r, s \in \Omega_0. \tag{3.1}$$

Proof. Using (2.11)–(2.13), (2.15) and Lemma 2.3, we have

$$\begin{aligned}
\delta_{r,s} &= \sum_{k\in\mathbb{Z}^+} \omega_r(\xi \oplus k)\overline{\omega_s(\xi \oplus k)} \\
&= \sum_{k\in\mathbb{Z}^+} m_r\big(p^{-1}(\xi \oplus k)\big)\hat{\omega}_0\big(p^{-1}(\xi \oplus k)\big)\overline{\hat{\omega}_0\big(p^{-1}(\xi \oplus k)\big)}\ \overline{m_s\big(p^{-1}(\xi \oplus k)\big)} \\
&= \sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi \oplus \ell)\big) \\
&\quad \left\{\sum_{k\in\mathbb{Z}^+} \hat{\omega}_0\big(p^{-1}(\xi \oplus k) \oplus \ell\big)\overline{\hat{\omega}_0\big(p^{-1}(\xi \oplus k) \oplus \ell\big)}\right\}\overline{m_s\big(p^{-1}(\xi \oplus \ell)\big)} \\
&= \sum_{\ell\in\Omega_0} m_r\big(p^{-1}(\xi \oplus \ell)\big)\overline{m_s\big(p^{-1}(\xi \oplus \ell)\big)}.
\end{aligned}$$

Theorem 3.2. *If $\{\omega_n : n \in \mathbb{Z}^+\}$ are the p-wavelet packets with respect to the scaling function $\varphi(x)$. Then, for $n \in \mathbb{Z}^+$, we have*

$$\langle \omega_n(.), \omega_n(.\ominus k)\rangle = \delta_{0,k},\ k \in \mathbb{Z}^+. \tag{3.2}$$

Proof. We prove this result by using induction on n. It follows from (2.11) and (2.13) that the claim is true for $n = 0$ and $n = 1, 2, ..., p-1$. Assume that (3.2) holds when $n < q$, where $q \in \mathbb{N}$. Then, we prove the result (3.2) for $n = q$. Let

$n = pr + s$, where $r \in \mathbb{Z}^+, s \in \Omega_0$ and $r < n$. Therefore, by induction assumption, we have

$$\langle \omega_r(.), \omega_r(. \ominus k)\rangle = \delta_{0,k} \iff \sum_{k\in\mathbb{Z}^+} \hat{\omega}_r(\xi \oplus k)\overline{\hat{\omega}_r(\xi \oplus k)} = 1, \; \xi \in \mathbb{R}^+.$$

Using Lemma 2.3, Lemma 3.1, and (2.15), we get

$$\begin{aligned}
\langle \omega_n(.), \omega_n(. \ominus k)\rangle &= \langle \hat{\omega}_n(.), \hat{\omega}_n(. \ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat{\omega}_{pr+s}(\xi)\,\overline{\hat{\omega}_{pr+s}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{m_s(p^{-1}\xi)}\,\overline{\hat{\omega}_r(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}^+}\int_{p([0,1]+k)} m_s(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{m_s(p^{-1}\xi)}\,\overline{\hat{\omega}_r(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p[0,1]} m_s(p^{-1}\xi)\left\{\sum_{k\in\mathbb{Z}^+} \hat{\omega}_r\big(p^{-1}(\xi\oplus k)\big)\,\overline{\hat{\omega}_r\big(p^{-1}(\xi\oplus k)\big)}\right\} \\
&\quad \times \overline{m_s(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p[0,1]} m_s(p^{-1}\xi)\overline{m_s(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]} \sum_{\ell\in\Omega_0} m_s\big(p^{-1}(\xi\oplus\ell)\big)\overline{m_s\big(p^{-1}(\xi\oplus\ell)\big)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]} \chi(k,\xi)d\xi \\
&= \delta_{0,k}.
\end{aligned}$$

Theorem 3.3. *Suppose $\{\omega_n : n \in \mathbb{Z}^+\}$ are the p-wavelet packets associated with the scaling function $\varphi(x)$. Then, for $r \in \mathbb{Z}^+$, we have*

$$\langle \omega_{pr+s_1}(.), \omega_{pr+s_2}(. \ominus k)\rangle = \delta_{0,k}\delta_{s_1,s_2}, \; s_1, s_2 \in \Omega_0, k \in \mathbb{Z}^+. \tag{3.3}$$

Proof. By Lemma 2.3, we have

$$\begin{aligned}
\langle \omega_{pr+s_1}, \omega_{pr+s_2}(. \ominus k)\rangle &= \langle \hat{\omega}_{pr+s_1}, \hat{\omega}_{pr+s_2}(. \ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat{\omega}_{pr+s_1}(\xi)\,\overline{\hat{\omega}_{pr+s_2}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi)\hat{\omega}_r(p^{-1}\xi)\,\overline{m_{s_2}(p^{-1}\xi)}\,\overline{\hat{\omega}_r(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= p\sum_{k\in\mathbb{Z}^+}\int_{([0,1]+k)} m_{s_1}(\xi)\hat{\omega}_r(\xi)\,\overline{m_{s_2}(\xi)}\,\overline{\hat{\omega}_r(\xi)}\chi(k,p\xi)d\xi \\
&= p\int_{[0,1]} m_{s_1}(\xi)\left\{\sum_{k\in\mathbb{Z}^+} \hat{\omega}_r(\xi\oplus k)\overline{\hat{\omega}_r(\xi\oplus k)}\right\}\overline{m_{s_2}(\xi)}\chi(k,p\xi)d\xi \\
&= \int_{p[0,1]} m_{s_1}(p^{-1}\xi)\,\overline{m_{s_2}(p^{-1}\xi)}\chi(k,\xi)d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[0,1]} \sum_{\ell\in\Omega_0} m_{s_1}\left(p^{-1}(\xi\oplus\ell)\right)\overline{m_{s_2}\left(p^{-1}(\xi\oplus\ell)\right)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]} \delta_{s_1,s_2}\chi(k,\xi)d\xi \\
&= \delta_{0,k}\delta_{s_1,s_2}.
\end{aligned}
$$

Theorem 3.4. *Let $\{\omega_n : n\in\mathbb{Z}^+\}$ be the p-wavelet packets associated with the scaling function $\varphi(x)$. Then, for $\ell, n\in\mathbb{Z}^+$, we have*

$$
\langle \omega_\ell(.), \omega_n(.\ominus k)\rangle = \delta_{\ell,n}\delta_{0,k},\ k\in\mathbb{Z}^+. \tag{3.4}
$$

Proof. For $\ell = n$, the result (3.4) follows by Theorem 3.2. When $\ell\neq n$, and $\ell, n\in\Omega_0$, the result (3.4) can be established from Theorem 3.3. Assuming that ℓ is not equal to n, and atleast one of $\{\ell, n\}$ does not belong to Ω_0, then we can rewrite ℓ, n as $\ell = pr_1 + s_1, n = pu_1 + v_1$, where $r_1, u_1\in\mathbb{Z}^+, s_1, v_1\in\Omega_0$.

Case 1. If $r_1 = u_1$, then $s_1\neq v_1$. Therefore, (3.4) follows by virtue of (2.15), Lemma 2.3 and (3.1), i.e.,

$$
\begin{aligned}
\langle \omega_\ell(.), \omega_n(.\ominus k)\rangle &= \langle \omega_{pr_1+s_1}, \omega_{pu_1+v_1}(.\ominus k)\rangle \\
&= \langle \hat\omega_{pr_1+s_1}, \hat\omega_{pu_1+v_1}(.\ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat\omega_{pr_1+s_1}(\xi)\,\overline{\hat\omega_{pu_1+v_1}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi)\hat\omega_{r_1}(p^{-1}\xi)\,\overline{\hat\omega_{u_1}(p^{-1}\xi)}\,\overline{m_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}^+}\int_{p([0,1]+k)} m_{s_1}(p^{-1}\xi)\hat\omega_{r_1}(p^{-1}\xi)\,\overline{\hat\omega_{u_1}(p^{-1}\xi)}\,\overline{m_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{p([0,1])} m_{s_1}(p^{-1}\xi)\left\{\sum_{k\in\mathbb{Z}^+}\hat\omega_{r_1}\left(p^{-1}(\xi\oplus k)\right)\overline{\hat\omega_{u_1}\left(p^{-1}(\xi\oplus k)\right)}\right\} \\
&\quad\times \overline{m_{v_1}(p^{-1}\xi)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]} \sum_{\ell\in\Omega_0} m_{s_1}\left(p^{-1}(\xi\oplus\ell)\right)\overline{m_{v_1}\left(p^{-1}(\xi\oplus\ell)\right)}\chi(k,\xi)d\xi \\
&= \int_{[0,1]} \delta_{s_1,v_1}\chi(k,\xi)d\xi \\
&= \delta_{0,k} = 0.
\end{aligned}
$$

Case 2. If $r_1\neq u_1$, order $r_1 = pr_2 + s_2, u_1 = pu_2 + v_2$, where $r_2, u_2\in\mathbb{Z}^+$ and $s_2, v_2\in\Omega_0$. If $r_2 = u_2$, then $s_2\neq v_2$. Similar to Case 1, (3.4) can be established. When $r_2\neq u_2$, we order $r_2 = pr_3 + s_3, u_2 = pu_3 + v_3$, where $r_3, u_3\in\mathbb{Z}^+$ and $s_3, v_3\in\Omega_0$. Thus, after taking finite steps (denoted by κ), we obtain $r_\kappa, u_\kappa\in\Omega_0$ and $s_\kappa, v_\kappa\in\Omega_0$. If $r_\kappa = u_\kappa$, then $s_\kappa\neq v_\kappa$. Similar to the Case 1, (3.4) follows. If $r_\kappa\neq u_\kappa$, then it follows from (2.11)–(2.13) that

$$
\langle \omega_{r_\kappa}(.), \omega_{u_\kappa}(.\ominus k)\rangle = 0,\ k\in\mathbb{Z}^+ \iff \sum_{k\in\mathbb{Z}^+}\hat\omega_{r_\kappa}(\xi\oplus k)\overline{\hat\omega_{u_\kappa}(\xi\oplus k)} = 0,\ \xi\in\mathbb{R}^+.
$$

Furthermore, we obtain

$$\begin{aligned}
\langle \omega_r(.), \omega_u(.\ominus k)\rangle &= \langle \hat{\omega}_r(.), \hat{\omega}_u(.\ominus k)\rangle \\
&= \langle \hat{\omega}_{pr_1+s_1}, \hat{\omega}_{pu_1+v_1}(.\ominus k)\rangle \\
&= \int_{\mathbb{R}^+} \hat{\omega}_{pr_1+s_1}(\xi)\,\overline{\hat{\omega}_{pu_1+v_1}(\xi)}\chi(k,\xi)d\xi \\
&= \int_{\mathbb{R}^+} m_{s_1}(p^{-1}\xi)m_{s_2}(p^{-2}\xi)\hat{\omega}_{r_2}(p^{-2}\xi)\,\overline{\hat{\omega}_{u_2}(p^{-2}\xi)}\,\overline{m_{v_1}(p^{-1}\xi)} \\
&\quad \times\ \overline{m_{v_2}(p^{-2}\xi)}\chi(k,\xi)d\xi \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \int_{\mathbb{R}^+} \left\{\prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi)\right\}\hat{\omega}_{r_\kappa}(p^{-\kappa}\xi)\,\overline{\hat{\omega}_{u_\kappa}(p^{-\kappa}\xi)}\left\{\prod_{\ell=1}^{\kappa}\overline{m_{v_\ell}(p^{-\ell}\xi)}\right\}\chi(k,\xi)d\xi \\
&= \sum_{k\in\mathbb{Z}^+}\int_{p^\kappa([0,1]+k)} \left\{\prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi)\right\}\left\{\hat{\omega}_{r_\kappa}\left(p^{-\kappa}\xi\right)\overline{\hat{\omega}_{u_\kappa}\left(p^{-\kappa}\xi\right)}\right\} \\
&\quad \times\left\{\prod_{\ell=1}^{\kappa}\overline{m_{v_\ell}(p^{-\ell}\xi)}\right\}\chi(k,\xi)d\xi \\
&= \int_{p^\kappa[0,1]} \left\{\prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi)\right\}\left\{\sum_{k\in\mathbb{Z}^+}\hat{\omega}_{r_\kappa}\left(p^{-\kappa}(\xi\oplus k)\right)\overline{\hat{\omega}_{u_\kappa}\left(p^{-\kappa}(\xi\oplus k)\right)}\right\} \\
&\quad \times\left\{\prod_{\ell=1}^{\kappa}\overline{m_{v_\ell}(p^{-\ell}\xi)}\right\}\chi(k,\xi)d\xi \\
&= \int_{p^\kappa[0,1]} \left\{\prod_{\ell=1}^{\kappa} m_{s_\ell}(p^{-\ell}\xi)\right\}.0.\left\{\prod_{\ell=1}^{\kappa}\overline{m_{v_\ell}(p^{-\ell}\xi)}\right\}\chi(k,\xi)d\xi = 0.
\end{aligned}$$

Therefore, for any $\ell, n \in \mathbb{Z}^+$, result (3.4) is established.

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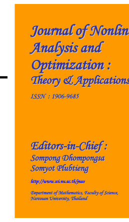

CONTINUITY OF FUZZY TRANSITIVE ORDERED SETS

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ABSTRACT. In the present paper we introduce and study the continuity for a set equipped with a transitive fuzzy binary order relation which we call a f-toset. Our work is inspired by the slogan: " Order theory is the study of transitive relations" due to S. Abramsky and A. Jung [1].

KEYWORDS : Fuzzy set; Fuzzy order relation; Continuous lattices; Continuous posets.
AMS Subject Classification: 06A11 03E72 06B35

1. INTRODUCTION

In crisp setting , S. Abramsky and A. Jung [1] introduced a method to construct canonical partially ordered set from a pre-ordered set and said: " Many notions from theory of partially ordered sets make sense even if reflexivity fails". Finally they sum up these considerations with the slogan: "Order theory is the study of transitive relations". In our opinion this slogan still valid in fuzzy setting . Thus From this point of view the present paper is devoted to introduce and study the continuity for a set with a transitive fuzzy binary order relation (so called a f-toset).

It is worth to mention that in crisp setting the continuous lattices were studied in [7] and types of continues posets (domains) were studied in [1, 7, 8, 10, 15].

Recently [13], the concept of continuity of some types of fuzzy directed complete posets was studied.

This paper consists of 3 Sections. In Section 2, some preliminaries and some basic concepts on f-toset are discussed. Section 3, is devoted to introduce and study the concept of continuous f-toset. Finally, a conclusion is given to compare some types of fuzzy posets.

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2. PRELIMINARIES AND SOME BASIC CONCEPTS

In this section we introduce the concept of fuzzy transitive ordered set and some of its basic concepts.

In this paper we use Claude Ponsard's definition of fuzzy order relation (see[2]).

Definition 2.1. Let X be a crisp set . A fuzzy order relation on X is a fuzzy subset of $X \times X$ satisfying the following three properties:

(i) $\forall\, x \in X,\ r(x,x) \in [0,1]$;

(ii) $\forall\, x,y \in X,\ r(x,y) + r(y,x) > 1$ implies $x = y$;

(iii) $\forall\, x,y,z \in X,\ r(x,y) \geq r(y,x)$ and $r(y,z) \geq r(z,y)$ implying $r(x,z) \geq r(z,x)$.

The pair (x,y) is called a fuzzy ordered set.

Definition 2.2. Let X be a crisp set . A fuzzy transitive order relation r on X is a fuzzy set of $X \times X$ satisfying (iii) in Definition 2.1. The pair (X,r) is called a fuzzy transitive order set (in brief f-toset).

Definition 2.3. Let (X,r) be a f-toset and let $A \subseteq X$. Then :

(1) The lower (resp. upper) bounded subset in X of A is denoted by $lb(A)$ (resp. $ub(A)$) and defined as follows :

$lb(A) = \{x \in X : \forall\, y \in A,\ r(y,x) \leq r(x,y)\}$ (resp. $ub(A) = \{x \in X : \forall\, y \in A,\ r(x,y) \leq r(y,x)\}$). Each element in $lb(A$) (resp. $ub(A)$) is called a lower (resp. an upper) bound of A;

(2) The subset of least (resp. largest) elements of A is denoted by $le(A)$ (resp. $la(A)$) and defined as follows :

$le(A) = \{x \in A : \forall\, y \in A,\ r(y,x) \leq r(x,y)\}$ (resp. $la(A) = \{x \in A : \forall\, y \in A,\ r(x,y) \leq r(y,x)\}$). Each element in is called a least (resp. largest) element of A;

(3) The infimum (resp. supremum) subset in X of A is denoted by $\inf(A)$ (resp. $\sup(A)$) and defined as follows:

$\inf(A) = la(lb(A))$ (resp. $\sup(A) = le(ub(A))$). Each element in inf(A) (resp. sup(A)) is called an infimum (resp. a supremum) of A;

(4) The lower (resp. upper) closure in X of A is denoted by $\downarrow(A)$ (resp. $\uparrow(A)$) and defined as follows:

$\downarrow(A) = \{x \in X : \exists\, y \in A\ s.t.\ r(y,x) \leq r(x,y)\}$ (resp. $\uparrow(A) = \{x \in X : \forall\, y \in A\ s.t.\ r(x,y) \leq r(y,x)\}$).

Remark 2.3. In any f-toset (X,r) one can remark that for any subset A of X, $la(A)$, $le(A)$, $\sup(A)$ and $\inf(A)$ need not be singletons.

Remark 2.4. In [8], the author considered the supremum (resp. infimum) of a subset A of fuzzy ordered set (X,r) as the unique least element (resp. unique largest element) of the set of upper bounds (resp. lower bounds) of A if it exists.

Now we introduce some propositions on the fuzzy lower and fuzzy upper closure of a subset in a f-toset without proof.

Proposition 2.1. Let (X,r) be a f-toset and let $A,B \subseteq X$. Then:

(1) $\downarrow(\phi) = \phi$ and $\downarrow(X) \subseteq X$;

(2) $\uparrow(\phi) = \phi$ and $\uparrow(X) \subseteq X$;

(3) $If\ A \subseteq B$ then $\downarrow(A) \subseteq \downarrow(B)$;

(4) $If\ A \subseteq B$ then $\uparrow(A) \subseteq \uparrow(B)$;

(5) $\downarrow\downarrow(A) \subseteq \downarrow(B)$;

(6) $\uparrow\uparrow(A) \subseteq \uparrow(B)$;

(7) $If\ A \subseteq \downarrow(B)$ then $\downarrow(A) \subseteq \downarrow(B)$;

(8) $If\ A \subseteq\uparrow (B)$ then $\uparrow (A) \subseteq\uparrow (B)$.

Proposition 2.2. Let $(X, \leq)$ be a f-toset and let $\{A_j : j \in J\}$ be a family of sub-sets of X. Then:

(1) $\downarrow (\cup_{j\in J} A_j) = \cup_{j\in J} \downarrow (A_j)$;
(2) $\uparrow (\cup_{j\in J} A_j) = \cup_{j\in J} \uparrow (A_j)$;
(3) $\uparrow (\cap_{j\in J} A_j) = \cap_{j\in J} \uparrow (A_j)$; and
(4) $\downarrow (\cap_{j\in J} A_j) = \cap_{j\in J} \downarrow (A_j)$.

Definition 2.3. Let $(X, \leq)$ be a f-toset and let $A, B \subseteq X$. A is called:

(1) a directed (resp. filtered) subset iff $A \neq \phi$ and for every distinct points x, y in A, $\exists\ z \in A \cap\ ub(\{x, y\})$ (resp. $z \in A \cap\ lb(\{x, y\})$);

(2) a cofinal in B iff $A \subseteq B \subseteq\downarrow (A)$.

Proposition 2.3. Let $(X, \leq)$ be a f-toset and let $A, B \subseteq X$. If B is directed subset and cofinal in A, then A is directed subset and $sup(A) = sup(B)$.

Proof. First, we prove that A is a directed subset. Since $B \subseteq A$, then $A \neq \phi$. Let $l, m \in A$ s.t. $l \neq m$. Then $\exists\ b_1, b_2 \in B$ s.t. $r(b_1, l) \leq r(l, b_1), r(b_2, m) \leq r(m, b_2)$ and $b \in ub(\{b_1, b_2\}) \cap A$. Hence A is a directed subset.

Second, one can deduce that $ub(A) = ub(B)$ (Indeed, since $B \subseteq A$, then $ub(A) \subseteq ub(B)$. Conversely, $y \notin ub(A) \Rightarrow \exists\ a \in A$ s.t. $r(y, a) \nleq r(a, y) \Rightarrow \exists\ a \in\downarrow (B)$ s.t. $r(y, a) \nleq r(a, y) \Rightarrow \exists\ b \in B$ s.t. $r(b, a) \leq r(a, b)$ and $r(y, a) \nleq r(a, y) \Rightarrow\ b \in B$ s.t. $r(y, b) \nleq r(b, y) \Rightarrow y \notin ub(B)$. Hence $ub(B) \subseteq ub(A)$.). Thus sup(A) = sup(B).

The concept of way below relation is extended in f-toset as follows:

Definition 2.4. Let (X, r) be a f-toset and let $x, y \in X$, we say x is way below (resp. y is way above) y (resp. x), written $x \ll y$ iff for every directed subset D of X if $y \in\downarrow$ $(sup(D))$, there exists $d \in D$ s.t. $r(d, x) \leq r(x, d)$. The family of the elements in X each of which way above (resp. way below) x is denoted and defined as follows:

$\Uparrow x = \{y \in X : x \ll y\}$ (resp. $\Downarrow x = \{y \in X : y \ll x\}$).

Proposition 2.4. In f-toset $(X, \leq)$ let $x, y, z \in X$. Then:

(1) If $r(y, x) \leq r(x, y)$ and $y \ll z$, then $x \ll z$;
(2) If $x \ll y$ and $r(z, y) \leq r(y, z)$, then $x \ll z$;
(3) If $\sup(\{y\}) \neq \phi$ and $x \ll y$, then $r(y, x) \leq r(x, y)$; and
(4) If $\sup(\{y\}) \neq \phi$ or $\sup(\{z\}) \neq \phi$, $x \ll y$ and $y \ll z$, then $x \ll z$.

Proof. (1) Let D be a directed subset of X s.t. $z \in\downarrow (\sup(D))$. Then $\exists\ d \in D$ s.t. $r(d, y) \leq r(y, d)$. Then $r(d, x) \leq r(x, d)$ and hence $x \ll z$.

(2) Let D be a directed subset of X s.t. $z \in\downarrow (\sup(D))$. Then $\exists\ k \in \sup(D)$ s.t. $r(k, z) \leq r(z, k)$. Thus $r(k, y) \leq r(y, k)$ and so $y \in\downarrow (\sup(D))$. Therefore $\exists\ l \in D$ s.t. $r(l, x) \leq r(x, l)$. Hence $x \ll z$.

(3) Let $D = \{y\}$ and assume that $x \ll y$. Then $\exists\ d \in D$ s.t. $r(d, x) \leq r(x, d)$ but $y = d$. Thus $r(y, x) \leq r(x, y)$.

(4) From (1) -(3) above we can prove (4).

The domain f-toset is defined as follows:

Definition 2.5. A f-toset (X, r) is called a domain f-toset iff for every directed subset A of X, $\sup(A) \neq \phi$.

3. CONTINUOUS F-TOSETS

First we introduce the following needed lammas without proof.

Lemma 3.1. Let (X, r) be a f-toset. If $\forall\ x \in X$, $\Downarrow x$ is a directed subset of X, then $\forall\ z \in X$, $D = \cup\{\Downarrow a : a \in\Downarrow z\}$ is a directed subset.

Lemma 3.2. Let (X, r) be a f-toset and let $x \in X$. Then $\forall\ x \in X$, $ub(\cup\{\Downarrow a : a \in\Downarrow$

$x\}) = ub(\cup\{\sup(\Downarrow a) : a \in\Downarrow x\})$. Thus $\sup(\cup\{\Downarrow a : a \in\Downarrow x\}) = \sup(\cup\{\sup(\Downarrow a) : a \in\Downarrow x\})$.

Now , we present the concept of continuity of a f-toset.

Definition 3.1. Let $(X, \leq)$ be a f-toset. It is said to be a continuous f-toset iff the following conditions are satisfied:
(1) $\sup(\{x\}) \neq \phi$;
(2) $\Downarrow x$ is a directed subset of X; and
(3) $x \in\downarrow (\sup(\cup\{\sup(\Downarrow a) : a \in\Downarrow x\}))$.

Theorem 3.1 (Interpolation). If (X, r) is a continuous f-toset, then the way below relation $'\ll'$ is interpolative, i.e., $x, z \in X$, $x \ll z$ implies that $\exists\ y \in X$ s.t. $x \ll y \ll z$.

Proof. From Lemmas 3.1 and 3.2 , $z \in\downarrow (\sup(\cup\{\Downarrow y : y \in\Downarrow z\}))$ and $\cup\{\Downarrow y : y \in\Downarrow z\}$ is directed. Then $\exists\ d \in\Downarrow y$ for some $y \in\Downarrow z$ s.t. $r(d, x) \leq r(x, d)$. From Proposition 2.4(1), we have that $x \ll y$. Hence $x \ll y \ll z$.

From Proposition 2.4(4) and Theorem 3.1 one can have the following result concerning with continuous information system . For the definition of continuous information system see [9].

Theorem 3.2. If (X, r), is a continuous f-toset, then $(X, \ll)$ is a continuous information system.

Lemma 3.3. For any f-toset (X, r), if the conditions
$(A)\ \forall\ x \in X,\ \sup(\{x\}) \neq \phi$ and
$(B)\ '\ll'$ is interpolative are satisfied,
then $\forall\ x \in X,\ \Downarrow x = \cup\{\Downarrow a : a \in\Downarrow x\}$.

Proof. First, let $z \in \cup\{\Downarrow a : a \in\Downarrow x\}$. Then $\exists\ a \in \Downarrow x$ s.t. $z \ll a$. From Proposition 2.4(4), $z \ll x$, i.e., $z \in\Downarrow x$. Second, let . Then $z \ll x$. Since $'\ll'$ is interpolative, then $\exists\ a \in X$ s.t. $z \ll a \ll x$, i.e., $z \in \cup\{\Downarrow a : a \in\Downarrow x\}$.

Applying Lemma 3.2, Lemma 3.3, Theorem 3.1 and Theorem 3.2 we introduce the following characterization of continuous f-tosets.

Theorem 3.3. (X, r) is a continuous f-toset iff the following conditions are satisfied:
(1) $\forall\ x \in X,\ \sup(\{x\}) \neq \phi$;
(2) $\forall x \in X, \Downarrow x$ is directed;
(3) $\ll$ is interpolative ; and
(4) $\forall\ x \in X,\ x \in\downarrow (\sup(\Downarrow x))$.

Proof. First of all, we note that conditions (1) and (2) above are common.
$\Rrightarrow$: From Theorem 3.1, $'\ll'$ is interpolative so that Condition (3) above is satisfied . From Lemma 3.2 and Lemma 3.3, Condition (4) above is satisfied.
$\Lleftarrow$: From Lemma 3.3, one can have that Condition (3) in Definition 3.1 is satisfied.

In the following we add more characterizations of continuous f-tosets

Theorem 3.4. (X, r) is a continuous f-toset iff the following conditions are satisfied:
(1) $\forall\ x \in X, \sup(\{x\}) \neq \phi$;
(2) $\ll$ is interpolative ; and
(3) $\forall\ x \in X, \exists$ a directed subset D of $\Downarrow x$ s.t. $x \in\downarrow (\sup(D))$.

Proof. $\Rrightarrow$: From Theorem 3.3, Conditions (1) and (2) above are satisfied. Condition (3) is satisfied if we put $D =\Downarrow x$.
$\Lleftarrow$: Now Conditions (1) and (3) in Theorem 3.3 are given in Theorem 3.4 as (1) and (2) above. We need to prove that D is cofinal in $\Downarrow x$. First $D \subseteq\Downarrow x$ and D is directed. Second, let $y \in\Downarrow x$. Since $x \in\downarrow (\sup(D))$, then $\exists\ d \in D$ s.t. $r(d, y) \leq r(y, d)$. So, $y \in\downarrow (D)$. Then from Proposition 2.3, $\Downarrow x$ is directed and $\sup(\Downarrow x) = \sup(D)$. Hence Conditions (2) and (4) in Theorem 3.3 are satisfied.

Theorem 3.5. (X, r) is a continuous f-toset iff the following conditions are satisfied:

(1) $\forall\, x \in X, \sup(\{x\}) \neq \phi$; and

(2) $\forall\, x \in X, \exists$ a directed subset D of $\cup\{\Downarrow\ a : a \in\Downarrow x\}$ s.t. $x \in\downarrow (\sup(D))$.

Proof. $\underline{\Rightarrow}$: From Theorem 3.4 and Lemma 3.3 one can have that $\Downarrow x = \cup\{\Downarrow a : a \in\Downarrow x\}$ so that from Condition (3) in Theorem 3.4 one have directly Condition (2) in above.

$\underline{\Leftarrow}$: Since D is cofinal in $\cup\{\Downarrow\ a\ :\ a\ \in\Downarrow\ x\}$ (Indeed, $D\ \subseteq\ \cup\{\Downarrow\ a\ :\ a\ \in\Downarrow\ x\}$. Let $z\ \in\ \cup\{\Downarrow\ a\ :\ a\ \in\Downarrow\ x\}$. Then $z\ \ll\ a$ for some $a\ \in\Downarrow\ x$ so that from Proposition 2.4(4) $z \ll x$. Since D is directed and $x \in\downarrow (sup(D))$, then $\exists\ d \in D$ s.t. $r(d, z) \leq r(z, d)$, i.e., $z\ \in\downarrow\ (D)$.), then from Proposition 2.3, $\cup\{\Downarrow\ a\ :\ a\ \in\Downarrow\ x\}$ is directed and $\sup(D) = \sup(\cup\{\Downarrow\ a\ :\ a\ \in\Downarrow\ x\})$. Hence condition (3) in the Theorem 3.4 is satisfied. Also one can prove that $'\ll'$ is interpolative (Indeed, let $x \ll z$ and from Condition (2) above $z\ \in\downarrow\ (\sup(\cup\{\Downarrow\ a\ :\ a\ \in\Downarrow\ z\}))$. Thus $\exists\ d\ \in\ \cup\{\Downarrow\ a\ :\ a\ \in\Downarrow\ z\}$ s.t. $r(d, x)\ \leq\ r(x, d)$ and $d\ \ll\ a\ \ll\ z$. So , from Proposition 2.4(1), $x\ \ll\ a$. Then $x\ \ll\ a\ \ll\ z$.). Then Condition (2) in Theorem 3.4 is satisfied. Hence (X, r) is a continuous f-toset .

Remark 3.1. From Lemma 3.1, one can write $\Downarrow\ x$ in Theorem 3.5 instead of $\cup\{\Downarrow a : a \in\Downarrow x\}$.

The concept of a base for a f-toset is introduced as follows:

Definition 3.2. Let $(X, \ll)$ be a f-toset. A subset B of X is called a base for X iff the following conditions are satisfied:

(1) $\forall x \in X, \sup(\{x\}) \neq \phi$; and

(2) $\forall x\ \in\ X,\ \exists$ a directed subset D of B s.t. $D\ \subseteq\ \cup\{\Downarrow\ a\ :\ a\ \in\Downarrow\ x\}$ and $x\ \in\downarrow (\sup(D))$.

Finally, we ‘give a characterization of continuous f-toset via the concept of the base of a f-toset.

Theorem 3.6. $(X, \ll)$ is a continuous f-toset iff it has a base.

Proof. $\underline{\Rightarrow}$: From Theorem 3.5, put $B = \cup_{x\in X}(\cup\{\Downarrow a : a \in\Downarrow x\})$.

$\underline{\Leftarrow}$: Condition (2) in Theorem 3.5 is satisfied directly from the Definition of the base of a f-toset.

Conclusion. (1) Recently [13], the concept of continuity of some types of fuzzy directed complete posets was studied. In Wei Yao's paper [13] he proved the equivalence between the fuzzy partial order in the sense of Bĕlohlăvek [2, 3] and the fuzzy order in the sense of Fan and Zhang [6, 14, 16]. In the present paper we study above the continuity of fuzzy partial poset due to Claude Ponsard [4] with regard S. Abramsky and A. Jung's slogan [1].

(2) First we recall the definition of Fan and Zhang [6, 14, 16] for $L = [0, 1]$ and $* = \wedge = \min$.

Definition [6, 14, 16]**.** A Fan-Zhang-fuzzy partial order on a set X is function $e : X \times X \longrightarrow [0, 1]$ satisfying

(a) $\forall\ x \in X,\ e(x, x) = 1$,

(b) $\forall\ x, y, z \in X,\ e(x, y) \wedge e(y, z) \leq e(x, z)$,

(c) $\forall\ x, y \in X,\ e(x, y) = e(y, x) = 1$ implies $x = y$.

(3) The following counterexamples illustrate that the concept of fuzzy partial order in the sense of Fan and Zhang[6, 14, 16] and the concept of fuzzy order in the sense of Cloude Ponsard [4] are independent notions.

Counterexample 1. Let $X = \{x, y, z\}$ and $R_1 : X \times X \longrightarrow [0, 1]$ defined as follows: $R_1(x, x) = R_1(y, y) = R_1(z, z) = R_1(y, x) = R_1(x, z) = R_1(z, x) = R_1(z, y) = \frac{1}{4}$ and $R_1(x, y) = R_1(y, z) = \frac{1}{2}$. Since $\frac{1}{4} = R_1(x, z) \ngeq R_1(x, y) \wedge R_1(y, z) = \frac{1}{2}$, then R_1 is not fuzzy partial order in the sense of Fan and Zhang. One can check that R_1 is a fuzzy partial order in the sense of Claude Ponsard [4] (Remark that

$\frac{1}{2} = R_1(x,y) \geq R_1(y,x) = \frac{1}{4}$ and $\frac{1}{2} = R_1(y,z) \geq R_1(z,y) = \frac{1}{4}$ implies $\frac{1}{4} = R_1(x,z) \geq R_1(z,x) = \frac{1}{4}$.).

Counterexample 2. Let $X = \{x, y, z\}$ and $R_2 : X \times X \longrightarrow [0,1]$ defined as follows: $R_2(x,x) = R_2(y,y) = R_2(z,z) = 1$, $R_2(x,y) = R_2(y,x) = R_2(x,z) = R_2(z,x) = R_2(z,y) = \frac{1}{4}$ and $R_2(y,z) = \frac{1}{2}$. Since $R_2(x,y) \geq R_2(x,z) \wedge R_2(z,y)$, $R_2(y,x) \geq R_2(y,z) \wedge R_2(z,x)$, $R_2(x,z) \geq R_2(x,y) \wedge R_2(y,z)$, $R_2(z,x) \geq R_2(z,y) \wedge R_2(y,x)$, $R_2(z,y) \geq R_2(z,x) \wedge R_2(x,y)$, $R_2(y,z) \geq R_2(y,x) \wedge R_2(x,z)$. Then one can observe that R_2 is a fuzzy partial order in the sense of Fan and Zhang. Since,

then R_1 is not fuzzy partial order in the sense of Fan and Zhang. One can check that R_1 is a fuzzy partial order in the sense of Claude Ponsard [4] (Remark that $\frac{1}{2} = R_1(x,y) \geq R_1(y,x) = \frac{1}{4}$ and $\frac{1}{4} = R_1(y,z) \geq R_1(z,y) = \frac{1}{4}$ implies $\frac{1}{4} = R_1(x,z) \geq R_1(z,x) = \frac{1}{4}$.). $\frac{1}{4} = R_2(x,y) \geq R_2(y,x) = \frac{1}{4}$ and $\frac{1}{4} = R_2(z,x) \geq R_2(x,z) = \frac{1}{4}$ but $\frac{1}{4} = R_2(z,y) \ngeq R_2(y,z) = \frac{1}{2}$, then R_2 is not fuzzy partial order in the sense of Claude Ponsard.

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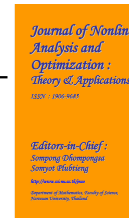

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AN OTHER APPROACH FOR THE PROBLEM OF FINDING A COMMON FIXED POINT OF A FINITE FAMILY OF NONEXPANSIVE MAPPINGS

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ABSTRACT. The purpose of this paper is to give a Tikhonov regularization method and some regularization inertial proximal point algorithm for the problem of finding a common fixed point of a finite family of nonexpansive mappings in a uniformly convex and uniformly smooth Banach space E, which admits a weakly sequentially continuous normalized duality mapping j from E to E^*.

KEYWORDS : Accretive operators; Uniformly smooth and uniformly convex Banach space; Sunny nonexpansive retraction; Weak sequential continuous mapping; Regularization.
AMS Subject Classification: 47H06 47H09 47H10 47J25

1. INTRODUCTION

Let E be a Banach space. We consider the following problem

$$\text{Finding an element } x^* \in S = \cap_{i=1}^N F(T_i), \tag{1.1}$$

where $F(T_i)$ is the set of fixed points of nonexpansive mappings $T_i : C \longrightarrow C$ and C is a closed convex nonexpansive retract subset of a uniformly convex and uniformly smooth Banach space E.

It is well-known that, numerous problems in mathematics and physical sciences can be recast in terms of a fixed point problem for nonexpansive mappings. For instance, if the nonexpansive mappings are projections onto some closed and convex sets, then the fixed point problem becomes the famous convex feasibility problem. Due to the practical importance of these problems, algorithms for finding fixed points of nonexpansive mappings continue to be flourishing topic of interest in fixed point theory. This problem has been investigated by many researchers: see, for instance, Bauschke [7], O' Hara et al. [22], Jung [16], Chang et al. [10], Takahashi and Shimoji [27], Ceng et al. [9], Chidume et al. [11, 12], Plubtieng and Ungchittrakool [23], Kang et al. [17], N. Buong et al. [8] and others.

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On the other hand, the problem of finding a fixed point of a nonexpansive mapping $T: E \longrightarrow E$ is equivalent to the problem of finding a zero of $m-$accretive $A = I - T$. One of the methods to solve the problem $0 \in A(x)$, where A is maximal monotone in a Hilbert space H is proximal point algorithm. This algorithm suggested by Rockafellar [24], starting from any initial guess $x_0 \in H$, this algorithm generates a sequence $\{x_n\}$ given by

$$x_{n+1} = J_{c_n}^A(x_n + e_n), \tag{1.2}$$

where $J_r^A = (I + rA)^{-1}\ \forall r > 0$ is the resolvent of A on the space H. Rockafellar [24] proved the weak convergence of the algorithm (1.2) provided that the regularization sequence $\{c_n\}$ remains bounded away from zero and the error sequence $\{e_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \|e_n\| < \infty$. However Güler's example [15] shows that in infinite dimensional Hilbert space, proximal point algorithm (1.2) has only weak convergence.

Ryazantseva [25] extended the proximal point algorithm (1.2) for the case that A is an $m-$accretive mapping in a properly Banach space E and proved the weak convergence of the sequence generated by (1.2) to a solution of the equation $0 \in A(x)$ which is assumed to be unique. Then, to obtain the strong convergence for algorithm (1.2), Ryazantseva [26] combined the proximal algorithm with the regularization, named regularization proximal algorithm, in the form

$$c_n(A(x_{n+1}) + \alpha_n x_{n+1}) + x_{n+1} = x_n,\ x_0 \in E. \tag{1.3}$$

Under some conditions on c_n and α_n, the strong convergence of $\{x_n\}$ of (1.3) is guaranteed only when the dual mapping j is weak sequential continuous and strong continuous, and the sequence $\{x_n\}$ is bounded.

Attouch and Alvarez [6] considered an extension of the proximal point algorithm (1.2) in the form

$$c_n A(u_{n+1}) + u_{n+1} - u_n = \gamma_n(u_n - u_{n-1}),\ u_0,\ u_1 \in H, \tag{1.4}$$

which is called an inertial proximal point algorithm, where $\{c_n\}$ and $\{\gamma_n\}$ are two sequences of positive numbers. With this algorithm we also only obtained weak convergence of the sequence $\{x_n\}$ to a solution of problem $A(x) \ni 0$ in Hilbert spaces. Note that this algorithm was proposed by Alvarez in [2] in the context of convex minimization.

Then, Moudafi [19] applied this algorithm for variational inequalities, Moudafi and Elisabeth [20] studied this algorithm by using enlargement of a maximal monotone operator, and Moudafi and Oliny [21] considered convergence of a splitting inertial proximal method. The main result in these papers is also the weak convergence of the algorithm in Hilbert spaces.

In this paper, we introduced the algorithms in the forms

$$\sum_{i=1}^{N} A_i(x_n) + \alpha_n(x_n - y) = 0, \tag{1.5}$$

$$c_n(\sum_{i=1}^{N} A_i(u_{n+1}) + \alpha_n(u_{n+1} - y)) + u_{n+1} = Q_C(u_n + \gamma_n(u_n - u_{n-1})), \tag{1.6}$$

where $y,\ u_0,\ u_1 \in C$, and $Q_C: E \longrightarrow C$ is a sunny nonexpansive retraction from E onto C to solve the problem (1.1).

And also, we give some analogue regularization methods for the more general problems, such as: the problem of finding a common fixed point of a finite family of nonexpansive nonself - mapping on a closed and convex subset of E. Finally, the stability of the regularization algorithms are considered in this paper.

2. PRELIMINARIES

Let E be a Banach space with its dual space E^*. For the sake of simplicity, the norms of E and E^* are denoted by the same symbol $\|.\|$. We write $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. We use the symbols $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ and $\longrightarrow$ to denote the weak convergence, weak* convergence and strong convergence, respectively.

Definition 2.1. A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0,2]$ the inequalities $\|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon$ imply there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\frac{\|x+y\|}{2} \leq 1 - \delta.$$

The function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1}\|x+y\| :\ \|x\| = \|y\| = 1,\ \|x-y\| = \varepsilon\} \tag{2.1}$$

is called the modulus of convexity of the space E. The function $\delta_E(\varepsilon)$ defined on the interval $[0,2]$ is continuous, increasing and $\delta_E(0) = 0$. The space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0,\ \forall \varepsilon \in (0,2]$.
The function

$$\rho_E(\tau) = \sup\{2^{-1}\big(\|x+y\| + \|x-y\|\big) - 1 :\ \|x\| = 1,\ \|y\| = \tau\}, \tag{2.2}$$

is called the modulus of smoothness of the space E. The function $\rho_E(\tau)$ defined on the interval $[0,+\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$.

Definition 2.2. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0. \tag{2.3}$$

It is well known that every uniformly convex and uniformly smooth Banach space is reflexive. In what follows, we denote

$$h_E(\tau) := \frac{\rho_E(\tau)}{\tau}. \tag{2.4}$$

The function $h_E(\tau)$ is nondecreasing. In addition, we have the following estimate

$$h_E(K\tau) \leq LKh_E(\tau),\ \forall K > 1,\ \tau > 0, \tag{2.5}$$

where L is the Figiel's constant [3, 4, 13], $1 < L < 1.7$.

Definition 2.3. A mapping j from E onto E^* satisfying the condition

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\} \tag{2.6}$$

is called the normalized duality mapping of E.

In any smooth Banach space $J(x) = 2^{-1}\text{grad}\|x\|^2$ and, if E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E^* is stricly convex or E is smooth, then J is single valued. Suppose that J be single valued, then J is said to be weakly sequentially continuous if for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x,\ J(x_n) \overset{*}{\rightharpoonup} J(x)$. We denote the single valued normalized duality mapping by j.

Definition 2.4. An operator $A :\ D(A) \subseteq E \rightrightarrows E$ is called accretive if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0,\ \forall u \in A(x),\ v \in A(y). \tag{2.7}$$

Definition 2.5. A mapping $T: \ C \longrightarrow E$ is said to be nonexpansive on a closed and convex subset C of Banach space E if

$$\|T(x) - T(y)\| \leq \|x - y\|, \ \forall x, y \in C. \tag{2.8}$$

It is clear that, if $T: \ C \longrightarrow E$ is a nonexpansive, then $I - T$ is accretive operator.

Definition 2.6. Let G be a nonempty closed and convex subset of E. A mapping $Q_G: \ E \longrightarrow G$ is said to be

i) a retraction onto G if $Q_G^2 = Q_G$;

ii) a nonexpansive retraction if it also satisfies the inequality

$$\|Q_G x - Q_G y\| \leq \|x - y\|, \ \forall x, y \in E; \tag{2.9}$$

iii) a sunny retraction if for all $x \in E$ and for all $t \in [0, +\infty)$,

$$Q_G(Q_G x + t(x - Q_G x)) = Q_G x. \tag{2.10}$$

A closed and convex subset C of E is said to be a nonexpansive retract of E, if there exists a nonexpansive retraction from E onto C and is said to be a sunny nonexpansive retract of E, if there exists a sunny nonexpansive retraction from E onto C.

Proposition 2.7. [14] *Let C be a nonempty closed convex subset of a smooth Banach E. A mapping $Q_C: \ E \longrightarrow C$ is a sunny nonexpansive retraction if and only if*

$$\langle x - Q_C x, j(\xi - Q_C x)\rangle \leq 0, \ \forall x \in E, \ \forall \xi \in C. \tag{2.11}$$

Definition 2.8. Let $C_1, \ C_2$ be convex subsets of E. The quantity

$$\beta(C_1, C_2) = \sup_{u \in C_1} \inf_{v \in C_2} \|u - v\| = \sup_{u \in C_1} d(u, C_2)$$

is said to be semideviation of the set C_1 from the set C_2. The function

$$\mathcal{H}(C_1, C_2) = \max\{\beta(C_1, C_2), \ \beta(C_2, C_1)\}$$

is said to be a Hausdorff distance between C_1 and C_2.

Lemma 2.9. [5] *If E is a uniformly smooth Banach space, C_1 and C_2 are closed and convex subsets of E such that the Hausdorff $\mathcal{H}(C_1, C_2) \leq \delta$, and Q_{C_1} and Q_{C_2} are the sunny nonexpansive retractions onto the subsets C_1 and C_2, respectively, then*

$$\|Q_{C_1} x - Q_{C_2} x\|^2 \leq 16R(2r + d) h_E(\frac{16L\delta}{R}), \tag{2.12}$$

where L is Figiel's constant, $r = \|x\|, \ d = \max\{d_1, d_2\}$, and $R = 2(2r + d) + \delta$. Here $d_i = dist(\theta, C_i)$, $i = 1, 2$, and θ is the origin of the space E.

3. MAIN RESULTS

First, we need the following lemmas in the proof of our results.

Lemma 3.1. [3] *Let E be a uniformly convex and uniformly smooth Banach space. If $A = I - T$ with a nonexpansive mapping T then for all $x, y \in D(T)$, the domain of T,*

$$\langle Ax - Ay, j(x - y)\rangle \geq L^{-1} R^2 \delta_E \left(\frac{\|Ax - Ay\|}{4R} \right), \tag{3.1}$$

where $\|x\| \leq R, \ \|y\| \leq R$ and $1 < L < 1.7$ is Figiel constant.

Lemma 3.2 (demiclosedness principle)**.** [1] *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping, C a nonempty closed convex subset of E, and $T : C \longrightarrow E$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, $x_n \rightharpoonup x$ in E and $(I - T)x_n \longrightarrow y$ imply that $x \in C$ and $(I - T)x = y$.*

Lemma 3.3. [28] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n,\ \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0, 1)$ for each $n \geq 0$ such that (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose either (a) $\sigma_n = o(\alpha_n)$, or (b) $\sum_{n=1}^{\infty} |\sigma_n| < \infty$, or (c) $\limsup \dfrac{\sigma_n}{\alpha_n} \leq 0$. Then $a_n \to 0$ as $n \to \infty$.

Lemma 3.4. [18] *Let C be a closed convex subset of a strictly convex Banach space E and let $T : C \longrightarrow E$ be a nonexpansive mapping from C into E. Suppose that C is be sunny nonexpansive retract of E. If $F(T) \neq \emptyset$, then $F(T) = F(Q_C T)$, where Q_C is a sunny nonexpansive retraction from E onto C.*

Theorem 3.5. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $T_i : C \longrightarrow C$, $i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Then*

i) *For each $\alpha_n > 0$ the equation (1.5) has unique solution x_n;*
ii) *If the sequence of positive numbers $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $Q_S y$, where $Q_S : E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.*

Moreover, we have the following estimate

$$\|x_{n+1} - x_n\| \leq \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n} R_0\ \forall n \geq 0, \tag{3.2}$$

where $R_0 = 2\|y - Q_S y\|$.

Proof. i) For each $n \geq 0$, equation (1.5) defines unique sequence $\{x_n\} \subset E$, because for each n, the element x_n is unique fixed point of the contraction mapping $T : C \longrightarrow C$ defined by

$$T(x) = \frac{1}{N + \alpha_n} \sum_{i=1}^{N} T_i(x) + \frac{\alpha_n}{N + \alpha_n} y. \tag{3.3}$$

ii) From equation (1.5), we have

$$\langle \sum_{i=1}^{N} A_i(x_n), j(x_n - x^*)\rangle + \alpha_n \langle x_n - y, j(x_n - x^*)\rangle = 0,\ \forall x^* \in S. \tag{3.4}$$

By virtue of the property of $\sum_{i=1}^{N} A_i$ and j, we obtain

$$\langle \sum_{i=1}^{N} A_i(x_n), j(x_n - x^*)\rangle \geq 0,\ \forall x^* \in S.$$

Thus,

$$\langle x_n - y, j(x_n - x^*)\rangle \leq 0,\ \forall x^* \in S. \tag{3.5}$$

From inequality (3.5), we get

$$\|x_n - x^*\|^2 \le \langle y - x^*, j(x_n - x^*)\rangle \le \|y - x^*\|.\|x_n - x^*\|, \ \forall x^* \in S. \tag{3.6}$$

Therefore

$$\|x_n - x^*\| \le \|y - x^*\|, \ \forall n \ge 0, \ \forall x^* \in S, \tag{3.7}$$

that implies the boundedness of the sequence $\{x_n\}$. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists some subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ which converges weakly to a limit point $\overline{x}$. Since C is closed and convex, it is also weakly closed. Therefore $\overline{x} \in C$.

We will show that $\overline{x} \in S$. Indeed, for each $i \in \{1, 2, ..., N\}$, $x^* \in S$ and $R > 0$ satisfy $R \ge \max\{\sup \|x_n\|, \|x^*\|\}$, we have

$$\begin{aligned} \delta_E\Big(\frac{\|A_i(x_n)\|}{4R}\Big) &\le \frac{L}{R^2}\langle A_i(x_n), j(x_n - x^*)\rangle \\ &\le \frac{L}{R^2}\langle \sum_{k=1}^{N} A_k(x_n), j(x_n - x^*)\rangle \\ &\le \frac{L\alpha_n}{R^2}\|x_n - y\|.\|x_n - x^*\| \\ &\le \frac{L\alpha_n}{R^2}(R + \|y\|).\|y - x^*\| \longrightarrow 0, \ n \longrightarrow \infty. \end{aligned}$$

Since modulus of convexity δ_E is continuous and E is the uniformly convex Banach space, $A_i(x_n) \longrightarrow 0, \ n \longrightarrow \infty$. From Lemma 3.2, it implies that $A_i(\overline{x}) = 0$. Since $i \in \{1, 2, ..., N\}$ is an arbitrary element, we obtain $\overline{x} \in S$.

In inequality (3.6) replacing x_n by x_{n_k} and x^* by $\overline{x}$, using the weak continuity of j we obtain $x_{n_k} \longrightarrow \overline{x}$. From inequality (3.5), we get

$$\langle \overline{x} - y, j(\overline{x} - x^*)\rangle \le 0, \ \forall x^* \in S. \tag{3.8}$$

Now, we show that the inequality (3.8) has unique solution. Suppose that $\overline{x}_1 \in S$ is also its solution. Then

$$\langle \overline{x}_1 - y, j(\overline{x}_1 - x^*)\rangle \le 0, \ \forall x^* \in S. \tag{3.9}$$

In inequalities (3.8) and (3.9) replacing x^* by $\overline{x}_1$ and $\overline{x}$, respectively, we obtain

$$\begin{aligned} &\langle \overline{x} - y, j(\overline{x} - \overline{x}_1)\rangle \le 0, \\ &\langle y - \overline{x}_1, j(\overline{x} - \overline{x}_1)\rangle \le 0. \end{aligned}$$

Their combination gives $\|\overline{x} - \overline{x}_1\|^2 \le 0$, thus $\overline{x} = \overline{x}_1 = Q_S y$ and the sequence $\{x_n\}$ converges weakly to $\overline{x} = Q_S y$, because $Q_S y$ satisfies the inequality (3.8). Finally, from the first inequality in (3.6), implies that $x_n \longrightarrow Q_S y$.

Now, we prove the inequality (3.2). In equation (1.5) replacing n by $n+1$ we have

$$\sum_{i=1}^{N} A_i(x_{n+1}) + \alpha_{n+1}(x_{n+1} - y) = 0. \tag{3.10}$$

From (3.10) and (1.5), we get

$$\langle \alpha_{n+1}x_{n+1} - \alpha_n x_n, j(x_{n+1} - x_n)\rangle \le (\alpha_{n+1} - \alpha_n)\langle y, j(x_{n+1} - x_n)\rangle. \tag{3.11}$$

Therefore,

$$\begin{aligned} \alpha_n\|x_{n+1} - x_n\|^2 &\le (\alpha_{n+1} - \alpha_n)\langle y - x_{n+1}, j(x_{n+1} - x_n)\rangle \\ &\le |\alpha_{n+1} - \alpha_n|.\|y - x_{n+1}\|.\|x_{n+1} - x_n\| \\ &\le 2\|y - Q_S y\|.|\alpha_{n+1} - \alpha_n|.\|x_{n+1} - x_n\|. \end{aligned}$$

Thus,

$$\|x_{n+1} - x_n\| \leq \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n} R_0 \ \forall n \geq 0,$$

where $R_0 = 2\|y - Q_S y\|$. □

Theorem 3.6. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $T_i : C \longrightarrow C$, $i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \neq \emptyset$. If the sequences $\{c_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy*

i) $0 < c_0 < c_n,\ \alpha_n > 0, \alpha_n \longrightarrow 0,\ \dfrac{|\alpha_{n+1} - \alpha_n|}{\alpha_n^2} \longrightarrow 0,\ \sum_{n=0}^{\infty} \alpha_n = +\infty$;

ii) $\gamma_n \geq 0,\ \gamma_n \alpha_n^{-1} \|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u_n\}$ defined by (1.6) converges strongly to $Q_S y$, where $Q_S : E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Proof. First, for each $n \geq 1$, equation (1.6) defines unique sequence $\{u_n\} \subset E$, because for each n, the element u_{n+1} is unique fixed point of the contraction mapping $f : C \longrightarrow C$ defined by

$$f(x) = \frac{c_n}{c_n(N + \alpha_n) + 1} \sum_{i=1}^{N} T_i(x) + \frac{c_n \alpha_n}{c_n(N + \alpha_n) + 1} y + \frac{1}{c_n(N + \alpha_n) + 1} z, \quad (3.12)$$

where $z = Q_C(u_n + \gamma_n(u_n - u_{n-1})) \in C$.

Now, we rewrite equations (1.5) and (1.6) in their equivalent forms

$$d_n \sum_{i=1}^{N} A_i(x_n) + x_n - y = \beta_n(x_n - y), \quad (3.13)$$

$$d_n \sum_{i=1}^{N} A_i(u_{n+1}) + u_{n+1} - y = \beta_n[Q_C(u_n + \gamma_n(u_n - u_{n-1})) - y], \quad (3.14)$$

where $\beta_n = \dfrac{1}{1 + c_n \alpha_n}$ and $d_n = c_n \beta_n$.

From (3.13), (3.14) and by virtue of the property of $\sum_{i=1}^{N} A_i$, we get

$$\begin{aligned} \|u_{u+1} - x_n\| &\leq \beta_n \|Q_C(u_n + \gamma_n(u_n - u_{n-1})) - x_n\| \\ &= \beta_n \|Q_C(u_n + \gamma_n(u_n - u_{n-1})) - Q_C(x_n)\| \\ &\leq \beta_n \|u_n - x_n\| + \beta_n \gamma_n \|u_n - u_{n-1}\|. \end{aligned}$$

Thus,

$$\begin{aligned} \|u_{n+1} - x_{n+1}\| &\leq \|u_{n+1} - x_n\| + \|x_{n+1} - x_n\| \\ &\leq \beta_n \|u_n - x_n\| + \beta_n \gamma_n \|u_n - u_{n-1}\| + \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n} R, \end{aligned} \quad (3.15)$$

or equivalent to

$$\|u_{n+1} - x_{n+1}\| \leq (1 - b_n)\|u_n - x_n\| + \sigma_n,\ b_n = \frac{c_n \alpha_n}{1 + c_n \alpha_n}, \quad (3.16)$$

where $\sigma_n = \beta_n\gamma_n\|u_n - u_{n-1}\| + \dfrac{|\alpha_{n+1}-\alpha_n|}{\alpha_n}R$.
By the assumption, we have

$$\begin{aligned}\frac{\sigma_n}{b_n} &= \frac{1}{c_n}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_n}+1)\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2}R \\ &\le \frac{1}{c_0}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_0}+1)\frac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2}R \longrightarrow 0.\end{aligned}$$

Furthermore, since $\sum_{n=0}^{\infty}\alpha_n = +\infty$, $\sum_{n=0}^{\infty} b_n = +\infty$.
By Lemma 3.3, we obtain $\|u_n - x_n\| \longrightarrow 0$. Since $x_n \longrightarrow Q_S y$, $u_n \longrightarrow Q_S y$. □

Corollary 3.7. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let $T_i: E \longrightarrow E$, $i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \neq \emptyset$. If the sequences $\{c_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy*

i) $0 < c_0 < c_n,\ \alpha_n > 0, \alpha_n \longrightarrow 0,\ \dfrac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2} \longrightarrow 0,\ \sum_{n=0}^{\infty}\alpha_n = +\infty$;
ii) $\gamma_n \ge 0, \gamma_n\alpha_n^{-1}\|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u_n\}$ defined by

$$c_n(\sum_{i=1}^N A_i(u_{n+1}) + \alpha_n u_{n+1}) + u_{n+1} = u_n + \gamma_n(u_n - u_{n-1}),\ u_0,\ u_1 \in E$$

converges strongly to $Q_S\theta$, where $Q_S: E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Proof. Applying Theorem 3.6 for $C = E$ and $y = \theta$, we obtain the proof of this corollary. □

Corollary 3.8. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $f_i: C \longrightarrow E$, $i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(f_i) \neq \emptyset$. If the sequences $\{c_n\}$, $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy*

i) $0 < c_0 < c_n,\ \alpha_n > 0, \alpha_n \longrightarrow 0,\ \dfrac{|\alpha_{n+1}-\alpha_n|}{\alpha_n^2} \longrightarrow 0,\ \sum_{n=0}^{\infty}\alpha_n = +\infty$;
ii) $\gamma_n \ge 0, \gamma_n\alpha_n^{-1}\|u_n - u_{n-1}\| \longrightarrow 0$,

then the sequence $\{u_n\}$ defined by

$$c_n(\sum_{i=1}^N B_i(u_{n+1}) + \alpha_n(u_{n+1} - y)) + u_{n+1} = Q_C(u_n + \gamma_n(u_n - u_{n-1})), \tag{3.17}$$

converges strongly to $Q_S y$, where $B_i = I - Q_C f_i$, $i = 1, 2, ..., N$, Q_C is a sunny nonexpansive retraction from E onto C, Q_S is a sunny nonexpansive retraction from E onto S, and y, u_0, $u_1 \in C$.

Proof. By Lemma 3.4, we have $S = \cap_{i=1}^N F(Q_C f_i)$. Applying Theorem 3.6 for $T_i = Q_C f_i$, $i = 1, 2, ..., N$ we obtain the proof of this corollary. □

Finally, we study stability of the algorithms (1.5) and (1.6) with respect to perturbations of both operators T_i and constraint set C satisfying the following conditions:

(P1) Instead of C, there is a sequence of closed convex sunny nonexpasive retract subsets $C_n \subset E,\ n = 1, 2, 3, ...$ such that the Hausdorff $\mathcal{H}(C_n, C) \leq \delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers with the propertie

$$\delta_{n+1} \leq \delta_n,\ \forall n \geq 1. \tag{3.18}$$

(P2) On the each set C_n, there is a nonexpansive self-mapping $T_i^n :\ C_n \longrightarrow C_n,\ i = 1, 2, ..., N$ satisfying the conditions: there exists the increasing positive for all $t > 0$ function $g(t)$ and $\xi(t)$ such that $g(0) \geq 0,\ \xi(0) = 0$ and $x \in C_k,\ y \in C_m,\ \|x - y\| \leq \delta$, then

$$\|T_i^k x - T_i^m y\| \leq g(\max\{\|x\|, \|y\|\})\xi(\delta). \tag{3.19}$$

In this paper, we establish the convergence and stability of the Tikhonov regularization method (1.5) and the regularization inertial proximal point algorithm (1.6) in the forms

$$\sum_{i=1}^{N} A_i^n(z_n) + \alpha_n(z_n - Q_{C_n}y) = 0, \tag{3.20}$$

$$c_n(\sum_{i=1}^{N} A_i^n(u_{n+1}) + \alpha_n(u_{n+1} - Q_{C_n}y)) + u_{n+1} = Q_{C_n}(u_n + \gamma_n(u_n - u_{n-1})), \tag{3.21}$$

respectively, where $u_0,\ u_1$ and y are elements in E, and $A_i^n = I - T_i^n,\ i = 1, 2, ..., N$, with respect to perturbations of the set C, and $Q_{C_n} :\ E \longrightarrow C_n$ is the sunny nonexpansive retraction of E onto C_n.

Theorem 3.9. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $T_i :\ C \longrightarrow C,\ i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \neq \emptyset$.*

i) *For each $\alpha_n > 0$ equation (3.20) has unique solution z_n;*

ii) *If the conditions (P1) and (P2) are fulfilled and the sequences of positive numbers $\{\alpha_n\},\ \{\delta_n\}$ satisfy*

$$\alpha_n \longrightarrow 0,\ \frac{\delta_n + \xi(\delta_n)}{\alpha_n} \longrightarrow 0,\ \text{as } n \longrightarrow \infty, \tag{3.22}$$

then $\{z_n\}$ converges strongly to $Q_S(Q_C y)$, where $Q_S :\ E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Moreover, if $\{\alpha_n\}$ is a decreasing sequence, then we have the following estimate

$$\begin{aligned} \|z_{n+1} - z_n\| \leq 4\delta_n + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n} + \frac{\alpha_n - \alpha_{n+1}}{\alpha_n}R \\ + K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)},\ \forall n \geq 0, \end{aligned} \tag{3.23}$$

where $R,\ K,\ K_3,\ K_4$ are any constants.

Proof. i) For each $n \geq 0$, by an argument similar to the proof of Theorem 3.5, it follows that, the equation (3.20) has a unique solution z_n.
ii) Since the distance Hausdorff $\mathcal{H}(C_n, C) \leq \delta_n$, therefore for each solution x_n of equation (1.5) (note that, in the case that the element y in (1.5) is replaced by $Q_C y$),

there exists an element $u_n \in C_n$ such that $\|x_n - u_n\| \le \delta_n$.
From equations (1.5) and (3.20), we have

$$\sum_{i=1}^{N}(A_i^n(z_n) - A_i^n(u_n)) + \alpha_n(z_n - x_n) - \alpha_n(Q_{C_n}y - Q_C y) + \sum_{i=1}^{N}(A_i^n(u_n) - A_i(x_n)) = 0. \tag{3.24}$$

By virtue of the property of $\sum_{i=1}^{N} A_i^n$ and j, we get

$$\langle\sum_{i=1}^{N}(A_i^n(z_n) - A_i^n(u_n)), j(z_n - u_n)\rangle \ge 0,$$

that implies

$$\alpha_n\langle z_n - x_n, j(z_n - u_n)\rangle \le \alpha_n\langle Q_{C_n}y - Q_C y, j(z_n - u_n)\rangle + \langle\sum_{i=1}^{N}(A_i(x_n) - A_i^n(u_n)), j(z_n - u_n)\rangle. \tag{3.25}$$

Thus,

$$\begin{aligned}\alpha_n\|z_n - u_n\| &\le \alpha_n\|x_n - u_n\| + \alpha_n\|Q_{C_n}y - Q_C y\| + \sum_{i=1}^{N}\|A_i(x_n) - A_i^n(u_n)\| \\ &\le \alpha_n\delta_n + \alpha_n\|Q_{C_n}y - Q_C y\| + \sum_{i=1}^{N}\|A_i(x_n) - A_i^n(u_n)\|.\end{aligned}$$

Since $\mathcal{H}(C_n, C) \le \delta_n$, there exists constants $K_1 > 0$ and $K_2 > 1$ such that the inequalities

$$\|Q_{C_n}y - Q_C y\| \le K_1\sqrt{h_E(K_2\delta_n)} \le K_1\sqrt{LK_2}\sqrt{h_E(\delta_n)}$$

hold.
Next, for each $i \in \{1, 2, ..., N\}$, we have

$$\begin{aligned}\|A_i(x_n) - A_i^n(u_n)\| &\le \delta_n + g(\max\{\|x_n\|, \|u_n\|\})\xi(\delta_n) \\ &\le \delta_n + g(M)\xi(\delta_n),\end{aligned}$$

where $M = \max\{\sup\|x_n\|, \sup\|u_n\|\} < +\infty$.
Consequently,

$$\alpha_n\|z_n - u_n\| \le \alpha_n\delta_n + \alpha_n K_1\sqrt{LK_2}\sqrt{h_E(\delta_n)} + N(\delta_n + g(M)\xi(\delta_n)). \tag{3.26}$$

Thus,

$$\begin{aligned}\|z_n - x_n\| &\le \|z_n - u_n\| + \|x_n - u_n\| \\ &\le 2\delta_n + K_1\sqrt{LK_2}\sqrt{h_E(\delta_n)} + N\frac{\delta_n + g(M)\xi(\delta_n)}{\alpha_n}.\end{aligned} \tag{3.27}$$

Since $\alpha_n \longrightarrow 0$, $\dfrac{\delta_n + \xi(\delta_n)}{\alpha_n} \longrightarrow 0$, hence $\delta_n \longrightarrow 0$ and $h_E(\delta_n) \longrightarrow 0$. By inequality (3.27), we obtain $\|x_n - z_n\| \longrightarrow 0$. By Theorem 3.5, it implies that $x_n \longrightarrow Q_S(Q_C y)$, thus the sequence $\{z_n\}$ also converges strongly to $Q_S(Q_C y)$.

Finally, we prove the inequality (3.23). In equation (3.20) replacing n by $n+1$, we have

$$\sum_{i=1}^{N} A_i^{n+1}(z_{n+1}) + \alpha_n(z_{n+1} - Q_{C_{n+1}}y) = 0. \tag{3.28}$$

Since

$$\mathcal{H}(C_n, C_{n+1}) \leq \mathcal{H}(C_n, C) + \mathcal{H}(C, C_{n+1}) \leq 2\delta_n,$$

we assert that for any $z_{n+1} \in C_{n+1}$ there exists an element $v_n \in C_n$ such that $\|z_{n+1} - v_n\| \leq 2\delta_n$.

From equations (3.20) and (3.28), we obtain

$$\sum_{i=1}^{N}(A_i^n(z_n) - A_i^n(v_n)) + \alpha_n(z_n - Q_{C_n}y) - \alpha_{n+1}(z_{n+1} - Q_{C_{n+1}}y) + \sum_{i=1}^{N}(A_i^n(v_n) - A_i^{n+1}(z_{n+1})) = 0.$$

By virtue of the property of $\sum_{i=1}^{N} A_i^n$ and j, we get

$$\begin{aligned} \alpha_n\|z_n - v_n\| &\leq \alpha_{n+1}\|v_n - z_{n+1}\| + |\alpha_n - \alpha_{n+1}|.\|v_n - Q_{C_n}y\| \\ &+ \alpha_{n+1}\|Q_{C_n}y - Q_{C_{n+1}}y\| + \sum_{i=1}^{N}\|A_i^n(v_n) - A_i^{n+1}(z_{n+1})\| \end{aligned} \tag{3.29}$$

Since $\mathcal{H}(C_n, C_{n+1}) \leq 2\delta_n$, there exists constants $K_3 > 0$ and $K_4 > 1$ such that the inequalities

$$\|Q_{C_n}y - Q_{C_{n+1}}y\| \leq K_3\sqrt{h_E(K_4\delta_n)} \leq K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)} \tag{3.30}$$

hold.

Since $v_n \in C_n$, therefore

$$\|v_n - Q_{C_n}y\| \leq \|v_n - y\| \leq \sup\|z_n\| + \|y\| + 2\delta_1 := R. \tag{3.31}$$

Next, for each $i \in \{1, 2, ..., N\}$, we have

$$\begin{aligned} \|A_i^n(v_n) - A_i^{n+1}(z_{n+1})\| &\leq 2\delta_n + \|T_i^n(v_n) - T_i^{n+1}(z_{n+1})\| \\ &\leq 2\delta_n + g(\max\{\|v_n\|, \|z_{n+1}\|\})\xi(2\delta_n) \\ &\leq 2\delta_n + g(M')\xi(2\delta_n), \end{aligned} \tag{3.32}$$

where $M' = \max\{\sup\|v_n\|, \sup\|z_n\|\} < +\infty$.

Combining (3.29), (3.30), (3.31) and (3.32), we obtain

$$\|z_n - v_n\| \leq 2\delta_n + K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)} + R\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n}, \tag{3.33}$$

where $K = \max\{2N, Ng(M')\}$.

Consequently,

$$\|z_{n+1} - z_n\| \leq 4\delta_n + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n} + R\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} + K_3\sqrt{LK_4}\sqrt{h_E(\delta_n)}. \tag{3.34}$$

□

Next, we will prove the strong convergence and stability of regularization inertial proximal point algorithm (3.21) by the following theorem.

Theorem 3.10. *Let E be a uniformly convex and uniformly smooth Banach space which admits a weakly sequentially continuous normalized duality mapping j from E to E^*. Let C be a nonempty closed convex sunny nonexpansive retract of E and let $T_i : C \longrightarrow C$, $i = 1, 2, ..., N$ be nonexpansive mappings such that $S = \cap_{i=1}^N F(T_i) \neq \emptyset$. If the conditions (P1) and (P2) are fulfilled, and the sequences $\{\alpha_n\}$, $\{\delta_n\}$, $\{\tilde{c}_n\}$ and $\{\gamma_n\}$ satisfy*

i) $\alpha_n \searrow 0,\ \dfrac{\alpha_n - \alpha_{n+1}}{\alpha_n^2} \longrightarrow 0,$ *as* $n \longrightarrow \infty,\ \sum_{n=1}^\infty \alpha_n = +\infty,$

ii) $\dfrac{\delta_n + \xi(2\delta_n)}{\alpha_n^2} \longrightarrow 0,\ \dfrac{\sqrt{h_E(\delta_n)}}{\alpha_n} \longrightarrow 0,$ *as* $n \longrightarrow \infty,$

iii) $0 < c_0 < c_n,\ \gamma_n \geq 0,\ \gamma_n \alpha_n^{-1} \|u_n - u_{n-1}\| \longrightarrow 0,$ *as* $n \longrightarrow \infty,$

then the sequence $\{u_n\}$ defined by (3.21) converges strongly to $Q_S(Q_C y)$, where $Q_S : E \longrightarrow S$ is a sunny nonexpansive retraction from E onto S.

Proof. First, for each n by an argument similar to the proof of Theorem 3.6, it follows that, the equation (3.21) has unique solution $u_{n+1} \in C_n$.

Now, we rewrite equations (3.20) and (3.21) in their equivalent forms

$$d_n \sum_{i=1}^N A_i^n(z_n) + z_n - Q_{C_n} y = \beta_n (z_n - Q_{C_n} y), \tag{3.35}$$

$$d_n \sum_{i=1}^N A_i^n(u_{n+1}) + u_{n+1} - Q_{C_n} y = \beta_n [Q_{C_n}(u_n + \gamma_n(u_n - u_{n-1})) - Q_{C_n} y], \tag{3.36}$$

where $\beta_n = \dfrac{1}{1 + c_n \alpha_n}$ and $d_n = c_n \beta_n$.

From (3.35), (3.36) and by virtue of the property of $\sum_{i=1}^N A_i^n$, we have

$$\begin{aligned} \|u_{u+1} - z_n\| &\leq \beta_n \|Q_{C_n}(u_n + \gamma_n(u_n - u_{n-1})) - z_n\| \\ &= \beta_n \|Q_{C_n}(u_n + \gamma_n(u_n - u_{n-1})) - Q_{C_n}(z_n)\| \\ &\leq \beta_n \|u_n - z_n\| + \beta_n \gamma_n \|u_n - u_{n-1}\|. \end{aligned}$$

Consequently,

$$\begin{aligned} \|u_{n+1} - z_{n+1}\| &\leq \|u_{n+1} - z_n\| + \|z_{n+1} - z_n\| \\ &\leq \beta_n \|u_n - z_n\| + \beta_n \gamma_n \|u_n - u_{n-1}\| + 4\delta_n + K \frac{\delta_n + \xi(2\delta_n)}{\alpha_n} \\ &+ R \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} + K_3 \sqrt{L K_4} \sqrt{h_E(\delta_n)}, \end{aligned} \tag{3.37}$$

or equivalent to

$$\|u_{n+1} - z_{n+1}\| \leq (1 - b_n) \|u_n - z_n\| + \sigma_n, \tag{3.38}$$

where $b_n = \dfrac{c_n \alpha_n}{1 + \tilde{c}_n \alpha_n}$ and

$$\begin{aligned} \sigma_n &= \beta_n \gamma_n \|u_n - u_{n-1}\| + 4\delta_n + K \frac{\delta_n + \xi(2\delta_n)}{\alpha_n} + R \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} \\ &+ K_3 \sqrt{L K_4} \sqrt{h_E(\delta_n)}. \end{aligned}$$

By the assumption, we obtain

$$\begin{aligned}\frac{\sigma_n}{b_n} &= \frac{1}{c_n}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_n} + \alpha_n)\left[\frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2}R + 4\frac{\delta_n}{\alpha_n} + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n^2}\right] \\ &+ (\frac{1}{c_n} + \alpha_n)K_3\sqrt{LK_4}\frac{\sqrt{h_E(\delta_n)}}{\alpha_n} \\ &\leq \frac{1}{c_0}\alpha_n^{-1}\gamma_n\|u_n - u_{n-1}\| + (\frac{1}{c_0} + \alpha_n)\left[\frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2}R + 4\frac{\delta_n}{\alpha_n} + K\frac{\delta_n + \xi(2\delta_n)}{\alpha_n^2}\right] \\ &+ (\frac{1}{c_0} + \alpha_n)K_3\sqrt{LK_4}\frac{\sqrt{h_E(\delta_n)}}{\alpha_n} \longrightarrow 0,\ n \longrightarrow \infty.\end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n = +\infty$, $\sum_{n=0}^{\infty} b_n = +\infty$.
By Lemma 3.3, it implies that $\|u_n - z_n\| \longrightarrow 0$. Since $z_n \longrightarrow Q_S(Q_C y)$, $u_n \longrightarrow Q_S(Q_C y)$.

□

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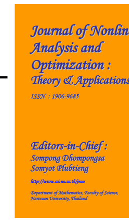

ON THE SEMILOCAL CONVERGENCE OF ULM'S METHOD

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ABSTRACT. We provide sufficient convergence conditions for the semilocal convergence of Ulm's method [9] to a locally unique solution of an equation in a Banach space setting. Our results compare favorably to recent ones by Ezquerro and Hernández [3] which have improved earlier ones [4], [6]–[10], since under the same computational cost we provide: larger convergence domain; finer error bounds on the distances involved, and an at least as precise information on the location of the solution.

KEYWORDS : Ulm's method; Newton's method; Banach space; Recurrence relations; Semilocal convergence; Fréchet derivative.
AMS Subject Classification: 65J15 65H10 65G99 47J25 49M15.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where, F is a Fréchet–differentiable operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time–invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear or nonlinear algebraic

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equations), or real or complex numbers (single algebraic equations with single unknowns). Excpet in special cases, the most commonly used solution methods are iterative–when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

In [9], Ulm introduced method

$$\begin{array}{l} B_{n+1} = 2\,B_n - B_n\,F'(x_n)\,B_n \qquad (x_0 \in \mathcal{D}), \quad B_0 \in \mathcal{L}(\mathcal{Y},\mathcal{X}), \\ x_{n+1} = x_n - B_n\,F(x_n) \qquad (n \geq 0) \end{array} \tag{1.2}$$

to generate a sequence $\{x_n\}$ approximating $x^\star$. Method (1.2) has some usefull properties: First it is like Newton's method, self–correcting. Second, it converges with Newton–like rate. Third, it is inversion free unlike Newton's method. Fourth, apart from solving equation (1.1), the method generates successive approximation: $B_n \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ to the inverse derivative $F'(x^\star)^{-1}$ which important especially when one is interested in solutions sensitive to small perturbations [2], [6].

Ulm [9], Moser [6], Hald [4], Zehnder [10], Petzeltova [7], Potra [8] and others [1], [2] have provided sufficient convergence conditions under various assumptions for the convergence of method (1.2) to $x^\star$.

Recently, Ezquerro and Hernández [3] provided a semilocal convergence analysis for method (1.2) using recurrence relations and conditions which are more general than the mentioned works (see also [5]). They also gave numerical examples where their results hold when the ones by the authors mentioned above do not hold.

Here we are motivated by optimization considerations and the work in [3]. In particular we also provide sufficient convergence conditions for method (1.2) using similar recurrence relations. However under the same computational cost as in [3], our approach has the following advantages:

(a) larger convergence domain;
(b) finer error estimations on the distances $\| x_{n+1} - x_n \|$, $\| x_n - x^\star \|$ $(n \geq 0)$;

and

(c) an at least as precise information on the location of the solution of $x^\star$.

2. Semilocal convergence analysis of method (1.2)

To make the paper as self-contained as possible, we re–introduce some of the notations used in [3]. We assume throughout this study:

($\mathcal{H}1$) $\| B_0 \| \leq c_0$,
($\mathcal{H}2$) $\| F(x_0) \| \leq \eta$,
($\mathcal{H}3$) $0 < \| I - F'(x_0)\,B_0 \| \leq a_0 < 1$,
($\mathcal{H}4$) ($\mathcal{H}4$) $\| F'(x) - F'(y) \| \leq \omega(\| x - y \|)$, for all x, $y \in \mathcal{D}$ and some continuous non–decreasing function such that

$$\omega(t\,r) \leq \omega(r)\,t^p \quad \text{for all} \quad r > 0,\ t \in [0,1],\ p \in [0,1].$$

It then follows from ($\mathcal{H}4$) that there exists a continuous and non–decreasing function $\omega_0 : (0,+\infty) \longrightarrow (0,+\infty)$ such that

$$\| F'(x) - F'(x_0) \| \leq \omega_0(\| x - x_0 \|) \quad \text{for all} \quad x \in \mathcal{D}$$

and

$$\omega_0(r) \leq \omega(r) \quad \text{for all} \quad r > 0. \tag{2.1}$$

Clearly $\dfrac{\omega(r)}{\omega_0(r)}$ can be arbitrarily large [1], [2]. Function ω_0 is not used in [3]. It turn out that the introduction of function ω_0 in case it is strictly smaller than ω is the reason for the finer convergence analysis than in [3] that follows:

Let us set

$$b_0 = c_0\,\omega_0(c_0\,\eta). \tag{2.2}$$

Define auxiliary scalar functions g and h by

$$g(x,y) = x + \frac{y}{1+p} \tag{2.3}$$

and

$$h(x,y) = 1 + x + y. \tag{2.4}$$

We shall show that method (1.2) is well defined. Note that if $x_1 \in \mathcal{D}$, then

$$\begin{aligned} \| I - F'(x_1)\,B_0 \| &= \| I - F'(x_0)\,B_0 + (F'(x_0) - F'(x_1))\,B_0 \| \\ &\leq a_0 + \omega_0(\| x_1 - x_0 \|)\ \| B_0 \| = a_0 + b_0; \end{aligned}$$

$$\begin{aligned} \| B_1 - B_0 \| &= \| B_0 - B_0\,F'(x_1)\,B_0 \| \\ &= \| B_0\,(I - F'(x_1)\,B_0) \| \\ &\leq c_0\,(a_0 + b_0); \end{aligned}$$

$$\begin{aligned} \| F(x_1) \| &\leq \| I - F'(x_0)\,B_0 \|\,\| F(x_0) \| + \int_0^1 \omega(t\ \| x_1 - x_0 \|)\,dt\ \| x_1 - x_0 \| \\ &\leq g(a_0, b_0)\ \| F(x_0) \|; \end{aligned}$$

$$\begin{aligned} \| B_1 \| &= \| 2\,B_0 - B_0\,F'(x_1)\,B_0 \| \\ &\leq \| B_0 \| + \| B_0 - B_0\,F'(x_1)\,B_0 \| \\ &\leq c_0 + c_0\,(a_0 + b_0) = c_0\,h(a_0, b_0); \end{aligned}$$

$$\| x_2 - x_1 \| \leq \| B_1 \|\,\| F(x_1) \| \leq g(a_0,b_0)\,h(a_0,b_0)\ \| F(x_0) \|;$$

$$\| x_2 - x_0 \| \leq (1 + g(a_0,b_0)\,h(a_0,b_0))\ \| B_0 \|\,\| F(x_0) \|;$$

and if $x_2 \in \mathcal{D}$, $g(a_0,b_0)\,h(a_0,b_0) < 1$, then we get

$$\| B_1 \|\,\| F'(x_2) - F'(x_1) \| \leq b_0\,g(a_0,b_0)^p\,h(a_0,b_0)^{1+p},$$

and

$$\| I - F'(x_1)\,B_1 \| \leq \| I - F'(x_1)\,B_0 \|^2 \leq (a_0 + b_0)^2,$$

so that

$$\begin{aligned} \| I - F'(x_2)\,B_1 \| &\leq \| I - F'(x_1)\,B_1 \| + \| F'(x_2) - F'(x_1) \|\,\| B_1 \| \\ &\leq (a_0 + b_0)^2 + b_0\,g(a_0,b_0)^p\,h(a_0,b_0)^{1+p}, \end{aligned}$$

and

$$\| B_2 - B_1 \| \leq c_0\,h(a_0,b_0)\left[(a_0+b_0)^2 + b_0\,g(a_0,b_0)^p\,h(a_0,b_0)^{1+p}\right].$$

Let us set

$$a_1 = (a_0 + b_0)^2, \quad b_1 = b_0\,g(a_0,b_0)^p\,h(a_0,b_0)^{1+p} \quad \text{and} \quad c_1 = c_0\,h(a_0,b_0).$$

Then we can define scalar sequences for all $n \geq 1$:

$$a_n = (a_{n-1} + b_{n-1})^2 \tag{2.5}$$

$$b_n = b_{n-1}\,g(a_{n-1},b_{n-1})^p\,h(a_{n-1},b_{n-1})^{1+p} \tag{2.6}$$

$$c_n = c_{n-1}\,h(a_{n-1},b_{n-1}). \tag{2.7}$$

Let us also define scalar sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ used in [3] as $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ respectively with

$$\gamma_0 = c_0, \quad \alpha_0 = a_0$$

but

$$\beta_0 = c_0\,\omega(c_0\,\eta).$$

Clearly in case function ω_0 is strictly smaller than ω, the our triplet $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ is finer than $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ used in [3].

We shall state the following results but only prove Theorem 2.6, since the rest of the proofs are similar to the corresponding ones in [3] (simply replace the triplet $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ by $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ respectively in the proofs given in [3]):

Lemma 2.1. *Let g and h be the scalar functions given by (2.3) and (2.4) respectively. If a_0 and b_0 satisfy*

$$g(a_0,b_0)^p\,h(a_0,b_0)^{1+p} < 1 \quad \text{and} \quad (a_0+b_0)^2 < a_0, \tag{2.8}$$

then the following hold true:

(a) *($g(a_0,b_0)\,h(a_0,b_0) < 1$,*
(b) *the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing.*

The next aim of the study is to prove that method (1.2) is well–defined, so that we present a system of recurrence relations in the next lemma from which we obtain the last. The proof of the lemma follows from a similar way that the mentioned above and using induction.

Lemma 2.2. *If a_0 and b_0 satisfy (2.8) and $\mathbb{B}(x_0, R\,c_0\,\eta) \subseteq \mathcal{D}$, where $R = \dfrac{1}{1-\Delta}$ and $\Delta = g(a_0,b_0)\,h(a_0,b_0)$, then the next recurrence relations are true for all $n \geq 1$:*

$(\mathcal{R}\mathbf{1})$ $\| F(x_n) \| \leq g(a_{n-1},b_{n-1})\ \| F(x_{n-1}) \|$,
$(\mathcal{R}\mathbf{2})$ $\| B_n \| \leq h(a_{n-1},b_{n-1})\ \| B_{n-1} \| \leq c_n$,
$(\mathcal{R}\mathbf{3})$ $\| x_{n+1}-x_n \| \leq g(a_{n-1},b_{n-1})\,h(a_{n-1},b_{n-1})\ \| B_{n-1} \|\,\| F(x_{n-1}) \|$,

$(\mathcal{R}\mathbf{4})$ $\| x_{n+1}-x_0 \| \leq \dfrac{1-\Delta^{n+1}}{1-\Delta}\ \| B_0 \|\,\| F(x_0) \| < R\,c_0\,\eta$,

$(\mathcal{R}\mathbf{5})$ $\| B_n \|\ \omega(\| x_{n+1}-x_n \|) \leq b_n$,
$(\mathcal{R}\mathbf{6})$ $\| I - F'(x_n)\,B_n \| \leq a_n$,
$(\mathcal{R}\mathbf{7})$ $\| I - F'(x_{n+1})\,B_n \| \leq a_n + b_n$,
$(\mathcal{R}\mathbf{8})$ $\| B_{n+1} - B_n \| \leq (a_n+b_n)\,c_n$.

Note that, from ($\mathcal{R}$**4**), we obtain $x_n \in \mathcal{D}$, for all $n \geq 0$, if the hypotheses of Lemma 2.2 are satisfied.

Remark 2.3. If $a_0 = 0$, then $B_0 = (F'(x_0))^{-1}$ and the first step of iteration (1.2) is the same as in Newton's method. In this case, we have

$$\begin{aligned}
&a_1 = b_0^2, \quad b_1 = b_0\,(1+b_0)\left(\frac{b_0\,(1+b_0)}{1+p}\right)^p, \quad c_1 = (1+b_0)\,c_0,\\
&a_n = (a_{n-1}+b_{n-1})^2,\ \ n \geq 2,\\
&b_n = b_{n-1}\,g(a_{n-1},b_{n-1})^p\,h(a_{n-1},b_{n-1})^{1+p},\ \ n \geq 2,\\
&c_n = c_{n-1}\,h(a_{n-1},b_{n-1}),\ \ n \geq 2.
\end{aligned}$$

These sequences $\{a_n\}$ and $\{b_n\}$, for $n \geq 0$, are also decreasing if

$$(1+p)\,b_0 + (1+b_0)\left(\frac{b_0\,(1+b_0)}{1+p}\right)^p < 1 \quad \text{and} \quad (b_0^2+b_1)^2 < b_0^2, \tag{2.9}$$

so that the recurrence relations appearing in Lemma 2.2 are also satisfied, excpet for ($\mathcal{R}4$), that now is

$$\| x_{n+1} - x_0 \| \leq \left(1 + f(b_0)\, \frac{1-\overline{\Delta}^n}{1-\overline{\Delta}}\right) \| B_0 \| \| F(x_0) \| < \overline{R}\, c_0\, \eta,$$

where

$$\overline{R} = 1 + f(b_0)\, \frac{b_0\,(1+b_0)}{(1+p)\,(1-\overline{\Delta})} \quad \text{and} \quad \overline{\Delta} = g(a_1, b_1)\, h(a_1, b_1).$$

Since the sequence $\{x_n\}$ is well–defined, the following aim is to see that $\{x_n\}$ is a Cauchy sequence. We then provide the following semilocal convergence result, which is also used to draw conclusions about the existence of a solution and the domain in which it is located.

Theorem 2.4. *Let $F : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathcal{Y}$ be a Fréchet-différentiable operator on a non-empty open convex domain $\mathcal{D}$. Let $x_0 \in \mathcal{D}$ and $B_0 \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. Suppose that conditions ($\mathcal{H}1$)-($\mathcal{H}4$), (2.8) and $\mathbb{B}(x_0, R\, c_0\, \eta) \subseteq \mathcal{D}$, are satisfied. Then the sequence $\{x_n\}$, defined by (1.2) and starting from x_0, remains in $\overline{\mathbb{B}}(x_0, R\, c_0\, \eta)$ and converges to a solution $x^\star$ of equation $F(x) = 0$.*

Remark 2.5. In the case $a_0 = 0$ ($B_0 = (F'(x_0))^{-1}$), the convergence of sequence (1.2) follows in the same way as in Theorem 2.4 with (2.9), except for R, that it now is $\overline{R}$.

In the next result we show the uniqueness of the solution $x^\star$ of equation $F(x) = 0$.

Theorem 2.6. *Suppose that conditions ($\mathcal{H}1$)-($\mathcal{H}4$) are satisfied and function ω is also strictly increasing. Then the solution $x^\star$ of equation $F(x) = 0$ is unique in the domain $\mathcal{D}_0 = \mathbb{B}(x_0, r^\star) \cap \mathcal{D}$, where $r^\star$ is the smallest positive root of the equation in the variable y:*

$$\int_{R\, c_0\, \eta}^{y} \omega_0(s)\, ds = \frac{1}{c_0}\,(1-a_0)\,(y - R\, c_0\, \eta). \tag{2.10}$$

Proof Let us assume $y^\star$ is a solution of $F(x) = 0$ in $\mathcal{D}_0$. According to Argyros ([1], [2]), we have the approximation

$$0 = F(y^\star) - F(x^\star) = \int_0^1 F'(x^\star + t\,(y^\star - x^\star))\, dt\, (y^\star - x^\star). \tag{2.11}$$

Let us set $\mathcal{M} = \displaystyle\int_0^1 F'(x^\star + t\,(y^\star - x^\star))\, dt$. We have:

$$\begin{aligned}
\| I - \mathcal{M}\, B_0 \| &\leq \| I - F'(x_0)\, B_0 \| + \| F'(x_0) - \mathcal{M} \| \| B_0 \| \\
&\leq a_0 + c_0 \int_0^1 \| F'(x_0) - F'(x^\star + t\,(y^\star - x^\star)) \| \; dt \\
&\leq a_0 + c_0 \int_0^1 \omega_0((1-t)\; \| x^\star - x_0 \| + t\; \| y^\star - x_0 \|)\, dt \\
&< a_0 + \frac{c_0}{r - R\, c_0\, \eta} \int_{R\, c_0\, \eta}^{r^\star} \omega_0(s)\, ds = 1,
\end{aligned}$$

it follows from the previous estimation and the Banach lemma of invertible operators [1], [2], that $\mathcal{M}^{-1}$ exists. In view of (2.11) we deduce $x^\star = y^\star$.

It is show in Theorem 2.4 that it is not necessary for x_0 to satisfy the conditions given by (2.8) to obtain the semilocal convergence of Ulm's method given by (1.2), since it suffices that they are satisfied for some iterate x_j of (1.2). So, we obtain the following corollary.

Corollary 2.7. *Under the conditions of Theorem 2.4, further assume: there exists $j \in \mathbb{N}$ such that*

$$g(a_j, b_j)^p\, h(a_j, b_j)^{1+p} < 1 \quad \text{and} \quad (a_j + b_j)^2 < a_j, \tag{2.12}$$

where, $a_j = \parallel I - F'(x_j)\, B_j \parallel$, $b_j = c_j\, \omega(c_j\, \overline{\eta})$, $c_j = \parallel B_j \parallel$, $\overline{\eta} = \parallel F(x_j) \parallel$ and $\mathbb{B}(x_0, R_j) \subseteq \mathcal{D}$, with $R_j = R\, c_0\, \eta + \sum_{i=0}^{i=j-1} \parallel x_{i+1} - x_i \parallel$, then sequence $\{x_n\}$, defined by (1.2) and starting from x_0, remains in $\overline{\mathbb{B}}(x_0, R_j)$ and converges to a solution $x^\star$ of equation $F(x) = 0$.

Proof The proof of Corollary 2.7 follows from the facts that the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing for all $n > j$ and the recurrence relations given in Lemma 2.2 now hold for all $n > j + 1$.

In order for us to show that the R–order of convergence of method (1.2) under hypotheses ($\mathcal{H}1$)–($\mathcal{H}4$) is $1 + p$, we first need a result concerning the behavior of certain functions.

Lemma 2.8. *Let g and h be the functions given by (2.3) and (2.4) respectively and define $\delta_1 = \dfrac{a_1}{a_0}$, $\delta_2 = \dfrac{b_1}{b_0}$ and $\delta = \max\{\delta_1, \delta_2\}$. If (2.8) is satisfied, then*

(a) *$g(\delta\, x, \delta\, y) = \delta\, g(x, y)$ and $h(\delta\, x, \delta\, y) < h(x, y)$, for all $\delta \in (0, 1)$,*

(b) *$a_n < \delta^{(1+p)^{n-1}}\, a_{n-1} < \delta^{\frac{(1+p)^n - 1}{p}}\, a_0$ and $b_n < \delta^{(1+p)^{n-1}}\, b_{n-1} < \delta^{\frac{(1+p)^n - 1}{p}}\, b_0$, for all $n \geq 1$.*

We show the following result on the R–order of convergence for method (1.2):

Theorem 2.9. *Under the conditions of Theorem 2.4, the method (1.2) has R-order of convergence at least $1 + p$. Moreover, the following a priori error estimates are obtained:*

$$\parallel x_n - x^\star \parallel \leq \frac{A^n\, \delta^{\frac{(1+p)^n - 1}{p^2}}}{1 - A\, \delta^{\frac{(1+p)^n}{p}}}\, c_0\, \eta, \tag{2.13}$$

where $A = \Delta\, \delta^{-1/p}$ and $\Delta = g(a_0, b_0)\, h(a_0, b_0)$.

Remark 2.10. Observe that if F' is Lipschitz continuous in $\mathcal{D}$, then $\omega(r) = K\, r$, $K \geq 0$. and method (1.2) is of R–order of convergence at least two.

Remark 2.11. If $a_0 = 0$ ($B_0 = (F'(x_0))^{-1}$), the R–order of sequence (1.2) follows exactly as in the previous theorem.

Taking now into account the estimates regarding consecutive points are good to distance $\parallel x_n - x^\star \parallel$ (see ($\mathcal{R}$**3**) in Lemma 2.2), we can for an element x_k ($k > n$) of the sequence $\{x_n\}$ such that $\parallel x_k - x^\star \parallel$ is smaller enough and $\parallel x_n - x^\star \parallel$ can be estimated from the distance between two consecutive points. So,

$$\parallel x_n - x^\star \parallel \leq \parallel x_{n+j} - x^\star \parallel + \sum_{i=1}^{i=j} \parallel x_{n+i} - x_{n+i-1} \parallel, \quad j \geq 1, \quad n \geq 1, \tag{2.14}$$

and the error given in (2.13) is then improved.

Remark 2.12. [3] To finish, as we have indicated in the introduction, we study the convergence of the sequence–operators $\{B_n\}$. Note that $\{B_n\}$ converges to the bounded right of $F'(x^\star)$. Indeed, from $(\mathcal{R}\mathbf{8})$, it follows

$$\| B_{k+1} - B_k \| \leq (a_k + b_k)\, c_k \leq (a_k + b_k)\, h(a_0, b_0)^k\, c_0,$$

since h is increasing in the both arguments $\{a_n\}$ and $\{b_n\}$ are decreasing sequences. In consequence,

$$\| B_{k+1} - B_k \| \leq \delta^{\frac{(1+p)^k - 1}{p}}\, (a_0 + b_0)\, h(a_0, b_0)^k\, c_0.$$

Therefore,

$$\begin{aligned} \| B_{n+m} - B_n \| &\leq \left(\sum_{k=0}^{k=m-1} \delta^{\frac{(1+p)^{n+k} - 1}{p}}\, h(a_0, b_0)^{n+k} \right) (a_0 + b_0)\, c_0 \\ &\leq \delta^{-1/p}\, (a_0 + b_0)\, c_0\, h(a_0, b_0)^{n+m-1}\, S \end{aligned}$$

where

$$S = \sum_{k=0}^{k=m-1} \delta^{\frac{(1+p)^{n+k}}{p}}.$$

Moreover,

$$S \leq \delta^{\frac{(1+p)^{n+m-1}}{p}} \left(\delta^{\frac{(1+p)^n}{p}\, (1-(1+p)^{m-1})}\, \frac{1 - \delta^{m\,(1+p)^n}}{1 - \delta^{(1+p)^n}} \right),$$

since $\delta^{\frac{(1+p)^k}{p}} \leq \delta^{\frac{(1+p)^n}{p}}\, \delta^{(1+p)^n\,(k-n)}$, for $k = n+1, n+2, \cdots, n+m-1$. Thus, $\{B_n\}$ is a Cauchy sequence and then $\lim_n B_n = B^\star$. On the other hand, $\| I - F'(x^\star)\, B_n \| \longrightarrow 0$ by letting $n \to \infty$ and taking into account that

$$\| I - F'(x_n)\, B_n \| \leq a_n \leq \delta^{2\,((1+p)^n - 1)/p}\, a_1,$$

$$\| B_n \| \leq h(a_0, b_0)^n\, c_0,$$

$$\| F'(x^\star) - F'(x_n) \| \leq \left(\frac{\Delta^n}{1 - \Delta} \right)^p \omega(\eta).$$

Consequently, $B^\star$ is the bounded right inverse of $F'(x^\star)$.

Remark 2.13. The sufficient convergence conditions given in [3] corresponding to (2.8) and (2.9) are given by

$$g(\alpha_0, \beta_0)^p\, h(\alpha_0, \beta_0)^{1+p} < 1 \quad \text{and} \quad (\alpha_0 + \beta_0)^2 < \alpha_0, \tag{2.15}$$

and

$$(1+p)\, \beta_0 + (1 + \beta_0) \left(\frac{\beta_0\, (1 + \beta_0)}{1+p} \right)^p < 1 \quad \text{and} \quad (\beta_0^2 + \beta_1^2)^2 < \beta_0^2, \tag{2.16}$$

respectively.

In case strict inequality holds in (2.1), conditions (2.15) and (2.16) imply (2.8) and (2.9) respectively but not necessary vice verca (unless if $\omega_0(r) = \omega(r)$ for all $r > 0$). Moreover due to the fact that

$$b_0 \leq \beta_0;$$

the rest of the advantages already stated at the introduction of this study hold true.

We provide a numerical example to show that our conditions (2.8) (or (2.9)) hold, whereas (2.15) (or (2.16)) do not.

Example 2.14. $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\mathcal{D} = [q, 2-q]$, $q \in [0,1]$, $x_0 = 1$ and define function F on $\mathcal{D}$ by

$$F(x) = x^3 - q. \tag{2.17}$$

Using (2.17), ($\mathcal{H}1$)-($\mathcal{H}4$), (2.2), (2.4) and (2.8), we get

$$\eta = 1 - q, \quad \omega(r) = 6\,(2-q)\,r, \quad \omega_0(r) = 3\,(3-q)\,r, \quad b_0 = 3\,c_0^2\,(3-q)\,(1-q)$$

and

$$\beta_0 = 6\,c_0^2 (2-q)\,(1-q).$$

Let $c_0 = B_0 = 8/30$ and $q = 0.55$.
Then we obtain

$$\alpha_0 = a_0 = 0.2, \quad \beta_0 = 0.2784, \quad b_0 = 0.2448 \quad \text{and} \quad (\alpha_0 + \beta_0)^2 = 0.22886656 > 0.2.$$

That is there is no guarantee that method (1.2) converges to $x^\star = \sqrt[3]{q} = 0.819321271$, since (2.15) is violated.

Howeover conditions (2.8) and (2.9) are satisfied since they become

$$0.672992926 < 1 \quad \text{and} \quad 0.19784704 < 0.2,$$

respectively.

Hence, the conclusions of Theorem 2.4 for equation apply and our method (1.2) converges to $x^\star$.

Remark 2.15. The earlier results on method (1.2), [4], [6], [7]–[10] require that operator F' satisfies the Lipschitz condition:

$$\| F'(x) - F'(y) \| \leq K \ \| x - y \| \quad \text{for all } x, y \in \mathcal{D}. \tag{2.18}$$

It follows from (2.18) that there exists K_0 such that

$$\| F'(x) - F'(x_0) \| \leq K_0 \ \| x - x_0 \| \quad \text{for all } x \in \mathcal{D}. \tag{2.19}$$

Clearly

$$K_0 \leq K \tag{2.20}$$

holds and $\dfrac{K}{K_0}$ can be arbitrarily large [1], [2].
In case strict inequality holds in (2.20), one can visit the results mentioned above and use (2.18) and (2.19) instead of only (2.18) in the convergence analysis of method (1.2). It then follows that the resulting approach will produce a finer convergence analysis for method (1.2) with advantages over earlier works as stated in the introduction of this study. Howeover we leave the details to the motivated reader.

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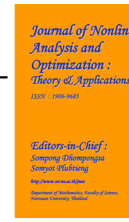

FIXED POINT THEOREMS IN MENGER SPACES USING THE (CLR_{ST}) PROPERTY AND APPLICATIONS

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ABSTRACT. In the present paper, we prove common fixed point theorems for two pairs of weakly compatible mappings in Menger spaces employing the (CLR_{ST}) property. Some examples are furnished which demonstrate the validity of the hypotheses and degree of generality of our results. We extend our main result to four finite families of self mappings. As applications to our results, we obtain the corresponding common fixed point theorems in metric spaces. Our results improve and extend the results of Cho et al. [4] and Pathak et al. [21] besides several known results.

KEYWORDS : t-norm; Menger space; Weakly compatible mappings; (E.A) property; Common property (E.A); (CLR_S) property; (CLR_{ST}) property; Fixed point.
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1. INTRODUCTION

In 1942, Karl Menger [15] introduced the notion of a probabilistic metric space (shortly, PM-space). The idea of Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of the distances. In fact the study of such spaces received an impetus with the pioneering work of Schweizer and Sklar [24]. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis especially due to its extensive applications in random differential as well as random integral equations.

In 1991, Mishra [17] extended the notion of compatibility (introduced by Jungck [8] in metric spaces) to PM-space. Cho et al. [4] studied the notion of compatible mappings of type (A) (introduced by Jungck et al. [9] in metric spaces) in Menger

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spaces which is equivalent to the concept of compatible mappings under some conditions. Further, Pathak et al. [21] improved and generalized the results of Cho et al. [4] by introducing the notion of weak compatible mappings of type (A) in Menger spaces. The fixed point theorems for contraction mappings in Menger spaces were obtained by many mathematicians (e.g. [16, 18, 20, 23]).

It is seen that most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. However, the study of common fixed points of non-compatible mappings is also equally interesting which was initiated by Pant [19] in metric spaces. In 2002, Aamri and El Moutawakil [1] defined the (E.A) property for self mappings whose class contains the class of non-compatible as well as compatible mappings. Kubiaczyk and Sharma [11] studied the common fixed points of weakly compatible mappings satisfying the property (E.A) in PM-spaces and used it to prove results on common fixed points. In the recent years, there are a number of results via (E.A) property in PM-spaces (e.g. [5, 7, 12, 13]). Inspired by Liu et al. [14], Ali et al. [2] (also, see [3]) defined the common property (E.A) for the existence of a common fixed point in Menger spaces and generalized several known results in Menger spaces as well as metric spaces. It is observed that (E.A) property and common property (E.A) require the closedness of the subspaces for the existence of fixed point. Recently, Sintunavarat and Kumam [27] coined the idea of “common limit in the range property” which never requires the closedness of the subspaces for the existence of fixed point (also see [26, 28]).

The aim of this paper is to prove common fixed point theorems for two pairs of weakly compatible mappings in Menger spaces employing the (CLR_{ST}) property. Illustrative examples are also furnished to support our results. We extend our main result to four finite families of mappings using the notion of pairwise commuting property of two finite families of mappings due to Imdad et al. [6]. As applications to our results, we obtain the corresponding common fixed point theorems in metric spaces.

2. PRELIMINARIES

Definition 2.1. [24] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

We denote by $\Im$ the set of all distribution functions while H always denotes the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.2. [24] A PM-space is an ordered pair $(X, \mathcal{F})$, where X is a non-empty set of elements and $\mathcal{F}$ is a mapping from $X \times X$ to $\Im$, the collection of all distribution functions. The value of $\mathcal{F}$ at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The functions $F_{x,y}$ are assumed to satisfy the following conditions:

(i) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$;
(ii) $F_{x,y}(0) = 0$;
(iii) $F_{x,y}(t) = F_{y,x}(t)$;
(iv) If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

Definition 2.3. [24] A mapping $\triangle : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (briefly, t-norm) if the following conditions are satisfied: for all $a, b, c, d \in [0, 1]$

(i) $\triangle(a, 1) = a$ for all $a \in [0, 1]$;
(ii) $\triangle(a, b) = \triangle(b, a)$;
(iii) $\triangle(a, b) \leq \triangle(c, d)$ for $a \leq c$, $b \leq d$;
(iv) $\triangle(\triangle(a, b), c) = \triangle(a, \triangle(b, c))$;

Examples of t-norms are $\triangle(a, b) = \min\{a, b\}$, $\triangle(a, b) = ab$ and $\triangle(a, b) = \max\{a + b - 1, 0\}$.

Definition 2.4. [24] A Menger space is a triplet $(X, \mathcal{F}, \triangle)$ where $(X, \mathcal{F})$ is a PM-space and t-norm $\triangle$ is such that the inequality

$$F_{x,z}(t + s) \geq \triangle\left(F_{x,y}(t), F_{y,z}(s)\right),$$

holds for all $x, y, z \in X$ and all $t, s > 0$.

Every metric space (X, d) can be realized as a Menger space by taking $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$.

Definition 2.5. [17] Two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be compatible if and only if $F_{ASx_n,SAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Definition 2.6. [4] Two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be compatible of type (A) if $F_{SAx_n,AAx_n}(t) \to 1$ and $F_{ASx_n,SSx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Remark 2.7. [4] If self mappings A and S are both continuous, then A and S are compatible if and only if they are compatible of type (A).

It is noted that Remark 2.7 is not true if self mappings A and S are not continuous on X. For examples, we refer to Jungck et al. [9].

Definition 2.8. [21] Two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be weak compatible of type (A) if

$$\lim_{n\to\infty} F_{ASx_n,SSx_n}(t) \geq \lim_{n\to\infty} F_{SAx_n,SSx_n}(t)$$

and

$$\lim_{n\to\infty} F_{SAx_n,AAx_n}(t) \geq \lim_{n\to\infty} F_{ASx_n,AAx_n}(t),$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Remark 2.9. [21] If self mappings A and S are both continuous. Then

(i) A and S are compatible of type (A) if and only if they are weak compatible of type (A).
(ii) A and S are compatible if and only if they are weak compatible of type (A).

It is noted that Remark 2.9 is not true if self mappings A and S are not continuous on X. For examples, we refer to Pathak et al. [21].

Definition 2.10. [10] Two self mappings A and S of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Sz$ (for $z \in X$), then $ASz = SAz$.

Remark 2.11. Two compatible self mappings are weakly compatible, but the converse is not true (see [25, Example 1]). Therefore the concept of weak compatibility is more general than that of compatibility.

Definition 2.12. [11] A pair (A, S) of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ is said to satisfy the (E.A) property, if there exists a sequence $\{x_n\}$ in X such that for all $t > 0$

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Here, it can be pointed out that weak compatibility and the (E.A) property are independent to each other (see [22, Example 2.2]).

Remark 2.13. From Definition 2.5, it is inferred that two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are non-compatible if and only if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$, but for some $t > 0$, $\lim_{n\to\infty} F_{ASx_n, SAx_n}(t)$ is either less than 1 or nonexistent.

Therefore, from Definition 2.12, it is straight forward to notice that every pair of non-compatible self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ satisfies the (E.A) property but not conversely (see [5, Example 1]).

Definition 2.14. [2] Two pairs (A, S) and (B, T) of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ are said to satisfy the common property (E.A), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some z in X.

On the lines of Sintunavarat and Kumam [27], we define the (CLR_S) property (with respect to mapping S) in Menger space as follows:

Definition 2.15. A pair (A, S) of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ is said to satisfy the (CLR_S) property with respect to mapping S if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X)$.

Now, we present examples of self mappings A and S satisfying the (CLR_S) property.

Example 2.16. Let $(X, \mathcal{F}, \triangle)$ be a Menger space with $X = [0, \infty)$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Define self mappings A and S on X by $A(x) = x + 2$ and $S(x) = 3x$ for all $x \in X$. Let a sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n\in\mathbb{N}}$ in X, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 3 = S(1),$$

that is, $3 \in S(X)$, which shows that A and S satisfy the (CLR_S) property.

Example 2.17. The conclusion of Example 2.16 remains true if the self mappings A and S are defined on X by $A(x) = \frac{x}{2}$ and $S(x) = \frac{2x}{3}$ for all $x \in X$. Let a sequence $\{x_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ in X. Since

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0 = S(0),$$

that is, $0 \in S(X)$, hence A and S satisfy the (CLR_S) property.

Remark 2.18. From the Examples 2.16-2.17, it is evident that a pair (A, S) satisfying the (E.A) property along with closedness of the subspace $S(X)$ always enjoys the (CLR_S) property.

With a view to extend the (CLR_S) property to two pair of self mappings, we define the (CLR_{ST}) property (with respect to mappings S and T) as follows.

Definition 2.19. Two pairs (A, S) and (B, T) of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ are said to satisfy the (CLR_{ST}) property (with respect to mappings S and T) if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Lemma 2.20. [17] *Let $(X, \mathcal{F}, \triangle)$ be a Menger space, where $\triangle$ is a continuous t-norm. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

3. RESULTS

In 1992, Cho et al. [4] proved the following fixed point theorem for compatible mappings of type (A) in Menger space.

Theorem 3.1. [4, Theorem 4.2] *Let $(X, \mathcal{F}, \triangle)$ be a complete Menger space with $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and A, B, S and T be mappings from X into itself such that*

(i) *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*
(ii) *the pairs (A, S) and (B, T) are compatible of type (A),*
(iii) *one of A, B, S and T is continuous,*
(iv) *there exists a constant $k \in (0, 1)$ such that*

$$(F_{Ax,By}(kt))^2 \geq \min\left\{ \begin{array}{c} (F_{Sx,Ty}(t))^2, F_{Sx,Ax}(t)F_{Ty,By}(t), F_{Sx,By}(2t)F_{Ty,Ax}(t), \\ F_{Ty,Ax}(t), F_{Sx,By}(2t)F_{Ty,By}(t) \end{array} \right\}, \tag{3.1}$$

for all $x, y \in X$ and $t \geq 0$. Then A, B, S and T have a unique common fixed point in X.

Further, Pathak et al. [21] improved and generalized the results of Cho et al. [4] by using the notion of weak compatible mappings of type (A) which is more general than compatible mappings of type (A).

The attempted improvements in this paper are four fold.

(i) The condition on containment of ranges amongst the involved mappings are relaxed.
(ii) Continuity requirements of all the involved mappings are completely relaxed.
(iii) The mappings of compatible of type (A) or weak compatible of type (A) are replaced by weakly compatible mappings which are more general among all existing weak commutativity concepts.
(iv) The condition on completeness of the whole space is relaxed.

Before proving our main result, we begin with the following observation.

Lemma 3.2. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm satisfying inequality (3.1) of Theorem 3.1. Suppose that*

(i) *the pair* (A,S) *satisfies the* (CLR_S) *property* $\Big($*or the pair* (B,T) *satisfies the* (CLR_T) *property*$\Big)$,

(ii) $A(X) \subset T(X)$ $\Big($*or* $B(X) \subset S(X)\Big)$,

(iii) $T(X)$ $\Big($*or* $S(X)\Big)$ *is a closed subset of* X,

(iv) $B(y_n)$ *converges for every sequence* $\{y_n\}$ *in* X *whenever* $T(y_n)$ *converges* $\Big($*or* $A(x_n)$ *converges for every sequence* $\{x_n\}$ *in* X *whenever* $S(x_n)$ *converges*$\Big)$.

Then the pairs (A,S) *and* (B,T) *share the* (CLR_{ST}) *property.*

Proof Suppose the pair (A,S) satisfies the (CLR_S) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \tag{3.2}$$

where $z \in S(X)$. As $A(X) \subset T(X)$ (wherein $T(X)$ is a closed subset of X), for each $\{x_n\} \subset X$ there corresponds a sequence $\{y_n\} \subset X$ such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z, \tag{3.3}$$

where $z \in S(X) \cap T(X)$. Thus in all, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$ as $n \to \infty$. Now we are required to show that $By_n \to z$ as $n \to \infty$. On using inequality (3.1) with $x = x_n$, $y = y_n$, we get

$$(F_{Ax_n,By_n}(kt))^2 \quad \geq \quad \min\left\{\begin{array}{c} (F_{Sx_n,Ty_n}(t))^2, F_{Sx_n,Ax_n}(t)F_{Ty_n,By_n}(t), \\ F_{Sx_n,By_n}(2t)F_{Ty_n,Ax_n}(t), F_{Ty_n,A_n}(t), \\ F_{S_n,By_n}(2t)F_{Ty_n,By_n}(t) \end{array}\right\}.$$

Let $By_n \to l(\neq z)$ for $t > 0$ as $n \to \infty$. Then, passing to limit as $n \to \infty$, we get

$$\begin{aligned} (F_{z,l}(kt))^2 &\geq \min\left\{\begin{array}{c} (F_{z,z}(t))^2, F_{z,z}(t)F_{z,l}(t), F_{z,l}(2t)F_{z,z}(t), \\ F_{z,z}(t), F_{z,l}(2t)F_{z,l}(t) \end{array}\right\} \\ &= (F_{z,l}(t))^2. \end{aligned}$$

Owing to Lemma 2.20, we have $z = l$ which contradicts. Hence the pairs (A,S) and (B,T) share the (CLR_{ST}) property.

Remark 3.3. In general, the converse of Lemma 3.2 is not true. For a counter example, one can see Example 3.5.

Now we prove a common fixed point theorem for two pairs of self mappings in Menger space.

Theorem 3.4. *Let* A, B, S *and* T *be self mappings of a Menger space* $(X, \mathcal{F}, \triangle)$, *where* $\triangle$ *is a continuous t-norm satisfying inequality (3.1) of Theorem 3.1. If the pairs* (A,S) *and* (B,T) *share the* (CLR_{ST}) *property, then* (A,S) *and* (B,T) *have a coincidence point each. Moreover,* A, B, S *and* T *have a unique common fixed point provided both the pairs* (A,S) *and* (B,T) *are weakly compatible.*

Proof Since the pairs (A,S) and (B,T) share the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(X) \cap T(X)$. Since $z \in S(X)$, there exists a point $u \in X$ such that $Su = z$. We assert that $Au = Su$. On using inequality (3.1) with $x = u$, $y = y_n$, we get

$$(F_{Au,By_n}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Su,Ty_n}(t))^2, F_{Su,Au}(t)F_{Ty_n,By_n}(t), \\ F_{Su,By_n}(2t)F_{Ty_n,Au}(t), \\ F_{Ty_n,Au}(t), F_{Su,By_n}(2t)F_{Ty_n,By_n}(t) \end{array} \right\}.$$

Taking limit $n \to \infty$, we obtain

$$\begin{aligned} (F_{Au,z}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{z,z}(t))^2, F_{z,Au}(t)F_{z,z}(t), F_{z,z}(2t)F_{z,Au}(t), \\ F_{z,Au}(t), F_{z,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Au,z}(t))^2. \end{aligned}$$

On employing Lemma 2.20, we have $Au = z$. Therefore $Au = Su = z$ and hence u is a coincidence point of (A, S).

Also $z \in T(X)$, there exists a point $v \in X$ such that $Tv = z$. We show that $Bv = Tv$. On using inequality (3.1) with $x = u$, $y = v$, we get

$$\begin{aligned} (F_{Au,Bv}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Su,Tv}(t))^2, F_{Su,Au}(t)F_{Tv,Bv}(t), F_{Su,Bv}(2t)F_{Tv,Au}(t), \\ F_{Tv,Au}(t), F_{Su,Bv}(2t)F_{Tv,Bv}(t) \end{array} \right\} \\ (F_{z,Bv}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{z,z}(t))^2, F_{z,z}(t)F_{z,Bv}(t), F_{z,Bv}(2t)F_{z,z}(t), \\ F_{z,z}(t), F_{z,Bv}(2t)F_{z,Bv}(t) \end{array} \right\} \\ &= (F_{z,Bv}(t))^2. \end{aligned}$$

Appealing to Lemma 2.20, we have $z = Bv$. Therefore $Bv = Tv = z$ and hence v is a coincidence point of (B, T).

Since the pair (A, S) is weakly compatible, therefore $Az = ASu = SAu = Sz$. Putting $x = z$ and $y = v$ in inequality (3.1), we have

$$\begin{aligned} (F_{Az,Bv}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Sz,Tv}(t))^2, F_{Sz,Az}(t)F_{Tv,Bv}(t), F_{Sz,Bv}(2t)F_{Tv,Az}(t), \\ F_{Tv,Az}(t), F_{Sz,Bv}(2t)F_{Tv,Bv}(t) \end{array} \right\} \\ (F_{Az,z}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Az,z}(t))^2, F_{Az,Az}(t)F_{z,z}(t), F_{Az,z}(2t)F_{z,Az}(t), \\ F_{z,Az}(t), F_{Az,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Az,z}(t))^2. \end{aligned}$$

In view of Lemma 2.20, we have $Az = z = Sz$ which shows that z is a common fixed point of the pair (A, S). Also the pair (B, T) is weakly compatible, therefore $Bz = BTv = TBv = Tz$. On using inequality (3.1) with $x = u$, $y = z$, we have

$$\begin{aligned} (F_{Au,Bz}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Su,Tz}(t))^2, F_{Su,Au}(t)F_{Tz,Bz}(t), F_{Su,Bz}(2t)F_{Tz,Au}(t), \\ F_{Tz,Au}(t), F_{Su,Bz}(2t)F_{Tz,Bz}(t) \end{array} \right\} \\ (F_{z,Bz}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{z,Bz}(t))^2, F_{z,z}(t)F_{Bz,Bz}(t), F_{z,Bz}(2t)F_{Bz,z}(t), \\ F_{Bz,z}(t), F_{z,Bz}(2t)F_{Bz,Bz}(t) \end{array} \right\} \\ &= (F_{z,Bz}(t))^2. \end{aligned}$$

Owing to Lemma 2.20, we have $Bz = z = Tz$ which shows that z is a common fixed point of the pair (B, T). Therefore z is a common fixed point of both the pairs (A, S) and (B, T). The uniqueness of common fixed point is an easy consequence of inequality (3.1).

The following example illustrates Theorem 3.4.

Example 3.5. Let $(X, \mathcal{F}, \triangle)$ be a Menger space, where $X = [1, 15)$, with t-norm $\triangle$ is defined by $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Define the self mappings A, B, S and T by

$$A(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 14, & \text{if } x \in (1, 3]. \end{cases} \quad B(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 5, & \text{if } x \in (1, 3]. \end{cases}$$

$$S(x) = \begin{cases} 1, & \text{if } x = 1; \\ 6, & \text{if } x \in (1, 3]; \\ \frac{x+1}{4}, & \text{if } x \in (3, 15). \end{cases} \quad T(x) = \begin{cases} 1, & \text{if } x = 1; \\ 11, & \text{if } x \in (1, 3]; \\ x-2, & \text{if } x \in (3, 15). \end{cases}$$

Taking $\{x_n\} = \{3 + \frac{1}{n}\}$, $\{y_n\} = \{1\}$ or $\{x_n\} = \{1\}$, $\{y_n\} = \{3 + \frac{1}{n}\}$, it is clear that both the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in S(X) \cap T(X).$$

Then $A(X) = \{1, 14\} \nsubseteq [1, 13) = T(X)$ and $B(X) = \{1, 5\} \nsubseteq [1, 4) \cup \{6\} = S(X)$. Thus, all the conditions of Theorem 3.4 are satisfied for some fixed $k \in (0, 1)$ and 1 is the unique common fixed point of the pairs (A, S) and (B, T). Here, it is worth noting that in this example $S(X)$ and $T(X)$ are not closed subsets of X. Also, all the involved mappings are even discontinuous at their unique common fixed point 1.

Remark 3.6. Theorem 3.4 improves the results of Cho et al. [4, Theorem 4.2] and Pathak et al. [21, Theorem 4.2] .

Theorem 3.7. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm satisfying inequality (3.1) of Theorem 3.1 and conditions (i)-(iv) of Lemma 3.2. Then A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof In view of Lemma 3.2, the pairs (A, S) and (B, T) share the (CLR_{ST}) property, that is, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(X) \cap T(X)$. The rest of the proof can be completed on the lines of the proof of Theorem 3.4, therefore details are omitted.

Example 3.8. In the setting of Example 3.5, replace the self mappings A, B, S and T by the following besides retaining the rest:

$$A(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 10, & \text{if } x \in (1, 3]. \end{cases} \quad B(x) = \begin{cases} 1, & \text{if } x \in \{1\} \cup (3, 15); \\ 4, & \text{if } x \in (1, 3]. \end{cases}$$

$$S(x) = \begin{cases} 1, & \text{if } x = 1; \\ 4, & \text{if } x \in (1, 3]; \\ \frac{x+1}{4}, & \text{if } x \in (3, 15). \end{cases} \quad T(x) = \begin{cases} 1, & \text{if } x = 1; \\ 10 + x, & \text{if } x \in (1, 3]; \\ x-2, & \text{if } x \in (3, 15). \end{cases}$$

Clearly, both the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 1 \in S(X) \cap T(X).$$

Notice that $A(X) = \{1, 10\} \subset [1, 13] = T(X)$ and $B(X) = \{1, 4\} \subset [1, 4] = S(X)$. Also the remaining conditions of Theorem 3.7 can be easily verified for some fixed $k \in (0, 1)$ while 1 is the unique common fixed point of the pairs (A, S) and (B, T). Here, it is worth noting that Theorem 3.4 can not be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of X. Also, all the involved mappings are even discontinuous at their unique common fixed point 1.

By choosing A, B, S and T suitably, we can deduce corollaries for a pair as well as for a triode of self mappings. The details of two possible corollaries for a triode of mappings are not included. As a sample, we obtain the following natural result for a pair of self mappings with an independent proof.

Theorem 3.9. *Let A and S be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Suppose that*

(i) *the pair (A, S) satisfies the (CLR_S) property,*
(ii) *there exists a constant $k \in (0, 1)$ such that*

$$(F_{Ax,Ay}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Sx,Sy}(t))^2, F_{Sx,Ax}(t)F_{Sy,Ay}(t), F_{Sx,Ay}(2t)F_{Sy,Ax}(t), \\ F_{Sy,Ax}(t), F_{Sx,Ay}(2t)F_{Sy,Ay}(t) \end{array} \right\}, \tag{3.4}$$

for all $x, y \in X$ and $t > 0$. Then (A, S) has a coincidence point. Moreover if the pair (A, S) is weakly compatible then it has a unique common fixed point in X.

Proof Since the pair (A, S) satisfies the (CLR_S) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = z$. We assert that $Au = Su$. On using inequality (3.4) with $x = u$, $y = x_n$, we get

$$(F_{Au,Ax_n}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Su,Sx_n}(t))^2, F_{Su,Au}(t)F_{Sx_n,Ax_n}(t), F_{Su,Ax_n}(2t)F_{Sx_n,Au}(t), \\ F_{Sx_n,Au}(t), F_{Su,Ax_n}(2t)F_{Sx_n,Ax_n}(t) \end{array} \right\}.$$

Taking limit $n \to \infty$, we have

$$\begin{aligned} (F_{Au,z}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{z,z}(t))^2, F_{z,Au}(t)F_{z,z}(t), F_{z,z}(2t)F_{z,Au}(t), \\ F_{z,Au}(t), F_{z,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Au,z}(t))^2. \end{aligned}$$

On employing Lemma 2.20, we have $Au = z$. Therefore $Au = Su = z$ and hence u is a coincidence point of (A, S). Since the pair (A, S) is weakly compatible, therefore $Az = ASu = SAu = Sz$. Putting $x = u$, $y = x_n$ in inequality (3.4), we get

$$(F_{Az,Ax_n}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{Sz,Sx_n}(t))^2, F_{Sz,Az}(t)F_{Sx_n,Ax_n}(t), F_{Sz,Ax_n}(2t)F_{Sx_n,Az}(t), \\ F_{Sx_n,Az}(t), F_{Sz,Ax_n}(2t)F_{Sx_n,Ax_n}(t) \end{array} \right\}.$$

Taking limit $n \to \infty$, we have

$$\begin{aligned} (F_{Az,z}(kt))^2 &\geq \min \left\{ \begin{array}{c} (F_{Az,z}(t))^2, F_{Az,Az}(t)F_{z,z}(t), F_{Az,z}(2t)F_{z,Az}(t), \\ F_{z,Az}(t), F_{Az,z}(2t)F_{z,z}(t) \end{array} \right\} \\ &= (F_{Az,z}(t))^2. \end{aligned}$$

In view of Lemma 2.20, we have $Az = z = Sz$. Therefore z is a common fixed point of the pair (A, S). The uniqueness of common fixed point is an easy consequence of inequality (3.4).

Example 3.10. Let $(X, \mathcal{F}, \triangle)$ be a Menger space, where $X = [2, 15)$, with t-norm $\triangle$ is defined by $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Now we define the self mappings A and S by

$$A(x) = \begin{cases} 2, & \text{if } x \in \{2\} \cup (3, 15); \\ 9, & \text{if } x \in (2, 3]. \end{cases} \quad S(x) = \begin{cases} 2, & \text{if } x = 2; \\ 10, & \text{if } x \in (2, 3]; \\ \frac{x+1}{2}, & \text{if } x \in (3, 15). \end{cases}$$

Taking $\{x_n\} = \{3 + \frac{1}{n}\}_{n\in\mathbb{N}}$ or $\{x_n\} = \{2\}$, it is clear that the pair (A, S) satisfies the (CLR_S) property

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2 \in S(X).$$

It is noted that $A(X) = \{2, 9\} \nsubseteq [2, 8) \cup \{10\} = S(X)$. Thus, all the conditions of Theorem 3.9 are satisfied and 2 is a unique common fixed point of the pair (A, S). Also, all the involved mappings are even discontinuous at their unique common fixed point 2.

4. APPLICATION

The following definition is essentially contained in Imdad et al. [6].

Definition 4.1. [6] Two families of self mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ are said to be pairwise commuting if

(i) $A_iA_j = A_jA_i$ for all $i, j \in \{1, 2, \dots, m\}$,
(ii) $S_kS_l = S_lS_k$ for all $k, l \in \{1, 2, \dots, n\}$,
(iii) $A_iS_k = S_kA_i$ for all $i \in \{1, 2, \dots, m\}$ and $k \in \{1, 2, \dots, n\}$.

Now, we utilize Definition 4.1 (which is indeed a natural extension of commutativity condition to two finite families) to prove a common fixed point theorem for four finite families of weakly compatible mappings in Menger space (as an application of Theorem 3.4).

Theorem 4.2. *Let $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm with $A = A_1A_2\dots A_m, B = B_1B_2\dots B_n, S = S_1S_2\dots S_p$ and $T = T_1T_2\dots T_q$ satisfying inequality (3.1) of Theorem 3.1 such that the pairs (A, S) and (B, T) share the (CLR_{ST}) property, then (A, S) and (B, T) have a point of coincidence each.*

Then $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ have a unique common fixed point provided the pairs of families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_h\})$ commute pairwise wherein $i \in \{1, 2, \dots, m\}, k \in \{1, 2, \dots, p\}, r \in \{1, 2, \dots, n\}$ and $h \in \{1, 2, \dots, q\}$.

Proof Owing to pairwise commuting property, we can prove that $AS = SA$ as

$$\begin{aligned} AS &= (A_1A_2\dots A_m)(S_1S_2\dots S_p) = (A_1A_2\dots A_{m-1})(A_mS_1S_2\dots S_p) \\ &= (A_1A_2\dots A_{m-1})(S_1S_2\dots S_pA_m) = (A_1A_2\dots A_{m-2})(A_{m-1}S_1S_2\dots S_pA_m) \\ &= (A_1A_2\dots A_{m-2})(S_1S_2\dots S_pA_{m-1}A_m) = \dots = A_1(S_1S_2\dots S_pA_2\dots A_{m-1}A_m) \\ &= (S_1S_2\dots S_p)(A_1A_2\dots A_m) = SA. \end{aligned}$$

Similarly, we can also easily prove that $BT = TB$ so that the pairs (A, S) and (B, T) are weakly compatible. Now using Theorem 3.4, we conclude that A, B, S and T have a unique common fixed point z in X.

Now, we prove that w remains the fixed point of all the component mappings. To do this, consider

$$\begin{aligned} A(A_iw) &= ((A_1A_2\dots A_m)A_i)\,w = (A_1A_2\dots A_{m-1})(A_mA_i)w \\ &= (A_1A_2\dots A_{m-1})(A_iA_m)w = (A_1A_2\dots A_{m-2})(A_{m-1}A_iA_m)w \\ &= (A_1A_2\dots A_{m-2})(A_iA_{m-1}A_m)w = \dots = A_1(A_iA_2\dots A_m)w \\ &= (A_1A_i)(A_2\dots A_m)w \\ &= (A_iA_1)(A_2\dots A_m)w = A_i(A_1A_2\dots A_m)w = A_iw. \end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} A(S_k w) &= S_k(Aw) = S_k w, S(S_k w) = S_k(Sw) = S_k w, S(A_i w) = A_i(Sw) = A_i w, \\ B(B_r w) &= B_r(Bw) = B_r w, B(T_h w) = T_h(Bw) = T_h w, T(T_h w) = T_h(Tw) = T_h w, \\ T(B_r w) &= B_r(Tw) = B_r w, \end{aligned}$$

which shows that (for all i, r, k and h) $A_i w$ and $S_k w$ are other fixed point of the pair (A, S) whereas $B_r w$ and $T_h w$ are other fixed points of the pair (B, T).

Now appealing to the uniqueness of common fixed points of mappings A, B, S and T, we get

$$w = A_i w = S_k w = B_r w = T_h w,$$

for all $i \in \{1, 2, \ldots, m\}, k \in \{1, 2, \ldots, p\}, r \in \{1, 2, \ldots, n\}, h \in \{1, 2, \ldots, q\}$, which shows that w is the unique common fixed point of $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$.

By setting $A_1 = A_2 = \ldots = A_m = A, B_1 = B_2 = \ldots = B_n = B, S_1 = S_2 = \ldots = S_p = S$ and $T_1 = T_2 = \ldots = T_q = T$ in Theorem 4.2, we deduce the following:

Corollary 4.3. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. Suppose that*

(i) *the pairs (A^m, S^p) and (B^n, T^q) share the (CLR_{S^p,T^q}) property,*
(ii) *there exists a constant $k \in (0, 1)$ such that*

$$(F_{A^m x, B^n y}(kt))^2 \geq \min \left\{ \begin{array}{c} (F_{S^p x, T^q y}(t))^2, F_{S^p x, A^m x}(t) F_{T^q y, B^n y}(t), \\ F_{S^p x, B^n y}(2t) F_{T^q y, A^m x}(t), \\ F_{T^q y, A^m x}(t), F_{S^p x, B^n y}(2t) F_{T^q y, B^n y}(t) \end{array} \right\}, \quad (4.1)$$

for all $x, y \in X, t > 0$, m, n, p and q are fixed positive integers. Then A, B, S and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

Remark 4.4. The results similar to Theorem 4.2 and Corollary 4.3 can be obtained in respect of Theorem 3.9.

Remark 4.5. Theorem 4.2 and Corollary 4.3 extend the results of Cho et al. [4] and Pathak et al. [21] to four finite families of self mappings.

5. CORRESPONDING RESULTS IN METRIC SPACES

In this section, as a sample, we utilize Theorem 3.4 to derive corresponding common fixed point theorem in metric space.

Theorem 5.1. *Let A, B, S and T be self mappings of a metric space (X, d). Suppose that*

(i) *the pairs (A, S) and (B, T) share the (CLR_{ST}) property,*
(ii) *there exists a constant $k \in (0, 1)$ such that*

$$(d(Ax, By))^2 \leq k \max \left\{ \begin{array}{c} (d(Sx, Ty))^2, d(Sx, Ax) d(Ty, By), \\ \frac{1}{2} d(Sx, By) d(Ty, Ax), d(Ty, Ax), \\ \frac{1}{2} d(Sx, By) d(Ty, By) \end{array} \right\}, \quad (5.1)$$

for all $x, y \in X$. Then the pairs (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof Define $F_{x,y}(t) = H(t - d(x, y))$ and $\triangle(a, b) = \min\{a, b\}$, for all $a, b \in [0, 1]$. Then metric space (X, d) can be realized as a Menger space $(X, \mathcal{F}, \triangle)$. It is straightforward to notice that Theorem 5.1 satisfies all the conditions of Theorem 3.4. Also inequality (5.1) of Theorem 5.1 implies inequality (3.1) of Theorem 3.1. For any

$x, y \in X$ and $t > 0$, $F_{Ax,By}(kt) = 1$ if $kt > d(Ax, By)$ which confirms the verification of inequality (3.1) of Theorem 3.1. Otherwise, if $kt \leq d(Ax, By)$, then

$$t \leq \max \left\{ \begin{array}{c} (d(Sx,Ty))^2, d(Sx,Ax)d(Ty,By), \frac{1}{2}d(Sx,By)d(Ty,Ax), \\ d(Ty,Ax), \frac{1}{2}d(Sx,By)d(Ty,By) \end{array} \right\},$$

which shows that inequality (3.1) of Theorem 3.1 is satisfied. Thus, all the conditions of Theorem 3.4 are satisfied so that conclusions follow immediately from Theorem 3.4.

Remark 5.2. The results similar to Theorem 5.1 can also be outlined in respects of Theorem 3.7, Theorem 3.9, Theorem 4.2 and Corollary 4.3.

Remark 5.3. Theorem 5.1 improves the results of Cho et al. [4, Theorem 4.3] and Pathak et al. [21, Theorem 4.3].

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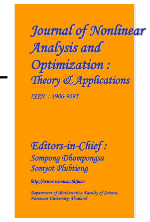

POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH P-LAPLACIAN

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ABSTRACT. In this paper, we study the existence of positive solution to boundary value problem for fractional differential equation with a one-dimensional p-Laplacian operator

$$\begin{cases} D^{\sigma}_{0+}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases}$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative of order $1 < \sigma \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$ and f is a lower semi-continuous function. By using Krasnoselskii's fixed point theorems in a cone, the existence of one positive solution and multiple positive solutions for nonlinear singular boundary value problems is obtained.

KEYWORDS : Cone; Multi point boundary value problem; Fixed point theorem; Riemann-Liouville fractional derivative.
AMS Subject Classification: 34B07 34D05 34L20.

1. INTRODUCTION

The purpose of this paper is to study the existence of positive solutions for the following m-point boundary value problem for fractional differential equation with p-Laplacian

$$\begin{cases} D^{\sigma}_{0+}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1.1}$$

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where D^{α}_{0+} is the Riemann-Liouville fractional derivative of order $1 < \sigma \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $m > 2$ ($m \in \mathbb{N}$), $a, b, c, d \geq 0$, $\rho = ac + bc + ad > 0$, $\xi_i \in (0, 1)$, $a_i, b_i \in (0, +\infty)$ ($i = 1, 2, \ldots, m-2$), $g \in C((0, 1); [0, +\infty))$ and $0 < \int_0^1 g(r)dr < \infty$, and f is a nonnegative, lower semi-continuous function defined on $[0, +\infty)$.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details, see [5, 8, 9] and the references therein. In [12], Liu, and Jia investigated the existence of multiple solutions for problem:

$$\begin{cases} {}^cD^{\sigma}_{0+}(p(t)u'(t)) + q(t)f(t, u(t)) = 0, \quad t > 0, \quad 0 < \sigma < 1, \\ p(0)u'(0) = 0, \\ \lim_{t\to\infty} u(t) = \int_0^{+\infty} g(t)u(t)dt, \end{cases}$$

where ${}^cD^{\sigma}_{0+}$ is the standard Caputo derivative of order σ. Some existence results were given for the problem (1.1) with $\sigma = 2$ by Yanga et al. [24] and Zhao et al. [25].

The solution of differential equations of fractional order is much involved. Some analytical methods are presented, such as the popular Laplace transform method [20, 21], the Fourier transform method [15], the iteration method [22] and Green function method [14, 23]. Numerical schemes for solving fractional differential equations are introduced, for example, in [3, 4, 17]. Recently, a great deal of effort has been expended over the last years in attempting to find robust and stable numerical as well as analytical methods for solving fractional differential equations of physical interest. The Adomian decomposition method [18], homotopy perturbation method [19], homotopy analysis method [2], differential transform method [16] and variational method [6] are relatively new approaches to provide an analytical approximate solution to linear and nonlinear fractional differential equations.
The existence of solutions of initial value problems for fractional order differential equations have been studied in the literature [1, 11, 20, 22] and the references therein.

In this paper, We show that the problem (1.1) has positive solutions by using Krasnoselskii's fixed point theorems in a cone.

The paper has been organized as follows. In Sect. 2, we give some preliminary facts and provide basic properties which are needed later. We also state the Krasnoselakii's fixed point theorem. In Sect. 3, we establish the existence of at least one or multiple positive solutions result for problem (1.1). In Section 4 we give an example as application.

2. PRELIMINARIES

In this section, we present some notation and preliminary lemmas that will be used in the proofs of the main results.

We work in the space $C([0, 1])$ with respect to the norm $||u|| = \max_{0\leq t\leq 1} |u(t)|$. For convenience, we make the following assumptions:

(H1) $f \in C([0, +\infty); [0, +\infty))$;

(H1*) f is a nonnegative, lower semi-continuous function defined on $[0, +\infty)$, i.e. $\exists I \subset [0, +\infty)$; $\forall x_n \in I$, $x_n \to x_0$ ($n \to \infty$), one has $f(x_0) \leq \underline{\lim}_{n\to\infty} f(x_n)$.

Moreover, f has only a finite number of discontinuity points in each compact subinterval of $[0, +\infty)$.

(H2) $g \in C((0,1); [0, +\infty))$ and $0 < \int_0^1 g(r)dr < +\infty$. Moreover, $g(t)$ does not vanish identically on any subinterval of $[0, 1]$;

(H3) $a, b, c, d \geq 0$, $\rho = ac + bc + ad > 0$, $\xi_i \in (0, 1)$, $a_i, b_i \in (0, +\infty)$ ($i = 1, 2, \ldots, m-2$), $\rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) > 0$ and $\Delta < 0$, where

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix}$$

and

$$\psi(t) = b + at, \quad \varphi(t) = c + d - ct, \quad t \in [0, 1], \tag{2.1}$$

are linearly independent solutions of the equation $x''(t) = 0$, $t \in [0, 1]$. Obviously, ψ is non-decreasing on $[0, 1]$ and φ is non-increasing on $[0, 1]$.

Definition 2.1. Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:
(1) $x \in P, \mu \geq 0$ implies $\mu x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ have absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.4. *([7]). The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L(0, 1)$.*

Lemma 2.5. *([7]). Let $\alpha > 0$. Then the differential equation*

$$D_{0+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, there $n - 1 < \alpha \leq n$.

Lemma 2.6. *([7]). Let $\alpha > 0$. Then the following equality holds for $u \in L(0, 1)$, $D_{0+}^{\alpha} u \in L(0, 1)$;*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

$c_i \in \mathbb{R}$, $i = 1, \ldots, n$, there $n - 1 < \alpha \leq n$.

In the following, we present the Green function of fractional differential equation boundary value problem.

Let $y(t) = \phi_p(u''(t))$, then the problem

$$\begin{cases} D^{\sigma}_{0+}(\phi_p(u''(t))) - g(t)f(u(t)) = 0, & t \in (0,1), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \end{cases}$$

is turned into problem

$$\begin{cases} D^{\sigma}_{0+}y(t) - g(t)f(u(t)) = 0, & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2.2}$$

Lemma 2.7. *If (H1) and (H2) hold, then the boundary value problem (2.2) has a unique solution*

$$y(t) = -\int_0^1 H(t,s)g(s)f(u(s))ds, \tag{2.3}$$

where

$$H(t,s) = \begin{cases} \frac{t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \le s \le t \le 1, \\ \frac{t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

proof. According to Lemma 2.6, we can obtain that

$$y(t) = I^{\sigma}_{0+}(g(t)f(u(t))) - c_1 t^{\sigma-1} - c_2 t^{\sigma-2} = \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}g(s)f(u(s))ds - c_1 t^{\sigma-1} - c_2 t^{\sigma-2}.$$

By the boundary conditions of (2.2), there are $c_2 = 0$ and $c_1 = \frac{1}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1}g(s)f(u(s))ds$.

Thus, the unique solution of problem (2.2) is

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}g(s)f(u(s))ds - \frac{t^{\sigma-1}}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1}g(s)f(u(s))ds \\ &= -\int_0^t \frac{t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1}}{\Gamma(\sigma)}g(s)f(u(s))ds - \int_t^1 \frac{t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}g(s)f(u(s))ds \\ &= -\int_0^1 H(t,s)g(s)f(u(s))ds. \end{aligned}$$

□

Lemma 2.8. *If (H3) holds, then for $y \in C[0,1]$, the boundary value problem*

$$\begin{cases} u''(t) = \phi_q(y(t))), & t \in (0,1), \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{2.5}$$

has a unique solution

$$u(t) = -\Big[\int_0^1 G(t,s)\phi_q(y(s))ds + A(\phi_q(y(s)))\psi(t) + B(\phi_q(y(s)))\varphi(t)\Big], \tag{2.6}$$

where

$$G(t,s)=\frac{1}{\rho}\begin{cases}\varphi(t)\psi(s), & 0\le s\le t\le 1,\\ \varphi(s)\psi(t), & 0\le t\le s\le 1,\end{cases} \tag{2.7}$$

$$A(\phi_q(y(s)))=\frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\phi_q(y(s))ds & \rho-\sum_{i=1}^{m-2}a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2}b_i\int_0^1 G(\xi_i,s)\phi_q(y(s))ds & -\sum_{i=1}^{m-2}b_i\varphi(\xi_i)\end{vmatrix} \tag{2.8}$$

$$B(\phi_q(y(s)))=\frac{1}{\Delta}\begin{vmatrix}-\sum_{i=1}^{m-2}a_i\psi(\xi_i) & \sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\phi_q(y(s))ds\\ \rho-\sum_{i=1}^{m-2}b_i\psi(\xi_i) & \sum_{i=1}^{m-2}b_i\int_0^1 G(\xi_i,s)\phi_q(y(s))ds\end{vmatrix} \tag{2.9}$$

proof. The proof is similar to that of Lemma 5.5.1 in [13], so we omit it here. □
we assume that $\theta\in(0,\frac{1}{2})$. Furthermore, for convenience, we set

$$\Lambda_1=\min\Big\{\frac{\varphi(1-\theta)}{\varphi(0)},\frac{\psi(\theta)}{\psi(1)}\Big\},\qquad \Gamma=\min\{\Lambda_1,\frac{\Lambda_2}{\Lambda_3}\},$$
$$\Lambda_2=\min\{\min_{\theta\le t\le 1-\theta}\varphi(t),\min_{\theta\le t\le 1-\theta}\psi(t),1\},\qquad \Lambda_3=\max\{1,||\varphi||,||\psi||\}.$$

Lemma 2.9. *Let $\rho,\Delta\neq 0$ and $\theta\in(0,\frac{1}{2})$, then we have the following results:*

$$0\le G(t,s)\le G(s,s),\quad \text{for } t,s\in[0,1], \tag{2.10}$$

and

$$G(t,s)\ge\Lambda_1 G(s,s),\quad \text{for } t\in\Big[\theta,1-\theta\Big] \text{ and } s\in[0,1]. \tag{2.11}$$

proof. The inequality (2.10) is obvious. In following, we are going to verify the inequality (2.11). Indeed, when $t\in[\theta,1-\theta]$, $s\in[0,1]$, we have

$$\begin{aligned}\frac{G(t,s)}{G(s,s)} &= \begin{cases}\frac{\varphi(t)}{\varphi(s)}, & 0\le s\le t\le 1-\theta,\\ \frac{\psi(t)}{\psi(s)}, & \theta\le t\le s\le 1,\end{cases}\\ &\ge \begin{cases}\frac{\varphi(1-\theta)}{\varphi(0)}, & 0\le s\le t\le 1-\theta,\\ \frac{\psi(\theta)}{\psi(1)}, & \theta\le t\le s\le 1,\end{cases}\\ &\ge \Lambda_1.\end{aligned}$$

This completes the proof. □

Proposition 2.10. *For $t,s\in[0,1]$, we have*

$$0\le H(t,s)\le H(s,s)\le\frac{1}{\Gamma(\sigma)}\big(\frac{1}{4}\big)^{\sigma-1}.$$

Proposition 2.11. *Let $\theta\in(0,\frac{1}{2})$, then there exists a positive function $\varrho\in C(0,1)$ such that*

$$\min_{\theta\le t\le 1-\theta}H(t,s)\ge\varrho(s)H(s,s),\qquad s\in(0,1).$$

proof. For $\theta\in(0,\frac{1}{2})$, we define

$$g_1(t,s)\ =\ t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1},\qquad 0\le s\le t\le 1,$$

$$g_2(t,s) \quad = \quad t^{\sigma-1}(1-s)^{\sigma-1} \qquad\qquad 0 \le t \le s \le 1.$$

Then

$$\begin{aligned} \frac{d}{dt}g_1(t,s) &= (\sigma-1)\Big(t^{\sigma-2}(1-s)^{\sigma-1} - (t-s)^{\sigma-2}\Big) \\ &= (\sigma-1)t^{\sigma-2}\Big((1-s)^{\sigma-1} - (1-\frac{s}{t})^{\sigma-2}\Big) \\ &\le (\sigma-1)t^{\sigma-2}\Big((1-s)^{\sigma-1} - (1-s)^{\sigma-2}\Big). \end{aligned}$$

which implies that $g_1(\cdot,s)$ is nonincreasing for all $s \in (0,1]$. Also, we have $g_2(\cdot,s)$ is nondecreasing for all $s \in (0,1)$. Then, we have

$$\begin{aligned} \min_{\theta\le t\le 1-\theta} H(t,s) &= \begin{cases} \frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, & s\in(0,\theta], \\ \min\{\frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, \frac{g_2(\theta,s)}{\Gamma(\sigma)}\}, & s\in[\theta,1-\theta], \\ \frac{g_2(\theta,s)}{\Gamma(\sigma)}, & s\in[1-\theta,1). \end{cases} \\ &= \begin{cases} \frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, & s\in(0,\mu], \\ \frac{g_2(\theta,s)}{\Gamma(\sigma)}, & s\in[\mu,1). \end{cases} \\ &= \begin{cases} \frac{(1-\theta)^{\sigma-1}(1-s)^{\sigma-1}-(1-\theta-s)^{\sigma-1}}{\Gamma(\sigma)}, & s\in(0,\mu], \\ \frac{\theta^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)} & s\in[\mu,1), \end{cases} \end{aligned}$$

where $\theta < \mu < 1-\theta$ is solution of equation

$$(1-\theta)^{\sigma-1}(1-\mu)^{\sigma-1} - (1-\theta-\mu)^{\sigma-1} = \theta^{\sigma-1}(1-\mu)^{\sigma-1}.$$

It follows from the monotonicity of g_1 and g_2 that

$$\max_{0\le t\le 1} H(t,s) = H(s,s) = \frac{s^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, \quad s\in(0,1).$$

Therefore, we set

$$\varrho(s) = \begin{cases} \frac{(1-\theta)^{\sigma-1}(1-s)^{\sigma-1}-(1-\theta-s)^{\sigma-1}}{s^{\sigma-1}(1-s)^{\sigma-1}}, & s\in(0,\mu], \\ \left(\frac{\theta}{s}\right)^{\sigma-1} & s\in[\mu,1). \end{cases}$$

Thus, we complete the proof. □

From Lemmas 2.7 and 2.8, we know that $u(t)$ is a solution of the problem (1.1) if and only if

$$u(t) = \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t), \tag{2.12}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Lemma 2.12. *Let (H1), (H2) and (H3) hold. Then the solution u of the problem (1.1) satisfies*

(i) $u(t) \ge 0$, *for* $t \in [0,1]$,

and

(ii) $\min_{\theta\le t\le 1-\theta} u(t) \ge \Gamma||u||$.

proof. (i) By Lemma 2.9, Proposition 2.10, (2.3), (2.6)-(2.9) and the property of function ϕ_q it is obvious that we have

$$G(t,s) \geq 0, \quad \phi_q(W(s)) \geq 0, \quad A(\phi_q(W(s))) \geq 0, \quad B(\phi_q(W(s))) \geq 0,$$

so we get $u(t) \geq 0$.

(ii) From Lemma 2.9 and (2.12), for $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)\phi_q(W(s))ds + \frac{\Lambda_2}{\Lambda_3} \cdot \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))] \\ &\geq \Gamma\Big[\int_0^1 G(s,s)\phi_q(W(s))ds + \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))]\Big] \\ &\geq \Gamma||u||. \end{aligned}$$

Therefore, we get $\min_{\theta \leq t \leq 1-\theta} u(t) \geq \Gamma||u||$. □

Then, choose a cone K is $C^1([0,1])$, by

$$K = \{u \in C[0,1] \mid u(t) \geq 0, \min_{\theta \leq t \leq 1-\theta} u(t) \geq \Gamma\|u\|\}.$$

Define an operator T by

$$(Tu)(t) = \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t), \quad (2.13)$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

It is clear that the existence of a positive solution for the system (1.1) is equivalent to the existence of nontrivial fixed point of T in K.

Lemma 2.13. *Suppose that the conditions (H1), (H2) and (H3) hold, then* $T(K) \subseteq K$ *and* $T : K \to K$ *is completely continuous.*

proof. For any $u \in K$, by (2.13), we obtain $(Tu)(t) \geq 0$ and, for $t \in [0,1]$,

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t) \\ &\leq \int_0^1 G(s,s)\phi_q(W(s))ds + \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))]. \end{aligned}$$

Thus, $||Tu|| \leq \int_0^1 G(s,s)\phi_q(W(s))ds + \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))]$.

On the other hand, for $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t) \end{aligned}$$

$$\begin{aligned}
&\geq \Lambda_1 \int_0^1 G(s,s)\phi_q(W(s))ds + \frac{\Lambda_2}{\Lambda_3}\cdot\Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))] \\
&\geq \Gamma\Big[\int_0^1 G(s,s)\phi_q(W(s))ds + \Lambda_3[A(\phi_q(W(s))) + B(\phi_q(W(s)))]\Big] \\
&\geq \Gamma||Tu||.
\end{aligned}$$

Therefore, we get $TK \subseteq K$

By conventional arguments and Ascoli-Arzela theorem, one can prove $T: K \to K$ is completely continuous, so we omit it here. □

Our approach is based on the following Guo-Krasnoselskii fixed point theorem of cone expansion-compression type [10].

Theorem 2.14. *Let E be a Banach space and $K \subsetneqq E$ a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T: K\bigcap(\overline{\Omega}_2\backslash\Omega_1) \to K$ be a completely continuous operator. In addition suppose either*
(A) $\|Tu\| \leq \|u\|, \forall u \in K\cap\partial\Omega_1$ and $\|Tu\| \geq \|u\|, \forall u \in K\cap\partial\Omega_2$ or
(B) $\|Tu\| \geq \|u\|, \forall u \in K\cap\partial\Omega_1$ and $\|Tu\| \leq \|u\|, \forall u \in K\cap\partial\Omega_2$
holds. Then T has a fixed point in $K\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3. MAIN RESULTS

We define $\Omega_l = \{u \in K : ||u|| < l\}$, $\partial\Omega_l = \{u \in K : ||u|| = l\}$, where $l > 0$.

If $u \in \partial\Omega_l$, for $t \in [\theta, 1-\theta]$, we have $\Gamma l \leq u \leq l$.

For convenience, we introduce the following notations. Let

$$\begin{aligned}
f_l &= \inf\Big\{\frac{f(u)}{\phi_p(l)}\Big|u \in [\Gamma l, l]\Big\}, \qquad f^l = \sup\Big\{\frac{f(u)}{\phi_p(l)}\Big|u \in [0, l]\Big\},\\
f_\varrho &= \liminf_{u\to\varrho}\frac{f(u)}{\phi_p(u)}, \quad (\varrho := 0^+ \text{ or } +\infty),\\
f^\varrho &= \limsup_{u\to\varrho}\frac{f(u)}{\phi_p(u)}, \quad (\varrho := 0^+ \text{ or } +\infty),\\
\eta &= \min_{\theta\leq s\leq 1-\theta}\varrho(s),\\
\frac{1}{\omega} &= \big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\Big[\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big) + \Lambda_3\widetilde{A} + \Lambda_3\widetilde{B}\Big],\\
\frac{1}{M} &= \big(\frac{\eta}{\Gamma(\sigma)}\big)^{q-1}\theta^{2(\sigma-1)(q-1)}\Big[\frac{\Lambda_1}{\rho}\varphi(1-\theta)\psi(\theta)\phi_q\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big) + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}\Big].
\end{aligned}$$

We always assume that (H1) hold in the following theorems.

Theorem 3.1. *Suppose that there exist constants $r, R > 0$ with $r < \Gamma R$ for $r < R$, such that the following two conditions*

(H4) $f^r \leq \phi_p(\omega)$,

and

(H5) $f_R \geq \phi_p(M)$,

hold. Then the problem (1.1) has at least one positive solution $u \in K$ such that

$$0 < r \leq ||u|| \leq R.$$

proof. Case 1. We shall prove that the result holds when (H1) is satisfied. Without loss of generality, we suppose that $r < \Gamma R$ for $r < R$.

By (H4), Proposition 2.10, (2.8) and (2.9), for $u \in \Omega_r$, we have

$$\begin{aligned} A(\phi_q(W)) &\leq \frac{\big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r}{\Delta}\left| \begin{array}{cc} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 g(\tau)d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 g(\tau)d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{array}\right|, \\ &= \big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r\widetilde{A}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} B(\phi_q(W)) &\leq \frac{\big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r}{\Delta}\left| \begin{array}{cc} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 g(\tau)d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 g(\tau)d\tau)ds \end{array}\right| \\ &= \big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r\widetilde{B}. \end{aligned} \tag{3.2}$$

Therefore, by (H4), Lemma 2.9, (2.13), (3.1) and (3.2), for $t \in [0,1]$ and $u \in \Omega_r$, we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t) \\ &\leq \big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big) \\ &\quad +\big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r\widetilde{A}\psi(t) + \big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r\widetilde{B}\varphi(t) \\ &\leq \big(\frac{1}{\Gamma(\sigma)}\big)^{q-1}\big(\frac{1}{4}\big)^{(\sigma-1)(q-1)}\omega r\Big[\Big(\int_0^1 G(s,s)ds\Big)\phi_q\Big(\int_0^1 g(\tau)d\tau\Big) + \Lambda_3\widetilde{A} + \Lambda_3\widetilde{B}\Big] \\ &= r = ||u||. \end{aligned}$$

This implies that $||Tu|| \leq ||u||$ for $u \in \Omega_r$.

on the other hand, by (H5), (2.13), Proposition 2.11, (2.8) and (2.9), for $u \in \Omega_R$, we have

$$\begin{aligned} A(\phi_q(W)) &\geq \frac{\big(\frac{\eta}{\Gamma(\sigma)}\big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR}{\Delta}\left| \begin{array}{cc} \sum_{i=1}^{m-2} a_i \int_\theta^{1-\theta} G(\xi_i,s)\phi_q(\int_\theta^{1-\theta} g(\tau)d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_\theta^{1-\theta} G(\xi_i,s)\phi_q(\int_\theta^{1-\theta} g(\tau)d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{array}\right|, \\ &= \big(\frac{\eta}{\Gamma(\sigma)}\big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR\widehat{A}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} B(\phi_q(W)) &\geq \frac{\big(\frac{\eta}{\Gamma(\sigma)}\big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR}{\Delta}\left| \begin{array}{cc} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_\theta^{1-\theta} G(\xi_i,s)\phi_q(\int_\theta^{1-\theta} g(\tau)d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_\theta^{1-\theta} G(\xi_i,s)\phi_q(\int_\theta^{1-\theta} g(\tau)d\tau)ds \end{array}\right| \\ &= \big(\frac{\eta}{\Gamma(\sigma)}\big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR\widehat{B}. \end{aligned} \tag{3.4}$$

Therefore, by (H5), Lemma 2.9, (2.13), (3.3) and (3.4), for $t \in [0,1]$ and $u \in \Omega_R$, we have

$$(Tu)(t) = \int_0^1 G(t,s)\phi_q(W(s))ds + A(\phi_q(W(s)))\psi(t) + B(\phi_q(W(s)))\varphi(t)$$

$$
\begin{aligned}
&\geq \Big(\frac{\eta}{\Gamma(\sigma)}\Big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR\Big[\frac{\Lambda_1}{\rho}\varphi(1-\theta)\psi(\theta)\phi_q\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big)+\Lambda_2\widehat{A}+\Lambda_2\widehat{B}\Big]\\
&= R=||u||.
\end{aligned}
$$

This implies that $||Tu||\geq||u||$ for $u\in\Omega_R$.

Therefore, by Theorem 2.14, it follows that T has a fixed-point u in $K\cap(\overline{\Omega_R}\setminus\Omega_r)$. This means that the problem (1.1) has at least one positive solution $u\in K$ such that $0<r\leq||u||\leq R$.

Case 2. When (H1*) holds, by applying the linear approaching method on the domain of discontinuous points of f we can establish sequence $\{f_j\}_{j=1}^{\infty}$ satisfying the following two conditions

(i) $f_j\in C[0,\infty)$ and $0\leq f_j\leq f_{j+1}$ on $[0,\infty)$,

and

(ii) $\lim_{j\to\infty} f_j=f$, $j=1,2,\ldots,$ is pointwisely convergent on $[0,\infty)$.

By virtue of proof of Case 1, we know that when $f=f_j$, the problem (1.1) has a positive solution $u_j(t)$ where

$$
\begin{aligned}
u_j(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds\\
&\quad+\frac{\psi(t)}{\Delta}\begin{vmatrix}\sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2}a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2}b_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2}b_i\varphi(\xi_i)\end{vmatrix}\\
&\quad+\frac{\varphi(t)}{\Delta}\begin{vmatrix}-\sum_{i=1}^{m-2}a_i\psi(\xi_i) & \sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds\\ \rho-\sum_{i=1}^{m-2}b_i\psi(\xi_i) & \sum_{i=1}^{m-2}b_i\int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds\end{vmatrix}\\
&= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds+\psi(t)A_j+\varphi(t)B_j,
\end{aligned}
$$

for all $t\in[0,1]$ and $r\leq||u_j||\leq R$, r,R are independent of j.

By uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$,$\varphi(t)$ and $\psi(t)$ on $[0,1]$, for any $\epsilon>0$ (small enough), there exists $\delta>0$ such that for $t_1,t_2\in[0,1]$ and $|t_1-t_2|<\delta$, one has $|G(t_1,s)-G(t_2,s)|<\epsilon$, $|\varphi(t_1)-\varphi(t_2)|<\epsilon$ and $|\psi(t_1)-\psi(t_2)|<\epsilon$. Thus, for $t_1,t_2\in[0,1]$ and $|t_1-t_2|<\delta$, one has

$$
\begin{aligned}
|u_j(t_1)-u_j(t_2)| &\leq \int_0^1|G(t_1,s)-G(t_2,s)|\cdot\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds\\
&\quad+A_j|\psi(t_1)-\psi(t_2)|+B_j|\varphi(t_1)-\varphi(t_2)|\\
&\leq \Big(\frac{1}{\Gamma(\sigma)}\Big)^{q-1}\Big(\frac{1}{4}\Big)^{(\sigma-1)(q-1)}\cdot\max_{||u_j||\leq R}f_j(u_j)\cdot\phi_q\Big(\int_0^1 g(\tau)d\tau\Big)\cdot\epsilon+A_j\cdot\epsilon+B_j\cdot\epsilon.
\end{aligned}
$$

So we get that $\{u_j\}_{j=1}^{\infty}$ are equicontinuous on $[0,1]$. Thus, by the Arzela-Asoli theorem, we know that there exists a convergent subsequence of $\{u_j\}_{j=1}^{\infty}$. For convenience, we denote this convergent subsequence with $\{u_j\}_{j=1}^{\infty}$. Without loss of generality, we suppose $\lim_{j\to\infty}u_j(t)=u(t)$, $\forall t\in[0,1]$, and $r\leq||u||\leq R$. By the Fatou's Lemma and Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
&\lim_{j\to\infty}u_j(t)\\
&\geq\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty}f_j(u_j(\tau))d\tau\Big)ds
\end{aligned}
$$

$$+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=0.5}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}$$
$$+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=0.5}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds\end{vmatrix}$$

i.e.

$$u(t)\geq\int_0^1 G(t,s)\phi_q(W(s))ds+A(\phi_q(W(s)))\psi(t)+B(\phi_q(W(s)))\varphi(t), \tag{3.5}$$

where $W(s)=\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$. On the other hand, by the conditions (i) and (ii), we have

$$\begin{aligned} u_j(t) \quad \leq & \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau\Big)ds \\ & +\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix} \\ & +\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds\end{vmatrix}, \end{aligned}$$

By the lower semi-continuity of f, taking limits in above inequality as $j\to\infty$, we have

$$\begin{aligned} u(t) \quad \leq & \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds \\ & +\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix} \\ & +\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)\phi_q(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds\end{vmatrix}, \end{aligned}$$

i.e

$$u(t)\leq\int_0^1 G(t,s)\phi_q(W(s))ds+A(\phi_q(W(s)))\psi(t)+B(\phi_q(W(s)))\varphi(t), \tag{3.6}$$

where $W(s)=\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

By (3.5) and (3.6), we have

$$u(t)=\int_0^1 G(t,s)\phi_q(W(s))ds+A(\phi_q(W(s)))\psi(t)+B(\phi_q(W(s)))\varphi(t),$$

where $W(s)=\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Therefore $u(t)$ is a positive solution of the problem (1.1). This completes the proof of Theorem 3.1. □

Similarly, we can obtain the following conclusion.

Theorem 3.2. *Suppose that there exist constants $r, R>0$ with $r<\Gamma R$ for $r<R$, such that the following two conditions*

(H4)* $f^r < \phi_p(\omega)$,
and
(H5)* $f_R > \phi_p(M)$,
hold. Then the problem (1.1) has at least one positive solution $u \in K$ *such that*

$$0 < r < ||u|| < R.$$

Theorem 3.3. *Assume that one of the following two conditions*
(H6) $f^0 \leq \phi_p(\omega)$, $\quad f_\infty \geq \phi_p(\frac{M}{\Gamma})$,
and
(H7) $f_0 \geq \phi_p(\frac{M}{\Gamma})$, $\quad f^\infty \leq \phi_p(\omega)$
is satisfied. Then the problem (1.1) has at least one positive solution.

proof. We need to do is to prove that the results of Theorem 3.3 hold when f is nonnegative and continuous on $[0, \infty)$. And by the similar proof process of Theorem 3.1 we can prove the results of Theorem 3.3 when f is nonnegative and lower semi-continuous on $[0, \infty)$.

We show that (H6) implies (H4) and (H5). Suppose that (H6) holds, then there exist r and R with $0 < r < \Gamma R$, such that

$$\frac{f(u)}{\phi_p(u)} \leq \phi_p(\omega), \quad 0 < u \leq r$$

and

$$\frac{f(u)}{\phi_p(u)} \geq \phi_p(\frac{M}{\Gamma}), \quad u \geq \Gamma R.$$

Hence, we obtain

$$f(u) \leq \phi_p(\omega)\phi_p(u) \leq \phi_p(\omega)\phi_p(r) = \phi_p(r\omega), \quad 0 < u \leq r$$

and

$$f(u) \geq \phi_p(\frac{M}{\Gamma})\phi_p(u) \geq \phi_p(\frac{M}{\Gamma})\phi_p(\Gamma R) = \phi_p(MR), \quad u \geq \Gamma R.$$

Thus, (H4) and (H5) holds.

Therefore, by Theorem 3.1, the problem (1.1) has at least one positive solution.

Now suppose that (H7) holds, then there exist $0 < r < R$ with $Mr < \omega R$ such that

$$\frac{f(u)}{\phi_p(u)} \geq \phi_p(\frac{M}{\Gamma}), \quad 0 < u \leq r. \tag{3.7}$$

and

$$\frac{f(u)}{\phi_p(u)} \leq \phi_p(\omega), \quad u \geq R. \tag{3.8}$$

By (3.7), it follows that

$$f(u) \geq \phi_p(\frac{M}{\Gamma})\phi_p(u) \geq \phi_p(\frac{M}{\Gamma})\phi_p(\Gamma r) = \phi_p(Mr), \quad \Gamma r \leq u \leq r.$$

So, the condition (H5) holds for r.

For (3.8), we consider two cases.

(i) If $f(u)$ is bounded, there exists a constant $D > 0$ such that $f(u) \leq D$, for $0 \leq u < \infty$. By (3.8), there exists a constant $\lambda \geq R$ with $Mr < \omega R \leq \lambda\omega$ satisfying $\phi_p(\lambda) \geq \max\{\phi_p(R), \frac{D}{\phi_p(\omega)}\}$ such that $f(u) \leq D \leq \phi_p(\lambda\omega)$ for $0 \leq u \leq \lambda$. This means that the condition (H4) holds for λ.

(ii) If $f(u)$ is unbounded, there exist $\lambda_1 \geq R$ with $Mr < \omega R \leq \lambda_1\omega$ such that $f(u) \leq f(\lambda_1)$ for $0 \leq u \leq \lambda_1$. This yields $f(u) \leq f(\lambda_1) \leq \phi_p(\lambda_1\omega)$ for $0 \leq u \leq \lambda_1$. Thus, condition (H4) holds for λ_1.

Therefore, by Theorem 3.1, the problem (1.1) has at least one positive solution. Theorem 3.3 is proved. □

Remark 3.4. It is obvious that Theorem 3.3 holds if f satisfies conditions $f^0 = 0$, $f_\infty = +\infty$ or $f_0 = +\infty$, $f^\infty = 0$.

In this section, we give some conclusions about the existence of multiple positive solutions. We always suppose that (H1*), (H2) and (H3) hold in the following theorems.

Theorem 3.5. *Assume that one of the following two conditions*

(H8) $f^r < \phi_p(\omega)$,

and

(H9) $f_0 \geq \phi_p(\frac{M}{\Gamma})$, $\quad f_\infty \geq \phi_p(\frac{M}{\Gamma})$

are satisfied. Then the problem (1.1) has at least two positive solutions such that

$$0 < ||u_1|| < r < ||u_2||.$$

proof. By the proof of Theorem 3.3, we can take $0 < r_1 < r < \Gamma r_2$ such that $f(u) \geq \phi_p(r_1 M)$ for $\Gamma r_1 \leq u \leq r_1$ and $f(u) \geq \phi_p(r_2 M)$ for $\Gamma r_2 \leq u \leq r_2$. Therefore, by Theorems 3.2 and 3.3, it follows that problem (1.1) has at least two positive solutions such that $0 < ||u_1|| < r < ||u_2||$. ■

Theorem 3.6. *Assume that one of the following two conditions*

(H10) $f_R > \phi_p(M)$,

and

(H11) $f^0 \leq \phi_p(\omega)$, $\quad f^\infty \leq \phi_p(\omega)$,

are satisfied. Then the problem (1.1) has at least two positive solutions such that

$$0 < ||u_1|| < R < ||u_2||.$$

Theorem 3.7. *Assume (H6) (or (H7)) holds, and there exist constants* $r_1, r_2 > 0$ *with* $r_1 M < r_2\omega$ *(or* $r_1 < \Gamma r_2$*) such that (H8) holds for* $r = r_2$ *(or* $r = r_1$*) and (H10) holds for* $R = r_1$ *(or* $R = r_2$*). Then the problem (1.1) has at least three positive solutions such that*

$$0 < ||u_1|| < r_1 < ||u_2|| < r_2 < ||u_3||.$$

The proofs of Theorems 3.6 and 3.7 are similar to that of Theorem 3.5, so we omit it here.

Theorem 3.8. *Let* $n = 2k+1$, $k \in \mathbb{N}$. *Assume (H6) (or (H7)) holds. If there exist constants* $r_1, r_2, \ldots, r_{n-1} > 0$ *with* $r_{2i} < \Gamma r_{2i+1}$, *for* $1 \leq i \leq k-1$ *and* $r_{2i-1}M < r_{2i}\omega$ *for* $1 \leq i \leq k$ *(or with* $r_{2i-1} < \Gamma r_{2i}$, *for* $1 \leq i \leq k$ *and* $r_{2i}M < r_{2i+1}\omega$ *for* $1 \leq i \leq k-1$*)*

such that (H10) (or (H8)) holds for r_{2i-1}, $1 \leq i \leq k$ *and (H8) (or (H10)) holds for* r_{2i}, $1 \leq i \leq k$. *Then the problem (1.1) has at least* n *positive solutions* $u_1, \ldots, u_n$ *such that*

$$0 < ||u_1|| < r_1 < ||u_2|| < r_2 < \cdots < ||u_{n-1}|| < r_{n-1} < ||u_n||.$$

4. APPLICATION

Example 4.1. Consider the following singular boundary value problems with a p-Laplacian operator

$$\begin{cases} D_{0+}^{\frac{3}{2}}(\phi_p(u''(t))) - t^{-\frac{1}{2}} f(u(t)) = 0, \quad t \in (0,1), \\ \phi_p(u''(0)) = \phi_p(u''(1)) = 0, \\ u(0) - u'(0) = \frac{1}{2}u(\frac{1}{2}), \\ u(1) + u'(1) = \frac{1}{2}u(\frac{1}{2}), \end{cases} \tag{4.1}$$

where $p = \frac{3}{2}$,

$$f(u) = \begin{cases} e^{-u}, & 0 \leq u \leq 10, \\ (n+1)e^{-u}, & n < u \leq n+1, \quad n = 10, 11, \ldots, 20, \\ e^{\sqrt{u}}, & u > 21. \end{cases}$$

We note that

$$a = b = c = d = 1, \quad \rho = 3, \quad q = 3, \quad m = 3, \quad \xi_1 = \frac{1}{2}, \quad \sigma = \frac{3}{2},$$
$$a_1 = b_1 = \frac{1}{2}, \quad f_0 = +\infty, \quad f_\infty = +\infty, \quad \Delta = -\frac{9}{2}, \quad g(t) = t^{-\frac{1}{2}}.$$

Let $\theta = \frac{1}{3}$, then

$$\Lambda_1 = \frac{2}{3}, \quad \Lambda_2 = 1, \quad \Lambda_3 = 2, \quad \Gamma = \frac{1}{2},$$
$$\omega = \frac{9\pi}{131}, \quad M = \frac{729\pi}{944(3 - 2\sqrt{2})\eta^2},$$

where $\eta = \min_{\frac{1}{3} \leq s \leq \frac{2}{3}} \varrho(s)$.

By calculating, we can let $\mu = \frac{2\sqrt{2}-1}{2\sqrt{2}}$. So, $f_\infty > \phi_p(\frac{M}{\Gamma})$ and $f_0 > \phi_p(\frac{M}{\Gamma})$. We choose $r = 10$, then

$$f^r = \sup\left\{\frac{f(u)}{\phi_p(r)}\Big| u \in [0, r]\right\} = 0.316227 < 0.464462 = \phi_p(\omega).$$

Thus, (H8) and (H9) hold. Obviously, (H1*), (H2) and (H3) hold. By Theorem 3.5, the problem (4.1) has at least two positive solutions $u_1, u_2 \in K$ such that $0 < ||u_1|| < 4 < ||u_2||$.

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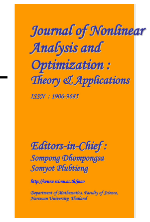

ON SET-VALUED MIXED VECTOR VARIATIONAL-LIKE INEQUALITIES IN BANACH SPACES

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ABSTRACT. In this paper we introduce the concept of generalized η-pseudomonotone mappings and generalized version of vector mixed variational-like inequalities in Banach spaces. Utilizing Ky Fan's Lemma and Nadler's Lemma, we derive the solvability for this class of vector mixed variational-like inequalities involving generalized η-pseudomonotone mappings. The results presented in this work are extensions and improvements of some earlier and recent results in the literature.

KEYWORDS : H-uniform continuity; Hausdorff metric; Complete continuity; Generalized η-pseudomonotone.
AMS Subject Classification: 47H17 90C29 49J40.

1. INTRODUCTION

Vector variational inequality theory was initially introduced and studied by Giannessi [8] in the setting of finite dimensional Euclidean spaces. Ever since it has been widely studied and generalized in infinite dimensional spaces since it covers many diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance, etc., as special cases. For details we refer [2,4,9-11,14-15,18,20-22] and references therein.

In recent past, a number of authors have studied generalizations of monotonicity such as pseudomonotonicity, relaxed monotonicity, quasimonotonicity and semimonotonicity; see [1,3,5,7,11-13,19] and the references therein. Bai *et al.* [1] introduced η-α-pseudomonotonicity and established some existence results for variational-like inequalities in reflexive Banach spaces. Recently, Zeng and Yao [21] considered and studied the solvability for a class of generalized vector variational inequalities in reflexive Banach spaces. They proved the solvability for this class of generalized vector variational inequalities with monotonicity assumption. Also they removed the boundedness assumption of K and extended to the general case of a nonempty closed and convex subset K.

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Inspired and motivated by the work of Bai *et al.*[1], Usman *et al.*[18] and Zeng and Yao [21], in this work we introduce the concept of generalized η-pseudomonotone mappings. Further, a more general vector mixed variational-like inequality problem for set-valued mappings which is a extension of the corresponding vector variational-like inequalities in [2,14-15], is considered. Furthermore, utilizing the Ky Fan's Lemma and the Nadler's Lemma, we establish some solvability results for this class of generalized vector mixed variational-like inequality problem involving generalized η-pseudomonotone mappings. The results presented in this work extend and unify corresponding results of [1,7,10,18,21].

2. PRELIMINARIES

Throughout the paper unless otherwise stated, let X and Y be two real Banach spaces, $K \subset X$ be a nonempty, closed and convex subset of X and $P \subset Y$ be a nonempty subset of Y. $P \neq Y$ be a closed, convex and pointed cone. The partial order $\leq_P$ in Y, induced by the pointed cone P is defined by declaring $x \leq_P y$ if and only if $y - x \in P$ for all x, y in Y. An ordered Banach space is a pair (Y, P) with the partial order induced by P. The weak order $\not\leq_{\text{int}\,P}$ in an ordered Banach space (Y, P) with $\text{int}\,P \neq \emptyset$ is defined as $x \not\leq_{\text{int}\,P} y$ if and only if $y - x \notin \text{int}\,P$ for all x, y in Y, where $\text{int}\,P$ denotes the interior of P. Let $L(X, Y)$ be the space of all continuous linear mappings from X into Y. Let $P : K \longrightarrow 2^Y$ be a set-valued mapping such that for each $x \in K$, $P(x)$ is a proper, closed, convex cone with $\text{int}\,P(x) \neq \emptyset$ and let $P_- = \bigcap_{x \in K} P(x)$.

Let $A : L(X, Y) \longrightarrow L(X, Y)$ be a mapping $\eta : X \times X \longrightarrow X$ and $f : K \times K \longrightarrow Y$ are the two bi-mappings and $V : K \longrightarrow 2^Y$ and $H : K \times Y \longrightarrow 2^{L(X,Y)}$ be set-valued mappings. In this paper we consider the following generalized vector mixed variational-like inequality problem (for short, GVMVLIP): Find $x \in K$, $z \in V(x)$ and $\xi \in H(x, z)$ such that

$$\langle A\xi, \eta(y, x)\rangle + f(y, x) \not\leq_{\text{int}\,P(x)} 0, \ \forall y \in K. \tag{2.1}$$

Some special cases of GVMVLIP (2.1)

(I) If $f \equiv 0$ and $A \equiv I$, the identity mapping of $L(X, Y)$, then GVMVLIP (2.1) reduces to the following generalized vector pre-variational inequality problem of finding $x \in K$ $z \in V(x)$ and $\xi \in H(x, z)$ such that

$$\langle \xi, \eta(y, x)\rangle \not\leq_{\text{int}\,P(x)} 0, \ \forall y \in K,$$

which was introduced and considered in real topological vector spaces by Chadli *et al.* [2] in 2004.

(II) If $V \equiv 0$, $H \equiv T : K \longrightarrow 2^{L(X,Y)}$ and $P(x) = P$, $\forall x \in K$, then GVMVLIP (2.1) reduces to the following generalized mixed vector variational-like inequality problem of finding $x \in K$ and $u \in T(x)$ such that

$$\langle Au, \eta(y, x)\rangle + f(y, x) \not\leq_{\text{int}\,P} 0, \ \forall y \in K,$$

which was introduced and studied by Usman *et al.* [18] in 2009.

(III) If $V \equiv 0$, $H \equiv T : K \longrightarrow 2^{L(X,Y)}$ and $A \equiv I$, the identity mapping of $L(X, Y)$, then GVMVLIP (2.1) reduces to the following generalized vector variational-type inequality problem of finding $x \in K$ such that for all $y \in K$, there exists $s_o \in T(x)$ such that

$$\langle s_0, \eta(y, x)\rangle + f(y, x) \not\leq_{\text{int}\,P(x)} 0, \ \forall y \in K,$$

which was introduced and considered in Hausdorff topological vector spaces by Lee *et al.* [14] in 2000.

(IV) If we take $T : K \longrightarrow L(X, Y)$ and $P(x) = P$, $\forall x \in K$ in (II), then it reduces to the following generalized weak vector variational-like inequality of finding $x \in K$ such that

$$\langle Tx, \eta(y, x)\rangle + f(y, x) \nleq_{\text{int}\, P} 0, \ \forall y \in K,$$

which was studied by Lee *et al.* [15] in 2008.

First, we recall the following concepts and results which are needed in the sequel.

Definition 2.1. A mapping $f : K \longrightarrow Y$ is said to be

(i) *P_--convex*, if $f(tx+(1-t)y) \leq_{P_-} tf(x)+(1-t)f(y), \ \forall x, y \in K, \ t \in [0, 1]$;
(ii) *P_--concave*, if $-f$ is P_--convex.

Definition 2.2. [22] Let $P : K \longrightarrow 2^Y$ be a set-valued mapping such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $T : K \longrightarrow L(X, Y)$ and $\eta : K \times K \longrightarrow X$ be two mappings. T is said to be *η-pseudomonotone*, if for any $x, y \in K$

$$\langle T(x), \eta(y, x)\rangle \geq_{P_-} 0 \implies \langle T(y), \eta(x, y)\rangle \leq_{P_-} 0, \text{ where } P_- = \bigcap_{x\in K} P(x).$$

Remark that, if $\eta(y, x) = y-x, \ \forall x, y \in K$, then η-pseudomonotonicity of T reduces to pseudomonotonicity of T.

Lemma 2.3. [4] *Let (Y, P) be an ordered Banach space with a closed, convex and pointed cone P with int $P \neq \emptyset$. Then $\forall x, y, z \in Y$, we have*

(i) $z \nleq_{\text{int}\, P} x$ *and* $x \geq_P y \Rightarrow z \nleq_{\text{int}\, P} y$*;*
(ii) $z \ngeq_{\text{int}\, P} x$ *and* $x \leq_P y \Rightarrow z \ngeq_{\text{int}\, P} y$*.*

Definition 2.4. A mapping $g : X \longrightarrow Y$ is said to be *completely continuous* if and only if the weak convergence of x_n to x in X implies the strong convergence of $g(x_n)$ to $g(x)$ in Y.

Lemma 2.5. [6] *Let K be a subset of a topological vector space X and let $F : K \longrightarrow 2^X$ be a KKM mapping. If for each $x \in K$, $F(x)$ is closed and for at least one $x \in K$, $F(x)$ is compact, then*

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

Lemma 2.6. [16]*Let X, Y and Z are real topological vector spaces, K be nonempty subset of X. Let $H : K \times Y \longrightarrow 2^Z$, $V : K \longrightarrow 2^Y$ be set-valued mapping. If both H, V are upper semicontinuous with compact values, then the set-valued mapping $T : K \longrightarrow 2^Z$ defined by*

$$T(x) = \bigcup_{z\in V(x)} H(x, z) = H(x, V(x))$$

is upper semicontinuous with compact values.

Lemma 2.7. [17]*Let $(X, \|.\|)$ be a normed vector space and H be a Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X, induced by a metric d in terms of $d(u, v) = \|u - v\|$, defined by*

$$H(U, V) = \max\{\sup_{u\in U} \inf_{v\in V} \|u - v\|, \sup_{v\in V} \inf_{u\in U} \|u - v\|\},$$

for U and V in $CB(X)$. If U and V are compact sets in X, then for each $u \in U$, there exists $v \in V$ such that $\|u - v\| \leq \|H(U, V)\|$.

Definition 2.8. A nonempty, compact set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ is called H-uniformly continuous if for any given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in K$ with $\|x - y\| < \delta$, there holds $H(Tx, Ty) < \epsilon$, where H is the Hausdorff metric defined on $CB(L(X, Y))$.

3. EXISTENCE RESULTS FOR GVMVLIP (2.1)

Now we shall derive the solvability for the GVMVLIP (2.1) involving generalized η-pseudomonotone mappings under some quite mild conditions by using Ky Fan's Lemma [6] and Nadler's Lemma [17].

First, we give the concept of generalized η-pseudomonotone mappings.

Definition 3.1. Let $f : K \times K \longrightarrow Y$ and $\eta : X \times X \longrightarrow X$ are the two bi-mappings, let $A : L(X, Y) \longrightarrow L(X, Y)$ be the mapping, $V : K \longrightarrow 2^Y$ and $H : K \times Y \longrightarrow 2^{L(X,Y)}$ are the set-valued mappings. Then H, V are said to be *generalized η-pseudomonotone* mappings with respect to A, if for any $x \in K$, $z_1 \in V(x)$ and $\xi_1 \in H(x, z_1)$, we have

$$\langle A\xi_1, \eta(y, x)\rangle + f(y, x) \not\leq_{\text{int}\, P(x)} 0, \text{ implies that}$$

$$\langle A\xi_2, \eta(y, x)\rangle + f(y, x) - \alpha(x, y) \not\leq_{\text{int}\, P(x)} 0, \ \forall y \in K, \ z_2 \in V(y), \ \xi_2 \in H(y, z_2),$$

where $\alpha : X \times X \longrightarrow Y$ is a mapping such that $\lim_{t \longrightarrow 0^+} \frac{\alpha(x, ty+(1-t)x)}{t} = 0$.

Remark 3.2. (i) If $f, V \equiv 0$, $H \equiv T$: $K \longrightarrow L(X, Y)$ and $A \equiv I$, the identity mapping of $L(X, Y)$ and $\alpha(x, y) = \alpha(y - x)$, where $\alpha : X \longrightarrow \mathbb{R}$ with $\alpha(\lambda z) = \lambda^p \alpha(z)$ for $\lambda > 0$, $p > 1$ and if $P(x) = \mathbb{R}_+$, $\forall x \in K$, then Definition 3.1 reduces to

$$\langle Ty, \eta(y, x)\rangle \geq 0 \text{ implies } \langle Tx, \eta(y, x)\rangle \geq \alpha(y - x), \ \forall x, y \in K.$$

Then T is said to be relaxed η-α-pseudomonotone, introduced and studied by Bai *et al.* [1].

(ii) In the case (i), if we take $\eta(y, x) = y - x$, for all $x, y \in K$ and $\beta \equiv 0$, then it reduces to

$$\langle Ty, y - x\rangle \geq 0 \text{ implies } \langle Tx, y - x\rangle \geq 0, \ \forall x, y \in K.$$

Then T is said to be pseudomonotone; see for example [5, 11, 13]

Now we prove Minty's type Lemma for GVMVLIP (2.1) with the help of generalized η-pseudomonotone mappings.

Lemma 3.3. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(X, Y) \longrightarrow L(X, Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty set-valued mapping. Suppose the following conditions hold:*

(i) *$f : K \times K \longrightarrow Y$ be a P_--convex in first argument with the condition $f(x, y) + f(y, x) = 0$, $\forall x, y \in K$;*
(ii) *$\langle A\xi, \eta(., y)\rangle : K \longrightarrow Y$ is P_--convex for each $(\xi, y) \in L(X, Y) \times K$ is fixed;*
(iii) *$\langle A\xi, \eta(x, x)\rangle = 0$, $\forall (\xi, y) \in L(X, Y) \times K$;*

(iv) *Let* $H : K \times Y \longrightarrow 2^{L(X,Y)}$, $V : K \longrightarrow 2^Y$ *be two upper semicontinuous mappings with compact values such that* H *and* V *are generalized* η*-pseudomonotone with respect to* A*. If the set-valued mapping* $T : K \longrightarrow 2^{L(X,Y)}$ *defined by*

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is H*-uniformly continuous.*

Then following two problems are equivalent:
(A) *there exists* $x_0 \in K,\ z_0 \in V(x_0)$ *and* $\xi_0 \in H(x_0, z_0)$ *such that*

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y, x_0) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K. \tag{3.1}$$

(B) *there exists* $x_0 \in K$ *such that*

$$\langle A\xi, \eta(y, x_0)\rangle + f(y, x_0) - \alpha(x_0, y) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K,\ z \in V(y),\ \xi \in H(y, z). \tag{3.2}$$

Proof. Suppose that there exists $x_0 \in K,\ z_0 \in V(x_0)$ and $\xi_0 \in H(x_0, z_0)$ such that

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y, x_0) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K.$$

Since $H,\ V$ are generalized η-pseudomonotone with respect to A, we have

$$\langle A\xi, \eta(y, x_0)\rangle + f(y, x_0) - \alpha(x_0, y) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K,\ z \in V(y),\ \xi \in H(y, z).$$

Conversely, suppose that there exists $x_0 \in K$ such that

$$\langle A\xi, \eta(y, x_0)\rangle + f(y, x_0) - \alpha(x_0, y) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall y \in K,\ z \in V(y),\ \xi \in H(y, z).$$

For any given $y \in K$, we know that $y_t = ty + (1-t)x_0 \in K$, for each $t \in (0,1)$, we have $y_t \in K$ as K is convex. Hence for each $\xi_t \in T(y_t) = H(y_t, V(y_t))$

$$\langle A\xi_t, \eta(y_t, x_0)\rangle + f(y_t, x_0) - \alpha(x_0, y_t) \not\leq_{\text{int}\, P(x_0)} 0. \tag{3.3}$$

Since f is P_--convex in first argument, it follows that

$$f(y_t, x_0) \leq_{P_-} tf(y, x_0) + (1-t)f(x_0, x_0) = tf(y, x_0). \tag{3.4}$$

From assumptions (ii) and (iii) on η, we have

$$\begin{aligned}\langle A\xi_t, \eta(y_t, x_0)\rangle &= \langle A\xi_t, \eta(ty + (1-t)x_0, x_0)\rangle \\ &\leq_{P_-} t\langle A\xi_t, \eta(y, x_0)\rangle + (1-t)\langle A\xi_t, \eta(x_0, x_0)\rangle \\ &= t\langle A\xi_t, \eta(y, x_0)\rangle.\end{aligned} \tag{3.5}$$

It follows from inclusions (3.3)-(3.5) and Lemma 2.3 that for $t > 0$ and $p > 1$

$$t[\langle A\xi_t, \eta(y, x_0)\rangle + f(y, x_0)] - \alpha(x_0, y_t) \not\leq_{\text{int}\, P(x_0)} 0,\ \forall v_t \in T(y_t),\ t \in (0,1).$$

$$\langle A\xi_t, \eta(y, x_0)\rangle + f(y, x_0) - \frac{\alpha(x_0, y_t)}{t} \not\leq_{\text{int}\, P(x_0)} 0,\ \forall v_t \in T(y_t),\ t \in (0,1). \tag{3.6}$$

We remark that according to Lemma 2.6, the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is upper semicontinuous with compact values. Hence $T(y_t)$ and $T(x_0)$ are compact and from Lemma 2.7, it follows that for each fixed $\xi_t \in T(y_t)$, there exists an $\zeta_t \in T(x_0)$ such that

$$\|\xi_t - \zeta_t\| \leq H(T(y_t), T(x_0)).$$

Since $T(x_0)$ is compact, without loss of generality, we may assume that $\zeta_t \longrightarrow \xi_0 \in T(x_0)$ as $t \longrightarrow 0^+$. Since T is H-uniformly continuous and $\|y_t - x_0\| = t\|y - x_0\| \longrightarrow 0$ as $t \longrightarrow 0^+$ so $H(T(y_t), T(x_0)) \longrightarrow 0$ as $t \longrightarrow 0^+$. Thus one has

$$\begin{aligned}\|\xi_t - \xi_0\| &\leq \|\xi_t - \zeta_t\| + \|\zeta_t - \xi_0\| \\ &\leq \|H(T(y_t), T(x_0))\| + \|\zeta_t - \xi_0\| \longrightarrow 0.\end{aligned}$$

Since A is continuous mapping, therefore letting $t \longrightarrow 0^+$, we obtain

$$\begin{aligned}\|\langle A\xi_t, \eta(y, x_0)\rangle - \langle A\xi_0, \eta(y, x_0)\rangle\| &= \|\langle A\xi_t - A\xi_0, \eta(y, x_0)\rangle\| \\ &\leq \|A\xi_t - A\xi_0\|\|\eta(y, x_0)\| \longrightarrow 0\end{aligned}$$

Also by inclusion (3.6), we deduce that

$$\langle A\xi_t, \eta(y, x_0)\rangle + f(y, x_0) - \frac{\alpha(x_0, y_t)}{t} \in Y\backslash(-\text{int } P(x_0)).$$

Since $Y\backslash(-\text{int } P(x_0))$ is closed and letting $t \longrightarrow 0^+$, we have

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y, x_0) \in Y\backslash(-\text{int } P(x_0)),$$

and so

$$\langle A\xi_0, \eta(y, x_0)\rangle + f(y, x_0) \not\leq_{\text{int } P(x_0)} 0.$$

Next we claim that there holds

$$\langle A\xi_0, \eta(v, x_0)\rangle + f(v, x_0) \not\leq_{\text{int } P(x_0)} 0, \ \forall v \in K.$$

Indeed, let v be an arbitrary element in K and let $v_t = tv + (1-t)x_0 \in K$, for each $t \in (0, 1)$. Then one has $\|y_t - v_t\| = t\|y - v\| \longrightarrow 0$ as $t \longrightarrow 0^+$. Hence from H-uniform continuity of T it follows that $H(Ty_t, Tv_t) \longrightarrow 0$ as $t \longrightarrow 0^+$. Let $\{\xi_t\}_{t\in(0,1)}$ be any net choosen such that $\xi_t \longrightarrow \xi_0$ as $t \longrightarrow 0^+$. Since Ty_t and Tv_t are compact, from Lemma 2.7, it follows that for each fixed $\xi_t \in Ty_t$ there exists a $\gamma_t \in Tv_t$ such that

$$\|\xi_t - \gamma_t\| \leq H(Ty_t, Tv_t).$$

Consequently

$$\begin{aligned}\|\gamma_t - \xi_0\| &\leq \|\xi_t - \gamma_t\| + \|\xi_t - \xi_0\| \\ &\leq H(Ty_t, Tv_t) + \|\xi_t - \xi_0\| \longrightarrow 0 \text{ as } t \longrightarrow 0^+.\end{aligned}$$

Note that A is continuous mapping, therefore letting $t \longrightarrow 0^+$, we obtain

$$\begin{aligned}\|\langle A\gamma_t, \eta(v, x_0)\rangle - \langle A\xi_0, \eta(v, x_0)\rangle\| &= \|\langle A\gamma_t - A\xi_0, \eta(v, x_0)\rangle\| \\ &\leq \|A\gamma_t - A\xi_0\|\|\eta(v, x_0)\| \longrightarrow 0.\end{aligned}$$

Replacing y, y_t and ξ_t in inclusion (3.6) by v, v_t and γ_t, respectively, one deduces that

$$\langle A\gamma_t, \eta(v, x_0)\rangle + f(v, x_0) - \frac{\alpha(x_0, v_t)}{t} \not\leq_{\text{int } P(x_0)} 0, \ \forall t \in (0, 1),$$

which implies that

$$\langle A\gamma_t, \eta(v, x_0)\rangle + f(v, x_0) - \frac{\alpha(x_0, v_t)}{t} \in Y\backslash(-\text{int } P(x_0)).$$

Since $Y\backslash(-\text{int } P(x_0))$ is closed and letting $t \longrightarrow 0^+$, one has that

$$\langle A\xi_0, \eta(v, x_0)\rangle + f(v, x_0) \in Y\backslash(-\text{int } P(x_0)),$$

and hence

$$\langle A\xi_0, \eta(v, x_0)\rangle + f(v, x_0) \not\leq_{\text{int } P(x_0)} 0.$$

Thus, according to arbitrariness of v the assertion is valid.

Since, $\xi_0 \in T(x_0) = \bigcup_{z \in V(x_0)} H(x_0, z) = H(x_0, V(x_0))$, it follows that there exists $z_0 \in V(x_0)$ such that $\xi_0 \in H(x_0, z_0)$. Therefore, (3.1) holds. This completes the proof. □

Now, with the help of above Minty's type Lemma, we have following existence theorem for GVMVLIP (2.1).

Theorem 3.4. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space X and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(X,Y) \longrightarrow L(X,Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact set-valued mapping. Suppose the following conditions hold:*

- (i) *$f : K \times K \longrightarrow Y$ be affine in first argument with the condition $f(x,y) + f(y,x) = 0, \ \forall x, y \in K$ and completely continuous in second argument;*
- (ii) *$\langle A\xi, \eta(x,x)\rangle = 0$, for each $x \in K$ and $\xi \in L(X,Y)$;*
- (iii) *for each $(\xi, y) \in L(X,Y) \times K$ fixed, $\langle A\xi, \eta(.,y)\rangle : K \longrightarrow Y$ is affine;*
- (iv) *for each $y \in K$ fixed, $\eta(y,.) : K \longrightarrow X$ is completely continuous;*
- (v) *for each fixed $y \in K, \ \alpha(.,y)$ is weakly lower semicontinuous.*

Suppose additionally that $H : K \times Y \longrightarrow 2^{L(X,Y)}$, $V : K \longrightarrow 2^Y$ be two upper semicontinuous mappings with compact values such that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is H-uniformly continuous, then there exists $x^\star \in K, \ z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi^\star, \eta(y, x^\star)\rangle + f(y, x^\star) \not\leq_{\text{int}\, P(x^\star)} 0, \ \forall y \in K.$$

Proof. We divide the proof into four steps.

Step I. We claim that for every finite subset E of K, there exists $\bar{x} \in \text{co}E$, $\bar{z} \in V(\bar{x})$ and $\bar{\xi} \in H(\bar{x}, \bar{z})$ such that

$$\langle A\bar{\xi}, \eta(y, \bar{x})\rangle + f(y, \bar{x}) \not\leq_{\text{int}\, P(\bar{x})} 0, \ \forall y \in \text{co}E.$$

Indeed, let E be any finite subset of K and let us define a vector set-valued mapping $F : \text{co}E \longrightarrow 2^{\text{co}E}$ as follows:

$$F(y) = \{x \in \text{co}E : \exists z \in V(x), \xi \in H(x,z) \text{ such that } \langle A\xi, \eta(y,x)\rangle + f(y,x) \not\leq_{\text{int}\, P(x)} 0\},$$

for all $y \in \text{co}E$. From assumption (ii), one has $F(y) \neq \emptyset$ since $y \in F(y)$. The set $F(y)$ is also closed. Indeed, let $\{x_n\} \subseteq F(y)$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Hence for each n, there exists $z_n \in V(x_n)$ and $\xi_n \in H(x_n, z_n)$ such that

$$\langle A\xi_n, \eta(y, x_n)\rangle + f(y, x_n) \not\leq_{\text{int}\, P(x_n)} 0.$$

Since V is upper semicontinuous with compact values, $V(\text{co}E)$ is compact. Therefore, without loss of generality one deduces that $z_n \longrightarrow z \in V(x)$ as $n \longrightarrow \infty$. On the other hand, since H is upper semicontinuous with compact values $H(\text{co}E, V(\text{co}E))$ is compact. It follows without loss of generality that $\xi_n \longrightarrow \xi \in H(x,z)$. Now, let $\{y_1, ..., y_n\} \subseteq \text{co}E$ and let us verify $\text{co}\{y_1, ..., y_n\} \subseteq \bigcup_{i=1}^{n} F(y_i)$. Let

$x \in \text{co}\{y_1, ..., y_n\}$, $x = \sum_{i=1}^{n} t_i x_i$ with $t_i \geq 0, i = 1, ..., n$ and $\sum_{i=1}^{n} t_i = 1$.
Utilizing assumptions (i)-(iii), we have

$$\begin{aligned} 0 &= \langle A\xi, \eta(x,x)\rangle + f(x,x) \\ &= \langle A\xi, \eta(\sum_{i=1}^{n} t_i y_i, x)\rangle + f(\sum_{i=1}^{n} t_i y_i, x) \\ &= \sum_{i=1}^{n} t_i[\langle A\xi, \eta(y_i, x)\rangle + f(y_i, x)] \end{aligned}$$

which hence implies that

$$\sum_{i=1}^{n} t_i[\langle A\xi, \eta(y_i, x)\rangle + f(y_i, x)] \not\leq_{\text{int}\, P(x)} 0.$$

Therefore there exists $i \in \{1, ..., n\}$ such that

$$\langle A\xi, \eta(y_i, x)\rangle + f(y_i, x) \not\leq_{\text{int}\, P(x)} 0.$$

Thus $x \in F(y_i) \subseteq \bigcup_{j=1}^{n} F(y_j)$. Consequently, from Lemma 2.5, we know that $\bigcap_{y \in \text{co}E} F(y) \neq \emptyset$. Let $\bar{x} \in \bigcap_{y \in \text{co}E} F(y)$. Then for each fixed $y \in coE$ there exists $\xi_y \in T\bar{x} = H(\bar{x}, V(\bar{x}))$ such that

$$\langle A\xi_y, \eta(y, \bar{x})\rangle + f(y, \bar{x}) \not\leq_{\text{int}\, P(\bar{x})} 0.$$

Let $y_t = \bar{x} + t(y - \bar{x}),\ \forall t \in (0,1)$. Then, observe that

$$\begin{aligned} \langle A\xi_y, \eta(y_t, \bar{x})\rangle + f(y_t, \bar{x}) &= \langle A\xi, \eta(\bar{x} + t(y - \bar{x}), \bar{x})\rangle + f(\bar{x} + t(y - \bar{x}), \bar{x}) \\ &= t[\langle A\xi_y, \eta(y, \bar{x})\rangle + f(y, \bar{x})]. \end{aligned}$$

Hence

$$\langle A\xi_y, \eta(y_t, \bar{x})\rangle + f(y_t, \bar{x}) \not\leq_{\text{int}\, P(\bar{x})} 0.$$

Since H and V are generalized η-pseudomonotone with respect to A, we have

$$\langle A\xi_t, \eta(y_t, \bar{x})\rangle + f(y_t, \bar{x}) - \alpha(\bar{x}, y_t) \not\leq_{\text{int}\, P(\bar{x})} 0,\ \forall \xi_t \in Ty_t,\ t \in (0,1). \tag{3.7}$$

If it was false then for some $t_0 \in (0,1)$ and some $\xi_{t_0} \in Ty_{t_0}$

$$\langle A\xi_{t_0}, \eta(y_{t_0}, \bar{x})\rangle + f(y_{t_0}, \bar{x}) - \alpha(\bar{x}, y_{t_0}) \leq_{\text{int}\, P(\bar{x})} 0.$$

Consequently

$$\langle A\xi_y, \eta(y_{t_0}, \bar{x})\rangle + f(y_{t_0}, \bar{x}) = \langle A\xi_{t_0}, \eta(y_{t_0}, \bar{x})\rangle + f(y_{t_0}, \bar{x}) - \alpha(\bar{x}, y_{t_0}) \leq_{\text{int}\, P(\bar{x})} 0,$$

which hence implies that

$$\langle A\xi_y, \eta(y_{t_0}, \bar{x})\rangle + f(y_{t_0}, \bar{x}) \leq_{\text{int}\, P(\bar{x})} 0.$$

Which leads to a contradiction and hence (3.7) is valid. Now observe that

$$\begin{aligned} \langle A\xi_t, \eta(y_t, \bar{x})\rangle + f(y_t, \bar{x}) - \alpha(\bar{x}, y_t) &= \langle A\xi_t, \eta(ty + (1-t)\bar{x}, \bar{x})\rangle + f(ty + (1-t)\bar{x}, \bar{x}) - \alpha(\bar{x}, ty + (1-t)\bar{x}) \\ &= t[\langle A\xi_t, \eta(y, \bar{x})\rangle + f(y, \bar{x}) - \frac{\alpha(\bar{x}, ty + (1-t)\bar{x})}{t}]. \end{aligned}$$

Which together with (3.7) implies that

$$\langle A\xi_t, \eta(y, \bar{x})\rangle + f(y, \bar{x}) - \frac{\alpha(\bar{x}, y_t)}{t} \not\leq_{\text{int}\, P(\bar{x})} 0,\ \forall \xi_t \in Ty_t,\ t \in (0,1). \tag{3.8}$$

We remark that according to Lemma 2.6, the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x,z) = H(x, V(x))$$

is upper semicontinuous with compact values. Hence $T(y_t)$ and $T(\bar{x})$ are compact and from Lemma 2.7, it follows that for each fixed $\xi_t \in T(y_t)$, there exists an $\zeta_t \in T(\bar{x})$ such that

$$\|\xi_t - \zeta_t\| \leq H(T(y_t), T(\bar{x})).$$

Since $T(\bar{x})$ is compact, without loss of generality, we may assume that $\zeta_t \longrightarrow \bar{\xi} \in T(\bar{x})$ as $t \longrightarrow 0^+$. Since T is H-uniformly continuous and $\|y_t - \bar{x}\| = t\|y - \bar{x}\| \longrightarrow 0$ as $t \longrightarrow 0^+$, so $H(T(y_t), T(\bar{x})) \longrightarrow 0$ as $t \longrightarrow 0^+$. Thus one has

$$\begin{aligned}\|\xi_t - \bar{\xi}\| &\leq \|\xi_t - \zeta_t\| + \|\zeta_t - \bar{\xi}\| \\ &\leq H(T(y_t), T(\bar{x})) + \|\zeta_t - \bar{\xi}\| \longrightarrow 0.\end{aligned}$$

Since A is continuous mapping, therefore letting $t \longrightarrow 0^+$, we obtain

$$\begin{aligned}\|\langle A\xi_t, \eta(y,\bar{x})\rangle - \langle A\bar{\xi}, \eta(y,\bar{x})\rangle\| &= \|\langle A\xi_t - A\bar{\xi}, \eta(y,\bar{x})\rangle\| \\ &\leq \|A\xi_t - A\bar{\xi}\| \|\eta(y,\bar{x})\rangle\| \longrightarrow 0.\end{aligned}$$

Also by inclusion (3.8), we deduce that

$$\langle A\xi_t, \eta(y,\bar{x})\rangle + f(y,\bar{x}) - \frac{\alpha(\bar{x}, y_t)}{t} \in Y \backslash (-\text{int } P(\bar{x})).$$

Since $Y \backslash (-\text{int } P(\bar{x}))$ is closed and letting $t \longrightarrow 0^+$, we have that

$$\langle A\bar{\xi}, \eta(y,\bar{x})\rangle + f(y,\bar{x}) \in Y \backslash (-\text{int } P(\bar{x})),$$

and so

$$\langle A\bar{\xi}, \eta(y,\bar{x})\rangle + f(y,\bar{x}) \not\leq_{\text{int } P(\bar{x})} 0.$$

Next we claim that there holds

$$\langle A\bar{\xi}, \eta(v,\bar{x})\rangle + f(v,\bar{x}) \not\leq_{\text{int } P(\bar{x})} 0, \ \forall v \in \text{co}E.$$

Indeed, let v be an arbitrary element in $\text{co}E$ and set $v_t = tv + (1-t)\bar{x} \in K$, for each $t \in (0,1)$. Then one has $\|y_t - v_t\| = t\|y - v\| \longrightarrow 0$ as $t \longrightarrow 0^+$. Hence from H-uniform continuity of T it follows that $H(Ty_t, Tv_t) \longrightarrow 0$ as $t \longrightarrow 0^+$. Let $\{\xi_t\}_{t \in (0,1)}$ be any net choosen as above such that $\xi_t \longrightarrow \bar{\xi}$ as $t \longrightarrow 0^+$. Since Ty_t and Tv_t are compact, from Lemma 2.6, it follows that for each fixed $\xi_t \in Ty_t$ there exists a $\gamma_t \in Tv_t$ such that

$$\|\xi_t - \gamma_t\| \leq H(T(y_t), T(v_t)).$$

Consequently

$$\begin{aligned}\|\gamma_t - \bar{\xi}\| &\leq \|\xi_t - \gamma_t\| + \|\xi_t - \bar{\xi}\| \\ &\leq H(T(y_t), T(v_t)) + \|\xi_t - \bar{\xi}\| \longrightarrow 0 \text{ as } t \longrightarrow 0^+.\end{aligned}$$

Since A is continuous mapping, therefore letting $t \longrightarrow 0^+$, we obtain

$$\begin{aligned}\|\langle A\gamma_t, \eta(v,\bar{x})\rangle - \langle A\bar{\xi}, \eta(v,\bar{x})\rangle\| &= \|\langle A\gamma_t - A\bar{\xi}, \eta(v,\bar{x})\rangle\| \\ &\leq \|A\gamma_t - A\bar{\xi}\| \|\eta(v,\bar{x})\rangle\| \longrightarrow 0.\end{aligned}$$

Replacing y, y_t and ξ_t in inclusion (3.8) by v, v_t and γ_t, respectively, one deduces that

$$\langle A\gamma_t, \eta(v,\bar{x})\rangle + f(v,\bar{x}) - \frac{\alpha(\bar{x}, v_t)}{t} \not\leq_{\text{int } P(\bar{x})} 0, \ \forall t \in (0,1),$$

which implies that

$$\langle A\gamma_t, \eta(v, \bar{x})\rangle + f(v, \bar{x}) - \frac{\alpha(\bar{x}, v_t)}{t} \in Y\backslash(-\text{int } P(\bar{x})).$$

Since $Y\backslash(-\text{int } P(\bar{x}))$ is closed and letting $t \longrightarrow 0^+$, one has that

$$\langle A\bar{\xi}, \eta(v, \bar{x})\rangle + f(v, \bar{x}) \in Y\backslash(-\text{int } P(\bar{x})),$$

and hence

$$\langle A\bar{\xi}, \eta(v, \bar{x})\rangle + f(v, \bar{x}) \not\leq_{\text{int } P(\bar{x})} 0.$$

Thus, according to arbitrariness of v the assertion is valid.

Since, $\bar{\xi} \in T(\bar{x}) = \bigcup_{z \in V(\bar{x})} H(\bar{x}, z) = H(\bar{x}, V(\bar{x}))$, it follows that there exists $\bar{z} \in V(\bar{x})$ such that $\bar{\xi} \in H(\bar{x}, \bar{z})$. Therefore, the assertion of Step I is valid.

Step II. We claim that for every finite subset of E, there exists $\bar{x} \in \text{co}E$ such that

$$\langle A\xi, \eta(y, \bar{x})\rangle + f(y, \bar{x}) - \alpha(\bar{x}, y) \not\leq_{\text{int } P(\bar{x})} 0, \ \forall y \in \text{coE}, z \in V(y), \ \xi \in H(y, z).$$

Indeed, the assertion follows imediately from Step I and Lemma 2.7.

Step III. We claim that there exists $x^\star \in K$ such that

$$\langle A\xi, \eta(y, x^\star)\rangle + f(y, x^\star) - \alpha(x^\star, y) \not\leq_{\text{int } P(x^\star)} 0, \ \forall y \in K, \ z \in V(y), \ \xi \in H(y, z).$$

Indeed, since X is reflexive and K is nonempty, bounded, closed and convex subset of X, so K is compact with respect to the weak topology of X. Let $\mathcal{F}$ be the family of all finite subsets of K. For each $E \in \mathcal{F}$, consider the following set:

$$M_E = \{x \in K : \langle A\xi, \eta(y, x)\rangle + f(y, x) - \alpha(x, y) \not\leq_{\text{int } P(x)} 0, \ \forall y \in \text{coE}, \ z \in V(y), \ \xi \in H(y, z)\}.$$

From Step II, one has $M_E \neq \emptyset$ for each $E \in \mathcal{F}$. We shall prove that $\bigcap_{E \in \mathcal{F}} \overline{M_E}^w \neq \emptyset$, where $\overline{M_E}^w$ denotes the closure of E with respect to the weak topology of X. For this, it suffices to show that the family $\{\overline{M_E}^w\}_{E \in \mathcal{F}}$ has the finite intersection property. Let $E, F \in \mathcal{F}$ and set $G = E \cup F \in \mathcal{F}$. Then $M_G \subseteq M_E \cap M_F$ and it follows that $\overline{M_E}^w \cap \overline{M_F}^w \neq \emptyset$. This shows that the family $\{\overline{M_E}^w\}_{E \in \mathcal{F}}$ has the finite intersection property. Since K is compact with respect to weak topology of X, it follows that $\bigcap_{E \in \mathcal{F}} \overline{M_E}^w \neq \emptyset$. Let $x^\star \in \bigcap_{E \in \mathcal{F}} \overline{M_E}^w$ and for an arbitrary $y \in K$ fixed, consider $F = \{y, x^\star\}$. Since $x^\star \in \overline{M_F}^w$, there exists $\{x_n\} \subseteq \overline{M_F}^w$ such that $\{x_n\} \subseteq K$, $x_n \longrightarrow x^\star$ and for each n

$$\langle A\xi, \eta(v, x_n)\rangle + f(v, x_n) - \alpha(x_n, v) \not\leq_{\text{int } P(x_n)} 0, \ \forall v \in \text{coE}, z \in V(v), \ \xi \in H(v, z).$$

In particular, whenever $v = y$, one derive for each n

$$\langle A\xi, \eta(y, x_n)\rangle + f(y, x_n) - \alpha(x_n, y) \not\leq_{\text{int } P(x_n)} 0, \ \forall z \in V(y), \ \xi \in H(y, z).$$

$$\langle A\xi, \eta(y, x_n)\rangle + f(y, x_n) - \alpha(x_n, y) \notin -\text{int } P(x_n), \ \forall z \in V(y), \ \xi \in H(y, z).$$

Since for each fixed $y \in K$, $\eta(y, .)$ and $f(y, .)$ are completelty continuous and for each fixed $y \in K$, $\alpha(., y)$ is lower semicontinuous, we conclude that for each $y \in K$, $z \in V(y)$ and $\xi \in H(y, z)$ fixed,

$$\langle A\xi, \eta(y, x_n)\rangle + f(y, x_n) - \alpha(x_n, y) \longrightarrow \langle A\xi, \eta(y, x^\star)\rangle + f(y, x^\star) - \alpha(x^\star, y) \text{ as } n \longrightarrow \infty.$$

Since $Y\backslash(-\text{int } P(x_n))$ is closed,

$$\langle A\xi, \eta(y, x^\star)\rangle + f(y, x^\star) - \alpha(x^\star, y) \in Y\backslash(-\text{int } P(x^\star)), \ \forall y \in K, \ z \in V(y), \ \xi \in H(y, z)$$

Thus, the assertion of Step III is proved.

Step IV. We claim that there exists $x^\star \in K, z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi, \eta(y, x^\star)\rangle + f(y, x^\star) \nleq_{\text{int}\, P(x^\star)} 0, \ \forall y \in K.$$

Indeed, the assertion follows immediately from Step III and Lemma 3.3. This completes the proof. □

If the boundedness of K is dropped off, then we have the following theorem under certain coercivity condition:

Theorem 3.5. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space X with $0 \in K$ and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(X, Y) \longrightarrow L(X, Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(X,Y)}$ be a nonempty compact set-valued mapping. Suppose the following conditions hold:*

- (i) *$f : K \times K \longrightarrow Y$ be affine in first argument with the condition $f(x, y) + f(y, x) = 0, \ \forall x, y \in K$ and completely continuous in second argument;*
- (ii) *$\langle A\xi, \eta(x, x)\rangle = 0$ for each $x \in K$ and $\xi \in L(X, Y)$;*
- (iii) *for each $(\xi, y) \in L(X, Y) \times K$ fixed, $\langle A\xi, \eta(., y)\rangle : K \longrightarrow Y$ is affine;*
- (iv) *for each $y \in K$ fixed, $\eta(y, .) : K \longrightarrow X$ is completely continuous;*
- (v) *for each fixed $y \in K$ $\alpha(., y)$ is weakly lower semicontinuous;*
- (vi) *there exists some $r > 0$ such that* **(a)** *$H : K_r \times Y \longrightarrow 2^{L(X,Y)}, V : K_r \longrightarrow 2^Y$ are two upper semicontinuous with compact convex values where $K_r = \{x \in K : \|x\| \leq r\}$, and*

(**b**) $\langle A\xi, \eta(0, x)\rangle + f(0, x) \leq_{\text{int}\, P(x)} 0, \ \forall z \in V(x), \ \xi \in H(x, z)$ and $x \in K$ with $\|x\| = r$. (3.9)

Suppose additionally that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(X,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is H-uniformly continuous, then there exists $x^\star \in K, \ z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi^\star, \eta(y, x^\star)\rangle + f(y, x^\star) \nleq_{\text{int}\, P(x^\star)} 0, \ \forall y \in K.$$

Proof. One can readly see that all conditions of Theorem 3.4 are fulfilled for a nonempty, bounded, closed and convex subset $K_r = K \cap B_r$, where $B_r = \{x \in X : \|x\| \leq r\}$. Thus according to Theorem 3.4, there exist $x_r \in K_r, z_r \in V(x_r)$ and $\xi_r \in H(x_r, z_r)$ such that

$$\langle A\xi_r, \eta(v, x_r)\rangle + f(v, x_r) \nleq_{\text{int}\, P(x_r)} 0, \ \forall v \in K_r. \tag{3.10}$$

Putting $v = 0$ in the above inclusion, one has

$$\langle A\xi_r, \eta(0, x_r)\rangle + f(0, x_r) \nleq_{\text{int}\, P(x_r)} 0. \tag{3.11}$$

Combining (3.9) with (3.11), we know that $\|x_r\| < r$. For any $y \in K$, choose $t \in (0, 1)$ small enough such that $(1 - t)x_r + ty \in K_r$. Putting $v = (1 - t)x_r + ty$ in (3.10), one has

$$\langle A\xi_r, \eta((1 - t)x_r + ty, x_r)\rangle + f((1 - t)x_r + ty, x_r) \nleq_{\text{int}\, P(x_r)} 0.$$

Since the mappings $f(., x_r)$ and $\eta(., x_r)$ are affine, we have

$$\langle A\xi_r, \eta((1 - t)x_r + ty, x_r)\rangle + f((1 - t)x_r + ty, x_r) = t[\langle A\xi_r, \eta(y, x_r)\rangle + f(y, x_r)].$$

Consequently, we have

$$\langle A\xi_r, \eta(y, x_r)\rangle + f(y, x_r) \not\leq_{\text{int}\, P(x_r)} 0, \ \forall y \in K.$$

This completes the proof. □

If $X = \mathbb{R}^n$, then complete continuity is equivalent to continuity. Also bounded and closed subset is equivalent to compact subset. By Theorem 3.4 and Theorem 3.5, we can obtain the following results:

Corollary 3.6. *Let K be a nonempty, compact and convex subset of a real reflexive Banach space $\mathbb{R}^n$ and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(\mathbb{R}^n, Y) \longrightarrow L(\mathbb{R}^n, Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(\mathbb{R}^n,Y)}$ be a nonempty compact set-valued mapping. Suppose the following conditions hold:*

- (i) *$f : K \times K \longrightarrow Y$ be affine in first argument with the condition $f(x, y) + f(y, x) = 0, \ \forall x, y \in K$ and continuous in second argument;*
- (ii) *$\langle A\xi, \eta(x, x)\rangle = 0$ for each $x \in K$ and $\xi \in L(\mathbb{R}^n, Y)$;*
- (iii) *for each $(\xi, y) \in L(\mathbb{R}^n, Y) \times K$ fixed, $\langle A\xi, \eta(., y)\rangle : K \longrightarrow Y$ is affine;*
- (iv) *for each $y \in K$ fixed, $\eta(y, .) : K \longrightarrow \mathbb{R}^n$ is continuous;*
- (v) *for each fixed $y \in K$ $\alpha(., y)$ is weakly lower semicontinuous.*

Suppose additionally that $H : K \times Y \longrightarrow 2^{L(\mathbb{R}^n,Y)}$, $V : K \longrightarrow 2^Y$ be two upper semicontinuous mappings with compact values such that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(\mathbb{R}^n,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is H-uniformly continuous, then there exists $x^\star \in K, \ z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi^\star, \eta(y, x^\star)\rangle + f(y, x^\star) \not\leq_{\text{int}\, P(x^\star)} 0, \ \forall y \in K.$$

Corollary 3.7. *Let K be a nonempty, closed and convex subset of a real reflexive Banach space $\mathbb{R}^n$ with $0 \in K$ and Y be a real Banach space. Let $P : K \longrightarrow 2^Y$ be such that for each $x \in K, P(x)$ is a proper, closed, convex cone with int $P(x) \neq \emptyset$. Let $A : L(\mathbb{R}^n, Y) \longrightarrow L(\mathbb{R}^n, Y)$ is a continuous mapping and $T : K \longrightarrow 2^{L(\mathbb{R}^n,Y)}$ be a nonempty compact set-valued mapping. Suppose the following conditions hold:*

- (i) *$f : K \times K \longrightarrow Y$ be affine in first argument with the condition $f(x, y) + f(y, x) = 0, \ \forall x, y \in K$ and continuous in second argument;*
- (ii) *$\langle A\xi, \eta(x, x)\rangle = 0$ for each $x \in K$ and $\xi \in L(\mathbb{R}^n, Y)$;*
- (iii) *for each $(\xi, y) \in L(\mathbb{R}^n, Y) \times K$ fixed, $\langle A\xi, \eta(., y)\rangle : K \longrightarrow Y$ is affine;*
- (iv) *for each $y \in K$ fixed, $\eta(y, .) : K \longrightarrow \mathbb{R}^n$ is continuous;*
- (v) *for each fixed $y \in K$ $\alpha(., y)$ is weakly lower semicontinuous;*
- (vi) *there exists some $r > 0$ such that* **(a)** *$H : K_r \times Y \longrightarrow 2^{L(\mathbb{R}^n,Y)}$, $V : K_r \longrightarrow 2^Y$ are two upper semicontinuous with compact convex values where $K_r = \{x \in K : \|x\| \leq r\}$, and*

(**b**) $\langle A\xi, \eta(0, x)\rangle + f(0, x) \leq_{\text{int}\, P(x)} 0, \ \forall z \in V(x), \ \xi \in H(x, z)$ and $x \in K$ with $\|x\| = r$.

Suppose additionally that H and V are generalized η-pseudomonotone with respect to A. If the set-valued mapping $T : K \longrightarrow 2^{L(\mathbb{R}^n,Y)}$ defined by

$$T(x) = \bigcup_{z \in V(x)} H(x, z) = H(x, V(x))$$

is H-uniformly continuous, then there exists $x^\star \in K,\ z^\star \in V(x^\star)$ and $\xi^\star \in H(x^\star, z^\star)$ such that

$$\langle A\xi^\star, \eta(y, x^\star)\rangle + f(y, x^\star) \not\leq_{int\, P(x^\star)} 0,\ \forall y \in K.$$

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BEST APPROXIMATION FOR CONVEX SUBSETS OF 2-INNER PRODUCT SPACES

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ABSTRACT. In this paper, we study the concept of best approximation in 2-inner product spaces. We get some characteristic theorems for the elements of best approximation for convex subsets of 2-inner product spaces. Finally we get some properties of the metric projection map in this spaces.

KEYWORDS : 2-Inner product space; 2-Normed space; b-Best approximation; b-Proximinal; b-Chebyshev; b-Metric projection; b-Dual cone
AMS Subject Classification: 41A65 41A15

1. INTRODUCTION

Recently, some results on best approximation theory in linear 2-normed spaces have been obtained by Y. J. Cho, S. Elumalai, S. S. Kim, R. Ravi, Sh. Rezapour and others (see [1], [4], [6], [12], [13], [15]). These papers are based on the research works in normed linear spaces made by I. Singer ([14]), T. D. Narang ([11]), S. S. Dragomir ([2]) and others. In this paper we want to investigate the concept of best approximation in 2-inner product spaces. The concept of 2-inner product spaces has been investigated by R. Ehret in 1969([3]), and has been developed extensively in different subjects by others.([10], [8])

Definition 1.1. Let X be a linear space of dimension greater than 1 over filed $\mathbb{R}$ of real numbers.
Suppose that $\langle ., .|.\rangle$ is a $\mathbb{R}$-valued function defined on $X \times X \times X$ satisfying the following conditions:
a) $\langle x, x|z\rangle \geq 0$ and $\langle x, x|z\rangle = 0$ if and only if x and z are linearly dependent.
b) $\langle x, x|z\rangle = \langle z, z|x\rangle$
c) $\langle y, x|z\rangle = \langle x, y|z\rangle$
d) $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$ for any scalar $\alpha \in \mathbb{R}$
e) $\langle x + x', y|z\rangle = \langle x, y|z\rangle + \langle x', y|z\rangle$

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$\langle .,.|.\rangle$ is called a *2-inner product* and $(X, \langle .,.|.\rangle)$ is called a *2-inner product space*(or a *2-per-Hilbert space*). A concept which is closely related to 2-inner product space and introduced by Gähler in 1965, is 2-normed space [5].

Definition 1.2. Let X be a linear space of dimension greater than 1 over filed $\mathbb{R}$ of real numbers. Suppose $\|.,.\|$is a real-valued function on $X \times X$ satisfying the following conditions:
a) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.
b) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.
c) $\|\lambda x, y\| = |\lambda|\|x, y\|$ for all $\lambda \in \mathbb{R}$ and $x, y \in X$.
d) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

Then $\|.,.\|$ is called a *2-norm* on X and $(X, \|.,.\|)$ is called a *linear 2-normed space*. It is easy to show that the 2-norm $\|.,.\|$ is non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X, p_b(x) = \|x, b\|; x \in X$ is a semi-norm on X and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X.
Let $(X, \langle .,.|.\rangle)$ be a 2-inner product space, then
(i) We can define a 2-norm on $X \times X$ by $\|x, y\| = \sqrt{\langle x, x|y\rangle}$.
(ii) Let $0 \neq b \in X$ and $x, y \in X\setminus < b >$. An element $x \in X$ is said to be *b-orthogonal* to an element $y \in X$, and we write $x \perp_b y$, if $\langle x, y|b\rangle = 0$.
(iii) For all $x, y, b \in X$, we have the Cauchy-Schwartz inequality

$$\langle x, y|b\rangle^2 \leq \|x, b\|^2\|y, b\|^2.$$

Let $(X, \|.,.\|)$ be a 2-normed space and V_1 and V_2 be two linear subspaces of X. A 2-functional $f : V_1 \times V_2 \to \mathbb{R}$ is called a bilinear 2-functional on $V_1 \times V_2$, whenever for all $x_1, x_2 \in V_1, y_1, y_2 \in V_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$;
i) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(y_1, y_2)$,
ii) $f(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1\lambda_2 f(x_1, y_1)$.
A bilinear 2-functional $f : V_1 \times V_2 \to \mathbb{R}$ is said to be bounded if there exists a non-negative real number M (called a Lipschitz constant for f) such that $|f(x, y)| \leq M\|x, y\|$ for all $x \in V_1$ and $y \in V_2$. Also, the norm of a bilinear 2-functional f is defined by

$$\|f\| = \inf\{M \geq 0 : M \text{ is a Lipschitz constant for } f\}$$

It is known that

$$\begin{aligned}\|f\| &= \sup\{|f(x, y)| : (x, y) \in V_1 \times V_2, \|x, y\| \leq 1\}\\ &= \sup\{|f(x, y)| : (x, y) \in V_1 \times V_2, \|x, y\| = 1\}\\ &= \sup\{|f(x, y)|/\|x, y\| : (x, y) \in V_1 \times V_2, \|x, y\| > 0\}.\end{aligned}$$

Definition 1.3. [7] A 2-functional $F : V_1 \times V_2 \to \mathbb{R}$ is said to be a convex 2-functional if

$$\begin{aligned}F(a\lambda x + (a - a\lambda)x', b\mu y + (b - b\mu)y') &\leq ab|\lambda\mu|F(x, y) + a|\lambda|(b - b\mu)F(x, y')\\ &+ (a - a\lambda)b|\mu|F(x', y) + (a - a\lambda)(b - b\mu)F(x', y')\end{aligned}$$

for all $|\lambda| \leq 1, |\mu| \leq 1$ and $a, b \geq 0$.

Definition 1.4. Let $(X, \langle .,.|.\rangle)$ be a 2-inner product space, and $b \in X$.

1) A sequence $\{x_n\}$ of X is said to be *b-convergent* and denote by $x_n \xrightarrow{b} x$, if there exists an element $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, b\| = 0$ for all $x \in X$.

2) A subset E of X is said *b-closed*, if for each sequence $\{x_n\}$ in E such that $x_n \xrightarrow{b} x$, we have that $x \in E$.

Definition 1.5. [13] Let $(X, \|.,.\|)$ be a 2-normed space, G a nonempty subset of X, $0 \neq b \in X$, then $g_0 \in G$ is called a *b-best approximation* to x from G if

$$\|x - g_0, b\| = \inf\{\|x - g, b\| : g \in G\}.$$

The set of all b-best approximations of x in G is denoted by $P_{G,b}(x)$. The mapping $P_{G,b} : X \longrightarrow 2^G$ is called the *b-metric projection* onto G.
If each $x \in X \setminus (G+ < b >)$ has at least (resp. exactly) one b-best approximation in G, then G is called a *b-proximinal* (resp. *b-chebyshev*) set.

Definition 1.6. 1) A nonempty subset K of the 2-inner product space X, is called convex if $\lambda x + (1 - \lambda)y \in K$ whenever $x, y \in K$ and $0 \leq \lambda \leq 1$.
2) A nonempty subset C of the 2-inner product space X, is called a convex cone if $\alpha x + \beta y \in C$ whenever $x, y \in C$ and $0 \leq \alpha, \beta \in \mathbb{R}$.
3) A nonempty subset M of the 2-inner product space X, is called a linear subspace if $\alpha x + \beta y \in M$ whenever $x, y \in M$ and $\alpha, \beta \in \mathbb{R}$.
4) A nonempty subset V of the 2-inner product space X, is called affine if $\alpha x + (1 - \alpha)y \in V$ whenever $x, y \in V$, and $\alpha \in \mathbb{R}$.

Definition 1.7. Let $(X, \langle ., .|.\rangle)$ be a 2-inner product space over the real number field $\mathbb{R}$. The *b-dual cone* (or *b-negative polar*) of S is the set

$$S_b^\circ := \{x \in X | \langle x, y|b\rangle \leq 0 \text{ for all } y \in S\}$$

The *b-orthogonal complement* of S is the set

$$S_b^\perp = S_b^\circ \cap (-S_b^\circ) = \{x \in X | \langle x, y|b\rangle = 0 \text{ for all } y \in S\}$$

2. CHARACTERIZATION THEOREMS FOR ELEMENTS OF B-BEST APPROXIMATION FOR CONVEX SUBSETS OF A 2-INNER PRODUCT SPACE

In this section we investigate some characteristic theorems for elements of b-best approximation for convex subsets of a 2-inner product space X. It is known that every nonempety b-closed convex set in a b-Hilbert space is b-chebyshev(see [9], [15]). Now we have the following theorem.

Theorem 2.1. *Let $(X, \langle ., .|.\rangle)$ be a 2-inner product space, and K a b-closed convex subset of X with $X \neq K$. If $x_0 \in X \setminus (K+ < b >)$ and $g_0 \in K$, then the following statements are equivalent.*
(1) $g_0 = P_{K,b}(x_0)$
(2) $x_0 - g_0 \in (K - g_0)_b^\circ$
(3) we have that

$$\inf_{g \in K} \langle g - x_0, g_0 - x_0|b\rangle = \|g_0 - x_0, b\|^2$$

Proof. $(1) \Rightarrow (2)$: This implication is proved in ([15]).
$(2) \Rightarrow (3)$: By (2) we have that for all $g \in K$,

$$\begin{aligned} 0 \leq \langle x_0 - g_0, g_0 - g|b\rangle &= \langle x_0 - g_0, g_0 - x_0 + x_0 - g|b\rangle \\ &= \langle x_0 - g_0, x_0 - g|b\rangle - \langle x_0 - g_0, x_0 - g_0|b\rangle \\ &= \langle x_0 - g_0, x_0 - g|b\rangle - \|x_0 - g_0, b\|^2. \end{aligned}$$

Hence

$$\|x_0 - g_0, b\|^2 \leq \langle x_0 - g_0, x_0 - g|b\rangle$$

for all $g \in K$, Thus

$$\begin{aligned}\|x-g_0,b\|^2 &\leq \inf_{g\in K}\langle x_0-g_0,x_0-g|b\rangle\\ &\leq \langle x_0-g_0,x_0-g_0|b\rangle = \|x-g_0,b\|^2\end{aligned}$$

Therefore

$$\|x_0-g_0,b\|^2 = \inf_{g\in K}\langle x_0-g_0,x_0-g|b\rangle.$$

$(3) \Rightarrow (1)$: If (3) holds, then by Cauchy-Schwartz inequality in 2-inner product spaces, for all $g \in K$ we have

$$\|x_0-g_0,b\|^2 \leq \langle x_0-g_0,x_0-g|b\rangle \leq \|x_0-g_0,b\|\|x_0-g,b\|.$$

Thus $\|x_0-g_0,b\| \leq \|x_0-g,b\|$ for all $g \in K$. That is $g_0 = P_{K,b}(x_0)$.

□

Lemma 2.1. *Let X be a 2-inner product space over the real field $\mathbb{R}$. Then:*
(1) If S is a nonempty subset of X, then S_b° is a b-closed convex cone and $S_b^\perp$ is a b-closed subspace.
(2) If C is a convex cone in X, then $(C-y)_b^\circ = C_b^\circ \cap y_b^\perp$ for each $y \in C$.
(3) If M is a subspace of X, then $M_b^\circ = M_b^\perp$.
(4) If C is a b-chebyshev convex cone in X, then $C_b^{\circ\circ} = C$.
(5) If M is a b-chebyshev subspace in X, then $M_b^{\circ\circ} = M_b^{\perp\perp} = M$.

Proof. (1) Let $x_n \in S_b^\circ$ and $x_n \xrightarrow{b} x$. Then for each $y \in S$,

$$\langle x,y|b\rangle = \lim\langle x_n,y|b\rangle \leq 0$$

implies $x \in S_b^\circ$ and S_b° is b-closed. Let $x,z \in S_b^\circ$ and $\alpha,\beta \geq 0$. Then, for each $y \in S$,

$$\langle \alpha x+\beta z,y|b\rangle = \alpha\langle x,y|b\rangle + \beta\langle z,y|b\rangle \leq 0$$

so $\alpha x+\beta z \in S_b^\circ$ and S_b° is a convex cone. Similarly we can prove $S_b^\perp$ is a b-closed subspace.
(2) We have $x \in (C-y)_b^\circ$ if and only if $\langle x,c-y|b\rangle \leq 0$ for all $c \in C$. Taking $c = 0$ and $c = 2y$, it follows that the last statement is equivalent to $\langle x,y|b\rangle = 0$ and $\langle x,c|b\rangle \leq 0$ for all $c \in C$. That is, $x \in C_b^\circ \cap y_b^\perp$.
(3) If M is a subspace, then $-M = M$ implies

$$M_b^\circ = M_b^\circ \cap (-M)_b^\circ = M_b^\perp.$$

(4) Let C be a b-chebyshev convex cone, and $x \in C\setminus < b >$. Then for any $y \in C_b^\circ, \langle x,y|b\rangle \leq 0$. Hence $x \in C_b^{\circ\circ}$. That is $C \subseteq C_b^{\circ\circ}$. Now remains to verify $C_b^{\circ\circ} \subseteq C$. If not, choose $x \in C_b^{\circ\circ}\setminus C$ and let $y_0 \in P_{C,b}(x)$. By (2) and theorem (2.1) we have

$$x-y_0 \in (C-y_0)_b^\circ = C_b^\circ \cap y_{0_b}^\perp.$$

Thus

$$0 < \|x-y_0,b\|^2 = \langle x-y_0,x-y_0|b\rangle = \langle x-y_0,x|b\rangle \leq 0$$

which is absurd. Therefore $C_b^{\circ\circ} \subseteq C$.
(5) It is clear by (3),(4). □

Now we investigate theorem (2.1) in the case of b-closed convex cone.

Theorem 2.2. *Let $(X,\langle .,.|.\rangle)$ be a 2-inner product space, C a b-closed convex cone in X with $X \neq C$. If $x_0 \in X\setminus(C+ < b >)$ and $g_0 \in C$, then the following statements are equivalent:*
(1) $g_0 = P_{C,b}(x_0)$,

(2) $x_0 - g_0 \in C_b^\circ \cap {g_0}_b^\perp$,
(3) $x_0 - g_0 \in C_b^\circ$ *and* $\langle x_0, g_0|b\rangle = \|g_0, b\|^2$

Proof. The equivalence of (2) and (3) is clear. The equivalence of (1) and (2)is also clear by lemma (2.2). □

Remark 2.2. If M is a linear subspace then by lemma(2.2), $M_b^\circ = M_b^\perp$ and so condition (2) of above theorem reduces to the classical condition that

$$g_0 = P_{M,b}(x_0) \Longleftrightarrow x - g_0 \perp_b M \Longleftrightarrow \langle x - g_0, g|b\rangle = 0 \text{ for all } g \in M \text{ ; (see [13]).}$$

Remark 2.3. Let V be an affine set in the 2-inner product space X, i.e. $V = M + v$, where M is a subspace and v is any element of V. Then we have that

$$g_0 = P_{V,b}(x_0) \Longleftrightarrow x_0 - g_0 \perp_b M \Longleftrightarrow \langle x_0 - g_0, g - v|b\rangle = 0 \text{ for all } g \in V.$$

Moreover

$$P_{V,b}(x + e) = P_{V,b}(x) \text{ for all } x \in X, e \in M_b^\perp.$$

If $(X, \langle ., .\rangle)$ is an inner product space, then the function

$$\langle x, y|z\rangle = \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle y, z\rangle & \langle z, z\rangle \end{vmatrix} = \|z\|^2\langle x, y\rangle - \langle x, z\rangle\langle y, z\rangle$$

for all $x, y, z \in X$ defined a 2-inner product on $X \times X \times X$.

Example 2.4. Let $X = \mathbb{R}^2$, $C = \{(y_1, y_2) \in \mathbb{R}^2; |y_2| \le y_1\}$ and $b = (b_1, b_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. It is simple to verify that C is a b-closed convex cone. Now if $(x_1, x_2) \in C+ < b >$, then

$$P_{C,b}(x_1, x_2) = \begin{cases} \big(\{\frac{x_2b_1 - x_1b_2}{b_2 - b_1}, \frac{x_2b_1 - x_1b_2}{b_2 - b_1}\} + < b > \big) \bigcap C & \text{if } x_2 > 0, b_1 \neq b_2 \\ \big((x_1, x_2) + < b > \big) \bigcap C & \text{if } b_1 = b_2 \\ \big(\{\frac{x_2b_1 - x_1b_2}{b_2 + b_1}, \frac{x_1b_2 - x_2b_1}{b_2 + b_1}\} + < b > \big) \bigcap C & \text{if } x_2 < 0, b_1 \neq -b_2 \\ \big((x_1, x_2) + < b > \big) \bigcap C & \text{if } b_1 = -b_2 \end{cases}$$

and if $(x_1, x_2) \in \mathbb{R}^2 \setminus (C+ < b >)$, then $P_{C,b}(x_1, x_2) = \{(0, 0)\}$.

Let $(X, \langle ., .|.\rangle)$ be a 2-inner product space over the real field $\mathbb{R}$ and let $F : X\times < b >\longrightarrow \mathbb{R}$ be a continues convex 2-functional on $X\times < b >$. Denote by

$$A_b(r) := \{x \in X|F(x, b) \le r\}$$

Assume that r is a real number that $A_b(r) \neq \emptyset$. It is clear that $A_b(r)$ is a b-closed convex subset of X.

Theorem 2.3. *Let* $(X, \langle ., .|.\rangle)$ *be a 2-inner product space over the real field* $\mathbb{R}$ *and let* $F : X\times < b >\longrightarrow \mathbb{R}$ *be a convex 2-functional on* $X\times < b >$. *Let* $x_0 \in X \setminus (A_b(r)+ < b >)$ *and* $g_0 \in A_b(r)$. *Then the following are equivalent.*
(1) $g_0 = P_{A_b(r),b}(x_0)$;
(2) $F(x, b) \ge r + \frac{F(x_0,b) - r}{\|x_0 - g_0, b\|^2}\langle x - g_0, x_0 - g_0|b\rangle$ *for all* $x \in A_b(r)$ *where* $r = F(g_0, b)$.

Proof. (1) ⇒ (2) Assume that $g_0 = P_{A_b(r),b}(x_0)$. Since $x_0 \in X \setminus (A_b(r)+ < b >)$ we have $F(x_0, b) > r$. Let $x \in A_b(r)$, then $F(x, b) \le r$. Set $\alpha = F(x_0, b) - r$, $\beta = r - F(x, b)$. Then $\alpha > 0$, $\beta \ge 0$ and $0 < \alpha + \beta = F(x_0, b) - F(x, b)$. Consider the element $u = \frac{\alpha x + \beta x_0}{\alpha + \beta}$. By convexity of F, for $y = y' = b, \lambda = \frac{\alpha}{\alpha+\beta}, \mu = 1$ and $a = b = 1$ in definition of convex 2-functional we have that:

$$F(u, b) = F(\frac{\alpha x + \beta x_0}{\alpha + \beta}, b) \le \frac{\alpha F(x, b) + \beta F(x_0, b)}{\alpha + \beta}$$
$$= \frac{(F(x_0, b) - r)F(x, b) + (r - F(x, b))F(x_0, b)}{F(x_0, b) - F(x, b)} = r$$

That is $u \in A_b(r)$. As $g_0 = P_{A_b(r),b}(x_0)$, we have by theorem (2.1) that $x_0 - g_0 \in (A_b(r) - g_0)_b^\circ$, so, $\langle g - g_0, x_0 - g_0|b\rangle \leq 0$ for all $g \in A_b(r)$. In particular, $\langle u - g_0, x_0 - g_0|b\rangle \leq 0$. That is

$$\begin{aligned} 0 &\geq \langle u - g_0, x_0 - g_0|b\rangle = \langle \frac{\alpha x + \beta x_0}{\alpha+\beta} - g_0, x_0 - g_0|b\rangle \\ &= \frac{1}{\alpha+\beta}\langle \alpha x + \beta x_0 - (\alpha+\beta)g_0, x_0 - g_0|b\rangle \\ &= \frac{\alpha}{\alpha+\beta}\langle x - g_0, x_0 - g_0|b\rangle + \frac{\beta}{\alpha+\beta}\langle x_0 - g_0, x_0 - g_0|b\rangle \\ &= \frac{(F(x_0,b) - r)}{F(x_0,b) - F(x,b)}\langle x - g_0, x_0 - g_0|b\rangle + \frac{(r - F(x,b))}{F(x_0,b) - F(x,b)}\|x_0 - g_0, b\|^2 \end{aligned}$$

Thus

$$F(x,b) \geq \frac{(F(x_0,b) - r)}{\|x_0 - g_0, b\|^2}\langle x - g_0, x_0 - g_0|b\rangle$$

for all $x \in A_b(r)$. Since above theorem is true for all $x \in A_b(r)$ so for $x = g_0$ we have that $F(g_0,b) \geq r$. But since $g_0 \in A_b(r)$ we have $F(g_0,b) \leq r$. Thus $F(g_0,b) = r$.
$(2) \Longrightarrow (1)$: Assume that (2) holds. Then for all $x \in A_b(r)$,

$$0 \geq F(x,b) - r \geq \frac{(F(x_0,b) - r)}{\|x_0 - g_0, b\|^2}\langle x - g_0, x_0 - g_0|b\rangle$$

Since $F(x_0,b) - y > 0$, we have that

$$\langle x - g_0, x_0 - g_0|b\rangle \leq 0$$

for all $x \in A_b(r)$. That is, $x_0 - g_0 \in (A_b(r) - g_0)^\circ$, whence, $g_0 = P_{A_b(r),b}(x_0)$. □

Corollary 2.5. *Let $f : X\times < b > \longrightarrow \mathbb{R}$ be a continues sublinear 2-functional on the 2-inner product space $(X, \langle ., .|.\rangle)$. Put $K_b(f) := \{x \in X|f(x,b) \leq 0\}$. Let $x_0 \in X \setminus (K_b(f)+ < b >)$ and $g_0 \in K_b(f)$. Then the following statements are equivalent:*
(1) $g_0 = P_{K_b(f),b}(x_0)$;
(2) $f(x,b) \geq \frac{f(x_0,b)}{\|x_0-g_0,b\|^2}\langle x - g_0, x_0 - g_0|b\rangle$ for all $x \in K_b(f)$.

Proof. It is sufficient that in above theorem taking $F = f$ and $r = 0$. □

It is clear that $X = K_b(f) \cup K_b(-f)$ and $\ker(f) = \{x \in X|f(x,b) = 0\} = K_b(f) \cap K_b(-f)$. If in the above corollary replacing f with $-f$, then we have:

Corollary 2.6. *Let $f : X\times < b > \longrightarrow \mathbb{R}$ be a continuous sublinear 2-functional on the 2-inner product space $(X, \langle ., .|.\rangle)$. Let $x_0 \in X \setminus (\ker(f)+ < b >)$ and $g_0 \in \ker(f)$. Then the following statements are equivalent:*
(1) $g_0 = P_{\ker(f),b}(x_0)$;
(2) $f(x,b) = \frac{f(x_0,b)}{\|x_0-g_0,b\|^2}\langle x, x_0 - g_0|b\rangle$ for all $x \in \ker(f)$.

Remark 2.7. For another proof of above corollary see [15].

3. B-METRIC PROJECTION IN 2-INNER PRODUCT SPACES

In this section we investigate some properties of the b-metric projection onto convex cone and get some consequence, specially we show that every 2-inner product space is direct sum of any b-chebyshev subspace and its b-orthogonal complement.

Proposition 3.1. *Let K be a convex b-chebyshev set and $K\cap < b >= \emptyset$. Then*
(1) $P_{K,b}$ is idempotent i.e.

$$P_{K,b}(P_{K,b}(x)) = P_{K,b}(x)$$

for every $x \in X$.
(2) $P_{K,b}$ is firmly nonexpansive i.e.

$$\langle x - y, P_{K,b}(x) - P_{K,b}(y)|b\rangle \geq \|P_{K,b}(x) - P_{K,b}(y), b\|^2$$

for all $x, y \in X$.
(3) $P_{K,b}$ is monotone i.e.

$$\langle x - y, P_{K,b}(x) - P_{K,b}(y)|b\rangle \geq 0$$

for all $x, y \in X$.
(4) $P_{K,b}$ is strictly nonexpansive i.e.

$$\|x - y, b\|^2 > \|P_{K,b}(x) - P_{K,b}(y), b\|^2 + \|x - P_{K,b}(x) - (y - P_{K,b}(y)), b\|^2$$

for all $x, y \in X$.
(5) $P_{K,b}$ is nonexpansive i.e.

$$\|P_{K,b}(x) - P_{K,b}(y), b\| \leq \|x - y, b\|$$

for all $x, y \in X$.
(6) $P_{K,b}$ is uniformly continuous.

Proof. (1) It is clear.
(2) We have:

$$\begin{aligned}\langle x - y, P_{K,b}(x) - P_{K,b}(y)|b\rangle &= \langle x - P_{K,b}(x), P_{K,b}(x) - P_{K,b}(y)|b\rangle \\ &+ \langle P_{K,b}(x) - P_{K,b}(y), P_{K,b}(x) - P_{K,b}(y)|b\rangle \\ &+ \langle P_{K,b}(y) - y, P_{K,b}(x) - P_{K,b}(y)|b\rangle\end{aligned}$$

The first and third terms on the right are nonnegative by theorem (2.1), and the second term is $\|P_{K,b}(x) - P_{K,b}(y), b\|^2$. This verifies (2).
(3) It is immediate consequence of (2).
(4) Using (2) we obtain for each $x, y \in X$ that:

$$\begin{aligned}\|x - y, b\|^2 &= \|(x - P_{K,b}(x)) + (P_{K,b}(x) - P_{K,b}(y)) + (P_{K,b}(y) - y), b\|^2 \\ &= \|P_{K,b}(x) - P_{K,b}(y), b\|^2 + \|(x - P_{K,b}(x)) - (y - P_{K,b}(y))\|^2 \\ &+ 2\langle P_{K,b}(x) - P_{K,b}(y), x - P_{K,b}(x) - (y - P_{K,b}(y))|b\rangle \\ &= \|P_{K,b}(x) - P_{K,b}(y), b\|^2 + \|(x - P_{K,b}(x)) - (y - P_{K,b}(y))\|^2 \\ &+ 2\langle P_{K,b}(x) - P_{K,b}(y), x - y|b\rangle - 2\|P_{K,b}(x) - P_{K,b}(y), b\|^2 \\ &\geq \|P_{K,b}(x) - P_{K,b}(y), b\|^2 + \|x - P_{K,b}(x) - (y - P_{K,b}(y)), b\|^2\end{aligned}$$

This proves (4).
(5) It is follows immediately from (4).
(6) It is follows immediately from (5). □

Theorem 3.1. *Let C be a b-chebyshev convex cone in the 2-inner product space X and $C\cap < b >= \emptyset$. Then C_b° is a b-chebyshev convex cone and*
(1) For each $x \in X$,

$$x = P_{C,b}(x) + P_{C_b^\circ,b}(x) \text{ and } P_{C,b}(x) \perp_b P_{C_b^\circ,b}(x).$$

Moreover, this representation is unique in the sense that if $x = y + z$ for some $y \in C$ and $z \in C_b^\circ$ with $y \perp_b z$, then $y = P_{C,b}(x)$ and $z = P_{C_b^\circ,b}(x)$.

(2) $\|x,b\|^2 = \|P_{C,b}(x),b\|^2 + \|P_{C_b^\circ,b}(x),b\|^2$ *for all* $x \in X\setminus < b >$.
(3) $C_b^\circ = \{x \in X | P_{C,b}(x) = 0\}$ *and* $C = \{x \in X | P_{C_b^\circ,b}(x) = 0\} = \{x \in X | P_{C,b}(x) = x\}$.
(4) $\|P_{C,b}(x),b\| \le \|x,b\|$ *for all* $x \in X$*; moreover,* $\|P_{C,b}(x),b\| = \|x,b\|$ *if and only if* $x \in C$.
(5) $C_b^{\circ\circ} = C$.
(6) $P_{C,b}$ *is positively homogeneous. i.e.*

$$P_{C,b}(\lambda x) = \lambda P_{C,b}(x) \text{ for all } x \in X, \lambda \ge 0.$$

Proof. (1) Let $x \in X$ and $c_0 = x - P_{C,b}(x)$. By theorem (2.3) $c_0 \in C_b^\circ$ and $c_0 \perp_b (x - c_0)$. For every $y \in C_b^\circ$,

$$\langle x - c_0, y|b\rangle = \langle P_{C,b}(x), y|b\rangle \le 0.$$

Hence $x - c_0 \in (C_b^\circ)^\circ$. By theorem (2.3), we get that $c_0 = P_{C_b^\circ,b}(x)$. This proves that C_b° is b-chebyshev convex cone, $x = P_{C,b}(x) + P_{C_b^\circ,b}(x)$ and $P_{C,b}(x) \perp_b P_{C_b^\circ,b}(x)$. Now we verify the uniqueness of this representation. Let $x = y + z$, where $y \in C, z \in C_b^\circ$ and $y \perp_b z$. For each $c \in C$,

$$\langle x - y, c|b\rangle = \langle z, c|b\rangle \le 0$$

and

$$\langle x - y, y|b\rangle = \langle z, y|b\rangle = 0.$$

By theorem (2.3) $y = P_{C,b}(x)$. Similarly $z = P_{C_b^\circ,b}(x)$.
(2) It is clear by (1) and Pythagorean theorem in 2-inner product spaces.
(3) By using (1) we have that:

$$x \in C_b^\circ \iff x = P_{C_b^\circ,b}(x) \iff P_{C,b}(x) = 0,$$

and

$$x \in C \iff x = P_{C,b}(x) \iff P_{C_b^\circ,b}(x) = 0.$$

(4) From (2), it is clear that $\|P_{C,b}(x),b\| \le \|x,b\|$ for all $x \in X$. Also from (2), $\|P_{C,b}(x),b\| = \|x,b\|$ if and only if $P_{C_b^\circ,b}(x) = 0$, which from (3) is equivalent to $x \in C$.
(5) By (3) we have:

$$C_b^{\circ\circ} = (C_b^\circ)^\circ = \{x \in X | P_{C_b^\circ,b}(x) = 0\} = C.$$

(6) Let $x \in X$ and $\lambda \ge 0$. Then $x = P_{C,b}(x) + P_{C_b^\circ,b}(x)$ and $\lambda x = \lambda P_{C,b}(x) + \lambda P_{C_b^\circ,b}(x)$. Since both C and C_b° are convex cones, $\lambda P_{C,b}(x) \in C$ and $\lambda P_{C_b^\circ,b}(x) \in C_b^\circ$. By (1) and the uniqueness of representation for λx, we see that $P_{C,b}(\lambda x) = \lambda P_{C,b}(x)$. □

Corollary 3.2. *Let* M *be a b-chebyshev subspace of the 2-inner product space* X*. Then* $M_b^\perp$ *is a b-chebyshev subspace and:*
(1) $x = P_{M,b}(x) + P_{M_b^\perp,b}(x)$*, for each* $x \in X$*. Moreover, this representation is unique in the sense that if* $x = y + z$ *where* $y \in M$ *and* $z \in M_b^\perp$*, then* $y = P_{M,b}(x)$ *and* $z = P_{M_b^\perp,b}(x)$.
(2) $\|x,b\|^2 = \|P_{M,b}(x),b\|^2 + \|P_{M_b^\perp,b}(x),b\|^2$ *for all* $x \in X\setminus < b >$.
(3) $M_b^\perp = \{x \in X | P_{M,b}(x) = 0\}$ *and* $M = \{x \in X | P_{M_b^\perp,b}(x) = 0\} = \{x \in X | P_{M,b}(x) = x\}$.
(4) $\|P_{M,b}(x),b\| \le \|x,b\|$ *for all* $x \in X$*; moreover,* $\|P_{M,b}(x),b\| = \|x,b\|$ *if and only if* $x \in M$.
(5) $M_b^{\perp\perp} = M$.

Theorem 3.2. *Let M be a b-chebyshev subspace of the 2-inner product space X and $M\cap < b >= \emptyset$. Then:*
(1) $P_{M,b}$ is a bounded linear 2-functional and $\|P_{M,b}\| = 1$ (in the case $M = \{0\}$, we have $\|P_{M,b}\| = 0$).
(2) $P_{M,b}$ is self-adjoint i.e.

$$\langle P_{M,b}(x), y|b\rangle = \langle x, P_{M,b}(y)|b\rangle \text{ for all } x, y \in X.$$

(3) For every $x \in X$,

$$\langle P_{M,b}(x), x|b\rangle = \|P_{M,b}(x), b\|^2.$$

(4) $P_{M,b}$ is nonnegative i.e.

$$\langle P_{M,b}(x), x|b\rangle \geq 0 \text{ for every } x \in X.$$

Proof. (1) Let $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. By remark (2.4), $x - P_{M,b}(x)$ and $y - P_{M,b}(y)$ are in $M_b^{\perp}$. since $M_b^{\perp}$ is subspace,

$$\alpha x + \beta y - (\alpha P_{M,b}(x) + \beta P_{M,b}(y)) = \alpha(x - P_{M,b}(x)) + \beta(y - P_{M,b}(y)) \in M_b^{\perp}.$$

Since $\alpha P_{M,b}(x) + \beta P_{M,b}(y) \in M$, remark (2.4) implies that $\alpha P_{M,b}(x) + \beta P_{M,b}(y) = P_{M,b}(\alpha x+\beta y)$. Thus $P_{M,b}$ is linear.From corollary (3.3 [4]) we get $\|P_{M,b}(x)\| \leq \|x, b\|$ for all $x \in X$. So $\|P_{M,b}$ is bounded and $\|P_{M,b}\| \leq 1$. Since $P_{M,b}y = y$ for all $y \in M$ and $\|y, b\| = \|P_{M,b}(y)\| \leq \|P_{M,b}\|\|y, b\|$, implies that $\|P_{M,b}\| \geq 1$ therefore $\|P_{M,b}\| = 1$.
(2) By remark (2.4), for each $x, y \in X$ we have $\langle P_{M,b}(x), y - P_{M,b}(y)|b\rangle = 0$, and hence

$$\langle P_{M,b}(x), y|b\rangle = \langle P_{M,b}(x), P_{M,b}(y)|b\rangle. \ (*)$$

By replacing x and y in above relation we obtain

$$\begin{aligned} \langle x, P_{M,b}(y)|b\rangle &= \langle P_{M,b}(y), x|b\rangle \\ &= \langle P_{M,b}(y), P_{M,b}(x)|b\rangle \\ &= \langle P_{M,b}(x), P_{M,b}(y)|b\rangle \\ &= \langle P_{M,b}(x), y|b\rangle. \end{aligned}$$

(3) Tacking $y = x$ in $(*)$.
(4) It is clear from (3). □

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A GENERAL ITERATIVE ALGORITHM FOR MONOTONE OPERATORS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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ABSTRACT. Let $VI(A, H)$ be the set of all solutions of the following variational inequality problem:

$$\text{find} \quad u \in H \quad \text{such that } \langle v - u, Au \rangle \geq 0, \quad \text{for all } v \in H.$$

Where H is a Hilbert space, $A : H \to H$ is a Lipschitz continuous and monotone operator. Assume that $F : H \to H$ is a Lipschitz continuous and strongly monotone operator. Let $f : H \to H$ be a Lipschitz continuous mapping. In this paper, we consider a demiclosed, demicontractive mapping T on H such that $Fix(T) \cap VI(A, H) \neq \emptyset$.

For finding an element x^* which solves the following variational inequality problem: find an $x^* \in Fix(T) \cap VI(A, H)$ such that

$$\langle v - x^*, \mu F x^* - \gamma f x^* \rangle \geq 0, \qquad \text{for all } v \in Fix(T) \cap VI(A, H),$$

when μ and γ are positive real numbers which satisfy appropriate conditions, we introduce a new general iterative algorithm and obtain strong convergence results.

KEYWORDS : Demicontractive mapping; Viscosity method; Monotone operator; Variational inequality, Fixed point.
AMS Subject Classification: 58E35 47H09 47H05 47H10.

1. INTRODUCTION

Many problems arising in engineering sciences and structural analysis, are reduced to variational inequalities and fixed point problems, and iterative algorithms to solve these problems have been proposed.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. Recall that a mapping $F : H \to H$ is called $\eta-$strongly monotone operator if there is a positive real number η such that

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \qquad \text{for all } x, y \in H.$$

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Assume that $f : H \to H$ is a α-contraction: that is, there is a constant $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in H$. Let T be a nonexpansive mapping on H. i.e. $|T(x) - T(y)\| \leq \|x - y\|$ for $x, y \in H$. We use $Fix(T)$ to denote the set of all fixed points of T.

The viscosity approximation method of selecting a particular fixed point of given nonexpansive mapping was proposed by Moudafi [9]. Particularly, he introduced the following process: Let $x_1 \in H$ be arbitrary and

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} T(x_n) \qquad n \geq 0, \tag{1.1}$$

where f is a contraction with the coefficient $\alpha \in [0, 1)$, T is a nonexpansive mapping on H and $\{\varepsilon_n\}$ is a sequence in $(0, 1)$ such that

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \Sigma_{n=0}^{\infty}\varepsilon_n = \infty, \text{ and } \lim_{n\to\infty}\left(\frac{1}{\varepsilon_n} - \frac{1}{\varepsilon_{n+1}}\right) = 0.$$

It is showed that the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution $x^* \in Fix(T)$ of the variational inequality:

$$\langle (f - I)x^*, x - x^* \rangle \leq 0, \qquad \text{for all } x \in Fix(T).$$

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping on a real Hilbert space H :

$$\min\{\frac{1}{2}\langle Bx, x\rangle \ - \langle x, b\rangle : x \in C\}, \tag{1.2}$$

where C is the set of all fixed points of a nonexpansive mapping T on H and b is a given point in H, B is a strongly positive bounded linear map on H : That is, there is a constant $\gamma \geq 0$ with the following property

$$\langle Bx, x\rangle \geq \gamma\|x\|^2, \qquad \text{for all } x \in H. \tag{1.3}$$

In [17], Xu proved that the sequence $\{x_n\}$ generated by the recursive relation

$$x_{n+1} = \alpha_n b + (1 - \alpha_n B)Tx_n, \qquad n \geq 0, \tag{1.4}$$

converges strongly to the unique solution of the the quadratic minimization problem (1.2) under suitable hypotheses on $\{\alpha_n\}$. In 2006, Marino and Xu combined the iterative method (1.4) with the viscosity approximation method (1.1) and consider the following iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B)Tx_n, \qquad n \geq 0. \tag{1.5}$$

They showed that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution $\tilde{x}$ of the variational inequality

$$\langle (\gamma f - B)\tilde{x}, x - \tilde{x}\rangle \leq 0 \qquad \text{for all } x \in C, \tag{1.6}$$

which is an optimal condition for the minimization problem

$$\min\{\frac{1}{2}\langle Bx, x\rangle \ - h(x) : x \in C\},$$

where h is a potential function for γf.

In 2009, Mainge [6] generalized the moudafi's scheme (1.1), and proved strong convergence results for quasi-nonexpansive mapping in Hilbert spaces.

In 2010, Tian defined the following iterative scheme

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n \mu F)T(x_n) \tag{1.7}$$

where $f : H \to H$ is a contraction and $F : H \to H$ is a κ-Lipschitzian continuous and $\eta-$strongly monotone operator with $\kappa > 0$, $\eta > 0$. He offered some strong convergence results for the case that T is a nonexpansive mapping on H.

In [14], Tian extended the algorithm (1.7) and acquired a more general result: suppose that T is a nonexpansive mapping on H, f is a $L-$Lipschitzian continuous operator with $L > 0$ and $F : H \to H$ is a κ-Lipschitzian continuous and $\eta-$strongly monotone operator with $\kappa > 0$, $\eta > 0$. Assume that $0 < \mu < 2\eta/\kappa^2$, $\quad 0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L$, and the sequence $\{\alpha_n\}$ satisfies the following conditions,

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \Sigma_{n=0}^{\infty}\alpha_n = \infty \quad \text{and} \quad \Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty,$$

then, the sequence $\{x_n\}$ defined by the recursive relation

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n \mu F)T(x_n), \qquad \text{for all } n \geq 0, \tag{1.8}$$

converges strongly to the unique solution $x^* \in Fix(T)$ of the following variational inequality:

$$\langle(\gamma f - \mu F)x^*, x - x^*\rangle \leq 0, \qquad \text{for all } x \in Fix(T). \tag{1.9}$$

Currently, Tian and Jin [15] considered the following iterative algorithm. Let $x_0 = x$ be an arbitrary element in H,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n \mu F)T_w(x_n), \qquad \text{for all } n \geq 0, \tag{1.10}$$

where $w \in (0, \frac{1}{2})$, $T_w := (1 - w)I + wT$, T is a quasi-nonexpansive mapping on H and the sequence $\{\alpha_n\}$ satisfies the following two conditions:

(i) $\lim \alpha_n = 0$.
(ii) $\Sigma_{n=0}^{\infty}\alpha_n = \infty$.

They obtained strong convergence results over the class of quasi-nonexpansive mappings in Hilbert spaces.

Before introducing our work in this paper, we need to offer a few background on the Korpelevich extragradient method. Note that in this paper, we denote by $VI(A, C)$ the set of solutions of the following variational inequality problem:

$$\text{find } u \in C \text{ such that } \quad \langle v - u, Au\rangle \geq 0, \qquad \text{for all } v \in C, \tag{1.11}$$

where C is a nonempty closed convex set in H and $A : H \to H$ is a monotone mapping on C : that is,

$$\langle Ax - Ay, x - y\rangle \geq 0, \qquad \text{for all } x, y \in C.$$

It should be noted that $VI(A, C)$ is closed and convex (see [4] and [1]).

In 1976, Korpelevich [3] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda A x_n), \\ x_{n+1} &= P_C(x_n - \lambda A y_n), \end{aligned} \tag{1.12}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{\theta})$, C is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and $\theta-$Lipschitzian continuous mapping of C into $\mathbb{R}^n$. Korpelevich proved that if $VI(A, C)$ is nonempty, then both sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.12), converge strongly to a point $z \in VI(A, C)$.

The following iterative algorithm which is based on Korpelevich's extragradient method [3] and Mann's iteration [7] was introduced by Nadezhkina and Takahashi

[10], when T is a nonexpansive and A is monotone and $\theta-$Lipschitz continuous:

$$\begin{aligned} &x_0 \in H, \\ &y_n = P_C(x_n - \lambda_n Ax_n), \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \lambda_n Ay_n) \quad n \geq 0, \end{aligned} \tag{1.13}$$

where $\lambda_n \subset [a,b]$ for some $a, b \in (0, \frac{1}{\theta})$ and $\alpha_n \subset [c,d]$ for some $c, d \in (0,1)$. They proved that both sequences $\{x_n\}$ and $\{y_n\}$ given by (1.13) converge weakly to the same point in $Fix(T) \cap VI(A,C)$.

The next algorithm in this direction was introduced by Zeng and Yao [20]. They proved the following Theorem:

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, $\theta-$Lipschitz continuous mapping and $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \cap VI(A,C) \neq \varnothing$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by*

$$\begin{aligned} &x_0 \in H, \\ &y_n = P_C(x_n - \lambda_n Ax_n), \\ &x_{n+1} = \alpha_n x_0 + (1-\alpha_n)TP_C(x_n - \lambda_n Ay_n) \quad n \geq 0, \end{aligned} .$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions

(H1) $\{\alpha_n\} \subset [0,1), \quad \sum_{n\geq 0} \alpha_n = \infty, \quad \alpha_n \to 0;$
(H2) $\{\theta\lambda_n\} \subset [a,b] \quad (where\ 0 < a \leq b < 1).$

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $P_{Fix(T)\cap VI(A,C)}x_0$ provided that $\|x_{n+1} - x_n\| \to 0$.

Suppose that $T : H \to H$ is a demicontractive mapping. In [4], for finding a solution of the following variational inequality problem:

$$\text{find } x^* \in Fix(T) \cap VI(A,H) \text{ such that}$$
$$\langle v - x^*, Fx^* \rangle \geq 0, \text{ for all } v \in Fix(T) \cap VI(A,H), \tag{1.14}$$

Mainge suggested a new iterative algorithm. Particularly, he proved the following excellent criterion:

Theorem 1.2. ([4]) *Assume that $A : H \to H$ is monotone on C and $\theta-$Lipschitz continuous on H. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A,C) \neq \varnothing$. Let $F : H \to H$ be a $L-$Lipschitzian, $\eta-strongly$ monotone operator with $L > 0, \eta > 0$, and assume that the following conditions hold:*

(H1) $w \in (0, \frac{1-\beta}{2}];$
(H2) $\{\alpha_n\} \subset [0,1),\ \alpha_n \to 0;$
(H3) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1);$
(H4) $\sum_{n\geq 0} \alpha_n = \infty.$

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned} &x_0 \in H, \\ &y_n = P_C(x_n - \lambda_n Ax_n), \\ &t_n = P_C(x_n - \lambda_n Ay_n), \\ &x_{n+1} = [(1-w)I + wT]v_n, \qquad v_n := t_n - \alpha_n F(t_n), \end{aligned} \tag{1.15}$$

converge strongly to x^*, the unique solution of (1.14).

In this paper, motivated by the above-mentioned works, we consider a demicontractive mapping T on H such that $Fix(T) \cap VI(A,H) \neq \varnothing$. For finding an element x^* which solves the following variational inequality problem:

$$\begin{aligned} &\text{find } x^* \in Fix(T) \cap VI(A,H) \text{ such that} \\ &\langle v - x^*, \mu F x^* - \gamma f x^* \rangle \geq 0, \text{ for all } v \in Fix(T) \cap VI(A,H), \end{aligned} \tag{1.16}$$

we introduce the following iterative algorithm

$$\begin{aligned} &x_0 \in H, \\ &y_n = x_n - \lambda_n A x_n, \\ &t_n = x_n - \lambda_n A y_n, \\ &x_{n+1} = \alpha_n \gamma f(t_n) + (1 - \alpha_n \mu F)[(1-w)I + wT](t_n), \end{aligned} .$$

and prove that under appropriate assumptions, the sequences $\{t_n\}$, $\{y_n\}$ and $\{x_n\}$ converge strongly to the same point x^* which is the unique solution of (1.16)

We note that the class of demicontractive mappings includes important operators such as quasi-nonexpansive mappings and the strictly pseudocontractive mappings with fixed points. Hence, our algorithm, which deals with demicontractive mappings and is based on the extragradient, visosity and Hybrid steepest descent method, enables us to obtain more extended results.

2. PRELIMINARIES

Throughout this paper, we denote $x_n \to x$ (respectively, $x_n \rightharpoonup x$) the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. Let C be a closed convex subset of a Hilbert space H. Let us recall that a mapping $T : H \to H$ is called

(i) quasi-nonexpansive if $\|Tx - q\| \leq \|x - q\|$ for all $(x,q) \in H \times Fix(T)$.
(ii) demicontractive if there is $\beta \in [0,1)$ such that $\|Tx - q\|^2 \leq \|x - q\|^2 + \beta\|x - Tx\|^2$ for all $(x,q) \in H \times Fix(T)$.

Also recall that $T : H \to H$ is demiclosed at the origin if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, we have $x \in Fix(T)$.

Lemma 2.1. ([4]) *Let $T : H \to H$ be a $\beta-$demicontractive mapping and let $T_w := (1-w)I + wT$. Then T_w is a quasi-nonexpansive mapping on H if $w \in [0, 1-\beta]$. Besides we have*

$$\|T_w x - q\|^2 \leq \|x - q\|^2 - w(1 - \beta - w)\|x - Tx\|^2.$$

Recall that the projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

The following lemma characterizes the projection P_C.

Lemma 2.2. ([12]) *Let C be a closed convex subset of a real Hilbert space H, $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if*

$$\langle x - y, y - z \rangle \geq 0, \qquad \textit{for all } z \in C. \tag{2.1}$$

Lemma 2.3. *Let H be a real Hilbert space. Then, the following simple well-known result holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \qquad x, y \in H.$$

Lemma 2.4. ([17]) *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the condition*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n \qquad n \geq 1$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$.
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *Let $f : H \to H$ be a L-Lipschitzian continuous operator with the coefficient $L > 0$, $F : H \to H$ be a κ-Lipschitzian continuous and $\eta-$strongly monotone operator with $\kappa > 0, \eta > 0$. Then, for $0 < \gamma \leq \frac{\mu\eta}{L}$,*

$$\langle x-y, (\mu F - \gamma f)x - (\mu F - \gamma f)y\rangle \geq (\mu\eta - \gamma L)\|x-y\|^2.$$

That is, $\mu F - \gamma f$ is strongly monotone with coefficient $\mu\eta - \gamma L$.

Lemma 2.6. *Let $F : H \to H$ be a $\kappa-$Lipschitzian, $\eta - strongly$ monotone operator with $\kappa > 0, \eta > 0$, and $f : H \to H$ be a $L-$Lipschitzian mapping and assume that $0 < \mu < 2\eta/\kappa^2,\ 0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L$. Then we have*

$$\|(1-\alpha_n\mu F)x - (1-\alpha_n\mu F)y\| \leq (1-\alpha_n\tau)\|x-y\| \quad \text{for all } x,y \in H.$$

Lemma 2.7. ([6]) *Let $\{\Gamma_n\}$ be a sequence of nonnegative real numbers which is not decreasing at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}_{n\geq 0}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also, let $\{\tau(n)\}_{n\geq 0}$ be a sequence of integers defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq 0}$ is a nondecreasing sequence, $\lim_{n\to\infty}\tau(n) = \infty$, and for all $n \geq 0$, $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ and $\Gamma_n < \Gamma_{\tau(n)+1}$.

The following lemma helps us to prove the main result of this paper in the next section.

Lemma 2.8. ([4, Lemma 4.2]) *Let A be a $\theta-$Lipschitzian continuous and monotone mapping on a real Hilbert space H. Assume that $VI(A,C) \neq \varnothing$. Let $\{t_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences in H such that*

$$y_n = P_C(x_n - \lambda_n A x_n), \qquad t_n = P_C(x_n - \lambda_n A y_n).$$

Then, we have the following inequalities

$$\|y_n - t_n\| \leq \theta\lambda_n\|x_n - y_n\| \qquad and \qquad \|t_n - u\|^2 \leq \|x_n - u\|^2 - (1-\theta^2\lambda_n^2)\|x_n - y_n\|^2,$$

where u is any element in $VI(A,C)$.

Let C be a closed convex subset of a real Hilbert space H and let $A : C \to H$ be a monotone mapping. Let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0 \text{ for all } u \in C\}$, and define $B : H \to 2^H$ by

$$Bv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \varnothing & \text{otherwise}, \end{cases} \tag{2.2}$$

then, B is maximal monotone: that is, the graph of B defined by $G(B) = \{(x,y) \in H \times H;\quad y \in B(x)\}$ is not contained in the graph of any other monotone mapping. Furthermore, we have the well-known result that $0 \in Bv$ if and only if $v \in VI(A,C)$ (see, for instance, [20] and [11]). Thus,

$$\langle u-v, -w\rangle \geq 0, \qquad \text{for all } (v,w) \in G(B) \Rightarrow u \in VI(A,C). \tag{2.3}$$

The following lemma gives us sufficient conditions ensuring that the weak cluster points of the sequence $\{t_n\}$, defined in the Lemma, belong to $VI(A,C)$. The lemma was proven implicitly in [20] and in [4], but, we bring the proof for the sake of completeness.

Lemma 2.9. *Let C be a closed convex subset of a real Hilbert space H and let $A : C \to H$ be a $\theta-$Lipschitzian continuous and monotone mapping. Let $\{\lambda_n\} \subset [\delta, \infty)$ (for some $\delta > 0$) and let $\{t_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences in C such that*

$$y_n = P_C(x_n - \lambda_n A x_n), \qquad t_n = P_C(x_n - \lambda_n A y_n).$$

Assume that

(i) *$\{t_{n_k}\}$ converges weakly to some u in C;*
(ii) *$\|x_{n_k} - y_{n_k}\| \to 0$ and $\|t_{n_k} - y_{n_k}\| \to 0$.*

Then, u belongs to the set $VI(A,C)$.

Proof. Assume that B is the mapping defined as in the (2.2). By the above comments, since A is monotone and Lipschitz continuous on C, it follows that B is maximal monotone. Hence, we can use the property (2.3). Suppose that $\{t_{n_k}\}$ converges weakly to some u in C. We show that $\langle u - v, -w\rangle \geq 0$ for all $(v,w) \in G(B)$. For this purpose, let (v,w) be an arbitrary element of $G(B)$. It follows from the definition that $w \in Av + N_C v$. Hence, $w - Av \in N_C v$ and $\langle v - z, w - Av\rangle \geq 0$ for all $z \in C$. Since $\{t_n\} \subset C$, we deduce that

$$\langle v - t_{n_k}, w\rangle \geq \langle v - t_{n_k}, Av\rangle. \tag{2.4}$$

Using (2.1) we have $\langle x_{n_k} - \lambda_{n_k} A y_{n_k} - t_{n_k}, t_{n_k} - v\rangle \geq 0$. Hence, we have

$$\begin{aligned}\langle v - t_{n_k}, w\rangle &\geq \langle v - t_{n_k}, Av\rangle - \tfrac{1}{\lambda_{n_k}}\langle t_{n_k} - v, x_{n_k} - \lambda_{n_k} A y_{n_k} - t_{n_k}\rangle \\ &= \langle v - t_{n_k}, Av - At_{n_k}\rangle + \langle v - t_{n_k}, At_{n_k} - Ay_{n_k}\rangle \\ &\quad -\langle v - t_{n_k}, \tfrac{t_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle,\end{aligned}$$

Now, since A is monotone, we deduce that

$$\langle v - t_{n_k}, w\rangle \geq \langle v - t_{n_k}, At_{n_k} - Ay_{n_k}\rangle - \langle v - t_{n_k}, \frac{t_{n_k} - x_{n_k}}{\lambda_{n_k}}\rangle. \tag{2.5}$$

Using (ii) and the fact that A is Lipschitz continuous it follows that $\|At_{n_k} - Ay_{n_k}\| \to 0$. On the other hand, since $\{t_{n_k}\}$ converges weakly to u, it follows that $\{t_{n_k}\}$ is bounded. Let $k \to \infty$ in (2.5) and note that $\lambda_{n_k} \geq \delta > 0$, hence, we deduce that $\langle v - u, w\rangle \geq 0$. Now (2.3) implies that $u \in VI(A,C)$. □

3. MAIN RESULTS

In this section, we assume that H is a real Hilbert space. Let F be a κ-Lipschitzian continuous and $\eta-$ strongly monotone operator with $\kappa > 0$, $\eta > 0$, let A be a monotone and $\theta-$Lipschizian operator on H, let T be a $\beta-$demicontractive mapping on H, and let $f : H \to H$ be a L-Lipschitzian continuous operator. Assume that $Fix(T) \neq \varnothing$. Suppose that $w \in (0, 1-\beta)$. We note that $Fix(T) = Fix(T_w)$. It follows from the Lemma 2.1 that $Fix(T)$ is closed and convex.

Lemma 3.1. *Let A be a $\theta-$Lipschitzian continuous and monotone mapping on H. Assume that $VI(A,H) \neq \varnothing$. Let $\{t_n\}$, $\{y_n\}$ and $\{x_n\}$ be sequences in H such that*

$$y_n = x_n - \lambda_n A x_n, \qquad t_n = x_n - \lambda_n A y_n.$$

Let x^* be the solution of the variational inequality (1.16). Assume that $T : H \to H$ is demiclosed on H and $Fix(T) \neq \varnothing$. Suppose that $\{t_n\}$ is a bounded sequence in

H and $\|Tt_n - t_n\| \to 0$. Suppose that $\|y_n - t_n\| \to 0$ and $\|x_n - y_n\| \to 0$. Then, we have

$$. \qquad \liminf_{n\to\infty}(\mu F - \gamma f)x^*, t_n - x^*\rangle \geq 0.$$

Proof. Since $\{t_n\}$ is bounded, we can take a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that

$$. \qquad \liminf_{n\to\infty}\langle(\mu F - \gamma f)x^*, t_n - x^*\rangle = \lim_{j\to\infty}\langle(\mu F - \gamma f)x^*, t_{n_j} - x^*\rangle,$$

and that $t_{n_j} \rightharpoonup \tilde{t}$. Since T is demiclosed on H, we have $\tilde{t} \in Fix(T)$. Now, since $\|y_n - t_n\| \to 0$ and $\|x_n - y_n\| \to 0$ it follows from the Lemma 2.9 that $\tilde{t}$ belongs to $VI(A, H)$. Thus, $\tilde{t} \in Fix(T) \cap VI(A, H)$. Since x^* is the solution of the variational inequality (1.16), we obtain that

$$\liminf_{n\to\infty}(\mu F - \gamma f)x^*, t_n - x^*\rangle = \langle(\mu F - \gamma f)x^*, \tilde{t} - x^*\rangle \geq 0.$$

□

Now, we are ready to prove the main result of this paper:

Theorem 3.1. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A, H) \neq \emptyset$. Let $F : H \to H$ be a $\kappa-$Lipschitzian, $\eta - strongly$ monotone operator with $\kappa > 0, \eta > 0$, and $f : H \to H$ be a $L-$Lipschitzian mapping and assume that the following conditions hold:*

- (H1) $0 < \mu < 2\eta/\kappa^2, \quad 0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L = \tau/L;$
- (H2) $w \in (0, 1 - \beta);$
- (H3) $\{\alpha_n\} \subset (0, 1), \ \alpha_n \to 0;$
- (H4) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1);$
- (H5) $\sum_{n\geq 0}\alpha_n = \infty.$

Then, the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned}
&x_0 \in H,\\
&y_n = x_n - \lambda_n Ax_n,\\
&t_n = x_n - \lambda_n Ay_n,\\
&x_{n+1} = \alpha_n\gamma f(t_n) + (1 - \alpha_n\mu F)[(1 - w)I + wT](t_n),
\end{aligned} \tag{3.1}$$

converge strongly to x^*, the unique solution of (1.16).

Proof. First, we show that $\{x_n\}$ is bounded.

Indeed, if $p \in Fix(T) \cap\ VI(A, H)$, by Lemmas 2.1, 2.6 and 2.8, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(t_n) + (1 - \alpha_n\mu F)T_w(t_n) - p\|\\
&= \|\alpha_n\gamma(f(t_n) - f(p)) + \alpha_n(\gamma f(p) - \mu Fp)\\
&\quad + (1 - \alpha_n\mu F)T_w(t_n) - (1 - \alpha_n\mu F)p\|\\
&\leq \alpha_n\gamma L\|t_n - p\| + \alpha_n\|\gamma f(p) - \mu Fp\| + (1 - \alpha_n\tau)\|t_n - p\|\\
&\leq (1 - \alpha_n(\tau - \gamma L))\|t_n - p\| + \alpha_n\|\gamma f(p) - \mu Fp\|\\
&\leq (1 - \alpha_n(\tau - \gamma L))\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu Fp\|
\end{aligned} \tag{3.2}$$

By induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma L}\}, \qquad \text{for all } n \geq 0.$$

Thus, $\{x_n\}$ is bounded. Lemma 2.8 implies the boundedness of the sequences $\{t_n\}$ and $\{f(t_n)\}$. Also from the definition, we deduce the boundedness of the sequence $\{y_n\}$.

Since $x^* \in Fix(T) \cap VI(A,H)$, we can use the Lemmas 2.3 and 2.1 to deduce that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n\mu F)T_w(t_n) - (1-\alpha_n\mu F)x^* - \alpha_n(\mu F x^* - \gamma f(t_n))\|^2 \\
&= \|((1-\alpha_n\mu F)T_w(t_n) - (1-\alpha_n\mu F)x^*) \\
&\quad - \alpha_n(\mu F x^* - \gamma f(x^*) + \gamma f(x^*) - \gamma f(t_n))\|^2 \\
&\le \|((1-\alpha_n\mu F)T_w(t_n) - (1-\alpha_n\mu F)x^*)\|^2 \\
&\quad - 2\alpha_n\langle \mu F x^* - \gamma f(x^*) + \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n\tau)^2\|(T_w(t_n) - x^*)\|^2 \\
&\quad - 2\alpha_n\langle \mu F x^* - \gamma f(x^*) + \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n\tau)^2\|t_n - x^*\|^2 - (1-\alpha_n\tau)^2(w(1-\beta-w)\|t_n - Tt_n\|^2) \\
&\quad - 2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\quad - 2\alpha_n\langle \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n\tau)^2(\|x_n - x^*\|^2 - (1-\theta^2\lambda_n^2))\|x_n - y_n\|^2) \\
&\quad - (1-\alpha_n\tau)^2(w(1-\beta-w)\|t_n - Tt_n\|^2) \\
&\quad - 2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^*\rangle - 2\alpha_n\langle \gamma f(x^*) \\
&\quad - \gamma f(t_n), x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.3}$$

Let $\Gamma_n := \|x_n - x^*\|^2$. Now, we consider two cases to prove that $x_n \to x^*$.

Case 1. There is n_0 such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge n_0$. It follows that $\lim_{n\to\infty}\Gamma_n$ exists and hence $\lim_{n\to\infty}\Gamma_n - \Gamma_{n+1} = 0$.

The inequality (3.3) implies that

$$\begin{aligned}
&(1-\alpha_n\tau)^2(w(1-\beta-w)\|t_n - Tt_n\|^2 + (1-\theta^2\lambda_n^2))\|x_n - y_n\|^2) \\
&\qquad \le (1-\alpha_n\tau)^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\qquad\quad - 2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\qquad\quad - 2\alpha_n\langle \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^*\rangle \\
&\qquad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\qquad\quad - 2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\qquad\quad - 2\alpha_n\langle \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.4}$$

Since $\{x_n\}$ and $\{f(t_n)\}$ are bounded and $\alpha_n \to 0$ and $\lim_{n\to\infty}\Gamma_n - \Gamma_{n+1} = 0$, from the inequality (3.4) it follows that $\|t_n - Tt_n\| \to 0$ and $\|x_n - y_n\| \to 0$. By considering the Lemma 2.8 and the assumption $H(4)$, we also obtain that $\|y_n - t_n\| \to 0$.

Notice that, since $\|t_n - Tt_n\| \to 0$ and T is demiclosed on H, every weak cluster point of $\{t_n\}$ belongs to $Fix(T)$. On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - t_n\| &= \|\alpha_n\gamma f(t_n) + (1-\alpha_n\mu F)T_w(t_n) - t_n\| \\
&= \|\alpha_n(\gamma f(t_n) - \mu F T_w(t_n)) + w(Tt_n - t_n)\| \\
&\le \alpha_n\|\gamma f(t_n) - \mu F(T_w t_n)\| + w\|Tt_n - t_n\| \to 0.
\end{aligned} \tag{3.5}$$

Hence we deduce that

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - t_n\| + \|t_n - y_n\| + \|y_n - x_n\| \to 0.$$

From (3.5) we have

$$\liminf_{n\to\infty} \langle \mu F x^* - \gamma f(x^*), t_n - x^* \rangle = \liminf_{n\to\infty} \langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle.$$

It can be checked that all of conditions of the Lemma 3.1 are satisfied, thus we may deduce that

$$\liminf_{n\to\infty} \langle \mu F x^* - \gamma f(x^*), t_n - x^* \rangle \ge 0.$$

Hence it follows that

$$\liminf_{n\to\infty} \langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \ge 0. \tag{3.6}$$

Now, (3.3) implies that for all $n > n_0$:

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n\tau)^2(\|x_n - x^*\|^2 - (1-\theta^2\lambda_n^2))\|x_n - y_n\|^2) \\
&\quad -(1-\alpha_n\tau)^2 w(1-\beta-w)\|t_n - Tt_n\|^2 \\
&\quad -2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\quad -2\alpha_n\langle \gamma f(x^*) - \gamma f(t_n), x_{n+1} - x^* \rangle \\
&\le (1-\alpha_n\tau)^2\|x_n - x^*\|^2 + 2\alpha_n\gamma L\|t_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad -2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\le (1-\alpha_n\tau)^2\|x_n - x^*\|^2 + 2\alpha_n\gamma L\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad -2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\le (1-2\alpha_n\tau + (\alpha_n\tau)^2))\|x_n - x^*\|^2 + 2\alpha_n\gamma L\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad -2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\le (1-2\alpha_n\tau + (\alpha_n\tau)^2))\|x_n - x^*\|^2 + 2\alpha_n\gamma L\|x_n - x^*\|^2 \\
&\quad -2\alpha_n\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle \\
&\le (1-2\alpha_n(\tau-\gamma L))\|x_n - x^*\|^2 + 2\alpha_n(\tau-\gamma L)(\tfrac{\alpha_n\tau^2\|x_n - x^*\|^2}{2(\tau-\gamma L)} \\
&\quad -\tfrac{\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle}{\tau-\gamma L})
\end{aligned}$$

Recall that by the the assumption $0 < \gamma < \tau/L$. Since $\{x_n\}$, $\{f(x_n)\}$ and $\{f(x_n)\}$ are bounded and $\alpha_n \to 0$, we deduce from (3.6) that

$$\limsup_{n\to\infty}\left(\frac{\alpha_n\tau^2\|x_n - x^*\|^2}{2(\tau-\gamma L)} - \frac{\langle \mu F x^* - \gamma f(x^*), x_{n+1} - x^* \rangle}{\tau-\gamma L})\right) \le 0,$$

Now, we can apply the Lemma 2.4 to conclude $x_n \to x^*$. Also, since $\|y_n - x_n\| \to 0$ and $\|t_n - y_n\| \to 0$, we have $y_n \to x^*$ and $t_n \to x^*$.

Case 2. Assume that there is a subsequence $\{\Gamma_{n_j}\}_{j\ge 0}$ of $\{\Gamma_n\}_{n\ge 0}$ such that $\Gamma_{n_j} < \Gamma_{n_{j+1}}$ for all $j > 0$. In this case, it follows from Lemma 2.7 that there is a subsequence $\{\Gamma_{\tau(n)}\}_{n\ge 0}$ of $\{\Gamma_n\}_{n\ge 0}$ such that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ and $\{\tau(n)\}$ is defined as in Lemma 2.7. In this case, first, we show that $\|t_{\tau(n)} - Tt_{\tau(n)}\| \to 0$. It follows from inequality (3.3) that

$$\begin{aligned}
&(1-\alpha_n\tau)^2(w(1-\beta-w)\|t_{\tau(n)} - Tt_{\tau(n)}\|^2 + (1-\theta^2\lambda_n^2)\|x_{\tau(n)} - y_{\tau(n)}\|^2) \\
&\qquad \le (1-\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2 \\
&\qquad\quad - 2\alpha_{\tau(n)}\langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^* \rangle \\
&\qquad\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x^* \rangle \qquad (3.7) \\
&\qquad \le \|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2 \\
&\qquad\quad - 2\alpha_{\tau(n)}\langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^* \rangle \\
&\qquad\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x^* \rangle.
\end{aligned}$$

Since $\{x_n\}$, and $\{t_n\}$, $\{f(t_n)\}$ are bounded and $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, it follows from the above inequality that

$$\|t_{\tau(n)} - Tt_{\tau(n)}\| \to 0 \quad \text{and} \quad \|x_{\tau(n)} - y_{\tau(n)}\| \to 0. \tag{3.8}$$

Now, we can apply the Lemma 2.8 to conclude that $\|y_{\tau(n)} - t_{\tau(n)}\| \to 0$. Also we have

$$\begin{aligned}\|x_{\tau(n)+1} - t_{\tau(n)}\| &= \|\alpha_{\tau(n)}\gamma f(t_{\tau(n)}) + (1-\alpha_{\tau(n)}\mu F)T_w(t_{\tau(n)}) - t_{\tau(n)}\|\\ &= \|\alpha_{\tau(n)}(\gamma f(t_{\tau(n)}) - \mu F T_w(t_{\tau(n)})) + w(Tt_{\tau(n)} - t_{\tau(n)})\|\\ &\le \alpha_{\tau(n)}\|\gamma f(t_{\tau(n)}) - \mu F(T_w t_{\tau(n)})\| + w\|Tt_{\tau(n)} - t_{\tau(n)}\| \to 0.\end{aligned}$$

Hence, we deduce that

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \le \|x_{\tau(n)+1} - t_{\tau(n)}\| + \|t_{\tau(n)} - y_{\tau(n)}\| + \|y_{\tau(n)} - x_{\tau(n)}\| \to 0.) \tag{3.9}$$

Since $\|x_{\tau(n)+1} - t_{\tau(n)}\| \to 0$, it follows that

$$\liminf_{n\to\infty}\langle \mu Fx^* - \gamma f(x^*), t_{\tau(n)} - x^*\rangle = \liminf_{n\to\infty}\langle \mu Fx^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle. \tag{3.10}$$

On the other hand, as $\|t_{\tau(n)} - Tt_{\tau(n)}\| \to 0$ and T is demiclosed on H, we can use Lemma 3.1 to deduce that

$$\liminf_{n\to\infty}\langle \mu Fx^* - \gamma f(x^*), t_{\tau(n)} - x^*\rangle \ge 0. \tag{3.11}$$

From (3.10) and (3.11), we deduce that

$$\liminf_{n\to\infty}\langle \mu Fx^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \ge 0. \tag{3.12}$$

Now, we use the inequality (3.3) to conclude that

$$\begin{aligned}\|x_{\tau(n)+1} - x^*\|^2 &\le (1-\alpha_{\tau(n)}\tau)^2\|x_{\tau(n)} - x^*\|^2\\ &\quad - (1-\alpha_{\tau(n)}\tau)^2(1-\theta^2\lambda^2_{\tau(n)})\|x_{\tau(n)} - y_{\tau(n)}\|^2\\ &\quad - (1-\alpha_{\tau(n)}\tau)^2 w(1-\beta-w)\|x_{\tau(n)} - Tx_{\tau(n)}\|^2\\ &\quad - 2\alpha_{\tau(n)}\langle \mu Fx^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle\\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x^*\rangle\\ &\le (1-2\alpha_{\tau(n)}\tau + \alpha^2_{\tau(n)}\tau^2)\|x_{\tau(n)} - x^*\|^2\\ &\quad - 2\alpha_{\tau(n)}\langle \mu Fx^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle\\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)} - x^*\rangle\\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x_{\tau(n)}\rangle \qquad (3.13)\\ &\le (1-2\alpha_{\tau(n)}\tau + \alpha^2_{\tau(n)}\tau^2)\|x_{\tau(n)} - x^*\|^2\\ &\quad - 2\alpha_{\tau(n)}\langle \mu Fx^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle\\ &\quad + 2\alpha_{\tau(n)}\gamma L\|t_{\tau(n)} - x^*\|\|x_{\tau(n)} - x^*\|\\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x_{\tau(n)}\rangle\\ &\le (1-2\alpha_{\tau(n)}\tau + \alpha^2_{\tau(n)}\tau^2)\|x_{\tau(n)} - x^*\|^2\\ &\quad - 2\alpha_{\tau(n)}\langle \mu Fx^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle\\ &\quad + 2\alpha_{\tau(n)}\gamma L\|x_{\tau(n)} - x^*\|^2\\ &\quad - 2\alpha_{\tau(n)}\langle \gamma f(x^*) - \gamma f(t_{\tau(n)}), x_{\tau(n)+1} - x_{\tau(n)}\rangle.\end{aligned}$$

Recall $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ for all $n \geq 0$. From the inequality (3.13) we have

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \\ &\leq 2\alpha_{\tau(n)}(-\tau\|x_{\tau(n)} - x^*\|^2 + \gamma L\|x_{\tau(n)} - x^*\|^2 \\ &\quad + \frac{\alpha_{\tau(n)}}{2}\tau^2\|x_{\tau(n)} - x^*\|^2 - \langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ &\quad + \|\gamma f(x^*) - \gamma f(t_{\tau(n)})\|\|x_{\tau(n)+1} - x_{\tau(n)}\|). \end{aligned} \tag{3.14}$$

Since $0 < \alpha_n < 1$, inequality (3.14) implies that

$$\begin{aligned} (\tau - \gamma L)\|x_{\tau(n)} - x^*\|^2 \leq (&\frac{\alpha_{\tau(n)}}{2}\tau^2\|x_{\tau(n)} - x^*\|^2 - \langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ &+ \|\gamma f(x^*) - \gamma f(t_{\tau(n)})\|\|x_{\tau(n)+1} - x_{\tau(n)}\|). \end{aligned} \tag{3.15}$$

Since $\{x_n\}$ and $\{f(t_n)\}$ are bounded, it follows from (3.9) and (3.12) that

$$\begin{aligned} \limsup_{n\to\infty}(&\frac{\alpha_{\tau(n)}}{2}\tau^2\|x_{\tau(n)} - x^*\|^2 - \langle \mu F x^* - \gamma f(x^*), x_{\tau(n)+1} - x^*\rangle \\ &+ \|\gamma f(x^*) - \gamma f(t_{\tau(n)})\|\|x_{\tau(n)+1} - x_{\tau(n)}\|) \leq 0. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we deduce that

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0.$$

Since $\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0$, it follows that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x^*\|^2 = 0.$$

On the other hand, from Lemma 2.7 we have $\Gamma_n < \Gamma_{\tau(n)+1}$ for all $n \geq 0$, and therefore, $x_n \to x^*$. Since $\|y_n - x_n\| \to 0$ and $\|t_n - y_n\| \to 0$, we have $y_n \to x^*$ and $t_n \to x^*$. □

If we take $f \equiv 0$ in the above Theorem, we have the following Theorem:

Theorem 3.2. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A,H) \neq \emptyset$. Let $F : H \to H$ be a $\kappa-$Lipschitzian, $\eta - strongly$ monotone operator with $\kappa > 0, \eta > 0$, and assume that the following conditions hold:*

- (H1) $0 < \mu < 2\eta/\kappa^2$;
- (H2) $w \in (0, 1-\beta)$;
- (H3) $\{\alpha_n\} \subset (0,1), \ \alpha_n \to 0$;
- (H4) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1)$;
- (H5) $\sum_{n\geq 0} \alpha_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned} &x_0 \in H, \\ &y_n = x_n - \lambda_n A x_n, \\ &t_n = x_n - \lambda_n A y_n, \\ &x_{n+1} = (1 - \alpha_n \mu F)[(1-w)I + wT](t_n), \end{aligned}$$

converge strongly to x^* which is the unique solution of the following variational inequality problem

find $x^* \in Fix(T) \cap VI(A,H)$ such that

$$\langle v - x^*, \mu F x^*\rangle \geq 0, \ \textit{for all } v \in Fix(T) \cap VI(A,H).$$

Corollary 3.2. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A, H) \neq \varnothing$. Let $u \in H$ be arbitrary chosen and assume that the following conditions hold:*

(H1) $w \in (0, 1-\beta)$;
(H2) $\{\alpha_n\} \subset (0, 1), \ \alpha_n \to 0$;
(H3) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1)$;
(H4) $\sum_{n \geq 0} \alpha_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned} &x_0 \in H, \\ &y_n = x_n - \lambda_n A x_n, \\ &t_n = x_n - \lambda_n A y_n, \\ &x_{n+1} = \alpha_n u + (1-\alpha_n)[(1-w)I + wT](t_n), \end{aligned}$$

converge strongly to x^*, which satisfies $x^* = P_{Fix(T) \cap VI(A,H)}(u)$.

Proof. In Theorem 3.1 set $F := I - u$. Note that F is 1-Lipschizian and 1-strongly monotone operator and the result follows. □

We can also drive the following corollary from the Theorem 3.1:

Corollary 3.3. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Suppose $T : H \to H$ is $\beta-$demicontractive, demiclosed with $Fix(T) \cap VI(A, C) \neq \varnothing$. Let $u \in H$ be arbitrary chosen and $f : H \to H$ be a $L-$Lipschitzian mapping and assume that the following conditions hold:*

(H1) $0 < \mu < 2, \quad 0 < \gamma < \mu(1 - \frac{\mu}{2})/L = \tau/L$;
(H2) $w \in (0, 1-\beta)$;
(H3) $\{\alpha_n\} \subset (0, 1), \ \alpha_n \to 0$;
(H4) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1)$;
(H5) $\sum_{n \geq 0} \alpha_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by following algorithm

$$\begin{aligned} &x_0 \in H, \\ &y_n = x_n - \lambda_n A x_n, \\ &t_n = x_n - \lambda_n A y_n, \\ &x_{n+1} = \alpha_n \gamma f(t_n) + \alpha_n \mu u + (1-\alpha_n \mu)[(1-w)I + wT](t_n), \end{aligned}$$

converge strongly to x^*, which satisfies $x^* = P_{Fix(T) \cap VI(A,H)}(u + \frac{\gamma}{\mu} f(x^*))$.

In the next Corollary, we show that Theorem 3.1 can be applied to approximating common zeroes of monotone operators:

Corollary 3.4. *Assume that $A : H \to H$ is a monotone and $\theta-$Lipschitz continuous mapping. Let $D : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}(0) \cap D^{-1}(0) \neq \varnothing$. Let J_r^D be the resolvent of D for each $r > 0$. Let $u \in H$ be arbitrary chosen and let $f : H \to H$ be a $L-$Lipschitzian mapping and assume that the following conditions hold:*

(H1) $0 < \mu < 2, \quad 0 < \gamma < \mu(1 - \frac{\mu}{2})/L = \tau/L$;
(H2) $w \in (0, 1)$;
(H3) $\{\alpha_n\} \subset (0, 1), \ \alpha_n \to 0$;
(H4) $\{\theta\lambda_n\} \subset [\delta_1, \delta_2] \quad (where\ 0 < \delta_1 \leq \delta_2 < 1)$;
(H5) $\sum_{n \geq 0} \alpha_n = \infty$.

Then the sequences $\{x_n\}$, $\{y_n\}$ *and* $\{t_n\}$ *generated by following algorithm*

$$\begin{aligned} &x_0 \in H, \\ &y_n = x_n - \lambda_n A x_n, \\ &t_n = x_n - \lambda_n A y_n, \\ &x_{n+1} = \alpha_n \gamma f(t_n) + \alpha_n \mu u + (1 - \alpha_n \mu)[(1-w)I + wJ_r^D](t_n), \end{aligned} \tag{3.17}$$

converge strongly to x^*, *which satisfies* $x^* = P_{A^{-1}(0)\cap D^{-1}(0)}(u + \frac{\gamma}{\mu} f(x^*))$.

Proof. Recall that J_r^D is a nonexpansive mapping (hence demiclosed and $0-$demicontractive). On the other hand, we have $A^{-1}(0) = VI(A, H)$ and $Fix(J_r^D) = D^{-1}(0)$. So we can apply Corollary 3.3 to conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ generated by the iterative method (3.17) converge strongly to x^*, which satisfies $x^* = P_{A^{-1}(0)\cap D^{-1}(0)}(u + \frac{\gamma}{\mu} f(x^*))$. □

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A RELATED FIXED POINT THEOREM IN n COMPLETE FUZZY METRIC SPACES

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ABSTRACT. We prove a related fixed point theorem for n mappings in n complete fuzzy metric spaces using an implicit relation which generalizes results of Aliouche and Fisher [1] and Rao et al. [13].

KEYWORDS : Fuzzy metric space; implicit relation; related fixed point
AMS Subject Classification: 47H10 54H25

1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets was introduced by L. Zadeh [16] in 1965. George and Veeramani [9] modified the concept of fuzzy metric spaces introduced by [11] in order to define the Hausdorff topology of fuzzy metric spaces which have very important applications in quantum particle physics particularly in connections with both string and $E-$infinity theory which were studied by El- Naschie [4, 5, 6, 7, 8] and [15]. They showed also that every metric space induces a fuzzy metric space.

Recently, Aliouche and Fisher [1], Aliouche et.al [2] and Rao et.al [13] proved some related fixed point theorems in metric spaces and fuzzy metric spaces.

Inspired by a work of Popa [12], we prove a related fixed point theorem in n complete fuzzy metric spaces using an implicit relation because it includes several contractive conditions.

Definition 1.1 ([14])**.** A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t-$norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

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Examples of a continuous $t-$norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2 ([9]). The triple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous $t-$norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Note that $M(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t.

Let $(X, M, *)$ be a fuzzy metric space.

1) For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and a radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

2) A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$.

Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable, see [9].

Example 1.3 ([9]). Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. Define for each $t \in (0, \infty)$ and all $x, y \in X$

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

Then (X, M) is a fuzzy metric space. It is called the standard fuzzy metric induced by the metric d.

Definition 1.4 ([9]). Let $(X, M, *)$ be a fuzzy metric space.

1) A sequence $\{x_n\}$ in X converges to x if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(x_n, x, t) > 1 - \epsilon$; i.e., $M(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$.

2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $M(x_n, x_m, t) > 1 - \epsilon$; i.e., $M(x_n, x_m, t) \to 1$ as $n, m \to \infty$ for all $t > 0$.

3) A fuzzy metric space (X, M, t) in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.5 ([10]). *For all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.*

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.7 ([10]). *M is a continuous function on $X^2 \times (0, \infty)$.*

We denote by Ψ the set of all function $\psi : [0,1]^4 \to [0,1]$ such that
(i) ψ is upper semi continuous in each coordinate variable,
(ii) ψ is decreasing in 3rd and 4th variable,
(iii) if either $\psi(u,v,1,u) \geq 0$ or $\psi(u,1,1,v) \geq 0$ or $\psi(u,1,v,1) \geq 0$ for all $u,v \in [0,1]$, then $u \geq v$.

Example 1.8. $\psi(t_1,t_2,t_3,t_4) = t_1 - \min\{t_2,t_3,t_4\}$

Example 1.9. $\psi(t_1,t_2,t_3,t_4) = t_1 - \phi(\min\{t_2,t_3,t_4\})$,
where $\phi :]0,1] \to]0,1]$ is a increasing and continuous function with $\phi(t) > t$ for $0 < t < 1$. For example $\phi(t) = \sqrt{t}$ or $\phi(t) = t^h$ for $0 < h < 1$.

We need the following lemma of [3].

Lemma 1.10. *Let $\{x_n\}$ be a sequence in fuzzy metric space $(X,M,*)$ with $M(x,y,t) \to 1$ as $t \to \infty$ for all $x,y \in X$. If there exists a number $k \in]0,1[$ such that*

$$M(x_{n+1},x_n,kt) \geq M(x_n,x_{n-1},t).$$

Then $\{x_n\}$ is a Cauchy sequence in X.

2. MAIN RESULTS

Theorem 2.1. *Let $(X_i,M_i,\theta_i)_{1\leq i\leq n}$, be n complete fuzzy metric spaces with $M_i(x,x_i,t) \longrightarrow 1$ as $t \longrightarrow \infty$ for all $x,x_i \in X_i$ and let $\{A_i\}_{i=1}^{i=n}$ be n-mappings such that $A_i : X_i \to X_{i+1}$ for all $i = 1,..,n-1$ and $A_n : X_n \to X_1$, satisfying the inequalities*

$$\phi_1 \begin{pmatrix} M_1(A_nA_{n-1}..A_2x_2, A_nA_{n-1}..A_2A_1x_1, kt), \\ M_2(x_2, A_1A_nA_{n-1}..A_2x_2, t), \\ M_1(x_1, A_nA_{n-1}..A_2x_2, t), M_1(x_1, A_nA_{n-1}..A_2A_1x_1, t) \end{pmatrix} \geq 0 \qquad (2.1)$$

for all $x_1 \in X_1$, $x_2 \in X_2$ and $t > 0$, in general, we have

$$\phi_i \begin{pmatrix} M_i(A_{i-1}A_{i-2}..A_1A_nA_{n-1}..A_{i+1}x_{i+1}, A_{i-1}..A_1A_n..A_ix_i, kt), \\ M_{i+1}(x_{i+1}, A_iA_{i-1}..A_1A_nA_{n-1}...A_{i+1}x_{i+1}, t), \\ M_i(x_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_{i+1}x_{i+1}, t), \\ M_i(x_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_ix_i, t), \end{pmatrix} \geq 0 \qquad (2.i)$$

for all $x_i \in X_i$, $x_{i+1} \in X_{i+1}$, $t > 0$ and $i = 2,...,n-1$ and

$$\phi_n \begin{pmatrix} M_n(A_{n-1}A_{n-2}...A_1x_1, A_{n-1}A_{n-2}...A_1A_nx_n, kt), \\ M_1(x_1, A_nA_{n-1}A_{n-2}..A_1x_1, t), \\ M_n(x_n, A_{n-1}A_{n-2}..A_1x_1, t), \\ M_n(x_n, A_{n-1}A_{n-2}..A_1A_nx_n, t) \end{pmatrix} \geq 0 \qquad (2.n)$$

for all $x_1 \in X_1$, $x_n \in X_n$ and $t > 0$, where $\phi_i \in \Psi$, $i = 1,2,...,n$ and $0 < k < 1$. Then $A_{i-1}A_{i-2}..A_1A_nA_{n-1}..A_i$ has a unique fixed point $p_i \in X_i$ for $i = 1,...,n$. Further, $A_ip_i = p_{i+1}$ for $i = 1,...,n-1$ and $A_np_n = p_1$.

Proof. Let $\left\{x_r^{(1)}\right\}, \left\{x_r^{(2)}\right\}, ..., \left\{x_r^{(i)}\right\}, ..., \left\{x_r^{(n)}\right\}$, $r \in \mathbb{N}$ be sequences in $X_1, X_2, ..., X_i, ...X_n$ respectively. Now let $x_0^{(1)}$ be an arbitrary point in X_1, we define the sequences $\left\{x_r^{(i)}\right\}_{r\in\mathbb{N}}$ for $i = 1,...,n$ by

$$x_r^{(1)} = (A_nA_{n-1}...A_1)^r x_0^{(1)},$$

$$x_r^{(i)} = A_{i-1}A_{i-2}..A_1(A_nA_{n-1}...A_1)^r x_0^{(1)} \text{ for } i = 2,...,n.$$

For $n=1,2,...$, we assume that $x_r^{(1)} \neq x_{r+1}^{(1)}$. Applying the inequality (2.1) for $x_2 = A_1 \left(A_n A_{n-1}...A_1\right)^{r-1} x_0^{(1)}$, $x_1 = \left(A_n A_{n-1}...A_1\right)^r x_0^{(1)}$ we get

$$\begin{aligned} &\phi_1 \left(\begin{array}{c} M_1\left(\left(A_nA_{n-1}...A_1\right)^r x_0^{(1)}, \left(A_nA_{n-1}...A_1\right)^{r+1} x_0^{(1)}, kt\right), \\ M_2\left(A_1\left(A_nA_{n-1}...A_1\right)^{r-1} x_0^{(1)}, A_1\left(A_nA_{n-1}...A_1\right)^{r} x_0^{(1)}, t\right), \\ M_1\left(\left(A_nA_{n-1}...A_1\right)^r x_0^{(1)}, \left(A_nA_{n-1}...A_1\right)^{r} x_0^{(1)}, t\right), \\ M_1\left(\left(A_nA_{n-1}...A_1\right)^r x_0^{(1)}, \left(A_nA_{n-1}...A_1\right)^{r+1} x_0^{(1)}, t\right) \end{array} \right) \\ = \;& \phi_1\left(M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, kt\right), M_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right), 1, M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, t\right)\right) \geq 0 \end{aligned}$$

From the implicit relation we have

$$M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, kt\right) \geq M_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right) \qquad (3.1)$$

Applying the inequality $(2.i)$ for $x_{i+1} = A_i...A_1\left(A_n...A_1\right)^{r-1} x_0^{(1)}$ and $x_i = A_{i-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}$, we obtain

$$\begin{aligned} &\phi_i \left(\begin{array}{c} M_i\left(A_{i-1}..A_1\left(A_n...A_1\right)^r x_0^{(1)}, A_{i-1}..A_1\left(A_n...A_1\right)^{r+1} x_0^{(1)}, kt\right), \\ M_{i+1}\left(x_{i+1} = A_i...A_1\left(A_n...A_1\right)^{r-1} x_0^{(1)}, A_i..A_1\left(A_n...A_1\right)^{r} x_0^{(1)}, t\right), \\ M_i\left(A_{i-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}, A_{i-1}..A_1\left(A_n...A_1\right)^{r} x_0^{(1)}, t\right), \\ M_i\left(A_{i-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}, A_{i-1}..A_1 A_{i-1}...A_1\left(A_n...A_1\right)^{r+1} x_0^{(1)}, t\right), \end{array} \right) \\ = \;& \phi_i \left(\begin{array}{c} M_i\left(x_r^{(i)}, x_{r+1}^{(i)}, kt\right), M_{i+1}\left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, t\right), \\ 1, M_i\left(x_r^{(i)}, x_{r+1}^{(i)}, t\right) \end{array} \right) \geq 0 \end{aligned}$$

and so

$$M_i\left(x_r^{(i)}, x_{r+1}^{(i)}, kt\right) \geq M_{i+1}\left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, t\right) \qquad (3.i)$$

for $i = 2,...,n-1$ and $r = 1,2,....$ Now applying the inequality $(2.n)$ for $x_n = A_{n-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}$ and $x_1 = \left(A_nA_{n-1}...A_1\right)^r x_0^{(1)}$we have

$$\begin{aligned} &\phi_n \left(\begin{array}{c} M_n\left(A_{n-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}, A_{n-1}...A_1\left(A_n...A_1\right)^{r+1} x_0^{(1)}, kt\right), \\ M_1\left(\left(A_n...A_1\right)^r x_0^{(1)}, \left(A_n...A_1\right)^{r+1} x_0^{(1)}, t\right), \\ M_n\left(x_n = A_{n-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}, A_{n-1}..A_1\left(A_n...A_1\right)^{r} x_0^{(1)}, t\right), \\ M_n\left(A_{n-1}...A_1\left(A_n...A_1\right)^r x_0^{(1)}, A_{n-1}A_{n-2}..A_1\left(A_nA_{n-1}...A_1\right)^{r+1} x_0^{(1)}, t\right) \end{array} \right) \\ = \;& \phi_n \left(\begin{array}{c} M_n\left(x_r^{(n)}, x_{r+1}^{(n)}, kt\right), M_1\left(x_{r-1}^{(1)}, x_r^{(1)}, t\right), \\ 1, M_n\left(x_r^{(n)}, x_{r+1}^{(n)}, t\right) \end{array} \right) \geq 0 \end{aligned}$$

and so

$$M_n\left(x_r^{(n)}, x_{r+1}^{(n)}, kt\right) \geq M_1\left(x_{r-1}^{(1)}, x_r^{(1)}, t\right) \qquad (3.n)$$

It now follows from $(3.1), (3.i)$ and $(3.n)$ that for large enough n we obtain

$$\begin{aligned} M_1\left(x_r^{(1)}, x_{r+1}^{(1)}, kt\right) &\geq M_2\left(x_{r-1}^{(2)}, x_r^{(2)}, t\right) \\ M_i\left(x_r^{(i)}, x_{r+1}^{(i)}, t\right) &\geq M_{i+1}\left(x_{r-1}^{(i+1)}, x_r^{(i+1)}, \frac{t}{k}\right) \\ &\geq ... \end{aligned}$$

$$\begin{aligned}
&\geq M_n\left(x^{(n)}_{r+i-n}, x^{(n)}_{r+i-n+1}, \frac{t}{k^{n-i}}\right)\\
&\geq M_1\left(x^{(1)}_{r+i-n-1}, x^{(1)}_{r+i-n}, \frac{t}{k^{n-i+1}}\right)\\
&\geq .....\\
&\quad M_1\left(x^{(1)}_{r+i-2n-1}, x^{(1)}_{r+i-2n}, \frac{t}{k^{2n-i+1}}\right)\\
&\geq ....\\
&\geq M_1\left(x^{(1)}_{r+i-mn-1}, x^{(1)}_{r+i-mn}, \frac{t}{k^{mn-i+1}}\right)\\
&\geq \min\left\{M_1\left(x^{(1)}_1, x^{(1)}_2, \frac{t}{k^{mn}}\right), ..., M_n\left(x^{(n)}_1, x^{(n)}_2, \frac{t}{k^{mn}}\right)\right\}
\end{aligned}$$

Since $0 < k < 1$. it follows from lemma 1.10 that $\left\{x^{(i)}_r\right\}$ is a Cauchy sequences in X_i with a limit p_i in X_i for $i = 1, 2, .., n$.

To prove that p_i is a fixed point of $A_{i-1}...A_1A_n...A_ip_i$ for $i = 2, ..., n-1$, suppose that $A_{i-1}...A_1A_n...A_ip_i \neq p_i$. Using the inequality $(2.i)$ for $x_i = p_i$ and $x_{i+1} = x^{(i+1)}_r$we obtain

$$\phi_i\left(\begin{array}{c} M_i\left(x^{(i)}_r, A_{i-1}..A_1A_n...A_ip_i, kt\right)\\ , M_{i+1}\left(x^{(i+1)}_r, x^{(i+1)}_{r+1}, t\right), M_i\left(p_i, x^{(i)}_r, t\right),\\ M_i\left(p_i, A_{i-1}..A_1A_n...A_ip_i, t\right)\end{array}\right) \geq 0$$

Letting $r \to \infty$ we have

$$\phi_i\left(\begin{array}{c} M_i\left(p_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_ip_i, kt\right), 1, 1,\\ d_i\left(p_i, A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_ip_i, t\right)\end{array}\right) \geq 0.$$

It follows from (iii) that $p_i = A_{i-1}A_{i-2}..A_1A_nA_{n-1}...A_ip_i$ in X_i for $i = 2, ..., n-1$ and $p_i = A_{i-1}p_{i-1} = ... = A_{i-1}A_{i-2}....A_2A_1p_1$.

For the case $i = 1$, we use (2.1) for $x_1 = p_1$ and $x_2 = A_1\left(A_nA_{n-1}...A_1\right)^{r-1}x^{(1)}_0 = x^{(2)}_r$giving

$$\phi_1\left(\begin{array}{c} M_1\left(x^{(1)}_r, A_nA_{n-1}...A_1p_1, kt\right),\\ M_2\left(x^{(2)}_r, x^{(2)}_{r+1}, t\right), M_1\left(p_1, x^{(1)}_r, t\right),\\ M_1\left(p_1, A_nA_{n-1}...A_1p_1, t\right)\end{array}\right) \geq 0$$

letting $r \to \infty$ we have

$$\phi_1\left(\begin{array}{c} M_1\left(p_1, A_nA_{n-1}...A_1p_1, kt\right), 1, 1,\\ M_1\left(p_1, A_nA_{n-1}...A_1p_1, t\right)\end{array}\right) \geq 0.$$

It follows from (iii) that $A_nA_{n-1}...A_2A_1p_1 = p_1$ in X_1.

Finally, if $i = n$, using the inequality $(2.n)$ for $x_n = p_n$ and $x_1 = x^{(1)}_r$ we get

$$\phi_n\left(\begin{array}{c} M_n\left(x^{(n)}_{r+1}, A_{n-1}A_{n-2}..A_1A_np_n, kt\right),\\ M_1\left(x^{(1)}_r, x^{(1)}_{r+1}, t\right), M_n\left(p_n, x^{(n)}_{r+1}, t\right),\\ M_n\left(p_n, A_{n-1}A_{n-2}..A_1A_np_n, t\right)\end{array}\right) \geq 0.$$

Letting $r \to \infty$ we have

$$\phi_n\left(\begin{array}{c} M_n\left(p_n, A_{n-1}A_{n-2}..A_1A_np_n, kt\right), 1, 1,\\ M_n\left(p_n, A_{n-1}A_{n-2}..A_1A_np_n, t\right)\end{array}\right) \geq 0.$$

and by (iii), $p_n = A_{n-1}A_{n-2}..A_1A_np_n$ in X_n and $p_n = A_{n-1}p_{n-1} = ... = A_{n-1}A_{n-2}....A_2A_1p_1$.

To prove the uniqueness, suppose that $A_{i-1}....A_1A_n...A_i$ has a second fixed point $z_1 \neq p_1$ in X_1. Using the inequality (2.1) for $x_{i+1} = A_iz_i$ and $x_i = p_i$ we get

$$\phi_i\begin{pmatrix} M_i\left(A_{i-1}....A_1A_n...A_iz_i, A_{i-1}....A_1A_n...A_ip_i, kt\right), \\ M_{i+1}\left(A_iz_i, A_{i-1}....A_1A_n...A_iz_i, t\right) \\ M_i\left(p_i, A_{i-1}....A_1A_n...A_iz_i, t\right), M_1\left(p_i, A_{i-1}....A_1A_n...A_ip_i, t\right) \end{pmatrix} \geq 0$$

and so

$$\phi_i\left(M_i\left(z_i, p_i, kt\right), 1, M_i\left(p_i, z_i, t\right), 1\right) \geq 0$$

which implies that $z_i = p_i$, proving the uniqueness of p_i in X_i for $i = 2, ..., n-1$. The uniqueness of p_1 in X_1 and p_n in X_n follow similarly.

Finally, we note that

$$A_ip_i = A_iA_{i-1}...A_1A_n...A_{i+1}\left(A_ip_i\right),$$

hence, p_i is a fixed point of $A_i...A_1A_n...A_{i+1}$. Since the fixed point is unique, it follows that $A_ip_i = p_{i+1}$ for all $i = 1, ..., n-1$. It follows similarly that $A_np_n = p_1$. This complete the proof of the theorem. □

Example 2.2. Let (M_i, X_i, θ_i) for $i = 1, ..., n$ be n fuzzy metric spaces, where $M_i\left(x_i, y_i, t\right) = \dfrac{t}{t + |x_i - y_i|}$ and $X_i = \{x_i : i-1 \leq x_i \leq i\}$ for $i = 1, ..., n$. Define $A_i : X_i \to X_{i+1}$ for $i = 1, ..., n-1$ and $A_n : X_n \to X_1$ by

$$A_1x_1 = \begin{cases} \frac{5}{4} & \text{if } 0 \leq x_1 < \frac{1}{2}, \\ \frac{3}{2} & \text{if } \frac{1}{2} \leq x_1 \leq 1 \end{cases},$$

$$A_ix_i = \begin{cases} i + \frac{1}{4} & \text{if } i-1 \leq x_i < i - \frac{3}{4}, \\ i + \frac{1}{2} & \text{if } i - \frac{3}{4} \leq x_i \leq i \end{cases} \quad \text{for all } i = 2, ..., n-1,$$

$$A_nx_n = \begin{cases} \frac{3}{4} & \text{if } n-1 \leq x_n < n - \frac{3}{4}, \\ 1 & \text{if } n - \frac{3}{4} \leq x_n \leq n \end{cases}$$

Let $\phi_1 = \phi_2 = ... = \phi_n = \phi$ and $\phi\left(t_1, t_2, t_3, t_4\right) = t_1 - \min\{t_2, t_3, t_4\}$. Note that there exists p_i in X_i such that $\left(A_{i-1}A_{i-2}..A_1A_n...A_i\right)p_i = p_i$ for $i = 1, ..., n$. For example If we put $i = n$, we get $\left(A_{n-1}A_{n-2}..A_1A_n\right)p_n = p_n$ if $p_n = n - \frac{1}{2}$ because

$$\begin{aligned} A_{n-1}A_{n-2}..A_1A_n\left(n - \frac{1}{2}\right) &= A_{n-1}A_{n-2}..A_1\left(1\right), \\ &= A_{n-1}A_{n-2}..A_2\left(\frac{3}{2}\right) \\ &\vdots \\ &= A_{n-1}A_{n-2}..A_{i+1}\left(i + \frac{1}{2}\right) \\ &\vdots \end{aligned}$$

$$\begin{aligned} &= A_{n-1}A_{n-2}\left(n-\frac{5}{2}\right) \\ &= A_{n-1}\left(n-\frac{3}{2}\right) \\ &= n-\frac{1}{2} \quad \text{car } n-\frac{7}{4}\leq n-\frac{3}{2}\leq n-1. \end{aligned}$$

Note that for all $i=1,....,n-1$ and $i-\frac{3}{4}\leq x_i<i$; $(i+1)-\frac{3}{4}\leq A_ix_i<i+1$ and $\frac{1}{2}\leq A_nx_n\leq 1$ with $n-\frac{3}{4}\leq x_n<n$, there exists $p_i=i-\frac{1}{2}$ such that $(A_{i-1}..A_1A_n...A_i)\left(i-\frac{1}{2}\right)=i-\frac{1}{2}$ for $i=1,...,n-1$.

The inequalities $(1.i)$ for all $i=1,...,n$ are satisfied since the value of the left hand side of each inequality is 1. In fact $M_i\left(A_{i-1}...A_1A_n..A_{i+1}x_{i+1},A_{i-1}...A_1A_n..A_ix_i,t\right)=1$ for $i=1,....,n-1$ because

(1) If $i-1\leq x_i<i-\frac{3}{4}$ we have

$$\begin{aligned} A_{i-1}...A_1A_n...A_ix_i &= A_{i-1}...A_1A_nA_{n-1}...A_{i+1}\left(i+\frac{1}{4}\right) \\ &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+2}\left((i+1)+\frac{1}{2}\right) \\ &\vdots \\ &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}\left((n-2)+\frac{1}{2}\right) \\ &= A_{i-1}A_{i-2}...A_1A_n\left(n-\frac{1}{2}\right) \\ &= A_{i-1}A_{i-2}...A_2A_1\left(1\right), \\ &= A_{i-1}A_{i-2}...A_2\left(1+\frac{1}{2}\right) \\ &\vdots \\ &= A_{i-1}\left((i-2)+\frac{1}{2}\right) \\ &= (i-1)+\frac{1}{2}=i-\frac{1}{2}. \end{aligned}$$

(2) If $i-\frac{3}{4}\leq x_i\leq i$, we get

$$\begin{aligned} A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_ix_i &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}..A_{i+1}\left(i+\frac{1}{2}\right) \\ &\vdots \\ &= A_{i-1}A_{i-2}...A_1A_nA_{n-1}\left((n-2)+\frac{1}{2}\right) \\ &= A_{i-1}A_{i-2}...A_1A_n\left(n-\frac{1}{2}\right), \\ &= A_{i-1}A_{i-2}...A_2A_1\left(1\right)= \end{aligned}$$

$$\vdots$$
$$A_{i-1}\left((i-2)+\frac{1}{2}\right) = (i-1)+\frac{1}{2} = i-\frac{1}{2}.$$

(3) if $i \leq x_{i+1} < i+\frac{1}{4}$ we get

$$\begin{aligned}
A_{i-1}...A_1A_n..A_{i+1}x_{i+1} &= A_{i-1}...A_1A_n..A_{i+2}\left((i+1)+\frac{1}{4}\right) \\
&\vdots \\
&= A_{i-1}...A_1A_n\left(n-\frac{1}{2}\right) \\
&= A_{i-1}...A_2A_1(1) = \\
&\vdots \\
&= A_{i-1}\left((i-2)+\frac{1}{2}\right) \\
&= (i-1)+\frac{1}{2} = i-\frac{1}{2}.
\end{aligned}$$

(4) If $i+\frac{1}{4} \leq x_{i+1} \leq i+1$ we obtain

$$\begin{aligned}
A_{i-1}...A_1A_n..A_{i+1}x_{i+1} &= A_{i-1}...A_1A_n..A_{i+2}\left((i+1)+\frac{1}{2}\right) \\
&\vdots \\
&= A_{i-1}...A_1A_n\left(n-\frac{1}{2}\right), \\
&= A_{i-1}...A_2A_1(1) = \\
&\vdots \\
&= A_{i-1}\left((i-2)+\frac{1}{2}\right) \\
&= (i-1)+\frac{1}{2} = i-\frac{1}{2}.
\end{aligned}$$

Thus, all the conditions of theorem 2.1 are satisfied.

If we take $n = 5$ in theorem 2.1, we get the following corollary.

Corollary 2.3. *Let* (X_i, M_i, θ_i), $i = 1, ..., 5$ *be* 5 *complete fuzzy metric spaces,* $A_i : X_i \to X_{i+1}$, $i = 1, 2, 3, 4$ *and* $A_5 : X_5 \to X_1$ *be* 5 *mappings satisfying*

$$\phi_1\left(\begin{array}{c} M_1(A_5A_4A_3A_2x_2, A_5A_4A_3A_2A_1x_1, kt), \\ M_1(x_1, A_5A_4A_3A_2x_2, t), \\ M_1(x_1, A_5A_4A_3A_2A_1x_1, t), \\ M_2(x_2, A_1A_5A_4A_3A_2x_2, t) \end{array}\right) \geq 0 \tag{4.1}$$

for all $x_1 \in X_1$ *and* $x_2 \in X_2$,

$$\phi_2\left(\begin{array}{c} M_2(A_1A_5A_4A_3x_3, A_1A_5A_4A_3A_2x_2, kt), \\ M_3(x_3, A_2A_1A_5A_4A_3x_3, t), \\ M_2(x_2, A_1A_5A_4A_3A_2x_2, t) \\ M_2(x_2, A_1A_5A_4A_3x_3, t) \end{array}\right) \geq 0 \tag{4.2}$$

for all $x_2 \in X_2$ *and* $x_3 \in X_3$

$$\phi_3 \begin{pmatrix} M_3\left(A_2A_1A_5A_4x_4, A_2A_1A_5A_4A_3x_3, kt\right), \\ M_4\left(x_4, A_3A_2A_1A_5A_4x_4, t\right), \\ M_3\left(x_3, A_2A_1A_5A_4A_3x_3, t\right), \\ M_3\left(x_3, A_2A_1A_5A_4x_4, t\right) \end{pmatrix} \geq 0 \tag{4.3}$$

for all $x_3 \in X_3$ *and* $x_4 \in X_4$

$$\phi_4 \begin{pmatrix} M_4\left(A_3A_2A_1A_5x_5, A_3A_2A_1A_5A_4x_4, kt\right), \\ M_5\left(x_5, A_4A_3A_2A_1A_5x_5, t\right), \\ M_4\left(x_4, A_3A_2A_1A_5A_4x_4, t\right), \\ M_4\left(x_4, A_3A_2A_1A_5x_5, t\right) \end{pmatrix} \geq 0 \tag{4.4}$$

for all $x_4 \in X_4$ *and* $x_5 \in X_5$

$$\phi_5 \begin{pmatrix} M_5\left(A_4A_3A_2A_1x_1, A_4A_3A_2A_1A_5x_5, kt\right), \\ M_1\left(x_1, A_5A_4A_3A_2A_1x_1, t\right), \\ M_5\left(x_5, A_4A_3A_2A_1A_5x_5, t\right), \\ M_5\left(x_5, A_4A_3A_2A_1x_1, t\right) \end{pmatrix} \geq 0 \tag{4.5}$$

for all $x_1 \in X_1$, $x_5 \in X_5$ *and for all* $t > 0$, *where* $0 < k < 1$. *Then*

(a_1) $A_5A_4A_3A_2A_1$ *has a unique fixed point* $w_1 \in X_1$,
(a_2) $A_1A_5A_4A_3A_2$ *has a unique fixed point* $w_2 \in X_2$,
(a_3) $A_2A_1A_5A_4A_3$ *has a unique fixed point* $w_3 \in X_3$,
(a_4) $A_3A_2A_1A_5A_4$ *has a unique fixed point* $w_4 \in X_4$,
(a_5) $A_4A_3A_2A_1A_5$ *has a unique fixed point* $w_5 \in X_5$,
Further, $A_1w_1 = w_2$, $A_2w_2 = w_3$, $A_3w_3 = w_4$, $A_4w_4 = w_5$ *and* $A_5w_5 = w_1$.

If we take $n = 2$ in theorem 2.1, we obtain theorem 2.9 of Rao et al. [13] and a fuzzy version of theorem 3 of [1].

The following example illustrates our corollary 2.3.

Example 2.4. Let (M_i, X_i, θ_i) for $i = 1, ..., 5$ be 5 fuzzy metric spaces where $M_i\left(x_i, y_i, t\right) = \dfrac{t}{t + |x_i - y_i|}$ and $X_i = \{x_i : i - 1 \leq x_i \leq i\}$ for $i = 1, ..., 5$. Define $A_i : X_i \to X_{i+1}$ for $i = 1, ..., 4$ and $A_5 : X_5 \to X_1$ by

$$A_1x_1 = \begin{cases} 1 & \text{if } x_1 \in [0, \frac{3}{4}[\\ \frac{3}{2} & \text{if } x_1 \in [\frac{3}{4}, 1] \end{cases}, \quad A_2x_2 = \begin{cases} \frac{5}{2} & \text{if } x_2 \in [1, \frac{3}{2}[\\ 3 & \text{if } x_2 \in [\frac{3}{2}, 2] \end{cases}$$

$$A_3x_3 = \begin{cases} \frac{13}{4} & \text{if } x_3 \in [2, \frac{5}{2}[\\ \frac{7}{2} & \text{if } x_3 \in [\frac{5}{2}, 3] \end{cases}, \quad A_4x_4 = \begin{cases} \frac{17}{4} & \text{if } x_4 \in [3, \frac{7}{2}[\\ \frac{9}{2} & \text{if } x_4 \in [\frac{7}{2}, 4] \end{cases}$$

$$A_5x_5 = \begin{cases} \frac{3}{4} & \text{if } x_5 \in [4, \frac{9}{2}[\\ 1 & \text{if } x_5 \in [\frac{9}{2}, 5] \end{cases}$$

Let $\phi_1\left(t_1, t_2, t_3, t_4\right) = t_1 - \min\left\{t_2, t_3, t_4\right\}$ and $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5$

Further, the inequalities $(4.1), (4.2), (4.3), (4.4), (4.5)$ are satisfied since the left hand side of each inequality is 1 and

$$A_5A_4A_3A_2A_1\left(1\right) = 1$$

$$
\begin{aligned}
A_1A_5A_4A_3A_2\left(\frac{3}{2}\right) &= \frac{3}{2} \\
A_2A_1A_5A_4A_3\left(\frac{5}{2}\right) &= \frac{5}{2} \\
A_3A_2A_1A_5A_4\left(\frac{7}{2}\right) &= \frac{7}{2} \\
A_4A_3A_2A_1A_5\left(\frac{9}{2}\right) &= \frac{9}{2}.
\end{aligned}
$$

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