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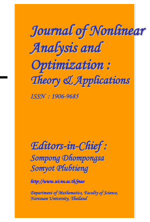

OPTION PRICING FOR A JUMP DIFFUSION MODEL WITH FRACTIONAL STOCHASTIC VOLATILITY◇

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ABSTRACT. An alternative stochastic volatility model with jumps is proposed, in which stock prices follow a jump diffusion model and their stochastic volatility follows a fractional stochastic volatility model. By using an approximate method, we find a formulation for the European-style option in terms of the characteristic function of tail probabilities.

KEYWORDS : Fractional Brownian motion; Approximate approach; Stochastic Volatility; Jump diffusion model.
AMS Subject Classification: 60G22.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. All processes that we shall consider in Section 1 and 2 will be defined in this space.

For $t \in [0, T]$ and $T < \infty$, a *geometric Brownian motion (gBm) model with jumps and with fractional stochastic volatility* is a model of the form

$$dS_t = S_t \left(\mu dt + \sqrt{v_t} dW_t \right) + S_{t-} Y_t dN_t, \tag{1.1}$$

where $\mu \in \Re$, $S = (S_t)_{t \in [0,T]}$ is a process representing a price of the underlying risky assets, $W = (W_t)_{t \in [0,T]}$ is the standard Brownian motion, $N = (N_t)_{t \in [0,T]}$ is a Poisson process with intensity λ, and $S_{t-} Y_t$ represents the amplitude of the jump which occurs at time t. We assume that the processes W and N are independent. The volatility process $v_t := \sigma_t^2$ in (1.1) is modeled by

$$dv_t = (\omega - \theta v_t)\, dt + \xi v_t dB_t, \tag{1.2}$$

where $\omega > 0$ is the mean long-term volatility, $\theta \in \Re$ is the rate at which the volatility reverts toward its long-term mean, $\xi > 0$ is the volatility of the volatility process,

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and $(B_t)_{t\in[0,T]}$ is a fractional Brownian motion. Assume that the proessess (S_t) and (v_t) are $\mathcal{F}_t$-measurable.

The notation S_{t-} means that whenever there is a jump, the value of the process before the jump is used on the left-hand side of the formula.

The fraction version of equation (1.1) is given by

$$dS_t = S_t\left(\mu dt + \sqrt{v_t}dB_t\right) + S_{t-}Y_t dN_t. \tag{1.3}$$

Recently, Intarasit and Sattayatham [1] showed that the process S_t in (1.3) can be approximated in $L_2(\Omega)$ by a semimartingale S_t^ε in the sense that $||S_t^\varepsilon - S_t||_{L_2(\Omega)} \to 0$ as $\varepsilon \to 0$, where S_t^ε satisfies the following equation

$$dS_t^\varepsilon = S_t^\varepsilon\left(\mu dt + \sqrt{v_t^\varepsilon}dW_t\right) + Y_t dN_t.$$

The purpose of this paper is to consider the problem of option pricing for the gBm model with jumps and with fractional stochastic volatility (1.1). But since the process S_t is a fractional process, we cannot apply Ito calculus directly. We shall thus work in another direction by finding a formula for option pricing for the process S_t^ε and using it as an approximation for pricing the model (1.1). In order to find such a formula, we shall work in the space of a risk-neutral probability measure. There are some authors who have investigated this problem before but not in the fractional case, for example Heston [2] and Yan and Hanson [3].

Recall that the fractional Brownian motion with *Hurst coeifficient* is a Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with zero mean, and the covariance function is given by

$$R(t,s) = E\left[B_t^H B_s^H\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

If $H = 1/2$, then $R(t,s) = min(t,s)$ and B_t^H is the usual standard Brownian motion. In the case $1/2 < H < 1$ the fractional Brownian motion exhibits statistical long-range dependency in the sense that $\rho_n := E[B_1^H\left(B_{n+1}^H - B_n^H\right)] > 0$ for all $n = 1, 2, 3, ...$ and $\sum\limits_{n=1}^{\infty}\rho_n = \infty$ ([4], page 2). Hence, in financial modeling, one usually assumes that $H \in (1/2, 1)$. Put $\alpha = 1/2 - H$. It is known that a fractional Brownian motion B_t^H can be decomposed as follows:

$$B_t^H = \frac{1}{\Gamma(1+\alpha)}\left\{Z_t + \int_0^t (t-s)^{-\alpha}dW_s\right\}$$

where Γ is the gamma function, $Z_t = \int_{-\infty}^0\left[(t-s)^{-\alpha} - (s)^{-\alpha}\right]dW_s$.

We suppose from now on that $0 < \alpha < 1/2$. The process Z_t has absolutely continuous trajectories, so it suffices to consider only the term

$$B_t = \int_0^t (t-s)^{-\alpha}dW_s, \tag{1.4}$$

that has a long-range dependence.

Note that B_t can be approximated by

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{-\alpha} dW_s \tag{1.5}$$

in the sense that B_t^ε converges to B_t in $L_2(\Omega)$ as $\varepsilon \to 0$, uniform with respect to $t \in [0,T]$ (see [5]).

Since $(B_t^\varepsilon)_{t\in[0,T]}$ is a continuous semimartingale then Itô calculus can be applied to the following stochastic differential equation (SDE)

$$dS_t^\varepsilon = S_t^\varepsilon(\mu dt + \sigma dB_t^\varepsilon),\ 0 \le t \le T.$$

Let S_t^ε be the solution of the above equation. Because of the convergence of B_t^ε to B_t in $L_2(\Omega)$ when $\varepsilon \to 0$, we shall define the *solution of a fractional stochastic differential equation* of the form

$$dS_t = S_t(\mu dt + \sigma dB_t),\ 0 \le t \le T,$$

to be a process S_t^* defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the process S_t^ε converges to S_t^* in $L_2(\Omega)$ as $\varepsilon \to 0$ and the convergence is uniform with respect to $t \in [0,T]$. This definition will be applied to the other similar fractional stochastic differential equations which will appear later.

The rest of the paper is organized as follows. A risk-neutral for gBm model with a compound Poisson process and stochastic volatility model is described in section 2. The risk-neutral for gBm model with a compound Poisson process and fractional stochastic volatility model is also introduced in this section. The relationship between the stochastic differential equation and the partial differential equation for the jump diffusion process with stochastic volatility is presented in section 3. In section 4, an option price formula is given. Finally the closed-form solution for a European call option in terms of characteristic function is given in section 5.

2. RISK-NEUTRAL FOR A GBM WITH JUMPS

In this section, a risk-neutral for a gBm model combining jumps with stochastic volatility is introduced. Its solution will also be discussed in this section.

Firstly, let us rewrite the model (1.1) into an integral form as follows:

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sqrt{v_s} S_s dW_s + \int_0^t S_{s-} Y_s dN_s. \tag{2.1}$$

Note that the last term on the right hand side of equation (2.1) is defined by

$$\int_0^t S_{s-} Y_s dN_s = \sum_{n=1}^{N_t} \Delta S_n,$$

where

$$\Delta S_n := S_{T_n} - S_{T_n -} = S_{n-} Y_n.$$

The assumption $Y_n > 0$ always leads to positive values of the stock prices. The process $(Y_n)_{n\in\mathbb{N}}$ is assumed to be independently identically distributed (i.i.d.) with density $\phi_Y(y)$ and $(T_n)_{n\in\mathbb{N}}$ is a sequence of jump time.

In order to solve equation (2.1) with an initial condition $S_{t(t=0)} = S_0$, we assume that $E[\int_0^T v_s S_s^2 ds] < \infty$. Then, by an application of Itô's formula for the jump process ([6], Theorem 8.14, page 275) on equation (2.1) with $f(S_t, t) = log(S_t)$, we get

$$\log S_t = \log S_0 + \mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s)\, dN_s,$$

or, equivalently,

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s)\, dN_s\right).$$

It is assumed that a risk-neutral probability measure $\mathcal{M}$ exists; the asset price S_t, under this risk-neutral measure, follows a jump-diffusion process, with zero-mean, risk-free rate r, and stochastic variance v_t,

$$dS_t = S_t\left((r - \lambda E_{\mathcal{M}}[Y_t])\, dt + \sqrt{v_t} dW_t\right) + S_{t-} Y_t dN_t. \tag{2.2}$$

It is only necessary to know that the risk-neutral measure exists (see, [6] page 321). Hence, all processes to be discussed after this will be the processes under the risk-neutral probability measure $\mathcal{M}$.

Using an initial condition $S_{t(t=0)} = S_0 \in L_2(\Omega)$, its solution is given by

$$S_t = S_0 \exp\left(\int_0^t (r - \lambda E_{\mathcal{M}}(Y_s))\, ds - \frac{1}{2}\int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1+Y_s)\, dN_s\right),$$

where v_t satisfies the following fractional SDE

$$dv_t = (\omega - \theta v_t)\, dt + \xi v_t dB_t, \tag{2.3}$$

with an initial condition $v_{t(t=0)} = v_0 \in L_2(\Omega)$.

For each $\varepsilon > 0$, consider an approximation model of equation (2.3);

$$dv_t^\varepsilon = (\omega - \theta v_t^\varepsilon)\, dt + \xi v_t^\varepsilon dB_t^\varepsilon. \tag{2.4}$$

By using the same initial condition as in equation (2.3), one can show that the solution v_t^ε of equation (2.4) converges in $L_2(\Omega)$ to the process

$$v_t = \left(v_0 + \omega \int_0^t \exp(\gamma s - \xi B_s)\, ds\right) \exp(\xi B_t - \gamma t)$$

for some real constant γ. ([1], Lemma 2). Hence, by definition, v_t is the solution of equation (2.3).

Now we consider an approximation model of equation (2.2);

$$dS_t^\varepsilon = S_t^\varepsilon\left((r - \lambda E_{\mathcal{M}}[Y_t])\, dt + \sqrt{v_t^\varepsilon} dW_t\right) + S_{t-}^\varepsilon Y_t dN_t, \tag{2.5}$$

and by using the same initial condition as in equation (2.2), we have

$$S_t^\varepsilon = S_0 \exp\left(\int_0^t (r - \lambda E_{\mathcal{M}}(Y_s))\, ds - \frac{1}{2}\int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1+Y_s)\, dN_s\right). \tag{2.6}$$

Again, we can prove that ([1], Theorem 3) S_t^ε converges to S_t in $L_2(\Omega)$ as $\varepsilon \to 0$ and uniformly on $t \in [0, T]$.

3. PARTIAL INTEGRO-DIFFERENTIAL EQUATION FOR JUMP DIFFUSION MODEL WITH STOCHASTIC VOLATILITY

Consider the process $\overrightarrow{X}_t = \left(X_t^1, X_t^2\right)$ where X_t^1 and X_t^2 are processes in $\Re$ and satisfy the following equations:

$$\begin{aligned} dX_t^1 &= f_1(t)\,dt + g_1(t)\,dW_t + X_{t-}^1 Y_t dN_t, \\ dX_t^2 &= f_2(t)\,dt + g_2(t)\,d\overline{W}_t, \end{aligned} \tag{3.1}$$

where f_1, g_1, f_2, and g_2 are all continuous functions from $[0,T]$ into $\Re$.

Since every compound Poisson process can be represented as an integral form of Poisson random measure ([6], page 77) then the last term on the right hand side of equation (3.1) can be written as follows

$$\int_0^t X_{s-}^1 Y_s dN_s = \sum_{n=1}^{N_t} X_{n-}^1 Y_n = \sum_{n=1}^{N_t} \left[X_{T_n}^1 - X_{T_n-}^1\right] = \int_0^t \int_{\Re} X_{s-}^1 z J_Z(ds\ dz)$$

where Y_n are i.i.d random variables with density $\phi_Y(y)$ and J_Z is a Poisson random measure of the process $Z_t = \sum_{n=1}^{N_t} Y_n$ with intensity measure $\lambda\phi_Y(dz)dt$.

Let $U(\overrightarrow{x})$ be a bounded real function on $\Re^2$ and twice continuously differentiable in $\overrightarrow{x} = (x_1, x_2) \in \Re^2$ and

$$u(\overrightarrow{x}, t) = E\left[U\left(\overrightarrow{X_T}\right) | \overrightarrow{X}_t = \overrightarrow{x}\right]. \tag{3.2}$$

By the two dimensional Dynkin's formula ([7], Theorem 7.7, page 203), u is a solution of the partial integro-differential equation (PIDE)

$$0 = \frac{\partial u(\overrightarrow{x}, t)}{\partial t} + \mathcal{A}u(\overrightarrow{x}, t) + \lambda \int_{\Re} \left[u(\overrightarrow{x} + \overrightarrow{y}, t) - u(\overrightarrow{x}, t)\right] \phi_Y(y) dy,$$

subject to the final condition $u(\overrightarrow{x}, T) = U(\overrightarrow{x})$ where $\overrightarrow{y} = (y, 0) \in \Re^2$. The notation $\mathcal{A}$ is defined by

$$\begin{aligned} \mathcal{A}u(\overrightarrow{x}, t) &= f_1(t)\frac{\partial u(\overrightarrow{x}, t)}{\partial x_1} + f_2(t)\frac{\partial u(\overrightarrow{x}, t)}{\partial x_2} \\ &\quad + \frac{1}{2}g_1^2(t)\frac{\partial^2 u(\overrightarrow{x}, t)}{\partial x_1^2} + \rho g_1(t) g_2(t)\frac{\partial^2 u(\overrightarrow{x}, t)}{\partial x_1 \partial x_2} + \frac{1}{2}g_2^2(t)\frac{\partial^2 u(\overrightarrow{x}, t)}{\partial x_2^2}, \end{aligned}$$

and the correlation ρ defined by $\rho = Corr\left[dW_t, d\overline{W}_t\right]$.

4. PRICING A EUROPEAN CALL OPTION

Let C denote the price at time t of a *European style call option* on the current price of the underlying asset S_t with strike price K and expiration time T.

The *terminal payoff of a European call option* on the underling stock S with strike price K is

$$\max(S_T - K, 0).$$

This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \leq K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assuming the risk-free interest rate r is constant over the lifetime of the option, the price of the European call at time t is equal to the discounted conditional expected payoff

$$C\left(S_t, v_t, t; K, T\right)$$

$$
\begin{aligned}
&= e^{-r(T-t)} E_{\mathcal{M}} \left[\max\left(S_T - K, 0\right) \mid S_t, v_t\right] \\
&= e^{-r(T-t)} \left(\int_K^{\infty} S_T P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T - K \int_K^{\infty} P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \right) \\
&= S_t \left(\frac{1}{e^{r(T-t)} S_t} \int_K^{\infty} S_T P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \right) - K e^{-r(T-t)} \int_K^{\infty} P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \\
&= S_t \left(\frac{1}{E_{\mathcal{M}} \left[S_T \mid S_t, v_t\right]} \int_K^{\infty} S_T P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \right) \\
&\quad - K e^{-r(T-t)} \int_K^{\infty} P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \\
&= S_t P_1 \left(S_t, v_t, t; K, T\right) - K e^{-r(T-t)} P_2 \left(S_t, v_t, t; K, T\right) \qquad (4.1)
\end{aligned}
$$

where $E_{\mathcal{M}}$ is the expectation with respect to the risk-neutral probability measure, $P_{\mathcal{M}}\left(S_T \mid S_t, v_t\right)$ is the corresponding conditional density given $\left(S_t, v_t\right)$, and

$$P_1\left(S_t, v_t, t; K, T\right) = \left(\int_K^{\infty} S_T P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T \right) / E_{\mathcal{M}} \left[S_T \mid S_t, v_t\right].$$

Note that P_1 is the risk-neutral probability that $S_T > K$ (since the integrand is nonnegative and the integral over $[0, \infty)$ is one), and finally, that

$$P_2\left(S_t, v_t, t; K, T\right) = \int_K^{\infty} P_{\mathcal{M}} \left(S_T \mid S_t, v_t\right) dS_T = \Pr\text{ob}\left(S_T > K \mid S_t, v_t\right)$$

is the risk-neutral in-the-money probability. Moreover, $E_{\mathcal{M}}\left[S_T \mid S_t, v_t\right] = e^{r(T-t)} S_t$ for $t \geq 0$.

Note that we do not have a closed form solution for these probabilities. However, these probabilities are related to characteristic functions which have closed form solutions as will be seen in Lemma 2.

We would like to compute the price of a European call option with strike price K and maturity T of the model (2.2) for which its fractional stochastic volatility satisfies equation (2.3).

To do this, consider the logarithm of S_t^{ε}, namely L_t^{ε}, i.e. $L_t^{\varepsilon} = \log\left(S_t^{\varepsilon}\right)$ where S_t^{ε} satisfies equation (2.6) and its inverse $S_t^{\varepsilon} = \exp\left(L_t^{\varepsilon}\right)$. Denote $\kappa = \log(K)$ the logarithm of the strike price.

We now refer to equation (2.4), since this approximate model is driven by a semimartingale B_t^{ε} and hence there is no opportunity of arbitrage. This is the advantage of our approximate approach and we will use this model for pricing the European call option instead of (2.3).

Note that we can write

$$dB_t^{\varepsilon} = \alpha \varphi_t^{\varepsilon} dt + \varepsilon^{\alpha} dW_t \qquad (4.2)$$

where $\varphi_t^{\varepsilon} = \int_0^t (t - u + \varepsilon)^{1-\alpha} dW_u$, $\alpha = 1/2 - H$ and $0 < \alpha < 1/2$ ([5], Lemma 2.1).

Substituting (4.2) into equation (2.4), we obtain

$$dv_t^{\varepsilon} = \left(\omega + \left(\alpha \xi \varphi_t^{\varepsilon} - \theta\right) v_t^{\varepsilon}\right) dt + \xi \varepsilon^{\alpha} v_t^{\varepsilon} dW_t. \qquad (4.3)$$

Consider the SDE (2.2) and (4.3). Define a function U on $\Re^2$ as follows:

$$U(x_1, x_2) = e^{-r(T-t)} max(e^{x_1} - \kappa, 0).$$

By virtue of equation (3.2),

$$\begin{aligned} u(\overrightarrow{x}, t) &= E\left[U\left(\overrightarrow{X_T}\right) | \overrightarrow{X}_t = \overrightarrow{x}\right] \\ &= e^{-r(T-t)} E_{\mathcal{M}}\left[\max\left(\exp\left(L_t^\varepsilon\right) - \kappa, 0\right) \mid L_t^\varepsilon = \ell^\varepsilon, v_t^\varepsilon = v^\varepsilon\right] \\ &:= C\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) \end{aligned}$$

satisfies the following PIDE:

$$\begin{aligned} 0 \quad = \quad & \frac{\partial C}{\partial t} + f_1 \frac{\partial C}{\partial \ell^\varepsilon} + f_2 \frac{\partial C}{\partial v^\varepsilon} + \frac{1}{2} g_1^2 \frac{\partial^2 C}{\partial (\ell^\varepsilon)^2} + \rho g_1 g_2 \frac{\partial^2 C}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2} g_2^2 \frac{\partial^2 C}{\partial (v^\varepsilon)^2} \\ & - rC + \lambda \int_{\Re} \left[C\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - C\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right] \phi_Y(y) dy. \qquad (4.4) \end{aligned}$$

In the current state variable, the last line of equation (4.1) becomes

$$C\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) = e^{\ell^\varepsilon} P_1\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) - e^{\kappa - r(T-t)} P_2\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right). \qquad (4.5)$$

The following lemma shows the relationship between P_1 and P_2 in the option value of the equation (4.5).

Lemma 4.1. *The functions P_1 and P_2 in the option value of the equation (4.5) satisfy the following PIDEs*

$$\begin{aligned} 0 = & \frac{\partial P_1}{\partial t} + A\left[P_1\right]\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) + v^\varepsilon \frac{\partial P_1}{\partial \ell^\varepsilon} + \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial P_1}{\partial v^\varepsilon} \\ & + \left(r - \lambda E_{\mathcal{M}}\left(Y_t\right)\right) P_1 + \lambda \int_{\Re} \left[\left(e^y - 1\right) P_1\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right)\right] \phi_Y(y) dy, \end{aligned}$$

subject to the boundary condition at expiration time $t = T$;

$$P_1\left(\ell^\varepsilon, v^\varepsilon, T; \kappa, T\right) = 1_{\ell^\varepsilon > \kappa}. \qquad (4.6)$$

Moreover, P_2 satisfies the equation

$$0 = \frac{\partial P_2}{\partial t} + A\left[P_2\right]\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) + rP_2,$$

subject to the boundary condition at expiration time $t = T$;

$$P_2\left(\ell^\varepsilon, v^\varepsilon, T; \kappa, T\right) = 1_{\ell^\varepsilon > \kappa}, \qquad (4.7)$$

where

$$\begin{aligned} A[f]\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right) := & \left(r - \lambda E\left(Y_t\right) - \frac{1}{2} v^\varepsilon\right) \frac{\partial f}{\partial \ell^\varepsilon} + \left(\omega + \left(\alpha \xi \varphi_t^\varepsilon - \theta\right) v^\varepsilon\right) \frac{\partial f}{\partial v^\varepsilon} \\ & + \frac{1}{2} v^\varepsilon \frac{\partial^2 f}{(\partial \ell^\varepsilon)^2} + \rho \xi \varepsilon^\alpha (v^\varepsilon)^{3/2} \frac{\partial^2 f}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{1}{2} \xi^2 \varepsilon^{2\alpha} (v^\varepsilon)^2 \frac{\partial^2 f}{\partial (v^\varepsilon)^2} \\ & - rf + \lambda \int_{\Re} \left[f\left(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T\right) - f\left(\ell^\varepsilon, v^\varepsilon, t; \kappa, T\right)\right] \phi_Y(y) dy. \end{aligned} \qquad (4.8)$$

Note that $1_{\ell^\varepsilon > \kappa} = 1$ if $\ell^\varepsilon > \kappa$ and otherwise $1_{\ell^\varepsilon > \kappa} = 0$.

Proof. We plan to substitute equation (4.5) into equation (4.4). Firstly, we compute

$$\begin{aligned}
\frac{\partial C}{\partial t} &= e^{\ell^\varepsilon}\frac{\partial P_1}{\partial t} - e^{\kappa - r(T-t)}\frac{\partial P_2}{\partial t} - re^{\kappa - r(T-t)}P_2 \\
\frac{\partial C}{\partial \ell^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial P_1}{\partial \ell^\varepsilon} + e^{\ell^\varepsilon}P_1 - e^{\kappa - r(T-t)}\frac{\partial P_2}{\partial \ell^\varepsilon} \\
\frac{\partial C}{\partial v^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial P_1}{\partial v^\varepsilon} - e^{\kappa - r(T-t)}\frac{\partial P_2}{\partial v^\varepsilon} \\
\frac{\partial^2 C}{(\partial \ell^\varepsilon)^2} &= e^{\ell^\varepsilon}\frac{\partial^2 P_1}{(\partial \ell^\varepsilon)^2} + 2e^{\ell^\varepsilon}\frac{\partial P_1}{\partial \ell^\varepsilon} + P_1 e^{\ell^{\varepsilon\varepsilon}} - e^{\kappa - r(T-t)}\frac{\partial^2 P_2}{(\partial \ell^\varepsilon)^2} \\
\frac{\partial^2 C}{\partial \ell^\varepsilon \partial v^\varepsilon} &= e^{\ell^\varepsilon}\frac{\partial^2 P_1}{\partial \ell^\varepsilon \partial v^\varepsilon} + e^{\ell^\varepsilon}\frac{\partial P_1}{\partial v^\varepsilon} - e^{\kappa - r(T-t)}\frac{\partial^2 P_2}{\partial \ell^\varepsilon \partial v^\varepsilon} \\
\frac{\partial^2 C}{\partial (v^\varepsilon)^2} &= e^{\ell^\varepsilon}\frac{\partial^2 P_1}{\partial (v^\varepsilon)^2} - e^{\kappa - r(T-t)}\frac{\partial^2 P_2}{\partial (v^\varepsilon)^2}
\end{aligned}$$

and

$$\begin{aligned}
& C(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - C(\ell^\varepsilon, v^\varepsilon, t; \kappa, T) \\
= & \left[e^{(\ell^\varepsilon + y)}P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - e^{\kappa - r(T-t)}P_2(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T)\right] \\
& - \left[e^{\ell^\varepsilon}P_1(\ell^\varepsilon, v^\varepsilon, t; \kappa, T) - e^{\kappa - r(T-t)}P_2(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)\right] \\
= & \left[\begin{array}{c} e^{\ell^\varepsilon}\left(e^y P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T)\right) \\ + \left(e^{\ell^\varepsilon}P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - e^{\ell^\varepsilon}P_1(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)\right) \end{array}\right] \\
& - e^{\kappa - r(T-t)}\left[P_2(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - P_2(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)\right] \\
= & e^{\ell^\varepsilon}(e^y - 1)P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) + e^{\ell^\varepsilon}\left(P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - P_1(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)\right) \\
& - e^{\kappa - r(T-t)}\left[P_2(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - P_2(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)\right].
\end{aligned}$$

Substitute all terms above in equation (4.4) and separate it by assumed independent terms P_1 and P_2. This gives two PIDEs for the risk-neutralized probability $P_j(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)$, $j = 1, 2$:

$$\begin{aligned}
0 = & \frac{\partial P_1}{\partial t} + \left(r - \lambda E_{\mathcal{M}}(Y_t) - \frac{1}{2}v^\varepsilon\right)\left(\frac{\partial P_1}{\partial \ell^\varepsilon} + P_1\right) + \left(\omega + (\alpha\xi\varphi_t^\varepsilon - \theta)v^\varepsilon\right)\frac{\partial P_1}{\partial v^\varepsilon} \\
& + \frac{1}{2}v^\varepsilon\left(\frac{\partial^2 P_1}{(\ell^\varepsilon)^2} + 2\frac{\partial P_1}{\partial \ell^\varepsilon} + P_1\right) + \rho\xi\varepsilon^\alpha(v^\varepsilon)^{3/2}\left(\frac{\partial^2 P_1}{\partial \ell^\varepsilon \partial v^\varepsilon} + \frac{\partial P_1}{\partial v^\varepsilon}\right) + \frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^\varepsilon)^2\frac{\partial^2 P_1}{\partial (v^\varepsilon)^2} \\
& - rP_1 + \lambda\int_{\Re}\left[\begin{array}{c} (e^y - 1)P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) \\ + \left(P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T) - P_1(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)\right) \end{array}\right]\phi_Y(y)dy \qquad (4.9)
\end{aligned}$$

subject to the boundary condition at the expiration time $t = T$ according to equation (4.6).

By using the notation in equation (4.8), PIDE (4.9) becomes

$$\begin{aligned}
0 = & \frac{\partial P_1}{\partial t} + \left(\begin{array}{c} A[P_1](\ell^\varepsilon, v^\varepsilon, t; \kappa, T) + v^\varepsilon\frac{\partial P_1}{\partial \ell^\varepsilon} + \rho\xi\varepsilon^\alpha(v^\varepsilon)^{3/2}\frac{\partial P_1}{\partial v^\varepsilon} + (r - \lambda E_{\mathcal{M}}(Y_t))P_1 \\ + \lambda\int_{\Re}\left[(e^y - 1)P_1(\ell^\varepsilon + y, v^\varepsilon, t; \kappa, T)\right]\phi_Y(y)dy \end{array}\right) \\
:= & \frac{\partial P_1}{\partial t} + A_1[P_1](\ell^\varepsilon, v^\varepsilon, t; \kappa, T).
\end{aligned}$$

For $P_2(\ell^\varepsilon, v^\varepsilon, t; \kappa, T)$:

$$0 = \frac{\partial P_2}{\partial t} + rP_2 + \left(r - \lambda E_{\mathcal{M}}(Y_t) - \frac{1}{2}v^\varepsilon\right)\left(\frac{\partial P_2}{\partial \ell^\varepsilon}\right) + \left(\omega + (\alpha\xi\varphi_t^\varepsilon - \theta)v^\varepsilon\right)\frac{\partial P_2}{\partial v^\varepsilon}$$

$$+\frac{1}{2}v^{\varepsilon}\frac{\partial^2 P_2}{\partial(\ell^{\varepsilon})^2}+\rho\xi\varepsilon^{\alpha}(v^{\varepsilon})^{3/2}\frac{\partial^2 P_1}{\partial\ell^{\varepsilon}\partial v^{\varepsilon}}+\frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^{\varepsilon})^2\frac{\partial^2 P_1}{\partial(v^{\varepsilon})^2}$$
$$-rP_2+\lambda\int_{\Re}\left[P_2\left(\ell^{\varepsilon}+y,v^{\varepsilon},t;\kappa,T\right)-P_2\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)\right]\phi_Y(y)dy \tag{4.10}$$

subject to the boundary condition at expiration time $t=T$ according to equation (4.7).

Again, by using the notation (4.8), PIDE (4.10) becomes

$$\begin{aligned}0&=\frac{\partial P_2}{\partial t}+\left(A\left[P_2\right]\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)+rP_2\right)\\&:=\frac{\partial P_2}{\partial t}+A_2\left[P_2\right]\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)\end{aligned}$$

The proof is now completed. □

5. A CLOSED-FORM SOLUTION FOR EUROPEAN CALL OPTIONS

For $j=1,2,$ the characteristic functions for $P_j\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right),$ with respect to the variable κ are defined by

$$f_j\left(\ell^{\varepsilon},v^{\varepsilon},t;x,T\right):=-\int_{-\infty}^{\infty}e^{ix\kappa}dP_j\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right),$$

with a minus sign to account for the negativity of the measure dP_j. Note that f_j also satisfies similar PIDEs

$$\frac{\partial f_j}{\partial t}+A_j[f_j]\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)=0, \tag{5.1}$$

with the respective boundary conditions

$$f_j\left(\ell^{\varepsilon},v^{\varepsilon},T;x,T\right)=-\int_{-\infty}^{\infty}e^{ix\kappa}dP_j\left(\ell^{\varepsilon},v^{\varepsilon},T;\kappa,T\right)=-\int_{-\infty}^{\infty}e^{ix\kappa}(-\delta(\ell^{\varepsilon}-\kappa)d\kappa)=e^{ix\ell^{\varepsilon}},$$

since

$$dP_j\left(\ell^{\varepsilon},v^{\varepsilon},T;\kappa,T\right)=d1_{\ell^{\varepsilon}>\kappa}=dH(\ell^{\varepsilon}-\kappa)=-\delta(\ell^{\varepsilon}-\kappa)d\kappa.$$

The following lemma shows how to calculate the functions P_1 and P_2 as they appeared in Lemma 1.

Lemma 5.1. *The functions P_1 and P_2 can be calculated by the inverse Fourier transforms of the characteristic function, i.e.*

$$P_j\left(\ell^{\varepsilon},v^{\varepsilon},t;\kappa,T\right)=\frac{1}{2}+\frac{1}{\pi}\int_{0+}^{+\infty}Re\left[\frac{e^{-ix\kappa}f_j\left(\ell^{\varepsilon},v^{\varepsilon},t;x,T\right)}{ix}\right]dx,$$

for $j=1,2,$ with $Re[\cdot]$ denoting the real component of a complex number.

By letting $\tau=T-t$. (i) The characteristic function f_1 is given by

$$f_1\left(\ell^{\varepsilon},v^{\varepsilon},t;x,t+\tau\right)=\exp\left(g_1\left(\tau\right)+v^{\varepsilon}h_1\left(\tau\right)+ix\ell^{\varepsilon}\right),$$

where

$$g_1\left(\tau\right)\quad=\quad\left[\left(r-\lambda E\left(Y_t\right)\right)ix-\lambda E_{\mathcal{M}}\left(Y_t\right)\right]\tau+\tau\lambda\int_{\Re}\left(e^{(ix+1)y}-1\right)\phi_Y(y)dy$$

$$-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1-\frac{(\Delta_1+\eta_1)+\left(1-e^{\Delta_1\tau}\right)}{2\Delta_1}\right)+(\Delta_1+\eta_1)\,\tau\right],$$

$$h_1(\tau)=\frac{\left(\eta_1^2-\Delta_1^2\right)\left(e^{\Delta_1\tau}-1\right)}{\xi^2\varepsilon^{2\alpha}v^\varepsilon\left(\eta_1+\Delta_1-(\eta_1-\Delta_1)e^{\Delta_1\tau}\right)},$$

$$\eta_1=\rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}\,(1+ix)+\left(\alpha\xi\varphi_t^\varepsilon-\theta\right),$$

and

$$\Delta_1=\sqrt{\eta_1^2-\xi^2\varepsilon^{2\alpha}v^\varepsilon ix\,(ix+1)}.$$

(ii) The characteristic function f_2 is given by

$$f_2\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right)=\exp\left(g_2(\tau)+v^\varepsilon h_2(\tau)+ix\ell^\varepsilon+r\tau\right),$$

where

$$\begin{aligned}g_2(\tau) &= \left[(r-\lambda E_{\mathcal{M}}(Y_t))\,iy-r\right]\tau+\tau\lambda\int_{\Re}\left(e^{ixy}-1\right)\phi_Y(y)dy\\ &\quad -\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1-\frac{(\Delta_2+\eta_2)+\left(1-e^{\Delta_2\tau}\right)}{2\Delta_1}\right)+(\Delta_2+\eta_2)\,\tau\right],\end{aligned}$$

$$h_2(\tau)=\frac{\left(\eta_2^2-\Delta_2^2\right)\left(e^{\Delta_2\tau}-1\right)}{\xi^2\varepsilon^{2\alpha}v^\varepsilon\left(\eta_2+\Delta_2-(\eta_2-\Delta_2)e^{\Delta_2\tau}\right)},$$

$$\eta_2=\rho\xi\varepsilon^\alpha\sqrt{v^\varepsilon}ix+\left(\alpha\xi\varphi_t^\varepsilon-\theta\right),$$

and

$$\Delta_2=\sqrt{\eta_2^2+\xi^2\varepsilon^{2\alpha}v^\varepsilon ix\,(ix-1)}.$$

Proof. Proof of (i). To solve for the characteristic explicitly, letting $\tau=T-t$ be the time-to-go, we conjecture that the function f_1 is given by

$$f_1\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right)=\exp\left(g_1(\tau)+v^\varepsilon h_1(\tau)+ix\ell^\varepsilon\right), \tag{5.2}$$

and the boundary condition $g_1(0)=0=h_1(0)$. This conjecture exploits the linearity of the coefficient in PIDE (5.1).

Note that the characteristic function of f_1 always exists. In order to substitute (5.2) into (5.1), firstly, we compute

$$\frac{\partial f_1}{\partial t}=\left(-g_1'(\tau)-v^\varepsilon h_1'(\tau)\right)f_1,\ \frac{\partial f_1}{\partial\ell^\varepsilon}=ixf_1,\ \frac{\partial f_1}{\partial v^\varepsilon}=h_1(\tau)f_1,$$

$$\frac{\partial^2 f_1}{(\partial\ell^\varepsilon)^2}=-x^2f_1,\ \frac{\partial^2 f_1}{\partial\ell^\varepsilon\partial v^\varepsilon}=ixh_1(\tau)f_1,\ \frac{\partial^2 f_1}{\partial(v^\varepsilon)^2}=h_1^2(\tau)f_1,$$

$$f_1\left(\ell^\varepsilon+y,v^\varepsilon,t;x,t+\tau\right)-f_1\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right)=\left(e^{ixy}-1\right)f_1\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right),$$

and

$$\begin{aligned}\left(e^y-1\right)f_1\left(\ell^\varepsilon+y,v^\varepsilon,t;x,t+\tau\right) &= \left(e^y-1\right)e^{g_1(\tau)+v^\varepsilon h_1(\tau)+ix(\ell^\varepsilon+y)}\\ &= \left(e^y-1\right)e^{ixy}f_1\left(\ell^\varepsilon,v^\varepsilon,t;x,t+\tau\right).\end{aligned}$$

Substituting all the above terms into equation (5.1) and after canceling the common factor of f_1, we get a simplified form as follows:

$$\begin{aligned}0 &= -g_1'(\tau)-v^\varepsilon h_1'(\tau)+\left(r-\lambda E_{\mathcal{M}}(Y_t)+\frac{1}{2}v^\varepsilon\right)ix\\ &\quad+\left(\left(\omega+\left(\alpha\xi\varphi_t^\varepsilon-\theta\right)v^\varepsilon\right)+\rho\xi\varepsilon^\alpha(v^\varepsilon)^{3/2}\right)h_1(\tau)\end{aligned}$$

$$-\frac{1}{2}v^{\varepsilon}x^2 + \rho\xi\varepsilon^{\alpha}(v^{\varepsilon})^{3/2}ixh_1(\tau) + \frac{1}{2}\xi^2\varepsilon^{2\alpha}(v^{\varepsilon})^2h_1^2(\tau)$$
$$-\lambda E_{\mathcal{M}}(Y_t) + \lambda\int_{\Re}\left(e^{(ix+1)y}-1\right)\phi_Y(y)dy.$$

By separating the order v^{ε} and ordering the remaining terms, we can reduce it to two ordinary differential equations (ODEs),

$$h_1'(\tau) = \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^{\varepsilon}h_1^2(\tau) + \left(\rho\xi\varepsilon^{\alpha}\sqrt{v^{\varepsilon}}(1+ix) + (\alpha\xi\varphi_t^{\varepsilon}-\theta)\right)h_1(\tau) + \frac{1}{2}ix - \frac{1}{2}x^2, \tag{5.3}$$

$$g_1'(\tau) = \omega h_1(\tau) + (r-\lambda E_{\mathcal{M}}(Y_t))\,ix - \lambda E_{\mathcal{M}}(Y_t) + \lambda\int_{\Re}\left(e^{(ix+1)y}-1\right)\phi_Y(y)dy. \tag{5.4}$$

Let $\eta_1 = \rho\xi\varepsilon^{\alpha}\sqrt{v^{\varepsilon}}(1+ix) + (\alpha\xi\varphi_t^{\varepsilon}-\theta)$ and substitute it to equation (5.3). We get

$$\begin{aligned} h_1'(\tau) &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^{\varepsilon}\left(h_1^2(\tau) + \frac{2\eta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}h_1(\tau) + \frac{1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}ix(ix+1)\right) \\ &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}\left(h_1(\tau) + \frac{2\eta_1 + \sqrt{4\eta_1^2 - 4\xi^2\varepsilon^{2\alpha}v^{\varepsilon}ix(ix+1)}}{2\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right) \\ &\quad\times\left(h_1(\tau) + \frac{2\eta_1 - \sqrt{4\eta_1^2 - 4\xi^2\varepsilon^{2\alpha}v^{\varepsilon}ix(ix+1)}}{2\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right) \\ &= \frac{1}{2}\xi^2\varepsilon^{2\alpha}v^{\varepsilon}\left(h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right)\left(h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right), \end{aligned}$$

where $\Delta_1 = \sqrt{\eta_1^2 - \xi^2\varepsilon^{2\alpha}v^{\varepsilon}ix(ix+1)}$.

By method of variable separation, we have

$$\frac{2dh_1(\tau)}{\left(h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right)\left(h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}\right)} = \xi^2\varepsilon^{2\alpha}v^{\varepsilon}d\tau.$$

Using partial fractions, we get

$$\frac{1}{\Delta_1}\left(\frac{1}{h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}} - \frac{1}{h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}}\right)dh_1(\tau) = d\tau.$$

Integrating both sides, we obtain

$$\log\left(\frac{h_1(\tau) + \frac{\eta_1-\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}}{h_1(\tau) + \frac{\eta_1+\Delta_1}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}}}\right) = \Delta_1\tau + C.$$

Using boundary condition $h_1(\tau=0)=0$ we get $C = \log\left(\frac{\eta_1-\Delta_1}{\eta_1+\Delta_1}\right)$.

Solving for h_1, we obtain

$$h_1(\tau) = \frac{\left(\eta_1^2-\Delta_1^2\right)\left(e^{\Delta_1\tau}-1\right)}{\xi^2\varepsilon^{2\alpha}v^{\varepsilon}\left(\eta_1+\Delta_1-(\eta_1-\Delta_1)e^{\Delta_1\tau}\right)}.$$

In order to solve $g_1(\tau)$ explicitly, we substitute h_1 into equation (5.4) and integrate with respect to τ on both sides. Then we get

$$g_1(\tau) = \left[(r-\lambda E_{\mathcal{M}}(Y_t))\,ix - \lambda E(Y_t)\right]\tau + \tau\lambda\int_{\Re}\left(e^{(ix+1)y}-1\right)\phi_Y(y)dy$$

$$-\frac{2\omega}{\xi^2\varepsilon^{2\alpha}v^\varepsilon}\left[\log\left(1-\frac{(\Delta_1+\eta_1)+\left(1-e^{\Delta_1\tau}\right)}{2\Delta_1}\right)+(\Delta_1+\eta_1)\,\tau\right].$$

Proof of (ii). The details of the proof are similar to case (i). Hence, we have

$$f_2\left(\ell^\varepsilon,v^\varepsilon,t;y,t+\tau\right)=\exp\left(g_2\left(\tau\right)+v^\varepsilon h_2\left(\tau\right)+iy\ell^\varepsilon+r\tau\right),$$

where $g_2\left(\tau\right), h_2\left(\tau\right), \eta_2$ and Δ_2 are as given in the Lemma.

We can thus evaluate the characteristic functions in closed form. However, we are interested in the risk-neutral probabilities P_j. These can be inverted from the characteristic functions by performing the following integration

$$\begin{aligned}P_j\left(S_t,v_t^\varepsilon,t;K,T\right) &= P_j\left(\ell^\varepsilon,v^\varepsilon,t;\kappa,T\right)\\ &= \frac{1}{2}+\frac{1}{\pi}\int_{0^+}^{+\infty}\boldsymbol{Re}\left[\frac{e^{-ix\kappa}f_j\left(\ell^\varepsilon,v^\varepsilon,t;x,T\right)}{ix}\right]dx\end{aligned}$$

for $j=1,2$, where $\ell^\varepsilon=\log S_t$, $v^\varepsilon=\log(v_t^\varepsilon)$, and $\kappa=\log(K)$.

To verify the above equation, firstly we note that

$$\begin{aligned}&E_{\mathcal{M}}\left[e^{ix(\log(S_t)-\log(K))}\mid\log(S_t)=L_t^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\\ =\ &E_{\mathcal{M}}\left[e^{ix(\ell^\varepsilon-\kappa)}\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\\ =\ &\int_{-\infty}^{+\infty}e^{ix(\ell^\varepsilon-\kappa)}dP_j\left(\ell^\varepsilon,v^\varepsilon,t;\kappa,T\right)\\ =\ &e^{-ix\kappa}\int_{-\infty}^{+\infty}e^{ix\ell^\varepsilon}dP_j\left(\ell^\varepsilon,v^\varepsilon,t;\kappa,T\right)\\ =\ &e^{-ix\kappa}\int_{-\infty}^{+\infty}e^{ix\kappa}(-\delta(\ell^\varepsilon-\kappa)d\kappa)\\ =\ &e^{-ix\kappa}f_j\left(\ell^\varepsilon,v_t^\varepsilon,t;x,T\right).\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{2}+\frac{1}{\pi}\int_{0^+}^{+\infty}\boldsymbol{Re}\left[\frac{e^{-ix\kappa}f_j\left(\ell^\varepsilon,v_t^\varepsilon,t;x,T\right)}{ix}\right]dx\\ =\ &\frac{1}{2}+\frac{1}{\pi}\int_{0^+}^{+\infty}\boldsymbol{Re}\left[\frac{E_{\mathcal{M}}\left[e^{ix(\log(S_t^\varepsilon)-\log(K))}\mid\log(S_t)=L_t^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]}{ix}\right]dx\\ =\ &E_{\mathcal{M}}\left[\frac{1}{2}+\frac{1}{\pi}\int_{0^+}^{+\infty}\boldsymbol{Re}\left[\frac{e^{ix(\ell^\varepsilon-\kappa)}}{ix}\right]dx\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\\ =\ &E_{\mathcal{M}}\left[\frac{1}{2}+\frac{1}{\pi}\int_{0^+}^{+\infty}\frac{\sin(x\left(\ell^\varepsilon-\kappa\right))}{x}dx\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\\ =\ &E_{\mathcal{M}}\left[\frac{1}{2}+sgn\left(\ell^\varepsilon-\kappa\right)\frac{1}{\pi}\int_{0^+}^{+\infty}\frac{\sin(x)}{x}dx\mid L_t^\varepsilon=\ell^\varepsilon,v_t^\varepsilon=v^\varepsilon\right]\end{aligned}$$

$$= E_{\mathcal{M}}\left[\frac{1}{2}+\frac{1}{2}sgn\left(\ell^{\varepsilon}-\kappa\right) \mid L_t^{\varepsilon}=\ell^{\varepsilon}, v_t^{\varepsilon}=v^{\varepsilon}\right]$$
$$= E_{\mathcal{M}}[1_{\ell^{\varepsilon}\geq\kappa} \mid L_t^{\varepsilon}=\ell^{\varepsilon}, v_t^{\varepsilon}=v^{\varepsilon}],$$

where we have used the Dirichlet formula $\int_{-\infty}^{+\infty}\frac{\sin(x)}{x}dx=1$, and the *sgn* function is defined as $sgn(x)=1$ if $x>0$, 0 if $x=0$ and -1 if $x<0$. □

In summary, we have just proved the following main theorem.

Theorem 5.1. *For each $\varepsilon>0$, the value of a European call option of SDE (2.5) is*

$$\widehat{C}\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right)=S_t^{\varepsilon}P_1\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right)-Ke^{-r(T-t)}P_2\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right),$$

where P_1 and P_2 are as given in Lemma 2, and

$$\widehat{C}\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right)=C\left(\log(S_t^{\varepsilon}), v^{\varepsilon}, t; \log(K), T\right).$$

Remark 5.2. In numerical computation, we firstly choose a real number $\varepsilon>0$ and then compute the value of $\widehat{C}\left(S_t^{\varepsilon}, v_t^{\varepsilon}, t; K, T\right)$ according to the formula as given in Theorem 3. The solution that we get is the value of a call option of the approximation model (2.5) and this value can be used as an approximating value of a call option of the fractional model (2.2) as ε approaches zero.

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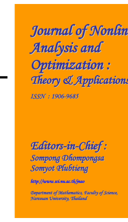

A VERSION OF KOLMOGOROV-ARNOLD REPRESENTATION THEOREM FOR DIFFERENTIABLE FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT. The most revised version of Kolmogorov-Arnold representation theorem shows that all continuous functions of several variables can be represented as superpositions which are constructed from the sums of continuous functions of one variables. In 2008, Kodama and Akashi show a relation between Kolmogorov-Arnold representation theorem and Vitušhkin theorem. In this paper, we discuss the problem asking what kind of differentiable functions of several variables can be represented as Kolmogorov-Arnold superpositions constructed from only several differentiable functions of one variable.

KEYWORDS : Kolmogorov-Arnold Representation; Differentiable Functions.

1. INTRODUCTION

Let $f(\cdot,\cdot,\cdot)$ be the function of three variable defined as

$$f(x,y,z) = xy + yz + zx, \quad x,y,z \in \mathbb{R},$$

where $\mathbb{R}$ is the set of all real numbers. Then, we can easily prove that there do not exist three functions $g(\cdot,\cdot)$, $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ of two variables satisfying the following equality:

$$f(x,y,z) = g(u(x,y), v(x,z)), \quad x,y,z \in \mathbb{R}.$$

This fact shows us that $f(\cdot,\cdot,\cdot)$ cannot be represented as any 1-time nested superposition constructed from three real-valued functions of two variables. Actually, it is clear that the following equality holds:

$$f(x,y,z) = x(y+z) + yz, \quad x,y,z \in \mathbb{R}.$$

This result shows us that $f(\cdot,\cdot,\cdot)$ can be represented as a 2-time nested superposition.

In 1957, Kolmogorov and Arnold [**4**] solved Hilbert's 13th problem asking whether all continuous real-valued functions of several real variables can be represented as

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superpositions of functions of fewer variables or not. If the most revised version with respect to this problem is applied to the set of all real-valued functions of three variables, for any continuous real-valued function $f(\cdot,\cdot,\cdot)$ of three real variables, we can choose a continuous real-valued function $g_f(\cdot)$ of one variable, which is dependent only on $f(\cdot,\cdot,\cdot)$, and a family $\{\phi_{ij}(\cdot); 0 \le i \le 6, 1 \le j \le 3\}$ of twenty one real-valued functions of one variable, which is independent of $f(\cdot,\cdot,\cdot)$ satisfying

$$f(x_1, x_2, x_3) = \sum_{i=0}^{6} g_f\left(\sum_{j=1}^{3} \phi_{ij}(x_j)\right).$$

The above result, which is called Kolmogorov-Arnold representation theorem [**4**,**5**], implies that any continuous real-valued function of three real variables can be represented as a 7-time nested superposition of continuous real-valued functions of one real variables.

In 2008, Kodama and Akashi [**3**] show a relation between Kolmogorov-Arnold representation theorem and Vitušhkin theorem. In this paper, we discuss the problem asking what kind of differentiable functions of several variables can be represented as Kolmogorov-Arnold superpositions constructed from only several differentiable functions of one variable.

2. SUPERPOSITION REPRESENTATIONS DERIVED FROM KOLMOGOROV-ARNOLD REPERSENTATION

In this section, we consider a generalized version of Kolmogorov-Arnold theorem and a necessary condition enabling to discriminate functions of three variables being able to be represented as Kolmogorov-Arnold superposition from the other functions. Here, a relation between Kolmogorov-Arnold representation and Vitušhkin theorem, can be shown as the following:

Proposition 1. Not all differentiable functions of three variables can be represented as Kolmogorov-Arnold superpositions constructed from only differentiable functions of one variable.

Proof. Vitušhkin theorem [**6**] assures that there exists a differentiable function $v(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any superposition constructed from several differentiable functions of two variables. Since Kolmogorov-Arnold theorem assures that all continuous functions of three variables can be represented as 7-time nested superposition constructed from several continuous functions of two variables. Therefore, if we apply Kolmogorov-Arnold representation to $v(\cdot,\cdot,\cdot)$, then we can obtain the following equality:

$$v(x_1, x_2, x_3) = \sum_{i=0}^{6} g_v\left(\sum_{j=1}^{3} \phi_{ij}(x_j)\right).$$

Here, if we can assume that all elements belonging to $\{g_v(\cdot); 0 \le i \le 6\}$ and $\{\phi_{ij}(\cdot); 0 \le i \le 6, 1 \le j \le 3\}$ are differentiable, then the above equality shows that $v(\cdot,\cdot,\cdot)$ can be represented as 7-time nested superposition. Now, we have a contradiction. □

Akashi [**2**] classified the concept of superposition irrepresentability, which plays an important role in Hilbert's 13th problem, into the following two concepts:

Strong superposition irrepresentability: There exists a function $s(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any finite-time nested superposition constructed from several functions of two variables.

Weak superposition irrepresentability: For any positive integer k, there exists a function $s_k(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any k-time nested superposition constructed from several functions of two variables.

For example, Vitušhkin proved that there exists a finitely differentiable function $s(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any finite-time nested superposition constructed from several finitely differentiable functions of two variables, and Hilbert proved that, for any positive integer k, there exists a polynomial $s_k(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any k-time nested superposition constructed from several polynomials of two variables. These results enables us to generalize Proposition 1 as the following:

Proposition 2. There exists a differentiable function $s(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as any finite-time nested superposition. Namely, for any positive integer n, for any family of differentiable functions $\{g_s^i(\cdot); 0 \le i \le n\}$ and for any family of differentiable functions $\{\phi_s^{ij}(\cdot); 0 \le i \le n, 1 \le j \le 3\}$, there exists a function $s(\cdot,\cdot,\cdot)$ of three variables satisfying the following inequality:

$$s(x_1,x_2,x_3) \ne \sum_{i=0}^{n} g_s^i\left(\sum_{j=1}^{3}\phi_s^{ij}(x_j)\right).$$

Proof. Let $v(\cdot,\cdot,\cdot)$ be one of the functions whose existence are proved by Vitušhkin **[6]** and assume that $v(\cdot,\cdot,\cdot)$ can be represented as the following:

$$v(x_1,x_2,x_3) = \sum_{i=0}^{n} g_v^i\left(\sum_{j=1}^{3}\phi_v^{ij}(x_j)\right).$$

Then, this equality shows that $v(\cdot,\cdot,\cdot)$ can be represented as a finite-time nested superposition, which contradicts Vitušhkin theorem. □

The above result shows that there exists a differentiable function $v(\cdot,\cdot,\cdot)$ of three variables, which cannot be represented as the Kolmogorov-Arnold superposition, if $g_v(\cdot)$ and $\phi_{01}(\cdot),\cdots,\phi_{63}(\cdot)$ are assumed to be differentiable. Actually, the above result cannot tell which functions of three variables can be represented as Kolmogorov-Arnold superpositions constructed from only differentiable functions of one variable. Therefore, we treat such a problem in the latter half of this paper.
Let $p(\cdot,\cdot,\cdot)$ and $q(\cdot,\cdot,\cdot)$ be differentiable functions of three variables which can be represented as n-time nested superpositions constructed from several differentiable functions of two variables, and let $r(\cdot,\cdot)$ be a differentiable functions of two variables. Then, the differentiable function $s(\cdot,\cdot,\cdot)$, which is defined as the following:

$$s(x_1,x_2,x_3) = r(p(x_1,x_2,x_3), q(x_1,x_2,x_3))$$

is said to be represented as an $n+1$-time nested superposition.

Proposition 3. Assume that there exists an element (x_1, x_2, x_3) belonging to $[0,1]^3$ satisfying $\prod_{i=1}^{3} s_i(x_1, x_2, x_3) \neq 0$ and that $s(\cdot,\cdot,\cdot)$ is differentiable and represented as the following superposition constructed from three differentiable functions $p(\cdot,\cdot)$, $q(\cdot,\cdot)$ and $r(\cdot)$:

$$s(\cdot,\cdot,\cdot) \;=\; r(p(\cdot,\cdot)+q(\cdot,\cdot)).$$

Then, for any constant c belonging to $[0,1]$, either $s_1(x_1,x_2,c)/s_2(x_1,x_2,c)$, $s_2(c,x_2,x_3)/s_3(c,x_2,x_3)$ or $s_3(x_1,c,x_3)/s_1(x_1,c,x_3)$ can be represented in the form of separation of two variables.

Proof. Without loss of generality, we can assume that $s(\cdot,\cdot,\cdot)$ is represented as the following:

$$s(x_1,x_2,x_3) \;=\; r(p(x_1,x_3)+q(x_2,x_3)).$$

Then, for any constant c belonging to $[0,1]$, we have

$$\frac{s_1(x_1,x_2,c)}{s_2(x_1,x_2,c)} \;=\; \frac{p_1(x_1,c)r'(p(x_1,c)+q(x_2,c))}{q_1(x_2,c)r'(p(x_1,c)+q(x_2,c))}.$$

This equality concludes the proof. □

Remark 1. Let $t(x_1,x_2,x_3)$ be the function of three variables defined as

$$t(x_1,x_2,x_3) \;=\; x_1^2x_2^3x_3^4.$$

This function can be represented as the following one-time nested superposition:

$$t(x_1,x_2,x_3) \;=\; \exp\left(\{\log x_1 + 3\log x_2\} + \{\log x_1 + 4\log x_3\}\right).$$

Let c be any constant belonging to $[0,1]$. Then, as for the function $t_1(\cdot,\cdot,c)/t_2(\cdot,\cdot,c)$, we have

$$\frac{t_1(x_1,x_2,c)}{t_2(x_1,x_2,c)} \;=\; \frac{2x_1}{3x_2}.$$

This equality shows that $t_1(\cdot,\cdot,c)/t_2(\cdot,\cdot,c)$ can be represented in the form of separation of two variables x_1 and x_2.

Remark 2. The condition presented in Proposition 3 is not a sufficient condition but a necessary one. For example, let $f_1(x_1,x_2,x_3)$ and $f_2(x_1,x_2,x_3)$ be the two functions of three variables defined as

$$f_1(x_1,x_2,x_3) = x_1x_2 + x_2x_3 + x_3x_1$$

and

$$f_2(x_1,x_2,x_3) = x_1x_2 + x_2x_3 + x_3x_1 + x_1^2 + x_2^2 + x_3^2\ ,$$

respectively. Then, the latter function satisfies the following representation:

$$f_2(x_1,x_2,x_3) = r(p(x_1,x_2)+q(x_1,x_3)),\quad 0 < x_1,x_2,x_3 < 1,$$

where $r(\cdot)$,$p(\cdot,\cdot)$ and $q(\cdot,\cdot)$ are defined as $r(x_1) = x_1^2$, $p(x_1,x_2) = x_1/2 + x_2$ and $q(x_1,x_3) = x_1/2 + x_3$, respectively, because we have

$$f_2(x_1,x_2,x_3) = \left\{\left(\frac{x_1}{2}+x_2\right)+\left(\frac{x_1}{2}+x_3\right)\right\}^2.$$

Nevertheless, it can be proved that we cannot find any functions $r(\cdot)$, $p(\cdot,\cdot)$ and $q(\cdot,\cdot)$ enabling the former function to be represented as

$$f_1(\cdot,\cdot,\cdot) = r(p(\cdot,\cdot) + q(\cdot,\cdot)).$$

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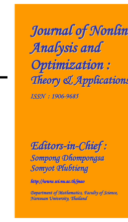

ON COMMON FIXED POINTS OF A NEW ITERATION FOR TWO NONSELF ASYMPTOTICALLY QUASI-NONEXPANSIVE-TYPE MAPPINGS IN BANACH SPACES

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ABSTRACT. Suppose that K is a nonempty closed convex subset of a real Banach space E which is also a nonexpansive retract of E. Let $T, S : K \to E$ be two nonself asymptotically quasi-nonexpansive-type mappings of E with $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in K$,

$$\begin{aligned} x_{n+1} &= P\left((1-a_n)x_n + a_n S(PS)^{n-1}\left((1-\beta_n)y_n + \beta_n S(PS)^{n-1}y_n\right)\right) \\ y_n &= P\left((1-b_n)x_n + b_n T(PT)^{n-1}\left((1-\gamma_n)x_n + \gamma_n T(PT)^{n-1}x_n\right)\right), \; n \geq 1, \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0,1]$. In this paper, we study the strongly converges to a common fixed point of the a new iterative scheme for two nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces. The results obtained in this paper extend and improve the recent ones announced by Tan and Xu [16], Shahzad [12], Thianwan [15], Kiziltunc et al. [17] and many others.

KEYWORDS : Nonself asymptotically quasi-nonexpansive-type mapping; strong convergence; common fixed points; iterative method; Banach space.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that E is a real Banach space and K is a nonempty closed convex subset of E and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$ denote the set of common fixed points of mappings T and S.

A mapping $T : K \to K$ is called *nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in K$.

A mapping $T : K \to K$ is called *quasi − nonexpansive mapping* if $F(T) \neq \varnothing$ and

$$\|Tx - x^*\| \leq \|x - x^*\| \tag{1.2}$$

for all $x \in K$ and $x^* \in F(T)$.

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A mapping $T : K \to K$ is called *asymptotically nonexpansive mapping* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.3}$$

for all $x, y \in K$ and $n \geq 1$.

A mapping $T : K \to K$ is called *asymptotically quasi − nonexpansive mapping* if $F(T) \neq \varnothing$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - x^*\| \leq k_n \|x - x^*\| \tag{1.4}$$

for all $x \in K, x^* \in F(T)$ and $n \geq 1$.

In [1], a mapping $T : K \to K$ is called *asymptotically nonexpansive − type mapping* if

$$\lim_{n\to\infty} \sup \left\{ \sup_{x,y \in K} \left\{ \|T^n x - T^n y\|^2 - \|x - y\|^2 \right\} \right\} \leq 0, \ n \geq 1. \tag{1.5}$$

In [1], a mapping $T : K \to K$ is called *asymptotically quasi − nonexpansive − type mapping* if $F(T) \neq \varnothing$ and

$$\lim_{n\to\infty} \sup \left\{ \sup_{x \in K,\, x^* \in F(T)} \left\{ \|T^n x - x^*\|^2 - \|x - x^*\|^2 \right\} \right\} \leq 0, \ n \geq 1. \tag{1.6}$$

From above definitions, if $F(T)$ is nonempty, a quasi-nonexpansive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings and asymptotically nonexpansive-type mappings are all special cases of asymptotically quasi-nonexpansive-type mappings. But the converse does not hold.

The class of asymptotically nonexpansive mappings is a natural generalization of the important class of nonexpansive mappings. Goebel and Kirk [2] proved that if K is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

Iterative techniques for asymptotically nonexpansive self-mappings in Banach spaces including Mann type and Ishikawa type iteration processes have been studied extensively by various authors, see for example [[2]-[6]]. However, if the domain of T, $D(T)$, is a proper subset of E (and this is the case in several applications), and T map $D(T)$ into E, then the iteration processes of Mann type and Ishikawa type studied by these authors, and their modifications introduced may fail to be well defined.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex set of a uniformly convex Banach space is a retract. A map $P : E \to K$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all $y \in R(P)$ in the range of P.

For nonself nonexpansive mappings, some authors (see, e.g., [[7],[8]]) have studied the strong and weak convergence theorems in Hilbert space or uniformly convex Banach spaces. The concept of nonself asymptotically nonexpansive mappings was introduced by Chidume [9] in 2003 as the generalization of asymptotically nonexpansive nonself-mappings. The nonself asymptotically nonexpansive mapping is defined as follows:

Definition 1.1. (see [9]) Let K a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A mapping $T : K \to E$ is called $nonself$ $asymptotically$ $nonexpansive$ if there exists sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ $(n \to \infty)$ such that

$$\left\| T (PT)^{n-1} x - T (PT)^{n-1} y \right\| \leq k_n \|x - y\| \tag{1.7}$$

for all $x,\, y \in K$ and $n \geq 1$. If T is a self-mapping, then P becomes the identity mapping so that (1.7) reduces (1.1).

T is said to be a $nonself$ $asymptotically$ $quasi-nonexpansive$ $mapping$, if $F(T) \neq \varnothing$ and there exists a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ $(n \to \infty)$ such that

$$\left\| T (PT)^{n-1} x - x^* \right\| \leq k_n \|x - x^*\| \tag{1.8}$$

for all $x \in K$, $x^* \in F(T)$ and $n \geq 1$.

In [10], T is said to be a $nonself$ $asymptotically$ $nonexpansive-type$ $mapping$, if

$$\lim_{n\to\infty} \sup \left\{ \sup_{x,\, y \in K} \left\{ \left\| T (PT)^{n-1} x - T (PT)^{n-1} y \right\| - \|x - y\| \right\} \right\} \leq 0, \ n \geq 1. \tag{1.9}$$

In [10], T is said to be a $nonself$ $asymptotically$ $quasi-nonexpansive-type$ $mapping$, if $F(T) \neq \varnothing$ and

$$\lim_{n\to\infty} \sup \left\{ \sup_{x \in K,\, x^* \in F(T)} \left\{ \left\| T (PT)^{n-1} x - x^* \right\| - \|x - x^*\| \right\} \right\} \leq 0, \ n \geq 1. \tag{1.10}$$

Remark 1.2. It follows from above Definition 1.1 that,

i. If $T : K \to E$ is a nonself asymptotically nonexpansive mapping, then T is a nonself asymptotically nonexpansive-type mapping;

ii. If $F(T) \neq \varnothing$ and $T : K \to E$ is a nonself asymptotically quasi-nonexpansive mapping, then T is a nonself asymptotically quasi-nonexpansive-type mapping;

iii. If $F(T) \neq \varnothing$ and $T : K \to E$ is a nonself asymptotically nonexpansive-type mapping, then T is a nonself asymptotically quasi-nonexpansive-type mapping.

Remark 1.3. Observe again that

$$\limsup_{n\to\infty} \left(\sup_{x \in K,\, x^* \in F(T)} \left\{ \left\| T (PT)^{n-1} x - x^* \right\|^2 - \|x - x^*\|^2 \right\} \right) \leq 0$$

implies

$$\limsup_{n\to\infty} (\sup_{x \in K,\, x^* \in F(T)} \left\{ \left(\left\| T (PT)^{n-1} x - x^* \right\| - \|x - x^*\| \right) (\left\| T (PT)^{n-1} x - x^* \right\| + \|x - x^*\| \right) \right\}) \leq 0$$

which implies

$$\limsup_{n\to\infty} \left(\sup_{x \in K,\, x^* \in F(T)} \left\{ \left\| T (PT)^{n-1} x - x^* \right\| - \|x - x^*\| \right\} \right) \leq 0\ .$$

Similarly, we can show that

$$\limsup_{n\to\infty} \left(\sup_{x \in K,\, x^* \in F(S)} \left\{ \left\| S (PS)^{n-1} x - x^* \right\| - \|x - x^*\| \right\} \right) \leq 0.$$

Suantai [11] defined a new three-step iterations which is an extension of Noor iterations and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in uniformly convex Banach spaces. Recently, Shahzad [12] extended Tan and Xu results [11, Theorem 1, p. 305] to the case of nonexpansive nonself-mapping in a uniformly convex Banach space. Peng [13] proved the convergence of finite steps iterative sequences with mean errors for asymptotically quasi-nonexpansive mappings in Banach spaces. In the same year, Yang [14] introduced a modified multistep iterative process for some common fixed point of a finite family of nonself asymptotically nonexpansive mappings on nonempty closed convex bounded subsets of a real uniformly convex Banach space. Thianwan [15] defined a weak and strong convergence theorems for new iterations with errors for nonexpansive nonself-mapping in a uniformly convex Banach space. In 2009, a new iterative scheme which is called the projection type Ishikawa iteration for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space was defined and constructed by Thianwan [18]. He gave some strong and weak convergence theorems of such iterations under some suitable conditions in a uniformly convex Banach space. In 2010, Wariam Chuayjan, Sornsak Thianwan and Boriboon Novaprateep [19] introduce and study a new type of multi-step iterative sequence with errors for a finite family of asymptotically quasi-nonexpansive-type nonself-mappings which can be viewed as an extension for Ishikawa type iterative schemes of Thianwan [18] and they proved strong convergence of a multi-step iterative scheme with errors to a common fixed point of a finite family of asymptotically quasi-nonexpansive-type nonself-mappings on nonempty closed convex subset of a real Banach space. In [1], Quan et al. proved approximation common fixed point of asymptotically quasi-nonexpansive-type mappings by the finite steps iterative sequences. In [10], Tian et al. introduced on the approximation problem of common fixed points for a finite family of nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces.

Inspired and motivated by this facts, I define and study the convergence theorems of a new two steps iterative sequences for nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces.

Let E be a Banach space and K be a nonempty closed convex subset of E, $P : E \to K$ the nonexpansive retraction of E onto K, and $T, S : K \to E$ be two nonself asymptotically quasi-nonexpansive-type mappings of E with sequences $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$, and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in K$,

$$\begin{aligned} x_{n+1} &= P\left((1-a_n)x_n + a_n S(PS)^{n-1}\left((1-\beta_n)y_n + \beta_n S(PS)^{n-1}y_n\right)\right), \quad (1.11) \\ y_n &= P\left((1-b_n)x_n + b_n T(PT)^{n-1}\left((1-\gamma_n)x_n + \gamma_n T(PT)^{n-1}x_n\right)\right), \ n \geq 1, \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

The purpose of this paper is to study the convergence theorems of a new two steps iterative sequences for nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces. The results of this paper can be viewed as an improve and extend the corresponding results of Tan and Xu [16], Shahzad [12], Thianwan [15], Kiziltunc et al. [17] and many others.

In the sequel, we need the following well known lemma to prove our main results.

Lemma 1.4. *(see [16]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)\, a_n + b_n, \qquad n \geq 1,$$

if $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ exists;

(ii) In particular, if $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

2. MAIN RESULTS

In this section, we prove the convergence theorem of two steps iterative sequences of the a new iterative scheme (1.11) for nonself asymptotically quasi-nonexpansive-type mappings in Banach spaces.

Theorem 2.1. *Let E be a real Banach space and K a nonempty closed subset of E which is also a nonexpansive retract with retraction P. Let $T, S : K \to E$ be two nonself asymptotically quasi-nonexpansive-type mappings of E and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose that $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0,1]$. Starting from an arbitrary $x_1 \in K$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1.11). Then $\{x_n\}$ strongly converges to a common fixed point of T and S in E if and only if*

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0. \tag{2.1}$$

Proof. We have that

$$\limsup_{n\to\infty} \left(\sup_{x\in K, x^*\in\mathcal{F}} \left\{ \left\| T(PT)^{n-1} x - x^* \right\| - \|x - x^*\| \right\} \right) \leq 0.$$

Similarly, we have that

$$\limsup_{n\to\infty} \left(\sup_{x\in K, x^*\in\mathcal{F}} \left\{ \left\| S(PS)^{n-1} x - x^* \right\| - \|x - x^*\| \right\} \right) \leq 0.$$

This implies that for any given $\varepsilon > 0$, there exists a positive integer n_0 such that for $n \geq n_0$ and $x^* \in \mathcal{F}$, we have that

$$\sup_{x\in K, x^*\in\mathcal{F}} \left\{ \left\| T(PT)^{n-1} x - x^* \right\|^2 - \|x - x^*\|^2 \right\} \leq \varepsilon$$

and similarly we have that

$$\sup_{x\in K, x^*\in\mathcal{F}} \left\{ \left\| S(PS)^{n-1} x - x^* \right\|^2 - \|x - x^*\|^2 \right\} \leq \varepsilon.$$

The necessity of (2.1) is obvious. Next we prove the sufficiency of (2.1). For $x^* \in \mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Since $\{x_n\}, \{y_n\} \subset E$, then we have

$$\left\| S(PS)^{n-1} y_n - x^* \right\| - \|y_n - x^*\| \leq \varepsilon,\ \forall x^* \in \mathcal{F},\ \forall n \geq n_0, \tag{2.2}$$

$$\left\| T(PT)^{n-1} x_n - x^* \right\| - \|x_n - x^*\| \leq \varepsilon,\ \forall x^* \in \mathcal{F},\ \forall n \geq n_0. \tag{2.3}$$

Set $\sigma_n = (1-\beta_n)\, y_n + \beta_n S(PS)^{n-1} y_n$ and $\xi_n = (1-\gamma_n)\, x_n + \gamma_n T(PT)^{n-1} x_n$. Thus for any $x^* \in \mathcal{F}$, using (1.11), (2.2) and (2.3) we have that

$$\begin{aligned} \|x_{n+1}-x^*\| &= \left\|P\left((1-a_n)x_n+a_nS(PS)^{n-1}(\sigma_n)\right)-x^*\right\| \\ &\leq \left\|(1-a_n)x_n+a_nS(PS)^{n-1}\left((1-\beta_n)y_n+\beta_nS(PS)^{n-1}y_n\right)-x^*\right\| \\ &\leq \left\|(1-a_n)x_n+a_nx^*-x^*+a_n\left(S(PS)^{n-1}(\sigma_n)-x^*\right)\right\| \\ &\leq a_n\varepsilon+a_n\|\sigma_n-x^*\|+(1-a_n)\|x_n-x^*\|. \end{aligned} \tag{2.4}$$

Consider the second term in right-hand side of (2.4), using (1.11) and (2.2), we have that

$$\begin{aligned} \|\sigma_n-x^*\| &= \left\|(1-\beta_n)y_n+\beta_nS(PS)^{n-1}y_n-x^*\right\| \\ &\leq (1-\beta_n)\|y_n-x^*\|+\beta_n\left\|S(PS)^{n-1}y_n-x^*\right\| \\ &\leq (1-\beta_n)\|y_n-x^*\|+\beta_n\varepsilon+\beta_n\|y_n-x^*\| \\ &= \|y_n-x^*\|+\beta_n\varepsilon. \end{aligned} \tag{2.5}$$

Using a similar method, consider the first term in right-hand side of (2.5), together with (1.11) and (2.3), we have that

$$\begin{aligned} \|y_n-x^*\| &= \left\|P\left((1-b_n)x_n+b_nT(PT)^{n-1}(\xi_n)\right)-x^*\right\| \\ &\leq \left\|(1-b_n)x_n+b_nT(PT)^{n-1}\left((1-\gamma_n)x_n+\gamma_nT(PT)^{n-1}x_n\right)-x^*\right\| \\ &\leq (1-b_n)\|x_n-x^*\|+b_n\left\|T(PT)^{n-1}\xi_n-x^*\right\| \\ &\leq b_n\varepsilon+b_n\|\xi_n-x^*\|+(1-b_n)\|x_n-x^*\|. \end{aligned} \tag{2.6}$$

Consider the second term in right-hand side of (2.6), using (1.11) and (2.3), we have that

$$\begin{aligned} \|\xi_n-x^*\| &= \left\|(1-\gamma_n)x_n+\gamma_nT(PT)^{n-1}x_n-x^*\right\| \\ &\leq (1-\gamma_n)\|x_n-x^*\|+\gamma_n\left\|T(PT)^{n-1}x_n-x^*\right\| \\ &\leq (1-\gamma_n)\|x_n-x^*\|+\gamma_n\varepsilon+\gamma_n\|x_n-x^*\| \\ &= \|x_n-x^*\|+\gamma_n\varepsilon. \end{aligned} \tag{2.7}$$

From (2.4), (2.5), (2.6) and (2.7), we have that

$$\begin{aligned} \|x_{n+1}-x^*\| &\leq a_n\varepsilon+a_n\|\sigma_n-x^*\|+(1-a_n)\|x_n-x^*\| \\ &\leq a_n(\|y_n-x^*\|+\beta_n\varepsilon)+a_n\varepsilon+(1-a_n)\|x_n-x^*\| \\ &\leq a_n((b_n\varepsilon+b_n\|\xi_n-x^*\|+(1-b_n)\|x_n-x^*\|)+\beta_n\varepsilon) \\ &\quad +a_n\varepsilon+(1-a_n)\|x_n-x^*\| \\ &\leq a_nb_n\varepsilon+a_nb_n\|\xi_n-x^*\|+a_n(1-b_n)\|x_n-x^*\|+a_n\beta_n\varepsilon \\ &\quad +a_n\varepsilon+(1-a_n)\|x_n-x^*\| \\ &\leq a_nb_n(\|x_n-x^*\|+\gamma_n\varepsilon)+(1-a_n)\|x_n-x^*\|+a_n\beta_n\varepsilon \\ &\quad +a_n(1-b_n)\|x_n-x^*\|+a_n\varepsilon+a_nb_n\varepsilon \\ &\leq a_nb_n\|x_n-x^*\|+(1-a_n)\|x_n-x^*\|+a_n\beta_n\varepsilon \\ &\quad +a_n(1-b_n)\|x_n-x^*\|+a_nb_n\gamma_n\varepsilon+a_n\varepsilon+a_nb_n\varepsilon \\ &\leq \|x_n-x^*\|+a_nb_n\gamma_n\varepsilon+a_n\varepsilon+a_nb_n\varepsilon+a_n\beta_n\varepsilon. \end{aligned} \tag{2.8}$$

Let $\varphi_n = a_n b_n \gamma_n \varepsilon + a_n \varepsilon + a_n b_n \varepsilon + a_n \beta_n \varepsilon$. Then $\sum_{n=1}^{\infty} \varphi_n < \infty$. Therefore, by (2.8), we have

$$\inf_{x^* \in \mathcal{F}} \|x_{n+1} - x^*\| \leq \inf_{x^* \in \mathcal{F}} \|x_n - x^*\| + \varphi_n, \quad \forall n \geq n_0. \tag{2.9}$$

It follows from (2.9) and $\sum_{n=1}^{\infty} \varphi_n < \infty$ that

$$d(x_{n+1}, \mathcal{F}) \leq d(x_n, \mathcal{F}) + \varphi_n. \tag{2.10}$$

By Lemma 1.4 and from (2.10), we know that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. Because $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, then we have that

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0. \tag{2.11}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence in E. In fact, from (2.9) that for any $n \geq n_0$, any $m \geq n_1$ and any $x^* \in \mathcal{F}$, we have that

$$\begin{aligned} \|x_{n+m} - x^*\| &\leq \|x_{n+m-1} - x^*\| + \varphi_{n+m-1} \\ &\leq \|x_{n+m-2} - x^*\| + (\varphi_{n+m-1} + \varphi_{n+m-2}) \\ &\leq \|x_{n+m-3} - x^*\| + (\varphi_{n+m-1} + \varphi_{n+m-2} + \varphi_{n+m-3}) \\ &\vdots \\ &\leq \|x_n - x^*\| + \sum_{k=n}^{n+m-1} \varphi_k. \end{aligned} \tag{2.12}$$

So by (2.12), we have that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - x^*\| + \|x_n - x^*\| \\ &\leq 2\|x_n - x^*\| + \sum_{k=n}^{\infty} \varphi_k. \end{aligned} \tag{2.13}$$

By the arbitrariness of $x^* \in \mathcal{F}$ and from (2.13), we have

$$\|x_{n+m} - x_n\| \leq 2d(x_n, \mathcal{F}) + \sum_{k=n}^{\infty} \varphi_k, \quad \forall n \geq n_0. \tag{2.14}$$

For any given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$, such that for any $n \geq n_1$, $d(x_n, \mathcal{F}) < \frac{\varepsilon}{4}$ and $\sum_{k=n}^{\infty} \varphi_k < \frac{\varepsilon}{2}$, we have $\|x_{n+m} - x_n\| < \varepsilon$, and so for any $m \geq 1$

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0. \tag{2.15}$$

This show that $\{x_n\}$ is Cauchy sequence in K. Since K is a closed subset of E, and so it is complete. Hence, there exists a $q \in K$ such that $x_n \to q$ as $n \to \infty$.

Finally, we have to prove that $q \in \mathcal{F}$. By contradiction, we assume that q is not in $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Since $\mathcal{F}$ is a closed set, $d(q, \mathcal{F}) > 0$. Hence for all $q \in \mathcal{F}$, we have that

$$\|q - x^*\| \leq \|q - x_n\| + \|x_n - x^*\|. \tag{2.16}$$

This implies that

$$d(q, \mathcal{F}) \leq \|q - x_n\| + d(x_n, \mathcal{F}). \tag{2.17}$$

From (2.16) and (2.17) (as $n \to \infty$), we have that $d(q, \mathcal{F}) \leq 0$. This is a contradiction. Hence $q \in \mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E be a real Banach space and K a nonempty closed subset of E which is also a nonexpansive retract with retraction P. Let T, $S : K \to E$ be two nonself asymptotically quasi-nonexpansive mappings of E and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose that $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$. Starting from an arbitrary $x_1 \in K$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1.11). Then $\{x_n\}$ strongly converges to a common fixed point of T and S in E if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Proof. Since T, $S : K \to E$ are two nonself asymptotically quasi-nonexpansive mappings , by the definition they are nonself asymptotically quasi-nonexpansive-type mappings. The conclusion of Theorem 2.2 can be proved from Theorem 2.1 immediately. □

Theorem 2.3. *Let E be a real Banach space and K a nonempty closed subset of E which is also a nonexpansive retract with retraction P. Let T, $S : K \to E$ be two nonself asymptotically nonexpansive mappings of E and $\mathcal{F} = F(T) \cap F(S) := \{x \in K : Tx = x = Sx\} \neq \varnothing$. Suppose that $\{a_n\}, \{b_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$. Starting from an arbitrary $x_1 \in K$, define the sequences $\{x_n\}$ and $\{y_n\}$ by the recursion (1.11). Then $\{x_n\}$ strongly converges to a common fixed point of T and S in E if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Proof. Since T, $S : K \to E$ are two nonself asymptotically nonexpansive mappings, taking $n = 1$ and $T = S$ in (1.11), we know that T, $S : K \to E$ are continuous nonself asymptotically nonexpansive mappings. Therefore, the conclusion of Theorem 2.3 can be proved from Theorem 2.1 immediately. □

Corollary 2.1. *Suppose the conditions in Theorem 2.1 are satisfied. Then the finite steps iterative sequence $\{x_n\}$ generated by the recursion (1.11) converges to common fixed point $x \in E$ iff there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to x.*

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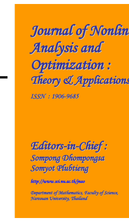

FIXED POINT THEOREMS FOR GENERALIZED WEAKLY CONTRACTIVE CONDITION IN ORDERED PARTIAL METRIC SPACES

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Abstract. In this paper, we establish some fixed point theorems for generalized weakly contractive condition in ordered partial metric spaces. The results extend the main theorems of Nashine and Altun [17] on the class of ordered partial metric ones. Also, some applications are given to illustrate our results.

Keywords: Partial metric, ordered set, fixed point, common fixed point.

AMS Subject Classification: 54H25, 47H10, 54E50.

1. INTRODUCTION AND PRELIMINARIES

The concept of partial metric space was introduced by Matthews [16] in 1994. In such spaces, the distance of a point to its self may not be zero. Specially, from the point of sequences, a convergent sequence need not have unique limit. Matthews [16] extended the well known Banach contraction principle to complete partial metric spaces. After that, many interesting fixed point results were established in such spaces. In this direction, we refer the reader to Valero [25], Oltra and Valero [23], Altun et al. [4], Romaguera [24], Altun and Erduran [2] and Aydi [6, 7, 8].

First, we recall some definitions and properties of partial metric spaces (see [2, 4, 16, 22, 23, 24, 25] for more details).

Definition 1.1. A partial metric on a non-empty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:
(p1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p2) $p(x, x) \leq p(x, y)$,
(p3) $p(x, y) = p(y, x)$,
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
A partial metric space is a pair (X, p) such that X is a non-empty set and p is a partial metric on X.

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Remark 1.2. It is clear that, if $p(x,y)=0$, then from (p1) and (p2), $x=y$. But if $x=y$, $p(x,y)$ may not be 0.

A basic example of a partial metric space is the pair $(\mathbb{R}_+,p)$, where $p(x,y)=\max\{x,y\}$ for all $x,y\in\mathbb{R}_+$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon),\ x\in X,\varepsilon>0\}$, where $B_p(x,\varepsilon)=\{y\in X : p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

If p is a partial metric on X, then the function $p^s : X\times X\to\mathbb{R}_+$ given by

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y), \tag{1.1}$$

is a metric on X.

Definition 1.3. Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then
(i) $\{x_n\}$ converges to a point $x\in X$ if and only if $p(x,x)=\lim\limits_{n\to+\infty}p(x,x_n)$,
(ii) $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim\limits_{n,m\to+\infty}p(x_n,x_m)$.

Definition 1.4. A partial metric space (X,p) is said to be complete if every Cauchy sequence (x_n) in X converges, with respect to τ_p, to a point $x\in X$, such that $p(x,x)=\lim\limits_{n,m\to+\infty}p(x_n,x_m)$.

Lemma 1.5. *Let (X,p) be a partial metric space. Then*
(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s),*
(b) *(X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim\limits_{n\to+\infty}p^s(x_n,x)=0$ if and only if*

$$p(x,x)=\lim_{n\to+\infty}p(x_n,x)=\lim_{n,m\to+\infty}p(x_n,x_m).$$

Definition 1.6. ([2]) Suppose that (X,p) is a partial metric space. A mapping $F:(X,p)\to(X,p)$ is said to be *continuous* at $x\in X$, if for every $\varepsilon>0$, there exists $\delta>0$ such that $F(B_p(x,\delta))\subseteq B_p(Fx,\varepsilon)$.

The following result is easy to check.

Lemma 1.7. *Let (X,p) be a partial metric space. $F:X\to X$ is continuous if and only if given a sequence $\{x_n\}\in\mathbb{N}$ and $x\in X$ such that $p(x,x)=\lim\limits_{n\to+\infty}p(x,x_n)$, then $p(Fx,Fx)=\lim\limits_{n\to+\infty}p(Fx,Fx_n)$.*

Remark 1.8. ([22]) Let (X,p) be a partial metric space and $F:(X,p)\to(X,p)$. If F is continuous on (X,p), then $F:(X,p^s)\to(X,p^s)$ is continuous.

On the other hand, fixed point problems of contractive mappings in metric spaces endowed with a partially order have been studied by many authors (see [1, 3, 5, 9, 10, 11, 12, 14, 15, 17, 18, 19, 20, 21]). In particular, Nashine and Altun [17] proved the following:

Theorem 1.9. *Let $(X,\leq)$ be a partially ordered set and (X,d) be a complete metric space. Suppose that $T:X\to X$ is a nondecreasing mapping such that for every two comparable elements $x,y\in X$*

$$\psi(d(Tx,Ty))\leq\psi(M(x,y))-\varphi(M(x,y)), \tag{1.2}$$

where

$$M(x,y) = ad(x,y) + bd(x,Tx) + cd(y,Ty) + e[d(y,Tx) + d(x,Ty)], \tag{1.3}$$

with $a > 0$*;* $b,\ c,\ e \geq 0$*,* $a+b+c+2e \leq 1$*, and* $\psi, \varphi : [0,+\infty) \to [0,+\infty)$*,* ψ *is a continuous, nondecreasing,* φ *is a lower semi-continuous functions and* $\psi(t) = 0 = \varphi(t)$ *if and only if* $t = 0$*. Also suppose, there exists* $x_0 \in X$ *with* $x_0 \leq Tx_0$*. Assume that :*
(i) T *is continuous, or*
(ii) if a nondecreasing sequence $\{x_n\}$ *converges to* x*, then* $x_n \leq x$ *for all* n*.*
Then, T *has a fixed point.*

The purpose of this paper is to extend Theorem 1.9 on the class of ordered partial metric spaces. Also, a common fixed point result is given.

2. MAIN RESULTS

Our first result is the following.

Theorem 2.1. *Let* $(X, \leq)$ *be a partially ordered set and* (X, p) *be a complete partial metric space. Suppose that* $T : X \to X$ *is a nondecreasing mapping such that for every two comparable elements* $x, y \in X$

$$\psi(p(Tx,Ty)) \leq \psi(\theta(x,y)) - \varphi(\theta(x,y)), \tag{2.1}$$

where

$$\theta(x,y) = ap(x,y) + bp(x,Tx) + cp(y,Ty) + e[p(y,Tx) + p(x,Ty)], \tag{2.2}$$

with $a, e > 0$*;* $b,\ c \geq 0$*,* $a+b+c+2e \leq 1$*, and* $\psi, \varphi : [0,+\infty) \to [0,+\infty)$*,* ψ *is a continuous, nondecreasing,* φ *is lower semi-continuous functions and* $\psi(t) = 0 = \varphi(t)$ *if and only if* $t = 0$*. Also suppose, there exists* $x_0 \in X$ *with* $x_0 \leq Tx_0$*. Assume that :*
(i) T *is continuous, or*
(ii) if a nondecreasing sequence $\{x_n\}$ *converges to* x *in* (X,p)*, then* $x_n \leq x$ *for all* n*.*
Then T *has a fixed point, say* z*. Moreover,* $p(z,z) = 0$*.*

Proof. If $Tx_0 = x_0$, then the proof is completed. Suppose $Tx_0 \neq x_0$. Now since $x_0 < Tx_0$ and T is nondecreasing we have

$$x_0 < Tx_0 \leq T^2x_0 \leq \ldots \leq T^nx_0 \leq T^{n+1}x_0 \leq \ldots$$

Put $x_n = T^nx_0$, hence $x_{n+1} = Tx_n$. If there exists $n_0 \in \{1, 2, \ldots\}$ such that $\theta(x_{n_0}, x_{n_0-1}) = 0$ then by definition (2.2), it is clear that $p(x_{n_0-1}, x_{n_0}) = p(x_{n_0}, Tx_{n_0-1}) = 0$, so $x_{n_0-1} = x_{n_0} = Tx_{n_0-1}$ and so we are finished. Now we can suppose

$$\theta(x_n, x_{n-1}) > 0, \tag{2.3}$$

for all $n \geq 1$. Let us check that

$$\lim_{n\to+\infty} p(x_{n+1}, x_n) = 0. \tag{2.4}$$

By (2.2), we have using condition (p4)

$$\begin{aligned}\theta(x_n, x_{n-1}) =& ap(x_n, x_{n-1}) + bp(x_n, Tx_n) + cp(x_{n-1}, Tx_{n-1}) \\ &+ e[p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})] \\ =& (a+c)p(x_n, x_{n-1}) + bp(x_n, x_{n+1}) + e[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)] \\ \leq& (a+c+e)p(x_n, x_{n-1}) + (b+e)p(x_n, x_{n+1}) \quad \text{[by (p4)]}.\end{aligned}$$

Now we claim that
$$p(x_{n+1},x_n) \le p(x_n,x_{n-1}), \tag{2.5}$$
for all $n \ge 1$. Suppose this is not true, that is, there exists $n_0 \ge 1$ such that $p(x_{n_0+1},x_{n_0}) > p(x_{n_0},x_{n_0-1})$. Now since $x_{n_0} \le x_{n_0+1}$, we can use the inequality (2.1), then we have
$$\begin{aligned}\psi(p(x_{n_0+1},x_{n_0})) =&\psi(p(Tx_{n_0},Tx_{n_0-1}))\\ \le&\psi(\theta(x_{n_0},x_{n_0-1}))-\varphi(\theta(x_{n_0},x_{n_0-1}))\\ \le&\psi((a+c+e)p(x_{n_0},x_{n_0-1})+(b+e)p(x_{n_0},x_{n_0+1}))\\ &-\varphi(\theta(x_{n_0},x_{n_0-1}))\\ \le&\psi((a+b+c+2e)p(x_{n_0},x_{n_0+1}))-\varphi(\theta(x_{n_0},x_{n_0-1}))\\ \le&\psi(p(x_{n_0},x_{n_0+1}))-\varphi(\theta(x_{n_0},x_{n_0-1})),\end{aligned}$$
which implies that $\varphi(\theta(x_{n_0},x_{n_0-1})) \le 0$, and by property of φ, giving that $\theta(x_{n_0},x_{n_0-1}) = 0$, this contradicts (2.3). Hence (2.5) holds, and so the sequence $\{p(x_{n+1},x_n)\}$ is nonincreasing and bounded below. Thus there exists $\rho \ge 0$ such that
$\lim_{n\to+\infty} p(x_{n+1},x_n) = \rho$. Assume that $\rho > 0$. By (2.2), we have
$$\begin{aligned}a\rho = \lim_{n\to+\infty} ap(x_n,x_{n-1}) \le& \limsup_{n\to+\infty}\theta(x_n,x_{n-1})\\ =&\limsup_{n\to+\infty}[(a+c)p(x_n,x_{n-1})+bp(x_n,x_{n+1})\\ &+e(p(x_{n-1},x_{n+1})+p(x_n,x_n))]\\ \le&\limsup_{n\to+\infty}[(a+c+e)p(x_n,x_{n-1})+(b+e)p(x_n,x_{n+1})].\end{aligned}$$
This implies
$$0 < a\rho \le \limsup_{n\to+\infty}\theta(x_n,x_{n-1}) \le (a+b+c+2e)\rho \le \rho,$$
and so there exist $\rho_1 > 0$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that
$$\lim_{k\to+\infty}\theta(x_{n(k)},x_{n(k)-1}) = \rho_1 \le \rho.$$
By the lower semi-continuity of φ we have
$$\varphi(\rho_1) \le \liminf_{k\to+\infty}\varphi(\theta(x_{n(k)},x_{n(k)+1})).$$
From (2.1), we have
$$\begin{aligned}\psi(p(x_{n(k)+1},x_{n(k)})) =& \psi(p(Tx_{n(k)},Tx_{n(k)-1}))\\ \le& \psi(\theta(x_{n(k)},x_{n(k)-1}))-\varphi(\theta(x_{n(k)},x_{n(k)-1})),\end{aligned}$$
and taking upper limit as $k \to +\infty$, we have using the properties of ψ and φ
$$\begin{aligned}\psi(\rho) \le&\psi(\rho_1)-\liminf_{k\to+\infty}\varphi(\theta(x_{n(k)},x_{n(k)+1}))\\ \le&\psi(\rho_1)-\varphi(\rho_1)\\ \le&\psi(\rho)-\varphi(\rho_1),\end{aligned}$$
that is, $\varphi(\rho_1) = 0$. Thus, by the property of φ, we have $\rho_1 = 0$, which is a contradiction. Therefore we have $\rho = 0$, that is (2.4) holds.
Now, we show that $\{x_n\}$ is a Cauchy sequence in the partial metric space (X,p). From Lemma 1.5, it is sufficient to prove that $\{x_n\}$ is a Cauchy sequence in the

metric space (X, p^s). Suppose to the contrary. Then there is a $\varepsilon > 0$ such that for an integer k there exist integers $m(k) > n(k) > k$ such that

$$p^s(x_{n(k)}, x_{m(k)}) > \varepsilon. \tag{2.6}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (2.6) and such that

$$p^s(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon. \tag{2.7}$$

Now, using (2.4)

$$\begin{aligned}\varepsilon < p^s(x_{n(k)}, x_{m(k)}) \leq & p^s(x_{n(k)}, x_{m(k)-1}) + p^s(x_{m(k)-1}, x_{m(k)}) \\ \leq & \varepsilon + p^s(x_{m(k)-1}, x_{m(k)}).\end{aligned}$$

Then by (2.4) it follows that

$$\lim_{k\to+\infty} p^s(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.8}$$

Also, by the triangle inequality, we have

$$|p^s(x_{n(k)}, x_{m(k)-1}) - p^s(x_{n(k)}, x_{m(k)})| \leq p^s(x_{m(k)-1}, x_{m(k)}).$$

By using (2.4), (2.8) we get

$$\lim_{k\to+\infty} p^s(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{2.9}$$

On the other hand, by definition of p^s,

$$p^s(x_{n(k)}, x_{m(k)}) = 2p(x_{n(k)}, x_{m(k)}) - p(x_{n(k)}, x_{n(k)}) - p(x_{m(k)}, x_{m(k)}),$$

$$p^s(x_{n(k)}, x_{m(k)-1}) = 2p(x_{n(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}) - p(x_{m(k)-1}, x_{m(k)-1}),$$

hence letting $k \to +\infty$, we find thanks to (2.8), (2.9) and the condition (p3) in (2.4)

$$\lim_{k\to+\infty} p(x_{n(k)}, x_{m(k)}) = \frac{\varepsilon}{2}, \tag{2.10}$$

$$\lim_{k\to+\infty} p(x_{n(k)}, x_{m(k)-1}) = \frac{\varepsilon}{2}. \tag{2.11}$$

In view of (2.2), we get

$$\begin{aligned} ap(x_{n(k)}, x_{m(k)-1}) \leq & \theta(x_{n(k)}, x_{m(k)-1}) \\ = & ap(x_{n(k)}, x_{m(k)-1}) + bp(x_{n(k)}, Tx_{n(k)}) + cp(x_{m(k)-1}, Tx_{m(k)-1}) \\ & + e[p(x_{m(k)-1}, Tx_{n(k)}) + p(x_{n(k)}, Tx_{m(k)-1})] \\ = & ap(x_{n(k)}, x_{m(k)-1}) + bp(x_{n(k)}, x_{n(k)+1}) + cp(x_{m(k)-1}, x_{m(k)}) \\ & + e[p(x_{m(k)-1}, x_{n(k)+1}) + p(x_{n(k)}, x_{m(k)})] \\ \leq & ap(x_{n(k)}, x_{m(k)-1}) + bp(x_{n(k)}, x_{n(k)+1}) + cp(x_{m(k)-1}, x_{m(k)}) \\ & + e[p(x_{m(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)}, x_{m(k)})]. \end{aligned}$$

Taking upper limit as $k \to +\infty$ and using (2.4), (2.10) and (2.11), we have

$$0 < a\frac{\varepsilon}{2} \leq \limsup_{k\to+\infty} \theta(x_{n(k)}, x_{m(k)-1}) \leq (a + 2e)\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}.$$

This implies that there exist $\varepsilon_1 > 0$ and a subsequence $\{x_{n(k(p))}\}$ of $\{x_{n(k)}\}$ such that

$$\lim_{p\to+\infty} \theta(x_{n(k(p))}, x_{m(k(p))-1}) = \varepsilon_1 \leq \frac{\varepsilon}{2}.$$

By the lower semi-continuity of φ we have

$$\varphi(\varepsilon_1) \leq \liminf_{k\to+\infty} \varphi(\theta(x_{n(k)}, x_{m(k)-1})).$$

Now by (2.1) we get

$$\begin{aligned}\psi(\frac{\varepsilon}{2}) =&\limsup_{p\to+\infty}\psi(p(x_{n(k(p))},x_{m(k(p))}))\\ \leq&\limsup_{p\to+\infty}\psi(p(x_{n(k(p))},x_{n(k(p))+1})+p(Tx_{n(k(p))},Tx_{m(k(p))-1}))\\ =&\limsup_{p\to+\infty}\psi(p(Tx_{n(k(p))},Tx_{m(k(p))-1}))\\ \leq&\limsup_{p\to+\infty}[\psi(\theta(x_{n(k(p))},x_{m(k(p))-1}))-\varphi(\theta(x_{n(k(p))},x_{m(k(p))-1}))]\\ =&\psi(\varepsilon_1)-\liminf_{p\to+\infty}\varphi(\theta(x_{n(k(p))},x_{m(k(p))-1}))\\ \leq&\psi(\varepsilon_1)-\varphi(\varepsilon_1)\\ \leq&\psi(\frac{\varepsilon}{2})-\varphi(\varepsilon_1),\end{aligned}$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in the metric space (X,p^s). From Lemma 1.5, (X,p^s) is a complete metric space. Then there is $z\in X$ such that

$$\lim_{n\to+\infty}p^s(x_n,z)=0.$$

Again, from lemma 1.5, we have thanks to (2.4) and the condition (p2)

$$p(z,z)=\lim_{n\to+\infty}p(x_n,z)=\lim_{n\to+\infty}p(x_n,x_n)=0. \tag{2.12}$$

We will prove that $Tz=z$.
1. Assume that (i) holds, that is, T is continuous. By (2.12), the sequence $\{x_n\}$ converges in (X,p) to z, and since T is continuous, hence the sequence $\{Tx_n\}$ converges to Tz, that is

$$p(Tz,Tz)=\lim_{n\to+\infty}p(Tx_n,Tz) \tag{2.13}$$

Again, thanks to (2.12),

$$p(z,Tz)=\lim_{n\to+\infty}p(x_n,Tz)=\lim_{n\to+\infty}p(Tx_{n-1},Tz)=p(Tz,Tz). \tag{2.14}$$

On the other hand, by (2.1), (2.14)

$$\psi(p(z,Tz))=\psi(p(Tz,Tz))\leq\psi(\theta(z,z))-\varphi(\theta(z,z)),$$

where from (2.12) and the condition (p2)

$$\theta(z,z)=ap(z,z)+(b+c+2e)p(z,Tz)=(b+c+2e)p(z,Tz)\leq p(z,Tz).$$

Thus,

$$\begin{aligned}\psi(p(z,Tz))\leq&\psi(\theta(z,z))-\varphi(\theta(z,z))\\ \leq&\psi(p(z,Tz))-\varphi(\theta(z,z)).\end{aligned}$$

It follows that $\varphi(\theta(z,z))=0$, so $\theta(z,z)=(b+c+2e)p(z,Tz)=0$, that is $p(z,Tz)=0$ because $e>0$. Hence $z=Tz$, that is, z is a fixed point of T.
2. Assume that (ii) holds. Then, we have $x_n\leq z$ for all n. Therefore, for all n, we can use the inequality (2.1) for x_n and z. Since

$$\begin{aligned}\theta(z,x_n)=&ap(z,x_n)+bp(z,Tz)+cp(x_n,Tx_n)+e[p(x_n,Tz)+p(z,Tx_n)]\\ =&ap(z,x_n)+bp(z,Tz)+cp(x_n,x_{n+1})+e[p(x_n,Tz)+p(z,x_{n+1})],\end{aligned}$$

hence, from (2.4), (2.12), $\lim_{n\to+\infty} \theta(z, x_n) = (b+e)p(z, Tz)$. We have

$$\begin{aligned}\psi(p(Tz,z)) &= \limsup_{n\to+\infty} \psi(p(Tz,x_{n+1}))\\ &= \limsup_{n\to+\infty} \psi(p(Tz,Tx_n))\\ &\leq \limsup_{n\to+\infty} \psi[(\psi(z,x_n)) - \varphi(\theta(z,x_n))]\\ &\leq \psi((b+e)p(Tz,z)) - \varphi((b+e)p(Tz,z))\\ &\leq \psi(p(Tz,z)) - \varphi((b+e)p(Tz,z)).\end{aligned}$$

Then, $\varphi((b+e)p(Tz,z)) = 0$, and since $e > 0$, hence by the property of φ we have $p(Tz,z) = 0$, so $Tz = z$. This completes the proof of Theorem 2.1. □

Remark 2.2. Theorem 2.1 holds for ordered partial metric spaces, so it is an extension of the result of Nashine and Altun [17] given in Theorem 1.9 which is verified just for ordered metric ones.

Corollary 2.3. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space. Suppose that $T : X \to X$ be a nondecreasing mapping such that for every two comparable elements $x, y \in X$*

$$p(Tx,Ty)) \leq \theta(x,y) - \varphi(\theta(x,y)), \tag{2.15}$$

where

$$\theta(x,y) = ap(x,y) + bp(x,Tx) + cp(y,Ty) + e[p(y,Tx) + p(x,Ty)], \tag{2.16}$$

with $a, e > 0$; b, $c \geq 0$, $a + b + c + 2e \leq 1$, and $\varphi : [0,+\infty) \to [0,+\infty)$, φ is a lower semi-continuous functions and $\varphi(t) = 0$ if and only if $t = 0$. Also suppose, there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Assume that:
(i) T is continuous, or
(ii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X, p), then $x_n \leq x$ for all n.
Then T has a fixed point, say z. Moreover, $p(z, z) = 0$.

Proof. It suffices to take $\psi(t) = t$ in Theorem. □

Corollary 2.4. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space. Suppose that $T : X \to X$ be a nondecreasing mapping such that for every two comparable elements $x, y \in X$*

$$p(Tx,Ty)) \leq k\theta(x,y), \tag{2.17}$$

where

$$\theta(x,y) = ap(x,y) + bp(x,Tx) + cp(y,Ty) + e[p(y,Tx) + p(x,Ty)], \tag{2.18}$$

with $k \in [0, 1)$, $a, e > 0$; b, $c \geq 0$ and $a + b + c + 2e \leq 1$. Also suppose, there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Assume that:
(i) T is continuous, or
(ii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X, p), then $x_n \leq x$ for all n.
Then T has a fixed point, say z. Moreover, $p(z, z) = 0$.

Proof. It suffices to take $\varphi(t) = (1 - k)t$ in Corollary 2.3. □

We give in the following a sufficient condition for the uniqueness of the fixed point of the mapping T.

Theorem 2.5. *Let all the conditions of Theorem 2.1 be fulfilled and let the following condition hold: for arbitrary two points $x, y \in X$ there exists $z \in X$ which is comparable with both x and y. If $(a + 2b + 2e) \leq 1$ or $(a + 2c + 2e) \leq 1$, then the fixed point of T is unique.*

Proof. Let u and v be two fixed points of T, i.e., $Tu = u$ and $Tv = v$. We have in mind, $p(u, u) = p(v, v) = 0$. Consider the following two cases:
1. u and v are comparable. Then we can apply condition (2.1) and obtain that

$$\psi(p(u, v)) = \psi(p(Tu, Tv)) \leq \psi(\theta(u, v)) - \varphi(\theta(u, v)),$$

where

$$\begin{aligned}\theta(u, v) =& ap(u, v) + bp(u, Tu) + cp(v, Tv) + e[p(u, Tv) + p(v, Tu)]\\ =&(a + 2e)p(u, v) + bp(u, u) + cp(v, v)\\ \leq&(a + b + c + 2e)p(u, v) \leq p(u, v).\end{aligned}$$

We deduce $\psi(p(u, v)) \leq \psi(p(u, v)) - \varphi(\theta(u, v))$, i,e, $\theta(u, v) = 0$, so $p(u, v) = 0$, meaning that $u = v$, that is the uniqueness of the fixed point of T.
2. Suppose now that u and v are not comparable. Choose an element $w \in X$ comparable with both of them. Then also $u = T^n u$ is comparable with $T^n w$ for each n (since T is nondecreasing). Applying (2.1), one obtains that

$$\begin{aligned}\psi(p(u, T^n w)) =&\psi(p(TT^{n-1}u, TT^{n-1}w))\\ \leq&\psi(\theta(T^{n-1}u, T^{n-1}w)) - \varphi(\theta(T^{n-1}u, T^{n-1}w))\\ =&\psi(\theta(u, T^{n-1}w)) - \varphi(\theta(u, T^{n-1}w))\end{aligned}$$

where

$$\begin{aligned}\theta(u, T^{n-1}w) =& ap(u, T^{n-1}w) + bp(u, TT^{n-1}u) + cp(T^{n-1}w, TT^{n-1}w)\\ &+e[p(u, TT^{n-1}w) + p(T^{n-1}w, Tu)]\\ =&ap(u, T^{n-1}w) + bp(u, u) + cp(T^{n-1}w, T^n w)\\ &+e[p(u, T^n w) + p(T^{n-1}w, u)]\\ =&(a + e)p(u, T^{n-1}w) + cp(T^{n-1}w, T^n w) + ep(u, T^n w)\\ \leq&(a + c + e)p(u, T^{n-1}w) + (c + e)p(u, T^n w).\end{aligned}$$

Similarly as in the proof of Theorem 2.1, it can be shown that, under the condition $(a + 2c + 2e) \leq 1$

$$p(u, T^n w) \leq p(u, T^{n-1}w).$$

Note that when we consider

$$\psi(p(T^n w, u)) \leq \psi(\theta(T^{n-1}w, u)) - \varphi(\theta(T^{n-1}w, u))$$

where

$$\begin{aligned}\theta(T^{n-1}w, u) =&(a + e)p(u, T^{n-1}w) + bp(T^{n-1}w, T^n w) + ep(u, T^n w)\\ \leq&(a + b + e)p(u, T^{n-1}w) + (b + e)p(u, T^n w),\end{aligned}$$

hence, one finds under $(a + 2b + 2e) \leq 1$ that

$$p(T^n w, u) \leq p(T^{n-1}w, u).$$

In each case, it follows that the sequence $\{p(u, f^n w)\}$ is nonincreasing and it has a limit $l \geq 0$. Adjusting again as in the proof of Theorem 2.1, one can finds that $l = 0$. In the same way it can be deduced that $p(v, T^n w) \to 0$ as $n \to +\infty$. Now, passing to the limit in $p(u, v) \leq p(u, T^n w) + p(T^n w, v)$, it follows that $p(u, v) = 0$, so $u = v$, and the uniqueness of the fixed point is proved. □

Example 2.6. Let $X = [0, +\infty)$ endowed with the usual partial order (which is a total order). Let $p(x, y) = \max(x, y)$. For any $x, y \in X$, we have $p^s(x, y) = |x - y|$. Then, (X, p^s) is a complete metric space, and so for (X, p). Take $T : X \to X$ be defined as

$$Tx = \frac{1}{5}x.$$

Letting $x_0 = 0$, we have $x_0 = 0 \leq 0 = Tx_0$. The mapping T is nondecreasing. Also, take $a = \frac{1}{2}$, $e = \frac{1}{8}$ and $b = c = 0$, so

$$\theta(x, y) = \frac{1}{2}p(x, y) + \frac{1}{8}[p(x, Ty) + p(y, Tx)] = \frac{1}{2}\max\{x, y\} + \frac{1}{8}[\max\{x, Ty\} + \max\{y, Tx\}].$$

Moreover, define

$$\psi(t) = t, \quad \varphi(t) = \frac{t}{2}.$$

Note that

$$\begin{aligned}\psi(p(Tx, Ty)) =& \frac{1}{5}\max\{x, y\} \leq \frac{1}{4}\max\{x, y\} \\ \leq& \frac{1}{4}\max\{x, y\} + \frac{1}{16}[\max\{x, Ty\} + \max\{y, Tx\}] = \frac{1}{2}\theta(x, y) \\ =& \psi(\theta(x, y)) - \varphi(\theta(x, y)).\end{aligned}$$

Thus, the inequality (2.1) is verified for each comparable x and y. All the hypotheses of Theorem 2.5 are verified. Here, T has a unique fixed point, which is $z = 0$.

Now we will give a common fixed point theorem for two maps. For this, we need the following definition, which is given in [13].

Definition 2.7. Let $(X, \leq)$ be a partially ordered set. Two mappings $S, T : X \to X$ are said to be weakly increasing if $Sx \leq TSx$ and $Tx \leq STx$ for all $x \in X$.

Note that, two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [3]

Theorem 2.8. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space. Suppose that $T, S : X \to X$ are two weakly increasing mappings such that for every two comparable elements $x, y \in X$*

$$\psi(p(Tx, Sy)) \leq \psi(u(x, y)) - \varphi(u(x, y)), \tag{2.19}$$

where

$$u(x, y) = ap(x, y) + bp(x, Tx) + cp(y, Sy) + e[p(y, Tx) + p(x, Sy)], \tag{2.20}$$

with $a, e > 0$; $b, c \geq 0$, $a + b + c + 2e \leq 1$, and $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, ψ is a continuous, nondecreasing, φ is a lower semi-continuous functions and $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Also suppose, there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Assume that :
(i) T is continuous, or
(ii) S is continuous, or
(iii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X, p), then $x_n \leq x$ for all n.
Then, S and T have a common fixed point.

Proof. Let x_0 be an arbitrary point of X. We can define a sequence $\{x_n\}$ in X as follows:

$$x_{2n+1} = Sx_{2n} \quad \text{and} \quad x_{2n+2} = Tx_{2n+1} \quad \text{for} \quad n \in \mathbb{N}.$$

Since S and T are weakly increasing, we have $x_1 = Sx_0 \leq TSx_0 = Tx_1 = x_2$ and $x_2 = Tx_1 \leq STx_1 = Sx_2 = x_3$. Continuing this process we have

$$x_1 \leq x_2 \leq ... \leq x_n \leq x_{n+1} \leq ...$$

The terms x_{2n-1} and x_{2n} are comparable then we can use the inequality (2.19) and we have

$$\psi(p(Tx_{2n-1}, Sx_{2n})) \leq \psi(u(x_{2n-1}, x_{2n})) - \varphi(u(x_{2n-1}, x_{2n})), \tag{2.21}$$

where

$$\begin{aligned} u(x_{2n-1}, x_{2n}) &= ap(x_{2n-1}, x_{2n}) + bp(x_{2n-1}, Tx_{2n-1}) + cp(x_{2n}, Sx_{2n}) \\ &\quad + e[p(x_{2n}, Tx_{2n-1}) + p(x_{2n-1}, Sx_{2n})] \\ &= (a+b)p(x_{2n-1}, x_{2n}) + cp(x_{2n}, x_{2n+1}) + e[p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})] \\ &\leq (a+b+e)p(x_{2n-1}, x_{2n}) + (c+e)p(x_{2n}, x_{2n+1}), \quad \text{using (p4)}. \end{aligned} \tag{2.22}$$

Now, we claim that

$$p(x_{n+1}, x_n) \leq p(x_n, x_{n-1}), \quad \forall\, n \in \mathbb{N}^*. \tag{2.23}$$

If $p(x_{2n+1}, x_{2n}) > p(x_{2n}, x_{2n-1})$ for some $n \in \{1, 2, ...\}$, then

$$u(x_{2n-1}, x_{2n}) \leq (a+b+c+2e)p(x_{2n+1}, x_{2n}) \leq p(x_{2n+1}, x_{2n}),$$

and so by (2.21) we have

$$\psi(p(x_{2n}, x_{2n+1})) \leq \psi(p(x_{2n+1}, x_{2n})) - \varphi(u(x_{2n-1}, x_{2n})),$$

so $u(x_{2n-1}, x_{2n}) = 0$, then from (2.22), we get

$$(a+b)p(x_{2n-1}, x_{2n}) + cp(x_{2n}, x_{2n+1}) + e[p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1})] = 0.$$

Having in mind that $e > 0$ and $a > 0$, hence

$$p(x_{2n-1}, x_{2n}) = p(x_{2n}, x_{2n}) + p(x_{2n-1}, x_{2n+1}) = 0. \tag{2.24}$$

Since $p(x_{2n}, x_{2n}) \leq p(x_{2n-1}, x_{2n})$, then $p(x_{2n}, x_{2n}) = 0$, so by (2.24),

$$p(x_{2n-1}, x_{2n+1}) = 0. \tag{2.25}$$

By assumption, we have $p(x_{2n+1}, x_{2n}) > 0 = p(x_{2n}, x_{2n-1})$. On the other hand, by property (p4)

$$\begin{aligned} 0 &< p(x_{2n+1}, x_{2n}) \leq p(x_{2n+1}, x_{2n-1}) + p(x_{2n-1}, x_{2n}) \\ &= p(x_{2n+1}, x_{2n-1}) + 0 = p(x_{2n+1}, x_{2n-1}), \end{aligned}$$

hence $p(x_{2n+1}, x_{2n-1}) > 0$, which is a contradiction with respect to (2.25). So we have $p(x_{2n+1}, x_{2n}) \leq p(x_{2n}, x_{2n-1})$ for all $n \in \mathbb{N}^*$. Similarly, we have

$$p(x_{2n+1}, x_{2n+2}) \leq p(x_{2n}, x_{2n+1}).$$

Therefore, (2.23) holds for any $n \in \mathbb{N}^*$. Hence, the sequence $\{p(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. Thus there exists $\rho \geq 0$ such that $\lim_{n\to+\infty} p(x_{n+1}, x_n) = \rho$. In particular, we give

$$\lim_{n\to+\infty} p(x_{2n}, x_{2n+1}) = \lim_{n\to+_i i} p(x_{2n-1}, x_{2n}) = \rho.$$

Suppose that $\rho > 0$. Therefore, from (2.22)

$$\begin{aligned} \limsup_{n\to+\infty} ap(x_{2n-1}, x_{2n}) &\leq \limsup_{n\to+\infty} u(x_{2n-1}, x_{2n}) \\ &\leq \limsup_{n\to+\infty} \{(a+b+e)p(x_{2n-1}, x_{2n}) + (c+e)p(x_{2n}, x_{2n+1})\}. \end{aligned}$$

This implies $0 < a\rho \leq \limsup_{n\to+\infty} u(x_{2n-1}, x_{2n}) \leq (a+b+c+2e)\rho \leq \rho$ and so there exist $\rho_1 > 0$ and a subsequence $\{u(x_{2n(k)-1}, x_{2n(k)})\}$ of $\{u(x_{2n-1}, x_{2n})\}$ such that

$$\lim_{k\to+\infty} u(x_{2n(k)-1}, x_{2n(k)}) = \rho_1 \leq \rho.$$

By the lower semi-continuity of φ we have

$$\varphi(\rho_1) \leq \liminf_{k\to+\infty} \varphi(u(x_{2n(k)-1}, x_{2n(k)})).$$

Now, by (2.19), we have

$$\begin{aligned}\psi(p(x_{2n(k)}, x_{2n(k)+1})) =&\psi(p(Tx_{2n(k)-1}, Sx_{2n(k)}))\\ \leq&\psi(u(x_{2n(k)-1}, x_{2n(k)})) - \varphi(u(x_{2n(k)-1}, x_{2n(k)})),\end{aligned}$$

and taking the upper limit as $k \to +\infty$, we have

$$\begin{aligned}\psi(\rho) \leq&\psi(\rho_1) - \liminf_{k\to+\infty} \varphi(u(x_{2n(k)-1}, x_{2n(k)}))\\ \leq&\psi(\rho_1) - \varphi(\rho_1)\\ \leq&\psi(\rho) - \varphi(\rho_1),\end{aligned}$$

so $\varphi(\rho_1) = 0$, which is a contradiction. Therefore, we have

$$\lim_{n\to+\infty} p(x_{n+1}, x_n) = \rho = 0. \tag{2.26}$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence in (X, p). Again, from Lemma 1.5, we need to check that $\{x_n\}$ is Cauchy in (X, p^s). To do this, it suffices to prove that $\{x_{2n}\}$ is Cauchy in (X, p^s). We proceed by contradiction. Then we can find an $\varepsilon > 0$ such that for each even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$p^s(x_{2n(k)}, x_{2m(k)}) \geq \varepsilon. \tag{2.27}$$

By choosing $2m(k)$ to be smallest number exceeding $2n(k)$ for which (2.27) holds, we may also assume

$$p^s(x_{2m(k)-2}, x_{2n(k)}) < \varepsilon. \tag{2.28}$$

Now, (2.27) and (2.28) imply

$$\begin{aligned}0 < \varepsilon &\leq p^s(x_{2n(k)}, x_{2m(k)})\\ &\leq p^s(x_{2n(k)}, x_{2m(k)-2}) + p^s(x_{2m(k)-2}, x_{2m(k)-1}) + p^s(x_{2m(k)-1}, x_{2m(k)})\\ &< \varepsilon + p^s(x_{2m(k)-2}, x_{2m(k)-1}) + p^s(x_{2m(k)-1}, x_{2m(k)}),\end{aligned}$$

and so thanks to (2.26)

$$\lim_{k\to+\infty} p^s(x_{2n(k)}, x_{2m(k)}) = \varepsilon. \tag{2.29}$$

Also, by the triangular inequality,

$$|p^s(x_{2n(k)}, x_{2m(k)-1}) - p^s(x_{2n(k)}, x_{2m(k)})| \leq p^s(x_{2m(k)-1}, x_{2m(k)}),$$

and

$$|p^s(x_{2n(k)+1}, x_{2m(k)-1}) - p^s(x_{2n(k)}, x_{2m(k)})| \leq p^s(x_{2m(k)-1}, x_{2m(k)}) + p^s(x_{2n(k)}, x_{2n(k)+1}).$$

Therefore we get

$$\lim_{k\to+\infty} p^s(x_{2n(k)}, x_{2m(k)-1}) = \varepsilon, \tag{2.30}$$

and

$$\lim_{k\to+\infty} p^s(x_{2n(k)+1}, x_{2m(k)-1}) = \varepsilon. \tag{2.31}$$

On the other hand, by definition of p^s, as in (2.10) and (2.11), we get from (2.26), (2.29), (2.30) and (2.31)

$$\lim_{k\to+\infty} p(x_{2n(k)}, x_{2m(k)}) = \frac{\varepsilon}{2}, \tag{2.32}$$

$$\lim_{k\to+\infty} p(x_{2n(k)}, x_{2m(k)-1}) = \frac{\varepsilon}{2}, \tag{2.33}$$

$$\lim_{k\to+\infty} p(x_{2n(k)+1}, x_{2m(k)-1}) = \frac{\varepsilon}{2}. \tag{2.34}$$

On the other hand, since $x_{2n(k)}$ and $x_{2m(k)-1}$ are comparable, we can use the condition (2.19) for these points. By, (2.26), (2.32), (2.33) and (2.34)

$$\begin{aligned}\lim_{k\to+\infty} u(x_{2m(k)-1}, x_{2n(k)}) &= \lim_{k\to+\infty}\Big(ap(x_{2m(k)-1}, x_{2n(k)}) + bp(x_{2m(k)-1}, Tx_{2m(k)-1}) \\ &+ cp(x_{2n(k)}, Sx_{2n(k)}) + e[p(x_{2n(k)}, Tx_{2m(k)-1}) \\ &+ p(x_{2m(k)-1}, Sx_{2n(k)})]\Big) \\ &= \lim_{k\to+\infty}\Big(ap(x_{2m(k)-1}, x_{2n(k)}) + bp(x_{2m(k)-1}, x_{2m(k)}) \\ &+ cp(x_{2n(k)}, x_{2n(k)+1}) + e[p(x_{2n(k)}, x_{2m(k)}) \\ &+ p(x_{2m(k)-1}, x_{2n(k)+1})]\Big) \\ &= (a+2e)\frac{\varepsilon}{2},\end{aligned}$$

then we have

$$\begin{aligned}\psi(\frac{\varepsilon}{2}) &= \limsup_{k\to+\infty} \psi(p(x_{2n(k)}, x_{2m(k)})) \\ &\le \limsup_{k\to+\infty} \psi(p(x_{2n(k)}, x_{2n(k)+1}) + p(x_{2n(k)+1}, x_{2m(k)})) \\ &\le \limsup_{k\to+\infty} \psi(p(x_{2n(k)}, x_{2n(k)+1}) + p(Sx_{2n(k)}, Tx_{2m(k)-1})) \\ &= \limsup_{k\to+\infty} \psi(p(Sx_{2n(k)}, Tx_{2m(k)-1})) \\ &\le \limsup_{k\to+\infty} [\psi(u(x_{2m(k)-1}, x_{2n(k)})) - \varphi(u(x_{2m(k)-1}, x_{2n(k)}))] \\ &= \psi((a+2e)\frac{\varepsilon}{2}) - \liminf_{k\to+\infty} \varphi(u(x_{2m(k)-1}, x_{2n(k)})) \\ &\le \psi((a+2e)\frac{\varepsilon}{2}) - \varphi((a+2e)\frac{\varepsilon}{2}) \\ &\le \psi(\frac{\varepsilon}{2}) - \varphi((a+2e)\frac{\varepsilon}{2}).\end{aligned}$$

This is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s), which is complete from Lemma 1.5. Then there is $z \in X$ such that

$$\lim_{n\to+\infty} p^s(x_n, z) = 0.$$

Again, from Lemma 1.5, we have thanks to (2.26) and the condition (p2)

$$p(z, z) = \lim_{n\to+\infty} p(x_n, z) = \lim_{n\to+\infty} p(x_n, x_n) = 0. \tag{2.35}$$

We will prove that $Tz = z$.
1. Assume that (i) holds, that is T is continuous in (X, p). In view of Remark 1.8, we have T is continuous in (X, p^s). Since the sequence $\{x_{2n+1}\}$ converges in (X, p^s) to z, hence $\{Tx_{2n+1}\}$ converges to Tz in (X, p^s), that is from Lemma 1.5

$$\begin{aligned} p(Tz, Tz) = \lim_{n\to+\infty} p(Tx_{2n+1}, Tz) &= \lim_{n\to+\infty} p(Tx_{2n+1}, Tx_{2n+1}) \\ &= \lim_{n\to+\infty} p(x_{2n+2}, x_{2n+2}) = 0 \quad [\text{by } (2.26)]. \end{aligned} \tag{2.36}$$

Again, thanks to (2.35)-(2.36),

$$p(z, Tz) = \lim_{n\to+\infty} p(x_{2n+2}, Tz) = \lim_{n\to+\infty} p(Tx_{2n+1}, Tz) = p(Tz, Tz). \tag{2.37}$$

It follows that $p(z, Tz) = 0$. Hence $z = Tz$, that is, z is a fixed point of T.
2. Assume that S is continuous. The proof of $Sz = z$ will be done similarly as in the first case (i).
3. Assume that (iii) holds. Then, we have $x_{2n} \le z$ for all n. Therefore, we can use the inequality (2.19) for x_{2n} and z.

$$\begin{aligned} \psi(p(Tz, x_{2n+1})) =& \psi(p(Tz, Sx_{2n})) \\ \le& \psi(u(z, x_{2n})) - \varphi(u(z, x_{2n})), \end{aligned}$$

where

$$\begin{aligned} u(z, x_{2n}) =& ap(z, x_{2n}) + bp(z, Tz) + cp(x_{2n}, Sx_{2n}) + e[p(x_{2n}, Tz) + p(z, Sx_{2n})] \\ =& ap(z, x_{2n}) + bp(z, Tz) + cp(x_{2n}, x_{2n+1}) + e[p(x_{2n}, Tz) + p(z, x_{2n+1})]. \end{aligned}$$

Thanks to (2.35), we get

$$\lim_{n\to+\infty} u(z, x_{2n}) = (b + e)p(z, Tz) \le p(z, Tz).$$

Therefore, taking the upper limit as $n \to +\infty$, we obtain using the properties of ψ and φ

$$\psi(p(Tz, z)) \le \psi(p(z, Tz)) - \varphi((b + e)p(z, Tz)),$$

giving that $p(z, Tz) = 0$, so $Tz = z$.

We have proved that z is a fixed point of a one mapping in each precedent case. Now we show that, such z is also a common fixed point of S and T . Indeed, without loss of generality, we take z be a fixed point of S. Now assume that $p(z, Tz) > 0$. If we use the inequality (2.19), for $x = y = z$, we have

$$\begin{aligned} \psi(p(Tz, z)) =& \psi(p(Tz, Sz)) \\ \le& \psi(u(z, z)) - \varphi(u(z, z)) \\ \le& \psi(p(Tz, z)) - \varphi((a + c + e)p(z, z) + (b + e)p(Tz, z)) \\ =& \psi(p(Tz, z)) - \varphi((b + e)p(Tz, z)). \end{aligned}$$

We have used (2.35) in the last identity, that is, $p(z, z) = 0$. It follows that $\varphi((b + e)p(Tz, z)) = 0$, so by a property of φ, we have $(b + e)p(Tz, z) = 0$ for $e > 0$, that is $p(Tz, z) = 0$, which is a contradiction because we assumed that $p(z, Tz) > 0$. Thus $p(z, Tz) = 0$ and so z is a common fixed point of S and T. The proof of Theorem 2.8 is completed. □

3. APPLICATION

In this section, we present some applications of previous results and we obtain some fixed point theorems for single mapping and pair of mappings satisfying a general contractive condition of integral type in ordered partial metric spaces. Take Γ to be the set of
$v : \mathbb{R}_+ \to \mathbb{R}_+$ which are Lebesgue integrable mappings, summable, nonnegative and satisfy

$$\int_0^{\varepsilon} v(t)dt > 0 \quad \text{for each} \quad \varepsilon > 0.$$

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space. Suppose that $T : X \to X$ be a nondecreasing mapping such that for every two comparable elements $x, y \in X$*

$$\int_0^{\psi(p(Tx,Ty))} v(t)dt \leq \int_0^{\psi(\theta(x,y))} v(t)dt - \int_0^{\varphi(\theta(x,y))} v(t)dt, \tag{3.1}$$

where

$$\theta(x, y) = ap(x, y) + bp(x, Tx) + cp(y, Ty) + e[p(y, Tx) + p(x, Ty)], \tag{3.2}$$

with $a, e > 0$; b, $c \geq 0$, $a + b + c + 2e \leq 1$, and $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, ψ is a continuous, nondecreasing, φ is a lower semi-continuous functions and $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Also suppose, there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Assume that :
(i) T is continuous, or
(ii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X, p), then $x_n \leq x$ for all n.
Then T has a fixed point.

Proof. Define $\triangle : \mathbb{R}_+ \to \mathbb{R}_+$ by $\triangle(x) = \int_0^x v(t)dt$, then $\triangle$ is continuous and nondecreasing with $\triangle(0) = 0$. Thus, equation (3.1) becomes

$$\triangle(\psi(p(Tx, Ty))) \leq \triangle(\psi(\theta(x, y))) - \triangle(\varphi(\theta(x, y)))$$

which further can be written as

$$\psi_1(p(Tx, Ty)) \leq \psi_1(\theta(x, y) - \varphi_1(\theta(x, y)),$$

where $\psi_1 = \triangle \circ \psi$ and $\varphi_1 = \triangle \circ \varphi$. Hence, Theorem 2.1 yields a fixed point. □

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space. Suppose that $T, S : X \to X$ are weakly increasing such that for every two comparable elements $x, y \in X$*

$$\int_0^{\psi(p(Tx,Sy))} v(t)dt \leq \int_0^{\psi(u(x,y))} v(t)dt - \int_0^{\varphi(u(x,y))} v(t)dt, \tag{3.3}$$

where

$$u(x, y) = ap(x, y) + bp(x, Tx) + cp(y, Sy) + e[p(y, Tx) + p(x, Sy)], \tag{3.4}$$

with $a, e > 0$; b, $c \geq 0$, $a + b + c + 2e \leq 1$, and $\psi, \varphi : [0, +\infty) \to [0, +\infty)$, ψ is a continuous, nondecreasing, φ is a lower semi-continuous functions and $\psi(t) = 0 = \varphi(t)$ if and only if $t = 0$. Assume that :
(i) T is continuous, or
(ii) S is continuous, or
(iii) if a nondecreasing sequence $\{x_n\}$ converges to x in (X, p), then $x_n \leq x$ for all n.
Then T and S have a common fixed point.

Proof. Define $\triangle : \mathbb{R}_+ \to \mathbb{R}_+$ by $\triangle(x) = \int_0^x v(t)dt$, then $\triangle$ is continuous and nondecreasing with $\triangle(0) = 0$. Thus, equation (3.3) becomes

$$\triangle(\psi(p(Tx, Sy))) \leq \triangle(\psi(u(x, y))) - \triangle(\varphi(u(x, y)))$$

which further can be written as

$$\psi_1(p(Tx, Sy)) \leq \psi_1(u(x, y) - \varphi_1(u(x, y)),$$

where $\psi_1 = \triangle \circ \psi$ and $\varphi_1 = \triangle \circ \varphi$. Hence, Theorem 2.8 yields a common fixed point. □

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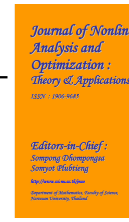

STRONG CONVERGENCE OF THREE-STEP ITERATIVE PROCESS WITH ERRORS FOR THREE MULTIVALUED MAPPINGS

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Abstract. In this paper, we introduced a three-step iterative process with errors for three multivalued mappings satisfying the condition (C) in uniformly convex Banach spaces and establish strong convergence theorems for the proposed process under some basic boundary conditions. Our results generalized recent known results in the literature.

Keywords: Fixed point; Condition (C); Three-step iteration process; Strong convergence.

AMS Subject Classification: 47H10, 47H09.

1. INTRODUCTION

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [9] and Nadler [10]. Since then the metric fixed point theory of multivalued mappings has been rapidly developed. The theory of multivalued mappings has applications in control theory, convex optimization, differential equations and economics. Theory of multivalued nonexpansive mappings is harder than the corresponding theory of singlevalued nonexpansive mappings. Different iterative processes have been used to approximate fixed points of multivalued nonexpansive mappings. In particular in 2005, Sastry and Babu [14] proved that the Mann and Ishikawa iteration process for multivalued maping T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. Panyanak [12] extended result of Sastry and Babu [14] to uniformly convex Banach spaces. Recently, Song and Wang [17] noted that there was a gap in the proof of the main result in [12]. They further revised the gap and also gave the affirmative answer to Panyanak's open question. Shahzad and Zegeye [16] extended and improved results already appeared in the papers [12, 14, 17]. Very recently, motivated by [16], Cholamjiak and Suantai [2, 3] introduced some new two-step

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iterative process for two multivalued mappings in Banach spaces and prove strong convergence of the proposed iterations.

Glowinski and Le Tallec [5] used three-step iterative process to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [5] that the three-step iterative process gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge et al. [6] studied the convergence analysis of three-step process of Glowinski and Le Tallec [5] and applied these process to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step process plays an important and significant part in solving various problems, which arise in pure and applied sciences.

Now the aim of this paper is to introduce a three-step iterative process with errors for multivalued mappings satisfying condition (C) and then prove some strong convergence theorems for such process in a uniformly convex Banach space. Both Mann and Ishikawa iterative processes for multivalued mappings can be obtained from this process as special cases by suitably choosing the parameters. Our results generalized recent known result in literature.

2. PRELIMINARIES

Recall that a Banach space X is said to be uniformly convex if for each $t \in [0, 2]$, the modulus of convexity of X given by:

$$\delta(t) = inf\{1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq t\}$$

satisfies the inequality $\delta(t) > 0$ for all $t > 0$.

A subset $E \subset X$ is called proximal if for each $x \in X$, there exists an element $y \in E$ such that

$$\| x - y \| = dist(x, E) = inf\{\| x - z \| : z \in E\}.$$

It is known that every closed convex subset of a uniformly convex Banach space is proximal.

We denote by $CB(E)$ and $P(E)$ the collection of all nonempty closed bounded subsets and nonempty proximal bounded subsets of E respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(X)$.

Let $T : X \longrightarrow 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denote by $F(T)$.

Definition 2.1. A multivalued mapping $T : X \longrightarrow CB(X)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \quad x, y \in X.$$

(ii) quasi nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq\| x - p \|$ for all $x \in X$ and all $p \in F(T)$.

In 2008, Suzuki [19] introduced a condition on mappings, called (C) which is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. Very recently, Abkar and Eslamian [1] used a modified Suzuki condition for multivalued mappings as follows:

Definition 2.2. A multivalued mapping $T : X \longrightarrow CB(X)$ is said to satisfy condition (C) provided that

$$\frac{1}{2}dist(x,Tx) \leq \|x-y\| \implies H(Tx,Ty) \leq \|x-y\|, \quad x,y \in X.$$

Lemma 2.3. *([1]) Let $T : X \longrightarrow CB(X)$ be a multivalued nonexpansive mapping, then T satisfies the condition (C).*

Lemma 2.4. *([4]) Let $T : X \longrightarrow CB(X)$ be a multivalued mapping which satisfies the condition (C) and has a fixed point. Then T is a quasi nonexpansive mapping.*

Lemma 2.5. *([4]) Let E be a nonempty subset of a Banach space X. Suppose $T : E \longrightarrow P(E)$ satisfies condition (C) then*

$$H(Tx,Ty) \leq 2dist(x,Tx) + \|x-y\|,$$

holds for all $x, y \in E$.

Lemma 2.6. *([20], Lemma1) Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\longrightarrow\infty} a_n$ exists. In particular , if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\longrightarrow\infty} a_n = 0$.

The following Lemma can be found in ([11], Lemma 1.4)

Lemma 2.7. *Let X ba a uniformly convex Banach space and let $B_r(0) = \{x \in X :\| x \|\leq r\}$, $r > 0$. Then there exist a continuous, strictly increasing, and convex function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\| \alpha x + \beta y + \gamma z + \eta w \|^2 \leq \alpha \| x \|^2 + \beta \| y \|^2 + \gamma \| z \|^2 + \eta\|w\|^2 - \alpha\beta\varphi(\| x - y \|),$$

for all $x, y, z, w \in B_r(0)$, and $\alpha, \beta, \gamma, \eta \in [0, 1]$ with $\alpha + \beta + \gamma + \eta = 1$.

3. MAIN RESULTS

In this section we use the following iteration process.
(A) Let X be a Banach space, E be a nonempty convex subset of X and $T_1, T_2, T_3 : E \longrightarrow CB(E)$ be three given mappings. Then, for $x_1 \in E$, we consider the following iterative process:

$$w_n = (1-a_n-b_n)x_n + a_n z_n + b_n s_n, \qquad n \geq 1,$$
$$y_n = (1-c_n-d_n-e_n)x_n + c_n u_n + d_n u'_n + e_n s'_n, \qquad n \geq 1,$$
$$x_{n+1} = (1-\alpha_n-\beta_n-\gamma_n)x_n + \alpha_n v_n + \beta_n v'_n + \gamma_n s''_n, \qquad n \geq 1,$$

where $z_n, u'_n \in T_1(x_n)$, $u_n, v'_n \in T_2(w_n)$ and $v_n \in T_3(y_n)$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0, 1]$ and $\{s_n\}, \{s'_n\}$ and $\{s''_n\}$ are bounded sequences in E.

Definition 3.1. A mapping $T : E \longrightarrow CB(E)$ is said to satisfy condition (I) if there is a non decreasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$ such that

$$dist(x,Tx) \geq g(dist(x,F(T)).$$

Let $T_i : E \longrightarrow CB(E), (i = 1, 2, 3)$ be three given mappings. The mappings T_1, T_2, T_3 are said to satisfy condition (II) if there exist a non decreasing function $g : [0,\infty) \longrightarrow [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for $r \in (0,\infty)$, such that

$$\frac{1}{3}sum_{i=1}^{3} dist(x,T_i x) \geq g(dist(x,\mathcal{F})),$$

where $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i)$.

Theorem 3.1. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow CB(E), (i = 1, 2, 3)$ be three multivalued mappings satisfying the condition (C) . Assume that $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, 3)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n + b_n, c_n+d_n+e_n, \alpha_n+\beta_n+\gamma_n \in [a, b] \subset (0, 1)$ and also $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that T_1, T_2 and T_3 satisfying the condition (II). Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. Let $p \in \mathcal{F}$. Then, by the boundedness of $\{s_n\}, \{s'_n\}$ and $\{s''_n\}$, we let

$$M = max\{sup_{n\geq1}\|s_n - p\|, sup_{n\geq1}\|s'_n - p\|, sup_{n\geq1}\|s''_n - p\|\}.$$

Using (A) and quasi nonexpansiveness of T_i (i=1,2,3) we have

$$\begin{aligned}
\| w_n - p \| &= \| (1 - a_n - b_n)x_n + a_n z_n + b_n s_n - p \| \\
&\leq (1 - a_n - b_n) \| x_n - p \| + a_n \| z_n - p \| + b_n\|s_n - p\| \\
&= (1 - a_n - b_n \| x_n - p \| + a_n dist(z_n, T_1(p)) + b_n\|s_n - p\| \\
&\leq (1 - a_n - b_n) \| x_n - p \| + a_n H(T_1(x_n), T_1(p)) + b_n\|s_n - p\| \\
&\leq (1 - a_n - b_n) \| x_n - p \| + a_n \| x_n - p \| + b_n\|s_n - p\| \\
&\leq (1 - b_n)\|x_n - p\| + b_n M \\
&\leq \|x_n - p\| + b_n M
\end{aligned}$$

and

$$\begin{aligned}
\| y_n - p \| &= \| (1 - c_n - d_n - e_n)x_n + c_n u_n + d_n u'_n + e_n s'_n - p \| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n \| u_n - p \| + d_n \| u'_n - p\| + e_n\|s'_n - p\| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n dist(u_n, T_2(p)) + d_n dist(u'_n, T_1(p)) + e_n\|s'_n - p\| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n H(T_2(w_n), T_2(p)) + d_n H(T_1(x_n), T_1(p)) + e_n\|s'_n - p\| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n \| w_n - p \| + d_n \| x_n - p \| + e_n\|s'_n - p\| \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \| + c_n \| x_n - p \| + d_n \| x_n - p \| + c_n b_n M + e_n M \\
&\leq \|x_n - p\| + b_n M + e_n M.
\end{aligned}$$

We also have

$$\begin{aligned}
\| x_{n+1} - p \| &= \| (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n v_n + \beta_n v'_n + \gamma_n s''_n - p \| \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n \| v_n - p \| + \beta_n \| v'_n - p\| + \gamma_n\|s''_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n dist(v_n, T_3(p)) + \beta_n dist(v'_n, T_2(p)) + \gamma_n\|s''_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n H(T_3(y_n), T_3(p)) + \beta_n H(T_2(w_n), T_2(p)) + \gamma_n\|s''_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p \| + \alpha_n \| y_n - p \| + \beta_n \| w_n - p \| + \gamma_n\|s''_n - p\| \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \| + \alpha_n\|x_n - p\| + \alpha_n b_n M + \alpha_n e_n M + \beta_n\|x_n - p\| + \beta_n b_n M + \gamma_n M \\
&\leq (1 - \gamma_n)\|x_n - p\| + M(b_n + e_n + \gamma_n) \\
&\leq \|x_n - p\| + \theta_n. \qquad (3.1)
\end{aligned}$$

where $\theta_n = M(b_n + e_n + \gamma_n)$. By assumption we have $\sum_{n=1}^{\infty} \theta_n < \infty$. Hence by Lemma 2.6 it follows that $\lim \| x_n - p \|$ exist for any $p \in \mathcal{F}$. Since the sequences $\{x_n\}, \{y_n\}$ and $\{w_n\}$ are bounded, we can find $r > 0$ depending on p such that

$x_n - p, y_n - p, w_n - p \in B_r(0)$ for all $n \geq 0$. Denote by

$$N = max\{sup_{n\geq1}\|s_n - p\|^2, sup_{n\geq1}\|s'_n - p\|^2, sup_{n\geq1}\|s''_n - p\|^2\}.$$

From Lemma 2.7, we get

$$\begin{aligned}
&\| w_n - p \|^2 = \| (1 - a_n - b_n)x_n + a_n z_n + b_n s_n - p \|^2 \\
&\leq (1-a_n-b_n) \| x_n-p \|^2 + a_n \| z_n-p \|^2 + b_n\|s_n-p\|^2 - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq (1-a_n-b_n \| x_n-p \|^2 + a_n dist(z_n, T_1(p))^2 + b_n\|s_n-p\|^2 - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq (1-a_n-b_n) \| x_n-p \|^2 + a_n H(T_1(x_n), T_1(p))^2 + b_n\|s_n-p\|^2 - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq (1-a_n-b_n) \| x_n-p \|^2 + a_n \| x_n-p \|^2 + b_n\|s_n-p\|^2 - a_n(1-a_n-b_n)\varphi(\|x_n-z_n\|) \\
&\leq (1 - b_n)\|x_n - p\|^2 + b_n N - a_n(1 - a_n - b_n)\varphi(\|x_n - z_n\|) \\
&\leq \|x_n - p\|^2 + b_n N - a_n(1 - a_n - b_n)\varphi(\|x_n - z_n\|)
\end{aligned}$$

It follows from Lemma 2.7 that

$$\begin{aligned}
&\| y_n - p \|^2 = \| (1 - c_n - d_n - e_n)x_n + c_n u_n + d_n u'_n + e_n s'_n - p \|^2 \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \|^2 + c_n \| u_n - p \|^2 + d_n \| u'_n - p\|^2 + e_n\|s'_n - p\|^2 \\
&- \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq (1-c_n-d_n-e_n) \| x_n-p \|^2 + c_n dist(u_n, T_2(p))^2 + d_n dist(u'_n, T_1(p))^2 + e_n\|s'_n-p\|^2 \\
&- \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq (1-c_n-d_n-e_n) \| x_n-p \|^2 + c_n H(T_2(w_n), T_2(p))^2 + d_n H(T_1(x_n), T_1(p))^2 + e_n\|s'_n-p\|^2 \\
&- \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \|^2 + c_n \| w_n - p \|^2 + d_n \| x_n - p \|^2 + e_n\|s'_n - p\|^2 \\
&- \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq (1 - c_n - d_n - e_n) \| x_n - p \|^2 + c_n \| x_n - p \|^2 + d_n \| x_n - p \|^2 + c_n b_n N + e_n N \\
&- \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|) \\
&\leq \|x_n - p\|^2 + b_n N + e_n N \\
&- \frac{1}{2}(1 - c_n - d_n - e_n)d_n\varphi(\|x_n - u'_n\|) - \frac{1}{2}(1 - c_n - d_n - e_n)c_n\varphi(\|x_n - u_n\|).
\end{aligned}$$

By another application of Lemma 2.7 we obtain that

$$\begin{aligned}
&\| x_{n+1} - p \|^2 = \| (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n v_n + \beta_n v'_n + \gamma_n s''_n - p \|^2 \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n) \| x_n - p \|^2 + \alpha_n \| v_n - p \|^2 + \beta_n \| v'_n - p\|^2 + \gamma_n\|s''_n - p\|^2 \\
&- \alpha_n(1 - \alpha_n - \beta_n - \gamma_n)\varphi(\|x_n - v_n\|) \\
&\leq (1-\alpha_n-\beta_n-\gamma_n) \| x_n-p \|^2 + \alpha_n dist(v_n, T_3(p))^2 + \beta_n dist(v'_n, T_2(p))^2 + \gamma_n\|s''_n-p\|^2 \\
&- \alpha_n(1 - \alpha_n - \beta_n - \gamma_n)\varphi(\|x_n - v_n\|) \\
&\leq (1-\alpha_n-\beta_n-\gamma_n) \| x_n-p \|^2 + \alpha_n H(T_3(y_n), T_3(p))^2 + \beta_n H(T_2(w_n), T_2(p)^2 + \gamma_n\|s''_n-p\| \\
&- \alpha_n(1 - \alpha_n - \beta_n - \gamma_n)\varphi(\|x_n - v_n\|) \\
&\leq (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p \|^2 + \alpha_n \| y_n - p \|^2 + \beta_n \| w_n - p \|^2 + \gamma_n\|s''_n - p\|^2
\end{aligned}$$

$$- \alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|x_n-v_n\|)$$
$$\le (1-\alpha_n-\beta_n-\gamma_n) \parallel x_n-p \parallel^2 + \alpha_n\|x_n-p\|^2 + \alpha_n b_n N + \alpha_n e_n N + \beta_n\|x_n-p\|^2 + \beta_n b_n N + \gamma_n N$$
$$- \alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|x_n-v_n\|) - \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)d_n\varphi(\|x_n-u_n'\|)$$
$$- \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)c_n\varphi(\|x_n-u_n\|) - a_n\beta_n(1-a_n-b_n)\varphi(\|x_n-z_n\|)$$
$$\le \|x_n-p\|^2 + N(b_n+e_n+\gamma_n) - \alpha_n(1-\alpha_n-\beta_n-\gamma_n)\varphi(\|x_n-v_n\|) - \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)d_n\varphi(\|x_n-u_n'\|)$$
$$- \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)c_n\varphi(\|x_n-u_n\|) - a_n\beta_n(1-a_n-b_n)\varphi(\|x_n-z_n\|).$$

So, we have

$$\begin{aligned}\frac{1}{2}a^2(1-b)\varphi(\|x_n-u_n'\|) &\\ &\le \frac{1}{2}\alpha_n(1-c_n-d_n-e_n)d_n\varphi(\|x_n-u_n'\|) \\ &\le \parallel x_n-p \parallel^2 - \parallel x_{n+1}-p \parallel^2 + N(b_n+e_n+\gamma_n).\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} a^2(1-b)\varphi(\|x_n-u_n'\|) \le \|x_1-p\|^2 + \sum_{n=1}^{\infty} N(b_n+e_n+\gamma_n) < \infty$$

from which it follows that $\lim_{n\longrightarrow\infty}\varphi(\|x_n-u_n'\|)=0$. Since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\longrightarrow\infty}\|x_n-u_n'\|=0.$$

Similarly we obtain that

$$\lim_{n\longrightarrow\infty}\|x_n-z_n\| = \lim_{n\longrightarrow\infty}\|x_n-u_n\| = \lim_{n\longrightarrow\infty}\|x_n-v_n\| = 0.$$

Hence we obtain $dist(x_n, T_1x_n) \le \|x_n-u_n'\| \longrightarrow 0$ as $n \longrightarrow \infty$. Also we have

$$\lim_{n\longrightarrow\infty}\|x_n-w_n\| = \lim_{n\longrightarrow\infty}(a_n\|z_n-x_n\| + b_n\|s_n-x_n\|) = 0.$$

and

$$\lim_{n\longrightarrow\infty}\|x_n-y_n\| = \lim_{n\longrightarrow\infty}(c_n\|u_n-x_n\| + d_n\|u_n'-x_n\| + e_n\|s_n'-x_n\|) = 0.$$

Therefore by Lemma 2.5 we have

$$\begin{aligned}dist(x_n,T_2(x_n)) &\le dist(x_n,T_2(w_n)) + H(T_2(w_n),T_2(x_n)) \\ &\le dist(x_n,T_2(w_n)) + 2\,dist(w_n,T_2(w_n)) + \|x_n-w_n\| \\ &\le 3\,\|x_n-w_n\| + 3\,dist(x_n,T_2(w_n)) \\ &\le 3\,\|x_n-w_n\| + 3\,\|x_n-u_n\| \longrightarrow 0 \qquad as \quad n\longrightarrow\infty.\end{aligned}$$

and

$$\begin{aligned}dist(x_n,T_3x_n) &\le dist(x_n,T_3(y_n)) + H(T_3(y_n),T_3(x_n)) \\ &\le dist(x_n,T_3(y_n)) + 2\,dist(y_n,T_3(w_n)) + \|x_n-y_n\| \\ &\le 3\,\|x_n-y_n\| + 3\,dist(x_n,T_3(y_n)) \\ &\le 3\,\|x_n-y_n\| + 3\,\|x_n-v_n\| \longrightarrow 0 \qquad as \quad n\longrightarrow\infty.\end{aligned}$$

Note that by our assumption $\lim_{n\longrightarrow\infty} dist(x_n, \mathcal{F}) = 0$. Hence there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\}$ in $\mathcal{F}$ such that $\|x_{n_k} - p_k\| < \frac{1}{2^k}$ for all k. Therefore by inequality 3.1 we get

$$\begin{aligned}\|x_{n_{k+1}} - p\| &\leq \|x_{n_{k+1}-1} - p\| + \theta_{n_{k+1}-1} \\ &\leq \|x_{n_{k+1}-2} - p\| + \theta_{n_{k+1}-2} + \theta_{n_{k+1}-1} \\ &\leq \dots \\ &\leq \|x_{n_k} - p\| + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i}\end{aligned}$$

for all $p \in \mathcal{F}$. This implies that

$$\begin{aligned}\|x_{n_{k+1}} - p\| &\leq \|x_{n_k} - p_k\| + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i} \\ &\leq \frac{1}{2^k} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i}.\end{aligned}$$

Now, we show that $\{p_k\}$ is a Cauchy sequence in E. Note that

$$\begin{aligned}\|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i} \\ &< \frac{1}{2^{k-1}} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i}.\end{aligned}$$

This implies that $\{p_k\}$ is a Cauchy sequence in E and hence converges to $q \in E$. Since for $i = 1, 2, 3$

$$dist(p_k, T_i(q)) \leq H(T_i(p_k), T_i(q)) \leq \|p_k - q\|$$

and $p_k \longrightarrow q$ as $n \longrightarrow \infty$, it follows that $dist(q, T_i(q)) = 0$ and thus $q \in \mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists, we conclude that $\{x_n\}$ converges strongly to q. □

Theorem 3.2. *Let E be a nonempty compact convex subset of uniformly convex Banach space X. Let $T_i : E \longrightarrow CB(E), (i = 1, 2, 3)$ be three multivalued mappings satisfying the condition (C) . Assume that $\mathcal{F} = \bigcap_{i=1}^{3} F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, 3)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n + b_n, c_n+d_n+e_n, \alpha_n+\beta_n+\gamma_n \in [a, b] \subset (0, 1)$ and also $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. As in the proof of Theorem 3.1, we have $\lim_{n\longrightarrow\infty} dist(T_i(x_n), x_n) = 0, (i = 1, 2, 3)$. Since E is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = w$ for some $w \in E$. By lemma 2.6, for $i = 1, 2, 3$ we have

$$\begin{aligned}dist(w, T_i(w)) &\leq \|w - x_{n_k}\| + dist(x_{n_k}, T_i(w)) \\ &\leq \|w - x_{n_k}\| + dist(x_{n_k}, T_i(x_{n_k})) + H(T_i(x_{n_k}), T_i(w)) \\ &\leq 3dist(x_{n_k}, T_i(x_{n_k})) + 2\|w - x_{n_k}\| \longrightarrow 0 \quad as \quad k \longrightarrow \infty,\end{aligned}$$

this implies that $w \in \mathcal{F}$. Since $\{x_{n_k}\}$ converges strongly to w and $\lim_{n \longrightarrow \infty} \|x_n - w\|$ exists (as in the proof of Theorem 3.1), this implies that $\{x_n\}$ converges strongly to w. □

Remark: If we put $T_1 = T_2 = T_3 = T$, then Theorem 3.1 also hold even if T is quasi-nonexpansive.

We now intend to remove the restriction that $T_i(p) = p$ for each $p \in \mathcal{F}$. We define the following iteration process.

(B): Let X be a Banach space, E be a nonempty convex subset of X and $T_i : E \longrightarrow P(E), (i = 1, 2, 3)$ be given mappings and

$$P_{T_i}(x) = \{y \in T_i(x) : \| x - y \| = dist(x, T_i(x))\}.$$

Then, for $x_1 \in E$, we consider the following iterative process:

$$w_n = (1 - a_n - b_n)x_n + a_n z_n + b_n s_n, \qquad n \geq 1,$$

$$y_n = (1 - c_n - d_n - e_n)x_n + c_n u_n + d_n u'_n + e_n s'_n, \qquad n \geq 1,$$

$$x_{n+1} = (1 - \alpha_n - \beta_n - \gamma_n)x_n + \alpha_n v_n + \beta_n v'_n + \gamma_n s''_n, \qquad n \geq 1,$$

where $z_n, u'_n \in P_{T_1}(x_n)$,$u_n, v'_n \in P_{T_2}(w_n)$ and $v_n \in P_{T_3}(y_n)$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0, 1]$ and $\{s_n\}, \{s'_n\}$ and $\{s''_n\}$ are bounded sequences in E.

Theorem 3.3. *Let E be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow P(E), (i = 1, 2, 3)$ be multivalued mappings such that P_{T_i} satisfing the condition (C). Let $\{x_n\}$ be the iterative process defined by (B), and $a_n + b_n, c_n + d_n + e_n, \alpha_n + \beta_n + \gamma_n \in [a, b] \subset (0, 1)$ and also $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that T_1, T_2 and T_3 satisfying the condition (II) and $\mathcal{F} \neq \emptyset$. Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. Let $p \in \mathcal{F}$. Then, for $i = 1, 2, 3$ we have $p \in P_{T_i}(p) = \{p\}$. Also, we have

$$\|z_n - p\| \leq dist(z_n, P_{T_1}(p)) \leq H(P_{T_1}(x_n), P_{T_1}(p)) \leq \|x_n - p\|$$

and

$$\|u_n - p\| \leq dist(u_n, P_{T_2}(p)) \leq H(P_{T_2}(w_n), P_{T_2}(p)) \leq \|w_n - p\|,$$

and

$$\|v_n - p\| \leq dist(v_n, P_{T_3}(p)) \leq H(P_{T_3}(y_n), P_{T_3}(p)) \leq \|y_n - p\|.$$

Now, by similar argument as in the proof of Theorem 3.1, $\lim_{n \longrightarrow \infty} \|x_n - q\|$ exists. Also we get a sequence $\{p_k\} \in \mathcal{F}$ which converges to some $q \in E$. Since for each $i = 1, 2, 3$

$$dist(p_k, T_i(q)) \leq dist(p_k, P_{T_i}(q)) \leq H(P_{T_i}(p_k), P_{T_i}(q)) \leq \| q - p_k \|,$$

and $p_k \longrightarrow q$ as $k \longrightarrow \infty$, it follows that $dist(q, T_i(q)) = 0$ for $i = 1, 2, 3$. Hence $q \in \mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n \longrightarrow \infty} \|x_n - q\|$ exists, we conclude that $\{x_n\}$ converges strongly to q. □

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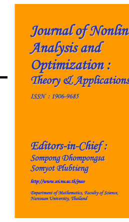

AN APPROACH FOR SIMULTANEOUSLY DETERMINING THE OPTIMAL TRAJECTORY AND CONTROL OF A HEATING SYSTEM$^\diamond$

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ABSTRACT. In the recent decade, a considerable number of optimal control problems have been solved successfully based on the properties of the measures. Even the method, has many useful benefits, in general, it is not able to determine the optimal trajectory and control at the same time; moreover, it rarely uses the advantages of the classical solutions of the involved systems. In this article, for a heating control system, we are going to present a new solution path. First, by considering all necessary conditions, the problem is represented in a variational format in which the trajectory is shown by a trigonometric series with the unknown coefficients. Then the problem is converted into a new one that the unknowns are the mentioned coefficients and a positive Radon measure. It is proved that the optimal solution is exited and it is also explained how the optimal pair would be identified from the results deduced by a finite linear programming problem. A numerical examples is also given.

KEYWORDS : Simultaneously Determining; Optimal Trajectory; Control of a Heating System.
AMS Subject Classification: 49QJ20, 49J45,49M25, 76D33.

1. INTRODUCTION

According to an idea of L. C. Young, by transferring the problem in to a theoretical measure optimization, in 1986 Rubio introduced a powerful method for solving optimal control problems ([11]). The important properties of the method (globality, automatic existence theorem a linear treatment even for extremely nonlinear problems, ...) caused it to be applied for the wide variety of problems. Even the method has been used frequency for solving several kinds of problems, like [3], [1], [4] and [7], but at least two important points were not considered in applying the method yet. Generally the method can not be able to produce the acceptable optimal trajectory and control directly at the same time; and moreover, the classical format of the system solution, usually is not taken into account. Therefore, there is no any possibility to use this important fact and their related literatures in the analysis of the system.

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In this article, we try to bring attention these two facts; for these perposes, an optimal control problem governed by a one-dimensional wave equation system (a heating system) with initial and boundary conditions and an integral criterion is considered as a sample. Regarding a general format of the classical solution, the problem is presented in a variational format and then by a doing deformation it is converted into a measure theoretical one with some positive coefficient. Next, extending the underlying space, using the density properties and applying some discretization scheme cause to approximate the optimal pair as a result of a finite linear programming. The approach would be improved if the number of constraints and nods are exceeded . In this manner, the optimal trajectory and control is determined at the same time.

2. THE CONTROL SYSTEM

For all $t \in [0, T] \subset R$ the deflection of the shell at an arbitrary point x in time t, is denoted by $u(t, x)$ which satisfies in the following equation (see [7] and [13]):

$$u_t = cu_{xx} \ (1)$$

where c is a constant that it dependents on physical structure of the heating. Since the heating at its boundary there is no any heating at these points and hence we have the following boundary conditions:

$$u(t, 0) = u(t, L), \ \forall \ 0 \leq x \leq L. \ (2)$$

If the initial deflection are denoted by $f(x)$, then the initial conditions of the system are defined as:

$$f(x) = u(0, x); \ (3)$$

According to [8], $u(t, x)$ belongs to the class of homogeneous Cauchy problems. Thus it can have a unique bounded classic solution on $D = [0, T] \times [0, L]$, if $f(x)$, and the different orders of their partial derivatives are continuous. Moreover, as mentioned in [7], the one-dimensional equation problem have the following Fourier series as the solution:

$$u(t, x) = \sum_{n=1}^{\infty} A_n e^{-k(\frac{n^2\pi^2}{L^2})t} sin \frac{n\pi}{L} x \ (4)$$

Where

$A_n = \frac{2}{L} \int_0^L f(x) sin \frac{n\pi}{L} x \ dx$ for $n = 1, 2, \ldots$

Convergence of the above series (to a bounded solution of the problem) indicates that one can approximate the solution by the finite number terms of the series.
Controlling the one-dimensional of heat system, needs to inter some power to the system somehow. This fact can be done by inserting a shock on a specified place of the shell. Without lose of the generality, suppose that the place for inserting the shock be $\frac{L}{2}$. Now, let $V \subseteq R$ be a bounded set, and $\vartheta = \vartheta(t) : [0, t] \to V$ be a Lebesgue-measurable control function. Moreover suppose
$f_0 = f_0(t, x, \vartheta(t)) : D \times V \to R$ be a continues function.

The aim is to find the optimal pair of the trajectory and control functions $u(t, x)$ and $\vartheta = \vartheta(t)$ simultaneously, as an optimal solution for the following control problem:

$Min: \ I(P) \equiv \int_D f_0(t,x,\vartheta)dA$

$S.to: \ u_t = cu_{xx}; \ (5-1)$

$u(t,0) = u(t,L) = 0; \ (5-2)$

$f(x) = u(0,x); \ (5-3) \ (5)$

$u_t|_{\frac{L}{2}} = \vartheta(t). \ (5-4)$

We remind that the objective functional $\int_D f_0(t,x,\vartheta)dA$ can explain error of the System or so on.

Definition: As a classical form, a pair $P \equiv (u,\vartheta)$ is called admissible if conditions (5-2) to (5-4) are satisfied, and u be a bounded solution of (5-1). The set of all admissible pairs is denoted by P .

Therefore, we wish to find the admissible minimizer pair for the functional $I(P)$ over P. It is necessary to indicate that the controllability and the observability of the above system were discussed in many references such as [3]. Thus, we can suppose that P is nonempty. In the next, we will try to find the solution of (5) according to the trigonometrical series and use of the embedding method, as mentioned in section one. For reaching to our purposes, we need to present the problem in a new formulation.

3. NEW REPRESENTATION OF THE PROBLEM

For a fixed N, the optimal trajectory of (5) can be approximated by the first N terms of a trigonometric series; i.e.:

$u(t,x) = \sum_{n=1}^{N} A_n e^{-k(\frac{n^2\pi^2}{L^2})t} sin\frac{n\pi}{L}x \ (6)$

where A_n for $n = 1, 2, ..., N$ are unknown real coefficients that must be determined under the conditions (5-2) to (5-4). Since this coefficients are unknown, the amount of the eliminated part of the solution in (4) (the tail of the series), can be considered in the calculated amount for unknowns. We define:

$$\overline{u}^n(t,x) = A_n e^{-k(\frac{n^2\pi^2}{L^2})t} sin\frac{n\pi}{L}x \ (7)$$

one can easily show that $\overline{u}^n_{xxxt}(t,x) = -\frac{n^2\pi^2}{L^2}\overline{u}^n_{xt}(t,x)$ and then we have:

$$u_{xxxt}(t,x,y) = \sum_{n=1}^{N} \overline{u}^n_{xxxt}(t,x) = \sum_{n=1}^{N} -\frac{n^2\pi^2}{L^2}\overline{u}^n_{xt}(t,x);$$

then, integrating over $[0,T] \times [0,\frac{L}{2}]$ gives

$$\int_0^T \int_0^{\frac{L}{2}} u_{xxxt}dxdt = \sum_{n=1}^{N} -\frac{n^2\pi^2}{L^2} \int_0^T \int_0^{\frac{L}{2}} \overline{u}^n_{xt}(t,x)dxdt;$$

By regarding the continuity of $\overline{u}^n(t,x)$and its partial derivatives, one can change the order of the integration. Then, by some simple calculations, the constraint (5-4) can be appeared as the following new format:

$$\int_0^T \vartheta(t)dt = \sum_{n=1}^{N} A_n sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1). \ (8)$$

Now, by considering the equations (6) and (8), the problem (5) can be represented as the new following exhibition:

$Min: \ I(P) = \int_D f_0(t, x, \vartheta(t))dA$

$S.to: \ f(x) = \sum_{n=1}^{N} A_n sin\frac{n\pi x}{L}$ (9)

$\int_0^T \vartheta(t)dt = \sum_{n=1}^{N} A_n sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1).$

Let $x_0 = 0, x_1, x_2, ..., x_m = L$ be belong to a dense subsequence of $[0, L]$. If $m \to \infty$ then obviously the solution of the following problem converges to the solution of (10). Thus, for a suitable numbers m, the solution of the problem (9) can be approximated by the solution of the following one:

$Min: \ I(P) = \int_0^T [\sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f_0(\vartheta, x_i, t)dx]dt = \int_0^T F_0(t, \vartheta)dt$

$S.to: \ f(x) = \sum_{n=1}^{N} A_n sin\frac{n\pi x}{L}$ (10)

$\int_0^T \vartheta(t)dt = \sum_{n=1}^{N} A_n sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1).$

4. METAMORPHOSIS

To show the existence of the optimal solution of (10) and introducing a simple linear treatment for obtaining it, we follow [2] and [10] by applying some new ideas. Hence, we do the metamorphosis step in this section, to deform the problem and define it in a new space in which it has many advantages.

Let $\Omega = [0, T] \times V$, then for each $(u, \vartheta) \in P$, $\Lambda_\vartheta : C(\Omega) \to R$ that $\Lambda_\vartheta(h) = \int_0^T h(t, \vartheta)dt$ be a positive continuous linear functional. Based on the Riesz Representation Theorem ([13]), there exists a positive Radon measure $\mu_\vartheta \in M^+(\Omega)$ (the space of all positive Radon measures on Ω) so that for all $h \in C(\Omega)$, $\mu_\vartheta(h) = \int_\Omega h \ d\mu = \Lambda_\vartheta(h)$. Therefore, problem (9) is changed into a new one in which its unknowns are the coefficients A_n ($n = 1, 2, ..., N$) and a positive Radon measure, say μ , produced by the Riesz Representation Theorem. To be sure that we are able to elastraite the global solution, like [10], we enlarge the underlying space and seek on a subset of $M^+(\Omega)$ which is defined by the last equations of (9). This means that instead of searching for the optimal measure, say μ^* , between the introduced measures from the Risez Representation Theorem, we seek in the set of all positive Radon measures in which they just satisfy in the conditions of (10); hence, the induced measures from the Risez theorem are belonged in this set and therefore our minimization is global. Thus we try to solve the following problem:

$Min: \ \mu(F_0)$

$S.to: \ f(x_i) = \sum_{n=1}^{N} A_n sin\frac{n\pi x_i}{L};$

$\mu(\vartheta) = \sum_{n=1}^{N} A_n sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1); \ (11)$

$i = 1, 2, \ldots, m;$

$\mu(\xi) = a_\xi, \ \forall \xi \in C^1(\Omega).$

where the unknown measure μ belongs to $M^+(\Omega)$, $C^1(\Omega)$ is a subset of functions in $C(\Omega)$ that depends only on variable t and a_ξ is the Lebesgue integral of ξ over $[0, T]$; indeed the last set of equations is added to guarantee this property of an admissible measure that its projection on the real line is the Lebesgue measure (see for instance [11] and [2]).
Now, suppose that A_n's are obtained by solving the following linear equations:

$$f(x_i) = \sum_{n=1}^{N} A_n sin\frac{n\pi x_i}{L}; \ i = 1, 2, \ldots, m$$

Then by substituting the obtained coefficients in the third equation of (11), the problem is converted into one in which the unknown is just the measure $\mu \in M^+(\Omega)$ which satisfied in the last two conditions of (11). Thus, if Q be the space of all measure in $M^+(\Omega)$ which satisfied the conditions of (11), as Rubio shown in [11], Q is compact in the sense of weak* topology; moreover $\mu \to \mu(F_0)$ is a continuous function. Since each continuous function has an infimum on a compact space there exists an optimal measure, say μ^*, which minimizes the objective function of (11) and together with the obtained unknown are satisfied in the conditions of (11).Thus , we have the following proposition.

Proposition 1: Problem (11) has the optimal solution.

Proof. see [9]. □

By regarding the result of Rosenblooms works in [10], the optimal measure has the form

$$\mu^* = \sum_{j=1}^{M} \alpha_j \delta(z_j) \ (12)$$

where $\delta(z_j)$ is an atomic measure with the support of the singleton set $\{z_j\}$, α_j is a nonnegative real coefficient, and z_j is a point belongs to Ω. Using (12) in (11), changes the problem into a nonlinear one in which its unknowns are the coefficients A_n, α_j, and the supporting points z_j for $n = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. We know that, by doing a discretization on Ω with nodes $z_j = (t_j, \vartheta_j), \ j = 1, 2, \ldots, M$ in a dense subset of $\omega \subseteq \Omega$, the supporting points can be determined; hence the problem can be converted into a linear one. But, regarding the last set of equations in (11), the number of constraints are still infinite. It would be more convenient if somehow we could change the problem into a finite linear programming one. Then in the next step of approximation, by choosing a dense countable subset of $C^1(\Omega)$ and then selecting a finite number of its elements as ξ_h for $h = 1, 2, ..., K$, the total number of the constraints of the problem would be finite. Therefore, the solution of (11) can be approximated by the following linear programming problem with variables $\alpha_j, \ j = 1, 2, \ldots, M$, and A_n^+, A_n^- that $A_n = A_n^+ - A_n^-$.

$Min: \ \sum_{j=1}^{M} \alpha_j F_0(t_j, \vartheta_j)$

$S.\,to: \ f(x_i) = \sum_{n=1}^{N}(A_n^+ - A_n^-) sin\frac{n\pi x_i}{L}; \ (11)$

$\sum_{j=1}^{M} \alpha_j \vartheta_j = \sum_{n=1}^{M}(A_n^+ - A_n^- sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1);$

$\sum_{j=1}^{M} \alpha_j \xi_h(t_j, \vartheta_j) = a_h, \ h = 1, 2, ..., K.$

$\alpha_j \geq 0, \ j = 1, 2, \ldots, M;, A_n^+, A_n^- \geq 0, \ n = 1, 2, ..., N;$

The density properties of the applied sets, indicate that if N, m, n, h chosen bigger and bigger, the optimal solution of (13) convergence into the solution of (10), or more precisely (5)(see [10]). Therefore, the solution of (5) can be approximated from the results of the finite linear programming problem (13).
To set up (13), as mentioned in [10] and some other literatures (like [4]), for $h = 1, 2, \ldots, K$ we choose $0 = t_0 \leq t_1 \leq \ldots t_K = T$ and $D_h = [t_{h-1}, t_h)$ for $h = 1, 2, ..., K$ and $D_h = [t_{K-1}, t_K]$; hence $\bigcup_{h=1}^{K} D_h = [0, T]$. Now we define the function ξ_h as follow:

$$\xi_k(t_j, \vartheta_j) = \begin{cases} 1 \ t_j \in D_h \\ 0 \ otherwise \end{cases}$$

although these class of functions are not continuous but, when $K \to \infty$ every functions in $C^1(\Omega)$ can be approximated by a finite linear combination of these functions (see [5]). In this manner, for an arbitrary function ξ_h , we have
$a_h = \int_0^T \xi_h \, dt = t_h - t_{h-1}$. Now by solving the linear programming problem (13), one can obtain the optimal coefficients $\alpha_j^*, \ A_n^*$ at the same time. Then, according to (6) and the explained method in [10], simultaneously the optimal trajectory and control functions can be determined, which is one of the main aim of this paper.

5. A NUMERICAL EXAMPLE

Based on the explained new approach, we incline to find the optimal pair of the trajectory and control for vibrating system in (5) defined by:

$$u_t = u_{xx}$$
$$u(t, 0) = u(t, L) = 0;$$
$$u(0, x) = x^2;$$

with the performance criterion defined by $F_0(t, \vartheta) = (\vartheta - t^2)^2$; indeed, here was supposed that $c = 1, \ t \in [0, 0.01], \ D = [0, 1] \times [0, 0.01], U = [-0.4, 0.902], \ f(x) = x^2$. Also we choose $N = 20, \ m = 14$. Then for discretization on Ω we chose 30 value for ϑ_l in U as $\vartheta_l = -0.4 + \frac{1.302l}{30}$, 30 value for t_j in $[0, 1]$ as $t_j = \frac{j}{30}$ and 14 value for x_i in $[0, 1]$ as $x_i = \frac{i}{m+1}$, for $l, j = 1, 2, ..., 30$ as also $i = 1, 2, ..., 14$. Therefore, for solving the problem, a similar linear programming problem like (13) with 980 variables was established as follow:

$Min: \ \sum_{j=1}^{900} \alpha_j (\vartheta - t^2)^2$

$s.\,to: \ \sum_{n=1}^{20}(A_n^+ - A_n^-) sin\frac{n\pi x_i}{2} = x_n^2 \ n = 1, 2, ..., 14;$
$\sum_{j=1}^{900} \alpha_j \vartheta_j - \sum_{n=1}^{20}(A_n^+ - A_n^- sin\frac{n\pi}{2}(e^{-k(\frac{n^2\pi^2}{L^2})T} - 1) = 0;$

$$\alpha_1+\alpha_2+\ldots+\alpha_{60}=0.1;\ \alpha_{61}+\alpha_{62}+\ldots+\alpha_{150}=0.1;\ \alpha_{631}+\alpha_{632}+\ldots+\alpha_{690}=0.1;$$
$$\alpha_{151}+\alpha_{152}+\ldots+\alpha_{270}=0.1;\ \alpha_{271}+\alpha_{272}+\ldots+\alpha_{330}=0.1;\ \alpha_{691}+\alpha_{692}+\ldots+\alpha_{780}=0.1;$$
$$\alpha_{331}+\alpha_{332}+\ldots+\alpha_{420}=0.1;\ \alpha_{421}+\alpha_{422}+\ldots+\alpha_{540}=0.1;\ \alpha_{781}+\alpha_{782}+\ldots+\alpha_{900}=0.1;$$
$$\alpha_{541}+\alpha_{542}+\ldots+\alpha_{630}=0.1;$$
$$A_n^+, A_n^-,\ \alpha_r \geq 0,\ r=1,\ldots,900,\ n=1,2,\ldots,20;$$

Then we applied the subroutine **DLPRS** from **IMSL** library of **Compaq Visual Fortran** to solve the above linear programming problem by Revised Simplex Method. The optimal value of the objective function was obtained as 0.0000022463. The optimal value of the variables were as follows:

$A_1^* = 0.37628;\ A_2^* = -0.31364$
$A_3^* = 0.19562;\ A_4^* = -0.14974$
$A_5^* = 0.11342;\ A_6^* = -0.091759$
$A_7^* = 0.073306;\ A_8^* = 0.060027$
$A_8^* = 0.0481107;\ A_{10}^* = 0$
$A_{11}^* = 0.28193;\ A_{12}^* = 0$
$A_{13}^* = 0;\ A_{14}^* = 0$
$A_{15}^* = 0;\ A_{16}^* = 0.0070069$
$A_{17}^* = 0;\ A_{18}^* = 0.021661$
$A_{19}^* = 0.25242\ A_{20}^* = 0.03849;$

$\alpha_1^* = \alpha_2^* = ... = \alpha_{10}^* = 0.1$

From the obtained optimal values, the nearly optimal piecewise-constant control was calculated as the explained manner in [11]. Also, by regarding (6) and the above obtained optimal coefficient, the trajectory function $u^*(t,x)$ was determined by :

$$u^*(t,x) = \sum_{n=1}^{20} A_n^* e^{-k(\frac{n^2\pi^2}{L^2})t} sin\frac{n\pi}{L}x$$

The obtained nearly optimal control and trajectory functions are plotted in figures 1 and 2 respectively (since the optimal trajectory is a variable function, it was plotted for some especifeid times).

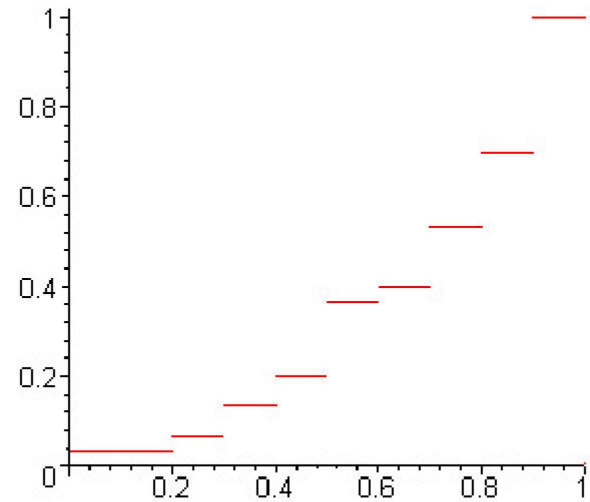

Figure 1:The Optimal Control

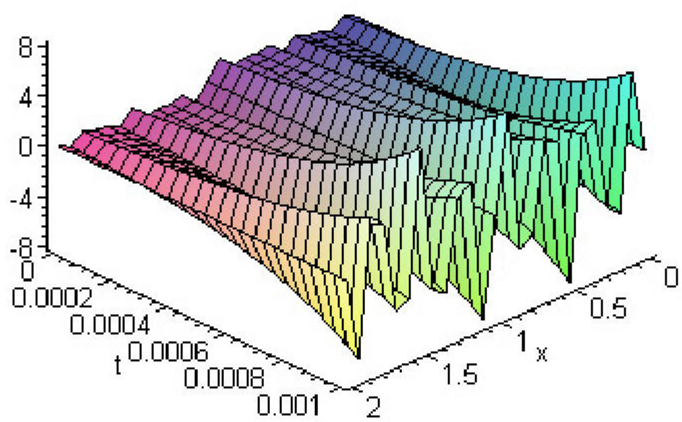

Figure 2: The Optimal Trajectory

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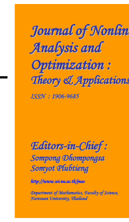

OPTIMALITY CONDITIONS FOR (G, α)-INVEX MULTIOBJECTIVE PROGRAMMING

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ABSTRACT. In this paper, a generalization of convexity, namely (G, α)-invexity, is considered in the case of nonlinear multiobjective programming problems where the functions constituting vector optimization problems are differentiable. Two auxiliary programming problems are constructed to present the modified Kuhn-Tucker necessary optimality conditions for (CVP). With the help of auxiliary programming problems (G-CVP), the relation between (CVP) and (G-CVP) is discussed; while with the help of (φ_G P), a new Kuhn-Tucker necessary condition for (CVP) is presented. Furthermore, the sufficiency of the introduced G-Karush-Kuhn-Tucker(G-Kuhn-Tucker) necessary optimality conditions, for nonconvex multiobjective programming probleminvolving (G, α)-invex functions, is proved.

KEYWORDS : (G, α)-invexity; Kuhn-Tucker constraint qualification; (weakly) efficient solution; G-Kuhn-Tucker necessary optimality conditions.
AMS Subject Classification: 90C29, 90C46.

1. INTRODUCTION

Convexity plays a central role in many aspects of mathematical programming including analysis of stability, sufficient optimality conditions and duality. Based on convexity assumptions, nonlinear programming problems can be solved efficiently. There have been many attempts to weaken the convexity assumptions in order to treat many practical problems. Therefore, many concepts of generalized convex functions have been introduced and applied to mathematical programming problems in the literature [1, 2, 10]. One of these concepts, invexity, was introduced by Hanson in [7]. Hanson has shown that invexity has a common property in mathematical programming with convexity that Karush-Kuhn-Tucker conditions are sufficient for global optimality of nonlinear programming under the invexity assumptions. Ben-Israel and Mond [6] introduced the concept of pre-invex functions which is a special case of invexity.

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Recently, Antczak extended further invexity to G-invexity [3] for scalar differentiable functions and introduced new necessary optimality conditions for differentiable mathematical programming problem. Antczak also applied the introduced G-invexity notion to develop sufficient optimality conditions and new duality results for differentiable mathematical programming problems. Furthermore, in the natural way, Antczak's definition of G-invexity was also extended to the case of differentiable vector-valued functions. In [4], Antczak defined vector G-invex (G-incave) functions with respect to η, and applied this vector G-invexity to develop optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints. He also established the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable vector optimization problems under the Kuhn-Tucker constraint qualification [4]. With this vector G-invexity concept, Antczak proved new duality results for nonlinear differentiable multiobjective programming problems[5]. A number of new vector duality problems such as G-Mond–Weir, G-Wolfe and G-mixed dual vector problems to the primal one were also defined in [5].

Motivated by [4, 5, 9], we present new classes of generalized convexity, namely vector (G,α)-invexity, in this paper. Basing on this new vector (G,α)-invexity, we have managed to deal with nonlinear programming problems under some assumptions. The rest of the paper is organized as follows: In section 2, we present concepts regarding vector (G,α)-invexity, and discuss In section 3, we firstly present G-Karush-Kuhn-Tucker necessary optimality conditions for mathematical programming problems with a different method from Antczak's one in [4]. Moreover, with the vector (G,α)-invexity assumption, we prove G-Karush-Kuhn-Tucker sufficient optimality conditions for mathematical programming problems.

2. VECTOR (G,α)-INVEX FUNCTIONS

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper. For any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T$, we define:

$$x > y \text{ if and only if } x_i > y_i, \text{ for } i = 1, 2, \cdots, n;$$
$$x \geqq y \text{ if and only if } x_i \geq y_i, \text{ for } i = 1, 2, \cdots, n;$$
$$x \geqslant y \text{ if and only if } x_i \geq y_i, \text{ for } i = 1, 2, \cdots, n, \text{ but } x \neq y;$$
$$x \ngtr y \text{ is the negation of } x > y.$$

We say that a vector $z \in \mathbb{R}^n$ is negative if $z \leqq 0$ and strictly negative if $z < 0$.

Let $f = (f_1, \cdots, f_k) : X \to \mathbb{R}^k$ be a vector-valued differentiable function defined on a nonempty set $X \subset \mathbb{R}^n$, $I_{f_i}(x)$, $i = 1, \cdots, k$, be the range of f_i, that is, the image of X under f_i. Let $G_f = (G_{f_1}, \cdots, G_{f_k}) : R \to \mathbb{R}^k$ be a vector-valued function such that any of its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain.

Definition 2.1. Let $f = (f_1, \cdots, f_k) : X \to \mathbb{R}^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset \mathbb{R}^n$, $I_{f_i}(x)$, $i = 1, \cdots, k$, be the range of f_i. If there exist a differentiable vector-valued function $G_f = (G_{f_1}, \cdots, G_{f_k}) : R \to \mathbb{R}^k$ such that any of its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain, a vector-valued function $\eta : X \times X \to \mathbb{R}^n$ and real function $\alpha_i : X \times X \to \mathbb{R}_+ (i \in K)$ such that, for all $x \in X (x \neq u)$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq \alpha_i(x,u) G'_{f_i}(f_i(u)) \nabla f_i(u) \eta(x,u), i = 1, \cdots, k \tag{2.1}$$

then f is said to be a (strictly) vector (G_f,α)-invex function at u on X (with respect to η) (or shortly, (G_f,α)-invex function at u on X), where $\alpha := (\alpha_1,\alpha_2,\cdots,\alpha_n)^T$. If (2.1) is satisfied for each $u \in X$, then f is vector (G_f,α)-invex on X with respect to η.

Remark 2.2. In order to define an analogous class of (strictly) vector (G_f,α)-incave functions with respect to η, the direction of the inequality in the definition of these functions should be changed to the opposite one.

Remark 2.3. (1) If f is (G_f,α)-invex function, then, by Definition 2.1 and definition of α-invex in [9], $G_f(f)(x) = (G_{f_1}(f_1(x)), G_{f_2}(f_2(x)),\cdots,G_{f_k}(f_k(x))$, $G_f(f)$ is α-invex.

(2) If $G_{f_i}(a) = a, \forall a \in \mathbb{R}$, then (G_f,α)-invex function is α-invex defined in [9].

(3) If $\alpha_i(x,u) = 1, \forall\, x,u \in X,\ i \in K$, then (G_f,α)-invex function is vector G_f-invex defined in [4]. Further, if $|K| = 1$, then (G_f,α)-invex function is G_f-invex defined in [3].

Hence, the (G_f,α)-invex function is a generalization of α-invex and G_f-invex function.

In general, a multiobjective programming problem is formulated as the following vector minimization problem:

$$\begin{aligned}(CVP)\quad & \min f(x) := (f_1(x), f_2(x),\cdots,f_k(x)),\\ & s.t.\ g(x) := (g_1(x), g_2(x),\cdots,g_m(x)) \leqq 0\\ & h(x) := (h_1(x),h_2(x),\cdots,h_p(x)) = 0\\ & x \in X\end{aligned}$$

where X is a nonempty set of $\mathbb{R}^n$, and f_i denotes a real-valued differentiable function on X. We denote by $K := \{1,2,\cdots,k\}$, $M := \{1,2,\cdots,m\}$ and $P := \{1,2,\cdots,p\}$.

Let $E_{CVP} = \{x \in X : g_j(x) \leq 0, j \in M, h_t(x) = 0, t \in P\}$ be the set of all feasible solutions for problem (CVP). Further, we denote by $J(\bar{x}) := \{j \in M : g_j(\bar{x}) = 0\}$ the set of constraint indices active at $\bar{x} \in E_{CVP}$.

For the convenience, we need the following vector minimization problem:

$$\begin{aligned}(G-CVP)\quad & \min G_f f(x) := (G_{f_1}(f_1(x), G_{f_2}(f_2(x)),\cdots,G_{f_k}(f_k(x)))\,,\\ & s.t.\ G_g g(x) := (G_{g_1}(g_1(x)), G_{g_2}(g_2(x)),\cdots,G_{g_m}(g_m(x))) \leqq G_g(0)\\ & G_h h(x) := \left(G_{h_1}(h_1(x)), G_{h_2}(h_2(x)),\cdots,G_{h_p}(h_p(x))\right) = G_h(0)\\ & x \in X\end{aligned}$$

where

$$G_g(0) := (G_{g_1}(0), G_{g_2}(0),\cdots,G_{g_m}(0))\,, G_h(0) := \left(G_{h_1}(0), G_{h_2}(0),\cdots,G_{h_p}(0)\right).$$

We denote by $E_{G-CVP} = \{x \in X : G_g g(x) \leq G_g(0), G_h h(x) = G_h(0)\}$, $J'(\bar{x}) := \{j \in M : G_{g_j} g_j(\bar{x}) = G_{g_j}(0)\}$. Then, it is easy to see that $E_{CVP} = E_{G-CVP}$ and $J(\bar{x}) = J'(\bar{x})$. So, we represent the set of all feasible solutions and the set of constraint active indices for either (CVP) or (G-CVP) by the notations E and $J(\bar{x})$, respectively.

Before studying optimality in multiobjective programming, one has to define clearly the concepts of optimality and solutions in multiobjective programming problem. Note that, in vector optimization problems there is a multitude of competing definitions and approaches. The dominated ones are now various scalarizations and (weak) Pareto optimality. The (weak) Pareto optimality in multiobjective programming associates the concept of a solution with some property that seems intuitively natural.

Definition 2.4. A feasible point $\bar{x}$ is said to be an efficient solution for a multiobjective programming problem (CVP) if and only if there exists no $x \in E$ such that

$$f(x) \leq f(\bar{x}).$$

Definition 2.5. A feasible point $\bar{x}$ is said to be a weakly efficient solution for a multiobjective programming problem (CVP) if and only if there exists no $x \in E$ such that

$$f(x) < f(\bar{x}).$$

Theorem 2.6. *Let $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Further, let $0 \in I_{g_j}(X)$, $j \in M$, and $0 \in I_{h_t}(X)$, $t \in M$. Then $\bar{x}$ is an efficient solution (a weakly efficient solution) for (CVP) if and only if $\bar{x}$ is an efficient solution (a weakly efficient solution) for (G-CVP).*

Proof Here, we only prove the case that $\bar{x}$ is an efficient solution, weakly efficient solution case can be proved in similar way.

"if" part, we prove that if $\bar{x}$ is an efficient solution for (G-CVP), then $\bar{x}$ is an efficient solution for (G-CVP). On the contrary, Let $\bar{x}$ be an efficient solution for (G-CVP) but not an efficient solution for (CVP). Then there exists $x_0 \in E$, such that

$$f(x_0) \leq f(\bar{x}).$$

That is

$$f_i(x_0) \leqslant f_i(\bar{x}), i = 1, \cdots, k$$

and there exists $i_0 \in K$ such that

$$f_{i_0}(x_0) < f_{i_0}(\bar{x}).$$

Note that the strictly monotonicity of G_{f_i}, $i = 1, \cdots, k$, we have

$$G_{f_i}(f_i(x_0)) \leqslant G_{f_i}(f_i(\bar{x}))$$

and

$$G_{f_i}(f_i(x_0)) < G_{f_i}(f_i(\bar{x})).$$

This contradict to the assumption that $\bar{x}$ be an efficient solution for (G-CVP).

The proof of "only if" part is similar to "if " part, we omitted it.

Example 2.7. We now consider the following multiobjective programming problem

$$\min f(x) := (f_1(x), f_2(x)) = \left(e^{x^2-4x}, \arctan x\right)$$
$$g(x) = \ln(x^2 - x + 1) \leq 0, x \in \mathbb{R}.$$

Let $G_{f_1}(f_1(t)) = \frac{1}{2}\ln t, G_{f_2}(f_2(t)) = \tan t, G_g(g(t)) = e^t$, and consider the following multiobjective programming problem:

$$\min G_f(f(x)) := (G_{f_1}(f_1(x)), G_{f_2}(f_2(x))) = \left(\frac{1}{2}\ln(e^{x^2-4x}), \tan(\arctan x)\right)$$
$$G_g(g(x)) = e^{\ln(x^2-x+1)} \leq e^0, x \in \mathbb{R}.$$

That is the following multiobjective programming problem:

$$\min G_f(f(x)) := (G_{f_1}(f_1(x)), G_{f_2}(f_2(x))) = \left(\frac{1}{2}(x^2 - 4x), x\right)$$

$$G_g(g(x)) = x^2 - x + 1 \leq 1, x \in [0, 1].$$

Note that $G_{f_1}(f_1(x)) = \frac{1}{2}(x^2 - 4x)$ is decreasing and $G_{f_2}(f_2(x)) = x$ is increasing on the interval $[0, 1]$, we can say $x = 0$ and $x = 1$ are efficient solution for G-CVP. Hence, by Theorem 2.6, $x = 0$ and $x = 1$ are efficient solution for CVP.

Definition 2.8. Let E be a set of all feasible solutions in the multiobjective programming problem (CVP) and $\bar{x} \in E$. The multiobjective programming problem (CVP) is said to satisfy the Kuhn-Tucker constraint qualification at $\bar{x}$ if,

$$C(E, \bar{x}) = \{d \in \mathbb{R}^n : \nabla g_j(\bar{x})d \leqq 0, j \in J(\bar{x}), \nabla h_t(\bar{x})d = 0, t \in P\}$$

where $C(E, \bar{x})$, the Bouligand tangent cone to E at $\bar{x}$, is defined as follows:

$$C(E, \bar{x}) = \left\{d \in \mathbb{R}^n : \exists x_k \in E, \lambda_k \in \mathbb{R}_+, \lim_{k\to\infty} x_k = \bar{x}, \lim_{k\to\infty} \lambda_k = 0, d = \lim_{k\to\infty} \frac{x_k - \bar{x}}{\lambda_k}\right\}$$

Proposition 2.9. *Let ϕ be a real strictly increasing and differentiable function defined on interval $(a, b) \subset \mathbb{R}$. Then*

$$\phi'(x) \geqslant 0, \forall\, x \in (a, b).$$

Denote $F_{CVP} =: \{d \in \mathbb{R}^n : \nabla g_j(\bar{x})d \leqq 0, j \in J(\bar{x}), \nabla h(\bar{x})d = 0\}$ and $F_{G-CVP} =: \{d \in \mathbb{R}^n : G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, j \in J(\bar{x}), G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, t \in P\}$.

Theorem 2.10. *Let $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Then $F_{CVP} \subset F_{G-CVP}$. Further, we assume that*

$$G'_{g_j}(g_j(\bar{x})) > 0, j \in J,$$
$$G'_{h_t}(h_t(\bar{x})) > 0, t \in P.$$

Then $F_{CVP} \supset F_{G-CVP}$.

Proof Since $G_{g_i}(j \in J)$, $G_{h_t}(t \in T)$ are strictly increasing functions, by Proposition 2.9, we have

$$G'_{g_j}(u) \geqslant 0, u \in I_{g_j}(X), j \in M$$
$$G'_{h_t}(u) \geqslant 0, u \in I_{h_t}(X), t \in P$$

Therefore, for $d \in F_{CVP}$, we have

$$G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, j \in M$$
$$G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, t \in P$$

That is $d \in F_{G-CVP}$. On the other hand, if $d \in F_{G-CVP}$, then $d \in F_{CVP}$ by assumption and the proof is complete.

Theorem 2.11. *Let $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Further, let $\bar{x} \in E$, $G'_{g_j}(g_j(\bar{x})) > 0, j \in J(\bar{x})$, and $G'_{h_t}(h_t(\bar{x})) > 0, t \in P$. Then the multiobjective programming problem (CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$ if only if the multiobjective programming problem (G-CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$.*

Proof From Definition 2.8 and Theorem 2.10, we get the desired result.

3. OPTIMALITY CONDITIONS IN MULTIOBJECTIVE PROGRAMMING

In [3], Antczak introduced the so-called G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable mathematical programming problem. In a natural way, he extended the so-called G-Karush-Kuhn-Tucker necessary optimality conditions to the vectorial case for differentiable multiobjective programming problems. In this section, we firstly prove G-Karush-Kuhn-Tucker necessary optimality which is Theorem 18 in [4] with a different technique. Moreover, we present a different G-Kuhn-Tucker necessary optimality conditions for differentiable multiobjective programming problems through a scalar assisted programming problem.

Theorem 3.1 (G-Karush-Kuhn-Tucker necessary optimality conditions)**.** *Let $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Let $\bar{x}$ be a weakly efficient solution (an efficient solution) for (CVP). Moreover, we assume that the multiobjective programming problem (CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Then, there exist $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\xi} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t=1}^{p} \bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0 \tag{3.1}$$

$$\bar{\xi}_j \left(G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x}))\right) \leqslant 0,\ j \in M, \forall x \in E \tag{3.2}$$

$$\bar{\lambda} \geq 0, \bar{\xi} \geqq 0 \tag{3.3}$$

Proof From Theorem 2.10, we have $F_{CVP} \subset F_{G-CVP}$. That is

$$\begin{cases} G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, & j \in J(\bar{x}) \\ G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, & t \in P \end{cases}$$

for all $d \in C(E, \bar{x})$. Since $\bar{x}$ is a weakly efficient solution (an efficient solution) for (CVP), then, by Theorem 2.6, $\bar{x}$ is a weakly efficient solution (an efficient solution) for (G-CVP). Note that X be a nonempty open set, therefore

$$\nabla(G_{f_i}(f_i))(\bar{x})d = G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})d \geq 0, \forall d \in \mathbb{R}^n, i \in K,$$

Therefore, the following system

$$\begin{cases} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) < 0, & i \in K \\ G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, & j \in J(\bar{x}) \\ G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, & t \in P \end{cases} \tag{3.4}$$

is inconsistent on $\mathbb{R}^n$.

Since the system (3.4) has no solution, then, from Motzkin's theorem [8], there exist $\lambda \in \mathbb{R}^k$, $\xi \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^p$ such that

$$\sum_{i \in K} \lambda_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \xi_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t \in P} \mu_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0$$

$$\xi_j \left(G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x}))\right) \leqslant 0,\ j \in J(\bar{x}), \forall x \in E$$

$$\lambda \geq 0, \xi \geqq 0.$$

Denote by $\bar{\lambda}_i = \lambda_i, i \in K$, $\bar{\xi}_j = \xi_j, j \in J(\bar{x})$, $\bar{\xi}_j = 0, j \in M/J(\bar{x})$, $\bar{\mu}_t = \mu_t, t \in P$, and we get the desired result.

Remark 3.2. In proof of Theorem 3.1, the method we used here is different from the one used by Antczak in [3].

Theorem 3.3 (G-Karush-Kuhn-Tucker necessary optimality conditions)**.** *Let $\bar{x}$ be a weakly efficient solution (an efficient solution) for (CVP), $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Moreover, we assume that the multiobjective programming problem (G-CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Then, there exist $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\xi} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t=1}^{p} \bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0 \tag{3.5}$$

$$\bar{\xi}_j \left(G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x}))\right) \leqslant 0, \ j \in M \tag{3.6}$$

$$\bar{\lambda} \geq 0, \bar{\xi} \geqq 0 \tag{3.7}$$

Proof Since $\bar{x}$ is a weakly efficient solution (an efficient solution) for (CVP), then, by Theorem 2.6, $\bar{x}$ is a weakly efficient solution (an efficient solution) for (G-CVP). Again, the multiobjective programming problem (G-CVP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Therefore, the system

$$\begin{cases} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})d < 0, & i \in K \\ G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, & j \in J, \\ G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, & t \in P \end{cases} \tag{3.8}$$

is inconsistent. The following part is similar to Theorem 3.1

In order to present a strongly G-Karush-Kuhn-Tucker necessary optimality conditions, namely G-Kuhn-Tucker necessary optimality conditions, we need the following scalar programming problem:

$$(\varphi_G P) \quad \min \quad \varphi_G(x) = \sum_{i=1}^{k} G_{f_i}(f_i(x)) \tag{3.9}$$

$$s.t. G_f(f(x)) \leqq G_f(f(\bar{x})) \tag{3.10}$$

$$G_g(g(x)) \leqq G_g(0) \tag{3.11}$$

$$G_h(h(x)) = G_h(0) \tag{3.12}$$

$$x \in E \tag{3.13}$$

Let $\bar{E} = \{x \in E : G_f(f(x)) \leqq G_f(f(\bar{x})), G_g(g(x)) \leqq G_g(0), G_h(h(x)) = G_h(0)\}$. we say that ($\varphi_G$P)satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$ if

$$C(\bar{E}, \bar{x}) = \{d \in \mathbb{R}^n : \nabla(G_f(f))(\bar{x})d \leqq 0, \nabla(G_g(g))(\bar{x})d \leqq 0, \nabla(G_h(h))(\bar{x})d = 0\}.$$

Theorem 3.4 (G-Kuhn-Tucker necessary optimality conditions)**.** *Let $\bar{x}$ be a weakly efficient solution (an efficient solution) for (CVP), $G_{f_i}(i \in K)$ be strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be strictly increasing function defined on $I_{g_j}(X)$ and $G_{h_t}(t \in P)$ be strictly increasing function defined on $I_{h_t}(X)$. Moreover, we assume that the scalar programming problem (φ_GP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Then, there exist $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\xi} \in \mathbb{R}^m$ and $\bar{\mu} \in \mathbb{R}^p$ such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t=1}^{p} \bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0 \tag{3.14}$$

$$\bar{\xi}_j \left(G_{g_j}(g_j(\bar{x})) - G_{g_j}(0)\right) \leqslant 0, \ j \in M \tag{3.15}$$

$$\bar{\lambda} > 0, \bar{\xi} \geqq 0 \tag{3.16}$$

Proof Since $\bar{x}$ is a weakly efficient solution (an efficient solution) for (CVP), then, by Theorem 2.6, $\bar{x}$ is a weakly efficient solution (an efficient solution) for (G-CVP). Further, we can prove that $\bar{x}$ is a optimal solution for (φ_G P). Therefore, from the Theorem 4.14 in [11],

$$\nabla(\varphi_G)(\bar{x})d \geq 0, \forall d \in C(\bar{E}, \bar{x}).$$

Again, the scalar programming problem (φ_GP) satisfies the Kuhn-Tucker constraint qualification at $\bar{x}$. Therefore, the following system:

$$\begin{cases} \nabla(\varphi_G)(\bar{x})d < 0, \\ \nabla(G_f(f))(\bar{x})d \leqq 0, \\ \nabla(G_g(g))(\bar{x})d \leqq 0, \\ \nabla(G_h(h))(\bar{x})d = 0 \end{cases} \tag{3.17}$$

is inconsistent. Note that

$$\nabla(\varphi_G)(\bar{x}) = \sum_{i=1}^{k} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}),$$

then the following system

$$\begin{cases} \left(\sum_{i=1}^{k} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})\right) d < 0 & \\ G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})d \leqq 0, & i \in K \\ G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})d \leqq 0, & j \in J, \\ G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})d = 0, & t \in P \end{cases} \tag{3.18}$$

is inconsistent. Hence, from Farkas's theorem, there exist $\lambda \in \mathbb{R}^k$, $\xi \in \mathbb{R}^{|J|}$ and $\mu \in \mathbb{R}^p$ such that

$$\sum_{i=1}^{k} G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{i\in K} \lambda_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j\in J(\bar{x})} \xi_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t\in P} \mu_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0$$

$$\sum_{i=1}^{k} \lambda_i \left(G_{f_i}(f_i(\bar{x})) - G_{f_i}(f_i(\bar{x}))\right) + \sum_{j\in J(\bar{x})} \xi_j \left(G_{g_j}(g_j(\bar{x})) - (G_{g_j}(0))\right) \leqslant 0,$$

$$\lambda \geqq 0, \xi \geqq 0.$$

That is

$$\sum_{i=1}^{k}(1+\lambda_i) G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j\in J(\bar{x})} \xi_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) \tag{3.19}$$

$$+ \sum_{t\in P} \mu_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x}) = 0$$

$$\xi_j \left(G_{g_j}(g_j(\bar{x})) - (G_{g_j}(0))\right) \leqslant 0, j \in J(\bar{x}), \lambda \geqq 0, \xi \geqq 0.$$

Denote by $\bar{\lambda}_i = 1 + \lambda_i$, $\bar{\xi}_j = \xi_j (j \in J(\bar{x}))$, $\bar{\xi}_j = 0 (j \in M/J(\bar{x}))$ and $\bar{\mu}_t = \mu_t$, we get the desired result.

Now, we establish the sufficient optimality conditions for multiobjective programming problems of such a type. In the following theorem, we assume that the functions constituting the considered vector optimization problem (CVP) belong to the introduced class of nonconvex functions. Then we prove that a feasible point $\bar{x}$, at which the G-Karush-Kuhn-Tucker necessary optimality conditions are fulfilled, is a weakly efficient solution.

Theorem 3.5 (G-Karush-Kuhn-Tucker sufficent optimality conditions)**.** *Let $\bar{x}$ be a feasible point for (CVP), $G_{f_i}(i \in K)$ be a differentiable real-valued strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be a differentiable real-valued strictly increasing function defined on $I_{g_j}(X)$, and $G_{h_t}(t \in P)$ be a differentiable real-valued strictly increasing function defined on $I_{h_t}(X)$, such that G-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\bar{x}$. Further, assume that f is vector (G_f, α)-invex with respect to η at $\bar{x}$ on X, g is vector (G_g, β)-invex with respect to the same function η at $\bar{x}$ on X, h_t $(t \in P^+(\bar{x}))$ is (G_{h_t}, γ_t)-invex with respect to the same function η at $\bar{x}$ on X, and h_t $(t \in P^-(\bar{x}))$ is (G_{h_t}, γ_t)-incave with respect to the same function η at $\bar{x}$ on X, where $P^+(\bar{x}) = \{t \in P : \bar{\mu}_t > 0\}$ and $P^-(\bar{x}) = \{t \in P : \bar{\mu}_t < 0\}$. Then $\bar{x}$ is a weakly efficient solution for (CVP).*

Proof Suppose, contrary to the result, that $\bar{x}$ is not a weakly efficient solution for (CVP). By Theorem 2.6, $\bar{x}$ is not a weakly efficient solution for (G-CVP). Hence, there exists $x_0 \in X$ such that

$$G_{f_i}(f_i(x_0)) < G_{f_i}(f_i(\bar{x})), i \in K \tag{3.20}$$

By the generalized invexity assumption of f, g and h, we have

$$G_{f_i}(f_i(x_0)) - G_{f_i}(f_i(\bar{x})) \geq \alpha_i(x_0, \bar{x})G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})\eta(x_0, \bar{x}), i \in K \tag{3.21}$$

$$G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(\bar{x})) \geq \beta_j(x_0, \bar{x})G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})\eta(x_0, \bar{x}), j \in M \tag{3.22}$$

$$G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x})) \geq \gamma_t(x_0, \bar{x})G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}), t \in P^+(\bar{x}) \tag{3.23}$$

$$G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x})) \leq \gamma_t(x_0, \bar{x})G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}), t \in P^-(\bar{x}) \tag{3.24}$$

Multiplying (3.22),(3.23) and (3.24) by $\xi_j(j \in M)$, $\mu_t(t \in P^+)$ and $\mu_t(t \in P^-)$, we get

$$\bar{\xi}_j(G_{g_j}(g_j(x_0)) - G_{g_j}(g_j(\bar{x}))) \geq \bar{\xi}_j\beta_j(x_0, \bar{x})G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})\eta(x_0, \bar{x}), j \in M \tag{3.25}$$

$$\bar{\mu}_t(G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x}))) \geq \bar{\mu}_t\gamma_t(x_0, \bar{x})G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}), t \in P^+ \tag{3.26}$$

$$\bar{\mu}_t(G_{h_t}(h_t(x_0)) - G_{h_t}(h_t(\bar{x}))) \geq \bar{\mu}_t\gamma_t(x_0, \bar{x})G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}), t \in P^- \tag{3.27}$$

From (3.2), (3.20), (3.21), (3.25), (3.26) and (3.27), we have

$$\begin{aligned}
&G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x})\eta(x_0, \bar{x}) < 0, i \in K \\
&\bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x})\eta(x_0, \bar{x}) \leq 0, j \in M \\
&\bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x} \leq 0, t \in P^+ \\
&\bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\eta(x_0, \bar{x}) \leq 0, t \in P^-
\end{aligned}$$

Note that $\lambda \geq 0$, then

$$\left(\sum_{i=1}^{k}\bar{\lambda}_i G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}) + \sum_{j=1}^{m}\bar{\xi}_j G'_{g_j}(g_j(\bar{x}))\nabla g_j(\bar{x}) + \sum_{t=1}^{p}\bar{\mu}_t G'_{h_t}(h_t(\bar{x}))\nabla h_t(\bar{x})\right)$$
$$\times \eta(x_0, \bar{x}) < 0$$

which contradicts the G-Karush-Kuhn-Tucker necessary optimality condition (3.1). Hence, $\bar{x}$ is a weakly efficient solution for (CVP), and the proof is complete.

Theorem 3.6 (G-Karush-Kuhn-Tucker sufficient optimality conditions)**.** *Let $\bar{x}$ be a feasible point for (CVP), $G_{f_i}(i \in K)$ be a differentiable real-valued strictly increasing function defined on $I_{f_i}(X)$, $G_{g_j}(j \in M)$ be a differentiable real-valued strictly increasing function defined on $I_{g_j}(X)$, and $G_{h_t}(t \in P)$ be a differentiable real-valued strictly increasing function defined on $I_{h_t}(X)$, such that G-Karush-Kuhn-Tucker necessary optimality conditions (3.1)-(3.3) are satisfied at $\bar{x}$. Further, assume that f is vector strictly (G_f, α)-invex with respect to η at $\bar{x}$ on X, g is vector (G_g, β)-invex with respect to the same function η at $\bar{x}$ on X, h_t $(t \in P^+(\bar{x}))$ is (G_{h_t}, γ_t)-invex with respect to the same function η at $\bar{x}$ on X, and h_t $(t \in P^-(\bar{x}))$ is (G_{h_t}, γ_t)-incave with respect to the same function η at $\bar{x}$ on X, where $P^+(\bar{x}) = \{t \in P : \bar{\mu}_t > 0\}$ and $P^-(\bar{x}) = \{t \in P : \bar{\mu}_t < 0\}$. Then $\bar{x}$ is an efficient solution for (CVP).*

Proof Proof for efficient optimality is similar to the proof of Theorem 3.5.

4. CONCLUSION

This paper represents a new type of generalized invexity, namely (G_f, α)-invexity. This new invexity unified the G-invexity and α-invexity presented in [4] and [9], respectively. We have constructed two auxiliary mathematical programmings (G-CVP) and (φ_G P), and have discussed the relations between programming (G-CVP) and (CVP). We have illustrated the relation result proved by a suitable example of the multiobjective programming problem (CVP) involving (G, α)-invex functions. With assisted mathematical programming (G-CVP), we have proved G-Karush-Kuhn-Tucker necessary optimality conditions for differentiable multiobjective programming problems with both inequality and equality constraints, through a easier method than presented in [4]. Furthermore, we have proved G-Kuhn-Tucker necessary optimality conditions for (CVP) with the help of auxiliary mathematical programming (φ_G P). As mentioned in [4], our statement of the so-called G-Kuhn-Tucker necessary optimality conditions established in this paper is more general than the classical Kuhn- Tucker necessary optimality conditions found in the literature. Also, we have proved the sufficiency of the introduced G-Karush-Kuhn-Tucker (G-Kuhn-Tucker) necessary optimality conditions for multiobjective programming problems involving (G, α)-invexity. More exactly, this result has been proved for such multiobjective programming problems in which the objective functions, the inequality constraints and the equality constraints (for which associated Lagrange multipliers are positive) are (G, α)-invex with respect to the same function η and the equality constraints (for which associated Lagrange multipliers are negative) are (G, α)-incave with respect to the same function η, but not necessarily with respect to the same function G. Hence, we can establish dual result, which is similar to ones presented in [5], for the programming problem (VCP).

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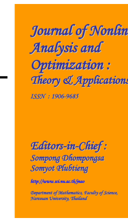

THE GENERALIZED B^r- DIFFERENCE RIESZ χ^2 SEQUENCE SPACES AND UNIFORM OPIAL PROPERTY

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ABSTRACT. We define the new generalized difference Riesz sequence spaces $\Lambda_r^{2q}(p, B^r)$ and $\chi_r^{2q}(p, B^r)$ which consist of all the sequences whose B^r- transforms are in the Riesz sequence spaces $r_\infty^q(p), r_c^q(p)$ and $r_0^q(p)$, respectively, introduced by Altay and Başar (2006). We examine some topological properties and compute the $\alpha-, \beta-$, and $\gamma-$ duals of the spaces $\Lambda_r^{2q}(p, B^r)$ and $\chi_r^{2q}(p, B^r)$. Fianlly, we determine the necessary and sufficient conditions on the matrix transform from the spaces $\Lambda_r^{2q}(p, B^r)$ and $\chi_r^{2q}(p, B^r)$ to the spaces Λ^2 and χ^2 and prove that sequence space $\chi_r^{2q}(p, B^r)$ have the uniform Opial property for $p_{mn} \geq 1$ for all $m, n \in \mathbb{N}$.

KEYWORDS : Gai Sequence; Analytic Sequence; Double Sequences; Riesz Sequence; Opial Property.
AMS Subject Classification: 40A05, 40C05, 40D05.

1. INTRODUCTION

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

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$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - \Upsilon|^{t_{mn}} = 1 \, for\, some\, \Upsilon \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and have given some inclusion relations.

Spaces are strongly summable sequences was discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary(single) sequence spaces to multiply sequence spaces.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and

$0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all\, finite\, sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.
If X is a sequence space, we give the following definitions:

(i)X' = the continuous dual of X;

(ii)$X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \, for\, each\, x \in X\right\}$;

(iii)$X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn}\, is\, convegent, \, for\, each\, x \in X\right\}$;

(iv)$X^\gamma = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn} x_{mn}\right| < \infty, for\, each\, x \in X\right\}$;

(v)$let\, X\, beanFK - space \supset \phi; \, then\, X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi)$X^\delta = \left\{a = (a_{mn}) : sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty, \, for\, each\, x \in X\right\}$;

$X^\alpha . X^\beta, X^\gamma$ are called $\alpha - (or Köthe - Toeplitz)$dual of $X, \beta - (or\, generalized - Köthe - Toeplitz)\, dual\, of X, \gamma -\, dual\, of\, X, \, \delta - dual\, of X\, respectively. X^\alpha$ is defined by Gupta and Kamptan [20]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by BaŞar and Altay in [42] and in the case $0 < p < 1$ by Altay and BaŞar in [43]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

A linear topological space X over the real field R is said to be a paranormed space if there is a subadditive function $g : X \to R$ such that $g(\theta) = 0, g(x) = g(-x)$ and scalar multiplication is continous; that is $|\alpha_{mn} - \alpha| \to 0$ and $g(x_{mn} - x) \to 0$ imply $g(\alpha_{mn} x_{mn} - \alpha x) \to 0$ for all α's in R and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that $p = (p_{mn})$ is a double analytic sequence of strictly positive real numbers with $supp_{mn} = H$ and $M = max(1, H)$.

Let λ and μ be two sequence spaces and $A = \left(a^{mn}_{k,\ell}\right)$ be an four dimensional infinite matrix of real numbers $\left(a^{mn}_{k,\ell}\right)$, where $m, n, k, \ell \in \mathbb{N}$. Then, we say A defines a matrix mapping from λ into μ and we denoe it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_{mn}) \in \lambda$ the sequence $Ax = \{(Ax)_{k\ell}\}$, the $A-$ transform of x, is in μ, where

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a^{mn}_{k\ell} x_{mn} \, (k, \ell \in \mathbb{N}) \tag{1.2}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.2) converges for each $k, \ell \in \lambda$. A sequence x is said to be $A-$ summable to α if Ax converges to α which is called as the $A-$ limit of x.

Let (q_{mn}) be a sequence of positive numbers and

$$Q_{k\ell} = \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} \, (k, \ell \in \mathbb{N}) \tag{1.3}$$

Then, the matrix $R^q = (r^{mn}_{k\ell})^q$ of the Riesz mean is given by

$$(r^{mn}_{k\ell})^q = \begin{cases} \frac{q_{mn}}{Q_{k\ell}} & \text{if } 0 \leq m, n \leq k, \ell \\ 0 & \text{if } (m, n) > k\ell \end{cases} \tag{1.4}$$

The double Riesz sequence spaces are defined as follows:

$\Lambda^{2q}_r(p) = \left\{x = (x_{mn}) \in w^2 : sup_{k\ell \in N} \left|\frac{1}{Q_{k\ell}} \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} (x_{mn})^{1/m+n}\right|^{p_{mn}} < \infty\right\}$,

$\chi^{2q}_r(p) = \left\{x = (x_{mn}) \in w^2 : lim_{k\ell \to \infty} \left|\frac{1}{Q_{k\ell}} \sum_{m=0}^{k} \sum_{n=0}^{\ell} q_{mn} ((m+n)! x_{mn})^{1/m+n}\right|^{p_{mn}} = 0\right\}$,

which are the sequence spaces of the sequences x whose R^q- transforms are in $\Lambda^2(p)$ and $\chi^2(p)$, respectively.

The main purpose of this paper is to introduce the B^r- difference Riesz sequence spaces $\Lambda^{2q}_r(p)$ and $\chi^{2q}_r(p)$ of the sequences whose $R^q B^r-$ transform are in $\Lambda^2(p)$ and $\chi^2(p)$, respectively and to investigate some topological and geometric properties of them. For simplicity, we take the matrix $R^q B^r = T$.

2. B^r- DIFFERENCE RIESZ DOUBLE SEQUENCE SPACES

Let us define the sequence $y = \{y_{k\ell}(q)\}$, which is used, as the $R^q B^r = T-$ transform of a sequence $x = (x_{mn})$, that is,

$$y_{k\ell}(q) = (Tx)_{k\ell} = \frac{1}{Q_{k\ell}} \sum_{m=0}^{k-1} \sum_{n=0}^{\ell-1} \left[\sum_{i=m}^{k} \sum_{j=n}^{\ell} b^{mn}_{k\ell} q_{ij} \left|(m+n)! x_{mn}\right|^{1/m+n}\right] + \frac{1}{Q_{k\ell}} q_{k\ell} \left|(k+\ell)! x_{k\ell}\right|^{1/k+\ell}, (k, \ell \in \mathbb{N}).$$

We define the B^r- difference Riesz sequence spaces
$\Lambda_r^{2q}(p, B^r) = \left\{x = (x_{mn}) \in w^2 : ((Tx)_{k\ell}) \in \Lambda^2(p)\right\},$
$\chi_r^{2q}(p, B^r) = \left\{x = (x_{mn}) \in w^2 : ((Tx)_{k\ell}) \in \chi^2(p)\right\}$
If $r = 1$ then they are reduced the spaces $\Lambda_r^{2q}(p, B)$ and $\chi_r^{2q}(p, B)$. If we take $B = \Delta$ then we have $\Lambda_r^{2q}(p, \Delta^r)$ and $\chi_r^{2q}(p, \Delta^r)$. If we take $B = \Delta$ and $r = 1$ then we have $\Lambda_r^{2q}(p, \Delta)$ and $\chi_r^{2q}(p, \Delta)$. If we take $p_{mn} = p = 1$ for all m, n then we have $\Lambda_r^{2q}(B^r)$ and $\chi_r^{2q}(B^r)$.
We have the following :

2.1. **DEFINTION.** Let $A = \left(a_{k,\ell}^{mn}\right)$ denotes a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k, \ell-$ th term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

2.2. **THEOREM.** $\chi_r^{2q}(p, B^r)$ is a complete metric space paranormed by g_B, defined by

$$g_B(x, y) = sup_{m,n\in\mathbb{N}} \left|\left((m+n)!\left((Tx)_{mn} - (Ty)_{mn}\right)\right)^{1/m+n}\right|^{p_{mn}/M},$$

g_B is a paranorm for the spaces $\Lambda_r^{2q}(p, B^r)$ only in the trivial case with $inf p_{mn} > 0$ when $\Lambda_r^{2q}(p, B^r) = \Lambda_r^{2q}(B^r)$.

Proof: We prove the theorem for the space $\chi_r^{2q}(p, B^r)$. The linearity of $\chi_r^{2q}(p, B^r)$ with respect to the coordinatewise addition and scalar multiplication that follow from the inequalities which are satisfied for $u, v \in \chi_r^{2q}(p, B^r)$

$$sup_{mn\in\mathbb{N}} \left|\frac{1}{Q_{mn}}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left[\sum_{i=j}^{m}\sum_{i=j}^{n} b_{mn}^{ij} q_{ij}\left|(i+j)!\left(u_{ij}+v_{ij}\right)\right|^{1/i+j}\right]\right|^{p_{mn}/M} +$$
$$\left|\frac{1}{Q_{mn}} q_{mn}\left|(m+n)!\left(u_{mn}+v_{mn}\right)\right|^{1/m+n}\right|^{p_{mn}/M} \leq$$
$$sup_{mn\in\mathbb{N}} \left|\frac{1}{Q_{mn}}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left[\sum_{i=j}^{m}\sum_{i=j}^{n} b_{mn}^{ij} q_{ij}\left|(i+j)!u_{ij}\right|^{1/i+j}\right]\right|^{p_{mn}/M} +$$
$$\left|\frac{1}{Q_{mn}} q_{mn}\left|(m+n)!u_{mn}\right|^{1/m+n}\right|^{p_{mn}/M} +$$
$$sup_{mn\in\mathbb{N}} \left|\frac{1}{Q_{mn}}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left[\sum_{i=j}^{m}\sum_{i=j}^{n} b_{mn}^{ij} q_{ij}\left|(i+j)!v_{ij}\right|^{1/i+j}\right]\right|^{p_{mn}/M} +$$
$$\left|\frac{1}{Q_{mn}} q_{mn}\left|(m+n)!v_{mn}\right|^{1/m+n}\right|^{p_{mn}/M}$$

and for any $\alpha \in \mathbb{R}$.

$$|\alpha|^{p_{mn}} \leq max\left\{1, |\alpha|^M\right\} \tag{2.1}$$

It is clear that $g_B(\theta) = 0$ and $g_B(-x) = g_B(x)$ for all $x \in \chi_r^{2q}(p, B^r)$. Again the inequality (2.1) yield the subadditivity of g_B and

$$g_B(\alpha u) \leq max\{1, |\alpha|\} g_B(u).$$

Let $\left\{x^{k\ell}\right\}$ be any sequence of the elements of the space $\chi_r^{2q}(p, B^r)$ such that

$$g_B\left(x^{k\ell} - x\right) \to 0$$

and $(\lambda_{k\ell})$ also be any sequence of scalars such that $\lambda_{k\ell} \to \lambda$, as $k, \ell \to \infty$. Then, since the inequality

$$g_B\left(x^{k\ell}\right) \leq g_B(x) + g_B\left(x^{k\ell} - x\right)$$

holds by sub additivity of g_B, $\left\{g_B\left(x^{k\ell}\right)\right\}$ is analytic, and thus we have

$$g_B\left(\lambda_{k\ell}x^{k\ell} - \lambda x\right) \leq |\lambda_{k\ell} - \lambda|^{1/M} g_B\left(x^{k\ell}\right) + |\lambda|^{1/M} g_B\left(x^{k\ell} - x\right),$$

which tends to zero as $k\ell \to \infty$. Hence, the scalar multiplication is continuous. Finally, it is clear to say that g_B is a paranorm on the space $\chi_r^{2q}(p, B^r)$. Moreover, we will prove the completeness of the space $\chi_r^{2q}(p, B^r)$. Let x^{ij} be a Cauchy sequence inthe $\chi_r^{2q}(p, B^r)$, where

$$x^{ij} = \left\{x_{mn}^{ij}\right\} = \begin{pmatrix} x_{01}^{ij} & x_{02}^{ij} & x_{03}^{ij}\cdots & x_{0n}^{ij} & 0 \\ x_{11}^{ij} & x_{12}^{ij} & x_{13}^{ij}\cdots & x_{1n}^{ij} & 0 \\ \vdots & & & & \\ x_{m1}^{ij} & x_{m2}^{ij} & x_{m3}^{ij}\cdots & x_{mn}^{ij} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \in \chi_r^{2q}(p, B^r).$$

Then, for a given $\epsilon > 0$, there exists a positive integer $k_0\ell_0(\epsilon)$ such that

$$g_B\left(x^{ij} - x^{pq}\right) < \epsilon$$

for all $i, j, p, q \geq k_0\ell_0(\epsilon)$. If we use the definition of g_B, we obtain for each fixed $m, n \in \mathbb{N}$ that

$$\left|\left(Tx^{ij}\right)_{mn} - (Tx^{pq})_{mn}\right| \leq sup_{mn\in\mathbb{N}}\left|\left((m+n)!\left(Tx^{ij}\right)_{mn} - (Tx^{pq})_{mn}\right)^{1/m+n}\right|^{p_{mn}/M} < \epsilon \tag{2.2}$$

for $i, j, p, q \geq k_0\ell_0(\epsilon)$ which leads us to the fact that

$$\begin{pmatrix} Tx_{mn}^{01} & Tx_{mn}^{02} & Tx_{mn}^{03}\cdots & Tx_{mn}^{oq} & 0 \\ Tx_{11}^{ij} & Tx_{12}^{ij} & Tx_{13}^{ij}\cdots & Tx_{1n}^{ij} & 0 \\ \vdots & & & & \\ Tx_{mn}^{p1} & Tx_{mn}^{p2} & Tx_{mn}^{p3}\cdots & Tx_{mn}^{pq} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

is a Cauchy sequence of real numbers for every fixed $m, n \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, so we write $\left(Tx^{ij}\right)_{mn} \to (Tx)_{mn}$ as $i, j \to \infty$. Hence, by using these infinitely many limits $(Tx)_{01}, (Tx)_{02}\cdots(Tx)_{mn}, 0, 0, \cdots$ we define the sequence

$$\begin{pmatrix} Tx_{01} & Tx_{02} & Tx_{03}\cdots & Tx_{on} & 0 \\ Tx_{11} & Tx_{12} & Tx_{13}\cdots & Tx_{1n} & 0 \\ \vdots & & & & \\ Tx_{m1} & Tx_{m2} & Tx_{m3}\cdots & Tx_{mn} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

From (2.2) with $p, q \to \infty$, we have

$$\left|\left(Tx^{ij}\right)_{mn} - (Tx)_{mn}\right| \leq \epsilon,\ i, j \geq k_0\ell_0(\epsilon) \tag{2.3}$$

for every fixed $m, n \in \mathbb{N}$. Since $x^{ij} = \left\{x_{mn}^{(ij)}\right\} \in \chi_r^{2q}(p, B^r)$,

$$\left|\left(Tx^{ij}\right)_{mn}\right|^{p_{mn}/M} < \epsilon,$$

for all $m, n \in \mathbb{N}$. Therefore, by (2.3), we obtain that

$$|(Tx)_{mn}|^{p_{mn}/M} \leq \left|(Tx)_{mn} - \left(Tx^{ij}\right)_{mn}\right|^{p_{mn}/M} + \left|\left(Tx^{ij}\right)_{mn}\right|^{p_{mn}/M} < \epsilon \tag{2.4}$$

for all $i, j \geq k_0\ell_0 (\epsilon)$. This shows that the sequence Tx belongs to the space $\chi^2 (p)$. Since $\left\{x^{(ij)}\right\}$ was an arbitrary Cauchy sequence, the space $\chi^{2q}_r (p, B^r)$ is complete. This completes the proof.

3. THE BASIS FOR THE SPACE $\chi^{2q}_r (p, B^r)$

In this section, we give sequence of the points of the spaces $\chi^{2q}_r (p, B^r)$ which form the basis for those space.

If a sequence space λ paranormed by h contains a sequence $(b_{k\ell})$ with the property that for every $x \in \lambda$, there is a unique sequence of scalars $(\alpha_{k\ell})$ such that

$$lim_{k\ell\to\infty} h\left(x - \sum_{m=0}^{k} \sum_{n=0}^{\ell} \alpha_{mn} \left((m+n)! x_{mn}\right)^{1/m+n}\right) = 0$$

then $(b_{k\ell})$ is a Schauder basis for $\lambda = 0$. The series $\sum\sum \alpha_{mn}\beta_{mn}$ which has the sum x is then called the expansion of x with respect to $(b_{k\ell})$ and written as $x = \sum\sum \alpha_{mn}\beta_{mn}$.

3.1. **THEOREM.** Let $\mu_{mn} (q) = (Tx)_{mn}$ for all $m, n \in \mathbb{N}$ and also $0 < p_{mn} \leq H < \infty$. Define the sequence $b_{mn} (q)$ of the elements of the space $\chi^{2q}_r (p, B^r)$ for every fixed $m, n \in \mathbb{N}$ then the sequence $\{b_{mn} (q)\}_{m,n\in\mathbb{N}}$ is a basis for the space $\chi^{2q}_r (p, B^r)$ and any $x \in \chi^{2q}_r (p, B^r)$ has a unique representation of the form

$$x = \sum_m \sum_n \mu_{mn} (q)\, b_{mn} (q) \tag{3.1}$$

Proof: It is clear that $\{b_{mn} (q)\} \subset \chi^{2q}_r (p, B^r)$, since

$$T b_{mn} (q) = \Im_{mn} \in \chi^2 (p) \,(for\, m, n \in \mathbb{N}) \tag{3.2}$$

for $0 < p_{mn} \leq H < \infty$, where $\Im_{mn}$ denotes the double sequence whose only nonzero term is 1 in the $(mn)^{th}$ place for each $m, n \in \mathbb{N}$. Let $x \in \chi^{2q}_r (p, B^r)$ be given. For every nonnegative integer r, s, we put

$$x^{[rs]} = \sum_{m=0}^{r} \sum_{n=0}^{s} \mu_{mn} (q)\, b_{mn} (q) \tag{3.3}$$

Then, we obtain by applying T to (3.3) with (3.2) that

$$Tx^{[rs]} = \sum_{m=0}^{r} \sum_{n=0}^{s} \mu_{mn} (q)\, T b_{mn} (q) = \sum_{m=0}^{r} \sum_{n=0}^{s} (T)_{mn} \Im_{mn},$$

$\left(R^q \left(x - x^{[rs]}\right)\right)_{ij} = \begin{cases} 0, & \text{if } 0 \leq i, j \leq r, s \\ (Tx)_{ij} & \text{if } (i, j) > (rs) \end{cases}$ Given $\epsilon > 0$, then there exists an integer $r_0 s_0$ such that

$$sup_{i,j\geq r,s} \left|(Tx)_{ij}\right|^{p_{mn}/M} < \frac{\epsilon}{2} \tag{3.4}$$

for all $r, s \geq r_0 s_0$. Hence,

$g_B \left(x - x^{[rs]}\right) = sup_{i,j\geq r,s} \left|(Tx)_{ij}\right|^{p_{mn}/M} \leq sup_{i,j\geq r_0,s_0} \left|(Tx)_{ij}\right|^{p_{mn}/M} < \frac{\epsilon}{2} < \epsilon$, for all $r, s \geq r_0 s_0$, which proves that $x \in \chi^{2q}_r (p, B^r)$ is represented as in (3.1).

To show the uniqueness of this representation, we suppose that there exists a representation

$$x = \sum_m \sum_n \lambda_{mn} (q)\, b_{mn} (q) .$$

Therefore the transformation $\chi^{2q}_r (p, B^r)$ to $\chi^2 (p)$ and also continuous we have $(Tx)_{k\ell} = \sum_m \sum_n \lambda_{mn} (q) \{T b_{mn} (q)\}_{k\ell} = \sum_m \sum_n \lambda_{mn} (q) \Im_{mn} = \lambda_{k\ell} (q) ; k, \ell \in \mathbb{N}$, which contradicts the fact that $(Tx)_{k\ell} = \mu_{mn} (q) \; for\, all\, m, n \in \mathbb{N}$. Hence the representation (3.1) of $x \in \chi^{2q}_r (p, B^r)$ is unique. This completes the proof.

4. UNIFORM OPIAL PROPERTY OF $\chi_r^{2q}(p, B^r)$ − DIFFERENCE RIESZ SEQUENCE SPACE

In this section, we investigate the uniform Opial property of the sequence spaces $\chi_r^{2q}(p, B^r)$.

The Opial propertyplays an important role in the study of weak convergence of iterates of mapping of Banach spaces and of the asympototic behavior of nonlinear semigroup. The Opial property is important because Banach spaces with this property have the weak fixed point property.

WE give the definition of uniform Opial property in a linear metric space and obtain that $\chi_r^{2q}(p, B^r)$ have uniform Opial property for $p_{mn} \geq 1$.

For a sequence $x = (x_{k\ell}) \in \chi_r^{2q}(p, B^r)$ and for $i, j \in \mathbb{N}$, we use the notation

$$x_{|ij} = \begin{pmatrix} x_{11} & x_{12} & x_{13}\cdots & x_{1j} & 0 \\ x_{21} & x_{22} & x_{13}\cdots & x_{2j} & 0 \\ \vdots & & & & \\ x_{i1} & x_{i2} & x_{i3}\cdots & x_{ij} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix}$$

and

$$x_{|\mathbb{N}-ij} = \begin{pmatrix} 0 & 0 & 0\cdots & 0 & 0 \\ x_{i+11} & x_{i+12} & x_{i+13}\cdots & x_{i+1j} & 0 \\ x_{i+21} & x_{i+22} & x_{i+23}\cdots & x_{i+2j} & 0 \\ x_{i+31} & x_{i+32} & x_{i+33}\cdots & x_{i+3j} & 0 \\ \vdots & & & & \\ x_{i+m1} & x_{i+m2} & x_{i+m3}\cdots & x_{i+mn} & 0 \end{pmatrix}$$

We know that every total paranormed space becomes a linear metric space with the metric given by

$$d(x, y) = g\left((m+n)!\,|x_{mn} - y_{mn}|^{1/m+n}\right).$$

It is clear that $\chi_r^{2q}(p, B^r)$ is total paranormed space with

$$d(x, y) = g_B\left((m+n)!\,|x_{mn} - y_{mn}|^{1/m+n}\right).$$

Now, we can give the definition of uniform Opial property in a linear metric space.

A linear metric space (X, d) has the uniform Opial property if for each $\epsilon > 0$ there exists $\tau > 0$ such that for any weakly gai sequence $\{x_{k\ell}\} \in S(0, r)$ and $x \in \chi_r^{2q}(p, B^r)$ with $d(x, 0) \geq \epsilon$ the following inequality holds:

$$r + \tau \leq lim\,inf_{k\ell\to\infty} d(x_{k\ell} + x, 0).$$

4.1. **LEMMA.** If $lim\,inf_{mn\to\infty} p_{mn} > 0$ then for any $L > 0$ and $\epsilon > 0$, there exists $\delta = \delta(\epsilon, L) > 0$ for $u, v \in \chi_r^{2q}(p, B^r)$ such that

$$d^M(u + v, 0) < d^M(u, 0) + \epsilon$$

whenever $d^M(u, 0) \leq L$ and $d^M(v, 0) \leq \delta$.

4.2. **THEOREM.** If $p_{mn} \geq 1$, then $\chi_r^{2q}(p, B^r)$ have uniform Opial property.

Proof: For any $\epsilon > 0$, we can find a positive number $\epsilon_0 \in (0, \epsilon)$ such that

$$r^M + \frac{\epsilon^M}{4} > (r + \epsilon_0)^M.$$

Take any $x \in \chi_r^{2q}(p, B^r)$ with $d^M(x, 0) \geq \epsilon^M$ and $(x_{k\ell})$ to be weakly gai sequence in $S(0, r)$. By this, we write

$$d^M(x_{k\ell}, 0) = r^M.$$

There exists $p_0 q_0 \in \mathbb{N}$ such that

$$d^M\left(x_{|\mathbb{N}-p_0 q_0}, 0\right) = \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |Tx_{mn}|^{p_{mn}} < \frac{\epsilon_0}{r}^M < \frac{\epsilon^M}{4} \tag{4.1}$$

Furthermore, we have

$$\epsilon^M \le d^M(x,0) = \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} |Tx_{mn}|^{p_{mn}} + \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |Tx_{mn}|^{p_{mn}},$$

$$\epsilon^M \le \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} |Tx_{mn}|^{p_{mn}} + \frac{\epsilon^M}{4}, \tag{4.2}$$

$$\frac{3\epsilon^M}{4} \le \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} |Tx_{mn}|^{p_{mn}}.$$

By $x_{k\ell} \to 0$, weakly, this implies that $x_{k\ell} \to 0$, coordinatewise, hence there exists $k_0 \ell_0 \in \mathbb{N}$ such that with (4.2)

$$\frac{3\epsilon^M}{4} \le \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} \left| T_{k\ell} x_{mn} + ((m+n)! x_{mn})^{1/m+n} \right|^{p_{mn}}, \tag{4.3}$$

for all $k\ell \ge n_0 \ell_0$. Lemma (4.1) asserts that

$$d^M(y+z,0) \le d^M(y,0) + \frac{\epsilon^M}{4}, \tag{4.4}$$

whenever $d^M(y,0) \le r^M$ and $d^M(z,0) \le \epsilon_0$. Again by $x_{k\ell} \to 0$, weakly, there exists $k_1 \ell_1 > k_0 \ell_0$ such that $d^M\left(x_{k\ell|p_0 q_0}, 0\right) < \epsilon_0$ for all $k\ell > k_1 \ell_1$, so by (4.4), we obtain that

$$d^M\left(x_{k\ell|\mathbb{N}-p_0 q_0} + x_{k\ell|p_0 q_0}, 0\right) < d^M\left(x_{k\ell|\mathbb{N}-p_0 q_0}, 0\right) + \frac{\epsilon^M}{4}, \tag{4.5}$$

hence,

$$d^M(x_{k\ell},0) - \frac{\epsilon^M}{4} < d^M\left(x_{k\ell|\mathbb{N}-p_0 q_0}, 0\right) = \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |T_{k\ell} x_{mn}|^{p_{mn}}, \tag{4.6}$$

$$r^M - \frac{\epsilon^M}{4} < \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |T_{k\ell} x_{mn}|^{p_{mn}}, \tag{4.7}$$

for all $k\ell > k_1 \ell_1$. This, together with (4.1),(4.2), implies that for any $k\ell > k_1 \ell_1$,

$$d^M(x_{k\ell}+x,0) = \sum_{m=0}^{p_0} \sum_{n=0}^{q_0} \left| T_{k\ell} x_{mn} + ((m+n)! x_{mn})^{1/m+n} \right|^{p_{mn}} +$$
$$\sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} \left| T_{k\ell} x_{mn} + ((m+n)! x_{mn})^{1/m+n} \right|^{p_{mn}} \ge$$
$$\sum_{m=0}^{p_0} \sum_{n=0}^{q_0} \left| T_{k\ell} x_{mn} + ((m+n)! x_{mn})^{1/m+n} \right|^{p_{mn}} +$$
$$\left| \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |T_{k\ell} x_{mn}|^{p_{mn}} \right| - \left| \sum_{m=p_0+1}^{\infty} \sum_{n=q_0+1}^{\infty} |T_{k\ell} x_{mn}|^{p_{mn}} \right|$$
$$> \frac{3\epsilon^M}{4} + \left(r^M + \frac{\epsilon^M}{4}\right) - \frac{\epsilon^M}{4} = r^M + \frac{\epsilon^M}{4} > (r+\epsilon_0)^M.$$

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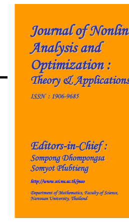

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A GENERAL ITERATIVE SCHEME FOR STRICT PSEUDONONSPREADING MAPPING RELATED TO OPTIMIZATION PROBLEM IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce a general iterative scheme for finding fixed points of a strictly pseudononspreading mapping. We show that, under some suitable conditions, the sequence which is generated by the proposed iterative scheme converges strongly to a fixed point of the mapping. Moreover, such a fixed point is a solution of a certain optimization problem that induced by a strongly positive bounded linear operator. Consequently, since the class of strictly pseudononspreading mapping is the largest one, the main results presented in this paper extend various results existing in the current literature.

KEYWORDS : Strictly pseudononspreading mapping; Nonspreading mapping; Strictly pseudo-contractive mapping; Optimization problem; Fixed point problem.

1. INTRODUCTION AND PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Recall that a mapping $T : D(T) \subset H \to H$ is said to be nonspreading if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in D(T). \tag{1.1}$$

It is worth to point out that the class of nonspreading mappings has been used for studying the resolvents of maximal monotone operators, which is one important problem (see [7, 8, 9]).

A mapping $T : D(T) \subset H \to H$ is said to be k-strictly pseudo-contraction if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2, \quad \forall x, y \in D(T). \tag{1.2}$$

Note that the class of k-strictly pseudo-contractions includes strictly the class of nonexpansive mappings (i.e., $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in D(T)$) as a subclass. In

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fact, T is a nonexpansive mapping if and only if T is a 0-strictly pseudo-contractive mapping. Moreover, strictly pseudo-contractive mappings is one of the most important class that have powerful applications among nonlinear mappings, as in solving inverse problem. Consequently, many authors have been devoting the studies on the problems of finding fixed points for strictly pseudo-contractions, see, for example, [1, 2, 5] and the references therein.

Recently, Osilike and Isiogugu [12] introduced a new class of mappings, so-called k-strictly pseudononspreading, that is, a mapping $T : D(T) \subset H \to H$ is said to be k-strictly pseudononspreading if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in D(T). \tag{1.3}$$

They showed that the class of nonspreading mappings is properly contained in the class of strictly pseudononspreading mappings. Moreover, by using an idea of mean convergence, they also introduced the following iterative process for k-strictly pseudononspreading mapping on a closed convex subset K of H. Let $T : K \longrightarrow K$ be a k-strictly pseudononspreading mapping and $\zeta \in [k, 1)$ be chosen. Starting with an arbitrary initial $x_0 \in K$, define the sequences $\{x_n\}$ and $\{y_n\}$ recursively by

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1 - \alpha_n) y_n, \\ y_n &= \frac{1}{n}\sum_{i=1}^{n-1} T_\zeta^i x_n, \end{aligned} \tag{1.4}$$

for all $n \geq 1$, where $\zeta \in [k, 1)$, $T_\zeta = \zeta I + (1 - \zeta)T$, and $\{\alpha_n\}$ is a sequence in $[0, 1)$. It is proved that, under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.4) strongly converges to $P_{F(T)}u$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

On the other hand, let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \ \text{ for all } x \in H. \tag{1.5}$$

The optimization problems are of very interesting and have been studying by many authors. A kind of optimization problem is the following :

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.6}$$

where $u \in H$ and $C = \bigcap_{i=1}^{\infty} C_i$, when $C_1, C_2, ...$ are infinitely many closed convex subsets on H such that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$. For more detailed accounts on optimization problems and related problems, we refer to [3, 4, 6, 14].

Let T be a nonexpansive mapping with a nonempty fixed point set. In 2003, Xu [14] considered the following iterative algorithm:

$$x_{n+1} = \alpha_n u + (I - \alpha_n A)Tx_n, \ n \geq 0, \tag{1.7}$$

where $x_0 \in H$ is chosen arbitrary and $\{\alpha_n\}$ is a sequence of real numbers. He proved that if the $\{\alpha_n\}$ satisfies certain conditions, then the sequence $\{x_n\}$ defined by (1.7) converges strongly to the unique solution of the optimization problem (1.6), when $C = F(T)$. Moreover, by using the viscosity approximation method introduced by Moudafi [11], Marino and Xu [10] considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, n \geq 0, \tag{1.8}$$

where $\gamma > 0$ and $f : H \longrightarrow H$ is a contractive mapping. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies certain conditions, then the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T),$$

which is, in fact, the optimality condition for the minimization problem:

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

In this paper, motivated by the above-mentioned results, we introduce a general iterative scheme for finding fixed points of a strictly pseudononspreading mapping and then prove that the sequence generated by the proposed iterative scheme converges strongly to a fixed point of such mapping, which is also a solution of the optimization problem (1.6). Additional results of the main result are also obtained. Our results improve and develop the corresponding results of Osilike and Isiogugu [12], Moudafi [11] and Marino and Xu [10].

To do so, we need the following well known results.

Lemma 1.1. [10] *Assume that A is a strongly positive linear bounded self-adjoint operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 1.2. [12] *Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a k-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Lemma 1.3. *Let H be a real Hilbert space. Then for any $x, y \in H$ we have*

(a) $\|\zeta x + (1-\zeta)y\|^2 = \zeta\|x\|^2 + (1-\zeta)\|y\|^2 - \zeta(1-\zeta)\|x-y\|^2$, *for each* $\zeta \in [0, 1]$.
(b) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.

Lemma 1.4. [13] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in (0, 1) and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\longrightarrow\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\longrightarrow\infty} a_n = 0$.

2. MAIN RESULTS

We start with an important useful lemma.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set. Let $\zeta \in [k, 1)$ be fixed and define $T_\zeta : C \to C$ by*

$$T_\zeta(x) = \zeta x + (1 - \zeta)Tx, \quad \forall x \in C. \tag{2.1}$$

Then $F(T) = F(T_\zeta)$.

Proof. This follows immediately from the fact that $x - T_\zeta x = (1 - \zeta)(x - Tx)$, for each $x \in C$. □

Now we are in a position to prove our main results.

Algorithm. Let $\gamma > 0$ be a constant and $T : H \to H$ be a k-strictly pseudononspreading mapping, $f : H \to H$ be a contraction and A be a strongly positive bounded linear operator on H. Let $\zeta \in [k, 1)$ and $\{\alpha_n\}$ be a sequence of real numbers in $[0, 1]$. For given $x_0 \in H$, we define a sequence $\{x_n\}$ in H by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n \tag{2.2}$$

where $y_n = \frac{1}{n}\sum_{i=0}^{n-1} T_\zeta^i x_n$ and T_ζ is defined by (2.1).

Theorem 2.2. *Let H be a real Hilbert space, and $T : H \longrightarrow H$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set. Let $f : H \longrightarrow H$ be a contraction with coefficient $\alpha \in (0, 1)$. Let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let the sequence $\{x_n\}$ be defined by (2.2). If the following control conditions are satisfied:*

(i) $0 < \gamma < \overline{\gamma}/\alpha$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$,
(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$ which solves the following optimization problem:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{2.3}$$

where h is a potential function for γf.

Proof. We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is a bounded sequence. Indeed, taking $p \in F(T)$ and $z \in H$, we see that

$$\begin{aligned}
\|T_\zeta z - p\|^2 &= \|[\zeta z + (1-\zeta)Tz] - [\zeta p + (1-\zeta)Tp]\|^2 \\
&= \|\zeta(z-p) + (1-\zeta)(Tz - Tp)\|^2 \\
&= \zeta\|z-p\|^2 + (1-\zeta)\|Tz - Tp\|^2 - \zeta(1-\zeta)\|z - Tz\|^2 \\
&\le \zeta\|z-p\|^2 + (1-\zeta)\left[\|z-p\|^2 + k\|z-Tz\|^2\right] - \zeta(1-\zeta)\|z-Tz\|^2 \\
&= \|z-p\|^2 + (1-\zeta)(k-\zeta)\|z-Tz\|^2 \\
&\le \|z-p\|^2.
\end{aligned}$$

Thus for each $z \in H$ and $p \in F(T)$, we have

$$\|T_\zeta^i z - p\| = \|T_\zeta(T_\zeta^{i-1}z) - p\| \le \|T_\zeta^{i-1}z - p\| \le \|T_\zeta^{i-2}z - p\| \le \cdots \le \|z-p\|, \tag{2.4}$$

for all $i \ge 0$. Consequently,

$$\begin{aligned}
\|y_n - p\| &= \left\|\frac{1}{n}\sum_{i=0}^{n-1} T_\xi^i x_n - p\right\| \\
&\le \frac{1}{n}\sum_{i=0}^{n-1} \left\|T_\xi^i x_n - p\right\| \\
&\le \frac{1}{n}\sum_{i=0}^{n-1} \|x_n - p\| = \|x_n - p\|.
\end{aligned} \tag{2.5}$$

On the other hand, by condition (ii), without loss of generality we may assume that $\alpha_n \le \|A\|^{-1}$. Thus, by using Lemma 1.1 and (2.5), we obtain

$$\|x_{n+1} - p\| = \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(y_n - p)\|$$

$$
\begin{aligned}
&\le (1-\alpha_n\overline{\gamma})\|y_n - p\| + \alpha_n\|\gamma f(x_n) - Ap\| \\
&\le (1-\alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| \\
&\le (1-\alpha_n(\overline{\gamma}-\gamma\alpha)\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\|. \qquad (2.6)
\end{aligned}
$$

Using (2.6) and induction, we know that

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma}-\gamma\alpha}\right\}, \quad \forall n \ge 0.$$

Therefore, $\{x_n\}$ is a bounded sequence. Consequently, $\{y_n\}$, $\{Ay_n\}$ and $\{f(x_n)\}$ are also bounded. Moreover, in view of (2.4), we know that $\|T_\zeta^n x_n - p\| \le \|x_n - p\|$ for all $n \ge 1$, this means $\{T_\zeta^n x_n\}$ is also bounded.

Step 2. We show

$$\limsup_{n\longrightarrow\infty}\langle \gamma f(x^*) - Ax^*, x_n - x^*\rangle \le 0,$$

where $x^* \in F(T)$ is the unique solution of the optimization problem (2.3).

Observe that, since $\{x_n\}$ is a bounded sequence, we can find a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ and $q \in H$ such that $x_{n_j+1} \rightharpoonup q$ as $j \to \infty$, and

$$\limsup_{n\longrightarrow\infty}\langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle = \lim_{j\longrightarrow\infty}\langle \gamma f(x^*) - Ax^*, x_{n_j+1} - x^*\rangle. \qquad (2.7)$$

On the other hand, since $\{Ay_n\}, \{f(x_n)\}$ are bounded sequences and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = \lim_{n\to\infty}\alpha_n\|\gamma f(x_n) - Ay_n\| = 0. \qquad (2.8)$$

Consequently, for arbitrary bounded linear functional g on H we have

$$
\begin{aligned}
|g(y_{n_j}) - g(q)| &\le |g(y_{n_j}) - g(x_{n_j+1})| + |g(x_{n_j+1}) - g(q)| \\
&\le \|g\|\|y_{n_j} - x_{n_j+1}\| + |g(x_{n_j+1}) - g(q)|. \qquad (2.9)
\end{aligned}
$$

Thus, from (2.8) and (2.9), we conclude that $y_{n_j} \rightharpoonup q$ as $j \to \infty$.

Now for each $j \ge 1$ and $i = 0, 1, 2, ..., n_j - 1$, we put $w_{n_j}^i := T_\zeta^i x_{n_j}$. Then we see that

$$
\begin{aligned}
\|T_\zeta^{i+1}x_{n_j} - T_\zeta q\|^2 &= \|T_\zeta w_{n_j}^i - T_\zeta q\|^2 \\
&= \|\zeta(w_{n_j}^i - q) + (1-\zeta)(Tw_{n_j}^i - Tq)\|^2 \\
&= \zeta\|w_{n_j}^i - q\|^2 + (1-\zeta)\|Tw_{n_j}^i - Tq)\|^2 \\
&\quad -\zeta(1-\zeta)\|w_{n_j}^i - Tw_{n_j}^i - (q - Tq)\|^2 \\
&\le (1-\zeta)\left[\|w_{n_j}^i - q\|^2 + k\|w_{n_j}^i - Tw_{n_j}^i - (q-Tq)\|^2\right] \\
&\quad \zeta\|w_{n_j}^i - q\|^2 + 2(1-\zeta)\langle w_{n_j}^i - Tw_{n_j}^i, q - Tq\rangle \\
&\quad -\zeta(1-\zeta)\|w_{n_j}^i - Tw_{n_j}^i - (q - Tq)\|^2 \\
&= \|w_{n_j}^i - q\|^2 + 2(1-\zeta)\langle w_{n_j}^i - Tw_{n_j}^i, q - Tq\rangle \\
&\quad -(1-\zeta)(\zeta - k)\|w_{n_j}^i - Tw_{n_j}^i - (q - Tq)\|^2 \\
&\le \|w_{n_j}^i - q\|^2 + 2(1-\zeta)\langle w_{n_j}^i - Tw_{n_j}^i, q - Tq\rangle \\
&= \|w_{n_j}^i - q\|^2 + \frac{2}{(1-\zeta)}\langle w_{n_j}^i - T_\zeta w_{n_j}^i, q - T_\zeta q\rangle \\
&= \|w_{n_j}^i - T_\zeta q + T_\zeta q - q\|^2 + \frac{2}{(1-\zeta)}\langle w_{n_j}^i - T_\zeta w_{n_j}^i, q - T_\zeta q\rangle
\end{aligned}
$$

$$= \|w_{n_j}^i - T_\zeta q\|^2 + \|T_\zeta q - q\|^2 + 2\langle w_{n_j}^i - T_\zeta q, q - T_\zeta q\rangle$$
$$+\frac{2}{(1-\zeta)}\langle w_{n_j}^i - T_\zeta w_{n_j}^i, q - T_\zeta q\rangle.$$

Using this one, by summing from $i = 0$ to $n_j - 1$ and dividing by n_j, we know that

$$\begin{aligned}\frac{1}{n_j}\|T_\zeta^{n_j} x_{n_j} - T_\zeta q\|^2 \quad &\leq \quad \frac{1}{n_j}\|x_{n_j} - T_\zeta q\|^2 + \|T_\zeta q - q\|^2 + \langle y_{n_j} - T_\zeta q, T_\zeta q - q\rangle \\ &\quad + \frac{2}{n_j(1-\zeta)}\langle x_{n_j} - T_\zeta^{n_j} x_{n_j}, q - T_\zeta q\rangle. \end{aligned} \tag{2.10}$$

Since $\{x_n\}, \{T_\zeta^n x_n\}$ are bounded sequences and $y_{n_j} \rightharpoonup q$ as $j \to \infty$, we see that (2.10) gives

$$0 \leq \|T_\zeta q - q\|^2 + 2\langle q - T_\zeta q, T_\zeta q - q\rangle = -\|T_\zeta q - q\|^2.$$

This implies that $q \in F(T_\zeta) = F(T)$. Consequently, since x^* is the solution of optimization problem (2.3), from (2.7) we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\langle(\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle &= \limsup_{j\longrightarrow\infty}\langle(\gamma f(x^*) - Ax^*, x_{n_j+1} - x^*\rangle \\ &= \langle \gamma f(x^*) - Ax^*, q - x^*\rangle \\ &\leq 0.\end{aligned}$$

Step 3. We prove $\{x_n\}$ converges strongly to $x^* \in F(T)$, where x^* is the unique solution of the optimization problem (2.3).

Consider,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \quad &= \quad \|(1-\alpha_n A)(y_n - x^*) + \alpha_n(\gamma f(x_n) - Ax^*)\|^2 \\ &\leq \quad \|(1-\alpha_n A)(y_n - x^*)\|^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*, x_{n+1} - x^*\rangle \\ &\leq \quad (1-\alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + \alpha_n\gamma\alpha\{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\} \\ &\quad\quad +2\alpha_n\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle,\end{aligned}$$

this implies that,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \quad &\leq \quad \frac{1 - 2\alpha_n\overline{\gamma} + \alpha_n^2\overline{\gamma}^2 + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle u + \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \\ &\leq \quad \left[1 - \frac{2\alpha_n(\overline{\gamma} - \gamma\alpha)}{1-\alpha_n\gamma\alpha}\right]\|x_n - x^*\|^2 \\ &\quad + \frac{2\alpha_n(\overline{\gamma} - \gamma\alpha)}{1-\alpha_n\gamma\alpha}\left[\frac{\alpha_n\overline{\gamma}^2 R}{2(\overline{\gamma} - \gamma\alpha)} + \frac{1}{\overline{\gamma} - \gamma\alpha}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle\right] \\ &= \quad (1-\kappa_n)\|x_n - x^*\|^2 + \kappa_n\sigma_n,\end{aligned}$$

where

$$R = \sup\{\|x_n - x^*\|^2 : n \geq 1\}, \quad \kappa_n = \frac{2((1+\mu)\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha},$$

$$\sigma_n = \frac{\alpha_n\overline{\gamma}^2 R}{2(\overline{\gamma} - \gamma\alpha)} + \frac{1}{(1+\mu)\overline{\gamma} - \gamma\alpha}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle.$$

It is easy to see that $\sum_{n=1}^{\infty}\kappa_n = \infty$ and $\limsup_{n\longrightarrow\infty}\sigma_n \leq 0$. Hence, by Lemma 1.4, we conclude that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Setting $A =: I$, the identity mapping, in Theorem 2.2, we obtain the following result.

Corollary 2.3. *Let H be a real Hilbert space and K be a closed convex subset of H. Let $T : K \longrightarrow K$ be a k-strictly pseudononspreading mapping on with a nonempty fixed point set. Let $f : K \longrightarrow K$ be a contraction with coefficient $\alpha \in (0,1)$. Let $x_0 \in K$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)y_n, \\ y_n &= \frac{1}{n}\sum_{i=1}^{n-1} T_\zeta^i x_n, \end{aligned} \tag{2.11}$$

for all $n \geq 1$, where T_ζ is defined by (2.1). If the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.

Proof. Since the identity mapping is a 1- strongly positive bounded linear operator and $\alpha \in (0,1)$, we can set $\gamma = 1$ in (2.2). Consequently, (2.2) reduces to (2.11) and the required result is followed from Theorem 2.2 immediately. □

Remark 2.4. Let u be a fixed element in K and setting $f := u$, a constant mapping. Then our Corollary 2.3 recovers the results presented in [12].

In view of Theorem 2.2, we can obtain the following results as special cases.

Theorem 2.5. *Let H be a real Hilbert space, and $T : H \longrightarrow H$ be a nonspreading mapping with a nonempty fixed point set. Let $f : H \longrightarrow H$ be a contraction with coefficient $\alpha \in (0,1)$. Let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let $\gamma > 0$ and $\{\alpha_n\} \subset [0,1]$ be a sequence of real numbers. Let $x_0 \in H$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= \alpha_n \gamma f(x_n) + (I-\alpha_n A)y_n, \\ y_n &= \frac{1}{n}\sum_{i=1}^{n-1} T^i x_n, \end{aligned} \tag{2.12}$$

for all $n \geq 1$. If the following control conditions are satisfied:

(i) $0 < \gamma < \overline{\gamma}/\alpha$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$,
(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$ which solves the following optimization problem (2.3).

Proof. Since every nonspreading mapping is a 0-strictly pseudononspreading mapping, by choosing $\zeta = 0$ in (2.2), we see that (2.2) is reduced to (2.12). Consequently, by Theorem 2.2, the proof is completed. □

Immediately, form Theorem 2.5, we also have the following results.

Corollary 2.6. *Let H be a real Hilbert space and K be closed convex subset of H. Let $T : K \longrightarrow K$ be a nonspreading mapping with a nonempty fixed point set and $f : K \longrightarrow K$ be a contraction mapping. Let $\{\alpha_n\} \subset [0,1]$ be a sequence of real numbers and define a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n,$$

$$y_n = \frac{1}{n}\sum_{i=1}^{n-1} T^i x_n, \tag{2.13}$$

for all $n \geq 1$. If the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.

Corollary 2.7. *Let H be a real Hilbert space and K be closed convex subset of H. Let $T : K \longrightarrow K$ be a nonspreading mapping with a nonempty fixed point set. Let $u \in H$ be fixed and $\{\alpha_n\} \subset [0, 1]$ be a sequence of real numbers. Let $x_0 \in K$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \\ y_n &= \frac{1}{n}\sum_{i=1}^{n-1} T^i x_n, \end{aligned} \tag{2.14}$$

for all $n \geq 1$. If the following control conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F(T)$.

Remark 2.8. It is worth to noting that, since the class of strictly pseudononspreading mappings contains a large number of mappings, our results extend and improve various related results existing in this present time.

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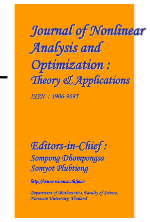

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APPROXIMATION METHOD FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS FOR A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS

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ABSTRACT. In this research, we introduce an iterative scheme by using the concept of K-mapping for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings and the set of solution of a generalized mixed equilibrium problem in a Hilbert space. Then, we prove strong convergence of the purposed iterative algorithm to a common element of the two sets. Moreover, we also give a numerical result of the studied method.

KEYWORDS : Generalized mixed equilibrium problems; Nonexpansive mapping; K-mapping; Strong convergence; Fixed point.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear mapping, $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a function and $F : C \times C \to \mathbb{R}$ be a bifunction. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$, $f : C \to C$ be a contraction if $\|fx - fy\| \leq a\|x - y\|$ where $a \in (0, 1)$. The set of fixed points of T (i.e. $F(T) = \{x \in H : Tx = x\}$) denoted by $F(T)$. Goebel and Kirk [2] showed that $F(T)$ is always closed convex and also nonempty provided T has a bounded trajectory.

A bounded linear operator A on H is called *strongly positive* with coefficient $\bar{\gamma}$ if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \ \forall x \in H.$$

For a bifunction $F : C \times C \longrightarrow \mathbb{R}$, equilibrium problem for F is to find $x \in C$ such that

$$F(x, y) \geq 0 \ \forall y \in C. \tag{1.1}$$

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The set of solutions of (1.1) is denoted by $EP(F)$. Many problems in physics, optimization, and economics are seeking some elements of $EP(F)$, see [4], [5]. Several iterative methods have been proposed to solve the equilibrium problem, see for instance, [1], [4], [5], [6], [10], [12].

In 2005, Combettes and Hirstoaga [5] introduced an iterative scheme of finding the best approximation to the initial data when EP(F) is nonempty and proved a strong convergence theorem.

A mapping A of C into H is called *inverse-strongly monotone*, see [3], if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2 \;\forall x,y\in C.$$

The variational inequality problem is to find a point $u\in C$ such that

$$\langle v-u, Au\rangle \;\forall v\in C. \tag{1.2}$$

The set of solutions of the variational inequality is denoted by $VI(C,A)$.

In 2008, Ceng and Yao [10] considered the following mixed equilibrium problem:

$$\text{Find } x\in C \text{ such that } F(x,y)+\varphi(y)\geq\varphi(x),\;\forall y\in C, \tag{1.3}$$

where $\varphi : C\to R$ is a function.
The set of solutions of (1.3) is denoted by $MEP(F,\varphi)$.

In 2008, Peng and Yao [6] considered the following generalized mixed equilibrium problem:

$$\text{Find } x\in C \text{ such that } F(x,y)+\langle Ax, y-x\rangle+\varphi(y)\geq\varphi(x),\;\forall y\in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $GMEP(F,\varphi,A)$. It is easy to see that x is a solution of problem (1.4) implies that $x\in dom\varphi=\{x\in C:\varphi(x)<+\infty\}$.

In the case of $A\equiv 0$, $GMEP(F,\varphi,A)=MEP(F,\varphi)$. In the case of $F\equiv\varphi\equiv 0$, then $GMEP(F,\varphi,A)=V(C,A)$. In the case of $A\equiv\varphi\equiv 0$, then $GMEP(F,\varphi,A)=EP(F)$.

In 2008, Peng and Yao [6] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.4), the set of fixed points of a nonexpansive mappings and the set of solutions of a variational inequality for a monotone, Lipschitz continuous mapping and obtained a strong convergence theorem.

In 2009, Peng and Yao [12] introduced iterative schemes by using the concept of *W-mapping* for finding a common element of the set of solutions of $GMEP(F,\varphi,A)$ and the set of common fixed point of infinitely family of nonexpansive mappings of C into itself.

The concept of W-mapping was first introduced by Shimoji and Takahashi [7]. They defined the mapping W_n as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1}+(1-\lambda_n)U_{n,n+1},\\
U_{n,n-1} &= \lambda_{n-1}T_{n-1}U_{n,n}+(1-\lambda_{n-1})I,\\
U_{n,n-2} &= \lambda_{n-2}T_{n-2}U_{n,n-1}+(1-\lambda_{n-2})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1}+(1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1}T_{k-1}U_{n,k}+(1-\lambda_{k-1})I,
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,2} + (1-\lambda_2)I, \\
W_n &= U_{n,1} = \lambda_1 T_1 U_{n,1} + (1-\lambda_1)I,
\end{aligned} \tag{1.5}$$

where $\{\lambda_n\} \subseteq [0,1]$ and $\{T_n\}_{i=1}^{\infty}$ is a sequence of nonexpansive mappings of C into itself. This mapping is called the $W-$*mapping* generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$. They proved that if X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^{\infty} F(T_i)$ where $0 < \lambda_i \leq d < 1$ for every $i \in \mathbb{N}$.

Recently, A. Kangtunyakarn and S. Suantai [13] introduced the concept of K-mapping and employed this mapping for finding a common element of the set the of the solution of an equilibrium problem and the set of common fixed points of a finite family of nonexpansive mapping, they defined $K_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,0} &= I, \\
U_{n,1} &= \lambda_1 T_1 U_{n,0} + (1-\lambda_1)U_{n,0}, \\
U_{n,2} &= \lambda_2 T_2 U_{n,1} + (1-\lambda_2)U_{n,1}, \\
U_{n,3} &= \lambda_3 T_3 U_{n,2} + (1-\lambda_3)U_{n,2}, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k-1} + (1-\lambda_k)U_{n,k-1}, \\
U_{n,k+1} &= \lambda_{k+1} T_{k+1} U_{n,k} + (1-\lambda_{k+1})U_{n,k}, \\
&\vdots \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n-2} + (1-\lambda_{n-1})U_{n,n-2}, \\
K_n &= U_{n,n} = \lambda_n T_n U_{n,n-1} + (1-\lambda_n)U_{n,n-1},
\end{aligned}$$

such a mapping K_n is called the *K-mapping* generated by $T_1, T_2, \ldots, T_n$ and $\lambda_1, \lambda_2, \ldots \lambda_n$.

In this research, we introduce K-mapping defined as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \alpha_n T_n U_{n,n+1} + (1-\alpha_n)U_{n,n+1}, \\
U_{n,n-1} &= \alpha_{n-1} T_{n-1} U_{n,n} + (1-\alpha_{n-1})U_{n,n}, \\
U_{n,n-2} &= \alpha_{n-2} T_{n-2} U_{n,n-1} + (1-\alpha_{n-2})U_{n,n-1}, \\
&\vdots \\
U_{n,k} &= \alpha_k T_k U_{n,k+1} + (1-\alpha_k)U_{n,k+1}, \\
U_{n,k-1} &= \alpha_{k-1} T_{k-1} U_{n,k} + (1-\alpha_{k-1})U_{n,k}, \\
&\vdots \\
U_{n,2} &= \alpha_2 T_2 U_{n,3} + (1-\alpha_2)U_{n,3}, \\
K_n &= U_{n,1} = \alpha_1 T_1 U_{n,2} + (1-\alpha_1)U_{n,2}.
\end{aligned}$$

Such a mapping K_n is called K-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$.

In 2007, S. Takahashi and W. Takahashi [15] modified the following iterative solution for finding a common element of the set of fixed point of a nonexpansive mapping and the solution of equilibrium problem by:

Let H be Hilbert space, C be a nonempty closed convex subset of H, f be a contraction of H into itself, S be a a nonexpansive mapping from C to H, $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{1.6}$$

In 2008, S. Takahashi and W. Takahashi [14] modified the following iterative solution for finding a common element of the set of fixed point of a nonexpansive mapping and the solution of equilibrium problems by:
Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)z_n], \ \forall n \in \mathbb{N}. \end{cases} \tag{1.7}$$

Motivated by this two works, we introduce an iterative scheme for finding a common element if the set of common fixed point of a countable family of nonexpansive mappings and the set of solution of generalized mixed equilibrium problems.

2. PRELIMINARIES

In this section, we give some useful lemmas that will be used for the main result in the next section.

Let C be closed convex subset of a Hilbert space H, let P_C be the metric projection of H onto C i.e., for $x \in H$, P_Cx satisfies the property

$$\|x - P_Cx\| = \min_{y \in C} \|x - y\|.$$

The following characterizes the projection P_C.

Lemma 2.1. *(See [9]) Given $x \in H$ and $y \in C$. Then $P_Cx = y$ if and only if there holds the inequality*
$\langle x - y, y - z\rangle \geq 0 \ \forall z \in C.$

Lemma 2.2. *(See [11]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \ \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\displaystyle\sum_{n=1}^{\infty} \alpha_n = \infty,$

(2) $\displaystyle\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\delta_n| < \infty.$

Then $\lim_{n\to\infty} s_n = 0.$

Lemma 2.3. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, \ x + y\rangle \ \forall x, y \in H.$$

Theorem 2.4. *(See [16]) A Banach space X is said to satisfy Opial's condition if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

For solving the generalized mixed equilibrium problem, let us give the following assumptions for the bifunction F, the function φ and the set C:

$(A1)$ $F(x,x) = 0\ \forall x \in C$,

$(A2)$ F is monotone, i.e. $F(x,y) + F(y,x) \le 0\ \forall x, y \in C$,

$(A3)$ for each $y \in C,\ x \mapsto F(x,y)$ is weakly upper semicontinuous,

$(A4)$ for $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous,

$(B1)$ $\forall x \in H$, and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y \in C \bigcap dom(\varphi)$ such that for any $z \in C \backslash D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(x),$$

$(B2)$ C is bounded set.

Lemma 2.5. *(See [6]) Let C be a nonempty closed convex subset of a Hilbert space H, $F : C \times C \to \mathbb{R}$ be a function such that satisfy $(A1)-(A4)$ and $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$T_r(x) = \{z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \ge \varphi(z),\ \forall y \in C\}$

for all $x \in H$. Then the following conclusions hold:

(1) for each $x \in H$, $T_r(x) \ne \emptyset$,

(2) T_r is is single-valued,

(3) T_r is firmly nonexpansive, i.e.

$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle\ \forall x, y \in H$,

(4) $F(T_r) = MEP(F, \varphi)$,

(5) $MEP(F, \varphi)$ is closed and convex.

Lemma 2.6. *In a strictly convex Banach space E, if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|$$

for all $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$.

By using the same argument as in [13] (Lemma 2.7 and Lemma 2.8), we obtain that the two following lemmas.

Lemma 2.7. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ for each $n \in \mathbb{N}$. Then for every $x \in C$ $\lim_{n\to\infty} K_n x$ exists.*

Let $K : C \to C$ be defined by $Kx = \lim_{n\to\infty} K_n x$. Such a mapping K is called K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$.

Lemma 2.8. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K be the K-mapping generated by $T_n, T_{n-1}, \ldots$ and $\lambda_n, \lambda_{n-1}, \ldots$ for each $n \in \mathbb{N}$. Then $F(K) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Lemma 2.9. *Let C be a closed convex subset of Hilbert space, let $\{T_i\}_{i=1}^{\infty}$ be a infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$ for* *every* $i = 1, 2, \ldots$. *Let* K_n *be the K-mapping generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ *for each* $n \in \mathbb{N}$. *Then*

$$\sum_{n=1}^{\infty} \sup\{\|K_{n+1}x - K_n x\| : x \in B\} < \infty \tag{2.1}$$

for every bounded subset B *of* C.

Proof. Let B be a bounded subset of C. Then for $n \in \mathbb{N}$, $x \in B$, we have

$$\begin{aligned}
\|K_{n+1}x - K_n x\| &= \|U_{n+1,n+1}x - U_{n,n}x\| \\
&= \|\lambda_1 T_1 U_{n+1,2}x + (1-\lambda_1)U_{n+1,2}x - (\lambda_1 T_1 U_{n,2}x + (1-\lambda_1)U_{n,2}x)\| \\
&= \|\lambda_1(T_1 U_{n+1,2}x_2 - T_1 U_{n,2}x) + (1-\lambda_1)(U_{n+1,2}x - U_{n,2}x)\| \\
&\leq \lambda_1\|U_{n+1,2}x - U_{n,2}x\| + (1-\lambda_1)\|U_{n+1,2}x - U_{n,2}x\| \\
&= \|U_{n+1,2}x - U_{n,2}x\| \\
&\;\;\vdots \\
&= \|U_{n+1,n+1}x - U_{n,n+1}x\| \\
&= \|\lambda_{n+1}T_{n+1}x + (1-\lambda_{n+1})Ix - Ix\| \\
&= \lambda_{n+1}\|T_{n+1}x - x\| \\
&\leq \lambda_{n+1}M.
\end{aligned} \tag{2.2}$$

where $M = \sup_{n\in\mathbb{N}} \sup\{\|T_{n+1}x - x\| : x \in B\}$.
This implies $\sup\{\|K_{n+1}x - K_n x\| \, x \in B\} \leq \lambda_{n+1}M$. By the assumption $\sum_{i=1}^{\infty} \lambda_i < \infty$, we obtain

$$\sum_{n=1}^{\infty} \sup\{\|K_{n+1}x - K_n x\| : x \in B\} < \infty.$$

□

Lemma 2.10. *Let* C *be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^{\infty}$ *be a infinite family of nonexpansive mappings of* C *into itself with* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and let* $\lambda_1, \lambda_2, \ldots$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, \ldots$ *with* $\sum_{i=1}^{\infty} \lambda_i < \infty$. *Let* K_n *be the* $K-$*mapping generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ *for each* $n \in \mathbb{N}$ *and* $\{x_n\} \subset C$ *a bounded sequence. Then*

$$\|Kx_n - K_n x_n\| \to 0 \text{ as } n \to \infty. \tag{2.3}$$

Proof. By Lemma 2.7, we have $Kx_n = \lim_{m\to\infty} K_m x_n$. For $m \geq n$, by (2.2) we have

$$\begin{aligned}
\|K_m x_n - K_n x_n\| &= \|K_m x_n - K_{m-1}x_n\| + \|K_{m-1}x_n - K_{m-2}x_n\| \\
&\quad + \ldots + \|K_{n+1}x_n - K_n x_n\| \\
&\leq (\lambda_m + \lambda_{m-1} + \ldots + \lambda_{n+1})M \\
&\leq \sum_{i=n+1}^{m} \lambda_i M,
\end{aligned}$$

where $M = \sup\{\|T_k x_n - x_n\| : k \in \mathbb{N}, n \in \mathbb{N}\}$. It follows that

$$\|Kx_n - K_n x_n\| = \lim_{m\to\infty} \|K_m x_n - K_n x_n\| \leq \sum_{i=n+1}^{\infty} \lambda_i M.$$

This implies $\lim_{n\to\infty} \|Kx_n - K_n x_n\| = 0$. □

3. MAIN RESULT

In this section, we modify iterative scheme (1.6) by using concept of K-mapping and prove strong convergence of the sequences $\{z_n\}$ and $\{x_n\}$ under some control conditions.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space, $F : C\times C \to \mathbb{R}$ be a bifunction satisfying the condition (A1)-(A4) and (B1) or (B2) and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semi-continuous and convex function. Let A be an α-inverse strongly monotone mapping of C into H, $f : C \to C$ be a contraction map with coefficient $0 < a < 1$, $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\Omega = \bigcap_{i=1}^{\infty} F(T_i)\bigcap GMEP(F, A, \varphi) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ for each $n \in \mathbb{N}$. Let $x_1 \in C$ and $\{z_n\}$ and $\{x_n\}$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)K_n z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where

$$\{\alpha_n\} \subset [0, 1] \text{ and } \{r_n\} \subset [0, 2\alpha] \text{ satisfy the following conditions :}$$

$$(i)\ 0 < a \leq r_n \leq b < 2\alpha,$$

$$(ii)\ \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$$

$$(iii)\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

*Then $\{x_n\}$ and $\{z_n\}$ converge strongly to $z_0 \in \Omega$,
where $z_0 = P_\Omega f(z_0)$.*

Proof. First, we show that $(I - r_nA)$ is nonexpansive. Let $x, y \in C$. Since A is $\alpha-$inverse strongly monotone and $r_n < 2\alpha \ \forall n \in \mathbb{N}$, we have

$$\begin{aligned} \|(I - r_nA)x - (I - r_nA)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha r_n\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + r_n(r_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2, \end{aligned} \tag{3.2}$$

Thus $(I - r_nA)$ is nonexpansive.
Since

$$F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \ \forall y \in C,$$

we obtain

$$F(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - (I - r_nA)x_n\rangle \geq 0 \ \forall y \in C.$$

By Lemma 2.5, we have $z_n = T_{r_n}(x_n - r_nAx_n) \ \forall n \in \mathbb{N}$.
Let $z \in \bigcap_{i=1}^{\infty} F(T_i)\bigcap GMEP(F, A, \varphi)$. Then $F(z, y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \geq 0$

for all $y \in C$, so

$$F(z,y) + \frac{1}{r_n}\langle y-z, z-z+r_nAz\rangle + \varphi(y) - \varphi(z) \geq 0 \text{ for all } y \in C.$$

Again by Lemma 2.5, we have $z = T_{r_n}(z - r_nAz)$. Since $I - r_nA$ and T_{r_n} are nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(K_nz_n - z)\| \\
&= \|\alpha_n f(x_n) + \alpha_n f(z) - \alpha_n f(z) - \alpha_n z + (1-\alpha_n)(K_nz_n - z)\| \\
&\leq \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|K_nz_n - z\| \\
&\leq \alpha_n(a\|x_n - z\| + \|f(z) - z\|) + (1-\alpha_n)\|z_n - z\| \\
&= \alpha_n(a\|x_n - z\| + \|f(z) - z\|) \\
&\quad +(1-\alpha_n)\|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)z\| \\
&\leq \alpha_n(a\|x_n - z\| + \|f(z) - z\|) + (1-\alpha_n)\|x_n - z\| \\
&= \alpha_n a\|x_n - z\| + (1-\alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&= (1 - \alpha_n + \alpha_n a)\|x_n - z\| + \alpha_n(1-a)\frac{\|f(z) - z\|}{(1-a)} \\
&\leq \max\{\|x_n - z\|, \frac{\|f(z) - z\|}{(1-a)}\}. \qquad (3.3)
\end{aligned}$$

By induction, we have $\|x_n - z\| \leq \max\{\|x_1 - z\|, \frac{\|f(z)-z\|}{(1-a)}\}$ $\forall n \in \mathbb{N}$, this implies $\{x_n\}$ bounded. It follows that $\{z_n\}$ and $\{K_nz_n\}$ are also bounded. Next, we will show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Putting $u_n = x_n - r_nAx_n$. Then, we have $z_{n-1} = T_{r_{n-1}}u_{n-1}$ and $z_n = T_{r_n}u_n$. By definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)K_nz_n - \alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})K_{n-1}z_{n-1}\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1}f(x_{n-1}) \\
&\quad +(1-\alpha_n)K_nz_n - (1-\alpha_n)K_{n-1}z_{n-1} + (1-\alpha_n)K_{n-1}z_{n-1} \\
&\quad -(1-\alpha_{n-1})K_{n-1}z_{n-1}\| \\
&\leq \alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad +(1-\alpha_n)\|K_nz_n - K_{n-1}z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|K_{n-1}z_{n-1}\| \\
&\leq \alpha_n a\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad +(1-\alpha_n)\|K_nz_n - K_{n-1}z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|K_{n-1}z_{n-1}\| \\
&= \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)\|K_nz_n - K_{n-1}z_{n-1}\| \\
&\quad +|\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|K_{n-1}z_{n-1}\|) \\
&\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)[\|K_nz_n - K_nz_{n-1}\| \\
&\quad +\|K_nz_{n-1} - K_{n-1}z_{n-1}\|] + |\alpha_n - \alpha_{n-1}|L \\
&\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)[\|z_n - z_{n-1}\| \\
&\quad +\|K_nz_{n-1} - K_{n-1}z_{n-1}\|] + |\alpha_n - \alpha_{n-1}|L, \qquad (3.4)
\end{aligned}$$

where $L = \sup\{\|f(x_n)\| + \|K_nz_n\| : n \in \mathbb{N}\}$. Since T_{r_n} and $I - r_nA$ are nonexpansive, we have

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \|T_{r_n}u_n - T_{r_{n-1}}u_{n-1}\| \\
&= \|T_{r_n}u_n - T_{r_n}u_{n-1} + T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\|
\end{aligned}$$

$$\begin{aligned} &\leq \|T_{r_n}u_n - T_{r_n}u_{n-1}\| + \|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\| \\ &\leq \|u_n - u_{n-1}\| + \|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\| \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|u_n - u_{n-1}\| &= \|x_n - r_nAx_n - x_{n-1} + r_{n-1}Ax_{n-1}\| \\ &= \|x_n - r_nAx_n - r_nAx_{n-1} + r_nAx_{n-1} - x_{n-1} + r_{n-1}Ax_{n-1}\| \\ &= \|(I - r_nA)x_n - (I - r_nA)x_{n-1} + (r_{n-1} - r_n)Ax_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\|. \end{aligned} \tag{3.6}$$

By Lemma 2.5, we have

$$F(T_{r_{n-1}}u_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - u_{n-1}\rangle + \varphi(y) - \varphi(T_{r_{n-1}}u_{n-1}) \geq 0,$$

$\forall y \in C$ and

$$F(T_{r_n}u_{n-1}, y) + \frac{1}{r_n}\langle y - T_{r_n}u_{n-1}, T_{r_n}u_{n-1} - u_{n-1}\rangle + \varphi(y) - \varphi(T_{r_n}u_{n-1}) \geq 0, \ \forall y \in C.$$

In particular, we have

$$\begin{aligned} &F(T_{r_{n-1}}u_{n-1}, T_{r_n}u_{n-1}) + \frac{1}{r_{n-1}}\langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - u_{n-1}\rangle \\ &+\varphi(T_{r_n}u_{n-1}) - \varphi(T_{r_{n-1}}u_{n-1}) \geq 0, \ \forall y \in C \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &F(T_{r_n}u_{n-1}, T_{r_{n-1}}u_{n-1}) + \frac{1}{r_n}\langle T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1}, T_{r_n}u_{n-1} - u_{n-1}\rangle \\ &+\varphi(T_{r_{n-1}}u_{n-1}) - \varphi(T_{r_n}u_{n-1}) \geq 0, \ \forall y \in C. \end{aligned} \tag{3.8}$$

Summing up (3.7) and (3.8) and using (A2), we obtain

$$\begin{aligned} &\frac{1}{r_n}\langle T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1}, T_{r_n}u_{n-1} - u_{n-1}\rangle \\ &+\frac{1}{r_{n-1}}\langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - u_{n-1}\rangle \geq 0 \ \forall y \in C. \end{aligned}$$

It then follows that

$$\langle T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1}, \frac{T_{r_n}u_{n-1} - u_{n-1}}{r_n} - \frac{T_{r_{n-1}}u_{n-1} - u_{n-1}}{r_{n-1}}\rangle \geq 0, \ \forall y \in C.$$

This implies that

$$\begin{aligned} 0 &\leq \langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - u_{n-1} - \frac{r_{n-1}}{r_n}(T_{r_n}u_{n-1} - u_{n-1})\rangle \\ &= \langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1} + T_{r_n}u_{n-1} - u_{n-1} \\ &\quad -\frac{r_{n-1}}{r_n}(T_{r_n}u_{n-1} - u_{n-1})\rangle \\ &= \langle T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}, T_{r_{n-1}}u_{n-1} - T_{r_n}u_{n-1} + (1 - \frac{r_{n-1}}{r_n})(T_{r_n}u_{n-1} - u_{n-1})\rangle. \end{aligned}$$

It follows that

$$\|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\|^2 \leq |1 - \frac{r_{n-1}}{r_n}|\|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\|(\|T_{r_n}u_{n-1}\| + \|u_{n-1}\|).$$

Hence, we obtain

$$\|T_{r_n}u_{n-1} - T_{r_{n-1}}u_{n-1}\| \leq |1 - \frac{r_{n-1}}{r_n}|L' = \frac{1}{r_n}|r_n - r_{n-1}|L' \leq \frac{1}{a}|r_n - r_{n-1}|L', \tag{3.9}$$

where $L' = \sup_{n\in\mathbb{N}} \|T_{r_n} u_{n-1}\| + \sup_{n\in\mathbb{N}} \|u_{n-1}\|$. From (3.5), (3.6), and (3.9), we have

$$\begin{aligned}\|z_n - z_{n-1}\| &\leq \|u_n - u_{n-1}\| + \|T_{r_n} u_{n-1} - T_{r_{n-1}} u_{n-1}\| \\ &\leq \|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| + \frac{1}{a}|r_n - r_{n-1}|L'. \end{aligned} \tag{3.10}$$

By (3.4) and (3.10), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)(\|z_n - z_{n-1}\| + \|K_n z_{n-1} - K_{n-1} z_{n-1}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}|L \\ &\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)(\|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| \\ &\quad + \frac{1}{a}|r_n - r_{n-1}|L' + \|K_n z_{n-1} - K_{n-1} z_{n-1}\|) + |\alpha_n - \alpha_{n-1}|L \\ &\leq \alpha_n a\|x_n - x_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| \\ &\quad + \frac{1}{a}|r_n - r_{n-1}|L' + \|K_n z_{n-1} - K_{n-1} z_{n-1}\| + +|\alpha_n - \alpha_{n-1}|L \\ &= (1 - \alpha_n(1-a))\|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| \\ &\quad + \frac{1}{a}|r_n - r_{n-1}|L' + \|K_n z_{n-1} - K_{n-1} z_{n-1}\| + |\alpha_n - \alpha_{n-1}|L \\ &= (1 - \alpha_n(1-a))\|x_n - x_{n-1}\| + |r_{n-1} - r_n|\|Ax_{n-1}\| \\ &\quad + \frac{1}{a}|r_n - r_{n-1}|L' + \sup_{z\in\{z_n\}} \|K_n z - K_{n-1} z\| + |\alpha_n - \alpha_{n-1}|L. \end{aligned} \tag{3.11}$$

From the conditions (ii), (iii), Lemma 2.9 and Lemma 2.2, we can conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.12}$$

By monotonicity of A and nonexpansiveness of T_{r_n}, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(K_n z_n - z)\|^2 \\ &\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(z - r_n Az)\|^2 \\ &\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|x_n - r_n Ax_n - z + r_n Az)\|^2 \\ &= \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\|(x_n - z) - r_n(Ax_n - Az)\|^2 \\ &= \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(\|x_n - z\|^2 + r_n^2\|Ax_n - Az\|^2 \\ &\quad -2r_n\langle x_n - z, Ax_n - Az\rangle) \\ &\leq \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(\|x_n - z\|^2 + r_n^2\|Ax_n - Az\|^2 \\ &\quad -2\alpha r_n\|Ax_n - Az\|^2) \\ &= \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(\|x_n - z\|^2 \\ &\quad + r_n(r_n - 2\alpha)\|Ax_n - Az\|^2) \\ &\leq \alpha_n\|f(x_n) - z\|^2 + \|x_n - z\|^2 \\ &\quad + r_n(1-\alpha_n)(r_n - 2\alpha)\|Ax_n - Az\|^2, \end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned} r_n(1-\alpha_n)(2\alpha - r_n)\|Ax_n - Az\|^2 &\leq \alpha_n\|f(x_n) - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ &= \alpha_n\|f(x_n) - z\|^2 + (\|x_n - z\| - \|x_{n+1} - z\|) \\ &\quad (\|x_n - z\| + \|x_{n+1} - z\|) \end{aligned}$$

$$\begin{aligned} &= \alpha_n\|f(x_n)-z\|^2+(\|x_n-x_{n+1}\|) \\ &\quad (\|x_n-z\|+\|x_{n+1}-z\|). \end{aligned}$$

This implies by condition (iii) and (3.12) that

$$\lim_{n\to\infty}\|Ax_n-Az\|=0. \tag{3.14}$$

By nonexpansiveness of T_{r_n} and $I-r_nA$, we have

$$\begin{aligned} \|z_n-z\|^2 &= \|T_{r_n}(x_n-r_nAx_n)-T_{r_n}(z-r_nAz)\|^2 \\ &\leq \langle (x_n-r_nAx_n)-(z-r_nAz), z_n-z\rangle \\ &= \frac{1}{2}(\|(x_n-r_nAx_n)-(z-r_nAz)\|^2+\|z_n-z\|^2 \\ &\quad -\|(x_n-r_nAx_n)-(z-r_nAz)-(z_n-z)\|^2) \\ &\leq \frac{1}{2}(\|x_n-z\|^2+\|z_n-z\|^2-\|(x_n-z_n)-r_n(Ax_n-Az)\|^2) \\ &= \frac{1}{2}(\|x_n-z\|^2+\|z_n-z\|^2-\|x_n-z_n\|^2 \\ &\quad +2r_n\langle x_n-z_n, Ax_n-Az\rangle - r_n^2\|Ax_n-Az\|^2). \end{aligned} \tag{3.15}$$

It follows that

$$\|z_n-z\|^2\leq\|x_n-z\|^2 \tag{3.16}$$

and

$$\|z_n-z\|^2 \leq \|x_n-z\|^2-\|x_n-z_n\|^2+2r_n\|x_n-z_n\|\|Ax_n-Az\|. \tag{3.17}$$

By (3.17), we have

$$\begin{aligned} \|x_{n+1}-z\|^2 &= \|\alpha_n(f(x_n)-z)+(1-\alpha_n)(K_nz_n-z)\|^2 \\ &\leq \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\|z_n-z\|^2 \\ &\leq \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)(\|x_n-z\|^2-\|x_n-z_n\|^2 \\ &\quad +2r_n\|x_n-z_n\|\|Ax_n-Az\|) \\ &\leq \alpha_n\|f(x_n)-z\|^2+\|x_n-z\|^2-(1-\alpha_n)\|x_n-z_n\|^2 \\ &\quad +2r_n\|x_n-z_n\|\|Ax_n-Az\|. \end{aligned}$$

This implies

$$\begin{aligned} (1-\alpha_n)\|x_n-z_n\|^2 &\leq \alpha_n\|f(x_n)-z\|^2+(\|x_n-x_{n+1}\|)(\|x_n-z\|+\|x_{n+1}-z\|) \\ &\quad +2r_n\|x_n-z_n\|\|Ax_n-Az\|. \end{aligned}$$

Again by the condition (iii), (3.12) and (3.14), we obtain

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.18}$$

Since $x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)K_nz_n$, we have $x_{n+1}-K_nz_n=\alpha_n(f(x_n)-K_nz_n)$. This implies that

$$\lim_{n\to\infty}\|x_{n+1}-K_nz_n\|=0. \tag{3.19}$$

By (3.12), (3.18) and (3.19), we have

$$\|K_nz_n-z_n\|\leq\|K_nz_n-x_{n+1}\|+\|x_{n+1}-x_n\|+\|x_n-z_n\|\to 0 \text{ as } n\to\infty. \tag{3.20}$$

Putting $z_0=P_\Omega f(z_0)$. Then

$$\langle f(z_0)-z_0, z_0-z\rangle\geq 0\ \forall z\in\bigcap_{i=1}^{\infty}F(T_i)\cap GMEP(F,A,\varphi). \tag{3.21}$$

We shall show that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0, x_n-z_0\rangle \le 0. \tag{3.22}$$

To show this inequality, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0, x_n-z_0\rangle = \lim_{k\to\infty}\langle f(z_0)-z_0, x_{n_k}-z_0\rangle. \tag{3.23}$$

Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k\to\infty$ where $\omega\in C$. We first show $\omega \in GMEP(F,A,\varphi)$. Then we have $z_{n_k} \rightharpoonup \omega$ as $k\to\infty$. From $z_n = T_{r_n}(x_n - r_nAx_n)$, we obtain

$$F(z_n,y)+\varphi(y)-\varphi(z_n)+\langle Ax_n, y-z_n\rangle + \frac{1}{r_n}\langle y-z_n, z_n-x_n\rangle \ge 0,\ \forall y\in C.$$

From $(A2)$, we have
$\varphi(y)-\varphi(z_n)+\langle Ax_n, y-z_n\rangle + \frac{1}{r_n}\langle y-z_n, z_n-x_n\rangle \ge F(y,z_n)$.
Then

$$\varphi(y)-\varphi(z_{n_k})+\langle Ax_{n_k}, y-z_{n_k}\rangle + \frac{1}{r_{n_k}}\langle y-z_{n_k}, z_{n_k}-x_{n_k}\rangle \ge F(y,z_{n_k}),\ \forall y\in C. \tag{3.24}$$

For $t\in[0,1]$ and $y\in C$, put $z_t = ty+(1-t)\omega$. Then, $z_t\in C$. So, from (3.24), we have

$$\begin{aligned}
\langle z_t - z_{n_k}, Az_t\rangle &\ge \langle z_t - z_{n_k}, Az_t\rangle - \langle z_t - z_{n_k}, Ax_{n_k}\rangle + F(z_t, z_{n_k}) - \varphi(z_t) + \varphi(z_{n_k})\\
&\quad -\langle z_t - z_{n_k}, \frac{z_{n_k}-x_{n_k}}{r_{n_k}}\rangle\\
&= \langle z_t - z_{n_k}, Az_t - Az_{n_k}\rangle + \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle\\
&\quad +F(z_t, z_{n_k}) - \varphi(z_t) + \varphi(z_{n_k})\\
&\quad -\langle z_t - z_{n_k}, \frac{z_{n_k}-x_{n_k}}{r_{n_k}}\rangle\\
&\ge \alpha\|Az_t - Az_{n_k}\|^2 + \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle\\
&\quad +F(z_t, z_{n_k}) - \varphi(z_t) + \varphi(z_{n_k})\\
&\quad -\langle z_t - z_{n_k}, \frac{z_{n_k}-x_{n_k}}{r_{n_k}}\rangle\\
&\ge \langle z_t - z_{n_k}, Az_{n_k} - Ax_{n_k}\rangle + F(z_t, z_{n_k}) - \varphi(z_t) + \varphi(z_{n_k})\\
&\quad -\langle z_t - z_{n_k}, \frac{z_{n_k}-x_{n_k}}{r_{n_k}}\rangle.
\end{aligned}$$

It follows from (3.18), $(A4)$ and weakly lower semicontinuity of φ that

$$\langle z_t-\omega, Az_t\rangle + \varphi(z_t) - \varphi(\omega) \ge F(z_t,\omega). \tag{3.25}$$

From $(A1)$, $(A4)$ and (3.25), we also have

$$\begin{aligned}
0 &= F(z_t,z_t) \le tF(z_t,y) + (1-t)F(z_t,\omega)\\
&\le tF(z_t,y) + (1-t)(\langle z_t-\omega, Az_t\rangle + \varphi(z_t) - \varphi(\omega))\\
&\le tF(z_t,y) + (1-t)(t\langle y-\omega, Az_t\rangle + t\varphi(y) + (1-t)\varphi(\omega) - \varphi(\omega))\\
&= tF(z_t,y) + (1-t)(t\langle y-\omega, Az_t\rangle + t\varphi(y) - t\varphi(\omega))\\
&= tF(z_t,y) + (1-t)t(\langle y-\omega, Az_t\rangle + \varphi(y) - \varphi(\omega)),
\end{aligned}$$

hence

$$0 \le F(z_t,y) + (1-t)(\langle y-\omega, Az_t\rangle + \varphi(y) - \varphi(\omega)).$$

Letting $t \to 0$, we have

$$0 \le F(\omega, y) + \langle y - \omega, A\omega\rangle + \varphi(y) - \varphi(\omega)$$

for all $y \in C$ and hence $\omega \in GMEP(F, A, \varphi)$.

Next, we show that $\omega \in \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $\omega \ne K\omega$. By using the Opial property, (3.20) and Lemma 2.10 we have

$$\begin{aligned}\liminf_{k\to\infty} \|z_{n_k} - \omega\| &< \liminf_{k\to\infty} \|z_{n_k} - K\omega\| \\ &\le \liminf_{k\to\infty} (\|z_{n_k} - K_{n_k} z_{n_k}\| + \|K_{n_k} z_{n_k} - K_{n_k}\omega\| + \|K_{n_k}\omega - K\omega\|) \\ &\le \liminf_{k\to\infty} \|z_{n_k} - \omega\|,\end{aligned}$$

which is a contradiction. Thus $K\omega = \omega$, so $\omega \in F(K)$. By Lemma 2.8, we obtain that $\omega \in \bigcap_{i=1}^{\infty} F(T_i)$. Hence $\omega \in \bigcap_{i=1}^{\infty} F(T_i) \bigcap GMEP(F, A, \varphi)$.
Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \bigcap_{i=1}^{\infty} F(T_i) \bigcap GMEP(F, A, \varphi)$, by (3.21), we have

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0\rangle = \langle f(z_0) - z_0, \omega - z_0\rangle \le 0. \tag{3.26}$$

From Lemma 2.3 and (3.16) we have

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\alpha_n(f(x_n) - z_0) + (1-\alpha_n)(K_n z_n - z_0)\|^2 \\ &\le (1-\alpha_n)^2 \|K_n z_n - z_0\|^2 + 2\alpha_n \langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n)^2 \|z_n - z_0\|^2 + 2\alpha_n \langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n \langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\ &\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + 2\alpha_n a \|x_n - z_0\| \|x_{n+1} - z_0\| \\ &\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\alpha_n)^2 \|x_n - z_0\|^2 + \alpha_n a\{\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\} \\ &\quad + 2\alpha_n \langle f(z_0) - z_0, x_{n+1} - z_0\rangle.\end{aligned} \tag{3.27}$$

This implies

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &\le \frac{(1-\alpha_n)^2 + \alpha_n a}{1 - \alpha_n a} \|x_n - z_0\|^2 + \frac{2\alpha_n}{1 - \alpha_n a} \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &= \frac{1 - 2\alpha_n + \alpha_n a}{1 - \alpha_n a} \|x_n - z_0\|^2 + \frac{\alpha_n^2}{1 - \alpha_n a} \|x_n - z_0\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n a} \langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le (1 - \frac{2(1-a)\alpha_n}{1 - \alpha_n a}) \|x_n - z_0\|^2 \\ &\quad + \frac{2(1-a)\alpha_n}{1 - \alpha_n a} \{\frac{\alpha_n M}{2(1-a)} + \frac{1}{1-a} \langle f(z_0) - z_0, x_{n+1} - z_0\rangle\},\end{aligned} \tag{3.28}$$

where $M = \sup\{\|x_n - z_0\|^2 : n \in \mathbb{N}\}$. Put $\beta_n = \frac{2(1-a)\alpha_n}{1 - a\alpha_n}$ and $\delta_n = \frac{2(1-a)\alpha_n}{1-\alpha_n a} \{\frac{\alpha_n M}{2(1-a)} + \frac{1}{1-a} \langle f(z_0) - z_0, x_{n+1} - z_0\rangle\}$. Then

$$\|x_{n+1} - z_0\|^2 \le (1 - \beta_n)\|x_n - z_0\|^2 + \delta_n \ \forall n \in \mathbb{N}. \tag{3.29}$$

It follows from assumption (iii) that $\sum_{n=1}^{\infty} \beta_n = \infty$. By (3.26) and $\lim_{n\to\infty} \alpha_n = 0$, we have $\limsup_{n\to\infty} \frac{\delta_n}{\beta_n} \le 0$. By Lemma 2.2, we obtain that $\|x_n - z_0\| \to 0$ as $n \to \infty$, This completes the proof. □

By setting $A \equiv 0$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let C be a bounded closed convex subset of a real Hilbert space, $F : C \times C \to \mathbb{R}$ be a bifunction sastisfing the condition (A1)-(A4) and $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semi-continuous and convex function and $f : C \to C$ be a contraction map with coefficient $0 < a < 1$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\Omega = \bigcap_{i=1}^{\infty} F(T_i) \bigcap MEP(F,\varphi) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Let K_n be the K-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ for each $n \in \mathbb{N}$. Let $x_1 \in C$ and $\{z_n\}$ and $\{x_n\}$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) K_n z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3.30}$$

where

$\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [0, 2\alpha]$ sastisfy the following conditions:

(i) $0 < a \leq r_n \leq b < 2\alpha$,

(ii) $\displaystyle\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

(iii) $\displaystyle\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\displaystyle\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ and $\{z_n\}$ converge strongly to $z_0 \in \Omega$, where $z_0 = P_\Omega f(z_0)$.

4. NUMERICAL RESULT

In this section, we give a numerical example of using the iterative method introduced in our main result. Let $H = \mathbb{R}$ and $C = [-5, 5]$. For $n \in \mathbb{N}$, let $T_n : C \longrightarrow C$ be defined by

$$T_n x = \begin{cases} x & , \text{if } x \geq 0. \\ -\frac{n}{n+1} x & , \text{if } x < 0. \end{cases} \tag{4.1}$$

and Let $F : C \times C \to \mathbb{R}$, $\varphi : C \to \mathbb{R}$, $f : C \to C$ and $A : C \to C$ be defined by

$F(y,z) = 2y^2 + 2zy - 4z^2$,
$\varphi(y) = -y^2$,
$Ax = x$,
$f(x) = \frac{x}{2}$.

It can be shown that that $GMEP(F, \varphi, A) = \{0\}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences generated by (3.1). Then we have

$$F(y, z_n) + \varphi(y) - \varphi(z_n) + < Ax_n, y - z_n > + \frac{1}{r_n} < y - z_n, z_n - x_n > \geq 0,$$

$$2y^2 + 2z_n y - 4z_n^2 - y^2 + z^2 + Ax_n y - Ax_n z + \frac{1}{r_n}(yz_n - yx_n - z_n^2 + z_n x_n) \geq 0,$$

$$r_n y^2 + 2r_n z_n y - 3r_n z^2 + r_n x_n y - r_n x_n z_n + yz_n - yx_n - z_n^2 + z_n x_n \geq 0,$$

$$r_n y^2 + 2r_n z_n y + z_n y - r_n x_n y - x_n y + r_n x_n z_n + x_n z_n - 3r_n z_n^2 - z_n^2 \geq 0,$$

$$r_n y^2 + [(2r_n + 1z_n) - (r_n + 1)]\, y + (r_n x_n z_n + x_n z_n - 3r_n z^2 z_n^2 - z_n^2) \geq 0.$$

Let $G(y) = r_n y^2 + [(2r_n + 1z_n) - (r_n + 1)]\, y + (r_n x_n z_n + x_n z_n - 3r_n z^2 z_n^2 - z_n^2)$
$G(y)$ is a quadratic function of y with coeffient $a = r_n$, $b_n = (2r_n + 1z_n) - (r_n + 1)$, $c = r_n x_n z_n + x_n z_n - 3r_n z^2 z_n^2 - z_n^2$
Determine the discriminant Δ of G as follows

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= [(2r_n + 1)z_n - (r_n + 1)x_n]^2 - 4r_n(r_n x_n z_n + x_n z_n - 3r_n z_n^2 - z_n^2) \\
&= r_n^2 x_n^2 + 2r_n x_n^2 + x_n^2 - 8r_n^2 x_n z_n - 10r_n x_n z_n + 2x_n z_n + 16r_n^2 z_n^2 + +8r_n z_n^2 + z_n^2 \\
&= \left[(r_n + 1)^2 x_n^2 - 2(r_n + 1)(4r_n + 1)x_n z_n + (4r_n + 1)^2 z_n^2\right] \\
&= [(r_n + 1)x_n - (4r_n + 1)z_n]^2
\end{aligned}$$

We know that $G(y) \geq 0\ \forall y \in C$. If it has most one solution in C, then $\Delta \leq 0$, so $z_n = \frac{(r_n+1)}{4r_n+1} x_n$. Now (3.1) becomes

$$x_{n+1} = \alpha_n \frac{x_n}{2} + (1 - \alpha_n) K_n \left((\frac{r_n + 1}{4r_{n+1}}) x_n \right). \tag{4.2}$$

Now, we set $\alpha_n = \frac{1}{10n}$, $r_n = \frac{n}{n+1}$, $\lambda_n = \frac{1}{2^n}$ and $x_1 = 1$. The following table shows numerical results of $\{x_n\}$ and $\{z_n\}$.

n	x_n	z_n
1	1.000000000	0.500000000
2	0.500000000	0.227272727
3	0.228409091	0.099928977
4	0.100404829	0.043030641
5	0.043209935	0.018281126
6	0.018347603	0.007694156
7	0.007718817	0.003216174
8	0.003225368	0.001337349
⋮	⋮	⋮
18	0.000000441	0.000000179
19	0.000000179	0.000000072
20	0.000000072	0.000000028

Table 1:

We observe that $\{x_n\}$ and $\{z_n\}$ converge to $0 \in \bigcap_{i=1}^{\infty} F(T_i) \cap GMEP(F, A, \varphi)$ and $x_{20} = 0.000000072$ and $z_{20} = 0.000000028$ are approximate solutions with accuracy at 7 decimal places.

Remark 4.1. If we use W_n instead of K_n in (3.1) we also obtain a strong convergence theorem as Theorem 3.1. The next table give comparisons of numerical results among algorithm 1, algorithm 2 and algorithm 3.

When the initial point are $x_1 = 1$ for algorithm 1 and algorithm 2, $x_1 = 0.005$ for algorithm 3. We set $\alpha_n = \frac{1}{10n}$, $r_n = \frac{n}{n+1}$, $\lambda_n = \frac{1}{2^n}$.

	Using K-mapping and $f(x) = \frac{x}{2}$	Using W-mapping and $f(x_n) = \frac{x}{2}$	Using K-mapping and $f(x) = 0.005$
n	x_n	x_n	x_n
1	-1.000000000	-1.000000000	0.005000000
2	-0.162500000	-0.162500000	0.002750000
3	-0.014295691	-0.028914536	0.001437500
4	-0.000927083	-0.004701357	0.000077461
5	-0.000050769	-0.000073748	0.000044867
6	-0.000002513	-0.000011301	0.000028603
7	-0.000000116	-0.000001704	0.000020128
8	-0.000000005	-0.000000254	0.000015410

Table 2:

Algorithm 1:

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)K_n z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.3}$$

Algorithm 2:

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)W_n z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.4}$$

Algorithm 3:

$$\begin{cases} F(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)K_n z_n, \ \forall n \in \mathbb{N}. \end{cases}$$

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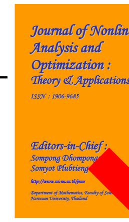

THE STABILITY OF GAUSS MODEL HAVING ONE-PREY AND TWO-PREDATORS

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ABSTRACT. Scientists are interesting to find conditions to continuously use of living resources at the same time. In the present paper some Gauss predator-prey models in which tree ecologically interacting species has been considered and the behavior of their solutions in the stability aspect have been investigated. The main aim is to present a mathematical analysis for the above models as global and local stability. Finally, stability of some examples of Gauss model with two preys and on predator are discussed.

KEYWORDS : Equilibrium point; Gauss Model; Predator-Prey System; locally asymptotically Stability.
AMS Subject Classification: 93A30, 9[illegible], 34D23, 65H17.

1. INTRODUCTION

The problem of predator-prey is well-known and an old problem in mathematical biology. Gauss is one of scientist that was studied predator-prey problem and he was obtained many results to interpret and analyze this problem. In (1934) Gauss and in (1936) Gauss and Smaragdov was studied generalization of the following model as a model for predator-prey interactions:

$$\begin{cases} \frac{dx}{dt} = ax - yp(x) \\ \\ \frac{dy}{dt} = y(-\gamma + cp(x)) \end{cases} \tag{1.1}$$

Above model states that the prey growth is enhanced by its own presence and its increase growth is limited at predators present, but the predator growth is decreased by its own presence and its growth rate is enhanced at preys present.
More general form of this model as an intermediate model of predator-prey interactions is as follow:

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$$\begin{cases} \frac{dx}{dt} = xg(x) - yp(x) \\ \\ \frac{dy}{dt} = y(-\gamma + q(x)) \end{cases} \tag{1.2}$$

Here the function $g(x)$ is the specific growth rate of the prey in the absence of any predators and represents the relative increase of preys in unit of time. The function $p(x)$ is efficiency of predator on particular prey and expresses the number of prey consumed by a predator in a unit of time. The function $q(x)$ is the predator response function with respect to that particular prey. The second statement (1.2) describes the growth rate of the predator population and the function $q(x) - \gamma$ gives the total increase of the predator population. It is clear that in the absence of prey the predator population declines.
Some of properties of $p(x)$, $q(x)$ and $g(x)$ will be studied in continuance. First,

$$g(0) = \alpha \geq 0 \text{ , } g(x) \text{ is continuous and differentiable}$$
$$\text{for } x \geq 0 \text{ , } g_x \leqslant 0$$

When the environment has a carrying capacity, there

$$k > 0 \; ; \; g(k) = 0$$

This last assumption is biologically realistic.
The term $p(x)$ will have the following properties:

$$p(0) = 0 \text{ , } p(x) \text{ is continuous and differentiable}$$
$$\text{for } x \geq 0 \text{ , } \frac{dp(x)}{dx} > 0$$

As a consequence, we have:

$$\lim_{x \to \infty} p(x) = p_\infty \text{ , } 0 < p_\infty \leq \infty$$

For definiteness we let

$$\frac{dp(x)}{dx}(0) > \beta$$

In Gauss model (1.1) $q(x) = cp(x)$. It will be helpful that we think $q(x)$ is in manner of $p(x)$. Essentially, $q(x)$ will have properties similar to $p(x)$, namely,

$$q(0) = 0 \text{ , } q(x) \text{ is continuous and differentiable}$$
$$\text{for } x \geq 0 \text{ , } \frac{dq(x)}{dx} > 0$$
$$\lim_{x \to \infty} q(x) = q_\infty \text{ , } 0 < q_\infty \leq \infty$$
$$\frac{dq(x)}{dx}(0) > \delta$$

System (1.2) always has two equilibrium points $(0, 0)$ and $(0, k)$.The prey isocline is $\frac{xg(x)}{p(x)}$.

2. THE PREDATOR-PREY GAUSS MODEL WITH TWO PREY AND ONE PREDATOR

Let x and y are density of preys species and z is density of predator species. Following system represents Gauss model with having two preys and one predator:

$$\begin{cases} \frac{dx}{dt} = a_1 x - zp_1(x) \\ \\ \frac{dy}{dt} = a_2 y - zp_2(y) \\ \\ \frac{dz}{dt} = -c_1 z + c_2 zp_1(x) + c_3 zp_2(x) \end{cases} \tag{2.1}$$

In above system two preys species live in an ecosystem independently and each species is bait of special predator z and all of coefficients a_1, a_2, c_1, c_2 and c_3 are positive constant. In this system preys enhance in absences of predator species and this increasing is limited by terms $-zp_1(x)$ and $-zp_2(x)$ respectively. In absence of preys density of predators populations decrease and preys have positive efficiency on predator population.

For example, let two species rabbit and rat live in an ecosystem and each of two species is baits of fox species. More than assumptions of Gauss model, assume that p_1 and p_2 has properties of $p(x)$ in Gauss model.

In system (2.1) following properties are holds:

i. Equilibrium point $(0,0,0)$ is stable point for system (2.1), in other hand when density of populations is zero density of population species will not change in different time.

ii. If population density of one of preys species is zero, then system (2.1) convert to system (1.1).

iii. If population density of predator species is zero, then system (2.1) convert to system with having two species that live in an ecosystem independently.

iv. If population density of two species are zero, then system (2.1) convert to equation of growth rate.

v. Orbit of solutions system (2.1) is $intR_3^+ = int\{x_i | x_i \geqslant 0, i = 1,2,3\}$.

The terms $p_1(x)$ and $p_2(y)$ has properties described as follows:

$$p_1(0) = 0 \text{ , } p_1(x) \text{ is continuous and differentiable}$$
$$\text{for } x \geq 0 \text{ , } \frac{dp_1(x)}{dx} > 0$$
$$p_2(0) = 0 \text{ , } p_2(y) \text{ is continuous and differentiable}$$
$$\text{for } y \geq 0 \text{ , } \frac{dp_2(y)}{dy} > 0$$

3. LOCAL STABILITY

We using the linearization method to study the stability of system (2.1). For this means, we access the jacobian matrix, which may be found as the follow:

$$J|_{(x,y,z)} = \begin{pmatrix} a_1 - z\frac{dp_1(x)}{dx} & 0 & -p_1(x) \\ 0 & a_2 - z\frac{dp_2(y)}{dy} & -p_2(y) \\ c_2 z\frac{dp_1(x)}{dx} & c_3 z\frac{dp_2(y)}{dy} & -c_1 + c_2 p_1(x) + c_3 p_2(y) \end{pmatrix}$$

Now let $(\overline{x}, \overline{y}, \overline{z})$ be equilibrium point of system (2.1). Then

$$A = J|_{(\overline{x},\overline{y},\overline{z})} = \begin{pmatrix} a_1 - \overline{z}\frac{dp_1(\overline{x})}{dx} & 0 & -p_1(\overline{x}) \\ 0 & a_2 - \overline{z}\frac{dp_2(\overline{y})}{dy} & -p_2(\overline{y}) \\ c_2\overline{z}\frac{dp_1(\overline{x})}{dx} & c_3 z\frac{dp_2(\overline{y})}{dy} & -c_1 + c_2 p_1(\overline{x}) + c_3 p_2(\overline{y}) \end{pmatrix}$$

So if $trA < 0$ and $detA > 0$, then system (2.1) is locally asymptotically stable.

Let $p_1(\overline{x}), p_2(\overline{y}) > 0$

$$A_1 = a_1 - \overline{z}\frac{dp_1(\overline{x})}{dx}$$
$$A_2 = a_2 - \overline{z}\frac{dp_2(\overline{y})}{dy}$$
$$A_3 = -c_1 + c_2 p_1(\overline{x}) + c_3 p_2(\overline{y})$$
$$A_4 = c_2\overline{z}\frac{dp_1(\overline{x})}{dx}$$
$$A_5 = c_3 z\frac{dp_2(\overline{y})}{dy}$$

So

$$A = J|_{(\bar{x},\bar{y},\bar{z})} = \begin{pmatrix} A_1 & 0 & -p_1(\bar{x}) \\ 0 & A_2 & -p_2(\bar{y}) \\ A_4 & A_5 & A_3 \end{pmatrix}$$

Therefore

$$detA = A_1A_2A_3 - p_2A_1A_5 + p_1A_2A_4$$

Now if $A_1, A_3 < 0$ and $A_2 > 0$ Then $detA > 0$, if $A_2 < A_1 + A_3$ too, then $trA < o$ and so system (2.1) is locally asymptotically stable.
So following proposition is proved.

Proposition 3.1. *Let $p_1(\bar{x}), p_2(\bar{y}) > 0$, the system (2.1) is locally asymptotically stable in equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ provided $A_1, A_3 < 0$ and $A_2 > 0$ and $A_2 < A_1 + A_3$* .

4. GLOBAL STABILITY

In this section, we will prove the global stability of the system (2.1) by constructing a suitable Lyapunove function.

Theorem 4.1. *The system (2.1) is globally asymptotically stable in equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ provided $x < \bar{x}$, $y < \bar{y}$ and $z > \bar{z}$.*

Proof:
Let us consider a suitable Lyapunove function:

$$v(x, y, z) = hx + ky + z$$

where $h = c_2$ and $k = c_3$. Obviously v is positive definite. Now the time derivative of v along the solution of (2.1) is given by:

$$\frac{dv}{dt} = h\frac{dx}{dt} + k\frac{dy}{dt} + \frac{dz}{dt}$$

Now by substituting $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ from system (2.1) $\frac{dv}{dt}$ is given by:

$$\frac{dv}{dt} = h\frac{dx}{dt} + k\frac{dy}{dt} + \frac{dz}{dt}$$

$\frac{dv}{dt} = h[a_1x - zp_1(x) - a_1\bar{x} + \bar{z}p_1(\bar{x})] + k[a_2y - zp_2(y) - a_2\bar{y} + \bar{z}p_2(\bar{y})] + [-c_1z + c_2zp_1(x) + c_3zp_2(y) + c_1\bar{z} - c_2\bar{z}p_1(\bar{x}) - c_3\bar{z}p_2(\bar{y})] = ha_1(x - \bar{x}) - h(zp_1(x) - \bar{z}p_1(\bar{x})) + ka_2(y - \bar{y}) - h(zp_2(y) - \bar{z}p_2(\bar{y})) - c_1(z - \bar{z}) - c_2(zp_1(x) - \bar{z}p_1(\bar{x})) + c_3(zp_2(y) - \bar{z}p_2(\bar{y}))$
As regarding to $h = c_2$ and $k = c_3$

$$\frac{dv}{dt} = c_2a_1(x - \bar{x}) + c_3a_2(y - \bar{y}) - c_1(z - \bar{z})$$

Therefore $\frac{dv}{dt} < 0$ if $x < \bar{x}$, $y < \bar{y}$ and $z > \bar{z}$.

4.1. Analysis of Example 1.

Consider following system:

$$\begin{cases} \frac{dx}{dt} = a_1x - b_1xz \\ \frac{dy}{dt} = a_2y - b_2yz \\ \frac{dz}{dt} = -c_1z + b_1c_2zx + b_2c_3zy \end{cases} \quad (4.1)$$

In the above system all of coefficients $a_1, a_2, b_1, b_2, c_1, c_2$ and c_3 are positive constant. In system (4.1) efficiency of the predator species on preys species and so efficiency of the preys species on predator species are linear.
Points $(0, 0, 0), (0, \frac{c_1}{b_2c_3}, \frac{a_2}{b_2})$ and $(\frac{c_1}{b_1c_2}, 0, \frac{a_1}{b_1})$ are equilibrium points of system (4.1), in which we analyzing stability of this point by using Jacobian matrix. Now assume that $b_1 = b_2 = b$ and $a_1 = a_1 = a$, thus intersection points of two lines $bz = a$ and $c_2bx + c_3by$ are equilibrium points of system (4.1) too. In order to analyzing

this points, which we denote these by $(\overline{x}, \overline{y}, \overline{z})$ using the Lyapunov function. First Jacobian matrix of system (4.1) is found out as follow:

$$J|_{(x,y,z)} = \begin{pmatrix} a_1 - b_1 z & 0 & -b_1 x \\ 0 & a_2 - b_2 z & -b_2 y \\ b_1 c_2 z & b_2 c_3 z & -c_1 + b_2 c_2 x + b_3 c_3 y \end{pmatrix}$$

So

$$J|_{(0,0,0)} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & -c_1 \end{pmatrix}$$

Thus equilibrium point $(0,0,0)$ is a saddle point for system(4.1).

$$J|_{(0, \frac{c_1}{b_2 c_3}, \frac{a_2}{b_2})} = \begin{pmatrix} a_1 - \frac{b_1 a_2}{b_2} & 0 & 0 \\ 0 & 0 & [illegible] \\ \frac{a_2 b_1 c_2}{b_2} & a_2 c_3 & 0 \end{pmatrix}$$

So $a_1 - \frac{b_1 a_2}{b_2}$ and $\pm i\sqrt{a_2 c_1}$ are eigenvalues of above matrix, thus equilibrium point $(0, \frac{c_1}{b_2 c_3}, \frac{a_2}{b_2})$ is hyperbolic point for system (4.1).
If $\frac{a_1}{a_2} < \frac{b_1}{b_2}$ system (4.1) is stable in equilibrium point $(\frac{c_1}{b_1 c_2}, 0, \frac{a_1}{b_1})$. Because

$$J|_{(\frac{c_1}{b_1 c_2}, 0, \frac{a_1}{b_1})} = \begin{pmatrix} 0 & 0 & -a_1 \\ 0 & a_2 - \frac{a_1 b_2}{b_1} & 0 \\ a_1 c_2 & \frac{a_1 b_2 c_3}{b_1} & 0 \end{pmatrix}$$

So zero and $a_2 - \frac{a_1 b_2}{b_1}$ are eigenvalues of above matrix, thus system (4.1)in equilibrium point $(\frac{c_1}{b_1 c_2}, 0, \frac{a_1}{b_1})$ is stable if $\frac{a_1}{a_2} < \frac{b_1}{b_2}$.
so the following proposition has been proved:

Proposition 4.1. *Following statements for system (4.1) are held:*
i. *Equilibrium point $(0,0,0)$ is a saddle point for system(4.1).*
ii. *Equilibrium point $(0, \frac{c_1}{b_2 c_3}, \frac{a_2}{b_2})$ is hyperbolic point for system (4.1).*
iii. *If $\frac{a_1}{a_2} < \frac{b_1}{b_2}$ system (4.1) is stable in equilibrium point $(\frac{c_1}{b_1 c_2}, 0, \frac{a_1}{b_1})$.*

Theorem 4.2. *The system (4.1) is stable in equilibrium point $(\overline{x}, \overline{y}, \overline{z})$.*

Proof:
Consider Lyapunov function
$v(x,y,z) = c_2 \int_{\overline{x}}^{x} \frac{s-\overline{x}}{s} ds + c_3 \int_{\overline{y}}^{y} \frac{t-\overline{y}}{t} dt + \int_{\overline{z}}^{z} \frac{v-\overline{z}}{v} dv$
Now by differentiate of above Lyapunov function with respect to variable t $\frac{dv}{dt}$ is found out as follow:
$\frac{dv}{dt} = c_2 \frac{x-\overline{x}}{x} \frac{dx}{dt} + c_3 \frac{y-\overline{y}}{y} \frac{dy}{dt} + \frac{z-\overline{z}}{z} \frac{dz}{dt}$
Now by instituted $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ from system (4.1) $\frac{dv}{dt}$ is found out as follow:
$\frac{dv}{dt} = c_2(x-\overline{x})(a_1 - b_1 z) + c_3(y-\overline{y})(a_2 - b_2 z) + (z-\overline{z})(-c_1 + b_1 c_2 x + b_2 c_3 y)$
As regarding, $b_1 = b_2 = b$ and $a_1 = a_1 = a$ and $bz = a$ and $c_2 bx + c_3$

$$\frac{dv}{dt} = 0$$

and proof is completed.

4.2. **Analysis of Example 2.** Consider following system of system (2.1):

$$\begin{cases} \frac{dx}{dt} = x(1 - \frac{z}{1+x}) \\ \frac{dy}{dt} = y(1 - \frac{z}{1+y}) \\ \frac{dz}{dt} = z(1 - \frac{z}{v_1 x} - \frac{z}{v_2 y}) \end{cases} \tag{4.2}$$

It is clear that x, y and $z \neq 0$. To obtaining equilibrium points of system (4.2) let first equation of above system is zero, in other hand $1 + x = z$. Now let tw[illegible] equation of above system is zero, in other hand $1+y = z$. two terms above conclu[illegible] $x = y$. Now constitute x and y in third equation of system (4.2) and simplify z [illegible] given by:

$$z = \frac{v_1 v_2}{v_1 + v_2} = h$$

Therefor equilibrium point(s) of system (4.2)is given by $(x, [illegible] hx)$.
Jacobian of system(4.2) is as follow:

$$J|_{(x,y,z)} = \begin{pmatrix} 1 - \frac{z}{(1+x)^2} & 0 & -\frac{x}{1+x} \\ 0 & 1 - \frac{z}{(1+y)^2} & [illegible] \\ \frac{z^2}{v_1 x^2} & \frac{z^2}{v_2 y^2} & 1 [illegible] 2\frac{z}{v_1 x} - 2\frac{z}{v[illegible]} \end{pmatrix}$$

Now by substituting equilibrium point(s) and simpli[illegible]ng, jacobia[illegible] matrix is given by:

$$A = J|_{(x,x,hx)} = \begin{pmatrix} 1 - \frac{hx}{(1+[illegible]} & 0 & [illegible] \\ 0 & 1 [illegible] & -\frac{x}{1+x} \\ \frac{h^2}{v_1} & \frac{h^2}{v_2} & -1 \end{pmatrix}$$

$$tr[illegible] = 1 - [illegible]$$

And by simplifying

$$detA = [illegible] + \frac{2[illegible]}{(1+x)[illegible]} - 1 - h^2[\frac{x^2}{(1+x)^4} + \frac{x^2}{(1+x)^3}]$$

If $trA < 0$ and $detA > 0$ th[illegible]ystem [illegible]) is locally asymptotically stable. Let $1 < \frac{hx}{(1+x)^2}$ and $1 > [illegible]$ since [illegible]ptions of proposition(3.1) is true, thus the system (4.2) is lo[illegible]lly asy[illegible]ptotically stable.
Therefor followi[illegible] pro[illegible]sitio[illegible]s proved.

P[illegible]posi[illegible] 4.2. *[illegible] $1 < [illegible]$ and $1 > \frac{hx}{(1+x)^2}$, thus the system (4.2) is locally [illegible]symptoti[illegible]lly stabl[illegible]*

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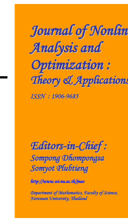

STUDY ON EXISTENCE THEORY FOR INTEGRAL EQUATIONS OF MIXED TYPE

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ABSTRACT. This article presents an existence of monotonic solution for the nonlinear mixed Volterra-Fredholm integral equation acting on $L_1[0,1]$. By using the techniques of the De Blasi measure of weak noncompactness, we employ the basic fixed point theorems such as Darbo's theorem to obtain the mentioned aims in Banach spaces.

KEYWORDS : Measure of weak noncompactness; Nonlinear mixed integral equation; Darbo's theorem; Banach algebra.
AMS Subject Classification: 35B30, 35B40.

1. INTRODUCTION

In the present paper, we try to prove the existence of monotonic solution of the following mixed Volterra-Fredholm integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_0^1 g(x,t,\epsilon,\tau)h(\epsilon,\tau,u(\epsilon,\tau))\, d\epsilon\, d\tau \ (x,t) \in [0,t]\times[0,1], \quad (1.1)$$

in $L_1[0,1]$.

Here $u(x,t)$, is an unknown function, and the functions $f(x,t)$, $g(x,t,\epsilon,\tau)$ and $h(\epsilon,\tau,u(\epsilon,\tau))$ are analytic on $D=[0,1]\times[0,T]$ and $t\in[0,1]$.

This equation arises in the theory of parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic and varies physical and biological problems.

The aim of this paper is to use Theorem 2.1 and the fixed point theorem with respect to a measure of weak noncompactness in the Banach algebras for solving Eq.1.1.

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2. BACKGROUND CONCEPTS

Let X be a banach space and θ its zero vector. Denote by B_r the closed ball in X centered at θ and with radius r. Let H be a nonempty bounded subset of X, De-Blasi measure of weak noncompactness β is defined in [1] as

$\beta(H) = inf\{ r > 0 :$ there exists a weakly compact subset W of X such that $H \subset W + B_r\}$.

β has the following useful properties (see [2] for a proof):

i) $\beta(H) = 0$ if and only if H is relatively weakly compact;

ii) $\beta(H_1) \leq \beta(H_2)$ if $H_1 \subseteq H_2$;

iii) $\beta(H) = \beta(\overline{co}(H))$;

iv) $\beta(rH) = \mid r \mid \beta(H),\ r \in R$;

v) If (H_n) is a decreasing sequence of nonempty, bounded, closed and convex of X with $\beta(H_n) \to 0$, then $H = \bigcap H_n$ is nonempty (obviously, it is closed, convex and relatively weakly compact by i) and ii)).

The Appell and De Pascal formula [3]

$$\beta(H) = \lim_{\epsilon \to 0}\{\sup_{\psi \in H}\{sup\ [\int_D |\psi(t)|dt\ :\ D \subset [0,1]\times[0,1],\ meas(D) \leq \epsilon\ ]\}\},$$

where $meas(D)$ is the Lebesgue measure of D, so easy in calculation.

Another useful measure is the Hausdorff measure χ of noncompactness is defined as [4]

$\chi(H) = inf\{ r > 0 :$ there exists a compact subset W of X such that $H \subset W + B_r\}$.

Both β and χ have several properties, also these two measure are connected in the following theorem.

Theorem 2.1. [5]*Let H be an arbitrary nonempty and bounded subset of $L_1[0,1]$. If H is compact in measure then $\beta(H) = \chi(H)$.*

Consider the following integral operator

$$Gu(x,t) = \int_0^t \int_0^1 g(x,t,\epsilon,\tau)u(\epsilon,\tau)\ d\tau d\epsilon,\ (x,t) \in [0,1]\times[0,1]$$

generated by the kernel $g : [0,1]\times[0,1]\times[0,1]\times[0,1] \to R$ which is assumed to be measurable with respect to all its variables. Then we have the following proposition [6].

Proposition 2.2. [6]*If the linear integral operator G transforms the space $L_1[0,1]$ into itself, then G is continuous on $L_1[0,1]$.*

3. MAIN RESULT

In this section we discuss the solvability of the nonlinear integral equation of mixed type

$$u(x,t) = f(x,t) + \int_0^t \int_0^1 g(x,t,\epsilon,\tau)h(\epsilon,\tau,u(\epsilon,\tau))\ d\epsilon\ d\tau,\ (x,t) \in [0,1]\times[0,1].$$

For simplicity we suppose the operator

$$(Tu)(x,t) = f(x,t) + \int_0^t \int_0^1 g(x,t,\epsilon,\tau)h(\epsilon,\tau,u(\epsilon,\tau))\, d\epsilon d\tau,\ (x,t) \in [0,1] \times [0,1]. \tag{3.1}$$

In what follows, we formulate the assumptions under which Eq.3.1 will be investigated. Namely we assume the following hypotheses:

H_1) $f : [0,1] \times [0,1] \to R$ is a continuous function.

H_2) $h : [0,1] \times [0,1] \times R \to R$ is continuous and satisfies in sublinear condition, so that there exist the nonnegative constants c and d such that

$$h(\epsilon,\tau,u(\epsilon,\tau)) < c + d \mid u(\epsilon,\tau) \mid .$$

H_3) $g : [0,1] \times [0,1] \times [0,1] \times [0,1] \to R$ is continuous function and measurable and the integral operator G maps L_1 into itself.

H_4) There exists a positive constant, satisfies $d \parallel G \parallel < 1$.

Theorem 3.1. *Let the assumptions* $(H1) - (H4)$ *be satisfied. Eq.(1) has at least one solution in* $L_1[0,1]$.

Proof. In view the assumptions and Proposition 2.2 ,we deduce that the operator T maps continuously the space L_1 into itself. Further applying our assumptions we derive the following estimate:

$$\parallel (Tu)(x,t) \parallel = \int_0^1 \int_0^1 |f(x,t) + \int_0^t \int_0^1 g(x,t,\epsilon,\tau)h(\epsilon,\tau,u(\epsilon,\tau))\, d\epsilon d\tau|\, dxdt$$

$$\leq \int_0^1 \int_0^1 f(x,t)dxdt + \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mid g(x,t,\epsilon,\tau) \mid\mid h(\epsilon,\tau,u(\epsilon,\tau)) \mid\ d\epsilon d\tau dxdt$$

$$\leq \parallel f \parallel + \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mid g(x,t,\epsilon,\tau) \mid [c + d \mid u(\epsilon,\tau) \mid]\, d\epsilon d\tau dxdt$$

In the assumption (H_3), g is continuous function in L_1. Then it is bounded, namely $\mid g \mid \leq M$ (M is positive constant). We have

$$\parallel (Tu)(x,t) \parallel \leq \parallel f \parallel + Mc + d \parallel G \parallel \parallel u(\epsilon,\tau) \parallel$$

Hence the operator T maps the ball B_r into itself where

$$r = \frac{\parallel f \parallel + Mc}{1 - d \parallel G \parallel} > 0.$$

Next, let Q_r be a subset of B_r consisting of all functions that are a.e nonnegative and nondecreasing on $[0,1]$, then as in [7] we deduce that Q_r is nonempty, closed, convex, bounded and compact in measure. Moreover, by using the assumptions and Proposition 2.2 we deduce that the operator T transforms Q_r continuously into Q_r.

Finally, we show that the operator T is contraction to the De-Blasi measure of weak non-compactness. For this, let $U \subset Q_r$, $\epsilon > 0$, then for a measurable subset $D \subset [0,1] \times [0,1]$ with $meas.(D) \leq \epsilon$ and for arbitrary $u \in H$ we have

$$\int_D |(Hu)(x,t)|dt \leq \parallel f \parallel_{L_1(D)} + \int_D \int_0^1 \int_0^1 \mid g(x,t,\epsilon,\tau) \mid [c + d \mid u(\epsilon,\tau) \mid]\, d\epsilon d\tau ds$$

$$\leq \| f \|_{L_1(D)} + cMmeas(D) + d \int_D \int_0^1 \int_0^1 | g(x,t,\epsilon,\tau) || u(\epsilon,\tau) | d\epsilon d\tau ds$$

$$\leq \| f \|_{L_1(D)} + cMmeas(D) + d \| G \|_D \| u(\epsilon,\tau) \|_{L_1(D)},$$

where $\| G \|_D$ is the norm of the operator $G : L_1(D) \to L_1(D)$.

Since

$$\| f \|_{L_1(D)} = \lim_{\epsilon \to 0} \{ \ sup \ \{ \int_D f(x,t) ds \ : D \subset [0,1] \times [0,1], meas(D) \leq \epsilon \} \} = 0.$$

Thus as $\epsilon \to 0$ and take the supremum for all $u \in U, \ D \subset [0,1] \times [0,1]$, we see that $\beta(TU) \leq d \| G \| \beta(U)$.

Since $U \subset Q_r$ and Q_r is compact in measure, then we have

$\chi(TU) \leq d \| G \| \chi(U)$.

Using all properties of Q_r and applying Darbo fixed point [8] we deduce that our operator T has a fixed point which is the solution of Eq.3.1 and the proof is complete. □

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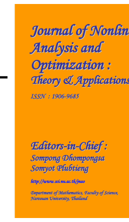

STUDY OF HARMONIC MULTIVALENT MEROMORPHIC FUNCTIONS BY USING GENERALIZED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In this paper we have studied a class of complex valued multivalent meromorphic harmonic and orientation preserving functions by using the generalized hypergeometric functions in the punctured unit disk and we have obtained coefficient estimates, distortion theorem . Other interesting properties are also investigated.

KEYWORDS : Multivalent functions; Meromorphic functions; Harmonic functions; Distortion theorem; Starlike functions.
AMS Subject Classification: 30C45.

1. INTRODUCTION

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain D , we can write
$f = h + \overline{g}$, where h and g are analytic in D.We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for $f$to be univalent and orientation preserving in D that $|h'(z)| > |g'(z)|$ in D (see [4]).

W. Hengartner and G. Schober [2], considered harmonic sense preserving univalent mappings defined on $\overline{\mathcal{N}U} = \{z : |z| > 1\}$ that map ∞ to ∞ and represented by
$f(z) = h(z) + \overline{g(z)} + A \log |z|$ where $h(z) = \alpha z + \sum\limits_{n=0}^{\infty} a_n z^{-n}, g(z) = \underline{z} + \sum\limits_{n=1}^{\infty} b_n z^{-n}$
are analytic in $\overline{\mathcal{N}U}$ and $|\alpha| > |\underline{!} \geq 0, A \in \mathbb{C}$, further $\frac{\overline{f_{\overline{z}}}}{f_z}$ is analytic and $\left|\frac{\overline{f_{\overline{z}}}}{f_z}\right| < 1$. Jahangiri [6] , O Ahuja and Jahangiri [1] and Murugusundaramoorthy [7] have studied classes of meromorphic harmonic functions.

Let us denoted the family $\Sigma_p(H)$ consisting of all harmonic sense-preserving multivalent meromorphic mapping in $\mathcal{N}U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$

$$f(z) = h(z) + \overline{g(z)} \tag{1-1}$$

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where

$$h(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1} z^{(n+p-1)}, g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{(n+p-1)},\ |b_p| < 1. \quad (1\text{-}2)$$

Also , we denote by $\overline{\Sigma_p(H)}$ the subfamily of $\Sigma_p(H)$ consisting of harmonic functions $f = h + \overline{g(z)}$ of the form

$$f(z) = z^{-p} - \sum_{n=1}^{\infty} a_{n+p-1} z^{(n+p-1)} + \overline{\sum_{n=1}^{\infty} b_{n+p-1} z^{(n+p-1)}},\ (|b_p| < 1). \quad (1\text{-}3)$$

The β- Convolution of$\phi(z)$ and $\psi(z)$ where

$$\phi(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1} z^{(n+p-1)} + \sum_{n=1}^{\infty} b_{n+p-1} z^{(n+p-1)} \quad (1\text{-}4)$$

and

$$\psi(z) = z^{-p} + \sum_{n=1}^{\infty} c_{n+p-1} z^{(n+p-1)} + \sum_{n=1}^{\infty} d_{n+p-1} z^{(n+p-1)} \quad (1\text{-}5)$$

is defined by

$$(\phi \otimes_\beta \psi)(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{a_{n+p-1} c_{n+p-1}}{(n+p-1)^\beta} \overline{z^{(n+p-1)}} + \sum_{n=1}^{\infty} \frac{b_{n+p-1} d_{n+p-1}}{(n+p-1)^\beta} \overline{z^{(n+p-1)}}. \quad (1\text{-}6)$$

The 0-convolution of ϕ and ψ is the familiar Hadamard product,also the 1-convolution of ϕ and ψ is named integral convolution.
For real or complex numbers $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2 ...., \beta_s (\beta_j \neq 0, -1, -2, -3, ...; j = 1, 2, ..., s)$, we define the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by

$${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n .....(\alpha_q)_n}{(\beta_1)_n ......(\beta_s)_n} \frac{z^n}{n!} \quad (1\text{-}7)$$

$$(q \leq s+1; q, s \in N_0 = N \cup \{0\}; z \in \mathcal{N}U),$$

where $(x)_k$ is the pochhammer symbol, defined , in term of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}.$$

Corresponding to a function

$$\mathcal{H}_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-p}\ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z), \quad (1\text{-}8)$$

we consider a linear operator $H_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$ defined by the convolution

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) * H_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \frac{1}{z^p(1-z)^{\mu+p}}.(\mu > -p)$$

Let $H_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \Sigma_p(H) \to \Sigma_p(H)$defined by

$$H^\mu_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = H_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z). \quad (1\text{-}9)$$

$$(\alpha_j, \beta_j \neq 0, -1, -2, -3, ...; i = 1, ..., q; j = 1, 2, ..., s, \mu > -p; f \in \Sigma_p(H); z \in \mathcal{N}U^*)$$

For notational simplicity , we use shorter notation

$$H^\mu_{p,q,s}(\alpha_1) = H_p(\alpha_1, ..., \alpha_q; \beta_1, .., \beta_s)$$

Thus, from (1-9) we deduce that after simple calculations , we obtain

$$z(H^\mu_{p,q,s}(\alpha_1) f(z)) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\mu+p)_{p+n} (\beta_1)_{n+p} .....(\beta_s)_{n+p}}{(\alpha_1)_{p+n} ......(\alpha_q)_{n+p}} a_{n+p-1} z^{n+p-1}. \quad (1\text{-}10)$$

We note that the linear operator $H^{\mu}_{p,q,s}$ is closely related to the Choi- Saigo-Srivastava operator [3].In view of relationship 1-10 for harmonic function $f = h + \overline{(g)}$ given by 1-1 , we define the operator

$$H^{\mu}_{p,q,s}f(z) = H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)}.$$

A function $f \in \Sigma_p(H)$ is said to be in the subclass $JH(\alpha)$ of meromorphic harmonic α-starlike function in $\mathcal{N}U^*$ if its satisfies the condition

$$Re\left\{-\frac{z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}}{H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)}}\right\} > \alpha$$

where $(0 \geq \alpha < p)$ and $H^{\mu}_{p,q,s}f(z)$ is given by 1-10. We also let $\overline{JH(\alpha)} = JH(\alpha) \cap \overline{\Sigma_p(H)}$.
In this paper we obtain coefficient conditions for the classes $\overline{JH(\alpha)}$and $JH(\alpha)$

2. Coefficient Estimates

Theorem 2.1 : Let $f = h + \overline{g}$ where g and h are given by (1-2) if

$$\sum_{n=1}^{\infty}(1-\alpha-n-p)|a_{n+p-1}|T^{\mu}_p(n) + (n+p-1-\alpha)|b_{n+p-1}|T^{\mu}_p(n) \leq 1, \qquad (2\text{-}1)$$

then $f \in JH(\alpha)$, where $T^{\mu}_p(n) = \frac{(\mu+p)_{p+n}(\beta_1)_{n+p}.....(\beta_s)_{n+p}}{(\alpha_1)_{p+n}......(\alpha_q)_{n+p}}$
Proof :Suppose the condition (2-1) holds true, we show that

$$Re\left\{-\frac{z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}}{H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)}}\right\} = Re\left\{\frac{A(z)}{B(z)}\right\} > \alpha,$$

where
$A(z) = -z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'} = -(-pz^{-p-1} + \sum_{n=1}^{\infty} T^{\mu}_p(n)(n+p-1)a_{n+p-1}z^{n+p-2}) + z(\sum_{n=1}^{\infty} T^{\mu}_p(n)(n+p-1)\overline{b_{n+p-1}z^{n+p-2}})$
and
$B(z) = H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)} = z^{-p} + \sum_{n=1}^{\infty} T^{\mu}_p(n)a_{n+p-1}z^{n+p-1}$
$+ \sum_{n=1}^{\infty} T^{\mu}_p(n)\overline{b_{n+p-1}z^{n+p-1}}$.

Using the fact that $Rew > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$, its suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| > 0. \qquad (2\text{-}2)$$

Substituting for $A(z)$ and $B(z)$ in (2-2), and performing elementary calculations, we find that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)|$$

$> 2|z|^{-p} - 2\sum_{n=1}^{\infty}[(1-\alpha-n-p)|a_{n+p-1}| + (n+p-1-\alpha)|b_{n+p-1}|]T^{\mu}_p(n)|z|^{n+p-1}$
$> 2[1 - \sum_{n=1}^{\infty}[(1-\alpha-n-p)|a_{n+p-1}| + (n+p-1-\alpha)|b_{n+p-1}|]T^{\mu}_p(n) \geq 0,$
which implies that $f \in JH(\alpha)$.
Theorem 2.2 : For $0 \leq \alpha < p$, $f = h + \overline{g} \in \overline{JH(\alpha)}$ if and only if

$$\sum_{n=1}^{\infty}(1-\alpha-n-p)|a_{n+p-1}|T^{\mu}_p(n) + (n+p-1-\alpha)|b_{n+p-1}|T^{\mu}_p(n) \leq 1, \qquad (2\text{-}3)$$

where $T^{\mu}_p(n) = \frac{(\mu+p)_{p+n}(\beta_1)_{n+p}.....(\beta_s)_{n+p}}{(\alpha_1)_{p+n}......(\alpha_q)_{n+p}}$

Proof : Since$\overline{JH(\alpha)} \subset JH(\alpha)$, we need to prove the "only if" part. To this end, for functions f of the form (1-3) , we have

$$Re\left\{-\frac{z(H^{\mu}_{p,q,s}h(z))' - \overline{z(H^{\mu}_{p,q,s}g(z))'}}{H^{\mu}_{p,q,s}h(z) + \overline{H^{\mu}_{p,q,s}g(z)}}\right\} > \alpha$$

implies that

$$Re\left\{\frac{z^{-p} - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)(1-\alpha-n-p)a_{n+p-1}z^{n+p-1} - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)\overline{b_{n+p-1}z^{n+p-1}}}{z^{-p} - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)a_{n+p-1}z^{n+p-1} + \sum_{n=1}^{\infty} T^{\mu}_{p}(n)\overline{b_{n+p-1}z^{n+p-1}}}\right\} > 0.$$

The above required condition must hold for all values of z in $\mathcal{N}U^*$. Upon choosing the values of z on the positive real axis where $0 = r < 1$, we must have

$$Re\left\{\frac{1 - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)(1-\alpha-n-p)a_{n+p-1}r^{n+2p-1} - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)\overline{b_{n+p-1}r^{n+2p-1}}}{1 - \sum_{n=1}^{\infty} T^{\mu}_{p}(n)a_{n+p-1}r^{n+2p-1} + \sum_{n=1}^{\infty} T^{\mu}_{p}(n)\overline{b_{n+p-1}r^{n+2p-1}}}\right\} > 0. \tag{2-4}$$

If the condition (2-3) does not hold , then the numerator in (2-4) is negative for r sufficiently close to 1 . Here , there exist $z_0 = r_0$ in (0,1) for which the quotient of (2-4) is negative .This contradicts the required condition for $f(z) \in \overline{JH(\alpha)}$.
The following result gives the distortion bounds

Theorem 2.3 :Let the function f be in the class $\overline{JH(\alpha)}$. Then , for $0 < |z| = r < 1$, we have

$$r^{-p} - \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \leq |f(z)| \leq r^{-p} + \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)}.$$

Proof :In view of Theorem 2.2 ,for $0 < |z| = r < 1$,

$$|f(z)| = |z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1}z^{n+p-1} - \overline{\sum_{n=1}^{\infty} b_{n+p-1}z^{n+p-1}}|$$

$$\leq r^{-p} + r^p \sum_{n=1}^{\infty}(a_{n+p-1} + b_{n+p-1})$$

$$\leq r^{-p} + \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \sum_{n=1}^{\infty}(p-\alpha)T^{\mu}_{p}(1)(a_{n+p-1} + b_{n+p-1})$$

$$\leq r^{-p} + \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \times \sum_{n=1}^{\infty} T^{\mu}_{p}(n)[(1-\alpha-n-p)a_{n+p-1} + (n+p-1-\alpha)b_{n+p-1}]$$

$$\leq r^{-p} + \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)}$$

and

$$|f(z)| = |z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1}z^{n+p-1} - \overline{\sum_{n=1}^{\infty} b_{n+p-1}z^{n+p-1}}|$$

$$\geq r^{-p} - r^p \sum_{n=1}^{\infty}(a_{n+p-1} + b_{n+p-1})$$

$$\geq r^{-p} - \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \sum_{n=1}^{\infty}(p-\alpha)T^{\mu}_{p}(1)(a_{n+p-1} + b_{n+p-1})$$

$$\geq r^{-p} - \frac{r^p}{(p-\alpha)T^{\mu}_{p}(1)} \times \sum_{n=1}^{\infty} T^{\mu}_{p}(n)[(1-\alpha-n-p)a_{n+p-1} + (n+p-1-\alpha)b_{n+p-1}]$$

$$\geq r^{-p} - \frac{r^p}{(p-\alpha)T_p^\mu(1)}$$

Theorem 2.4 :Let $\phi(z)$ and $\psi(z)$ have the form (1-5) and (1-6), respectively be in $\overline{JH(\alpha)}$ .Then the β- Convolution of$\phi(z)$ and $\psi(z)$ belongs to $\overline{JH(\alpha)}$ where $\beta \geq \max\left\{(\log(n+p-1))^{-1}\log\frac{1}{1-\alpha-n-p}, (\log(n+p-1))^{-1}\log\frac{1}{n+p-1-\alpha}\right\}$.

Proof : Since $\phi(z), \psi(z) \in \overline{JH(\alpha)}$ we can write

$$\sum_{n=1}^{\infty}[(1-\alpha-n-p)a_{n+p-1} + (n+p-1-\alpha)b_{n+p-1}]T_p^\mu(n) \leq 1 \qquad \text{(2-5)}$$

$$\sum_{n=1}^{\infty}[(1-\alpha-n-p)c_{n+p-1} + (n+p-1-\alpha)d_{n+p-1}]T_p^\mu(n) \leq 1,$$

also we have $c_{n+p-1} \leq \frac{1}{1-\alpha-n-p}, d_{n+p-1} \leq \frac{1}{n+p-1-\alpha}$,we must show

$$\sum_{n=1}^{\infty}[\frac{(1-\alpha-n-p)a_{n+p-1}c_{n+p-1}T_p^\mu(n)}{(n+p-1)^\beta} + \frac{(n+p-1-\alpha)b_{n+p-1}d_{n+p-1}T_p^\mu(n)}{(n+p-1)^\beta} \leq 1. \qquad \text{(2-6)}$$

For this purpose we have

$$\sum_{n=1}^{\infty}[\frac{(1-\alpha-n-p)a_{n+p-1}c_{n+p-1}}{(n+p-1)^\beta} + \frac{(n+p-1-\alpha)b_{n+p-1}d_{n+p-1}}{(n+p-1)^\beta}]T_p^\mu(n)$$
$$\leq \sum_{n=1}^{\infty}\frac{a_{n+p-1}+b_{n+p-1}}{(n+p-1)^\beta},$$

thus in view of (2-5) the (2-6) holds true if both the following hold true

$$(n+p-1)^\beta \geq \frac{1}{1-\alpha-n-p}, (n+p-1)^\beta \geq \frac{1}{n+p-1-\alpha}.$$

Equivalently if

$$\beta \geq \max\left\{(\log(n+p-1))^{-1}\log\frac{1}{1-\alpha-n-p}, (\log(n+p-1))^{-1}\log\frac{1}{n+p-1-\alpha}\right\}.$$

Theorem 2.5: Let $0 \leq \alpha \leq \gamma < 1$ and $\phi \in \overline{JH(\alpha)}, \psi \in \overline{JH(\gamma)}$. Then

$$\phi * \psi \in \overline{JH(\gamma)} \subset \overline{JH(\alpha)}$$

Proof :Its clear that$\overline{JH(\gamma)} \subset \overline{JH(\alpha)}$, also we have

$$\sum_{n=1}^{\infty}[(1-\gamma-n-p)a_{n+p-1} + (n=p-1-\gamma)b_{n+p-1}]T_p^\mu(n) \leq 1,$$

$$\sum_{n=1}^{\infty}[(1-\alpha-n-p)c_{n+p-1} + (n+p-1-\alpha)d_{n+p-1}]T_p^\mu(n) \leq 1,$$

and consequently we can write $c_{n+p-1} \leq \frac{1}{1-\alpha-n-p}, d_{n+p-1} \leq \frac{1}{n+p-1-\alpha}$, therefore we obtain

$$\sum_{n=1}^{\infty}[(1-\gamma-n-p)a_{n+p-1}c_{n+p-1} + (n+p-1-\gamma)b_{n+p-1}d_{n+p-1}]T_p^\mu(n)$$

$$\leq \sum_{n=1}^{\infty}[\frac{(1-\gamma-n-p)}{(1-\alpha-n-p)}a_{n+p-1}]T_p^{\mu}(n)$$
$$+\sum_{n=1}^{\infty}[\frac{(n+p-1-\gamma)}{(n+p-1-\alpha)}b_{n+p-1}]T_p^{\mu}(n) \leq 1$$

and this shows that

$$\phi * \psi \in \overline{JH(\gamma)}$$

Theorem 2.6: Let $f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n+p-1,j}z^{n+p-1} - \sum_{n=1}^{\infty} b_{n+p-1,j}\overline{z^{n+p-1}}$ belong to $\overline{JH(\alpha)}$,for $j = 1, 2, ...$ Then $F(z) = \sum_{j=1}^{\infty} \sigma_j f_j(z)$belongs to $\overline{JH(\alpha)}$,where $\sum_{j=1}^{\infty} \sigma_j = 1, si_j \geq 0$.

Proof :We have $F(z) = z^{-p}+\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} \sigma_j a_{n+p-1,j})z^{n+p-1}-\sum_{n=1}^{\infty}(\sum_{j=1}^{\infty} \sigma_j b_{n+p-1,j})\overline{z^{n+p-1}}$, therefore

$$\sum_{n=1}^{\infty} T_p^{\mu}(n)(1-\alpha-n-p)(\sum_{j=1}^{\infty}\sigma_j a_{n+p-1}) + T_p^{\mu}(n)(n+p-1-\alpha)(\sum_{j=1}^{\infty}\sigma_j b_{n+p-1})$$
$$= \sum_{j=1}^{\infty}\sigma_j \sum_{n=1}^{\infty} T_p^{\mu}(n)(1-\alpha-n-p)a_{n+p-1} + T_p^{\mu}(n)(n+p-1-\alpha)b_{n+p-1}$$
$$\leq \sum_{j=1}^{\infty}\sigma_j = 1.$$

Then $F(z) \in \overline{JH(\alpha)}$.

Corollary 2.7: $\overline{JH(\alpha)}$ is convex set.

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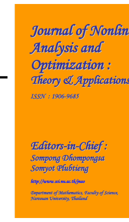

CONVERGENCE THEOREMS OF HYBRID METHODS FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF AN INFINITE FAMILY OF LIPSCHITZIAN QUASI-NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. We use a hybrid iterative method to find a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in real Hilbert spaces. We also show that our main strong convergence theorem for finding that common element can be deduced for nonexpansive mappings and applied for strict pseudo-contraction mappings. Our results extend the work by Cho et al. (2009) [4].

KEYWORDS : Generalized mixed equilibrium problem; Variational inequality; Equilibrium problem; Fixed point; Quasi-nonexpansive mappings; Strict-pseudo contraction.
AMS Subject Classification: 47H05, 47H10.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Let $B : C \to H$ be a nonlinear mapping, $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a function and $f : C \times C \to \mathbb{R}$ be a bifunction. Peng and Yao [10] considered the following *generalized mixed equilibrium problem*:

$$\text{Finding } u \in C \text{ such that } f(u,y) + \varphi(y) + \langle Bu, y-u\rangle \geq \varphi(u), \quad \forall y \in C. \tag{1.1}$$

In this paper, we denote the set of solutions of (1.1) by *GMEP* (f,φ,B). It is obvious that if u is a solution of (1.1), it implies that $u \in \text{ dom}\,\varphi = \{u \in C : \varphi(u) < +\infty\}$.

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If $B = 0$ in (1.1), we obtain the following *mixed equilibrium problem* [3]:

$$\text{Finding } u \in C \text{ such that } f(u,y) + \varphi(y) \geq \varphi(u), \quad \forall y \in C. \tag{1.2}$$

We denote the set of solutions of (1.2) by *MEP* (f, φ).

If $\varphi = 0$ in (1.1), we obtain the following *generalized equilibrium problem* [16]:

$$\text{Finding } u \in C \text{ such that } f(u,y) + \langle Bu, y-u\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

We denote the set of solutions of (1.3) by *GEP* (f, B).

If $\varphi = 0$ and $B = 0$ in (1.1), we obtain the following *equilibrium problem* [2]:

$$\text{Finding } u \in C \text{ such that } f(u,y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

We denote the set of solutions of (1.4) by *EP* (f).

If $f(x,y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *generalized variational inequality problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) + \langle Bu, y-u\rangle \geq \varphi(u), \quad \forall y \in C. \tag{1.5}$$

We denote the set of solutions of (1.5) by *GVI* (C, φ, B).

If $\varphi = 0$ and $f(x,y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *variational inequality problem* (see also [1, 5]):

$$\text{Finding } u \in C \text{ such that } \langle Bu, y-u\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

We denote the set of solutions of (1.6) by *VI* (C, B).

If $B = 0$ and $f(x,y) = 0$ for all $x, y \in C$ in (1.1), we obtain the following *minimization problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) \geq \varphi(u), \quad \forall y \in C. \tag{1.7}$$

We denote the set of solutions of (1.7) by *MP*(C, φ).

In 1994, Blum and Oettli showed that the formulation of (1.4) covered monotone inclusive problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems and Nash equilibria in noncooperative games. Several problems in physics, optimization and economics can be reduced to find solutions of (1.4). The existence of equilibrium problems has been discovered by many authors (see, for example, [1, 6, 8, 15] and the references therein). Also, some solution methods have been studied by some authors (see, for example, [6, 15, 13] and the references therein).

In 2003, Takahashi and Toyoda [16] introduced the method for finding an element of $F(S) \cap VI(C, A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to H$ is a nonexpansive mapping and $A : C \to H$ is an inverse-strongly monotone mapping. Their iteration is the following:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A x_n), n \geq 0,$$

where $x_0 \in C$, $\{\alpha_n\}$ is a sequence in (0,1), $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ and P_C is the metric projection from H onto C. They proved that, if $F(S) \cap VI(C, A) \neq \emptyset$, $\{x_n\}$ converges weakly to a point $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

Later, Takahashi and Takahashi [15] studied the contraction method for finding $F(S) \cap EP(f)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to H$

is a nonexpansive mapping, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions. Their algorithm is the following:

$$
\begin{aligned}
&x_1 \in H,\\
&f(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C,\\
&x_{n+1} = \alpha_n f_1(x_n) + (1 - \alpha_n)Sy_n \quad \forall n \geq 1,
\end{aligned}
$$

where $\{\alpha_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and some appropriate conditions. They proved that, if $F(S) \cap EP(f) \neq \emptyset$, $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $z \in F(S) \cap EP(f)$, where $z = P_{F(S)\cap EP(f)}f(z)$.

Recently, Cho et al. [4] introduced a hybrid projection method for finding $F := F(S) \cap VI(C, B) \cap GEP(f, A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to C$ is a k-strict pseudo-contraction with a fixed point, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions, $A : C \to H$ is an α-inverse-strongly monotone mapping and $B : C \to H$ is an β-inverse-strongly monotone mapping. Their iterative scheme is the following:

$$
\begin{aligned}
&x_1 \in C,\\
&C_1 = C,\\
&f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\
&z_n = P_C(u_n - \lambda_n Bu_n),\\
&y_n = \alpha_n x_n + (1 - \alpha_n)S_k z_n,\\
&C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\},\\
&x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 1,
\end{aligned}
$$

where $S_k x = kx + (1 - k)Sx$ for all $x \in C$, $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$ and $\{r_n\} \subset (0, 2\alpha)$ and some appropriate conditions. They proved that, if $F \neq \emptyset$, $\{x_n\}$ converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

In this paper, motivated by the above result, we prove a strong convergent theorem of a hybrid projection iterative method defined by (3.1) for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces. Our main result can be deduced for nonexpansive mappings applied for strict pseudo-contraction mappings. It is clear that our result generalizes the work by [4].

2. PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle\cdot, \cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Let $B : C \to H$. $x_n \to x$ means $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ implies $\{x_n\}$ converges weakly to x. We denote the set of fixed points of T by $F(T)$, i.e. $F(T) = \{x \in C : Tx = x\}$.

Recall the following definitions:

(1) A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(2) A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if
$$\|Tx - p\| \leq \|x - p\|, \quad \forall p \in F(P).$$
(3) A mapping $T : C \to C$ is said to be *Lipschitzian* if there is a positive constant L such that
$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$
(4) A mapping $T : C \to C$ is said to be *strictly pseudo-contractive* with the coefficient $k \in [0, 1)$ if
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$
(5) B is said to be *monotone* if
$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C.$$
(6) B is said to be α*-strongly monotone* if there exists a constant $\alpha > 0$ such that
$$\langle Bx - By, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$
(7) B is said to be α*-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that
$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$
(8) A set-valued mapping $T : H \to 2^H$ is said to be *monotone* if, for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.
(9) A monotone mapping $T : H \to 2^H$ is said to be *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

In the other words, a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let $B : C \to H$ be a monotone mapping and $N_c v$ be the normal cone to C at $v \in C$, i.e., $N_c v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}$. Define a mapping T on C by
$$Tv = \begin{cases} Bv + N_c v & \text{if } v \in C \\ \emptyset & \text{if } v \notin C. \end{cases} \tag{2.1}$$
Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Bv, u - v\rangle \geq 0$ for all $u \in C$ (see [14]).

Let B be a β-inverse-strongly monotone mapping of C into H. It is easy to show that B is $\frac{1}{\beta}$-Lipschitz. For $\lambda \in (0, 2\beta]$, it is known that $I - \lambda B$ is a nonexpansive mapping of C into H.

Let C be a nonempty closed convex subset of H. Therefore, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the *metric projection* of H on C. We know that for $x \in H$ and $z \in C, z = P_C x$ is equivalent to $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$. It is also known that a Hilbert space H satisfies the *Opial condition*, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality
$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$
holds for every $y \in H$ and $x \neq y$.

Let A be a monotone mapping from C into H. In the context of the variational inequality problem, it is known that
$$u \in \text{VI}(C, A) \Rightarrow u = P_C(u - \lambda Au), \text{ for all } \lambda > 0,$$
and
$$u = P_C(u - \lambda Au), \text{ for some } \lambda > 0 \Rightarrow u \in \text{VI}(C, A).$$

Let C be a nonempty closed subset of a Hilbert space H. Let $\{T_n\}$ and Γ be two families of nonlinear mappings of C into itself such that $F(\Gamma) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, where $F(\Gamma) = \bigcap_{T\in\Gamma} F(T)$. $\{T_n\}$ is said to satisfy the *NST-condition* ([9]) with Γ if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|z_n - T z_n\| = 0 \text{ for all } T \in \Gamma.$$

In the case $\Gamma \in \{T\}$, i.e., Γ consists of one mapping T, $\{T_n\}$ is said to satisfy the NST-condition with T.

For solving the generalized mixed equilibrium problem, we may assume the following conditions for the bifunction f, the function φ and the set C:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semi-continuous;
(A4) for each $x \in C, y \mapsto f(x, y)$ is weakly upper semicontinuous;
(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \bigcap \mathrm{dom}(\varphi)$ such that for any $z \in C \backslash D_x$,

$$f(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) C is a bounded set.

The following lemmas are useful for proving some convergence results in next two sections.

Lemma 2.1. *([10, 11, 12]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C\}$$

for all $x \in H$. Then the following conclusions hold:

(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) $F(T_r) =$ MEP (F, φ);
(4) MEP (F, φ) is closed and convex.

Lemma 2.2. *([7]). Let $A : C \to H$ be α-inverse-strongly monotone. If $r \in (0, 2\alpha)$, then we have $I - rA$ is nonexpansive.*

Lemma 2.3. *([17]). Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a k-strict pseudo-contraction. Define a mapping $S : C \to C$ by $Sx = \alpha x + (1 - \alpha)Tx$ for all $x \in C$ and $\alpha \in [k, 1)$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

We would like to mention the following remark since our result is very interesting. It shows that a monotone mapping maps all points in a generalized mixed equlibrium problem to the same point.

Remark 2.4. Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying *(A2)* and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function. Let A be a monotone mapping of C into H. Then $Au = Av$ for all $u, v \in GMEP(f, \varphi, A)$.

Proof. Let $u, v \in GMEP(f, \varphi, A)$. We then get

$$f(u, y) + \varphi(y) + \langle Au, y - u\rangle \geq \varphi(u), \quad \forall y \in C \tag{2.2}$$

and

$$f(v, y) + \varphi(y) + \langle Av, y - v\rangle \geq \varphi(v), \quad \forall y \in C. \tag{2.3}$$

By letting $y = v$ in (2.2) and $y = u$ in (2.3), we get

$$f(u, v) + \varphi(v) + \langle Au, v - u\rangle \geq \varphi(u) \tag{2.4}$$

and

$$f(v, u) + \varphi(u) + \langle Av, u - v\rangle \geq \varphi(v). \tag{2.5}$$

By (2.4), (2.5) and the condition (A2), we have

$$\langle Av - Au, u - v\rangle \geq f(u, v) + f(v, u) + \langle Au, v - u\rangle + \langle Av, u - v\rangle \geq 0. \tag{2.6}$$

From A is a α-inverse-strongly monotone mapping,

$$0 \leq \alpha\|Au - Av\|^2 \leq \langle Au - Av, u - v\rangle \leq 0.$$

That is $Au = Av$. □

By letting $f = 0$ and $\varphi = 0$ in Lemma 2.4, we obtain the following remark.

Remark 2.5. Let C be a closed convex subset of a real Hilbert space H and A be a monotone mapping of C into H. Then $Au = Av$ for all $u, v \in VI(C, A)$.

3. MAIN RESULT

In this section, we show a strong convergent theorem of hybrid methods for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces under some appropriate conditions.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of Lipschitzian quasi-nonexpansive mappings of C into itself such that $\lim_{n\to\infty} \|S_n x - S_n y\| \leq L_n\|x - y\|$ for all $x, y \in C$, $\sup_n L_n = L$, $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n Bu_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n)S_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset [0,1)$, $\{\lambda_n\} \subset (0,2\beta)$, $\{r_n\} \subset (0,2\alpha)$,

$$0 \le \alpha_n \le a < 1, \quad 0 < b \le \lambda_n \le c < 2\beta, \quad \text{and} \quad 0 < d \le r_n \le e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (3.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

Proof. We divide our proof into 5 steps.

Step 1: We show that $F \subset C_n$ and C_n is closed and convex for all $n \ge 1$.

From the assumption, $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \ge 1$. Next, we show that C_{m+1} is closed and convex. For any $w \in C_m$, we see that

$$\|y_m - w\| \le \|x_m - w\|$$

is equivalent to

$$\|x_m\|^2 - \|y_m\|^2 - 2\langle w, x_m - y_m\rangle \ge 0.$$

Therefore, C_{m+1} is closed and convex.

Since A is α-inverse-strongly monotone and B is β-inverse-strongly monotone, by Lemma 2.2, we get that $I - r_n A$ and $I - \lambda_n B$ are nonexpansive.

By nonexpansiveness of T_{r_n} and $I - r_n A$, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(p - r_n Ap)\|^2 \\
&\le \|(x_n - r_n A x_n) - (p - r_n Ap)\|^2 \\
&= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - 2r_n\alpha\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2. \qquad (3.2)
\end{aligned}$$

We are now ready to show that $F \subset C_n$ for each $n \ge 1$. From the assumption, we have that $F \subset CC_1$. Suppose $F \subset C_m$ for some $m \ge 1$. For any $w \in F \subset C_m$, by nonexpansiveness of $I - \lambda_m B$, we have

$$\begin{aligned}
\|y_m - w\| &= \|\alpha_m x_m + (1-\alpha_m) S_m z_m - w\| \\
&\le \alpha_m\|x_m - w\| + (1-\alpha_m)\|z_m - w\| \\
&= \alpha_m\|x_m - w\| + (1-\alpha_m)\|P_C(I - \lambda_m B)u_m - P_C(I - \lambda_m B)w\| \\
&\le \alpha_m\|x_m - w\| + (1-\alpha_m)\|u_m - w\| \\
&\le \alpha_m\|x_m - w\| + (1-\alpha_m)\|x_m - w\| \\
&= \|x_m - w\|.
\end{aligned}$$

That is $w \in C_{m+1}$. By mathematical induction, we conclude that $F \subset C_n$ for each $n \ge 1$.

Step 2: We show that $\{x_n\}$ is bounded.

Since $x_n = P_{C_n} x_1$ and $x_{n+1} = P_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we get

$$\begin{aligned}
0 &\le \langle x_1 - x_n, x_n - x_{n+1}\rangle \qquad (3.3) \\
&= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\
&\le -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|.
\end{aligned}$$

Thus

$$\|x_1 - x_n\| \le \|x_1 - x_{n+1}\|. \tag{3.4}$$

Since $x_n = P_{C_n}x_1$, for any $w \in F \subset C_n$, we have

$$\|x_1 - x_n\| \le \|x_1 - w\|. \tag{3.5}$$

In particular, we obtain

$$\|x_1 - x_n\| \le \|x_1 - P_F x_1\|. \tag{3.6}$$

By (3.4) and (3.6), we get that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. It implies that $\{x_n\}$ is bounded.

Step 3: We show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ for all $S \in \mathcal{S}$.

By using (3.3), we obtain that

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\| \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle \\ &\quad + \|x_1 - x_{n+1}\|^2 \\ &\le \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2\end{aligned}$$

and so

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.7}$$

Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$$

and then

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \le 2\|x_n - x_{n+1}\|.$$

By (3.7), we get

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.8}$$

On the other hand, we have

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)S_n z_n\| = (1-\alpha_n)\|x_n - S_n z_n\|.$$

It follows from (3.8) and the assumption $0 \le \alpha_n \le a < 1$ that

$$\lim_{n\to\infty}\|x_n - S_n z_n\| = 0. \tag{3.9}$$

For any $w \in F$, we have

$$\begin{aligned}\|y_n - w\|^2 &= \|\alpha_n x_n + (1-\alpha_n)S_n z_n - w\|^2 \\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|S_n z_n - w\|^2 \\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|z_n - w\|^2.\end{aligned} \tag{3.10}$$

From (3.2), we obtain

$$\begin{aligned}\|y_n - w\|^2 &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|P_C(I - \lambda_n B)u_n - w\|^2 \\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|(I - \lambda_n B)u_n - (I - \lambda_n B)w\|^2 \\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|u_n - w\|^2 + \lambda_n^2\|Bu_n - Bw\|^2 \\ &\quad -2\lambda_n\langle u_n - w, Bu_n - Bw\rangle) \\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|u_n - w\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2) \\ &\le \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2)\end{aligned}$$

$$\leq \|x_n - w\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2.$$

We then have

$$\begin{aligned}(1-a)b(2\beta - c)\|Bu_n - Bw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &= (\|x_n - w\| - \|y_n - w\|)(\|x_n - w\| + \|y_n - w\|) \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|).\end{aligned}$$

By (3.8), we obtain that

$$\lim_{n\to\infty} \|Bu_n - Bw\| = 0. \tag{3.11}$$

On the other hand, since P_C is firmly nonexpansive and $I - \lambda_n B$ is nonexpansive, we have

$$\begin{aligned}\|z_n - w\|^2 &= \|P_C(I-\lambda_n B)u_n - P_C(I-\lambda_n B)w\|^2 \\ &\leq \langle (I-\lambda_n B)u_n - (I-\lambda_n B)w, z_n - w\rangle \\ &= \frac{1}{2}\{\|(I-\lambda_n B)u_n - (I-\lambda_n B)w\|^2 + \|z_n - w\|^2 \\ &\quad -\|(I-\lambda_n B)u_n - (I-\lambda_n B)w - (z_n - w)\|^2\} \\ &\leq \frac{1}{2}\{\|u_n - w\|^2 + \|z_n - w\|^2 \\ &\quad -\|u_n - z_n - \lambda_n(Bu_n - Bw)\|^2\} \qquad (3.12) \\ &= \frac{1}{2}\{\|u_n - w\|^2 + \|z_n - w\|^2 - \|u_n - z_n\|^2 \\ &\quad +2\lambda_n\langle u_n - z_n, Bu_n - Bw\rangle - \lambda_n^2\|Bu_n - Bw\|^2\}.\end{aligned}$$

From (3.2), it implies that

$$\begin{aligned}\|z_n - w\|^2 &\leq \|u_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, Bu_n - Bw\rangle \\ &\quad -\lambda_n^2\|Bu_n - Bw\|^2 \\ &\leq \|x_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Bu_n - Bw\|. \qquad (3.13)\end{aligned}$$

By (3.10) and (3.13), we get

$$\begin{aligned}(1-\alpha_n)\|u_n - z_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\quad +2(1-\alpha_n)\lambda_n\|u_n - z_n\|\|Bu_n - Bw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad +2(1-\alpha_n)\lambda_n\|u_n - z_n\|\|Bu_n - Bw\|.\end{aligned}$$

By (3.8), (3.11) and the assumption $0 \leq \alpha_n \leq a < 1$, we get

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.14}$$

Also, by (3.10) and (3.12), we obtain that

$$\begin{aligned}\|y_n - w\|^2 &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|u_n - w\|^2 \qquad (3.15) \\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(w - r_nAw)\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|(x_n - r_nAx_n) - (w - r_nAw)\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 - 2r_n\langle x_n - w, Ax_n - Aw\rangle \\ &\quad +r_n^2\|Ax_n - Aw)\|^2) \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 \\ &\quad +r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2) \\ &\leq \|x_n - w\|^2 + (1-\alpha_n)r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2.\end{aligned}$$

From the assumptions $0 \leq \alpha_n \leq a < 1$ and $0 < d \leq r_n \leq e < 2\alpha$, we have

$$\begin{aligned}(1-a)d(2\alpha - e)\|Ax_n - Aw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|).\end{aligned}$$

By (3.8), we obtain

$$\lim_{n\to\infty} \|Ax_n - Aw\| = 0. \tag{3.16}$$

On the other hand, by using Lemma 2.1, we have T_{r_n} is firmly nonexpansive. Then we get

$$\begin{aligned}\|u_n - w\|^2 &= \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)w\|^2 \\ &\leq \langle (I - r_nA)x_n - (I - r_nA)w, u_n - w\rangle \\ &= \frac{1}{2}(\|(I - r_nA)x_n - (I - r_nA)w\|^2 + \|u_n - w\|^2 \\ &\quad -\|(I - r_nA)x_n - (I - r_nA)w - (u_n - w)\|^2) \\ &\leq \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|(x_n - u_n) - r_n(Ax_n - Aw)\|^2) \\ &= \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|x_n - u_n\|^2 \\ &\quad +2r_n\langle x_n - u_n, Ax_n - Aw\rangle - r_n^2\|Ax_n - Aw\|^2)\end{aligned}$$

and so

$$\begin{aligned}\|u_n - w\|^2 &\leq \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Aw\rangle \\ &\quad -r_n^2\|Ax_n - Aw\|^2 \\ &\leq \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Aw\|. \end{aligned} \tag{3.17}$$

By (3.15) and (3.17), we get

$$\begin{aligned}\|y_n - w\|^2 &\leq \|x_n - w\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 \\ &\quad +2(1-\alpha_n)r_n\|x_n - u_n\|\|Ax_n - Aw\|,\end{aligned}$$

which implies that

$$\begin{aligned}(1-\alpha_n)\|x_n - u_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Aw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad +2r_n\|x_n - u_n\|\|Ax_n - Aw\|.\end{aligned}$$

From the assumptions $0 \leq \alpha_n \leq a < 1$, $0 < d \leq r_n \leq e < 2\alpha$, (3.8) and (3.16), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.18}$$

On the other hand, we have

$$\begin{aligned}\|x_n - S_nx_n\| &\leq \|S_nx_n - S_nz_n\| + \|S_nz_n - x_n\| \\ &\leq L\|x_n - z_n\| + \|S_nz_n - x_n\| \\ &\leq L\|x_n - u_n\| + L\|u_n - z_n\| + \|S_nz_n - x_n\|.\end{aligned}$$

Using (3.9), (3.14) and (3.18), we conclude that

$$\lim_{n\to\infty} \|x_n - S_nx_n\| = 0. \tag{3.19}$$

From the assumption $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0, \quad \forall S \in \mathcal{S}. \tag{3.20}$$

Since $\{x_n\}$ is bounded, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ.

Step 4: We show that $\xi \in F = F(\mathcal{S}) \cap VI(C,B) \cap GMEP(f,\varphi,A)$.

First, we show that $\xi \in F(\mathcal{S})$. Suppose that $\xi \neq S\xi$ for some $S \in \mathcal{S}$. From Opial's condition and (3.20), we obtain

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - \xi\| &< \liminf_{i\to\infty} \|x_{n_i} - S\xi\| \\ &= \liminf_{i\to\infty} \|x_{n_i} - Sx_{n_i} + Sx_{n_i} - S\xi\| \\ &\leq \liminf_{i\to\infty} \|x_{n_i} - \xi\|,\end{aligned}$$

which give us a contradiction. Hence, $\xi \in F(\mathcal{S})$.

Now, we prove that $\xi \in VI(C,B)$. Let T be the maximal monotone mapping defined by (2.1):

$$Tx = \begin{cases} Bx + N_c x & \text{if } x \in C \\ \emptyset & \text{if } x \notin C. \end{cases}$$

For any given $(x,y) \in G(T)$, we get $y - Bx \in N_C x$. By $z_n \in C$ and the definition of N_C, we have

$$\langle x - z_n, y - Bx\rangle \geq 0. \tag{3.21}$$

On the other hand, since $z_n = P_C(I - \lambda_n B)u_n$, we obtain

$$\langle x - z_n, z_n - (I - \lambda_n B)u_n\rangle \geq 0$$

and then

$$\langle x - z_n, \frac{z_n - u_n}{\lambda_n} + Bu_n\rangle \geq 0. \tag{3.22}$$

Since $u_n - z_n \to 0$ as $n \to \infty$, we have that

$$Bu_n - Bz_n \to 0 \text{ as } n \to \infty. \tag{3.23}$$

From (3.21), (3.22) and the β-inverse monotonicity of B, we obtain

$$\begin{aligned}\langle x - z_{n_i}, y\rangle &\geq \langle x - z_{n_i}, Bx\rangle \\ &\geq \langle x - z_{n_i}, Bx\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i}\rangle \\ &= \langle x - z_{n_i}, Bx - Bz_{n_i}\rangle + \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle \\ &\quad - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &= \langle x, Bz_{n_i} - Bu_{n_i}\rangle - \langle z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle x, Bz_{n_i} - Bu_{n_i}\rangle - \|z_{n_i}\|\|Bz_{n_i} - Bu_{n_i}\| \\ &\quad - \frac{1}{\lambda_{n_i}}\|x - z_{n_i}\|\|z_{n_i} - u_{n_i}\|.\end{aligned} \tag{3.24}$$

By (3.14) and (3.18), we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $x_{n_i} \rightharpoonup \xi$, we obtain $z_{n_i} \rightharpoonup \xi$. From (3.14), (3.23) and (3.24), we obtain

$$\langle x-\xi, y\rangle = \lim_{n_i\to\infty}\langle x - z_{n_i}, y\rangle \geq 0.$$

Since T is maximal monotone, we obtain that $0 \in T\xi$. It follows that $\xi \in VI(C,B)$.

Next, we show that $\xi \in GMEP(f,\varphi,A)$. For any $y \in C$,

$$f(u_n,y)+\varphi(y)+\langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq \varphi(u_n).$$

From the condition (A2), we get that

$$\varphi(y)+\langle Ax_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq f(y,u_n)+\varphi(u_n).$$

Replacing n by n_i, we obtain

$$\varphi(y)+\langle Ax_{n_i}, y-u_{n_i}\rangle + \langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle \geq f(y,u_{n_i})+\varphi(u_{n_i}). \tag{3.25}$$

For any t with $0 < t \leq 1$ and $y \in C$, put $\rho_t = ty + (1-t)\xi$. Since $y \in C$ and $\xi \in C$, we obtain $\rho_t \in C$. It follows from (3.25) and the monotonicity of A that

$$\begin{aligned}
\langle \rho_t - u_{n_i}, A\rho_t\rangle &\geq \langle \rho_t - u_{n_i}, A\rho_t\rangle - \langle Ax_{n_i}, \rho_t - u_{n_i}\rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle \\
&\quad + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\
&= \langle \rho_t - u_{n_i}, A\rho_t - Au_{n_i}\rangle + \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\
&\quad - \langle \rho_t - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\
&\geq \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle \\
&\quad + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t).
\end{aligned} \tag{3.26}$$

Since A is Lipschitzian, by (3.18), we get $Au_{n_i} - Ax_{n_i} \to 0$ as $i \to \infty$. From (A3) and (3.26), we arrive at

$$\langle \rho_t - \xi, A\rho_t\rangle \geq f(\rho_t,\xi)+\varphi(\xi)-\varphi(\rho_t). \tag{3.27}$$

From (A1), (A3), (3.27) and convexity of φ, we have that

$$\begin{aligned}
0 &= f(\rho_t,\rho_t) \leq tf(\rho_t,y)+(1-t)f(\rho_t,\xi) \\
&\leq tf(\rho_t,y)+(1-t))\big(\langle \rho_t-\xi, A\rho_t\rangle + \varphi(\rho_t)-\varphi(\xi)\big) \\
&\leq tf(\rho_t,y)+(1-t)t\big(\langle y-\xi, A\rho_t\rangle + \varphi(y)-\varphi(\xi)\big),
\end{aligned}$$

which implies that

$$f(\rho_t,y)+(1-t)(\langle y-\xi, A\rho_t\rangle + \varphi(y)-\varphi(\xi)) \geq 0.$$

Letting $t \to 0$, by (A4), we arrive at

$$f(\xi,y)+\langle y-\xi, A\xi\rangle + \varphi(y)-\varphi(\xi) \geq 0.$$

This shows that $\xi \in GMEP(f,\varphi,A)$.

Step 5: We show that $x_n \to P_F x_1$.

Let $\overline{x} = P_F x_1$ Since $\overline{x} = P_F x_1 \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x_1$, we get

$$\|x_1 - x_{n+1}\| \leq \|x_1 - \overline{x}\|.$$

On the other hand, we have

$$\|x_1 - \overline{x}\| \quad \leq \quad \|x_1 - \xi\|$$

$$\begin{aligned} &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \|x_1 - \overline{x}\|. \end{aligned}$$

Therefore, we get

$$\|x_1 - \xi\| = \lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - \overline{x}\|.$$

This implies that $\overline{x} = \xi$. Since H has the Kadec-Klee property and $x_1 - x_{n_i} \rightharpoonup x_1 - \overline{x}$, it follows that $x_{n_i} \to \overline{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \to \overline{x}$ as $n \to \infty$. The proof is now complete. □

4. DEDUCED THEOREMS AND APPLICATIONS

Theorem 3.1 can be reduced to many different results. By putting $S_n = S$ for all $n \geq 1$ in Theorem 3.1, we obtain the following theorem:

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a L-Lipschitzian quasi-nonexpansive mapping such that $F = F(S) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \forall y \in C, \\ z_n = P_C(u_n - \lambda_n Bu_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n)Sz_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (4.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

When $\{S_n\}$ and $\mathcal{S}$ are families of nonexpansive mappings, we get the following theorem:

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (3.1), where $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,*

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (3.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

Now we show how to apply Theorem 4.2 for families of strict pseudo-contraction mappings.

Theorem 4.3. *Let* C *be a closed convex subset of a real Hilbert space* H, $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1)-(A4) and* $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and* B *be a* β*-inverse-strongly monotone mapping of* C *into* H, *respectively. Let* $\{R_n\}$ *and* $\mathcal{R}$ *be families of* $k-$*strict pseudo-contraction mappings of* C *into itself such that* $\bigcap_{n=1}^{\infty} F(R_n) = F(\mathcal{R})$ *and* $F = F(\mathcal{R}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. *Define a mapping* $S_n : C \to C$ *by* $S_n x = kx + (1-k)R_n x$ *for all* $x \in C$. *Suppose that* $\{R_n\}$ *satisfies the NST-condition with* $\mathcal{R}$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$ *be a sequence generated by the algorithm (3.1), where* $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta), \{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (3.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

Proof. By Lemma 2.3, we obtain that S_n is nonexpansive for all positive integer n. We also get that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S} = \{kI + (1-k)T : T \in \mathcal{R}\}$. The proof is now complete because of the direct result of Theorem 4.2. □

By putting $S_n = S$ for all $n \geq 1$ in Theorem 4.3, we obtain the following corallary.

Corollary 4.4. *Let* C *be a closed convex subset of a real Hilbert space* H, $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying (A1)-(A4) and* $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and* B *be a* β*-inverse-strongly monotone mapping of* C *into* H, *respectively. Let* $R : C \to C$ *be a* $k-$*strict pseudo-contraction such that* $F = F(R) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. *Define a mapping* $S : C \to C$ *by* $Sx = kx + (1-k)Rx$ *for all* $x \in C$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$ *be a sequence generated by the following algorithm (4.1), where* $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta), \{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (4.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

Remark 4.5. By letting $\varphi = 0$ in Corollary 4.4, we obtain Theorem 2.1 of [4].

Remark 4.6. Since Theorems 3.1, 4.1, 4.2, 4.3 and Corollary 4.4 are for finding a common element of the set of fixed points, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem, we can reduce each theorem or corollary by letting $B = 0$, $A = 0$, $\varphi = 0$ or $f(x, y) = 0$.

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