Vol. 3 No. 1 (2012)

Journal of Nonlinear Analysis and Optimization: Theory & Applications

Editors-in-Chief:

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About the Journal

Journal of Nonlinear Analysis and Optimization: Theory & Applications is a peer-reviewed, open-access international journal, that devotes to the publication of original articles of current interest in every theoretical, computational, and applicational aspect of nonlinear analysis, convex analysis, fixed point theory, and optimization techniques and their applications to science and engineering. All manuscripts are refereed under the same standards as those used by the finest-quality printed mathematical journals. Accepted papers will be published in two issues annually in March and September, free of charge.

This journal was conceived as the main scientific publication of the Center of Excellence in Nonlinear Analysis and Optimization, Naresuan University, Thailand.

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Official Website: https://ph03.tci-thaijo.org/index.php/jnao

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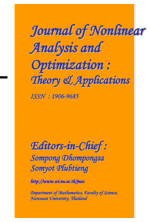
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 1-12
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



AN OUTER APPROXIMATION METHOD UTILIZING THE RELATION OF CONNECTIONS AMONG VERTICES FOR SOLVING A DC PROGRAMMING PROBLEM

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ABSTRACT. In this paper, we improve the outer approximation method proposed by Tuy [10] for solving a dc programming problem. The improved algorithm has the global convergence by generating a sequence of polytopes approximating a compact convex set from outside. Moreover, by incorporating a procedure for calculating the vertex sets utilizing the relations of connections among vertices by edges, the improved algorithm can calculate an approximate solution effectively.

KEYWORDS : Global optimization; Dc programming problem; Outer approximation method.

1. INTRODUCTION

Dc programming is one of the important subjects in global optimization and has been studied by Avriel and Williams [1], Hillestad and Jacobsen [2], Meyer [3], Rosen [5], Tuy [7] and Ueing [11]. It is known that many global optimization problems can be transformed into dc programming problems. In particular, Avriel and Williams [1] and Zaleesky [12] have shown that dc programming problems often occur in certain engineering design and economic management applications. Moreover, iterative solution methods for solving dc programming problems have been proposed by Thoai[6], Tuy [9, 10] and many other researchers. Many algorithm of them all are based on outer approximation methods. Outer approximation is one of the powerful procedures in global optimization and can solve various global optimization problems. The algorithms have the global convergence by generating a sequence of convex polyhedral sets. Therefore, it is shown that every accumulation

[1] Supported by the Commission on Higher Education and the Thailand Research Fund (Project No. MRG5380247).
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Article history : Received 24 December 2011. Accepted 27 December 2011.

point of the sequence of the provisional solutions generated by the algorithm is a globally optimal solution.

In this paper, we consider a problem (DC) to minimize a dc (difference of two convex) function over a compact convex set. To calculate an approximate solution of (DC) effectively, we improve the outer approximation method proposed by Tuy [10]. The proposed algorithm has the global convergence by generating a sequence of polytopes approximating the intersection of a closed half space and the epigraph of one convex function constructing the objective function of (DC) from outside. Moreover, we propose a procedure for calculating all verteices of polytopes by utilizing the relation of connections among vertices by edges. By incorporating the proposed procedure for calculating the vertex sets into the improved outer approximation method, the efficiency of the algorithm is upgraded.

The organization of this paper is as follows: In Section 2, we consider a dc programming problem. In Section 3, we explain the outer approximation algorithm proposed by Tuy [10]. In Section 4, we improve Tuy's outer approximation algorithm. In Section 5, we propose a procedure for calculating the vertex sets utilizing the relation of connections among vertices by edges. In Section 6, to verify the effectively of the algorithm integrated the outer approximation method and the procedure for calculating the vertex sets proposed in Sections 4 and 5, we show computational experiments.

Throughout this paper, we use the following notation: $\mathbb{R}$ denotes the set of all real numbers. For a subset $X \subset \mathbb{R}^n$, $\operatorname{int} X$, $\operatorname{bd} X$ and $\operatorname{co} X$ denote the interior, the boundary and the convex hull of X, respectively. For a finite set $X \subset \mathbb{R}^n$, $|X|$ denotes the number of elements of X. Given a convex polyhedral set (or polytope) $X \subset \mathbb{R}^n$, $V(X)$ denotes the set of all vertices of X. For vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, we use two kinds of symbols for open and closed line segments: $]\boldsymbol{a}, \boldsymbol{b}[:= \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \boldsymbol{a} + \delta(\boldsymbol{b} - \boldsymbol{a}),\ 0 < \delta < 1\}$ and $[\boldsymbol{a}, \boldsymbol{b}] := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = \boldsymbol{a} + \delta(\boldsymbol{b} - \boldsymbol{a}),\ 0 \leq \delta \leq 1\}$. Given a vector $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a}^\top$ denotes the transposed vector of $\boldsymbol{a}$. Given a convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\partial f(\boldsymbol{x})$ denotes the subdifferential of f at $\boldsymbol{x}$, that is, $\partial f(\boldsymbol{x}) := \{\boldsymbol{u} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq \langle \boldsymbol{u}, \boldsymbol{y} - \boldsymbol{x} \rangle + f(\boldsymbol{x}),\ \boldsymbol{y} \in \mathbb{R}^n\}$.

2. DC PROGRAMMING PROBLEM

Let us consider the following dc programming problem:

$$(\mathrm{DC}) \begin{cases} \text{minimize} & f(\boldsymbol{x}) - g(\boldsymbol{x}) \\ \text{subject to} & h_j(\boldsymbol{x}) \leq 0,\ j = 1, \ldots, m, \end{cases}$$

where $f, g, h_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ ($j = 1, \ldots, m$) are continuously differentiable convex functions. Since the objective function of (DC) is defined as the difference of two convex functions, the objective function is called dc function. Let $X := \{\boldsymbol{x} \in \mathbb{R}^n : h_j(\boldsymbol{x}) \leq 0,\ j = 1, \ldots, m\}$. Then, from the convexity of h_j, X is a convex set.

For (DC), we assume the following conditions.

(A1): $\operatorname{int} X = \{\boldsymbol{x} \in \mathbb{R}^n : h_j(\boldsymbol{x}) < 0,\ j = 1, \ldots, m\} \neq \emptyset$.

(A2): X is compact.

(A3): A real number r is given and satisfies $r \geq \max\{\|\boldsymbol{x} - \boldsymbol{y}\| : \boldsymbol{x}, \boldsymbol{y} \in X\}$.

From Assumption (A2) and the continuity of the objective function, (DC) has a globally optimal solution. Now, we notice that (DC) is a convex (concave) programming problem if g (f) is linear. Then, the useful solution methods have been already proposed. Therefore, we add the following assumption for (DC).

(A4): f and g are not linear.

3. TUY'S OUTER APPROXIMATION ALGORITHM

In order to calculate an approximate solution of (DC), the following outer approximation algorithm has been proposed by Tuy [10].

Algorithm OA

Step 0:

Step 0-1: Find a feasible solution $\boldsymbol{y}^1 \in X$ and set $\omega_1 := f(\boldsymbol{y}^1) - g(\boldsymbol{y}^1)$. Go to Step 0-2.

Step 0-2: Construct a polytope $S \subset \mathbb{R}^n$ satisfying $S \supset X$. Calculate the vertex set $V(S)$. Choose $\boldsymbol{x}' \in V(S)$ satisfying $f(\boldsymbol{x}') = \max\{f(\boldsymbol{x}) : \boldsymbol{x} \in V(S)\}$. Set $\tilde{t} := f(\boldsymbol{x}')$ and $D(\boldsymbol{y}^1) := \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \boldsymbol{x} \in X,\ f(\boldsymbol{x}) - t \leq \omega_1,\ t \leq \tilde{t}\}$. Go to Step 0-3.

Step 0-3: Construct a polytope $P_1 \subset \mathbb{R}^{n+1}$ satisfying $P_1 \supset D(\boldsymbol{y}^1)$ and $P_1 \subset \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : t \leq \tilde{t}\}$. Calculate the vertex set $V(P_1)$. Go to Step 0-4.

Step 0-4: Find $(\bar{\boldsymbol{y}}, \bar{t}) \in \operatorname{int} D(\boldsymbol{y}^1)$. Set a tolerance $\tau \geq 0$ and $k = 1$. Go to Step 1.

Step 1: Choose $(\boldsymbol{x}^k, t_k) \in \arg\min\{-g(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in V(P_k)\}$. Go to Step 2.

Step 2: If $-g(\boldsymbol{x}^k) + t_k \geq -\tau$, then stop: $\boldsymbol{y}^k$ is an approximate solution of (DC). Otherwise, go to Step 3.

Step 3: Set $\boldsymbol{y}^{k+1}, \omega_{k+1}, P_{k+1}$ as follows:

$$\boldsymbol{y}^{k+1} := \begin{cases} \boldsymbol{x}^k & \text{if } f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) < 0 \text{ and } \boldsymbol{x}^k \in X, \\ \boldsymbol{y}^k & \text{otherwise,} \end{cases}$$

$$\omega_{k+1} := \begin{cases} f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) & \text{if } f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) < 0 \text{ and } \boldsymbol{x}^k \in X, \\ \omega_k & \text{otherwise,} \end{cases}$$

$$P_{k+1} := P_k \cap \{(\boldsymbol{x}, t) \in R^n \times R : \ell_k(\boldsymbol{x}, t) \leq 0\},$$

where

$$\ell_k(\boldsymbol{x}, t) := \langle \boldsymbol{d}^k, (\boldsymbol{x}^\top, t)^\top - (\boldsymbol{z}^k, \theta_k)^\top \rangle,$$
$$\boldsymbol{d}^k \in \partial\phi(\boldsymbol{z}^k, \theta_k),$$
$$(\boldsymbol{z}^k, \theta_k) \in [(\boldsymbol{x}^k, t_k), (\bar{\boldsymbol{y}}, \bar{t})] \cap \operatorname{bd} D(\boldsymbol{y}^k).$$

Calculate the vertex set $V(P_{k+1})$. Set $k \leftarrow k + 1$ and return to Step 1.

The sequences $\{D(\boldsymbol{y}^k)\}$ and $\{P_k\}$ satisfy the following conditions.

$$D(\boldsymbol{y}^1) \supset D(\boldsymbol{y}^2) \supset \cdots \supset D(\boldsymbol{y}^k) \supset D(\boldsymbol{y}^{k+1}) \supset \cdots,$$
$$P_1 \supset P_2 \supset \cdots \supset P_k \supset P_{k+1} \supset \cdots,$$
$$P_k \supset D(\boldsymbol{y}^k) \text{ for each } k = 1, 2, \ldots.$$

Moreover, we have the following inequality.

$$\omega_1 \geq \omega_2 \geq \cdots \geq \omega_k \geq \omega_{k+1} \geq \cdots \geq \min(\mathrm{DC}),$$

where $\min(\mathrm{DC})$ denotes the optimal value of (DC). By approximating $\{D(\boldsymbol{y}^k)\}$ by $\{P_k\}$ from outside, Algorithm OA generates the provisional solution sequence $\{\boldsymbol{y}^k\}$. Moreover, it is shown that every accumulation point $\{\boldsymbol{y}^k\}$ is a globally optimal solution of (DC). Hence, by setting $\tau > 0$, Algorithm OA calculates an approximate solution of (DC). However, $\{P_k\}$ often do not converge effectively, because the shape of $D(\boldsymbol{y}^k)$ approximated by P_k is changed as the number of iterations increases. Therefore, we improve Algorithm OA by fixing the target approximated by $\{P_k\}$ in Section 4. By such the improvement, the convergence of $\{P_k\}$ becomes more efficient.

4. IMPROVEMENT OF TUY'S OUTER APPROXIMATION ALGORITHM

In order to enhance the computational efficiency of the algorithm, we improve Tuy's outer approximation algorithm as follows:

Algorithm IOA

Step 0:

Step 0-1: Find a feasible solution $\boldsymbol{y}^1 \in X$ and set $\omega_1 := f(\boldsymbol{y}^1) - g(\boldsymbol{y}^1)$. Go to Step 0-2.

Step 0-2: Construct a polytope $S \subset \mathbb{R}^n$ satisfying $S \supset X$. Calculate the vertex set $V(S)$. Choose $\boldsymbol{x}' \in V(S)$ satisfying $f(\boldsymbol{x}') = \max\{f(\boldsymbol{x}) : \boldsymbol{x} \in V(S)\}$. Set $\tilde{t} := f(\boldsymbol{x}')$ and $D := \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \boldsymbol{x} \in X,\ f(\boldsymbol{x}) \leq t \leq \tilde{t}\}$. Go to Step 0-3.

Step 0-3: Construct a polytope $P_1 \subset \mathbb{R}^{n+1}$ satisfying $P_1 \supset D$ and $P_1 \subset \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : t \leq \tilde{t}\}$. Calculate the vertex set $V(P_1)$. Go to Step 0-4.

Step 0-4: Find $(\bar{\boldsymbol{y}}, \bar{t}) \in \operatorname{int} D$. Set a tolerance $\tau \geq 0$ and $k = 1$. Go to Step 1.

Step 1: Choose $(\boldsymbol{x}^k, t_k) \in \arg\min\{-g_k(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in V(P_k)\}$, where $g_k(\boldsymbol{x}) := g(\boldsymbol{x}) + \omega_k$. Go to Step 2.

Step 2: If $-g_k(\boldsymbol{x}^k) + t_k \geq -\tau$, then stop: $\boldsymbol{y}^k$ is an approximate solution of (DC). If $\phi(\boldsymbol{x}^k, t_k) \leq \tau$, then stop: $\boldsymbol{x}^k$ is an approximate solution of (DC), where

$$\phi(\boldsymbol{x}, t) := \max\{h_1(\boldsymbol{x}), \ldots, h_m(\boldsymbol{x}), f(\boldsymbol{x}) - t\}.$$

Otherwise, go to Step 3.

Step 3: Set $\boldsymbol{y}^{k+1}, \omega_{k+1}, P_{k+1}$ as follows:

$$\boldsymbol{y}^{k+1} := \begin{cases} \boldsymbol{x}^k & \text{if } f(\boldsymbol{x}^k) - g_k(\boldsymbol{x}^k) < 0 \text{ and } \boldsymbol{x}^k \in X, \\ \boldsymbol{y}^k & \text{otherwise,} \end{cases}$$

$$\omega_{k+1} := \begin{cases} f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) & \text{if } f(\boldsymbol{x}^k) - g_k(\boldsymbol{x}^k) < 0 \text{ and } \boldsymbol{x}^k \in X, \\ \omega_k & \text{otherwise,} \end{cases}$$

$$P_{k+1} := P_k \cap \{(\boldsymbol{x}, t) \in R^n \times R : \ell_k(\boldsymbol{x}, t) \leq 0\},$$

where

$$\ell_k(\boldsymbol{x}, t) := \langle \boldsymbol{d}^k, (\boldsymbol{x}^\top, t)^\top - (\boldsymbol{z}^k, \theta_k)^\top \rangle,$$
$$\boldsymbol{d}^k \in \partial\phi(\boldsymbol{z}^k, \theta_k),$$
$$(\boldsymbol{z}^k, \theta_k) \in [(\boldsymbol{x}^k, t_k), (\bar{\boldsymbol{y}}, \bar{t})] \cap \operatorname{bd} D.$$

Calculate the vertex set $V(P_{k+1})$. Set $k \leftarrow k + 1$ and return to Step 1.

From Assumption (A4), $\operatorname{int} D \neq \emptyset$ and hence $(\bar{\boldsymbol{y}}, \bar{t}) \in \operatorname{int} D$ can be found. For each k, since $D = \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \phi(\boldsymbol{x}, t) \leq 0, t \leq \tilde{t}\}$ and $\phi(\boldsymbol{x}, t)$ is convex on $\mathbb{R}^{n+1}$, we have $D \subset \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \ell_k(\boldsymbol{x}, t) \leq 0\}$. Hence,

$$P_1 \supset P_2 \supset \cdots \supset P_k \supset \cdots \supset D.$$

By the definition of $\boldsymbol{y}^k$ at Steps 0-1 and 3, $\{\boldsymbol{y}^k\} \subset X$. At iteration k, if $f(\boldsymbol{x}^k) - g_k(\boldsymbol{x}^k) < 0$,

$$f(\boldsymbol{x}^k) - g_k(\boldsymbol{x}^k) = f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) - \omega_k = f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) - (f(\boldsymbol{y}^k) - g(\boldsymbol{y}^k)) < 0,$$

that is, $f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) < f(\boldsymbol{y}^k) - g(\boldsymbol{y}^k)$. Therefore, for each k, we have the following inequalities.

$$f(\boldsymbol{y}^k) - g(\boldsymbol{y}^k) \geq f(\boldsymbol{y}^{k+1}) - g(\boldsymbol{y}^{k+1}) \geq \min(\text{DC}),$$
$$\omega^k \geq \omega^{k+1} \geq \min(\text{DC}).$$

Moreover, the following theorems hold.

Theorem 4.1. *Assume that* $-g_k(\boldsymbol{x}^k) + t_k \geq 0$ *at iteration* k *of Algorithm* IOA. *Then,* $\boldsymbol{y}^k$ *is a globally optimal solution of* (DC).

Proof. At iteration k of Algorithm IOA, it follows from the definition of $\boldsymbol{y}^k$ that $\boldsymbol{y}^k \in X$. Hence, in order to complete the proof, we shall show that $f(\boldsymbol{y}^k) - g(\boldsymbol{y}^k) \leq f(\boldsymbol{x}) - g(\boldsymbol{x})$ for each $\boldsymbol{x} \in X$. Let $\boldsymbol{x}' \in X$. Then, $(\boldsymbol{x}', f(\boldsymbol{x}')) \in D$. Since $D \subset P_k$ and g_k is convex, from the definition of $(\boldsymbol{x}^k, t_k)$, we have

$$\begin{aligned} -g_k(\boldsymbol{x}') + f(\boldsymbol{x}') &\geq \min\{-g_k(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in D\} \\ &\geq \min\{-g_k(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in P_k\} \\ &= \min\{-g_k(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in V(P_k)\} \\ &= -g_k(\boldsymbol{x}^k) + t_k \geq 0 \end{aligned}$$

Hence,

$$f(\boldsymbol{x}') - g_k(\boldsymbol{x}') = f(\boldsymbol{x}') - g(\boldsymbol{x}') - \omega_k = f(\boldsymbol{x}') - g(\boldsymbol{x}') - (f(\boldsymbol{y}^k) - g(\boldsymbol{y}^k)) \geq 0.$$

Therefore,

$$f(\boldsymbol{x}') - g(\boldsymbol{x}') \geq f(\boldsymbol{y}^k) - g(\boldsymbol{y}^k).$$

Consequently, $\boldsymbol{y}^k$ is a globally optimal solution of (DC). □

Theorem 4.2. *Assume that* $\phi(\boldsymbol{x}^k, t_k) \leq 0$ *at iteration* k *of Algorithm* IOA. *Then,* $\boldsymbol{x}^k$ *is a globally optimal solution of* (DC).

Proof. Since $\phi(\boldsymbol{x}^k, t_k) \leq 0$, $(\boldsymbol{x}^k, t_k) \in D$. Hence,

$$\begin{aligned} -g_k(\boldsymbol{x}^k) + t_k &\geq \min\{-g_k(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in D\} \\ &\geq \min\{-g_k(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in P_k\} \\ &= \min\{-g_k(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in V(P_k)\} \\ &= -g_k(\boldsymbol{x}^k) + t_k. \end{aligned}$$

This implies that $-g_k(\boldsymbol{x}^k) + t_k \geq \min\{-g_k(\boldsymbol{x}) + t : (\boldsymbol{x}, t) \in D\}$. Since $(\boldsymbol{x}^k, f(\boldsymbol{x}^k)) \in D$, $-g_k(\boldsymbol{x}^k) + t_k \leq -g_k(\boldsymbol{x}^k) + f(\boldsymbol{x}^k)$, that is, $t_k \leq f(\boldsymbol{x}^k)$. Moreover, since $(\boldsymbol{x}^k, t_k) \in D$, $t_k \geq f(\boldsymbol{x}^k)$. Hence, $t_k = f(\boldsymbol{x}^k)$. Since $-g_k(\boldsymbol{x}) + t \geq -g_k(\boldsymbol{x}) + f(\boldsymbol{x})$ for each $(\boldsymbol{x}, t) \in D$, we have

$$\begin{aligned} -g(\boldsymbol{x}^k) + f(\boldsymbol{x}^k) - \omega_k &= \min\{-g(\boldsymbol{x}) + t - \omega_k : (\boldsymbol{x}, t) \in D\} \\ &= \min\{-g(\boldsymbol{x}) + f(\boldsymbol{x}) - \omega_k : \boldsymbol{x} \in X\}. \end{aligned}$$

Since ω_k is constant, $f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) = \min\{f(\boldsymbol{x}) - g(\boldsymbol{x}) : \boldsymbol{x} \in X\}$. Consequently, $\boldsymbol{x}^k$ is a globally optimal solution of (DC). □

Corollary 4.3. *Assume that* $\phi(\boldsymbol{x}^k, t_k) > 0$ *at iteration* k *of Algorithm* IOA. *Then, there exists* $(\boldsymbol{z}^k, \theta_k) \in [(\boldsymbol{x}^k, t_k), (\bar{\boldsymbol{y}}, \bar{t})] \cap \operatorname{bd} D$. *Moreover,* $(\boldsymbol{x}^k, t_k) \notin P_{k+1}$.

Proof. We note $D = \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \phi(\boldsymbol{x}, t) \leq 0,\ t \leq \tilde{t}\}$. Since $(\bar{\boldsymbol{y}}, \bar{t}) \in \operatorname{int} D$, $\phi(\bar{\boldsymbol{y}}, \bar{t}) < 0$. Hence, from the assumption of this corollary, there exists $(\boldsymbol{z}^k, \theta_k) \in [(\boldsymbol{x}^k, t_k), (\bar{\boldsymbol{y}}, \bar{t})]$ such that $\phi(\boldsymbol{z}^k, \theta_k) = 0$, that is, $(\boldsymbol{z}^k, \theta_k) \in \operatorname{bd} D$. Moreover, from the convexity of ϕ,

$$\begin{aligned} \ell_k(\bar{\boldsymbol{y}}, \bar{t}) &= \langle \boldsymbol{d}^k, (\bar{\boldsymbol{y}}^\top, \bar{t})^\top - (\boldsymbol{z}^k, \theta_k)^\top \rangle = \phi(\boldsymbol{z}^k, \theta_k) + \langle \boldsymbol{d}^k, (\bar{\boldsymbol{t}}^\top, \bar{t})^\top - (\boldsymbol{z}^k, \theta_k)^\top \rangle \\ &\leq \phi(\bar{\boldsymbol{y}}, \bar{t}) < 0. \end{aligned}$$

We note that there exists $\mu > 0$ satisfying

$$((\boldsymbol{x}^k)^\top, t_k)^\top - (\boldsymbol{z}^k, \theta_k)^\top = -\mu((\bar{\boldsymbol{y}}^\top, \bar{t})^\top - (\boldsymbol{z}^k, \theta_k)^\top).$$

Therefore,

$$\ell_k(\boldsymbol{x}^k, t_k) = \langle \boldsymbol{d}^k, ((\boldsymbol{x}^k)^\top, t_k)^\top - (\boldsymbol{z}^k, \theta_k)^\top \rangle = -\mu \langle \boldsymbol{d}^k, (\bar{\boldsymbol{y}}^\top, \bar{t})^\top - (\boldsymbol{z}^k, \theta_k)^\top \rangle > 0.$$

Consequently, $(\boldsymbol{x}^k, t_k) \notin P_{k+1}$. □

Theorem 4.4. *Assume that* $\tau = 0$ *and that* $\{(\boldsymbol{x}^k, t_k)\}$ *is an infinite sequence generated by Algorithm* IOA. *Then, every accumulation point of* $\{(\boldsymbol{x}^k, t_k)\}$ *is contained in* D.

Proof. Since $\{(\boldsymbol{x}^k, t_k)\} \subset P_1$ and P_1 is bounded, $\{(\boldsymbol{x}^k, t_k)\}$ has an accumulation point. Moreover, since $\{(\boldsymbol{z}^k, \theta_k)\} \subset \mathrm{bd}\ D$ and $\mathrm{bd}\ D$ is bounded, $\{(\boldsymbol{z}^k, \theta_k)\}$ has an accumulation point. Furthermore, since $\{\boldsymbol{d}^k\} \subset \cup_{(x,t)\in \mathrm{bd}\ D} \partial\phi(\boldsymbol{x}, t)$ and $\cup_{(x,t)\in \mathrm{bd}\ D} \partial\phi(\boldsymbol{x}, t)$ is bounded (see Theorem 24.7 in [4]), $\{\boldsymbol{d}^k\}$ has an accumulation point. Let $(\hat{\boldsymbol{x}}, \hat{t})$, $(\hat{\boldsymbol{z}}, \hat{\theta})$ and $\hat{\boldsymbol{d}}$ be accumulation points of $\{(\boldsymbol{x}^k, t_k)\}$, $\{(\boldsymbol{z}^k, \theta_k)\}$ and $\{\boldsymbol{d}^k\}$, respectively. Without loss of generality, we can assume that $(\boldsymbol{x}^k, t_k) \longrightarrow (\hat{\boldsymbol{x}}, \hat{t})$ and $(\boldsymbol{z}^k, \theta_k) \longrightarrow (\hat{\boldsymbol{z}}, \hat{\theta})$ as $k \longrightarrow +\infty$. Then, from the upper semi-continuity of $\partial\phi$, $\hat{d} \in \partial\phi(\hat{\boldsymbol{z}}, \hat{t})$ (see Theorem 24.4 in [4]).

In order to obtain a contradiction, we suppose that $(\hat{\boldsymbol{x}}, \hat{t}) \notin D$. Since $\{(\boldsymbol{x}^k, t_k)\} \subset P_1 \subset \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : t \le \tilde{t}\}$ and $D = \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \phi(\boldsymbol{x}, t) \le 0,\ t \le \tilde{t}\}$, $\hat{t} \le \tilde{t}$ and hence $\phi(\hat{\boldsymbol{x}}, \hat{t}) > 0$. Then, there exists $\mu > 0$ such that $(\hat{\boldsymbol{x}}, \hat{t}) - (\hat{\boldsymbol{z}}, \hat{\theta}) = -\mu((\bar{\boldsymbol{y}}, \bar{t}) - (\hat{\boldsymbol{z}}, \hat{\theta}))$. Since $\langle \hat{\boldsymbol{d}}, (\bar{\boldsymbol{y}}^\top, \bar{t})^\top - (\hat{\boldsymbol{z}}^\top, \hat{\theta})^\top \rangle \le \phi(\bar{\boldsymbol{y}}, \bar{t}) < 0$,

$$\langle \hat{\boldsymbol{d}}, (\hat{\boldsymbol{x}}^\top, \hat{t})^\top - (\hat{\boldsymbol{z}}^\top, \hat{\theta})^\top \rangle = -\mu \langle \hat{d}, (\bar{\boldsymbol{y}}^\top, \bar{t})^\top - (\hat{\boldsymbol{z}}^\top, \hat{\theta})^\top \rangle > 0.$$

Moreover, we have

$$\begin{aligned} \lim_{k\longrightarrow+\infty} \ell_k(\boldsymbol{x}^k, t_k) &= \lim_{k\longrightarrow+\infty} \langle \boldsymbol{d}^k, ((\boldsymbol{x}^k)^\top, t_k)^\top - ((\boldsymbol{z}^k)^\top, \theta_k)^\top \rangle \\ &= \langle \hat{\boldsymbol{d}}, (\hat{\boldsymbol{x}}^\top, \hat{t})^\top - (\hat{\boldsymbol{z}}^\top, \hat{\theta})^\top \rangle > 0. \end{aligned}$$

Hence, there exists $k_1, k_2 > 0$ such that $\ell_{k_2}(\hat{\boldsymbol{x}}, \hat{t}) > 0$ and $\ell_{k_2}(\boldsymbol{x}^k, t_k) > 0$ for each $k \ge k_1$. This implies that $\{(\boldsymbol{x}^k, t_k)\}_{k\ge k_1} \cap P_{k_2+1} = \emptyset$. However, from the definitions of P_k and $(\boldsymbol{x}^k, t_k)$, $\{(\boldsymbol{x}^k, t_k)\}_{k>\max\{k_1,k_2\}} \subset P_{k_2+1}$. This is a contradiction. Consequently, $(\hat{\boldsymbol{x}}, \hat{t}) \in D$. □

Theorem 4.5. *Assume that* $\tau = 0$ *and that* $\{(\boldsymbol{x}^k, t_k)\}$ *is an infinite sequence generated by Algorithm* IOA. *Then, every accumulation point of* $\{\boldsymbol{x}^k\}$ *is a globally optimal solution of* (DC).

Proof. In the same way of Theorem 4.4, we can assume that $(\boldsymbol{x}^k, t_k) \longrightarrow (\hat{\boldsymbol{x}}, \hat{t})$ as $k \longrightarrow +\infty$. Since $\{\boldsymbol{y}^k\} \subset X$ and X is bounded, $\{\boldsymbol{y}^k\}$ has an accumulation point. Hence, without loss of generality, we can assume that $\boldsymbol{y}^k \longrightarrow \hat{\boldsymbol{y}}$ as $k \longrightarrow +\infty$. Moreover, since $\omega_k = f(\boldsymbol{y}^k) - g(\boldsymbol{y}^k)$, from the continuity of f and g, we can set $\hat{\omega} := \lim_{k\longrightarrow+\infty} \omega_k$. Then, by the definitions of g_k and D, we have

$$\begin{aligned} \hat{t} - g(\hat{\boldsymbol{x}}) - \hat{\omega} &= \lim_{k\longrightarrow+\infty} t_k - g(\boldsymbol{x}^k) - \omega_k \\ &= \lim_{k\longrightarrow+\infty} \min\{t - g(\boldsymbol{x}) - \omega_k : (\boldsymbol{x}, t) \in V(P_k)\} \\ &= \lim_{k\longrightarrow+\infty} \min\{t - g(\boldsymbol{x}) - \omega_k : (\boldsymbol{x}, t) \in P_k\} \\ &\le \lim_{k\longrightarrow+\infty} \min\{t - g(\boldsymbol{x}) - \omega_k : (\boldsymbol{x}, t) \in D\} \\ &= \lim_{k\longrightarrow+\infty} \min\{f(\boldsymbol{x}) - g(\boldsymbol{x}) - \omega_k : (\boldsymbol{x}, t) \in D\} \\ &= \min(\mathrm{DC}) - \hat{\omega}. \end{aligned}$$

Hence, $\hat{t} - g(\hat{\boldsymbol{x}}) \leq \min(\mathrm{DC})$. Moreover, from Theorem 4.4, $(\hat{\boldsymbol{x}}, \hat{t}) \in D$. By the definition of D, $\hat{t} \geq f(\hat{\boldsymbol{x}})$. Therefore, $\hat{t} - g(\hat{\boldsymbol{x}}) \geq f(\hat{\boldsymbol{x}}) - g(\hat{\boldsymbol{x}}) \geq \min(\mathrm{DC})$. Thus, we have

$$\hat{t} - g(\hat{\boldsymbol{x}}) = f(\hat{\boldsymbol{x}}) - g(\hat{\boldsymbol{x}}) = \min(\mathrm{DC}).$$

Consequently, $\hat{\boldsymbol{x}}$ is a globally optimal solution of (DC). □

Corollary 4.6. *Assume that $\tau = 0$ and that $\{\boldsymbol{y}^k\}$ is an infinite sequence generated by Algorithm* IOA. *Then, every accumulation point of $\{\boldsymbol{y}^k\}$ is a globally optimal solution of* (DC).

Proof. In the same way of Theorem 4.5, we can assume that $\boldsymbol{y}^k \longrightarrow \hat{\boldsymbol{y}}$ as $k \longrightarrow +\infty$. Since $\{\boldsymbol{y}^k\} \subset X$ and X is compact, $\hat{\boldsymbol{y}} \in X$. Hence, $f(\hat{\boldsymbol{y}}) - g(\hat{\boldsymbol{y}}) \geq \min(\mathrm{DC})$. Moreover, by Theorem 4.5 and the definition of $\boldsymbol{y}^k$, we have

$$\min(\mathrm{DC}) == \lim_{k \longrightarrow +\infty} f(\boldsymbol{x}^k) - g(\boldsymbol{x}^k) \geq \lim_{k \longrightarrow +\infty} f(\boldsymbol{y}^{k+1}) - g(\boldsymbol{y}^{k+1}) = f(\hat{\boldsymbol{y}}) - g(\hat{\boldsymbol{y}}).$$

Therefore, $f(\hat{\boldsymbol{y}}) - g(\hat{\boldsymbol{y}}) = \min(\mathrm{DC})$. Consequently, $\hat{\boldsymbol{y}}$ is a globally optimal solution of (DC). □

From Theorems 4.1 and 4.2, it is shown that the stopping criteria of Algorithm IOA are valid. In the case where infinite sequences $\{\boldsymbol{x}^k\}$ and $\{\boldsymbol{y}^k\}$ are generated by Algorithm IOA, by Theorem 4.5 and Corollary 4.6, every accumulation point of $\{\boldsymbol{x}^k\}$ and $\{\boldsymbol{y}^k\}$ is a globally optimal solution of (DC). Moreover, by setting $\tau > 0$, it follows from Theorem 4.4 that Algorithm IOA terminates within a finite number of iterations.

5. PROCEDURE FOR CALCULATING THE VERTEX SET

At Step 3 of Algorithm IOA proposed in Section 4, the vertex set $V(P_k)$ is calculated. In the classical methods, to obtain the vertex set, many systems of linear equations are solved. However, it is known that such procedures are inefficient. In this section, we propose a procedure for calculating the vertex set $V(P_k)$ utilizing the relation of connections among vertices by edges. Hence, in order to construct such a procedure, we introduce the following definitions on convex analysis.

Definition 5.1. ([8]) Let P be an n-dimensional polyhedral set and let H be a supporting hyperplane of P. Then, the intersection $F := H \cap P$ is called a *face* of P. If $\dim F = 1$, F is called an *edge* of P. If $\dim F = n - 1$, F is called a *facet* of P.

We remember that P_k is given at iteration k of Algorithm IOA. Let $(\boldsymbol{v}(i), t(i))$ ($i \in \Delta_k \subset \{1, \ldots, \alpha(k)\}$) be vertices of P_k and let F_j ($j \in \Gamma_k \subset \{1, \ldots, \beta(k)\}$) be facets of P_k, where

- $\alpha(k)$ and $\beta(k)$ denote the numbers of vertices and facets generated by iteration k, respectively,
- Δ_k is the index set of all vertices of P_k, that is, $\{(\boldsymbol{v}(i), t(i)) : i \in \Delta_k\} = V(P_k)$
- Γ_k is the index set of all facets of P_k, that is, $F_j \cap P_k \neq \emptyset$ for each $j \in \Gamma_k$ and $F_{j'} \cap P_k = \emptyset$ for any $j' \in \{1, \ldots, \beta(k)\} \backslash \Gamma_k$.

For each $i \in \Delta_k$, we set the index sets $\mathcal{V}_i$ and $\mathcal{F}_i$ as follows:

$$\mathcal{V}_i := \{j : \mathrm{co}\,\{(\boldsymbol{v}(i), t(i)), (\boldsymbol{v}(j), t(j))\} \text{ is an edge of } P_k,\ j \in \Delta_k \backslash \{i\}\},$$

$$\mathcal{F}_i := \{j : (\boldsymbol{v}(i), t(i)) \in F_j,\ j \in \Gamma_k\}.$$

The procedure proposed in this section consists of the following three parts.

Section 5.1: Search for all vertices of P_k to remove.

Section 5.2: Calculation of all vertices of P_{k+1} to generate.
Section 5.3: Check for the availability of F_i ($i \in \Gamma_k$).
Section 5.4: Update the index sets $\mathcal{V}_i$.

5.1. **SEARCH FOR ALL VERTICES TO REMOVE.** Since $P_{k+1} = P_k \cap \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \ell_k(\boldsymbol{x}, t) \leq 0\}$ and $\ell_k(\boldsymbol{v}(i_k), t(i_k)) > 0$, there exist vertices of P_k which are not contained in P_{k+1}. Hence, to construct the vertex set $V(P_{k+1})$, it is necessary to search for all $(\boldsymbol{v}, t) \in V(P_k)$ satisfying $\ell_k(\boldsymbol{v}, t) > 0$. For this reason, we propose the following procedure.

Procedure A

Step 0: Set $\mathcal{M} := \mathcal{M}' = \{i_k\}$. Go to Step 1.
Step 1: If $\mathcal{M}' = \emptyset$, then stop. Otherwise, choose $j \in \mathcal{M}'$ and go to Step 2.
Step 2:
- **Step 2-0:** Set $\mathcal{T} := \mathcal{V}_j$ and go to Step 2-1.
- **Step 2-1:** If $\mathcal{T} = \emptyset$, then go to Step 3. Otherwise, choose $q \in \mathcal{T}$ and go to Step 2-2.
- **Step 2-2:** If $\ell_k(\boldsymbol{v}(q), t(q)) > 0$ and $q \notin \mathcal{M}$, set $\mathcal{M} \leftarrow \mathcal{M} \cup \{q\}$ and $\mathcal{M}' \leftarrow \mathcal{M}' \cup \{q\}$. Go to Step 2-3.
- **Step 2-3:** Set $\mathcal{T} \leftarrow \mathcal{T} \setminus \{q\}$ and return to Step 1.

Step 3: Set $\mathcal{M}' \leftarrow \mathcal{M}' \setminus \{j\}$ and return to Step 1.

Procedure A attain the following.

- $\mathcal{M}$ listed the indices of all vertex of P_k which are not contained in P_{k+1}. Hence, for each $i \in \mathcal{M}$,

$$(\boldsymbol{v}(i), t(i)) \in V(P_k) \text{ and } (\boldsymbol{v}(i), t(i)) \notin V(P_{k+1}).$$

5.2. **CALCULATION OF ALL VERTICES TO GENERATE.** Let $(\boldsymbol{v}(i_1), t(i_1)), (\boldsymbol{v}(i_2), t(i_2)) \in V(P_k)$ satisfy

$$i_2 \in \mathcal{V}_{i_1},\ \ell_k(\boldsymbol{v}(i_1), t(i_1)) > 0 \text{ and } \ell_k(\boldsymbol{v}(i_2), t(i_2)) < 0.$$

Then, co $\{(\boldsymbol{v}(i_1), t(i_1)), (\boldsymbol{v}(i_2), t(i_2))\}$ is an edge of P_k and there exists a vertex $(\boldsymbol{v}', t')$ of P_{k+1} generated at iteration k of Algorithm IOA, that is, $(\boldsymbol{v}', t') \in V(P_{k+1}) \setminus V(P_k)$. Moreover, the following assertions hold.

- $(\boldsymbol{v}', t') \in F_j$ for all $j \in \mathcal{F}_{i_1} \cap \mathcal{F}_{i_2}$.
- $(\boldsymbol{v}', t') \in F_{\beta(k)+1}$.

Here, $F_{\beta(k)+1} := P_{k+1} \cap \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \ell_k(\boldsymbol{x}, t) = 0\} = P_k \cap \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \ell_k(\boldsymbol{x}, t) = 0\}$. We note that $F_{\beta(k)+1}$ is a unique facet of P_{k+1} at iteration k of Algorithm IOA. Hence, we can set $\beta(k+1) := \beta(k) + 1$.

In order to calculate all vertices generated at iteration k of Algorithm IOA, we proposed the following procedure.

Procedure B

Step 0: Set $\mathcal{W} := \emptyset$, $\alpha(k+1) := \alpha(k)$ and $\mathcal{M}' := \mathcal{M}$, where $\mathcal{M}$ is generated by Procedure A. Go to Step 1.
Step 1: If $\mathcal{M}' = \emptyset$, then set $\Delta_{k+1} := (\Delta_k \cup \{\alpha(k)+1, \ldots, \alpha(k+1)\}) \setminus \mathcal{M}$ and $\beta(k+1) := \beta(k) + 1$, and stop. Otherwise, choose $j \in \mathcal{M}'$ and go to Step 2.
Step 2:
- **Step 2-0:** Set $\mathcal{T} := \mathcal{V}_j$ and go to Step 2-1.
- **Step 2-1:** If $\mathcal{T} = \emptyset$, then go to Step 3. Otherwise, choose $q \in \mathcal{T}$ and go to Step 2-2.

Step 2-2: If $\ell_k(\boldsymbol{v}(q), t(q)) < 0$, then go to Step 2-3. If $\ell_k(\boldsymbol{v}(q), t(q)) = 0$, then go to Step 2-4. Otherwise, go to Step 2-5.

Step 2-3: Set $(\boldsymbol{v}(\alpha(k+1)+1), t(\alpha(k+1)+1)$, $\mathcal{F}_{\alpha(k+1)+1}$ and $\mathcal{V}_{\alpha(k+1)+1}$ as follows:

$$\begin{aligned}
(\boldsymbol{v}(\alpha(k+1)+1), t(\alpha(k+1))) &:= (1-\lambda)(\boldsymbol{v}(j), t(j)) + \lambda(\boldsymbol{v}(q), t(q)),\\
\mathcal{F}_{\alpha(k+1)+1} &:= (\mathcal{F}_j \cap \mathcal{F}_q) \cup \{\beta(k)+1\},\\
\mathcal{V}_{\alpha(k+1)+1} &:= \{q\},\\
\mathcal{V}_q &\leftarrow (\mathcal{V}_q \backslash \{j\}) \cup \{\alpha(k+1)+1\},\\
\mathcal{W} &\leftarrow \mathcal{W} \cap \{\alpha(k+1)+1\},\\
\alpha(k+1) &\leftarrow \alpha(k+1)+1,
\end{aligned}$$

where

$$\lambda := \frac{\ell_k(\boldsymbol{v}(q), t(q))}{\ell_k(\boldsymbol{v}(j), t(j)) - \ell_k(\boldsymbol{v}(q), t(q))}.$$

The renewal of $\mathcal{V}_{\alpha(k+1)+1}$ are incomplete at this step. The index set $\mathcal{V}_{\alpha(k+1)+1}$ will be corrected by Procedure D proposed in Section 5.4. Go to Step 2.

Step 2-4: Update $\mathcal{F}_q$ and $\mathcal{V}_q$ as follows:

$$\begin{aligned}
\mathcal{F}_q &\leftarrow \mathcal{F}_q \cup \{\beta(k)+1\},\\
\mathcal{V}_q &\leftarrow \mathcal{V}_q \backslash \{j\},\\
\mathcal{W} &\leftarrow \mathcal{W} \cup \{q\}.
\end{aligned}$$

Go to Step 2-5.

Step 2-5: Set $\mathcal{T} \leftarrow \mathcal{T} \backslash \{q\}$, and return to Step 2-1.

Step 3: Set $\mathcal{M}' \leftarrow \mathcal{M}' \backslash \{j\}$ and return to Step 1.

By Procedure B, we have

- Δ_{k+1}, $\alpha(k+1)$ and $\beta(k+1)$ are calculated. Hence, $V(P_{k+1}) := \{(\boldsymbol{v}(i), t(i)) : i \in \Delta_{k+1}\}$.
- $\{(\boldsymbol{v}(i), t(i)) : i = \alpha(k)+1, \ldots, \alpha(k+1)\} = V(P_{k+1}) \backslash V(P_k)$ is calculated.
- $\mathcal{W}$ listed the indices of all vertices of P_{k+1} contained in $\{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \ell_k(\boldsymbol{x}, t) = 0\}$ is constructed. Hence, $\mathcal{W} = \{\alpha(k)+1, \ldots, \alpha(k+1)\} \cup \{i \in \Delta_k : \ell_k(\boldsymbol{v}(i), t(i)) = 0\}$. We note that for each $i \in \mathcal{W}$, $\mathcal{V}_i$ needs to be generated or updated.
- For each $i \in \Delta_k \backslash \{i \in \Delta_k : \ell_k(\boldsymbol{v}(i), t(i)) = 0\}$, $\mathcal{V}_i$ is updated completely.
- For each $i \in \Delta_{k+1}$, $\mathcal{F}_i$ is updated.

5.3. **CHECK FOR THE AVAILABILITY OF FACETS.** In this section, we propose a procedure for checking the availability of facets F_i ($i \in \Gamma_k \cup \{\beta(k+1)\}$) for P_{k+1}. Then, it is clear that the following theorem holds.

Theorem 5.2. *Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polyhedral set and let $F(1), \ldots, F(\beta)$ be facets of P. Then, for each $i \in \{1, \ldots, \beta\}$ and $j \in \{1, \ldots, \beta\} \backslash \{i\}$, $F_i \cap P_k \not\subset F_j$.*

By Theorem 5.2, we propose a procedure for omitting $i \in \Gamma_k$ satisfying $F_i \cap V(P_{k+1}) \subset F_j$ for some $j \in (\Gamma_k \cup \{\beta(k+1)\}) \backslash \{i\}$ from $\Gamma_k \cup \{\beta(k+1)\}$ as follows.

Procedure C

Step 0: Set $\Gamma_{k+1} := \Gamma_k \cup \{\beta(k+1)\}$ and $i := 1$. Go to Step 1.

Step 1: If $i = \beta(k+1)$, then stop. If $i \notin \Gamma_{k+1}$, go to Step 4. Otherwise, go to Step 2.

Step 2: Set $\Psi_i := \{q \in \Delta_{k+1} : i \in \mathcal{F}_q\}$. Go to Step 3.
Step 3:
Step 3-0: Set $j := i + 1$ and go to Step 3-1.
Step 3-1: If $j = \beta(k+1) + 1$, then go to Step 4. If $j \notin \Gamma_{k+1}$, then go to Step 3-4. Otherwise, go to Step 3-2.
Step 3-2: Set $\Psi_j := \{q \in \Delta_{k+1} : j \in \mathcal{F}_q\}$. Go to Step 3-3.
Step 3-3: If $\Psi_i \subset \Psi_j$, then set $\Gamma_{k+1} \leftarrow \Gamma_{k+1} \backslash \{i\}$, $\mathcal{F}_q \leftarrow \mathcal{F}_q \backslash \{i\}$ for each $q \in \Delta_{k+1}$, and go to Step 4. Otherwise, go to Step 3-4.
Step 3-4: Set $j \leftarrow j + 1$ and return to Step 3-1.
Step 4: Set $i \leftarrow i + 1$ and return to Step 1.

From Procedure C, for each $i \in \Gamma_{k+1}$,

$$\dim\,(P_{k+1} \cap F_i) = n - 1.$$

5.4. **UPDATE THE INDEX SETS.** Procedure B generates $V(P_k)$ by utilizing $\mathcal{V}_i$ ($i \in \mathcal{M}$), where $\mathcal{M}$ is constructed by Procedure A. Hence, in order to generate $V(P_{k+2})$ at iteration $k+1$ of Algorithm IOA in the same way, it is necessary to construct $\mathcal{V}_{\alpha(k)+1}, \ldots, \mathcal{V}_{\alpha(k+1)}$, and to update $\mathcal{V}_i$ for each $i \in \{i \in \Gamma_k : \ell_k(\boldsymbol{v}(i), t(i)) = 0\}$. Then, the following theorem holds.

Theorem 5.3. *Let $P \subset \mathbb{R}^n$ be an n-dimensional convex polyhedral set and let $\{F_i :\in \Gamma\}$ be the set of all facets of P, where Γ is the index set. The line segment $\mathrm{co}\,\{\boldsymbol{v}', \boldsymbol{v}''\}$ ($\boldsymbol{v}', \boldsymbol{v}'' \in V(P)$, $\boldsymbol{v}' \neq \boldsymbol{v}''$) is an edge of P if and only if there exist $i_1, \ldots, i_{n-1} \in \Gamma$ such that $\mathrm{co}\,\{\boldsymbol{v}', \boldsymbol{v}''\} \subset F_{i_j}$ for each $j = 1, \ldots, n-1$.*

By Procedure C, $\{F_i : i \in \Gamma_{k+1}\}$ is the set all facets of P_{k+1}. Hence, to construct $\mathcal{V}_i$ for each $i \in \mathcal{W}$ ($\mathcal{W}$ is generated by Procedure B), we propose the following procedure.

Procedure D
Step 0: Set $j = 1$ and go to Step 1.
Step 1: If $j = \alpha(k+1)$, then stop. If $j \notin \mathcal{W}$, then go to Step 3. Otherwise, go to Step 2.
Step 2:
Step 2-0: Set $q := j + 1$ and go to Step 2-1.
Step 2-1: If $q = \alpha_{k+1}$, then go to Step 3. If $q \notin \mathcal{W}$, then go to Step 2-3. Otherwise, go to Step 2-2.
Step 2-2: If $|\mathcal{F}_j \cap \mathcal{F}_q| = n - 1$, set $\mathcal{V}_j \leftarrow \mathcal{V}_j \cup \{q\}$ and $\mathcal{V}_q \leftarrow \mathcal{V}_q \cup \{j\}$. Go to Step 2-3.
Step 2-3: Set $q \leftarrow q + 1$ and return to Step 2-1.
Step 3: Set $j \leftarrow j + 1$ and return to Step 1.

By Procedures B and D, all index sets $\mathcal{V}_i$ ($i \in \Delta_{k+1}$) are updated completely.

6. COMPUTATIONAL EXPERIMENTS

In order to investigate the efficiency of Algorithm IOA, we did numerical experiments for the following problem:

$$\begin{cases} \text{minimize} & \left(\dfrac{1}{2}\displaystyle\sum_{i=1}^{n} f_i^1 x_i^2 - \sum_{j=1}^{n} f_j^2 x_j + f\right) - \left(\dfrac{1}{2}\displaystyle\sum_{i=1}^{n} g_i^1 x_i^2 - \sum_{j=1}^{n} g_j^2 x_j + g\right) \\ \text{subject to} & \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} a_i (x_i - b_i)^2 - c \leq 0,\ \boldsymbol{x} \in \mathbb{R}^n, \end{cases} \tag{6.1}$$

where $f_j^i, g_j^i, a_j, b_j, f, g, c \in [1, 10]$ ($i = 1, 2,\ j = 1, \dots, n$) are defined by a random number generator. For each n satisfying $1 \leq n \leq 8$, we generated 60 problems in the form of (6.1). The numerical experiments were performed by a computer (CPU: Xeron MP 3.33GHz–8MB×3, RAM: 6GB). Algorithm IOA was encoded by the C language on Linux. The computational result of Algorithms IOA for such problems are written in Table 1, where the tolerance is $\tau = 0.001$. Here, $|V(P_{\text{last}})|$ in Table 1 denotes the number of vertices of P_k at the last iteration.

TABLE 1. Computational results of Algorithm IOA

n	Number of iteration		CPU-time(sec)		$\|V(P_{\text{last}})\|$	
	Average	Standard deviation	Average	Standard deviation	Average	Standard deviation
1	3.379	1.495	0.009	0.032	6.379	1,473
2	15.917	5.460	0.000	0.001	34.933	10.535
3	50.950	87.218	0.002	0.011	256.017	511.184
4	68.617	26.764	0.009	0.009	907.633	448.863
5	151.879	98.777	0.351	0.608	7166.828	6302.436
6	227.076	159.698	10.464	26.017	40650.830	41346.200
7	318.729	172.307	586.560	1005.900	233899.100	186763.300
8	243.350	75.510	4626.178	4006.415	540451.800	264155.460

The computational result shows that the proposed algorithm is effective for not so large size problems.

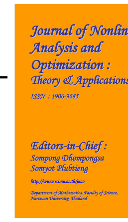
7. CONCLUSIONS

In this paper, we have improved Tuy's outer approximation algorithm for solving (DC). Both Tuy's algorithm and the improved algorithm generate the sequences of polytopes to guarantee the global convergence. The sequence of polytopes generated by Tuy's algorithm approximates the sequence of compact convex sets. On the other hands, the improved algorithm generates the sequence of polytopes approximating a compact convex set from outside. By fixing the target set approximated by the sequence of polytopes, the efficiency of the proposed algorithm has been upgraded. Moreover, to improve the computational efficiency of the algorithm, we have proposed a procedure for calculating the vertex sets. By utilizing the relation of connections among vertices by edges, the proposed procedure calculates the vertex sets without solving systems of linear equations.

REFERENCES

1. M. Avriel and A.C. Williams, Complementary Geometric Programming, SIAM J. Appl. Math. 19 (1970) 125–141.
2. R.J. Hillestad and S.E. Jacobsen, Reverse Convex Programming, Appl. Math. Optim. 6 (1980) 63–78.
3. R. Meyer, The Validity of a Family of Optimization Methods, SIAM J. Control 8 (1970) 42–54.
4. R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, N.J., 1970.
5. J.B. Rosen, Iterative Solution of Nonlinear Optimal Control Problems, SIAM J. Control 4 (1966) 223–244.
6. N.V. Thoai, A Modified Version of Tuy's Method for Solving DC Programming, Optim. 19 (1988) 665–674.
7. H. Tuy, Convex Programs with an Additional Reverse Convex Constraint, J. Optim. Theory Appl. 52 (1987) 463–486.
8. H. Tuy, Convex Analysis and Global Optimization, Kluwer Academic Publishers, 1998.

9. H. Tuy, Canonical DC Programming: Outer Approximation Methods Revisited, Oper. Res. Let. 18 (1995) 99-106.
10. H. Tuy, On Global Optimality Conditions and Cutting Plane Algorithms, J. Optim. Theory Appl. 118 (2003) 201–216.
11. U.A. Ueing, A Combinatorial Method to Compute a Global Solution of Certain Nonconvex Optimization Problems, Numerical Methods for Nonlinear Optimization Edited by F.A. Lootsma, Academic Press, New York (1972) 223–230.
12. A.B. Zaleesky, Nonconvexity of Feasible Domains and Optimization of Management Decision (in Russian), Ekonomika i Matematitcheskie Metody 16 (1980) 1069–1081.



Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 13-23
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

ITERATIVE ALGORITHM FOR A SYSTEM OF MULTI-VALUED VARIATIONAL INCLUSIONS INVOLVING (B, ϕ)-MONOTONE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this paper, we introduce a new class of resolvent mappings for (B, ϕ)-monotone mappings in Banach space, which is a natural and important generalization of a class of resolvent mappings studied in [X.-P. Luo, N.-J. Huang; A new class of variational inclusions with B-monotone operators in Banach spaces, J. Comput. Appl. Math. 233 (2010), 1888-1896]. We study some properties of this new class of resolvent mappings and by making use of it, we discuss the existence and iterative approximation of solutions of a system of multi-valued variational inclusions. The method and results presented in this paper improve and generalize many known results in the literature.

KEYWORDS : System of multi-valued variational inclusions; (B, ϕ)-monotone mappings; Iterative algorithm and convergence analysis.
AMS Subject Classification: 47J22, 49J40.

1. INTRODUCTION

In 1968, Brézis [5] initiated the study of the existence theory of a class of variational inequalities later known as variational inclusions, using resolvent (proximal) mapping. Variational inclusions include variational inequalities as special cases. For applications of variational inclusions, see [4, 10]. In 1994, Hassouni and Moudafi [15] discussed iterative approximation of solutions for an important class of variational inclusions using resolvent mapping. Since then various resolvent mappings have been introduced and used to develop the iterative methods for studying the existence and iterative approximation of solutions of variational inclusions, see for example [1–3, 6–9, 11–14, 16, 18–26, 29–35].

Very recently Luo and Huang [25, 26] introduced and studied the classes of (H, ϕ)-η-monotone mappings and B-monotone mappings in Banach space, respectively, and discussed their properties. Using these classes of resolvent mappings for

[1] Supported by University Grants Commission, Government of India, under the Major Research Project No. F.36-7/2008 (SR)..
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Article history : Received 5 January 2012. Accepted 11 March 2012.

(H, ϕ)-η-monotone mappings and B-monotone mappings, they studied the convergence analysis of the iterative algorithms for some classes of variational inclusions.

In this paper, we introduce a class of resolvent mappings for (B, ϕ)-monotone mappings in Banach space, which is a natural and important generalization of the class of resolvent mappings studied in [26]. We study some properties of this new class of resolvent mappings and by making use of it, we discuss the existence and iterative approximation of solutions of a system of multi-valued variational inclusions. The method and results presented in this paper improve and generalize many known results in the literature.

2. PRELIMINARIES

Let X be a real Banach space with the topological dual space X^* and $\langle \cdot, \cdot \rangle$ denote the dual pair between X and its dual X^* and 2^X denote the family of all nonempty subsets of X. The *normalized duality* mapping $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) = \{f \in X^*, \ \langle f, x\rangle = \|f\|\|x\|, \ \|f\| = \|x\|\}, \ \forall x \in X.$$

The *modulus of smoothness* of X is the function $\rho_X : [0, \infty) \longrightarrow [0, \infty)$ defined by

$$\rho_X(t) = \sup\Big\{\frac{(\|x+y\| + \|x-y\|)}{2} - 1 : \ x, y \in X, \|x\| = 1, \|y\| = t\Big\}.$$

A Banach space X is called *uniformly smooth* if

$$\lim_{t \longrightarrow 0} \frac{\rho_X(t)}{t} = 0.$$

We denote by $CB(X)$ the family of all nonempty, closed and bounded subsets of X and $\mathcal{D}(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{D}(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\}, \ A, B \in CB(X).$$

Definition 2.1. [31] Let $A : X \longrightarrow X^*$ be a single-valued mapping. A is said to be:

(i) *monotone*, if

$$\langle A(x) - A(y), x - y\rangle \geq 0, \ \forall x, y \in X;$$

(ii) *strictly monotone*, if

$$\langle A(x) - A(y), x - y\rangle > 0, \ \forall x, y \in X,$$

and equal to 0 if and only if $x = y$;

(iii) γ-*strongly monotone*, if there exists a constant $\gamma > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq \gamma\|x - y\|, \ \forall x, y \in X;$$

(iv) m-*relaxed monotone*, if there exists a constant $m > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq -m\|x - y\|, \ \forall x, y \in X;$$

(v) δ-*Lipschitz continuous*, if there exists a constant $\delta > 0$ such that

$$\|A(x) - A(y)\| \leq \delta\|x - y\|, \ \forall x, y \in X.$$

Definition 2.2. [26] Let $B : X \longrightarrow X^*$, $\phi : X^* \longrightarrow X^*$, $f, g : X \longrightarrow X$ be single-valued mappings, and let $M : X \times X \longrightarrow 2^{X^*}$ be a multi-valued mapping. Then

(i) $M(f,\cdot)$ is said to be α-*strongly monotone* with respect to f, if there exists a constant $\alpha > 0$ such that

$$\langle u-v, x-y\rangle \geq \alpha\|x-y\|^2,\ \forall x,y,w \in X, u \in M(f(x),w), v \in M(f(y),w);$$

(ii) $M(\cdot,g)$ is said to be β-*relaxed monotone* with respect to g if there exists a constant $\beta > 0$ such that

$$\langle u-v, x-y\rangle \geq -\beta\|x-y\|^2,\ \forall x,y,w \in X, u \in M(w,g(x)), v \in M(w,g(y));$$

(iii) $M(\cdot,\cdot)$ is said to be $\alpha\beta$-*symmetric monotone* with respect to f and g if $M(f,\cdot)$ is α-strongly monotone with respect to f and $M(\cdot,g)$ is β-relaxed monotone with respect to g with $\alpha \geq \beta$ and $\alpha = \beta$ if and only $x = y$ $\forall x,y \in X$.

Lemma 2.3. *[28] Let X be a real Banach space and let $J : X \longrightarrow 2^{X^*}$ be the normalized duality mapping. Then for any given $x,y \in X$, the following inequality holds:*

$$\|x+y\|^2 \ \leq\ \|x\|^2 + 2\langle y, j(x+y)\rangle,\ \forall j(x+y) \in J(x+y).$$

Lemma 2.4. *[27] Let X be a complete metric space; let $C(X)$ be the family of all nonempty compact subsets of X and let $W : X \longrightarrow C(X)$ be a multi-valued mapping. Then for any given $x,y \in X,\ u \in W(x)$, there exists $v \in W(y)$ such that*

$$d(u,v) \leq \mathcal{D}(W(x),W(y)),$$

where $\mathcal{D}(\cdot,\cdot)$ is Hausdorff metric on $C(X)$.

Lemma 2.5. *[27] Let X be a complete metric space and let $W : X \longrightarrow CB(X)$ be a multi-valued mapping. Then for any $\epsilon > 0$ and for any given $x,y \in X,\ u \in W(x)$, there exists $v \in W(y)$ such that*

$$d(u,v) \leq (1+\epsilon)\mathcal{D}(W(x),W(y)),$$

where $\mathcal{D}(\cdot,\cdot)$ is Hausdorff metric on $CB(X)$.

3. (B,ϕ)-MONOTONE MAPPINGS

First, we define the notion of (B,ϕ)-monotone mappings.

Definition 3.1. Let $B : X \longrightarrow X^*,\ \phi : X^* \longrightarrow X^*,\ f,g : X \longrightarrow X$ be single-valued mappings, and let $M : X \times X \longrightarrow 2^{X^*}$ be a multi-valued mapping. Then M is said to be (B,ϕ)-monotone if $\phi \circ M$ be $\alpha\beta$-*symmetric monotone* with respect to f and g, and $(B + \phi \circ M(f,g))(X) = X^*$.

Remark 3.2. (i) If $\phi \circ M(f,g) = \lambda M(f,g)$, for $\lambda > 0$ and M be $\alpha\beta$-*symmetric monotone* with respect to f and g, then (B,ϕ)-monotone mapping reduces to the B-monotone mapping considered in [26].

(ii) If $\phi \circ M(f,g) = \lambda M$, for $\lambda > 0$ and M be relaxed monotone then (B,ϕ)-monotone mapping reduces to the A-monotone mapping considered in [11].

Theorem 3.3. *Let $f,g : X \longrightarrow X$ and $\phi : X^* \longrightarrow X^*$ be single-valued mappings; let $B : X \longrightarrow X^*$ be a strictly monotone mapping and let $M : X \times X \longrightarrow 2^{X^*}$ be a (B,ϕ)-monotone mapping. If $\langle u-v, x-y\rangle \geq 0$ holds for all $(y,v) \in$ Graph $(\phi \circ M(f,g))$, then $u \in \big(\phi \circ M(f,g)\big)(x)$, where Graph$(\phi \circ M(f,g)) = \{(x,x^*) \in X \times X^* :\ x^* \in (\phi \circ M(f,g))(x)\}$.*

Proof. Suppose that there exists (x_0, u_0) such that

$$\langle u_0 - v, x_0 - y\rangle \geq 0,\ \forall (y, v) \in \textit{Graph}\ (\phi \circ M(f, g)). \tag{3.1}$$

Since M is (B, ϕ)-monotone, we know that $(B+\phi\circ M(f, g))(X) = X^*$ and so there exists $(x_1, u_1) \in \textit{Graph}\ (\phi \circ M(f, g))$ such that

$$B(x_1) + u_1 = B(x_0) + u_0. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$0 \leq \langle u_0 - u_1, x_0 - x_1\rangle = -\langle B(x_0) - B(x_1), x_0 - x_1\rangle.$$

But the strictly monotonicity of B implies that $x_1 = x_0$. By (3.2) we also observe that $u_1 = u_0$. Hence $(x_0, u_0) \in \textit{Graph}\ (\phi \circ M(f, g))$, that is, $u_0 \in \big(\phi \circ M(f, g)\big)(x_0)$. This completes the proof. □

Theorem 3.4. *Let $f, g : X \longrightarrow X$ and $\phi : X^* \longrightarrow X^*$ be single-valued mappings; let $B : X \longrightarrow X^*$ be a strictly monotone mapping and let $M : X \times X \longrightarrow 2^{X^*}$ be a (B, ϕ)-monotone mapping. Then $(B + \phi \circ M(f, g))^{-1}$ be a single-valued mapping.*

Proof. For any $x^* \in X^*$, let $x, y \in (B + \phi \circ M(f, g))^{-1}(x^*)$, then it follows that

$$x^* - B(x) \in \phi \circ M(f(x), g(x))$$
$$x^* - B(y) \in \phi \circ M(f(y), g(y)).$$

Pick any given $w \in \phi \circ M(f(y), g(x))$, since $\phi \circ M$ be *$\alpha\beta$-symmetric monotone* with respect to f and g, we have

$$(\alpha - \beta)\|x - y\|^2 \leq \langle x^* - B(x) - w + w - (x^* - B(y)), x - y\rangle.$$

It follows from $\alpha \geq \beta$ and the strictly monotonicity of B that $x = y$. Thus $(B + \phi \circ M(f, g))^{-1}$ be a single-valued mapping. This completes the proof. □

Based on Theorems 3.3-3.4, we can define the following resolvent mapping $R^B_{M(\cdot,\cdot),\phi}$.

Definition 3.5. Let X be a reflexive Banach space with the dual space X^*. Let $f, g : X \longrightarrow X$, $\phi : X^* \longrightarrow X^*$ be single-valued mappings; let $B : X \longrightarrow X^*$ be a σ-strongly monotone mapping and let $M : X \times X \longrightarrow 2^{X^*}$ be a (B, ϕ)-monotone mapping. A resolvent mapping $R^B_{M(\cdot,\cdot),\phi} : X^* \longrightarrow X$ is defined by $R^B_{M(\cdot,\cdot),\phi}(x^*) = (B + \phi \circ M(f, g))^{-1}(x^*),\ \forall x^* \in X^*$.

Theorem 3.6. *Let X be a reflexive Banach space with the dual space X^*. Let $f, g : X \longrightarrow X$, $\phi : X^* \longrightarrow X^*$ be single-valued mappings; let $B : X \longrightarrow X^*$ be a σ-strongly monotone mapping and let $M : X \times X \longrightarrow 2^{X^*}$ be a (B, ϕ)-monotone mapping. Then the resolvent mapping $R^B_{M(\cdot,\cdot),\phi} : X^* \longrightarrow X$ is Lipschitz continuous with constant $\frac{1}{(\alpha-\beta+\sigma)}$, i.e.,*

$$\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\| \leq \frac{1}{(\alpha - \beta + \sigma)}\|x^* - y^*\|,\ \forall x^*, y^* \in X^*.$$

Proof. Let $x^*, y^* \in X^*$. It follows that

$$R^B_{M(\cdot,\cdot),\phi}(x^*) = (B + \phi \circ M(f, g))^{-1}(x^*)$$
$$R^B_{M(\cdot,\cdot),\phi}(y^*) = (B + \phi \circ M(f, g))^{-1}(y^*)$$

and so

$$x^* - B(R^B_{M(\cdot,\cdot),\phi}(x^*)) \in \phi \circ M\big(f(R^B_{M(\cdot,\cdot),\phi}(x^*)), g(R^B_{M(\cdot,\cdot),\phi}(x^*))\big)$$

$$y^* - B(R^B_{M(\cdot,\cdot),\phi}(y^*)) \in \phi \circ M\big(f(R^B_{M(\cdot,\cdot),\phi}(y^*)), g(R^B_{M(\cdot,\cdot),\phi}(y^*))\big).$$

Pick any given $w \in \phi \circ M(f(R^B_{M\left(\cdot,\cdot\right),\phi}(y^*)), g(R^B_{M(\cdot,\cdot),\phi}(x^*))\big)$. Since $\phi \circ M$ be $\alpha\beta$-*symmetric monotone* with respect to f and g, we have

$$\begin{aligned}
&(\alpha-\beta)\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\|^2 \\
&\qquad \le \langle (x^* - B(R^B_{M(\cdot,\cdot),\phi}(x^*))) - w + w \\
&\qquad\quad -(y^* - B(R^B_{M(\cdot,\cdot),\phi}(y^*))), R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\rangle, \\
&\qquad = \langle x^* - y^* - B(R^B_{M(\cdot,\cdot),\phi}(x^*)) \\
&\qquad\quad +B(R^B_{M(\cdot,\cdot),\phi}(y^*)), R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\rangle.
\end{aligned}$$

Since B be a σ- strongly monotone mapping, then

$$\begin{aligned}
&\|x^* - y^*\|\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\| \\
&\qquad \ge \langle x^* - y^*, R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\rangle \\
&\qquad \ge (\alpha-\beta)\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\| \\
&\qquad\quad +\langle B(R^B_{M(\cdot,\cdot),\phi}(x^*) - B(R^B_{M(\cdot,\cdot),\phi}(y^*), R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\rangle \\
&\qquad \ge (\alpha-\beta+\sigma)\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\|^2
\end{aligned}$$

and so

$$\|R^B_{M(\cdot,\cdot),\phi}(x^*) - R^B_{M(\cdot,\cdot),\phi}(y^*)\| \le \frac{1}{(\alpha-\beta+\sigma)}\|x^* - y^*\|, \ \forall x^*, y^* \in X^*.$$

This completes the proof. □

4. SYSTEM OF MULTI-VALUED VARIATIONAL INCLUSIONS

Throughout the rest of this paper, unless otherwise stated for each $i = 1, 2$, we assume that X_i be a real Banach space with norm $\|\cdot\|_i$ and denote the duality pairing between X_i and X_i^* by $\langle\cdot,\cdot\rangle_i$. Let $A_i : X_i \longrightarrow X_i^*,\ p_i, f_i, g_i : X_i \longrightarrow X_i,\ F_i : X_i \times X_i \longrightarrow X_i$ be single-valued mappings and $M_i : X_i \times X_i \longrightarrow 2^{X_i^*},\ W_i : X_1 \longrightarrow CB(X_1)$ and let $V_i : X_2 \longrightarrow CB(X_2)$ be multi-valued mappings. We shall study the following system of multi-valued variational inclusions (in short, SMVI). For given $\theta_i \in X_i^*$, the zero element, find $(x_1, x_2) \in (X_1, X_2),\ w_i \in W_i(x_1),\ v_i \in V_i(x_2)$ such that

$$\begin{cases} \theta_1 \in A_1(x_1 - p_1(x_1)) + F_1(w_1, v_1) + M_1(f_1(x_1), g_1(x_1)) \\ \\ \theta_2 \in A_2(x_2 - p_2(x_2)) + F_2(w_2, v_2) + M_2(f_2(x_2), g_2(x_2)). \end{cases} \tag{4.1}$$

We remark that by giving suitable choices of mappings $A_i, p_i, F_i, M_i, f_i, g_i, W_i, V_i$ and of spaces $X_i,\ X_i^*,\ (i = 1, 2)$, we can observe that SMVI (4.1) reduces to many new and previously known systems of variational inclusions, systems of variational inequalities, variational inclusions and variational inequalities in Banach spaces as well as in Hilbert spaces, see for example, Kazmi and Khan [18], Kazmi and Khan [19], Kazmi and Bhat [17], Luo and Haung [25], Wang and Ding [30].

Theorem 4.1. *For each* $i = 1, 2$, *let* $\phi_i : X_i^* \longrightarrow X_i^*$ *be a single-valued mapping satisfying* $\phi_i(x_i + y_i) = \phi_i(x_i) + \phi_i(y_i)$ *and* $ker(\phi_i) = \{\theta_i\}$, *where* $ker(\phi_i) = \{x_i \in X_i^* : \phi_i(x_i) = \theta_i\}$. *Let* $A_i : X_i \longrightarrow X_i^*,\ p_i, f_i, g_i : X_i \longrightarrow X_i,\ F_i : X_i \times X_i \longrightarrow X_i$ *be single-valued mappings and let* $M_i : X_i \times X_i \longrightarrow 2^{X_i^*}$, $W_i : X_1 \longrightarrow CB(X_1)$ *and* $V_i : X_2 \longrightarrow CB(X_2)$ *be multi-valued mappings. Let* $B_i : X_i \longrightarrow X_i^*$ *be a* σ_i*-strongly monotone mapping. Then* $(x_1, x_2, w_1, w_2, v_1, v_2)$ *is a solution of SMVI (4.1) if and only if*

$$x_i = R^{B_i}_{M_i(\cdot,\cdot),\phi_i}[B_i(x_i) - \phi_i \circ (A_i(x_i - p_i(x_i)) + F_i(w_i, v_i))] \tag{4.2}$$

where $w_i \in W_i(x_1),\ v_i \in V_i(x_2)$ *and* $R^{B_i}_{M_i(\cdot,\cdot),\phi_i} = (B_i + \phi_i \circ M_i(f_i, g_i))^{-1}$.

Proof. By definition of $R^{B_i}_{M_i(\cdot,\cdot),\phi_i}$, we know that (4.2) holds if and only if

$$B_i(x_i) - \phi_i \circ (A_i(x_i - p_i(x_i)) + F_i(w_i, v_i)) \in (B_i + \phi_i \circ M_i(f_i, g_i))(x_i)$$

which is equivalent to

$$-\phi_i \circ (A_i(x_i - p_i(x_i)) + F_i(w_i, v_i)) \in \phi_i \circ M_i(f_i(x_i), g_i(x_i)).$$

It follows from $\phi_i(x_i + y_i) = \phi_i(x_i) + \phi_i(y_i)$ that (4.2) holds if and only if

$$\theta_i \in \phi_i \circ [A_i(x_i - p_i(x_i)) + F_i(w_i, v_i) + M_i(f_i(x_i), g_i(x_i))].$$

Since $ker(\phi_i) = \{\theta_i\}$, (4.2) holds if and only if

$$\theta_i \in A_i(x_i - p_i(x_i)) + F_i(w_i, v_i) + M_i(f_i(x_i), g_i(x_i)).$$

□

Based on Theorem 4.1, we construct the following iterative algorithm for solving SMVI (4.1):

Iterative Algorithm 4.1. For each $i = 1, 2$, given $x_i^0 \in X_i,\ w_i^0 \in W_i(x_1^0),\ v_i^0 \in V_i(x_2^0)$, compute the sequences $\{x_i^n\}, \{w_i^n\}, \{v_i^n\}$ defined by the iterative schemes:

$$x_i^{n+1} = R^{B_i}_{M_i(\cdot,\cdot),\phi_i}\Big[B_i(x_i^n) - \phi_i \circ [A_i(x_i^n - p_i(x_i^n)) + F_i(w_i^n, v_i^n)]\Big],$$

$$w_i^n \in W_i(x_1^n),\ \|w_i^{n+1} - w_i^n\| \le (1 + n^{-1})\mathcal{D}_1\Big(W_i(x_1^{n+1}), W_i(x_1^n)\Big),$$

$$v_i^n \in V_i(x_2^n),\ \|v_i^{n+1} - v_i^n\| \le (1 + n^{-1})\mathcal{D}_2\Big(V_i(x_1^{n+1}), V_i(x_1^n)\Big),$$

for all $n = 0, 1, 2, ...,$.

Now we give the sufficient conditions which guarantee the convergence of the iterative sequences generated by Iterative Algorithm 4.1.

Theorem 4.2. *For each* $i = 1, 2$, *let* X_i *be a reflexive Banach space with dual space* X_i^*. *Let* $\phi_i : X_i^* \longrightarrow X_i^*$ *be a* λ_i*-Lipschitz continuous satisfying* $\phi_i(x_i + y_i) = \phi_i(x_i) + \phi_i(y_i)$ *and* $ker(\phi_i) = \{\theta_i\}$. *Let* $f_i, g_i : X_i \longrightarrow X_i$ *be single-valued mappings; let* $B_i : X_i \longrightarrow X_i^*$ *be a* σ_i*-strongly monotone and* δ_i*-Lipschitz continuous mapping and let* $M_i : X_i \times X_i \longrightarrow 2^{X_i^*}$ *be a* (B_i, ϕ_i)*-monotone mapping. Let* $A_i : X_i \longrightarrow X_i^*$ *be* γ_i*-Lipschitz continuous mapping; let* $p_i : X_i \longrightarrow X_i$ *be a* m_i*-strongly accretive and* ξ_i*-Lipschitz continuous mapping; let* $W_i : X_1 \longrightarrow CB(X_1)$ *be a* $\mathcal{D}_1$*-Lipschitz continuous mapping with respect to* $\lambda_{W_i} > 0$ *and let* $V_i : X_2 \longrightarrow CB(X_2)$ *be a* $\mathcal{D}_2$*-Lipschitz continuous mapping with respect to* $\lambda_{V_i} > 0$ *respectively. Let the mapping*

$F_i : X_i \times X_i \longrightarrow X_i$ be a $\bar{\alpha}_i$-Lipschitz continuous in the first argument and $\bar{\beta}_i$-Lipschitz continuous in the second argument. Suppose that the following conditions are satisfied:

$$k_1 = \frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big[\delta_1+\lambda_1\bar{\alpha}_1\lambda_{W_1}+\lambda_1\gamma_1\sqrt{1-2m_1+2\xi_1(2+\xi_1)}+\lambda_1\bar{\beta}_1\lambda_{V_1}\Big]<1,$$
$$k_2 = \frac{1}{(\alpha_2-\beta_2+\sigma_2)}\Big[\delta_2+\lambda_2\bar{\beta}_2\lambda_{V_2}+\lambda_2\gamma_2\sqrt{1-2m_2+2\xi_2(2+\xi_2)}+\lambda_2\bar{\alpha}_2\lambda_{W_1}\Big]<1. \tag{4.3}$$

Then, for each $i=1,2$, the iterative sequences $\{x_i^n\},\{w_i^n\},\{v_i^n\}$ generated by Iterative Algorithm 4.1, converge strongly to x_1,x_2,w_1,w_2,v_1,v_2, respectively and $(x_1,x_2,w_1,w_2,v_1,v_2)$ is a solution of SMVI (4.1).

Proof. By Iterative Algorithm 4.1 and Theorem 3.6, we have

$$\begin{aligned}\Big\|x_1^{n+1}-x_1^n\Big\|_1 &= \Big\|R^{B_1}_{M_1(\cdot,\cdot),\phi_1}\Big[B_1(x_1^n)-\phi_1\circ[A_1(x_1^n-p_1(x_1^n))+F_1(w_1^n,v_1^n)]\Big]\\ &\quad -R^{B_1}_{M_1(\cdot,\cdot),\phi_1}\Big[B_1(x_1^{n-1})-\phi_1\circ[A_1(x_1^{n-1}-p_1(x_1^{n-1}))+F_1(w_1^{n-1},v_1^{n-1})]\Big]\Big\|_1\\ &\le \frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big\|\Big[B_1(x_1^n)-\phi_1\circ[A_1(x_1^n-p_1(x_1^n))+F_1(w_1^n,v_1^n)]-B_1(x_1^{n-1})\\ &\quad +\phi_1\circ[A_1(x_1^{n-1}-p_1(x_1^{n-1}))+F_1(w_1^{n-1},v_1^{n-1})]\Big]\Big\|_1\\ &\le \frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big(\|B_1(x_1^n)-B_1(x_1^{n-1})\|_1+\lambda_1\|A_1(x_1^n-p_1(x_1^n))-A_1(x_1^{n-1}-p_1(x_1^{n-1}))\|_1\\ &\quad +\lambda_1\|F_1(w_1^n,v_1^n)-F_1(w_1^{n-1},v_1^{n-1})\|_1\Big).\end{aligned} \tag{4.4}$$

By Lipschitz continuity of $\phi_1, A_1, B_1, F_1, W_1, V_1$, we have

$$\|B_1(x_1^n)-B_1(x_1^{n-1})\|_1 \le \delta_1\|x_1^{n+1}-x_1^n\|_1, \tag{4.5}$$

and

$$\begin{aligned}&\|F_1(w_1^n,v_1^n)-F_1(w_1^{n-1},v_1^{n-1})\|_1\\ &\le \|F_1(w_1^n,v_1^n)-F_1(w_1^{n-1},v_1^n)\|_1+\|F_1(w_1^{n-1},v_1^n)-F_1(w_1^{n-1},v_1^{n-1})\|_2\\ &\le \bar{\alpha}_1\|w_1^n-w_1^{n-1}\|_1+\bar{\beta}_1\|v_1^n-v_1^{n-1}\|_2\\ &\le \bar{\alpha}_1(1+n^{-1})\mathcal{D}_1\Big(W_1(x_1^n),W_1(x_1^{n-1})\Big)+\bar{\beta}_1(1+n^{-1})\mathcal{D}_2\Big(V_1(x_2^n),V_1(x_2^{n-1})\Big)\\ &\le \bar{\alpha}_1(1+n^{-1})\lambda_{W_1}\|x_1^n-x_1^{n-1}\|_1+\bar{\beta}_1(1+n^{-1})\lambda_{V_1}\|x_2^n-x_2^{n-1}\|_2\\ &\le (1+n^{-1})\Big[\bar{\alpha}_1\lambda_{W_1}\|x_1^n-x_1^{n-1}\|_1+\bar{\beta}_1\lambda_{V_1}\|x_2^n-x_2^{n-1}\|_2\Big],\end{aligned} \tag{4.6}$$

$$\|A_1(x_1^n-p_1(x_1^n))-A_1(x_1^{n-1}-p_1(x_1^{n-1}))\|_1 \le \gamma_1\|x_1^n-x_1^{n-1}-(p_1(x_1^n)-p_1(x_1^{n-1}))\|_1. \tag{4.7}$$

Since p_1 is a m_1-strongly accretive and ξ_1-Lipschitz continuous mapping, then we have

$$\begin{aligned}&\|x_1^n-x_1^{n-1}-(p_1(x_1^n)-p_1(x_1^{n-1}))\|_1^2\\ &\le \|x_1^n-x_1^{n-1}\|_1^2-2\Big\langle p_1(x_1^n)-p_1(x_1^{n-1}),j\Big((x_1^n-x_1^{n-1})-(p_1(x_1^n)-p_1(x_1^{n-1}))\Big)\Big\rangle\\ &\le \|x_1^n-x_1^{n-1}\|_1^2-2\Big\langle p_1(x_1^n)-p_1(x_1^{n-1}),j(x_1^n-x_1^{n-1})\Big\rangle\end{aligned}$$

$$+2\Big\langle p_1(x_1^n)-p_1(x_1^{n-1}),-j\Big((x_1^n-x_1^{n-1})-(p_1(x_1^n)-p_1(x_1^{n-1}))\Big)+j((x_1^n-x_1^{n-1})\Big\rangle$$
$$\leq \|x_1^n-x_1^{n-1}\|_1^2-2m_1\|x_1^n-x_1^{n-1}\|_1^2+2\|p_1(x_1^n)-p_1(x_1^{n-1}\|_1$$
$$\Big[\|x_1^n-x_1^{n-1}\|_1+\|p_1(x_1^n)-p_1(x_1^{n-1}\|_1+\|x_1^n-x_1^{n-1}\|_1\Big]$$
$$\leq \|x_1^n-x_1^{n-1}\|_1^2-2m_1\|x_1^n-x_1^{n-1}\|_1^2+2\xi_1(2+\xi_1)\|x_1^n-x_1^{n-1}\|_1^2$$
$$\leq \Big(1-2m_1+2\xi_1(2+\xi_1)\Big)\|x_1^n-x_1^{n-1}\|_1^2$$
$$\|x_1^n-x_1^{n-1}-(p_1(x_1^n)-p_1(x_1^{n-1}))\|_1 \leq \sqrt{1-2m_1+2\xi_1(2+\xi_1)}\|x_1^n-x_1^{n-1}\|_1. \quad (4.8)$$

It follows from (4.4)-(4.8), we have

$$\Big\|x_1^{n+1}-x_1^n\Big\|_1 \leq \frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big[\delta_1\|x_1^n-x_1^{n-1}\|_1+\lambda_1(1+n^{-1})\Big(\bar{\alpha}_1\lambda_{W_1}\|x_1^n-x_1^{n-1}\|_1$$
$$+\bar{\beta}_1\lambda_{V_1}\|x_2^n-x_2^{n-1}\|_2\Big)+\lambda_1\gamma_1\sqrt{1-2m_1+2\xi_1(2+\xi_1)}\|x_1^n-x_1^{n-1}\|_1\Big].$$

Hence we have

$$\Big\|x_1^{n+1}-x_1^n\Big\|_1 \leq \frac{1}{(\alpha_1-\beta_1+\sigma_1)}\Big[\Big(\delta_1+\lambda_1\bar{\alpha}_1\lambda_{W_1}(1+n^{-1})+\lambda_1\gamma_1\sqrt{1-2m_1+2\xi_1(2+\xi_1)}\Big)$$
$$\times\|x_1^n-x_1^{n-1}\|_1+\lambda_1\bar{\beta}_1\lambda_{V_1}(1+n^{-1})\|x_2^n-x_2^{n-1}\|_2\Big)\Big]. \quad (4.9)$$

Next

$$\Big\|x_2^{n+1}-x_2^n\Big\|_2=\Big\|R^{B_2}_{M_2(\cdot,\cdot),\phi_2}\Big[B_2(x_2^n)-\phi_2\circ[A_2(x_2^n-p_2(x_2^n))+F_2(w_2^n,v_2^n)]\Big]$$
$$-R^{B_2}_{M_2(\cdot,\cdot),\phi_2}\Big[B_2(x_2^{n-1})-\phi_2\circ[A_2(x_2^{n-1}-p_2(x_2^{n-1}))+F_2(w_2^{n-1},v_2^{n-1})]\Big]\Big\|_2$$
$$\leq \frac{1}{(\alpha_2-\beta_2+\sigma_2)}\Big\|\Big[B_2(x_2^n)-\phi_2\circ[A_2(x_2^n-p_2(x_2^n))+F_2(w_2^n,v_2^n)]-B_2(x_2^{n-1})$$
$$+\phi_2\circ[A_2(x_2^{n-1}-p_2(x_2^{n-1}))+F_2(w_2^{n-1},v_2^{n-1})]\Big]\Big\|_2$$
$$\leq \frac{1}{(\alpha_2-\beta_2+\sigma_2)}\Big(\|B_2(x_2^n)-B_2(x_2^{n-1})\|_2+\lambda_2\|A_2(x_2^n-p_2(x_2^n))-A_2(x_2^{n-1}-p_2(x_2^{n-1}))\|_2$$
$$+\lambda_2\|F_2(w_2^n,v_2^n)-F_2(w_2^{n-1},v_2^{n-1})\|_2\Big). \quad (4.10)$$

By Lipschitz continuity of $\phi_2, A_2, B_2, F_2, W_2, V_2$, we have

$$\|B_2(x_2^n)-B_2(x_2^{n-1})\|_2 \leq \delta_2\|x_1^{n+1}-x_1^n\|_2, \quad (4.11)$$

and

$$\|F_2(w_2^n,v_2^n)-F_2(w_2^{n-1},v_2^{n-1})\|_2$$
$$\leq \|F_2(w_2^n,v_2^n)-F_2(w_1^{n-1},v_2^n)\|_1+\|F_2(w_2^{n-1},v_2^n)-F_2(w_2^{n-1},v_2^{n-1})\|_2$$
$$\leq \bar{\alpha}_2\|w_2^n-w_2^{n-1}\|_1+\bar{\beta}_2\|v_2^n-v_2^{n-1}\|_2$$
$$\leq \bar{\alpha}_2(1+n^{-1})\mathcal{D}_1\Big(W_2(x_1^n),W_2(x_1^{n-1})\Big)+\bar{\beta}_2(1+n^{-1})\mathcal{D}_2\Big(V_2(x_2^n),V_2(x_2^{n-1})\Big)$$
$$\leq \bar{\alpha}_2(1+n^{-1})\lambda_{W_2}\|x_1^n-x_1^{n-1}\|_1+\bar{\beta}_2(1+n^{-1})\lambda_{V_2}\|x_2^n-x_2^{n-1}\|_2$$
$$\leq (1+n^{-1})\Big[\bar{\alpha}_2\lambda_{W_2}\|x_1^n-x_1^{n-1}\|_1+\bar{\beta}_2\lambda_{V_2}\|x_2^n-x_2^{n-1}\|_2\Big] \quad (4.12)$$
$$\|A_2(x_2^n-p_2(x_2^n))-A_2(x_2^{n-1}-p_2(x_2^{n-1}))\|_2 \leq \gamma_2\|x_2^n-x_2^{n-1}-(p_2(x_2^n)-p_2(x_2^{n-1}))\|_2. \quad (4.13)$$

Since p_2 is a m_2-strongly accretive and ξ_2-Lipschitz continuous mapping, then we have

$$\|x_2^n - x_2^{n-1} - (p_2(x_2^n) - p_2(x_2^{n-1}))\|_2 \leq \sqrt{1 - 2m_2 + 2\xi_2(2 + \xi_2)}\|x_2^n - x_2^{n-1}\|_2. \quad (4.14)$$

It follows from (4.10)-(4.14), we have

$$\|x_2^{n+1} - x_2^n\|_2 \leq \frac{1}{(\alpha_2 - \beta_2 + \sigma_2)}\Big[\delta_2\|x_2^n - x_2^{n-1}\|_2 + \lambda_2(1+n^{-1})\Big(\bar{\alpha}_2\lambda_{W_2}\|x_1^n - x_1^{n-1}\|_1 + \bar{\beta}_2\lambda_{V_2}\|x_2^n - x_2^{n-1}\|_2\Big) + \lambda_2\gamma_2\sqrt{1 - 2m_2 + 2\xi_2(2 + \xi_2)}\|x_2^n - x_2^{n-1}\|_2\Big]$$

$$\|x_2^{n+1} - x_2^n\|_2 \leq \frac{1}{(\alpha_1 - \beta_1 + \sigma_1)}\Big[\Big(\delta_2 + \lambda_2\bar{\beta}_2\lambda_{V_2}(1+n^{-1}) + \lambda_2\gamma_2\sqrt{1 - 2m_2 + 2\xi_2(2 + \xi_2)}\Big) \times\|x_2^n - x_2^{n-1}\|_2 + \lambda_2\bar{\alpha}_2\lambda_{W_2}(1+n^{-1})\|x_1^n - x_1^{n-1}\|_1\Big)\Big]. \quad (4.15)$$

From (4.9) and (4.15), we have

$$\begin{aligned}\|(x_1^{n+1}, x_2^{n+1}) - (x_1^n, x_2^n)\|_* &= \|x_1^{n+1} - x_1^n\|_1 + \|x_2^{n+1} - x_2^n\|_2 \\ &\leq k_1^n\|x_1^n - x_1^{n-1}\|_1 + k_2^n\|x_2^n - x_2^{n-1}\|_2 \\ &\leq \max\{k_1^n, k_2^n\}\Big(\|x_1^n - x_1^{n-1}\|_1 + \|x_2^n - x_2^{n-1}\|_2\Big) \\ &\leq \max\{k_1^n, k_2^n\}\Big[\|(x_1^n, x_2^n) - (x_1^{n-1}, x_2^{n-1})\|_*\Big],\end{aligned} \quad (4.16)$$

where $X^* = X_1 \times X_2$ is a reflexive Banach space with norm $\|\cdot\|_* = \|\cdot\|_1 + \|\cdot\|_2$.

Letting $n \longrightarrow \infty$, we obtain $\max\{k_1^n, k_2^n\} \longrightarrow \max\{k_1, k_2\}$, where

$$k_1 = m_1 + \frac{1}{(\alpha_1 - \beta_1 + \sigma_1)}\lambda_1\bar{\beta}_1\lambda_{V_1};\ k_2 = m_2 + \frac{1}{(\alpha_2 - \beta_2 + \sigma_2)}\lambda_2\bar{\alpha}_2\lambda_{W_2}; \quad (4.17)$$

$$m_1 = \frac{1}{(\alpha_1 - \beta_1 + \sigma_1)}\Big(\delta_1 + \lambda_1\bar{\alpha}_1\lambda_{W_1} + \lambda_1\gamma_1\sqrt{1 - 2m_1 + 2\xi_1(2 + \xi_1)}\Big); \quad (4.18)$$

$$m_2 = \frac{1}{(\alpha_2 - \beta_2 + \sigma_2)}\Big(\delta_2 + \lambda_2\bar{\beta}_2\lambda_{V_2} + \lambda_2\gamma_2\sqrt{1 - 2m_2 + 2\xi_2(2 + \xi_2)}\Big). \quad (4.19)$$

By (4.3), it follows that $0 < \max\{k_1, k_2\} < 1$ and $\{(x_1^n, x_2^n)\}$ is a Cauchy sequence. Thus there exists $(x_1^*, x_2^*) \in X^*$ such that $(x_1^n, x_2^n) \longrightarrow (x_1^*, x_2^*)$ as $n \longrightarrow \infty$. Now we claim that $w_i^n \longrightarrow w_i \in W_i(x_1)$. In fact it follows from the Lipschitz continuity of W_i and Iterative Algorithm 4.1

$$\begin{aligned}\|w_i^{n+1} - w_i^n\| &\leq (1 + n^{-1})\mathcal{D}_1\Big(W_i(x_1^{n+1}), W_i(x_1^n)\Big) \\ &\leq (1 + n^{-1})\lambda_{W_i}\|x_1^{n+1} - x_1^n\|.\end{aligned} \quad (4.20)$$

Since $\{x_i^n\}$ is a Cauchy sequence, it follows that $\{w_i^n\}$ is also a Cauchy sequence. In a similar way, one can show that $\{v_i^n\}$ is a Cauchy sequence. Thus there exist $w_i \in X_1,\ v_i \in X_2$ such that $w_i^n \longrightarrow w_i,\ v_i^n \longrightarrow v_i$ as $n \longrightarrow \infty$. Further

$$\begin{aligned}d_i(w_i, W_i(x_1)) &\leq \|w_i - w_i^n\| + d_i(w_i^n, W_i(x_1)) \\ &\leq \|w_i - w_i^n\| + \mathcal{D}_1\Big(W_i(x_1^n), W_i(x_1)\Big) \\ &\leq \|w_i - w_i^n\| + \lambda_{W_i}\|x_1^n - x_1\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

Since $W_i(x_1) \in CB(X_1)$. It follows that $w_i \in W_i(x_1)$. Similarly we can show that $v_i \in V_i(x_2)$. By the continuity of $\phi_i, A_i, B_i, F_i, W_i, V_i, R^{B_i}_{M_i(\cdot,\cdot),\phi_i}$ and Iterative Algorithm 4.1, we have

$$x_i = R^{B_i}_{M_i(\cdot,\cdot),\phi_i}\Big[B_i(x_i) - \phi_i \circ [A_i(x_i - p_i(x_i)) + F_i(w_i, v_i)]\Big].$$

By Theorem 4.1, $(x_1, x_2, w_1, w_2, v_1, v_2)$ is a solution of problem SMVI (4.1). This is completes the proof. □

Remark 4.3. (i) If $\phi \circ M(f, g) = \lambda M(f, g)$, for $\lambda > 0$ and M be $\alpha\beta$-*symmetric monotone* with respect to f and g, then Theorem 3.3-3.6 reduce to Theorem 3.3-3.4 given in [26].

(ii) If $\phi \circ M(f, g) = \lambda M$, for $\lambda > 0$ and M be relaxed monotone then Theorem 3.3-3.6 reduce to Theorem 3.3-3.4 given in [11].

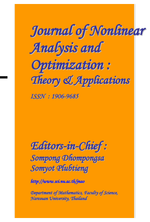
(iii) For each $i \in \{1, 2\}$, if $g_i = I_i$, identity operator on X_i; $\phi \circ M_i(f_i, g_i) = \lambda M_i(f_i, I_i)$, for $\lambda > 0$ and M be $\alpha_i\beta_i$-*symmetric monotone* with respect to f_i and I_i, then Theorem 4.2 is a generalization of Theorem 5.1 given in [11].

(iv) The method presented in this paper can be used to extend the results given in [30, 34, 35].

References

1. M. Alimohammady, J. Balooee, Y.J. Cho, M. Roohi, A new system of nonlinear fuzzy variational inclusions involving (A, η)-accretive mappings in uniformly smooth Banach spaces, J. Inequal. Appl. Vol. **2009**, Article ID 806727, (2009) 33 pages.
2. M. Alimohammady, J. Balooee, Y.J. Cho, M. Roohi, Iterative algorithms for a new class of extended general nonconvex set-valued variational inequalities, Nonlinear Anal. **73** (2010), 3907-3923.
3. M. Alimohammady, J. Balooee, Y.J. Cho, M. Roohi, New perturbed finite step iterative algorithms for a system of extended generalized nonlinear mixed-quasi variational inclusions, Comput. Math. Appl. **60** (2010), 2953-2970.
4. J.P. Aubin and A. Cellina, *Differential Inclusions,* Springer-Verlag, Berlin, 1984.
5. H. Brézis, Équations et inéquations non linéares dans les espaces vectoriels en dualité, Ann. Inst. Fourier **18**(1) (1968), 115-175.
6. Y.J. Cho, H.Y. Lan, A new class of generalized nonlinear multi-valued quasi-variational-like inclusions with H-monotone mappings, Math. Inequal. Appl. **10** (2007), 389–401.
7. Y.J. Cho, H.Y. Lan, Generalized nonlinear random (A, η)-accretive equations with random relaxed cocoercive mappings in Banach spaces, Comput. Math. Appl. 55(2008), 2173-2182.
8. Y.J. Cho, X. Qin, Systems of generalized nonlinear variational inequalities and its projection methods, Nonlinear Anal. **69** (2008), 4443-4451.
9. Y.J. Cho, X.L. Qin, M.J. Shang, Y. F. Su, Generalized nonlinear variational inclusions involving (A, η)-monotone mappings in Hilbert spaces, Fixed Point Theory Appl. Vol.**2007**, Article ID 29653, (2007), 6 pages.
10. V.E. Demyanov, G.E. Stavroutakis, L.N. Polyakov and P.D. Panagiotopolous, *Quasi-differentiability and Nonsmooth Modelling in Mechanics, Engineering and Economics*, Kluwer Academic Publishers, Dordrecht, 1996.
11. H.R. Feng and X.P. Ding, A new system of generalized nonlinear quasi-variational-like inclusions with A-monotone operator in Banach spaces, J. Comput. Math. Appl. **225** (2009), 365-373.
12. Y.P. Fang and N.J. Huang, H-monotone operator and resolvent operator technique for variational inclusions, Appl. Math. Comput. **145** (2003), 795-803.
13. Y.P. Fang and N.J. Huang, H-Accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces, Appl. Math. Lett. **17** (2004), 647-653.

14. Y.P. Fang and N.J. Huang, Iterative algorithms for a system of variational inclusion involving H-accretive operators in Banach spaces, Acta Math. Hungar. **108** (2005), 183-195.
15. A. Hassouni and A. Moudafi, A perturbed algorithm for variational inequalities, J. Math. Anal. Appl. **185** (1994), 706-712.
16. N.J. Huang and Y.P. Fang, Generalized m-accretive mappings in Banach spaces, J. Sichuang Univ. **38**(4) (2001), 591-592.
17. K.R. Kazmi and M.I. Bhat, Convergence and stability of iterative algorithms for some classes of general variational inclusions in Banach spaces, Southeast Asian Bull. Math. **32** (2008), 99-116.
18. K.R. Kazmi and F.A. Khan, Iterative approximation of a unique solution of a system of Variational-like inclusions in real q_i-uniformly smooth Banach spaces, Nonlinear Anal. **67**(3) (2007), 917-929.
19. K.R. Kazmi and F.A. Khan, Iterative approximation of a solution of multi-valued variational-like inclusion in Banach spaces : A P-η-proximal-point mapping approach, J. Math. Anal. Appl. **325**(1) (2007), 665-674.
20. H.Y. Lan, Y.J. Cho, Nan-jing Huang, Stability of iterative procedures for a class of generalized nonlinear quasi-variational-like inclusions involving maximal η-monotone mappings, Fixed Point Theory and Applications, Edited by Y.J. Cho, J.K. Kim and S.M. Kang, **6** (2006), 107-116.
21. H.Y. Lan, Y.J. Cho, R.U. Verma, On solution sensitivity of generalized relaxed cocoercive implicit quasivariational jnclusions with A-monotone mappings, J. Comput. Anal. Appl. **8** (2006), 75-87.
22. H.Y. Lan, Y.J. Cho, R.U. Verma, Nonlinear relaxed cocoercive variational inclusions involving (A,η)-accretive mappings in Banach spaces, Comput. Math. Appl. **51** (2006), 1529-1538.
23. H.Y. Lan, J.I. Kang, Y.J. Cho, Nonlinear (A,η)-monotone operator inclusion systems involving non-monotone set-valued mappings, Taiwan. J. Math. **11** (2007), 683-701.
24. H.Y. Lan, J.K. Kim, Y.J. Cho, On a new system of nonlinear A-monotone multivalued variational inclusions, J. Math. Anal. Appl. **327** (2007), 481-493.
25. X.P. Luo and N.J. Huang, (H,ϕ)-η-Monotone operators in Banach spaces with an application to variational inclusions, Appl. Math. Comput. **216** (2010), 1131-1139.
26. X.P. Luo and N.J. Huang, A new class of variational inclusions with B-monotone operators in Banach space, J. Comput. Appl. Math. **233** (2010), 1888-1896.
27. S.B. Nadler, Multivalued contraction mapping, Pacific J. Math. **30**(3) (1969), 457-488.
28. P.D. Panagiotopoulos, *Inequality Problem in Mechanics and Applications*. Birkhauser, Boston, 1985.
29. J.H. Sung, L.W. Zhang and X.T. Xiao, An algorithm based on resolvent operators for solving variational inequalities in Hilbert spaces, Nonlinear Anal. **69**(10) (2008), 3344-3357.
30. Z.B. Wang and X.P. Ding, $(H(.,.),\eta)$-Accretive operators with an application for solving set-valued variational inclusions in Banach spaces, J. Comput. Math. Appl. **59** (2010), 1559-1567.
31. F.Q. Xia and N.J. Huang, Variational inclusions with a general H-monotone operator in Banach spaces, Comput. Math. Appl. **54** (2007), 24-30.
32. Y. Yao, Y.J. Cho, Y. Liou, Iterative algorithms for variational inclusions, mixed equilibrium problems and fixed point problems approach to optimization problems, Central European J. Math. **9** (2011), 640-656.
33. Y. Yao, Y.J. Cho, Y. Liou, Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems, Europ. J. Operat. Research **212** (2011), 242-250.
34. Y.Z. Zau and N.J. Huang, $H(\cdot,\cdot)$-accretive operator with an application for solving variational inclusions in Banach spaces, Appl. Math. Comput. **204** (2008), 809-816.
35. Y.Z. Zau and N.J. Huang, A new system of variational inclusions involving $H(.,.)$-accretive operators in Banach spaces, Appl. Math. Comput. **212** (2009), 135-144.



Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 25-32
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

EXISTENCE RESULTS FOR A QUASILINEAR BOUNDARY VALUE PROBLEM INVESTIGATED VIA DEGREE THEORY

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ABSTRACT. In this article we prove the existence of at least one weak solution for the quasilinear problem

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x) + h(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p > 1$, $\Omega \subset \mathbb{R}^N$ is a non-empty bounded domain with Lipschitz boundary ($\Omega \in C^{0,1}$), λ is a positive parameter and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Carathéodory function. The approach is fully based on the degree theory.

KEYWORDS : p-Laplacian; Principal eigenvalue; (S_+) condition; Topological degree.
AMS Subject Classification: 35J60; 35B30; 35B40.

1. INTRODUCTION

The aim of this paper is to establish the existence of at least one weak solution for the following quasilinear problem

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x) + h(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p > 1$, $\Omega \subset \mathbb{R}^N$ is a non-empty bounded domain with Lipschitz boundary ($\Omega \in C^{0,1}$), λ is a positive parameter and $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Carathéodory function.

On the Sobolev space $W_0^{1,p}(\Omega)$, we consider the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p dx \right)^{\frac{1}{p}}.$$

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Article history : Received 4 January 2012. Accepted 30 March 2012.

By a (weak) solution of the problem (1.1), we mean any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \lambda\int_\Omega |u(x)|^{p-2}u(x)v(x)dx - \int_\Omega h(x,u(x))v(x) = 0$$

for all $v \in W_0^{1,p}(\Omega)$. It is well known that the eigenvalue problem

$$\begin{cases} -\Delta_p u(x) = \lambda|u(x)|^{p-2}u(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has a principal eigenvalue (i.e., the least one) $\lambda_1 > 0$ which is simple and characterized variationally by

$$\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^p\, dx}{\int_\Omega |u(x)|^p\, dx}.$$

Let X be a reflexive real Banach space and X^* its dual. Here and in the sequel we denote by $\langle f, u\rangle := f(u)$ the value of the linear form $f \in X^*$ for an element $u \in X$. If X is a Hilbert space, then according to the Riesz Representation Theorem, $\langle f, u\rangle = (u, f)$.

Definition 1.1. The operator $T : X \to X^*$ is said to satisfy the (S_+) condition, if the assumptions

$$u_n \rightharpoonup u_0 \quad \text{(weakly) in } X \qquad \text{and} \qquad \limsup_{n\to\infty}\langle T(u_n), u_n - u_0\rangle \leq 0$$

imply

$$u_n \to u_0 \quad \text{(strongly) in } X.$$

It is clear that if $T : X \to X^*$ satisfies the (S_+) condition and $K : X \to X^*$ be a compact operator, then the sum $T + K : X \to X^*$ satisfies the (S_+) condition. We say that $T : X \to X^*$ is demicontinuous, if T maps strongly convergent sequences in X to weakly convergent sequences in X^*.

The main aim of the present paper is to prove the existence of at least one weak solution of (1.1) via degree theory. Various applications of degree theory for solutions of nonlinear boundary value problems are already available, see for instance [1, 2, 4–6, 8]. For other basic notations and definitions we refer to [3].

2. MAIN RESULTS

First we here recall for reader's convenience the following Theorem of [9] which is our main tool to prove the results.

Theorem 2.1 (Skrypnik [9]). *Let $T : X \to X^*$ be a bounded and demicontinuous operator satisfying the (S_+) condition. Let $\mathcal{D} \subset X$ be an open, bounded and nonempty set with the boundary $\partial\mathcal{D}$ such that $T(u) \neq 0$ for $u \in \partial\mathcal{D}$. Then there exists an integer*

$$\deg(T, \mathcal{D}, 0)$$

(called the degree of the mapping T) such that

(i) $\deg(T, \mathcal{D}, 0) \neq 0$ *implies that there exists an element $u_0 \in \mathcal{D}$ such that*

$$T(u_0) = 0.$$

(ii) *If $\mathcal{D}$ is symmetric with respect to the origin and T satisfies $T(u) = -T(-u)$ for any $u \in \partial\mathcal{D}$, then*

$$\deg(T, \mathcal{D}, 0)$$

is an odd number (and thus different from zero).

(iii) (*Homotopy invariance property*) *Let* T_λ *be a family of bounded and demicontinuous mappings which satisfy the* (S_+) *condition and which depend continuously on a real parameter* $\lambda \in [0,1]$, *and let* $T_\lambda(u) \neq 0$ *for any* $u \in \partial\mathcal{D}$ *and* $\lambda \in [0,1]$. *Then*

$$\deg(T_\lambda, \mathcal{D}, 0)$$

is constant with respect to $\lambda \in [0,1]$. *In particular, we have*

$$\deg(T_0, \mathcal{D}, 0) = \deg(T_1, \mathcal{D}, 0).$$

We introduce the operators $J, G, S : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ in the following way

$$\langle J(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx,$$

$$\langle G(u), v\rangle := \int_\Omega |u(x)|^{p-2}u(x)v(x)dx,$$

$$\langle S(u), v\rangle := \int_\Omega h(x, u(x))v(x)dx$$

for any $u, v \in W_0^{1,p}(\Omega)$. First we sketch the properties of operators J, G and S.

Lemma 2.2. *The operators* J, G *and* S *are well defined. Also we have the following properties of* J, G *and* S.

(a) J, G *and* S *are bounded and continuous (and so demicontinuous) operators;*
(b) G *and* S *are compact operators;*
(c) J *satisfies the* (S_+) *condition;*
(d) J *is invertible and its inverse is continuous.*

Proof. The fact that J, G and S are well defined follows the standard procedure. The first two statements follows from the Hölder inequality, the boundedness of h and the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$. Let us prove the third statement. Indeed, let $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ and

$$\limsup_{n\to\infty}\langle J(u_n), u_n - u_0\rangle \leq 0.$$

Then $\lim_{n\to\infty}\langle J(u_0), u_n - u_0\rangle = 0$, and so

$$\begin{aligned}
0 \quad &\geq \quad \limsup_{n\to\infty}\langle J(u_n) - J(u_0), u_n - u_0\rangle \\
&= \quad \limsup_{n\to\infty}\int_\Omega \Big(|\nabla u_n(x)|^{p-2}\nabla u_n(x) - |\nabla u_0(x)|^{p-2}\nabla u_0(x)\Big)\Big(\nabla u_n(x) - \nabla u_0(x)\Big)dx \\
&\geq \quad \limsup_{n\to\infty}\Bigg\{\int_\Omega |\nabla u_n(x)|^p dx - \left(\int_\Omega |\nabla u_n(x)|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla u_0(x)|^p dx\right)^{\frac{1}{p}} \\
&\qquad - \left(\int_\Omega |\nabla u_0(x)|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla u_n(x)|^p dx\right)^{\frac{1}{p}} - \int_\Omega |\nabla u_0(x)|^p dx\Bigg\} \\
&= \quad \limsup_{n\to\infty}\Big[\|u_n\|^{p-1} - \|u_0\|^{p-1}\Big]\Big[\|u_n\| - \|u_0\|\Big] \geq 0,
\end{aligned}$$

where the last inequality follows from the fact that $s \mapsto |s|^{p-1}$ is strictly increasing on $(0,\infty)$. Hence $\|u_n\| \to \|u_0\|$, and due to the uniform convexity of $W_0^{1,p}(\Omega)$ we have $u_n \to u_0$ in $W_0^{1,p}(\Omega)$. Thus J satisfies the (S_+) condition.
Finally, we prove the fourth statement. Indeed, the strict monotonicity of $s \mapsto |s|^{p-2}$ implies that

$$\langle J(u) - J(v), u - v\rangle > 0 \qquad \text{for} \qquad u \neq v.$$

Hence J is injective. To prove that J^{-1} is continuous we proceed via contradiction. Suppose there exists a sequence $\{f_n\}_{n=1}^{\infty}$, $f_n \to f$ in $(W_0^{1,p}(\Omega))^*$ and

$$\|J^{-1}(f_n) - J^{-1}(f)\| \geq \delta \qquad \text{for a} \quad \delta > 0.$$

Let $u_n := J^{-1}(f_n)$ and $u := J^{-1}(f)$. It follows that

$$\|f_n\|\|u_n\| \geq \langle f_n, u_n\rangle = \langle J(u_n), u_n\rangle = \|u_n\|^p, \qquad \text{i.e.,} \qquad \|u_n\|^{p-1} \leq \|f_n\|.$$

We may then assume $u_n \rightharpoonup \tilde{u}$ in $W_0^{1,p}(\Omega)$ due to the reflexivity of $W_0^{1,p}(\Omega)$. Hence

$$\langle J(u_n) - J(\tilde{u}), u_n - \tilde{u}\rangle = \langle J(u_n) - J(u), u_n - \tilde{u}\rangle + \langle J(u) - J(\tilde{u}), u_n - \tilde{u}\rangle \to 0$$

since $J(u_n) \to J(u)$ in $(W_0^{1,p}(\Omega))^*$. Then we have

$$0 = \lim_{n\to\infty} \langle J(u_n) - J(\tilde{u}), u_n - \tilde{u}\rangle \geq \lim_{n\to\infty} \left[\|u_n\|^{p-1} - \|\tilde{u}\|^{p-1}\right]\left[\|u_n\| - \|\tilde{u}\|\right] \geq 0,$$

i.e., $\|u_n\| \to \|\tilde{u}\|$. Hence $u_n \to \tilde{u}$ follows due to the fact that $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space. Since J is continuous and injective, $\tilde{u} = u$, a contradiction. □

We state our main result as follows.

Theorem 2.3. *Let $\lambda < \lambda_1$ and let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a bounded Carathéodory function. Then the problem (1.1) has at least one weak solution.*

Proof. We set

$$T = J - \lambda G - S,$$

such that J, G and S are as above. Then existence of a weak solution of (1.1) is equivalent to the existence of a solution of the operator equation

$$T(u) = 0. \tag{2.1}$$

Our plan is to use the degree argument to prove the existence of a solution of (2.1). By Lemma 2.2, the operator T is a bounded and demicontinuous operator satisfying the (S_+) condition.
The operator J satisfies

$$\langle J(u), u\rangle = \|u\|^p.$$

Moreover, J and G are odd mappings and $(p-1)$-homogeneous, i.e.,

$$J(tu) = t^{p-1}J(u) \quad \text{and} \quad G(tu) = t^{p-1}G(u) \quad \text{for any} \quad t > 0, \quad u \in W_0^{1,p}(\Omega).$$

Our sketch is the following. The existence of at least one solution of (2.1) would follow from

$$\deg(J - \lambda G - S, B(0; R), 0) \neq 0 \tag{2.2}$$

if we found a ball $B(0; R)$ for which (2.2) is valid. To prove (2.2) we use the homotopy invariance property of the degree (Theorem 2.1(iii)) and connect the operator $J - \lambda G - S$ with the operator $J - \lambda G$ on the boundary of a ball $B(0; R)$ with a sufficiently large radius $R > 0$. Once this is done we finally use

$$\deg(J - \lambda G, B(0; R), 0) \neq 0. \tag{2.3}$$

(The value of the degree in (2.3) is an odd number according to Theorem 2.1(ii)). So, to complete the proof, we have to find an admissible homotopy connecting $J - \lambda G - S$ and $J - \lambda G$. We define a homotopy

$$T_\tau(u) := J(u) - \lambda G(u) - \tau S(u), \qquad \tau \in [0, 1], \qquad u \in W_0^{1,p}(\Omega).$$

It is enough to prove that there exists $R > 0$ such that for all $u \in W_0^{1,p}(\Omega)$, $\|u\| = R$ and $\tau \in [0, 1]$ we have

$$T_\tau(u) \neq 0. \tag{2.4}$$

Assume, by contradiction, that no such $R > 0$ exists, i.e., we can find sequence $\{u_n\}_{n=1}^{\infty} \subset W_0^{1,p}(\Omega)$ and $\{\tau_n\}_{n=1}^{\infty} \subset [0,1]$ such that $\|u_n\| \to \infty$ and

$$J(u_n) - \lambda G(u_n) - \tau_n S(u_n) = 0. \tag{2.5}$$

We set $v_n := \frac{u_n}{\|u_n\|^{p-1}}$, divide (2.5) by $\|u_n\|^{p-1}$ and use that J and G are $(p-1)$-homogenous to get

$$J(v_n) - \lambda G(v_n) - \tau_n \frac{S(u_n)}{\|u_n\|^{p-1}} = 0. \tag{2.6}$$

Due to the reflexivity of $W_0^{1,p}(\Omega)$ and the compactness of the interval $[0,1]$, passing to suitable subsequence, we may assume that

$$v_{n_k} \rightharpoonup v \quad \text{in } \ W_0^{1,p}(\Omega) \qquad \text{and} \qquad \tau_{n_k} \to \tau \in [0,1].$$

Let $M := \sup_{x\in\Omega,\, s\in\mathbb{R}} |h(x,s)|$. We have

$$\int_\Omega \frac{|h(x,u_{n_k}(x))|}{\|u_{n_k}\|^{p-1}}|v(x)|dx \le M \int_\Omega \frac{|v(x)|}{\|u_{n_k}\|^{p-1}}dx \le M_1 \frac{\|v\|}{\|u_{n_k}\|^{p-1}} \to 0 \qquad \text{as} \quad k\to\infty,$$

where $M_1 > 0$ is a constant. To summarize, since G is compact, we have

$$\tau_{n_k}\frac{S(u_{n_k})}{\|u_{n_k}\|^{p-1}} \to 0, \tag{2.7}$$

$$\lambda G(v_{n_k}) \to \lambda G(v), \tag{2.8}$$

in $(W_0^{1,p}(\Omega))^*$ as $k\to\infty$.

So, putting together (2.6)-(2.8) we also obtain that

$$J(v_{n_k}) \to \lambda G(v)$$

in $(W_0^{1,p}(\Omega))^*$ as $k\to\infty$, i.e.,

$$v_{n_k} \to J^{-1}(\lambda G(v))$$

in $W_0^{1,p}(\Omega)$ as $k\to\infty$ (Remember that J is invertible and its inverse is continuous). Since at the same time $v_{n_k} \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, we have

$$v_{n_k} \to v \qquad \text{in} \quad W_0^{1,p}(\Omega)$$

and

$$J(v) - \lambda G(v) = 0 \qquad \text{in} \quad (W_0^{1,p}(\Omega))^* \qquad \text{for a} \quad \tau\in[0,1]. \tag{2.9}$$

Since $\|v_{n_k}\| = 1$ for all $k = 1,2,\dots$, we have $\|v\| = 1$. However, this contradicts the assumption $\lambda < \lambda_1$. It proves that (2.4) holds, i.e., the homotopy T_τ is admissible. This completes the proof. □

3. THE CASE $p = 2$ AND $\lambda = \lambda_1$

Let us assume that $p = 2$ and consider the eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

It is known that the eigenvalues of (3.1) form an increasing sequence

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots, \qquad \lambda_n \to \infty.$$

In fact, it is also possible to prove that λ_1 has multiplicity 1 (i.e., $\lambda_1 < \lambda_2$) and the corresponding eigenfunction $\varphi_1 \in W_0^{1,2}(\Omega)$ is positive in Ω. Moreover, we have

$$\int_\Omega \nabla\varphi_1(x)\nabla v(x)dx = \lambda_1 \int_\Omega \varphi_1(x)v(x)dx, \tag{3.2}$$

for any $v \in W_0^{1,2}(\Omega)$.

Now, We formulate the following Theorem.

Theorem 3.1. *Let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a bounded Carathéodory function and satisfy the following conditions:*

(i) $\lim_{s\to+\infty} h(x,s) = h(x,+\infty)$, $\lim_{s\to-\infty} h(x,s) = h(x,-\infty)$, *for a.a.* $x \in \Omega$;

(ii) $h(x,-\infty) < h(x,s) < h(x,+\infty)$, *for a.a.* $x \in \Omega$ *and for all* $s \in \mathbb{R}$.

Then, the problem

$$\begin{cases} -\Delta u(x) = \lambda_1 u(x) + h(x,u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

has at least one weak solution if and only if

$$\int_\Omega h(x,-\infty)\varphi_1(x)dx < 0 < \int_\Omega h(x,+\infty)\varphi_1(x)dx. \tag{3.4}$$

Proof. For the sufficiency part we will follow a scheme similar to the proof of Theorem 2.3, but now $J, G, S : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ and

$$(J(u),v) := \int_\Omega \nabla u(x)\nabla v(x)dx = (u,v),$$

$$(G(u),v) := \int_\Omega u(x)v(x)dx,$$

$$(S(u),v) := \int_\Omega h(x,u(x))v(x)dx$$

for any $u, v \in W_0^{1,2}(\Omega)$. For $\delta > 0$ so small that $\lambda_1 + \delta < \lambda_2$ we define a homotopy

$$T_\tau(u) := u - \lambda_1 G(u) - (1-\tau)\delta G(u) - \tau S(u), \qquad \tau \in [0,1], \quad u \in W_0^{1,2}(\Omega).$$

Performing all steps as in the proof of Theorem 2.3 we arrive at an analogue of (2.9), namely,

$$v - \big[\lambda_1 + (1-\tau)\delta\big]G(v) = 0, \qquad \|v\| = 1, \qquad \text{for a} \quad \tau \in [0,1],$$

This is a contradiction if $\tau \neq 1$, since $\lambda_1 + (1-\tau)\delta$ is not an eigenvalue ($\lambda_1 < \lambda_1 + (1-\tau)\delta < \lambda_2$) and $v \neq 0$.

Let us assume $\tau = 1$, i.e., $\tau_{n_k} \to 1$. Now, however, we have no contradiction, since λ_1 is an eigenvalue and

$$v - \lambda_1 G(v) = 0$$

has a solution with $\|v\| = 1$. Another step is necessary to reach a contradiction and to prove that the homotopy T_τ is admissible. We have to revise the last step when passing to the limit in

$$v_n - \lambda_1 G(v_n) - (1-\tau_n)\delta G(v_n) - \tau_n \frac{S(u_n)}{\|u_n\|} = 0$$

and employ special properties of S. Namely,

$$u_{n_k} - \lambda_1 G(u_{n_k}) - (1-\tau_{n_k})\delta G(u_{n_k}) - \tau_{n_k} S(u_{n_k}) = 0$$

is equivalent to the integral identity

$$\int_\Omega \nabla u_{n_k}(x)\nabla w(x)dx = \big[\lambda_1 + (1-\tau_{n_k})\delta\big]\int_\Omega u_{n_k}(x)w(x)dx + \tau_{n_k}\int_\Omega h(x,u_{n_k}(x))w(x)dx \tag{3.5}$$

for all $w \in W_0^{1,2}(\Omega)$. Taking $w = \varphi_1$ in (3.5) and using the fact that

$$\int_\Omega \nabla u_{n_k}(x)\nabla\varphi_1(x)dx = \lambda_1 \int_\Omega u_{n_k}(x)\varphi_1(x)dx,$$

(see (3.2)), we obtain

$$(\tau_{n_k}-1)\delta\int_\Omega u_{n_k}(x)\varphi_1(x)dx = \tau_{n_k}\int_\Omega h(x,u_{n_k}(x))\varphi_1(x)dx. \tag{3.6}$$

As above, $v_{n_k} := \frac{u_{n_k}}{\|u_{n_k}\|} \to v$ in $W_0^{1,2}(\Omega)$ and $v = \kappa\varphi_1$ with a $\kappa \neq 0$. Assume that $\kappa > 0$. Since $v_{n_k} \to \kappa\varphi_1$ in $W_0^{1,2}(\Omega)$, by the compact embedding $W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, we have $v_{n_k} \to \kappa\varphi_1$ in $L^2(\Omega)$. Hence (at least for a subsequence) $v_{n_k}(x) \to \kappa\varphi_1 > 0$ a.e. in Ω, i.e., $u_{n_k}(x) \to +\infty$ a.e. in Ω. Passing to the limit in (3.6) and using $\tau_{n_k} \to 1_-$ and the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_\Omega h(x,+\infty)\varphi_1(x)dx = \lim_{k\to\infty}(\tau_{n_k}-1)\delta\int_\Omega u_{n_k}(x)\varphi_1(x)dx \le 0.$$

This contradicts the second inequality in (3.4). Similarly, if $\kappa < 0$, then (at least for a subsequence) $u_{n_k}(x) \to -\infty$ a.e. in Ω. Passing to the limit in (3.6), we obtain

$$\int_\Omega h(x,-\infty)\varphi_1(x)dx = \lim_{k\to\infty}(\tau_{n_k}-1)\delta\int_\Omega u_{n_k}(x)\varphi_1(x)dx \ge 0.$$

This contradicts the first inequality in (3.4). This proves that T_τ is admissible, and so (3.4) is sufficient for the existence of a weak solution of (3.3).

To prove that (3.4) is also necessary we proceed as follows. Let u_0 be a weak solution of (3.3), i.e.,

$$\int_\Omega \nabla u_0(x)\nabla v(x)dx = \lambda_1\int_\Omega u_0(x)v(x)dx + \int_\Omega h(x,u_0(x))v(x)dx,$$

for any $v \in W_0^{1,2}(\Omega)$. Take $v = \varphi_1$, then

$$\int_\Omega \nabla u_0(x)\nabla\varphi_1(x)dx = \lambda_1\int_\Omega u_0(x)\varphi_1(x)dx + \int_\Omega h(x,u_0(x))\varphi_1(x)dx.$$

Using (3.2), we have

$$\int_\Omega h(x,u_0(x))\varphi_1(x)dx = 0.$$

By assumption (ii),

$$h(x,-\infty) < h(x,u_0(x)) < h(x,+\infty). \tag{3.7}$$

Multiply (3.7) by $\varphi_1(>0)$ and integrate. Then

$$\int_\Omega h(x,-\infty)\varphi_1(x)dx < 0 < \int_\Omega h(x,+\infty)\varphi_1(x)dx,$$

and we have the result. □

Similarly to the proof of Theorem 3.1, we can prove the following

Theorem 3.2. *Let $h : \Omega\times\mathbb{R} \to \mathbb{R}$ be a bounded Carathéodory function and satisfy the following conditions:*

(i) $\lim_{s\to+\infty} h(x,s) = h(x,+\infty)$, $\quad\lim_{s\to-\infty} h(x,s) = h(x,-\infty)$, *for a.a.* $x\in\Omega$;

(ii) $h(x,+\infty) < h(x,s) < h(x,-\infty)$, *for a.a.* $x \in \Omega$ *and for all* $s \in \mathbb{R}$.

Then, the problem (3.3) *has at least one weak solution if and only if*

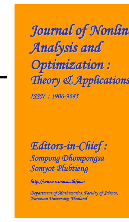
$$\int_\Omega h(x,+\infty)\varphi_1(x)dx < 0 < \int_\Omega h(x,-\infty)\varphi_1(x)dx. \tag{3.8}$$

Remark 3.3. It is possible to solve the problem (3.3) directly by means of the Leray-Schauder degree theory as well, since the operator J in the proof of Theorem 3.1 is just an identity on $W_0^{1,2}(\Omega)$.

References

1. F.E. Browder, *Nonlinear elliptic boundary value problems and the generalized topological degree,* Bull. Amer. Math. Soc. 76 (1970) 999-1005.
2. Y.Q. Chen, D. O'Regan, *Generalized degree theory for semilinear operator equations,* Glasgow Math. J. 48 (2006) 65-73.
3. K. Deimling, *Nonlinear Functional Analysis,* Springer Verlag, Berlin-Heidelberg-New York, 1985.
4. D. O'Regan, Y.J. Cho, Y.Q. Chen, *Topological Degree Theory and Applications,* Series in Mathematical Analysis and Applications, vol. 10, Chapman & Hall/CRC, Boca Raton-London-New York, 2006.
5. T. Pruszko, *Topological degree methods in multi-valued boundary value problems,* Nonlinear Anal. 9 (1981) 959-973.
6. T. Pruszko, *Some applications of the topological degree theory to multi-valued boundary value problems*, Dissert. Math. 229 (1984) 1-52.
7. I. Rachunkova, S. Stanek, *Topological degree method in functional boundary value problems,* Nonlinear Anal. 27 (1996) 153-166.
8. S.B. Robinson, E.M. Landesman, *A general approach to solvability conditions for semisimilar elliptic boundary value problems at resonance,* Differential Integral Equations 8(6) (1995) 1555-1569.
9. I.V. Skrypnik, *Nonlinear Elliptic Boundary Value Problems,* Teubner, Leipzig, 1986.

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 33-43
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR FVPS WITH GENERALIZED BOUNDARY CONDITIONS

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ABSTRACT. This paper presents the necessary and sufficient optimality conditions for fractional variational problems with generalized boundary conditions. The Lagrangian that we consider depends on left Caputo fractional derivatives of order α and which also depends on the boundary points $y(a)$ and $y(b)$,the terminal time b, and the end point $y(b)$ may be not specified . Examples are presented to demonstrate the application of the formulations.

KEYWORDS : Necessary and sufficient optimality conditions; Caputo fractional derivatives; Generalized boundary conditions; Fractional variational problems.

1. INTRODUCTION

Fractional calculus is the branch of mathematics that generalizes the derivative and the integral of a function to a noninteger order. The study of fractional problems of the calculus of variations is a subject of current strong research due to its many applications in science, engineering, mechanics, chemistry, biology, economics and control theory (see [3, 4, 8–10, 13]).

The fractional calculus of variations was introduced by Riewe in [11, 12], where he developed Hamiltonian, and other concepts of classical mechanics using fractional calculus.

Klimek presented a fractional sequential mechanics model with symmetric fractional derivatives [5] and stationary conservation laws for fractional differential equations with variable coefficients [6].

In [1], Agrawal combined the calculus of variations and the concepts of fractional derivatives to obtain the fractional variational problems.

In [2], Agrawal considered the functional which had unspecified end points. The extremal function satisfied the terminal condition $y(a) = y_a$ and intersected the curve $z = c(x)$ for the first time at b , i.e. $y(b) = c(b)$. Here c(x) was the specified curve.

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[1] This research is supported by the Nature Science Foundation of Education Committee of Hu Bei Province (Q20091107), Hubei Province Key Laboratory of Systems Science in Metallurgical Process(C201015)and WUST(2008RC01).

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Article history : Received 05 January 2012. Accepted 28 March 2012.

In [7], the two authors discussed the necessary optimality conditions for the functionals of the form:$J(y) = I^{\gamma}_{a+}L(x, y(x), ^R D^{\alpha}_{a+}y, y(a))$.

In this paper, we will develop the theory of fractional variational calculus further by proving the necessary and sufficient optimality conditions for more general problems. The functional that we consider has the form:

$$J(y) = \int_a^b L(x, y(x), ^C D^{\alpha}_{a+}y(x), y(a), y(b))dx \longrightarrow min \qquad \text{(VP)}$$

where $L(x, y, u, v, w)$ is a function with continuous first and second (partial) derivatives with respect to all its arguments. And $y(x)$ has continuous left Caputo fractional derivatives of order α, here $0 < \alpha < 1$. The initial time $x = a$ is specified, while the initial point $y(a)$, the terminal time b , and the end point $y(b)$ may be not specified. In some cases, the end point $y(x)$ may intersect a specified curve at the terminal time.

2. PRELIMINARIES

2.1. **REVIEW ON FRACTIONAL CALCULUS.** Let $f : [a, b] \longrightarrow$ R be a function, α a positive real number, n the integer satisfying that $n - 1 \leq \alpha < n$,and Γ the Euler gamma function.

2.1.1.The left and right Riemann-Liouville fractional integrals of order α are defined by

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt$$

respectively;

2.1.2. The left and right Riemann-Liouville fractional derivatives of order α are defined by

$$D^{\alpha}_{a+}f(x) = \frac{d^n}{dx^n}I^{n-\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt$$

and

$$D^{\alpha}_{b-}f(x) = (-1)^n\frac{d^n}{dx^n}I^{n-\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (t-x)^{n-\alpha-1}f(t)dt$$

respectively;

2.1.3. The left and right Caputo fractional derivatives of order α are defined by

$$^C D^{\alpha}_{a+}f(x) = I^{n-\alpha}_{a+}\frac{d^n}{dx^n}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt$$

and

$$^C D^{\alpha}_{b-}f(x) = (-1)^n I^{n-\alpha}_{b-}\frac{d^n}{dx^n}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_x^b (-1)^n(t-x)^{n-\alpha-1}f^{(n)}(t)dt$$

respectively.

2.2. **INTEGRATION BY PARTS FOR CAPUTO FRACTIONAL DERIVATIVES.**

$$\int_a^b g(x) \cdot^C D_{a+}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{b-}^{\alpha} g(x)dx + \sum_{j=0}^{n-1} [D_{b-}^{\alpha+j-n} g(x) \cdot D_{b-}^{n-1-j} f(x)]_a^b$$

and

$$\int_a^b g(x) \cdot^C D_{b-}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{a+}^{\alpha} g(x)dx + \sum_{j=0}^{n-1} [(-1)^{n+j} D_{a+}^{\alpha+j-n} g(x) \cdot D_{a+}^{n-1-j} f(x)]_a^b$$

Therefore, if $0 < \alpha < 1$, we obtain

$$\int_a^b g(x) \cdot^C D_{a+}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{b-}^{\alpha} g(x)dx + [I_{b-}^{1-\alpha} g(x) \cdot f(x)]_a^b$$

and

$$\int_a^b g(x) \cdot^C D_{b-}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{a+}^{\alpha} g(x)dx - [I_{a+}^{1-\alpha} g(x) \cdot f(x)]_a^b$$

Moreover, if f is a function such that $f(a) = f(b) = 0$,we have the simpler formulas:

$$\int_a^b g(x) \cdot^C D_{a+}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{b-}^{\alpha} g(x)dx$$

and

$$\int_a^b g(x) \cdot^C D_{b-}^{\alpha} f(x)dx = \int_a^b f(x) \cdot D_{a+}^{\alpha} g(x)dx$$

3. Necessary optimality conditions

We will discuss the necessary optimality conditions for the problem (VP) in different cases of boundary conditions.

Case1. The initial point y(a) is specified .

Theorem 3.1. *If the terminal time $x = b$ and the end point y(b) are specified as well. Assume that $Y(x)$ is the desired function, which satisfies that $Y(a) = Y_a$ and $Y(b) = Y_b$. Then, $Y(x)$ satisfies the fractional E-L equation:*

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

Proof. Define a family of curves:

$$y(x) = Y(x) + \varepsilon\eta(x) \tag{3.1}$$

where $\eta(x)$ is a variation of $y(x)$.
Substituting (3.1) into (VP), we obtain

$$J(\varepsilon) = \int_a^b L(x, Y + \varepsilon\eta, ^C D_{a+}^{\alpha}(Y + \varepsilon\eta), Y(a) + \varepsilon\eta(a), Y(b) + \varepsilon\eta(b))dx$$

For simplicity, we write $L(x, Y + \varepsilon\eta, ^C D_{a+}^{\alpha}(Y + \varepsilon\eta), Y(a) + \varepsilon\eta(a), Y(b) + \varepsilon\eta(b))$ as $\widehat{L}$,
and write $L(x, Y, ^C D_{a+}^{\alpha} Y, Y(a), Y(b))$ as L,
Find the expression for $dJ/d\varepsilon$

$$\frac{dJ}{d\varepsilon} = \int_a^b \partial_2 \widehat{L} \cdot \eta + \partial_3 \widehat{L} \cdot^C D_{a+}^{\alpha} \eta + \partial_4 \widehat{L} \cdot \eta(a) + \partial_5 \widehat{L} \cdot \eta(b)dx$$

Notice that $J(\varepsilon) \geq J(0)$,then

$$\frac{dJ}{d\varepsilon} \mid_{\varepsilon=0} = \int_a^b \partial_2 L \cdot \eta + \partial_3 L {}^C D_{a+}^{\alpha} \eta + \partial_4 L \cdot \eta(a) + \partial_5 L \cdot \eta(b) dx = 0$$

Using integration by parts

$$\begin{aligned} &\int_a^b \eta(\partial_2 L + D_{b-}^{\alpha} \partial_3 L) dx + \eta(I_{b-}^{1-\alpha} \partial_3 L) \mid_a^b + \int_a^b \eta(a) \partial_4 L + \eta(b) \partial_5 L dx \\ &= \int_a^b \eta\left(\frac{\partial L}{\partial Y} + D_{b-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y}\right) dx + \eta\left(I_{b-}^{1-\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y}\right) \mid_a^b \\ &+ \int_a^b \eta(a) \frac{\partial L}{\partial Y(a)} dx + \int_a^b \eta(b) \frac{\partial L}{\partial Y(b)} dx = 0 \end{aligned} \tag{3.2}$$

Since $\eta(x)$ is arbitrary, $I_{b-}^{1-\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y}) \mid_{x=b} = 0$, $\eta(a) = 0$, and $\eta(b) = 0$, which gives the fractional E-L equation in the form

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0 \tag{3.3}$$

□

Theorem 3.2. *If the terminal time $x = b$ is specified, while the end point $y(b)$ is unspecified. Assume that $Y(x)$ is the desired function. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.*

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

and

$$\int_a^b \frac{\partial L}{\partial Y(b)} dx = 0$$

Proof. Using the same way as theorem 3.1, we get

$$\begin{aligned} \frac{dJ}{d\varepsilon} \mid_{\varepsilon=0} &= \int_a^b \eta\left(\frac{\partial L}{\partial Y} + D_{b-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y}\right) dx + \eta\left(I_{b-}^{1-\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y}\right) \mid_a^b \\ &+ \int_a^b \eta(a) \frac{\partial L}{\partial Y(a)} dx + \int_a^b \eta(b) \frac{\partial L}{\partial Y(b)} dx = 0 \end{aligned}$$

Since $\eta(x)$ is arbitrary, which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

Since $I_{b-}^{1-\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} \mid_{x=b} = 0$ and $\eta(a) = 0$,we have

$$\int_a^b \eta(b) \frac{\partial L}{\partial Y(b)} dx = \eta(b) \int_a^b \frac{\partial L}{\partial Y(b)} dx = 0$$

We know that $\eta(b)$ is arbitrary,which gives the transversality condition as

$$\int_a^b \frac{\partial L}{\partial Y(b)} dx = 0 \tag{3.4}$$

□

Theorem 3.3. *If the terminal time $x = b$ and the end point $y(b)$ are both unspecified, and the function $y(x)$ intersects the curve $z = c(x)$ for the first time at $x = b$, i.e. $y(b) = c(b)$. Here $z = c(x)$ is the specified curve.*

Assume that $Y(x)$ is the desired function, which intersects the curve $z = c(x)$ at $x = B$, i.e. $Y(B) = c(B)$. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

and

$$[(Dc - DY) \int_a^B \frac{\partial L}{\partial Y(B)} dx + L] \mid_{x=B} = 0$$

Proof. Define a family of curves

$$y(x) = Y(x) + \varepsilon\eta(x) \tag{3.5}$$

We further define a set of end points:

$$b = B + \varepsilon\delta(b) \tag{3.6}$$

Where $\delta(b)$ is a variation of b.
Substituting (3.5) and (3.6) into (VP), then the equation (VP) can be rewritten as

$$J(\varepsilon) = \int_a^{B+\varepsilon\delta(b)} L[x, Y + \varepsilon\eta, {}^C D_{a+}^{\alpha}(Y + \varepsilon\eta), Y(a) + \varepsilon\eta(a), Y(B + \varepsilon\delta(b)) + \varepsilon\eta(B + \varepsilon\delta(b))]dx$$

For simplicity, we write $L[x, Y+\varepsilon\eta, {}^C D_{a+}^{\alpha}(Y+\varepsilon\eta), Y(a)+\varepsilon\eta(a), Y(B+\varepsilon\delta(b))+\varepsilon\eta(B+\varepsilon\delta(b))]$ as $\widehat{L}$, and write $L[x, Y, {}^C D_{a+}^{\alpha} Y, Y(a), Y(B)]$ as L. Then

$$\frac{dJ}{d\varepsilon} = \int_a^{B+\varepsilon\delta(b)} \partial_2\widehat{L} \cdot \eta + \partial_3\widehat{L} \cdot {}^C D_{a+}^{\alpha}\eta + \partial_4\widehat{L} \cdot \eta(a) + \partial_5\widehat{L} \cdot \eta(B + \varepsilon\delta(b))dx$$
$$+\delta(B + \varepsilon\delta(b))L[B + \varepsilon\delta(b), Y + \varepsilon\eta, {}^C D_{a+}^{\alpha}(Y + \varepsilon\eta),$$
$$Y(a) + \varepsilon\eta(a), Y(B + \varepsilon\delta(b)) + \varepsilon\eta(B + \varepsilon\delta(b))]$$

Notice that $J(\varepsilon) \geq J(0)$, then we have

$$\frac{dJ}{d\varepsilon} \mid_{\varepsilon=0} = \int_a^B \partial_2 L \cdot \eta + \partial_3 L \cdot {}^C D_{a+}^{\alpha}\eta + \partial_4 L \cdot \eta(a) + \partial_5 L \cdot \eta(B)dx$$
$$+L(B, Y, {}^C D_{a+}^{\alpha} Y, Y(a), Y(B))\delta(B)$$

Using integration by parts

$$\frac{dJ}{d\varepsilon} \mid_{\varepsilon=0} = \int_a^B \eta(\partial_2 L + D_{B-}^{\alpha}\partial_3 L)dx + \eta \cdot (I_{B-}^{1-\alpha}\partial_3 L) \mid_a^B + \int_a^B \partial_4 L \cdot \eta(a)dx$$
$$+ \int_a^B \partial_5 L \cdot \eta(B)dx + L(B, Y, {}^C D_{a+}^{\alpha} Y, Y(a), Y(B))\delta(B) = 0$$

Since $\eta(x)$ is arbitrary, which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y} + D_{B-}^{\alpha} \frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

Since $I_{B-}^{1-\alpha}\partial_3 L \mid_{x=B} = 0$ and $\eta(a) = 0$, we obtain

$$\eta(B) \int_a^B \frac{\partial L}{\partial Y(B)} dx + \delta(B)L(B, Y, {}^C D_{a+}^{\alpha} Y, Y(a), Y(B)) = 0 \tag{3.7}$$

We notice that

$$y(b) = c(b) \tag{3.8}$$

By using (3.5) and (3.6), the equation (3.8) becomes

$$Y(B+\varepsilon\delta(b))+\varepsilon\eta(B+\varepsilon\delta(b))=c(B+\varepsilon\delta(b)) \tag{3.9}$$

Differentiating equation (3.9) with respect to ε and then setting $\varepsilon=0$,we get

$$DY(B)\cdot\delta(B)+\eta(B)=Dc(B)\cdot\delta(B) \tag{3.10}$$

Here $D(\cdot)=\frac{d(\cdot)}{dx}$
We further get

$$\eta(B)=\delta(B)D(c(B)-Y(B)) \tag{3.11}$$

Substituting (3.11) into (3.7), then (3.7) becomes the form

$$\delta(B)[(Dc-DY)\int_a^B\frac{\partial L}{\partial Y(B)}dx+L]\mid_{x=B}=0$$

Since $\delta(B)$ is arbitrary, then the transversality condition is given below

$$[(Dc-DY)\int_a^B\frac{\partial L}{\partial Y(B)}dx+L]\mid_{x=B}=0 \tag{3.12}$$

□

Case 2. The initial point $y(a)$ is unspecified.

Theorem 3.4. *If the terminal time $x=b$ and the end point $y(b)$ are both specified. Assume that $Y(x)$ is the desired function, which satisfies that $Y(b)=Y_b$. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.*

$$\frac{\partial L}{\partial Y}+D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}=0$$

and

$$\int_a^b\frac{\partial L}{\partial Y(a)}dx-\frac{1}{\Gamma(1-\alpha)}\int_a^b(t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^{\alpha}Y(t)}dt=0$$

Proof. Using the same way as theorem 3.1, we get

$$\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0}=\int_a^b\eta(\frac{\partial L}{\partial Y}+D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y})dx+\eta(I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y})\mid_a^b$$
$$+\int_a^b\eta(a)\frac{\partial L}{\partial Y(a)}dx+\int_a^b\eta(b)\frac{\partial L}{\partial Y(b)}dx=0$$

Since $\eta(x)$ is arbitrary, which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y}+D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}=0$$

Since $\eta(b)=0$ and $I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}\mid_{x=b}=0$,we have

$$\eta(a)[\int_a^b\frac{\partial L}{\partial Y(a)}dx-I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}\mid_{x=a}]=0 \tag{3.13}$$

We know that $\eta(a)$ is arbitrary, therefore (3.13) becomes

$$\int_a^b\frac{\partial L}{\partial Y(a)}dx-I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}\mid_{x=a}=0 \tag{3.14}$$

We further get the transversality conditions as follow

$$\int_a^b \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^{\alpha} Y(t)}dt = 0$$

□

Theorem 3.5. *If the terminal time $x = b$ is specified, while the end point $y(b)$ is unspecified. Assume that $Y(x)$ is the desired function. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.*

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

and

$$\int_a^b \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^{\alpha} Y(t)}dt = 0$$

$$\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0$$

Proof. Using the same way as theorem 3.1, we get

$$\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0}= \int_a^b \eta(\frac{\partial L}{\partial Y} + D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha} Y})dx + \eta(I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha} Y})\mid_a^b$$
$$+\int_a^b \eta(a)\frac{\partial L}{\partial Y(a)}dx + \int_a^b \eta(b)\frac{\partial L}{\partial Y(b)}dx = 0$$

Since $\eta(x)$ is arbitrary,which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

Since $I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha} Y}\mid_{x=b}= 0$,we have

$$\eta(a)[\int_a^b \frac{\partial L}{\partial Y(a)}dx - I_{b-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha} Y}\mid_{x=a}] + \eta(b)\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0$$

We know that $\eta(a)$ and $\eta(b)$ are both arbitrary, the transversality conditions are given below

$$\int_a^b \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^{\alpha} Y(t)}dt = 0$$

$$\int_a^b \frac{\partial L}{\partial Y(b)}dx = 0$$

□

Theorem 3.6. *If the terminal time $x = b$ and the end point $y(b)$ are both unspecified, and the function $y(x)$ intersects the curve $z = c(x)$ for the first time at $x = b$, i.e. $y(b) = c(b)$, where $z = c(x)$ is the specified curve.*

Assume that $Y(x)$ is the desired function, which intersects the curve $z = c(x)$ at $x = B$, i.e. $Y(B) = c(B)$. Then, $Y(x)$ satisfies the following fractional E-L equation and the transversality condition.

$$\frac{\partial L}{\partial Y} + D_{b-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha} Y} = 0$$

and

$$[(Dc - DY)\int_a^B \frac{\partial L}{\partial Y(B)}dx + L]\mid_{x=B}= 0$$

$$\int_a^B \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^B (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^{\alpha}Y(t)}dt = 0$$

Proof. Using the same way as theorem 3.1, we obtain

$$\frac{dJ}{d\varepsilon}\mid_{\varepsilon=0}= \int_a^B \eta(\partial_2 L + D_{B-}^{\alpha}\partial_3 L)dx + \eta\cdot(I_{B-}^{1-\alpha}\partial_3 L)\mid_a^B + \int_a^B \partial_4 L\cdot\eta(a)dx$$

$$+\int_a^B \partial_5 L\cdot\eta(B)dx + L(B,Y,^C D_{a+}^{\alpha}Y, Y(a), Y(B))\delta(B) = 0$$

Since $\eta(x)$ is arbitrary,which gives the fractional E-L equation as

$$\frac{\partial L}{\partial Y} + D_{B-}^{\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y} = 0$$

Since $I_{B-}^{1-\alpha}\partial_3 L\mid_{x=B}= 0$,we get

$$\eta(a)(\int_a^B \frac{\partial L}{\partial Y(a)}dx - I_{B-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}\mid_{x=a}) + \eta(B)\int_a^B \frac{\partial L}{\partial Y(B)}dx$$

$$+\delta(B)L(B,Y,^C D_{a+}^{\alpha}Y, Y(a), Y(B)) = 0 \tag{3.15}$$

From theorem 3.3, we know that

$$\eta(B) = \delta(B)D(c(B) - Y(B)) \tag{3.16}$$

Substituting (3.16) into (3.15), then (3.15) becomes the following form

$$\eta(a)(\int_a^B \frac{\partial L}{\partial Y(a)}dx - I_{B-}^{1-\alpha}\frac{\partial L}{\partial^C D_{a+}^{\alpha}Y}\mid_{x=a}) + \delta(B)[(Dc - DY)\int_a^B \frac{\partial L}{\partial Y(B)}dx + L]\mid_{x=B}= 0$$

Since $\eta(a)$ and $\eta(B)$ are both arbitrary, the transversality conditions are given below

$$[(Dc - DY)\int_a^B \frac{\partial L}{\partial Y(B)}dx + L]\mid_{x=B}= 0$$

and

$$\int_a^B \frac{\partial L}{\partial Y(a)}dx - \frac{1}{\Gamma(1-\alpha)}\int_a^B (t-a)^{-\alpha}\frac{\partial L(t)}{\partial^C D_{a+}^{\alpha}Y(t)}dt = 0$$

□

4. SUFFICIENT CONDITIONS

In this section, we prove the sufficient optimality conditions of the following functional

$$J(y) = \int_a^b L(x, y(x),^C D_{a+}^{\alpha}y(x), y(a), y(b))dx \longrightarrow min \tag{4.0}$$

Where $0 < \alpha < 1$,the initial time a and the terminal time b are both specified, while, the boundary points $y(a)$ and $y(b)$ are both free.

Some conditions of convexity are in order. Given a function $L = L(x, y, u, v, w)$,we say that L is jointly convex in (y, u, v, w), if $\frac{\partial L}{\partial y}, \frac{\partial L}{\partial u}, \frac{\partial L}{\partial v}, \frac{\partial L}{\partial w}$ exist and are continuous and verify the following condition:

$$L(x, y+y_1, u+u_1, v+v_1, w+w_1) - L(x, y, u, v, w)$$

$$\geq \frac{\partial L}{\partial y} \cdot y_1 + \frac{\partial L}{\partial u} \cdot u_1 + \frac{\partial L}{\partial v} \cdot v_1 + \frac{\partial L}{\partial w} \cdot w_1$$

for all $(x, y, u, v, w), (x, y+y_1, u+u_1, v+v_1, w+w_1) \in [a, b] \times R^4$

Theorem 4.1. *Let $L(x, y, u, v, w)$ be jointly convex in (y, u, v, w). If y satisfies the fractional E-L equation*

$$\frac{\partial L}{\partial Y} + D^{\alpha}_{b-} \frac{\partial L}{\partial^C D^{\alpha}_{a+} Y} = 0 \tag{4.1}$$

and the transversality conditions

$$\int_a^b \frac{\partial L}{\partial Y(a)} dx - \frac{1}{\Gamma(1-\alpha)} \int_a^b (t-a)^{-\alpha} \frac{\partial L(t)}{\partial^C D^{\alpha}_{a+} Y(t)} dt = 0 \tag{4.2}$$

$$\int_a^b \frac{\partial L}{\partial Y(b)} dx = 0 \tag{4.3}$$

Then, y is a global minimizer to the functional (4.0).

Proof. Since L is jointly convex in (y, u, v, w), for any admissable function $y+\eta$, we have

$$J(y+\eta) - J(y) = \int_a^b L(x, y+\eta, {}^C D^{\alpha}_{a+}(y+\eta), y(a)+\eta(a), y(b)+\eta(b))$$

$$-L(x, y, {}^C D^{\alpha}_{a+} y, y(a), y(b)) dx$$

$$\geq \int_a^b \frac{\partial L}{\partial y} \cdot \eta + \frac{\partial L}{\partial^C D^{\alpha}_{a+} y} {}^C D^{\alpha}_{a+} \eta + \frac{\partial L}{\partial y(a)} \cdot \eta(a) + \frac{\partial L}{\partial y(b)} \cdot \eta(b) dx$$

Using integration by parts

$$J(y+\eta) - J(y) \geq \int_a^b \eta\left(\frac{\partial L}{\partial y} + D^{\alpha}_{b-}\left(\frac{\partial L}{\partial^C D^{\alpha}_{a+} y}\right)\right) dx + \eta(a)\left[\int_a^b \frac{\partial L}{\partial y(a)}\right.$$

$$\left. - I^{1-\alpha}_{b-} \frac{\partial L}{\partial^C D^{\alpha}_{a+} y} \, |_{x=a} \right] + \eta(b) \int_a^b \frac{\partial L}{\partial y(b)} dx = 0$$

Since y satisfies (4.1)-(4.3), thus we obtain $J(y+\eta) - J(y) \geq 0$.
We can similarly prove the sufficient optimality conditions of the functional (VP) with other different boundary conditions. □

5. ILLUSTRATIVE EXAMPLES

Consider the following functional:

$$J(y) = \int_a^b y^2(x) + ({}^C D^{\alpha}_{a+} y(x))^2 + y(a)^2 + y(b)^2 dx \longrightarrow min$$

Where the initial time $x = a$ is specified. We will discuss its E-L equations and transversality conditions in different cases of boundary conditions.

Case1. The initial point is specified i.e. $y(a) = y_a$

Example 5.1. If the terminal time $x = b$ and the end point $y(b)$ are both specified.

This problem becomes a specified boundary conditions problem. Assume $y(x)$ is the desired function, we get the generalized fractional E-L equation in the following form:

$$y(x) + D^{\alpha}_{b-}({}^C D^{\alpha}_{a+} y(x)) = 0$$

Example 5.2. If the terminal time $x = b$ is specified, while the end point $y(b)$ is unspecified.We get the generalized fractional E-L equation and the transversality condition, respectively, in the following form:

$$y(x) + D^{\alpha}_{b-}({}^C D^{\alpha}_{a+} y(x)) = 0$$

and

$$y(b)(b-a) = 0$$

Example 5.3. If the terminal time $x = b$ and the end point $y(b)$ are both unspecified in advance, and the function $y(x)$ intersects the curve $c(x) = x^2$,for the first time at $x = b$,i.e $y(b) = c(b) = b^2$.

Assume that $y(x)$ is the desired function, and it satisfies $y(b) = b^2$.

For this problem, we get the generalized fractional E-L equation and the transversality condition, respectively, in the following form:

$$y(x) + D^{\alpha}_{b-}({}^C D^{\alpha}_{a+} y(x)) = 0$$

and

$$2(2b - Dy(b))y(b)(b-a) + L(b, y, {}^C D^{\alpha}_{a+} y, y(a), y(b)) = 0$$

Case2. The initial point is unspecified.

Example 5.4. If the terminal time $x = b$ and the end point $y(b)$ are both specified. We get the generalized fractional E-L equation in the following form:

$$y(x) + D^{\alpha}_{b-}({}^C D^{\alpha}_{a+} y(x)) = 0$$

and

$$y(x) + D^{\alpha}_{b-}({}^C D^{\alpha}_{a+} y(x)) = 0$$

$$y(a)(b-a) - \frac{1}{\Gamma(1-\alpha)} \int_a^b (t-a)^{-\alpha} \cdot {}^C D^{\alpha}_{a+} y(t) dt = 0$$

Example 5.5. If the terminal time $x = b$ is specified, while the end point $y(b)$ is unspecified.We get the generalized fractional E-L equation in the following form:

$$y(x) + D^{\alpha}_{b-}({}^C D^{\alpha}_{a+} y(x)) = 0$$

and

$$y(b)(b-a) = 0$$

$$y(a)(b-a) - \frac{1}{\Gamma(1-\alpha)} \int_a^b (t-a)^{-\alpha} \cdot {}^C D^{\alpha}_{a+} y(t) dt = 0$$

Example 5.6. If the terminal time $x = b$ and the end point $y(b)$ are both unspecified in advance, and the function $y(x)$ intersects the curve $c(x) = x^2$, for the first time at $x = b$, i.e $y(b) = c(b) = b^2$.

Assume that $y(x)$ is the desired function, and it satisfies $y(b) = b^2$.

For this problem, we get the generalized fractional E-L equation and the transversality condition, respectively, in the following form:

$$y(x) + D_{b-}^{\alpha}({}^{C}D_{a+}^{\alpha}y(x)) = 0$$

and

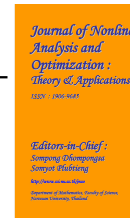
$$y(a)(b-a) - \frac{1}{\Gamma(1-\alpha)}\int_a^b (t-a)^{-\alpha} \cdot {}^{C}D_{a+}^{\alpha}y(t)dt = 0$$

$$2(2b - Dy(b))y(b)(b-a) + L(b, y, {}^{C}D_{a+}^{\alpha}y, y(a), y(b)) = 0$$

References

1. Agrawal, O P.: Formulation of Euler-Lagrange equations for fractional variational problems, *J.Math. Anal. Appl.* 272, 368-79(2002).
2. Agrawal, O P.: Generalized Euler-Lagrange equations and transversality conditions for fvps in terms of the caputo derivative, *J. Vib. Control.* 13, 1217-1237(2007).
3. Atanackovic T M, Oparnica Lj and Pilipovic S: On a nonlinear distributed order fractional differential equation, *J. Math. Anal. Appl.* 328 590-608(2007).
4. Hilfer R.: Applications of Fractional Calculus in Physics.World Scientific, *Singapore.* (2000).
5. Klimek M.: Fractional sequential mechanics-models with symmetric fractional derivative.Czech, *J. Phys.* 51,1348-1354 (2001).
6. Klimek M.: Stationary conservation laws for fractional differential equations with variable coefficients. *J. Phys. A,Math. Gen.* 35, 6675-6693 (2002).
7. Mohamed A. E. Herzallah, Dumitru Baleanu: Fractional-Order Variational Calculus with Generalized Boundary Conditions, Adv. Difference equations, 2011(2011), Article ID 357580, 9 pages.
8. Magin R.L.: Fractional calculus in bioengineering Crit,*Rev. Biomed. Eng.* 321-377(2004).
9. Miller K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations, *Wiley, NewYork* (1993).
10. Podlubny, I.: Fractional Differential Equations,*Academic Press, New York* (1999).
11. Riewe F.: Nonconservative Lagrangian and Hamiltonian mechanics,*Phys. Rev. E.* 53(2), 1890 -1899 (1996)
12. Riewe F.: Mechanics with fractional derivatives, *Phys. Rev.E.* 55(3), 3582-3592(1997).
13. Samko, S.G., Kilbas, A.A., Marichev, O.I.:Fractional Integrals and Derivatives Theory and Applications. *Gordon and Breach, Longhorne.* (1993).
14. Mohamed A.E. Herzallah,Dumitru Baleanu:Fractional-order Euler-Lagrange equations and formulation of Hamiltonian equations.Springer Science+Business Media B.V. (2009).

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 45-53
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



BEST APPROXIMATION IN INTUITIONISTIC FUZZY n-NORMED LINEAR SPACES

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ABSTRACT. The central issues that are addressed by the theory of best approximation are related to the questions of existence, uniqueness, characterizations and qualitative properties of the minimizing functions. The theory can also estimate the rapidity of convergence of a sequence of functions converging to a minimizing function. The aim of this article is to introduce and study the notions of best approximation, proximal set, Chebyshev set and approximatively compact set in the new setup of intuitionistic fuzzy n-normed linear spaces.

KEYWORDS : Intuitionistic fuzzy n-normed linear space; Best approximation; n-normed space.
AMS Subject Classification: 40A05; 60B99.

1. INTRODUCTION

Since the introduction of fuzzy set theory by Zadeh [40] in 1965, fuzzy logic became an important area of research in various branches of mathematics such as metric and topological spaces [8, 12, 16], theory of functions [15, 38], approximation theory [1], etc. Fuzzy set theory also found applications for modeling uncertainty and vagueness in various fields of science and engineering. The notion of intuitionistic fuzzy set (IFS) introduced by Atanassov [4] has triggered a lot of debate (for details, see [6, 7, 13]) regarding the use of the terminology "intuitionistic" and the term is considered to be a misnomer on the following account:

- The algebraic structure of IFSs in not intuitionistic, since negation is involutive in IFS theory.
- Intuitionistic logic obeys the law of contradiction, IFSs do not.

Also IFSs are considered to be equivalent to interval-valued fuzzy sets and they are particular cases of L-fuzzy sets. In response to this debate, Attanassov justified the terminology in [2]. Apart from the terminological issues, research in intuitionistic fuzzy setting remains well motivated as IFSs give us a very natural tool for modeling imprecision in real life situations and found its application in various area of science

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Article history : Received 05 January 2012. Accepted 17 March 2012.

and engineering, e.g., this theory has been extensively used in decision making problems [3] and E-infinity theory of high energy physics [22].

The theory of 2-norm and n-norm on a linear space was introduced by S. Gähler [10, 11], which was developed by S.S. Kim and Y.J. Cho [18], R. Malceski [20], A. Misiak [21], H. Gunawan and M. Mashadi [14]. Fuzzy norm on a linear space was first introduced by Katsaras [17] and studied by various authors from different points of view [5, 9, 16, 19, 39]. Vijayabalaji and Narayanan [36] extended n-normed linear space to fuzzy n-normed linear space. Saadati and Park [24] introduced the notion of intuitionistic fuzzy normed space while the notion of intuitionistic fuzzy n-normed linear space was introduced by Vijayabalaji et al. [37].

The best approximation problems were introduced by P. L. Chebyshev in 1853. Such problems deal with the search for a function in a prescribed class which has the least deviation from a given function, as measured in a prescribed metric. The theory also takes into account the continuity properties of the metric projection and can estimate the rapidity of convergence of a sequence of functions converging to a minimizing function. Some works in approximation theory can be found in [27, 28, 34, 35]. Some interesting works on the theory of best approximation has been done by Sintunavarat, W. and his coauthors in [29–33]. In the present paper, we propose to define and study the notions of t-best approximation, t-proximal set, t-Chebyshev set, t-approximatively compact set and prove some useful results related to those concepts in an intuitionistic fuzzy n-normed linear space. Most of the results in this article are closely linked with the notion of convergence of a sequence in intuitionistic fuzzy n-normed linear space and a new and unambiguous definition of the same has been given in [25, 26]. Here we develop the results based on that new definition.

2. PRELIMINARIES

Throughout this paper $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and the set of natural numbers respectively. First we recall some definitions.

Definition 2.1. [14] Let $n \in \mathbb{N}$ and X be a real linear space of dimension $d \geq n$ (d may be infinite). A real valued function $\|.\|$ on $\underbrace{X \times X \times \cdots \times X}_{n} = X^n$ is called an n-norm on X if it satisfies the following properties:

(i) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(ii) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation,
(iii) $\|x_1, x_2, \ldots, \alpha x_n\| = |\alpha|\|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$,
(iv) $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$, and the pair $(X, \|.\|)$ is called an n-normed linear space.

Definition 2.2. [25] An intuitionistic fuzzy n-normed linear space (in short IFnNLS) is the five-tuple $(X, \mu, \nu, *, \circ)$, where X is a linear space of dimension $d \geq n$ over a field F, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, μ, ν are fuzzy sets on $X^n \times (0, \infty)$, μ denotes the degree of membership and ν denotes the degree of non-membership of $(x_1, x_2, \ldots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions for every $x_1, x_2, \ldots, x_n \in X$ and $s, t > 0$:

(i) $\mu(x_1, x_2, \ldots, x_n, t) + \nu(x_1, x_2, \ldots, x_n, t) \leq 1$,
(ii) $\mu(x_1, x_2, \ldots, x_n, t) > 0$,
(iii) $\mu(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(iv) $\mu(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

(v) $\mu(x_1, x_2, \ldots, cx_n, t) = \mu(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,
(vi) $\mu(x_1, x_2, \ldots, x_n, s) * \mu(x_1, x_2, \ldots, x_n^{'}, t) \leq \mu(x_1, x_2, \ldots, x_n + x_n^{'}, s + t)$,
(vii) $\mu(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(viii) $\lim_{t\to\infty} \mu(x_1, x_2, \ldots, x_n, t) = 1$ and $\lim_{t\to 0} \mu(x_1, x_2, \ldots, x_n, t) = 0$,
(ix) $\nu(x_1, x_2, \ldots, x_n, t) < 1$,
(x) $\nu(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(xi) $\nu(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,
(xii) $\nu(x_1, x_2, \ldots, cx_n, t) = \nu(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in F$,
(xiii) $\nu(x_1, x_2, \ldots, x_n, s) \circ \nu(x_1, x_2, \ldots, x_n^{'}, t) \geq \nu(x_1, x_2, \ldots, x_n + x_n^{'}, s + t)$,
(xiv) $\nu(x_1, x_2, \ldots, x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(xv) $\lim_{t\to\infty} \nu(x_1, x_2, \ldots, x_n, t) = 0$ and $\lim_{t\to 0} \nu(x_1, x_2, \ldots, x_n, t) = 1$.

Definition 2.3. [25] Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. We say that a sequence $x = \{x_k\}$ in X is convergent to $L \in X$ with respect to the intuitionistic fuzzy n-norm $(\mu, \nu)^n$ if, for every $\epsilon \in (0, 1)$, $t > 0$ and $y_1, y_2, \ldots, y_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that $\mu(y_1, y_2, \ldots, y_{n-1}, x_k - L, t) > 1 - \epsilon$ and $\nu(y_1, y_2, \ldots, y_{n-1}, x_k - L, t) < \epsilon$ for all $k \geq k_0$. It is denoted by $x_k \overset{(\mu,\nu)^n}{\to} L$ as $k \to \infty$.

Definition 2.4. The closure of a subset B in an IFnNLS $(X, \mu, \nu, *, \circ)$ is denoted by $\overline{B}$ and defined by the set of all $x \in X$ such that there exists a sequence $\{x_k\}$ in B such that $x_k \overset{(\mu,\nu)^n}{\to} x$. We say that B is closed whenever $\overline{B} = B$.

3. MAIN RESULTS

Now we obtain our main results.

Definition 3.1. Let A be a non-empty subset of an IFnNLS $(X, \mu, \nu, *, \circ)$. For $x \in X, t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X, denote $\mu^Y(x, t) = \mu(y_1, y_2, \ldots, y_{n-1}, x, t)$ and $\nu^Y(x, t) = \nu(y_1, y_2, \ldots, y_{n-1}, x, t)$. Let

$$\mu^Y(x - A, t) = \sup\{\mu^Y(x - y, t) : y \in A\},$$
$$\nu^Y(x - A, t) = \inf\{\nu^Y(x - y, t) : y \in A\}.$$

An element $u \in A$ is said to be a t-best approximation to x from A if

$$\mu^Y(x - u, t) = \mu^Y(x - A, t) \text{ and } \nu^Y(x - u, t) = \nu^Y(x - A, t).$$

By $P_A^Y(x, t)$, we denote the set of elements of t-best approximation of x by elements of the set A, i.e.,

$$P_A^Y(x, t) = \{y \in A : \mu^Y(x - A, t) = \mu^Y(x - y, t) \text{ and } \nu^Y(x - A, t) = \nu^Y(x - y, t)\}.$$

Definition 3.2. Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. For $\alpha \in (0, 1)$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X we define the open ball $B_x^Y(\alpha, t)$ and the closed ball $B_x^Y[\alpha, t]$ with center $x \in X$ and radius $t > 0$ as follows:

$$B_x^Y(\alpha, t) = \{y \in X : \mu^Y(x - y, t) > 1 - \alpha \text{ and } \nu^Y(x - y, t) < \alpha\},$$
$$B_x^Y[\alpha, t] = \{y \in X : \mu^Y(x - y, t) \geq 1 - \alpha \text{ and } \nu^Y(x - y, t) \leq \alpha\}.$$

Definition 3.3. Let A be a non-empty subset of an IFnNLS $(X, \mu, \nu, *, \circ)$. Then A is said to be a t-proximal set if $P_A^Y(x, t)$ is non-empty for every $x \in X \setminus A$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X. A is called a t-Chebyshev set if $P_A^Y(x, t)$ contains exactly one element for every $x \in X$ and some linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X. Also A is called a t-quasi-Chebyshev set if $P_A^Y(x, t)$ is a compact set.

Example 3.4. Let $X = \mathbb{R}^n$ $(n \geq 2)$ with

$$\|x_1, x_2, \ldots, x_n\| = abs\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Define $\mu, \nu : X^n \times (0, \infty) \to [0, 1]$ by

$$\mu(x_1, x_2, \ldots, x_n, t) = \frac{1}{e^{\|x_1, x_2, \ldots, x_n\|/t}} \text{ and } \nu(x_1, x_2, \ldots, x_n, t) = 1 - \frac{1}{e^{\|x_1, x_2, \ldots, x_n\|/t}}.$$

Also let $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. Then $(X, \mu, \nu, *, \circ)$ is an IFnNLS. Let

$$A = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -1 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2, x_3 = \ldots = x_n = 0\}.$$

Consider $x_1 = (0, 3, 0, \ldots, 0), x_2 = (0, 2, 0, \ldots, 0), x_3 = (0, 0, 1, \ldots, 0), \ldots, x_n = (0, 0, \ldots, 1) \in \mathbb{R}^n$. Then for every $t > 0$,

$$\begin{aligned} &\mu((-1, 1, 0, \ldots, 0) - (0, 3, 0, \ldots, 0), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \mu((1, 1, 0, \ldots, 0) - (0, 3, 0, \ldots, 0), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \frac{1}{e^{2/t}}. \end{aligned}$$

Again

$$\begin{aligned} &\mu((0, 3, 0, \ldots, 0) - A, (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \sup\{\mu((0, 3, 0, \ldots, 0) - (u_1, u_2, \ldots, u_n), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \\ &\quad \ldots, (0, 0, \ldots, 1), t) : -1 \leq u_1 \leq 1, 0 \leq u_2 \leq u_1^2, u_3 = \ldots = u_n = 0\} \\ &= \frac{1}{e^{2/t}}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\nu((-1, 1, 0, \ldots, 0) - (0, 3, 0, \ldots, 0), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \nu((1, 1, 0, \ldots, 0) - (0, 3, 0, \ldots, 0), (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, \\ &\quad (0, 0, \ldots, 1), t) \\ &= \nu((0, 3, 0, \ldots, 0) - A, (0, 2, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1), t) \\ &= 1 - \frac{1}{e^{2/t}}. \end{aligned}$$

Thus for every $t > 0$, $y_0 = (-1, 1, 0, \ldots, 0), y_1 = (1, 1, 0, \ldots, 0)$ are t-best approximations to $(0, 3, 0, \ldots, 0)$ from A. Therefore, A is a t-proximal set but not a t-Chebyshev set.

Saadati and Park [23] investigated several properties of intuitionistic fuzzy topological spaces. Every IFnNLS X induces a topology τ such that for some $A \subseteq X$, $A \in \tau$ if and only if for every $x \in A$ and a linearly independent set $Y = \{y_1, y_2, \ldots, y_{n-1}\} \subseteq X$, there exist $t > 0$ and $\alpha \in (0, 1)$ such that $B_x^Y(\alpha, t) \subseteq A$. It is not difficult to see that the family $\{B_x^Y(\frac{1}{n}, \frac{1}{n}) : n = 1, 2, \ldots\}$ is a countable local basis at x and consequently τ is a first countable topology.

Lemma 3.5. *Let A be a non-empty subset of an IFnNLS $(X, \mu, \nu, *, \circ)$ and $x \in X$. Then $x \in \overline{A}$ if and only if for all $t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X,*

$$\mu^Y(x - A, t) = 1 \text{ and } \nu^Y(x - A, t) = 0.$$

Proof Let $x \in \overline{A}$. As X is first countable, there exists a sequence $\{x_k\}$ in A such that $x_k \overset{(\mu,\nu)^n}{\rightarrow} x$ as $k \to \infty$. Then for every $t > o, \lambda \in (0, 1)$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X there exists $n_0 \in \mathbb{N}$ such that $\mu^Y(x - x_k, t) > 1 - \lambda$ and $\nu^Y(x - x_k, t) < \lambda$ for all $k \geq n_0$. Thus

$$1 - \lambda < \mu^Y(x - x_k, t) \leq \mu^Y(x - A, t) \leq 1$$

and

$$\lambda > \nu^Y(x - x_k, t) \geq \mu^Y(x - A, t) \geq 0,$$

for all $k \geq n_0$. Hence $\mu^Y(x - A, t) = 1$ and $\nu^Y(x - A, t) = 0$.

Conversely, suppose $\mu^Y(x - A, t) = 1$ and $\nu^Y(x - A, t) = 0$ for all $t > 0$ and linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X. We know that $\{B^Y(x, \lambda, t) : t > o, \lambda \in (0, 1), Y = \{y_1, y_2, \ldots, y_{n-1}\}\}$ is a local base at x. By definition, there exists a sequence $\{x_k\} \subseteq A$ such that $\mu^Y(x - x_k, \frac{1}{k}) \geq 1 - \frac{1}{k}$ and $\nu^Y(x - x_k, \frac{1}{k}) \leq \frac{1}{k}$, which implies that $\{x_k\} \subseteq B^Y(x, \frac{1}{k}, \frac{1}{k})$. Given $t > 0$ and $\lambda \in (0, 1)$, choose $k \in \mathbb{N}$ such that $t, \lambda > \frac{1}{k}$, then $B^Y(x, \frac{1}{k}, \frac{1}{k}) \subseteq B^Y(x, \lambda, t)$. So we have $B^Y(x, \lambda, t) \cap A \neq \phi$. Hence $x \in \overline{A}$.

Theorem 3.6. *Let A be a non-empty subset of an IFnNLS $(X, \mu, \nu, *, \circ)$. Then for all $x, y \in X, t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X, we have*

(i) $P^Y_{A+y}(x + y, t) = P^Y_A(x, t) + y$,
(ii) $P^Y_{\alpha A}(\alpha x, |\alpha| t) = \alpha P^Y_A(x, t)$, $\alpha \in \mathbb{R} \setminus \{0\}$,
(iii) *A is t-proximal (t-Chebyshev) if and only if $A + y$ is t-proximal (t-Chebyshev).*

Proof (i) Let $u \in P^Y_{A+y}(x + y, t)$. Then

$$\mu^Y(x - A, t) = \mu^Y(x + y - (A + y), t) = \mu^Y(x - (u - y), t)$$

and

$$\nu^Y(x - A, t) = \nu^Y(x + y - (A + y), t) = \nu^Y(x - (u - y), t),$$

which implies that $u - y \in P^Y_A(x, t)$. Hence $u \in P^Y_A(x, t) + y$.

The converse is obvious.

(ii) Let $u \in P^Y_{\alpha A}(\alpha x, |\alpha| t)$ for some $x \in X, t > 0, \alpha \in \mathbb{R} \setminus \{0\}$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X. Then

$$\mu^Y(x - A, t) = \mu^Y(\alpha x - \alpha A, |\alpha| t) = \mu^Y(\alpha x - u, |\alpha| t) = \mu^Y(x - \frac{u}{\alpha}, t)$$

and similarly, $\nu^Y(x - A, t) = \nu^Y(x - \frac{u}{\alpha}, t)$, which implies $\frac{u}{\alpha} \in P^Y_A(x, t)$ and hence $u \in \alpha P^Y_A(x, t)$.

The converse is obvious.

(iii) follows from (i).

Lemma 3.7. *Let $(X, \|.\|)$ be an n-normed space and μ, ν be fuzzy sets on $X^n \times (0, \infty)$ such that $\mu(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}$ and $\nu(x_1, x_2, \ldots, x_n, t) = \frac{\|x_1, x_2, \ldots, x_n\|}{t + \|x_1, x_2, \ldots, x_n\|}$, also let $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. Let A be a non-empty subset of the IFnNLS $(X, \mu, \nu, *, \circ)$. Then $u \in A$ is a best approximation to*

$x \in X$ in the n-normed space $(X, \|.\|)$ if and only if u is a t-best approximation to x in the IFnNLS $(X, \mu, \nu, *, \circ)$ for each $t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ of X.

Proof Since u is a best approximation to $x \in X$, for a linearly independent set $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ we have $\|y_1, y_2, \ldots, y_{n-1}, x - u\| = d^Y(x, A)$, where d is the metric induced by the n-norm $\|.\|$. Then

$$\begin{aligned} \mu^Y(x - A, t) = \frac{t}{t + d^Y(x, A)} &= \frac{t}{t + \|y_1, y_2, \ldots, y_{n-1}, x - u\|} \\ &= \mu^Y(x - u, t), \end{aligned}$$

and

$$\begin{aligned} \nu^Y(x - A, t) = \frac{d^Y(x, A)}{t + d^Y(x, A)} &= \frac{\|y_1, y_2, \ldots, y_{n-1}, x - u\|}{t + \|y_1, y_2, \ldots, y_{n-1}, x - u\|} \\ &= \nu^Y(x - u, t). \end{aligned}$$

Hence the result.

Remark 3.8. We recall that a set A is said to be countably compact, if every countable open cover has a finite subcover, or equivalently, if for every decreasing sequence $A_1 \supset A_2 \supset \ldots$ of non-empty closed subsets of A we have $\cap_{n=1}^{\infty} A_n \neq \phi$.

Theorem 3.9. *Let $(X, \mu, \nu, *, \circ)$ be an IFnNLS. If A is a non-empty subset of $X, \lambda \in (0, 1)$ and $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ is a linearly independent subset of X such that $A \cap B_x^Y[\lambda, t]$ is countably compact, then A is t-proximal.*

Proof For every $n \in \mathbb{N}$,

$$\begin{aligned} &0 < 1 - \mu^Y(x - A, t) + \frac{\mu^Y(x - A, t)}{n + 1} < 1, \\ &\text{and } 0 < \nu^Y(x - A, t) - \frac{\nu^Y(x - A, t)}{n + 1} < 1. \end{aligned}$$

Put $A_n^t = A \cap B_x^Y[1 - \mu^Y(x - A, t) + \frac{\mu^Y(x-A,t)}{n+1}, t] \cap B_x^Y[\nu^Y(x - A, t) - \frac{\nu^Y(x-A,t)}{n+1}, t]$ $(n = 1, 2, \ldots)$. We have $\ldots \supset A_n^t \supset A_{n+1}^t \supset \ldots$ and each A_n^t is non-empty. Since for every $n \in \mathbb{N}$, $\mu^Y(x - A, t)(1 - \frac{1}{n+1}) < \mu^Y(x - A, t)$ and $1 - \nu^Y(x - A, t) + \frac{\nu^Y(x-A,t)}{n+1} > \nu^Y(x - A, t)$, there exists $a_n^t \in A$ such that $\mu^Y(x - A, t)(1 - \frac{1}{n+1}) < \mu^Y(x - a_n^t, t)$ and $1 - \nu^Y(x - A, t) + \frac{\nu^Y(x-A,t)}{n+1} > \nu^Y(x - a_n^t, t)$. Hence $a_n^t \in A_n^t$. Since each A_n^t is countably compact and closed, it follows that there exists an $a_0 \in \cap_{n=1}^{\infty} A_n^t$. Then we have

$$\begin{aligned} &\mu^Y(x - A, t) \geq \mu^Y(x - a_0, t) \geq \mu^Y(x - A, t)(1 - \frac{1}{n + 1}) \quad (n = 1, 2, \ldots) \\ &\Rightarrow \mu^Y(x - A, t) = \mu^Y(x - a_0, t), \end{aligned}$$

and

$$\begin{aligned} &\nu^Y(x - A, t) \leq \nu^Y(x - a_0, t) \leq \nu^Y(x - A, t)(1 - \frac{1}{n + 1}) \quad (n = 1, 2, \ldots) \\ &\Rightarrow \nu^Y(x - A, t) = \nu^Y(x - a_0, t), \end{aligned}$$

whence $a_0 \in P_A^Y(x, t)$.

Definition 3.10. A non-empty subset A of an IFnNLS $(X, \mu, \nu, *, \circ)$ is said to be t-approximatively compact if for each $x \in X, t > 0$, a linearly independent subset $Y = \{y_1, y_2, \dots, y_{n-1}\}$ of X and each sequence $\{y_k\}$ in A with $\mu^Y(x - y_k, t) \to \mu^Y(x - A, t)$ and $\nu^Y(x - y_k, t) \to \nu^Y(x - A, t)$ there exists a subsequence $\{y_{k_n}\}$ of $\{y_k\}$ converging to an element u in A.

Lemma 3.11. *If A is*

(i) *approximately compact in an n-normed space $(X, \|.\|)$, then for each $t > 0$, A is t-approximatively compact in the induced IFnNLS $(X, \mu, \nu, *, \circ)$.*

(ii) *a compact subset of an IFnNLS, then A is t-approximatively compact for each $t > 0$.*

Theorem 3.12. *For $t > 0$, let A be a non-empty t-approximatively compact subset of an IFnNLS $(X, \mu, \nu, *, \circ)$, then A is a t-proximal set.*

Proof For $x \in X, t > 0$ and a linearly independent subset $Y = \{y_1, y_2, \dots, y_{n-1}\}$ of X there exists a sequence $\{y_k\} \subset A$ such that $\mu^Y(x - y_k, t) \to \mu^Y(x - A, t)$ and $\nu^Y(x - y_k, t) \to \nu^Y(x - A, t)$. Since A is a t-approximatively compact set, there exists a subsequence $\{y_{k_n}\}$ of $\{y_k\}$ and $u \in A$ such that $y_{k_n} \overset{(\mu,\nu)^n}{\to} u$. Thus we have $\mu^Y(x - y_{k_n}, t) \to \mu^Y(x - u, t)$ and $\nu^Y(x - y_{k_n}, t) \to \nu^Y(x - u, t)$.Hence $\mu^Y(x - u, t) \geq \mu^Y(x - A, t)$ and $\nu^Y(x - u, t) \leq \nu^Y(x - A, t)$. Consequently u is a t-best approximation to x from A.

Theorem 3.13. *If for some $t > 0$, A is a t-approximatively compact subset of an IFnNLS $(X, \mu, \nu, *, \circ)$, then A is closed in X.*

Proof Let $x \in \overline{A}$. Then for a linearly independent subset $Y = \{y_1, y_2, \dots, y_{n-1}\}$ of X, $\mu^Y(x - A, t) = 1$ and $\nu^Y(x - A, t) = 0$. Since A is t-approximatively compact, there exists $y \in A$ such that $\mu^Y(x - y, t) = \mu^Y(x - A, t) = 1$ and $\nu^Y(x - y, t) = \nu^Y(x - A, t) = 0$. Hence $x \in A$.

Theorem 3.14. *If A is a t-approximatively compact subset of an IFnNLS $(X, \mu, \nu, *, \circ)$, then A is a t-quasi-Chebyshev set.*

Proof Let $\{y_k\}$ be a sequence in $P_A^Y(x, t)$. Since A is t-approximatively compact, there exists subsequence $\{y_{k_n}\}$ of $\{y_k\}$ and $u \in A$ such that $y_{k_n} \overset{(\mu,\nu)^n}{\to} u$. Then $\mu^Y(x - y_{k_n}, t) \to \mu^Y(x - u, t)$ and $\nu^Y(x - y_{k_n}, t) \to \nu^Y(x - u, t)$. On the other hand, $\mu^Y(x - y_{k_n}, t) \to \mu^Y(x - A, t)$ and $\nu^Y(x - y_{k_n}, t) \to \nu^Y(x - A, t)$. Therefore $\mu^Y(x - u, t) = \mu^Y(x - A, t)$ and $\nu^Y(x - u, t) = \nu^Y(x - A, t)$ and so $u \in P_A^Y(x, t)$. Hence $P_A^Y(x, t)$ is compact.

Conclusion. Most of the results in this paper run parallel to those of classical ones or related works in best approximation theory, but in the proofs a different approach has been adopted as the convergence of a sequence in an IFnNLS is defined in a different way (see Definition 2.3) than it has been defined in [37]. Results obtained here are more general than previous works done in this field and can give tools to deal with convergence related problems arising in science and engineering.

Acknowledgements Research of the second author is supported by National Institute of Technology Silchar, under order No. NITS/D-SW/34/07 (Ministry of Human Resource Development, Government of India).

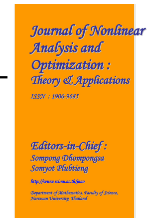
Also thanks to the reviewer for his valuable suggestions.

References

1. G. A. Anastassiou, Fuzzy approximation by fuzzy convolution type operators, Comput. Math. Appl. 48 (2004) 1369-1386.
2. K. Atanassov, Answer to D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade's paper "Terminological difficulties in fuzzy set theorey–the case of intuitionistic fuzzy sets, Fuzzy Sets Syst. 156 (2005) 496-499.
3. K. Atanassov, G. Pasi and R. Yager, Intuitionistic fuzzy interpretations of multi-person multicriteria decision making, Proceedings of 2002 First International IEEE Symposium Intelligent Systems, 1 (2002) 115-119.
4. K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets. Syst. 20 (1986) 87-96.
5. T. Bag and S. K. Samanta, A comparative study of fuzzy norms on a linear space, Fuzzy Sets. Syst. 159 (2008) 670-684.
6. G. Cattaneo and D. Ciucci, Basic intuitionistic principles in fuzzy set theories and its extensions (A terminological debate on Atanassov IFS), Fuzzy Sets. Syst. 157 (2006) 3198-3219.
7. D. Dubois, S. Gottwald, P. Hajek, J. Kacprzyk and H. Prade, Terminological difficulties in fuzzy set theory-The case of "Intuitionistic Fuzzy Sets", ,Fuzzy Sets. Syst. 156 (2005) 485-491.
8. M. A. Erceg, Metric spaces in fuzzy set theory, J. Math. Anal. Appl. 69 (1979) 205-230.
9. C. Felbin, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets Syst. 48 (1992) 239-248.
10. S. Gähler, Lineare 2-normietre Räume, Math. Nachr. 28 (1965) 1-43.
11. S. Gähler, Untersuchungen uber verallgemeinerte m-metrische Räume, I, II, III, Math. Nachr. 40 (1969) 165-189.
12. A. George and P. Veeramani, On some result in fuzzy metric space, Fuzzy Sets Syst. 64 (1994) 395-399.
13. P. Grzegorzewski and E. Mrówka, Some notes on (Atanassov's) intuitionistic fuzzy sets, Fuzzy Sets. Syst. 156 (2005) 492-495.
14. H. Gunawan and M. Mashadi, On n-normed spaces, Int. J. Math. Math. Sci. 27 (2001) 631-639.
15. G. Jäger, Fuzzy uniform convergence and equicontinuity, Fuzzy Sets Syst. 109 (2000) 187-198.
16. O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets Syst. 12 (1984) 215-229.
17. A. K. Katsaras, Fuzzy topological vector spaces, Fuzzy Sets Syst. 12 (1984) 143-154.
18. S. S. Kim and Y. J. Cho, Strict convexity in linear n-normed spaces, Demonst. Math. 29 (1996) 739-744.
19. J. Madore, Fuzzy physics, Ann. Phys. 219 (1992) 187-198.
20. R. Malceski, Strong n-convex n-normed spaces, Mat. Bilt. 21 (1997) 81-102.
21. A. Misiak, n-inner product spaces, Math. Nachr. 140 (1989) 299-319.
22. M. S. El Naschie, On the unification of heterotic strings, M-theory and ϵ^{∞}-theory, Chaos, Solitons & Fractals. 11 (2000) 2397-2408.
23. R. Saadati and J. H. Park, Intuitionistic fuzzy Euclidean normed spaces, Commun. Math. Anal. 12 (2006) 85-90.
24. R. Saadati and J. H. Park, On the intuitionistic fuzzy topological spaces, Chaos, Solitons & Fractals. 27 (2006) 331-344.
25. M. Sen and P. Debnath, Lacunary statistical convergence in intuitionistic fuzzy n-normed linear spaces, Math. Comp. Modelling. 54 (2011) 2978-2985.

26. M. Sen and P. Debnath, Statistical convergence in intuinionistic fuzzy n-normed linear spaces, Fuzzy Inf. Eng. 3 (2011) 259-273.
27. M. Shams and S. M. Vaezpour, Best approximation on probabilistic normed spaces, Chaos, Solitons & Fractals. 41 (2009) 1661-1667.
28. I. Singer, Best approximation in normed linear spaces by elements of linear subspaces, Springer-Verlag, 1970.
29. W. Sintunavarat, Y. J. Cho and P. Kuman, Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces, Fixed Point Th. Appl. 81 2011.
30. W. Sintunavarat and P. Kuman, Common fixed points for R-weakly commuting in fuzzy metric spaces, Annali dell'Universitá di Ferrara. (Accepted).
31. W. Sintunavarat and P. Kuman, Fixed point theorems for a generalized intuitionistic fuzzy contraction in intuitionistic fuzzy metric spaces, Thai J. Math., (in press).
32. W. Sintunavarat and P. Kuman, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math. (2011) DOI: 10.1155/2011/637958.
33. W. Sintunavarat and P. Kuman, Common fixed point theorems for generalized JH-operator classes and invariant approximations, J. Inequal. Appl. 67 (2011).
34. S. M. Vaezpour and F. Karimi, t-best approximation in fuzzy normed spaces, Iran. J. Fuzzy Syst. 5 (2) (2008) 93-99.
35. P. Veeramani, Best approximation in fuzzy metric spaces, J. Fuzzy Math. 9 (2001) 75-80.
36. S. Vijayabalaji and A. Narayanan, Fuzzy n-normed linear space, J. Math. Math. Sci. 24 (2005) 3963-3977.
37. S. Vijayabalaji, N. Thillaigovindan and Y. B. Jun, Intuitionistic fuzzy n-normed linear space, Bull. Korean. Math. Soc. 44 (2007) 291-308.
38. K. Wu, Convergences of fuzzy sets based on decomposition theory and fuzzy polynomial function, Fuzzy Sets Syst. 109 (2000) 173-185.
39. J. Z. Xiao and X. H.Zhu, Fuzzy normed spaces of operators and its completeness, Fuzzy Sets Syst. 133 (2003) 389-399.
40. L. A. Zadeh, Fuzzy sets, Inform. Cont. 8 (1965) 338-353.

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 55-66
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



COMMON FIXED POINT THEOREMS OF INTEGRAL TYPE IN MENGER PM SPACES

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ABSTRACT. In this paper, we propose integral type common fixed point theorems in Menger spaces satisfying common property $(E.A)$. Our results generalize several previously known results in Menger as well as metric spaces. Some related results are also derived besides furnishing an illustrative example.

KEYWORDS : Menger space; Common property $(E.A)$; Weakly compatible pair of mappings and t-norm.

1. INTRODUCTION AND PRELIMINARIES

Menger [24] initiated the study of probabilistic metric space (often abbreviated as PM space) in 1942 and by now the theory of probabilistic metric spaces has already made a considerable progress in several directions (see[29]). The idea of Menger was to use distribution functions (instead of non-negative real numbers) as values of a probabilistic metric. This new notion (i.e.PM space) can cover even those situations wherein one can not exactly ascertain a distance between two points, but can only know the possibility of a possible value for the distance (between a pair of points). This probabilistic generalization of metric space is well utilized in the investigations of physiological thresholds besides physical quantities particularly in connections with both string and E-infinity theory (cf.[10]).

In 1986, Jungck [18] introduced the notion of compatible mappings and utilized the same to improve commutativity conditions in common fixed point theorems. This concept has been frequently employed to prove existence theorems on common fixed points. However, the study of common fixed points of non-compatible mappings is also equally interesting which was initiated by Pant [29]. Recently, Aamri and Moutawakil [1] and Liu et al. [23] respectively defined the property

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Article history : Received 15 November 2010. Accepted 27 December 2011.

$(E.A)$ and the common property $(E.A)$ and proved interesting common fixed point theorems in metric spaces. Most recently, Kubiaczyk and Sharma [21] adopted the property $(E.A)$ in PM spaces and used it to prove results on common fixed points wherein authors claim their results for strict contractions which are merely for contractions. Recently, Imdad et al. [16] adopted the common property $(E.A)$ in PM spaces and proved some coincidence and common fixed point results in Menger spaces.

The theory of fixed points in PM spaces is a part of probabilistic analysis and continues to be an active area of mathematical research. Thus far, several authors studied fixed point and common fixed point theorems in PM spaces which include [2, 3, 7, 12, 13, 15, 16, 20, 27, 28, 30, 31, 33, 34, 35] besides many more. In 2002, Branciari [5] obtained a fixed point result for a mapping satisfying an integral analogue of Banach contraction principle. The authors of the papers [4, 9, 16, 32, 37, 38] proved a host of fixed point theorems involving relatively more general integral type contractive conditions. In an interesting note, Suzuki [36] showed that Meir-Keeler contractions of integral type are still Meir-Keeler contractions.

The aim of this paper is to prove integral type fixed point theorems in Menger PM spaces satisfying common property $(E.A)$. Our results substantially improve the corresponding theorems contained in [5, 8, 16, 32, 38] along with some other relevant results in Menger as well as metric spaces. Some related results are also derived besides furnishing an illustrative example.

Definition 1.1. [33] A mapping $F : \Re \to \Re^+$ is called distribution function if it is non-decreasing, left continuous with $\inf\{F(t) : t \in \Re\} = 0$ and $\sup\{F(t) : t \in \Re\} = 1$.

Let L be the set of all distribution functions whereas H be the set of specific distribution functions (also known as Heaviside function) defined by

$$H(x) = \begin{cases} 0, \text{ if } x \leq 0 \\ 1, \text{ if } x > 0. \end{cases}$$

Definition 1.2. [24] Let X be a non-empty set. An ordered pair $(X, \mathcal{F})$ is called a PM space if $\mathcal{F}$ is a mapping from $X \times X$ into L satisfying the following conditions:

(i) $F_{p,q}(x) = H(x)$ if and only if $p = q$,
(ii) $F_{p,q}(x) = F_{q,p}(x)$,
(iii) $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$, then $F_{p,r}(x + y) = 1$, for all $p, q, r \in X$ and $x, y \geq 0$.

Every metric space (X, d) can always be realized as a PM space by considering $\mathcal{F} : X \times X \to L$ defined by $F_{p,q}(x) = H(x - d(p, q))$ for all $p, q \in X$. So PM spaces offer a wider framework (than that of the metric spaces) and are general enough to cover even wider statistical situations.

Definition 1.3. [33] A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if

(i) $\Delta(a, 1) = a, \Delta(0, 0) = 0$,
(ii) $\Delta(a, b) = \Delta(b, a)$,
(iii) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a, d \geq b$,
(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$ for all $a, b, c \in [0, 1]$.

Example 1.4. The following are the four basic t-norms:

(i) The minimum t-norm: $T_M(a, b) = \min\{a, b\}$.

(ii) The product t-norm: $T_P(a,b) = a.b$.

(iii) The Lukasiewicz t-norm: $T_L(a,b) = \max\{a+b-1, 0\}$.

(iv) The weakest t-norm, the drastic product:

$$T_D(a,b) = \begin{cases} \min\{a,b\}, \text{ if } \max\{a,b\} = 1 \\ \\ 0, \text{ otherwise.} \end{cases}$$

In respect of above mentioned t-norms, we have the following ordering:

$T_D < T_L < T_P < T_M$.

Definition 1.5. [24] A Menger PM space $(X, \mathcal{F}, \Delta)$ is a triplet where $(X, \mathcal{F})$ is a PM space and Δ is a t-norm satisfying the following condition:

$$F_{p,r}(x+y) \geq \Delta(F_{p,q}(x), F_{q,r}(y)).$$

Definition 1.6. [12] A sequence $\{p_n\}$ in a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to be convergent to a point p in X if for every $\epsilon > 0$ and $\lambda > 0$, there is an integer $M(\epsilon, \lambda)$ such that $F_{p_n,p}(\epsilon) > 1 - \lambda$, for all $n \geq M(\epsilon, \lambda)$.

Lemma 1.7. [33, 26] *Let $(X, \mathcal{F}, \Delta)$ be a Menger space with a continuous t-norm Δ with $\{x_n\}, \{y_n\} \subset X$ such that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y. If $F_{x,y}(.)$ is continuous at the point t_0, then $\lim\limits_{n \longrightarrow \infty} F_{x_n,y_n}(t_0) = F_{x,y}(t_0)$.*

Definition 1.8. Let (A, S) be a pair of maps from a Menger PM space $(X, \mathcal{F}, \Delta)$ into itself. Then the pair of maps (A, S) is said to be weakly commuting if

$$F_{ASx,SAx}(t) \geq F_{Ax,Sx}(t),$$

for each $x \in X$ and $t > 0$.

Definition 1.9. [28] A pair (A, S) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to be compatible if $F_{ASp_n,SAp_n}(x) \longrightarrow 1$ for all $x > 0$, whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \longrightarrow t$, for some t in X as $n \longrightarrow \infty$.

Clearly, a weakly commuting pair is compatible but every compatible pair need not be weakly commuting.

Definition 1.10. [11] A pair (A, S) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to be non-compatible if and only if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = t \in X \text{ for some } t \in X,$$

implies that $\lim\limits_{n \longrightarrow \infty} F_{ASx_n,SAx_n}(t_0)$ (for some $t_0 > 0$) is either less than 1 or non-existent.

Definition 1.11. [21] A pair (A, S) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ is said to satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = t \in X.$$

Clearly, a pair of compatible mappings as well as non-compatible mappings satisfies the property $(E.A)$.

Inspired by Liu et al. [23], Imdad et al. [16] defined the following:

Definition 1.12. Two pairs (A, S) and (B, T) of self mappings of a Menger PM space $(X, \mathcal{F}, \Delta)$ are said to satisfy the common property $(E.A)$ if there exist two sequences $\{x_n\}, \{y_n\}$ in X and some t in X such that

$$\lim_{n\longrightarrow\infty} Ax_n = \lim_{n\longrightarrow\infty} Sx_n = \lim_{n\longrightarrow\infty} Ty_n = \lim_{n\longrightarrow\infty} By_n = t.$$

Definition 1.13. [19] A pair (A, S) of self mappings of a nonempty set X is said to be weakly compatible if the pair commutes on the set of their coincidence points i.e. $Ap = Sp$ (for some $p \in X$) implies $ASp = SAp$.

Definition 1.14. [15] Two finite families of self mappings $\{A_i\}$ and $\{B_j\}$ are said to be pairwise commuting if:

(i) $A_iA_j = A_jA_i,\ i, j \in \{1, 2...m\}$,
(ii) $B_iB_j = B_jB_i,\ i, j \in \{1, 2...n\}$,
(iii) $A_iB_j = B_jA_i,\ i \in \{1, 2...m\}, j \in \{1, 2...n\}$.

2. MAIN RESULTS

The following lemma is crucial in the proof of succeeding theorems.

Lemma 2.1. *Let $(X, \mathcal{F}, \Delta)$ be a Menger space. If there exists some $k \in (0, 1)$ such that for all $p, q \in X$ and all $x > 0$,*

$$\int_0^{F_{p,q}(kx)} \phi(t)dt \geq \int_0^{F_{p,q}(x)} \phi(t)dt, \tag{1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a summable nonnegative Lebesgue integrable function such that $\int_\epsilon^1 \phi(t)dt > 0$ for each $\epsilon \in [0, 1)$, then $p = q$.

Proof. As

$$\int_0^{F_{p,q}(kx)} \phi(t)dt \geq \int_0^{F_{p,q}(x)} \phi(t)dt$$

implies

$$\int_0^{F_{p,q}(x)} \phi(t)dt \geq \int_0^{F_{p,q}(k^{-1}x)} \phi(t)dt,$$

one can inductively write (for $m \in N$)

$$\begin{aligned}\int_0^{F_{p,q}(x)} \phi(t)dt \geq \int_0^{F_{p,q}(k^{-1}x)} \phi(t)dt \geq ... &\geq \int_0^{F_{p,q}(k^{-m}x)} \phi(t)dt \\ &\geq ... \to \int_0^1 \phi(t)dt \text{ as } m \to \infty.\end{aligned}$$

Therefore

$$\int_0^{F_{p,q}(x)} \phi(t)dt - \int_0^1 \phi(t)dt \geq 0$$

and henceforth

$$\int_0^{F_{p,q}(x)} \phi(t)dt - \left(\int_0^{F_{p,q}(x)} \phi(t)dt + \int_{F_{p,q}(x)}^1 \phi(t)dt\right) \geq 0$$

or

$$\int_{F_{p,q}(x)}^1 \phi(t)dt \leq 0$$

which amounts to say that $F_{p,q}(x) \geq 1$ for all $x \geq 0$. Thus, we get $p = q$. □

Remark 2.2. By setting $\phi(t) = 1$ (for each $t \geq 0$) in (1.1) of Lemma 2.1, we have

$$\int_0^{F_{p,q}(kx)} \phi(t)dt = F_{p,q}(kx) \geq F_{p,q}(x) = \int_0^{F_{p,q}(x)} \phi(t)dt,$$

which shows that Lemma 2.1 is a generalization of the Lemma 2 (contained in [35])

In what follows, Δ is a continuous t-norm (in the product topology).

Lemma 2.3. *Let A, B, S and T be four self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ which satisfy the following conditions: (i) the pair (A, S) (or (B, T)) satisfies the property $(E.A)$,*

(ii) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges,

(iii) for any $p, q \in X$ and for all $x > 0$,

$$\int_0^{F_{Ap,Bq}(kx)} \phi(t)dt \geq \int_0^{m(x,y)} \phi(t)dt \tag{2.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a non-negative summable Lebesgue integrable function such that $\int_\epsilon^1 \phi(t)dt > 0$ for each $\epsilon \in [0, 1)$, where $0 < k < 1$ and

$$m(x, y) = \min\{F_{Sp,Tq}(x), F_{Sp,Ap}(x), F_{Tq,Bq}(x), F_{Sp,Bq}(x), F_{Tq,Ap}(x)\},$$

(iv) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$).

Then the pairs (A, S) and (B, T) share the common property $(E.A)$.

Proof. Suppose that the pair (A, S) enjoys the property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \longrightarrow \infty} Ax_n = \lim_{n \longrightarrow \infty} Sx_n = t, \text{ for some } t \in X.$$

Since $A(X) \subset T(X)$, for each x_n there exists $y_n \in X$ such that $Ax_n = Ty_n$, and henceforth

$$\lim_{n \to \infty} Ty_n = \lim_{n \to \infty} Ax_n = t.$$

Thus in all, we have $Ax_n \to t, Sx_n \to t$ and $Ty_n \to t$. Now we assert that $By_n \to t$. To accomplish this, using (2.1), with $p = x_n, q = y_n$, one gets

$$\int_0^{F_{Ax_n,By_n}(kx)} \phi(t)dt \geq \int_0^{m(x,y)} \phi(t)dt$$

$$\geq \int_0^{\min(F_{Sx_n,Ty_n}(x), F_{Sx_n,Ax_n}(x), F_{Ty_n,By_n}(x), F_{Sx_n,By_n}(x), F_{Ty_n,Ax_n}(x))} \phi(t)dt.$$

Let $l = \lim_{n \longrightarrow \infty} B(y_n)$. Also, let $x > 0$ be such that $F_{t,l}(\cdot)$ is continuous in x and kx. Then, on making $n \to \infty$ in the above inequality, we obtain

$$\int_0^{F_{t,l}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,t}(x), F_{t,t}(x), F_{t,l}(x), F_{t,l}(x), F_{t,t}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,l}(kx)} \phi(t)dt \geq \int_0^{F_{t,l}(x)} \phi(t)dt.$$

This implies that $l = t$ (in view of Lemma 2.1) which shows that the pairs (A, S) and (B, T) share the common property $(E.A)$. $\square$

Remark 2.4. The converse of Lemma 2.3 is not true in general. For a counter example, one can see Example 3.4 furnished in the end of this paper.

Theorem 2.5. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ which satisfy the inequality (2.1) together with*

(i) the pairs (A, S) and (B, T) share the common property $(E.A)$,

(ii) $S(X)$ and $T(X)$ are closed subsets of X.

Then the pairs (A, S) and (B, T) have a point of coincidence each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof. Since the pairs (A, S) and (B, T) share the common property $(E.A)$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \quad \text{for some } t \in X.$$

Since $S(X)$ is a closed subset of X, hence $\lim_{n\to\infty} Sx_n = t \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = t$. Now, we assert that $Au = Su$. To prove this, on using (2.1) with $p = u, q = y_n$, one gets

$$\int_0^{F_{Au,By_n}(kx)} \phi(t)dt \geq \int_0^{\min(F_{Su,Ty_n}(x), F_{Su,Au}(x), F_{Ty_n,By_n}(x), F_{Su,By_n}(x), F_{Ty_n,Au}(x))} \phi(t)dt$$

which on making $n \to \infty$, reduces to

$$\int_0^{F_{Au,t}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,t}(x), F_{t,Au}(x), F_{t,t}(x), F_{t,t}(x), F_{t,Au}(x))} \phi(t)dt$$

or

$$\int_0^{F_{Au,t}(kx)} \phi(t)dt \geq \int_0^{F_{Au,t}(x)} \phi(t)dt.$$

Now appealing to Lemma 2.1, we have $Au = t$ and henceforth $Au = Su$. Therefore, u is a coincidence point of the pair (A, S).

Since $T(X)$ is a closed subset of X, therefore $\lim_{n\to\infty} Ty_n = t \in T(X)$ and hence one can find a point $w \in X$ such that $Tw = t$. Now we show that $Bw = Tw$. To accomplish this, on using (2.1) with $p = x_n, q = w$, we have

$$\int_0^{F_{Ax_n,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{Sx_n,Tw}(x), F_{Sx_n,Ax_n}(x), F_{Tw,Bw}(x), F_{Sx_n,Bw}(x), F_{Tw,Ax_n}(x))} \phi(t)dt$$

which on making $n \to \infty$, reduces to

$$\int_0^{F_{t,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,t}(x), F_{t,t}(x), F_{t,Bw}(x), F_{t,Bw}(x), F_{t,t}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,Bw}(kx)} \phi(t)dt \geq \int_0^{F_{t,Bw}(x)} \phi(t)dt.$$

On employing Lemma 2.1, we have $Bw = t$ and henceforth $Tw = Bw$. Therefore, w is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$, therefore

$$At = ASu = SAu = St.$$

Again, on using (2.1) with $p = t, q = w$, we have

$$\int_0^{F_{At,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{St,Tw}(x),F_{St,At}(x),F_{Tw,Bw}(x),F_{St,Bw}(x),F_{Tw,At}(x))} \phi(t)dt$$

or

$$\int_0^{F_{At,t}(kx)} \phi(t)dt \geq \int_0^{\min(F_{At,t}(x),F_{t,t}(x),F_{t,t}(x),F_{At,t}(x),F_{t,At}(x))} \phi(t)dt$$

or

$$\int_0^{F_{At,t}(kx)} \phi(t)dt \geq \int_0^{F_{At,t}(x)} \phi(t)dt.$$

Appealing to Lemma 2.1, we have $At = St = t$ which shows that t is a common fixed point of the pair (A, S).

Also the pair (B, T) is weakly compatible and $Bw = Tw$, hence

$$Bt = BTw = TBw = Tt.$$

Next, we show that t is a common fixed point of the pair (B, T). In order to accomplish this, using (2.1) with $p = u, q = t$, we have

$$\int_0^{F_{Au,Bt}(kx)} \phi(t)dt \geq \int_0^{\min(F_{Su,Tt}(x),F_{Su,Au}(x),F_{Tt,Bt}(x),F_{Su,Bt}(x),F_{Tt,Au}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,Bt}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,Bt}(x),F_{t,t}(x),F_{Bt,Bt}(x),F_{t,Bt}(x),F_{Bt,t}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,Bt}(kx)} \phi(t)dt \geq \int_0^{F_{t,Bt}(x)} \phi(t)dt.$$

Using Lemma 2.1, we have $Bt = t$ which shows that t is a common fixed point of the pair (B, T). Hence t is a common fixed point of both the pairs (A, S) and (B, T). Uniqueness of common fixed point is an easy consequence of the inequality (2.1). This completes the proof. □

Remark 2.6. Theorem 2.5 extends the main result of Ciric [8] to Menger spaces. Theorem 2.5 also generalizes the main result of Kubiaczyk and Sharma [21] for two pairs of mappings without conditions on containments amongst range sets of the involved mappings.

Theorem 2.7. *The conclusions of Theorem 2.5 remain true if the condition (ii) of Theorem 2.5 is replaced by the following: (iii)′ $\overline{A(X)} \subset T(X)$ and $\overline{B(X)} \subset S(X)$.*

Corollary 2.8. *The conclusions of Theorems 2.5 and 2.7 remain true if the condition (ii) (of Theorem 2.5) and (iii)′ (of Theorem 2.7)are replaced by the following: (iv) $A(X)$*

and $B(X)$ are closed subsets of X provided $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Theorem 2.9. *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ satisfying the inequality (2.1). Suppose that*

(i) the pair (A, S) (or (B, T)) has property $(E.A)$,

(ii) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges,

(iii) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$),

(iii) $S(X)$ *(or* $T(X)$*) is a closed subset of* X.

Then the pairs (A,S) *and* (B,T) *have a point of coincidence each. Moreover,* A,B,S *and* T *have a unique common fixed point provided both the pairs* (A,S) *and* (B,T) *are weakly compatible.*

Proof. In view of Lemma 2.3, the pairs (A,S) and (B,T) share the common property $(E.A)$, i.e. there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \text{ for some } t \in X.$$

If $S(X)$ is a closed subset of X, then proceeding on the lines of Theorem 2.5, one can show that the pair (A,S) has a coincidence point, say u, i.e. $Au = Su = t$. Since $A(X) \subset T(X)$ and $Au \in A(X)$, there exists $w \in X$ such that $Au = Tw$. Now, we assert that $Bw = Tw$.
On using (2.1) with $p = x_n, q = w$, one gets

$$\int_0^{F_{Ax_n,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{Sx_n,Tw}(x),F_{Sx_n,Ax_n}(x),F_{Tw,Bw}(x),F_{Sx_n,Bw}(x),F_{Tw,Ax_n}(x))} \phi(t)dt$$

which on making $n \to \infty$, reduces to

$$\int_0^{F_{t,Bw}(kx)} \phi(t)dt \geq \int_0^{\min(F_{t,t}(x),F_{t,t}(x),F_{t,Bw}(x),F_{t,Bw}(x),F_{t,t}(x))} \phi(t)dt$$

or

$$\int_0^{F_{t,Bw}(kx)} \phi(t)dt \geq \int_0^{F_{t,Bw}(x)} \phi(t)dt.$$

Owing to Lemma 2.1, we have $t = Bw$ and hence $Tw = Bw$ which shows that w is a coincidence point of the pair (B,T). Rest of the proof can be completed on the lines of the proof of Theorem 2.5. This completes the proof.

By choosing A, B, S and T suitably, one can deduce corollaries involving two or three mappings. As a sample, we deduce the following natural result for a pair of self mappings. □

Corollary 2.10. *Let* A *and* S *be self mappings on a Menger space* $(X, \mathcal{F}, \Delta)$*. Suppose that*

(i) the pair (A,S) *enjoys the property* $(E.A)$,

(ii) for all $p, q \in X$ *and for all* $x > 0$,

$$\int_0^{F_{Ap,Aq}(kx)} \phi(t)dt \geq \int_0^{m(x,y)} \phi(t)dt \tag{2.2}$$

where $m(x,y) = \min\{F_{Sp,Sq}(x), F_{Sp,Ap}(x), F_{Sq,Aq}(x), F_{Sp,Aq}(x), F_{Sq,Ap}(x)\}, 0 < k < 1$

(iii) $S(X)$ *is a closed subset of* X.

Then A *and* S *have a coincidence point. Moreover, if the pair* (A,S) *is weakly compatible, then* A *and* S *have a unique common fixed point.*

As an application of Theorem 2.5, we have the following result for four finite families of self mappings. While proving our result, we utilize Definition 1.14 which is a natural extension of commutativity condition to two finite families of mappings.

Theorem 2.11. *Let $\{A_1, A_2, ...A_m\}, \{B_1, B_2, ...B_p\}, \{S_1, S_2, ...S_n\}$ and $\{T_1, T_2, ...T_q\}$ be four finite families of self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ with $A = A_1A_2...A_m, B = B_1B_2...B_p, S = S_1S_2...S_n$ and $T = T_1T_2...T_q$ satisfying the condition (2.1). If $S(X)$ and $T(X)$ are closed subsets of X, and the pairs (A, S) and (B, T) share the common property $(E.A)$, then*

(i) *the pair (A, S) as well as (B, T) has a coincidence point,*
(ii) *A_i, B_k, S_r and T_t have a unique common fixed point provided the pair of families $(\{A_i\}, \{S_r\})$ and $(\{B_k\}, \{T_t\})$ commute pairwise.*

Proof. The proof follows on the lines of Theorem 4.1 due to Imdad and Ali [14] and Theorem 3.1 due to Imdad et al. [15]. □

Remark 2.12. By restricting four families as $\{A_1, A_2\}, \{B_1, B_2\}, \{S_1\}$ and $\{T_1\}$ in Theorem 2.11 we get improved version of results due to Chugh and Rathi [7], Kutukcu and Sharma [22], Rashwan and Hedar [30], Singh and Jain [35] and others. Theorem 2.11 also generalizes the main result of Razani and Shirdaryazdi [31] for any finite number of mappings.

By setting $A_1 = A_2 = ... = A_m = G, B_1 = B_2 = ... = B_p = H, S_1 = S_2 = ...S_n = I$ and $T_1 = T_2 = ... = T_q = J$ in Theorem 2.11, we deduce the following:

Corollary 2.13. *Let G, H, I and J be self mappings of a Menger space $(X, \mathcal{F}, \Delta)$ such that the pairs (G^m, I^n) and (H^p, J^q) share the common property $(E.A)$ and also satisfies the condition*

$$\int_0^{F_{G^m x, H^p y}(kz)} \phi(t)dt \geq \int_0^{m(x,y)} \phi(t)dt$$

(where $m(x, y) = \min\{F_{I^n x, J^q y}(z), F_{I^n x, G^m x}(z), F_{I^n x, H^p y}(z), F_{J^q y, H^p y}(z), F_{J^q y, G^m x}(z)\}$) for all $x, y \in X$, $\forall\ z > 0$ where $k \in (0, 1)$ and m, n, p and q are fixed positive integers. If $I^n(X)$ and $J^q(X)$ are closed subsets of X, then G, H, I and J have a unique common fixed point provided $GI = IG$ and $HJ = JH$.

Remark 2.14. Corollary 2.13 is a slight but partial generalization of Theorem 2.5 as the commutativity requirements (i.e. $GI = IG$ and $HJ = JH$) in this corollary are relatively stronger as compared to weak compatibility (in Theorem 2.5). Corollary 2.13 also presents a generalized and improved form of a result due to Bryant [6] in Menger PM spaces.

3. RELATED RESULTS AND AN EXAMPLE

In this section, we utilize Theorem 2.5 and Theorem 2.9 [16, 14] as means to derive corresponding common fixed point theorems in metric spaces.

Theorem 3.1. *Let A, B, S and T be self mappings of a metric space (X, d). Suppose that*

(i) the pairs (A, S) and (B, T) share the common property $(E.A)$,

(ii) $S(X)$ and $T(X)$ are closed subsets of X,

(iii) for all $x, y \in X$

$$\int_0^{d(Ax,By)} \phi(t)dt \leq k \int_0^{m(x,y)} \phi(t)dt \tag{3.1}$$

where $m(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty)\}$, and $0 < k < 1$.

Then the pairs (A,S) and (B,T) have a point of coincidence each. Moreover, A,B,S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Proof. Define $F_{x,y}(t) = H(t - d(x,y))$ and $\Delta(a,b) = \min\{a,b\}$. Then $(X,\mathcal{F},\Delta)$ is a Menger space induced by the metric space (X,d). It is straight forward to notice that the conditions (i) and (ii) of Theorem 3.1 respectively imply conditions (i) and (ii) of Theorem 2.5. Also inequality (3.1) of Theorem 3.1 implies inequality (2.1) of Theorem 2.5. To accomplish this notice that (for any $x,y \in X$ and $t > 0$), $F_{Ax,By}(kt) = 1$ provided $kt > d(Ax,By)$ which amounts to say that (2.1) holds. Otherwise, if $kt \leq d(Ax,By)$, then

$$t \leq \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Sx,By), d(Ax,Ty)\},$$

and hence in all the cases, condition (2.1) holds. Thus, all the conditions of Theorem 2.5 are satisfied and conclusions follow immediately in view of Theorem 2.5. □

Remark 3.2. Theorem 3.1 improves the main result of Ciric [8] and several other similar common fixed point theorems especially those contained in [14, 15, 22, 28]) as we never require any condition on the containment of ranges amongst involved mappings besides weakening the completeness of the space to closedness of suitable subsets along with improvement in commutativity considerations. Here, one may also notice that all the involved mappings can be discontinuous (at the same time).

Remark 3.3. Similarly, we can also apply our other results (i.e. Theorems 2.7-2.11 and Corollaries 2.8-2.13) to derive the corresponding common fixed point theorems in metric spaces but here details are avoided.

We conclude this paper by furnishing an illustrative example to demonstrate the validity of the hypotheses of Theorem 2.5.

Example 3.4. Consider $X = [-1,1]$ and define $F_{x,y}(t) = H(t - |x-y|)$ for all $x,y \in X$. Then $(X,\mathcal{F},\Delta)$ is a Menger PM space with $\Delta(a,b) = \min\{a,b\}$. Define self mappings A,B,S and T on X as

$$A(x) = \begin{cases} \frac{3}{5}, & \text{if } x \in \{-1,1\} \\ \frac{x}{4}, & \text{if } x \in (-1,1), \end{cases} \qquad B(x) = \begin{cases} \frac{3}{5}, & \text{if } x \in \{-1,1\} \\ \frac{-x}{4}, & \text{if } x \in (-1,1), \end{cases}$$

$$S(x) = \begin{cases} \frac{1}{2}, & \text{if } x = -1 \\ \frac{x}{2}, & \text{if } x \in (-1,1) \\ \frac{-1}{2}, & \text{if } x = 1 \end{cases} \quad \text{and } T(x) = \begin{cases} \frac{-1}{2}, & \text{if } x = -1 \\ \frac{-x}{2}, & \text{if } x \in (-1,1) \\ \frac{1}{2}, & \text{if } x = 1. \end{cases}$$

Then with sequences as $\{x_n = \frac{1}{n}\}$ and $\{y_n = \frac{-1}{n}\}$ in X, we have

$$\lim_{n\longrightarrow\infty} Sx_n = \lim_{n\longrightarrow\infty} Ax_n = \lim_{n\longrightarrow\infty} By_n = \lim_{n\longrightarrow\infty} Ty_n = 0$$

which shows that pairs (A,S) and (B,T) share the common property (E.A). By a routine calculation, one can verify the contraction condition (2.1) with $k = \frac{1}{2}$. Also,

$$A(X) = B(X) = \left\{\frac{3}{5}\right\} \cup \left(-\frac{1}{4}, \frac{1}{4}\right) \not\subset \left[-\frac{1}{2}, \frac{1}{2}\right] = S(X) = T(X).$$

Thus, all the conditions of Theorem 2.5 are satisfied and 0 is a unique common fixed point of the pairs (A,S) and (B,T) which is their coincidence point as well.

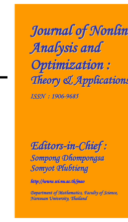
Here it is worth noting that majority of earlier established theorems (with rare possible exceptions) cannot be used in the context of this example as Theorem 2.5 never requires any condition on the containment of ranges of the involved mappings. Also the completeness condition is replaced by the closedness of the subspaces. Moreover, the continuity requirements of all the involved mappings are completely relaxed whereas most of the earlier theorems require the continuity of at least one involved mapping.

References

1. A. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270(2002), 181-188.
2. J. Ali, M. Imdad and D. Bahuguna, Common fixed point theorems in Menger spaces with common property (E.A), Comp. Math. Appl., 60(2010), 3152-3159.
3. J. Ali, M. Imdad, D. Miheţ, and M. Tanveer, Common fixed points of strict contractions in Menger spaces, Acta Math. Hung., 132(4)(2011), 367-386.
4. A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl., 322(2)(2006), 796-802.
5. A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci., 29(9)(2002), 531-536.
6. V.W. Bryant, A remark on fixed point theorem for iterated mappings, Amer. Math. Monthly, 75 (1968), 399-400.
7. R. Chugh and S. Rathi, Weakly compatible maps in probabilistic metric spaces, J. Indian Math. Soc., 72 (2005), 131-140.
8. Lj.B. Cirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., 45 (1974), 267-273.
9. A. Djoudi and A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type, J. Math. Anal. Appl., 329 (1) (2007), 31-45.
10. MS El Naschie, A review of applications and results of E-infinity theory, Int. J. Nonlinear Sci. Numer. Simul. 8(2007), 11-20.
11. J.X. Fang and Y. Gao, Common fixed point theorems under strict contractive conditions in Menger spaces, Nonlinear Analysis, 70 (2009). 184-193.
12. O. Hadzić and E. Pap, Fixed point theory in probabilistic metric spaces, Kluwer Academic Publishers, Dordrecht, 2001.
13. T.L. Hicks, Fixed point theory in probabilistic metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 13 (1983), 63-72.
14. M. Imdad and J. Ali, Jungck's common fixed point theorem and E.A property, Acta Math. Sinica, 24 (2008), 87-94.
15. M. Imdad, Javid Ali and M. Tanveer, Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces, Chaos, Solitons & Fractals, 42 (2009), 3121-3129.
16. M. Imdad, M. Tanveer and M. Hasan, Some commn fixed point theorems in Menger PM spaces, Fixed Point Theory Appl., Vol. 2010, 14 pages.
17. M. Imdad, M. Tanveer and M. Hasan, Erratum to "Some common fixed point theorems in Menger PM Spaces", Fixed Point Theory Appl., 2011, 2011:28.
18. G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9 (4) (1986), 771-779.
19. G. Jungck, Common fixed points for noncontinuous nonself maps on nonmetric spaces. Far East J. Math. Sci., 4 (2) (1996), 199-215.
20. J.K. Kohli and S. Vashistha, Common fixed point theorems in probabilistic metric spaces, Acta Math Hung., 115 (2007), 37-47.
21. I. Kubiaczyk and S. Sharma, Some common fixed point theorems in Menger space under strict contractive conditions, Southeast Asian Bull. Math., 32 (2008), 117-124.
22. S. Kutukcu and S. Sharma, Compatible maps and common fixed points in Menger probabilistic metric spaces, Commun. Korean Math. Soc., 24 (2009), 17-27.
23. Y. Liu, Jun Wu and Z. Li, Common fixed points of single-valued and multi-valued maps, Int. J. Math. Math. Sci., 19 (2005), 3045-3055.
24. K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. USA, 28 (1942), 535-537.
25. K. Menger, Probabilistic geometry, Proc. Nat. Acad. Sci. USA, 37 (1951), 226-229.

26. D. Miheţ, A generalization of a contraction principle in probabilistic metric spaces (II), Int. J. Math. Math. Sci 5 (2005), 729-736.
27. D. Miheţ, Fixed point theorems in fuzzy metric spaces using property E.A., Nonlinear Anal., 73 (2010), 2184-2188.
28. S.N. Mishra, Common fixed points of compatible mappings in PM-spaces, Math. Japon., 36 (1991), 283-289.
29. R.P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl., 188 (1994), 436-440.
30. R.A. Rashwan and A. Hedar, On common fixed point theorems of compatible maps in Menger spaces, Demonst. Math., 31 (1998), 537-546.
31. A. Razani and M. Shirdaryazdi, A common fixed point theorem of compatible maps in Menger space, Chaos, Solitions and Fractals, 32 (2007), 26-34.
32. B. E. Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type, Internat. J. Math. Math. Sci., 63(2003), 4007-4013.
33. B. Schweizer and A. Sklar, Probabilistic metric spaces, Elsevier, North Holand, New York, 1983.
34. V.M. Sehgal and A.T. Bharucha-Reid, Fixed point of contraction mappings on probabilistic metric spaces, Math. Systems Theory, 6 (1972), 97-102.
35. B. Singh and S. Jain, A fixed point theorem in Menger spaces through weak compatibility, J. Math. Anal. Appl., 301 (2005), 439-448.
36. T. Suzuki, Meir-Keeler contractions of integral type are still Meir-Keeler contractions, Internat. J. Math. Math. Sci., (2007), Article ID 39281, 6 pages.
37. D. Turkoglu and I. Altun, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying an implicit relation, Matematica Mexicana. Boletin.Tercera Serie., 13 (1) (2007), 195-205.
38. P. Vijayaraju, B.E. Rhoades and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, Internat. J. Math. Math. Sci., 15(2005), 2359-2364.

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 67-77
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



STRONG CONVERGENCE THEOREMS FOR STRONGLY RELATIVELY NONEXPANSIVE SEQUENCES AND APPLICATIONS

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ABSTRACT. The aim of this paper is to establish strong convergence theorems for a strongly relatively nonexpansive sequence in a smooth and uniformly convex Banach space. Then we employ our results to approximate solutions of the zero point problem for a maximal monotone operator and the fixed point problem for a relatively nonexpansive mapping.

KEYWORDS : Strongly relatively nonexpansive sequence; Common fixed point; Strong convergence theorem.
AMS Subject Classification: 47H09, 47H10, 41A65.

1. INTRODUCTION

Let E be a smooth and uniformly convex Banach space, E^* the dual of E, $A \subset E \times E^*$ a maximal monotone operator with a zero point, and $\{r_n\}$ a sequence of positive real numbers. Assume that $\{x_n\}$ is a sequence defined as follows: $x_1 \in E$ and

$$x_{n+1} = J^{-1}\left(\frac{1}{n}Jx + \left(1 - \frac{1}{n}\right)J(J + r_nA)^{-1}Jx_n\right)$$

for $n \in \mathbb{N}$, where J and J^{-1} are the duality mappings of E and E^*, respectively. It is known [9] that if $r_n \longrightarrow \infty$, then $\{x_n\}$ converges strongly to some zero point of A. However, we have not known whether $\{x_n\}$ converges strongly or not without the assumption that $r_n \longrightarrow \infty$. In §5 we present an affirmative answer to this problem; see Theorem 5.2 and Remark 5.3.

Furthermore, a more general result is proved; see Theorem 4.1, which is a strong convergence theorem for a strongly relatively nonexpansive sequence introduced

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Article history : Received 09 February 2012. Accepted 22 March 2012.

in [3]. In the proofs of Theorem 4.1, we use modifications of ideas developed in [10, 16]. In particular, Lemma 3.2 due to Maingé [10] is a fundamental tool; see also Example 3.3 and Lemma 3.4.

In §5, using Theorem 4.1, we also show Theorem 5.5 which is a strong convergence theorem for a relatively nonexpansive mapping in the sense of Matsushita and Takahashi [11].

2. PRELIMINARIES

Throughout the present paper, E denotes a real Banach space with norm $\|\cdot\|$, E^* the dual of E, $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$, and $\mathbb{N}$ the set of positive integers. The norm of E^* is also denoted by $\|\cdot\|$. Strong convergence of a sequence $\{x_n\}$ in E to $x \in E$ is denoted by $x_n \longrightarrow x$ and weak convergence by $x_n \rightharpoonup x$. The (normalized) duality mapping of E is denoted by J, that is, it is a set-valued mapping of E into E^* defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$.

Let S_E denote the unit sphere of E, that is, $S_E = \{x \in E : \|x\| = 1\}$. The norm $\|\cdot\|$ of E is said to be Gâteaux differentiable if the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. In this case E is said to be smooth and it is known that the duality mapping J of E is single-valued. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S_E$ the limit (2.1) is attained uniformly for $x \in S_E$. A Banach space E is said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in S_E$. In this case it is known that J is uniformly norm-to-norm continuous on each bounded subset of E; see [17] for more details.

A Banach space E is said to be strictly convex if $x, y \in S_E$ and $x \neq y$ imply $\|x + y\| < 2$. A Banach space E is said to be uniformly convex if for any $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in S_E$ and $\|x - y\| \geq \epsilon$ imply $\|x + y\|/2 \leq 1 - \delta$. It is known that E is reflexive and strictly convex if E is uniformly convex; E is uniformly smooth if and only if E^* is uniformly convex; see [17] for more details.

In the rest of this section, unless otherwise stated, we assume that E is a smooth, strictly convex, and reflexive Banach space. In this case it is known that the duality mapping J of E is single-valued and bijective, and J^{-1} is the duality mapping of E^*.

We deal with a real-valued function ϕ on $E \times E$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$; see [1, 8]. From the definition of ϕ, it is clear that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \tag{2.2}$$

for all $x, y \in E$. Since $\|\cdot\|^2$ is convex,

$$\phi\Big(w, J^{-1}\big(\lambda Jx + (1-\lambda)Jy\big)\Big) \leq \lambda\phi(w, x) + (1-\lambda)\phi(w, y) \tag{2.3}$$

holds for all $x, y, w \in E$ and $\lambda \in [0, 1]$. It is known that

$$\phi(x, J^{-1}x^*) \leq \phi(x, J^{-1}(x^* - y^*)) + 2\langle J^{-1}x^* - x, y^*\rangle \tag{2.4}$$

holds for all $x \in E$ and $x^*, y^* \in E^*$; see [9, Lemma 3.2].

Lemma 2.1. *([8, Proposition 2]) Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. If $\phi(x_n, y_n) \longrightarrow 0$, then $x_n - y_n \longrightarrow 0$.*

Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then it is obvious from the definition of ϕ that $\phi(x_n, y_n) \longrightarrow 0$ if $x_n - y_n \longrightarrow 0$. From this fact and Lemma 2.1, we deduce the following: If E is a uniformly convex and uniformly smooth Banach space, then

$$x_n - y_n \longrightarrow 0 \Leftrightarrow Jx_n - Jy_n \longrightarrow 0 \Leftrightarrow \phi(x_n, y_n) \longrightarrow 0. \tag{2.5}$$

In the rest of this section, we assume that C is a nonempty closed convex subset of E.

Let $T\colon C \longrightarrow E$ be a mapping. The set of fixed points of T is denoted by $F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T [6, 14] if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Tx_n \longrightarrow 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. A mapping T is said to be of type (r) if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$; T is said to be relatively nonexpansive [11, 12] if T is of type (r) and $F(T) = \hat{F}(T)$. We know that if $T\colon C \longrightarrow E$ is of type (r), then $F(T)$ is closed and convex; see [12, Proposition 2.4].

It is known that, for each $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\phi(x_0, x) = \min\{\phi(y, x) : y \in C\}.$$

Such a point x_0 is denoted by $Q_C(x)$ and Q_C is called the generalized projection of E onto C; see [1, 8]. It is known that

$$\langle z - Q_C(x), Jx - JQ_C(x)\rangle \leq 0 \tag{2.6}$$

or equivalently

$$\phi(z, Q_C(x)) + \phi(Q_C(x), x) \leq \phi(z, x) \tag{2.7}$$

holds for all $x \in E$ and $z \in C$. It is obvious from (2.7) that the generalized projection Q_C is of type (r).

Let A be a set-valued mapping of E into E^*, which is denoted by $A \subset E \times E^*$. The effective domain of A is denoted by $\mathrm{dom}(A)$ and the range of A by $R(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$ and $R(A) = \bigcup_{x \in \mathrm{dom}(A)} Ax$. A set-valued mapping $A \subset E \times E^*$ is said to be a monotone operator if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be maximal if $A = A'$ whenever $A' \subset E \times E^*$ is a monotone operator such that $A \subset A'$. It is known that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex, where $A^{-1}0 = \{x \in E : Ax \ni 0\}$.

Let $A \subset E \times E^*$ be a maximal monotone operator and $r > 0$. Then it is known that $R(J + rA) = E^*$; see [15]. Thus a single-valued mapping $L_r = (J + rA)^{-1}J$ of E onto $\mathrm{dom}(A)$ is well defined and is called the resolvent of A. It is also known that $F(L_r) = A^{-1}0$ and

$$\phi(u, L_r x) + \phi(L_r x, x) \leq \phi(u, x) \tag{2.8}$$

for all $x \in E$ and $u \in F(L_r)$; see [7, 9]. It is obvious from (2.8) that the resolvent L_r of A is of type (r) for all $r > 0$ whenever $A^{-1}0$ is nonempty.

The following lemma is well known; see [2, 18].

Lemma 2.2. *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers, $\{\gamma_n\}$ a sequence of real numbers, and $\{\alpha_n\}$ a sequence in $[0, 1]$. Suppose that $\xi_{n+1} \leq (1 - \alpha_n)\xi_n + \alpha_n\gamma_n$ for every $n \in \mathbb{N}$, $\limsup_{n \longrightarrow \infty} \gamma_n \leq 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\xi_n \longrightarrow 0$.*

3. EVENTUALLY INCREASING FUNCTIONS AND STRONGLY RELATIVELY NONEXPANSIVE SEQUENCES

In this section, we provide some needed lemmas about an eventually increasing function and a strongly relatively nonexpansive sequence.

A function $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ is said to be *eventually increasing* if $\lim_{n\longrightarrow\infty} \tau(n) = \infty$ and $\tau(n) \le \tau(n+1)$ for all $n \in \mathbb{N}$. By definition, we easily obtain the following:

Lemma 3.1. *Let $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ be an eventually increasing function and $\{\alpha_n\}$ a sequence of real numbers such that $\alpha_n \longrightarrow 0$. Then $\alpha_{\tau(n)} \longrightarrow 0$.*

We need the following lemma:

Lemma 3.2. *(Maingé [10, Lemma 3.1]) Let $\{\xi_n\}$ be a sequence of real numbers. Suppose that there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\xi_{n_i} < \xi_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exist $N \in \mathbb{N}$ and a function $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\tau(n) \le \tau(n+1)$, $\xi_{\tau(n)} \le \xi_{\tau(n)+1}$, and $\xi_n \le \xi_{\tau(n)+1}$ for all $n \ge N$ and $\lim_{n\longrightarrow\infty} \tau(n) = \infty$.*

Under the assumptions of Lemma 3.2, we can not choose a strictly increasing function τ; see the following example:

Example 3.3. Let $\{\xi_n\}$ be a sequence of real numbers define by

$$\xi_n = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ 1/n & \text{if } n \text{ is even.} \end{cases}$$

Then the following hold:

(1) There exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\xi_{n_i} < \xi_{n_i+1}$ for all $i \in \mathbb{N}$;

(2) there does not exist a subsequence $\{\xi_{m_k}\}$ of $\{\xi_n\}$ such that $\xi_{m_k} \le \xi_{m_k+1}$ and $\xi_k \le \xi_{m_k+1}$ for all $k \in \mathbb{N}$.

Proof. Define $n_i = 2i - 1$ for each $i \in \mathbb{N}$. Then it is clear that

$$\xi_{n_i} = \xi_{2i-1} = 0 < \frac{1}{2i} = \xi_{2i} = \xi_{n_i+1}$$

for every $i \in \mathbb{N}$. Thus (1) holds.

Let $\{\xi_{m_k}\}$ be a subsequence of $\{\xi_n\}$. Suppose that $\xi_{m_k} \le \xi_{m_k+1}$ for all $k \in \mathbb{N}$. Then it is easy to check that m_k is odd and $m_k + 1$ is even for every $k \in \mathbb{N}$. We now assume that $\xi_k \le \xi_{m_k+1}$ for all $k \in \mathbb{N}$. Then it follows that

$$\frac{1}{k} = \xi_k \le \xi_{m_k+1} = \frac{1}{m_k + 1}$$

if k is even. This implies that $k \ge m_k + 1 \ge k + 1$, which is a contradiction. □

Using Lemma 3.2, we obtain the following:

Lemma 3.4. *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers which is not convergent. Then there exist $N \in \mathbb{N}$ and an eventually increasing function $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\xi_{\tau(n)} \le \xi_{\tau(n)+1}$ for all $n \in \mathbb{N}$ and $\xi_n \le \xi_{\tau(n)+1}$ for all $n \ge N$.*

Proof. Since $\{\xi_n\}$ is not convergent, for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m \ge n$ and $\xi_m < \xi_{m+1}$, and hence there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\xi_{n_i} < \xi_{n_i+1}$ for every $i \in \mathbb{N}$. Lemma 3.2 implies that there exist $N \in \mathbb{N}$ and a function $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\sigma(n) \le \sigma(n+1)$, $\xi_{\sigma(n)} \le \xi_{\sigma(n)+1}$, and $\xi_n \le \xi_{\sigma(n)+1}$ for every $n \ge N$ and $\lim_{n\longrightarrow\infty} \sigma(n) = \infty$. Let us define $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ by $\tau(n) = \sigma(N)$ for $n \in \{1, 2, \dots, N\}$ and $\tau(n) = \sigma(n)$ for $n > N$, which completes the proof. □

In the rest of this section, unless otherwise stated, we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E.

Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Then

- $\{T_n\}$ is said to be a *strongly relatively nonexpansive sequence* [3] if each T_n is of type (r) and $\phi(T_n x_n, x_n) \longrightarrow 0$ whenever $\{x_n\}$ is a bounded sequence in E and $\phi(p, x_n) - \phi(p, T_n x_n) \longrightarrow 0$ for some point $p \in F$;
- $\{T_n\}$ satisfies the *condition (Z)* if every weak cluster point of $\{x_n\}$ belongs to F whenever $\{x_n\}$ is a bounded sequence in C such that $T_n x_n - x_n \longrightarrow 0$.

Let $A \subset E \times E^*$ be a maximal monotone operator with a zero point and $\{r_n\}$ a sequence of positive real numbers. Then (2.8) shows that the sequence $\{L_{r_n}\}$ of resolvents of A is a strongly relatively nonexpansive sequence; see [3] for more details.

In order to prove our main result in §4, we need the following lemmas:

Lemma 3.5. *Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ an eventually increasing function, and $\{z_n\}$ a bounded sequence in C such that $\phi(p, z_n) - \phi(p, T_{\tau(n)} z_n) \longrightarrow 0$ for some $p \in F$. If $\{T_n\}$ is a strongly relatively nonexpansive sequence, then $\phi(T_{\tau(n)} z_n, z_n) \longrightarrow 0$.*

Proof. Suppose that $\phi(T_{\tau(n)} z_n, z_n) \not\longrightarrow 0$. Then there exist $\epsilon > 0$ and a strictly increasing function $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\tau \circ \sigma$ is also strictly increasing and

$$\phi(T_{\tau\circ\sigma(n)} z_{\sigma(n)}, z_{\sigma(n)}) \geq \epsilon \tag{3.1}$$

for all $n \in \mathbb{N}$. Set $\mu = \tau \circ \sigma$ and $R(\mu) = \{\mu(n) : n \in \mathbb{N}\}$. Define a sequence $\{y_n\}$ in C as follows: For each $n \in \mathbb{N}$,

$$y_n = \begin{cases} z_{\sigma\circ\mu^{-1}(n)} & \text{if } n \in R(\mu); \\ p & \text{if } n \notin R(\mu). \end{cases}$$

It is clear that $\{y_n\}$ is bounded,

$$\phi(p, y_n) - \phi(p, T_n y_n) = \phi(p, z_{\sigma\circ\mu^{-1}(n)}) - \phi(p, T_{\tau(\sigma\circ\mu^{-1}(n))} z_{\sigma\circ\mu^{-1}(n)})$$

for $n \in R(\mu)$, and $\phi(p, y_n) - \phi(p, T_n y_n) = 0$ for $n \notin R(\mu)$. Since $\sigma \circ \mu^{-1}$ is strictly increasing, it follows that $\phi(p, y_n) - \phi(p, T_n y_n) \longrightarrow 0$, so $\phi(T_n y_n, y_n) \longrightarrow 0$ because $\{T_n\}$ is a strongly relatively nonexpansive sequence. Therefore, noting that $y_{\mu(n)} = z_{\sigma(\mu^{-1}(\mu(n)))} = z_{\sigma(n)}$ and μ is strictly increasing, we have

$$\phi(T_{\tau\circ\sigma(n)} z_{\sigma(n)}, z_{\sigma(n)}) = \phi(T_{\mu(n)} y_{\mu(n)}, y_{\mu(n)}) \longrightarrow 0,$$

which contradicts to (3.1). □

Lemma 3.6. *Let $\{T_n\}$ be a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ an eventually increasing function, and $\{z_n\}$ a bounded sequence in C such that $T_{\tau(n)} z_n - z_n \longrightarrow 0$. Suppose that $\{T_n\}$ satisfies the condition (Z). Then every weak cluster point of $\{z_n\}$ belongs to F.*

Proof. Let z be a weak cluster point of $\{z_n\}$. Then there exists a strictly increasing function $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $z_{\sigma(n)} \rightharpoonup z$ as $n \longrightarrow \infty$ and $\tau\circ\sigma$ is strictly increasing. Set $\mu = \tau \circ \sigma$ and $R(\mu) = \{\mu(n) : n \in \mathbb{N}\}$. Define a sequence $\{y_n\}$ in C as follows: For each $n \in \mathbb{N}$,

$$y_n = \begin{cases} z_{\sigma\circ\mu^{-1}(n)} & \text{if } n \in R(\mu); \\ p & \text{if } n \notin R(\mu), \end{cases}$$

where p is a point in F. Then it is clear that $\{y_n\}$ is bounded,

$$y_n - T_n y_n = z_{\sigma\circ\mu^{-1}(n)} - T_{\tau(\sigma\circ\mu^{-1}(n))} z_{\sigma\circ\mu^{-1}(n)}$$

for $n \in R(\mu)$, and $y_n - T_n y_n = 0$ for $n \mathbb{N}otin R(\mu)$. Since $z_n - T_{\tau(n)} z_n \longrightarrow 0$ and $\sigma\circ\mu^{-1}$ is strictly increasing, it follows that $y_n - T_n y_n \longrightarrow 0$. Noting that μ is strictly increasing and $y_{\mu(n)} = z_{\sigma\circ\mu^{-1}(\mu(n))} = z_{\sigma(n)}$ for every $n \in \mathbb{N}$, we know that $\{z_{\sigma(n)}\}$ is a subsequence of $\{y_n\}$, and hence z is a weak cluster point of $\{y_n\}$. Since $\{T_n\}$ satisfies the condition (Z), we conclude that $z \in F$. □

Lemma 3.7. *Let $\{T_n\}$ be a sequence of mappings of C into E, F a nonempty closed convex subset of E, $\{z_n\}$ a bounded sequence in C such that $z_n - T_n z_n \longrightarrow 0$, and $u \in E$. Suppose that every weak cluster point of $\{z_n\}$ belongs to F. Then*

$$\limsup_{n\longrightarrow\infty} \langle T_n z_n - w, Ju - Jw\rangle \le 0,$$

where $w = Q_F(u)$.

Proof. Since $z_n - T_n z_n \longrightarrow 0$ and $\{z_n\}$ is bounded, there exists a weakly convergent subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\begin{aligned}\limsup_{n\longrightarrow\infty} \langle T_n z_n - w, Ju - Jw\rangle &= \limsup_{n\longrightarrow\infty} \langle z_n - w, Ju - Jw\rangle \\ &= \lim_{i\longrightarrow\infty} \langle z_{n_i} - w, Ju - Jw\rangle.\end{aligned}$$

Let z be the weak limit of $\{z_{n_i}\}$. By assumption, we see that $z \in F$. Thus (2.6) shows that

$$\lim_{i\longrightarrow\infty} \langle z_{n_i} - w, Ju - Jw\rangle = \langle z - w, Ju - Jw\rangle \le 0,$$

which is the desired result. □

4. STRONG CONVERGENCE THEOREMS FOR STRONGLY RELATIVELY NONEXPANSIVE SEQUENCES

In this section, we prove the following strong convergence theorem:

Theorem 4.1. *Let E be a smooth and uniformly convex Banach space, C a nonempty closed convex subset of E, $\{S_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} F(S_n)$ is nonempty, and $\{\alpha_n\}$ a sequence in $[0,1]$ such that $\alpha_n \longrightarrow 0$. Let u be a point in E and $\{x_n\}$ a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = Q_C J^{-1}\big(\alpha_n Ju + (1-\alpha_n) JS_n x_n\big) \tag{4.1}$$

for $n \in \mathbb{N}$. Suppose that

- *$\{S_n\}$ is a strongly relatively nonexpansive sequence;*
- *$\{S_n\}$ satisfies the condition (Z);*
- *$\alpha_n > 0$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.*

Then $\{x_n\}$ converges strongly to $w = Q_F(u)$.

First, we show some lemmas; then we prove Theorem 4.1. In the rest of this section, we set

$$y_n = J^{-1}\big(\alpha_n Ju + (1-\alpha_n) JS_n x_n\big)$$

for $n \in \mathbb{N}$, so (4.1) is reduced to $x_{n+1} = Q_C(y_n)$.

Lemma 4.2. *Both $\{x_n\}$ and $\{S_n x_n\}$ are bounded, and moreover, the following hold:*

1. *$y_n - S_n x_n \longrightarrow 0$;*
2. *$\phi(w, x_{n+1}) \le \alpha_n \phi(w, u) + \phi(w, S_n x_n)$ for every $n \in \mathbb{N}$;*
3. *$\phi(w, x_{n+1}) \le (1-\alpha_n)\phi(w, x_n) + 2\alpha_n \langle y_n - w, Ju - Jw\rangle$ for every $n \in \mathbb{N}$.*

Proof. Since Q_C and S_n are of type (r) and $w \in F(S_n) \subset C$, it follows from (2.3) that

$$\begin{aligned} \phi(w, x_{n+1}) &\leq \phi(w, y_n) \\ &\leq \alpha_n \phi(w, u) + (1-\alpha_n)\phi(w, S_n x_n) \\ &\leq \alpha_n \phi(w, u) + (1-\alpha_n)\phi(w, x_n) \end{aligned} \tag{4.2}$$

for every $n \in \mathbb{N}$. Thus, by induction on n, we have

$$\phi(w, S_n x_n) \leq \phi(w, x_n) \leq \max\{\phi(w, x_1), \phi(w, u)\}.$$

Therefore, by virtue of (2.2), it turns out that $\{x_n\}$ and $\{S_n x_n\}$ are bounded.

By $\alpha_n \longrightarrow 0$, it is clear that $Jy_n - JS_n x_n = \alpha_n(Ju - JS_n x_n) \longrightarrow 0$. This shows that

$$y_n - S_n x_n = J^{-1}Jy_n - J^{-1}JS_n x_n \longrightarrow 0$$

because E^* is uniformly smooth and J^{-1} is uniformly continuous on every bounded set. Thus (1) holds.

(2) follows from (4.2).

Since S_n is of type (r), it follows from (4.2), (2.4), and (2.3) that

$$\begin{aligned} \phi(w, x_{n+1}) &\leq \phi(w, y_n) \\ &\leq \phi\Big(w, J^{-1}\big(\alpha_n Ju + (1-\alpha_n)JS_n x_n - \alpha_n(Ju - Jw)\big)\Big) \\ &\quad + 2\langle y_n - w, \alpha_n(Ju - Jw)\rangle \\ &\leq (1-\alpha_n)\phi(w, S_n x_n) + \alpha_n \phi(w, w) + 2\alpha_n \langle y_n - w, Ju - Jw\rangle \\ &\leq (1-\alpha_n)\phi(w, x_n) + 2\alpha_n \langle y_n - w, Ju - Jw\rangle \end{aligned} \tag{4.3}$$

for every $n \in \mathbb{N}$. Therefore, (3) holds. □

Lemma 4.3. *Suppose that*

$$\limsup_{n \longrightarrow \infty} \big(\phi(w, x_n) - \phi(w, x_{n+1})\big) \leq 0. \tag{4.4}$$

Then $\{x_n\}$ converges strongly to w.

Proof. We first show that $S_n x_n - x_n \longrightarrow 0$. Since S_n is of type (r), it follows from (2) in Lemma 4.2 that

$$0 \leq \phi(w, x_n) - \phi(w, S_n x_n) \leq \phi(w, x_n) - \phi(w, x_{n+1}) + \alpha_n \phi(w, u)$$

for every $n \in \mathbb{N}$, so $\phi(w, x_n) - \phi(w, S_n x_n) \longrightarrow 0$ by (4.4) and $\alpha_n \longrightarrow 0$. Since $\{S_n\}$ is a strongly relatively nonexpansive sequence and $\{x_n\}$ is bounded by Lemma 4.2, $\phi(S_n x_n, x_n) \longrightarrow 0$. Using Lemma 2.1, we conclude that $S_n x_n - x_n \longrightarrow 0$.

We know that $y_n - S_n x_n \longrightarrow 0$ by (1) in Lemma 4.2 and $\{S_n\}$ satisfies the condition (Z) by assumption, so Lemma 3.7 implies that

$$\limsup_{n \longrightarrow \infty} \langle y_n - w, Ju - Jw\rangle = \limsup_{n \longrightarrow \infty} \langle S_n x_n - w, Ju - Jw\rangle \leq 0.$$

It follows from (3) in Lemma 4.2 that

$$\phi(w, x_{n+1}) \leq (1-\alpha_n)\phi(w, x_n) + 2\alpha_n \langle y_n - w, Ju - Jw\rangle$$

for every $n \in \mathbb{N}$. Therefore, noting that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and using Lemma 2.2, we conclude that $\phi(w, x_n) \longrightarrow 0$, and hence $x_n \longrightarrow w$ by Lemma 2.1. □

Lemma 4.4. *The real number sequence $\{\phi(w, x_n)\}$ is convergent.*

Proof. We assume, to obtain a contraction, that $\{\phi(w, x_n)\}$ is not convergent. Then Lemma 3.4 implies that there exist $N \in \mathbb{N}$ and an eventually increasing function $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\phi(w, x_{\tau(n)}) \leq \phi(w, x_{\tau(n)+1}) \tag{4.5}$$

for every $n \in \mathbb{N}$ and

$$\phi(w, x_n) \leq \phi(w, x_{\tau(n)+1}) \tag{4.6}$$

for every $n \geq N$.

We show that $S_{\tau(n)}x_{\tau(n)} - x_{\tau(n)} \longrightarrow 0$. Since $S_{\tau(n)}$ is of type (r), it follows from (4.5), (2) in Lemma 4.2, and Lemma 3.1 that

$$\begin{aligned} 0 &\leq \phi(w, x_{\tau(n)}) - \phi(w, S_{\tau(n)}x_{\tau(n)}) \\ &\leq \phi(w, x_{\tau(n)+1}) - \phi(w, S_{\tau(n)}x_{\tau(n)}) \\ &\leq \alpha_{\tau(n)}\phi(w, u) \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow \infty$. Since $\{x_{\tau(n)}\}$ is bounded and $\{S_n\}$ is a strongly relatively nonexpansive sequence, it follows from Lemma 3.5 that $\phi(S_{\tau(n)}x_{\tau(n)}, x_{\tau(n)}) \longrightarrow 0$, so we conclude that $S_{\tau(n)}x_{\tau(n)} - x_{\tau(n)} \longrightarrow 0$ by Lemma 2.1.

Finally, we obtain a contradiction that $\phi(w, x_n) \longrightarrow 0$. From (3) in Lemma 4.2 and (4.5), we know that

$$\begin{aligned} \phi(w, x_{\tau(n)+1}) &\leq (1 - \alpha_{\tau(n)})\phi(w, x_{\tau(n)}) + 2\alpha_{\tau(n)}\langle y_{\tau(n)} - w, Ju - Jw\rangle \\ &\leq (1 - \alpha_{\tau(n)})\phi(w, x_{\tau(n)+1}) + 2\alpha_{\tau(n)}\langle y_{\tau(n)} - w, Ju - Jw\rangle \end{aligned} \tag{4.7}$$

for every $n \in \mathbb{N}$, where $y_{\tau(n)} = J^{-1}\big(\alpha_{\tau(n)}Ju + (1 - \alpha_{\tau(n)})JS_{\tau(n)}x_{\tau(n)}\big)$ for $n \in \mathbb{N}$. Thus, by $\alpha_{\tau(n)} > 0$, (4.7) shows that

$$\phi(w, x_{\tau(n)+1}) \leq 2\langle y_{\tau(n)} - w, Ju - Jw\rangle \tag{4.8}$$

for every $n \in \mathbb{N}$. Since $\{S_n\}$ satisfies the condition (Z), it follows from Lemma 3.6 that every weak cluster point of $\{x_{\tau(n)}\}$ belongs to F. Using (4.8), (1) in Lemma 4.2, and Lemma 3.7, we have

$$\begin{aligned} \limsup_{n\longrightarrow\infty} \phi(w, x_{\tau(n)+1}) &\leq 2 \limsup_{n\longrightarrow\infty}\langle y_{\tau(n)} - w, Ju - Jw\rangle \\ &= 2 \limsup_{n\longrightarrow\infty}\langle S_{\tau(n)}x_{\tau(n)} - w, Ju - Jw\rangle \leq 0. \end{aligned}$$

Therefore, by virtue of (4.6), we conclude that

$$\limsup_{n\longrightarrow\infty} \phi(w, x_n) \leq \limsup_{n\longrightarrow\infty} \phi(w, x_{\tau(n)+1}) \leq 0,$$

and hence $\phi(w, x_n) \longrightarrow 0$, which is a contradiction. □

Proof of Theorem 4.1. Using Lemmas 4.3 and 4.4, we get the conclusion. □

5. APPLICATIONS

In this section, we study the zero point problem for a maximal monotone operator and the fixed point problem for a relatively nonexpansive mapping. We employ Theorem 4.1 to approximate solutions of these problems.

To prove the first theorem, we need the following lemma:

Lemma 5.1. *([4, Lemma 3.5]) Let E be a strictly convex and reflexive Banach space whose norm is uniformly Gâteaux differentiable, $\{r_n\}$ a sequence of positive real numbers, and L_{r_n} the resolvent of a maximal monotone operator $A \subset E \times E^*$. Suppose that $\inf_n r_n > 0$ and $A^{-1}0$ is nonempty. Then $\{L_{r_n}\}$ satisfies the condition (Z).*

We adopt a modified proximal point algorithm introduced by Kohsaka and Takahashi [9] in the following theorem:

Theorem 5.2. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, $A \subset E \times E^*$ a maximal monotone operator, $\{\alpha_n\}$ a sequence in $(0,1]$, and $\{r_n\}$ a sequence of positive real numbers. Suppose that $A^{-1}0$ is nonempty, $\alpha_n \longrightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\inf_n r_n > 0$. Let u be a point in E and $\{x_n\}$ a sequence defined by $x_1 \in E$ and*

$$x_{n+1} = J^{-1}\big(\alpha_n Ju + (1-\alpha_n) J L_{r_n} x_n\big) \tag{5.1}$$

for $n \in \mathbb{N}$, where $L_{r_n} = (J + r_n A)^{-1} J$. Then $\{x_n\}$ converges strongly to $Q_{A^{-1}0}(u)$.

Proof. Set $S_n = L_{r_n}$ for $n \in \mathbb{N}$. It is known that $F(S_n) = A^{-1}0$ and L_{r_n} is a type (r) self-mapping of E for each $n \in \mathbb{N}$. Hence $\bigcap_{n=1}^{\infty} F(S_n) = A^{-1}0$ is nonempty. It is also known that $\{S_n\}$ is a strongly relatively nonexpansive sequence by [3, Example 3.2] and $\{S_n\}$ satisfies the condition (Z) by Lemma 5.1. It is clear that Q_E is the identity mapping on E. Therefore, Theorem 4.1 implies the conclusion. □

Remark 5.3. Theorem 5.2 is similar to [9, Theorem 3.3]. In [9, Theorem 3.3], E is assumed to be smooth and uniformly convex and $\{\alpha_n\}$ in $[0,1]$ while $\{r_n\}$ is assumed to diverge to infinity.

To prove the next theorem, we need the following lemma:

Lemma 5.4. *([3, Lemma 2.1]) Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a uniformly convex Banach space E and $\{\lambda_n\}$ a sequence in $[0,1]$ such that $\liminf_{n\longrightarrow\infty} \lambda_n > 0$. Suppose that*

$$\lambda_n \|x_n\|^2 + (1-\lambda_n)\|y_n\|^2 - \|\lambda_n x_n + (1-\lambda_n) y_n\|^2 \longrightarrow 0.$$

Then $(1-\lambda_n)(x_n - y_n) \longrightarrow 0$.

The following is a strong convergence theorem for a relatively nonexpansive mapping; see [11, 12] for other convergence theorems and see also [3].

Theorem 5.5. *([13, Theorem 3.4]) Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, $T\colon C \longrightarrow E$ a relatively nonexpansive mapping, $\{\alpha_n\}$ a sequence in $(0,1]$, and $\{\beta_n\}$ a sequence in $[0,1]$. Suppose that $\alpha_n \longrightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $0 < \liminf_{n\longrightarrow\infty} \beta_n \le \limsup_{n\longrightarrow\infty} \beta_n < 1$. Let u be a point in E and $\{x_n\}$ a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = Q_C J^{-1}\Big(\alpha_n Ju + (1-\alpha_n)\big(\beta_n J x_n + (1-\beta_n) J T x_n\big)\Big) \tag{5.2}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $Q_{F(T)}(u)$.

Proof. Set $S_n = J^{-1}\big(\beta_n J + (1-\beta_n) JT\big)$ for $n \in \mathbb{N}$. Then it is easy to check that each S_n is a mapping of type (r) and $\bigcap_{n=1}^{\infty} F(S_n) = F(T)$; see [5, Corollary 3.8]. Moreover, it is clear that (5.2) coincides with (4.1). To finish the proof, it is enough to show that $\{S_n\}$ is a strongly relatively nonexpansive sequence and $\{S_n\}$ satisfies the condition (Z).

Let $\{y_n\}$ be a bounded sequence in C such that $\phi(p, y_n) - \phi(p, S_n y_n) \longrightarrow 0$ for some $p \in \bigcap_{n=1}^{\infty} F(S_n)$. Since T is of type (r), we have

$$\begin{aligned}
&\beta_n \|J y_n\|^2 + (1-\beta_n)\|JTy_n\|^2 - \|J S_n y_n\|^2 \\
&\quad = \beta_n \phi(p, y_n) + (1-\beta_n)\phi(p, Ty_n) - \phi(p, S_n y_n) \\
&\quad \le \beta_n \phi(p, y_n) + (1-\beta_n)\phi(p, y_n) - \phi(p, S_n y_n) \\
&\quad = \phi(p, y_n) - \phi(p, S_n y_n) \longrightarrow 0.
\end{aligned}$$

Using Lemma 5.4 and (2.5), it turns out that

$$Jy_n - JS_ny_n = (1-\beta_n)(Jy_n - JTy_n) \longrightarrow 0,$$

and hence $\phi(S_ny_n, y_n) \longrightarrow 0$. Thus $\{S_n\}$ is a strongly relatively nonexpansive sequence.

Let $\{z_n\}$ be a bounded sequence in C such that $z_n - S_nz_n \longrightarrow 0$. Then it follows from (2.5) that

$$(1-\beta_n)(Jz_n - JTz_n) = Jz_n - JS_nz_n \longrightarrow 0,$$

so we conclude that $z_n - Tz_n \longrightarrow 0$ by $\limsup_{n\longrightarrow\infty} \beta_n < 1$ and (2.5). Since T is relatively nonexpansive, every weak cluster point of $\{z_n\}$ belongs to $F(T)$. This means that $\{S_n\}$ satisfies the condition (Z). Consequently, Theorem 4.1 implies the conclusion. □

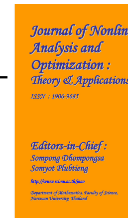
Remark 5.6. In [13, Theorem 3.4], $\{\alpha_n\}$ and $\{\beta_n\}$ are assumed to be sequences in $(0, 1)$.

References

1. Y. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math. 178, Dekker, New York, 1996, 15–50.
2. K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2007), 2350–2360.
3. K. Aoyama, F. Kohsaka and W. Takahashi, Strongly relatively nonexpansive sequences in Banach spaces and applications, J. Fixed Point Theory Appl. 5 (2009), 201–224.
4. K. Aoyama, F. Kohsaka and W. Takahashi, Proximal point methods for monotone operators in Banach spaces, Taiwanese J. Math. 15 (2011), 259–281.
5. K. Aoyama, F. Kohsaka and W. Takahashi, Strong convergence theorems by shrinking and hybrid projection methods for relatively nonexpansive mappings in Banach spaces, Nonlinear analysis and convex analysis, Yokohama Publ., Yokohama, 2009, 7–26.
6. Y. Censor and S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, Optimization 37 (1996), 323–339.
7. S. Kamimura, F. Kohsaka and W. Takahashi, Weak and strong convergence theorems for maximal monotone operators in a Banach space, Set-Valued Anal. 12 (2004), 417–429.
8. S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002), 938–945.
9. F. Kohsaka and W. Takahashi, Strong convergence of an iterative sequence for maximal monotone operators in a Banach space, Abstr. Appl. Anal. 2004, 239–249.
10. P.-E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008), 899–912.
11. S. Matsushita and W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2004, 37–47.

12. S. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theory 134 (2005), 257–266.
13. W. Nilsrakoo and S. Saejung, Strong convergence theorems by Halpern-Mann iterations for relatively nonexpansive mappings in Banach spaces, Appl. Math. Comput. 217 (2011), 6577–6586.
14. S. Reich, A weak convergence theorem for the alternating method with Bregman distances, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math. 178, Dekker, New York, 1996, 313–318.
15. R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970), 75–88.
16. S. Saejung, Halpern's iteration in Banach spaces, Nonlinear Anal. 73 (2010), 3431–3439.
17. W. Takahashi, Nonlinear functional analysis, Yokohama Publishers, Yokohama, 2000.
18. H.-K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. (2) 66 (2002), 240–256.

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 79-84
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



ASYMPTOTIC BEHAVIOR OF INFINITE PRODUCTS OF PROJECTION AND NONEXPANSIVE OPERATORS WITH COMPUTATIONAL ERRORS

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ABSTRACT. We study the asymptotic behavior of infinite products of orthogonal projections and other (possibly nonlinear) nonexpansive operators in Hilbert space in the presence of computational errors.

KEYWORDS : Hilbert space; Infinite product; Nonexpansive operator; Orthogonal projection.
AMS Subject Classification: 46C05, 47H09, 47H10.

1. INTRODUCTION

Consider m closed linear subspaces $S_1, S_2, \ldots, S_m$ of a given Hilbert space H and let S denote their intersection. Let the infinite product $\prod_{j=1}^{\infty} P_j := \cdots P_3P_2P_1$ only consist of orthogonal projections $P_{S_k},\ 1 \le k \le m$, onto these subspaces. We are concerned with the asymptotic behavior of the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_n = \prod_{j=1}^{n} P_j x_0 = P_n P_{n-1} ... P_1 x_0,\ n = 1, 2, \ldots.$$

Denoting the norm of H by $||\cdot||$, we recall that the classical theorems of J. von Neumann [10] and I. Halperin [9] declare that, for any $x_0 \in H$,

$$\lim_{n \longrightarrow \infty} ||(P_{S_m} P_{S_{m-1}} \cdots P_{S_1})^n x_0 - P_S x_0|| = 0. \tag{1.1}$$

We observe that the iterative process here is strongly cyclical and this condition is, in fact, essential for the proof of (1.1). The convergence in (1.1) may not be uniform (on bounded subsets of initial points) and these theorems do not provide any rate of convergence.

Except for some earlier partial results, general necessary and sufficient conditions for uniform convergence in (1.1) were found much later (see [6] for $m = 2$ and [2] for the general case). In addition, some estimates of the rate of this convergence were also obtained, mainly by using the notion of angles between subspaces [8].

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Article history : Received 5 January 2012. Accepted 18 March 2012.

Traditionally, these estimates were asserted for cyclical products, although this restriction was not as essential as it was for proving (1.1) in [9]. As a matter of fact, these estimates were stated for one cycle, but then immediately generalized to the power n. Consequently, the possibility of concatenating different fragmentary products of projections into an infinite one was not considered.

We have recently [15–17] used a geometric approach (in particular, angles between subspaces) to establish useful estimates for proving convergence of infinite products involving not only orthogonal projections, but also other (possibly nonlinear) nonexpansive operators in Hilbert space. In [18] we provide sufficient conditions for the strong and uniform (on bounded subsets of initial points) convergence of such infinite products by applying new estimates of the inclination [1] of a finite tuple of closed linear subspaces.

In view of the diverse applications of infinite products (see, for instance, [7] and the references therein), it is natural to ask if these results continue to hold in the presence of computational errors. In the present paper we give affirmative answers to this question. Our main results, Theorems 2.1 and 2.2, are formulated in Section 2. Their proofs are given in Section 3.

Previous studies concerning inexact powers and infinite products of operators can be found, for example, in [3–5, 11–14, 19].

2. MAIN RESULTS

For each $x \in H$ and each $B \subset H$, set

$$\rho(x, B) = \inf\{||x - y|| : \ y \in B\}.$$

We consider an iterative process, presented as an infinite product $\prod_{i=1}^{\infty} A_i \mathcal{P}_i$, where all A_i are quasi-nonexpansive, possibly nonlinear, operators of arbitrary nature and each $\mathcal{P}_i$ is a finite product of all the projections $P_{S_1}, P_{S_2}, \dots, P_{S_m}$ in any order and amount (that is, with possible repetitions). Here $S_1, S_2, \dots, S_m$ are assumed to be closed linear subspaces of a given Hilbert space H. By S we denote the intersection of $S_1, S_2, \dots, S_m$; the case $S = \{0\}$ is permitted as well. We assume that all the subspaces S_j are different. Recall that the principal question studied in this paper concerns the asymptotic behavior of the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_n = \prod_{i=1}^{n} A_i \mathcal{P}_i x_0, \ n = 1, 2, \dots,$ in the presence of computational errors.

We are now ready to state our two main results.

Theorem 2.1. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of self-mappings of a Hilbert space H. Assume that for all integers $n \geq 1$,*

$$A_n(S) \subset S \tag{2.1}$$

and

$$||A_n y - A_n x|| \leq ||y - x|| \text{ for all } y \in H \text{ and all } x \in S. \tag{2.2}$$

Assume further that for all integers $n \geq 1$ and all $x \in H$,

$$||\mathcal{P}_n x - P_S x|| \leq q_n ||x - P_S x|| \tag{2.3}$$

with the factors $q_n \in (0, 1]$ satisfying

$$\lim_{n \longrightarrow \infty} \prod_{i=1}^{n} q_i = 0 \tag{2.4}$$

and

$$M_0 := \sup\{1 + \sum_{p=2}^{k} \prod_{i=p}^{k} q_i : \ k = 2, 3, \dots\} < \infty. \tag{2.5}$$

Let the positive numbers M, ϵ, δ and a natural number n_0 satisfy

$$\prod_{i=1}^{n_0} q_i M \leq \epsilon/2 \tag{2.6}$$

and

$$\delta M_0 \leq \epsilon/2. \tag{2.7}$$

Finally, assume that $\{x_i\}_{i=0}^{\infty} \subset H$,

$$||x_0|| \leq M, \tag{2.8}$$

and that for any integer $n \geq 1$,

$$||x_n - A_n \mathcal{P}_n x_{n-1}|| \leq \delta. \tag{2.9}$$

Then

$$\rho(x_i, S) \leq \epsilon \text{ for all integers } i \geq n_0.$$

Theorem 2.2. *Assume that (2.1) and (2.2) hold for all integers $n \geq 1$, and that (2.3) holds for all integers $n \geq 1$ and all points $x \in H$ with the factors $q_n \in (0, 1]$ satisfying both (2.4) and (2.5). Assume further that $\{\delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ and*

$$\lim_{i \longrightarrow \infty} \delta_i = 0. \tag{2.10}$$

Let $\epsilon, M > 0$ be given. Then there exists an integer $n_1 \geq 1$ such that for each sequence $\{x_n\}_{n=0}^{\infty} \subset H$ satisfying

$$||x_0|| \leq M \tag{2.11}$$

and

$$||x_n - A_n \mathcal{P}_n x_{n-1}|| \leq \delta_n \text{ for all integers } n \geq 1, \tag{2.12}$$

the following inequality holds:

$$\rho(x_n, S) \leq \epsilon \text{ for all integers } n \geq n_1. \tag{2.13}$$

Before proceeding to the proofs of these theorems in Section 3, we observe that both (2.4) and (2.5) clearly hold if for all $i = 1, 2, \ldots,$

$$q_i \leq q < 1$$

for some constant q.

More generally, both (2.4) and (2.5) hold if there are a real number $q \in (0, 1)$ and a natural number p such that

$$q_{ip} \leq q \text{ for all natural numbers } i.$$

In this connection, recall that the number

$$l(S_1, S_2, \ldots, S_m) = \inf_{x \notin S} \max_{1 \leq j \leq m} \rho(x, S_j) \rho(x, S)^{-1}$$

is called the inclination of the m-tuple $(S_1, S_2, \ldots, S_m)$. Clearly, $0 \leq l \leq 1$.

By [1], for any set of integers $\{i_1, \ldots, i_N\} = \{1, 2, \ldots, m\}$ and any $x \in H$,

$$||P_{S_{i_N}} P_{S_{i_{N-1}}} \cdots P_{S_{i_1}} x - P_S x|| \leq (1 - l^2 N^{-2})^{1/2} ||x - P_S x||.$$

Thus (2.4) and (2.5) hold if there is a natural number p such that

$$\sup\{N_{ip} : \ i = 1, 2, \ldots\} < \infty,$$

where N_k is the number of operators in the product $\mathcal{P}_k$.

3. PROOFS OF THEOREMS 2.1 AND 2.2

We begin with the following lemma.

Lemma 3.1. *Assume that both (2.1) and (2.2) hold for each integer $n \geq 1$, (2.3) holds for all integers $n \geq 1$ and all points $x \in H$, δ is a positive number, and the sequence $\{x_n\}_{n=0}^{\infty} \subset X$ satisfies*

$$||x_n - A_n \mathcal{P}_n x_{n-1}|| \leq \delta \tag{3.1}$$

for all integers $n \geq 1$. Then for any integer $k \geq 2$,

$$\rho(x_k, S) \leq (\prod_{i=1}^{k} q_i)\rho(x_0, S) + \delta(1 + \sum_{p=2}^{k}(\prod_{i=p}^{k} q_i)). \tag{3.2}$$

Proof. In view of (3.1), for any integer $n \geq 1$,

$$\begin{aligned} \rho(x_n, S) &\leq ||x_n - A_n \mathcal{P}_n x_{n-1}|| + \rho(A_n \mathcal{P}_n x_{n-1}, S) \\ &\leq \delta + ||A_n \mathcal{P}_n x_{n-1} - A_n P_S x_{n-1}||, \end{aligned}$$

and in view of (2.1) and (2.2),

$$\begin{aligned} \rho(x_n, S) \leq \delta + ||\mathcal{P}_n x_{n-1} - P_S x_{n-1}|| &\leq \delta + q_n ||x_{n-1} - P_S x_{n-1}|| \\ &\leq \delta + q_n \rho(x_{n-1}, S). \end{aligned} \tag{3.3}$$

By (3.3),

$$\rho(x_1, S) \leq \delta + q_1 \rho(x_0, S),\ \ \rho(x_2, S) \leq q_1 q_2 \rho(x_0, S) + q_2\delta + \delta. \tag{3.4}$$

We now show by induction that for any integer $k \geq 2$, inequality (3.2) holds. To this end, we first note that in view of (3.4), inequality (3.2) certainly holds for $k = 2$.

Assume that $n \geq 2$ is an integer and that (3.2) holds for $k = n$. Thus

$$\rho(x_n, S) \leq (\prod_{i=1}^{n} q_i)\rho(x_0, S) + \delta(1 + \sum_{p=2}^{n}(\prod_{i=p}^{n} q_i)). \tag{3.5}$$

By (3.3) and (3.5),

$$\begin{aligned} \rho(x_{n+1}, S) &\leq \delta + q_{n+1}\rho(x_n, S) \\ &\leq \delta + (\prod_{i=1}^{n+1} q_i)\rho(x_0, S) + \delta q_{n+1}(1 + \sum_{p=2}^{n}(\prod_{i=p}^{n} q_i)) \\ &= (\prod_{i=1}^{n+1} q_i)\rho(x_0, S) + \delta(1 + q_{n+1} + \sum_{p=2}^{n}(\prod_{i=p}^{n+1} q_i)) \\ &= (\prod_{i=1}^{n+1} q_i)\rho(x_0, S) + \delta(1 + \sum_{p=2}^{n+1}(\prod_{i=p}^{n+1} q_i)), \end{aligned}$$

so that (3.2) holds for $k = n+1$. Thus we have shown by induction that inequality (3.2) holds for all integers $k \geq 2$, as claimed. Lemma 3.1 is proved. ☐

Completion of the proof of Theorem 2.1. By Lemma 3.1, (2.8), (2.5), (2.6) and (2.7), for any integer $k \geq n_0$, we have

$$\begin{aligned} \rho(x_k, S) &\leq (\prod_{i=1}^{k} q_i)\rho(x_0, S) + \delta(1 + \sum_{p=2}^{k}(\prod_{i=p}^{k} q_i)) \\ &\leq (\prod_{i=1}^{n_0} q_i)M + \delta M_0 \leq \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Theorem 2.1 is proved.

Proof of Theorem 2.2. Let M_0 be defined by (2.5). Choose a positive number $\bar{\delta}$ such that

$$\bar{\delta} M_0 < \epsilon/2. \tag{3.6}$$

By (2.10), there is a natural number n_0 such that

$$\delta_i < \bar{\delta} \text{ for all integers } i \geq n_0. \tag{3.7}$$

Choose

$$\widehat{\delta} = \sup\{\delta_i : \ i = 1, 2, \dots\} \tag{3.8}$$

and set

$$M_1 = \widehat{\delta} M_0 + M. \tag{3.9}$$

By (2.4), there is a natural number $n_1 > n_0$ such that

$$M_1 \prod_{i=n_0+1}^{n_1} q_i < \epsilon/2. \tag{3.10}$$

Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset H$ satisfies (2.11) and (2.12). By (3.8), (2.11), (2.12), Lemma 3.1 and (3.9),

$$\rho(x_{n_0}, S) \leq \prod_{i=1}^{n_0} q_i \rho(x_0, S) + \widehat{\delta} M_0 < M_1. \tag{3.11}$$

For each integer $i \geq 1$, define

$$y_{i-1} = x_{i-1+n_0}, \ \tilde{A}_i = A_{i+n_0}, \ \tilde{\mathcal{P}}_i = \mathcal{P}_{i+n_0}. \tag{3.12}$$

By (3.11), (3.12), (2.12), (2.5), (3.6), (3.7), (3.10) and by Lemma 3.1 applied to $\{y_i\}_{i=0}^{\infty}$, $\{\tilde{A}_i\}_{i=1}^{\infty}$ and $\{\tilde{\mathcal{P}}_i\}_{i=1}^{\infty}$, we have for all integers $k \geq n_1$,

$$\rho(x_k, S) = \rho(y_{k-n_0}, S)$$

$$\leq (\prod_{p=n_0+1}^{k} q_p)\rho(y_0, S) + \widehat{\delta}(1 + \sum_{p=n_0+1}^{k} (\prod_{i=p}^{k} q_i))$$

$$\leq \prod_{p=n_0+1}^{n_1} q_p M_1 + \bar{\delta} M_0 < \epsilon/2 + \epsilon/2 = \epsilon,$$

as asserted. Theorem 2.2 is proved.

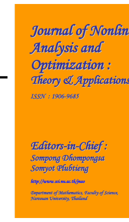
Acknowledgments. This research was supported by the Israel Science Foundation (Grant No. 647/07), the Fund for the Promotion of Research at the Technion and by the Technion VPR Research Fund.

References

1. C. Badea, S. Grivaux and V. Müller, The rate of convergence in the method of alternating projections, Preprint, Institute of Mathematics, AS CR, Prague, 2010.
2. H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, SIAM Rev. 38 (1996), 367–426.
3. D. Butnariu, S. Reich and A. J. Zaslavski, Convergence to fixed points of inexact orbits of Bregman-monotone and of nonexpansive operators in Banach spaces, Fixed Point Theory and its Applications. Yokohama Publishers, Yokohama, 2006, 11–32.

4. D. Butnariu, S. Reich and A. J. Zaslavski, Asymptotic behavior of inexact orbits for a class of operators in complete metric spaces, J. Appl. Anal. 13 (2007), 1–11.
5. D. Butnariu, S. Reich and A. J. Zaslavski, Stable convergence theorems for infinite products and powers of nonexpansive mappings, Numerical Func. Anal. Optim. 29 (2008), 304–323.
6. F. Deutsch, Best approximation in inner product spaces, Springer. New York, 2001.
7. J. Dye and S. Reich, Random products of nonexpansive mappings, Res. Notes Math. Ser. 244 (1992), Longman Sci. Tech. Harlow, 106–118.
8. K. Friedrichs, On certain inequalities and characteristic value problems for analytic functions and for functions of two variables, Trans. Amer. Math. Soc. 41 (1937), 321–364.
9. I. Halperin, The product of projection operators, Acta Sci. Math.(Szeged) 23 (1962), 96-99.
10. J. von Neumann, On rings of operators, Reduction theory, Ann. of Math. 50 (1949), 401–485.
11. A. M. Ostrowski, The round-off stability of iterations, Z. Angew. Math. Mech. 47 (1967), 77-81.
12. E. Pustylnik, S. Reich and A. J. Zaslavski, Inexact orbits of nonexpansive mappings, Taiwanese J. Math. 12 (2008), 1511–1523.
13. E. Pustylnik, S. Reich and A. J. Zaslavski, Convergence to compact sets of inexact orbits of nonexpansive mappings in Banach and metric spaces, Fixed Point Theory Appl. 2008, Article ID 528614, 1–10.
14. E. Pustylnik, S. Reich and A. J. Zaslavski, Inexact infinite products of nonexpansive mappings, Numerical Func. Anal. Optim. 30 (2009), 632–645.
15. E. Pustylnik, S. Reich and A. J. Zaslavski, Convergence of infinite products of nonexpansive operators in Hilbert space, J. Nonlinear Convex Anal. 11 (2011), 461–474.
16. E. Pustylnik, S. Reich and A. J. Zaslavski, New possibilities regarding the alternating projections method, J. Nonlinear Anal. Optim. 2 (2011), 33–37.
17. E. Pustylnik, S. Reich and A. J. Zaslavski, Convergence of non-cyclic iterative projection methods, J. Math. Anal. Appl. 380 (2011), 759-767.
18. E. Pustylnik, S. Reich and A. J. Zaslavski, Convergence of non-periodic infinite products of orthogonal projections and nonexpansive operators in Hilbert space, J. Approximation Theory 164 (2012), 611–624.
19. S. Reich and A. J. Zaslavski, Inexact powers and infinite products of nonlinear operators, Int. J. Math. Stat. 6 (2010), 89-109.

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 85-92
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



A NEW ITERATIVE METHOD FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this paper, a new iterative method for finding a common fixed point of a countable nonexpansive mappings in a Hilbert space, is introduced. Then a strong convergence theorem for a countable family of nonexpansive mappings is proved. This theorem improve and extend some recent results of Tian (2010) and Xu (2004).

KEYWORDS : Fixed point; Nonexpansive mapping; Iterative method; Variational inequality; Viscosity approximation.

1. INTRODUCTION

Let H be a real Hilbert space. A mapping S of H into itself is called nonexpansive, if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in H$. Let $Fix(S)$ denote the fixed points set of S. We assume $Fix(S) \neq \emptyset$, it is well known, $Fix(S)$ is closed and convex. Recall that a contraction on H is a self-mapping f of H such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in H$, where $\alpha \in (0, 1)$ is a constant. Let A be a bounded linear operator on H. A is strongly positive; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in H$.

Moudafi [5] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Sx_n, \ n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Xu [9] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution x^* in $Fix(S)$ of the variational inequality:

$$\langle (I - f)x^*, x^* - x\rangle \leq 0, \ \text{for all } x \in Fix(S). \tag{1.2}$$

Note that iterative methods for nonexpansive mappings can be used to solve a convex minimization problem. See, e.g., [2, 8, 10] and references therein. A typical

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Article history : Received 17 March 2012. Accepted 25 March 2012.

problem is that of minimizing a quadratic function over the set of the fixed points of nonexpansive mapping on a real Hilbert space

$$\min_{x\in Fix(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.3}$$

where b is a given point in H.

In [8], it is proved, the sequence $\{x_n\}$ defined by the iterative method below with an arbitrary initial $x_0 \in H$

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Sx_n,\ n \geq 0, \tag{1.4}$$

converges strongly to the unique solution of the minimization problem (1.3) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Combining the iterative method (1.1) and (1.4), Marino and Xu [4] consider the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Sx_n,\ n \geq 0. \tag{1.5}$$

It is proved, if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0,\ \text{for all } x \in Fix(S).$$

which is the optimality condition for the minimization problem

$$\min_{x\in Fix(S)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Yamada [10] introduced the following hybrid iterative method for solving the variational inequality

$$x_{n+1} = Sx_n - \mu\lambda_n F(Sx_n),\ n \geq 0, \tag{1.6}$$

where F is $k - Lipschitzian$ and η-strongly monotone operator with $k > 0, \eta > 0$ and $0 < \mu < 2\eta/k^2$, then, if $\{\lambda_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution of the variational inequality

$$\langle Fx^*, x - x^*\rangle \geq 0,\ \text{for all } x \in Fix(S).$$

Tian [7] combined the iterative method (1.5) with the Yamada's method (1.6) and considered the following iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Sx_n,\ n \geq 0. \tag{1.7}$$

He proved, if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution $x^* \in Fix(S)$ of the variational inequality

$$\langle(\gamma f - \mu F)x^*, x - x^*\rangle \leq 0,\ \text{for all } x \in Fix(S).$$

In this paper, motivated by Tian [7], we prove a strong convergence theorem for a countable family of nonexpansive mappings in a Hilbert space. Our result improve and extend the corresponding results in recent works.

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle .,.\rangle$ and the norm $\|.\|$. Weak and strong convergence is denoted by notation $\rightharpoonup$ and $\to$, respectively. In a real Hilbert space H,

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2,$$

for all $x,y\in H$ and $\lambda\in\mathbb{R}$. Let C be a nonempty closed convex subset of H. Then, for any $x\in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x-P_C(x)\|\leq\|x-y\|,\quad \text{for all } y\in C.$$

Such a P_C is called the metric projection of H onto C. It is known, P_C is nonexpansive. Further, for $x\in H$ and $z\in C$,

$$z=P_C(x)\ \Leftrightarrow\ \langle x-z,z-y\rangle\geq 0,\quad \text{for all } y\in C.$$

Now, we collect some lemmas which will be used in the main result.

Lemma 2.1. *Let H be a real Hilbert space. Then, for all $x,y\in H$,*

(I) $\|x+y\|^2\leq\|x\|^2+2\langle y,x+y\rangle$;
(II) $\|x+y\|^2\geq\|x\|^2+2\langle y,x\rangle$.

Lemma 2.2. [1] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}\leq(1-\gamma_n)a_n+\gamma_n v_n+r_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{r_n\}$ is a sequence of nonnegative real numbers and $\{v_n\}$ is a sequence in $\mathbb{R}$ such that

(I) $\sum_{n=1}^{\infty}\gamma_n=\infty$;
(II) $\limsup_{n\to\infty}v_n\leq 0$;
(III) $\sum_{n=1}^{\infty}r_n<\infty$.

Then, $\lim_{n\to\infty}a_n=0$.

Lemma 2.3. [3] *Let C be a nonempty closed convex subset of H and $S:C\to C$ a nonexpansive mapping with $Fix(S)\neq\emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-S)(x_n)\}$ converges strongly to y, then $(I-S)x=y$.*

Lemma 2.4. [1] *Let C be a nonempty closed convex subset of H. Suppose*

$$\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\|:z\in C\}<\infty.$$

Then, for each $y\in C$, $\{T_ny\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty=\lim_{n\to\infty}T_ny$, for all $y\in C$. Then $\lim_{n\to\infty}\sup\{\|Tz-T_nz\|:z\in C\}=0$.

Theorem 2.5. [7] *Let H be a real Hilbert space, $S:H\to H$ be a nonexpansive mapping with $Fix(S)\neq\emptyset$, $f:H\to H$ be a contraction with coefficient $0<\alpha<1$ and F be a k-Lipschitzian and η-strongly monotone operator on H with $k>0,\eta>0$. Let $0<\mu<2\eta/k^2$ and $0<\gamma<\mu(\eta-\frac{\mu k^2}{2})/\alpha=\tau/\alpha$. Then the unique fixed point $x_t\in H$ of the contraction $x\mapsto t\gamma f(x)+(I-t\mu F)Sx$ converges strongly to a fixed point q of S as $t\to 0$ which solves the following variational inequality*

$$\langle(\mu F-\gamma f)q,q-z)\rangle\leq 0,\ \textit{for all } z\in Fix(S).$$

Theorem 2.6. [7] *Let H be a real Hilbert space, $S : H \to H$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$, $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$ and F be a k-Lipschitzian and η-strongly monotone operator on H with $k > 0, \eta > 0$. Let $0 < \mu < 2\eta/k^2$ and $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. Suppose $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:*

(I) $\lim_{n\to\infty} \alpha_n = 0$;
(II) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(III) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Let $x_0 \in H$. Then the sequence $\{x_n\}$ defined by (1.7) converges strongly to q that is obtained in Theorem 2.5.

3. MAIN RESULTS

In this section, we prove the following strong convergence theorem for finding a common element of fixed points set of a countable family of nonexpansive mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space H. Let $\{S_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on H which satisfies $\bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$. Let f be a contraction of H into itself with coefficient $0 < \alpha < 1$ and F a k-Lipschitzian and η-strongly monotone operator on H with $k > 0, \eta > 0$. Let $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$ and $\tau < 1$. Define a sequence $\{x_n\} \subset H$ as follows: $x_1 = x \in H$ and*

$$\begin{cases} y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) S_n x_n, \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n S_n y_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:

(I) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(II) $\lim_{n\to\infty} \beta_n = 0$ *or* $\beta_n \in [0,b)$ *for some* $b \in (0,1)$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(III) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Suppose $\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty$ for any bounded subset K of H. Let S be a mapping of H into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in H$ and suppose $Fix(S) = \bigcap_{n=1}^{\infty} Fix(S_n)$. Then the sequences $\{x_n\}$ defined by (3.1) converge strongly to $q \in Fix(S)$ which is a unique solution of the following variational inequality

$$\langle (\mu F - \gamma f)q, q - z \rangle \leq 0, \text{ for all } z \in Fix(S).$$

Proof. Let $Q = P_{\bigcap_{n=1}^{\infty} Fix(S_n)}$. So

$$\begin{aligned} \|Q(I - \mu F + \gamma f)(x) - Q(I - \mu F + \gamma f)(y)\| &\leq \|(I - \mu F + \gamma f)(x) - \\ &\quad\ (I - \mu F + \gamma f)(y)\| \\ &\leq \|(I - \mu F)(x) - (I - \mu F)(y)\| \\ &\quad\ + \gamma \|f(x) - f(y)\| \\ &\leq (1 - \tau)\|x - y\| + \gamma\alpha\|x - y\| \\ &= (1 - (\tau - \gamma\alpha))\|x - y\|, \end{aligned}$$

for all $x, y \in H$. Therefore, $P_{\bigcap_{n=1}^{\infty} Fix(S_n)}(I - \mu F + \gamma f)$ is a contraction of H into itself, which implies, there exists a unique element $q \in H$ such that $q = Q(I - \mu F + \gamma f)(q) = P_{\bigcap_{n=1}^{\infty} Fix(S_n)}(I - \mu F + \gamma f)(q)$ or equivalently

$$\langle (\mu F - \gamma f)q, q - z) \rangle \leq 0, \text{ for all } z \in Fix(S). \tag{3.2}$$

We proceed with following steps:
Step 1. $\{x_n\}$ and $\{y_n\}$ are bounded. Let $p \in Fix(S)$. Then, from (3.1), we have

$$\begin{aligned}
\|x_{n+1}-p\| = \quad & \|(1-\beta_n)(y_n-p)+\beta_n(S_ny_n-p)\| \leq \|y_n-p\| \\
= \quad & \|\alpha_n(\gamma f(x_n)-\mu F(p))+(I-\alpha_n\mu F)(S_nx_n-p)\| \\
\leq \quad & (1-\alpha_n\tau)\|x_n-p\|+\alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu F(p)\| \\
\leq \quad & (1-\alpha_n(\tau-\gamma\alpha))\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu F(p)\| \\
\leq \quad & \max\{\|x_n-p\|, \tfrac{\|\gamma f(p)-\mu F(p)\|}{\tau-\gamma\alpha}\}.
\end{aligned}$$

By induction,

$$\|x_n-p\| \leq \max\{\|x_1-p\|, \frac{\|\gamma f(p)-\mu F(p)\|}{\tau-\gamma\alpha}\}, \quad n \geq 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{f(x_n)\}$, $\{(FS_n)x_n\}$ and $\{S_ny_n\}$. Without loss of generality, we may assume $\{x_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{(FS_n)x_n\}$, $\{S_ny_n\} \subset K$, where K is a bounded set of H.

Step 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Since K is bounded, $\{S_ny_n-y_n\}$, $\{f(x_n)\}$ and $\{(FS_n)x_n\}$ are bounded. Let

$$M_1 = \sup\{\|S_ny_n-y_n\|, \|f(x_n)\|, \|(\mu FS_n)x_n\| : n \in \mathbb{N}\}.$$

From the definition of $\{x_n\}$, it is easily seen

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| = \quad & \|(1-\beta_{n+1})y_{n+1}+\beta_{n+1}S_{n+1}y_{n+1}-(1-\beta_n)y_n-\beta_nS_ny_n\| \\
= \quad & \|(1-\beta_{n+1})y_{n+1}+\beta_{n+1}S_{n+1}y_{n+1}-(1-\beta_{n+1})y_n-\beta_nS_ny_n \\
& +(1-\beta_{n+1})y_n-(1-\beta_n)y_n-\beta_{n+1}S_ny_n+\beta_{n+1}S_ny_n\| \\
= \quad & \|(1-\beta_{n+1})(y_{n+1}-y_n)+\beta_{n+1}(S_{n+1}y_{n+1}-S_ny_n) \\
& +(\beta_{n+1}-\beta_n)(S_ny_n-y_n)\| \\
\leq \quad & (1-\beta_{n+1})\|y_{n+1}-y_n\|+\beta_{n+1}\|S_{n+1}y_{n+1}-S_ny_n\| \\
& +|\beta_{n+1}-\beta_n|M_1 \\
\leq \quad & (1-\beta_{n+1})\|y_{n+1}-y_n\|+\beta_{n+1}\|S_{n+1}y_{n+1}-S_ny_{n+1}\| \\
& +\beta_{n+1}\|y_{n+1}-y_n\|+|\beta_{n+1}-\beta_n|M_1 \\
\leq \quad & \|y_{n+1}-y_n\|+\|S_{n+1}y_{n+1}-S_ny_{n+1}\|+|\beta_{n+1}-\beta_n|M_1,
\end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$. From (3.1), we obtain

$$\begin{aligned}
\|y_{n+1}-y_n\| = \quad & \|\alpha_{n+1}\gamma f(x_{n+1})+(I-\alpha_{n+1}\mu F)S_{n+1}x_{n+1}-\alpha_n\gamma f(x_n) \\
& -(I-\alpha_n\mu F)S_nx_n\| \\
= \quad & \|(I-\alpha_{n+1}\mu F)(S_{n+1}x_{n+1}-S_nx_n)-(\alpha_{n+1}-\alpha_n)\mu F(S_nx_n) \\
& +\alpha_{n+1}\gamma(f(x_{n+1})-f(x_n))+(\alpha_{n+1}-\alpha_n)\gamma f(x_n)\| \\
\leq \quad & (1-\alpha_{n+1}\tau)\|S_{n+1}x_{n+1}-S_nx_n\|+\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\| \\
& +|\alpha_{n+1}-\alpha_n|\|\gamma f(x_n)-\mu F(S_nx_n)\| \\
\leq \quad & (1-\alpha_{n+1}\tau)(\|S_{n+1}x_{n+1}-S_{n+1}x_n\|+\|S_{n+1}x_n-S_nx_n\|) \\
& +\alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|(\gamma+1)M_1 \\
\leq \quad & (1-\alpha_{n+1}(\tau-\gamma\alpha))\|x_{n+1}-x_n\|+M_1(\gamma+1)|\alpha_{n+1}-\alpha_n| \\
& +\|S_{n+1}x_n-S_nx_n\|,
\end{aligned} \tag{3.4}$$

for all $n \in N$. Using (3.4) in (3.3), we have

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| \leq \quad & (1-\alpha_{n+1}(\tau-\gamma\alpha))\|x_{n+1}-x_n\| \\
& +2\sup\{\|S_{n+1}z-S_nz\| : z \in K\} \\
& +M_2(|\alpha_{n+1}-\alpha_n|+|\beta_{n+1}-\beta_n|),
\end{aligned} \tag{3.5}$$

where $M_2 = M_1(\gamma+1)$. Assume $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Setting $r_n = M_2(|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) + 2\sup\{\|S_{n+1}z - S_n z\| : z \in K\}$

$$\sum_{n=1}^{\infty} r_n = M_2 \sum_{n=1}^{\infty} (|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|) + 2\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty.$$

Therefore, it follows from Lemma 2.2, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Now, suppose $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$. From (3.5), we get

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| \leq \quad & (1 - \alpha_{n+1}(\tau - \gamma\alpha))\|x_{n+1} - x_n\| \\ & +2\sup\{\|S_{n+1}z - S_n z\| : z \in K\} + M_2|\beta_{n+1} - \beta_n| \\ & +\alpha_{n+1} M_2 |1 - \tfrac{\alpha_n}{\alpha_{n+1}}|. \end{aligned}$$

Setting $r_n = M_2|\beta_{n+1} - \beta_n| + 2\sup\{\|S_{n+1}z - S_n z\| : z \in K\}$, we have

$$\sum_{n=1}^{\infty} r_n = M_2 \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| + 2\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in K\} < \infty.$$

Therefore, it follows from Lemma 2.2, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Step 3. $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Indeed, from (3.1), we obtain

$$\|x_{n+1} - y_n\| = \beta_n \|y_n - S_n y_n\|.$$

If $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$. If $\beta_n \in [0, b)$ for some $b \in (0,1)$, we have

$$\begin{aligned} \|x_{n+1} - y_n\| = \quad & \beta_n \|y_n - S_n y_n\| \\ \leq \quad & \beta_n(\|y_n - S_n x_n\| + \|S_n x_n - S_n y_n\|) \\ \leq \quad & b(\|y_n - S_n x_n\| + \|x_n - y_n\|) \\ \leq \quad & b(\|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| + \|y_n - S_n x_n\|). \end{aligned}$$

Hence

$$\|x_{n+1} - y_n\| \leq \frac{b}{1-b}(\|x_n - x_{n+1}\| + \|y_n - S_n x_n\|).$$

So, by Step 2 and

$$\lim_{n\to\infty} \|y_n - S_n x_n\| = \lim_{n\to\infty} \alpha_n \|\gamma f(x_n) - \mu F(S_n x_n)\| = 0, \tag{3.6}$$

we have $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$. This implies $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Step 4. $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Since

$$\|x_n - S_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - S_n x_n\|,$$

it follows from Step 2, Step 3 and (3.6) that $\lim_{n\to\infty} \|x_n - S_n x_n\| = 0$. By

$$\begin{aligned} \|x_n - Sx_n\| \leq \quad & \|Sx_n - S_n x_n\| + \|S_n x_n - x_n\| \\ \leq \quad & \sup\{\|Sz - S_n z\| : z \in \{x_n\}\} + \|x_n - S_n x_n\| \end{aligned}$$

and Lemma 2.4, we have $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Step 5. We claim $\limsup_{n\to\infty} \langle (\gamma f - \mu F)q, y_n - q \rangle \leq 0$, where $q = P_{\bigcap_{n=1}^{\infty} Fix(S_n)}(I - \mu F + \gamma f)(q)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - \mu F)q, x_n - q \rangle = \lim_{i\to\infty} \langle (\gamma f - \mu F)q, x_{n_k} - q \rangle.$$

Since $\{x_{n_k}\}$ is bounded in H, without loss of generality, we assume $x_{n_k} \rightharpoonup z \in H$. It follows from Step 4 and Lemma 2.3 that $z \in Fix(S)$. So, from (3.2), we obtain

$$\limsup_{n\to\infty} \langle (\gamma f - \mu F)q, y_n - q \rangle = \limsup_{n\to\infty} \langle (\gamma f - \mu F)q, x_n - q \rangle = \langle (\gamma f - \mu F)q, z - q) \rangle \leq 0.$$

Step 6. $\{x_n\}$ converges strongly to q. From (3.1) and Lemma 2.1, we get

$$\begin{aligned}
\|x_{n+1}-q\|^2 = & \ \|(1-\beta_n)(y_n-q)+\beta_n(S_ny_n-q)\|^2 \\
\leq & \ \|y_n-q\|^2 = \|\alpha_n\gamma f(x_n)+(I-\alpha_n\mu F)S_nx_n-q\|^2 \\
= & \ \|\alpha_n(\gamma f(x_n)-\mu F(q))+(I-\alpha_n\mu F)(S_nx_n-q)\|^2 \\
\leq & \ \|(I-\alpha_n\mu F)(S_nx_n-q)\|^2+2\alpha_n\langle\gamma f(x_n)-\mu F(q),y_n-q\rangle \\
\leq & \ (1-\alpha_n\tau)^2\|x_n-q\|^2+2\alpha_n\langle\gamma f(x_n)-\gamma f(q),y_n-q\rangle \\
& \ +2\alpha_n\langle\gamma f(q)-\mu F(q),y_n-q\rangle \\
\leq & \ (1-\alpha_n\tau)^2\|x_n-q\|^2+2\alpha_n\gamma\alpha\|x_n-q\|(\|y_n-x_n\|+\|x_n-q\|) \\
& \ +2\alpha_n\langle\gamma f(q)-\mu F(q),y_n-q\rangle \\
\leq & \ (1-2\alpha_n(\tau-\gamma\alpha))\|x_n-q\|^2+(\alpha_n\tau)^2\|x_n-q\|^2 \\
& \ +2\alpha_n\gamma\alpha\|x_n-q\|\|y_n-x_n\|+2\alpha_n\langle\gamma f(q)-\mu F(q),y_n-q\rangle \\
\leq & \ (1-2\alpha_n(\tau-\gamma\alpha))\|x_n-q\|^2+2\alpha_n(\tau-\gamma\alpha)\{\tfrac{(\alpha_n\tau^2)M_3^2}{2(\tau-\gamma\alpha)} \\
& \ +\tfrac{\gamma\alpha M_3}{\tau-\gamma\alpha}\|y_n-x_n\|+\tfrac{1}{\tau-\gamma\alpha}\langle\gamma f(q)-\mu F(q),y_n-q\rangle\} \\
= & \ (1-\delta_n)\|x_n-q\|^2+\delta_n\theta_n,
\end{aligned}$$

where $M_3=\sup\{\|x_n-q\| : n\geq 1\}$, $\delta_n=2\alpha_n(\tau-\gamma\alpha)$ and $\theta_n=\frac{(\alpha_n\tau^2)M_3^2}{2(\tau-\gamma\alpha)}+\frac{\gamma\alpha M_3}{\tau-\gamma\alpha}\|y_n-x_n\|+\frac{1}{\tau-\gamma\alpha}\langle\gamma f(q)-\mu F(q),y_n-q\rangle$. It is easy to see, $\lim_{n\to\infty}\delta_n=0$, $\sum_{n=1}^{\infty}\delta_n=\infty$ and $\limsup_{n\to\infty}\theta_n\leq 0$. Hence by Lemma 2.2, $\{x_n\}$ converges strongly to q. From Step 4, Step 6 and Lemma 2.3, we have $q\in Fix(S)$. This completes the proof. □

Taking $F=A$ (A is a strongly positive bounded linear operator on H), $\mu=1$ in Theorem 3.1, we get

Corollary 3.2. *we have $\{x_n\}$ generated by*

$$\begin{cases} y_n=\alpha_n\gamma f(x_n)+(I-\alpha_nA)S_nx_n, \\ x_{n+1}=(1-\beta_n)y_n+\beta_nS_ny_n, \quad n\geq 1, \end{cases}$$

converges strongly to $q\in Fix(S)$ which solves the variational inequality

$$\langle(A-\gamma f)q,q-z)\rangle\leq 0, \text{ for all } z\in Fix(S).$$

Taking $F=I$, $\mu=1$, $\gamma=1$ in Theorem 3.1, we get

Corollary 3.3. *we have $\{x_n\}$ generated by*

$$\begin{cases} y_n=\alpha_nf(x_n)+(1-\alpha_n)S_nx_n, \\ x_{n+1}=(1-\beta_n)y_n+\beta_nS_ny_n, \quad n\geq 1, \end{cases}$$

converges strongly to $q\in Fix(S)$ which solves the variational inequality

$$\langle(I-f)q,q-z)\rangle\leq 0, \text{ for all } z\in Fix(S).$$

Remark 3.4. Theorem 3.1 can be obtained without assumption $\tau<1$. Therefore, Theorem 3.1 is a generalization of Theorem 2.6.

Proof. We only use the assumption $\tau<1$ for finding $q\in H$ which solves the variational inequality (3.2) in Theorem 3.1. It is needed to prove Step 5 of the proof of Theorem 3.1. So, we just retrieve Step 5 of the proof of Theorem 3.1. By Theorem 2.5, there exists $q\in Fix(S)$ such that

$$\langle(\mu F-\gamma f)q,q-z)\rangle\leq 0, \text{ for all } z\in Fix(S).$$

Thus the Step 5 in the proof of Theorem 3.1 is obtained. The rest of the proof is similar to the original one. □

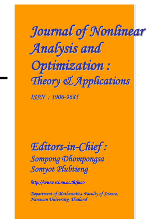
Remark 3.5. Corollary 3.3 is a generalization of [9, Theorem 3.2].

Acknowledgements The authors would like to thanks the referees for their valuable comments.

References

1. K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in Banach space, Nonlinear Anal. 67(2007), 2350-2360.
2. F. Deutech, I. Yamada, Minimizing certain convex functions over the intersection of the fixed poiont sets of nonexpansive mappings, Nummer. Funct. Anal. Optim. 19(1998), 33-56.
3. K. Geobel, W.A. Kirk, Topics in Metric Fixed Point theory, Cambridge Studies in Advanced Mathematics, 28. Cambridge University Press, Cambridge, 1990.
4. G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hibert spaces, J. Math. Anal. Appl. 318(2006), 43-52.
5. A. Moudafi, Viscosity approximation methods for fixed point problems, J. Math. Anal. Appl. 241(2000), 46-55.
6. J.W. Peng, J.C. Yao, A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings, Nonlinear Anal. doi:10.1016/j.na.2009.05.028.
7. M. Tian, A general iterative alghoritm for nonexpansive mappings in Hilbert spaces, Nonlinear Anal. 73(2010), 689-694.
8. H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116(2003), 659-678.
9. H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. **298** (2004), 279-291.
10. I. Yamada, The hybrid steepest descent for variational inequality problems over tha intersection of fixed points sets of nonexpansive mappings in: D. Butnariu, Y. Censor, S. Reich (Eds.), Inherently Parallel Alghoritms in Feasibility and Optimization and Their Application, Elservier, New York, 2001, pp.473-504.

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 93-99
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



EXISTENCE RESULTS FOR A P-LAPLACIAN EQUATION WITH DIFFUSION, STRONG ALLEE EFFECT AND CONSTANT YIELD HARVESTING

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ABSTRACT. We study a reaction-diffusion equation involving p-Laplacian with strong Allee effect type growth and constant yield harvesting (semipositone) in heterogeneous bounded habitats. An existence result under suitable assumptions is presented. We obtain our results via the method of sub-super solutions.

KEYWORDS : Semipositone; Sub-super solution; Allee effect; Harvesting.
AMS Subject Classification: 35J60; 35B35; 35B40.

1. Introduction

The aim of this paper is to investigate the following the nonlinear boundary value problem

$$\begin{cases} -\Delta_p u = a(x)u + b(x)u^2 - h(x)u^3 - c\alpha(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 1)$ with boundary $\partial\Omega$ of class C^1, $\Delta_p z := \text{div}(|\nabla z|^{p-2}\nabla z)$ is the p-Laplacian operator, $p > 1$, a, b, h and α are c^μ functions (Hölder continuous) such that h and b are strictly positive functions on Ω and a is negative at least for some $x \in \Omega$ (strong allee effect). Let a_0, a_1, b_0, b_1, h_0 and h_1 are defined as $a_0 := -\inf_{x\in\overline{\Omega}} a(x),\ a_1 := \sup_{x\in\overline{\Omega}} a(x),\ b_0 := \inf_{x\in\overline{\Omega}} b(x),\ h_1 := \sup_{x\in\overline{\Omega}} h(x)$ and $b_1 := \sup_{x\in\overline{\Omega}} b(x), h_0 := \inf_{x\in\overline{\Omega}} h(x)$. Here $c\alpha$ with $\alpha : \Omega \longrightarrow [0,1]$ and $c \geq 0$ a parameter represents the constant yield harvesting. We prove the existence of positive solution under certain condition. Our approach is based on the method of sub and supersolutions. Recently, In the case when $p = 2$ the problem (1.1) have been studied by R.Shivaji et al. (see [13]). The main purpose of this paper is to extend it to the p-Laplacian case with constant yield harvesting.

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Article history : Received 4 January 2012. Accepted 11 March 2012.

A typical model of reaction diffusion equations that describes the spatiotemporal distribution and abundance of organisms is

$$u_t = d\Delta u + uf(x,u)$$

where $u(x,t)$ is the populations density, $d > 0$ is the diffusion coefficient, Δu is the Laplacian of u with respect to the x variable, and $f(x,u)$ is the per capita growth rate, which is affected by the heterogeneous environment. Such ecology models were first studied by Skellam (see [12]). A classic example is Fishers equation (see [4]) with

$$f(x,u) = (1-u).$$

Similar reaction diffusion biological models have been studied by Kolmogoroff, Petrovsky, and Piscounoff (see [9]) earlier. Since then reaction diffusion models have been used to describe various spatiotemporal phenomena in biology, physics, chemistry and ecology, Fife (see [3]), Okuba and Levin(see [5]), Murray (see [6]), and Cantrell and Cosner (see [2]). Since the pioneer work by Skellam (see [12]), the logistic growth rate

$$f(x,u) = m(x) - b(x)u$$

has been used in population dynamics to model the crowding effect (see Oruganti, Shi and Shivaji (see [10])). A more general logistic type model can be characterized by a declining growth rate per capita function, i.e., $f(x,u)$ is decreasing with respect to u. In this paper, we consider the dispersal and evolution of species on a bounded domain Ω (in R^N) when the per capita growth rate is

$$f(x,u) = a(x)u + b(x)u^2 - h(x)u^3.$$

Note that (1.1)is a semipositone problem due to the presence of the constant yield harvesting term. It is well known in the literature that the study of positive solutions to semipositone problems is mathematically challenging (see [1],[8], [10]). (see [10])where such a model was discussed for the logistic growth case with constant coefficients. Here we deal with the more difficult strong Allee effect growth.We prove the existence of positive solution via the method of sub-super solutions.

2. Preliminaries

Here and in what follows, $W_0^{1,p}(\Omega)$, $p > 1$, denotes the usual Sobolev space, We give the definition of sub-super solution of (1.1).

Definition 2.1. we say that (ψ) (*resp.* ϕ in $W_0^{1,p}(\Omega)$ are called a subsolution (*resp.* supersolution) of (1.1), if ψ satisfies

$$\begin{cases} \int_\Omega |\nabla\psi(x)|^{p-2}\nabla\psi(x)\nabla w(x)dx \le \int_\Omega \Big(a(x)\psi(x) + b(x)\psi^2 - h(x)\psi^3 - c\alpha(x)\Big)w(x)dx \\ \psi \le 0 \end{cases} \tag{2.1}$$

$$\begin{cases} \int_\Omega |\nabla\phi(x)|^{p-2}\nabla\phi(x)\nabla w(x)dx \ge \int_\Omega \Big(a(x)\phi(x) + b(x)\phi^2 - h(x)\phi^3 - c\alpha(x)\Big)w(x)dx \\ \phi \ge 0 \end{cases} \tag{2.2}$$

for all non-negative test functions $w \in W_0^{1,p}(\Omega)$.

Now, if there exists a subsolution and a supersolution ψ and ϕ, respectively, such that $\psi(x) \le 0 \le \phi(x)$ for all $x \in \Omega$, then (1.1) has a positive solution $u \in W_0^{1,p}(\Omega)$ such that $\psi(x) \le u(x) \le \phi(x)$ for all $x \in \Omega$. We shall obtain the existence of positive weak solution to the problem (1.1) by constructing a subsolution ψ and a positive supersolution ϕ.

Proposition 2.2.

: (a) *If* $\lambda_1 \geq \frac{b_1^2+4a_0h_0}{4h_0}$ *and* $c>0$, *then (1.1) has no positive solution.*
: (b) *for* c *large, the problem (1.1) has no positive solution.*

Proof. (a) Let λ_1 be the first eigenvalue of- Δ_p with Dirichlet boundary conditions and ϕ_1 the corresponding eigenfunction

$$\begin{cases} -\Delta_p \phi = \lambda_1 |\phi|^{p-2}\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

with $\|\phi_1\|_\infty = 1$. let u be a positive solution of (1.1). Then by using the green identity, we have

$$0 = \int (u\Delta_p\phi_1 - \phi_1\Delta_p u) = \int \phi_1(a(x)u + b(x)u^2 - h(x)u^3 - c\alpha(x)) - \lambda_1|\phi_1|^{p-2}u\phi_1 dx.$$

Let

$$\tilde{f}(s) = a_1 + b_1 s - h_0 s^2$$

and

$$f^*(s) = u\tilde{f}(u).$$

Then

$$\tilde{f}_0 = \sup_{u\in[0,\infty]} \tilde{f}(u) = \lambda_1 \geq \frac{b_1^2 + 4a_0h_0}{4h_0}$$

and

$$a(x)u + b(x)u^2 - h(x)u^3 - c\alpha(x) \leq f^*(u) - c\alpha(x) \quad \text{for } x \geq 0.$$

Rewriting we have

$$0 = \int (u\Delta_p\phi_1 - \phi_1\Delta_p u) \leq \int \phi_1 f^*(u) - \lambda_1|\phi_1|^{p-2}u\phi_1 - c\alpha(x)\phi_1 dx.$$

But for $\lambda_1 \geq \frac{b_1^2+4a_0h_0}{4h_0}$,

$$\int \phi_1 f^*(u) - \lambda_1|\phi_1|^{p-2}u\phi_1 - c\alpha(x)\phi_1 dx = \int (\tilde{f}(u) - \lambda_1|\phi_1|^{p-2})u\phi_1 dx - c\int \alpha(x)\phi_1 dx \leq 0$$

which is a contradiction. Hence (1.1) has no positive solution.
We now prove (b). We observe that

$$\begin{aligned} c\int \alpha(x)\phi_1 dx &= \int \phi_1\Delta_p u + \int \phi_1(a(x)u + b(x)u^2 - h(x)u^3)dx \\ &\leq -\lambda_1 \int |\phi_1|^{p-2}u + \tilde{f}_0 \int \phi_1 \\ &\leq \tilde{f}_0 \int \phi_1. \end{aligned}$$

Clearly is not satiesfies for c large and Hence part (b) of Proposition holds. □

3. Existence results

In this section we prove the existence of solution by comparison method. we first prove an existence result for

$$\begin{cases} -\Delta_p u = -a_0 u + b_0 u^2 - h_1 u^3 - c\alpha(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Let λ_1 be the principal eigenvalue of the operator $-\Delta_p$ with Dirichlet boundary conditions and ϕ_1 the corresponding eigenfunction with ϕ_1 and $\|\phi_1\|_\infty = 1$. Hence there exist $\eta \geq 0$ and $\mu \in (0,1]$ and $k > 0$ such that

$$\begin{cases} |\nabla\phi_1|^p - \lambda_1\phi_1^p \geq k & \text{in } \overline{\Omega}_\eta, \\ \phi_1 \geq \mu & \text{in } \Omega \setminus \overline{\Omega}_\eta, \end{cases}$$

where $\overline{\Omega}_\eta = \{x \in \Omega : d(x, \partial\Omega) \leq \eta\}$. To discuss our existence result, it is known that $\phi_1 > 0$ in Ω and $\frac{\partial\phi_1}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outward unit normal to $\partial\Omega$.
Let

$$b_0 > 2\sqrt{a_0 h_1}$$

and

$$g(s) = -a_0 s + b_0 s^2 - h_1 s^3.$$

The zeros of g are 0,

$$r := \frac{b_0 - \sqrt{b_0^2 - 4a_0h_1}}{2h_1},$$

and

$$R := \frac{b_0 + \sqrt{b_0^2 - 4a_0h_1}}{2h_1},$$

and hence

$$g(s) := -h_1 s(s-R)(s-r).$$

Let r^* be the first positive zero of g'. In fact

$$r^* = \frac{b_0 - \sqrt{b_0^2 - 3a_0h_1}}{3h_1} < \frac{b_0}{3h_1}.$$

But g is convex on $(0, \frac{b_0}{3h_1})$. Hence

$$\sigma := -\inf_{s\in[0,R]} g(s) < a_0[\frac{b_0 - \sqrt{b_0^2 - 3a_0h_1}}{3h_1}] = a_0 r^*$$

We first note that

$$\begin{aligned} \frac{\sigma}{R^{p-1}} &< a_0\Big(\frac{b_0 - \sqrt{b_0^2 - 3a_0h_1}}{3h_1}\Big)\Big(\frac{b_0 + \sqrt{b_0^2 - 4a_0h_1}}{2h_1}\Big)^{1-p} \\ &= \frac{(2h_1)^{p-1}a_0}{[b_0 + \sqrt{b_0^2 - 4a_0h_1}]^{p-1}}\,\frac{b_0 - \sqrt{b_0^2 - 3a_0h_1}}{3h_1} \\ &= \frac{(2h_1)^{p-1}a_0^2}{b_0 + \sqrt{b_0^2 - 3a_0h_1}}\Big[b_0 + \sqrt{b_0^2 - 4a_0h_1}\Big]^{1-p} \end{aligned}$$

and thus RHS tends to zero as b_0 tends to infinity. Hence there exist $b_0^{(1)} := b_0^{(1)}(a_0, h_1, \Omega)$ such that for every $b_0 > b_0^{(1)}$, we have $m > R^{\frac{\sigma}{p-1}}$. Next we also note that

$$\frac{R}{r} = \frac{b_0 + \sqrt{b_0^2 - 4a_0h_1}}{b_0 - \sqrt{b_0^2 - 4a_0h_1}} = \frac{[b_0 + \sqrt{b_0^2 - 4a_0h_1}]^2}{4a_0h_1} \to \infty \quad \text{as} \quad b_0 \to \infty.$$

Hence there exists $b_0^{(2)} := b_0^{(2)}(a_0, h_1, \Omega)$ such that for every $b_0 > b_0^{(2)}$, we have

$$[R\frac{p-1}{p}\mu^{\frac{p}{p-1}}, R\frac{p-1}{p}] \subset (r, R)$$

and

$$k_\mu := \inf_{s\in[R\frac{p-1}{p}\mu^{\frac{p}{p-1}}, R\frac{p-1}{p}]} g(s) > 0.$$

Eventually

$$\begin{aligned}\frac{k_\mu}{R^{p-1}} &= \frac{\min\{g(R\frac{p-1}{p}\mu^{\frac{p}{p-1}}), g(R\frac{p-1}{p})\}}{R^{p-1}} \\ &= \min\{h_1 R\frac{p-1}{p}\mu^{\frac{p}{p-1}}(R\frac{p-1}{p}\mu^{\frac{p}{p-1}} - R)(R\frac{p-1}{p}\mu^{\frac{p}{p-1}} - r), \\ & \quad h_1 R\frac{p-1}{p}(R\frac{p-1}{p} - R)(R\frac{p-1}{p} - r)\} \qquad (3.2)\end{aligned}$$

tends to infinity as b_0 tends to infinity. Thus there exists $b_0^{(3)} := b_0^{(3)}(a_0, h_1, \Omega) > b_0^{(2)}$ such that such that for every $b_0 > b_0^{(3)}$ we have

$$\lambda_1 < \frac{k_\mu}{R^{p-1}}.$$

For a given $a_0 > 0, h_1 > 0$, define $\tilde{b_0} := \max\{b_0^{(3)}, b_0^{(1)}\} := \tilde{b_0}(a_0, h_1, \Omega)$. Then we have

Lemma 3.1. *Let $b_0 > \tilde{b_0}$ and $c^* := c^*(a_0, h_1, \Omega, b_0) := \min\{mR^{p-1} - \sigma, k_\mu - R^{p-1}\lambda_1\}$ for $c \le c^*$, (3.1) has a positive solution.*

Proof. We now construct subsolution

$$\psi := R(\frac{p}{p-1})\phi_1^{\frac{p}{p-1}}$$

is a sub-solution of (3.1). For $x \in \overline{\Omega}_\eta$ then

$$\nabla\psi = R\phi_1^{\frac{1}{p-1}}\nabla\phi_1$$

and ψ will be a subsolution if

$$\begin{cases} \int_\Omega |\nabla\psi(x)|^{p-2}\nabla\psi(x)\nabla w(x)dx \le \int_\Omega \Big(a(x)\psi(x) + b(x)\psi^2 - h(x)\psi^3 - c\alpha(x)\Big)w(x)dx \\ \psi \le 0 \end{cases} \qquad (3.3)$$

But

$$\begin{aligned}\int_\Omega |\nabla\psi(x)|^{p-2}\nabla\psi(x)\nabla w(x)dx &= \int_\Omega \phi(x)|\nabla\phi(x)|^{p-2}\nabla\phi(x)\nabla w(x)dx \\ &= R^{p-1}[\int_\Omega |\nabla\phi(x)|^{p-2}\nabla\phi(x)\nabla(\phi(x)w(x))]dx \\ &- R^{p-1}[\int_\Omega |\nabla\phi(x)|^p w(x)]dx \\ &= R^{p-1}[\int_\Omega (\lambda_1\phi(x)^p - |\nabla\phi(x)|^p)w(x)]dx.\end{aligned}$$

Now for $x \in \overline{\Omega}_\eta$,

$$R^{p-1}(\lambda_1\phi(x)^p - |\nabla\phi(x)|^p) \le -R^{p-1}m \le -\sigma - c \le -a_0\psi + b_0\psi^2 - h_1\psi^3 - c\alpha(x).$$

Next for $x \in \Omega \setminus \overline{\Omega}_\eta$,

$$R^{p-1}(\lambda_1\phi(x)^p - |\nabla\phi(x)|^p) \le -R^{p-1}\lambda_1 \le k_\mu - c \le -a_0\psi + b_0\psi^2 - h_1\psi^3 - c\alpha(x).$$

Hence

$$\psi := R(\frac{p}{p-1})\phi_1^{\frac{p}{p-1}}$$

is a sub-solution of (3.1). We also note that $\phi = R$ is a super-solution. Hence (3.1) has a positive solution. □

Our main result is the following theorem.

Theorem 3.1. *There exists positive constants* $\tilde{b_0} := \tilde{b_0}^{(1)}(a_0, h_1, \Omega)$ *and*

$$c^* := c^*(a_0, h_1, \Omega, b_0)$$

such that for $b_0 \geq \tilde{b}_0$ *and* $c \leq c^*$ *the problem (1.1) has a positive solution. Further* c^* *is an increasing function of* b_0 *and*

$$\lim_{b_0 \to \infty} c^* = \infty.$$

Proof. We observe that $\psi := R(\frac{p}{p-1})\phi_1^{\frac{p}{p-1}}$ is a positive sub-solution for (1.1), since ψ satisfies

$$\begin{cases} -\Delta_p \psi \leq g(\psi) - c\alpha(x) \leq a(x)\psi + b(x)\psi^2 - h(x)\psi^3 - c\alpha(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

It is easy to see that $\phi = M$ where $M > 0$ is sufficiently large so that $\phi \geq \psi$ is also satisfied. Hence (1.1) has a positive solution. We recall here that $\sigma = -g(r^*, b_0)$. Differentiating σ with respect to b_0, we have

$$\begin{aligned} \frac{d\sigma}{db_0} &= -\frac{\partial g(r^*, b_0)}{\partial r^*}\frac{dr^*}{db_0} - \frac{\partial g}{\partial b_0} \\ &= -\frac{\partial g}{\partial b_0} \\ &= -{r^*}^2 < 0. \end{aligned}$$

Also R is an increasing function of b_0. Thus $R^{p-1}m - \sigma$ is an increasing function of b_0. Next since $\frac{r}{R}$ decreases as b_0 increases, we deduce from (5) that $\frac{k_\mu}{R^{p-1}}$ increases as b_0 increases. Therefore $k_\mu - R^{p-1}\lambda_1 = R^{p-1}[\frac{k_\mu}{R^{p-1}} - \lambda_1]$ also increases as b_0. Hence by the definition of c^*, it is clear that c^* is an increasing function of b_0. Finally since $R \to \infty$, $\frac{\sigma}{R^{p-1}} \to 0$ and $\frac{k_\mu}{R^{p-1}} \to \infty$ as $b_0 \to \infty$, it is easy to see that $\lim_{b_0 \to \infty} c^* = \infty$. This completes the proof of Theorem. □

References

[1] H. Berestycki, L.A. Caffarelli, L. Nirenberg, *Inequalities for second order elliptic equations with applications to unbounded domains.* A celebration of John F. Nash Jr., Duke Math. J. 81 (1996)467-494.

[2] R.S. Cantrell, C. Cosner, Spatial Ecology via ReactionDiffusion Equation, Wiley Ser.Math. Comput. Biol. John Wiley. Sons Ltd, 2003.

[3] Paul C. Fife, Mathematical Aspects of Reacting and Diffusing Systems, Lecture Notes in Biomath., vol. 28, Springer-Verlag, 1979.

[4] R.A. Fisher, The wave of advance of advantageous genes, Ann. Eugenics 7 (1937) 353-369.

[5] Akira Okuba, Simon Levin, Diffusion and Ecological Problems: Modern Perspectives, second ed.Interdiscip. Appl. Math., vol. 14, Springer-Verlag, New York, 2001.

[6] J.D. Murray, Mathematical Biology. I. An Introduction, third ed., Interdiscip. Appl. Math., vol. 17, Springer-Verlag, New York, 2002;
J.D. Murray, Mathematical Biology. II. Spatial Models and Biomedical Applications, third ed., Interdiscip. Appl. Math., vol. 18, Springer-Verlag, New York, 2003.

[7] W.E. Kunin, Density and reproductive success in wild populations of Diplotaxis erucoides (Grassicaceae), Oecologia 91 (1992) 129.

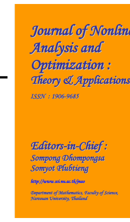
[8] P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, SIAM Rev. 24 (1982) 441–467.

[9] A. Kolmogoroff, I. Petrovsky, N. Piscounoff, Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem, Moscow Univ. Bull. Math. 1 (1937) 1-25 (in French)

[10] S. Oruganti, J. Shi, R. Shivaji, Diffusive logistic equations with constant yield harvesting, I: Steady states, Trans. Amer. Math. Soc. 354 (2002) 3601–3619.

[11] J.R. Philip, Sociality and sparse populations, Ecology 38 (1957) 107–111.
[12] J.G. Skellam, Random dispersal in theoretical populations, Biometrika 38 (1951) 196-218.
[13] Jaffar Ali, R. Shivaji, Kellan Wampler ,Population models with diffusion, strong Allee effect and constant yield harvesting,J. Math. Anal. Appl. 352 (2009) 907-913.

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 3, No. 1, (2012), 101-110
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



APPROXIMATION METHODS FOR FINITE FAMILY OF GENERALIZED NONEXPANSIVE MULTIVALUED MAPPINGS IN BANACH SPACES

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ABSTRACT. In the present paper, we define and study two new finite-step iterative process for a finite family of generalized nonexpansive multivalued mappings in Banach spaces. Several convergence theorems of the proposed iterations are established for this mappings.

KEYWORDS : Common fixed point; Generalized nonexpansive multivalued mapping; Iterative process, Convergence theorem.
AMS Subject Classification: 47H10,47H09.

1. INTRODUCTION

Let D be a nonempty subset of a Banach space X. Let $T : D \longrightarrow D$ be a singlevalued mapping. The Mann iteration process (see [1]), starting from $x_0 \in D$, is the sequence $\{x_n\}$ defined by

$$x_{n+1} = a_n x_n + (1 - a_n) T x_n, \qquad a_n \in [0, 1], n \geq 0,$$

where a_n satisfies certain conditions. The Ishikawa iteration process (see [2]), starting from $x_0 \in D$, is the sequence $\{x_n\}$ defined by

$$\begin{cases} y_n = b_n x_n + (1 - b_n) T x_n, & b_n \in [0, 1], n \geq 0, \\ x_{n+1} = a_n x_n + (1 - a_n) T y_n, & a_n \in [0, 1], n \geq 0, \end{cases}$$

where a_n and b_n satisfy certain conditions.

Recently, Agarwal et al. [3] introduced the following iterative process which is both faster than and independent of the Ishikawa process.

$$\begin{cases} x_0 = x \in D, \\ y_n = b_n x_n + (1 - b_n) T x_n, & b_n \in [0, 1], n \geq 0, \\ x_{n+1} = a_n T x_n + (1 - a_n) T y_n, & a_n \in [0, 1], n \geq 0. \end{cases}$$

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Article history : Received 7 September 2011. Accepted 17 October 2011.

In the recent years, iterative process for approximating fixed point of nonexpansive multivalued mappings have been investigated by various authors (see, e.g., [4–11]) using the Mann iteration process or the Ishikawa iteration process.

In this paper, inspired by [3] we introduce two new iterative process for a finite family of generalized nonexpansive multivalued mappings in Banach spaces. Weak and strong convergence theorems of the proposed iteration processes to a common fixed point of a finite family of generalized nonexpansive multivalued mappings in uniformly convex Banach spaces are established. Our results are new even for single valued mappings.

2. PRELIMINARIES

A Banach space X is said to satisfy Opial's condition if $x_n \longrightarrow z$ weakly as $n \longrightarrow \infty$ and $z \neq y$ imply that

$$\limsup_{n\longrightarrow\infty} \|x_n - z\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|.$$

All Hilbert spaces, all finite dimensional Banach spaces and $\ell^p(1 \leq p < \infty)$ have the Opial property.

A subset $D \subset X$ is called proximal if for each $x \in X$, there exists an element $y \in D$ such that

$$\| x - y \| = \text{dist}(x, D) = \inf\{\| x - z \| : z \in D\}.$$

It is known that every closed convex subset of a uniformly convex Banach space is proximal.

We denote by $CB(D), K(D)$ and $P(D)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of D respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} \text{dist}(x, B), \sup_{y\in B} \text{dist}(y, A)\},$$

for all $A, B \in CB(X)$.

Let $T : X \longrightarrow 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denote by $F(T)$.

Definition 2.1. A multivalued mapping $T : X \longrightarrow CB(X)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in X.$$

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq\| x - p \|$ for all $x \in X$ and all $p \in F(T)$.

J. Garcia et al. [12] introduced a new condition on singlevalued mappings, called condition (E), which is weaker than nonexpansiveness. Very recently, Abkar and Eslamian [13] used a modified condition for multivalued mappings as follows:

Definition 2.2. A multivalued mapping $T : X \longrightarrow CB(X)$ is said to satisfy condition (E_μ) provided that

$$\text{dist}(x, Ty) \leq \mu\, \text{dist}(x, Tx) + \|x - y\|, \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Lemma 2.3. *Let $T : X \longrightarrow CB(X)$ be a multivalued nonexpansive mapping. Then T satisfies the condition (E_1).*

The proof of the following Lemma is similar to that of Theorem 3.3 in [10], hence we omit the details.

Lemma 2.4. *([10]) Let D be a nonempty closed convex subset of uniformly convex Banach space X with the Opial property. Let $T : D \longrightarrow K(D)$ be a multivalued mapping satisfying the condition (E). If $x_n \longrightarrow x$ weakly and $\lim_{n\longrightarrow\infty} \operatorname{dist}(x_n, Tx_n) = 0$, then $x \in Tx$.*

The following Lemma can be found in [14].

Lemma 2.5. *Let X be a Banach space. Then X is uniformly convex if and only if for any given number $r > 0$ there exists a continuous, strictly increasing and convex function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\| \alpha x + (1-\alpha)y \|^2 \leq \alpha \| x \|^2 + (1-\alpha) \| y \|^2 - \alpha(1-\alpha)\varphi(\| x - y \|),$$

for all $x, y \in B_r(0) = \{x \in X : \| x \| \leq r\}$, and $\alpha \in [0,1]$.

Definition 2.6. Let D be a nonempty closed subset of a Banach space X. A mapping $T : D \longrightarrow CB(D)$ is hemi-compact if for any bounded sequence $\{x_n\}$ in D such that $\operatorname{dist}(Tx_n, x_n) \longrightarrow 0 \quad as \quad n \longrightarrow \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow p \in D$. We note that if D is compact, then every multivalued mapping $T : D \longrightarrow CB(D)$ is hemi-compact.

3. CONVERGENCE THEOREMS

Let D be a nonempty convex subset of a Banach space X, let $T_1, T_2, ..., T_m$ be finite multivalued mappings from D into $CB(D)$. Let $\{x_n\}$ be the sequence defined by $x_1 \in D$ and

$$(\mathbf{A}) : \quad \begin{cases} y_{n,1} = (1-a_{n,1})x_n + a_{n,1}z_{n,1}, \\ y_{n,2} = (1-a_{n,2})z_{n,1} + a_{n,2}z_{n,2}, \\ \ldots \\ y_{n,m-1} = (1-a_{n,m-1})z_{n,m-2} + a_{n,m-1}z_{n,m-1}, \\ x_{n+1} = (1-a_{n,m})z_{n,m-1} + a_{n,m}z_{n,m}, \qquad n \geq 1, \end{cases}$$

where $z_{n,1} \in T_1(x_n)$, $z_{n,k} \in T_k(y_{n,k-1})$ for $k = 2, ..., m$, and $\{a_{n,i}\} \in [0,1]$.

Definition 3.1. A mapping $T : D \longrightarrow CB(D)$ is said to satisfy condition (I) if there is a nondecreasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$ such that

$$\operatorname{dist}(x, Tx) \geq g(\operatorname{dist}(x, F(T)).$$

Let $T_i : D \longrightarrow CB(D), (i = 1, 2, ..., m)$ be finite given mappings. The mappings $T_1, T_2, ..., T_m$ are said to satisfy condition (II) if there exists a nondecreasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$, such that

$$\max_{1\leq i\leq m} \operatorname{dist}(x, T_i x) \geq g(\operatorname{dist}(x, \mathcal{F})),$$

where $\mathcal{F} = \bigcap_{i=1}^m F(T_i)$.

Theorem 3.2. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : D \longrightarrow CB(D), (i = 1, 2, ..., m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfy the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., m)$, for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,i} \in [\alpha, \beta] \subset (0,1)$(i=1,2,...,m). Assume further that the mappings $T_1, T_2, ..., T_m$ satisfying the condition (II). Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, .., T_m$.*

Proof. Let $p \in \mathcal{F}$. It follows from (A) and the quasi-nonexpansivness of $T_i, (i = 1, 2, ..., m)$ that

$$\begin{aligned} \| y_{n,1} - p \| &= \| (1-a_{n,1})x_n + a_{n,1}z_{n,1} - p \| \\ &\leq (1-a_{n,1}) \| x_n - p \| + a_{n,1} \| z_{n,1} - p \| \\ &= (1-a_{n,1}) \| x_n - p \| + a_{n,1}\text{dist}(z_{n,1}, T_1(p)) \\ &\leq (1-a_{n,1}) \| x_n - p \| + a_{n,1}H(T_1(x_n), T_1(p)) \\ &\leq (1-a_{n,1}) \| x_n - p \| + a_{n,1} \| x_n - p \| \\ &\leq \|x_n - p\| \end{aligned}$$

and

$$\begin{aligned} \| y_{n,2} - p \| &= \| (1-a_{n,2})z_{n,1} + a_{n,2}z_{n,2} - p \| \\ &\leq (1-a_{n,2}) \| z_{n,1} - p \| + a_{n,2} \| z_{n,2} - p \| \\ &= (1-a_{n,2})\text{dist}(z_{n,1}, T_1(p)) + a_{n,2}\text{dist}(z_{n,2}, T_2(p)) \\ &\leq (1-a_{n,2})H(T_1(x_n), T_1(p)) + a_{n,2}H(T_2(y_{n,1}), T_2(p)) \\ &\leq (1-a_{n,2}) \| x_n - p \| + a_{n,2} \| y_{n,1} - p \| \\ &\leq \|x_n - p\|. \end{aligned}$$

Continuing the above process, we get

$$\|y_{n,k} - p\| \leq \|x_n - p\| \quad \text{for} \ \ k = 3, ..., m-1.$$

In particular

$$\begin{aligned} \| x_{n+1} - p \| &= \| (1-a_{n,m})z_{n,m-1} + a_{n,m}z_{n,m} - p \| \\ &\leq (1-a_{n,m}) \| z_{n,m-1} - p \| + a_{n,m} \| z_{n,m} - p \| \\ &= (1-a_{n,m})\text{dist}(z_{n,m-1}, T_{m-1}(p)) + a_{n,m}\text{dist}(z_{n,m}, T_m(p)) \\ &\leq (1-a_{n,m})H(T_{m-1}(y_{n,m-2}), T_{m-1}(p)) + a_{n,m}H(T_m(y_{n,m-1}), T_m(p)) \\ &\leq (1-a_{n,m}) \| y_{n,m-2} - p \| + a_{n,m} \| y_{n,m-1} - p \| \\ &\leq (1-a_{n,m}) \| x_n - p \| + a_{n,m} \| x_n - p \| \\ &\leq \|x_n - p\|. \end{aligned}$$

Thus $\lim_{n\longrightarrow\infty} \| x_n - p \|$ exists for any $p \in \mathcal{F}$. Since the sequences $\{x_n\}$ and $\{y_{n,i}\}$, $(i = 1, ..., m-1)$ are bounded, we can find $r > 0$ depending on p such that $x_n - p$, $y_{n,i} - p \in B_r(0)$ for all $n \geq 0$. From Lemma 2.5, we get

$$\begin{aligned} \| y_{n,1} - p \|^2 &= \| (1-a_{n,1})x_n + a_{n,1}z_{n,1} - p \|^2 \\ &\leq (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1} \| z_{n,1} - p \|^2 \\ &\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\ &= (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1}\text{dist}(z_{n,1}, T_1(p))^2 \\ &\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\ &\leq (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1}H(T_1(x_n), T_1(p))^2 \\ &\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\ &\leq (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1} \| x_n - p \|^2 \\ &\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\ &\leq \|x_n - p\|^2 - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \end{aligned}$$

and

$$\| y_{n,2} - p \|^2 = \| (1-a_{n,2})z_{n,1} + a_{n,2}z_{n,2} - p \|^2$$

$$
\begin{aligned}
&\leq (1-a_{n,2}) \parallel z_{n,1}-p \parallel^2 + a_{n,2} \parallel z_{n,2}-p \parallel^2 \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&= (1-a_{n,2})\mathrm{dist}(z_{n,1},T_1(p))^2 + a_{n,2}\mathrm{dist}(z_{n,2},T_2(p))^2 \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\leq (1-a_{n,2})H(T_1(x_n),T_1(p))^2 + a_{n,2}H(T_2(y_{n,1}),T_2(p))^2 \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\leq (1-a_{n,2}) \parallel x_n-p \parallel^2 + a_{n,2} \parallel y_{n,1}-p \parallel^2 \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\leq \|x_n-p\|^2 - a_{n,1}a_{n,2}(1-a_{n,1})\varphi(\|x_n-z_{n,1}\|) \\
&\quad -a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|).
\end{aligned}
$$

Applying Lemma 2.5 again, we have

$$
\begin{aligned}
\parallel y_{n,3}-p \parallel^2 &= \parallel (1-a_{n,3})z_{n,2}+a_{n,3}z_{n,3}-p \parallel^2 \\
&\leq (1-a_{n,3}) \parallel z_{n,2}-p \parallel^2 + a_{n,3} \parallel z_{n,3}-p \parallel^2 \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&= (1-a_{n,3})\mathrm{dist}(z_{n,2},T_2(p))^2 + a_{n,3}\mathrm{dist}(z_{n,3},T_3(p))^2 \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&\leq (1-a_{n,3})H(T_2(y_{n,1}),T_2(p))^2 + a_{n,3}H(T_3(y_{n,2}),T_3(p))^2 \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&\leq (1-a_{n,3}) \parallel y_{n,1}-p \parallel^2 + a_{n,3} \parallel y_{n,2}-p \parallel^2 \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&\leq (1-a_{n,3})\|x_n-p\|^2 + a_{n,3}\|x_n-p\|^2 \\
&\quad -a_{n,3}a_{n,2}a_{n,1}(1-a_{n,1})\varphi(\|x_n-z_{n,1}\|) \\
&\quad -a_{n,3}a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|) \\
&\leq \|x_n-p\|^2 - a_{n,1}a_{n,2}a_{n,3}(1-a_{n,1})\varphi(\|x_n-z_{n,1}\|) \\
&\quad -a_{n,3}a_{n,2}(1-a_{n,2})\varphi(\|z_{n,1}-z_{n,2}\|) \\
&\quad -a_{n,3}(1-a_{n,3})\varphi(\|z_{n,2}-z_{n,3}\|)
\end{aligned}
$$

By continuing this process we obtain

$$
\begin{aligned}
\parallel x_{n+1}-p \parallel^2 &= \parallel (1-a_{n,m})z_{n,m-1}+a_{n,m}z_{n,m}-p \parallel^2 \\
&\leq (1-a_{n,m}) \parallel z_{n,m-1}-p \parallel^2 + a_{n,m} \parallel z_{n,m}-p \parallel^2 \\
&\quad -a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|) \\
&= (1-a_{n,m})\mathrm{dist}(z_{n,m-1},T_{m-1}(p))^2 + a_{n,m}\mathrm{dist}(z_{n,m},T_m(p))^2 \\
&\quad -a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|) \\
&\leq (1-a_{n,m})H(T_{m-1}(y_{n,m-2}),T_{m-1}(p))^2 \\
&\quad +a_{n,m}H(T_m(y_{n,m-1}),T_m(p))^2 \\
&\quad -a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|) \\
&\leq (1-a_{n,m}) \parallel y_{n,m-2}-p \parallel^2 + a_{n,m} \parallel y_{n,m-1}-p \parallel^2 \\
&\quad -a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|) \\
&\leq \|x_n-p\|^2 - a_{n,m}a_{n,m-1}a_{n,m-2}...a_{n,1}(1-a_{n,1})\varphi(\|x_n-z_{n,1}\|)
\end{aligned}
$$

$$-\cdots - a_{n,m}a_{n,m-1}(1-a_{n,m-1})\varphi(\|z_{n,m-2}-z_{n,m-1}\|)$$
$$-a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1}-z_{n,m}\|).$$

Thus we have

$$a_{n,m}a_{n,m-1}(1-a_{n,m-1})\varphi(\|z_{n,m-2}-z_{n,m-1}\|) \le \|x_n-p\|^2 - \| x_{n+1}-p \|^2 .$$

It follows from our assumption that

$$\alpha^2(1-\beta)\varphi(\|z_{n,m-2}-z_{n,m-1}\|) \le \|x_n-p\|^2 - \| x_{n+1}-p \|^2 .$$

Since $\lim_{n\longrightarrow\infty}\|x_n-p\|$ exists for all $p\in\mathcal{F}$, taking the limit as $n\longrightarrow\infty$ yields that

$$\lim_{n\longrightarrow\infty}\varphi(\|z_{n,m-2}-z_{n,m-1}\|)=0.$$

Since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\longrightarrow\infty}\|z_{n,m-2}-z_{n,m-1}\|=0.$$

Similarly, we can obtain that

$$\lim_{n\longrightarrow\infty}\|x_n-z_{n,1}\|=0,$$

and

$$\lim_{n\longrightarrow\infty}\|z_{n,k}-z_{n,k-1}\|=0 \qquad \text{for } \ k=2,...,m.$$

Using (A) we have

$$\lim_{n\longrightarrow\infty}\|x_n-y_{n,1}\|=a_{n,1}\lim_{n\longrightarrow\infty}\|x_n-z_{n,1}\|=0.$$

Also we have

$$\|x_n-z_{n,2}\|\le\|x_n-z_{n,1}\|+\|z_{n,1}-z_{n,2}\|\longrightarrow 0, \qquad n\longrightarrow\infty.$$

Hence

$$\lim_{n\longrightarrow\infty}\|x_n-y_{n,2}\|=(1-a_{n,2})\lim_{n\longrightarrow\infty}\|x_n-z_{n,1}\|+a_{n,2}\lim_{n\longrightarrow\infty}\|x_n-z_{n,2}\|=0.$$

Continuing the above process , for $k=2,...,m$ we have

$$\|x_n-z_{n,k}\|\le\|x_n-z_{n,k-1}\|+\|z_{n,k-1}-z_{n,k}\|\longrightarrow 0, \qquad n\longrightarrow\infty$$

and also for $k=2,...,m-1$

$$\lim_{n\longrightarrow\infty}\|x_n-y_{n,k}\|=(1-a_{n,k})\lim_{n\longrightarrow\infty}\|x_n-z_{n,k-1}\|+a_{n,k}\lim_{n\longrightarrow\infty}\|x_n-z_{n,k}\|=0.$$

Also we have

$$\text{dist}(x_n,T_1x_n)\le\|x_n-z_{n,1}\|\longrightarrow 0.$$

For $k=2,...,m$, since T_i satisfies the condition (E) we have

$$\begin{aligned}\text{dist}(x_n,T_kx_n) &\le \|x_n-y_{n,k-1}\|+\text{dist}(y_{n,k-1},T_kx_n)\\ &\le \|x_n-y_{n,k-1}\|+\mu\,\text{dist}(y_{n,k-1},T_ky_{n,k-1})+\|x_n-y_{n,k-1}\|\\ &\le \|x_n-y_{n,k-1}\|+\mu\,\text{dist}(x_n,T_ky_{n,k-1})+\mu\,\|x_n-y_{n,k-1}\|\\ &\quad +\|x_n-y_{n,k-1}\|\\ &\le (\mu+2)\|x_n-y_{n,k-1}\|+\mu\|x_n-z_{n,k}\|\longrightarrow 0, \qquad n\longrightarrow\infty.\end{aligned}$$

Note that by our assumption $\lim_{n\longrightarrow\infty}\text{dist}(x_n,\mathcal{F})=0$. Next we show that $\{x_n\}$ is a Cauchy sequence in D. Let $\varepsilon>0$ be arbitrarily chosen. Since $\lim_{n\longrightarrow\infty}\text{dist}(x_n,\mathcal{F})=0$, there exists N such that for all $n\ge N$, we have

$$\text{dist}(x_n,\mathcal{F})<\frac{\varepsilon}{4}.$$

In particular, $\inf\{\|x_N - p\| : p \in \mathcal{F}\} < \varepsilon/4$. Thus there exist a $q \in \mathcal{F}$ such that

$$\|x_N - q\| < \frac{\varepsilon}{2}.$$

Hence for $m, n \geq N$ we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq 2\|x_N - q\| < 2(\frac{\varepsilon}{2}) = \varepsilon.\end{aligned}$$

This implies that $\{x_n\}$ is Cauchy sequence in D and hence converges to some $w \in D$. For $i = 1, ..., m$, since T_i satisfies the condition (E), we have

$$\begin{aligned}\text{dist}(w, T_i w) &\leq \|w - x_n\| + \text{dist}(x_n, T_i w) \\ &\leq 2\|w - x_n\| + (\mu)\text{dist}(x_n, T_i x_n) \longrightarrow 0, \qquad n \longrightarrow \infty\end{aligned}$$

from which it follows that $\text{dist}(w, T_i w) = 0$ which in turn implies that $w \in T_i(w)$. Thus $w \in \mathcal{F}$. This completes the proof. □

Theorem 3.3. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : D \longrightarrow CB(D), (i = 1, 2, ..., m)$ be a finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,i} \in [\alpha, \beta] \subset (0, 1)$(i=1,2,...,m). If one of the mappings $T_i, (i = 1, 2, ..., m)$ is hemi-compact, then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, .., T_m$.*

Proof. As in the proof of Theorem 3.2, we have $\lim_{n\longrightarrow\infty} \text{dist}(T_i x_n, x_n) = 0, (i = 1, 2, ..., m)$. Without loss of generality, we assume that T_1 is hemi-compact. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = w$ for some $w \in D$. Since T_i satisfies the condition (E) for $i = 1, 2, ..., m$, we have

$$\begin{aligned}\text{dist}(w, T_i(w)) &\leq \|w - x_{n_k}\| + \text{dist}(x_{n_k}, T_i(w)) \\ &\leq (\mu)\,\text{dist}(x_{n_k}, T_i(x_{n_k})) + 2\|w - x_{n_k}\| \longrightarrow 0, \qquad \text{as} \quad k \longrightarrow \infty.\end{aligned}$$

This implies that $w \in \mathcal{F}$. Since $\{x_{n_k}\}$ converges strongly to w and $\lim_{n\longrightarrow\infty} \|x_n - w\|$ exists (as in the proof of Theorem 3.2), it follows that $\{x_n\}$ converges strongly to w. □

Theorem 3.4. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property. Let $T_i : D \longrightarrow K(D), (i = 1, 2, ..., m)$ be finite family of quasi-nonexpansive multivalued mappings satisfying the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., m)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_{n,i} \in [\alpha, \beta] \subset (0, 1)$(i=1,2,...,m). Then $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, .., T_m$.*

Proof. As in the proof of Theorem 3.2, $\{x_n\}$ is bounded and for $i = 1, 2, ..., m$,

$$\lim_{n\longrightarrow\infty} \text{dist}(x_n, T_i(x_n)) = 0.$$

Consider $w_1, w_2 \in D$, two weak cluster points of the sequence $\{x_n\}$. Then there exist two subsequences $\{y_n\}$ and $\{z_n\}$ of $\{x_n\}$ such that $y_n \longrightarrow w_1$ weakly and $z_n \longrightarrow w_2$ weakly. By Lemma 2.4 we have $w_1, w_2 \in \mathcal{F}$. As in the proof of Theorem 3.2, $\lim_{n\longrightarrow\infty} \|x_n - w_1\|$ and $\lim_{n\longrightarrow\infty} \|x_n - w_2\|$ exist. We claim that $w_1 = w_2$. Indeed, assume to the contrary that $w_1 \neq w_2$. By the Opial property we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|x_n - w_1\| &= \lim_{n\longrightarrow\infty} \|y_n - w_1\| < \lim_{n\longrightarrow\infty} \|y_n - w_2\| \\ &= \lim_{n\longrightarrow\infty} \|x_n - w_2\| = \lim_{n\longrightarrow\infty} \|z_n - w_2\|\end{aligned}$$

$$< \lim_{n\longrightarrow\infty} \|z_n - w_1\| = \lim_{n\longrightarrow\infty} \|x_n - w_1\|,$$

which is a contradiction, hence $w_1 = w_2$. Thus the sequence $\{x_n\}$ has at most one weak cluster point. Since D is weakly sequentially compact, we deduce that the sequence $\{x_n\}$ has exactly one weak cluster point $w \in D$, that is $x_n \longrightarrow w$ weakly. By Lemma 2.4 we obtain that $w \in \mathcal{F}$. □

We now intend to remove the restriction that $T_i(p) = p$ for each $p \in \mathcal{F}$. We define the following iteration process.

Let D be a nonempty convex subset of a Banach space X, and let $T_1, T_2, ..., T_m$ be finite multivalued mappings from D into $P(D)$, and

$$P_{T_i}(x) = \{y \in T_i(x) :\| x - y \|= \text{dist}(x, T_i(x))\}.$$

Then, for $x_1 \in D$, we consider the following iterative process:

$$(\mathbf{B}): \quad \begin{cases} y_{n,1} = (1 - a_{n,1})x_n + a_{n,1}z_{n,1}, \\ y_{n,2} = (1 - a_{n,2})z_{n,1} + a_{n,2}z_{n,2}, \\ ... \\ y_{n,m-1} = (1 - a_{n,m-1})z_{n,m-2} + a_{n,m-1}z_{n,m-1}, \\ x_{n+1} = (1 - a_{n,m})z_{n,m-1} + a_{n,m}z_{n,m}, \qquad n \geq 1, \end{cases}$$

where $z_{n,1} \in P_{T_1}(x_n)$, $z_{n,k} \in P_{T_k}(y_{n,k-1})$ for $k = 2, ..., m$, and $\{a_{n,i}\} \in [0, 1]$.

Theorem 3.5. *Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : D \longrightarrow P(D), (i = 1, 2, ..., m)$ be multivalued mappings such that P_{T_i} is quasi-nonexpansive and satisfies the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iterative process defined by (B), and $a_{n,i} \in [\alpha, \beta] \subset (0, 1)$ (i=1,2,...,m). Assume that the mappings $T_1, T_2, ..., T_m$ satisfy the condition (II). Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_m$.*

Proof. Let $p \in \mathcal{F}$, then, $P_{T_i}(p) = \{p\}$ for $i = 1, 2, ..., m$. It follows from (B) that

$$\begin{aligned} \| y_{n,1} - p \| &= \| (1 - a_{n,1})x_n + a_{n,1}z_{n,1} - p \| \\ &\leq (1 - a_{n,1}) \| x_n - p \| + a_{n,1} \| z_{n,1} - p \| \\ &= (1 - a_{n,1}) \| x_n - p \| + a_{n,1}\text{dist}(z_{n,1}, P_{T_1}(p)) \\ &\leq (1 - a_{n,1}) \| x_n - p \| + a_{n,1}H(P_{T_1}(x_n), P_{T_1}(p)) \\ &\leq (1 - a_{n,1}) \| x_n - p \| + a_{n,1} \| x_n - p \| \\ &\leq \|x_n - p\|. \end{aligned}$$

Continuing the above process, we get

$$\|y_{n,k} - p\| \leq \|x_n - p\| \quad \text{for} \;\; k = 2, ..., m - 1.$$

It follows that

$$\begin{aligned} \| x_{n+1} - p \| &= \| (1 - a_{n,m})z_{n,m-1} + a_{n,m}z_{n,m} - p \| \\ &\leq (1 - a_{n,m}) \| z_{n,m-1} - p \| + a_{n,m} \| z_{n,m} - p \| \\ &= (1 - a_{n,m})\text{dist}(z_{n,m-1}, P_{T_{m-1}}(p)) + a_{n,m}\text{dist}(z_{n,m}, P_{T_m}(p)) \\ &\leq (1 - a_{n,m})H(P_{T_{m-1}}(y_{n,m-2}), P_{T_{m-1}}(p)) \\ &\quad + a_{n,m}H(P_{T_m}(y_{n,m-1}), P_{T_m}(p)) \\ &\leq (1 - a_{n,m}) \| y_{n,m-2} - p \| + a_{n,m} \| y_{n,m-1} - p \| \\ &\leq (1 - a_{n,m}) \| x_n - p \| + a_{n,m} \| x_n - p \| \\ &\leq \|x_n - p\|. \end{aligned}$$

Thus $\lim_{n\longrightarrow\infty} \| x_n - p \|$ exists for any $p \in \mathcal{F}$. Since the sequences $\{x_n\}$ and $\{y_{n,i}\}$, $i = 1, ..., m-1$ are bounded, we can find $r > 0$ depending on p such that $x_n - p$, $y_{n,i} - p \in B_r(0)$ for all $n \geq 0$. From Lemma 2.5, we get

$$\begin{aligned}
\| y_{n,1} - p \|^2 &= \| (1-a_{n,1})x_n + a_{n,1}z_{n,1} - p \|^2 \\
&\leq (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1} \| z_{n,1} - p \|^2 \\
&\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&= (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1}\text{dist}(z_{n,1}, P_{T_1}(p))^2 \\
&\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&\leq (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1}H(P_{T_1}(x_n), P_{T_1}(p))^2 \\
&\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&\leq (1-a_{n,1}) \| x_n - p \|^2 + a_{n,1} \| x_n - p \|^2 \\
&\quad - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&\leq \|x_n - p\|^2 - a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|).
\end{aligned}$$

It follows from lemma 2.5 that

$$\begin{aligned}
\| x_{n+1} - p \|^2 &= \| (1-a_{n,m})z_{n,m-1} + a_{n,m}z_{n,m} - p \|^2 \\
&\leq (1-a_{n,m}) \| z_{n,m-1} - p \|^2 + a_{n,m} \| z_{n,m} - p \|^2 \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|) \\
&= (1-a_{n,m})\text{dist}(z_{n,m-1}, P_{T_{m-1}}(p))^2 + a_{n,m}\text{dist}(z_{n,m}, P_{T_m}(p))^2 \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|) \\
&\leq (1-a_{n,m})H(P_{T_{m-1}}(y_{n,m-2}), P_{T_{m-1}}(p))^2 \\
&\quad + a_{n,m}H(P_{T_m}(y_{n,m-1}), P_{T_m}(p))^2 \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|) \\
&\leq (1-a_{n,m}) \| y_{n,m-2} - p \|^2 + a_{n,m} \| y_{n,m-1} - p \|^2 \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|) \\
&\leq \|x_n - p\|^2 - a_{n,m}a_{n,m-1}a_{n,m-2}...a_{n,1}(1-a_{n,1})\varphi(\|x_n - z_{n,1}\|) \\
&\quad - \cdots - a_{n,m}a_{n,m-1}(1-a_{n,m-1})\varphi(\|z_{n,m-2} - z_{n,m-1}\|) \\
&\quad - a_{n,m}(1-a_{n,m})\varphi(\|z_{n,m-1} - z_{n,m}\|).
\end{aligned}$$

Now, by a similar argument as in the proof of Theorem 3.2 we have $\lim_{n\longrightarrow\infty} \|x_n - z_{n,k}\| = 0$ for $k = 2, ..., m-1$ and $\lim_{n\longrightarrow\infty} \|x_n - y_{n,k}\| = 0$ for $k = 2, ..., m$. Also we have

$$\text{dist}(x_n, T_1x_n) \leq \text{dist}(x_n, P_{T_1}x_n) \leq \|x_n - z_{n,1}\| \longrightarrow 0.$$

For $k = 2, ..., m$, since P_{T_i} satisfies the condition (E) we have

$$\begin{aligned}
\text{dist}(x_n, T_kx_n) &\leq \text{dist}(x_n, P_{T_k}x_n) \leq \|x_n - y_{n,k-1}\| + \text{dist}(y_{n,k-1}, P_{T_k}x_n) \\
&\leq \|x_n - y_{n,k-1}\| + \mu\,\text{dist}(y_{n,k-1}, P_{T_k}y_{n,k-1}) + \|x_n - y_{n,k-1}\| \\
&\leq \|x_n - y_{n,k-1}\| + \mu\,\text{dist}(x_n, P_{T_k}y_{n,k-1}) + \mu\,\|x_n - y_{n,k-1}\| \\
&\quad + \|x_n - y_{n,k-1}\| \\
&\leq (\mu+2)\|x_n - y_{n,k-1}\| + \mu\|x_n - z_{n,k}\| \longrightarrow 0, \qquad n \longrightarrow \infty.
\end{aligned}$$

Note that by our assumption $\lim_{n\longrightarrow\infty} \text{dist}(x_n, \mathcal{F}) = 0$. By a similar argument as in the proof of Theorem 3.2 we show that $\{x_n\}$ is Cauchy sequence in D and hence converges to $w \in D$. For $i = 1, ..., m$, since P_{T_i} satisfies the condition (E), we have

$$\text{dist}(w, T_iw) \leq \text{dist}(w, P_{T_i}w)$$

$$\begin{aligned} &\leq \quad \|w - x_n\| + \text{dist}(x_n, P_{T_i}w) \\ &\leq \quad 2\|w - x_n\| + (\mu)\text{dist}(x_n, P_{T_i}x_n) \longrightarrow 0, \qquad n \longrightarrow \infty \end{aligned}$$

from which it follows that $\text{dist}(w, T_i w) = 0$, which in turn implies that $w \in T_i(w), (i = 1, 2, ..., m)$. Thus $w \in \mathcal{F}$. This completes the proof. □

References

1. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953) 506-510.
2. S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974) 147-150.
3. R. P. Agarwal, D. Oregan, D. R. Sahu, *Iterative construction of fixed point of nearly asymptotically nonexpansive mappings.*, J. Nonlinear convex Anal. **8** (2007) 61-79.
4. K. P. R. Sastry, G. V. R. Babu, *Convergence of Ishikawa iterates for a multivalued mapping with a fixed point*, Czechoslovak Math. J. **55** (2005) 817-826.
5. B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comput. Math. Appl. **54** (2007) 872-877.
6. Y. Song, H. Wang, Erratum to *Mann and Ishikawa iterative processes for multi-valued mappings in Banach spaces*, Comput. Math. Appl. **55** (2008) 2999-3002.
7. Y. Song, H. Wang, *Convergence of iterative algorithms for multivalued mappings in Banach spaces*, Nonlinear Analysis. **70** (2009) 1547-1556.
8. N. Shahzad, H. Zegeye, *On Mann and Ishikawa iteration schemes for multivalued maps in Banach space.* Nonlinear Analysis. **71** (2009) 838-844.
9. M. Abbas, S. H. Khan, A. R. Khan, R. P. Agarwal , *Common fixed points of two multivalued nonexpansive mappings by one-step iterative scheme*, Appl. Math. Lett., **24** (2011) 97-102.
10. M. Eslamian, A. Abkar, *One-step iterative process for a finite family of multivalued mappings*, Math. Comput. Modell . **54** (2011) 105-111.
11. W. Cholamjiak, S. Suantai, *Approximation of common fixed point of two quasi-nonexpansive multi-valued maps in Banach spaces*, Comput. Math. Appl. **61** (2011) 941-949.
12. J. Garcia-Falset, E. Llorens-Fuster, T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings*, J. Math. Anal. Appl. **375** (2011) 185-195.
13. A. Abkar, M. Eslamian, *Common fixed point results in CAT(0) spaces*, Nonlinear Anal., **74** (2011) 1835-1840.
14. H. K. Xu, *Inequalities in Banach spaces with application*, Nonlinear Anal. **16** (1991) 1127-1138.