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ON GENERALIZED VARIATIONAL INEQUALITY PROBLEMS$^\diamond$

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ABSTRACT. In this paper, we introduce and study the generalized implicit vector variational inequality problems with set valued mappings in topological vector spaces. We establish existence theorems for the solution set of these problems to be nonempty compact and convex. Our results extend the results by Fang and Huang [Existence results for generalized implicit vector variational inequalities with multivalued mappings, Indian J. Pure and Appl. Math. 36(2005), 629-640].

KEYWORDS : Implicit vector variational inequality; Set valued mapping; Affine mapping,
C-pseudomonotone; Strongly C-pseudomonotone.

1. INTRODUCTION

The vector variational inequality, as an important generalization of the scalar variational inequality, has been shown to have wide applications to vector optimization problems and vector equilibrium problems (see [1, 2, 4, 10]). The first result on the vector variational inequality is paper [8] by Giannessi, which studies the vector variational inequality in the setting of a finite dimensional case. Later on, Chen and Yang [4] studied the vector variational inequalities in infinite dimensional spaces. The theory of vector variational inequalities can be used to the study vector complementarity problems and multi-objective programming problems. For details, we refer to [1, 2, 8, 9, 10].
Throughout this paper, unless otherwise specified, we always let X and Y be real Hausdorff topological vector spaces, $K \subseteq X$ a nonempty convex set, $C : K \longrightarrow 2^Y$ with pointed closed cone convex values (we recall that a subset A of Y is convex

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cone and pointed whenever $A + A \subseteq Y, tA \subseteq A$, for $t \geq 0$, and $A \cap -A = \{0\}$ respectively) where 2^Y denotes all the subsets of Y. Denote by $L(X,Y)$ the set of all continuous linear mappings from X into Y. For any given $l \in L(X,Y),\ x \in X$, let $\langle l, x\rangle$ denote the value of l at x. Let $T : K \longrightarrow 2^{L(X,Y)}$ and $G : X \times Y \longrightarrow X$ be two mappings. Finally let $A : K \times K \longrightarrow 2^{L(X,Y)}$ be a set-valued mapping. We need the following definitions and results in the sequel.

Definition 1.1. Let X and Y be two topological spaces. A set-valued mapping $G : X \longrightarrow 2^Y$ is called:

(i) **upper semi-continuous** (u.s.c.) at $x \in X$ if for each open set V containing $G(x)$, there is an open set U containing x such that for each $t \in U,\ G(t) \subseteq V;\ G$ is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) **upper hemicontinuous** if the restriction of G on straight lines is upper semi-continuous.

(iii) **lower semi-continuous** (l.s.c.) at $x \in X$ if for each open set V with $G(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U,\ G(t) \cap V \neq \emptyset;\ G$ is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iv) **closed** if the graph of G, i.e., the set $\{(x,y) : x \in X, y \in G(x)\}$, is a closed set in $X \times Y$.

(v) **compact** if the closure of range G, i.e., $cl\ G(X)$, is compact, where $G(X) = \bigcup_{x\in X} G(x)$.

(vi) **continuous** if G is both lower semi-continuous and upper semi-continuous.

Lemma 1.2. *(([10],[12]). Let X and Y be two topological spaces. Suppose that $G : X \longrightarrow 2^Y$, is a set-valued mapping. Then the following statements are true:*

(a) *If G is closed and compact, then G is u.s.c.*
(b) *If for any $x \in X,\ G(x)$ is compact, then G is u.s.c. on X if and only if for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \longrightarrow x$ and for every $y_\alpha \in G(x_\alpha)$, there exist $y \in G(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \longrightarrow y$.*
(c) *If G is lower semicontinuous then for any closed $C \subseteq Y$ any net $\{x_\alpha\} \subseteq X$ converges to x and $G(x_\alpha) \subset C$ for all α imply that $G(x) \subseteq C$.*

Definition 1.3. We say that the mapping $G : K \longrightarrow 2^Y$ is C-upper sign-continuous if, for all $x, y \in K$, the following implication holds:

$$G((1-t)x + ty) \cap C((1-t)x + ty) \neq \emptyset,\ \forall t \in]0,1[\Rightarrow G(x) \cap C(x) \neq \emptyset.$$

Remark 1.4. Let $f : K \times K \longrightarrow \Re$ be a real mapping. If we define $G(x) = \{f(x,y)\}$, for all $x, y \in K$, and $C(x) = [0,\infty)$, then Definition 1.3 reduces to the upper sign-continuous introduced by Bianchi and Pini in [1]. The upper sign continuity notion was first introduced by Hadjisavvas [10] for a single valued mapping in the framework of variational inequality problems. It is clear that if G is C-upper sign-continuous then G is upper hemicontinuous but the simple example $G : \Re \longrightarrow 2^{\Re}$ defined by

$$G(x) = \begin{cases} \{0\} & \text{if } x = 0 \\ \Re & \text{if } x \neq 0 \end{cases}$$

and $C(x) = [0, \infty)$, for $x \in \Re$, shows that the converse dose not hold in general. We note that the map G is upper semicontinuous at each nonzero element of real numbers.

Definition 1.5. Let E be a topological vector space. A mapping $F : M \subseteq E \longrightarrow 2^E$ is said to be a KKM mapping, if, for any finite set $A \subseteq M$,

$$\text{co}A \subseteq F(A),$$

where co$A$ denotes the convex hull of A.

Lemma 1.6. *([5]). Let M be a nonempty subset of a Hausdorff topological vector space E and $F : M \longrightarrow 2^M$ be a KKM mapping. If $F(x)$ is closed in E, for every $x \in M$, and compact, for some $x \in M$, then*

$$\bigcap_{x \in M} F(x) \neq \emptyset.$$

Definition 1.7. Let $T : K \longrightarrow 2^{L(X,Y)}$ be a set valued mapping. Then T is said to be

(i) strongly C-pseudomonotone with respect to G if for any given $x, y \in K$,

$$\langle Tx, G(x, y)\rangle \nsubseteq -\text{int}C(x) \Rightarrow \langle Ty, G(y, x)\rangle \subseteq -\text{int}C(x).$$

(i) C-pseudomonotone with respect to G if for any given $x, y \in K$,

$$\langle Tx, G(x, y)\rangle \nsubseteq -C(x) \setminus \{0\} \Rightarrow \langle Ty, G(y, x)\rangle \subseteq -C(y).$$

Remark 1.8. (a) It is clear from Definition 1.7 that strongly C-pseudomonotone with respect to G implies C-pseudomonotone with respect to G but the simple example $X = \Re^2$, $Y = \Re$, K any closed convex subset of X, $T(x) = \{(0, 0)\}$, for all $x \in K$ and G is an arbitrary function from $X \times Y$ to X shows that the converse is not valid in general.

(b) If we define $G(x, y) = y - g(x)$ where $g : K \longrightarrow K$ is a mapping, then Definition 1.7 collapses to the Definition 2.3 in [8].

Example 1.9. Let $X = K = \Re$, $Y = \Re^2$ and $C(x) = P = \{(x, y) \in \Re^2 : x \geq 0,\ y \geq 0\}$, for all $x, y \in K$. Let us define

$$T(x) = \{\begin{pmatrix} x \\ x^2 \end{pmatrix}\},\ G(x, y) = y - x \text{ and } g(x) = x.$$

Then, obviously, $T(x) \subset L(X, Y)$. If we take $y < x$, $x < 0$, then $\langle T(x), y - x\rangle = \{(y-x)(x, x^2)\} \nsubseteq -\text{int}P$ since $(y-x)x > 0$ and $\langle T(y), y-x\rangle = \{(y-x)(y, y^2)\} \nsubseteq P$ because $(y - x)y^2 < 0$ and so T is not strongly C- pseudomonotone mapping with respect to g in the sense of Huang and Fang [8]. While, if $\langle T(y), y-x\rangle \in -\text{int}P$ then $(x - y)(y, y^2) \in \text{int}P$. Thus $x - y > 0$ and $y > 0$ which imply that $\langle T(x), x - y\rangle = (x - y)(x, x^2) \in \text{int}P$ and so $\langle T(x), y - x\rangle \in -\text{int}P$. This shows that T is strongly C-pseudomonotone with respect to G in our sense.

2. MAIN RESULTS

In this section, we consider the following generalized implicit vector variational inequality problems in the topological vector space setting:

(IVVI$_1$) find $x \in K$ such that $\langle Tx, G(x, y)\rangle \nsubseteq -\text{int}C(x),\ \forall y \in K,$

and

(IVVI$_2$) find $x \in K$ such that $\langle Tx, G(x,y)\rangle \nsubseteq -C(x) \setminus \{0\},\ \forall y \in K.$

Clearly, a solution of problem (IVVI_2) is also a solution of problem (IVVI$_1$). Note that if $G(x,y) = 0$, for all $(x,y) \in X \times Y$, then the solution set of (IVVI$_1$) and (IVVI$_2$) are equal to the set K. Moreover, if $G(x,y) = 0$, for all $y \in Y$, then x is a solution of (IVVI$_1$) and (IVVI$_2$).

Example 2.1. Let $X = \Re,\ K = [1,2],\ Y = \Re^2$ and $C(x) = \{(t,s) \in \Re^2 : 0 \le s \le tx\}$, for all $x \in K$. Let $T(x) = \{(t,tx) \in \Re^2 : t \in \Re\}$ and $G(x,y) = 1$, for all $x, y \in K$. One can check that $x = 1$ is a solution (IVVI$_1$) and is not a solution of (IVVI$_2$).

Example 2.2. Consider the following linear programming:

$$\min\{\sum_{i=1}^{n} c_i x_i : x = (x_i)_{i=1}^n \in \Re^n \text{ with } x \ge 0,\ Ax = b\},$$

where A (is a matrix), $c = (c_i)_{i=1}^n \in \Re^n,\ b \in \Re^m$ are fixed.

Now if we define $X = \Re^n,\ K = \{x \in \Re^n : x \ge 0,\ Ax = b\}$(note K is closed and convex), $Y = \Re,\ C(x) = [0,\infty), G(x,y) = y - x$ and define $T(x) = \{c\}$, for all $x \in K$, then the above linear programming is a special case of (IVVI$_1$).

The following lemmas play a key role in this section. Furthermore, they improve Lemmas 2.3 and 2.4, respectively, of [8]. Precisely, they extend Lemmas 2.3 and 2.4 of [8] from Banach spaces to topological vector spaces, omit the closedness of $C : K \longrightarrow 2^Y$, replace special case $(x,y) \longrightarrow y - g(x)$, where $g : K \longrightarrow K$ is a mapping, with a general mapping $(x,y) \longrightarrow G(x,y)$, and upper sign-continuityupper with hemicontinuity .

Lemma 2.3. *Suppose that*

(1) *for each fixed y, the mapping $x \longrightarrow \langle Tx, G(x,y)\rangle$ is upper sign-continuous ;*
(2) *T is C-pseudomonotone with respect to G;*
(3) *$\langle Tx, G(x,x)\rangle \cap C(x) \neq \emptyset$ for every $x \in K$;*
(4) *$G(x,y)$ is affine in the second variable.*

Then for any given $y \in K$, the following are equivalent:

(i) *$\langle Ty, G(y,z)\rangle \nsubseteq -C(y) \setminus \{0\},\ \forall z \in K$;*
(ii) *$\langle Tz, G(z,y)\rangle \subseteq -C(z),\ \forall z \in K$*

Proof. The fact $(i) \Rightarrow (ii)$ directly follows from the definition of C-pseudomonotonicity with respect to G. Now let (ii) hold and z be an arbitrary element of K. Hence (ii) implies that

$$\langle Tz_t, G(z_t,y)\rangle \subseteq -C(z_t),\ \forall z \in K, \forall t \in]0,1[, \tag{2.1}$$

where $z_t = y + t(z - y)$.

We claim that, for all $t \in]0,1[$,

$$\langle Tz_t, G(z_t,z)\rangle \cap C(z_t) \neq \emptyset, \tag{2.2}$$

Otherwise, we have, for some $t \in]0,1[$,

$$\langle Tz_t, G(z_t,z)\rangle \subseteq Y \setminus C(z_t). \tag{2.3}$$

From (4) , (2.1) and (2.3) we get

$$\langle Tz_t, G(z_t, z_t)\rangle = \langle Tz_t, (1-t)G(z_t, y) + tG(z_t, z)\rangle =$$
$$(1-t)\langle Tz_t, G(z_t, y)\rangle + t\langle Tz_t, G(z_t, z)\rangle \subseteq -C(z_t) + Y \setminus C(z_t) \subseteq Y \setminus C(z_t),$$

which is a contradiction (with condition (3)). Thus,

$$\langle Tz_t, G(z_t, z)\rangle \cap C(z_t) \neq \emptyset, \ \forall t \in]0,1[,$$

and so by (1) we deduce that

$$\langle Ty, G(y, z)\rangle \cap C(y) \neq \emptyset.$$

This completes the proof. □

Remark 2.4. (a) We can omit condition (3) of Lemma 2.3 when $G(x,x) = 0, \forall x \in K$. Moreover, we can replace (4) by $C(x)$- convexity of $G(x,y)$ in the second variable, that is, for all $x, z_1, z_2 \in K$ and $t \in]0,1[$, the following implication holds:

$$G(x, (1-t)z_1 + tz_2) \subseteq (1-t)G(x, z_1) + tG(x, z_2) - C(x).$$

We note that even in the real line convexity of G with respect second variable is weaker than to be affine G with respect second variable. To see this consider $G(x,y) = y^2$, where $x, y \in \Re$. Hence we obtain another version of Lemma 2.3, for $G(x,x) = 0, \forall x \in K$, as follows:

Lemma 2.5. *Suppose that*

(1) *for each fixed* y*, the mapping* $x \longrightarrow \langle Tx, G(x,y)\rangle$ *is upper sign-continuous ;*
(2) T *is* C*-pseudomonotone with respect to* G*;*
(3) $G(x,y)$ *is* $C(x)$*-convex in the second variable.*

Then for any given $y \in K$*, the following are equivalent:*

(i) $\langle Ty, G(y,z)\rangle \nsubseteq -C(y) \setminus \{0\}, \ \forall z \in K$*;*
(ii) $\langle Tz, G(z,y)\rangle \subseteq -C(z), \ \forall z \in K$.

We can get, by a similar argument given in lemma 2.5, the following result which is another version of Lemma 2.5 when G is strongly C-pseudomonotone in the second variable and $G(x,x) = 0$, for all $x \in K$.

Lemma 2.6. *Assume that*

(1) *for each fixed* $y \in K$*, the mapping* $x \longrightarrow \langle Tx, G(x,y)\rangle$ *is upper sign-continuous ;*
(2) T *is strongly* C*-pseudomonotone with respect to* G*;*
(3) $G(x,y)$ *is* $C(x)$*-convex in the second variable.*

Then for any given $y \in K$*, the following are equivalent:*

(i) $\langle Ty, G(y,z)\rangle \nsubseteq -\mathit{int}\, C(y), \ \forall z \in K$*;*
(ii) $\langle Tz, G(z,y)\rangle \subseteq -C(z), \ \forall z \in K$

In the following we establish an existence result for (IVVI_2) .

Theorem 2.7. *Suppose all the assumptions of Lemma 2.3 (or Lemma 2.5) hold and , for each fixed* $x \in K$*, the mapping* $y \longrightarrow G(x,y)$ *is continuous. If there exist a compact convex subset* D *of* K *and a compact subset* B *of* K *such that*

$$(C) \qquad \forall x \in K \backslash B \ \exists z \in D : \ \langle Tz, G(z,x)\rangle \nsubseteq -C(z),$$

then the solution set of (IVVI$_2$) is nonempty compact and convex.

Proof. Define $F_1, F_2 : K \longrightarrow 2^K$ by

$$F_1(z) = \{x \in K : \langle Tx, G(x,z)\rangle \nsubseteq -C(x) \setminus \{0\}\},$$

$$F_2(z) = \{x \in K : \langle Tz, G(z,x)\rangle \subseteq -C(z)\}.$$

We claim that F_1 is a KKM mapping. If it is not the case, then there exist $z_1, ..., z_n \in K$ and $t_i > 0$ with $\Sigma_{i=1}^n t_i = 1$ such that $x = \Sigma_{i=1}^n t_i z_i \notin \cup_{i=1}^n F_1(z_i)$, i.e.,

$$\langle Tx, G(x,z_i)\rangle \subseteq -C(x) \setminus \{0\},\ i = 1, 2, ..., n.$$

It follows from the condition (4) of Lemma 2.3 (or condition 3 of Lemma 2.5) that

$$\langle Tx, G(x,x)\rangle = \langle Tx, \Sigma_{i=1}^n t_i G(x,z_i)\rangle \subseteq$$

$$\Sigma_{i=1}^n t_i \langle Tx, \Sigma_{i=1}^n G(x,z_i)\rangle \subseteq -C(x) \setminus \{0\},$$

and so

$$\langle Tx, G(x,x)\rangle \subseteq -C(x) \setminus \{0\}.$$

This and $C(x) \cap (-C(x) \setminus \{0\}) = \emptyset$ (note that the mapping C has pointed closed cone convex values) imply that

$$\langle Tx, G(x,x) \cap C(x) = \{\emptyset\},$$

which contradicts condition (3) of Lemma 2.3. Hence F_1 is a KKM mapping and so F_2 is also a KKM mapping (note, by (2) of Lemma 2.3 we have $F_1(z) \subseteq F_2(z)$ for every $z \in K$). Further, by the continuity of the mapping $y \longrightarrow G(x,y)$ for each fixed $x \in K$, that $F_2(z)$ is closed in K, for every $z \in K$. Now $F_2|_{co(A\cup D)}$ (the restriction of F_2 on compact and convex subset $co(A \cup D)$ of K where A is finite subset of K) satisfies all the assumptions of Lemma 1.6 and hence

$$\cap_{z\in co(A\cup D)} F_2(z) \neq \emptyset. \qquad (2.4)$$

By condition (C) we have

$$\cap_{z\in D} F_2(z) \subseteq B. \qquad (2.5)$$

From (2.4) and (2.5) we inform that the family $\{F_2(z)\}_{z\in K}$ has finite intersection property and so

$$\cap_{z\in K} F_2(z) \neq \emptyset.$$

Then there exists $x \in K$ such that

$$\langle Tz, G(z,x)\rangle \subseteq -C(z),\ \forall z \in K.$$

Now from Lemma 2.3 (or Lemma 2.5) we get

$$\langle Tx, G(x,z)\rangle \nsubseteq -C(x) \setminus \{0\},\ \forall z \in K,$$

and so x is a solution of (IVVI_2). By Lemma 2.3 (or Lemma 2.5) the solution set of (IVVI_2) equals to the set

$$S = \{x \in K : \langle Tz, G(z,x)\rangle \subseteq -C(z), \forall z \in K\},$$

which is convex by (4) of Lemma 2.3 (or (3) of Lemma 2.5). Also by our assumption (that is condition (C)) S is a closed subset of B and hence compact. This completes the proof. □

Remark 2.8. (a) One can see, by the definitions of (IVVI_1), (IVVI_2) and $-int\ C(x) \subseteq -C(x)$, for all $x \in K$, that any solution of (IVVI_2)is a solution of (IVVI_1). So we can consider Theorem 2.7 as an existence theorem for the solution of (IVVI_1).

(b) To be continuity of G in the second variable in Theorem 2.7 can be replaced by the lower semi-continuity of the mapping

$$z \longrightarrow \{x \in K : \langle Tz, G(z,x)\rangle \subseteq -C(z)\}.$$

(c) We can drop condition (C) of Theorem 2.7 when K is compact. Hence Theorem 2.7 improves Lemma 2.7 of [8]. Because, in Lemma 2.4 of [8], the authors supposed that K is a bounded closed subset of a reflexive Banach space X which means K is compact in the W^*- topology on X.

The next result guarantees, under suitable conditions, the solution set of (IVVI_1) is nonempty compact and convex. We omit its proof, since it is similar to the proof of Theorem 2.7.

Theorem 2.9. *Suppose all the assumptions of Lemma 2.6 hold and , for each fixed* $x \in K$, *the mapping* $y \longrightarrow G(x,y)$ *is continuous (or the mapping* $z \longrightarrow \{x \in K : \langle Tz, G(z,x)\rangle \subseteq -C(z)\}$ *is lower semi-continuous). If there exist a compact convex subset* D *of* K *and a compact subset* B *of* K *such that*

$$\forall x \in K\backslash B\ \exists z \in D :\ \langle Tz, G(z,x)\rangle \nsubseteq -C(z),$$

then the solution set of (IVVI_1*) is nonempty compact and convex*

3. APPLICATION

In this section, using Theorems 2.7 and 2.9, we establish some existence theorems for the following two generalized implicit vector variational inequality problems in the locally convex topological vector spaces. Our results extend Theorems 3.1 and 3.2 of [8] from the reflexive Banach spaces to locally convex spaces. Moreover, we do not use the notion demi-C-continuity on our maps. : Now we recall

generalized implicit vector variational inequality problems as follow:

find $u \in K$ such that

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\},\ \forall v \in K,$$

and

find $u \in K$ such that

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -\text{int}\ C(u),\ \forall v \in K.$$

In order to prove our existence theorems we need the following result.

Theorem 3.1. *(Kakutani-Fan-Glicksberg)([7]). Let* X *be a locally convex Hausdorff space,* $D \subseteq X$ *a nonempty, convex compact subset. Let* $T : D \longrightarrow 2^D$ *be upper semicontinuous with nonempty, closed convex* $T(x)$, *for all* $x \in D$. *Then* T *has a fixed point in* D.

Theorem 3.2. *Assume that the following conditions hold*

(i) *for each fixed* $(w,y) \in K \times K$, *the mapping* $x \longrightarrow \langle A(w,x), G(x,y)\rangle \cap C(x)$ *is upper sign-continuous with compact values;*
(ii) *for each fixed* $w \in K$ *the mapping* $x \longrightarrow A(w,x)$ *is* C*-pseudomonotone with respect to* G*;*

(iii) *for each fixed* $w \in K\langle A(w,x), G(x,x)\rangle \cap C(x) \neq \emptyset$;
(iv) $G(x,y)$ *is affine in the second variable;*
(v) *for each finite dimensional subspace* M *of* X *with* $K_M = K \cap M \neq \emptyset$, *there exist compact subset* B_M *and compact convex subset* D_M *of* K_M *such that* $\forall (w,x) \in K_M \times (K_M \backslash B_M)$, $\exists z \in D_M$ *such that* $\langle A(w,z), G(z,x)\rangle \nsubseteq -C(z)$;
(vi) *for each fixed* $v \in K$, *the mapping* $(x,y) \longrightarrow \langle A(x,y), G(v,y)\rangle$*is lower semicontinuous.*

Then there exits $u \in K$ *such that*

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\}, \ \forall v \in K.$$

Proof. Let $M \subset X$ be a finite dimensional subspace with $K_M = K \cap M \neq \emptyset$. For each fixed $w \in K$, consider the problem of finding $u \in K_M$ such that

$$\langle A(w,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\}, \ \forall v \in K_M. \tag{2.6}$$

By Theorem 2.7, the solution set of problem (2.6) is nonempty compact and convex subset of K_M and so the mapping $F : K_M \longrightarrow 2^{K_M}$ defined by

$$F(w) = \{u \in K_M : \ \langle A(w,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\}, \ \forall v \in K_M\}$$

has nonempty compact and convex values. Lemma 2.3 implies

$$F(w) = \{u \in K_M : \ \langle A(w,u), G(v,u)\rangle \subseteq -C(v), \ \forall v \in K_M\}.$$

Further, F has closed graph. Indeed, let $(w_\alpha, u_\alpha) \in K_M \times F(w_\alpha)$ converge to $(w,u) \in K_M \times K_M$. Then $\langle A(w_\alpha,u_\alpha), G(v,u_\alpha)\rangle \subseteq -C(v)$, for all α and $v \in K_M$. Now from (vi) we get $\langle A(w,u), G(v,u)\rangle \subseteq -C(v)$ and hence $u \in F(w)$. This shows that the graph of F is closed and so since the values of F are compact we deduce from Lemma 1.2 that F is upper semi-continuous on K_M. Therefore, by using the Kakutani-Fan-Glicksberg fixed point theorem (that is Theorem 3.1), F has a fixed point $w_0 \in K_M$, i.e., there exists $w_0 \in K_M$ such that

$$\langle A(w_0,v), G(v,w_0)\rangle \subseteq -C(v), \ \forall v \in K_M.$$

Set $\mathcal{M} = \{M \subset X : M$ is a finite dimensional subspace with $K_M \neq \emptyset\}$ and for $M \in \mathcal{M}$ and

$$W_M = \{u \in K_M \ : \ \langle A(u,v), G(v,u)\rangle \subseteq -C(v), \ \forall v \in K_M\}, \ \forall\ M \in \mathcal{M}.$$

Clearly, W_M is nonempty and by (vi),(v)is a compact subset of B_M. For each finite subset $\{M_i\}_{i=1}^n$ of $\mathcal{M}$, from the definition of W_M, we have $W_{\bigcup_i M_i} \subset \bigcap_{i=1}^n W_{M_i}$, so $\{W_M : M \in \mathcal{M}\}$ has the finite intersection property. Hence, there is $u \in \bigcap_{M\in\mathcal{M}} W_M$. We show that

$$\langle A(u,v), G(v,u)\rangle \subseteq -C(v), \ \forall v \in K.$$

Indeed, for each $v \in K$, there is $M \in \mathcal{M}$ such that $v \in K_M$. Since W_M is closed and $u \in W_M$, there exists a net $\{u_\alpha\} \subset W_M$ such that u_α converges to u. It follows that

$$\langle A(u_\alpha,v), G(v,u_\alpha)\rangle \subseteq -C(v).$$

Since $C(v)$ is closed, G is continuous in the second variable, u_α converges to u one has

$$\langle A(u,v), G(v,u)\rangle \subseteq -C(v), \ \forall v \in K.$$

Now Lemma 2.3 implies

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -C(u) \setminus \{0\}, \ \forall v \in K.$$

The proof is complete. □

Remark 3.3. In Theorem 3.2, we can omit (iii) if $G(x,x) = 0$, for all $x \in K$, and also condition (v) when K is compact. Hence we get Theorem 3.1 of [8] without assuming K is a bounded subset of a reflexive Banach space X. Moreover, in Theorem 3.2 $C : K \longrightarrow 2^Y$ does not need to have closed graph as supposed in Theorem 3.1 in [8].

Using Theorem 2.7 and the proof given for Theorem 3.2, we obtain the following theorem.

Theorem 3.4. *Assume that the following conditions hold*

- (i) *for each fixed $(w,y) \in K \times K$, the mapping $x \longrightarrow \langle A(w,x), G(x,y)\rangle \cap C(x)$ is upper sign-continuous with compact values;*
- (ii) *for each fixed $w \in K$ the mapping $x \longrightarrow A(w,x)$ is strongly C-pseudomonotone with respect to G;*
- (iii) *for each fixed $w \in K$, $\langle A(w,x), G(x,x)\rangle \nsubseteq -intC(x)$;*
- (iv) *$G(x,y)$ is affine and continuous in the second variable;*
- (v) *for each finite dimensional subspace M of X with $K_M = K \cap M \neq \emptyset$, there exist compact subset B_M and compact convex subset D_M of K_M such that $\forall (w,x) \in K_M \times (K_M \backslash B_M)$, $\exists z \in D_M$ such that $\langle A(w,z), G(z,x)\rangle \nsubseteq -C(z)$;*
- (vi) *for each fixed $v \in K$, the mapping $(x,y) \longrightarrow \langle A(x,y), G(v,y)\rangle$ is lower semicontinuous.*

Then there exits $u \in K$ such that

$$\langle A(u,u), G(u,v)\rangle \nsubseteq -\textit{int}\, C(u),\ \forall v \in K.$$

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STOCHASTIC MODEL FOR GOLD PRICES AND ITS APPLICATION FOR NO-ARBITRAGE GOLD DERIVATIVE PRICING

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ABSTRACT. In this paper, we develop a one-factor model of stochastic behavior of gold prices. The gold prices are assumed to follow an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes instantaneous convenience yields which follow an ordinary differential equation. Moreover, we derive closed-form solutions for no-arbitrage prices of gold futures and European gold options under the no-arbitrage assumptions.

KEYWORDS : Gold futures; European options; No-arbitrage prices; Instantaneous convenience yields.

1. INTRODUCTION

Gold has been a valuable metal throughout the ages because of its versatility. It can be used in many applications. One of its primary uses is as jewelry and adornment. It is even used in aerospace, dentistry, electronics. People can also use gold as the standard value for the money of each country. Furthermore, gold is extremely important to the development of industry and technology of each country because it is considered as a liquid asset and has low fluctuations. Therefore it is used as an assurance when issuing bank notes. We can see that in many countries they have gold for a guaranteed source of investment funds. Moreover, gold investment is a good way for getting high return in the long run. Nowadays,

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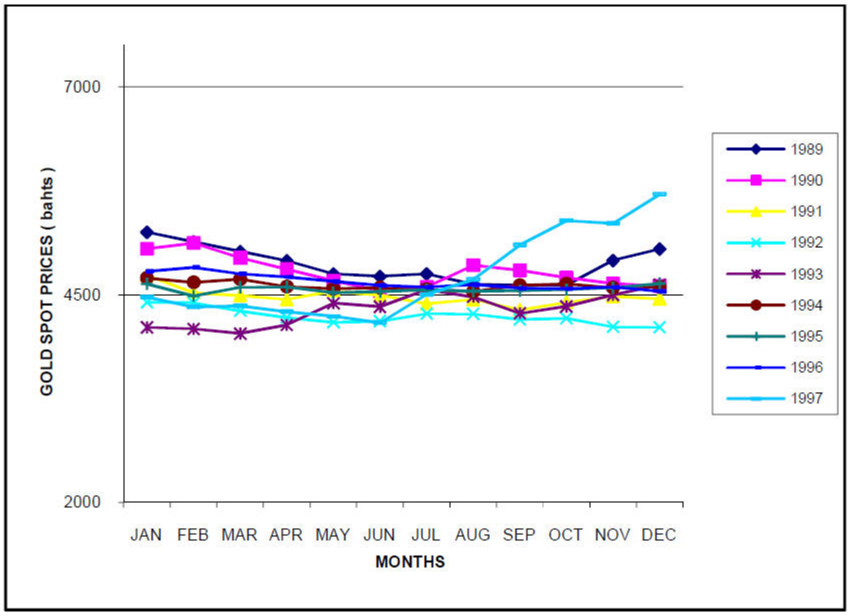

FIGURE 1. Monthly averages of gold spot prices over nine years

the investment in commodity derivatives markets which have gold as an underlying asset is receiving widespread attention. Gold derivatives markets have witnessed a tremendous growth in recent years. In order to provide closed-form solutions for gold derivatives such as futures and options, one can treat gold spot prices as a random walk. In other words, gold spot prices are assumed to follow a stochastic differential equation (SDE). Using the SDE theory, the closed-form solutions can be obtained by solving the associated partial differential equations.

By considering gold as one of those commodities, one can choose a model proposed by Brennan-Schwartz (1985) [1] to describe the dynamics of gold spot prices To follow the model, gold spot prices are assumed to follow a Geometric Brownian Motion (GBM) and the convenience yield, which is an important factor influencing commodity prices, is described in the same way as a dividend yield. Nevertheless, this specification is inappropriate because it does not take into account the mean-reversion property of commodity prices. Schwartz (1997) [2] introduced variation of this model in which the convenience yield is mean reverting and intervenes in the commodity price dynamics. Besides the mean reversion property of commodity prices, the other main empirical characteristic that makes commodities noticeable different from stocks, bonds and other financial assets, is seasonality in prices. Many commodities, such as agricultural commodities or natural gas, exhibit seasonality in prices, due to harvest cycles in the case of agricultural commodities and change consumptions in the case of natural gas. In addition, gold also have seasonal variation in prices as well (see Figure 1.). We use the gold spot prices data [6] during January 1989 to December 1997 in plotting graph to observe the tendency of gold spot prices shown in Figure 1.. It shows seasonal variation of gold prices. From analyzing the graph, we can see that gold spot prices at the beginning of each year are high. They drop in the middle and at the end of each years, they are high again. Thus, the graph looks like sine wave. Therefore, a model of gold spot prices introduced in this paper will take to account a seasonal variation in gold prices.

The remaining of this paper is organizes as follows. In section 2, we present a one-factor model, which is an extension of Schwartz model 1 [2]. In section 3, we present the no-arbitrage assumptions and derive for closed-form solutions for no-arbitrage prices of gold futures and European gold options. Finally, We conclude the paper in section 4.

2. STOCHASTIC MODEL FOR GOLD PRICES

In this section, we develop a one-factor model of stochastic behavior gold prices. The gold prices are assumed to follow an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes instantaneous convenience yields which follow an ordinary differential equation (ODE). Our model can be written as follow: under an equivalent martingale measure $\mathbb{Q}$,

$$dS_t = (r - \delta(t))S_t dt + \sigma S_t dW_t, S_0 = s_0 > 0, \tag{2.1}$$

$$\frac{d\delta(t)}{dt} = \kappa\left(\alpha(t) - \delta(t)\right), \delta\left(0\right) = \delta_0, \tag{2.2}$$

$$\alpha\left(t\right) = \alpha_0 + \alpha_1 sin(2\pi(t - t_\alpha)), \tag{2.3}$$

where $(S_t(\omega))_{t\in[0,T]}$ is a gold spot price process, r is a risk-free interest rate, $\delta(t)$ is instantaneous convenience yield at time T which is assumed to follow the ordinary differential equation (2) and has mean reversion property, $(W_t(\omega))_{t\in[0,T]}$ is a-one dimensional standard Brownian motion under a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a filtration $(\mathcal{F})_{t\geq 0}$, κ is the speed of adjustment of the gold prices, α_0 is a long run mean, σ is the volatility of gold prices, $\alpha(t)$ is seasonal variation in in convenience yields. The parameters α_1 and T denoted, respectively, the annual seasonality parameters and the seasonality centering parameter, representing the time of annual peak of equilibrium price in a year.

From (2), the closed-form solution of the ODE (2) is

$$\delta(t) = \alpha_0 + C(t) + (\delta_0 - \alpha_0 - C(0))e^{-\kappa t}, \tag{2.4}$$

where

$$C(t) = \frac{-2\alpha_1\kappa\pi cos(2\pi(t - t_\alpha)) + \alpha_1\kappa^2 sin(2\pi(t - t_\alpha))}{\kappa^2 + 4\pi^2}. \tag{2.5}$$

From Kloeden and Platen [4 page 120], the strong solution of the SDE (1) can be expressed as

$$S_t = S_{t_0} e^{\int_{t_0}^{t} (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(t - t_0)}Z}, \tag{2.6}$$

for all $0 \leq t_0 \leq t \leq T$, where Z is the standard normal random variable.

Proposition 2.1. *The gold spot price S_t modeled by (1) is neither negative nor zero for all $t \geq 0$. Moreover, for a fixed $t_0 \geq 0, ln(S_t/S_{t_0})$ is normal distributed with mean $\int_{t_0}^{t} (r - \frac{\sigma^2}{2} - \delta(s))ds$ and variance $\sigma^2(t - t_0)$.*

3. VALUATION OF GOLD DERIVATIVES

In order to price futures and options of the commodity (gold in this case), we assume the following assumptions hold.

The no-arbitrage assumptions

1. The market is arbitrage-free, that is for any portfolio $\varphi = (\varphi_i), V_\varphi(0) = 0$

and $V_\varphi(T) \geq 0, \mathbb{P}-$a.s. for all time $T > 0$ imply $V_\varphi(T) = 0, \mathbb{P}-$ a.s., where $V_\varphi(t) \equiv V_\varphi(t, S_t, \varphi_t)$ denotes the value of the portfolio φ at time t and $\mathbb{P}$ denotes an original probability measures. Namely, if a portfolio requires a null investment and is riskless (there is no possible loss at the time horizon T), then its terminal value at time T has to be zero.
2. The market participants are subject to no transaction costs when they trade.
3. The market participants are subject to the same or no tax rate on all net trading profits.
4. The market participants can borrow/lend money at the same risk-free rate of interest.

A. GOLD FUTURES PRICING

Form Rujivan [5], under the no-arbitrage assumptions in a futures market, the no-arbitrage price (fair-price) of gold futures at a current time t, denoted by $F^T(t, S_t)$, satisfies,

$$F^T(t, S_t) = E_{\mathbb{Q}}[S_T|\mathcal{F}_t], \tag{3.1}$$

where the expectation is taken under the equivalent martingale measure $\mathbb{Q}$ conditioned on $\mathcal{F}_t$.

The relation (7) implies that F^T solves the partial differential equation:

$$-\frac{\partial F^T}{\partial t} = (r - \delta(t))S\frac{\partial F^T}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial F^T}{\partial S^2} \tag{3.2}$$

subject to a terminal condition

$$F^T(T, S) = S, \tag{3.3}$$

for all $S \geq 0$.

We now solve the PDE (8) subject to (9) to obtained closed-form solutions for no-arbitrage gold futures price.

Theorem 3.1. *(Determination of gold futures prices)*
For given and fixed maturity date T, no-arbitrage gold futures prices to the PDE (8) can be expressed as

$$F^T(t, S_t) = S_t e^{A(T-t)} \tag{3.4}$$

where

$$A(\tau) = r\tau - \int_0^\tau \delta(T - s)ds, \tag{3.5}$$

for all $\tau \geq 0$.

Proof. To avoid confusion about the notations, we omit writing the subscript t of S_t and write $F^T = F^T(t, S_t)$.
Let $\tau = T - t$, and we calculate

$$\frac{\partial F^T}{\partial t} = -(F^T A'(\tau)), \quad \frac{\partial F^T}{\partial S} = \frac{\partial F^T}{S}, \quad \frac{\partial^2 F^T}{\partial S^2} = 0,$$

where $' = \frac{d}{d\tau}$.
Replacing the above partial derivatives into (8), we then obtain the following ODE,

$$A'(\tau) = (r - \delta(T - \tau)) \tag{3.6}$$

and the terminal condition implies that

$$A(0) = 0. \tag{3.7}$$

Then, we solve (12) subject to (13) to obtain (11). □

B. EUROPEAN GOLD OPTIONS PRICING

In this section, we consider European options written on gold spot prices. We first consider a European call option. Let t and K be respectively, a maturity date and strike price of the call option. From [5], under the no-arbitrage assumptions, the call option price must equal to the present value of the expected payoff of the call option under the equivalent martingale measure $\mathbb{Q}$,

$$C(T,t,S_t,K) = e^{-r(T-t)}E_{\mathbb{Q}}[max(0,S_T-K)|\mathcal{F}_t], \tag{3.8}$$

Theorem 3.2. *(No-arbitrage prices of European call options for gold) The no-arbitrage European gold call options prices with strike price K is*

$$C(T,t,S_t,K) = S_t e^{-\int_t^T \delta(s)ds}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{3.9}$$

where

$$d_1 = \frac{ln(\frac{S_t}{K}) + r(T-t) - \int_t^T \delta(s)ds + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}} \tag{3.10}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{3.11}$$

Proof. We omit writing the martingale measure $\mathbb{Q}$ and the filtration $\mathcal{F}_t$ in this proof to avoid confusion about notations. From (14), we obtain

$$\begin{aligned} C(T,t,S_t,K) &= e^{-r(T-t)}E[max(0,S_T-K)], \\ &= e^{-r(T-t)}E[(S_T-K)^+], \\ &= e^{-r(T-t)}E[I(S_T-K)], \\ &= e^{-r(T-t)}I[S_T] - Ke^{-r(T-t)}E[I], \end{aligned} \tag{3.12}$$

where I is the indicator random variable for the event that the option finishes in the money, that is,

$$I = \begin{cases} 1, & \text{if } \quad S_T > K, \\ 0, & \text{if } \quad S_T \leq K. \end{cases} \tag{3.13}$$

Next, we will show that

$$I = \begin{cases} 1, & \text{if } \quad Z > \sigma\sqrt{T-t} - d_1, \\ 0, & \text{if } \quad otherwise. \end{cases} \tag{3.14}$$

where d_1 is given in (16).
Using (6) and (19) , we obtain

$$\begin{aligned} S_T > K &\Leftrightarrow S_t e^{\int_t^T (r-\frac{\sigma^2}{2}-\delta(s))ds + \sigma\sqrt{T-t}Z} > K \\ &\Leftrightarrow \int_t^T (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{T-t}Z > ln(\frac{K}{S_t}) \\ &\Leftrightarrow Z > \frac{ln(\frac{K}{S_t}) - r(T-t) + \int_t^T \delta(s)ds + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ &\Leftrightarrow Z > \sigma\sqrt{T-t} - d_1 \end{aligned}$$

and we also obtain (20). From (20), we have

$$E[I] = P(S_T > K)$$

$$\begin{aligned}
&= P(Z > \sigma\sqrt{T-t} - d_1) \\
&= P(Z < d_1 - \sigma\sqrt{T-t}) \\
&= \Phi(d_1 - \sigma\sqrt{T-t}) \\
&= \Phi(d_2), \qquad (3.15)
\end{aligned}$$

where Φ is the standard normal distribution function.
Using (6) and (20) with $a = \sigma\sqrt{T-t} - d_1$,
we then obtain,

$$\begin{aligned}
e^{-r(T-t)}E[IS_T] &= e^{r(T-t)} \int_a^\infty S_t e^{r(T-t) - \int_t^T \delta(s)ds + \frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{T-t}y} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{1}{2}(y^2 - 2\sigma\sqrt{T-t}y + \sigma^2(T-t))} dy \\
&= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{\frac{-(y-\sigma\sqrt{T-t})^2}{2}} dy \\
&= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_{-d_1}^\infty e^{\frac{-x^2}{2}} dx \\
&= S_t e^{-\int_t^T \delta(s)ds} P(Z > -d_1) \\
&= S_t e^{-\int_t^T \delta(s)ds} P(Z < d_1) \\
&= S_t e^{-\int_t^T \delta(s)ds} \Phi(d_1) \qquad (3.16)
\end{aligned}$$

Replacing (21) and (22) into (18), we then reach (15). □

Theorem 3.3. *(No-arbitrage prices of European put options for gold) The no-arbitrage European gold put options prices with strike price K is*

$$P(T, t, S_t, K) = Ke^{-r(T-t)}\Phi(-d_2) - S_t(1 - e^{-\int_t^T \delta(s)ds}\Phi(d_1)), \qquad (3.17)$$

where d_1 and d_2 satisfy, respectively, (16) and (17).

Proof. The put-call parity [3] equation satisfies

$$C(T, t, S_t, K) + Ke^{-r(T-t)} = P(T, t, S_t, K) + S_t. \qquad (3.18)$$

Replacing (15) into (24) , we then obtain

$$P(T, t, S_t, K) = (Ke^{-r(T-t)} - Ke^{-r(T-t)}\Phi(d_2)) - (S_t - S_t e^{-\int_t^T \delta(s)ds}\Phi(d_1)).$$

Finally, we rearrange the equation above to obtain the no- arbitrage European gold put options prices. □

4. CONCLUSION

This paper has developed a one-factor model for gold prices which is an extension of the model proposed by Schwartz. We have provided closed-form solutions for no-arbitrage prices of gold futures and European gold options under the no-arbitrage assumptions. Moreover, one can use our model to predict gold prices in the futures if the unknown parameters are estimated using historical data of gold spot prices. Finally, closed-form no-arbitrage prices of American options for gold will be derived in our future work

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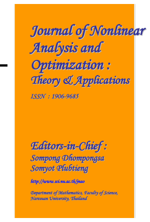

NIELSEN THEORY AND SOME SELECTED APPLICATIONS TO DIFFERENTIAL AND INTEGRAL EQUATIONS; A BRIEF SURVEY$^{\diamond}$

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ABSTRACT. In this paper, we first give a quick introduction to Nielsen theory in the traditional topological setting. We then turn to the applications of Nielsen theory, and survey some results in nonlinear analysis.

KEYWORDS : Nielsen Theory; Nielsen Number; Nielsen Theory in Banach Space; Applications of Nielsen Theory.

1. INTRODUCTION

Nielsen theory occupies a prominent place within topological fixed point theory and is currently one of the most active research areas of algebraic topology. Nielsen theory, named in honor of its founder, Danish mathematician Jakob Nielsen (1890 - 1959), is concerned with finding the minimum number of solutions to certain equations involving maps, minimized among all the maps in a given homotopy class. The first part of this paper focuses on the traditional definition of Nielsen fixed point classes and the Nielsen number itself.

Not only has the Nielsen theory been widely studied in the topological setting, there are many areas of mathematics that have used the idea of Nielsen theory to solve existing problems. In the second part of the paper, we focus on the applications of Nielsen theory in nonlinear analysis.

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This work originated with the suggestion of Professor Dr. Sompong Dhompongsa and people in his group when the author presented previous works concerned with calculating the Nielsen number on various topological spaces. They wanted to know if there is any way that the author's speciality in topological fixed point and Nielsen theory could be applied to their work in analysis. The author hopes that this brief survey will demonstrate the possibility of more collaborations between these two areas.

2. NIELSEN THEORY: THE INTRODUCTION

Let X be a connected compact polyhedron then X has a universal cover $p : \widetilde{X} \to X$. For any self-map $f : X \to X$, a *lift* $\tilde{f}$ of f is a map from $\widetilde{X}$ to itself such that $p \circ \tilde{f} = f \circ p$. Also, recall that a *deck transformation* is a homeomorphism $\alpha : \tilde{X} \to \tilde{X}$ such that $p \circ \alpha = p$.

2.1. **The Minimum Number.** The main object of study in topological (Nielsen) fixed point theory is concerned with finding the "minimum number" of the fixed points of $f : X \to X$.

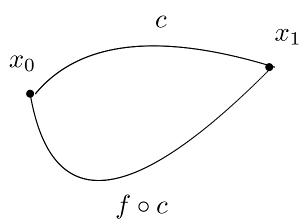
$$\text{MF}[f] = \{ \text{ min } \sharp Fix(g) : g \simeq f\}$$

where $Fix(g) = \{x : g(x) = x\}$ and $\simeq$ denotes homotopy. Note that MF[f] = 0 would mean there is a map homotopic to f that has no fixed points.

Main Problem: Determine MF[f] from information about f.

2.2. **Fixed Point Classes.** Nielsen theory depends on the concept of the "fixed point class" which partitions $Fix(f)$ into equivalence classes. There are two ways of describing the equivalence relation.

Definition 2.1. Definition of Fixed Point Classes (via path homotopy): Two fixed points x_0 and x_1 of $f : X \to X$ are in the same fixed point class if and only if there exists a path c from x_0 to x_1 such that $c \simeq f \circ c$.



Definition 2.2. Definition of Fixed Point Classes (via lifting classes): Two lifts $\tilde{f}$ and $\tilde{f}'$ of f are said to be *conjugate* if there exists $\gamma \in \mathfrak{D}$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. A lifting class is denoted as

$$[\tilde{f}] = \{\gamma \circ \tilde{f} \circ \gamma^{-1} | \gamma \in \mathfrak{D}\}.$$

Two fixed points x_0 and x_1 of $f : X \to X$ are in the same fixed point class if and only if $x_0, x_1 \in p(Fix(\tilde{f}))$. If $\tilde{f}, \tilde{f}'$ are conjugate lifts, then $p(Fix(\tilde{f})) = p(Fix(\tilde{f}'))$ so fixed point classes depend on conjugacy classes of lifts.

For a chosen base point x_0 and a chosen lift $\tilde{f}$ of f, the $\tilde{f}_\pi$-conjugacy class of $\alpha \in \pi_1(X, x_0)$ is called ***the coordinate of the lifting class*** (see [20] for more details) $[\alpha \circ \tilde{f}]$. This α is can be obtained geometrically.

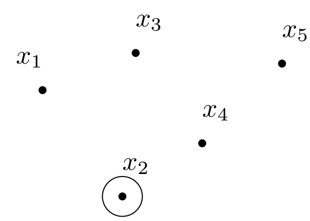
Suppose $x_0 \in p(Fix(\tilde{f}))$ and $\bar{x}_0 \in p^{-1}(x_0)$ and $\bar{x}_0 \in Fix(\tilde{f})$ is the constant path at

x_0 where $\tilde{f}$ is the chosen lift of f. Then the coordinate of the class of a fixed point x of f is the $\tilde{f}_\pi$-conjugacy class of $\alpha = [c * (f \circ c)^{-1}] \in \pi_1(X, x_0)$, where c is any path from x_0 to x. In other words, $x \in p(Fix(\alpha \circ \tilde{f}))$.

2.3. **The Nielsen Number.** The fixed point index is an indispensable tool of fixed point theory. The index of each fixed point allow us to algebraically count fixed points in an open set. Roughly, we can think of the index of $\mathbb{F}$, a fixed point class, with respect to f as follows. Suppose $\mathbb{F}$ is finite and for $x_2 \in \mathbb{F}$, pick a sphere centered at x_2 such that this sphere is small enough to exclude other fixed points and the sphere is mapped by f into a Euclidean neighborhood of x_2. The index of x_2 is the degree of the map

$$\frac{id - f}{|id - f|}$$

restricted to the sphere.



Definition 2.3. The *index* of a fixed point class $\mathbb{F}$ is the sum of the indices of all fixed points in $\mathbb{F}$.

A fixed point class $\mathbb{F}$ is said to be *essential* if the index of $\mathbb{F}$ with respect to f is not zero.

Definition 2.4. The *Nielsen Number* of f, denoted $N(f)$, is the number of essential fixed point classes of f.

Theorem 2.1. *If g is homotopic to f, then $N(g) = N(f)$ and therefore*

$$N(f) \leq MF[f]$$

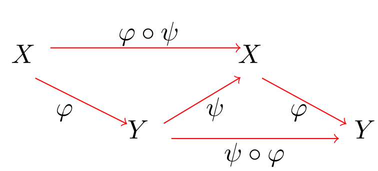
In fact, if X is not a surface, then there exists a map $g \simeq f$ such that it has exactly $N(f)$ fixed points.

2.4. **Properties of the Nielsen number.**

Homotopy invariant: The Nielsen number is a homotopy invariant.

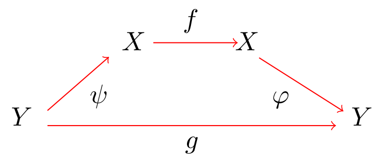
Commutativity: Let φ be a map from X to Y and let ψ be a map from Y to X, then

$$N(\varphi \circ \psi) = N(\psi \circ \varphi)$$



Homotopy-type Invariant: Let φ be a map from X to Y and let ψ be a map from Y to X such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity. Let f be a self-map on X and suppose g be a self-map on Y is homotopic to $\varphi \circ f \circ \psi$ then

$$N(g) = N(f)$$



2.5. **Geometric and Algebraic Aspects of Nielsen Theory.** The interest in the Nielsen number $N(f)$ comes from the fact that it usually does solve the "minimum number" MF[f] problem, but not always. However it is difficult to calculate the Nielsen number $N(f)$ from the definition.

Thus the "classical" Nielsen fixed point theory has two aspects

Geometric Aspect: determining when $N(f) =$ MF[f].
Algebraic Aspect: calculating $N(f)$.

3. APPLICATIONS OF NIELSEN THEORY TO DIFFERENTIAL AND INTEGRAL EQUATIONS

This section of this review paper based on the subsection of the same title by Professor Robert F. Brown in [6].

At the 1950 International Congress of Mathematicians, Leray suggested that the work of Nielsen what gives a lower bound for the number of fixed points of a map should be extended to analytic problems because solutions to analytic problems can often be formulated as fixed points of functions and the existence of multiple solutions is often significant. However the setting of Nielsen theory was originally concerned with maps on finite polyhedra or compact ANRs, not the appropriate setting for analytic problems. In fact, Leray himself obtained a result in 1959, generalizing global Lefschetz fixed point theory that was the first step in extending Nielsen theory to the analytic setting [22].

3.1. **Selected applications of Nielsen Theory.**

(i) Let E, F be Banach spaces, $L : E \to F$ an isomorphism, $H : E \times \mathbb{R}^n \to F$ a completely continuous map and $B : E \to \mathbb{R}^n$ a continuous linear mapping onto a Euclidean space. Brown in [8] applied the Nielsen theory to help finding solutions to $Ly = H(y, \lambda)$ that satisfies $By = 0$. These solutions can be represented as the fixed points of a map $T : E \times \mathbb{R}^n \to E \times \mathbb{R}^n$. If T satisfies a condition called μ-retractibility, then Nielsen fixed point theory may be extended to produce lower bounds for the number of fixed points of such maps.

(ii) Brown and Zezza in [9] applied Nielsen theory to some problems in control theory. They studied the controllability of perturbed linear control processes whose control space can be reduced to a finite-dimensional one. Their techniques produce a lower bound on the number of controls that achieve a given target, using Nielsen theory to detect when there is more than one solution.

(iii) Fečkan [11] used Nielsen theory to establish the existence of multiple periodic solutions to problems in which the operator L is not an isomorphism. Suppose $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is 1-periodic in the first variable and bounded. For the system $y' = \varepsilon h(t, y)$, with small ε, if the map $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ is μ-retractible onto a compact, locally contractible subset S by a retraction ρ, then $N(\rho\Phi|_S)$, the Nielsen number of the map $\rho\Phi|_S$, is a lower bound for the number of 1-periodic solution to the system.
Fečkan also used this approach to study some nth order boundary value problems in [12].

(iv) Suppose that $L : X \to Y$ be a linear map between two Banach spaces X, Y. We say that L is a Fredholm operator if the image of L is closed in Y and the kernel and cokernel of L are finite dimensional. The index of a Fredholm map is defined as

$$i(L) = dimKer(L) - dim(Y/Im(L))$$

Feckan [15] applied Nielsen theory to the problem of the form $Lu = F(u)$ for L Fredholm with positive index.
Many of his results applying Nielsen theory to ordinary differential equations on Banach spaces are summarized in [10]. He also mentions there that his techniques could be used for some problems concerning partial differential equations as well.

(v) Borisovich, Kucharski and Marzantowicz [4] applied Nielsen theory to nonlinear integration equations of the *Urysohn type*.
Let X be the Banach space of pairs of continuous functions on $[0, 1]$. Let $G : X \to X$ be defined by

$$G(u, v)(t) = (u(t), v(t))$$

where $u(t)$ and $v(t)$ are of Urysohn type, i.e.

$$\begin{aligned} u(t) &= \int_0^1 K_1(t, s, u(s), v(s))v^\beta(s)ds \\ v(t) &= \int_0^1 K_2(t, s, u(s), v(s))u^\alpha(s)ds. \end{aligned}$$

Let A be the subset of K consisting of pairs (u, v) of functions, each of which takes only non-negative or only non-positive values and are not both the zero function. They found that the Nielsen number of G restricted to A was 2 which means that the system has at least two non-zero solutions.

(vi) There has been extensive work on the application of Nielsen theory to nonlinear analysis focusing on differential inclusions, multivalued functions and boundary value problems due mainly to Andres, Gorniewicz and Jeierski. See [1] for more in-depth information on this area.

(vii) In 2003, Andres and Vath [2] developed two definitions of the Nielsen numbers for the more general setting of noncompact maps. One definition is based on Nielsen's original idea for fixed point classes. The second definition is based on the idea of classes due to Wecken. They focused on a Nielsen number for coincidence points of two continuous maps $p, q : \Gamma \to X$ which is the lower bound of the number of coincidence points ($p(x) = q(x)$ for $x \in \Gamma$). Notice that if q is the identity map, this gives the classical Nielsen number. In particular, their definition can be used in the Banach space setting.

(viii) In 2007, Andres and Vath [3] used the definitions that they defined in [2] to calculate the Lefschetz and the Nielsen numbers of iterated function systems or dynamical systems in hyperspaces. Their success was due to the fact that hyperspaces are topologically simple. Their result stated that the Lefschetz and the Nielsen numbers can be calculated as easily as just counting fixed points of a map of a finite set of, typically, very small cardinality.

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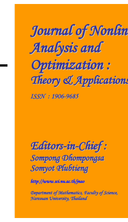

SQUARE ROOT AND 3RD ROOT FUNCTIONAL EQUATIONS IN C^*-ALGEBRAS: AN FIXED POINT APPROACH

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ABSTRACT. In this paper, we introduce a square root functional equation and a 3rd root functional equation. Using fixed point methods, we prove the Hyers-Ulam stability of the square root functional equation and of the 3rd root functional equation in C^*-algebras.

KEYWORDS : Hyers-Ulam stability; C^*-algebra; Convex cone; Fixed point, Square root functional equation; 3rd root functional equation.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations was originated from a question of Ulam [27] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [26] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [26] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M. Rassias [23]–[25] followed the innovative approach of the Th.M. Rassias' theorem [26] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of several functional equations have been extensively investigated by a

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number of authors and there are many interesting results concerning this problem (see [5, 8, 9, 13, 16, 17]).

Definition 1.1. [7] Let A be a C^*-algebra and $x \in A$ a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$.
The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [7]).

It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [11]).

In this paper, we introduce a *square root functional equation*

$$S\left(x + y + x^{\frac{1}{4}} y^{\frac{1}{2}} x^{\frac{1}{4}} + y^{\frac{1}{4}} x^{\frac{1}{2}} y^{\frac{1}{4}}\right) = S(x) + S(y) \tag{1.1}$$

and a *3rd root functional equation*

$$T\left(x + y + 3x^{\frac{1}{3}} y^{\frac{1}{3}} x^{\frac{1}{3}} + 3y^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}}\right) = T(x) + T(y) \tag{1.2}$$

for all $x, y \in A^+$. Each solution of the square root functional equation is called a *square root mapping* and each solution of the 3rd root functional equation is called a *3rd root mapping*.

Note that the functions $S(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $T(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ in the set of non-negative real numbers are solutions of the functional equations (1.1) and (1.2), respectively.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [2, 6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4], [19]–[22]).

Using the fixed point method, we prove the Hyers-Ulam stability of the functional equations (1.1) and (1.2) in C^*-algebras.

Throughout this paper, let A^+ and B^+ be the sets of positive elements in C^*-algebras A and B, respectively.

2. STABILITY OF THE SQUARE ROOT FUNCTIONAL EQUATION

In this section, we investigate the square root functional equation in C^*-algebras.

Lemma 2.1. [15] *Let $S : A^+ \to B^+$ be a square root mapping satisfying* (1.1). *Then S satisfies*

$$S(4^n x) = 2^n S(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$.

We prove the Hyers-Ulam stability of the square root functional equation in C^*-algebras. Note that the fundamental ideas in the proofs of the main results in Sections 2 and 3 are contained in [2, 3, 4].

Theorem 2.1. *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2} \varphi\left(4x, 4y\right) \tag{2.1}$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying

$$\left\| f\left(x + y + x^{\frac{1}{4}} y^{\frac{1}{2}} x^{\frac{1}{4}} + y^{\frac{1}{4}} x^{\frac{1}{2}} y^{\frac{1}{4}}\right) - f(x) - f(y) \right\| \leq \varphi(x, y) \tag{2.2}$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to A^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \leq \frac{L}{2 - 2L} \varphi(x, x) \tag{2.3}$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (2.2), we get

$$\|f(4x) - 2f(x)\| \leq \varphi(x, x) \tag{2.4}$$

for all $x \in A^+$.

Consider the set

$$X := \{g : A^+ \to B^+\}$$

and introduce the generalized metric on X:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu \varphi(x, x), \ \forall x \in A^+\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (X, d) is complete (see [18]).

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{4}\right)$$

for all $x \in A^+$.

Let $g, h \in X$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varphi(x, x)$$

for all $x \in A^+$. Hence

$$\|Jg(x) - Jh(x)\| = \|2g\left(\frac{x}{4}\right) - 2h\left(\frac{x}{4}\right)\| \leq L\varphi(x, x)$$

for all $x \in A^+$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in X$.

It follows from (2.4) that

$$\|f(x) - 2f\left(\frac{x}{4}\right)\| \le \frac{L}{2}\varphi(x,x)$$

for all $x \in A^+$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.2, there exists a mapping $S : A^+ \to B^+$ satisfying the following:

(1) S is a fixed point of J, i.e.,

$$S\left(\frac{x}{4}\right) = \frac{1}{2}S(x) \tag{2.5}$$

for all $x \in A^+$. The mapping S is a unique fixed point of J in the set

$$M = \{g \in X : d(f,g) < \infty\}.$$

This implies that S is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - S(x)\| \le \mu\varphi(x,x)$$

for all $x \in A^+$;

(2) $d(J^n f, S) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\longrightarrow\infty} 2^n f\left(\frac{x}{4^n}\right) = S(x)$$

for all $x \in A^+$;

(3) $d(f,S) \le \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,S) \le \frac{L}{2-2L}.$$

This implies that the inequality (2.3) holds.

By (2.1) and (2.2),

$$\begin{aligned} &2^n\left\|f\left(\frac{x+y+x^{\frac14}y^{\frac12}x^{\frac14}+y^{\frac14}x^{\frac12}y^{\frac14}}{4^n}\right) - f\left(\frac{x}{4^n}\right) - f\left(\frac{y}{4^n}\right)\right\| \\ \le\ & 2^n\varphi\left(\frac{x}{4^n},\frac{y}{4^n}\right) \le L^n\varphi(x,y) \end{aligned}$$

for all $x, y \in A^+$ and all $n \in \mathbb{N}$. So

$$\left\|S\left(x+y+x^{\frac14}y^{\frac12}x^{\frac14}+y^{\frac14}x^{\frac12}y^{\frac14}\right) - S(x) - S(y)\right\| = 0$$

for all $x, y \in A^+$. Thus the mapping $S : A^+ \to B^+$ is a square root mapping, as desired. □

Corollary 2.2. *Let $p > \frac{1}{2}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\begin{aligned} &\left\|f\left(x+y+x^{\frac14}y^{\frac12}x^{\frac14}+y^{\frac14}x^{\frac12}y^{\frac14}\right) - f(x) - f(y)\right\| \\ \le\ & \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \end{aligned} \tag{2.6}$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to B^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \le \frac{2\theta_1 + \theta_2}{4^p - 2}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 2^{1-2p}$ and we get the desired result. □

Theorem 2.2. *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 2L\varphi\left(\frac{x}{4}, \frac{y}{4}\right)$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying (2.2). *Then there exists a unique square root mapping $S : A^+ \to A^+$ satisfying* (1.1) *and*

$$\|f(x) - S(x)\| \leq \frac{1}{2 - 2L}\varphi(x,x)$$

for all $x \in A^+$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g\left(4x\right)$$

for all $x \in A^+$.

It follows from (2.4) that

$$\left\|f(x) - \frac{1}{2}f(4x)\right\| \leq \frac{1}{2}\varphi(x,x)$$

for all $x \in A^+$. So $d(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.3. *Let $0 < p < \frac{1}{2}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (2.6). *Then there exists a unique square root mapping $S : A^+ \to B^+$ satisfying* (1.1) *and*

$$\|f(x) - S(x)\| \leq \frac{2\theta_1 + \theta_2}{2 - 4^p}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 2^{2p-1}$ and we get the desired result. □

3. STABILITY OF THE 3RD ROOT FUNCTIONAL EQUATION

In this section, we investigate the 3rd root functional equation in C^*-algebras.

Lemma 3.1. [15] *Let $T : A^+ \to B^+$ be a 3rd root mapping satisfying* (1.2). *Then T satisfies*

$$T(8^n x) = 2^n T(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$.

We prove the Hyers-Ulam stability of the 3rd root functional equation in C^*-algebras.

Theorem 3.1. *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{2} \varphi(8x, 8y)$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying

$$\left\| f\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) - f(x) - f(y) \right\| \le \varphi(x, y) \quad (3.1)$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to A^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \le \frac{L}{2 - 2L} \varphi(x, x)$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (3.1), we get

$$\|f(8x) - 2f(x)\| \le \varphi(x, x) \quad (3.2)$$

for all $x \in A^+$.

Let (X, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{8}\right)$$

for all $x \in A^+$.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{8}\right)$$

for all $x \in A^+$.

It follows from (3.2) that

$$\|f(x) - 2f\left(\frac{x}{8}\right)\| \le \frac{L}{2}\varphi(x, x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let $p > \frac{1}{3}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\left\| f\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) - f(x) - f(y) \right\|$$
$$\le \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \quad (3.3)$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \le \frac{2\theta_1 + \theta_2}{8^p - 2}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 2^{1-3p}$ and we get the desired result. □

Theorem 3.2. *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{8}, \frac{y}{8}\right)$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying (3.1). *Then there exists a unique 3rd root mapping $T : A^+ \to A^+$ satisfying* (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{1}{2 - 2L}\varphi(x, x)$$

for all $x \in A^+$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(8x)$$

for all $x \in A^+$.

It follows from (3.2) that

$$\left\|f(x) - \frac{1}{2}f(8x)\right\| \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in A^+$. So $d(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.3. *Let $0 < p < \frac{1}{3}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (3.3). *Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying* (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{2\theta_1 + \theta_2}{2 - 8^p}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 2^{3p-1}$ and we get the desired result. □

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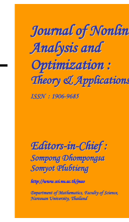

NEW POSSIBILITIES REGARDING THE ALTERNATING PROJECTIONS METHOD

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ABSTRACT. We provide a short survey of some recent results of ours regarding the convergence of infinite products of nonexpansive operators, some of which are orthogonal projections, while others may even be nonlinear. It turns out that the standard requirement of cyclic order of the projections may be replaced with the positivity of the angles between the given subspaces. Moreover, one of these angles may even be unknown, provided the corresponding projection appears in the infinite product sufficiently rarely.

KEYWORDS : Alternating projection; Angle; Infinite product; Nonexpansive operator.

1. INTRODUCTION

The well-known method of alternating projections is used for finding points in the intersection of a finite number of closed and convex sets via individual approximations from each one of these sets. If all the sets are closed linear subspaces of a given Hilbert space H, then we can use the best approximations which in this case are the standard orthogonal projections. Taking an initial point $x_0 \in H$ and projecting it successively onto the given subspaces, we may expect that the sequence of iterations $\{x_n = P_n P_{n-1} \cdots P_1 x_0\}$ will converge to a point x^* in the intersection of all the subspaces.

We are interested in either strong or uniform convergence. The only necessary condition for this is that every subspace is involved in the iteration process infinitely often. The various sufficient conditions presented in the literature require

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a special order of the projections and/or additional information regarding interrelations between the given subspaces. For example, many iteration procedures were studied for the case where the orthogonal projections are arranged cyclically in a prescribed repeated order. In this connection, one should mention, first and foremost, the celebrated pioneering works by J. von Neumann [4] and I. Halperin [3]. For instance, in the case of three projections P_U, P_V, P_W onto three given subspaces U, V, W, respectively, Halperin's theorem ensures the strong convergence of the sequence $\{x_n = (P_U P_V P_W)^n x_0\}$ to $P_{U\cap V\cap W} x_0$ for any $x_0 \in H$.

The uniform convergence of iterations cannot be obtained by only using the order of the projections, because any projection operator has norm 1, which is insufficient for achieving uniform convergence. Nevertheless, the product of two projections $P_U P_V$ could have norm less than 1 in the case of a positive angle between the subspaces U and V. The concept of an "angle between subspaces" in an arbitrary Hilbert space has many different definitions applied to different problems and situations. We adopt the one given by K. Friedrichs in [2] and widely used for studying projection methods in the monograph [1] by F. Deutsch. Namely,

$$\theta(U,V) = \inf\{|\theta(x,y)| : \ x \in U^\circ, \ y \in V^\circ, \ x,y \neq 0\},$$

where $U^\circ = U \cap (U\cap V)^\perp, \ V^\circ = V \cap (U \cap V)^\perp$ and

$$\theta(x,y) = \arccos \frac{\langle x,y\rangle}{\|x\|\cdot\|y\|}$$

is the usual angle between two vectors in a space with inner product $\langle\cdot\,,\cdot\rangle$. The uniform norm convergence of the sequence $\{x_n = (P_U P_V)^n x_0\}$ to $P_{U\cap V} x_0$ immediately follows from the estimate

$$\|P_U P_V x - P_{U\cap V} x\| \leq \cos\theta(U,V)\, \|x - P_{U\cap V} x\| \qquad \text{for any } x \in H.$$

The angle $\theta(U,V)$ is positive (and, consequently, $\cos\theta(U,V) < 1$) whenever at least one of the subspaces U, V is finite dimensional. Otherwise the angle may be zero even in rather standard situations.

Example. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis in the separable Hilbert space H and let the numerical sequences $\{\alpha_n\}$ and $\{\beta_n\}$ be such that $\alpha_n^2 + \beta_n^2 = 1$ for each n with $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$. We consider two infinite dimensional subspaces U and V with the bases $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$, respectively, defined by

$$u_{2n-1} = \frac{1}{\sqrt{2}}(e_{3n} - e_{3n-1}), \quad u_{2n} = \frac{1}{\sqrt{2}}(e_{3n} + e_{3n-1}), v_n = \alpha_n e_{3n-2} + \beta_n e_{3n}\,, \quad n = 1, 2, \dots .$$

It is a simple task to check that both bases $\{u_n\}$ and $\{v_n\}$ are orthonormal and that $U \cap V = \{0\}$, so that $U = U^\circ$ and $V = V^\circ$. Hence we may use the vectors $e_{3n} = \frac{1}{\sqrt{2}}(u_{2n-1} + u_{2n})$ from the subspace U and v_n from the subspace V, obtaining that $\cos\theta(U,V) \geq \langle e_{3n}, v_n\rangle = \beta_n \longrightarrow 1$ as $n \longrightarrow \infty$, that is, $\theta(U,V) = 0$.

Observe, in addition, that for any of the basis vectors $u_m \in U^\circ$ and $v_n \in V^\circ$, the angles $\theta(u_m, v_n) > \frac{\pi}{4}$ because $\langle u_m, v_n\rangle < \frac{1}{\sqrt{2}}$ for all combinations of m, n. This means that, generally speaking, the conclusion that $\theta(U,V) = 0$ cannot be obtained by only considering the basis vectors, making the problem of deciding whether $\theta(U,V) > 0$ rather difficult in general.

2. UNIFORM CONVERGENCE

In the case of three subspaces U, V, W the pairwise angles between them are insufficient for getting uniform convergence. The known proofs for the projections arranged as $P_U P_V P_W$ require either the angles $\alpha = \theta(U, V)$ and $\beta = \theta(U \cap V, W)$ or the angles $\alpha = \theta(V, W)$ and $\beta = \theta(U, V \cap W)$. The corresponding estimate is

$$\|P_U P_V P_W x - P_{U\cap V\cap W}x\| \leq q\|x - P_{U\cap V\cap W}x\| \qquad \text{for any } \ x \in H,$$

with $q = (1 - \sin^2 \alpha \sin^2 \beta)^{1/2}$ in the first case and with

$$q = \max \left[\cos \frac{\beta}{2},\ (1 + \sin^2 \frac{\beta}{2} \tan^2 \alpha)^{-1/2}\right]$$

in the second one. We see that uniform convergence may be ascertained only when both α and β are positive. In particular, uniform convergence occurs if at least one of the subspaces is finite dimensional.

Using angles between subspaces, we are able to consider short fragments of the iteration process independently of other fragments, which may be further combined in arbitrary order, making the cyclic order of projections unnecessary:

$$\cdots (P_U P_V P_W)(P_V P_W P_U)(P_W P_V P_U)(P_V P_W P_U) \cdots$$

Moreover, we may insert between these fragments other nonexpansive, even nonlinear, operators if this is needed for improving the calculation process or for obtaining solutions with additional properties. Recall that an operator A is called *nonexpansive* if $\|Ax - Ay\| \leq \|x - y\|$ for any $x, y \in H$.

Generally speaking, the approximation process can be presented as an infinite product of operators $\prod_{n=1}^{\infty} A_n \mathcal{P}_n$, where all A_n are nonexpansive and all $\mathcal{P}_n$ are linear contractions, obtained as compositions of the projections P_U, P_V, P_W in any suitable order.

Theorem 2.1. *Let the subspace $F = U \cap V \cap W$ be invariant under all the nonexpansive operators A_n acting on a Hilbert space H and participating in a given infinite product. Let $q < 1$ be such that*

$$\|\mathcal{P}_n x - P_F x\| \leq q\|x - P_F x\| \qquad \text{for any } x \in H,\ n = 1, 2, \ldots.$$

Then, for any initial point $x_0 \in H$, the corresponding partial products form a sequence

$$\{x_n = A_n \mathcal{P}_n A_{n-1} \mathcal{P}_{n-1} \cdots A_1 \mathcal{P}_1 x_0\}$$

such that

$$\lim_{n \longrightarrow \infty} \|x_n - P_F x_n\| = 0,$$

uniformly over any bounded set of initial points x_0. If, in addition, all $x \in F$ are fixed points for each A_n, then $\{x_n\}$ is strongly convergent to some $x^* \in F$.

In particular, the operators A_n could be the separate projections P_U, P_V, P_W or their pairwise products, and thus the triple products $\mathcal{P}_n = P_U P_V P_W$ (or in another order) need not follow cyclically and may be located in the iterative process arbitrarily far from each other. In practice this means that the order of the projections may be *essentially random* (not cyclic and even not “almost" cyclic); the only condition for uniform convergence to the best approximation from $U \cap V \cap W$ is that the triple products with known positivity of the corresponding angles appear infinitely many times, for instance,

$$\cdots P_V P_U P_V (P_W P_U P_V) P_W P_V P_W P_V (P_W P_V P_U) P_V P_U \cdots .$$

3. STRONG CONVERGENCE

Rather surprisingly, a similar assertion can be proved in the case where only one angle (say $\theta(U,V)$) is known to be positive and no properties of the angles involving the third subspace W are given. Of course, the convergence of iterations in this case may be only strong, but it holds without imposing any cyclic arrangement of the projections.

Theorem 3.1. *Let U, V, W be three subspaces of a Hilbert space H such that the angle $\theta(U,V)$ is positive. Let the nonexpansive operators A_n, $n = 1, 2, \ldots,$ be such that all elements of the subspace $U \cap V$ are fixed points of each A_n. Let a sequence of natural numbers $\{k_n\}$ be such that*

$$\sum_{n=1}^{\infty} q^{k_n} < \infty , \qquad \text{where} \quad q = \cos\theta(U,V).$$

Then, for any $x \in H$, there exists $x^* \in U \cap V \cap W$ such that

$$\lim_{n \longrightarrow \infty} \|P_W A_n (P_U P_V)^{k_n} P_W A_{n-1} (P_U P_V)^{k_{n-1}} \cdots P_W A_1 (P_U P_V)^{k_1} x - x^*\| = 0.$$

Note that Theorem 3.1 admits an interesting new application to Numerical Analysis. Suppose we are interested in finding the point $P_{U\cap V\cap W} x_0$ for some given $x_0 \in H$. By Halperin's theorem we may use the iterations $x_n = (P_W P_U P_V)^n x_0$ which converge to the sought-after point. Suppose the subspace W is such that any computation of the projection P_W is difficult in comparison with other projections. Omitting all A_n, we see that, in the case where $\theta(U,V) > 0$, Theorem 3.1 enables another iteration process, namely,

$$x_N = P_W (P_U P_V)^{k_n} P_W (P_U P_V)^{k_{n-1}} \cdots P_W (P_U P_V)^{k_1} x_0, \quad N = n + \sum_{i=1}^{n} k_i ,$$

which converges to the same point $P_{U\cap V\cap W} x_0$ for arbitrarily quickly increasing k_n and, correspondingly, arbitrarily rare computations of P_W.

We now give a brief sketch of the proof of Theorem 3.1. We will use the following result from [5, p. 1512]. Here $\rho(y,F) := \inf\{\|y - z\| : z \in F\}$.

Proposition 3.1. *Let $T :\ H \longrightarrow H$ be a nonexpansive operator and let a set $F \subset H$ be such that, for any given $x \in H$, the sequence $\{\rho(T^n x, F)\}$ converges to 0 as $n \longrightarrow \infty$. Let a sequence $\{x_n\}_{n=0}^{\infty} \subset H$ be such that, for each $n = 0, 1, 2, \ldots$,*

$$\|x_{n+1} - T x_n\| \le \gamma_n, \qquad \sum_{n=1}^{\infty} \gamma_n < \infty.$$

Then $\rho(x_n, F) \longrightarrow 0$ as $n \longrightarrow \infty$. If, in addition, $\{T^n x\}$ is strongly convergent for each $x \in H$, then $\{x_n\}$ strongly converges to some $x^* \in F$.

According to the von Neumann theorem for two projections,

$$\lim_{n \longrightarrow \infty} \|(P_W P_{U\cap V})^n x - P_{U\cap V\cap W} x\| = 0, \qquad \forall x \in H,$$

which corresponds to the hypotheses of Proposition 3.1 if we put $T = P_W P_{U\cap V}$ and $F = U \cap V \cap W$. Next, for arbitrary $x \in H$, we define $x_1 = x$, $x_{n+1} = P_W A_n (P_U P_V)^{k_n} x_n$. The inequality

$$\|P_U P_V x - P_{U\cap V} x\| \le q\, \|x - P_{U\cap V} x\|, \quad q = \cos\theta(U, V),$$

which has already been indicated above, can be readily generalized by induction to

$$\|(P_U P_V)^n x - P_{U\cap V} x\| \le q^n\, \|x - P_{U\cap V} x\|,$$

and by the postulated properties of the operators A_n, we have $A_n P_{U\cap V} x = P_{U\cap V} x$ for each n. Consequently,

$$\begin{gathered}\|x_{n+1} - T x_n\| = \|P_W A_n (P_U P_V)^{k_n} x_n - P_W P_{U\cap V} x_n\| \\ \le \|A_n (P_U P_V)^{k_n} x_n - A_n P_{U\cap V} x_n\| \le \|(P_U P_V)^{k_n} x_n - P_{U\cap V} x_n\| \\ \le q^{k_n}\, \|x_n - P_{U\cap V} x_n\| \le q^{k_n}\, \|x_n\| \le q^{k_n}\, \|x\|,\end{gathered}$$

where we have used the fact that the sequence $\{\|x_n\|\}$ is decreasing.

Setting

$$\gamma_n = q^{k_n}\, \|x\|,$$

we obtain the last assumption of Proposition 3.1.

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DOOB'S DECOMPOSITION OF FUZZY SUBMARTINGALES VIA ORDERED NEAR VECTOR SPACES[◇]

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ABSTRACT. We use ideas from measure-free martingale theory and Rådström's completion of a near vector space to derive a Doob decomposition of submartingales in ordered near vector spaces, which is a generalization of result noted by Daures, Ni and Zhang, and an analogue of the Doob decomposition of submartingales in the fuzzy setting, as noted by Shen and Wang.

1. INTRODUCTION

The aim of this paper is to complete the work of [11] and extend those results to the fuzzy setting. In [11] we focussed on the Doob's decomposition of submartingales and used the notion of near vector spaces to overcome the problems faced.

In this paper, we again concern ourselves with Doob's decomposition of submartingales. This decomposition was extended from the classical setting of real valued martingales to set-valued martingales by Daures, Ni and Zhang (see [12, 13]) and also by Shen and Wang (see [16]).

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When modelling events which involve inherent uncertainty due to incomplete information, one method is to use fuzzy sets. This leads to the concept of fuzzy martingales. In the fuzzy setting we immediately encounter the same type of problem the one faces in the set-valued setting. That is that neither the spaces of submartingales nor the range spaces of the submartingales are vector spaces. Since they are in fact near vector spaces we can apply the near vector ideas developed in [10, 14].

Our aim is to build on the work done in [10] and [11] and once again use ideas from measure-free martingale theory (see [1, 2, 3, 7, 15, 17]), together with Rådström's completion of a near vector space, to give an elementary proof for a version of Doob's decomposition of fuzzy submartingales.

After introducing the necessary preliminaries and notation, we consider Doob's decomposition of a submartingale in an ordered vector space. From this, and with the aid of Rådström's completion of a near vector space (see [10]), we obtain a Doob decomposition of a submartingale in an ordered near vector space. We then specialize the ordered near vector space to the appropriate fuzzy set-valued space of submartingales that are integrable. As special cases, we obtain the Daures, Ni and Zhang result by using the fact that martingales which are integrably bounded are integrable (see [12]). We also derive an analogue of the Doob decomposition of fuzzy submartingales, as noted by Shen and Wang.

2. PRELIMINARIES

Let (Ω, Σ, μ) be a finite measure space. If Σ_0 is a sub σ-algebra of Σ, denote by $L^0(\Omega, \Sigma_0, \mu)$ the set of Σ_0-measurable functions $f : \Omega \longrightarrow \mathbb{R}$. If $f \in L^1(\Omega, \Sigma_0, \mu)$ is a random variable, we denote by $\mathbb{E}[f|\Sigma_0]$ the conditional expectation of f with respect to Σ_0. If (Σ_i) an increasing sequence of sub σ-algebras of Σ, then (f_i, Σ_i) is a martingale (submartingale) provided that

$$f_i \in L^0(\Omega, \Sigma_i, \mu) \text{ and } f_i = (\leq)\, \mathbb{E}[f_{i+1}|\Sigma_i]$$

for all $i \in \mathbb{N}$. The following well-known result relates submartingales to martingales:

Theorem 2.1. (Doob's Decomposition) *If (Σ_i) an increasing sequence of sub σ-algebras of Σ, and (f_i, Σ_i) is a submartingale, then (f_i, Σ_i) has a unique decomposition*

$$f_i(\omega) = M_i(\omega) + A_i(\omega) \text{ a.e.}$$

where (M_i, Σ_i) is a set-valued martingale and (A_i) is a predictable (i.e., A_i is Σ_{i-1}-measurable for all $i \geq 2$), increasing sequence such that

(a) $A_1(\omega) = 0$ *a.e.,*
(b) $A_j(\omega) = \sum_{i=1}^{j-1} \Big(\mathbb{E}[f_{i+1}|\Sigma_i](\omega) - f_i(\omega) \Big)$ *a.e. for $j \geq 2$,*
(c) $M_j(\omega) = f_j(\omega) - A_j(\omega)$ *a.e. for all $j \in \mathbb{N}$.*

Daures, Ni and Zhang proved an analogue of Doob's decomposition for set-valued submartingales (see [3, 13]). Before we state our main result, as can be found in [12], we first recall some terminology from [6, 12].

Our main focus is on the application of near vector spaces to fuzzy submartingales so we present an overview of the important notions associated with fuzzy sets and fuzzy random variables. For a more comprehensive treatment of fuzzy sets and the application of fuzzy sets in functional analysis the reader is referred to [12].

Definition 2.1. Let X be a set and I the unit interval $[0,1]$. A *fuzzy set* on X (*fuzzy-subset of* X) is a map from X into I. That is, if A is a fuzzy subset of X then $A \in I^X$, where I^X denotes the collection of all maps from X into I.

I^X is naturally equipped with an order structure induced by I. If $A, B \in I^X$ then we say that A is a *fuzzy subset* of B if $A(x) \leq B(x)$ for all $x \in X$.
For a given fuzzy set we associate collections of crisp subsets of X with it.

If $A \in I^X$ and $\alpha \in I$ we define,

$$A^\alpha = \{x \in X : A(x) > \alpha\};$$

$$A_\alpha = \{x \in X : A(x) \geq \alpha\}.$$

These crisp sets are referred to as α-*levels* (or *cuts*), strong and weak respectively. We call A^0 the *support* of A and denote it by $\mathrm{supp}(A)$. We have the following useful relationship between fuzzy sets and their corresponding α-cuts.

Lemma 2.2. *Let A and B be fuzzy sets on a set X. Then for all $\alpha \in (0,1]$:*

(i) $A = B \iff A_\alpha = B_\alpha$ *for all* $\alpha \in (0,1]$, *and*
(ii) $A \leq B \iff A_\alpha \subseteq B_\alpha$.

The following theorems enable us to decompose a given fuzzy set into a supremum of a collection of crisp sets.

Theorem 2.2. *For a $A \in I^X$ and $x \in X$ we have*

$$A(x) = \sup_{\alpha \in (0,1]} \{\alpha \chi_{A_\alpha}(x)\}.$$

If $A \in \mathcal{P}(X)$ and $\alpha \in I$ we define

$$\alpha\chi_A(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

So

$$\alpha\chi_{\{x\}} = \begin{cases} \alpha & \text{on } x \\ 0 & \text{elsewhere} \end{cases}$$

We call $\alpha\chi_{\{x\}}$ a *fuzzy point* with support at x and *value* α. We will denote the set of fuzzy points in I^X by $\tilde{X}$. A fuzzy point is clearly a generalization of a point in ordinary set theory.
If A, B are crisp subsets of a vector space X and t a scalar we have
$t \cdot A = \{ta : a \in A\}$ and $A + B = \{a + b : a \in A, b \in B\}$. We define addition and scalar multiplication of fuzzy sets in the natural way which is a direct consequence of the image of a fuzzy mapping.

Definition 2.3. Let X be a vector space. For $A, B \in I^X, t \in K$ and $x \in X$

(a) [Addition] $(A + B)(x) = \sup_{x_1 + x_2 = x}\{A(x_1) \wedge A(x_2)\}$.
(b) [Scalar multiplication] $t \cdot A(x) = A(\frac{x}{t})$ for $t \neq 0$. If $t = 0$:

$$t \cdot A(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \sup A & \text{if } x = 0. \end{cases}$$

Denote the power set of X by $\mathcal{P}(X)$. It is well-known that the set $\mathcal{P}_0(X) = \mathcal{P}(X)\backslash\emptyset$ does not, in general, form a vector space with respect to the above defined operations.

A crisp subset A of a vector space X is said to be convex of for any $a, b \in A$ and any $k \in [0, 1]$ we have that $ka + (1 - k)b \in A$. We have an analagous notion in the fuzzy setting.

Definition 2.4. Let A be a fuzzy subset of a vector space X. Then A is *(fuzzy)convex* if $A(kx + (1 - k)y) \geq A(x) \wedge A(y)$ whenever $x, y \in X$ and $0 \leq k \leq 1$.

It is once again a simple matter to confirm that if a fuzzy set A is convex then for each $\alpha \in [0, 1)$, A_α is convex in the classical sense.
In [11] we considered the certain collections of sets.

$$\mathrm{f}(X) := \{A \in \mathcal{P}_0(X) : A \text{ is closed}\}.$$

For all $A, C \in \mathrm{f}(X)$, define

$$A \oplus C = \overline{A + C},$$

where the closure is taken with respect to the norm on X. Then $\mathrm{f}(X)$ is closed under $\oplus$.
In [10] we introduced the following notation:

$$\begin{aligned} \mathrm{cf}(X) : &= \{A \in \mathrm{f}(X) : A \text{ is convex}\}, \\ \mathrm{bf}(X) : &= \{A \in \mathrm{f}(X) : A \text{ is bounded}\}, \\ \mathrm{cbf}(X) : &= \{A \in \mathrm{bf}(X) : A \text{ is convex}\}. \end{aligned}$$

We define $\mathrm{F}(X)$ as the collection of fuzzy sets $A : X \to I$ such that

(a) A is uppersemicontinuous,
(b) supp(A) is compact,
(c) $\{x \in X : A(x) = 1\} \neq \emptyset$.

Lemma 2.5. *Let* $A \subseteq \mathrm{F_c}(X)$ *if and only if* $A_\alpha \in \mathrm{cf}(X)$*for all* $\alpha \in I$.

If $A \in \mathcal{P}_0(X)$ and $x \in X$, the *distance* between x and A is defined by

$$d(x, A) = \inf\{\|x - y\|_X : y \in A\}.$$

Define d_H for all $A, B \in \mathrm{cf}(X)$ by

$$d_H(A, B) = \sup_{a \in A} d(a, B) \vee \sup_{b \in B} d(b, A).$$

Then d_H is a metric on $\mathrm{cf}(X)$, which is called the *Hausdorff metric*, and $(\mathrm{cf}(X), d_H)$ is a complete metric space (cf. [12]). In the special case where $B = \{0\}$, let

$$\|A\|_H = d_H(A, \{0\});$$

in general $\|\cdot\|_H$ is not a norm. Furthermore, $\mathrm{cbf}(X)$ is a closed subspace of $\mathrm{bf}(X)$ (cf. [12]).
We generalize the Hausdorff metric to the fuzzy setting in the natural way by defining d_∞ as

$$d_\infty(F_1, F_2) = d_H(\text{supp}(F_1), \text{supp}(F_2)),$$

for fuzzy random variables F_1 and F_2.
In [11] we defined the following operation the $\ominus$ operation on crisp sets A and B in the following way: $A \ominus B := \{x \in X : x + B \subseteq A\}$. We can naturally extend this to fuzzy sets in the following way.

Definition 2.6. Let $A, B \in I^X$. We define $A \ominus B$ as

$$A \ominus B = \sup_{\alpha \in (0,1]} \{\alpha \chi_{A_\alpha \ominus B_\alpha}(x)\}.$$

The following definition provides us with the fundamental notions of random variables in the fuzzy setting.

Definition 2.7. (a) A *fuzzy set-valued random variable* (f.r.v.) or a *fuzzy random set* is a function $F : \Omega \to \mathrm{F}(X)$ such that $F_\alpha(\omega) = \{x \in X : F(\omega)(x) \geq \alpha\}$ is a set-valued random variable for all $\alpha \in (0, 1]$. We denote by $\mathbf{M}[\Sigma, \mathrm{F}(X)]$ the collection of all Σ-measurable fuzzy random variables. We denote by $\mathbf{M}[\Sigma, \mathrm{F_c}(X)]$ the collection of measurable and integrable functions $f : \Sigma \to \mathrm{F_c}(X)$ respectively.

(b) The *expectation* of a fuzzy random variable F, denoted $E(F)$, is defined by

$$E(F) = \int_\Omega \text{supp}(F) d\mu.$$

(c) Let Σ_0 be a sub σ-algebra of Σ. Then the *conditional expectation* of F relative to Σ_0 is defined as $\mathcal{E}[F|\Sigma_0] = \mathcal{E}[\text{supp}(F)|\Sigma_0]$.

(d) A *selection* of $F \in \mathbf{M}[\Sigma, \mathrm{F}(X)]$ is a function $f \in L^1(\mu, \tilde{\mathbb{R}})$ such that $f(\omega) \leq F(\omega)$ for all $\omega \in \Omega$ a.e. We denote the set of selections of F by S^1_F and we say that F is *integrable* if $S^1_F \neq \emptyset$. We denote by $\mathcal{L}[\Sigma, \mathrm{F}(X)]$ the collection $\{F \in \mathbf{M}[\Sigma, \mathrm{F}(X)] : S^1_F \neq \emptyset\}$ and $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ denotes the set $\{F \in \mathcal{L}[\Sigma, \mathrm{F}(X)] : F(\omega) \in \mathrm{F_c}(X), \forall \omega \in \Omega\}$.

Hiai and Umegaki proved in [6] that, if $F \in \mathcal{L}[\Sigma, \mathrm{f}(X)]$, then there exists a unique $G \in \mathbf{M}[\Sigma_0, \mathrm{f}(X)]$ such that

$$S^1_G(\Sigma_0) = \overline{\{\mathbb{E}[f|\Sigma_0] : f \in S^1_F\}},$$

where the closure is taken in $L^1(\Sigma, \mu, X)$, and $\mathbb{E}[f|\Sigma_0]$ denotes the conditional expectation of $f \colon \Omega \longrightarrow X$ with respect to Σ_0. As is customary, we denote G by $\mathcal{E}[F|\Sigma_0]$ and call $\mathcal{E}[F|\Sigma_0]$ the *conditional expectation* of $F \colon \Omega \longrightarrow \mathrm{f}(X)$ relative to Σ_0 (cf. [6, 12]). It is well-known that the following properties hold:

Theorem 2.3. (see [6, 12]) *Let Σ_0 be a sub σ-algebra of Σ. If $F \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$, then the conditional expectation $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}[\Sigma_0, \mathrm{F_c}(X)]$ of F with respect to Σ_0 has the following properties:*

- (E1) *If $F_1, F_2 \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$, then $\mathcal{E}[F_1 + F_2|\Sigma_0] = \mathcal{E}[F_1|\Sigma_0] \oplus \mathcal{E}[F_2|\Sigma_0]$.*
- (E2) *If $F \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ and $\lambda \in \mathbb{R}_+$, then $\mathcal{E}[\lambda F|\Sigma_0] = \lambda \mathcal{E}[F|\Sigma_0]$.*
- (E3) *If $F_1, F_2 \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$, then $F_1 \leq F_2$ implies $\mathcal{E}[F_1|\Sigma_0] \leq \mathcal{E}[F_2|\Sigma_0]$.*
- (E4) *If $F \in \mathcal{L}[\Sigma_0, \mathrm{F_c}(X)]$, then $\mathcal{E}[F|\Sigma_0] = F$.*
- (E5) *If $\Sigma_0 \subseteq \Sigma_2 \subseteq \Sigma$ and $F \in \mathcal{L}[\Sigma_0, \mathrm{F_c}(X)]$, then $\mathcal{E}\left[\, \mathcal{E}[F|\Sigma_2] \,|\, \Sigma_0\right] = \mathcal{E}[F|\Sigma_0]$.*

If $F \in \mathbf{M}[\Sigma, \mathrm{F}(X)]$, then F is called *integrably bounded* provided that there exists $\rho \in L^1(\mu)$ such that $\|x\|_X \leq \rho(\omega)$ for all $x \in F(\omega)$ and for all $\omega \in \Omega$. In this case, $F(\omega) \in \mathrm{F_c}(X)$ a.e. and $\|F(\omega)\|_H = \sup\{\|x\|_X : x \in \text{supp}(F)(\omega)\} \leq \rho(\omega)$ for all $\omega \in \Omega$.

Let $\mathcal{L}^1[\Sigma, \mathrm{F}(X)]$ denote the set of all equivalence classes of a.e. equal $F \in \mathbf{M}[\Sigma, \mathrm{F}(X)]$ which are integrably bounded. If $\Delta \colon \mathcal{L}^1[\Sigma, \mathrm{F}(X)] \times \mathcal{L}^1[\Omega, \mathrm{F}(X)] \longrightarrow \mathbb{R}_+$ is defined by

$$\Delta(F_1, F_2) = \int_\Omega d_H\big(\text{supp}(F_1(\omega)), \text{supp}(F_2(\omega))\big) d\mu,$$

then $(\mathcal{L}^1[\Sigma, \mathrm{F}(X)], \Delta)$ is a complete metric space. Define addition $+$, scalar multiplication $\cdot$ and an order relation pointwise on $\mathcal{L}^1[\Sigma, \mathrm{F}(X)]$.
Let

$$\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)] = \{F \in \mathcal{L}^1[\Sigma, \mathrm{F}(X)] : F(\omega) \in \mathrm{F_c}(X) \text{ a.e.}\}.$$

Note that $\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)] \subseteq \mathcal{L}^1[\Sigma, \mathrm{F}(X)]$ and for Σ_0 a sub σ-algebra of Σ we have $\mathcal{E}[F|\Sigma_0] \in \mathcal{L}^1[\Sigma_0, \mathrm{F}(X)]$ for all $F \in \mathcal{L}^1[\Sigma, \mathrm{F}(X)]$.
We generalize the following from [12]:

Definition 2.8. Let $(F_i) \subseteq \mathcal{L}[\Sigma, \mathrm{F}(X)]$ and (Σ_i) an increasing sequence of sub σ-fields of Σ. Then $(F_i, \Sigma_i)_{i\in\mathbb{N}}$ is called a *martingale* (respectively, *submartingale*) in $\mathcal{L}[\Sigma, \mathrm{F}(X)]$ provided that $F_i \in \mathbf{M}[\Sigma_i, \mathrm{F}(X)]$ and $F_i(\omega) = (\leq)\ \mathcal{E}[F_{i+1}|\Sigma_i](\omega)$ a.e. for all $i \in \mathbb{N}$.

Let $X^* = \{x^* : X \longrightarrow \mathbb{R} : x^* \text{ is linear and continuous}\}$. For every bounded subset C of X and each $x^* \in X^*$, let

$$s(x^*, C) := \sup\{x^*(x) : x \in \mathrm{supp}(C)\}.$$

We are now in a position to state a fuzzy version of Doob decomposition theorem of Daures, Ni and Zhang, as can be found in [12]:

Theorem 2.4. *Let (F_i, Σ_i) be a set-valued submartingale in $\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]$; i.e., (F_i, Σ_i) be a submartingale in $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ and $(F_i) \subseteq \mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]$. If there exists $B \in \Sigma$ with $\mu(B) = 0$ such that for any $\omega \notin B$ and all $i \in \mathbb{N}$*

(i) $s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) - s(\,\cdot\,, F_{i-1}(\omega))$ *and*
(ii) $s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)])$

are convex functions on X^*, then (F_i, Σ_i) can be decomposed as

$$F_i(\omega) = M_i(\omega) + A_i(\omega) \text{ for all } \omega \notin B$$

where (M_i, Σ_i) is a fuzzy martingale and (A_i) is a fuzzy set-valued predictable increasing sequence such that for all $\omega \notin B$

(a) $A_1(\omega) = 0$,
(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\right)}$ *for all* $j \geq 2$,
(c) $M_1(\omega) = F_1(\omega)$, *and*
(d) $M_j(\omega) = \overline{\left(\sum_{i=2}^{j} [F_i(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega)\right) + F_1(\omega)}$ *for all* $j \geq 2$.

The proof of Theorem original crisp version of 2.4, as given in [12], exploits the properties of the the functions $s(\cdot, C)$ where $C \in \mathrm{F_c}(X)$.
To achieve our aim, we derive a Doob decomposition theorem for submartingales in ordered near vector spaces as considered in [10]. As a consequence and as an intermediate step, we obtain a Doob decomposition for fuzzy submartingales which are integrable in §4. The latter then yields the Daures, Ni and Zhang result (see [3, 13]) as a special case, as integrably bounded functions are integrable (see [12, p.31]). It also yields an analogue of the Doob decomposition of fuzzy submartingales, as noted by Shen and Wang (see [16]).

3. DOOB'S DECOMPOSITION IN AN ORDERED NEAR VECTOR SPACE

It was noted in [10] that if X is a Banach space, then $(\mathrm{cf}(X), +, \cdot)$ is a near vector space. For the convenience of the reader, we recall the definition from [10].

Let S be a nonempty set. As in [10], S is said to be a *near vector space*, provided that addition $+\colon S \times S \longrightarrow S$ is defined such that $(S,+)$ is a cancellative commutative semigroup; i.e., for all $x, y, z \in S$:

$$x+z = y+z \Rightarrow x = y, \quad x+y = y+x, \quad (x+y)+z = x+(y+z),$$

and multiplication $\cdot : \mathbb{R}_+ \times S \longrightarrow S$ by positive scalars is defined such that for all $x, y \in S$ and $\lambda, \delta \in \mathbb{R}_+$:

$$\lambda x + \lambda y = \lambda(x+y),\ (\lambda+\delta)x = \lambda x + \delta x,\ (\lambda\delta)x = \lambda(\delta x),\ 1x = x.$$

Let S be a (near) vector space. If $(S, \leq)$ is a partially ordered set such that $\leq$ is compatible with addition and multiplication by positive scalars; i.e., for all $x, y \in S$ and $\alpha \in \mathbb{R}_+$,

$$x \leq y \ \Rightarrow\ [x+z \leq y+z \text{ and } \alpha x \leq \alpha y],$$

then S is called an *ordered* (*near*) *vector space*.
It was noted in [10] that if X is a Banach space, then $(\mathrm{cf}(X), \subseteq, +, \cdot)$ is an ordered near vector space.
Rådström proved the following result in [14, Theorem 1]:

Theorem 3.1. *If S is a near vector space, then there exist a vector space $R(S)$ and a map $j\colon S \longrightarrow R(S)$ such that*

(a) *j is injective,*
(b) *$j(\alpha x + \beta y) = \alpha j(x) + \beta j(y)$ for all $\alpha, \beta \in \mathbb{R}_+$ and $x, y \in S$,*
(c) *$R(S) = j(S) - j(S) := \{j(x) - j(y) : x, y \in S\}$.*

An outline of the proof of the previous theorem can be found in [10].
Let S be an ordered near vector space. Define an order $\leq$ on $R(S)$ by

$$[x, y] \leq [x_1, y_1] \Longleftrightarrow x + y_1 \leq y + x_1,$$

Then $R(S)$ is an ordered vector space and $j\colon S \longrightarrow R(S)$ is an order embedding (see also [10]); i.e.,

$$s \leq t \Leftrightarrow j(s) \leq j(t).$$

Let S be an ordered near vector space which also satisfies

(Z) there exists $0 \in S$ such that $x + 0 = x$ for all $x \in S$ and $\lambda 0 = 0$ for all $\lambda \in \mathbb{R}_+$.

Then S is said to be an *ordered near vector space with a zero*.
If X is a separable Banach space, then

(a) $(\mathbf{M}[\Sigma, \mathrm{F_c}(X)], +, \cdot, \leq)$,
(b) $(\mathcal{L}[\Sigma, \mathrm{F_c}(X)], +, \cdot, \leq)$, and
(c) $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)], +, \cdot, \leq)$

are ordered near vector space with with $\chi_{\{0\}}$ as zero where χ_A is the characteristic function of A. In fact, $(\mathcal{L}[\Sigma, \mathrm{F_c}(X)], +, \cdot, \leq)$ is a sub ordered near vector space of $(\mathbf{M}[\Sigma, \mathrm{F_c}(X)], +, \cdot, \leq)$ and $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)], +, \cdot, \leq)$ is a sub ordered near vector space of $(\mathcal{L}[\Sigma, \mathrm{F_c}(X)], +, \cdot, \leq)$.
It is clear that if S is an ordered near vector space S with a zero, then there exists a subtraction operation on $R(S)$, but this does not guarantee the existence of a subtraction operation on S under which S is closed.
To overcome this problem, we consider the following:

Definition 3.1. Let S be an ordered near vector space with a zero and define $\sqsubseteq$ by

$$y \sqsubseteq x \Longleftrightarrow \exists\, z \in S\, [0 \leq z \text{ and } y + z = x].$$

Then, by the cancellation law, z is unique in Definition 3.1 and we define

$$z := x - y.$$

Also,

$$x \sqsubseteq y \Rightarrow x \leq y \text{ for all } x, y \in S$$

and it follows that $\sqsubseteq$ is a partial ordering on S. We call $\sqsubseteq$ the *ordering associated with* $\leq$.
Also note that, for all $x \in S$,

$$0 \sqsubseteq x \Longleftrightarrow 0 \leq x;$$

i.e.,

$$S_+ := \{x \in S : 0 \leq x\} = \{x \in S : 0 \sqsubseteq x\}.$$

It is readily verified that $(S, \sqsubseteq)$ is an ordered near vector space with 0 as zero. Furthermore, if we consider the Rådström completion $R(S)$ of $(S, +, \cdot, \sqsubseteq)$, then

$$\begin{aligned} y \sqsubseteq x \quad &\Longleftrightarrow \quad \exists\, z \in S \ (0 \leq z \text{ and } [z, 0] = [x, y]) \\ &\Longleftrightarrow \quad \exists\, x - y \in S \ (0 \leq x - y \text{ and } [x - y, 0] = [x, y]) \,. \end{aligned}$$

Our strategy is now as follows. We first consider Doob's decomposition of a submartingale in an ordered vector space. Then we use this ordered vector space result to obtain a Doob decomposition of a submartingale in an ordered near vector space. We specialize the ordered near vector space to the appropriate fuzzy set-valued space of submartingales that are integrable and obtain the Daures, Ni and Zhang result as a special case from the latter for integrably bounded martingales.

We now define martingales in terms of projections rather than sub σ-algebras. By considering martingales in this way, we can apply the theory of martingales to near vector spaces.

Definition 3.2. Let S be any nonempty set, (T_i) a commuting sequence (i.e., $T_iT_j = T_jT_i = T_i$ for all $i \leq j$) of projections on S and $(f_i) \subseteq S$. Then

(a) (f_i, T_i) is a *martingale* in S, provided that $f_i = T_i f_j$ for all $i \leq j$.

If, in addition, $(S, \leq)$ is a partially ordered set and each T_i is order preserving; i.e., $u \leq v \Rightarrow T_i u \leq T_i v$ for all $u, v \in S$, then

(b) (f_i, T_i) is called a *submartingale* in S, provided that $f_i \in \mathcal{R}(T_i)$ for all i (where $\mathcal{R}(T_i)$ is the range of T_i) and $f_i \leq (\geq)\, T_i f_j$ for all $i \leq j$.

As was noted in [9], it follows from Theorem 2.3 that if (Σ_i) is a filtration and if we set

$$T_i(F) = \mathcal{E}[F|\Sigma_i] \text{ for all } F \in \mathcal{L}[\Sigma, \mathrm{F_c}(X)] \Big(F \in \mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]\Big) \text{ and } i \in \mathbb{N},$$

then (T_i) is a commuting sequence of order preserving projections on the ordered near vector space $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)])$ and the range $\mathcal{R}(T_i)$ of T_i is $\mathcal{L}[\Sigma_i, \mathrm{F_c}(X)]$ $(\mathcal{L}^1[\Sigma_i, \mathrm{F_c}(X)])$ for each i. Furthermore, if $(F_i) \subseteq \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)])$ and (Σ_i) is an increasing sequence of sub σ-fields of Σ, then (F_i, T_i) is a *martingale* (respectively, *submartingale*) in the ordered near vector space $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)])$ in the sense of Definition 3.2.
The following result, which is based on a vector lattice version in [7], is the first step in achieving our aim of proving the Daures, Ni and Zang result in an elementary way:

Theorem 3.2. *Let E be an ordered vector space, $(f_i) \subseteq E$ and (T_i) a commuting sequence of positive linear projections on E. If (f_i, T_i) is a submartingale,*

(i) $A_1 = 0$,
(ii) $A_j = \sum_{i=1}^{j-1} \left(T_i f_{i+1} - f_i\right)$ *for all* $j \geq 2$ *and*
(iii) $M_j = f_j - A_j$ *for all* $j \in \mathbb{N}$,

then the decomposition $f_i = M_i + A_i, i \in \mathbb{N}$, *is the unique decomposition of* (f_i, T_i) *with* (M_j, T_j) *a martingale in* E, $(A_j) \subseteq E$ *a positive and increasing sequence and* $A_{j+1} \in \mathcal{R}(T_j)$ *for all* $j \in \mathbb{N}$.

Let S be an ordered near vector space. A map $T\colon S \longrightarrow S$ is called $\mathbb{R}_+$-*linear* provided that $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$ for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$. It was shown in [10] that if S is an ordered near vector space and $T\colon S \longrightarrow S$ is an order preserving $\mathbb{R}_+$-linear map, then $\hat{T}$, defined by $\hat{T}[x, y] = [Tx, Ty]$ for all $[x, y] \in R(S)$, is an order preserving linear map from $R(S)$ to $R(S)$.
Let (f_i, T_i) be a submartingale in an ordered near vector space S, where (T_i) is a commuting sequence of order preserving $\mathbb{R}_+$-linear idempotents on S. Then $([f_i, 0], \hat{T}_i)$ is a submartingale in $R(S)$ and $(\hat{T}_i)$ is a commuting sequence of order preserving linear projections on $R(S)$.
We need the following notion:

Definition 3.3. Let S be an ordered near vector space with a zero, $(f_i) \subseteq S$, (T_i) a commuting sequence of order preserving $\mathbb{R}_+$-linear idempotents on S and (f_i, T_i) We call (f_i, T_i) a $\sqsubseteq$-*submartingale* in S if $f_i \in \mathcal{R}(T_i)$ for all i and $f_j \sqsubseteq T_j(f_i)$ for all $j \leq i$.

Theorem 3.3. *Let* S *be an ordered near vector space with a zero,* $(f_i) \subseteq S$ *and* (T_i) *a commuting sequence of increasing* $\mathbb{R}_+$*-linear projections on* S*. If* (f_i, T_i) *is a* $\sqsubseteq$*-submartingale,*

(a) $A_1 = 0$,
(b) $A_j = \sum_{i=1}^{j-1} [T_i f_{i+1} - f_i, 0]$ *for all* $j \geq 2$,
(c) $M_1 = [f_1, 0]$ *and*
(d) $M_j = [f_1, 0] + \sum_{i=1}^{j-1} [f_{i+1}, T_i f_{i+1}]$ *for all* $j \in \mathbb{N}$,

then the decomposition $[f_i, 0] = M_i + A_i$ *for all* $i \in \mathbb{N}$, *is the unique decomposition of* $([f_i, 0], \hat{T}_i)$ *with* (M_j, T_j) *a martingale in* $R(S)$, $(A_j) \subseteq R(S)$ *a positive and increasing sequence and* $A_{j+1} \in \mathcal{R}(T_j)$ *for all* $j \in \mathbb{N}$.

4. THE DAURES-NI-ZHANG VERSION OF DOOB'S DECOMPOSITION IN THE FUZZY SETTING

Let X be a Banach space. We first specialize our above discussion on the associated ordering to the ordered near vector space $(\mathrm{F_c}(X), +, \cdot, \leq)$.
The ordering $\sqsubseteq$ on $\mathrm{F_c}(X) \times \mathrm{F_c}(X)$ associated with $\leq$ is given by

$$A \sqsubseteq B \iff \exists\, C \in \mathrm{F_c}(X)\ (\chi_{\{0\}} \leq C \text{ and } A + C = B).$$

- $0 \in A \ominus B \Leftrightarrow B \subseteq A$ for all $A, B \in \mathcal{P}_0(X)$,
- if supp(A) is bounded, then $A \ominus A = \chi_{\{0\}}$,
- if $A \in \mathrm{F}(X)$, then $A \ominus B \in \mathrm{F}(X)$,
- if A is convex, so is $A \ominus B$ provided that $A \ominus B \neq \chi_\emptyset$,
- if supp(A) and supp(B) are bounded, then supp($A \ominus B$) is also bounded,

The following theorem and two subsequent corollaries follow from the results in [11].

Theorem 4.1. *Let* X *be a Banach space. If* $A, B \in \mathrm{F_c}(X)$*, then there exists* $C \in \mathrm{F_c}(X)$ *such that* $B + C = A$ *if and only if* $B + (A \ominus B) = A$*. Moreover, in this case,* $A \ominus B$ *is the unique* C *satisfying* $A = C + B$.

Corollary 4.1. *Let X be a Banach space and $A, B \in \mathrm{F_c}(X)$. Then the following statements are equivalent:*

(i) *There exists $C \in \mathrm{F_c}(X)$ such that $B + C = A$.*
(ii) $B + (A \ominus B) = A$.
(iii) $[s(\,\cdot\,, A) - s(\,\cdot\,, B)]_\alpha$ *is a convex function on X^* for each $\alpha \in I$.*

Corollary 4.2. *Let X be a Banach space. Then, for all $A, B \in \mathrm{F_c}(X)$, the following statements are equivalent:*

(i) $B \sqsubseteq A$.
(ii) $\chi_{\{0\}} \leq A \ominus B$ *and* $B + A \ominus B = A$.
(iii) $B \leq A$ *and* $s(\,\cdot\,, A) - s(\,\cdot\,, B)$ *is a convex function on X^*.*

Proof. From [12, p.159] we have that for each $\alpha \in (0, 1]$,

$$\{0\} = [\chi_{\{0\}}]_\alpha \subseteq A_\alpha \ominus B_\alpha \Leftrightarrow B_\alpha \subseteq A_\alpha$$
$$\Leftrightarrow B \leq A \text{ and } \chi_{\{0\}} \subseteq A \ominus B$$

by Lemma 2.2. By applying Theorem 4.1 we complete the proof. □

We use our main result Theorem 3.3 to obtain:

Theorem 4.2. *Let (F_i, Σ_i) be a fuzzy submartingale in $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$. If there exists $B \in \Sigma$ with $\mu(B) = 0$ such that for any $\omega \notin B$ and all $i \in \mathbb{N}$*

(i) $[s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) - s(\,\cdot\,, F_{i-1}(\omega))]_\alpha$ *and*
(ii) $[s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)])]_\alpha$

are convex functions on X^* for all $\alpha \in I$, then (F_i, Σ_i) has a decomposition

$$F_i(\omega) = M_i(\omega) + A_i(\omega) \text{ for all } \omega \notin B,$$

where (M_i, Σ_i) is a fuzzy martingale and (A_i) is a predictable increasing sequence $\mathbf{M}[\Sigma, \mathrm{F_c}(X)]$ in such that for all $\omega \notin B$

(a) $A_1(\omega) = \chi_{\{0\}}$,
(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\right)}$ *for all* $j \geq 2$,
(c) $M_1(\omega) = F_1(\omega)$, *and*
(d) $M_j(\omega) = \overline{\left(\sum_{i=2}^{j} [F_i(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega)\right) + F_1(\omega)}$ *for all* $j \geq 2$.

Proof. The proof of this theorem is very similar to the proof of Theorem 4.4 from [11]. We want to apply Theorem 3.3 to the ordered near vector space $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$. It was noted earlier that $(\mathcal{E}[\,\cdot\,|\Sigma_i])$ is a commuting sequence of increasing $\mathbb{R}_+$-linear projections on $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ such that $\mathcal{R}(\mathcal{E}[\,\cdot\,|\Sigma_i]) = \mathcal{L}[\Sigma_i, \mathrm{cf}(E)]$ for all $i \in \mathbb{N}$. We first verify that (F_i, Σ_i) is a fuzzy $\sqsubseteq$-submartingale.
As (F_i, Σ_i) is a fuzzy submartingale, it follows from $F_i(\omega) \subseteq \mathcal{E}[F_{i+1}|\Sigma_i](\omega)$ a.e. for all $\omega \in \Omega$ and $i \in \mathbb{N}$ that

$$\chi_{\{0\}} \leq \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega) \text{ a.e. for all } \omega \in \Omega \text{ and } i \in \mathbb{N}.$$

Also, $s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) - s(\,\cdot\,, F_{i-1}(\omega))$ for all $\omega \notin B$ and all $i \in \mathbb{N}$ means that

$$F_i(\omega) + \left(\mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega)\right) = \mathcal{E}[F_{i+1}|\Sigma_i](\omega)$$

for all $\omega \notin B$ and all $n \in \mathbb{N}$; consequently,

$$F_i(\omega) \sqsubseteq \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \text{ a.e. for all } \omega \notin B \text{ and } i \in \mathbb{N}.$$

Hence, (F_i, Σ_i) be a set-valued $\sqsubseteq$-submartingale.

Let $\mathcal{A}_1(\omega) = 0$ for all $\omega \notin B$ and, for all $j \geq 2$,

$$\mathcal{A}_j = \sum_{i=1}^{j-1} \Big[\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i, \chi_{\{0\}}\Big] = \Big[\overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big)}, \chi_{\{0\}}\Big],$$

$\mathcal{M}_1 = [F_1, 0]$ and, for all $j \geq 2$,

$$\mathcal{M}_j = [F_1, 0] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i]\Big].$$

Then it follows from Theorem 3.3 that in the Rådström's completion $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$ of $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$, we have that the submartingale $([F_i, \chi_{\{0\}}], \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ has a unique decomposition

$$[F_i(\omega), \chi_{\{0\}}(\omega)] = \mathcal{M}_i(\omega) + \mathcal{A}_i(\omega) \text{ for all } \omega \notin B \text{ and } i \in \mathbb{N},$$

where with $(\mathcal{M}_j, \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ a martingale in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$, $(\mathcal{A}_j) \subseteq \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ a positive and increasing sequence and $\mathcal{A}_{j+1} \in \mathcal{L}[\Sigma_j, \mathrm{F_c}(E)]$ for all $j \in \mathbb{N}$.
From the assumption $s(\,\cdot\,, F_n(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)])$ for all $\omega \notin B$ and all $i \geq 2$, we get that $F_i = \mathcal{E}[F_i|\Sigma_{i-1}(\omega)] + \Big(F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]\Big)$. Hence, in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$, it follows that

$$\Big[F_i, \mathcal{E}[F_i|\Sigma_{i-1}]\Big] = \Big[F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}], \chi_{\{0\}}\Big].$$

But then, for all $j \geq 2$,

$$\begin{aligned}
\mathcal{M}_j &= [F_1, \chi_{\{0\}}] + \sum_{i=1}^{j-1} \Big[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i]\Big] \\
&= [F_1, \chi_{\{0\}}] + \sum_{i=1}^{j-1} \Big[F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i], \{0\}\Big] \\
&= \Big[\overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i]\Big) + F_1}, \ \chi_{\{0\}}\Big]
\end{aligned}$$

Let

$$A_1 = 0 \text{ and } A_j = \overline{\sum_{i=1}^{j-1} \Big(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i\Big)} \text{ for all } j \geq 2,$$

$$M_1 = 0 \text{ and } M_j = \overline{\sum_{i=1}^{j-1} \Big(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i]\Big) + F_1} \text{ for all } j \geq 2.$$

Then (F_i, Σ_i) has a decomposition

$$F_i = M_i + A_i \text{ for all } i \in \mathbb{N},$$

with the required properties. □

We are now in a position to prove Theorem 2.4, the fuzzy version of the Doob decomposition as noted by Daures, Ni and Zhang, using Theorem 4.2.
Proof of Theorem 2.4. As $\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]$ is a sub ordered near vector space of $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ (see [9]), we may in Corollary 4.2 replace $\mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ by $\mathcal{L}^1[\Sigma, \mathrm{F_c}(X)]$, which completes the proof of Theorem 2.4. □

5. THE SHEN-WANG VERSION OF DOOB'S DECOMPOSITION IN THE FUZZY SETTING

If $\{0\} \neq E$ is a Banach lattice, then the canonical embedding $E \hookrightarrow \mathrm{F_c}(E)$, given by $x \mapsto \chi_{\{x\}}$, is not order preserving if $\mathrm{cf}(E)$ is endowed with the usual fuzzy ordering. We want to relate the ordering on E to an appropriate ordering on $\mathrm{F_c}(E)$. We, therefore, consider

$$\mathrm{F_c}(E_+): \quad = \quad \{A \in \mathrm{F_c}(E_+) : A \text{ is convex}\}.$$

For all $F, G \in \mathrm{F_c}(E)$, define

$$F \preceq G \Leftrightarrow \exists H \in \mathrm{F_c}(E_+)\,(F + H = G).$$

Direct verification yields that

- if $F \in \mathrm{F_c}(E)$, then $\chi_{\{0\}} \preceq F$ if and only if $0 \leq f$ for all $f \in F$,
- $(\mathrm{F_c}(E), \preceq)$ is a partially ordered set and $(\mathrm{F_c}(E), +, \cdot, \preceq)$ is an ordered near vector space.

It is also clear that the ordering $\sqsubseteq$ associated with $\preceq$ on $\mathrm{F_c}(E)$ coincides with $\preceq$. We extend the ordering $\preceq$ pointwise to the spaces $\mathcal{L}[\Sigma, \mathrm{F_c}(E)]$ and $\mathcal{L}^1[\Sigma, \mathrm{F_c}(E)]$. Then $(\mathcal{L}[\Sigma, \mathrm{F_c}(E)], +, \cdot, \preceq)$ and $(\mathcal{L}^1[\Sigma, \mathrm{F_c}(E)], +, \cdot, \preceq)$ are ordered near vector spaces.
The next result shows that conditional expectations are $\preceq$-preserving:

Lemma 5.1. *Let E be a Banach lattice, (Ω, Σ, μ) a finite measure space space and Σ_0 a sub σ-algebra of Σ. Then the following holds:*

(E3$'$) *If $F_1, F_2 \in \mathcal{L}[\Sigma, \mathrm{F_c}(E)]$, then $F_1 \preceq F_2$ implies $\mathcal{E}[F_1|\Sigma_0] \preceq \mathcal{E}[F_2|\Sigma_0]$.*

Proof. Once again the proof of this theorem is very similar to the proof of Theorem 5.2 from [11]. Let $F_1 \preceq F_2$. Select $H \in \mathcal{L}[\Sigma, \mathrm{F_c}(E)]$ for which $H(\omega) \in \mathrm{F_c}(E_+)$ a.e. and $F_1 + H = F_2$. Then $\mathcal{E}[F_1|\Sigma_0] + \mathcal{E}[H|\Sigma_0] = \mathcal{E}[F_2|\Sigma_0]$. To conclude that $\mathcal{E}[F_1|\Sigma_0] \preceq \mathcal{E}[F_2|\Sigma_0]$, it suffices to show that $\chi_{\{0\}} \preceq \mathcal{E}[H|\Sigma_0]$.
If $h \in L^1(\Omega, \Sigma, \mu)$ such that $h(\omega) \in H(\omega)$ a.e., then, as $H(\omega) \in \mathrm{F_c}(E_+)$ a.e., it follows that $h(\omega) \geq 0$ a.e.; consequently, $\mathbb{E}[h|\Sigma_0](\omega) \geq 0$ a.e. and

$$S^1_H(\Sigma_0) = \{h \in L^1(\Omega, \Sigma_0, \mu) : 0 \leq h(\omega) \in H(\omega) \text{ a.e.}\}.$$

But then $\chi_{\{0\}} \preceq \overline{\{\mathbb{E}[h|\Sigma_0] : h \in S^1_H(\Sigma_0)\}}$. From the definition of $\mathcal{E}[H|\Sigma_0]$, it follows that $\chi_{\{0\}} \preceq \mathcal{E}[H|\Sigma_0]$, and the proof is complete. □

The following version of Doob's decomposition is similar to a result noted by Shen and Wang (see [16]). Their result differs from the one below mainly in the assumption (1) in Theorem 5.1. This assumption yields an explicit description of the martingale involved in the decomposition, which they do not obtain in their result.

Theorem 5.1. *Let E be a Banach lattice, (F_i, Σ_i) be a $\preceq$-submartingale in $\mathcal{L}[\Sigma, \mathrm{F_c}(E)]$ (alternatively, $\mathcal{L}^1[\Sigma, \mathrm{F_c}(E)]$). If there exists $B \in \Sigma$ with $\mu(B) = 0$ and, for each $i \geq 2$,*

$$s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)]) \text{ for all } \omega \notin B$$

is a convex functions on X^*, then there is a decomposition of (F_i, Σ_i) as

$$F_i(\omega) = M_i(\omega) + A_i(\omega) \text{ for all } \omega \notin B$$

where (M_i, Σ_i) is a set-valued martingale in $\mathcal{L}[\Sigma, \mathrm{F_c}(E)]$ (alternatively, $\mathcal{L}^1[\Sigma, \mathrm{F_c}(E)]$) and (A_i) is a set-valued predictable $\preceq$-increasing sequence such that for all $\omega \notin B$

(a) $A_1(\omega) = 0$,
(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_k](\omega) \ominus F_k(\omega) \right)}$ *for all* $j \geq 2$,
(c) $M_1(\omega) = F_1(\omega)$, *and*
(d) $M_j(\omega) = \overline{\left(\sum_{i=2}^{j} [F_k(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega) \right) + F_1(\omega)}$ *for all* $j \geq 2$.

Moreover, the decomposition is unique.

Proof. The proof is very similar to that of Theorem 4.2, although we are considering the ordering $\preceq$ instead of $\subseteq$. The details follow:
From Lemma 5.1, we know that $\mathcal{E}[\,\cdot\,|\Sigma_i]$ is $\preceq$-preserving. Hence, in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$ we have that $([F_i, \chi_{\{0\}}], \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ has a unique decomposition

$$[F_i, \chi_{\{0\}}] = \mathcal{M}_i + \mathcal{A}_i \text{ for all } i \in \mathbb{N},$$

where $\mathcal{A}_1 = 0$ and, for all $j \geq 2$,

$$\mathcal{A}_j = \sum_{i=1}^{j-1} \left[\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i, \chi_{\{0\}} \right] = \left[\overline{\sum_{i=1}^{j-1} \left(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i \right)}, \chi_{\{0\}} \right],$$

$\mathcal{M}_1 = [F_1, 0]$ and, for all $j \geq 2$,

$$\mathcal{M}_j = [F_1, 0] + \sum_{i=1}^{j-1} \left[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i] \right]$$

with $(\mathcal{M}_j, \widehat{\mathcal{E}[\cdot\,|\Sigma_i]})$ a martingale in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$, $(\mathcal{A}_j) \subseteq \mathcal{L}[\Sigma, \mathrm{F_c}(X)]$ a positive and increasing sequence and $\mathcal{A}_{j+1} \in \mathcal{L}[\Sigma_j, \mathrm{F_c}(E)]$ for all $j \in \mathbb{N}$.
From the assumption $s(\,\cdot\,, F_n(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}(\omega)])$ for all $\omega \notin B$ and all $i \geq 2$, we get that $F_i = \mathcal{E}[F_i|\Sigma_{i-1}(\omega)] + \left(F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}(\omega)] \right)$. Hence, in $R(\mathcal{L}[\Sigma, \mathrm{F_c}(X)])$, it follows that

$$\left[F_i, \mathcal{E}[F_i|\Sigma_{i-1}] \right] = \left[F_i \ominus \mathcal{E}[F_i|\Sigma_{i-1}], \chi_{\{0\}} \right].$$

But then, for all $j \geq 2$,

$$\begin{aligned} \mathcal{M}_j &= [F_1, \chi_{\{0\}}] + \sum_{i=1}^{j-1} \left[F_{i+1}, \mathcal{E}[F_{i+1}|\Sigma_i] \right] \\ &= [F_1, \chi_{\{0\}}] + \sum_{i=1}^{j-1} \left[F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i], \chi_{\{0\}} \right] \\ &= \left[\overline{\sum_{i=1}^{j-1} \left(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i] \right) + F_1}, \chi_{\{0\}} \right] \end{aligned}$$

Let

$$A_1 = 0 \text{ and } A_j = \overline{\sum_{i=1}^{j-1} \left(\mathcal{E}[F_{i+1}|\Sigma_i] \ominus F_i \right)} \text{ for all } j \geq 2,$$

$$M_1 = 0 \text{ and } M_j = \overline{\sum_{i=1}^{j-1} \left(F_{i+1} \ominus \mathcal{E}[F_{i+1}|\Sigma_i] \right) + F_1} \text{ for all } j \geq 2.$$

Then (F_i, Σ_i) has a decomposition

$$F_i = M_i + A_i \text{ for all } i \in \mathbb{N},$$

with the desired properties. ☐

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EXISTENCE AND UNIQUENESS OF NASH EQUILIBRIUM IN A SUPPLY CHAIN NEWSVENDOR GAME

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ABSTRACT. This paper characterizes properties of Nash equilibriums for a supply chain newsvendor game with resource constraints. The newsvendor game has been modelled as a Stackelberg game in $\mathbb{R}^N$ and also been reformulated as a fixed-point problem for proving the existence and uniqueness.

KEYWORDS : Fixed-point; Contraction mapping; Lipschitz condition; Lagrangian; Kuhn-Tucker condition; Lagrange-Burmann expansion.

1. INTRODUCTION

The classic newsvendor (or newsboy or single-period) model is a mathematical model in operations management and applied economics. It used to determine optimal inventory levels that reap the most profits under selling price p and uncertain demand D with cumulative distribution function F_D. Please refer to the review to this topic over the past 40 years [8]. Determining the fulfillment amount is elaborate, as order too much will lead to waste and order too little will results to business lost. Since the inventory at hand is fixed, every demand above this quantity will be lost, that is, the expected sales $S(q) = E\min(q, D)$. A standard newsvendor problem solves the profit maximization problem

$$\max_q \pi = \max_q E[p\min(q, D) - cq]$$

where $c < p$ denotes the procurement cost for the good and E is the expectation operator with respect to the random variable D. Rewrite the function $\min(a, b) =$

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$b - (b-a)^+$ for $(a)^+ = \max(a, 0)$. The celebrated form of solution for the optimal quantity to the classic newsvendor problem read as

$$q^* = \arg\max_q \pi = F_D^{-1}(\frac{p-c}{p}) \tag{1.1}$$

where F_D^{-1} denotes the inverse function of the well-defined cumulative distribution function F of D. The ratio $\frac{p-c}{p}$ sometimes is referred to a critical ratio. This paper considers a newsvendor selling N products in a supply chain while procuring them from a supplier, the problem is often modelled as a non-cooperative game. The expected profit functions for the retailer can be written as

$$\pi_r(q, w) = \sum_{i=1}^{N} p_i E \min(q_i, D_i) - w_i q_i, \tag{1.2}$$

while the one for the supplier is

$$\pi_s(q, w) = \sum_{i=1}^{N} (w_i - c_i) q_i. \tag{1.3}$$

The retailer determine an optimal order quantity $q = (q_1, \dots, q_N)$ within a strategic space $Q \subset \mathbb{R}^N$ while the supplier determine an optimal wholesale price $w = (w_1, \dots, w_N)$ within a strategic space $W \subset \mathbb{R}^N$ for maximizing their own profits. c represents the unit production cost. Through the rest of the paper, we use bold face letter for vector representation, e.g., $c = (c_1, \dots, c_N)$. The non-cooperative game between two vertical players in a supply chain is

$$\begin{bmatrix} \max_{q \in Q} \pi_r(q, w) \\ \max_{w \in W} \pi_s(q, w) \end{bmatrix} \tag{1.4}$$

In certain situations, the order quantities and wholesale prices need to be confined to particular conditions. For example the constraint conditions can be generalized to, for example, a minimum service level $q > R_i$ [3], or resource limitation $\sum_{i \in \mathbb{C}_j} q_i < C_j$ [1, 6] for multiple product newsvendor problems. In this paper, we consider the resource constraint,

$$\sum_{i=1}^{N} q_i \le \bar{C}. \tag{1.5}$$

2. PRELIMINARIES

For ease of derivation, the expected sales $S(q_i) = E[\min(q_i, D_i)]$ rewrite to $E[D_i - (D_i - q_i)^+]$. By using integral by part, we can further write

$$\begin{aligned} S(q_i) &= \int_0^\infty x f_i(x) dx - \int_{q_i}^\infty (x - q_i) f_i(x) dx \\ &= q_i F_i(q_i) - \int_0^{q_i} F_i(x) dx + q_i(1 - F_i(q_i)) \\ &= q_i - \int_0^{q_i} F_i(x) dx \end{aligned}$$

The problem (1.4) with constraint (1.5) can be solved by the Lagrangian,

$$L_r(q, w, \lambda) = \pi_r(q, w) + \lambda \left(\sum_{i=1}^{N} q_i - \bar{q} \right). \tag{2.1}$$

The best response function for the retailer has been solved by the Karush-Kuhn-Tucker condition, that is,

$$q_i(w_i, \lambda) = F_i^{-1}\left(\frac{p_i - w_i - \lambda}{p_i}\right) \tag{2.2}$$

$$\sum_{i=1}^{N} q_i - \bar{C} \leq 0 \tag{2.3}$$

$$\lambda\left(\sum_{i=1}^{N} q_i - \bar{C}\right) = 0 \tag{2.4}$$

Since the supplier possesses the freedom to pricing its goods, the supply chain game is deemed a Stackelberg game with supplier as the leader and the retailer as the follower. Substituting the best follower response function q, the supplier profit (1.3) becomes

$$\pi_s(q, w, \lambda) = \sum_{i=1}^{N}(w_i - c_i)F_i^{-1}\left(\frac{p_i - w_i - \lambda}{p_i}\right), \tag{2.5}$$

and the corresponding first order condition

$$\frac{\partial \pi_s}{\partial w_i} = F_i^{-1}\left(\frac{p_i - w_i - \lambda}{p_i}\right) - (w_i - c_i)\nabla F_i^{-1}\left(\frac{p_i - w_i - \lambda}{p_i}\right) = 0 \tag{2.6}$$

Define $w_i' = \frac{p_i - w_i - \lambda}{p_i}$ and $g_i(w_i') = (p_i - c_i - \lambda - p_i w_i')\nabla F_i^{-1}(w_i')$. Equ. (2.6) becomes a fixed point problem

$$w_i' = F(g_i(w_i')) = F_i \circ g_i(w_i'), \tag{2.7}$$

where the operator $\circ$ represents the function composition.

3. MAIN RESULTS

Before proving the nature of equilibriums, we need some definitions. A function $f : X \times Y \to \mathbb{R}$ has *increasing difference* in $(x, y) \in X \times Y \subset \mathbb{R} \times \mathbb{R}$ if for all $x' \geq x$ and $y' \geq y$, f satisfies

$$f(x', y') - f(x, y') \geq f(x', y) - f(x, y).$$

The definition of increasing difference implies that the incremental gain or payoff to choosing a higher x is greater when y is larger. A game is a *supermodular game* if the strategy spaces are compact and the best response function $u(s_i, s_{-i})$ is continuous and has increasing difference in (s_i, s_{-i}).

Lemma 3.1. [11] *Given a twice differentiable function f, it has increasing difference in (x, y) if and only if* $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$.

Lemma 3.2. [11] *A supermodular game admits at least one pure strategy Nash equilibrium.*

The existence of Nash equilibriums have been proved in many reports by either the convexity or supermodularity of payoff functions[7, 4, 9, 10].

It is also essential to apply fixed point theorems for proving the existence of Nash equilibriums, for example, Brouwer, Kakutani and Tarski's fixed point theorems [2].

Theorem 3.1. *The game (1.4) admits at least one pure strategy Nash equilibrium, if F is non-decreasing.*

Proof. Consider the game (1.4) with the fixed point problem (2.7). The composition mapping $F \circ g$ is continuous and it maps a convex and compact set W into a closed convex subset of W. By the Kakutani fixed point theorem, there exists a point $w^0 \in W$ such that $w^0 \in F \circ g(w)$. □

In order to prove uniqueness, we the following definitions. A mapping $f : W \to W$ is a contraction if there exists a Lipschitz constant $0 < k < 1$, such that

$$||f(x) - f(y)|| \leq k||x - y||, \forall x, y \in W.$$

This Lipschitz condition amounts to the condition $||f'(w)|| < 1$ for all $w \in W$. It is well-known that a contraction has a unique fixed point.

Define the coefficient of relative risk aversion $\epsilon(w)$ as $\frac{-\nabla^2 F(w)}{\nabla F(w)}$ and it is a measure of risk attitude which is relative to the strength of preference [5].

Theorem 3.2. *If the accumulative distribution function F for the uncertain demand is well-defined and*

$$\nabla^2 F_i < \frac{g_i}{1 + p_i g_i}, \forall w_i \in W, \tag{3.1}$$

the game (1.4) has a unique solution.

Proof. Taking derivative to the composition (2.7),

$$\begin{aligned} \|\nabla(F_i \circ g_i)(w_i)\| &= ||(\nabla g_i \cdot (\nabla F_i) \circ g_i)(w_i)|| \\ &= ||((-p_i \nabla F_i^{-1} + g_i \nabla^2 F_i^{-1}) \cdot (\nabla F_i) \circ g_i)(w_i)|| \\ &\leq ||(-p_i \nabla F_i^{-1} + g_i \nabla^2 F_i^{-1})|| \cdot ||((\nabla F_i) \circ g_i)(w)|| \end{aligned}$$

By the Lagrange inversion theorem and Lagrange-Bürmann formula, the derivative of an inversion reads as

$$\nabla \left(F^{-1}\right)(w) = \frac{1}{\nabla F(F^{-1}(w))}.$$

Let $F^{-1}(w) = a$. Given $\nabla^2 F < 0$, therefore,

$$\begin{aligned} &||\frac{-p_i}{\nabla F_i(a)} + \frac{g_i}{\nabla^2 F_i(a)}|| \leq 1, \text{and} \\ &||\nabla F_i||||g_i(w)|| \leq 1 \end{aligned}$$

It is necessary that

$$|| - p_i g_i + \frac{g_i}{\nabla^2 F_i(a)}|| \leq 1, \ \forall a \in W.$$

If the condition (3.1) holds, $\|\nabla(F_i \circ g_i)(w_i)\| < 1$. Therefore the game has a unique solution. □

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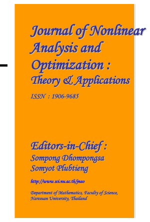

A TOPOLOGY IN A VECTOR LATTICE AND FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS

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ABSTRACT. In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

KEYWORDS : Fixed-point; Contraction mapping; Lipschitz condition; Lagrangian; Kuhn-Tucker condition; Lagrange-B¨urmann expansion.

1. INTRODUCTION

There are many fixed point theorems in a topological vector space, for instance, Kirk's fixed point theorem in a Banach space, and so on; see for example [8].

In this paper we consider fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum $\vee$ and the infimum $\wedge$, and also an order is introduced from these operators; see also [6, 9] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [2] one method is introduced in the case of the vector lattice with unit.

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In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice and in the previous paper [5] we show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem. The purpose of this paper is to introduce a topology in a vector lattice and to show a fixed point theorem for a nonexpansive mapping and also common fixed point theorems for commutative family of nonexpansive mappings in a vector lattice.

2. TOPOLOGY IN A VECTOR LATTICE

First we introduce a topology in a vector lattice introduced by [2]; see also [4, 5].

Let X be a vector lattice. $e \in X$ is said to be an unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of X. In the case where X is the set of real numbers $\mathbf{R}$, $\mathcal{K}_\mathbf{R}$ is the set of positive real numbers. Let X be a vector lattice with unit and let Y be a subset of X. Y is said to be open if for any $x \in Y$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_\mathbf{R}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset Y$. Let $\mathcal{O}_X$ be the class of open subsets of X. Y is said to be closed if $Y^C \in \mathcal{O}_X$. For $e \in \mathcal{K}_X$ and for an interval $[a, b]$ we consider the following subset

$$[a, b]^e = \{x \mid \text{ there exists some } \varepsilon \in \mathcal{K}_\mathbf{R} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

By the definition of $[a, b]^e$ it is easy to see that $[a, b]^e \subset [a, b]$. Every mapping from $X \times \mathcal{K}_X$ into $(0, \infty)$ is said to be a gauge. Let Δ_X be the class of gauges in X. For $x \in X$ and $\delta \in \Delta_X$, $O(x, \delta)$ is defined by

$$O(x, \delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.$$

$O(x, \delta)$ is said to be a δ-neighborhood of x. Suppose that for any $x \in X$ and for any $\delta \in \Delta_X$ there exists $U \in \mathcal{O}_X$ such that $x \in U \subset O(x, \delta)$.

For a subset Y of X we denote by $\mathrm{cl}(Y)$ and $\mathrm{int}(Y)$, the closure and the interior of Y, respectively. Let X and Y be vector lattices with unit, $x_0 \in Z \subset X$ and f a mapping from Z into Y. f is said to be continuous in the sense of topology at x_0 if for any $V \in \mathcal{O}_Y$ with $f(x_0) \in V$ there exists $U \in \mathcal{O}_X$ with $x_0 \in U$ such that $f(U \cap Z) \subset V$.

Let X be a vector lattice with unit. X is said to be Hausdorff if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $x_1 \in O_1$, $x_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. A subset Y of X is said to be compact if for any open covering of Y there exists a finite sub-covering. A subset Y of X is said to be normal if for any closed subsets F_1 and F_2 with $F_1 \cap F_2 \cap Y = \emptyset$ there exists $O_1, O_2 \in \mathcal{O}_X$ such that $F_1 \subset O_1$, $F_2 \subset O_2$ and $O_1 \cap O_2 \cap Y = \emptyset$.

A vector lattice is said to be Archimedean if it holds that $x = 0$ whenever there exists $y \in X$ with $y \geq 0$ such that $0 \leq rx \leq y$ for any $r \in \mathcal{K}_\mathbf{R}$.

Let X be a vector lattice with unit and Y a vector lattice, $x_0 \in Z \subset X$ and f a mapping from Z into Y. f is said to be continuous at x_0 if there exists $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the conditions (U1), $(\mathrm{U2})^d$ and $(\mathrm{U3})^s$ such that for any $e \in \mathcal{K}_X$ there exists $\delta \in \mathcal{K}_\mathbf{R}$ such that for any $x \in Z$ if $|x - x_0| \leq \delta e$, then $|f(x) - f(x_0)| \leq v_e$; where

- (U1) $v_e \in Y$ with $v_e > 0$;
- $(\mathrm{U2})^d$ $v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;
- $(\mathrm{U3})^s$ For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_\mathbf{R}$ such that $v_{\theta(e)e} \leq \frac{1}{2} v_e$.

Let X be an Archimedean vector lattice. Then there exists a positive homomorphism f from X into $\mathbf{R}$, that is, f satisfies the following conditions:

- (H1) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for any $x, y \in X$ and for any $\alpha, \beta \in \mathbf{R}$;
- (H2) $f(x) \geq 0$ for any $x \in X$ with $x \geq 0$;

see [5]*Example 3.1. Suppose that there exists a homomorphism f from X into $\mathbf{R}$ satisfying the following condition instead of (H2):

(H2)^s $f(x) > 0$ for any $x \in X$ with $x > 0$.

Example 2.1. We consider of a sufficient condition to satisfy (H2)^s. Let X be a Hilbert lattice with unit, that is, X has an inner product $\langle \cdot, \cdot \rangle$ and for any $x, y \in X$ if $|x| \leq |y|$, then $\langle x, x \rangle \leq \langle y, y \rangle$. For any $e \in \mathcal{K}_X$ let f be a function from X into $\mathbf{R}$ defined by $f(x) = \langle x, e \rangle$. Then f satisfies (H1) and (H2)^s clearly.

3. FIXED POINT THEOREM FOR A NONEXPANSIVE MAPPING

Let X be a vector lattice and Y a subset of X. A mapping f from Y into Y is said to be nonexpansive if $|f(x) - f(y)| \leq |x - y|$ for any $x, y \in Y$. In this section we consider a fixed point theorem for a nonexpansive mapping.

Lemma 3.1. *Let X be a Hausdorff Archimedean vector lattice with unit and K a non-empty compact convex subset of X. Then*

$$c(K) = \left\{ x \,\middle|\, x \in K, \bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| \right\}$$

is non-empty compact convex.

Proof. For any $x \in K$ and for any $e \in \mathcal{K}_X$ let

$$F(x, e) = \left\{ y \,\middle|\, y \in K, |x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + e \right\}.$$

Then $F(x, e)$ is non-empty compact convex. Let $C(e) = \bigcap_{x \in K} F(x, e)$. Since $\bigcap_{i=1}^{n} F(x_i, e) \neq \emptyset$ for any $x_1, \ldots, x_n \in K$, $C(e)$ is non-empty compact convex. Since $C(e_1) \supset C(e_2)$ for any $e_1, e_2 \in \mathcal{K}_X$ with $e_1 \geq e_2$, $\bigcap_{e \in \mathcal{K}_X} C(e)$ is non-empty compact convex. Moreover $c(K) = \bigcap_{e \in \mathcal{K}_X} C(e)$. Indeed $c(K) \subset \bigcap_{e \in \mathcal{K}_X} C(e)$ is clear. Let $x \in C(e)$ for any $e \in \mathcal{K}_X$. Then

$$|x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + e$$

for any $y \in K$. Therefore

$$\bigvee_{y \in K} |x - y| \leq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y| + \bigwedge_{e \in \mathcal{K}_X} e = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.$$

By definition

$$\bigvee_{y \in K} |x - y| \geq \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|.$$

Therefore

$$\bigvee_{y \in K} |x - y| = \bigwedge_{x \in K} \bigvee_{y \in K} |x - y|,$$

that is, $x \in c(K)$. □

Let X be a Hausdorff Archimedean vector lattice with unit and Y a subset of X. We say that Y has the normal structure if for any compact convex subset K, which contains two points at least, of Y there exists $x \in K$ such that

$$\bigvee_{y \in K} |x - y| < \bigvee_{x, y \in K} |x - y|.$$

Lemma 3.2. *Let X be a Hausdorff Archimedean vector lattice with unit and K a non-empty compact convex subset, which contains two points at least, of X. Suppose that K has the normal structure. Then*

$$\bigvee_{x,y\in c(K)} |x-y| < \bigvee_{x,y\in K} |x-y|.$$

Proof. Since K has the normal structure, there exists $z \in K$ such that

$$|x-y| \leq \bigvee_{y\in K} |x-y| = \bigwedge_{x\in K}\bigvee_{y\in K} |x-y| \leq \bigvee_{y\in K} |z-y| < \bigvee_{x,y\in K} |x-y|$$

for any $x, y \in c(K)$. Therefore

$$\bigvee_{x,y\in c(K)} |x-y| < \bigvee_{x,y\in K} |x-y|.$$

□

Theorem 3.1. *Let X be a Hausdorff Archimedean vector lattice with unit and K a non-empty compact convex subset of X. Suppose that K has the normal structure. Then every nonexpansive mapping from K into K has a fixed point.*

Proof. Let f be a nonexpansive mapping from K into K and $\{K_\lambda \mid \lambda \in \Lambda\}$ the family of non-empty compact convex subsets of K satisfying that $f(K_\lambda) \subset K_\lambda$. By Zorn's lemma there exists a minimal element K_0 of $\{K_\lambda \mid \lambda \in \Lambda\}$. Assume that K_0 contains two points at least. By Lemma 3.1 $c(K_0)$ is non-empty compact convex. Let $x \in c(K_0)$. For any $y \in K_0$, we obtain that

$$|f(x)-f(y)| \leq |x-y| \leq \bigvee_{y\in K_0} |x-y| = \bigwedge_{x\in K_0}\bigvee_{y\in K_0} |x-y|.$$

Let

$$M = \left\{ y \;\middle|\; y \in K, |f(x)-y| \leq \bigwedge_{x\in K_0}\bigvee_{y\in K_0} |x-y| \right\}.$$

Then $f(K_0) \subset M$ and hence $f(K_0 \cap M) \subset K_0 \cap M$. Since K_0 is a minimal element, it holds that $K_0 \subset M$. Therefore

$$\bigvee_{y\in K_0} |f(x)-y| \leq \bigwedge_{x\in K_0}\bigvee_{y\in K_0} |x-y|.$$

By definition, we have

$$\bigvee_{y\in K_0} |f(x)-y| \geq \bigwedge_{x\in K_0}\bigvee_{y\in K_0} |x-y|.$$

Therefore

$$\bigvee_{y\in K_0} |f(x)-y| = \bigwedge_{x\in K_0}\bigvee_{y\in K_0} |x-y|,$$

that is, $f(x) \in c(K_0)$. Since K_0 is a minimal element, it holds that $c(K_0) = K_0$ and hence

$$\bigvee_{x,y\in c(K_0)} |x-y| = \bigvee_{x,y\in K_0} |x-y|.$$

However by Lemma 3.2

$$\bigvee_{x,y\in c(K_0)} |x-y| < \bigvee_{x,y\in K_0} |x-y|.$$

It is a contradiction. Therefore K_0 only contains a unique point. The point is a fixed point. □

4. FIXED POINT THEOREM FOR THE COMMUTATIVE FAMILY OF NONEXPANSIVE MAPPINGS

For any nonexpansive mapping f from K into K let $F_K(f)$ be the set of fixed points of f.

Lemma 4.1. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a subset of X and f a nonexpansive mapping from Y into Y. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying the condition* (H2)^s. *Then $F_Y(f)$ is closed.*

Proof. Assume that $F_Y(f)$ is not closed. Then for any $\delta \in \Delta_X$ there exists $x \in F_Y(f)^C$ such that $O(x,\delta) \not\subset F_Y(f)^C$. Take $y_\delta \in O(x,\delta) \cap F_Y(f)$. Then $f(y_\delta) = y_\delta$. Note that every nonexpansive mapping is continuous and hence by [5]*Lemma 3.2 it is also continuous in the sense of topology. Since $\{y_\delta \mid \delta \in \Delta_X\}$ is convergent to x in the sense of topology, $\{f(y_\delta) \mid \delta \in \Delta_X\}$ is convergent to $f(x)$ in the sense of topology. Since X is Hausdorff, $f(x) = x$. It is a contradiction. Therefore $F_Y(f)$ is closed. □

Lemma 4.2. *Let X be a vector lattice. If $|x-z| = |x-w|$, $|y-z| = |y-w|$ and $|x-z| + |y-z| = |x-y|$, then $z = w$.*

Proof. Note that $|a+b| = |a-b|$ if and only if $|a| \wedge |b| = 0$. Since

$$|x-z| = \left|x - \frac{1}{2}(z+w) - \frac{1}{2}(z-w)\right|$$

and

$$|x-w| = \left|x - \frac{1}{2}(z+w) + \frac{1}{2}(z-w)\right|,$$

it holds that $\left|x - \frac{1}{2}(z+w)\right| \wedge \frac{1}{2}|z-w| = 0$. In the same way it holds that $\left|y - \frac{1}{2}(z+w)\right| \wedge \frac{1}{2}|z-w| = 0$. Note that $(a+b) \wedge c \le a \wedge c + b \wedge c$ for any $a, b, c \ge 0$. Therefore

$$\begin{aligned} |x-y| \wedge \frac{1}{2}|z-w| &\le \left(\left|x - \frac{1}{2}(z-w)\right| + \left|\frac{1}{2}(z-w) - y\right|\right) \wedge \frac{1}{2}|z-w| \\ &\le \left|x - \frac{1}{2}(z-w)\right| \wedge \frac{1}{2}|z-w| + \left|y - \frac{1}{2}(z+w)\right| \wedge \frac{1}{2}|z-w| \\ &= 0. \end{aligned}$$

Assume that $z \ne w$. Note that, if $|b| \wedge |c| = 0$, then $||a| - |b|| \wedge |c| = |a| \wedge |c|$. Therefore

$$\begin{aligned} (|x-z| + |y-z|) \wedge \frac{1}{2}|z-w| &\ge |x-z| \wedge \frac{1}{2}|z-w| \\ &\ge \left|\left|x - \frac{1}{2}|z-w|\right| - \frac{1}{2}|z-w|\right| \wedge \frac{1}{2}|z-w| \\ &= \frac{1}{2}|z-w| > 0. \end{aligned}$$

It is a contradiction. Therefore $z = w$. □

Lemma 4.3. *Let X be a Hausdorff Archimedean vector lattice with unit, Y a subset of X and f a nonexpansive mapping from Y into Y. Then $F_Y(f)$ is convex.*

Proof. Let $x, y \in F_Y(f)$ and $0 \leq \alpha \leq 1$. Then

$$\begin{aligned} |x - f((1-\alpha)x + \alpha y)| &= |f(x) - f((1-\alpha)x + \alpha y)| \\ &\leq |x - ((1-\alpha)x + \alpha y)| = \alpha|x-y|, \\ |y - f((1-\alpha)x + \alpha y)| &= |f(y) - f((1-\alpha)x + \alpha y)| \\ &\leq |y - ((1-\alpha)x + \alpha y)| = (1-\alpha)|x-y|. \end{aligned}$$

Since

$$\begin{aligned} |x-y| &\leq |x - f((1-\alpha)x + \alpha y)| + |y - f((1-\alpha)x + \alpha y)| \\ &\leq |x - ((1-\alpha)x + \alpha y)| + |y - ((1-\alpha)x + \alpha y)| = |x-y|, \end{aligned}$$

it holds that

$$\begin{aligned} |x - f((1-\alpha)x + \alpha y)| &= |x - ((1-\alpha)x + \alpha y)|, \\ |y - f((1-\alpha)x + \alpha y)| &= |y - ((1-\alpha)x + \alpha y)|, \end{aligned}$$

and hence

$$|x - f((1-\alpha)x + \alpha y)| + |y - f((1-\alpha)x + \alpha y)| = |x-y|.$$

By Lemma 4.2 $f((1-\alpha)x + \alpha y) = (1-\alpha)x + \alpha y$, that is, $F_Y(f)$ is convex. □

Theorem 4.1. *Let X be a Hausdorff Archimedean vector lattice with unit, K a compact convex subset of X and $\{f_i \mid i = 1, \ldots, n\}$ the finite commutative family of nonexpansive mappings from K into K. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying the condition* (H2)^s *and K has the normal structure. Then $\bigcap_{i=1}^n F_K(f_i)$ is non-empty.*

Proof. Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be the family of non-empty compact convex subsets of K satisfying that $f_i(K_\lambda) \subset K_\lambda$ for any i. By Zorn's lemma there exists a minimal element K_0 of $\{K_\lambda \mid \lambda \in \Lambda\}$. Assume that K_0 contains two points at least. By Theorem 3.1 $F_{K_0}(f_1 \circ \cdots \circ f_n)$ is non-empty. Moreover by Lemma 4.1 and Lemma 4.3 $F_{K_0}(f_1 \circ \cdots \circ f_n)$ is compact convex. It holds that $f_i(F_{K_0}(f_1 \circ \cdots \circ f_n)) = F_{K_0}(f_1 \circ \cdots \circ f_n)$ for any i. It is shown as follows. Let $x \in F_{K_0}(f_1 \circ \cdots \circ f_n)$. Since

$$f_i(x) = f_i((f_1 \circ \cdots \circ f_n)(x)) = (f_1 \circ \cdots \circ f_n)(f_i(x))$$

for any i, $f_i(x) \in F_{K_0}(f_1 \circ \cdots \circ f_n)$, that is, $f_i(F_{K_0}(f_1 \circ \cdots \circ f_n)) \subset F_{K_0}(f_1 \circ \cdots \circ f_n)$. Next let $x_i = (f_1 \circ \cdots \circ f_{i-1} \circ f_{i+1} \circ \cdots \circ f_n)(x)$. Since

$$(f_1 \circ \cdots \circ f_n)(x_i) = (f_1 \circ \cdots \circ f_{i-1} \circ f_{i+1} \circ \cdots \circ f_n)(x) = x_i,$$

it holds that $x_i \in F_{K_0}(f_1 \circ \cdots \circ f_n)$. Moreover $f_i(x_i) = x$. Therefore $F_{K_0}(f_1 \circ \cdots \circ f_n) \subset f_i(F_{K_0}(f_1 \circ \cdots \circ f_n))$. Since K has the normal structure, there exists $x_0 \in K_0$ such that

$$\bigvee_{y \in K_0} |x_0 - y| < \bigvee_{x, y \in K_0} |x - y|.$$

Let

$$A = \left\{ x \;\middle|\; x \in K_0, \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| \right\}.$$

A is non-empty and convex clearly. Moreover since X is Archimedean, A is closed and hence compact. Let $x \in A$. Then for any i and for any $y \in F_{K_0}(f_1 \circ \cdots \circ f_n)$

$$|f_i(x) - y| = |f_i(x) - f_i(y_i)| \leq |x - y_i|$$

$$\leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y|$$
$$\leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y|$$

and hence $f_i(a) \in A$, that is, $f_i(A) \subset A$. Since K_0 is minimal, $A = K_0$. Therefore

$$\bigvee_{x,y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y| \leq \bigvee_{y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x_0 - y| < \bigvee_{x,y \in F_{K_0}(f_1 \circ \cdots \circ f_n)} |x - y|.$$

It is a contradiction. Therefore K_0 only contains a unique point. The point is a common fixed point of $\{f_i \mid i = 1, \ldots, n\}$. □

Theorem 4.2. *Let X be a Hausdorff Archimedean vector lattice with unit, K a compact convex subset of X and $\{f_i \mid i \in I\}$ the commutative family of nonexpansive mappings from K into K. Suppose that there exists a homomorphism from X into $\mathbf{R}$ satisfying the condition* (H2)^s *and K has the normal structure. Then $\bigcap_{i \in I} F_K(f_i)$ is non-empty.*

Proof. By Theorem 4.1 $\bigcap_{k=1}^{n} F_K(f_{i_k})$ is non-empty for any finite set $i_1, \ldots, i_n \in I$. Since K is compact, $\bigcap_{i \in I} F_K(f_i)$ is non-empty. □

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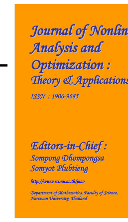

CONVERGENCE THEOREM IN CAT(O) SPACE

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ABSTRACT. Some new iterative process for multivalued mappings in CAT(0) spaces are introduced. Strong and $\triangle$-convergence theorems for such iterative process are established.

KEYWORDS : Fixed point; Strong convergence; $\triangle$-convergence; Quasi nonexpansive multivalued mapping; Condition (E).

1. INTRODUCTION

Fixed point theory in CAT(0) spaces was first studied by W. A. Kirk (see [11, 12].) He showed that every nonexpansive single valued mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single valued and multivalued mappings in CAT(0) spaces has been developed and a number of papers have appeared (see for example, [1, 3, 6, 15]). In [6], Dhompongsa and Panyanak obtained $\triangle$-convergence theorems for the Mann and Ishikawa iterations for nonexpansive single valued mappings in CAT(0) spaces. Very recently, Khan and Abbas [10] introduced a new iterative process for nonexpansive single valued mappings and proved convergence theorems for such iterative process in CAT(0) spaces. On the other hand some authors introduced and studied Mann and Ishikawa iteration for multivalued mappings in Hilbert spaces as well as in Banach spaces (see [16, 17, 18, 19].) The purpose of this paper is to introduce some iterative process for quasi nonexpansive multivalued mappings and prove $\triangle$-convergence and strong convergence theorems for such iterative process in CAT(0) spaces.

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2. PRELIMINARIES

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(r) = y$ and $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, r]$. In particular, the mapping c is an isometry and $d(x, y) = r$. The image of c is called a geodesic segment joining x and y which when unique is denoted by $[x, y]$. For any $x, y \in X$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $z = (1 - \alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$. The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y. A subset D of X is called convex if D includes every geodesic segment joining any two points of D.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, $d(x, y) \leq d_{R^2}(\overline{x}, \overline{y})$.

The following properties of a CAT(0) space are useful (see [2]):

(i) A CAT(0) space X is uniquely geodesic;

(ii) For any $x \in X$ and any closed convex subset $D \subset X$ there is a unique closest point to $x \in D$.

A notion of $\triangle$-convergence in CAT(0) spaces based on the fact that in Hilbert spaces a bounded sequence is weakly convergent to its unique asymptotic center has been studied in [13]. Let $\{x_n\}$ be a bounded sequence in X and D be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(D, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in D\},$$

where

$$r(x, \{x_n\}) = \limsup_{n \longrightarrow \infty} d(x_n, x),$$

and the set

$$A = A(D, \{x_n\}) = \{x \in D : r(x, \{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to D. Similarly, the set A is called the *asymptotic center* of $\{x_n\}$ relative to D.

In a CAT(0) space, the asymptotic center $A = A(D, \{x_n\})$ of (x_n) consists of exactly one point whenever D is closed and convex. A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is $\triangle$-convergent to x and given $y \in X$ with $x \neq y$,

$$\lim sup_{n \longrightarrow \infty} d(x, x_n) < \lim sup_{n \longrightarrow \infty} d(y, x_n).$$

Thus every CAT(0) space X satisfies the Opial property.

Lemma 2.1. *([13]) Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.*

Lemma 2.2. *([4]) If D is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.3. *([6]) If $\{x_n\}$ is a bounded sequence in complete CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.4. *([6]) Let (X,d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \qquad \textit{and} \qquad d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z.

Let (X, d) be a geodesic metric space. We denote by $CB(X)$ the collection of all nonempty closed bounded subsets of X, we also write $K(X)$ to denote the collection of all nonempty compact subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\},$$

for all $A, B \in CB(X)$ where $dist(x, B) = \inf_{y\in B} d(x, y)$.
Let $T : X \longrightarrow 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T is denoted by $F(T)$.

Definition 2.5. A multivalued mapping $T : X \longrightarrow CB(X)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in X.$$

(ii) quasi nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq d(x, p)$ for all $x \in X$ and all $p \in F(T)$.

In [7], Garcia-Falset et al. introduced condition (E) for single valued mappings. The current authors in [1] stated this condition for multivalued mappings as follows:

Definition 2.6. A multivalued mapping $T : X \longrightarrow CB(X)$ is said to satisfy condition (E_μ) provided that

$$dist(x, Ty) \leq \mu\, dist(x, Tx) + d(x, y), \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Lemma 2.7. *Let $T : X \longrightarrow CB(X)$ be a multivalued nonexpansive mapping, then T satisfies the condition (E_1).*

The following lemmas can be found in [6].

Lemma 2.8. *Let X be a CAT(0) space. Then for all $x, y, z \in X$ and all $t \in [0, 1]$ we have*

(i) $d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$,
(ii) $d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2$.

3. MAIN RESULTS

In this section we use the following iteration process.
(A): Let X be a CAT(0) space, D be a nonempty convex subset of X and $T : D \longrightarrow CB(D)$ be a given mapping. Then, for $x_1 \in D$, and $a_n, b_n \in [0, 1]$, we consider the following iterative process:

$$y_n = (1 - b_n)x_n \oplus b_n z_n, \quad n \geq 1,$$
$$x_{n+1} = (1 - a_n)u_n \oplus a_n w_n, \quad n \geq 1,$$

where $z_n, u_n \in Tx_n$ and $w_n \in Ty_n$.

Theorem 3.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow CB(D)$ be a quasi nonexpansive multivalued mapping such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n, b_n \in [a, b] \subset (0, 1)$. Assume that*

$$\lim_{n \longrightarrow \infty} dist(x_n, T(x_n)) = 0 \implies \liminf_{n \longrightarrow \infty} dist(x_n, F(T)) = 0.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$. Then, using (A) and quasi nonexpansiveness of T we have

$$\begin{aligned} d(y_n, p) &= d((1 - b_n)x_n \oplus b_n z_n, p) \\ &\leq (1 - b_n)d(x_n, p) + b_n d(z_n, p) \\ &= (1 - b_n)d(x_n, p) + b_n dist(z_n, Tp) \\ &\leq (1 - b_n)d(x_n, p) + b_n H(Tx_n, Tp) \\ &\leq (1 - b_n)d(x_n, p) + b_n d(x_n, p) = d(x_n, p). \end{aligned}$$

We also have

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - a_n)u_n \oplus a_n w_n, p) \\ &\leq (1 - a_n)d(u_n, p) + a_n d(w_n, p) \\ &= (1 - a_n)dist(u_n, Tp) + a_n dist(w_n, Tp) \\ &\leq (1 - a_n)H(Tx_n, Tp) + a_n H(Ty_n, Tp) \\ &\leq (1 - a_n)d(x_n, p) + a_n d(y_n, p) \leq d(x_n, p). \end{aligned}$$

Thus, the sequence $\{d(x_n, p)\}$ is decreasing and bounded below. It now follows that $\lim_{n \longrightarrow \infty} d(x_n, p)$ exists for any $p \in F(T)$. From Lemma 2.8, we get

$$\begin{aligned} d(y_n, p)^2 &= d((1 - b_n)x_n \oplus b_n z_n, p)^2 \\ &\leq (1 - b_n)d(x_n, p)^2 + b_n d(z_n, p)^2 - b_n(1 - b_n)d(x_n, z_n)^2 \\ &= (1 - b_n)d(x_n, p)^2 + b_n dist(z_n, Tp)^2 - b_n(1 - b_n)d(x_n, z_n)^2 \\ &\leq (1 - b_n)d(x_n, p)^2 + b_n H(Tx_n, Tp)^2 - b_n(1 - b_n)d(x_n, z_n)^2 \\ &\leq (1 - b_n)d(x_n, p)^2 + b_n d(x_n, p)^2 - b_n(1 - b_n)d(x_n, z_n)^2 \\ &= d(x_n, p)^2 - b_n(1 - b_n)d(x_n, z_n)^2. \end{aligned}$$

By another application of Lemma 2.8 we obtain

$$\begin{aligned} d(x_{n+1}, p)^2 &= d((1 - a_n)u_n \oplus a_n w_n, p)^2 \\ &\leq (1 - a_n)d(u_n, p)^2 + a_n d(w_n, p)^2 - a_n(1 - a_n)d(u_n, w_n)^2 \\ &\leq (1 - a_n)dist(u_n, Tp)^2 + a_n dist(w_n, Tp)^2 \\ &\leq (1 - a_n)H(Tx_n, Tp)^2 + a_n H(Ty_n, Tp)^2 \\ &\leq (1 - a_n)d(x_n, p)^2 + a_n d(y_n, p)^2 \end{aligned}$$

$$\leq \quad (1-a_n)d(x_n,p)^2 + a_n d(x_n,p)^2 - a_n b_n(1-b_n)d(x_n,z_n)^2,$$

so that

$$a^2(1-b)d(x_n,z_n)^2 \leq a_n b_n(1-b_n)d(x_n,z_n) \leq d(x_n,p)^2 - d(x_{n+1},p)^2.$$

This implies that

$$\sum_{n=1}^{\infty} a^2(1-b)d(x_n,z_n)^2 \leq d(x_1,p)^2 < \infty,$$

and hence $\lim_{n\longrightarrow\infty} d(x_n,z_n) = 0$. Thus $dist(x_n,Tx_n) \leq d(x_n,z_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Note that by our assumption $\lim_{n\longrightarrow\infty} dist(x_n,F(T)) = 0$. Therefore, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence p_k in $F(T)$ such that for all $k \in \mathbb{N}$

$$d(x_{n_k},p_k) < \frac{1}{2^k}.$$

Since the sequence $\{d(x_n,p)\}$ is decreasing, we get

$$d(x_{n_{k+1}},p_k) \leq d(x_{n_k},p_k) < \frac{1}{2^k}.$$

Hence

$$d(p_{k+1},p_k) \leq d(x_{n_{k+1}},p_{k+1}) + d(x_{n_{k+1}},p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

Consequently, we conclude that $\{p_k\}$ is a Cauchy sequence in D and hence converges to $q \in D$. Since

$$dist(p_k,T(q)) \leq H(T(p_k),T(q)) \leq d(p_k,q)$$

and $p_k \longrightarrow q$ as $k \longrightarrow \infty$, it follows that $dist(q,T(q)) = 0$ and hence $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\longrightarrow\infty} d(x_n,q)$ exists, it follows that $\{x_n\}$ converges strongly to q. □

Theorem 3.2. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Suppose $T : D \longrightarrow K(D)$ satisfies the condition (E). If $\{x_n\}$ is a sequence in D such that $\lim_{n\longrightarrow\infty} dist(Tx_n,x_n) = 0$ and $\triangle - \lim_n x_n = v$. Then $v \in D$ and $v \in Tv$.*

Proof. Let $\triangle - \lim_n x_n = v$. We note that by Lemma 2.2, $v \in D$. For each $n \geq 1$, we choose $z_n \in Tv$ such that$d(x_n,z_n) = dist(x_n,Tv)$.

By the compactness of Tv there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{k\longrightarrow\infty} z_{n_k} = w \in Tv$. Since T satisfies the condition (E) we have

$$dist(x_{n_k},Tv) \leq \mu\, dist(x_{n_k},T(x_{n_k})) + d(x_{n_k},v).$$

for some $\mu \geq 1$. Note that

$$d(x_{n_k},w) \leq d(x_{n_k},z_{n_k}) + d(z_{n_k},w) \leq \mu\, dist(x_{n_k},T(x_{n_k})) + d(x_{n_k},v) + d(z_{n_k},w).$$

Thus

$$\limsup_{k\longrightarrow\infty} d(x_{n_k},w) \leq \limsup_{k\longrightarrow\infty} d(x_{n_k},v).$$

From the Opial property of CAT(0) space X, we have $v = w \in Tv$. □

Now, we are ready to prove a $\triangle$-convergence theorem.

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow K(D)$ be a quasi nonexpansive multivalued mapping satisfying condition (E) and such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n, b_n \in [a,b] \subset (0,1)$. Then $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Proof. As in the proof of Theorem 3.1 we have $\lim_{n\longrightarrow\infty} dist(Tx_n, x_n) = 0$. Now we let $W_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset F(T)$. Let $u \in W_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.1 and 2.2 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_n v_n = v \in D$. Since $\lim_{n\longrightarrow\infty} dist(Tv_n, v_n) = 0$, by Theorem 3.2 we have $v \in F(T)$, and $\lim_{n\longrightarrow\infty} d(x_n, v)$ exists. Hence $u = v \in F(T)$ by Lemma 2.3. This shows that $W_w(x_n) \subset F(T)$. Next we show that $W_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_w(x_n) \subset F(T)$ and $d(x_n, v)$ converges, by Lemma 2.3 we have $x = u$. □

By using Theorem 3.3 along with Lemma 2.7 we obtain the following corollary.

Corollary 3.1. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow K(D)$ be a multivalued nonexpansive mapping such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n, b_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Theorem 3.4. *Let D be a nonempty compact convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow CB(D)$ be a quasi nonexpansive multivalued mapping satisfying condition (E) and such that $F(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (A), and $a_n, b_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. As in the proof of Theorem 3.1 we have $\lim_{n\longrightarrow\infty} dist(Tx_n, x_n) = 0$. Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = w$ for some $w \in D$. Since T satisfies the condition (E), for some $\mu \geq 1$ we have

$$\begin{aligned} dist(w, Tw) &\leq d(w, x_{n_k}) + dist(x_{n_k}, Tw) \\ &\leq \mu\, dist(x_{n_k}, T(x_{n_k})) + 2d(w, x_{n_k}) \longrightarrow 0 \qquad as \quad k \longrightarrow \infty. \end{aligned}$$

This implies that $w \in F(T)$. Since $\{x_{n_k}\}$ converges strongly to a point w and $\lim_{n\longrightarrow\infty} d(x_n, w)$ exists (as the proof of Theorem 3.1 shows), it follows that $\{x_n\}$ converges strongly to w. □

We now define the following iteration process.
(B): Let $T : D \longrightarrow P(D)$ be a given mapping and

$$P_T(x) = \{y \in Tx : \| x - y \| = dist(x, Tx)\}.$$

For fixed $x_1 \in D$, and $a_n, b_n \in [0, 1]$, we consider the iterative process defined by:

$$y_n = (1 - b_n)x_n \oplus b_n z_n, \qquad n \geq 1,$$

$$x_{n+1} = (1 - a_n)u_n \oplus a_n w_n, \qquad n \geq 1,$$

where $z_n, u_n \in P_T x_n$ and $w_n \in P_T y_n$.

Theorem 3.5. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \longrightarrow CB(D)$ be a multivalued mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the iterative process defined by (B), and $a_n, b_n \in [a, b] \subset (0, 1)$. Assume that*

$$\lim_{n\longrightarrow\infty} dist(x_n, T(x_n)) = 0 \implies \liminf_{n\longrightarrow\infty} dist(x_n, F(T)) = 0.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$. Then $p \in P_T p = \{p\}$. Hence using (B) and Lemma 2.8 we have

$$\begin{aligned} d(y_n, p) &= d((1-b_n)x_n \oplus b_n z_n, p) \\ &\le (1-b_n)d(x_n, p) + b_n d(z_n, p) \\ &= (1-b_n)d(x_n, p) + b_n dist(z_n, P_T p) \\ &\le (1-b_n)d(x_n, p) + b_n H(P_T x_n, P_T p) \\ &\le (1-b_n)d(x_n, p) + b_n d(x_n, p) = d(x_n, p), \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_n)u_n \oplus a_n w_n, p) \\ &\le (1-a_n)d(u_n, p) + a_n d(w_n, p) \\ &= (1-a_n)dist(u_n, P_T p) + a_n dist(w_n, P_T p) \\ &\le (1-a_n)H(P_T x_n, P_T p) + a_n H(P_T y_n, P_T p) \\ &\le (1-a_n)d(x_n, p) + a_n d(y_n, p) \le d(x_n, p). \end{aligned}$$

Hence, the sequence $\{d(x_n, p)\}$ is decreasing and bounded below. It now follows that $\lim_{n\longrightarrow\infty} d(x_n, p)$ exists for any $p \in F(T)$. From Lemma 2.8, we get

$$\begin{aligned} d(y_n, p)^2 &= d((1-b_n)x_n \oplus b_n z_n, p)^2 \\ &\le (1-b_n)d(x_n, p)^2 + b_n d(z_n, p)^2 - b_n(1-b_n)d(x_n, z_n)^2 \\ &= (1-b_n)d(x_n, p)^2 + b_n dist(z_n, P_T p)^2 - b_n(1-b_n)d(x_n, z_n)^2 \\ &\le (1-b_n)d(x_n, p)^2 + b_n H(P_T x_n, P_T p)^2 - b_n(1-b_n)d(x_n, z_n)^2 \\ &\le (1-b_n)d(x_n, p)^2 + b_n d(x_n, p)^2 - b_n(1-b_n)d(x_n, z_n)^2 \\ &= d(x_n, p)^2 - b_n(1-b_n)d(x_n, z_n)^2. \end{aligned}$$

By applying Lemma 2.8 we infer that

$$\begin{aligned} d(x_{n+1}, p)^2 &= d((1-a_n)u_n \oplus a_n w_n, p)^2 \\ &\le (1-a_n)d(u_n, p)^2 + a_n d(w_n, p)^2 - a_n(1-a_n)d(u_n, w_n)^2 \\ &\le (1-a_n)dist(u_n, P_T p)^2 + a_n dist(w_n, P_T p)^2 \\ &\le (1-a_n)H(P_T x_n, P_T p)^2 + a_n H(P_T y_n, P_T p)^2 \\ &\le (1-a_n)d(x_n, p)^2 + a_n d(y_n, p)^2 \\ &\le (1-a_n)d(x_n, p)^2 + a_n d(x_n, p)^2 - a_n b_n(1-b_n)d(x_n, z_n)^2. \end{aligned}$$

As in the proof of Theorem 3.1, $\lim_{n\longrightarrow\infty} dist(x_n, F(T)) = 0$. Therefore, we can choose a subsequence $\{x_{n_k}\}$ and a sequence p_k in $F(T)$ such that for all $k \in \mathbb{N}$

$$d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

Again, $\{p_k\}$ is a Cauchy sequence in D and hence converges to some $q \in D$. Since

$$dist(p_k, Tq) \le dist(p_k, P_T q) \le H(P_T p_k, P_T q) \le d(p_k, q)$$

and $p_k \longrightarrow q$ as $k \longrightarrow \infty$, it follows that $dist(q, Tq) = 0$ and hence $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\longrightarrow\infty} d(x_n, q)$ exists, we conclude that $\{x_n\}$ converges strongly to q.

□

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T-BEST APPROXIMATION IN FUZZY AND INTUITIONISTIC FUZZY METRIC SPACES

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ABSTRACT. In this paper we introduce and extend some notions in [18] and introduce a notion of t-best approximatively compact sets, t-best approximation points, t-proximinal sets and t-boundedly compact sets in both fuzzy and intuitionistic fuzzy metric spaces. The results obtained in this paper are related to the corresponding results in metric spaces and fuzzy metric spaces and fuzzy normed space. Many examples are also given.

KEYWORDS : Best approximation; Topology; Intuitionistic fuzzy metric spaces.

1. INTRODUCTION AND PRELIMINARIES

It is well known that the notion of fuzzy metric spaces plays a fundamental role in fuzzy topology, so many authors have introduced and studied several notions of fuzzy metric spaces from different points of view. In particular, following Menger [9], Kramosil and Michalek [8] generalized the concept of probabilistic metric space and studied an interesting notion of fuzzy metric space with the help of continuous t-norm. Later on, in order to construct a Hausdorff topology on the fuzzy metric space George and Veeramani [6] modified the concept of fuzzy metric space introduced by Kramosil and Michalek and obtained several classical theorems on this new structure. Actually, this topology is first countable and metrizable [3]. Further results in the topology of fuzzy metric spaces, in the sense of [6] may be found in [2, 3, 5, 7, 11, 15]. Park [10] extended the notion of fuzzy metric space proposed by George and Veeramani [6] and introduced the notion of intuitionistic fuzzy metric space which is based both on the idea of intuitionistic fuzzy set and the concept

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of fuzzy metric spaces. The topology generated by intuitionistic fuzzy metric space coincides with the topology generated by fuzzy metric space and, hence, topological results for intuitionistic fuzzy metric space are immediate consequences of the corresponding for fuzzy metric space. Some results in the topology of the intuitionistic fuzzy metric spaces may be found in [4, 10, 12].

Best approximation in fuzzy metric spaces has been discussed by Veeramani in [18]. In this paper , we start with the definitions of intuitionistic fuzzy metric spaces [10] and fuzzy normed spaces [13] and conclude some useful results, to be used in the next section. In section 2, we define the notion of t-approximatively compact sets in fuzzy and intuitionistic fuzzy metric spaces and introduce the notions of t-proximinal sets, t-boundedly compact sets, and t-best approximation points, this notions are the generalization of [13, 18]. For this notions we also point out some results about the relationship between metric spaces, fuzzy metric spaces and intuitionistic fuzzy metric spaces. To define the intuitionistic fuzzy metric space we have to state several concepts as follows:

Definition 1.1. [14]. A binary operation $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

- **(a):** $*$ is commutative and associative;
- **(b):** $*$ is continuous;
- **(c):** $a*1=a$ for all $a\in[0,1]$;
- **(d):** $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ and $a,b,c,d\in[0,1]$.

Definition 1.2. [14]. A binary operation $\Diamond:[0,1]\times[0,1]\longrightarrow[0,1]$ is continuous t-conorm if $\Diamond$ satisfies the following conditions:

- **(a):** $\Diamond$ is commutative and associative;
- **(b):** $\Diamond$ is continuous;
- **(c):** $a\Diamond 0=a$ for all $a\in[0,1]$;
- **(d):** $a\Diamond b\leq c\Diamond d$ whenever $a\leq c$ and $b\leq d$ and $a,b,c,d\in[0,1]$.

Definition 1.3. [10]. A 5-tuple $(X,M,N,*,\Diamond)$ is said to be an *intuitionistic fuzzy metric space* if X is a arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M,N are fuzzy sets on $X\times X\times(0,\infty)$, satisfying the following conditions, for all $x,y,z\in X, s,t>0$

- **(a):** $M(x,y,t)+N(x,y,t)\leq 1$;
- **(b):** $M(x,y,t)>0$;
- **(c):** $M(x,y,t)=1$ if and only if $x=y$;
- **(d):** $M(x,y,t)=M(y,x,t)$;
- **(e):** $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$;
- **(f):** $M(x,y,.):(0,\infty)\to(0,1]$ is continuous;
- **(g):** $N(x,y,t)>0$;
- **(h):** $N(x,y,t)=0$ if and only if $x=y$;
- **(i):** $N(x,y,t)=N(y,x,t)$;
- **(j):** $N(x,y,t)\Diamond N(y,z,s)\geq N(x,z,t+s)$;
- **(k):** $N(x,y,.):(0,\infty)\to(0,1]$ is continuous.

The functions $M(x,y,t)$, $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively. A *fuzzy metric space* is a triple $(X,M,*)$ such that conditions (b)-(f) are satisfied[6].

Definition 1.4. [13] The 3-tuple $(X,N,*)$ is said to be a fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X\times(0,1)$ satisfying the following conditions for every $x,y\in X$ and $t,s>0$,

(a): $N(x,t) > 0$;
(b): $N(x,t) = 1$ if and only if $x = 0$;
(c): $N(\alpha x,t) = N(x,t/|\alpha|)$, for all $\alpha \neq 0$;
(d): $N(x,t) * N(y,s) \leq N(x+y,t+s)$;
(e): $N(x,.) : (0,\infty) \to [0,1]$ is continuous;
(f): $\lim_{t\to\infty} N(x,t) = 1$

Remark 1.5. [13] Every fuzzy normed space $(X,N,*)$ induces a fuzzy metric space $(X,M,*)$ by defining $M(x,y,t) = N(x-y,t)$ and is therefore a topological space

If $(X,M,*)$ is a fuzzy metric space then George and Veeramani proved [6] that every fuzzy metric spaces $(X,M,*)$ generates a Hausdorff first countable topology τ_M on X which has as a base the family of open balls of the form $\{B_M(x,r,t); x \in X, r \in (0,1), t > 0\}$, where $B_M(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}$. Furthermore, for each x in X, $\{B(x,1/n,1/n); n \in \mathbb{N}\}$ is a neighborhood local base at x for the topology τ_M, so τ_M is the first countable. Recently, George and Romaguera proved [3] that if $(X,M,*)$ is a fuzzy metric space, then $\{U_n; n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}_n$ on X compatible with τ_M, where $U_n = \{(x,y) \in X \times X; M(x,y,1/n) > 1-(1/n)\}$ for all $n \in \mathbb{N}$. Therefore, $(X,M,*)$ is a metrizable space. Analogously, Saadati and Park [12] proved for intuitionistic fuzzy metric spaces $(X,M,N,*,\Diamond)$ that the topology $\tau_{(M,N)}$ on X for which open balls are of the form $\{B_{(M,N)}(x,r,t); x \in X, r \in (0,1), t > 0\}$ where $B_{(M,N)}(x,r,t) = \{y \in X : M(x,y,t) > 1-r, N(y,x,t) < r\}$ is metrizable.

Remark 1.6. [4, Proposition 1] For each $x \in X$, $r \in (0,1)$ and $t > 0$, we have $B_M(x,r,t) = B_{(M,N)}(x,r,t)$, thus, the two topologies τ_M and $\tau_{(M,N)}$ are equivalent in X, and the results obtained in the topology of intuitionistic fuzzy metric spaces become immediate consequences of the corresponding results of the topology of fuzzy metric spaces.

Remark 1.7. In a fuzzy metric space $(X,M,*)$ for each x in X, $0 < r < 1$ and $t > 0$, the set $B_M[x,r,t]$ defined as

$$B_M[x,r,t] = \{y \in X : M(x,y,t) \geq 1-r\}$$

is a closed set [6]. By a similar proof [18, Preposition.1] we deduce for each $x \in X$, $0 < r < 1$ and $t > 0$ in an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$, the set $B_{(M,N)}[x,r,t]$ defined as

$$B_{(M,N)}[x,r,t] = \{y \in X : M(x,y,t) \geq 1-r, N(y,x,t) \leq r\}$$

is equal to $B_M[x,r,t]$. Consequently, the set $B_{(M,N)}[x,r,t]$ is a closed set in the topology $\tau_{(M,N)}$ on X.

Remark 1.8. [4, Proposition 2] Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space, then the triple $(X,M_N,\star)$ is a fuzzy metric on X where M_N is defined on $X \times X \times (0,\infty)$ by $M_N(x,y,t) = 1-N(x,y,t)$ and $\star$ is the continuous t-norm defined by $a \star b = 1-[(1-a)\Diamond(1-b)]$.

In sequel we simply show the fuzzy metric space $(X,M_N,\star)$ by $(X,N,\star)$.

As a conclusion from above remark and definition of intuitionistic fuzzy metric spaces, we derive the following theorem.

Theorem 1.1. *The 5-tuple $(X,M,N,*,\Diamond)$ is an intuitionistic fuzzy metric space if and only if the triples $(X,M,*)$ and $(X,N,\star)$ are fuzzy metric spaces on X and for each $x,y \in X$ and $t \in (0,\infty)$,*

$$M(x,y,t) + N(x,y,t) \leq 1.$$

Remark 1.9. [10] Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated , ie, $x\Diamond y = 1 - [(1-x) * (1-y)]$ for any $x, y \in [0,1]$. We call the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, *the spacial intuitionistic fuzzy metric space* when $M \neq 1 - N$.

Remark 1.10. [11, Proposition.1] Let $(X, M, *)$ be a fuzzy metric space then M is a continuous function on $X \times X \times (0, \infty)$.

By using remarks 1.8 and 1.10 we derive the following:

Theorem 1.2. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space, then M and N are continues functions on $X \times X \times (0, \infty)$.*

The followings are examples for intuitionistic fuzzy metric space:

Example 1.11. [10] Let (X, d) be a metric space. Define $a * b = ab$ and $a\Diamond b = min\{1, a+b\}$ for all $a, b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X \times X \times (0, \infty)$ defined as

$$M_d(x,y,t) = \frac{ht^n}{ht^n + md(x,y)} \quad \text{and} \quad N_d(x,y,t) = \frac{md(x,y)}{ht^n + md(x,y)}$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space and called *induced intuitionistic fuzzy metric space*. The fuzzy metric space $(X, M_d, *)$ is called, *induced fuzzy metric space* [6].

Note the above example holds when t-norm is $a * b = \min\{a, b\}$ and t-conorm is $a\Diamond b = \max\{a, b\}$ and hence the 5-tuple $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = m = n = 1$, we get

$$M_d(x,y,t) = \frac{t}{t + d(x,y)} \quad \text{and} \quad N_d(x,y,t) = \frac{d(x,y)}{t + d(x,y)}$$

The fuzzy metric space $(X, M_d, N_d, *, \Diamond)$ is called, *the standard intuitionistic fuzzy metric space.*

Example 1.12. Let $X = \mathbb{N}$. Define $a * b = \max\{0, a+b-1\}$ and $a\Diamond b = a + b - ab$ for all $a, b \in [0,1]$ and let M and N be fuzzy sets on $X \times X \times (0, \infty)$ as

$$M(x,y,t) = \begin{cases} \frac{x+t}{y+t} & x \leq y \\ \\ \frac{y+t}{x+t} & y \leq x \end{cases} \quad \text{and} \quad N(x,y,t) = \begin{cases} \frac{y-x}{y+t} & x \leq y \\ \\ \frac{x-y}{x+t} & y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$, then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space, but if we choose t-norm $a * b$ by $\min\{a, b\}$ and t-conorm $a\Diamond b$ by $\max\{a, b\}$, then $(X, M, N, *, \Diamond)$ is not an intuitionistic fuzzy metric space.

Note that, in the above example, t-norm $*$ and t-conorm $\Diamond$ are not associated, and there exists no metric d on X that induces standard intuitionistic fuzzy metric space on X.

In all above examples, M is related to N by $M = 1 - N$. The following is an example of a spacial intuitionistic fuzzy metric space in which $M \neq 1 - N$.

Example 1.13. Let $X = \mathbb{R}^n$ and give d_2 the Euclidean distance on X and d_∞ the max-distance on X.i.e. for each $x = (x_1, x_2, \ldots, x_n)$,$y = (y_1, y_2, \ldots, y_n) \in X$ give

$$d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$
$$d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$$

Define $a * b = ab$ and $a\Diamond b = 1 - [(1-a) * (1-b)]$ and let M_{d_2} and N_{d_∞} be fuzzy sets on $X \times X \times (0, \infty)$ as follows

$$M_{d_2}(x,y,t) = \frac{t}{t+d_2(x,y)} \quad \text{and} \quad N_{d_\infty}(x,y,t) = \frac{d_\infty(x,y)}{t+d_\infty(x,y)}$$

for all $x, y \in X$ and $t > 0$, we show that $M_{d_2}(x,y,t) + N_{d_\infty}(x,y,t) \leq 1$, the inequality $d_\infty(x,y) \leq d_2(x,y)$ holds, thus

$$N_{d_\infty}(x,y,t) = \frac{d_\infty(x,y)}{t+d_\infty(x,y)} \leq \frac{d_2(x,y)}{t+d_2(x,y)}$$

consequently

$$M_{d_2}(x,y,t) + N_{d_\infty}(x,y,t) \leq \frac{t}{t+d_2(x,y)} + \frac{d_2(x,y)}{t+d_2(x,y)} = 1.$$

By theorem 1.1 $(X, M_{d_2}, N_{d_\infty}, *, \Diamond)$ is an intuitionistic fuzzy metric space.

2. BEST APPROXIMATION

We begin this section with the concept of t-best approximation points in fuzzy metric spaces introduced by Veeramani [18].
Our reference for best approximation in metric spaces is [16].

Definition 2.1. Let A be a nonempty subset of fuzzy metric space $(X, M, *)$. For $x \in X$ and $t > 0$, define

$$M(A,x,t) = \sup\{M(x,y,t) : y \in A\}$$

An element $y_0 \in A$ is said to be a *t-best approximation point to* x from A if

$$M(y_0,x,t) = M(A,x,t).$$

We denote by $P_A^M(x,t)$ the set of t-best approximation points to x. For $t > 0$ a subset A of a fuzzy metric space $(X, M, *)$ is called t-*proximinal* if for every point $x \in X$, $P_A^M(x,t) \neq \emptyset$.

Example 2.2. [18] Let $X = \mathbb{N}$, define $a * b = ab$ for all $a, b \in [0,1]$, let M be a fuzzy set on $X \times X \times (0,\infty)$ as follows

$$M(x,y,t) = \begin{cases} \frac{x+t}{y+t} & x \leq y \\ \frac{y+t}{x+t} & y \leq x \end{cases}$$

for all $x, y \in X$ and $t > 0$, then $(X, M, *)$ is a fuzzy metric space. Let $A = \{2, 4, 6, \ldots\}$ then we conclude

$$M(A,3,t) = \max\{\tfrac{2+t}{3+t}, \tfrac{3+t}{4+t}\} = \tfrac{3+t}{4+t} = M(3,4,t)$$

Hence, for each $t > 0$, 4 is t-best approximation point to 3 from A. As $M(3,4,t) > M(2,3,t)$, 2 is not a t-best approximation point to 3, so $P_A^M(3,t) = \{4\}$.

A fuzzy metric space $(X, M, *)$ is called strong fuzzy metric space [18, Defination 2.1], if for each x in X and $t > 0$, the map $y \to M(x,y,t)$ is a continuous map on X. Since by remark 1.10 the map $y \to M(x,y,t)$ is always continuous thus every strong fuzzy metric space $(X, M, *)$ is a fuzzy metric space and we can omit the notion of strong fuzzy metric spaces which is used in [18].

Definition 2.3. [18] For $t > 0$, a nonempty subset A of a fuzzy metric space $(X, M, *)$ is said to be t-*approximatively compact* if for each x in X and each sequence y_n in A with $M(y_n, x, t) \to M(A, x, t)$, there exists a subsequence y_{n_k} of y_n converging to an element y_0 in A.

Definition 2.4. [18] For $t > 0$, a nonempty closed subset A of a fuzzy metric space $(X, M, *)$ is said to be *t-boundedly compact* if for each x in X and $0 < r < 1$, the set $B[x, r, t] \cap A$ is a compact subset of X.

Remark 2.5. [18] Let (X,d) be a metric space and $A \subseteq X$, then A is a approximatively compact set in the metric space (X,d) if and only if for any $t > 0$, A is a t-approximatively compact set in the induced fuzzy metric space $(X, M_d, *)$.

Veeramani proved that every nonempty t-approximatively compact subset of a fuzzy metric space is t-proximinal and every t-boundedly compact subset of fuzzy metric space is t-approximatively compact [18, Theorem 2.10,Theorem 2.16 respectively]. Also it can be easily proved that every t-proximinal set is a closed set. Thus each of the following properties in fuzzy metric spaces implies the next one:compact, t-boundedly compact, t-approximatively compact, t-proximinal and closed.

Now we define the notions of t-approximatively compact sets and t-best approximation points in intuitionistic fuzzy metric space$(X, M, N, *, \diamondsuit)$.

Definition 2.6. Let A be a subset of intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$, for $x \in X, t > 0$ let

$$M(A,x,t) = \sup\{M(y,x,t) : y \in A\} \quad \text{and} \quad N(A,x,t) = \inf\{N(y,x,t) : y \in A\}$$

We say A is a t-*proximinal* subset of X (respect to (M,N)) if for each x in X there exist two elements $y_1, y_2 \in A$ such that $M(y_1,x,t) = M(A,x,t)$ and $N(y_2,x,t) = N(A,x,t)$. In this case we say y_1, y_2 are t-*best approximation points* to x (respect to (M,N) from A). We denote by $P_A^{(M,N)}(x,t)$ the set of $\{a \in A; M(a,x,t) = M(A,x,t), N(a,x,t) = N(A,x,t)\}$. If $(X, M, N, *, \diamondsuit)$ be a non-spacial intuitionistic fuzzy metric space, then we have $M = 1 - N$ and if we define $P_A^N(x,t)$ by $\{a \in A; N(a,x,t) = N(A,x,t)\}$then $P_A^M(x,t) = P_A^N(x,t)$ and we can choose $y_1 = y_2$. In this case we say y_1 is a t-*best approximation point* to x (respect to (M,N) from A). Notice by theorem 1.1, $(X, N, \star)$ is a fuzzy metric space and we have $P_A^{(M,N)}(x,t) = P_A^M(x,t) \cap P_A^N(x,t)$.

Next examples illustrate the last definition.

Example 2.7. Take $X = \mathbb{R}^2$, let $(X, M_{d_2}, N_{d_\infty}, *, \diamondsuit)$ be an intuitionistic fuzzy metric space defined in example 1.13 and take $a = (0, 5/4), b = (1,1) \in X$. Define $A \subseteq X$ by a line that connect a to b. i.e. $A = \{\lambda a + (1-\lambda)b; \lambda \in [0,1]\}$. We observe that $d_2(A, x_0) = \inf\{\sqrt{x^2+y^2}; (x,y) \in A\} = 5/4$ and we find there exists exactly one element $y_1 = (0,5/4)$ in A such that $d_2(A,x_0) = d_2(y_1,x_0)$. On the other hand, we observe that $d_\infty(A,x_0) = \inf\{\max\{|x|,|y|\}, (x,y) \in A\} = 1$ and we find there exists exactly one element $y_2 = (1,1)$ in A such that $d_\infty(A,x_0) = d_\infty(y_2,x_0)$, consequently, for every $t > 0$

$$\begin{aligned} M_{d_2}(A,x_0,t) &= \sup\{M_{d_2}(y,x_0,t); y \in A\} = \sup\{\frac{t}{t + d_2(x_0,y)}; y \in A\} \\ &= \frac{t}{t + \inf\{d_2(x_0,y); y \in A\}} = \frac{t}{t + d_2(x_0,y_1)} \end{aligned}$$

Consequently, there exists a unique point $y_1 = (0,5/4)$ in A such that $M_{d_2}(A,x_0,t) = M_{d_2}(y_1,x_0,t)$ and by similar reasoning we find there exists a unique point $y_2 = (1,1)$ in A such that $N_{d_\infty}(A,x_0,t) = N_{d_\infty}(y_2,x_0,t)$, so $y_1 = (0,5/4)$ and $y_2 = (1,1)$ are t-best approximation points to $x = (0,0)$ (respect to (M_{d_2}, N_{d_∞})) and $P_A^{(M_{d_2},N_{d_\infty})}(x,t) = \emptyset$.

Example 2.8. In the example 2.7, if we replace the fuzzy set N_{d_∞} by N_{d_2}, then $y_1 = (0,5/4)$ is a t-best approximation point to $x = (0,0)$ respect to (M_{d_2}, N_{d_2}) and $P_A^{(M_{d_2},N_{d_2})}(x,t) = \{(0,5/4)\}$.

Remark 2.9. **(a):** For any $t > 0$, A is a t-proximinal subset of X in fuzzy metric space $(X, M, *)$ if and only if A is a t-proximinal subset of X in fuzzy metric space $(X, M, 1-M, *, \Diamond)$. If A is a subset of X, then for each $x \in X$, $P_A^M(x,t) = P_A^{(M,1-M)}(x,t)$.

(b): Suppose $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and A a subset of X then for every x in X, $y_1, y_2 \in A$ are t-best approximation points to x (respect to (M, N)) in the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ if and only if y_1 and y_2 are t-best approximation points to x in the fuzzy metric spaces $(X, M, *)$ and $(X, N, \star)$ respectively.

Example 2.10. Consider the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ in the example 1.12, we have $M = 1 - N$. Let $A = \{2, 4, 6, \dots\}$, we conclude from the above remark and example 2.2

$$M(A, 3, t) = \max\{\tfrac{2+t}{3+t}, \tfrac{3+t}{4+t}\} = \tfrac{3+t}{4+t} = M(3, 4, t)$$

and

$$N(A, 3, t) = 1 - M(A, 3, t) = 1 - M(3, 4, t) = N(3, 4, t)$$

Hence for each $t > 0$, 4 is t-best approximation point to 3. As $M(3, 4, t) > M(2, 3, t)$, 2 is not a t-best approximation point to 3 and $P_A^M(3,t) = P_A^{(M,N)}(3,t) = \{4\}$.

Remark 2.11. Let (X, d) be a metric space and A a nonempty subset of M, then the following are equivalent.

(a): $y_0 \in A$ is a t-best approximation point to $x \in X$ in the metric space (X, d).

(b): $y_0 \in A$ is a t-best approximation point to $x \in X$ in the induced fuzzy metric space $(X, M_d, *)$.

(c): $y_0 \in A$ is a t-best approximation point to $x \in X$ in the induced intuitionistic fuzzy metric space $(X, M_d, N_d, *,$
$\Diamond)$.

Definition 2.12. For $t > 0$, a nonempty subset A of an intuitionistic fuzzy metric spaces $(X, M, N, *, \Diamond)$ is said to be t-*approximatively compact* if for each x in X and sequences each x_n and y_n in X with $M(y_n, x, t) \to M(A, x, t)$ and $N(x_n, x, t) \to N(A, x, t)$, there exist subsequences x_{n_k} of x_n and y_{n_k} of y_n converging to elements x_0 and $y_0 \in A$, respectively.

Remark 2.13. **(a):** If A be a compact subset of X in intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then for each $t > 0$, A is a t-approximatively compact set.

(b): A is a approximatively compact subset of X in metric space (X, d) if and only if for each $t > 0$, A is a t-approximatively compact subset of X in the induced intuitionistic fuzzy metric space $(X, M_d, N_d, *, \Diamond)$.

(c): For each $t > 0$, if A is a t-approximatively compact subset of X in intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then A will be a t-approximatively compact subset of X in fuzzy metric space $(X, M, *)$.

Theorem 2.1. *Let A be a t-approximatively compact subset of X in intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ then A is closed.*

Proof. Let $t > 0$ and A be a t-approximatively compact subset of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, then by remark 2.13, A is a t-approximatively compact set in fuzzy metric space $(X, M, *)$, thus by using [18, Theorem 2.11] A is a closed set in fuzzy metric space $(X, M, *)$ and since the two topologies τ_M

and $\tau_{(M,N)}$ coincides in X, A is a closed set in intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$. □

Theorem 2.2. *For $t > 0$, let A be a nonempty t-approximatively compact subset of X in an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ then A is a t-proximinal set.*

Proof. For each x in X, we have

$$M(A, x, t) = \sup\{M(y, x, t); y \in A\} \quad \text{and} \quad N(A, x, t) = \inf\{N(y, x, t); y \in A\}$$

consequently there exist sequences x_n and y_n in A such that

$$M(y_n, x, t) \to M(A, x, t) \quad \text{and} \quad N(x_n, x, t) \to N(A, x, t)$$

since A is a t-approximatively compact set, subsequences y_{n_k} of y_n and x_{n_k} of x_n and points $x_0, y_0 \in A$ exist such that $x_{n_k} \to x_0$ and $y_{n_k} \to y_0$. By theorem 1.2, M and N are continuous functions thus we have

$$M(y_{n_k}, x, t) \to M(y_0, x, t) \quad \text{and} \quad N(x_{n_k}, x, t) \to N(x_0, x, t)$$

so we conclude

$$M(y_0, x, t) = M(A, x, t) \quad \text{and} \quad N(x_0, x, t) = N(A, x, t)$$

consequently, y_0, x_0 are t-best approximation points to x from A (respect to (M, N)), i.e. A is a t-proximinal set. □

Definition 2.14. For $t > 0$, a nonempty closed subset A of an intuitionistic fuzzy metric $(X, M, N, *, \lozenge)$ is said to be t-*boundedly compact* if for each x in X and $0 < r < 1$, the set $B_{(M,N)}[x, r, t] \cap A$ is a compact subset of X.

Theorem 2.3. *Let $(X, M, N, *, \lozenge)$ is an intuitionistic fuzzy metric space, If A is a nonempty t-boundedly compact subset of X then A is a t-approximatively compact set.*

Proof. By remark 1.7 we have $B_M[x, r, t] = B_{(M,N)}[x, r, t]$, thus A is a t-boundedly compact set in the fuzzy metric space $(X, M, *)$ if and only if A is a t-boundedly compact set in the intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$. The proof is an immediate consequence of [18, Theorem 2.16]. □

Remark 2.15. Since in an intuitionistic fuzzy metric space a set is compact if and only if it is sequentially compact, thus for each $t > 0$, if A is a t-approximatively compact set then for each x in X the set $P_A^{(M,N)}(x, t)$ is a compact set.

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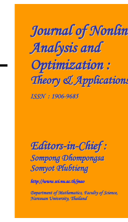

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FIXED POINT AND MEAN CONVERGENCE THEOREMS FOR A FAMILY OF λ-HYBRID MAPPINGS

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ABSTRACT. The aim of this paper is to prove fixed point and mean convergence theorems for a sequence of λ-hybrid mappings in Hilbert spaces.

KEYWORDS : λ-hybrid mapping; Fixed point; Mean convergence theorem.

1. INTRODUCTION

In this paper, we show fixed point and mean convergence theorems for a sequence of λ-hybrid mappings in Hilbert spaces. Particularly, we focus on pointwise convergent sequences of such mappings.

According to [2] and §2, every nonexpansive mapping [5, 6, 10] is a 1-hybrid mapping and every nonspreading mapping introduced by Kohsaka and Takahashi [7] is a 0-hybrid mapping. Thus our results may be regarded as generalizations of results of [1] and [8]. Akatsuka, Aoyama, and Takahashi [1] showed a mean convergence theorem for a pointwise convergent sequence of nonexpansive mappings; Kurokawa and Takahashi [8] proved some mean convergence theorems for nonspreading mappings in Hilbert spaces.

Moreover, since the convex combination of the identity mapping and a strictly pseudononspreading mapping introduced by Osilike and Isiogugu [9] is λ-hybrid for some real number λ, our mean convergence theorem is a generalization of [9]. Osilike and Isiogugu [9] showed some mean convergence theorems for strictly pseudononspreading mappings in Hilbert spaces.

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2. PRELIMINARIES

Throughout the present paper, H denotes a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, C a nonempty closed convex subset of H, I the identity mapping on H, and $\mathbb{N}$ the set of positive integers. Strong convergence of a sequence $\{x_n\}$ in H to x is denoted by $x_n \longrightarrow x$ and weak convergence by $x_n \rightharpoonup x$. The metric projection of H onto C is denoted by P_C, that is, for each $x \in H$, $P_C x$ is the unique point in C such that $\|P_C x - x\| = \min\{\|y - x\| : y \in C\}$. It is known that P_C is nonexpansive and

$$\langle y - P_C x, x - P_C x \rangle \leq 0 \tag{2.1}$$

for all $x \in H$ and $y \in C$; see [10].

Let D be a nonempty subset of H. The set of fixed points of a mapping $T\colon D \longrightarrow H$ is denoted by $F(T)$. A mapping $T\colon D \longrightarrow H$ is said to be quasi-nonexpansive if $F(T)$ is nonempty and $\|Tx - z\| \leq \|x - z\|$ for all $x \in D$ and $z \in F(T)$. Let λ be a real number. A mapping $T\colon D \longrightarrow H$ is said to be λ-hybrid [2] if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\lambda\langle x - Tx, y - Ty\rangle$$

or equivalently

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle$$

for all $x, y \in D$. Let κ be a real number with $\kappa \in [0, 1)$. A mapping $T\colon D \longrightarrow H$ is said to be κ-strictly pseudononspreading [9] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle + \kappa\|x - Tx - (y - Ty)\|^2$$

for all $x, y \in D$. It is known that

- T is λ-hybrid for every $\lambda \in [0, 1]$ if T is a firmly nonexpansive mapping [3, 4, 5, 6];
- T is 1-hybrid if and only if T is nonexpansive;
- T is 0-hybrid if and only if T is nonspreading in the sense of [7];
- T is $1/2$-hybrid if and only if T is hybrid in the sense of [11];
- $F(T)$ is closed and convex if $T\colon C \longrightarrow H$ is a quasi-nonexpansive mapping;
- T is quasi-nonexpansive if T is a λ-hybrid mapping with a fixed point.

The following lemma plays an important role in the present paper.

Lemma 2.1. *Let H be a Hilbert space, D a nonempty subset of H, γ and κ real numbers, and $T\colon D \longrightarrow H$ a mapping such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\gamma\langle x - Tx, y - Ty\rangle + \kappa\|x - Tx - (y - Ty)\|^2 \tag{2.2}$$

for all $x, y \in D$. Let $T_\alpha\colon D \longrightarrow H$ be a mapping defined by $T_\alpha = \alpha I + (1 - \alpha)T$, where α is a real number with $\alpha < 1$. Then

$$\begin{aligned} \|T_\alpha x - T_\alpha y\|^2 &+ \frac{\alpha - \kappa}{1 - \alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2 \\ &\leq \|x - y\|^2 + \frac{2\gamma}{1 - \alpha}\langle x - T_\alpha x, y - T_\alpha y\rangle \end{aligned} \tag{2.3}$$

for all $x, y \in D$. Moreover, if $\kappa \leq \alpha$, then T_α is $(1 - \alpha - \gamma)/(1 - \alpha)$-hybrid.

Proof. Let $x, y \in D$ be fixed. Since $1 - \alpha > 0$ and $I - T = (I - T_\alpha)/(1 - \alpha)$, it follows from (2.2) that

$$\begin{aligned} &(1 - \alpha)\|Tx - Ty\|^2 \\ &\quad\leq (1 - \alpha)\big(\|x - y\|^2 + 2\gamma\langle x - Tx, y - Ty\rangle + \kappa\|x - Tx - (y - Ty)\|^2\big) \end{aligned}$$

$$= (1-\alpha)\|x-y\|^2 + \frac{2\gamma}{1-\alpha}\langle x - T_\alpha x, y - T_\alpha y\rangle + \frac{\kappa}{1-\alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2$$

and hence

$$\begin{aligned} &\|T_\alpha x - T_\alpha y\|^2 \\ &= \alpha\|x-y\|^2 + (1-\alpha)\|Tx - Ty\|^2 - \alpha(1-\alpha)\|x - Tx - (y - Ty)\|^2 \\ &= \alpha\|x-y\|^2 + (1-\alpha)\|Tx - Ty\|^2 - \frac{\alpha}{1-\alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2 \\ &\le \|x-y\|^2 + \frac{2\gamma}{1-\alpha}\langle x - T_\alpha x, y - T_\alpha y\rangle - \frac{\alpha-\kappa}{1-\alpha}\|x - T_\alpha x - (y - T_\alpha y)\|^2. \end{aligned}$$

Thus (2.3) holds. Now we suppose that $\kappa \le \alpha$. Then $(\alpha-\kappa)/(1-\alpha) \ge 0$, and (2.3) yields that

$$\|T_\alpha x - T_\alpha y\|^2 \le \|x-y\|^2 + 2\left(1 - \frac{1-\alpha-\gamma}{1-\alpha}\right)\langle x - T_\alpha x, y - T_\alpha y\rangle.$$

Therefore, T_α is $(1-\alpha-\gamma)/(1-\alpha)$-hybrid. □

Lemma 2.1 implies the following lemma.

Lemma 2.2. *Let H be a Hilbert space, D a nonempty subset of H, λ a real number, and $T\colon D \longrightarrow H$ a λ-hybrid mapping. Let $T_\alpha\colon D \longrightarrow H$ be a mapping defined by $T_\alpha = \alpha I + (1-\alpha)T$, where α a real number with $0 \le \alpha < 1$. Then T_α is $(\lambda-\alpha)/(1-\alpha)$-hybrid.*

Proof. Assuming that $\gamma = 1-\lambda$ and $\kappa = 0$ in Lemma 2.1, we obtain the conclusion. □

Using Lemma 2.2, we can show the following corollary.

Corollary 2.3. *Let H be a Hilbert space and D a nonempty convex subset of H. Suppose that every nonspreading self-mapping on D has a fixed point. If $\lambda \in [0,1)$, then every λ-hybrid mapping $T\colon D \longrightarrow D$ has a fixed point.*

Proof. Let $\lambda \in [0,1)$ and let $T\colon D \longrightarrow D$ be a λ-hybrid mapping. Then it follows from Lemma 2.2 that $T_\lambda = \lambda I + (1-\lambda)T$ is a nonspreading mapping of D into itself. Hence, by assumption, we know that $F(T_\lambda)$ is nonempty. On the other hand, it obviously holds that $F(T_\lambda) = F(T)$. Thus $F(T)$ is nonempty. □

Remark 2.4. It is known that every nonspreading self-mapping on C has a fixed point if C is a nonempty bounded closed convex subset of H; see [7, Theorem 4.1].

Lemma 2.1 also implies the following lemma, which was essentially proven in [9].

Lemma 2.5. *Let H be a Hilbert space, D a nonempty subset of H, κ and β real numbers with $0 \le \kappa \le \beta < 1$, $T\colon D \longrightarrow H$ a κ-strictly pseudononspreading mapping, and $T_\beta\colon D \longrightarrow H$ the mapping defined by $T_\beta = \beta I + (1-\beta)T$. Then T_β is $-\beta/(1-\beta)$-hybrid.*

Proof. Assuming that $\alpha = \beta$ and $\gamma = 1$ in Lemma 2.1, we obtain the conclusion. □

We need the following lemmas in order to prove our results in the remainder sections.

Lemma 2.6. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Hilbert space H and $\{\eta_n\}$ a sequence of real numbers. Suppose that $\{x_n\}$ is bounded and both $\{y_n\}$ and $\{\eta_n\}$ are convergent. Then*

$$\frac{1}{n}\sum_{k=1}^{n}\eta_k\langle x_{k+1}-x_k, y_k\rangle \longrightarrow 0$$

as $n \longrightarrow \infty$.

Proof. Let y and η be the limits of $\{y_n\}$ and $\{\eta_n\}$, respectively. Since $\{x_n\}$ is bounded, it follows that $\langle x_{n+1}-x_n, y_n-y\rangle \longrightarrow 0$ and hence

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\langle x_{k+1}-x_k, y_k\rangle &= \frac{1}{n}\sum_{k=1}^{n}\langle x_{k+1}-x_k, y\rangle + \frac{1}{n}\sum_{k=1}^{n}\langle x_{k+1}-x_k, y_k-y\rangle \\ &= \frac{1}{n}\langle x_{n+1}-x_1, y\rangle + \frac{1}{n}\sum_{k=1}^{n}\langle x_{k+1}-x_k, y_k-y\rangle \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$. Therefore, since $\{\langle x_{n+1}-x_n, y_n\rangle\}$ is bounded, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\eta_k\langle x_{k+1}-x_k, y_k\rangle = \frac{1}{n}\sum_{k=1}^{n}\eta\langle x_{k+1}-x_k, y_k\rangle + \frac{1}{n}\sum_{k=1}^{n}(\eta_k-\eta)\langle x_{k+1}-x_k, y_k\rangle \longrightarrow 0$$

as $n \longrightarrow \infty$. □

The following lemma was essentially shown in [1, Lemma 3.1], where $\{\xi_n\}$ was assumed to be convergent to 0. For the sake of completeness, we give the proof.

Lemma 2.7. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T\colon C \longrightarrow H$ a mapping. Let $\{x_n\}$ be a sequence in C, $\{\xi_n\}$ a sequence of real numbers, $\{z_n\}$ a sequence in C defined by $z_n = (1/n)\sum_{k=1}^{n} x_k$ for $n \in \mathbb{N}$, and z a weak cluster point of $\{z_n\}$. Suppose that*

$$\xi_n \le \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$ and $(1/n)\sum_{k=1}^{n}\xi_k \longrightarrow 0$ as $n \longrightarrow \infty$. Then z is a fixed point of T.

Proof. By assumption, it is clear that

$$\begin{aligned}\xi_k &\le \|x_k - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz + Tz - z\|^2 - \|x_{k+1} - Tz\|^2 \\ &= \|x_k - Tz\|^2 - \|x_{k+1} - Tz\|^2 + 2\langle x_k - Tz, Tz - z\rangle + \|Tz - z\|^2\end{aligned}$$

for every $k \in \mathbb{N}$. Summing these inequalities from $k = 1$ to n and dividing by n, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\xi_k &\le \frac{1}{n}\left(\|x_1 - Tz\|^2 - \|x_{n+1} - Tz\|^2\right) \\ &\quad + 2\left\langle \frac{1}{n}\sum_{k=1}^{n} x_k - Tz, Tz - z\right\rangle + \|Tz - z\|^2 \\ &\le \frac{1}{n}\|x_1 - Tz\|^2 + 2\langle z_n - Tz, Tz - z\rangle + \|Tz - z\|^2\end{aligned}$$

for every $n \in \mathbb{N}$. Since z is a weak cluster point of $\{z_n\}$, there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$. Replacing n by n_i in the above inequality, we obtain

$$\frac{1}{n_i}\sum_{k=1}^{n_i}\xi_k \le \frac{1}{n_i}\|x_1 - Tz\|^2 + 2\langle z_{n_i} - Tz, Tz - z\rangle + \|Tz - z\|^2.$$

Since $(1/n_i)\sum_{k=1}^{n_i}\xi_k \longrightarrow 0$ and $z_{n_i} \rightharpoonup z$, we conclude that

$$0 \leq 2\langle z - Tz, Tz - z\rangle + \|Tz - z\|^2 = -\|Tz - z\|^2$$

and hence $Tz = z$. □

Lemma 2.8 (Takahashi and Toyoda [12]). *Let F be a nonempty closed convex subset of a Hilbert space H, P the metric projection of H onto F, and $\{x_n\}$ a sequence in H such that $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in F$ and $n \in \mathbb{N}$. Then $\{Px_n\}$ converges strongly to some point in F.*

3. FIXED POINT THEOREMS

In this section, we study existence of fixed points of λ-hybrid mappings.

The following theorem is a generalization of [1, Theorem 3.2] and [2, Theorem 4.1].

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{\lambda_n\}$ a sequence of real numbers such that $\lambda_n \longrightarrow \lambda$, and $T_n\colon C \longrightarrow C$ a λ_n-hybrid mapping for $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent and T denotes the pointwise limit of $\{T_n\}$, that is, $Tx = \lim_{n\longrightarrow\infty} T_n x$ for $x \in C$. Then the following hold:

(i) *The mapping T is λ-hybrid and $\bigcap_{n=1}^{\infty} F(T_n) \subset F(T)$;*
(ii) *if $\{x_n\}$ is bounded, then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T.*

Proof. We first prove (1). Let $x, y \in C$ be fixed. Since each T_n is λ_n-hybrid, it follows that

$$\|T_n x - T_n y\|^2 \leq \|x - y\|^2 + 2(1 - \lambda_n)\langle x - T_n x, y - T_n y\rangle$$

for every $n \in \mathbb{N}$. Taking the limit $n \longrightarrow \infty$, we have

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty\rangle.$$

Thus T is λ-hybrid. Furthermore, let $u \in \bigcap_{n=1}^{\infty} F(T_n)$. Since T_n is pointwise convergent, $Tu = \lim_{n\longrightarrow\infty} T_n u = u$ and hence $u \in F(T)$.

We next prove (2). Assume that $\{x_n\}$ is bounded. Then $\{z_n\}$ is also bounded and thus there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z \in C$. It is enough to show that z is a fixed point of T. Since T_n is λ_n-hybrid and $x_{n+1} = T_n x_n$, we have

$$\begin{aligned}
\|x_{n+1} - Tz\|^2 &= \|x_{n+1} - T_n z + T_n z - Tz\|^2 \\
&= \|x_{n+1} - T_n z\|^2 + \|T_n z - Tz\|^2 + 2\langle x_{n+1} - T_n z, T_n z - Tz\rangle \\
&\leq \|x_n - z\|^2 + 2(1 - \lambda_n)\langle x_n - x_{n+1}, z - T_n z\rangle \\
&\quad + \|T_n z - Tz\|(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|).
\end{aligned}$$

Therefore, we conclude that

$$\mu_n + \epsilon_n \leq \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2$$

for every $n \in \mathbb{N}$, where $\mu_n = 2(1 - \lambda_n)\langle x_{n+1} - x_n, z - T_n z\rangle$ and

$$\epsilon_n = -\|T_n z - Tz\|(\|T_n z - Tz\| + 2\|x_{n+1} - T_n z\|).$$

Since $\{x_n\}$ is bounded and both $\{\lambda_n\}$ and $\{T_n z\}$ are convergent, Lemma 2.6 shows that $(1/n)\sum_{k=1}^{n}\mu_k \longrightarrow 0$, and hence $(1/n)\sum_{k=1}^{n}(\mu_k+\epsilon_k) \longrightarrow 0$. Thus Lemma 2.7 implies that z is a fixed point of T. □

A direct consequence of Theorem 3.1 is as follows:

Corollary 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, λ a real number, $T\colon C \longrightarrow C$ a λ-hybrid mapping, and $\{\alpha_n\}$ a sequence in $[0,1)$ such that $\alpha_n \longrightarrow 0$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Suppose that $\{x_n\}$ is bounded. Then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T.

Proof. Put $T_n = \alpha_n I + (1-\alpha_n)T$ for $n \in \mathbb{N}$. Then Lemma 2.2 shows that $T_n\colon C \longrightarrow C$ is $(\lambda-\alpha_n)/(1-\alpha_n)$-hybrid. It is clear that $(\lambda-\alpha_n)/(1-\alpha_n) \longrightarrow \lambda$ and T is the pointwise limit of $\{T_n\}$. Therefore, Theorem 3.1 implies the conclusion. □

In particular, assuming that $\alpha_n = 0$ for each $n \in \mathbb{N}$ in Corollary 3.1, we obtain the following:

Corollary 3.2. *([2, Theorem 4.1]). Let H, C, λ, and T be the same as in Corollary 3.1. Let x be a point in C and $\{z_n\}$ a sequence in C defined by*

$$z_n = \frac{1}{n}\sum_{k=1}^{n} T^{k-1}x$$

for $n \in \mathbb{N}$, where T^0 is the identity mapping on C. Suppose that $\{T^n x\}$ is bounded. Then T has a fixed point and every weak cluster point of $\{z_n\}$ is a fixed point of T.

4. MEAN CONVERGENCE THEOREMS

In this section, we prove some mean convergence theorems for a family of λ-hybrid mappings.

We first prove the following lemma, which is a variant of [2, Lemma 5.1].

Lemma 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T_n\colon C \longrightarrow C$ a quasi-nonexpansive mapping for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ has a common fixed point. Let F be the set of common fixed points of $\{T_n\}$ and P the metric projection of H onto F. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_n$$

for $n \in \mathbb{N}$. Then the following hold:

- (i) *The sequence $\{x_n\}$ is bounded and $\{Px_n\}$ converges strongly;*
- (ii) *if each weak cluster point of $\{z_n\}$ belongs to F, then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$.*

Proof. We first prove (1). Since T_n is quasi-nonexpansive,

$$\|x_{n+1}-u\| = \|T_n x_n - u\| \le \|x_n - u\|$$

for all $u \in F$ and $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded and Lemma 2.8 implies that $\{Px_n\}$ converges strongly.

We next prove (2). Since $\{z_n\}$ is bounded by (1), there exists a weak cluster point z of $\{z_n\}$. Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z$ and w the strong limit of $\{Px_n\}$. It is enough to show that $z = w$. Since P is the metric projection of H onto F and $z \in F$, it follows from (2.1) that

$$\langle z - Px_k, x_k - Px_k \rangle \leq 0$$

for every $k \in \mathbb{N}$. Since each T_k is quasi-nonexpansive and $Px_k \in F$, it follows from the definition of P that

$$\|x_{k+1} - Px_{k+1}\| \leq \|x_{k+1} - Px_k\| = \|T_k x_k - Px_k\| \leq \|x_k - Px_k\|$$

for every $k \in \mathbb{N}$. Therefore

$$\begin{aligned} \langle z - w, x_k - Px_k \rangle &= \langle z - Px_k, x_k - Px_k \rangle + \langle Px_k - w, x_k - Px_k \rangle \\ &\leq \langle Px_k - w, x_k - Px_k \rangle \\ &\leq \|Px_k - w\| \|x_k - Px_k\| \\ &\leq \|Px_k - w\| \|x_1 - Px_1\| \end{aligned}$$

for every $k \in \mathbb{N}$. Summing these inequalities from $k = 1$ to n_i and dividing by n_i, we have

$$\left\langle z - w, z_{n_i} - \frac{1}{n_i} \sum_{k=1}^{n_i} Px_k \right\rangle \leq \frac{1}{n_i} \sum_{k=1}^{n_i} \|Px_k - w\| \|x_1 - Px_1\|.$$

Since $z_{n_i} \rightharpoonup z$ as $i \longrightarrow \infty$ and $Px_n \longrightarrow w$ as $n \longrightarrow \infty$, we obtain $\langle z - w, z - w \rangle \leq 0$ and hence $z = w$. This completes the proof. □

Using Theorem 3.1 and Lemma 4.1, we obtain the following:

Theorem 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{\lambda_n\}$ a sequence of real numbers such that $\lambda_n \longrightarrow \lambda$, and $T_n \colon C \longrightarrow C$ a λ_n-hybrid mapping for $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = T_n x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent, T denotes the pointwise limit of $\{T_n\}$, and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$, where P is the metric projection of H onto $F(T)$.

Proof. Since T_n is λ_n-hybrid and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, each T_n is quasi-nonexpansive. Thus it follows from Lemma 4.1 that $\{x_n\}$ is bounded. Hence Theorem 3.1 shows that every weak cluster point of $\{z_n\}$ belongs to $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, Lemma 4.1 implies the conclusion. □

The following corollary is a direct consequence of Theorem 4.1.

Corollary 4.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H, λ a real number, $T \colon C \longrightarrow C$ a λ-hybrid mapping with a fixed point, and $\{\alpha_n\}$ a sequence in $[0, 1)$ such that $\alpha_n \longrightarrow 0$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \text{ and } z_n = \frac{1}{n} \sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$, where P is the metric projection of H onto $F(T)$.

Proof. Put $T_n = \alpha_n I + (1-\alpha_n)T$ for $n \in \mathbb{N}$. Then Lemma 2.2 shows that each $T_n : C \longrightarrow C$ is $(\lambda-\alpha_n)/(1-\alpha_n)$-hybrid. It is clear that $(\lambda-\alpha_n)/(1-\alpha_n) \longrightarrow \lambda$ and T is the pointwise limit of $\{T_n\}$. It is also clear that $F(T_n) = F(T)$ for every $n \in \mathbb{N}$ and hence $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore, Theorem 4.1 implies the conclusion. □

Using Corollary 4.2, we immediately obtain the following weak convergence theorem for a strictly pseudononspreading mapping, which is a generalization of [8, Theorem 3.1].

Corollary 4.3. *(Osilike and Isiogugu* [9, Theorem 3.1]*) Let H, C, $\{\alpha_n\}$, and P be the same as in Corollary 4.2. Let κ and β be real numbers with $0 \leq \kappa \leq \beta < 1$ and $T : C \longrightarrow C$ a κ-strictly pseudononspreading mapping with a fixed point. Let $\{x_n\}$ and $\{z_n\}$ be sequences in C defined by $x_1 \in C$,*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\big(\beta x_n + (1-\beta)Tx_n\big), \text{ and } z_n = \frac{1}{n}\sum_{k=1}^{n} x_k$$

for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to the strong limit of $\{Px_n\}$.

Proof. Set $T_\beta = \beta I + (1-\beta)T$. Then it follows from Lemma 2.5 that $-\beta/(1-\beta)$-hybrid. Obviously, $F(T) = F(T_\beta)$. Thus Corollary 4.2 implies the conclusion. □

Assuming that $\alpha_n = 0$ for each $n \in \mathbb{N}$ in Corollary 4.2, we obtain the following:

Corollary 4.4. *(*[2, Theorem 5.2]*) Let H, C, λ, T, and P be the same as in Corollary 4.2. Let x be a point in C and $\{z_n\}$ a sequence in C defined by*

$$z_n = \frac{1}{n}\sum_{k=1}^{n} T^{k-1}x$$

for $n \in \mathbb{N}$, where T^0 is the identity mapping on C. Then $\{z_n\}$ converges weakly to the strong limit of $\{PT^n x\}$.

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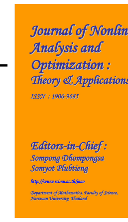

THE ALTERNATELY FIBONACCI COMPLEMENTARY DUALITY IN QUADRATIC OPTIMIZATION PROBLEM

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ABSTRACT. In this paper, we consider a pair of primal and dual quadratic optimization problems, and we compare optimal values and optimal points of both problems. The optimal values and optimal points of both problems have a triple Fibonacci property as follows. (i) The value of maximum and minimum are the same (duality). (ii) The maximum point and the minimum point are two-step alternate Fibonacci sequences (2-step alternately Fibonacci). (iii) Both the optimal points constitute alternately two consecutive positive numbers and two consecutive negative numbers of Fibonacci sequence (alternately Fibonacci complement). This triplet is called the *alternately Fibonacci complementary duality*. Moreover, we show a two-step alternate DA VINCI Code by using optimal points of their quadratic optimization problems, and we propose a method the *alternately Fibonacci section* to find optimal points for their problems.

KEYWORDS : Quadratic optimization problem; Fibonacci sequence; Alternately Fibonacci complementary duality.

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1. INTRODUCTION

A recent move “The DA VINCI Code" (2006) shows the following finite sequence:

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 1 \quad 3 \quad 2 \quad 1. \tag{1.1}$$

The code utilizes the Fibonacci sequence as a mysterious code [5], which is the first eight numbers

$$F_1,\ F_2,\ F_3,\ F_4,\ F_5,\ F_6,\ F_7,\ F_8$$

in the Fibonacci sequence (Table 1).

Definition 1.1. The *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation,

$$F_{n+2} - F_{n+1} - F_n = 0, \quad F_1 = 1,\ F_0 = 0. \tag{1.2}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

Table 1 Fibonacci sequence $\{F_n\}$

The many relationships between optimization theory and the code were studied in [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Iwamoto, Kira and Ueno [14] proposed the *Fibonacci complementary duality* for two pairs of primal and dual optimization problems. The Fibonacci complementary duality proved that there are some beautiful relations during optimal points of both optimization problems, and whose optimal values and optimal points are characterized by the Fibonacci sequence. Our results in this paper become paired their results in [14].

This paper considers a pair of primal and dual quadratic optimization problems, and we compare optimal values and optimal points of both problems. We show that both optimal solutions are characterized by an alternate Fibonacci sequence in the following sense. (i) The value of maximum and minimum are the same (duality). (ii) The maximum point and the minimum point are two-step alternate Fibonacci sequences (2-step alternately Fibonacci). “*Alternate*" means that a positive number and a negative number appear alternately for a sequence. (iii) Both the optimal points constitute alternately two consecutive positive numbers and two consecutive negative numbers of Fibonacci sequence (alternately Fibonacci complement). This triplet is called the *alternately Fibonacci complementary duality*. Moreover, this paper considers a pair of primal and dual optimizaiton problems of four variables together with their respective reversed problems. We show a two-step alternate DA VINCI Code by using optimal points of their problems, and we propose a method the *alternately Fibonacci section* to find optimal points for the problems.

This paper is organized as follows. In section 2, we establish the alternately Fibonacci complementary duality about two pairs of primal and dual quadratic optimization problems. Section 3 proves the two-step alternate DA VINCI Code by using optimal points of four variables for the problems. In section 4, we propose the alternately Fibonacci section to find optimal points for quadratic optimization problems.

2. THE ALTERNATELY FIBONACCI COMPLEMENTARY DUALITY

In this section, we consider a pair of primal and dual quadratic optimization problems

$$(\mathrm{P}_n)\qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=0}^{n-1}\left[(x_k+x_{k+1})^2+x_{k+1}^2\right] \\ \text{subject to} & \text{(i) } x\in R^n \\ & \text{(ii) } x_0=c, \end{array}$$

where $c \in R$, $x = (x_1, x_2, \ldots, x_n)$, and

$$(\mathrm{D}_n)\qquad \begin{array}{ll} \text{Maximize} & 2c\mu_0-\mu_0^2-\displaystyle\sum_{k=0}^{n-2}\left[(\mu_k+\mu_{k+1})^2+\mu_{k+1}^2\right]-\mu_{n-1}^2 \\ \text{subject to} & \text{(i) } \mu\in R^n \end{array}$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$.

Theorem 2.1. *For the problems* (P_n) *and* (D_n)*, let* $x = (x_0, x_1, \ldots, x_n)$ *be feasible of the primal problem* (P_n) *and* $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ *be feasible of the dual problem* (D_n)*. Then,* $\min(\mathrm{P}_n) \geq \max(\mathrm{D}_n)$.

Proof Let $x = (x_0, x_1, \ldots, x_n)$ be feasible of the primal problem (P_n), and $I(x)$ be the evaluated value

$$I(x) := \sum_{k=0}^{n-1}\left[(x_k+x_{k+1})^2+x_{k+1}^2\right]. \tag{2.1}$$

Let

$$u_k = x_k + x_{k+1} \qquad 0 \leq k \leq n-1. \tag{2.2}$$

Then we have for any $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1}) \in R^n$,

$$I(x) = \sum_{k=0}^{n-1}\left[u_k^2 + x_{k+1}^2 - 2\mu_k\left(x_{k+1}+x_k-u_k\right)\right]. \tag{2.3}$$

Since

$$\begin{aligned} I(x) &= 2x_0\mu_0 + \sum_{k=1}^{n-1}[x_k + 2(\mu_{k-1}+\mu_k)x_k] + x_n^2 + 2\mu_{n-1}x_n \\ &+ \sum_{k=0}^{n-1}\left(u_k^2 - 2\mu_k u_k\right), \end{aligned}$$

we have

$$\begin{aligned} I(x) &= 2x_0\mu_0 - \mu_0^2 - \sum_{k=0}^{n-2}\left[(\mu_k+\mu_{k+1})^2 + \mu_{k+1}^2\right] - \mu_{n-1}^2 \\ &\quad + \sum_{k=1}^{n-1}(x_k+\mu_{k-1}+\mu_k)^2 + (x_n+\mu_{n-1})^2 + \sum_{k=0}^{n-1}(u_k-\mu_k)^2 \\ &\geq 2x_0\mu_0 - \mu_0^2 - \sum_{k=0}^{n-2}\left[(\mu_k+\mu_{k+1})^2 + \mu_{k+1}^2\right] - \mu_{n-1}^2 \end{aligned}$$

for any $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$, $u = (u_0, \ldots, u_{n-1}) \in R^n$ satisfying (2.2) and any $\mu = (\mu_0, \ldots, \mu_{n-1}) \in R^n$. Let us take

$$J(\mu) := 2c\mu_0 - \mu_0^2 - \sum_{k=0}^{n-2}\left[(\mu_k+\mu_{k+1})^2 + \mu_{k+1}^2\right] - \mu_{n-1}^2. \qquad (2.4)$$

Then we have an inequality

$$I(x) \geq J(\mu) \qquad (2.5)$$

for any feasible $x = (x_0, x_1, \ldots, x_n) \in R^{n+1}$ of the primal problem (P_n) and any feasible $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1}) \in R^n$ of the dual problem (D_n).

Lemma 2.2. *(Lucas formula) Let $\{F_k\}$ be the Fibonacci sequence. For any $n \geq 1$, we have*

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}. \qquad (2.6)$$

Theorem 2.3. *The primal problem (P_n) has the minimum value* $m = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at the point*

$$\begin{aligned} \hat{x} &= (x_0,\ \hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &= \frac{c}{F_{2n+1}}\Big(F_{2n+1}, -F_{2n-1}, F_{2n-3}, \ldots, \\ &\qquad \ldots, (-1)^k F_{2n-2k+1}, \ldots, (-1)^{n-1}F_3, (-1)^n F_1\Big). \end{aligned}$$

Proof Since the objective function $I(x)$ (see also (2.1)) for (P_n) is convex and differentiable for any $x = (x_1, x_2, \ldots, x_n)$, the minimum point for (P_n) satisfies the first order optimality condition,

$$\frac{\partial I}{\partial x_k} \;=\; 0 \qquad 1 \le k \le n. \tag{2.7}$$

These conditions (2.7) are equivalent to the following n equations:

$$(\mathrm{AF})_{\mathrm{P}} \qquad (-1)^k \frac{x_k + x_{k+1}}{F_{2n-2k}} \;=\; (-1)^{k+1} \frac{x_{k+1}}{F_{2n-2k-1}} \qquad 0 \le k \le n-1. \tag{2.8}$$

The condition $(\mathrm{AF})_{\mathrm{P}}$ is called the *alternately Fibonacci condition* for (P_n). From the condition $(\mathrm{AF})_{\mathrm{P}}$,

$$x_{k+1} \;=\; -\frac{F_{2n-2k-1}}{F_{2n-2k+1}} x_k \qquad 0 \le k \le n-1. \tag{2.9}$$

Thus we have

$$\begin{aligned} \hat{x} \;&=\; (x_0,\ \hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &=\; \frac{c}{F_{2n+1}} \Big(F_{2n+1}\,, -F_{2n-1}\,, F_{2n-3}\,, \ldots\,, \\ &\qquad ,\ldots\,, (-1)^k F_{2n-2k+1}\,, \ldots\,, (-1)^{n-1} F_3\,, (-1)^n F_1 \Big). \end{aligned} \tag{2.10}$$

Next, we prove the minimum value $m = \dfrac{F_{2n}}{F_{2n+1}} c^2$. From (2.1) and (2.10),

$$\begin{aligned} F_{2n+1}^2 \frac{I(\hat{x})}{c^2} \;&=\; \left[(F_{2n+1} - F_{2n-1})^2 + (-F_{2n-1})^2 \right] \\ &+\; \left[(-F_{2n-1} + F_{2n-3})^2 + F_{2n-3}^2 \right] + \\ &\qquad \cdots + \left[\left\{ (-1)^{n-1} F_3 + (-1)^n F_1 \right\}^2 + \left\{ (-1)^n F_1 \right\}^2 \right] \\ &=\; \sum_{k=0}^{n-1} \left(F_{2n-2k}^2 + F_{2n-2k-1}^2 \right) \\ &=\; \sum_{k=1}^{2n} F_k^2 \\ &=\; F_{2n} F_{2n+1} \quad \text{(by Lucas formula)}. \end{aligned}$$

Consequently, we get

$$m \;=\; I(\hat{x}) \;=\; \frac{F_{2n}}{F_{2n+1}} c^2. \tag{2.11}$$

Theorem 2.4. *The dual problem* (D_n) *has the maximum value* $M = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at the point*

$$\begin{aligned}\mu^* &= (\mu_0^*,\ \mu_1^*,\ \mu_2^*,\ \mu_3^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*)\\ &= \frac{c}{F_{2n+1}}\Big(F_{2n}\,, -F_{2n-2}\,, F_{2n-4}\,, \ldots\,,\\ &\qquad\qquad , \ldots\,, (-1)^k F_{2n-2k}\,, \ldots\,, (-1)^{n-1}F_2\Big).\end{aligned}$$

Proof Since the objective function $J(\mu)$ (see also (2.4)) for (D_n) is concave and differentiable for any $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$, the maximum point for (D_n) satisfies the first order optimality condition,

$$\frac{\partial J}{\partial \mu_k} = 0 \qquad 0 \le k \le n-1. \tag{2.12}$$

These conditions (2.12) are equivalent to the following n equations:

$$(\mathrm{AF})_{\mathrm{D}} \quad \frac{c-\mu_0}{F_{2n-1}} = \frac{\mu_0+\mu_1}{F_{2n-1}}, \quad (-1)^k\frac{\mu_k+\mu_{k+1}}{F_{2n-2k-1}} = (-1)^{k+1}\frac{\mu_{k+1}}{F_{2n-2k-2}} \tag{2.13}$$

for $0 \le k \le n-2$. The condition $(\mathrm{AF})_{\mathrm{D}}$ is called the *alternately Fibonacci condition* for (D_n). From the condition $(\mathrm{AF})_{\mathrm{D}}$,

$$\mu_0 = \frac{F_{2n-2}+F_{2n-1}}{F_{2n-2}+F_{2n-1}+F_{2n-1}}c, \quad \mu_{k+1} = -\frac{F_{2n-2k-2}}{F_{2n-2k}}\mu_k \qquad 0 \le k \le n-2. \tag{2.14}$$

Thus we have

$$\begin{aligned}\mu^* &= (\mu_0^*,\ \mu_1^*,\ \mu_2^*,\ \mu_3^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*)\\ &= \frac{c}{F_{2n+1}}\Big(F_{2n}\,, -F_{2n-2}\,, F_{2n-4}\,, \ldots\,,\\ &\qquad\qquad , \ldots\,, (-1)^k F_{2n-2k}\,, \ldots\,, (-1)^{n-1}F_2\Big).\end{aligned} \tag{2.15}$$

Next, we prove the maximum value $M = \dfrac{F_{2n}}{F_{2n+1}}c^2$. From (2.4) and (2.15),

$$\begin{aligned}F_{2n+1}^2\frac{J(\mu^*)}{c^2} &= 2F_{2n}F_{2n+1} - F_{2n}^2 - \big[(F_{2n}-F_{2n-2})^2 + (-F_{2n-2})^2\big] -\\ &\qquad \cdots - \big[(-F_{2n-2}+F_{2n-4})^2 + F_{2n-4}^2\big]\\ &\qquad - \big[(F_4-F_2)^2 + (-F_2)^2\big] - (-F_2)^2\\ &= 2F_{2n}F_{2n+1} - \sum_{k=1}^{2n} F_k^2\\ &= 2F_{2n}F_{2n+1} - F_{2n}F_{2n+1} \quad \text{(by Lucas formula)}\\ &= F_{2n}F_{2n+1}.\end{aligned}$$

Consequently, we get

$$M \;=\; J(\mu^*) \;=\; \frac{F_{2n}}{F_{2n+1}}c^2. \tag{2.16}$$

There are the following triplet relations between the minimum point $\hat{x}$ of the primal problem (P_n) and the maximum point μ^* of the dual problem (D_n).

(i) (duality) The value of maximum and minimum are the same:

$$m = M \;=\; \frac{F_{2n}}{F_{2n+1}}\,c^2.$$

It is a quadratic function of c, whose coefficient is ratio of adjacent Fibonacci number. This is the first alternately Fibonacci complementary duality.

(ii) (2-step alternately Fibonacci) Both the minimum point $\hat{x}$ and the maximum point μ^* are two-step alternate Fibonacci sequence:

$$\begin{aligned}\hat{x} \;&=\; (x_0,\ \hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n)\\ &=\; \frac{c}{F_{2n+1}}\Big(\,F_{2n+1}\,,-F_{2n-1}\,,F_{2n-3}\,,\ldots\,,\ldots\,,(-1)^kF_{2n-2k+1}\,,\ldots\,,(-1)^nF_1\,\Big)\end{aligned}$$

and

$$\begin{aligned}\mu^* \;&=\; (\mu_0^*,\ \mu_1^*,\ \mu_2^*,\ \mu_3^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*)\\ &=\; \frac{c}{F_{2n+1}}\Big(\,F_{2n}\,,-F_{2n-2}\,,F_{2n-4}\,,\ldots\,,(-1)^kF_{2n-2k}\,,\ldots\,,(-1)^{n-1}F_2\,\Big).\end{aligned}$$

This is the second.

(iii) (alternately Fibonacci complement) Both the optimal points constitute alternately the (1-step) alternate two-run Fibonacci sequence:

$$\begin{aligned}&(x_0\,,\mu_0^*\,,\hat{x}_1\,,\mu_1^*\,,\ldots\,,\hat{x}_k\,,\mu_k^*\,,\ldots\,,\mu_{n-1}^*\,,\hat{x}_n)\\ =\;&\frac{c}{F_{2n+1}}\Big(\,F_{2n+1}\,,F_{2n}\,,-F_{2n-1}\,,-F_{2n-2}\,,\ldots\,,(-1)^kF_{2n-2k+1}\,,(-1)^kF_{2n-2k}\,,\\ &\qquad\qquad \ldots\,,(-1)^{n-1}F_2\,,(-1)^nF_1\,\Big).\end{aligned}$$

This is the third.

This triplet is called the *alternately Fibonacci complementary duality.*

Theorem 2.5. *If* (P_n) *has an optimal solution, then there is a feasible solution of* (D_n) *and the two objectives have the same values. Moreover,*

$$\begin{aligned}\hat{x} &= (x_0,\ \hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &= \frac{c}{F_{2n+1}}\Big(F_{2n+1}\,, -F_{2n-1}\,, F_{2n-3}\,, \ldots\,, \ldots\,, (-1)^k F_{2n-2k+1}\,, \ldots\,, (-1)^n F_1 \Big)\end{aligned}$$

is the optimal solution of (P_n)*, and*

$$\begin{aligned}\mu^* &= (\mu^*_0,\ \mu^*_1,\ \mu^*_2,\ \mu^*_3,\ \ldots,\ \mu^*_k,\ \ldots,\ \mu^*_{n-1}) \\ &= \frac{c}{F_{2n+1}}\Big(F_{2n}\,, -F_{2n-2}\,, F_{2n-4}\,, \ldots\,, (-1)^k F_{2n-2k}\,, \ldots\,, (-1)^{n-1} F_2 \Big)\end{aligned}$$

is the optimal solution of (D_n)*. Hence, the alternately Fibonacci complementary duality holds between* (P_n) *and* (D_n)*.*

Proof It is obvious to prove this theorem from the proofs of theorem 2.1, 2.3, and 2.4.

3. THE ALTERNATE DA VINCI CODE

We introduced the DA VINCI Code (see also (1.1)) in the introduction, and we showed that the code utilizes the Fibonacci sequence. In this section, we consider a pair of primal problem (P_4) and its dual (D_4) together with their respective reversed problems. We prove the two-step alternate DA VINCI Code by using optimal points of their problems. Two-step alternate DA VINCI Code is defined as the following sequence:

$$1 \quad -1 \quad -2 \quad 3 \quad 5 \quad -8 \quad -13 \quad 21. \tag{3.1}$$

Let us now consider the following primal quadratic optimization problem (P_4):

$$\begin{array}{lll} & \text{minimize} & \displaystyle\sum_{k=0}^{3}\left[(x_k + x_{k+1})^2 + x_{k+1}^2\right] \\ (\mathrm{P}_4) & \text{subject to} & \text{(i)}\ -\infty < x_k < \infty \qquad k = 1,2,3,4 \\ & & \text{(ii)}\ x_0 = c. \end{array}$$

By theorem 2.3, the primal problem (P_4) has the minimum value

$$m_4 = \frac{F_8}{F_9}c^2 \tag{3.2}$$

at the point

$$\hat{x} = (x_0\,, \hat{x}_1\,, \hat{x}_2\,, \hat{x}_3\,, \hat{x}_4) = \frac{c}{F_9}\,(F_9\,, -F_7\,,\ F_5\,, -F_3\,,\ F_1)\,. \tag{3.3}$$

From theorem 2.1, a dual problem (D_4) for the primal problem (P_4) is given by

$$(\mathrm{D}_4) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_0 - \mu_0^2 - \sum_{k=0}^{2} \left[(\mu_k + \mu_{k+1})^2 + \mu_{k+1}^2\right] - \mu_3^2 \\ \text{subject to} & \text{(i)} \ -\infty < \mu_k < \infty \quad k = 0, 1, 2, 3. \end{array}$$

By theorem 2.4, the dual problem (D_4) has the maximum value

$$M_4 = \frac{F_8}{F_9} c^2 \tag{3.4}$$

at the point

$$\mu^* = (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*) = \frac{c}{F_9} (F_8, -F_6, F_4, -F_2). \tag{3.5}$$

Let us now appreciate a triplet alternately Fibonacci complementary duality about both the optimal solutions for the primal problem (P_4) and the dual problem (D_4).

(i) (duality) The value of maximum and minimum are the same:

$$m_4 = M_4 = \frac{F_8}{F_9} c^2. \tag{3.6}$$

This is the first alternately Fibonacci complementary duality.

(ii) (2-step alternately Fibonacci) Both the minimum point

$$\hat{x} = \frac{c}{F_9} (F_9, -F_7, F_5, -F_3, F_1) \tag{3.7}$$

and the maximum point

$$\mu^* = \frac{c}{F_9} (F_8, -F_6, F_4, -F_2) \tag{3.8}$$

are 2-step alternate Fibonacci sequences, as was shown. This is the second.

(iii) (alternately Fibonacci complement) Both the optimal points constitute alternately the (1-step) alternate two-run Fibonacci sequence:

$$\begin{aligned} &(x_0, \ \mu_0^*, \ \hat{x}_1, \ \mu_1^*, \ \hat{x}_2, \ \mu_2^*, \ \hat{x}_3, \ \mu_3^*, \ \hat{x}_4) \\ = \ & \frac{c}{F_9}\Big(F_9, \ F_8, \ -F_7, \ -F_6, \ F_5, \ F_4, \ -F_3, \ -F_2, \ F_1\Big). \end{aligned} \tag{3.9}$$

This is the third.

Hence, the alternately Fibonacci complementary duality holds between the primal problem (P_4) and the dual problem (D_4).

Next we consider a reversed problem (RP_4) for the problem (P_4) as follows:

$$(\mathrm{RP}_4) \qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{4} \left[x_k^2 + (x_k + x_{k+1})^2\right] \\ \text{subject to} & \text{(i)} \ -\infty < x_k < \infty \qquad k = 1,2,3,4 \\ & \text{(ii)} \ x_5 = c, \end{array}$$

that is, a variable $(x_0, x_1, x_2, x_3, x_4)$ for (P_4) was replaced by a variable $(x_5, x_4, x_3, x_2, x_1)$. Moreover, its dual problem is the following problem:

$$(\mathrm{RD}_4) \qquad \begin{array}{ll} \text{Maximize} & -\mu_1^2 - \displaystyle\sum_{k=1}^{3} \left[\mu_k^2 + (\mu_k + \mu_{k+1})^2\right] - \mu_4^2 + 2c\mu_4 \\ \text{subject to} & \text{(i)} \ -\infty < \mu_k < \infty \qquad k = 1,2,3,4. \end{array}$$

From (3.2) and (3.3), the problem (RP_4) has the minimal value

$$m_4' \;=\; \frac{21}{34}c^2 \tag{3.10}$$

at the point

$$\tilde{x} \;=\; (\tilde{x}_1\,, \tilde{x}_2\,, \tilde{x}_3\,, \tilde{x}_4\,, x_5) \;=\; \frac{c}{34}(1\,, -2\,, 5\,, -13\,, 34)\,. \tag{3.11}$$

From (3.4) and (3.5), its dual problem (RD_4) has the maximum value

$$M_4' \;=\; \frac{21}{34}c^2 \tag{3.12}$$

at the point

$$\mu^\star \;=\; (\mu_1^\star\,, \mu_2^\star\,, \mu_3^\star\,, \mu_4^\star) \;=\; \frac{c}{34}(-1\,,\ 3\,, -8\,,\ 21)\,. \tag{3.13}$$

Both the reversed optimization problems (RP_4) and (RD_4) have the alternately Fibonacci complementary duality.

(i) (duality) The value of maximum and minimum are the same:

$$m_4' = M_4' \;=\; \frac{21}{34}\,c^2.$$

(ii) (2-step alternately Fibonacci) Both the minimum point

$$\tilde{x} \;=\; \frac{c}{34}(1\,, -2\,,\ 5\,, -13\,,\ 34) \tag{3.14}$$

and the maximum point

$$\mu^\star \;=\; \frac{c}{34}(-1\,,\ 3\,, -8\,,\ 21) \tag{3.15}$$

are 2-step alternate Fibonacci sequences, as was shown.

(iii) (alternately Fibonacci complement) Both the optimal points constitute alternately the (1-step) alternate two-run Fibonacci sequence:

$$\begin{aligned} &(\tilde{x}_1\,,\ \mu_1^\star\,,\ \tilde{x}_2\,,\ \mu_2^\star\,,\ \tilde{x}_3\,,\ \mu_3^\star\,,\ \tilde{x}_4\,,\ \mu_4^\star\,,\ x_5) \\ = \ &\frac{c}{34}\Big(1\,,-1\,,-2\,,\ 3\,,\ 5\,,-8\,,-13\,,\ 21\,,\ 34\Big). \end{aligned} \tag{3.16}$$

For (3.16), we take a constant $c = 34$, then the sequence $(\tilde{x}_1\,,\ \mu_1^\star\,,\ \tilde{x}_2\,,\ \mu_2^\star\,,\ \tilde{x}_3\,,\ \mu_3^\star\,,\ \tilde{x}_4\,,\ \mu_4^\star)$ constitutes a two-step alternate DA VINCI Code.

4. THE ALTERNATELY FIBONACCI SECTION

In this section, for the problem (P_4) and (D_4), we propose the *alternately Fibonacci sections*, which are a method to find optimal points for the quadratic optimization problems. The alternately Fibonacci sections are given by the alternately Fibonacci conditions $(\mathrm{AF})_\mathrm{P}$ and $(\mathrm{AF})_\mathrm{D}$.

First, let us now propose the alternately Fibonacci section for the primal problem (P_4). From (2.8) in the proof of theorem 2.3, the alternately Fibonacci condition $(\mathrm{AF})_\mathrm{P}$ for the problem (P_4) is given by the following four equations:

$$(\mathrm{AF})_\mathrm{P} \qquad \frac{c+x_1}{F_8} = \frac{x_1}{-F_7},\ \ \frac{x_1+x_2}{-F_6} = \frac{x_2}{F_5},\ \ \frac{x_2+x_3}{F_4} = \frac{x_3}{-F_3},\ \ \frac{x_3+x_4}{-F_2} = \frac{x_4}{F_1}. \tag{4.1}$$

The condition $(\mathrm{AF})_\mathrm{P}$ means that the following eight quantities:

$$c+x_1, \quad x_1, \quad x_1+x_2, \quad x_2, \quad x_2+x_3, \quad x_3, \quad x_3+x_4, \quad x_4 \tag{4.2}$$

are allocated on the basis of the alternate two-run Fibonacci sequence $F_8 : -F_7 : -F_6 : F_5 : F_4 : -F_3 : -F_2 : F_1$. We take an interval $[0, c]$, where $c > 0$. The first equation of $(\mathrm{AF})_\mathrm{P}$ means that an optimal point $-\hat{x}_1$ is an internally dividing point of the interval $[0, c]$ depending on a ratio $F_7 : F_8$. That is,

$$-\hat{x}_1 \ = \ \frac{F_7}{F_7+F_8}c \ = \ \frac{F_7}{F_9}c.$$

Next the second equation of $(\mathrm{AF})_\mathrm{P}$ means that an optimal point $\hat{x}_2$ is an internally dividing point of the interval $[0, -\hat{x}_1]$ depending on a ratio $F_5 : F_6$. That is,

$$\hat{x}_2 \ = \ \frac{F_5}{F_5+F_6}\,(-\hat{x}_1) \ = \ \frac{F_5}{F_7}\,(-\hat{x}_1)\,.$$

The third equation of $(\mathrm{AF})_{\mathrm{P}}$ means that an optimal point $-\hat{x}_3$ is an internally dividing point of the interval $[0, \hat{x}_2]$ depending on a ratio $F_3 : F_4$. That is,

$$-\hat{x}_3 \;=\; \frac{F_3}{F_3 + F_4}\hat{x}_2 \;=\; \frac{F_3}{F_5}\hat{x}_2.$$

At the last, the fourth equation of $(\mathrm{AF})_{\mathrm{P}}$ means that an optimal point $\hat{x}_4$ is an internally dividing point of the interval $[0, -\hat{x}_3]$ depending on a ratio $F_1 : F_2$. That is,

$$\hat{x}_4 \;=\; \frac{F_1}{F_1 + F_2}\,(-\hat{x}_3) \;=\; \frac{F_1}{F_3}\,(-\hat{x}_3)\,.$$

Thus we can find the optimal point $\hat{x} = (x_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)$ as follows,

$$\begin{aligned}
\hat{x}_1 &= -\frac{F_7}{F_9}c,\\
\hat{x}_2 &= \frac{F_5}{F_7}\cdot\frac{F_7}{F_9}c \;=\; \frac{F_5}{F_9}c,\\
\hat{x}_3 &= -\frac{F_3}{F_5}\cdot\frac{F_5}{F_7}\cdot\frac{F_7}{F_9}c \;=\; -\frac{F_3}{F_9}c,\\
\hat{x}_4 &= \frac{F_1}{F_3}\cdot\frac{F_3}{F_5}\cdot\frac{F_5}{F_7}\cdot\frac{F_7}{F_9}c, \;=\; \frac{F_1}{F_9}c.
\end{aligned}$$

Consequently, we get

$$\hat{x} \;=\; (x_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) \;=\; \frac{c}{F_9}\,(F_9\,, -F_7\,, F_5\,, -F_3\,, F_1)\,.$$

Let us now propose the alternately Fibonacci section for the dual problem (D_4). From (2.13) in the proof of theorem 2.4, the alternately Fibonacci condition $(\mathrm{AF})_{\mathrm{D}}$ for the problem (D_4) is given by the following four equations:

$$(\mathrm{AF})_{\mathrm{D}} \qquad \frac{c-\mu_0}{F_7} = \frac{\mu_0+\mu_1}{F_7} = \frac{\mu_1}{-F_6}, \quad \frac{\mu_1+\mu_2}{-F_5} = \frac{\mu_2}{F_4}, \quad \frac{\mu_2+\mu_3}{F_3} = \frac{\mu_3}{-F_2}. \tag{4.3}$$

The condition $(\mathrm{AF})_{\mathrm{D}}$ means that the following seven quantities:

$$\mu_0, \qquad \mu_0+\mu_1, \qquad \mu_1, \qquad \mu_1+\mu_2, \qquad \mu_2, \qquad \mu_2+\mu_3, \qquad \mu_3. \tag{4.4}$$

are allocated on the basis of the alternate two-run Fibonacci sequence $F_8 : F_7 : -F_6 : -F_5 : F_4 : F_3 : -F_2$. We take an interval $[0, c]$, where $c > 0$. The first equation of $(\mathrm{AF})_{\mathrm{D}}$ means that an optimal point μ_0^* is an internally dividing point of the interval $[0, c]$ depending on a ratio $F_8 : F_7$. That is,

$$\mu_0^* \;=\; \frac{F_8}{F_8 + F_7}c \;=\; \frac{F_8}{F_9}c.$$

Next the second equation of $(\mathrm{AF})_{\mathrm{D}}$ means that an optimal point $-\mu_1^*$ is an internally dividing point of the interval $[0, \mu_0^*]$ depending on a ratio $F_6 : F_7$. That is,

$$-\mu_1^* \;=\; \frac{F_6}{F_6 + F_7}\mu_0^* \;=\; \frac{F_6}{F_8}\mu_0^*.$$

The third equation of $(\mathrm{AF})_{\mathrm{D}}$ means that an optimal point μ_2^* is an internally dividing point of the interval $[0, -\mu_1^*]$ depending on a ratio $F_4 : F_5$. That is,

$$\mu_2^* \;=\; \frac{F_4}{F_4 + F_5}\left(-\mu_1^*\right) \;=\; \frac{F_4}{F_6}\left(-\mu_1^*\right).$$

At the last, the fourth equation of $(\mathrm{AF})_{\mathrm{D}}$ means that an optimal point $-\mu_3^*$ is an internally dividing point of the interval $[0, \mu_2^*]$ depending on a ratio $F_2 : F_3$. That is,

$$-\mu_3^* \;=\; \frac{F_2}{F_2 + F_3}\mu_2^* \;=\; \frac{F_2}{F_4}\mu_2^*.$$

Thus we can find the optimal point $\mu^* = (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*)$ as follows,

$$\begin{aligned}
\mu_0^* &= \frac{F_8}{F_9}c, \\
\mu_1^* &= -\frac{F_6}{F_8}\cdot\frac{F_8}{F_9}c \;=\; -\frac{F_6}{F_9}c, \\
\mu_2^* &= \frac{F_4}{F_6}\cdot\frac{F_6}{F_8}\cdot\frac{F_8}{F_9}c \;=\; \frac{F_4}{F_9}c, \\
\mu_3^* &= -\frac{F_2}{F_4}\cdot\frac{F_4}{F_6}\cdot\frac{F_6}{F_8}\cdot\frac{F_8}{F_9}c \;=\; -\frac{F_2}{F_9}c.
\end{aligned}$$

Consequently, we get

$$\mu^* \;=\; (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*) \;=\; \frac{c}{F_9}\left(F_8\,, -F_6\,, F_4\,, -F_2\right).$$

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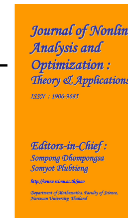

THE HAHN-BANACH THEOREM AND THE SEPARATION THEOREM IN A PARTIALLY ORDERED VECTOR SPACE

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ABSTRACT. In this paper, using the Bourbaki-Kneser fixed point theorem, we give a new proof of the Hahn-Banach theorem in a partially ordered vector space and the separation theorem in the Cartesian product of a vector space and a partially ordered vector space.

KEYWORDS : Fixed point theorem; Hahn-Banach theorem; Separation theorem; Partially ordered vector space.

1. INTRODUCTION

The Hahn-Banach theorem is one of the most fundamental theorems in the functional analysis theory and the separation theorem is one of the most fundamental theorems in the optimization theory. These theorems are known well in the case where the range space is the real number system. The following is the Hahn-Banach theorem: *Let p be a sublinear mapping from a vector space X to the real number system R, Y a vector subspace of X and q a linear mapping from Y to R such that $q \leq p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X to R such that $g \leq p$.*

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Moreover, the following is the separation theorem:

Let X be a normed space, X^* a its dual space, A, B subsets of X such that A is closed convex and B is compact convex subset with $A \cap B = \emptyset$, then there exists an $f \in X^* \setminus \{0\}$ such that $\inf\{f(y) \mid y \in B\} \geq \sup\{f(x) \mid x \in A\}$.

It is known that the Hahn-Banach theorem establishes in the case where the range space is a Dedekind complete Riesz space; see [4, 17, 19] and the separation theorem establishes in the Cartesian product space of a vector space and a Dedekind complete ordered vector space; see [6, 7, 15, 16].

The Hahn-Banach theorem is proved often using the Zorn lemma. For the proof of the Hahn-Banach theorem, there exist several approaches. For instance, Hirano, Komiya, and Takahashi [8] showed the Hahn-Banach theorem by using the Markov-Kakutani fixed point theorem [10] in the case where the range space is the real number system.

In this paper, in Section 3, using the Bourbaki-Kneser fixed point theorem, we give a new proof of the Hahn-Banach theorem and the Mazur-Orlicz theorem in the case where the range space is a Dedekind complete partially ordered vector space (Theorem 3.1 and Theorem 3.2). In Section 4, we give a new proof of the separation theorem in the Cartesian product of a vector space and a Dedekind complete partially ordered vector space (Theorem 4.1); see [6, 7, 15, 16]. The Bourbaki-Kneser fixed point theorem is proved without using the Zorn lemma; see [11]. Therefore the theorems above are proved without using the Zorn lemma.

2. PRELIMINARIES

Let R be the set of real numbers, N the set of natural numbers, I an indexed set, $(E, \leq)$ a partially ordered set and F a subset of E. The set F is called a *chain* if any two elements are comparable, that is, $x \leq y$ or $y \leq x$ for any x, $y \in F$. An element $x \in E$ is called a *lower bound* of F if $x \leq y$ for any $y \in F$. An element $x \in E$ is called the *minimum* of F if x is a lower bound of F and $x \in F$. If there exists a lower bound of F, then F is said to be *bounded from below*. An element $x \in E$ is called an *upper bound* of F if $y \leq x$ for any $y \in F$. An element $x \in E$ is called the *maximum* of F if x is an upper bound and $x \in F$. If there exists an upper bound of F, then F is said to be *bounded from above*. If the set of all lower bounds of F has the maximum, then the maximum is called an *infimum* of F and denoted by $\inf F$. If the set of all upper bounds of F has the minimum, then the minimum is called a *supremum* of F and denoted by $\sup F$. A partially ordered set E is said to be *complete* if every nonempty chain of E has an infimum; E is said to be *chain complete* if every nonempty chain of E which is bounded from below has an infimum; E is said to be *Dedekind complete* if every nonempty subset of E which is bounded from below has an infimum. A mapping f from E to E is called *decreasing* if $f(x) \leq x$ for every $x \in E$. For the further information of a partially ordered set; see [1, 4, 5, 14, 17].

In a complete partially ordered set, the following theorem is obtained; see [3, 11, 12].

Theorem 2.1 (Bourbaki-Kneser). *Let E be a complete partially ordered set. Let f be a decreasing mapping from E to E. Then f has a fixed point.*

Recently, T. C. Lim [13] proved a common fixed point theorem for the family of decreasing commutative mapping, which is a generalization of Theorem 2.1.

A partially ordered set E is called a partially ordered vector space if E is a vector space and $x+z \leq y+z$ and $\alpha x \leq \alpha y$ hold whenever $x, y, z \in E$, $x \leq y$, and α is a nonnegative real number. If a partially ordered vector space E is a lattice, that is, any two elements have a supremum and an infimum, then E is called a *Riesz space*.

Let X be a vector space and E a partially ordered vector space. A mapping f from X to E is said to be *concave* if

$$f(tx+(1-t)y) \geq tf(x)+(1-t)f(y)$$

for any $x, y \in X$ and $t \in [0,1]$. A mapping f from X to E is called *sublinear* if the following conditions are satisfied.
(S1) For any $x, y \in X$, $p(x+y) \leq p(x)+p(y)$.
(S2) For any $x \in X$ and $\alpha \geq 0$ in R, $p(\alpha x) = \alpha p(x)$.

3. THE HAHN-BANACH THEOREM

Lemma 3.1. *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E, K a nonempty convex subset of X and q a concave mapping from K to E such that $q \leq p$ on K. For any $x \in X$, let*

$$f(x) = \inf\{p(x+ty)-tq(y) \mid t \in [0,\infty) \text{ and } y \in K\}.$$

Then f is sublinear such that $f \leq p$. Moreover if g is a linear mapping from X to E, then $g \leq f$ is equivalent to $g \leq p$ on X and $q \leq g$ on K.

Proof. For any $x \in X$,

$$\{p(x+ty)-tq(y) \mid t \in [0,\,\infty) \text{ and } y \in K\}$$

is bounded from below. Indeed, since

$$p(x+ty)-tq(y) \geq p(ty)-p(-x)-tq(y) \geq -p(-x),$$

it is established. Since E is Dedekind complete, f is well-defined and we have $f(x) \geq -p(-x)$. By definition of f, we have $f(x) \leq p(x)$ and $f(\alpha x) = \alpha f(x)$ for any $\alpha \geq 0$. Thus (S2) is established. Let $x_1, x_2 \in X$. For any $y_1, y_2 \in K$ and $s, t > 0$, we have

$$\begin{aligned} p(x_1+sy_1)-sq(y_1)+p(x_2+ty_2)-tq(y_2) \\ \geq p(x_1+x_2+(s+t)w)-(s+t)q(w) \\ \geq f(x_1+x_2), \end{aligned}$$

where $w = \frac{1}{s+t}(sy_1+ty_2) \in K$. For $p(x_1+sy_1)-sq(y_1)$, take infimum with respect to $s > 0$ and $y_1 \in K$, we have

$$f(x_1)+p(x_2+ty_2)-tq(y_2) \geq f(x_1+x_2)$$

and for $p(x_2+ty_2)-tq(y_2)$, also take infimum with respect to $t > 0$ and $y_2 \in K$, we have

$$f(x_1)+f(x_2) \geq f(x_1+x_2).$$

Thus (S1) is established. Suppose that g is a linear mapping from X to E. If $g \leq f$, then we have $g \leq p$. Moreover for any $y \in K$, since

$$-g(y) = g(-y) \leq f(-y) \leq p(-y+y)-q(y) = -q(y),$$

we have $g \geq q$ on K. To prove the converse, suppose that $g \leq p$ on X and $q \leq g$ on K. For any $x \in X$, $y \in K$ and $t > 0$, we have

$$g(x) = g(x+ty)-tg(y) \leq p(x+ty)-tq(y).$$

This implies that $g \leq f$. □

The above lemma is proved in case where the range space is a Dedekind complete Riesz space, see [17, Lemma 1.5.1].

By Theorem 2.1 and Lemma 3.1, we can give a following lemma.

Lemma 3.2. *Let f be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Then there exists a linear mapping g from X to E such that $g \leq f$.*

Proof. Let E^X be the set of mappings of X into E. Define $f \leq g$ for $f, g \in E^X$ by $f(x) \leq g(x)$ for all $x \in X$. Then $(E^X, \leq)$ is a partially ordered vector space. Put $f^*(x) = -f(-x)$ for any $x \in X$. Let

$$Y = \{h \in E^X \mid h \text{ is sublinear, } f^* \leq h \leq f\}.$$

Then Y is an ordered set. Since E is Dedekind complete, E^X is also so. Consider an arbitrary chain $Z \subset Y$. Since E^X is Dedekind complete and Z is bounded from below, there exists a $g = \inf Z$ in E^X. Then g is sublinear. Since Y is bounded from below, it holds that $g \in Y$. Thus Y is complete. Let $K = \{y\}$. Then h is also a concave mapping from K to E. We define a decreasing operator S by

$$Sh(x) = \inf\{h(x + ty) - th(y) \mid t \in [0, \infty), y \in K\}$$

for any $h \in Y$. By Lemma 3.1, Sh is sublinear and S is a mapping from Y to Y. Thus by Theorem 2.1, we have a fixed point $g \in Y$. Then for any $x \in X$, we have $g(x) \leq g(x + y) - g(y)$ and

$$g(x) + g(y) \leq g(x + y) \leq g(x) + g(y).$$

Since

$$0 = g(0) = g(-\alpha x + \alpha x) = g(-\alpha x) + \alpha g(x)$$

for any $\alpha > 0$ and $x \in X$, we have $g(-\alpha x) = -\alpha g(x)$. Thus $g(\alpha x) = \alpha g(x)$ for any $\alpha \in R$ and $x \in X$. Therefore, g is linear. □

By Lemma 3.2 and Lemma 3.1, we can prove the Hahn-Banach theorem and the Mazur-Orlicz theorem in case where the range space is a Dedekind complete partially ordered vector space.

Theorem 3.1. *Let p be a sublinear mapping from a vector space X to a Dedekind complete ordered vector space E, Y a vector subspace of X and q a linear mapping from Y to E such that $q \leq p$ on Y. Then q can be extended to a linear mapping g defined on the whole space X to E such that $g \leq p$.*

Proof. By Lemma 3.1, there exists a sublinear mapping f such that $f \leq p$. By Lemma 3.2, there exists a linear mapping g such that $g \leq f$. Then putting $K = Y$ in Lemma 3.1, we have $g \leq p$ on X and $q \leq g$ on Y. Since q is linear, for any $y \in Y$, we have

$$g(-y) \leq f(-y) \leq p(-y + y) - q(y) = -q(y) = q(-y).$$

Then we have $g \leq q$ on Y. Thus $q = g$ on Y. Therefore, the assertion holds. □

We obtain the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space.

Theorem 3.2. *Let p be a sublinear mapping from a vector space X to a Dedekind complete partially ordered vector space E. Let $\{x_j \mid j \in I\}$ be a family of elements of X and $\{y_j \mid j \in I\}$ a family of elements of E. Then the following* (1) and (2) *are equivalent.*
(1) *There exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for any $x \in X$ and $y_j \leq f(x_j)$ for any $j \in I$.*
(2)*For any $n \in N$, $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $j_1, j_2, \ldots, j_n \in I$, we have*

$$\sum_{i=1}^{n} \alpha_i y_{j_i} \leq p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right).$$

Proof. The assertion from (1) to (2) is clear. For any $x \in X$, by (2), we have

$$-p(-x) \leq p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i}.$$

Put

$$p_0(x) = \inf\left\{ p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i} \;\middle|\; n \in N, \alpha_i \geq 0 \text{ and } j_i \in I \right\}.$$

Since E is Dedekind complete, p_0 is well-defined and p_0 is sublinear. Thus by Lemma 3.2, there exists a linear mapping f from X to E such that $f(x) \leq p_0(x)$ for any $x \in X$. Since $p_0(-x_j) \leq -y_j$, we have

$$y_j \leq -p_0(-x_j) \leq f(x_j).$$

Since $p_0(x) \leq p(x)$, we have $f(x) \leq p(x)$. Thus the assertion holds. □

4. THE SEPARATION THEOREM

Let X be a vector space, E a Dedekind complete partially ordered vector space and $X \times E$ the Cartesian product of X and E. Let A be a nonempty subset of X and $L(A)$ denotes the affine manifold spanned by A. We denotes the *algebraical relative interior* of A, that is,

$$Int(A) = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{For any } x' \in L(A) \text{ there exists } \varepsilon > 0 \text{ such that} \\ x + \lambda(x' - x) \in A \text{ for any } \lambda \in [0,\, \varepsilon) \end{array} \right\}.$$

If $L(A) = X$, then we write $I(A)$ instead of $Int(A)$. Let f be a linear mapping from X to E, g a linear mapping from E to E and u_0 a point in E. Then

$$H = \{(x,\, y) \in X \times E \mid f(x) + g(y) = u_0\}$$

is empty or an affine manifold in $X \times E$. Let A, B be nonempty subsets of $X \times E$. A nonempty subset $A \subset X \times E$ is said to be *cone* (with the vertex in $x_0 \in X \times E$) if $\lambda > 0$ implies $\lambda(A - x_0) \subset (A - x_0)$. It is said that an affine manifold H separates A and B if

$$H_- \supset A \text{ and } H_+ \supset B$$

where we set

$$H_- = \{(x,\, y) \in X \times E \mid f(x) + g(y) \leq u_0\}$$

and

$$H_+ = \{(x,\, y) \in X \times E \mid f(x) + g(y) \geq u_0\}.$$

The operator P_X defined by $P_X(x, y) = x$ for any $(x, y) \in X \times E$ is called the *projection* of $X \times E$ onto X. Then P_X is a linear mapping from $X \times E$ to X. We define

$$P_X(A) = \{x \in X \mid \text{ there exists } y \in E \text{ such that } (x, y) \in A\}.$$

Then we have

$$P_X(A + B) = P_X(A) + P_X(A)$$

for $A \neq \emptyset$ and $B \neq \emptyset$. The subset

$$C(A) = \{\lambda z \in X \times E \mid \lambda \geq 0,\ z \in A\}$$

is called the *cone spanned by* A. If A is convex, then $C(A)$ is convex. By Lemma 3.2, we obtain the separation theorem in the Cartesian product of a vector space and a Dedekind complete partially ordered vector space.

Theorem 4.1. *Let A and B be subsets of $X \times E$ such that $C(A - B)$ is convex, and $P_X(A - B)$ satisfies the following* (i) *and* (ii) *:*
(i) $0 \in I(P_X(A - B))$,
(ii) *if $(x, y_1) \in A$ and $(x, y_2) \in B$, then $y_1 \geq y_2$ holds.*
Then there exists a linear mapping f from X to E and a $y_0 \in E$ such that the affine manifold

$$H = \{(x, y) \in X \times E \mid f(x) - y = y_0\}$$

separates A and B.

Proof. By assumption (i) and the definition of $I(P_X(A - B))$, for any $x \in X$ there exists $\varepsilon > 0$ and for any $\lambda \in [0, \varepsilon)$, there exists $y \in E$ such that $(\lambda x, y) \in A - B$. Then there exist $x_1, x_2 \in X$ and $y_1, y_2 \in E$ such that

$$(\lambda x, y) = (x_1 - x_2, y_1 - y_2) = (x_1, y_1) - (x_2, y_2) \in A - B.$$

Define

$$E_x = \{y \in E \mid (x, y) \in C(A - B)\}, \text{ for any } x \in X.$$

Since $\lambda^{-1}(y_1 - y_2) \in E_x$ for any $\lambda \in (0, \varepsilon)$, we have $E_x \neq \emptyset$. Let $y \in E_0$ and $y \neq 0$, then there exists $\lambda > 0$, $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$ such that

$$(0, y) = \lambda\{(x_1, y_1) - (x_2, y_2)\}$$

and $x_1 = x_2$. By assumption (ii), we have $y = \lambda(y_1 - y_2) \geq 0$. We define $E_+ = \{y \in E \mid y \geq 0\}$. Then we have $y \in E_+$. Since $C(A - B)$ is convex cone, we have $E_x + E_{x'} \subset E_{x+x'}$ for any $x, x^{'} \in X$. We prove that for every $x \in X$ the subset E_x possesses a lower bound in E. Since E_x is nonempty, for any $x \in X$, there exists $y^{'} \in E$ with $-y^{'} \in E_{-x}$. Then we have

$$y - y^{'} \in E_x + E_{-x} \subset E_0 \subset E_+$$

for any $y \in E_x$. This implies $y^{'} \leq y$ for any $y \in E_x$. Since E is Dedekind complete, operator $p(x) = \inf\{y \mid y \in E_x\}$ is well defined. Then $p(x)$ is sublinear. By Lemma 3.2, there exists a linear mapping f from X to E such that $f(x) \leq p(x)$ for all $x \in X$. Then for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$, take $x = x_1 - x_2$, we have

$$f(x_1 - x_2) \leq p(x_1 - x_2) \leq y_1 - y_2.$$

Therefore,

$$f(x_1) - y_1 \leq f(x_2) - y_2.$$

Since E is Dedekind complete, there exists a $y_0 \in E$ such that

$$f(x_1) - y_1 \leq y_0 \leq f(x_2) - y_2$$

for any $(x_1, y_1) \in A$, $(x_2, y_2) \in B$. Thus the affine manifold H separates A and B. □

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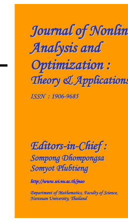

UNCONSTRAINED OPTIMIZATION IN A STOCHASTIC CELLULAR AUTOMATA SYSTEM

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ABSTRACT. This paper considers a stochastic cellular automata system which models a random dynamical system, and introduces a simple unconstrained optimization problem on such a system to capture hidden characteristics over time. To achieve this goal, we create a random metric which is applied to nearby and far-away locations of automata in order to find hidden characteristics in the automata system over time. Solving the random metric based unconstrained optimization problem, we found that solutions show high and low level fluctuations, depending on the choice of the perturbation parameter λ and the corresponding locations. The application of our method to cell concentration data reveals its consistency and adaptability. This work is an expanded version of our previous work [5].

KEYWORDS : Unconstrained optimization; Nonlinear dynamic; Time series analysis; Local autoregressive modeling; Probabilistic metric.

1. INTRODUCTION

In the last two decades, the analysis of various properties of dynamical systems has been a field of active research. Researchers have now agreed that dynamical models may be analyzed by estimating the frequency spectrum of the system. This may be done by non-parametric methods; for example, Fast Fourier Transform [10] or by fitting the autoregressive model, which is a parametric model [7]. However, in the presence of a high-dimensional system with hidden characteristics, the first

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approach is inappropriate because it does not explain the temporal events. In contrast, the second approach provides a more flexible framework for parsimonious dynamical modeling of time series data, which can be readily applied to prediction or classification. There are many dynamical phenomena in nature that may be treated as stochastic processes; for example financial markets, cell dynamics, hydrodynamics, economic, biological and psychological events. Although most of them may not be considered as stochastic at the microscopic level, they are to be considered as such at macroscopic level. By sampling the dynamical process at equally spaced intervals of time we characterize the stochastic processes and their predictive mechanisms [3, 6].

Without loss of generality, we consider a dynamical system with n particles randomly interacting over time, with on and off states respectively. Here, on and off states depend on the type of system being used: "on" may be an activated state whereas "off" may be a non-activated state. If we consider a biological process, for example, then "on" may be a high concentration and "off" a low concentration of some chemical or substance. We are interested in finding the optimal method of extracting hidden information among those particles, so as to be able to understand and predict the future behavior of the system by optimization of such information. As a first step we should be able to capture the nonlinearity in the dynamic and measurability of the random process which is involved. But, nonlinearity itself is not a property, but rather the absence of property [10], and therefore we have to be more specific as to our interpretation of nonlinearity. We adopt the notation $X_i = X_{i,t} = X_i(t)$, read as "the state X at location i at time t". Consequently we could build a class of parametric models based on the reconstructed state space (for details, see [11]), but here we prefer to explore the direct parametric model of the state dynamics by using the time series $X_i(t)$, $i = 1, \dots, N$ at various "ON" and "OFF" locations (analogously to [5]). Further, we consider as a modeling assumption that each particle in the system *de facto* interacts with "nearby" and/or "faraway" neighbors where interactions are captured by our designed metric. Furthermore, we define the state dynamic as a mixture of the dynamics at the i-th locations where the "on state of the system" is represented by X_i and those at the j-th locations represented by X_j (or the "off state of the system"). We shall explore the linear dependence among local states using a linear function f of the past states of the stochastic cellular automata system; this leads us to the stochastic equations

$$X_i = f\left(X_{i-1}, \dots, X_{i-p}\right) + \varepsilon_i \tag{1.1}$$

$$X_j = f\left(X_{j-1}, \dots, X_{j-q}\right) + \varepsilon_j \tag{1.2}$$

Above, p and q are the respective model orders and ε_i and ε_j represent the respective dynamical noises of the system. If $f(\cdot)$ is a linear function, we obtain the classic $AR(p)$ models. The correlation $\rho\left(X_i, X_j\right)$ between random states X_i and X_j is statistically given by

$$\rho\left(X_i, X_j\right) = \rho(i,j) = \begin{cases} \rho_{ij} & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}, \tag{1.3}$$

where

$$\rho(i,j) = \frac{E\left(x_i - Ex_i\right)\left(x_j - Ex_j\right)}{E\left(x_i - Ex_i\right)^2 E\left(x_j - Ex_j\right)^2} \tag{1.4}$$

and $i \cong [\Delta t]$ and $j \cong [\Delta t]$. The last approximation occurs because, as in [1, 4, 11], we may obtain a stationary model from any stationary nonlinear stochastic dynamical system model (as sampling intervals are fixed in time series models). In the next section we will explicitly define the function f which is important in designing our quantitative measure.

2. LOCAL AUTOREGRESSIVE MODELS

In order to design a quantitative measure for such a dynamical system, we assume that nonlinearity may be modeled by a nonlinear as well as a linear model with some added noise (as in [10]). Here we choose the following Local Autoregressive Models (LAMs) of orders p and q respectively (for on/off locations):

$$X_i = \sum_{k=1}^{p} a_k X_{i-k} + \varepsilon_i \qquad i - AR(p) \tag{2.1}$$

$$X_j = \sum_{l=1}^{q} b_l X_{j-l} + \varepsilon_j \qquad j - AR(q) \tag{2.2}$$

where $\varepsilon_i \sim N(0,1), \varepsilon_j \sim N(0,1)$ are Gaussian noises of the system. We define a bandwidth parameter h such that $X_i = \frac{\alpha X_{j-1}}{h}$ approaches zero for the largest occurring values of h. Observe that h is a variable parameter and α is a fixed parameter. In the next section we state three lemmas to support our method.

Lemma 2.1 (Kolmogorov-Smirnov)**.** *The distance between two independent random variables is the distance between their probability distribution functions:*

$$d\left(X_i, X_j\right) = \sup_{x_i, x_j} \left|P\left(X_i \le x_i\right) - P\left(X_j \le x_j\right)\right| \tag{2.3}$$

This lemma provides a useful way of checking equality of certain functions of independent identically distributed random variables, where x_i, x_j are values of random states X_i, X_j, and $P\left(X_i \le x_i\right)$ and $P\left(X_j \le x_j\right)$ are the respective probabilities of those values.

Lemma 2.2. *The distance between two random states X_i and X_j is equal to the p-th mean of the consecutive difference between the probable values of those random variables:*

$$d_p\left(X_i, X_j\right) = E_P\left|X_i - X_j\right|^p = \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(x_i p_i - x_j p_j\right)^p \tag{2.4}$$

This is known as the *mean difference value* in probability theory. The proofs to Lemmas 2.1 and 2.2 may be found in [8].

Lemma 2.3. *The distance between two independent and identically distributed random variables X_i and X_j, with given means and standard deviations is equal to the expected value of their consecutive differences.*

$$d_2\left(X_i, X_j\right) = E_P\left|X_i - X_j\right|^2; \qquad |i-j| \le 1 \tag{2.5}$$

The proof of Lemma 2.3 follows from our results in Sections 3 and 4.

In the case when there are some perturbations in the system (that is, there exists at least one pair (i, j) such that $\lambda_{ij} \neq 0$) and the states X_i and X_j are faraway from each other i.e. $|i-j| > 1$ we write:

$$d_2(X_i, X_j) = E_{P_0}|X_i - X_j|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}|P(\omega_i) - \lambda_{ij}P'(\omega_j)|; |i-j| > 1, i \leq N, j \leq N \tag{2.6}$$

where $P_0 = (P_i, P'_j)$ is a simple scalar product, P represents a probability at location i and P' represents a probability at location j. Observe that Lemma 2.3 is a consequence of Lemmas 2.1 and 2.2, and it will be used in the next section of the paper.

3. MEASURABILITY

3.1. **Mathematical Foundation.** After identifying the system (see [5]), we build a metric to measure the degree of interactions between the two state dynamics X_i and X_j. We develop an analogue of Kolmogorov distance over the sample distributions at locations i and j, called an α-*metric*. Technically, we assume that minimizing such a metric will provide a solution which corresponds to the hidden characteristics of the system; this is justified by the law of minimal entropy in physics. It is clear that nearby states (locations such that $|i-j| \leq 1$) will be less perturbed than faraway states (locations such that $|i-j| > 1$); thus our α-metric is

$$J(i,j,\lambda) = \begin{cases} E_{P_0}|X_i - X_j|^2; \quad |i-j| \leq 1 & \text{(i)} \\ E_{P_0}|X_i - X_j|^2 + \sum_{l=1}^{N}\lambda_{kl}|P(\omega_l) - \lambda_{kl}P'(\omega_l)|; \quad |i-j| > 1 & \text{(ii)} \end{cases} \tag{3.1}$$

where

$$\begin{aligned} E_{P_0}|X_i - X_j|^2 &= \frac{1}{N-1}\sum_{i=1}^{N}\sum_{j=1}^{N}|X_i - X_j|^2 P_0(X_i < x_i, X_j < x_j) \\ &= |x_{i1}p_1 - x_{j1}p'_1|^2 + |x_{i2}p_2 - x_{j2}p'_2|^2 + \ldots + |x_{in}p_n - x_{jn}p'_n|^2 \end{aligned}$$

and $X_i = X(\omega_i) = x_i$ with probability $P(\omega_i) = p_i \in (0,1)$, $X_j = X(\omega_j) = x_j$ with probability $P(\omega_j) = p_j \in (0,1)$ with

$$E_P X_i = \sum_{k=1}^{n} x_i p_k = \sum_{k=1}^{n} x_{ik}p_k = x_{i1}p_1 + x_{i2}p_2 + \ldots + x_{in}p_n \tag{3.2}$$

$$E_{P'} X_j = \sum_{l=1}^{n} x_j p_l = \sum_{l=1}^{n} x_{jl}p_l = x_{j1}p'_1 + x_{j2}p'_2 + \ldots + x_{jn}p'_n. \tag{3.3}$$

From (3.1), subequation (i) is the analogue of Kolmogorov distance (KD), and subequation (ii) is an adjusted version of KD called (AKD) and λ is the parameter of regularization of the interactions between particles (see [2]). We will discuss the following cases:

(a) If $\lambda = 0$, then (i) and (ii) are identical and $|i-j| \leq 1$. That is, J represents the probabilistic metric of nearest neighboring states.

(b) If $\lambda \neq 0$ then $J_1 = E_P |X_i - X_j|^2 + \lambda E_P |X_i - X_j|^2$ is the probabilistic metric of the faraway neighboring states. Finally, we wish to find the solution to the minimization problem

$$\widetilde{J} = \min_{x_i, x_j, \lambda} J(x_i, x_j, \lambda). \tag{3.4}$$

At each given time, (14) gives an unconstrained linear optimization problem on the random states X_i and X_j, which can be easily solved.

Proposition 3.1. *If there are perturbations in the system, the dynamic quadratic variation of both X_i and X_j is proportional to their probabilistic variation in the perturbed part of the system: $|X_i - X_j|^2 = \lambda |P_i - P_j|$.*

The above proposition is important because it simplifies the manipulation of the automata system at the perturbed locations. In this way, we may create a simulation using approximations of various equations, which we shall observe in the next subsection.

3.2. **Solving the Unconstrained Optimization Problem.** We wish to solve the equation (3.4), where

$$J = \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j=1}^{N} |X_i - X_j|^2 P(X_i < x_i, X_j < x_j) + \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_j |P(\omega_i) - \lambda_j P(\omega_j)|, \tag{3.5}$$

which we decompose in two parts in an intuitive way: $J = J_1 + J_2$. The first part, J_1, of this equation gives the approximation

$$\begin{aligned} & \sum_{i=1}^{N} \sum_{j=1}^{N} |X_i - X_j|^2 P(X_i < x_j, X_j < x_j) \\ \approx \quad & (x_{i1} p_1 - X_{j1} p'_1)^2 + (x_{i2} p_2 - X_{j2} p'_2)^2 + \ldots + (x_{in} p_n - X_{jn} p'_n)^2 \end{aligned} \tag{3.6}$$

and the second part may be approximated as

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_j |P(\omega_i) - \lambda_j P(\omega_j)| \approx \lambda_1 (p_1 - \lambda_1 p'_1)^2 + \lambda_2 (p_2 - \lambda_2 p'_2)^2 + \ldots + \lambda_n (p_n - \lambda_n p'_n)^2. \tag{3.7}$$

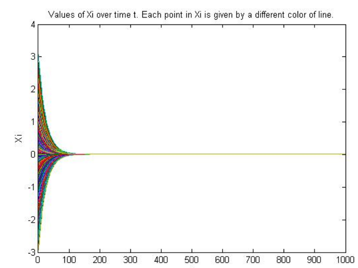
Taking the first partial derivative of J with respect to the i-th location, and putting the sum equal to zero implies that the optimal x_i is $\hat{x_i} = \left(\frac{p_1}{p'_1}, \frac{p_2}{p'_2}, \ldots, \frac{p_n}{p'_n}\right)$. Analogously, with respect to the j-th location we have $\hat{x_j} = \left(\frac{p'_1}{p_1}, \frac{p'_2}{p_2}, \ldots, \frac{p'_n}{p_n}\right)$. Partially differentiating all the components of J with respect to λ_m gives $\hat{\lambda} = \left(\frac{p_1}{2p'_1}, \frac{p_2}{2p'_2}, \ldots, \frac{p_n}{2p'_n}\right)$. Finally, to check whether our optimal solutions $\left(\hat{x_i}, \hat{x_j}, \hat{\lambda}\right)$ are consistent and can be validated in general, we apply our new concept to existing cell concentration data to obtain the results presented in the next section.

4. SIMULATION AND RESULTS

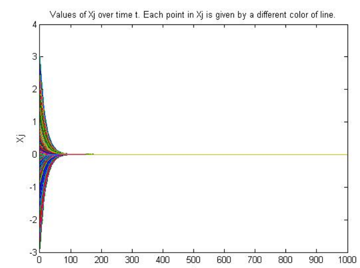
To validate our theoretical framework, we apply our concept to real existing biological data. We use the cellular concentration data available at [13] and apply our method to measure the optimal adjusted Kolmogorov distance between states at various locations over time. We let the initial states be $X_i(0)$, the initial cell concentration at the i-th locations (which we take to be the first column of the aforementioned data), and $X_j(0)$, the initial cell concentration at the j-th locations (we take this to be the second column of the data). The "on" and "off" states are respectively represented by increases and decreases of the concentration at a given location over time. First we plot the dynamic at each location and the α-metric in separate graphs.

We subdivide our cases studied into five classes based on the choice of the value of lambda, which is defined in this work as the perturbation parameter and is applied only to faraway states. All dynamics for cases (a) and (b) show fluctuations at the beginning but those fluctuations become rapidly damped over time. We now plot each scenario based on the choice of $\lambda_{ij} \in [0, 1]$. The results in this section are taken from [5].

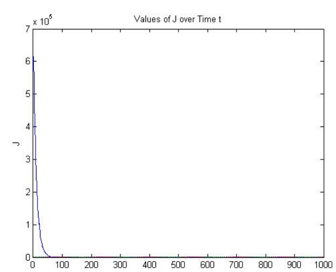
4.1. **Extremal Cases $\lambda_{ij} = 0$ or $\lambda_{ij} = 1$.** We take λ_{ij} to be either of the endpoints of the closed interval $[0, 1]$. Figs. 1 and 2 give typical plots for the dynamic of X_i, the dynamic of X_j and the dynamic of the α-metric over time, respectively.



(a)



(b)



(c)

FIGURE 1. Case 1: $\lambda_{ij} = 0$ for all i, j.

4.2. **Normal Cases $\lambda_{ij} \neq 0, 1$.** This time we take λ_{ij} to be to be one of three values not equal to the endpoints of the interval. Figs. 3, 4 and 5 give typical plots for the dynamic of X_i, the dynamic of X_j and the dynamic of the α-metric over time, respectively.

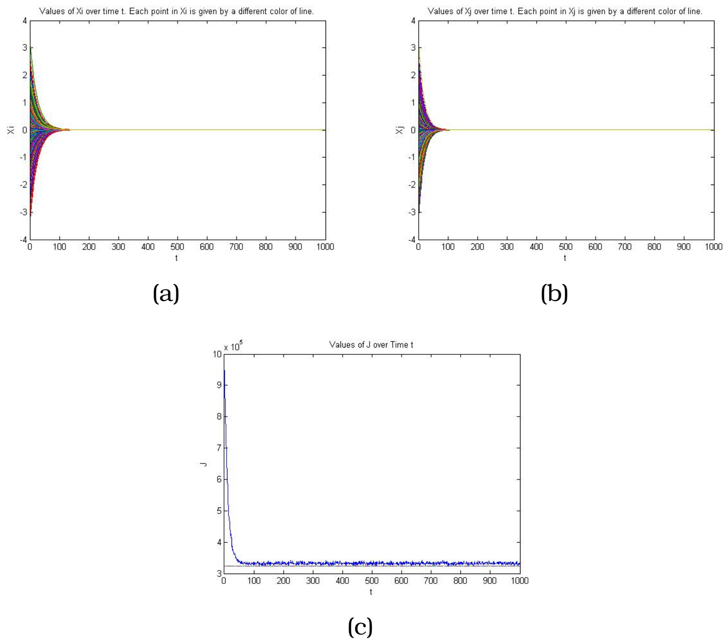


FIGURE 2. Case 2: $\lambda_{ij} = 1$ for all i, j.

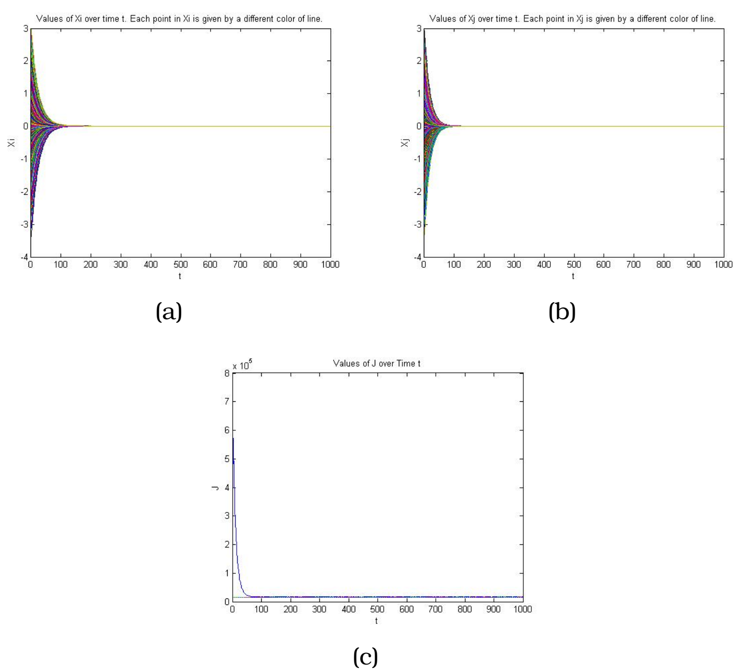


FIGURE 3. Case 3: $\lambda_{ij} = \lambda_N = N-$dimensional constant vector.

Comments: Figs. 1–5 are detailed plots of the dynamics of particles $X_i(t)$ and $X_j(t)$ over time and the α-metric J over 1000 timesteps. Observe that cases 1, 2

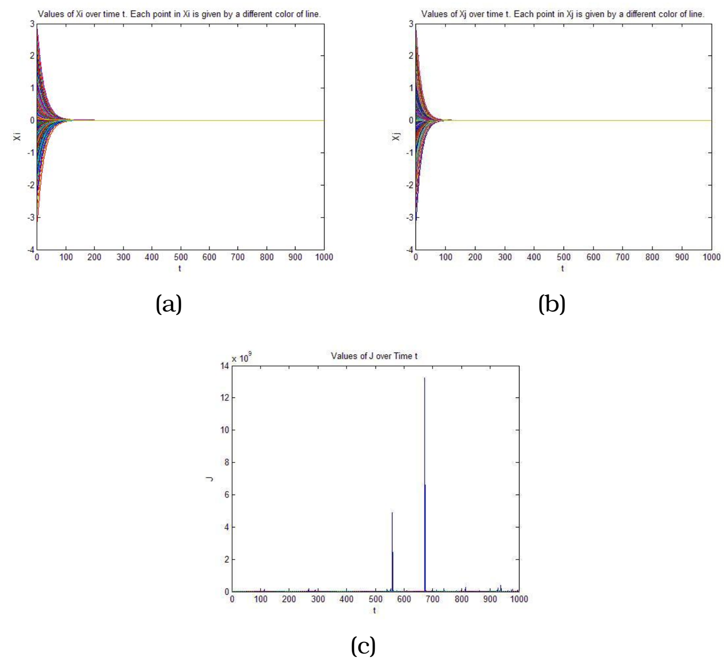


FIGURE 4. Case 4: $\lambda_{ij} = \lambda_i$.

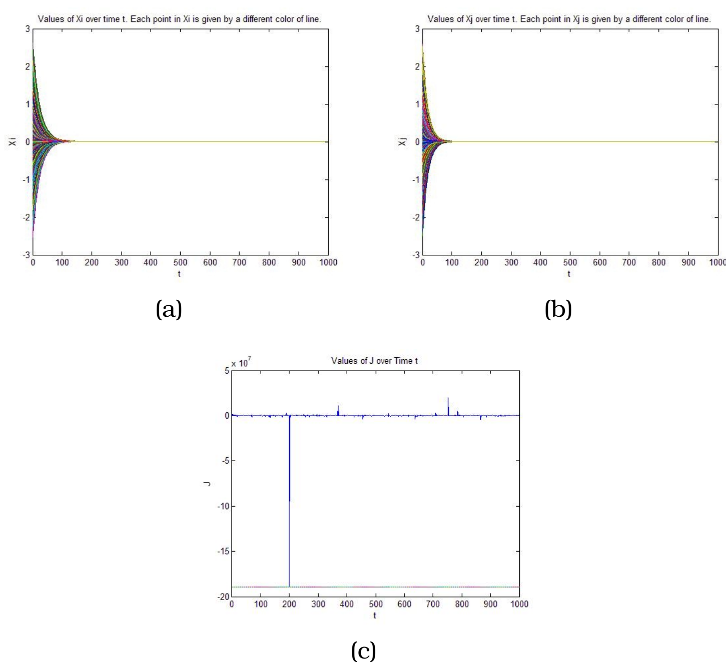


FIGURE 5. Case 5: $\lambda_{ij} = \lambda_j$.

and 3 exhibit low fluctuations in the α-metric J over time, but in the remaining cases there are very high fluctuations in J. However it is surprising to observe that the smallest value of J is achieved faster in case 5. This again proves that random fluctuations may be beneficial to dynamical systems in terms of optimality and stability. Additionally, this example shows that local stability drives global stability.

In the following tables, we give the optimal value $\widetilde{J}$ and the standard deviation of the α-metric J found in each case.

Extremal Cases	1. $\lambda_{ij} = 0$	2. $\lambda_{ij} = 1$
$\overline{J}$	4.765×10^4	3.226×10^5
SD	1.637×10^1	4.667×10^4

TABLE 1. This table shows the minimum value of the adjusted Kolmogorov distance for each perturbation λ_{ij} in the extremal cases. Comparing both cases, we observe that case 1 has the smallest optimal J and spread (or SD). This means that although case 1 has the best optimal J, the number of states reaching such optimality are very limited. In case 2, J attains an optimal value greater than that of case 1, but with a greater spread. These results confirm relative stability of the cell concentration stochastic cellular system, since optimal J exist in both cases and are comparable.

Normal Cases	3. $\lambda_{ij} = \lambda_N$	4. $\lambda_{ij} = \lambda_i$	5. $\lambda_{ij} = \lambda_j$
$\overline{J}$	1.565×10^4	5.232×10^5	-1.893×10^8
SD	4.960×10^4	4.465×10^8	6.047×10^6

TABLE 2. This table shows the minimum values of the adjusted Kolmogorov distance for each perturbation λ_{ij} in the regular cases together with the spread. Case 5 has the smallest optimal j (the fact it is so small is interesting in itself) with a medium spread, and case 4 has the medium optimal J but with very high spread. Finally case 3 has a low optimal J (comparable to case 1) and also the smallest spread among the normal cases. These results indicate that many states in case 3 reach the optimal level, and that states at the j-th locations are more robust than those at the i-th locations.

Further, if we compare the normal and extremal cases we observe that the spreads for the normal cases are at least those of the extremal cases, but comparing optimal values in both normal and extremal cases remains difficult in general (optimal J in case 1 is relatively stable, but is unstable in case 2).

5. DISCUSSION AND CONCLUSION

In this paper, we have designed a random metric (the so-called α-metric) to capture nonlinearity and important hidden events in a "random" dynamical system. To this end we have incorporated a Local Autoregressive Model of the dynamic into the "probabilistic" metric and solved an unconstrained optimization problem on a stochastic cellular automata system. We hypothesize the solution will correspond to some hidden characteristics of the state dynamics. Such information may prove to be useful for exploring such a system. Again our motivation here comes from the wish to optimally capture hidden but relevant information in a stochastic cellular automata system defined as a mixture of stochastic processes.

Further, the created random metric was applied to nearby and faraway locations in order to find hidden characteristics of the automata system over time. Solving the unconstrained optimization random metric based problem, we found that solutions show high and low level fluctuations depending on the choice of the perturbation parameter λ and the location. The application of our method to cell concentration data reveals its consistency and adaptability (Table 2). Finally this exercise confirms that reliable detection and quantitative description in stochastic systems with limited precision remains a difficult task, especially when the dynamic is characterized by some additional complexity. But the example here with cellular data shows appreciable results despite a rather high dimension; therefore our methodology which combines a random metric and unconstrained optimization has the merit of producing interesting results.

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THE FEICHTINGER CONJECTURE FOR EXPONENTIALS

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ABSTRACT. The Feichtinger conjecture for exponentials asserts that the following property holds for every fat Cantor subset B of the circle group: the set of restrictions to B of exponential functions can be covered by Riesz sets. In their seminal paper on the Kadison-Singer problem, Bourgain and Tzafriri proved that this property holds if the characteristic function of B has Sobolev regularity. Their probability based proof does not explicitly construct a Riesz cover. They also showed how to construct fat Cantor sets whose characteristic functions have Sobolev regularity. However, these fat Cantor sets are not convenient for numerical calculations. This paper addresses these concerns. It constructs a family of fat Cantor sets, parameterized by their Haar measure, whose characteristic functions have Sobolev regularity and their Fourier transforms are Riesz products. It uses these products to perform computational experiments that suggest that if the measure of one of these fat Cantor sets B is sufficiently close to one, then it may be possible to explicitly construct a Riesz cover for B using the Thue-Morse minimal sequence that arises in symbolic topological dynamics.

KEYWORDS : Beurling density; Fat Cantor set; Feichtinger conjecture for exponentials; Paley-Littlewood decomposition; Riesz cover; Riesz product; Sobolev regularity; Spectral envelope; Thue-Morse minimal sequence.

1. INTRODUCTION

We let $\mathbb{N} = \{1, 2, 3, \dots\}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$, and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denote the natural numbers, integers, reals, complex numbers and the circle group with Haar measure μ. Throuought this paper B denotes a Borel subset of $\mathbb{T}$ with $\mu(B) > 0$, F denotes a nonempty subset of $\mathbb{Z}$, and χ_B and χ_F denote their characteristic functions. F is

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an arithmetic set if $F = j + n\mathbb{Z}$ for $j \in \mathbb{Z}, n \in \mathbb{N}$. We define exponential functions $e_k(t) = e^{2\pi ikt}, k \in \mathbb{Z}, t \in \mathbb{T}$; $E(F) = \{\, e_k \,:\, k \in F\,\}$;

$P(F) =$ trigonometric polynomials spanned by $E(F)$ whose norm $||f||_2 = 1$;

$$\alpha(B,F) = \inf\{\int_{t\in B} |f(t)|^2\,dt \,:\, f \in P(F)\,\}. \tag{1.1}$$

(B,F) is a **Riesz pair** if $\alpha(B,F) > 0$, or equivalently if $E(F)\chi_B$ is a Riesz basic sequence [4]. $\{F_j \,:\, j = 1, ..., n\}$ is a **Riesz cover** for B if each (B, F_j) is a Riesz pair and $\cup_{j=1}^n F_j = \mathbb{Z}$. This paper studies the

Feichtinger Conjecture for Exponentials (FCE) Every B has a Riesz cover.

The FCE is a special case of the Feichtinger Conjecture (FC): Every unit norm Bessel sequence is a finite union of Riesz basic sequences, which was formulated in ([2], Conjecture 1.1). Casazza and Crandel [3] proved that the FC is equivalent to a yes answer to the following problem which has remained open since it was formulated in 1959 [9]:

Kadison-Singer Problem (KSP) Does every pure state on the C*-algebra $\ell^\infty(\mathbb{Z})$ admit a unique extension to the C*-algebra of bounded operators $B(\ell^2(\mathbb{Z}))$?

A **fat Cantor** is a set that is homeomorphic to Cantor's ternary set and has positive Haar measure. Lemma 2.2 shows that the FCE is equivalent to the assertion: every fat Cantor set has a Riesz cover. In their seminal paper on the KSP, Bourgain and Tzafriri proved a result ([1], Theorem 4.1) that implies B has a Riesz cover whenever χ_B is in the Sobolev space $H^s(\mathbb{T})$ for some $s > 0$. However, their existence proof does not explicitly construct Riesz covers. They also proved a result ([1], Corollary 4.2) that implies $\chi_B \in H^s(\mathbb{T})$ for all $s < \frac{1}{2}$ whenever $\mathbb{T}\backslash B = \cup_{j=1}^\infty O_j$ where O_j are pairwise disjoint open intervals that satisfy $\mu(O_j) < 2^{-j}$, thus showing the existence of fat Cantor sets that have Riesz covers. This is surprising since Lemma 2.3 shows that if B is a fat Cantor set then (B,F) is not a Riesz pair for a class of sets F that includes the class of arithmetic sets.
This paper has four main results:

(i) Construction of **ternary fat Cantor sets** such that the Fourier transforms of their characteristic functions are Riesz products described by Equation 3.1.

(ii) Proof of Theorem 3.1 which shows that ternary fat Cantor sets satisfy $\chi_B \in H^s(\mathbb{T})$ for every $s < 1 - \frac{\log 2}{\log 3} \approx 0.3691$ so they have Riesz covers. Ternary fat Cantor sets differ from those constructed by Bourgain and Tzafriri because the lengths of the open intervals O_j removed have algebraic decay $j^{-\log 3/\log 2}$ rather than exponential decay 2^{-j}. The proof uses Lemma 2.5 which provides a refinement of the standard Paley-Littlewood decomposition that Bourgain and Tzafriri used to prove ([1], Corollary 4.2).

(iii) Computation of estimates of $\alpha(B,F)$, where B is a ternary fat Cantor set and $\chi_F = \cdots 10010110.0110100110010110 \cdots$ is the Thue-Morse minimal sequence [17], [13]. These estimates suggest that $\{F, 1+F, 2+F\}$ is a Riesz cover for B if $\mu(B)$ is sufficiently close to 1.

(iv) Proof of Theorem 3.2 that shows the spectral envelope $S(F)$ of F (see Definition 2.1) is convex whenever χ_F is a minimal sequence.

Results (iii) and (iv) relate the FCE to the field of symbolic dynamics. We give a brief review of the concepts from this field that we use in this paper.

Let A be a finite set with the discrete topology. The symbolic dynamical system over A is the pair $(A^{\mathbb{Z}}, \sigma)$ where $A^{\mathbb{Z}}$ has the product topology (homeomorphic to Cantor's ternary set) and σ is the shift homeomorphism defined by

$$(\sigma\, b)(n) = b(n-1), b \in A^{\mathbb{Z}}, n \in \mathbb{Z}.$$

A sequence $b \in A^{\mathbb{Z}}$ is **minimal** if its orbit closure $\overline{\{\,\sigma^n(b) : n \in \mathbb{Z}\,\}}$ is a minimal closed shift-invariant set. Zorn's lemma ensures the existence of minimal sequences in any nonempty closed shift invariant subset. $F \subset \mathbb{Z}$ is **syndetic** if there exists $n \in \mathbb{N}$ such that $\cup_{j=0}^{n-1}(j+F) = \mathbb{Z}$, **thick** if for every $n \in \mathbb{N}$ there exists $k \in \mathbb{Z}$ such that $k + \{0, 2, 3, ..., n-1\} \subset F$, and **piecewise syndetic** if $F = F_s \cap F_t$ where F_s is syndetic and F_t is thick. Minimal sequences are characterized by a result of Gottschalk [6], [7] that says a sequence b is minimal if and only if for every finite $G \subset \mathbb{Z}$, the set $\{\,n \in \mathbb{Z} : \sigma^n(b)|_G = b|_G\,\}$ is syndetic. Gottschalk's result shows that the Thue-Morse sequence is a minimal sequence and it can be used to construct other minimal sequences. Clearly if F is nonempty and χ_F is a minimal sequence then F is syndetic. Therefore, if (B, F) is a Riesz pair and χ_F is a minimal sequence then B has a Riesz cover. Furstenberg in ([5],Theorem 1.23) used Gottschalk's result for symbolic dynamics over the set $\{1, ..., n\}$ to prove that if $\cup_{j=1}^{n} F_j = \mathbb{Z}$ then one of the sets F_j is piecewise syndetic. In ([12], Theorem 1.1) we used Furstenberg's result to prove that B has a Riesz cover if and only if there exists nonempty set F such that (B, F) is a Riesz pair and χ_F is a minimal sequence. This result shows that FCE holds if and only if for every Borel set $B \subset \mathbb{T}$ with $\mu(B) > 0$ there exists a nonempty $F \subset \mathbb{Z}$ such that χ_F is a minimal sequence and (B, F) is a Riesz pair. Paulsen [15] investigated the relationship between the Kadison-Singer Problem and syndetic sets and in [16] he used methods from operator algebras (completely positive maps and multiplicative domains) to independently derive the key results in our paper [12] that relate the FCE to syndetic sets.

2. PRELIMINARIES

$L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$ are Hilbert spaces with scalar products $(f, g) = \int_{x\in\mathbb{T}} f(x)\overline{g(x)}d\mu(x)$ and $(a, b) = \sum_{n\in\mathbb{Z}} a(n)\overline{b(n)}$ and associated norms $||\ \ ||_2$ and the Fourier transform $L^2(\mathbb{T}) \ni f \to \widehat{f} \in \ell^2(\mathbb{Z})$, defined by

$$\widehat{f}(k) = (f, e_k) = \int_{\mathbb{T}} f(x)\, e_k(-x)\, dx, \quad k \in Z,$$

is a unitary surjection. Sobolev spaces for $s \geq 0$ are defined by

$$H^s(\mathbb{T}) = \{\, f \in L^2(\mathbb{T}) : \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2\, |k|^{2s} < \infty\,\}.$$

A function $f \in L^2(\mathbb{T})$ is called Sobolev regular if $f \in H^s(\mathbb{T})$ for some $s > 0$. We observe that $L^1(\mathbb{T}) \subset L^2(\mathbb{T})$ is a Banach algebra under convolution and each exponential function e_k defines a **multiplicative linear functional** or **character** $L^1(\mathbb{T}) \ni f \to \widehat{f}(k) \in \mathbb{C}$. $C(\mathbb{T})$ denotes the Banach space of continuous complex valued functions on $\mathbb{T}$ with the infinity norm $||\,||_\infty$ and $M(\mathbb{T})$ denotes the Banach algebra of complex valued measures on $\mathbb{T}$ with the total variation norm and the convolution product. The Riesz Representation Theorem asserts that $M(\mathbb{T})$ is the linear dual of $C(\mathbb{T})$. For $t \in \mathbb{T}$, δ_t denotes the Dirac measure at t defined by $\delta_t(f) = f(t), f \in C(\mathbb{T})$. A measure ν is positive if $\nu(f) \geq 0$ whenever $f \geq 0$, a probability measure if it is positive and $\nu(1) = 1$, and discrete if it is a countable linear combination of Dirac measures. We identify $L^1(\mathbb{T})$ with the set of absolutely continuous measures in $M(\mathbb{T})$. The **maximal ideal space** $\widehat{M}(\mathbb{T})$ is the set of **generalized characters** which define multiplicative linear functionals on $M(\mathbb{T})$ via the Gelfand correspondence. For $\nu \in M(\mathbb{T})$ and $t \in \mathbb{T}$ we define translation $\tau_t\nu(f) = \nu(f(\cdot - t))$ and for $n \in \mathbb{N}$ we define the dilation $d_n\nu$ by $d_n\nu(f) = \sum_{k=0}^{n-1}\int_0^1 f(\frac{x+k}{n})d\nu(x)$. Clearly $d_n\nu$ has period $\frac{1}{n}$ since $\tau_{\frac{1}{n}}d_n\nu = d_n\nu$.

Definition 2.1. The **spectral envelope** of F is the set of probability measures on $\mathbb{T}$ given by

$$S(F) = \text{weak}^* - \text{closure}\,\{\,|f|^2\;:\;f \in P(F)\}. \tag{2.1}$$

If $j \in \mathbb{Z}, n \in \mathbb{N}$ then $S(j+F) = S(F)$ and $S(nF) = \{\,d_n\nu\;:\;\nu \in S(F)\,\}$. If $\nu \in S(F)$ and $t \in \mathbb{T}$ then $\tau_t\,\nu \in S(F)$. The functions $f_n = \frac{1}{\sqrt{n}}(e_1 + \cdots + e_n) \in P(\mathbb{Z})$ and the sequence of Fejer kernels $K_n = |f_n|^2$ converges weakly to $\delta_0 \in S(\mathbb{Z})$. If $\nu \in M(\mathbb{T})$ is a probability measure then $K_n * \nu$ are nonnegative trigonometric polynomials so the Riesz-Fejer spectral factorization theorem implies there exists $Q_n \in P(\mathbb{T})$ such that $K_n * \nu = |Q_n|^2$. Since $K_n * \nu$ converges weakly to ν, it follows that $\nu \in S(\mathbb{Z})$. Therefore $S(\mathbb{Z})$ consists of all probability measures and $S(j + n\mathbb{Z})$ contains the discrete measure $d_{\frac{1}{n}}\delta_0 = \sum_{k=0}^{n-1}\delta_{\frac{k}{n}}$.

If $B_0 \subset B$ are Borel sets and B does not have a Riesz cover then B_0 does not have a Riesz cover. Assume that a Borel set $B \subseteq \mathbb{T}$ contains an interval $[a,b]$ with $b > a$. Choose $n \in \mathbb{N}$ with $n(b-a) \geq 1$ and set $F_j = j + n\mathbb{Z}\;:\;j = 0,...,n-1$. If $f \in P(F_j)$ then $|f|^2$ has period $\frac{1}{n}$ and hence $\alpha(B, F_j) \geq 1/n$, $j = 0,...,n-1$. Therefore B has a Riesz cover consisting of arithmetic sets. If a Borel set B_1 with $\mu(B_1) > 0$ does not have a Riesz cover then, since μ is inner regular, there exists a closed $B_0 \subset B_1$ with $\mu(B_0) > 0$. Clearly B_0 is nowhere dense. A theorem of Brouwer ([10], Theorem 7.4) shows if $\mu(B) > 0$ and B is closed, nowhere dense, and perfect (has no isolated points) then B is a fat Cantor set.

Lemma 2.2. *If a Borel set B_1 with $\mu(B_1) > 0$ does not have a Riesz cover then there exists a fat Cantor set $B \subseteq B_1$ such that B does not have a Riesz cover. Therefore, the FCE is equivalent to the assertion that every fat Cantor set has a Riesz cover.*

Proof. By the preceding argument B_1 contains a nowhere dense closed set $B_0 \subset B_1$ with $\mu(B_0) > 0$. We use transfinite induction to define a collection of nonincreasing subsets $B_\gamma \subseteq B_0$ indexed by ordinals as follows: $B_{\gamma+}$ = set of limit points of B_γ and for limit ordinals $B_\beta = \cap_{\gamma<\beta}B_\gamma$. The Cantor-Baire Stationary Principle implies that there exists a countable ordinal γ such that $B_{\gamma+} = B_\gamma$. Let $B = B_\gamma$. Then B is perfect. A set of isolated points is countable so has Haar measure zero. Since we remove a countable number of such sets from B_0 to obtain B, $\mu(B) > 0$. Since B is also closed and nowhere dense Brouwer's theorem implies that B is a fat Cantor set. □ □

Lemma 2.3. *If B is closed, $\nu \in S(F)$, and $\nu(B) = 0$ then (B, F) is not a Riesz pair. If B is a fat Cantor set and $S(F)$ contains a discrete measure then (B, F) is not a Riesz pair. A fat Cantor set can not have a Riesz cover consisting of arithmetic sets.*

Proof. If B is closed then the map $S(F) \ni \nu \to \nu(B)$ is upper semi-continuous since
$\limsup_{\nu_j \to \nu} \nu_j(B) \leq \nu(B)$, hence

$$\alpha(B, F) = \inf \{\nu(B) \,:\, \nu \in S(F)\} \tag{2.2}$$

and this implies the first assertion. If $\nu \in S(F)$ is discrete then there exists sequences $c_j \geq 0, t_j \in \mathbb{T}, j \in \mathbb{N}$ such that $\nu = \sum_{j \in \mathbb{N}} c_j \delta_{t_j}$. The Baire Category Theorem implies there exists $t \notin \cup_{j \in \mathbb{N}}(B - t_j)$. Then $\tau_t\, \nu \in S(F)$ and

$$\tau_t\, \nu(B) = \sum_{j \in \mathbb{N}, t_j + t \in B} c_j = \sum_{j \in \mathbb{N}, t \in (B - t_j)} c_j = 0$$

and this implies the second assertion. The fact that spectral envelopes of arithmetic sets contain discrete measures implies the third assertion. □ □

Remark 2.4. In [12] we describe Bohr sets and show ([12], Theorem 2.1) that if F is a Bohr set then $S(F)$ contains discrete measures. Therefore, by Lemma 2.3, if B is a fat Cantor set and F is a Bohr set then (B, F) is not a Riesz pair.

Lemma 2.5. *Let $\sigma_j, j \in \mathbb{N}$ be a sequence of positive real numbers that satisfies the following two conditions:*

$$(i) \limsup_{j \to \infty} \frac{\sigma_{j+1}}{\sigma_j} < 1$$

and

$$(ii) \liminf_{j \to \infty} \frac{\sigma_{j+1}}{\sigma_j} > 0.$$

If $c > 0$, $p > 0$ and $f \in L^2(\mathbb{T})$ satisfies

$$||f(\cdot - \sigma_j) - f(\cdot)||_2^2 \leq c\, \sigma_j^p, \quad j \in \mathbb{N} \tag{2.3}$$

then $f \in H^s(\mathbb{T})$ for all $s \in (0, \frac{p}{2})$.

Proof. Equation 2.3 implies that $\widehat{f}$ satisfies

$$\sum_{k \in \mathbb{Z}} |\widehat{f}(k)|^2 \, \frac{4 \sin^2 \pi k \sigma_j}{\sigma_j^p} \leq c, \quad j \geq J. \tag{2.4}$$

Condition (ii) ensures that there exists $\theta \in (0, \pi)$ and $J \in \mathbb{N}$ such that

$$\theta\, \sigma_j \leq (\pi - \theta)\, \sigma_{j+1}, \quad j \geq J.$$

We set $c_1 = c(1 - \theta/\pi)^{2s}/(4 \sin^2 \theta)$ and observe that

$$\sum_{|k| \geq J} |\widehat{f}(k)|^2 |k|^{2s} \leq \sum_{j \in \mathbb{N}} \sum_{\theta \leq \pi\, |k|\, \sigma_j \leq \pi - \theta} |\widehat{f}(k)|^2\, |k|^{2s} \leq c_1 \sum_{j \in \mathbb{N}} \sigma_j^{p - 2s}. \tag{2.5}$$

Condition (i) ensures that this sum converges whenever $s \in (0, \frac{p}{2})$. □ □

3. MAIN RESULTS

Construction of Ternary Fat Cantor Sets

For every $\gamma \in (0, 1)$ the following construction gives a fat Cantor set $B \subset \mathbb{T}$ such that $\mu(B) = \gamma$. Start with the interval $S_0 = [-\frac{1}{2}, \frac{1}{2}]$ and remove the middle open interval having length $\frac{1}{3}(1 - \gamma)$ to obtain a set S_1 equal to the union of two disjoint equal length closed intervals. From each of these two intervals remove the middle open interval having length $\frac{1}{9}(1 - \gamma)$ to obtain a set S_4 equal to the union of four disjoint equal length closed intervals. Continue in this manner to construct a decreasing sequence of closed sets S_j each the union of 2^j closed intervals having length

$$L_j = \gamma\, 2^{-j} + (1-\gamma)\, 3^{-j}.$$

Construct $S = \cap_{j\in\mathbb{N}} S_j$ and $B = S + Z \subset \mathbb{T}$. Clearly $\mu(B) = \gamma$ and $B = -B$. Let x_j be the distance between the center of the rightmost interval in S_{j-1} and the rightmost interval in S_j, let $I_j = [-\frac{1}{2}L_j, \frac{1}{2}L_j]$, and define the sequence of discrete measures

$$\nu_j = \left(\frac{1}{2}\delta_{-x_1} + \frac{1}{2}\delta_{x_1}\right) * \left(\frac{1}{2}\delta_{-x_2} + \frac{1}{2}\delta_{x_2}\right) * \cdots * \left(\frac{1}{2}\delta_{-x_j} + \frac{1}{2}\delta_{x_j}\right).$$

Then

$$x_j = \frac{1}{2}(L_{j-1} - L_j), \quad j \in \mathbb{N}$$

and

$$\chi_{S_j} = 2^j\, \chi_{I_j} * \nu_j.$$

Since we have weakly convergent sequences $\chi_{S_j} \to \chi_B$ and $2^j\, \chi_{I_j} \to \mu(B)\, \delta_0$ it follows that

$$\mu(B)\, \nu_j \to \chi_B.$$

Therefore the Fourier transform of χ_B equals the Riesz product ([18], Section 7, Chapter 5)

$$\widehat{\chi_B}(k) = \mu(B) \prod_{j\in\mathbb{N}} \cos(2\pi x_j k), \quad k \in \mathbb{Z}. \tag{3.1}$$

Equation 3.1 provides an efficient method to compute $\widehat{\chi}_B$.

Theorem 3.1. *If B is a ternary fat Cantor set then $\chi_B \in H^s(\mathbb{T})$ for $s < 1 - \frac{\log 2}{\log 3}$ so B has a Riesz cover.*

Proof. Assume that B is a ternary fat Cantor set and set $\gamma = \mu(B)$. Set $\sigma_j = 3^{-j}(1-\gamma), j \in \mathbb{N}$ and $p = 1 - \frac{\log 2}{\log 3}$. Lemma 2.5 implies that it suffices to show that there exists $c > 0$ such that

$$||\chi_B(\cdot - \sigma_j) - \chi_B||_2^2 \leq c\, \sigma_j^p, \quad j \in \mathbb{N}. \tag{3.2}$$

The Borel subsets of $\mathbb{T}$ form an abelian group under the Boolean operation

$$B_1 \Delta B_2 = (B_1 \cup B_2) \backslash (B_1 \cap B_2)$$

with identity ϕ, $B_1 \Delta B_1 = \phi$, $\mu(B_1 \Delta B_2) \leq \mu(B_1) + \mu(B_2)$, and $||\chi_{B_1} - \chi_{B_2}||_2^2 = \mu(B_1 \Delta B_2)$. Observe that since S_j consists of the union of 2^j closed intervals separated by distance $\geq \sigma_j$,

$$\mu((S_j + \sigma_j)\Delta S_j) \leq 2(1-\gamma)\left(\frac{2}{3}\right)^j, \; j \in \mathbb{N}.$$

Furthermore

$$\mu(S_j \Delta B) = \mu(S_j \backslash B) = \sum_{k=j}^{\infty} 2^k \sigma_{k+1} = (1-\gamma)\left(\frac{2}{3}\right)^j.$$

Inequality 3.2 holds with $c = 4(1-\gamma)^{1-p}$ since $(B + \sigma_j)\, \Delta\, B = [(B + \sigma_j)\, \Delta\, (S_j + \sigma_j)]\, \Delta [B\, \Delta\, S_j]\, \Delta\, [(S_j + \sigma_j)\, \Delta S_j]$ implies that

$$||\chi_B(\cdot - \sigma_j) - \chi_B||_2^2 \leq 4(1-\gamma)\left(\frac{2}{3}\right)^j.$$

□

Theorem 3.2. *If χ_F is a minimal sequence then $S(E)$ is convex. Furthermore,*

$$\alpha(B, F) = \inf\{\, \nu(B) \,:\, \nu \in S_e(F)\,\} \tag{3.3}$$

where $S_e(F)$ is the set of extreme points in $S(E)$.

Proof. Set $Q(F) = \{\, |f|^2 \,:\, f \in P(F)\,\}$. Since $S(E)$ is the weak* closure of $Q(F)$, to prove that $S(F)$ is convex it suffices to show that the convex combination of any two elements in $Q(F)$ is in $S(E)$. Let $f, h \in P(F)$ and let $a, b \in [0, 1]$ satisfy $a^2 + b^2 = 1$. Gottshalk's theorem implies there exists a sequence $n_j \in \mathbb{N}$ converging to ∞ with $e_{n_j}\, h \in P(F)$, $j \in \mathbb{N}$. Define the sequence

$$g_j = \frac{|f + e_{n_j}\, h|^2}{||f + e_{n_j}\, h||_2}, \quad j \in \mathbb{N}.$$

Then $g_j \in Q(F), \quad j \in \mathbb{N}$ and the Riemann-Lebesgue lemma implies that

$$\lim_{j\to\infty} g_j = a^2|f|^2 + b^2|h|^2$$

thus proving that $S(F)$ is convex. $S_e(F)$ is nonempty since the the Krein-Milman theorem implies that $S(F)$ is the weak*-closure of the set of convex combinations of points in $S(F)$. Since $S(F)$ is separable, Choquet's theorem implies that every element $\nu \in S(F)$ is represented by a probability measure on $S_e(F) \subset S(F)$, from which Equation 3.3 follows.

□

Optimization Algorithm to Estimate $\alpha(B, F)$. We now describe a computational approach to estimate $\alpha(B, F)$ under the assumption that χ_F is a minimal sequence. Let $B(\ell^2(\mathbb{Z}))$ denote the C^*-algebra of bounded operators on the Hilbert space $\ell^2(\mathbb{Z})$. Define the Laurent operator $L_B \in B(\ell^2(\mathbb{Z}))$ by the Toeplitz matrix $[L_B](j, k) = \widehat{\chi}_B(k - j)$, $j, k \in \mathbb{Z}$ and define the restriction operator $R_F : \ell^2(\mathbb{Z}) \to \ell^2(F)$ by $R_F(a)(k) = a(k), \quad a \in \ell^2(\mathbb{Z})$, $k \in F$, so the adjoint $R_F^* : \ell^2(F) \to \ell^2(\mathbb{Z})$ is given by

$$R_F^*(b)(k) = b(k) \text{ if } k \in F, \text{ else} = 0.$$

The matrix $[R_F\, L_B\, R_F^*]$ for the operator $R_F\, L_B\, R_F^* \,:\, \ell^2(F) \to \ell^2(F)$ is a principle submatrix of the matrix $[L_B]$ for the Laurent operator L_B. Then

$$\alpha(B, F) = \inf \text{spec} R_F\, L_B\, R_F^* \tag{3.4}$$

where spec denotes the spectrum of the restricted operator. For finite $G \subset \mathbb{Z}$, $\alpha(B, G) = \min eig[R_G\, L_B\, R_G^*]$ since the later matrix is finite. For infinite F such that χ_F is a minimal sequence, define $F_n = [0, n] \cap F$, $n \in \mathbb{N}$. Gottschalk's theorem implies that for every finite $G \subset F$ there exists $m \in \mathbb{N}$ such that $G \subset F_n$ whenever $n \geq m$. Since F_n is an increasing sequence of sets the sequence $\alpha(B, F_n)$ is a nonincreasing sequence of nonnegative numbers. This implies the following result which provides an algorithm to approximate $\alpha(B, F)$.

$$\alpha(B, F) = \lim_{n\to\infty} \alpha(B, F_n). \tag{3.5}$$

Description of Numerical Experiments We used Equation 3.1 to compute $\widehat{\chi}_B$ for ternary Cantor sets. Figure 1 left, right shows the values $\widehat{\chi}_B(k), k = 1 : 4095$ for $\mu(B) = 0.25, 0.75$, respectively. We used Equation 3.5 to estimate $\alpha(B, F)$, where χ_F is the Thue-Morse minimal sequence, by $\alpha(B, F_{4095})$, where $F_{4095} = [0, 4095] \cap F$. For $\mu(B) = 0.25$, the computed value of $\alpha(B, F_{4095})$ is the negative number -1.2261×10^{-14} due to the fact that the true value is smaller than machine precision. For $\mu(B) = 0.75$ the computed value of $\alpha(B, F_{4095})$ is 0.085512 which is 385 trillion times machine precision! What explains this difference? We proved in ([12],Corollary 1.1) that if $B(B, F)$ is a Riesz pair then $D^+(F) \leq \mu(B)$ where the upper **Beurling density**

$$D^+(F) = \lim_{k\to\infty} \max_{a\in\mathbb{R}} \frac{|F \cap (a, a+k)|}{k}. \quad (3.6)$$

Here $|F\cap(a, a+k)|$ is the cardinality of $F\cap(a, a+k)$. This result was based on a deep result of Landau ([11], Theorem 3) in a form discussed by Olevskii and Ulanovskii [14]. Clearly, if χ_F is the Thue-Morse minimal sequence then $D^+(F) = \frac{1}{2}$, so for $\mu(B) < \frac{1}{2}$, $\alpha(B, F) = 0$. This means that trigonometric polynomials having frequencies in F can have their squared moduli **localized** on the set $\mathbb{T}\backslash B$. The coefficients of the most localized polynomial having frequencies in the finite set F_n are the entries of the eigenvectors corresponding to the eigenvalue $\alpha(B, F_n)$ of the restricted matrix. It is an open question if this happens for $\mu(B) \geq \frac{1}{2}$. The function in Figure 1 displays more intermittency than the function in Figure 2 because the gaps are larger. Perhaps this difference in intermittency can be used to explain the immense difference in the α values.

We used Equation 3.5 to compute $\alpha(B, F_n)$ as a function of $L = \log_2 n$ for ten ternary Cantor sets B with $\mu(B) \in \{0.5, 1.5, 2.5, ..., 9.5\}$. Figure 2 left shows the values of $\alpha(B, F_n)$ and Figure 2 right shows the values of $\log\, \alpha(B, F_n)$ for each of the ten sets. Both plots show that for $\mu(B) < \frac{1}{2}$,

$$\alpha(B, F) = \lim_{L\to\infty} \alpha(B, F_n) = 0.$$

However, Figure 1 shows that for $\mu(B) > \frac{1}{2}$, $\alpha(B, F_n)$ decreases as a function of L much slower and Figure 2 suggests that for $\mu(B) > \frac{1}{2}$, the sequence may not converge to 0 because $\log \alpha(B, F_n)$ appears to be a convex function of L. If this is the case then for $\mu(B) > \frac{1}{2}$, $\alpha(B, F) > 0$ and $\{F, 1+F, 2+F\}$ is a Riesz cover for B.

Suggestions for Further Research Theorem 3.2 shows that characterization of the set of extreme points $S_e(F)$ in the spectral envelopes of integer subsets F such that χ_F is a minimal sequence is crucial to understanding the FCE. For such a set F consider the dynamical system $(X(F), \sigma)$ where $X(F)$ is the orbit closure of χ_B. Then $X(F)$ has at least one shift invariant ergodic probability measure ζ.

Remark 3.1. If χ_F is the Thue-Morse minimal sequence then ζ is unique [8].

Define $X_1(F) = \{b \in X(F) : b(0) = 1\}$. For $g \in L^2(X, \zeta, \sigma)$ define $\sigma g(x) = g(\sigma\, x)$. Then the sequence $(\sigma^j\, g, g)$ is positive definite so by the Herglotz theorem there exists a positive measure $\nu_g \in M(\mathbb{T})$ such that $\nu_g(e_j) = (\sigma^j\, g, g), \quad j \in \mathbb{Z}$. Define the set

$$\Sigma(F, \zeta) = \{\nu_g : g \in L^2(X, \zeta),\ \text{support}(g) \subseteq X_1(F)\}.$$

The Birkhoff ergodic theorem can be used to show that $\Sigma(F, \zeta) \subset S(F)$. This fact, together with the fact that $P(F)$ contains no extreme points, suggests research to answer the

Question 3.2. Is $S_e(F) \subseteq \Sigma(F, \zeta)$?

The fact that generalized characters play a crucial role in characterizing the structure of the Banach algebra $M(\mathbb{T})$ suggests research to investigate their utility for characterizing spectral envelopes.

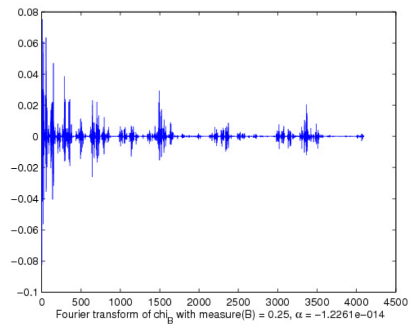

FIGURE 1. Fourier Transform of Characteristic Function of Ternary Fat Cantor Sets

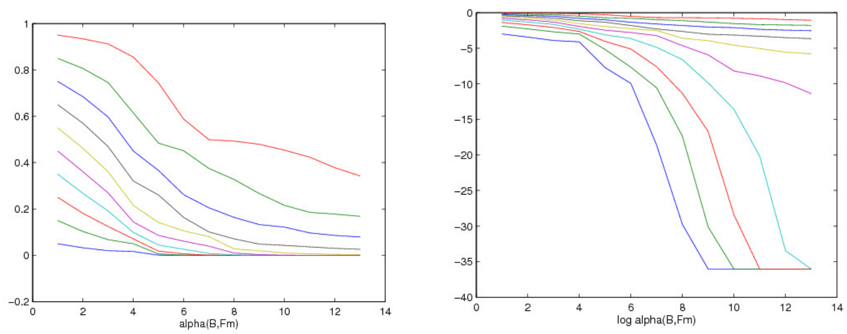

FIGURE 2. α(B,F), μ(B) = 0.05 : 0.1 : 0.95, as a function of (1 + $\log_2$ size F)

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A GENESIS OF GENERAL KKM THEOREMS FOR ABSTRACT CONVEX SPACES

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ABSTRACT. In a recent work of the author [42], we obtained three general KKM type theorems for abstract convex spaces. In this review, we show that two of them can be stated for intersectionally closed-valued KKM maps in the sense of Luc et al. [13]. Each of such KKM type theorems contains a large number of previously known particular forms. We recall some of them due to the author in order to give a short history on such theorems. Further comments on related works are given.

KEYWORDS : Abstract convex space; (partial) KKM principle; KKM space; Generalized (G-) convex space; Fixed point.

1. INTRODUCTION

Many problems in nonlinear analysis can be solved by showing the non-emptiness of the intersection of certain family of subsets of an underlying set. One of the remarkable results on such nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM theorem) in 1929 [10], which is concerned with certain types of multimaps later called the KKM maps. The KKM theory, first named this by the author [17], is the study of applications of equivalent formulations or generalizations of the KKM theorem.

Since 2006, we have introduced the new concepts of abstract convex spaces and KKM spaces which are adequate to establish the KKM theory. With such new concepts, we could generalize and simplify many known results in the theory on convex spaces, H-spaces, G-convex spaces, and others; see [27-33,39,40,42,43].

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In a recent work [42], we reviewed some known facts on abstract convex spaces and obtained three general KKM type theorems which are equivalent or can be extended to Theorems A, B, and C in this paper, resp. Each of them contains a large number of previously known particular forms which are generalizations, imitations, or modifications of the original KKM theorem due to many other authors. In the present paper, we recall some historically important previous particular versions of our KKM type theorems in [42] in order to give a short history on each of them. Moreover, further comments on related works are given.

2. ABSTRACT CONVEX SPACES

For the concepts of abstract convex spaces and KKM spaces, the reader may consult with our previous works [27-33,39,40,42,43].

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$]. Some authors use KKM instead of $\mathfrak{KC}$.

We have abstract convex subspaces as the following simple observation shows:

Proposition 2.3. ([42]) *For an abstract convex space $(E, D; \Gamma)$ and a nonempty subset D' of D, let X be a Γ-convex subset of E relative to D' and $\Gamma' : \langle D' \rangle \multimap X$ a map defined by*

$$\Gamma'_A := \Gamma_A \subset X \ for \ A \in \langle D' \rangle.$$

Then $(X, D'; \Gamma')$ itself is an abstract convex space called a subspace relative to D'.

Proposition 2.4. ([42]) *Let $(E, D; \Gamma)$ be an abstract convex space, $(X, D'; \Gamma')$ a subspace, and Z a topological space. If $F \in \mathfrak{KC}(E, D, Z)$, then $F|_X \in \mathfrak{KC}(X, D', \overline{F(X)})$.*

Definition 2.5. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E,D,E) \cap \mathfrak{KO}(E,D,E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a *KKM space* if it satisfies the KKM principle.

We had the following diagram for triples $(E,D;\Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space
$\Longrightarrow$ H-space $\Longrightarrow$ G-convex space $\Longleftrightarrow$ ϕ_A-space $\Longrightarrow$ KKM space
$\Longrightarrow$ Space satisfying the partial KKM principle
$\Longrightarrow$ Abstract convex space.

Example. There are plenty of examples of abstract convex spaces; see [31-33,39,40, 42,43]. Here we need only two classes of them:

(I) A *generalized convex space* or a *G-convex space* $(X,D;\Gamma)$ due to Park is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \longrightarrow \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$ and Δ_J its face corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

(II) A *ϕ_A-space* $(X,D;\{\phi_A\}_{A \in \langle D \rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \longrightarrow X$ (that is, singular n-simplexes) for $A \in \langle D \rangle$ with $|A| = n+1$. Every ϕ_A-space can be made into a G-convex space; see [28,41].

3. THE KKM THEOREM A

The following whole intersection property for the map-values of a KKM map is a standard form of the KKM type theorems:

Theorem A. *Let $(E,D;\Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E,D,E)$ [resp.,
$1_E \in \mathfrak{KO}(E,D,E)$], and $G : D \multimap E$ a multimap satisfying*

(1) G has closed [resp., open] values; and

(2) $\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).

Then $\{G(z)\}_{z \in D}$ has the finite intersection property.

Further, if

(3) $\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,

then we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Proof. The first part is a simple consequence of definition. For the second part, let $K := \bigcap_{z \in M} \overline{G(z)}$. Since $\{G(z) \mid z \in D\}$ has the finite intersection property, so does $\{K \cap \overline{G(z)} \mid z \in D\}$ in the compact set K. Hence it has the whole intersection property. □

Recall that Theorem A is a simple consequence of the definitions of the partial KKM principle or the KKM space.

From now on, in this section, we give historically well-known particular forms of Theorem A in the chronological order.

(I) Knaster, Kuratowski, and Mazurkiewicz in 1929 [10] obtained the following so-called KKM theorem from the Sperner combinatorial lemma, and applied it to a simple proof of the Brouwer fixed point theorem:

Theorem. ([10]) *Let* A_i $(0 \leq i \leq n)$ *be* $n+1$ *closed subsets of an* n*-simplex* $p_0 p_1 \cdots p_n$. *If the inclusion relation*

$$p_{i_0} p_{i_1} \cdots p_{i_k} \subset A_{i_0} \cup A_{i_1} \cup \cdots \cup A_{i_k}$$

holds for all faces $p_{i_0} p_{i_1} \cdots p_{i_k}$ $(0 \leq k \leq n, 0 \leq i_0 < i_1 < \cdots < i_k \leq n)$, *then* $\bigcap_{i=0}^{n} A_i \neq \emptyset$.

(II) A milestone on the history of the KKM theory was erected by Ky Fan in 1961 [3]. He extended the KKM theorem to arbitrary topological vector spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space.

Lemma. ([3]) *Let* X *be an arbitrary set in a topological vector space* Y. *To each* $x \in X$, *let a closed set* $F(x)$ *in* Y *be given such that the following two conditions are satisfied:*

(i) *The convex hull of any finite subset* $\{x_1, x_2, \cdots, x_n\}$ *of* X *is contained in* $\bigcup_{i=1}^{n} F(x_i)$.

(ii) $F(x)$ *is compact for at least one* $x \in X$.

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem and was shown to have numerous applications; see [24].

(III) In 1987, W.K. Kim [8] and M.-H. Shih and K.-K. Tan [51] independently showed that any simplex is a KKM space:

Theorem. ([8,51]) *Let* $X = \{x_1, \ldots, x_n\}$ *be the set of vertices of a simplex* S^{n-1} *in* $E = \mathbb{R}^n$ *and let* $F : X \multimap E$ *be an open-valued KKM map. Then* $\bigcap_{i=1}^{n} F(x_i) \neq \emptyset$.

However, the main idea was given in the earlier work of Fan [4, Theorem 2] as a matching theorem for closed coverings. Later, results of Kim [8,9] were generalized by the present author [15,16]. Moreover, Lassonde [12] refined Kim's idea and gave some applications.

(IV) The following shows that G-convex spaces are KKM spaces:

Theorem. ([25,26]) *Let* $(X, D; \Gamma)$ *be a* G*-convex space and* $F : D \multimap X$ *a map such that*

(1) F *has closed* [*resp., open*] *values; and*

(2) F *is a KKM map.*

Then $\{F(z)\}_{z \in D}$ *has the finite intersection property.*

Further, if

(3) $\bigcap_{z \in M} \overline{F(z)}$ *is compact for some* $M \in \langle D \rangle$,

then we have

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset.$$

(V) In 2008, we showed that any ϕ_A-space is a KKM space [34-36]:

Definition. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ and a topological vector space Z, let $F : X \multimap Z$ be a multimap. Then a map $G : D \multimap X$ satisfying

$$F(\phi_A(\Delta_J)) \subset G(J) \text{ for each } A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map w.r.t. F*.

Theorem. ([34-36]) *For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Then $\{G(z)\}_{z\in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D\rangle$ with $|N| = n+1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.)*

Further, if

(3) $\bigcap_{z\in M} \overline{G(z)}$ is compact for some $M \in \langle D\rangle$,

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

4. THE KKM THEOREM B

Recall that the main conclusions of most results in the preceding section are of the form

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

for a multimap $G : D \multimap E$.

Consider the following related four conditions:

(a) $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$ implies $\bigcap_{z\in D} G(z) \neq \emptyset$.

(b) $\bigcap_{z\in D} \overline{G(z)} = \overline{\bigcap_{z\in D} G(z)}$ (G is *intersectionally closed-valued* [13]).

(c) $\bigcap_{z\in D} \overline{G(z)} = \bigcap_{z\in D} G(z)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

In [13], the authors noted that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and gave examples of multimaps satisfying (b) but not (c). Therefore it is a proper time to deal with condition (b) instead of (c) in the KKM theory.

Example. The following maps G are intersectionally closed-valued, but not transfer closed-valued:

(1) $G(z)$ = (0, 1) for every $z \in$ [0, 1] is a constant multimap from D = $[0, 1]$ to E = [0, 1]; see [13].

(2) Each $G(z)$ is a convex set in a Euclidean space and has a relative interior point in common; see Rockafellar [50, Theorem 6.5].

(3) For a given subset E of a topological vector space with $x^* \in E$, each $G(z), z \in D$, is a nicely star-shaped at x^*; see [13].

From the partial KKM principle we have a whole intersection property of the Fan type as follows:

Theorem B. *Let $(E, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle [that is, $1_E \in \mathfrak{KC}(E, D, E)$] and $G : D \multimap E$ a map such that*

(1) *$\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D\rangle$]; and*

(2) *there exists a nonempty compact subset K of E such that either*

(i) $\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ *for some* $M \in \langle D \rangle$; *or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$L_N \cap \bigcap_{z \in D'} \overline{G(z)} \subset K.$$

Then we have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$;

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. As in [33,42,43], from the hypothesis, we must have $K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

(α) Since G is transfer closed-valued,

$$K \cap \bigcap_{z \in D} G(z) = K \cap \bigcap_{z \in D} \overline{G(z)} \neq \emptyset.$$

(β) Since G is intersectionally closed-valued,

$$\overline{\bigcap_{z \in D} G(z)} = \bigcap_{z \in D} \overline{G(z)} \neq \emptyset.$$

This implies the conclusion. □

Recall that conditions (i) and (ii) in Theorem B are usually called the *compactness conditions* or the *coercivity conditions*, and (ii) has numerous variations or particular forms appeared in a very large number of literature. Note that Theorem B can be easily deduced from the compact case of Theorem A, and hence it seems to be not a big problem to treat the case (ii).

From now on, in this section, we give some important forerunners of Theorem B:

(I) According to Lassonde [11], a *convex space* is a nonempty convex set (in a vector space) equipped with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite subset $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$.

In 1983, Lassonde gave the following:

Theorem. ([11]) *Let D be an arbitrary set in a convex space X, Y any topological space, and* $F : D \longrightarrow 2^Y$ *a multifunction having the following properties*

(i) *For each* $x \in D$, $F(x)$ *is compactly closed in Y.*

(ii) *For some continuous map* $s : X \longrightarrow Y$, *the multifunction* $G : D \longrightarrow 2^X$ *given by* $G(x) = s^{-1}(F(x))$ *is KKM.*

(iii) *For some c-compact set* $L \subset X$, $\bigcap\{F(x) \mid x \in L \cap D\}$ *is compact.*

Then $\bigcap\{F(x) \mid x \in D\} \neq \emptyset$.

Note that (iii) is a compactness condition implying condition (ii) of Theorem B.

In our work in 1989 [14], from this, we deduced a general Fan-Browder fixed point theorem with its various applications to analytic alternatives, section properties, fixed points, minimax and variational inequalities.

More general results for H-spaces $(X, D; \Gamma)$ are deduced in 1992 [18].

Note that the continuous maps s were later extended to acyclic maps, admissible maps, better admissible maps, and $\mathfrak{KC}$-maps by the author; see Section 5.

(II) In 1984, Fan obtained the following:

Theorem. ([4]) *In a topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \dots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$. If there is a nonempty subset X_0 of X such that the intersection $\bigcap_{x \in X_0} F(x)$ is compact and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} \neq \emptyset$.*

(III) A particular form of Theorem B for H-spaces $(X, D; \Gamma)$ was obtained and applied in 1993 [19].

(IV) In 2000 [26], for a G-convex space $(X \supset D; \Gamma)$, we had a particular form of Theorem B as follows:

Theorem. ([26]) *Let $(X \supset D; \Gamma)$ be a G-convex space, K a nonempty compact subset of X, and $F : D \multimap X$ a multimap such that*

(1) *F is transfer closed-valued;*

(2) *$\overline{F}$ is a KKM map; and*

(3) *for each $N \in \langle D \rangle$, there exists a compact G-convex subspace L_N of X containing N such that*

$$L_N \cap \bigcap \{\overline{F(z)} \mid z \in L_N \cap D\} \subset K.$$

Then $K \cap \bigcap \{F(z) \mid z \in D\} \neq \emptyset$.

Recall that this theorem generalizes earlier works of Tian, Ding, Chang et al., Lin and Park; see [26]. For H-spaces, the preceding theorem was given in [18,19]. Note that many particular forms are still happening. Moreover, condition (3) was first due to S.-Y. Chang [2] as a generalizations of Fan's original conditions, and had been adopted by the present author since 1992. However, still many authors are using particular forms of (3).

(V) In 2001 [49], Park and Lee defined generalized KKM maps on G-convex spaces. In 2007 [7], H. Kim and the author showed that a generalized KKM map G is a KKM map on a new G-convex space $(X, I; \Gamma^G)$ and, from Theorem B, deduced a KKM type theorem [42, Theorem 3.7] for generalized KKM maps with closed values.

(VI) When G is closed-valued or transfer closed-valued in condition (1) of Theorem B, we obtained the following already:

Theorem. ([42]) *Let $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $G : D \multimap X$ a map such that*

(1) *G is transfer closed-valued;*

(2) *$\overline{G}$ is a KKM map; and*

(3) *there exists a nonempty compact subset K of X such that either*

(i) *$K \supset \bigcap \{\overline{G(z)} \mid z \in M\}$ for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*

$$K \supset L_N \cap \bigcap \{\overline{G(z)} \mid z \in D'\}.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

This is an an equivalent form of [33, Theorem 8.2], subsumes a very large number of particular KKM type theorems in the literature, and has a number of equivalent formulations for abstract convex spaces satisfying the partial KKM principle as in [33]. Since the preceding theorem can be easily deduced from Theorem B, we do not need to think about the 'transfer' case.

5. THE KKM THEOREM C

Theorem B can be extended to $F \in \mathfrak{KC}(X, D, Z)$ instead of $1_X \in \mathfrak{KC}(X, D, X))$ as the following generalized form of [42, Theorem 2.10] shows:

Theorem C. *Let* $(X, D; \Gamma)$ *be an abstract convex space,* Z *a topological space,* $F \in \mathfrak{KC}(X, D, Z)$, *and* $G : D \multimap Z$ *a map such that*

(1) $\overline{G}$ *is a KKM map w.r.t.* F*; and*

(2) *there exists a nonempty compact subset* K *of* Z *such that either*

(i) $K \supset \bigcap\{\overline{G(y)} \mid y \in M\}$ *for some* $M \in \langle D\rangle$*; or*

(ii) *for each* $N \in \langle D\rangle$, *there exists a* Γ*-convex subset* L_N *of* X *relative to some* $D' \subset D$ *such that* $N \subset D'$, $F(L_N)$ *is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap\{\overline{G(z)} \mid z \in D'\}.$$

Then we have

$$\overline{F(X)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(X)} \cap K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$*; and*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. Case (i): Since $F(\Gamma_N) \subset \overline{G}(N)$ for each $N \in \langle D\rangle$ by (1), we have

$$F(\Gamma_N) \subset F(X) \cap \overline{G}(N) \subset \overline{F(X)} \cap \overline{G}(N) =: G'(N),$$

where $G'(y) := \overline{F(X)} \cap \overline{G(y)}$ is closed for each $y \in D$. Then, by Proposition 2.4 on $(X, D', \overline{F(X)})$, $\{G'(y) \mid y \in D\}$ has the finite intersection property. Since the requirement (i) implies

$$\overline{F(X)} \cap K \supset \overline{F(X)} \cap \bigcap_{y\in M} \overline{G(y)} = \bigcap_{y\in M} G'(y),$$

$\bigcap_{y\in M} G'(y)$ is compact. Therefore $\bigcap\{G'(y) \mid y \in D\} \neq \emptyset$ by Theorem A and hence

$$\overline{F(X)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Case (ii): Suppose that

$$\overline{F(X)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} = \emptyset.$$

Since $\overline{F(X)} \cap K$ is compact, $\overline{F(X)} \cap K \subset \bigcup\{Z \setminus \overline{G(y)} \mid y \in N\}$ for some $N \in \langle D\rangle$. Let L_N be the Γ-convex subset of X in (ii). Define $G' : D' \multimap \overline{F(L_N)}$ by

$G'(y) := \overline{G(y)} \cap \overline{F(L_N)}$ for $y \in D'$. For each $A \in \langle D' \rangle$, define $\Gamma'_A := \Gamma_A \cap L_N$. Then $(L_N, D'; \Gamma')$ is an abstract convex space. Moreover,

$$(F|_{L_N})(\Gamma'_A) \subset F(\Gamma_A) \cap F(L_N) \subset \overline{G}(A) \cap \overline{F(L_N)} = G'(A)$$

for each $A \in \langle D' \rangle$ by (2); and hence $G' : D' \multimap \overline{F(L_N)}$ is a KKM map w.r.t. $F|_{L_N}$ on the abstract convex space $(L_N, D'; \Gamma')$ with closed values in $\overline{F(L_N)}$. Since $F \in \mathfrak{KC}(X, D, Z)$, by Proposition 2.4, we have $F|_{L_N} \in \mathfrak{KC}(L_N, D', \overline{F(L_N)})$ and hence, $\{G'(y) \mid y \in D'\} = \{\overline{G(y)} \cap \overline{F(L_N)} \mid y \in D'\}$ has the finite intersection property. Since we assumed that $\overline{F(L_N)}$ is compact, each $G'(y)$ is compact. Hence $\bigcap\{G'(y) \mid y \in D'\} \neq \emptyset$ by Theorem A and there exists a

$$z \in \bigcap_{y \in D'} G'(y) = \overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K$$

by (ii). Since $z \in K$ and $z \in \overline{F(L_N)}$, we have $z \in \bigcup\{Z \setminus \overline{G(y)} \mid y \in N\}$ by our assumption. So $z \notin \overline{G(y)}$ for some $y \in N \subset D'$, and hence $z \notin \bigcap\{\overline{G(y)} \mid y \in D'\}$. This contradicts $z \in \bigcap\{G'(y) \mid y \in D'\}$. Therefore, we must have

$$\overline{F(X)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

(α) Since G is transfer closed-valued,

$$\overline{F(X)} \cap K \cap \bigcap_{y \in D} G(y) = \overline{F(X)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

(β) Since G is intersectionally closed-valued,

$$\overline{\bigcap_{y \in D} G(y)} = \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

This implies the conclusion. □

Note that Theorem C can be reformulated to the equivalent forms of coincidence theorems, matching theorems, analytic alternatives, minimax inequalities, geometric and section properties as in [7,33,48].

Let $cc(E)$ denote the set of nonempty closed convex subsets of a convex space E, $kc(E)$ the set of nonempty compact convex subsets of E, and $ka(E)$ the set of compact acyclic subsets of a topological space E.

Now we give a brief genesis of Theorem C as follows:

(I) In 1992, for a convex space and an acyclic map (that is, a u.s.c. map with compact acyclic values), we have the following KKM theorem [17, Theorem 3]:

Theorem. ([17]) *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, $F : X \longrightarrow ka(Y)$ a u.s.c. multimap, and K a nonempty compact subset of Y. Let $G : D \multimap Y$ be a multimap such that*

(1) *for each $x \in D$, $G(x)$ is closed;*

(2) *for each $N \in \langle D \rangle$, $F(\operatorname{co} N) \subset G(N)$; and*

(3) *there exists an $L_N \in kc(X)$ containing N such that $x \in L_N \setminus F^+(K)$ implies $\bigcap\{G(z) \mid z \in L_N \cap D\} \subset Y \setminus F(x)$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{G(x) \mid x \in D\} \neq \emptyset$.

For $X = Y$ and $F = 1_X$, condition (2) simply states that $G : D \multimap X$ is a KKM map. In [18] we showed that many of the key results of a large number of other papers are consequences of the preceding theorem.

(II) In 1994 [21], we introduced an *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ between topological spaces X and Y as the one such that, for each T and each compact subset K of X, there exists a map $\Gamma \in \mathfrak{A}_c(K, Y)$ satisfying $\Gamma(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(iii) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Theorem. ([21]) *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : D \multimap Y$ be a multimap such that*

(1) *for each $x \in D$, $G(x)$ is closed in Y;*

(2) *for any $N \in \langle D \rangle$, $F(\mathrm{co}\, N) \subset G(N)$; and*

(3) *there exist a nonempty compact subset K of Y and, for each $N \in \langle D \rangle$, a compact D-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{Gx \mid x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx \mid x \in D\} \neq \emptyset$.

Note that, if F is single-valued, the Hausdorffness assumption on Y is not necessary.

(III) The origin of the class $\mathfrak{KC}$ and Theorem C is the following [47, Theorem 3]:

Theorem. ([47]) *Let $(X \supset D; \Gamma)$ be a G-convex space, Y a Hausdorff space, and $F \in \mathfrak{A}_c^\kappa(X, Y)$. Let $G : D \multimap Y$ be a map such that*

(1) *for each $x \in D$, $G(x)$ is closed in Y;*

(2) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$; and*

(3) *there exist a nonempty compact subset K of Y such that either*

(i) *$\bigcap\{G(x) \mid x \in M\} \subset K$ for some $M \in \langle D \rangle$; or*

(ii) *for each $N \in \langle D \rangle$, a compact G-convex subset L_N of X containing N such that $F(L_N) \cap \bigcap\{G(x) \mid x \in L_N \cap D\} \subset K$.*

Then $\overline{F(X)} \cap K \cap \bigcap\{G(x) \mid x \in D\} \neq \emptyset$.

This was due to Park and Kim [47,48], where this had been reformulated to more than a dozen foundational results in the KKM theory. The class $\mathfrak{A}_c^\kappa$ in the above theorem can be replaced by the extended class $\mathfrak{B}$ for G-convex spaces.

This was given originally under the assumption that $D \subset X$, which is redundant in view of condition (ii) of Theorem C. In the preceding theorem, the admissible class $\mathfrak{A}_c^\kappa$ is a subclass of $\mathfrak{KC}$; and note that $F(L_N)$ is compact since L_N is compact and F can be regarded as a composition of u.s.c. maps having compact values (by the definition of $\mathfrak{A}_c^\kappa$).

In 1997 [48], we gave ten equivalent formulation of the preceding theorem in the form of coincidence theorems, matching theorems, analytic alternatives, minimax inequalities, geometric and section properties. Similarly, we can make equivalent formulations of Theorem C. Moreover, in [48], a large number of particular forms of the preceding theorem are listed. Note also that, after [48], there have appeared too many similar works on G-convex spaces and modifications of the preceding theorem to trace out all of them.

(IV) The KKM theorem in (III) was modified by Kalmoun and Rihai [5] in 2001 as follows: For a transfer closed-valued map G in (1) and by considering $\overline{G}$ instead of G in (2) and (3), they deduced the same conclusion as in Theorem C. By applying this modification, they obtained an existence theorem for generalized vector equilibrium problems and applied it to greatest element, fixed point, and vector saddle point problems within the frame of G-convex spaces. A particular form of their KKM theorem was applied in 2003 [6] to existence for vector equilibrium, mixed variational inequalities, greatest elements for a binary relation, and the Fan-Browder fixed point theorem.

(V) Let X be a convex space and Y a Hausdorff space. In 1997 [22], we introduced a new "better" admissible class $\mathfrak{B}$ of multimaps as follows:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ such that, for any polytope P in X and any continuous map $f : F(P) \longrightarrow P$, $f(F|_P)$ has a fixed point.

The following KKM theorem is due to the author [22, Theorem 3]:

Theorem. ([22]) *Let X be a convex space, Y a Hausdorff space, $F \in \mathfrak{B}(X,Y)$ a compact map, and $S : X \multimap Y$ a map. Suppose that*
(1) *for each $x \in X$, $S(x)$ is closed; and*
(2) *for each $N \in \langle X \rangle$, $F(\operatorname{co} N) \subset S(N)$.*
Then $\overline{F(X)} \cap \bigcap\{S(x) \mid x \in X\} \neq \emptyset$.

This KKM theorem was applied in [23] to a minimax inequality related to admissible multimaps, from which we deduced generalized versions of lopsided saddle point theorems, fixed point theorems, existence of maximizable linear functionals, the Warlas excess demand theorem, and the Gale-Nikaido-Debreu theorem.

In 2010 [1], the preceding theorem is shown to be equivalent to some existence theorems of variational inclusion problems. These were applied to existence theorems of common fixed point, generalized maximal element theorems, a general coincidence theorems and a section theorem.

(VI) There have also appeared a large number of the so-called generalized KKM maps in the literature. In fact, a number of authors tried to generalize the concept of KKM maps on particular cases of ϕ_A-spaces. All such KKM maps are known to be the ones for certain G-convex spaces; see [34].

6. COMMENTS ON RELATED WORKS

There are several hundred published papers on generalizations of the KKM theorem. Recently, in order to upgrade the KKM theory, we have tried to criticize some inappropriate results of other authors. We give abstracts of a few examples of such papers by the present author:

(I) In [34], we introduced basic results in the KKM theory on abstract convex spaces and the KKM maps. These were applied to various modifications of the concepts of generalized convex spaces and KKM type maps. We discuss the nature of those modifications and criticize recently appeared 'generalizations' of our previous works due to many other authors.

(II) Basic results in the KKM theory on abstract convex spaces and the KKM maps are applied to ϕ_A-spaces which unify various imitations of G-convex spaces in [37]. We also showed that basic theorems on ϕ_A-spaces can be applied to correct and improve results on the so-called R-KKM maps on the so-called L-convex spaces in a work of C.M. Chen.

(III) In [38], we introduced a new concept of abstract convex minimal spaces which was used to establish typical results in the KKM theory. Since any minimal space can be made into a topological space, results on abstract convex minimal spaces can be deduced from the theory on abstract convex spaces. In this way, the KKM type theorems were used to obtain coincidence theorems, the Fan-Browder type fixed point theorems, the Fan intersection theorem, and the Nash equilibrium theorem on abstract convex minimal spaces.

(IV) Recently, some authors adopted the concept of the so-called *generalized R-KKM maps* which were used to rewrite known results in the KKM theory. In [44], we showed that those maps are simply KKM maps on G-convex spaces. Consequently, results on generalized R-KKM maps follow the corresponding previous ones on G-convex spaces.

(V) In a paper by Hou Jicheng, *On some KKM type theorems*, Advances in Mathematics, 36(1) (2007), 86–88, the author claimed that some previous KKM type theorems are false by giving a counterexample. In [45], we showed that the counterexample does not work and, consequently, the results are correct. Moreover, we claimed that the artificial concept like transfer compactly closed-valued maps can be destroyed. Finally, we introduced a theorem generalizing the main target of Hou.

(VI) In the KKM theory, instead of the concepts of closure, interior, closed-valued multimap, l.s.c. multimap, finite open cover, etc., resp., some authors adopt ccl, cint, transfer compactly closed-valued map, transfer compactly l.s.c. multimap, transfer compactly local intersection property, etc., resp. In [46], we showed that such inappropriate and artificial concepts can be invalidated. For example, by giving finer topologies on the underlying space, we can invalidate "compactly"-attached terminology. In such ways, we obtained simpler formulations of some KKM type theorems, some Fan-Browder type fixed point theorems, and others.

This is why we eliminated such useless terminology in this paper.

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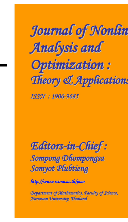

WEAK CONVERGENCE THEOREMS FOR GENERALIZED HYBRID MAPPINGS IN BANACH SPACES

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ABSTRACT. Let E be a real Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. In this paper, we first deal with some properties for generalized hybrid mappings in a Banach space. Then, we prove weak convergence theorems of Mann's type for such mappings in a Banach space satisfying Opial's condition.

KEYWORDS : Banach space; Nonexpansive mapping; Nonspreading mapping; Hybrid mapping; Fixed point; Weak convergence.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty subset of H. Then a mapping $T : C \longrightarrow H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$$

for all $x, y \in C$; see, for instance, Browder [3] and Goebel and Kirk [7]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium

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problem in a Hilbert space; see, for instance, [2] and [6]. Recently, Kohsaka and Takahashi [15], and Takahashi [21] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \longrightarrow H$ is called nonspreading [15] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.1}$$

for all $x, y \in C$. A mapping $T : C \longrightarrow H$ is called hybrid [21] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \tag{1.2}$$

for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [14], Iemoto and Takahashi [10] and Takahashi and Yao [23]. Motivated by these mappings and results, Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced a broad class of nonlinear mappings in a Hilbert space called λ-hybrid which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings. Furthermore, Kocourek, Takahashi and Yao [12] introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings in a Hilbert space. They called such a class the class of generalized hybrid mappings and then proved general fixed point theorems and some convergence theorems in a Hilbert space; see also [25] and [8]. Hsu, Takahashi and Yao [9] extended this class of generalized hybrid mappings in a Hilbert space to Banach spaces and they also called such a class the class of generalized hybrid mappings. Further, they proved general fixed point theorems in a Banach space; see also [13].

In this paper, we first deal with some properties for generalized hybrid mappings in a Banach space. Then, we prove weak convergence theorems for such mappings in a Banach space satisfying Opial's condition.

2. PRELIMINARIES

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \longrightarrow x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \longrightarrow E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \longrightarrow E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty closed convex subset of a strictly convex Banach space E and $T : C \longrightarrow C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [11]. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is norm to weak* uniformly continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm to norm continuous. If E is uniformly smooth, J is norm to norm uniformly continuous on each bounded subset of E. For more details, see [19, 20]. The following results are also in [19, 20].

Theorem 2.1. *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j\rangle,$$

where $j \in Jy$.

Theorem 2.2. *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

The following result was proved by Xu [26].

Theorem 2.3 (Xu [26])**.** *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a Banach space. Then, E satisfies Opial's condition [17] if for any $\{x_n\}$ of E such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n \longrightarrow \infty} \|x_n - x\| < \liminf_{n \longrightarrow \infty} \|x_n - y\|.$$

Let E be a Banach space and let $A \subset E \times E$. Then, A is accretive if for $(x_1, y_1), (x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$, where J is the duality mapping of E. An accretive operator $A \subset E \times E$ is called m-accretive if $R(I + rA) = E$ for all $r > 0$, where I is the identity operator and $R(I + rA)$ is the range of $I + rA$. An accretive operator $A \subset E \times E$ is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$, where $\overline{D(A)}$ is the closure of the domain $D(A)$ of A. An m-accretive operator satisfies the range condition.

3. GENERALIZED HYBRID MAPPINGS IN BANACH SPACES

Let E be a Banach space and let C be a nonempty subset of E. Then, a mapping $T : C \longrightarrow E$ is said to be firmly nonexpansive [4] if

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle,$$

for all $x, y \in C$, where $j \in J(Tx - Ty)$. It is known that the resolvent of an accretive operator satisfying the range condition in a Banach space is a firmly nonexpansive mapping. In fact, let $C = \overline{D(A)}$ and $r > 0$. Define the resolvent J_r of A as follows:

$$J_r x = \{z \in D(A) : x \in z + rAz\}$$

for all $x \in \overline{D(A)}$. It is known that such $J_r x$ is a singleton; see [19]. We have that for $x_1, x_2 \in \overline{D(A)}$, $x_1 = z_1 + ry_1,\ y_1 \in Az_1$ and $x_2 = z_2 + ry_2,\ y_2 \in Az_2$. Since A is accretive, we have that $\langle y_1 - y_2, j\rangle \geq 0$, where $j \in J(z_1 - z_2)$. So, we have

$$\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j\rangle \geq 0.$$

Furthermore, we have that

$$\begin{aligned} &\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j\rangle \geq 0 \\ &\qquad \Longleftrightarrow \langle x_1 - z_1 - (x_2 - z_2), j\rangle \geq 0 \\ &\qquad \Longleftrightarrow \langle x_1 - x_2, j\rangle \geq \|z_1 - z_2\|^2. \end{aligned}$$

From $z_1 = J_r x_1$ and $z_2 = J_r x_2$, we have that J_r is a firmly nonexpansive mapping; see also [4], [5] and [24]. From [9] we know that the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings are deduced from the class of firmly nonexpansive mappings in a Banach space. In general, Hsu, Takahashi and Yao [9] defined a class of nonlinear mappings in a Banach space containing the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings as follows: Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.1}$$

for all $x, y \in C$. They also called such a mapping an (α, β)-generalized hybrid mapping in a Banach space. We note that an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. As in [9], we have the following result in a Banach space; see [9] for the proof.

Theorem 3.1. *Let C be a nonempty subset of a Banach space E and let T be a generalized hybrid mapping of C into E, i.e., there are $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.2}$$

for all $x, y \in C$. Then, the following hold:

(i) *If $\alpha + \beta < 1$, then $T = I$, where $Ix = x$ for all $x \in C$;*
(ii) *if $\alpha = 0$ and $\beta = 1$, then T satisfies that $\|Tx - y\| = \|Ty - x\|$ for all $x, y \in C$;*
(iii) *if $\alpha = 0$ and $\beta > 1$, then T satisfies that*

$$2\|x - y\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$;
(iv) *if $\beta = t\alpha + 1$, $-1 \leq t < \infty$ and $\alpha > 0$, then T satisfies that*

$$2\|Tx - Ty\|^2 + 2t\|x - y\|^2 \leq (t+1)\|Tx - y\|^2 + (t+1)\|Ty - x\|^2$$

for all $x, y \in C$. In particular, T is nonexpansive for $t = -1$, nonspreading for $t = 0$, and hybrid for $t = -\frac{1}{2}$;

(v) *if* $\beta = t\alpha + 1$, $-\infty < t < -1$ *and* $\alpha < 0$, *then* T *satisfies that*

$$2\|Tx - Ty\|^2 + 2t\|x - y\|^2 \geq (t + 1)\|Tx - y\|^2 + (t + 1)\|Ty - x\|^2$$

for all $x, y \in C$.

Furthermore, we have the following result.

Theorem 3.2. *Let* E *be a Banach space, let* C *be a nonempty subset of* E *and let* $\lambda \in [0, 1]$. *Then the following hold:*

(i) *A generalized hybrid mapping with a fixed point is quasi-nonexpansive;*
(ii) *a firmly nonexpansive mapping is* $(2 - \lambda, 1 - \lambda)$*-generalized hybrid.*

Proof. We show (i). Since $T : C \longrightarrow E$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{3.3}$$

for all $x, y \in C$. Let $u \in F(T)$. Then we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1 - \alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1 - \beta)\|u - y\|^2 \tag{3.4}$$

and hence $\|u - Ty\|^2 \leq \|u - y\|^2$. This implies that T is quasi-nonexpansive. We next show (ii). Let T be a firmly nonexpansive mapping of C into E. Then we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle.$$

From Theorem 2.1 we have that

$$\begin{aligned}\|Tx-Ty\|^2 &\leq \langle x - y, j\rangle \\ &\iff 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\ &\Longrightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 \\ &\iff \|Tx - Ty\|^2 \leq \|x - y\|^2 \\ &\iff \|Tx - Ty\| \leq \|x - y\|.\end{aligned}$$

So, for $\lambda \in [0, 1]$ we have

$$\lambda\|Tx - Ty\|^2 \leq \lambda\|x - y\|^2. \tag{3.5}$$

Futhermore, we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned}\|Tx-Ty\|^2 &\leq \langle x - y, j\rangle \\ &\iff 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\ &\iff 0 \leq 2\langle x - Tx, j\rangle + 2\langle Ty - y, j\rangle \\ &\Longrightarrow 0 \leq \|x - Ty\|^2 - \|Tx - Ty\|^2 + \|Tx - y\|^2 - \|Tx - Ty\|^2 \\ &\iff 0 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\ &\iff 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.\end{aligned}$$

Thus, for $\lambda \in [0, 1]$ we have

$$2(1 - \lambda)\|Tx - Ty\|^2 \leq (1 - \lambda)\|x - Ty\|^2 + (1 - \lambda)\|y - Tx\|^2. \tag{3.6}$$

Therefore, we have from (3.5) and (3.6) that

$$(2 - \lambda)\|Tx - Ty\|^2 \leq (1 - \lambda)\|x - Ty\|^2 + (1 - \lambda)\|y - Tx\|^2 + \lambda\|x - y\|^2$$

and hence

$$(2 - \lambda)\|Tx - Ty\|^2 + (\lambda - 1)\|x - Ty\|^2 \leq (1 - \lambda)\|y - Tx\|^2 + \lambda\|x - y\|^2.$$

This implies that T is a $(2 - \lambda, 1 - \lambda)$-generalized hybrid mapping. □

Using Takahashi and Jeong's result [22], Hsu, Takahashi and Yao [9] also proved the following fixed point theorem for generalized hybrid mappings in a Banach space.

Theorem 3.3. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a generalized hybrid mapping. Then the following are equivalent:*

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Using Theorem 3.3, they also proved the following fixed point theorems in a Banach space.

Theorem 3.4. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Theorem 3.5. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Theorem 3.6. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

4. SOME PROPERTIES OF GENERALIZED HYBRID MAPPINGS

Let E be a Banach space. Let C be a nonempty subset of E. Let $T : C \longrightarrow C$ be a mapping. Then, $p \in C$ is called an asymptotic fixed point of T [18] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $I - T$ of C into E is said to be demiclosed on C if $\hat{F}(T) = F(T)$.

Theorem 4.1. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta \in \mathbb{R}$ and let T be an (α, β)-generalized hybrid mapping of C into itself such that $\alpha \geq 1$ and $\beta \geq 0$. Then $\hat{F}(T) = F(T)$, i.e., $I - T$ is demiclosed.*

Proof. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{4.1}$$

for all $x, y \in C$. The inclusion $F(T) \subset \hat{F}(T)$ is obvious. Thus we show $\hat{F}(T) \subset F(T)$. Let $u \in \hat{F}(T)$ be given. Then we have a sequence $\{x_n\}$ of C such that

$x_n \rightharpoonup u$ and $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$. Since $T : C \longrightarrow C$ is a generalized hybrid mapping, we obtain that

$$\alpha\|Tx_n - Tu\|^2 + (1-\alpha)\|x_n - Tu\|^2 \leq \beta\|Tx_n - u\|^2 + (1-\beta)\|x_n - u\|^2. \quad (4.2)$$

From $\alpha \geq 1$, $\beta \geq 0$ and (4.2), we have

$$\begin{aligned}\alpha\|Tx_n - Tu\|^2 \leq \beta(\|Tx_n - x_n\| + \|x_n - u\|)^2 + (1-\beta)\|x_n - u\|^2 \\ + (\alpha - 1)(\|x_n - Tx_n\| + \|Tx_n - Tu\|)^2.\end{aligned}$$

So, we have that

$$\begin{aligned}(\alpha - (\alpha - 1))\|Tx_n - Tu\|^2 \leq (\beta + (1-\beta))\|x_n - u\|^2 + (\beta + \alpha - 1)\|x_n - Tx_n\|^2 \\ + 2(\beta + \alpha - 1)(\|x_n - u\| + \|Tx_n - Tu\|)\|Tx_n - x_n\|\end{aligned}$$

and hence

$$\begin{aligned}\|Tx_n - Tu\|^2 \leq \|x_n - u\|^2 + (\beta + \alpha - 1)\|x_n - Tx_n\|^2 \\ + 2(\beta + \alpha - 1)(\|x_n - u\| + \|Tx_n - Tu\|)\|Tx_n - x_n\|.\end{aligned} \quad (4.3)$$

From $x_n \rightharpoonup u$, we obtain that $\{x_n\}$ is bounded. From $\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$ we also have that $\{Tx_n\}$ is bounded. So, we can take a positive constant M such that

$$\sup\{\|x_n - u\| + \|Tx_n - Tu\| : n \in \mathbb{N}\} \leq M. \quad (4.4)$$

Suppose $Tu \neq u$. Then we have from Opial's condition, (4.3) and (4.4) that

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \|x_n - u\|^2 &< \liminf_{n\longrightarrow\infty} \|x_n - Tu\|^2 \\ &= \liminf_{n\longrightarrow\infty} \|x_n - Tx_n + Tx_n - Tu\|^2 \\ &= \liminf_{n\longrightarrow\infty} \|Tx_n - Tu\|^2 \\ &\leq \liminf_{n\longrightarrow\infty}(\|x_n - u\|^2 + (\beta + \alpha - 1)\|x_n - Tx_n\|^2 \\ &\quad + 2(\beta + \alpha - 1)M\|Tx_n - x_n\|) \\ &= \liminf_{n\longrightarrow\infty} \|x_n - u\|^2.\end{aligned}$$

This is a contradiction. So, we have $Tu = u$ and hence $\hat{F}(T) \subset F(T)$. This completes the proof. □

Remark. We do not know that the demiclosedness property for a generalized hybrid mapping holds or not in a uniformly convex Banach space.

Using Theorem 4.1, we can prove the following theorems in a Banach space.

Theorem 4.2. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. In Theorem 4.1, a $(1, 0)$-generalized hybrid mapping of C into itself is nonexpansive. Further, $\alpha = 1 \geq 1$ and $\beta = 0 \geq 0$. By Theorem 4.1, $I - T$ is demiclosed on C. □

Theorem 4.3. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. In Theorem 4.1, a (2, 1)-generalized hybrid mapping of C into itself is nonspreading. Further, $\alpha = 2 > 1$ and $\beta = 1 > 0$. By Theorem 4.1, $I - T$ is demiclosed on C. □

Theorem 4.4. *Let E be a Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let $T : C \longrightarrow C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2 + \|x - y\|^2, \quad \forall x, y \in C.$$

Then, $I - T$ is demiclosed on C.

Proof. In Theorem 4.1, a $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping of C into itself is hybrid. Further, $\alpha = \frac{3}{2} > 1$ and $\beta = \frac{1}{2} > 0$. By Theorem 4.1, $I - T$ is demiclosed on C. □

Next, we have the following property of the fixed point set of a generalized hybrid mapping in a Banach space.

Theorem 4.5. *Let E be a strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be a generalized hybrid mapping of C into itself. Then $F(T)$ is closed and convex.*

Proof. Let $T : C \rightarrow C$ be a generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{4.5}$$

for all $x, y \in C$. If $F(T)$ is empty, then $F(T)$ is closed and convex. If $F(T)$ is nonempty, then we have from Theorem 3.2 that T is quasi-nonexpansive. From Itoh and Takahashi [11], we have that $F(T)$ is closed and convex. □

Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \longrightarrow C$ is called asymptotically regular if for any $x \in C$,

$$T^{n+1}x - T^n x \longrightarrow 0.$$

Theorem 4.6. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let T be a generalized hybrid mapping of C into itself with $F(T) \neq \emptyset$ and let γ be a real number with $0 < \gamma < 1$. Define a mapping $S : C \longrightarrow C$ by*

$$S = \gamma I + (1 - \gamma)T.$$

Then, for any $x \in C$, $S^{n+1}x - S^n x$ converges strongly to 0, i.e., S is asymptotically regular.

Proof. Let $T : C \rightarrow C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Then, from Theorem 3.2 we have that T is quasi-nonexpansive. Using that T is quasi-nonexpansive, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|S^{n+1}x - u\| &= \|SS^n x - u\| \\
&= \|\gamma S^n x + (1 - \gamma)TS^n x - u\| \\
&= \|\gamma(S^n x - u) + (1 - \gamma)(TS^n x - u)\| \\
&\le \gamma\|S^n x - u\| + (1 - \gamma)\|TS^n x - u\|
\end{aligned}$$

$$\leq \gamma\|S^n x - u\| + (1-\gamma)\|S^n x - u\|$$
$$= \|S^n x - u\|.$$

So, we have that $\lim_{n\longrightarrow\infty}\|S^n x - u\|$ exists. Then, $\{S^n x\}$ is bounded. Since T is quasi-nonexpansive, $\{TS^n x\}$ is also bounded. Let

$$r = \max\{\sup_{n\in\mathbb{N}}\|S^n x - u\|, \sup_{n\in\mathbb{N}}\|TS^n x - u\|\}.$$

Then, from Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0,\infty) \longrightarrow [0,\infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$. So, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|S^{n+1} - u\|^2 &= \|SS^n x - u\|^2 \\
&= \|\gamma S^n x + (1-\gamma)TS^n x - u\|^2 \\
&\leq \gamma\|S^n x - u\|^2 + (1-\gamma)\|TS^n x - u\|^2 - \gamma(1-\gamma)g(\|S^n x - TS^n x\|) \\
&\leq \gamma\|S^n x - u\|^2 + (1-\gamma)\|S^n x - u\|^2 - \gamma(1-\gamma)g(\|S^n x - TS^n x\|) \\
&= \|S^n x - u\|^2 - \gamma(1-\gamma)g(\|S^n x - TS^n x\|) \\
&\leq \|S^n x - u\|^2
\end{aligned}$$

and hence

$$\gamma(1-\gamma)g(\|S^n x - TS^n x\|) \leq \|S^n x - u\|^2 - \|S^{n+1}x - u\|^2.$$

Since $\lim_{n\longrightarrow\infty}\|S^n x - u\|^2$ exists and $0 < \gamma < 1$, we have

$$\lim_{n\longrightarrow\infty} g(\|S^n x - TS^n x\|) = 0.$$

From the properties of g, we have $\lim_{n\longrightarrow\infty}\|S^n x - TS^n x\| = 0$. From

$$\|S^{n+1}x - TS^n x\| = \|\gamma S^n x + (1-\gamma)TS^n x - TS^n x\| = \gamma\|S^n x - TS^n x\|,$$

we have that

$$\begin{aligned}
\|S^{n+1}x - S^n x\| &= \|S^{n+1}x - TS^n x + TS^n x - S^n x\| \\
&\leq \|S^{n+1}x - TS^n x\| + \|TS^n x - S^n x\| \\
&= \gamma\|S^n x - TS^n x\| + \|TS^n x - S^n x\| \longrightarrow 0.
\end{aligned}$$

This completes the proof. □

5. WEAK CONVERGENCE THEOREMS

In this section, we first prove a weak convergence theorem of Mann's type [16] for generalized hybrid mappings in a Banach space satisfying Opial's condition.

Theorem 5.1. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let $\alpha, \beta \in \mathbb{R}$ and let T be an (α,β)-generalized hybrid mapping of C into itself such that $\alpha \geq 1$ and $\beta \geq 0$. Let $\{\gamma_n\}$ be a sequence of real numbers with $0 < a \leq \gamma_n \leq b < 1$ and define a sequence $\{x_n\}$ of C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n)Tx_n, \quad \forall n \in \mathbb{N}.$$

If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to some element $z \in F(T)$.

Proof. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping., i.e.,

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Since $F(T) \neq \emptyset$, we know from Theorem 3.2 that T is quasi-nonexpansive. Using the fact that T is quasi-nonexpansive, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - u\| &= \|\gamma_n x_n + (1-\gamma_n)Tx_n - u\| \\ &= \|\gamma_n(x_n - u) + (1-\gamma_n)(Tx_n - u)\| \\ &\le \gamma_n\|x_n - u\| + (1-\gamma_n)\|Tx_n - u\| \\ &\le \gamma_n\|x_n - u\| + (1-\gamma_n)\|x_n - u\| \\ &= \|x_n - u\|.\end{aligned}$$

So, we have that $\lim_{n\longrightarrow\infty} \|x_n - u\|$ exists. Then, $\{x_n\}$ is bounded. Since T is quasi-nonexpansive, $\{Tx_n\}$ is also bounded. Let

$$r = \max\{\sup_{n\in\mathbb{N}} \|x_n - u\|, \sup_{n\in\mathbb{N}} \|Tx_n - u\|\}.$$

Then, from Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \longrightarrow [0, \infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \le \lambda \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$. So, we have that for any $u \in F(T)$, $x \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\gamma_n x_n + (1-\gamma_n)Tx_n - u\|^2 \\ &= \|\gamma_n(x_n - u) + (1-\gamma_n)(Tx_n - u)\|^2 \\ &\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|Tx_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\ &\le \gamma_n\|x_n - u\|^2 + (1-\gamma_n)\|x_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\ &= \|x_n - u\|^2 - \gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \\ &\le \|x_n - u\|^2\end{aligned}$$

and hence

$$\gamma_n(1-\gamma_n)g(\|x_n - Tx_n\|) \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Since $\lim_{n\longrightarrow\infty} \|x_n - u\|^2$ exists, we have from $0 < a \le \gamma_n \le b < 1$ that

$$\lim_{n\longrightarrow\infty} g(\|x_n - Tx_n\|) = 0.$$

From the properties of g, we have

$$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0. \tag{5.1}$$

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $u \in C$. Using Theorem 4.1 and (5.1), we have $Tu = u$. Let us show that the entire sequence $\{x_n\}$ converges weakly to some point of $F(T)$. To show it, let us take two subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. Suppose $u \neq v$. From $u, v \in F(T)$, we know that $\lim_{n\longrightarrow\infty} \|x_n - u\|$ and $\lim_{n\longrightarrow\infty} \|x_n - v\|$ exist. Since E satisfies Opial's condition, we have that

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|x_n - u\| &= \lim_{i\longrightarrow\infty} \|x_{n_i} - u\| \\ &< \lim_{i\longrightarrow\infty} \|x_{n_i} - v\|\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n \longrightarrow \infty} \|x_n - v\| \\
&= \lim_{j \longrightarrow \infty} \|x_{n_j} - v\| \\
&< \lim_{j \longrightarrow \infty} \|x_{n_j} - u\| \\
&= \lim_{n \longrightarrow \infty} \|x_n - u\|.
\end{aligned}
$$

This is a contradiction. So, we must have $u = v$. This implies that $\{x_n\}$ converges weakly to a point of $F(T)$. □

Remark. We do not know that such a weak convergence theorem for a generalized hybrid mapping holds or not in a uniformly convex Banach space which has a Fréchet differentiable norm.

Using Theorem 5.1, we obtain the following result.

Theorem 5.2. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex subset of E. Let T be a generalized hybrid mapping of C into itself with $F(T) \neq \emptyset$ and let γ be a real number with $0 < \gamma < 1$. Define a mapping $S : C \longrightarrow C$ by*

$$S = \gamma I + (1 - \gamma)T.$$

Then, for any $x \in C$, $S^n x$ converges weakly to an element $z \in F(T)$.

Proof. Putting $\gamma_n = \gamma$ for all $n \in \mathbb{N}$ and $S = \gamma I + (1 - \gamma)T$, we have that for any $x \in C$,

$$x_2 = Sx_1 = Sx, x_3 = S^2 x_1 = S^2 x, \ldots$$

in Theorem 5.1. So, we have from Theorem 5.1 that $S^n x$ converges weakly to an element $z \in F(T)$. This completes the proof. □

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EXISTENCE OF SYSTEMS OF GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS IN PRODUCT FC-SPACES◇

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ABSTRACT. In this paper, three classes of systems of generalized vector quasi-equilibrium problems are introduced and studied in product FC-spaces without convexity structure. We prove some new equilibrium existence theorems for three classes of systems of generalized vector quasi-equilibrium problems in noncompact product FC-spaces. These results improve and generalize some recent results in literature to product FC-spaces without any convexity structure.

KEYWORDS : Systems of generalized vector quasi-equilibrium problem; $C_i(x) - FC-$ diagonal quasi-convex; $C_i(x) - FC-$ quasi-convex; $C_i(x) - FC-$ quasi-convex-like; FC-spaces.

1. INTRODUCTION

Let X be a convex subset of a real topological vector space E (in short t.v.s.) and $F : X \times X \to R$ be a given function with $F(x, x) \geq 0$ for all $x \in X$. By equilibrium problem, Blum and Oettli [1] considered the problem of finding $u \in X$ such that $F(u, y) \geq 0$ for all $y \in X$. This problem contains optimization problems, Nash type equilibria problems, variational inequality problems, complementary problems and fixed point problems as special case. In 1980, Giannessi [2] introduced the vector variational inequality problem in finite dimensional Euclidean spaces. From the above applications, generalized vector quasi-equilibrium problems, and system of generalized vector quasi-equilibrium problems have become important developed directions of vector variational inequality theory, for example, see [4-26].

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In 2000, Ansari et al. [4] introduced the system of vector equilibrium problems (for short, SVEP), that is, a family of equilibrium problems for vector-valued bifunctions defined on a product set, with applications in vector optimization problems and Nash equilibrium problem [5] for vector valued functions. Recently, Ansari et al. [6] introduced the following concept of system of vector quasi-equilibrium problems (in short, SVQEP) as follows (see also in [7-13]; Let I be any index set and for each $i \in I$, let X_i be a topological vector space. Consider a family of nonempty convex subsets $\{K_i\}_{i\in I}$ with $K_i \subset X_i$. We denote by $K = \prod_{i\in I} K_i$ and $X = \prod_{i\in I} X_i$. For each $i \in I$, let Y_i be a topological vector space and let $C_i : K \longrightarrow 2^{Y_i}$, $S_i : K \longrightarrow 2^{K_i}$ and $F_i : K \times K_i \longrightarrow 2^{Y_i}$ be multi-valued mappings. The system of vector quasi-equilibrium problems (in short, SVQEP), that is, to find $x \in K$ such that for each $i \in I$,

$$x_i \in S_i(x) \ : \ F_i(x, y_i) \notin -intC_i(x) \ \forall y_i \in S_i(x). \tag{1.1}$$

If $S_i(x) = K_i$ for all $x \in K$, then (SVQEP) reduces to (SVEP) (see [4]) and if the index set I is singleton, then (SVQEP) becomes the vector quasi-equilibrium problem which contains vector quasi-optimization problem and vector quasi-variational inequality problem as special cases (see [3]).

In 2010, Li and Li [14] considered three following problems. Let let X, Y and Z be three real topological spaces, let X and Y be Hausdorff spaces, $E \subset X$ and $D \subset Z$ be two nonempty subsets. Let $C : X \to 2^Y$ be set-valued mapping such that $C(x)$ be a proper, closed and convex cone of Y with nonempty interior. Let $S : E \to 2^E, T : E \to 2^D$ and $F : E \times D \times E \to 2^Y$ be three set-valued maps. Three classes of generalized vector quasi-equlilbrium problems: Find $\bar{x} \in E$ and $\bar{z} \in T(\bar{x})$ such that

(i) (GVQEP I) $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{z}) \nsubseteq -intC(\bar{x}), \forall y \in S(\bar{x})$.

(ii) (GVQEP II) $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{z}) \cap -intC(\bar{x}) = \emptyset, \forall y \in S(\bar{x})$.

(iii) (GVQEP III) $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{z}) \subset -C(\bar{x}), \forall y \in S(\bar{x})$.

Moreover, they obtained some existence results by using the well know Fan-KKM theorem without the compact assumption and unless otherwise specified.

On the other hand, it is well known that many existence theorems of maximal elements for set-valued mappings have been established in topological vector spaces, H-spaces and G-convex spaces by many authors. The notion of generalized convex (in short, G-convex) spaces was introduced by Park and Kim in [15, 16] . In 2005, Ding [17] was introduced the notion of a finitely continuous topological space (in short, FC-space). It is clear that the class of G-convex spaces is a subclass of FC-spaces. We emphasis that FC-space is a topological space without any convexity structure.

Motivated and inspired by research works mentioned above, in this paper, we introduce three classes of systems of generalized vector quasi-equilibrium problems in product FC-spaces. Let X and Y be two nonempty sets. We denote by 2^Y and $\langle X \rangle$ the family of all subsets of Y and the family of all nonempty finite subsets of X, respectively. Let I be any index set. For each $i \in I$, let X_i and Y_i be topological spaces and Z_i be a topological vector space. Let $X = \prod_{i\in I} X_i, Y = \prod_{i\in I} Y_i$ and for $i \in I$ and $z \in X, z_i = \pi_i(z)$ be the projection of z onto X_i. For each $i \in I$, let $A_i : Y \times X \to 2^{X_i}, T_i : Y \times X \to 2^{Y_i}, C_i : X \to 2^{Z_i}$ such that for each $z \in X, C_i(z)$ be a closed convex cone with nonempty interior, and $\Psi_i : X_i \times Y \times X \to 2^{Z_i}$ be set-valued mappings.

We consider the following three classes of systems of generalized vector quasi-equilibrium problems:

(i) (SGVQEP 1) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z}) \text{ and } \Psi_i(x_i, \hat{y}, \hat{z}) \nsubseteq -intC_i(\hat{z}), \forall x_i \in A_i(\hat{y}, \hat{z}).$$

(ii) (SGVQEP 2) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z}) \text{ and } \Psi_i(x_i, \hat{y}, \hat{z}) \cap -intC_i(\hat{z}) = \emptyset, \forall x_i \in A_i(\hat{y}, \hat{z}).$$

(iii) (SGVQEP 3) Find $(\hat{y}, \hat{z}) \in Y \times X$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{y}, \hat{z}), \hat{z}_i \in A_i(\hat{y}, \hat{z}) \text{ and } \Psi_i(x_i, \hat{y}, \hat{z}) \subset -C_i(\hat{z}), \forall x_i \in A_i(\hat{y}, \hat{z}).$$

Let V_0 be a topological vector space ordered by a proper closed convex cone D in V_0 and let $h : Y \times X \to 2^{V_0}$ is a set-valued mapping. Moreover, we introduce the notations of $C_i(z) - FC-$diagonal quasi-convex, $C_i(x) - FC-$quasi-convex and $C_i(x) - FC-$quasiconvex-like for set-valued mappings in FC-space. By using these notions and an existence theorem of maximal elements for a family of set-valued mappings, we prove some new existence theorems of solutions for the SGVQEP (1), SGVQEP (2) and SGVQEP (3) in noncompact product FC-spaces without convexity structure. These results improve and generalize some recent known results in literature to noncompact FC-spaces.

2. PRELIMINARIES

Let Δ_n be the standard n-dimensional simplex with vertices $\{e_0, e_1, \ldots, e_n\}$. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. The following notion was introduced by Ben-El-Mechaiekh et al. [18].

Definition 2.1. (X, Γ) is said to be a L-convex space if X is a topological space and $\Gamma : \langle X \rangle \to 2^X$ is a mapping such that for each $N \in \langle X \rangle$ with $|N| = n+1$, there exists a continuous mapping $\varphi_N : \Delta_n \to \Gamma(N)$ satisfying $A \in \langle N \rangle$ with $|A| = J+1$ implies $\varphi_N(\Delta_J) \subset \Gamma(A)$, where Δ_J is the face of Δ_N corresponding to A.

The following notion of a finitely continuous topological space (in short, FC-space) was introduced by Ding [17].

Definition 2.2. (X, φ_N) is said to be a FC-space if X is a topological space and for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to X$. A subset D of (X, φ_N) is said to be a FC-subspace of X if for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and for each $\{x_{i_0}, \ldots, x_{i_k}\} \subset N \cap D, \varphi_N(\Delta_k) \subset D$, where $\Delta_k = co(\{e_{i_j} : j = 0, \ldots, k\})$.

It is clear that any convex subset of a topological vector space, any H-space introduced by Horvath [19], any G-convex space introduced by Park and Kim [15, 16], and any L-convex spaces introduced by Ben-El-Mechaiekh et al. [18] are all FC-space.

By the definition of FC-subspaces of a FC-space, it is easy to see that if $\{B_i\} \in I$ is a family of FC-subspaces of a FC-space (Y, φ_N) and $\cap_{i \in I} B_i \neq \emptyset$, then $\cap_{i \in I} B_i$ is also a FC-subspace of (Y, φ_N) where I is any index set. For a subset A of (Y, φ_N), we can define the FC-hull of A as follows:

$$FC(A) = \bigcap \{B \subset Y : A \subset B \text{ and } B \text{ is } FC - subspace\ of\ Y\}.$$

Clearly, $FC(A)$ is the smallest FC-subspace of Y containing A and each FC-subspace of a FC-space is also a FC-space.

Lemma 2.3. [20] *Let (Y, φ_N) be a FC-space and A be a nonempty subset of Y. Then*

$$FC(A) = \bigcup\{FC(N) : N \in \langle A\rangle\}.$$

Lemma 2.4. [20] *Let X be a topological space, (Y, φ_N) be a FC-space and $G : X \to 2^Y$ be such that $G^{-1}(y) = \{x \in X : y \in G(x)\}$ is compactly open in X for each $y \in Y$. Then the mapping $FC(G) : X \to 2^Y$ defined by $FC(G)(x) = FC(G(x))$ for each $x \in X$ satisfies that $(FC(G))^{-1}(y)$ is also compactly open in X for each $y \in Y$.*

Lemma 2.5. [17] *Let I be any index set. For each $i \in I$, let (Y_i, φ_{N_i}) be a FC-space. Let $Y = \prod_{i\in I} Y_i$ and $\varphi_N = \prod_{i\in I} \varphi_{N_i}$. Then (Y, φ_N) is also a FC-space.*

Lemma 2.6. [20] *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a FC-space, $X = \prod_{i\in I} X_i$ and K be a compact subset of X. For each $i \in I$, let $G_i : X \to 2^{X_i}$ be such that*

- (i) *for each $i \in I$ and $x \in X, G_i(x)$ is a FC-subspace of X_i,*
- (ii) *for each $x \in X$, $\pi_i(x) \notin G_i(x)$ for all $i \in I$,*
- (iii) *for each $y_i \in X_i, G_i^{-1}(y_i)$ is compactly open in X*
- (iv) *for each $N_i \in \langle X_i\rangle$, there exists a nonempty compact FC-subspace L_{N_i} of X_i containing N_i and for each $x \in X \setminus K$, there exists $i \in I$ satisfying $L_{N_i} \cap G_i(x) \neq \emptyset$.*

Then there exists $\hat{x} \in K$ such that $G_i(\hat{x}) = \emptyset$, for each $i \in I$.

Lemma 2.7. [21] *Let X and Y be topological spaces and $G : X \to 2^Y$ be a set-valued mapping. Then G is lower semicontinuous in $x \in X$ if and only if for any $y \in G(x)$ and any net $\{x_\alpha\} \subset X$ satisfying $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in G(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 2.8. [22] *Let X, Y and Z be topological spaces. Let $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

- (i) *C has closed (resp., open) graph,*
- (ii) *for each $y \in Y, F(\cdot, y)$ is lower semicontinuous on each compact subset of X.*

Then the mapping $F^* : Y \to 2^X$ defined by $F^*(y) = \{x \in X : F(x, y) \subset C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \cap C(x) = \emptyset\}$) has compactly closed values.

Lemma 2.9. [22] *Let X, Y and Z be topological spaces. Let $F : X \times Y \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings such that*

- (i) *C has closed (resp., open) graph in $X \times Z$,*
- (ii) *for each $y \in Y, F(\cdot, y)$ is upper semicontinuous on each compact subset of X with nonempty compactly closed values.*

Then the mapping $F^* : Y \to 2^X$ defined by $F^*(y) = \{x \in X : F(x, y) \nsubseteq C(x)\}$ (resp., $F^*(y) = \{x \in X : F(x, y) \cap C(x) \neq \emptyset\}$) has compactly closed values.

3. THE EXISTENCE OF THE SYSTEM OF GENERALIZED VARIATIONAL INEQUALITY

Throughout this section, unless otherwise specified, we assume the following notations and assumptions. Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) and (Y_i, φ'_{N_i}) be $FC-$spaces, and Z_i be a nonempty set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : Y \times X \to 2^{X_i}, T_i : Y \times X \to 2^{Y_i}$and $C_i : Y \times X \to 2^{Z_i}$ such that for each $z \in X, C_i(z)$ be a closed convex cone with nonempty interior, $\Psi_i : X_i \times Y \times X \to 2^{Z_i}$ be set valued mappings. From Li and Li [14] and Ding [22], we first propose the following generalized convexity definitions.

Definition 3.1. Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) and (Y_i, φ'_{N_i}) be $FC-$spaces, and Z_i be a nonempty set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : Y \times X \to 2^{X_i}, T_i : Y \times X \to 2^{Y_i}$and $C_i : Y \times X \to 2^{Z_i}$ such that for each $z \in X, C_i(z)$ be a closed convex cone with nonempty interior, $\Psi_i : X_i \times Y \times X \to 2^{Z_i}$ be set valued mappings. For each $i \in I$ and $y \in Y, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is said to be

(i) Ψ_i is said to be $C_i(z) - FC-$ diagonal quasi-convex of weak type 1 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i\rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i_j}, y, z) \nsubseteq -intC_i(z),$$

(ii) Ψ_i is said to be $C_i(z) - FC-$ diagonal quasi-convex of weak type 2 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i\rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i_j}, y, z) \cap -intC_i(z) = \emptyset,$$

(iii) Ψ_i is said to be $C_i(z) - FC-$ diagonal quasi-convex of strong type 1 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i\rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i_j}, y, z) \subset C_i(z).$$

(iii) Ψ_i is said to be $C_i(z) - FC-$ diagonal quasi-convex of strong type 2 in first argument if each $N_i = \{x_{i,0}, \dots, x_{i,n}\} \in \langle X_i\rangle$ and for each $z \in X$ with $z_i \in FC(N_i)$, there exists $j \in \{0, \dots, k\}$ such that

$$\Psi_i(x_{i_j}, y, z) \cap C_i(z) \neq \emptyset.$$

Now, we establish an existence result for a solution of systems of generalized vector quasi-equilibrium problems (SGVQEP) as follows :

Theorem 3.2. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;*

(i) *for each $(y, z) \in Y \times X, T_i(y, z)$ and $A_i(y, z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
(ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
(iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is upper semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
(iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z)-FC-$diagonal quasi-convex of weak type 1 in first argument;*
(v) *the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;*
(vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i\rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \subset -intC_i(z)$.*

Then the solution set M_1 of SGVQEP(1) is nonempty and compact in $H \times K$, where $M_1 = \{(y,z) \in H \times K : y_i \in T_i(y,z), z_i \in A_i(y,z)$ *and* $\Psi_i(x_i,y,z) \nsubseteq -intC_i(z), \forall x_i \in A_i(y,z), i \in I\}$.

Proof. **Step I.** Show that M_1 is nonempty.
For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \to 2^{X_i}$ by

$$P_i(y,z) = \{x_i \in X_i : \Psi_i(x_i,y,z) \subset -intC_i(z)\}, \forall (y,z) \in Y \times X.$$

We now, show that for each $i \in I$ and $(y,z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y,z)). \tag{3.1}$$

If it is false, then there exist $i \in I$ and $(\bar{y},\bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y},\bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,o},\ldots,x_{i,n}\} \in \langle P_i(\bar{y},\bar{z})\rangle$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we have

$$\Psi_i(x_{i_j},\bar{y},\bar{z}) \subset -intC_i(\bar{z}), \forall j = 0,\ldots,n.$$

By (iv) and Definition 3.1.(i), there exists $j \in \{0,\ldots,n\}$ such that

$$\Psi_i(x_{i_j},\bar{y},\bar{z}) \nsubseteq -intC_i(\bar{z}), \forall j = 0,\ldots,n,$$

which is a contradiction. Then, for each $i \in I$ and $(y,z) \in Y \times X, z_i = \pi_i(z) \notin FC(P_i(y,z))$. Hence by condition (iii) and Lemma 2.9, for each $i \in I$ and $x_i \in X_i$, the set

$$P_i^{-1}(x_i) = \{(y,z) \in Y \times X : \Psi_i(x_i,y,z) \subset -intC_i(z)\}$$

is compactly open in $\in Y \times X$. From Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $\in Y \times X$ for each $x_i \in X_i$. By Lemma 2.5, it follows $i \in I, Y_i \times X_i$ is a FC-space and $Y \times X$ is also a FC-space for all $i \in I$.

Next, for each $i \in I$, we define a set-valued mapping $G_i : Y \times X \to 2^{Y_i \times X_i}$ by

$$G_i(y,z) = \begin{cases} T_i(y,z) \times [A_i(y,z) \cap FC(P_i(y,z))], \ if \ (y,z) \in W_i, \\ T_i(y,z) \times A_i(y,z), \ if (y,z) \notin W_i. \end{cases}$$

By condition (i), for each $i \in I$ and $(y,z) \in Y \times X$, $G_i(y,z)$ is an FC-subspace of $Y_i \times X_i$. From the definition of W_i and (3.2), $(y_i,z_i) \notin G_i(y,z)$, for each $i \in I$ and $(y,z) \in Y \times X$.

Then, for each $i \in I$ and $(u_i,v_i) \in Y_i \times X_i$, we have

$$\begin{aligned} G_i^{-1}(u_i,v_i) &= [T_i^{-1}(u_i) \bigcap A_i^{-1}(v_i) \bigcap (FC(P_i))^{-1}(v_i)] \\ &\quad \bigcup [(Y \times X \backslash W_i) \bigcap T_i^{-1}(u_i) \bigcap A_i^{-1}(v_i)]. \end{aligned}$$

Since $(FC(P_i))^{-1}(v_i)$ is compactly open in $Y \times X$ for each $v_i \in X_i$, it follows by the condition (ii) that $G_i^{-1}(u_i,v_i)$ is also compactly open in $Y \times X$. By (vi), for each $H_i = M_i \times N_i \in \langle Y_i \times X_i\rangle$ there exists compact FC-subspace $L_{H_i} = L_{M_i} \times L_{N_i}$ of $Y_i \times X_i$ containing H_i such that

$$G_i(y,z) \cap L_{H_i} \neq \emptyset.$$

Thus all conditions of Lemma 2.6 are satisfied. Hence by Lemma 2.6, there exists $(\hat{y},\hat{z}) \in H \times K$ such that $G_i(\hat{y},\hat{z}) = \emptyset$ for each $i \in I$. If $(\hat{y},\hat{z}) \notin W_j$ for some $j \in I$, then either $T_j(\hat{y},\hat{z}) = \emptyset$ or $A_j(\hat{y},\hat{z}) = \emptyset$, which contradicts the condition (i). Therefore, $(\hat{y},\hat{z}) \in W_i$ for all $i \in I$. This implies that for each $i \in I, \hat{y}_i \in T_i(\hat{y},\hat{z}), \hat{z}_i \in A_i(\hat{y},\hat{z})$ and $A_i(\hat{y},\hat{z}) \cap FC(P_i(\hat{y},\hat{z})) = \emptyset$ and hence $A_i(\hat{y},\hat{z}) \cap P_i(\hat{y},\hat{z}) = \emptyset$. Therefore, for each $i \in I, \hat{y}_i \in T_i(\hat{y},\hat{z}), \hat{z}_i \in A_i(\hat{y},\hat{z})$ and

$$\Psi_i(x_i,\hat{y},\hat{z}) \nsubseteq -intC_i(\hat{z}), \forall x_i \in A_i(\hat{y},\hat{z}).$$

Hence, $(\hat{y}, \hat{z}) \in M_1$,and M_1 is nonempty.
Step II. Show that M_1 is compact.
By condition (iii) and Lemma 2.9, we note that, for each $i \in I$ and $v_i \in X_i$, the set

$$\{(y, z) \in Y \times X : \Psi_i(v_i, y, z) \nsubseteq -intC_i(z)\}$$

is compactly closed in $Y \times X$. This implies that the set

$$\{(y, z) \in H \times K : \Psi_i(x_i, y, z) \nsubseteq -intC_i(z), \forall x_i \in A_i(y, z)\}$$

is closed in $H \times K$, for all $i \in I$. By condition (v), the set

$$W_i = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$$

is also closed in $H \times K$ for each $i \in I$. It follows that M_1 is closed in $H \times K$. Hence $H \times K$ is compact in $Y \times X$ and therefore M_1 is nonempty and compact . This completes the proof. □

Theorem 3.3. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;*

(i) *for each $(y, z) \in Y \times X, T_i(y, z)$ and $A_i(y, z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
(ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
(iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
(iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z)-FC-$diagonal quasi-convex of weak type 2 in first argument;*
(v) *the set $W_i = \{(y, z) \in Y \times X : y_i \in T_i(y, z), z_i \in A_i(y, z)\}$ is compactly closed in $Y \times X$;*
(vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i\rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y, z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y, z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y, z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i, y, z) \cap (-intC_i(z)) \neq \emptyset$.*

Then the solution set M_2 of SGVQEP(2) is nonempty and compact in $H \times K$, where $M_2 = \{(y, z) \in H \times K : y_i \in T_i(y, z), z_i \in A_i(y, z)$ and $\Psi_i(x_i, y, z) \cap (-intC_i(z)) = \emptyset, \forall x_i \in A_i(y, z), i \in I\}$.

Proof. For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \to 2^{X_i}$ by

$$P_i(y, z) = \{x_i \in X_i : \Psi_i(x_i, y, z) \cap (-intC_i(z)) \neq \emptyset\}, \forall (y, z) \in Y \times X.$$

We now show that, for each $i \in I$ and $(y, z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y, z)). \tag{3.2}$$

If it is false, then there exist $i \in I$ and $(\bar{y}, \bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y}, \bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,o}, \ldots, x_{i,n}\} \in \langle P_i(\bar{y}, \bar{z})\rangle$, such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we note that

$$\Psi_i(x_{i_j}, \bar{y}, \bar{z}) \subset -intC_i(\bar{z}), \forall j = 0, \ldots, n.$$

It follows by (iv) and Definition 3.1.(ii) that there exists $j \in \{0, \ldots, n\}$ such that

$$\Psi_i(x_{i_j}, \bar{y}, \bar{z}) \nsubseteq -intC_i(\bar{z}), \forall j = 0, \ldots, n,$$

which is a contradiction. Hence, for each $i \in I$ and $(y,z) \in Y \times X, z_i = \pi_i(z) \notin FC(P_i(y,z))$. By condition (iii) and Lemma 2.8, we note that, the set

$$P_i^{-1}(x_i) = \{(y,z) \in Y \times X : \Psi_i(x_i,y,z) \cap (-intC_i(z)) \neq \emptyset\}$$

is compactly open in $\in Y \times X$ for all $i \in I$ and $x_i \in X_i$. It follows from Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $\in Y \times X$ for each $x_i \in X_i$. Hence, by Lemma 2.5, for each $i \in I, Y_i \times X_i$ is a FC-space and $Y \times X$ is also an FC-space. By the similar argument as in the proof of Theorem 3.2, we obtain the desired result. □

Theorem 3.4. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$,*

(i) *for each $(y,z) \in Y \times X, T_i(y,z)$ and $A_i(y,z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
(ii) *for each $(u_i,v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
(iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y,z) \mapsto \Psi_i(x_i,y,z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
(iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z)-FC-$diagonal quasi-convex of strong type 1 in first argument;*
(v) *the set $W_i = \{(y,z) \in Y \times X : y_i \in T_i(y,z), z_i \in A_i(y,z)\}$ is compactly closed in $Y \times X$;*
(vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y,z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y,z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y,z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i,y,z) \nsubseteq -C_i(z)$.*

Then the solution set M_3 of SGVQEP(3) is nonempty and compact in $H \times K$, where $M_3 = \{(y,z) \in H \times K : y_i \in T_i(y,z), z_i \in A_i(y,z)$ and $\Psi_i(x_i,y,z) \subset -C_i(z), \forall x_i \in A_i(y,z), i \in I\}$.

Proof. For each $i \in I$, we define a set-valued mapping $P_i : Y \times X \to 2^{X_i}$ by

$$P_i(y,z) = \{x_i \in X_i : \Psi_i(x_i,y,z) \nsubseteq -C_i(z)\}, \forall (y,z) \in Y \times X.$$

We now show that, for each $i \in I$ and $(y,z) \in Y \times X$,

$$z_i = \pi_i(z) \notin FC(P_i(y,z)). \tag{3.3}$$

If it is false, then there exist $i \in I$ and $(\bar{y},\bar{z}) \in Y \times X$ such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(P_i(\bar{y},\bar{z}))$. Hence by Lemma 2.3, there exists $N_i = \{x_{i,o},\ldots,x_{i,n}\} \in \langle P_i(\bar{y},\bar{z}) \rangle$, such that $\bar{z}_i = \pi_i(\bar{z}) \in FC(N_i)$. Thus, we note that

$$\Psi_i(x_{i_j},\bar{y},\bar{z}) \nsubseteq -C_i(\bar{z}), \forall j = 0,\ldots,n.$$

It follows (iv) and Definition 3.1.(iii) that there exists $j \in \{0,\ldots,n\}$ such that

$$\Psi_i(x_{i_j},\bar{y},\bar{z}) \subset -C_i(\bar{z}), \forall j = 0,\ldots,n,$$

which is a contradiction. Hence, for each $i \in I$ and $(y,z) \in Y \times X, z_i = \pi_i(z) \notin FC(P_i(y,z))$. Then by condition (iii) and Lemma 2.8, for each $i \in I$ and $x_i \in X_i$, the set

$$P_i^{-1}(x_i) = \{(y,z) \in Y \times X : \Psi_i(x_i,y,z) \nsubseteq -C_i(z)\}$$

is compactly open in $\in Y \times X$. It follows from Lemma 2.4 that $(FC(P_i))^{-1}(x_i)$ is also compactly open in $\in Y \times X$ for each $x_i \in X_i$. Hence, by Lemma 2.5, for each $i \in I, Y_i \times X_i$ is a FC-space and $Y \times X$ is also an FC-space. By the similar argument as in the proof of Theorem 3.2, we obtain the desired result. □

Definition 3.5. For each $i \in I$ and $(y, z) \in Y \times X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is said to be

(i) $C_i(z) - FC-$ quasi-convex in first argument if each $(y, z) \in Y \times X$, $N_i = \{x_{i,0}, \ldots, x_{i,n}\} \in \langle X_i \rangle, \{x_{i,i_0}, \ldots, x_{i,i_k}\} \subset N_i$ and $x_i^* \in \varphi'_{N_i}(\Delta_k)$, there exists $j \in \{0, \ldots, k\}$ such that

$$\Psi_i(x_{i,i_j}, y, z) \subset \Psi_i(x_i^*, y, z) + C_i(z),$$

(ii) $C_i(z) - FC-$ quasi-convex-like in first argument if each $(y, z) \in Y \times X$, $N_i = \{x_{i,0}, \ldots, x_{i,n}\} \in \langle X_i \rangle, \{x_{i,i_0}, \ldots, x_{i,i_k}\} \subset N_i$ and $x_i^* \in \varphi'_{N_i}(\Delta_k)$, there exists $j \in \{0, \ldots, k\}$ such that

$$\Psi_i(x_i^*, y, z) \subset \Psi_i(x_{i,i_j}, y, z) - C_i(z).$$

Form Ding [23], we give the following lemma;

Lemma 3.6. [23] *For each $i \in I$, let Z_i be a topological vector space and $C_i : X \to 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X, C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I, (y, z) \in Y \times X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$quasiconvex-like in first argument, then, the set*

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \nsubseteq -intC_i(z)\}$$

is FC-subspace of X_i.

Lemma 3.7. [23] *For each $i \in I$, let Z_i be a topological vector space and $C_i : X \to 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X, C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I, (y, z) \in Y \times X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$quasiconvex in first argument, then, the set*

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \cap -intC_i(z) = \emptyset\}$$

is FC-subspace of X_i.

Lemma 3.8. [23] *For each $i \in I$, let Z_i be a topological vector space and $C_i : X \to 2^{Z_i}$ be a set-valued mapping, such that for each $z \in X, C_i(z)$ is closed convex cone in Z_i with nonempty interior. If for each $i \in I, (y, z) \in Y \times X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$quasiconvex-like in first argument, then, the set*

$$\{x_i \in X_i : \Psi_i(x_i, y, z) \subset -C_i(z)\}$$

is FC-subspace of X_i.

Theorem 3.9. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$, the following condition are satisfied;*

(i) *for each $(y, z) \in Y \times X, T_i(y, z)$ and $A_i(y, z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
(ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
(iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y, z) \mapsto \Psi_i(x_i, y, z)$ is upper semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
(iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$ quasi-convex-like in first argument and for each $(y, z) \in Y \times X, \Psi_i(x_i, y, z) \nsubseteq -intC_i(z)$;*

(v) *the set* $W_i = \{(y,z) \in Y \times X : y_i \in T_i(y,z), z_i \in A_i(y,z)\}$ *is compactly closed in* $Y \times X$*;*
(vi) *there exists nonempty compact subset* $H \times K$ *of* $Y \times X$*, and for each* $M_i \times N_i \in \langle Y_i \times X_i \rangle$*, there exists compact FC-subspace* $L_{M_i} \times L_{N_i}$ *containing* $M_i \times N_i$ *such that for each* $(y,z) \in Y \times X \setminus H \times K$*, there exist* $i \in I, \bar{u_i} \in T_i(y,z) \cap L_{M_i}$ *and* $\bar{v_i} \in A_i(y,z) \cap L_{N_i}$ *satisfying* $\Psi_i(\bar{v_i}, y, z) \subset -intC_i(z)$.

Then the solution set M_1 *of SGVQEP(1) is nonempty and compact in* $H \times K$*, where* $M_1 = \{(y,z) \in H \times K : y_i \in T_i(y,z), z_i \in A_i(y,z)$ *and* $\Psi_i(x_i,y,z) \nsubseteq -intC_i(z), \forall x_i \in A_i(y,z), i \in I\}$.

Proof. We first show that, Ψ_i is $C_i(z) - FC-$diagonal quasi-convex of SK-type 1 in first argument for all $i \in I$ and $z \in X$. If it is false, then there exist $i \in I$, $N_i = \{x_{i,0}, \ldots, x_{i,n}\} \in \langle X_i \rangle$ and for each $\bar{z} \in X$ with $\bar{z_i} \in FC(N_i)$, such that $\Psi_i(x_{i_j}, y, z) \subset -intC_i(z)$, for all $j \in \{0, \ldots, k\}$. Hence, we obtain $N_i \subset P_i(y, \bar{z})$. It follows from (iv) and Lemma 3.6 that for each $i \in I$ and $(y,\bar{z}) \in Y \times X$, $P_i(y,\bar{z})$ is an FC-subspace of X_i. Then we have $\bar{z_i} \in FC(N_i) \subset P_i(y,\bar{z})$, which contradicts the fact that for each $(y,z) \in Y \times X, z_i \notin P_i(y,z)$. Therefore, for each $i \in I$ and $z \in X$, Ψ_i is $C_i(z) - FC-$diagonal quasi-convex of SK-type 1 in first argument. Thus all conditions of Theorem 3.2 for the SGVQEP(1) are satisfied. Hence the conclusion of Theorem 3.9 hold from Theorem 3.2. This completes the proof. □

Theorem 3.10. *Let* K *and* H *be nonempty compact subsets of* X *and* Y*, respectively, and for each* $i \in I, C_i(z)$ *be closed convex cone with nonempty interior. Suppose that for each* $i \in I$*, the following condition are satisfied;*

(i) *for each* $(y,z) \in Y \times X, T_i(y,z)$ *and* $A_i(y,z)$ *are nonempty* $FC-$*subspaces of* Y_i *and* X_i*, respectively;*
(ii) *for each* $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ *and* $A_i^{-1}(v_i)$ *are compactly open in* $Y \times X$*;*
(iii) *the mapping* $z \mapsto intC_i(z)$ *has an open graph and for each* $x_i \in X_i$*, the mapping* $(y,z) \mapsto \Psi_i(x_i,y,z)$ *is lower semicontinuous on each compact subsets of* $Y \times X$ *with nonempty compactly closed values;*
(iv) *for each* $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ *is* $C_i(z) - FC-$ *quasi-convex in first argument and for each* $(y,z) \in Y \times X, \Psi_i(x_i,y,z) \cap (-intC_i(z)) = \emptyset$*;*
(v) *the set* $W_i = \{(y,z) \in Y \times X : y_i \in T_i(y,z), z_i \in A_i(y,z)\}$ *is compactly closed in* $Y \times X$*;*
(vi) *there exists nonempty compact subset* $H \times K$ *of* $Y \times X$*, and for each* $M_i \times N_i \in \langle Y_i \times X_i \rangle$*, there exists compact FC-subspace* $L_{M_i} \times L_{N_i}$ *containing* $M_i \times N_i$ *such that for each* $(y,z) \in Y \times X \setminus H \times K$*, there exist* $i \in I, \bar{u_i} \in T_i(y,z) \cap L_{M_i}$ *and* $\bar{v_i} \in A_i(y,z) \cap L_{N_i}$ *satisfying* $\Psi_i(\bar{v_i}, y, z) \cap (-intC_i(z)) \neq \emptyset$.

Then the solution set M_2 *of SGVQEP(2) is nonempty and compact in* $H \times K$*, where* $M_2 = \{(y,z) \in H \times K : y_i \in T_i(y,z), z_i \in A_i(y,z)$ *and* $\Psi_i(x_i,y,z) \cap (-intC_i(z)) = \emptyset, \forall x_i \in A_i(y,z), i \in I\}$.

Proof. We first show that ,Ψ_i is $C_i(z) - FC-$diagonal quasi-convex of SK-type 2 in first argument, for each $i \in I$ and $z \in X$. If it is false, then, there exist $i \in I$, $N_i = \{x_{i,0}, \ldots, x_{i,n}\} \in \langle X_i \rangle$ and for each $\bar{z} \in X$ with $\bar{z_i} \in FC(N_i)$, such that $\Psi_i(x_{i_j}, y, z) \cap -intC_i(z) \neq \emptyset$ for all $j \in \{0, \ldots, k\}$. Hence, we obtain $N_i \subset P_i(y,\bar{z})$. It follows from (iv) and Lemma 3.7 that for each $i \in I$ and $(y,\bar{z}) \in Y \times X$, $P_i(y,\bar{z})$ is an FC-subspace of X_i. Then we have $\bar{z_i} \in FC(N_i) \subset P_i(y,\bar{z})$, which contradicts the fact that for each $(y,z) \in Y \times X, z_i \notin P_i(y,z)$. Therefore, for each $i \in I$ and

$z \in X$, Ψ_i is $C_i(z) - FC-$diagonal quasi-convex of SK-type 2 in first argument. Thus all conditions of Theorem 3.3 for the SGVQEP(2) are satisfied. Hence the conclusion of Theorem 3.10 hold from Theorem 3.3. This completes the proof. □

By applying Lemma 2.6, Lemma 3.8 and the similar argument as in the proof of Theorem 3.9-3.10, we can easily prove the following results.

Theorem 3.11. *Let K and H be nonempty compact subsets of X and Y, respectively, and for each $i \in I, C_i(z)$ be closed convex cone with nonempty interior. Suppose that for each $i \in I$,*

- (i) *for each $(y,z) \in Y \times X, T_i(y,z)$ and $A_i(y,z)$ are nonempty $FC-$subspaces of Y_i and X_i, respectively;*
- (ii) *for each $(u_i, v_i) \in Y_i \times X_i, T_i^{-1}(u_i)$ and $A_i^{-1}(v_i)$ are compactly open in $Y \times X$;*
- (iii) *the mapping $z \mapsto intC_i(z)$ has an open graph and for each $x_i \in X_i$, the mapping $(y,z) \mapsto \Psi_i(x_i,y,z)$ is lower semicontinuous on each compact subsets of $Y \times X$ with nonempty compactly closed values;*
- (iv) *for each $z \in X, \Psi_i : X_i \times Y \times X \to 2^{Z_i}$ is $C_i(z) - FC-$quasi-convex-like in first argument and for each $(y,z) \in Y \times X, \Psi_i(x_i,y,z) \subset -C_i(z)$;*
- (v) *the set $W_i = \{(y,z) \in Y \times X : y_i \in T_i(y,z), z_i \in A_i(y,z)\}$ is compactly closed in $Y \times X$;*
- (vi) *there exists nonempty compact subset $H \times K$ of $Y \times X$, and for each $M_i \times N_i \in \langle Y_i \times X_i \rangle$, there exists compact FC-subspace $L_{M_i} \times L_{N_i}$ containing $M_i \times N_i$ such that for each $(y,z) \in Y \times X \setminus H \times K$, there exist $i \in I, \bar{u}_i \in T_i(y,z) \cap L_{M_i}$ and $\bar{v}_i \in A_i(y,z) \cap L_{N_i}$ satisfying $\Psi_i(\bar{v}_i,y,z) \nsubseteq -C_i(z)$.*

Then the solution set M_3 of SGVQEP(3) is nonempty and compact in $H \times K$, where $M_3 = \{(y,z) \in H \times K : y_i \in T_i(y,z), z_i \in A_i(y,z)$ and $\Psi_i(x_i,y,z) \subset -C_i(z), \forall x_i \in A_i(y,z), i \in I\}$.

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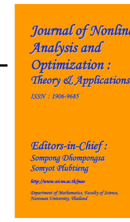

ON THE VECTOR MIXED QUASI-VARIATIONAL INEQUALITY PROBLEMS[◇]

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ABSTRACT. In this paper, we introduce and consider a new class of vector mixed quasi variational inequality and vector complementarity problem in a topological vector space. We show that under certain conditions the solution set of the vector mixed quasi complementarity problem equals to the solution set of the vector mixed quasi variational inequalities. Using the Ky Fan's lemma, we study the existence of a solution of the vector mixed quasi variational inequalities and vector mixed quasi complementarity problems. Moreover we discuss on some of our assumptions. Our results extend those of Farajzadeh et al [Mixed quasi complementarity problems in topological vector spaces, J. Global. Optim.,45 (2009) 229 - 235] to the vector case.

KEYWORDS : Complementarity problems; Mixed quasi-variational inequality; Existence results.

1. INTRODUCTION

Complementarity problems theory, which was introduced and studied by Lemke [14]and Cottle and Dantzig [5] in early 1960's, has enjoyed a vigorous and dynamics growth. Complementarity problems have been extended and generalized in various directions to study a large class of problems arising in industry, finance, optimization, regional, physical, mathematical and engineering sciences, see [1-19]. Equally important is the mathematical subject known as variational inequalities which was introduced in early 1960's. For the applications, physical formulation, numerical

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methods, dynamical system and sensitivity analysis of the mixed quasi variational inequalities, see [1,5,15] and the references therein. It has been shown that if the set involved in complementarity problems and variational inequalities is a convex cone together with suitable assumptions then both the complementarity problems and variational inequalities are equivalent, see Karamardian [13]. This equivalence has played a central and crucial role in suggesting new and unified algorithms for solving complementarity problem and its various generalizations and extensions, see [1-19] and the references therein for more details.

Inspired and motivated by the reference [9], we introduce and analyze a new class of vector mixed quasi variational inequality and vector mixed complementarity problem in topological vector spaces. These classes are quite general and unifies several classes of complementarity problems in a general framework. Under suitable conditions, we establish the equivalence between the vector mixed quasi complementarity problem and vector mixed quasi variational inequalities problem. This alternative equivalence is used to discuss several existence results for the solution of the these problems by using Fan's lemma in topological vector spaces. The vector mixed quasi variational inequalities include vector mixed quasi complementarity problems, f-complementarity problems, general complementarity problems, various classes of vector variational inequalities and related vector optimization problems as special cases.

2. PRELIMINARIES

Let X and Y be two real Hausdorff topological vector spaces and K be a nonempty subset of X. Denote by $L(X,Y)$ the space of all continuous linear mappings from X into Y, and $\langle t,x\rangle$ be the value of the linear continuous mapping $t\in L(X,Y)$ at x. Suppose that $C: K \longrightarrow 2^Y$ is a set valued map with nonempty pointed (that is, $C(u)\cap -C(u)=\{0\}$ for all $u\in K$) convex cone values, $F: K\times K\longrightarrow Y$, and $T: K\longrightarrow L(X,Y)$.

We consider the problem of finding $u\in K$ such that

$$\langle Tu,u\rangle + F(u,u) = 0, \quad \langle Tu,v\rangle + F(v,u)\in C(u), \quad \forall v\in K, \tag{2.1}$$

which we call it the *vector mixed quasi complementarity problem (VMQCP)*.

We note that if $Y=\Re$ (real numbers) and $C(u)=[0,\infty),\ F(v,u)=f(v), \forall u,v\in K$, then problem (1) is equivalent to finding $u\in K$ such that

$$\langle Tu,u\rangle + f(u) = 0, \quad \langle Tu,v\rangle + f(v)\geq 0, \quad \forall u,v\in K, \tag{2.2}$$

which is known as the f-complementarity problem, introduced and studied by Itoh et al [12]. Moreover if $F(v,u)=f(v), \quad \forall v\in K$, problem (1) reduces to the vector version of Itoh et al's problem introduced in [12]. For the applications and numerical methods of problem (2), see [16, 2, 10].

If $F(u,v)=0$, for all $u,v\in K$, and $K^*\equiv\{u\in X^*: \langle u,v\rangle\geq 0, \quad \forall v\in K\}$ is a polar (dual) cone of the convex cone K, then the mixed quasi complementarity problem (VMQCP) is equivalent to finding $u\in K$ such that

$$Tu\in K^* \quad \text{and} \quad \langle Tu,u\rangle = 0, \tag{2.3}$$

which is called the general complementarity problem. For the recent applications, numerical results and formulation of the complementarity problems, see [2,3,4,7,9,16] and the references therein.

Related to the vector mixed quasi complementarity problem (1), we consider the problem of finding $u \in K$, where K is a nonempty subset of X, such that

$$\langle Tu, v-u\rangle + F(v,u) - F(u,u) \in C(u), \quad \forall v \in K, \tag{2.4}$$

which we call it the *vector mixed quasi variational inequality* (VMQVIP). For the formulation, numerical results, existence results, sensitivity analysis and dynamical aspects of the scalar mixed quasi variational inequalities (that is $Y = \Re$), see [1,8,10,13-19] and the references therein.

It is obvious that any solution of (VMQCP) is a solution of (VMQVIP). The following example shows that the converse does not hold in general.

Example 2.1. Let $X = Y = \Re,\ K = [0, +\infty),\ F(x,y) = 1$, for all $x, y \in K,\ C(x) = [0, +\infty)$ for all $x \in K$ and define $T : K \longrightarrow \Re^* = \Re$ by

$$T(x) = \begin{cases} 0 & \text{if} \quad x = 0, \\ -1 & \text{otherwise.} \end{cases}$$

Then $u = 0$ is a solution of (VMQVIP), whereas (VMQCP) hasn't any solution. We now show that the problems (1) and (4),under some conditions, are equivalent, that is their solution sets are equal and this is the main motivation of our next result.

Theorem 2.2. *Let K be a nonempty subset of a topological vector space X with $2K \subseteq K$, and $0 \in K$. If $F(2v,u) = 2F(v,u)$, for all $u, v \in K$, then the solution sets of (VMQCP) and (VMVIP) are equal.*

Proof. It suffices to show that any solution of (VMQVIP) is a solution of (VMQCP). Let $u \in K$ be a solution of the vector mixed quasi variational inequality (4). Then by taking $v = 0$ and $v = 2u$ in (4), we have

$$\langle Tu, -u\rangle + F(0,u) - F(u,u) \in C(u),$$
$$\langle Tu, u\rangle + F(2u,u) - F(u,u) \in C(u),$$

which implies, using $F(0,u) = 0,\ F(2u,u) = 2F(u,u)$, and $C(u) \cap -C(u) = \{0\}$, that

$$\langle Tu, u\rangle + F(u,u) = 0. \tag{2.5}$$

Also, from (5) and (4), we have , for all $v \in K$,

$$\langle Tu, v\rangle + F(v,u) =$$
$$\langle Tu, v\rangle + F(v,u) - (\langle Tu, u\rangle + F(u,u)) =$$
$$\langle Tu, v-u\rangle + F(v,u) - F(u,u) \in C(u)$$

that is,

$$\langle Tu, v\rangle + F(v,u) \in C(u), \quad \forall v \in K. \tag{2.6}$$

This shows that $u \in K$ is a solution of (VMQCP). □

Remark 2.3. (a) If K is a closed convex cone, then $0 \in K$, and $2K \subseteq K$, but every convex set with $0 \in K$, does not so. For instance, $K = N \cup \{0\}$, the set N denotes the natural numbers, is not a convex cone while is a nonempty set with $2K \subseteq K$, and $0 \in K$.

(b) If F is positively homogeneous in the first variable then $F(2u,v) = 2F(u,v),\ \forall u, v \in K$. However the converse is not true, for instance, $F(u,v) = 0,$ for u rational and $F(u,v) = u,$ for u irrational, which is not positively homogeneous but satisfies $F(2u,v) = 2F(u,v)$ for $u, v \in K$.

In the rest of this section, we recall some definitions and Ky Fan's lemma, which will be used in the next section.
We shall denote by 2^A the family of all subsets of A and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of A. Let X be a nonempty set, Y a topological space, and $\Gamma : X \longrightarrow 2^Y$ a multi-valued map. Then, Γ is called transfer closed-valued if, for every $y \notin \Gamma(x)$, there exists $x' \in X$ such that $y \notin cl\Gamma(x')$, where cl denotes the closure of a set. It is well-known that, $\Gamma : X \longrightarrow 2^Y$ is transfer closed-valued if and only if

$$\bigcap_{x\in X} \Gamma(x) = \bigcap_{x\in X} cl\Gamma(x).$$

If $B \subseteq Y$ and $A \subseteq X$, then $\Gamma : A \longrightarrow 2^B$ is called transfer closed-valued if the set valued mapping $x \longrightarrow \Gamma(x) \cap B$ is transfer closed-valued. In this case where $X = Y$ and $A = B$, Γ is called transfer closed-valued on A.
Let X and Y are two topological vector space, K be a nonempty subset of Y, and $C \subseteq Y$ be nonempty and convex. The map $f : K \longrightarrow Y$ is said to be C-convex if for each $0 \leq \lambda \leq 1$ and $x_1, x_2 \in K$ we have

$$\lambda f(x_1) + (1-\lambda)f(x_2) - f(\lambda x_1 + (1-\lambda)x_2) \in C.$$

Let K be a nonempty convex subset of a topological vector space X and let K_0 be a subset of K. A set valued map $\Gamma : K_0 \longrightarrow 2^K$ is said to be a *KKM map* when

$$coA \subseteq \bigcup_{x\in A} \Gamma(x), \quad \forall\ A\ \in \mathcal{F}(K_0),$$

where co denotes the convex hull.

Lemma 2.4. *(Ky Fan [7]). Let K be a nonempty subset of a topological vector space X and $F : K \longrightarrow 2^X$ be a KKM mapping with closed values. Assume that there exist a nonempty compact convex subset B of K such that $D = \bigcap_{x\in B} F(x)$ is compact. Then*

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

3. MAIN RESULTS

In this section, we provide some existence theorems in order to guarantee the solution set of (VMQVIP) and (VMQCP) be nonempty and relatively compact.
Throughout this section ,unless otherwise specified, let X and Y be two real Hausdorff topological vector spaces and K be a nonempty convex subset of X. Denote by $L(X,Y)$ the space of all continuous linear mappings from X into Y, and $\langle t, x\rangle$ be the value of the linear continuous mapping $t \in L(X,Y)$ at x. Suppose that $C : K \longrightarrow 2^Y$ is a set valued map with nonempty convex cone values and $F : K \times K \longrightarrow Y$, $T : K \longrightarrow L(X,Y)$ are two mappings.

We need the following lemma for the next result.

Lemma 3.1. *Let X be a topological vector space and $E \subseteq X$ be compact and convex. Let $A = \{a_1, ..., a_n\}$ be a finite subset of X. Then $co(A \cup E)$ is compact.*

Proof. Let $\Delta_n = \{\Sigma_{i=1}^n \lambda_i e_i \ : \ \lambda_i \geq 0, \ \Sigma_{i=1}^n \lambda_i = 1\}$, where (e_i) is the standard base of $\mathbb{R}^n$. Define $\Lambda : \Delta_{n+1} \times \{a_1\} \times ... \times \{a_n\} \times E \longrightarrow co(A \cup E)$ by

$$\Lambda(\lambda_1, ..., \lambda_{n+1}, a_1, ..., a_n, e) = \Sigma_{i=1}^n \lambda_i a_i + \lambda_{n+1} e.$$

It is clear that Λ is an onto continuous mapping and $\Delta_{n+1} \times \{a_1\} \times ... \times \{a_n\} \times E$ is compact. Then $co(A \cup E) = \Lambda(\Delta_{n+1} \times \{a_1\} \times ... \times \{a_n\} \times E)$ is compact. The proof is complete. □

Theorem 3.2. *Assume that*

(a) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that*

(i) for each $x \in K \backslash B$, there exists $y \in D$ such that $\langle T(x), y - x\rangle + F(y, x) - F(x, x) \notin C(x)$;

(ii) for each fixed $y \in K$, the set valued mapping $G : K \longrightarrow 2^Y$ defined by

$$G(y) = \{x \in K : \langle Tx, y - x\rangle + F(y, x) - F(x, x) \in C(x)\}$$

is transfer closed on each compact convex subset K containing D

(b) *there exists a mapping $h : K \times K \longrightarrow Y$ such that*
(i) $h(x, x) \in C(x), \ \forall \ x \in K$;
(ii) $\langle T(x), y - x\rangle + F(y, x) - F(x, x) - h(x, y) \in C(x), \ \forall x, y \in K, ;$
(iii) the set $\{y \in K : h(x, y) \notin C(x)\}$ is convex, $\forall \ x \in K$.

Then (VMQVI) has a solution. Moreover, the solution set of (VMQVI) is relatively compact.

Proof. For each fixed $A \in \mathcal{F}(K)$, define the multi-valued maps $\Gamma_A, \widehat{\Gamma_A} : co(A \cup D) \longrightarrow 2^{co(A \cup D)}$ as follows:

$$\Gamma_A(y) = \{x \in co(A \cup D) : \langle T(x), y - x\rangle + F(y, x) - F(x, x) \in C(x)\},$$
$$\widehat{\Gamma_A}(y) = \{x \in co(A \cup D) : h(x, y) \in C(x)\}.$$

We show that $\widehat{\Gamma}$ is a KKM mapping. On the contrary, suppose there exists $M = \{x_1, x_2, ..., x_n\} \subseteq co(A \cup D)$, and $z \in coM$, such that $z \notin \cup_{i \in \{1,2,...,n\}} \widehat{\Gamma}(x_i)$. Then $h(z, x_i) \notin C(z)$ for $i = 1, 2, 3, ..., n$. It follows by (b)(iii) that, $h(z, z) \notin C(z)$ contradicting (b)(i). Hence $\widehat{\Gamma_A}$ is a KKM map and so Γ_A is a KKM map (since $\widehat{\Gamma}(y) \subseteq \Gamma(y)$, for all $y \in K$). By Lemma 2.1 the set $co(A \cup D)$ is compact and convex and hence by Lemma 1.4 the intersection $\bigcap_{x \in co(A \cup D)} cl\Gamma_A(x)$ is nonempty and so by the assumption (ii) of (a) we have

$$\bigcap_{x \in co(A \cup D)} cl\Gamma_A(x) = \bigcap_{x \in co(A \cup D)} \Gamma_A(x).$$

Hence $M_A = \bigcap_{x \in co(A \cup D)} \Gamma_A(x)$ is nonempty. We consider the family

$$\sum = \{M_A : A \in \mathcal{F}(K)\}.$$

It is clear that $\sum$ has finite intersection property (note if $A_1, ..., A_n \in \mathcal{F}(K)$ then $\cap_{i=1}^n M_{A_i} \supseteq M_{\cup_{i=1}^n A_i}$). Hence $\bigcap_{A \in \mathcal{F}(K)} M_A$ is nonempty (note (a)(i) implies $M_A \subseteq B$ for each $A \in \mathcal{F}(K)$ and the family $\sum$ has finite intersection property) and so there exists $\overline{x} \in \bigcap_A clM_A$. Now if x is an arbitrary element of K then we claim that

$$\langle T(\overline{x}), x - \overline{x}\rangle + F(x, \overline{x}) - F(\overline{x}, \overline{x}) \in C(\overline{x}),$$

that is $\overline{x}$ is a solution of (VMQVI).

To see this let $S = \{x, \overline{x}\}$. Then

$$\overline{x} \in (\bigcap_{A \in \mathcal{F}(K)} cl_K M_A) \cap co(D \cup S) \subseteq \bigcap_{y \in S} cl_{co(D \cup S)} \Gamma_S(y) \cap co(D \cup S) =$$

$$\bigcap_{y \in S} \Gamma_S(y) \cap co(D \cup S) \subseteq \Gamma_S(x),$$

and so this proves the assertion. It is obvious from (a)(i) that the solution set of (VMQVI) is a subset of B. □

Remark 3.3. Condition (a)(ii) of Theorem 2.2 holds when T, F are continuous mappings and the graph of the mapping C is closed. The following simple example shows that the continuity of the maps isn't necessary condition to hold (a)(ii). Let $X = Y = \Re$, and K be a non-singleton convex subset of X and let $T(x) = 0, F(x,x) = 0, F(x,y) = 1$ for $x \neq y$,. Then F is not continuous but the condition (a)(ii) is satisfied. can fail.

Corollary 3.4. *Assume that:*

(a) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that*

(i) for each $x \in K \backslash B$, there exists $y \in D$ such that $\langle T(x), y - x\rangle + F(y,x) - F(x,x) \notin C(x)$;

(ii) for each fixed $y \in K$, the set valued mapping $G : K \longrightarrow 2^Y$ defined by

$$G(y) = \{x \in K : \langle Tx, y - x\rangle + F(y,x) - F(x,x) \in C(x)\}$$

is transfer closed on each compact convex subset K containing D

(b) the set $\{y \in K : \langle Tx, y - x\rangle + F(y,x) - F(x,x) \notin C(x)\}$ is convex, $\forall\, x \in K$.

Then, (VMQVI) has a solution. Moreover, the solution set of (VMQVI) is relatively compact.

Proof. The result follows from Theorem 2.2 by defining $h(x,y) = \langle Tx, y - x\rangle + F(y,x) - F(x,x)$, for each $x, y \in K$.

Remark 3.5. The condition (b) of Corollary 2.4 holds if the function $y \longrightarrow F(x,y)$ is $C(x)-$ convex for each $x \in K$. To see this let $\lambda \in [0,1]$, and

$$\langle Tx, y_i - x\rangle + F(y_i, x) - F(x,x) \in Y \backslash C(x), \text{ for } i = 1, 2.$$

Since $Y \backslash C(x)$ is an open set then there exists a balanced neighborhood V of zero such that we have

$$V + (\langle Tx, y_1 - x\rangle + F(y,x) - F(x,x)) \in Y \backslash C(x).$$

Moreover, note $Y \backslash C(x)$ is a cone, for each positive integer n we get

$$\frac{\lambda}{n}(V + (\langle Tx, y_1 - x\rangle + F(y_1,x) - F(x,x)) \in Y \backslash C(x). \quad (2.1)$$

For sufficiently large positive integer n we have

$$\frac{1-\lambda}{\lambda n}(\langle Tx, y_2 - x\rangle + F(y_2,x) - F(x,x)) \in V \quad (2.2)$$

From (2.1) and (2.2) we get

$$\lambda(\langle Tx, y_1 - x\rangle + F(y_1, x) - F(x, x)) + (1-\lambda)(\langle Tx, y_2 - x\rangle + F(y_2, x) - F(x, x)) \in Y\backslash C(x) \qquad (2.3).$$

From the $C(x)-$ convexity we have

$$\lambda F(y_1, x) + (1-\lambda)F(y_2, x) - F(\lambda y_1 + (1-\lambda)y_2, x) \in C(x) \ (2.4).$$

Finally (2.3), (2.4) and $(Y\backslash C(x)) - C(x) \subseteq Y\backslash C(x)$ imply the result.

By combining Corollary 2.4 and Remark 2.5 we obtain the following result.

Theorem 3.6. *Assume that:*

(a) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that*

(i) for each $x \in K\backslash B$, there exists $y \in D$ such that $\langle T(x), y - x\rangle + F(y, x) - F(x, x) \notin C(x)$;

(ii) for each fixed $y \in K$, the set valued mapping $G : K \longrightarrow 2^Y$ defined by

$$G(y) = \{x \in K : \langle Tx, y - x\rangle + F(y, x) - F(x, x) \in C(x)\}$$

is transfer closed on each compact convex subset K containing D

(b) *the function $y \longrightarrow F(x, y)$ is $C(x)-$ convex, $\forall x \in K$;*

Then, (VMQVI) has a solution. Moreover, the solution set of (VMQVI) is compact.

Theorem 3.7. *Suppose that:*

(a) *there exist a nonempty compact subset B and a nonempty convex compact subset D of K such that*

(i) for each $x \in K\backslash B$, there exist $y \in D$ and an open neighborhood U_x of x in K such that

$$\langle T(z), y - z\rangle + F(y, z) - F(z, z) \notin C(z), \forall z \in U_x;$$

(ii) for each fixed $y \in K$, the set valued mapping $G : K \longrightarrow 2^Y$ defined by

$$G(y) = \{x \in K : \langle Tx, y - x\rangle + F(y, x) - F(x, x) \in C(x)\}$$

is transfer closed on each compact convex subset K containing D

(b) *the function $y \longrightarrow F(x, y)$ is $C(x)-$ convex, $\forall x \in K$;*

Then, (VMQVI) has a solution. Moreover, the solution set of (VMQVI) is nonempty and relative compact in K.

Proof. We define $\Gamma : K \longrightarrow 2^K$ by

$$\Gamma(y) = \{x \in K : \langle T(x), y - x\rangle + F(y, x) - F(x, x) \in C(x)\}.$$

One can see, by using (b) and Remark 2.5, Γ is a KKM map and so $cl\Gamma$ is a KKM mapping (Note $\Gamma(x) \subseteq cl\Gamma(x)$ for all $x \in K$). From (a)(i) we conclude $\bigcap_{x\in D} cl\Gamma(x) \subseteq B$ and hence Lemma 1.4 implies $\bigcap_{x\in D} cl\Gamma(x) \neq \emptyset$. Then for each nonempty finite subset A of K, by Lemmas 2.1, 1.4, we get $\bigcap_{x\in co(D\cup A)} cl\Gamma(x) \neq \emptyset$. Now we can conclude the proof as the same manner of the proof of Theorem 2.2. □

Theorem 3.8. *Suppose that all assumptions of one of the Theorems 2.2, 2.6 or 2.7 or Corollary 2.4 are satisfied. If,* $0 \in K$ *and* $F(2u, v) = 2F(u, v), \forall u, v \in K$, *then, (VMQCP) has a solution. Moreover, the solution set of (VMQCP) is relative compact.*

Proof. The result follows from Theorems 2.2 and 2.3. □

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A STRUCTURE THEOREM ON NON-HOMOGENEOUS LINEAR EQUATIONS IN HILBERT SPACES

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ABSTRACT. A very particular by-product of the result announced in the title reads as follows: Let $(X, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, $T : X \longrightarrow X$ a compact and symmetric linear operator, and $z \in X$ such that the equation $T(x) - \|T\|x = z$ has no solution in X. For each $r > 0$, set $\gamma(r) = \sup_{x\in S_r} J(x)$, where $J(x) = \langle T(x) - 2z, x\rangle$ and $S_r = \{x \in X : \|x\|^2 = r\}$. Then, the function γ is C^1, increasing and strictly concave in $]0, +\infty[$, with $\gamma'(]0, +\infty[) =]\|T\|, +\infty[$; moreover, for each $r > 0$, the problem of maximizing J over S_r is well-posed, and one has

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = z$$

where $\hat{x}_r$ is the only global maximum of $J_{|S_r}$.

KEYWORDS : Linear equation; Hilbert space; Eigenvalue; Well-posedness.

1. INTRODUCTION AND PRELIMINARIES

Here and in the sequel, $(X, \langle\cdot,\cdot\rangle)$ is real Hilbert space. For each $r > 0$, set

$$S_r = \{x \in X : \|x\|^2 = r\} .$$

In [1], we established the following result (with the usual conventions $\sup\emptyset = -\infty$, $\inf\emptyset = +\infty$):

Theorem A ([1], Theorem 1). *Let $J : X \longrightarrow \mathbf{R}$ be a sequentially weakly continuous C^1 functional, with $J(0) = 0$. Set*

$$\rho = \limsup_{\|x\|\longrightarrow+\infty} \frac{J(x)}{\|x\|^2}$$

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and

$$\sigma = \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} .$$

Let a, b satisfy

$$\max\{0, \rho\} \leq a < b \leq \sigma .$$

Assume that, for each $\lambda \in]a, b[$, the functional $x \longrightarrow \lambda\|x\|^2 - J(x)$ has a unique global minimum, say $\hat{y}_\lambda$. Let M_a (resp. M_b if $b < +\infty$ or $M_b = \emptyset$ if $b = +\infty$) be the set of all global minima of the functional $x \longrightarrow a\|x\|^2 - J(x)$ (resp. $x \longrightarrow b\|x\|^2 - J(x)$ if $b < +\infty$). Set

$$\alpha = \max\left\{0, \sup_{x \in M_b} \|x\|^2\right\} ,$$

$$\beta = \inf_{x \in M_a} \|x\|^2$$

and, for each $r > 0$,

$$\gamma(r) = \sup_{x \in S_r} J(x).$$

Finally, assume that J has no local maximum with norm less than β.

Then, the following assertions hold:

(a_1) *the function $\lambda \longrightarrow g(\lambda) := \|\hat{y}_\lambda\|^2$ is decreasing in $]a, b[$ and its range is $]\alpha, \beta[$;*

(a_2) *for each $r \in]\alpha, \beta[$, the point $\hat{x}_r := \hat{y}_{g^{-1}(r)}$ is the unique global maximum of $J_{|S_r}$ and every maximizing sequence for $J_{|S_r}$ converges to $\hat{x}_r$;*

(a_3) *the function $r \longrightarrow \hat{x}_r$ is continuous in $]\alpha, \beta[$;*

(a_4) *the function γ is C^1, increasing and strictly concave in $]\alpha, \beta[$;*

(a_5) *one has*

$$J'(\hat{x}_r) = 2\gamma'(r)\hat{x}_r$$

for all $r \in]\alpha, \beta[$;

(a_6) *one has*

$$\gamma'(r) = g^{-1}(r)$$

for all $r \in]\alpha, \beta[$.

We want to remark that, in the original statement of [1], one assumes that X is infinite-dimensional and that J has no local maxima in $X \setminus \{0\}$. These assumptions come from [2] whose results are applied to get (a_3), (a_4) and (a_5). The validity of the current formulation just comes from the proofs themselves given in [2] (see also [3]).

The aim of this very short paper is to show the impact of Theorem A in the theory of non-homogeneous linear equations in X.

2. MAIN RESULTS

Throughout the sequel, z is a non-zero point of X and $T : X \longrightarrow X$ is a continuous linear operator.

We are interested in the study of the equation

$$T(x) - \lambda x = z$$

for $\lambda > \|T\|$. In this case, by the contraction mapping theorem, the equation has a unique non-zero solution, say $\hat{v}_\lambda$. Our structure result just concerns such solutions.

As usual, we say that:

- T is compact if, for each bounded set $A \subset X$, the set $\overline{T(A)}$ is compact ;

- T is symmetric if
$$\langle T(x), u\rangle = \langle T(u), x\rangle$$
for all $x, u \in X$.

We also denote by V the set (possibly empty) of all solutions of the equation
$$T(x) - \|T\|x = z$$
and set
$$\theta = \inf_{x\in V} \|x\|^2 \ .$$
Of course, $\theta > 0$. Our result reads as follows:

Theorem 1. - *Assume that T is compact and symmetric .*

For each $\lambda > \|T\|$ and $r > 0$, set
$$g(\lambda) = \|\hat{v}_\lambda\|^2$$
and
$$\gamma(r) = \sup_{x\in S_r} J(x)$$
where
$$J(x) = \langle T(x) - 2z, x\rangle \ .$$
Then, the following assertions hold:
(b_1) *the function g is decreasing in $]\|T\|, +\infty[$ and*
$$g(]\|T\|, +\infty[) =]0, \theta[\ ;$$
(b_2) *for each $r \in]0, \theta[$, the point $\hat{x}_r := \hat{v}_{g^{-1}(r)}$ is the unique global maximum of $J_{|S_r}$ and every maximizing sequence for $J_{|S_r}$ converges to $\hat{x}_r$;*
(b_3) *the function $r \longrightarrow \hat{x}_r$ is continuous in $]0, \theta[$;*
(b_4) *the function γ is C^1, increasing and strictly concave in $]0, \theta[$;*
(b_5) *one has*
$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = z$$
for all $r \in]0, \theta[$;
(b_6) *one has*
$$\gamma'(r) = g^{-1}(r)$$
for all $r \in]0, \theta[$.

Before giving the proof of Theorem 1, we establish the following

Proposition 1. - *Let T be symmetric and let J be defined as in Theorem 1. Then, for $\tilde{x} \in X$, the following are equivalent:*
(i) *$\tilde{x}$ is a local maximum of J .*
(ii) *$\tilde{x}$ is a global maximum of J .*
(iii) *$T(\tilde{x}) = z$ and $\sup_{x\in X} \langle T(x), x\rangle \le 0$.*

Proof. First, observe that, since T is symmetric, the functional J is Gâteaux differentiable and its derivative, J', is given by
$$J'(x) = 2(T(x) - z)$$
for all $x \in X$ ([4], p. 235). By the symmetry of T again, it is easy to check that, for each $x \in X$, the inequality
$$J(\tilde{x} + x) \le J(\tilde{x}) \tag{1}$$
is equivalent to
$$\langle 2(T(\tilde{x}) - z) + T(x), x\rangle \le 0 \ . \tag{2}$$
Now, if (i) holds, then $J'(\tilde{x}) = 0$ (that is $T(\tilde{x}) = z$) and there is $\rho > 0$ such that (1) holds for all $x \in X$ with $\|x\| \le \rho$. So, from (2), we have $\langle T(x), x\rangle \le 0$ for the same

x and then, by linearity, for all $x \in X$, getting (iii). Vice versa, if (iii) holds, then (2) is satisfied for all $x \in X$ and so, by (1), $\tilde{x}$ is a global maximum of J, and the proof is complete. $\triangle$

Proof of Theorem 1. For each $x \in X$, we clearly have

$$J(x) \leq \|T(x) - 2z\|\|x\| \leq \|T\|\|x\|^2 + 2\|z\|\|x\|$$

and so

$$\limsup_{\|x\| \longrightarrow +\infty} \frac{J(x)}{\|x\|^2} \leq \|T\| \ . \tag{3}$$

Moreover, if $v \in X \setminus \{0\}$ and $\mu \in \mathbf{R} \setminus \{0\}$, we have

$$\frac{J(\mu v)}{\|\mu v\|^2} \geq -2\frac{\langle z, v\rangle}{\mu\|v\|^2} - \|T\|$$

and so

$$\limsup_{x \longrightarrow 0} \frac{J(x)}{\|x\|^2} = +\infty \ . \tag{4}$$

Moreover, the compactness of T implies that J is sequentially weakly continuous ([4], Corollary 41.9). Now, let $\lambda \geq \|T\|$. For each $x \in X$, set

$$\Phi(x) = \|x\|^2 \ .$$

Then, for each $x, v \in X$, we have

$$\langle \lambda\Phi'(x) - J'(x) - (\lambda\Phi'(v) - J'(v)), x - v\rangle = \langle 2\lambda(x - v) - 2(T(x) - T(v)), x - v\rangle \geq$$
$$2\lambda\|x - v\|^2 - 2\|T(x) - T(v)\|\|x - v\| \geq 2(\lambda - \|T\|)\|x - v\|^2 \ . \tag{5}$$

From (5) we infer that the derivative of the functional $\lambda\Phi - J$ is monotone, and so the functional is convex. As a consequence, the critical points of $\lambda\Phi - J$ are exactly its global minima. So, $\hat{v}_\lambda$ is the only global minimum of $\lambda\Phi - J$ if $\lambda > \|T\|$ and V is the set of all global minima of $\|T\|\Phi - J$. Now, assume that J has a local maximum, say w. Then, by Proposition 1, w is a global minimum of $-J$ and $\sup_{x \in X}\langle T(x), x\rangle \leq 0$. Since T is symmetric, this implies, in particular, that $\|T\|$ is not in the spectrum of T. So, V is a singleton. By Proposition 1 of [1], we have

$$\|w\|^2 \geq \theta \ .$$

In other words, J has no local maximum with norm less than θ. At this point, taking (3) and (4) into account, we see that the assumptions of Theorem A are satisfied (with $a = \|T\|$ and $b = +\infty$, and so $\alpha = 0$ and $\beta = \theta$), and the conclusion follows directly from that result. $\triangle$

Some remarks on Theorem 1 are now in order.

Remark 1. - Each of the two properties assumed on T cannot be dropped. Indeed, consider the following two counter-examples.

Take $X = \mathbf{R}^2$, $z = (1, 0)$ and $T(t, s) = (t + s, s - t)$ for all $(t, s) \in \mathbf{R}^2$. So, T is compact but not symmetric. In this case, we have

$$\hat{x}_r = (-\sqrt{r}, 0) \ ,$$
$$\gamma(r) = r + 2\sqrt{r}$$

for all $r > 0$. Hence, in particular, we have

$$T(\hat{x}_r) - \gamma'(r)\hat{x}_r = (1, \sqrt{r}) \neq z \ .$$

That is, (b_5) is not satisfied.

Now, take $X = l_2$, $z = \{w_n\}$, where $w_2 = 1$ and $w_n = 0$ for all $n \neq 2$, and $T(\{x_n\}) = \{v_n\}$ for all $\{x_n\} \in l_2$, where $v_1 = 0$ and $v_n = x_n$ for all $n \geq 2$.

So, T is symmetric but not compact. In this case, we have $\theta = +\infty$ and

$$\gamma(r) = r - 2\sqrt{r}$$

for all $r \geq 4$. Hence, γ is not strictly concave in $]0, +\infty[$.

Remark 2. - Note that the compactness of T serves only to guarantee that the functional $x \longrightarrow \langle T(x), x\rangle$ is sequentially weakly continuous. So, Theorem 1 actually holds under such a weaker condition.

Remark 3. - A natural question is: if assertions $(b_1) - (b_6)$ hold, must the operator T be symmetric and the functional $x \longrightarrow \langle T(x), x\rangle$ sequentially weakly continuous ?

Remark 4. - Note that if T, besides to be compact and symmetric, is also positive (i.e. $\inf_{x\in X}\langle T(x), x\rangle \geq 0$), then, by classical results, the operator $x \longrightarrow T(x) - \|T\|x$ is not surjective, and so there are $z \in X$ for which the conclusion of Theorem 1 holds with $\theta = +\infty$.

We conclude with an application of Theorem 1 to a classical Dirichlet problem.

So, let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. Let λ_1 be the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix a non-zero continuous function $\varphi : \overline{\Omega} \longrightarrow \mathbf{R}$.

For each $\mu \in]0, \lambda_1[$, let u_μ be the unique classical solution of the problem

$$\begin{cases} -\Delta u = \mu(u + \varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, set

$$\psi(\mu) = \int_\Omega |\nabla u_\mu(x)|^2 dx$$

and

$$\eta(r) = \sup_{u\in U_r} \Phi(u)$$

where

$$\Phi(u) = \int_\Omega |u(x)|^2 dx + 2\int_\Omega \varphi(x)u(x)dx$$

and

$$U_r = \left\{u \in H^1_0(\Omega) : \int_\Omega |\nabla u(x)|^2 dx = r\right\} .$$

Finally, denote by A the set of all classical solutions of the problem

$$\begin{cases} -\Delta u = \lambda_1(u + \varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and set

$$\delta = \inf_{u\in A} \int_\Omega |\nabla u(x)|^2 dx .$$

Then, by using standard variational methods, we can directly draw the following result from Theorem 1 :

Theorem 2. - *The following assertions hold:*

(c_1) *the function* ψ *is increasing in* $]0, \lambda_1[$ *and one has*

$$\psi(]0, \lambda_1[) =]0, \delta[\ ;$$

(c_2) *for each* $r \in]0,\delta[$, *the function* $w_r := u_{\psi^{-1}(r)}$ *is the unique global maximum of* $\Phi_{|U_r}$ *and each maximizing sequence for* $\Phi_{|U_r}$ *converges to* w_r *with respect to the topology of* $H^1_0(\Omega)$ *;*
(c_3) *the function* $r \longrightarrow w_r$ *is continuous in* $]0,\delta[$ *with respect to the topology of* $H^1_0(\Omega)$ *;*
(c_4) *the function* η *is* C^1*, increasing and strictly concave in* $]0,\delta[$ *;*
(c_5) *for each* $r \in]0,\delta[$, *the function* w_r *is the unique classical solution of the problem*

$$\begin{cases} -\Delta u = \frac{1}{\eta'(r)}(u+\varphi(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

(c_6) *one has*

$$\eta'(r) = \frac{1}{\psi^{-1}(r)}$$

for all $r \in]0,\delta[$.

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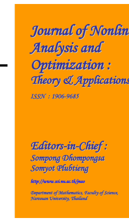

A COMPARATIVE STUDY ON CLASSICAL AND META-HEURISTIC OPTIMIZATION METHODS IN CELL FORMATION PROBLEM

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ABSTRACT. The Cell Formation (CF) problem determines the decomposition of manufacturing cells, in which parts are grouped into part families, and machines are allocated into machine cells to take advantages of minimum inter-cellular movements and the maximum number of parts flow. In this paper, we compare two classical and meta-heuristic optimization methods for solving the manufacturing CF problem. Hence, a dynamic integer model of CF with three sub-objective functions is considered. Also, a set of 20 test problems with various sizes is solved, once by using of Lingo software as a classical optimization method and another with proposed Modified Self-adaptive Differential Evolution (MSDE) algorithm as a meta-heuristic. The result of this comparative study indicates that MSDE algorithm performs more effective for all test problems. Furthermore, due to the fact that CF is a NP-hard problem, classical optimal method needs a long computational time and so not reliable.

KEYWORDS : Cell formation problem; Modified self-adaptive differential evolution; Classical optimization method.

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1. Introduction

1.1. **Cell formation problem.** With increasing market pressure for shorter product life cycle and need of responding to environmental issues, the dynamic aspects of manufacturing play a key role in designing and optimizing of a manufacturing system. Cellular manufacturing system (CMS) is a production approach aimed at increasing production efficiency and flexibility by utilizing the process similarities of the parts in order to minimize the intracellular cost of parts. It involves grouping similar parts into part families and the corresponding machines into machine cells. The success of CMS is rooted in their ability to reduce set-up times, reduced work-in-process inventories, reduced material handling costs, simplified scheduling, shorter lead-times, faster response to internal and external changes, better production control and etc. [1]. Also, some disadvantages have point out in prior research for the CMS, such as: reduced flexibility in compare of a job shop, reduced machine utilization due to dedication of machines to cells and impact of machine breakdown on due date [2]. There are three important steps in CMS design: 1) cell formation (CF), 2) machine layout and 3) cell layout, that Cell formation is the first major step in designing a CMS, which involves identification of machine cells and part families. Most of the CF studies have focused on independence of cells, and a few number of them point out the inter-cell movements [3]. In the last three decades of research in CF, researchers have mainly used zero-one machine component incidence matrix as the input data for the problem [4]. In addition, many procedures have been proposed for CF problem such as: data array reordering, hierarchical clustering, non-hierarchical clustering, heuristic and meta-heuristic algorithms, mathematical programming, graph theory, artificial neural networks and fuzzy sets [5].

1.2. **Differential evolution (DE) algorithm.** Evolutionary algorithms are population based stochastic optimization techniques that search for solution using a simplified model of natural genetics in non-continuous, non-convex, nonlinear, time-dependent or solution spaces. The DE is one of the most frequently used evolutionary algorithms, which was first successfully applied by Storn and Price in optimization of some well known nonlinear, non-differentiable and non-convex functions [6]. In recent years, DE has gradually become more popular and has been applied successfully to solve analytically non-solvable problems with non-continuous, nonlinear, noisy, flat, and multi-dimensional objective functions [7]. Due to its simplicity, effectiveness and robustness, DE has successfully applied in different practical applications [8], image processing [9], data clustering [10], optimal design [11] and scheduling [12] problems. The DE algorithm begins from a primary population which generated randomly and consists of four stages process named: initialization, mutation, crossover and selection. In mutation process of a DE algorithm, the weighted difference between two randomly selected population members is added to a third member to generate a mutated solution. Then, a crossover operator follows to combine the mutated solution with the target solution so as to generate a trial solution. Thereafter, a selection operator is applied to compare the fitness function value of both competing solutions, namely, target and trial solutions to determine who can survive for the next generation. This process is repeated over several generations resulting in an evolution of the population to optimal values. The convergence properties of DE are strongly related to its stochastic nature and DE uses random sequence for its parameters. Most of DE studies are related with the mutation operator and just two variants being currently used, so called binomial and exponential crossover.

This paper presents the MSDE algorithm, which is improved of DE algorithm. MSDE in compare of DE algorithm rectifies the mutation rules, thus it select randomly three chromosomes from solution area and uses the environs of the best selected chromosome for any mutant chromosome. In this paper, the performance of MSDE is compared as a meta-heuristic optimizing method with the classical one. Therefore, a dynamic integer model of CF with nonlinear objective function is first presented and then, it solved by Lingo software and MSDE algorithm to achieve the optimum solution. The results have compared from two aspects: the time to get optimum solution (CPU time) and the optimum objective function.

The remainder of this paper is organized as follows: In Section 2 we review on previous work on related areas. The dynamic integer CF model is presented in Section 3. Section 4 describes the MSDE algorithm for CF problem. In section 5 we show the computational results on test problems with various sizes, and Section 6 concludes the paper.

2. Literature review

In this section, we review the previous research on CF problem. Therefore, we can classify these studies by date of research. Before 1995, integer programming formulations of CFP solved by non-traditional methods such as simulated annealing (SA) [13], genetic algorithm (GA) [14], tabu search (TS) [15] and Neural network [16]. Since 1996 until 2000, researches have dedicated the use of multi-criteria CF problem on special areas of science. For instance, evaluate and compare the performance of CMS [17], CM scheduling problems [18] and fuzzy evolutionary algorithms in CF problem [19]. Most studies after year 2000, have been on designing of a CMS [20] and evaluate the performance of a CMS using machine reliability [21]. Also, the GA and TS algorithm and utilize of sequence data have considered for inter cellular movements [22]. In order to evaluate the goodness of the cell formation, a good number of performance measures have been proposed in the literature [23]. In 2006, Geonwook & Herman formed part families and designed machine cells by using of GA under demand changes [24]. Also, Wu et al., designed a CMS Concurrently using GA [25]. In 2007, Mahdavi et al., designed a new mathematical model for CMS based on cell utilization [26], Tavakkoli-Moghaddam et al., designed a facility layout problem in CMS with stochastic demands [27], and, Das et al., worked on machine reliability and preventive maintenance planning in a CMS[28]. Recently, Wu et al., used SA algorithm in a CMS [29] and Safaei et al., proposed a hybrid SA for solving an extended model of dynamic CMS [30].

The related previous work on DE algorithm was started by Wang & Chou in 1997. They used the DE algorithm in control problems and the time of distinguish algebraic equations problems [31]. Afterwards, DE utilized extensively for scientific approach, fuzzy logic and neural networks in 1998 [32]. In 2000, DE algorithms have been used in several areas such as electrical energy distribution [33], pharmacy [34], science of aviation and magnetic [35], chemical engineering [36] and environmental science [37]. Also, it applied in group technology and linear-programming models in 2001 [38], medical science in 2003, optimizing non-linear process in 2006 and in 2007 for optimizing functions with unknown parameters. It has been shown to perform better than GA [39] or particle swarm optimization (PSO) [40] over several numerical benchmarks [41]. Recently, some researchers succeeded to use this algorithm for optimizing of traveling salesman problem (TSP) [42], machines layout [43] and Flow Shop Scheduling [44] problems.

3. Dynamic CF model

In this section, we present a dynamic integer CF model including three sub-objective functions as cost parameters.

3.1. **Assumptions.** The following assumptions are considered in the proposed dynamic cell formation problem:

(i) The operating times for all part type operations on different machine types are known.
(ii) Each machine type can perform several operations (machine flexibility).
(iii) Operating cost of each machine type per hour is known.
(iv) Machines are available at the start of each period (no installation time).
(v) Investment or purchase cost of each type of machine in each period is known.
(vi) Bounds and quantity of machines in each cell are constant.
(vii) Inter-cell relocation costs are constant for all moves regardless of the distance traveled.
(viii) Machine relocation from one cell to another is performed between periods with no time.
(ix) Parts are moved between any two cells in batches and the inter-cell cost per batch between cells is known and constant.
(x) Batch size is constant for all productions in all periods.
(xi) The demand for each part type in different period is dynamic.
(xii) Setup times are not considered.

3.2. **Indexing sets.** j: index for operations j = 1,. . . ,O_p
p: index for parts p = 1,. . . ,P
m: index for machines m = 1,. . . ,M
c: index for cells c = 1,. . .,C
h: index for periods h = 1,. . .,H

3.3. **Input parameters.** C_m : purchase cost of machine type m
IC : inter-cell material handling cost per batch
D_{ph} : demand for product p in period h
B_{int} : batch size for inter-cell material handling
MC_m : relocation cost of machine type m
L_B : lower bound of cell size
U_B : upper bound of cell size
T_m : available time for each machine type m
t_{jp} : time required to perform operation j on product p

3.4. **Decision variables.**

$$B_{jpch} : \begin{cases} 1 & \text{if part type } p \text{ remains in cell } c \text{ after operation } j \text{ in period } h \\ 0 & \text{other wise} \end{cases}$$

$$X_{jpch} : \begin{cases} 1 & \text{if operation } j \text{ of part type } p \text{ is done in cell } c \text{ in period } h \\ 0 & \text{other wise} \end{cases}$$

N_{mch} : number of machines of type m devoted in cell c during period h
K^{+}_{mch} : number of machines of type m added in cell c during period h

$K^{-}{}_{mch}$: number of machines of type m removed from cell c during period h

3.5. **Mathematical formulation.**

$$Min \sum_{m=1}^{M}\sum_{c=1}^{C}\sum_{h=1}^{H}(C_m N_{mch}) \quad + \quad \sum_{j=1}^{Op}\sum_{p=1}^{P}\sum_{c=1}^{C}\sum_{h=1}^{H}(IC \times B_{jpch} \times \frac{D_{ph}}{B_{int}}) + \sum_{m=1}^{M}\sum_{c=1}^{C}\sum_{h=1}^{H} MC_m(k^{+}_{mch} + k^{-}_{mch}) \tag{3.1}$$

s.t:

$$\sum_{c=1}^{C} x_{jpch} = 1, \qquad \forall p, j, h, \tag{3.2}$$

$$\sum_{p=1}^{P}\sum_{j=1}^{O_p} D_{ph} t_{jp} x_{jpch} \leq T_m N_{mch}, \qquad \forall m, c, h, \tag{3.3}$$

$$l_B \leq \sum_{m=1}^{M} N_{mch} \leq u_B, \qquad \forall c, h, \tag{3.4}$$

$$N_{mch} = N_{mc(h-1)} + k^{+}_{mch} - k^{-}_{mch}, \qquad \forall m, c, h, \tag{3.5}$$

$$x_{jpch} + x_{(j+1)pch} - B_{jpch} \leq 1, \qquad \forall j, p, c, h, \tag{3.6}$$

$$B_{jpch}, x_{jpch} \in \{0, 1\},$$

$N_{mch}, k^{+}_{mch}, k^{-}_{mch} \geq 0$ and integer.

3.6. **Sub-objective functions.** *According to the proposed dynamic CF problem, multiple costs are considered in design of objective function. Meanwhile, it is not possible to consider all costs in the model due to the complexity and computational time required. Here, objective function consists of three cost parameters. The objective is to minimize the sum of the following costs:*

(i) *Machine annual cost: This cost includes investment and amortization cost per period and calculates based on the number of machines of each type used for specific period. Increasing of this cost could lead to the high total cost of companies.*

$$\min f_1 = \sum_{m=1}^{M}\sum_{c=1}^{C}\sum_{h=1}^{H}(C_m N_{mch}), \tag{3.7}$$

(ii) *Inter-cell handling costs: One of the major problems in design of a CF, is Inter-cell moves. The cost of transferring parts between cells incurred when parts cannot be produced completely by a machine type or in a single cell. Inter-cell moves decrease the efficiency of cellular manufacturing by complicating production control and increasing material handling requirements and flow time.*

$$\min f_2 = \sum_{j=1}^{Op}\sum_{p=1}^{P}\sum_{c=1}^{C}\sum_{h=1}^{H}(IC \times B_{jpch} \times \frac{D_{ph}}{B_{int}}), \tag{3.8}$$

(iii) *Machine relocation cost during different period: The cost of relocating machines from one cell to another between periods. In dynamic and stochastic production systems, the best CF design for one period may not be an efficient design for subsequent periods. By rearranging the manufacturing cells, the CF can continue operating efficiently as the product mix and demand change. Moving machines from cell to cell requires effort and can lead to the disruption of production.*

$$\min f_3 = \sum_{m=1}^{M}\sum_{c=1}^{C}\sum_{h=1}^{H} MC_m(k^{+}_{mch} + k^{-}_{mch}). \tag{3.9}$$

3.7. **Model constraints.** Primary constraints of model include size of cells and binary decision variables. Constraint (3.2) ensures that each part operation is assigned to one machine and one cell. Constraint (3.3) ensures that machines capacities have not exceeded and can satisfy the demand. Constraints (3.4) specify the lower and upper bounds of cells. Constraint (3.5) ensures that the number of machines in current period is equal to the aggregated number of machines in the previous plus the number of machines being moved in and subtracted by the number of machines being moved out. In other words, they ensure conservation of machines over the horizon. In constraint (3.6), if at least one of the operations of part p is proceed in cell c in period h, then the value of B_{jpch} will be equal to one; otherwise it is equal to zero.

4. Differential Evolution Algorithm

4.1. **MSDE optimization process.** MSDE algorithm rectifies the mutation rules, which has done by random selection of three chromosomes within solution space and using the environment of the best selected chromosome for any mutant chromosome. This algorithm also improves the scale factor of F_i by computing the mutation probability of P_r from normal distribution. This algorithm consists of four steps of initialization, mutation, crossovers and selection, which are repeated until receiving of near-optimal solution. One of the important characteristics of proposed MSDE algorithm in this paper is that refreshes itself in each generation number for k=1,. . . ,P_S by vacating of useless information, which can accelerate the process of algorithm. These steps are explained as follows:

A. Initialization:

The first step in MSDE algorithm is to create an initial population of candidate solutions by assigning random values to each decision variable of each chromosome for the population. In this step also the parameters of N_C (number of chromosomes) which i=1,2,. . . ,N_C, P_S and CR have adjusted and the scale factor of F_i initializes for the first population with normal distribution, N(0.5,0.15), where 0.5 is the mean and 0.15 is the standard deviation and will going to generate random number between (0,1].

B. Mutation:

The principal idea of MSDE is a new scheme for generating trial parameter vectors by adding the weighted difference vector between two population members to a third member. The mutation operator is in charge of introducing new parameters into the population. Therefore, it creates mutant vectors according to Eq. (4.1) by computing the scale factor of F_i according to Eq. (4.2), where (X_{r1}, X_{r2}, X_{r3}) are three chromosomes which selected randomly from S={$X_1{}^k$, $X_2{}^k$,..., $X_{Nc}{}^k$}and X_{rb} is the best of them. Also, (F_{r4}, F_{r5}, F_{r6}) are three factor scale vectors which selected randomly from current population that are not similar for any two chromosomes.

$$V_i^{(K)} = X_{rb} + F_i^{(K)}(X_{r2}^{(K)} - X_{r3}^{(K)}), \tag{4.1}$$

$$F_i^{(K)} = F_{r4}^{(K)} + N(0.05)(F_{r5}^{(K)} - F_{r6}^{(K)}), \qquad r_4 \neq r_5 \neq r_6, \tag{4.2}$$

C. Crossover:
The crossover operator creates the trial vectors, which are used in selection process. A trail vector ($U_{i,j}$) is a combination of a mutant vector and a parent vector that compared against the crossover constant called CR for each gene of population chromosomes. If the value of the random number is less or equal than CR, the parameter will select from the mutant vector ($V_{i,j}$), otherwise from the parent vector ($X_{i,j}$) as given in Eq. (4.3). The crossover operator maintains diversity in the population, preventing from local minimum convergence.

$$u_{i,j} = \begin{cases} V_{i,j} & if \;\; rand(0,1) \leq Pc \\ X_{i,j} & otherwise \end{cases} \qquad j = 1...n, \tag{4.3}$$

D. Selection:
The selection operator chooses the vectors that are going to compose the population in the next generation. This operator compares the fitness of the trial vector {f($U_{i,j}$)} corresponding to target vector {f($X_{i,j}$)} and selects the one that has better target function for k=1,. . . ,P_S according to Eq. (4.4).

$$f(U_{i,j}) < f(X_{i,j}). \tag{4.4}$$

Chromosome structure for CF problem
A chromosome or feasible solution proportional to the described CF model consists of the following genes in each period.

(i) (a) The gene related to the number of available machines in each cell in each period, is called [N_{mch}] matrix. The gene is limited to 0 up to U_B and has three dimensions of *m*, *c* and *h*.
(b) The gene related to the number of moved machines inside or outside of cell in each cell in each period, is called [K_{mch}] matrix. The gene is limited to L_B up to U_B. This gene could take positive or negative integer values and has three dimensions of *m*, *c* and *h*.
(c) The gene for distinguishing inter-cell transportation, is called [B_{jpch}] matrix. This gene is binary and has four dimensions of *j*, *p*, *c* and *h*, that by remaining of part *P* after operation *j* in cell, get the number 1 and lead to the inter-cell transportation cost, otherwise get the number zero and has no cost.

5. Computational results

In this section, in order to make the similar comparative condition of study for two computational optimizing methods, we utilized a 20 set of test problem, which generated randomly with various dimensions and then the proposed CF model would solved once by Lingo software as a classical method and another, by MSDE algorithm as a meta-heuristic optimizing method. It should be mentioned that due to increase of the reliability of assess process, we procure the crossover constant called P_c by several experiment. Therefore, we have selected the two experimental test problem first, and then have solved by different size of P_c in order to get the best quantity of P_c.

5.1. **Comparative result of optimizing methods.** Considering to the input parameters of proposed CF model, we have selected 20 test problems with different sizes up to the maximum number of defined. The proposed CF model have solved by optimizing methods in similar conditions and limitations in order to compare optimization methods exactly in two factors named, the time to get result and value of objective function. The coding language of MSDE algorithm has been C^{++} and utilized by the processor of 3.00 GB with 2.00 GB of RAM to compute the CF model. Table 1 dedicates the comparative result of test problems with Lingo 8 and MSDE algorithm separately.
The CF input parameters used in all problems are as follows:

(i) Purchase cost of machine type m (C_m) = [500,1000]
(ii) Inter-cell material handling cost per batch (IC) = 50
(iii) Demand for product p in period h (D_{ph}) = [100,200]
(iv) Batch size for inter-cell material handling (B_{int}) = 50
(v) Relocation cost of machine type m (MC_m) = [100,150]
(vi) The min/max capacity of cell: $LB = 1$, $UB = M/2$, where M = number of machines
(vii) Available time for each machine type m (T_m) = [120,600] s
(viii) Time required to perform operation j on product p (t_{jp}) = [5,30] s
(ix) $CR = 0.2$, $N_C = 1000$, $P_S = 100$

According to the table, the decline percent by using of MSDE has grown with large scale of input parameters. Decline percent is on range of 70% for small tests that has been received up to 97% for larger than 500000 dimensions tests.

Table 1. Comparative result of optimizing methods

No	***Problem Dimensions*** *j-p-m-c-h*	***Lingo result*** *runtime (s) function*		***MSDE result*** *runtime (s) function*		*Decline percent in MSDE*
1	*4-4-3-2-2*	*3*	*1730*	*4*	*1800*	*-0.33%*
2	*5-4-4-3-2*	*28*	*3900*	*8*	*4300*	*71%*
3	*6-5-4-3-3*	*50*	*10600*	*14*	*10600*	*72%*
4	*9-8-8-4-2*	*490*	*25100*	*26*	*25100*	*94.7%*
5	*10-10-8-4-3*	*2400*	*54700*	*49*	*54700*	*98%*
6	*12-10-10-3-3*	*2250*	*56300*	*44*	*48700*	*98%*
7	*14-12-10-4-3*	*2900*	*92500*	*75*	*93700*	*97.4%*
8	*15-15-12-4-4*	*5040*	*170500*	*126*	*170500*	*97.5%*
9	*18-18-17-3-3*	*10700*	*128200*	*99*	*129500*	*99%*
10	*20-20-18-4-3*	*>10h*	*-*	*164*	*229100*	*-*
11	*24-20-20-4-4*	*>10 h*	*-*	*251*	*373100*	*-*
12	*25-25-22-5-3*	*>10 h*	*-*	*320*	*458000*	*-*
13	*30-25-25-4-5*	*>10 h*	*-*	*458*	*736000*	*-*
14	*30-30-30-5-5*	*>10 h*	*-*	*678*	*1108400*	*-*
15	*35-32-30-5-5*	*>10 h*	*-*	*803*	*1383000*	*-*
16	*35-35-32-6-5*	*>10 h*	*-*	*1058*	*1818800*	*-*
17	*40-38-38-5-6*	*>10 h*	*-*	*1308*	*2257100*	*-*
18	*42-42-40-6-6*	*>10 h*	*-*	*1784*	*3148200*	*-*
19	*50-45-42-6-5*	*>10 h*	*-*	*1830*	*3347200*	*-*
20	*50-50-48-6-5*	*>10 h*	*-*	*2095*	*3724300*	*-*

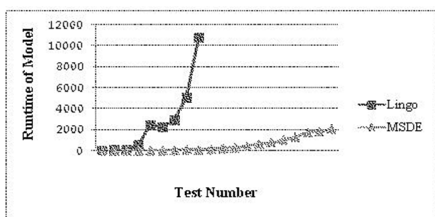

Diagram 1. Compare the runtime of CFP for two methods

As the diagram shows, the trend of runtime for CF problem has increased rapidly by Lingo software with large dimension parameters. These trends dedicate that although using of classical optimizing method cannot be able to solve nonlinear and non-differentiable functions efficiently, but evolutionary algorithm such as DE can be utilized effectively, specially for large scale problems.

5.2. **The convergence of MSDE algorithm.** Due to distinguish the behavior of MSDE algorithm at finding the optimum solution, we have proceeded to depict the diagrams which can construe the algorithm reliability for different positions of input parameters. Therefore, we selected the two test problems with different dimensions, as the problem with 6 operations, 5 parts, 4 machines, 3 cells and 3 periods and 1080 dimensions as a small test problem and the problem with 35 operations, 35 parts, 32 machines, 6 cells and 5 periods and 1176000 dimensions as a large dimension one. The convergence of MSDE algorithm has considered for two test problems by different quantity of generation number and calculated for

N_g= 100, 200, 400, 600, 800, 1000 according to the objective function. The results have shown in diagrams 2 and 3 for two problems separately.

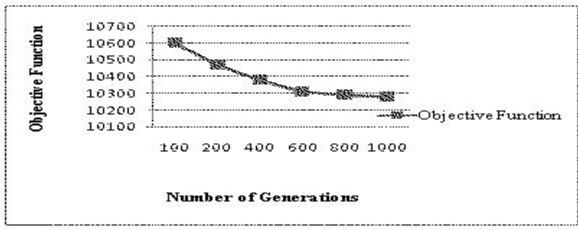

Diagram 2. Convergence diagram for test problem with (6,5,4,3,3) as (j,p,m,c,h)

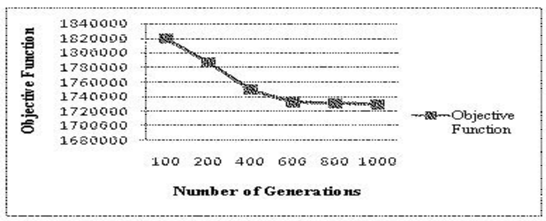

Diagram 3. Convergence diagram for test problem with (35,35,32,6,5) as (j,p,m,c,h)

6. Conclusion

The differential evolution algorithm is one the newest and important evolutionary algorithm, which can used property for optimizing of variety functions. According to the result, MSDE decreases highly the computational time. Whereas the cell formation problem is a hard-problem, so using classical optimizing method increases the computational time and cannot be reliable. Furthermore, due to the diversity of products and application of cellular manufacturing systems, evolutionary algorithms such as DE can be lucrative. Therefore, it is recommended that utilize evolutionary algorithm for solving non-linear and multi-criteria functions such as cell formation problem.

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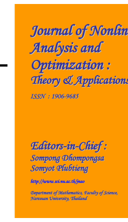

PARETO-EFFICIENT TARGET BY OBTAINING THE FACETS OF THE EFFICIENT FRONTIER IN DEA

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ABSTRACT. In this paper, we propose an algorithm to calculate an improvement target for each inefficient DMU in the CCR model by calculating all equations forming the facets of the efficient frontier. By introducing a parameter into the algorithm, we calculate a minimal distance point or a Pareto-efficient point on the efficient frontier as an improvement target. All improvement targets are obtained by solving quadratic mathematical problems.

KEYWORDS : DEA; Efficient frontier; Production possibility set; Pareto-efficient.

MSC : Primary 46N10; Secondary 93B15.

1. Introduction

DEA(Data Envelopment Analysis) is a non-parametric analytical methodology used for efficiency analysis of a DMU(Decision Making Unit) that consumes inputs to produce outputs. Each DMU is classified as either inefficient or efficient unit according to the optimal value of the CCR model defined in [3]. Moreover, the efficient DMUs are split between Pareto-efficient and Pareto-inefficient units depending on the positive optimal slackness of the CCR model. In the radial

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measure models, an improvement target can be obtained simply by using the optimal value. However, it is often diffcult to improve the values of inputs and outputs according to the improvement, because the improvements obtained by the radial measure models improve the only input (or output) values at the same rate. Therefore, Frei and Harker have proposed the minimal distance projection to the efficient frontier by using the Euclidean norm in [5]. Takeda and Nishino have proposed minimal norm problem to the efficient frontier from an inefficient DMU in [8]. Recently, improvement of efficiency for each inefficient DMU is one of the important subjects in DEA. Aparicio, Ruiz and Sirvent have formulated several mixed integer linear programs for typical norms to obtain a closest target on the efficient frontier in [1]. Further, Lozano and Villa have proposed a gradual efficiency improvement strategy in [7].

In this paper, we propose three kinds of improvement targets for each inefficient DMU in the CCR model. In order to calculate the targets, we use all equations forming the facets of the efficient frontier. The first and second targets are obtained by an algorithm with a parameter. By considering the convex combination of their targets and its projection to the efficient frontier, we suggest the third target as more flexible improvement.

The constitution of this paper is as follows. In Section 2, we introduce the CCR model and some definitions. In Section 3, we propose an algorithm to calculate a improvement target by introducing a parameter and a symmetric positive semidefinite matrix. In order to obtain improvements of DMUs, we use all equations forming the facets of the efficient frontiers. In Section 4, we show a numerical experiment.

Throughout this paper, we use the following notation: Let $\mathbb{R}^n$ be an n-dimensional Euclidean space. For a nutural number m, $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m : x_i \geqslant 0, i = 1, \ldots, m\}$ and $\mathbb{R}^m_- := \{x \in \mathbb{R}^m : x_i \leqslant 0, i = 1, \ldots, m\}$. For a vector $a \in \mathbb{R}^n$, $a^\top$ denotes the transposed vector of a. Let I_n be the unit matrix on $\mathbb{R}^n$. For a subset $S \subset \mathbb{R}^n$, $\dim S$ denotes the dimension of S. For a subset $S \subset \mathbb{R}^n$, $\operatorname{int} S$ and $\operatorname{bd} S$ denote the interior and boundary of S, respectively. For subsets S_1 and $S_2 \subset \mathbb{R}^n$, $S_1 + S_2 := \{a + b : a \in S_1, b \in S_2\}$.

2. CCR MODEL

In this section, we introduce the basic DEA model proposed by Charns, Cooper and Rhodes [3]. Through this paper, n denotes the number of DMUs. Each DMU consumes m different inputs to produce s different outputs. For each $j \in \{1, \ldots, n\}$, DMU(j) has an input vector $x(j) := (x(j)_1, \ldots, x(j)_m)^\top$ and an output vector

$y(j) := (y(j)_1, \ldots, y(j)_s)^\top$. Moreover, we assume the following conditions.

(A1): $x(j) > 0$, $y(j) > 0$ for each $j \in \{1, \ldots, n\}$.
(A2): $(x(j_1)^\top, y(j_1)^\top)^\top \neq (x(j_2)^\top, y(j_2)^\top)^\top$ for each $j_1, j_2 \in \{1, \ldots, n\}$ $(j_1 \neq j_2)$.
(A3): $n > m + s$.
(A4): $\dim(\{x(1), \ldots, x(n)\} \times \{y(1), \ldots, y(n)\}) = m + s$.

Almost DEA models have Assumption (A1). Assumptions (A2), (A3) and (A4) are necessary to execute an algorithm to calculate all facets forming the efficient frontier. However, they are satisfied for almost practical problems. Assumption (A4) means that the convex hull of all DMUs has an interior point.

The CCR model formulated by Charnes, Cooper and Rhodes [3] evaluates the ratio between weighted sums of inputs and outputs. The CCR model provides for constant returns to scale(CRS). Therefore, some researchers call the CCR model the CRS model. In order to calculate an efficiency of DMU$(k)(1 \leq k \leq n)$, the CCR model is formulated as follows:

$$
(\mathrm{CCR}(k)) \begin{cases} \text{maximize} & \dfrac{u^\top y(k)}{v^\top x(k)} \\ \text{subject to} & \dfrac{u^\top y(j)}{v^\top x(j)} \leq 1,\ j = 1, ..., n, \\ & u_r \geq 0,\ r = 1, ..., s, \\ & v_i \geq 0,\ i = 1, ..., m. \end{cases}
$$

Since Problem $(\mathrm{CCR}(k))$ is a fractional programming problem, we can not solve it easily. Therefore, we transform Problem $(\mathrm{CCR}(k))$ into the linear programming problem by setting the denominator of the objective function equals to 1:

$$
(\mathrm{CCRLP}(k)) \begin{cases} \text{maximize} & u^\top y(k) \\ \text{subject to} & v^\top x(k) = 1, \\ & u^\top y(j) - v^\top x(j) \leq 0,\ j = 1, \ldots, n, \\ & u_r \geq 0,\ r = 1, ..., s, \\ & v_i \geq 0,\ i = 1, ..., m. \end{cases}
$$

Then, the dual problem of Problem $(\mathrm{CCR}(k))$ is defined as a linear programming problem as follows:

$$
(\mathrm{CCRD}(k)) \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \theta x(k)_i - \sum_{j=1}^{n} \lambda_j x(j)_i \geq 0,\ i = 1, \ldots, m, \quad (1) \\ & \sum_{j=1}^{n} \lambda_j y(j)_r - y(k)_r \geq 0,\ r = 1, \ldots, s, \quad (2) \\ & \lambda_j \geq 0,\ j = 1, \ldots, n, \quad (3) \\ & \theta \in \mathbb{R}. \end{cases}
$$

Let $\theta^*_{\mathrm{CCR}}(k)$ denote the optimal value of (CCRD(k)). By conditions (2) and (3), we have that $(\lambda_1, \ldots, \lambda_n) \neq (0, \ldots, 0)$ and hence $\lambda_{\hat{j}} > 0$ for some $\hat{j} \in \{1, \ldots, n\}$. Then, it follows from (2) that $\theta^*_{\mathrm{CCR}} x(k)_i - \sum_{j=1}^{n} \lambda_j x(j)_i \geq \theta^*_{\mathrm{CCR}} x(k)_i - \lambda_{\hat{j}} x(\hat{j})_i \geq 0$. This implies that $\theta^*_{\mathrm{CCR}}(k) > 0$. Moreover, we note that $(\lambda^{'}, \theta^{'})$ is a feasible solution of (CCRD(k)) if $\theta^{'} = 1, \lambda^{'}_k = 1$ and $\lambda^{'}_j = 0$ for each $j \in \{1, \ldots, n\} \backslash \{k\}$. Therefore, $0 < \theta^*_{\mathrm{CCR}}(k) \leq 1$. By using the optimal value $\theta^*_{\mathrm{CCR}}(k)$ of (CCRD(k)), the efficiency of DMU(k) for the CCR model is defined as follows:

Definition 2.1. If $\theta^*_{\mathrm{CCR}}(k) = 1$ then $\mathrm{DMU}(k)$ is said to be CCR-efficient. Otherwise, $\mathrm{DMU}(k)$ is said to be CCR-inefficient.

Sometimes, there exists i (or r) such that $v_i = 0$ (or $u_r = 0$). This means the i (or r)th input(output) is not completely used to evaluate DMU(k). In order to resolve this shortage, Charns, Cooper and Rhodes have modified the CCR model by introducing a positive lower limit $(\varepsilon > 0)$ in [4]. Then the constraint conditions of Problems $(\mathrm{CCR}(k))$ and $(\mathrm{CCRLP}(k))$ are replaced as follows:

$$
\begin{matrix} v_i \geq 0,\ i = 1, \ldots, m, \\ u_r \geq 0,\ r = 1, \ldots, s. \end{matrix} \Rightarrow \begin{matrix} v_i \geq \varepsilon,\ i = 1, \ldots, m, \\ u_r \geq \varepsilon,\ r = 1, \ldots, s. \end{matrix}
$$

Then, Problem $(\mathrm{CCRD}(k))$ can be reformulated as follows:

$$(\mathrm{CCRD}\varepsilon(k))\begin{cases}\text{minimize} & \theta-\varepsilon\left(\sum_{i=1}^{m}s_{ix}+\sum_{r=1}^{s}s_{ry}\right)\\ \text{subject to} & \theta x(k)_i-\sum_{j=1}^{n}\lambda_j x(j)_i-s_{ix}=0,\ i=1,\dots,m,\\ & \sum_{j=1}^{n}\lambda_j y(j)_r-y(k)_r-s_{ry}=0,\ r=1,\dots,s,\\ & \lambda_j\geq 0,\ j=1,\dots,n,\\ & s_{ix}\geq 0,\ i=1,\dots,m,\\ & s_{ry}\geq 0,\ r=1,\dots,s,\\ & \theta\in\mathbb{R}.\end{cases}$$

By using an optimal solution $(\theta^*_{\mathrm{CCR}}(k), s^*_x, s^*_y)$ of Problem (CCRDε(k)), the efficiency of DMU(k) for the CCR model is more strictly evaluated.

Definition 2.2. If $\theta^*_{\mathrm{CCR}}(k)=1$ and $(s^*_x, s^*_y)=(0,0)$ then DMU(k) is said to be CCR-Pareto-efficient. If $\theta^*_{\mathrm{CCR}}(k)=1$ and $(s^*_x, s^*_y)\neq(0,0)$ then DMU(k) is said to be CCR-Pareto-inefficient. Otherwise, DMU(k) is said to be CCR-inefficient.

Let T_{CCR} be the production possibility set(PPS) of the CCR model defined in [3] as follows:

$$T_{\mathrm{CCR}}:=\left\{(x,y): x\geq\sum_{j=1}^{n}\lambda_j x(j),\ 0\leq y\leq\sum_{j=1}^{n}\lambda_j y(j)\text{ for some }\lambda\geq 0\right\}.$$

Definition 2.3. (Conical hull) Let E be a nonempty subset in $\mathbb{R}^n$. Then, $\mathrm{conic}\,E$ is called the conical hull of E if it is defined as follows.

$$\mathrm{conic}\,E:=\left\{x\in\mathbb{R}^n: x=\sum_{j=1}^{n}\lambda_j x(j),\ x(j)\in E,\ \lambda_j\geqslant 0,\ j=1,\dots,n\right\}.$$

By the definitions of T_{CCR} and conical hull, T_{CCR} is represented as follows:

$$T_{\mathrm{CCR}}=(\mathrm{conic}\,\{(x(1),y(1)),\dots,(x(n),y(n))\}+(\mathbb{R}^m_+\times\mathbb{R}^s_-))\cap(\mathbb{R}^m\times\mathbb{R}^s_+).$$

Hence, T_{CCR} is a closed convex set. We define the efficient frontier of the CCR model as follows:

$$F_{\mathrm{CCR}}=\mathrm{bd}(T_{\mathrm{CCR}}+(\mathbb{R}^m_+\times\mathbb{R}^s_-))\cap(\mathbb{R}^m\times\mathbb{R}^s_+).$$

We explain the efficiency of the CCR model by using Figure 1. There are six DMUs and each DMU have two inputs (x_1, x_2) and one output (y). By Definition 2.1, B,C,D and F are evaluated as CCR-efficient DMUs. Next, we consider a cone $(\mathbb{R}^2_-\times\mathbb{R}^1_+)$+DMU($k$) for each CCR-efficient DMU(k). For example, for C, we consider

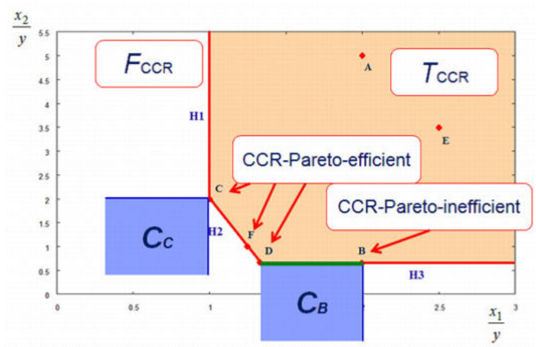

FIGURE 1. CCR-Pareto-efficiency

$C_C := (\mathbb{R}^2_- \times \mathbb{R}^1_+) + (x(C), y(C))$. Then, $(C_C \cap T_{\mathrm{CCR}}) \backslash \mathrm{C} = \phi$, hence C is CCR-Pareto-efficient DMU. Similarly, D and F are evaluated as CCR-Pareto-efficient DMUs. In contrast, let $C_B := (\mathbb{R}^2_- \times \mathbb{R}^1_+) + (x(B), y(B))$ then $(C_B \cap T_{\mathrm{CCR}}) \backslash \mathrm{B} \neq \phi$, hence B is CCR-Pareto-inefficient DMU.

3. IMPROVEMENTS FOR INEFFICIENT DMUS

In this section, we propose three types of improvements for making inefficient DMUs efficient in the CCR model with the minimal change of input and output values. The first improvement is unrestricted, that is, we consider only the minimal change of input and output values. The inefficient DMUs can become efficient units by the smallest change under the condition which the improvement target is feasible. However, the improvement is sometimes Pareto-inefficient in the CCR model. Therefore, we propose the second improvement by forcing the Pareto-efficiency of the CCR model. Moreover, we calculate the third improvement intermediate between the first and second improvements by considering the convex combination and a projection.

First, we define the norm depending on a symmetric positive semidefinite matrix $A \in \mathbb{R}^{(m+s)\times(m+s)}$ as follows.

$$||Z||_A := \sqrt{Z^\top A Z},\ Z \in \mathbb{R}^{m+s}.$$

Under this norm, we consider the minimal change of input and output values for each inefficient DMUs.

Example 3.1. In the case of $A = I_{m+s}$, $||\cdot||_A$ corresponds to the Euclidean norm. If A is defined by

$$A = M_k := \begin{pmatrix} \left(\dfrac{1}{P(k)_1}\right)^2 & & \underset{\sim}{0} \\ & \ddots & \\ \underset{\sim}{0} & & \left(\dfrac{1}{P(k)_{m+s}}\right)^2 \end{pmatrix},$$

then $||\cdot||_A$ means the norm which considered the ratio of input and output values.

Let N_c be the number of facets forming the efficient frontier of the CCR model and let S_c be the index set of all facets. Then, we note that $N_c < \infty$ and we can calculate the coefficients of equations forming the facets(see [9]). Let $W_j := (-p_j^\top, q_j^\top)^\top$ for each $j \in S_c$, where, $p_j, q_j \geq 0$, $p_j \in \mathbb{R}^m$ and $q_j \in \mathbb{R}^s$. By using W_j, we represent T_{CCR} and F_{CCR} as follows.

Theorem 3.2. $T_{\text{CCR}} = \displaystyle\bigcap_{j \in S_c} \{Z : W_j^\top Z \leq 0\}$.

Proof. Firstly, we shall show that $T_{\text{CCR}} \subset \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. For each $Z := (x^\top, y^\top)^\top \in T_{\text{CCR}}$, there exists $\lambda' \geq 0$ such that $x \geq \sum_{i=1}^n \lambda_i' x(i)$, $y \leq \sum_{i=1}^n \lambda_i' y(i)$. Since $W_j = (-p_j^\top, q_j^\top)^\top$, then $W_j^\top Z = -p_j^\top x + q_j^\top y \leq -p_j^\top \sum_{i=1}^n \lambda_i' x(i) + q_j^\top \sum_{i=1}^n \lambda_i' y(i)$. By the definition of F_{CCR}, $-p_j^\top x(i) + q_j^\top y(i) \leq 0$ for each $i \in \{1, \ldots, n\}$. Hence, $W_j^\top Z \leq 0$ and $(x^\top, y^\top)^\top \in \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. Therefore, $T_{\text{CCR}} \subset \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. Secondly, we shall show that $T_{\text{CCR}} \supset \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. For each $Z \in \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$, the following two cases occur.

(i): There exists $j \in S_c$ such that $W_j^\top Z = 0$.
(ii): There exist no $j \in S_c$ such that $W_j^\top Z = 0$.

In Case (i), by the definition of W_j, there exists $\lambda \geq 0$ such that $x = \sum_{i=1}^n \lambda_i x(i)$, $y = \sum_{i=1}^n \lambda_i y(i)$. Hence, $Z \in T_{\text{CCR}}$. In Case (ii), there exist $\delta > 0$ and $j \in S_c$ such that $W_j^\top(Z + \delta W_j) = 0$ and $W_k^\top(Z + \delta W_k) \leq 0$ for each $k \in S_c$. Let $Z' := Z + \delta W_j$. Then, $x \geq x'$ and $y \leq y'$. By definition of W_j, there exists $\lambda \geq 0$ such that $x' = \sum_{i=1}^n \lambda_i x(i)$, $y' = \sum_{i=1}^n \lambda_i y(i)$. Hence, $Z' \in T_{\text{CCR}}$ and $Z \in T_{\text{CCR}}$. Therefore, $T_{\text{CCR}} \supset \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. Consequently, $T_{\text{CCR}} = \bigcap_{j\in S_c}\{Z : W_j^\top Z \leq 0\}$. □

Theorem 3.3. $F_{\text{CCR}} = \Big(\displaystyle\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\Big) \cap T_{\text{CCR}}$.

Proof. Firstly, we shall show that $F_{\text{CCR}} \subset \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\text{CCR}}$. For each $Z' := (x'^\top, y'^\top)^\top \in F_{\text{CCR}}$, $(x'^\top, y'^\top)^\top \in T_{\text{CCR}}$. Let $(\theta^*_{\text{CCR}}(Z'), \lambda_1^*, \ldots, \lambda_n^*)$ be an optimal solution of the CCR model for Z', that is $\theta^*_{\text{CCR}}(Z')$ solves the following problem.

$$
(\mathrm{CCR(Z')})\begin{cases}
\text{minimize} & \theta \\
\text{subject to} & \theta x'_i - \sum_{j=1}^{n} \lambda_j x(j)_i \geq 0 \; i = 1, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y(j)_r - y'_r \geq 0 \; r = 1, \ldots, s, \\
& \lambda_j \geq 0 \; j = 1, \ldots, n, \\
& \theta \in \mathbb{R}.
\end{cases}
$$

Since $\theta^*_{\text{CCR}}(Z') = 1$, there exists i such that $x'_i = \sum_{j=1}^{n} \lambda_j^* x(j)_i$. Hence, $(x'^\top, y'^\top)^\top \in \mathrm{bd}(T_{\text{CCR}})$. By Theorem 3.2, there exists $j \in S_c$ such that $W_j^\top Z' = 0$. Hence, $Z' \in \bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}$. Therefore, $F_{\text{CCR}} \subset \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\text{CCR}}$. Secondly, we shall show that $F_{\text{CCR}} \supset \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\text{CCR}}$. For each $Z' \in \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\text{CCR}}$, by Theorem 3.2, $Z' \in \mathrm{bd}(T_{\text{CCR}})$. By definition of F_{CCR}, $Z' \in F_{\text{CCR}}$. Therefore, $F_{\text{CCR}} \supset \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\text{CCR}}$. Consequently, $F_{\text{CCR}} = \left(\bigcup_{j\in S_c}\{Z : W_j^\top Z = 0\}\right) \cap T_{\text{CCR}}$. □

We propose the following algorithm for obtaining the improvements $d^\alpha(k)$, where $\alpha \in \{0, 1\}$. Improvements for $\mathrm{DMU}(k)$ are obtained by the following algorithm:

Algorithm GIT:

Step 0: @

Select $\alpha \in \{0, 1\}$ (Choose the type of the improvment). Set $j := 1$ and go to Step 1.

Step 1: @

If $\alpha = 1$, then set

$$S'_c := \{l \in S_c : W_{li} \neq 0 \; i \in \{1, \ldots, m + s\}\} \text{ and } S := S'_c.$$

If $\alpha = 0$, then set

$$S := S_c.$$

Let N be the number of elements of S. Go to Step 2.

Step 2: @

Let $d_j^\alpha(k)$ be an optimal solution of Problem $(\mathrm{MIT}_j^\alpha(k))$ defined as follows:

$$(\mathrm{MIT}_j^\alpha(k)) \begin{cases} \text{minimize} & ||Z||_A \\ \text{subject to} & (Z+P(k))^\top W_j = 0, \\ & \alpha(Z+P(k))^\top W_o \leq 0 \text{ for each } o \in S, \end{cases}$$

where, j denote the jth element of S. If $j = N$, then go to Step 3. Otherwise, set $j \leftarrow j+1$ and go to Step 2.

Step 3: @
Select $j^{'} \in \arg\min\{||d_j^\alpha(k)||_A : j \in S\}$ and set $d^\alpha(k) := d_{j^{'}}^\alpha(k)$. This algorithm terminates.

We can execute Algorithm GIT using the existing nonlinear optimization techniques (e.g. [2]). The existence and properties of an optimal solution are proved by the following theorems.

Theorem 3.4. *For each* $\alpha \in \{0,1\}$*, Problem* $(\mathrm{MIT}_j^\alpha(k))$ *has an optimal solution.*

Proof. Let $B_j^\alpha(k)$ be the feasible sets of Problem $(\mathrm{MIT}_j^\alpha(k))$ $(\alpha \in \{0,1\})$. We show the case of $\alpha = 0$. For the case of $\alpha = 1$, we can complete the proof in a way similar to the case of $\alpha = 0$. By the definition of T_{CCR}, $0 \in T_{\mathrm{CCR}}$. Since T_{CCR} is closed, by Theorem 3.3, F_{CCR} is closed and $0 \in F_{\mathrm{CCR}}$. Hence, $Z = -P(k)$ is a feasible solution and $\{Z : (Z+P(k))^\top W_j = 0\}$ is closed. Therefore, $B_j^0(k)$ is nonempty and closed. Since $B_j^0(k)$ is nonempty, for each $(x^{'}, y^{'}) \in B_j^0(k)$, $\bar{B}_j^0(k) := B_j^0(k) \cap \{(x^\top, y^\top)^\top : ||(x^\top, y^\top)^\top||_A \leq ||(x^{'\top}, y^{'\top})^\top||_A\}$ is compact. Therefore, we note that Problem $(\mathrm{MIT}_j^0(k))$ is equivalent to the following problem.

$$(\overline{\mathrm{MIT}}_j^0(k)) \begin{cases} \text{minimize} & ||Z||_A \\ \text{subject to} & Z \in \bar{B}_j^0(k). \end{cases}$$

Since $\bar{B}_j^0(k)$ is compact, by the continuity of the objective function, Problem $(\overline{\mathrm{MIT}}_j^0(k))$ has an optimal solution. By the definition of $\bar{B}_j^0(k)$, an optimal solution of Problem $(\overline{\mathrm{MIT}}_j^0(k))$ is also an optimal solution of Problem $(\mathrm{MIT}_j^0(k))$. Therefore, Problem $(\mathrm{MIT}_j^0(k))$ has an optimal solution. □

Theorem 3.5. *For each CCR-inefficient* $\mathrm{DMU}(k)$*, let* $d^\alpha(k)$ $(\alpha \in \{0,1\})$ *be an optimal solution calculated by Algorithm* GIT*. Then,* $P(k) + d^\alpha(k) \in F_{\mathrm{CCR}}$.

Proof. We prove the case of $\alpha = 0$. In order to obtain a contradiction, we suppose that $P(k)+d^0(k) \notin F_{\mathrm{CCR}}$. By Theorem 3.3, $P(k)+d^0(k) \notin T_{\mathrm{CCR}}$, and by Theorem 3.2, there exists $j \in S_c$ such that $(P(k) + d^0(k))^\top W_j > 0$. Since $\mathrm{DMU}(k)$ is a CCR-inefficient DMU, $P(k) \in \mathrm{int}\, T_{\mathrm{CCR}}$. Hence, from Theorem 3.2, $P(k)^\top W_j < 0$ and $(\gamma(P(k) +$

$d^0(k)) + (1-\gamma)P(k))^\top W_j = (P(k) + \gamma d^0(k))^\top W_j = 0$, where $\gamma := -\frac{P(k)^\top W_j}{d^0(k)^\top W_j}$. Since $(P(k) + d^0(k))^\top W_j > 0$, we obtain $0 < \gamma < 1$. Therefore, $\gamma d^0(k)$ is a feasible solution of Problem $(\mathrm{MIT}_j^0(k))$. By the definition of $d_j^0(k)$, we have the following inequality: $||d_j^0(k)||_A \leq ||\gamma d^0(k)||_A < ||d^0(k)||_A$. This contradicts the optimality of $d^0(k)$ for Algorithm GIT. Consequently, $P(k) + d^0(k) \in F_{\mathrm{CCR}}$. For the case of $\alpha = 1$, we replace S_c by $S_c^{'}$ and can complete the proof in a way similar to the case of $\alpha = 0$. □

By Theorem 3.5, we note that $P(k) + d^0(k)$ is a CCR-efficient point for each CCR-inefficient $\mathrm{DMU}(k)$. Moreover, we obtain a Pareto-efficient point based on parameter $\alpha = 1$ as indicated by the following theorem.

Theorem 3.6. *For each CCR-inefficient* $\mathrm{DMU}(k)$*, let* $d^1(k)$ *be an optimal solution calculated by modified Algorithm* $\mathrm{GIT}(\alpha = 1)$*. Then,* $P(k) + d^1(k) \in F_{\mathrm{CCR}}$ *is a CCR-Pareto-efficient point.*

Proof. By Theorems 3.4 and 3.5, the existence of an optimal solution and $P(k) + d^1(k) \in F_{\mathrm{CCR}}$ are proved. In order to obtain a contradiction, we suppose that $P(k) + d^1(k)$ has positive slack, that is, there exist slack vectors $s^x \geq 0 \in \mathbb{R}^m$ and $s^y \geq 0 \in \mathbb{R}^s$ satisfying $(s^{x\top}, s^{y\top}) \neq (0, 0)$, and $P(k) + d^1(k) + (-s^{x\top}, s^{y\top})^\top \in F_{\mathrm{CCR}}$. Since $d^1(k)$ is an optimal solution of Problem $(\mathrm{MIT}_j^1(k))$ for some $j \in \{1, \ldots, N\}$, there exists $j \in S$ such that $(d^1(k) + P(k))^\top W_j = 0$. Then $(d^1(k)+P(k)+(-s^{x\top}, s^{y\top})^\top)^\top W_j = (-s^{x\top}, s^{y\top})W_j > 0$. By Theorems 3.2 and 3.3, this contradicts $P(k)+d^1(k)+(-s^{x\top}, s^{y\top})^\top \in F_{\mathrm{CCR}}$. Therefore, $P(k) + d^1(k)$ is a CCR-Pareto-efficient point. □

By Theorems 3.3 and 3.5, $P(k) + d^0(k)$ and $P(k) + d^1(k)$ are contained in T_{CCR}. Since T_{CCR} is a closed convex set, $d^\lambda(k) := \lambda(P(k) + d^0(k)) + (1-\lambda)(P(k) + d^1(k)) \in T_{\mathrm{CCR}}$ for each $\lambda \in (0, 1)$, where $d^0(k)$ and $d^1(k)$ are optimal solutions calculated by modified Algorithm $\mathrm{GIT}(\alpha = 0)$ and $(\alpha = 1)$, respectively. However, we note that $d^\lambda(k)$ is not always contained in F_{CCR}, since F_{CCR} is not convex set. In order to calculate a point on F_{CCR} based on $d^\lambda(k)$, we consider a projection. Let $\bar{\beta} := \min\{\beta : (P(k) + \beta(d^\lambda(k) - P(k)))^\top W_j = 0 \text{ for some } j \in S_c\}$. Then, by Theorems 3.2 and 3.3, $P(k) + \bar{\beta}(d^\lambda(k) - P(k)) \in F_{\mathrm{CCR}}$. We propose this point $P(k) + \bar{\beta}(d^\lambda(k) - P(k))$ as improvement intermediate between the two improvements which are obtained based on $d^0(k)$ and $d^1(k)$.

We explain the improvements proposed in this paper by using Figure 2. Now, we consider the improvements for E which is CCR-inefficient DMU. By using the optimal value of Problem (CCRD(k)), we obtain a traditional improvement target d_{CCR}. This improvement improves the only input values at the same rate, that is the output value is fixed. In contrast, we calculate three kinds of improvements which improve the input and output values. If the decision makers of E want to calculate a minimal distance target, then they select $\alpha = 0$ and obtain $d^0(\mathrm{E})$. If they want to calculate a Pareto-efficien target, then they select $\alpha = 1$ and obtain $d^1(\mathrm{E})$. In order to calculate a intermediate target, we consider a convex combination and project the point on F_{CCR} as shown in Figure 2.

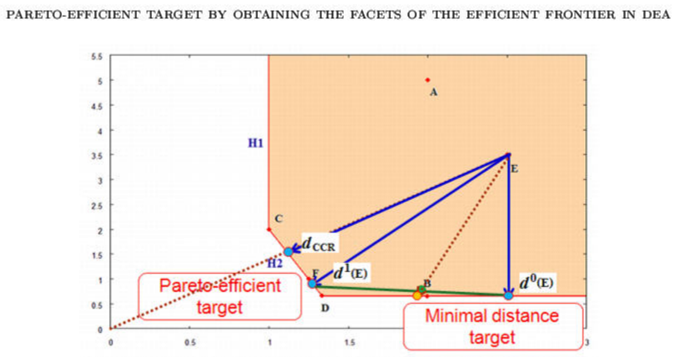

FIGURE 2. Improvement targets

4. EXAMPLE

In this section, we perform a numerical analysis for 10 Japanese banks by utilizing algorithms provided in this paper. As shown in Table 1, each bank has the ordinary profit as the single output. The number of employees and total assets are the two inputs used to generate the output.

In this example, we obtain four hyperplanes forming F_{CCR} by using Algorithm FFC as follows.

$$H_1 := \{(x_1, x_2, y) : -71.9x_1 - x_2 + 121.4y = 0\},$$
$$H_2 := \{(x_1, x_2, y) : -13.8x_1 - x_2 + 53.3y = 0\},$$
$$H_3 := \{(x_1, x_2, y) : -x_1 + 1.2y = 0\},$$
$$H_4 := \{(x_1, x_2, y) : -x_2 + 32.8y = 0\}.$$

By using the coefficients of the equations forming the hyperplanes, we can calculate CCR-efficiency scores without solving linear programming problem for each DMU(see [6, 9]). The CCR-efficiency scores are shown in the Table 2. Three banks (B1, B6 and B7) are evaluated as CCR-efficient DMUs and they do not have positive slack, hence, they do not have to think the improvement. The other bank's improvements are given by Tables 3 and 4. The minimal distance

TABLE 1. Inputs and Output values for 10 Japanese banks, 2008

Bank	Input1 (persons)	Input2 (one hundred million Japanese yen)	Output (one hundred million Japanese yen)
B1	3701	119895	3179
B2	3675	98359	2688
B3	3659	80955	2180
B4	3004	59600	1563
B5	2887	66373	1477
B6	2872	90984	2450
B7	2752	60770	1852
B8	2506	49008	1137
B9	2268	41151	1148
B10	2148	41158	1124

improvement target of each CCR-inefficient DMU is given in Table 3. The improvement shown in Table 4 is CCR-Pareto improvement. The two improvements for B2 coincide and the other DMUs have different two improvements. The improvements over the efficient frontier of the CCR model think decreasing inputs values and increasing output value.

TABLE 2. DEA analysis for 10 Japanese banks, 2008

Bank	CCR
B1	1.000000
B2	0.961359
B3	0.884268
B4	0.860520
B5	0.741447
B6	1.000000
B7	1.000000
B8	0.761275
B9	0.915398
B10	0.896108

5. CONCLUSIONS

In this paper, we have proposed Algorithm GIT for calculating three kinds of improvements for CCR-inefficient DMUs. In order to calculate the improvements, all equations forming F_{CCR} have been used.

TABLE 3. Minimal distance improvement ($A = M_k$)

Bank	Input1	Input2	Output
B1	-	-	-
B2	-32	-900	83
B3	0.00	-5290	126
B4	0.00	-4770	108
B5	0.00	-11665	190
B6	-	-	-
B7	-	-	-
B8	0.00	-7415	131
B9	0.00	-1896	48
B10	0.00	-2368	58

TABLE 4. Pareto-efficient improvement ($A = M_k$)

Bank	Input1	Input2	Output
B1	-	-	-
B2	-32	-900	83
B3	-215	-1529	201
B4	-324	-412	241
B5	-208	-7193	326
B6	-	-	-
B7	-	-	-
B8	-521	-3567	229
B9	-466	-1119	69
B10	-346	-1136	93

By using a property of the coefficients of them, we have calculated three kinds of improvements.

The first improvement turns to the closest point over F_{CCR}. Sometimes, the improvement have a positive slack, that is, it may be a CCR-Pareto-inefficient point. In order to calculate CCR-Pareto-efficient point on F_{CCR}, we have proposed the second improvement. The decision makers can select either a minimal distance improvement or Pareto-efficient improvement by choosing a parameter in Algorithm GIT. Moreover, to calculate more flexible improvements, we have suggested a method using the convex combination and a projection. In the convex combination, the decision makers can adjust which they emphasize, feasibility or efficiency.

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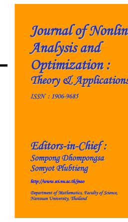

CONVERGENCE OF NONLINEAR PROJECTIONS AND SHRINKING PROJECTION METHODS FOR COMMON FIXED POINT PROBLEMS

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ABSTRACT. In this paper, we first study some properties of Mosco convergence for a sequence of nonempty sunny generalized nonexpansive retracts in Banach spaces. Next, motivated by the result of Kimura and Takahashi and that of Plubtieng and Ungchittrakool, we prove a strong convergence theorem for finding a common fixed point of generalized nonexpansive mappings in Banach spaces by using the shrinking projection method.

KEYWORDS : Shrinking projection method; Sunny generalized nonexpansive retraction; Generalized nonexpansive; Mosco convergence; Fixed point.

1. INTRODUCTION

Let E be a real Banach space and C a nonempty closed convex subset of E. A mapping $T : C \longrightarrow C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Iterative methods for approximation of fixed points of nonexpansive mappings have been studied by many researchers; see [5, 21, 25, 28, 29, 31, 34, 36] and others. In particular, Takahashi, Takeuchi and Kubota [34] established strong convergence of an iterative scheme with new type of hybrid method as follows:

Theorem 1.1 (Takahashi-Takeuchi-Kubota [34]). *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}$ be a*

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sequence $[0, a]$*, where* $0 < a < 1$*. For a point* $x \in H$ *chosen arbitrarily, generate a sequence* $\{x_n\}$ *by the following iterative scheme:* $x_1 \in C$*,* $C_1 = C$*, and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x \end{cases}$$

for every $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to* $P_{F(T)}x \in C$*, where* P_K *is the metric projection of* H *onto a nonempty closed convex subset* K *of* H*.*

This iterative method is also known as the shrinking projection method. We note that the original result is a convergence theorem to a common fixed point of a family of nonexpansive mappings with certain conditions.

On the other hand, relatively nonexpansive mappings and generalized nonexpansive mappings, which are generalizations of a nonexpansive mappings in Hilbert spaces, have been considered recently. Their properties and iterative schemes have been studied in [3, 8, 9, 10, 11, 12, 13, 14, 30, 22, 23, 18, 19, 20, 27] and others. Recently, Kimura and Takahashi [18] obtain a strong convergence theorem for finding a common fixed point of relatively nonexpansive mappings in a Banach space by using the shrinking projection method. The method for its proof is different from the original one; they use the concept of Mosco convergence of sequences of nonempty closed convex subsets of a Banach space. They also succeed in making conditions of the coefficients and the underlying space weaker.

In this paper, we study the shrinking projection method for generalized nonexpansive mappings in a Banach space. We first prove convergence theorems for a sequence of sunny generalized nonexpansive retractions, which is a generalization of the metric projections in Hilbert spaces. Next, using the technique developed by Kimura and Takahashi [18], we prove a strong convergence theorem for finding a common fixed point of a finite family of generalized nonexpansive mappings in Banach space by using the iterative scheme of [27]; see also [1, 11, 13, 16, 20] and others.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We denote strong convergence and weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ whenever $x, y \in E$ satisfies $\|x\| = \|y\| = 1$ and $x \neq y$. E is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$ implies $\|x+y\|/2 \leq 1 - \delta$. The following lemma holds.

Lemma 2.1 (Plubtieng-Ungchittrakool [27])**.** *Let* E *be a uniformly convex Banach space and* $\rho > 0$*. Then, there exists a continuous, strictly increasing, and convex function* $g : [0, \infty) \longrightarrow [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\left\|\sum_{i=1}^{r} \delta_i x_i\right\|^2 \leq \sum_{i=1}^{r} \delta_i \|x_i\|^2 - \delta_j \delta_k g(\|x_j - x_k\|) \tag{2.1}$$

for each $j, k \in \{1, 2, \ldots, r\}$*, where* $\{x_1, x_2, \ldots, x_r\} \subset E$ *satisfy* $\|x_i\| \leq \rho$ *for each* $i = 1, 2, \ldots, r$ *and* $\{\delta_1, \delta_2, \ldots, \delta_r\} \subset [0, 1]$ *satisfy* $\sum_{i=1}^{r} \delta_i = 1$*.*

A Banach space E is said to be smooth if

$$\lim_{t \longrightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in B = \{z \in E : \|z\| = 1\}$. In this case, the norm of E is said to be Gâteaux differentiable. The norm of E is said to be Fréchet differentiable if for each $x \in B$, the limit (2.2) is attained uniformly for $y \in B$. See [32] for more details. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E converges strongly to x_0 whenever it satisfies $x_n \rightharpoonup x_0$ and $\|x_n\| \longrightarrow \|x_0\|$.

The normalized duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in E$. We also know the following properties; see [4, 32, 33] for more details.

(i) $Jx \neq \emptyset$ for each $x \in E$;
(ii) if E is reflexive, then J is surjective;
(iii) if E is strictly convex, then J is one-to-one and satisfies that $\langle x - y, x^* - y^* \rangle > 0$ for each $x, y \in E$ with $x \neq y$, $x^* \in Jx$ and $y^* \in Jy$;
(iv) if E is smooth, then J is single-valued and norm-to-weak* continuous;
(v) if E is reflexive, smooth and strictly convex, then the duality mapping $J_* : E^* \longrightarrow E$ is the inverse of J, that is, $J_* = J^{-1}$;
(vi) if E has a Fréchet differentiable norm, then J is norm-to-norm continuous;
(vii) E is reflexive, strictly convex and has the Kadec-Klee property if and only if E^* has a Fréchet differentiable norm.

Let E be a reflexive Banach space and $\{C_n\}$ a sequence of nonempty closed convex subsets of E. We denote by $\text{s-Li}_n C_n$ the set of limit points of $\{C_n\}$, that is, $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \in C_n$ for each $n \in \mathbb{N}$ and $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Similarly, we denote by $\text{w-Ls}_n C_n$ the set of weak cluster points of $\{C_n\}$; $y \in \text{w-Ls}_n C_n$ if and only if there exists $\{y_{n_i}\} \subset E$ such that $y_{n_i} \in C_{n_i}$ for each $i \in \mathbb{N}$ and $y_{n_i} \rightharpoonup y$ as $i \longrightarrow \infty$. Using these definitions, we define Mosco convergence[24] of $\{C_n\}$. If C_0 satisfies

$$\underset{n}{\text{s-Li}}\, C_n = C_0 = \underset{n}{\text{w-Ls}}\, C_n,$$

then we say that $\{C_n\}$ is a Mosco convergent sequence to C_0. In this case, we denote it by

$$C_0 = \underset{n}{\text{M-lim}}\, C_n.$$

Notice that the inclusion $\text{s-Li}_n C_n \subset \text{w-Ls}_n C_n$ is always true. Thus, to prove $C_0 = \text{M-lim}_n C_n$ we only need to show $\text{w-Ls}_n C_n \subset C_0 \subset \text{s-Li}_n C_n$.

Let E be a smooth Banach space and consider the following function $V : E \times E \longrightarrow \mathbb{R}$ defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for each $x, y \in E$. We know the following properties; see [2, 7, 9, 15, 23] for more details:

(i) $(\|x\| - \|y\|)^2 \leq V(x, y) \leq (\|x\| + \|y\|)^2$ for each $x, y \in E$;
(ii) $V(x, y) + V(y, x) = 2\langle x - y, Jx - Jy \rangle$ for each $x, y \in E$;
(iii) $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy \rangle$ for each $x, y, z \in E$;
(iv) if E is additionally assumed to be strictly convex, then $V(x, y) = 0$ if and only if $x = y$.

The following lemma is due to [15].

Lemma 2.2 (Kamimura-Takahashi [15]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \longrightarrow \infty} V(x_n, y_n) = 0$, then $\lim_{n \longrightarrow \infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed subset of a smooth Banach space E. A mapping $T : C \longrightarrow C$ is said to be generalized nonexpansive [8, 9] if $F(T) \neq \emptyset$ and

$$V(Tx, p) \leq V(x, p)$$

for each $x \in C$ and $p \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \longrightarrow D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \longrightarrow D$ is said to be a retraction if $Rx = x$ for each $x \in D$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto D is uniquely determined if it exists; see [9]. Then, such a sunny generalized nonexpansive retraction of E onto D is denoted by R_D. A nonempty subset D of E is called a sunny generalized nonexpansive retract of E if there exists a sunny generalized nonexpansive retraction of E onto D. Obviously, the set of fixed points of a sunny generalized nonexpansive retraction of E onto D is D; see [8, 9] for more details. We recall the following results for sunny generalized nonexpansive retractions and sunny generalized nonexpansive retracts.

Lemma 2.3 (Ibaraki-Takahashi [8, 9]). *Let D be a nonempty subset of a smooth and strictly convex Banach space E. Let R_D be a retraction of E onto D. Then R_D is sunny and generalized nonexpansive if and only if*

$$\langle x - R_D x, JR_D x - Jy \rangle \geq 0.$$

for each $y \in D$.

Theorem 2.4 (Ibaraki-Takahashi [12], Inthakon-Dhompongsa-Takahashi [14]). *Let E be a reflexive, smooth and strictly convex Banach space and C a nonempty closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping of C into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Lemma 2.5 (Kohsaka-Takahashi [19]). *Let E be a smooth, reflexive, and strictly convex Banach space and D a nonempty sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction of E onto D, $x \in E$, and $z \in D$. Then, the following conditions are equivalent:*

(i) $z = Rx$;
(ii) $V(x, z) = \min_{y \in D} V(x, y)$.

Theorem 2.6 (Kohsaka-Takahashi [19]). *Let E be a smooth, reflexive, and strictly convex Banach space and D a nonempty subset of E. Then, the following conditions are equivalent:*

(i) *D is a sunny generalized nonexpansive retract of E;*
(ii) *D is a generalized nonexpansive retract of E;*
(iii) *JD is closed and convex.*

In this case, D is closed.

3. Convergence theorem for sunny generalized nonexpansive retractions

In this section, we prove weak and strong convergence theorems for a sequence of sunny generalized nonexpansive retractions. The sequence of ranges of these nonlinear projections is assumed to converge in the sense of Mosco; see [7, 13, 17, 35] for related results.

We remark that, using the relation between generalized nonexpansive retractions and generalized projections shown in [19] and the methods used in [7], we can prove the essential parts of the theorems in this section; see also [9]. For the sake of completeness, we will give a proof for all results.

Theorem 3.1. *Let E be a reflexive, smooth and strictly convex Banach space and let $\{D_n\}$ be a sequence of nonempty sunny generalized nonexpansive retracts of E. Let $u \in E$ and $\{u_n\}$ be a sequence of E converging strongly to u. If $D_0^* = \text{M-}\lim_n JD_n$ exists and is nonempty, then $\{JR_{D_n}u_n\}$ converges weakly to $JR_{D_0}u$, where $D_0 = J^{-1}D_0^*$.*

Proof. It is easy to prove that D_0^* is closed and convex if JD_n is a closed convex subset of E for each $n \in \mathbb{N}$. Let $x_n = R_{D_n}u_n$ for each $n \in \mathbb{N}$. Since $D_0^* = \text{M-}\lim_n JD_n$, we have, for each $y \in D_0$, there exists $\{y_n^*\} \subset E^*$ such that $y_n^* \longrightarrow Jy$ as $n \longrightarrow \infty$ and that $y_n^* \in JD_n$ for each $n \in \mathbb{N}$. From Lemma 2.3, we have

$$\langle u_n - x_n, Jx_n - y_n^* \rangle \geq 0.$$

Hence, we obtain that

$$\begin{aligned} 0 &\leq \langle u_n - x_n, Jx_n - Ju_n \rangle + \langle u_n - x_n, Ju_n - y_n^* \rangle \\ &\leq -\big(\|u_n\| - \|x_n\|\big)^2 + \big(\|u_n\| + \|x_n\|\big)\|Ju_n - y_n^*\| \end{aligned}$$

and thus

$$\big(\|u_n\| - \|x_n\|\big)^2 \leq \big(\|u_n\| + \|x_n\|\big)\|Ju_n - y_n^*\|. \tag{3.1}$$

Assume that $\{x_n\}$ is unbounded. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i \longrightarrow \infty} \|x_{n_i}\| = \infty$. Since $y_n^* \longrightarrow Jy$ and $u_n \longrightarrow u$, by (3.1) we get contradiction. Hence $\{x_n\}$ is bounded and so is $\{Jx_n\}$. Let $\{Jx_{n_i}\}$ be a subsequence of $\{Jx_n\}$ converging weakly to some $x_0^* \in E^*$. From the definition of D_0^*, we get $x_0^* \in \text{M-}\lim_n JD_n = D_0^*$.

Now we let $x_0 = J^{-1}x_0^*$ and prove that $R_{D_0}u = x_0$. From lower semicontinuity of the norm, we have

$$\begin{aligned} \liminf_{i \longrightarrow \infty} V(u_{n_i}, x_{n_i}) &= \liminf_{i \longrightarrow \infty} \big(\|u_{n_i}\|^2 - 2\langle u_{n_i}, Jx_{n_i} \rangle + \|Jx_{n_i}\|^2\big) \\ &\geq \|u\|^2 - 2\langle u, x_0^* \rangle + \|x_0^*\|^2 \\ &= \|u\|^2 - 2\langle u, Jx_0 \rangle + \|x_0\|^2 \\ &= V(u, x_0). \end{aligned}$$

On the other hand, we get from Lemma 2.5 that

$$\begin{aligned} \liminf_{n \longrightarrow \infty} V(u_n, x_n) &\leq \liminf_{n \longrightarrow \infty} V(u_n, J^{-1}y_n^*) = \lim_{n \longrightarrow \infty} \big(\|u_n\|^2 - 2\langle u_n, y_n^* \rangle + \|y_n^*\|^2\big) \\ &= \|u\|^2 - 2\langle u, Jy \rangle + \|Jy\|^2 = V(u, y), \end{aligned}$$

that is,

$$V(u, x_0) = \min_{y \in D_0} V(u, y).$$

Hence we get $R_{D_0}u = x_0$.

According to our consideration above, each subsequence $\{Jx_{n_i}\}$ of $\{Jx_n\}$ which converges weakly has the unique limit $JR_{D_0}u$. Therefore, the sequence $\{Jx_n\}$ itself converges weakly to $JR_{D_0}u$. □

Theorem 3.2. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm. Let $\{D_n\}$ be a sequence of nonempty sunny generalized nonexpansive retracts of E. Let $u \in E$ and let $\{u_n\}$ be a sequence of E converging strongly to u. If $D_0^* = \text{M-}\lim_n JD_n$ exists and is nonempty, then $\{JR_{D_n}u_n\}$*

converges strongly to $JR_{D_0}u$*, where* $D_0 = J^{-1}D_0^*$*. Moreover,* $\{R_{D_n}u_n\}$ *converges weakly to* $R_{D_0}u$*.*

Proof. We write $x_n = R_{D_n}u_n$ and $x_0 = R_{D_0}u$. By Theorem 3.1, we obtain $Jx_n \rightharpoonup Jx_0$ as $n \longrightarrow \infty$. We first prove that $Jx_n \longrightarrow Jx_0$ as $n \longrightarrow \infty$. Since E has a Fréchet differential norm, E^* has the Kadec-Klee property. Therefore, it is sufficient to prove that $\|Jx_n\| \longrightarrow \|Jx_0\|$ as $n \longrightarrow \infty$. Since $Jx_0 \in D_0^*$, there exists a sequence $\{y_n^*\} \subset E^*$ such that $y_n^* \longrightarrow Jx_0$ as $n \longrightarrow \infty$ and $y_n^* \in JD_n$ for each $n \in \mathbb{N}$. It follows that

$$\begin{aligned} V(u, x_0) &\leq \liminf_{n\longrightarrow\infty} V(u_n, x_n) \\ &\leq \limsup_{n\longrightarrow\infty} V(u_n, x_n) \\ &\leq \lim_{n\longrightarrow\infty} V(u_n, J^{-1}y_n^*) \\ &\leq V(u, x_0). \end{aligned}$$

Hence we obtain $V(u, x_0) = \lim_{n\longrightarrow\infty} V(u_n, x_n)$. Since $\lim_{n\longrightarrow\infty}\langle u_n, Jx_n\rangle = \langle u, Jx_0\rangle$, we get

$$\lim_{n\longrightarrow\infty} \|Jx_n\| = \|Jx_0\|.$$

Therefore we obtain that $\{Jx_n\}$ converges strongly to Jx_0.

Since E is reflexive and strictly convex, E^* is smooth and thus the duality mapping J^{-1} of E^* is norm-to-weak continuous. Hence, we obtain that

$$x_n = J^{-1}Jx_n \rightharpoonup J^{-1}Jx_0 = x_0,$$

which completes the proof. □

Theorem 3.3. *Let* E *be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let* $\{D_n\}$ *be a sequence of nonempty sunny generalized nonexpansive retracts of* E*. Let* $u \in E$ *and let* $\{u_n\}$ *be a sequence of* E *converging strongly to* u*. If* $D_0^* = \text{M-}\lim_n JD_n$ *exists and is nonempty, then* $\{R_{D_n}u_n\}$ *converges strongly to* $R_{D_0}u$*, where* $D_0 = J^{-1}D_0^*$*.*

Proof. By Theorem 3.2, we obtain that $\{JR_{D_n}u_n\}$ converges strongly to $JR_{D_0}u$. Since E is reflexive, strictly convex and has the Kadec-Klee property, E^* has a Fréchet differentiable norm. Therefore the duality mapping J^{-1} of E^* is norm-to-norm continuous. Hence, we have that

$$R_{D_n}u_n = J^{-1}JR_{D_n}u_n \longrightarrow J^{-1}JR_{D_0}u = R_{D_0}u,$$

which completes the proof. □

On the other hand, the following theorem shows that the strong convergence of sequences $\{R_{D_n}u\}$ implies the Mosco convergence of $\{JD_n\}$ under certain conditions.

Theorem 3.4. *Let* E *be a reflexive and strictly convex Banach space having a Fréchet differentiable norm. Let* $D_0, D_1, D_2, D_3, \ldots$ *be nonempty sunny generalized nonexpansive retracts of* E*. Suppose that* $\{R_{D_n}u\}$ *converges strongly to* $R_{D_0}u$ *for each* $u \in E$*, where* R_{D_n} *is the sunny generalized nonexpansive retractions of* E *onto* D_n *for each* $n \in \mathbb{N} \cup \{0\}$*. Then*

$$JD_0 = \text{M-}\lim_n JD_n.$$

Proof. For an arbitrary $u^* \in JD_0$, put $u = J^{-1}u^* \in D_0$. Since E has a Fréchet differentiable norm, we have

$$JR_{D_n}u \longrightarrow JR_{D_0}u = Ju = u^*$$

and that $JR_{D_n}u \in JD_n$ for all $n \in \mathbb{N}$. This means that $u^* \in$ s-$\mathrm{Li}_n\, JD_n$ and hence we have $JD_0 \subset$ s-$\mathrm{Li}_n\, JD_n$. Next we show that w-$\mathrm{Ls}_n\, JD_n \subset JD_0$. For any $z^* \in$ w-$\mathrm{Ls}_n\, JD_n$, there exists $\{z_i^*\}$ such that $\{z_i^*\}$ converges weakly to z^* as $i \longrightarrow \infty$ and that $z_i^* \in JD_{n_i}$ for each $i \in \mathbb{N}$. Using Lemma 2.3, we have that

$$\langle z - R_{D_{n_i}}z, JR_{D_{n_i}}z - z_i^*\rangle \geq 0.$$

where $z = J^{-1}z^*$. Tending $i \longrightarrow \infty$, we get

$$\langle z - R_{D_0}z, JR_{D_0}z - Jz\rangle \geq 0.$$

By the strict convexity of E, we have that J is strictly monotone. Hence we have $z^* = JR_{D_0}z \in JD_0$. This means that w-$\mathrm{Ls}_n\, JD_n \subset JD_0$, and consequently we obtain $JD_0 =$ M-$\lim_{n\longrightarrow\infty} JD_n$. □

Using Theorem 3.3 and 3.4, we obtain the following theorem.

Theorem 3.5. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $D_0, D_1, D_2, D_3, \ldots$ be nonempty sunny generalized nonexpansive retracts of E. If each R_{D_n} is the sunny generalized nonexpansive retractions of E onto D_n for each $n \in \mathbb{N} \cup \{0\}$, then*

$$JD_0 = \text{M-}\lim_n JD_n.$$

if and only if $\{R_{D_n}u_n\}$ converges strongly to $R_{D_0}u$ for every strongly convergent sequence $\{u_n\} \subset E$ having a limit $u \in E$.

4. Strong convergence theorem for generalized nonexpansive mappings

In this section, using the technique developed by Kimura and Takahashi [18], we prove a strong convergence theorem for finding a common fixed point of finite family of generalized nonexpansive mappings in Banach space. In this result, we adopt the iterative scheme used in [27].

Theorem 4.1. *Let E be a uniformly convex Banach space having a Fréchet differentiable norm, C a nonempty closed subset of E such that JC is closed and convex, and $T_1, T_2, \ldots, T_r$ generalized nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Suppose that $z \in \bigcap_{i=1}^r F(T_i)$ whenever both $\{z_n\}$ and $\{T_i z_n\}$ converge strongly to z for each $i = 1, 2, \ldots, r$. For a point $x \in E$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{cases} y_n = \gamma_n x_n + (1-\gamma_n)\displaystyle\sum_{i=1}^r \delta_n^{(i)} T_i x_n, \\ C_{n+1} = \{z \in C_n : V(y_n, z) \leq V(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}}x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\gamma_n\}, \{\delta_n^{(i)}\} \subset [0,1]$ satisfy the following conditions:

(i) $\liminf_{n\longrightarrow\infty} \gamma_n < 1$,
(ii) $\liminf_{n\longrightarrow\infty} \delta_n^{(i)} > 0$ *for each* $i = 1, 2, \ldots, r$,
(iii) $\sum_{i=1}^r \delta_n^{(i)} = 1$ *for each* $n \in \mathbb{N}$.

Then $\{x_n\}$ converges strongly to $R_{\bigcap_{i=1}^r F(T_i)}x$.

Proof. We first show that $\{x_n\}$ is well defined. Let $p \in \bigcap_{i=1}^{r} F(T_i)$. Then for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} V(y_n, p) &= V\left(\gamma_n x_n + (1-\gamma_n)\sum_{i=1}^{r} \delta_n^{(i)} T_i x_n, p\right) \\ &\le \gamma_n V(x_n, p) + (1-\gamma_n) V\left(\sum_{i=1}^{r} \delta_n^{(i)} T_i x_n, p\right) \\ &\le \gamma_n V(x_n, p) + (1-\gamma_n)\sum_{i=1}^{r} \delta_n^{(i)} V(T_i x_n, p) \\ &\le \gamma_n V(x_n, p) + (1-\gamma_n)\sum_{i=1}^{r} \delta_n^{(i)} V(x_n, p) \\ &\le \gamma_n V(x_n, p) + (1-\gamma_n) V(x_n, p) = V(x_n, p). \end{aligned}$$

Therefore $p \in C_n$ for all $n \in \mathbb{N}$ and hence $\bigcap_{i=1}^{r} F(T_i) \subset C_n$ for all $n \in \mathbb{N}$. This implies that C_n is nonempty for all $n \in \mathbb{N}$. Next, we show that JC_n is closed and convex for all $n \in \mathbb{N}$. From the definition of V, we may show that

$$\begin{aligned} C_{n+1} &= \{z \in C_n : V(y_n, z) \le V(x_n, z)\} \\ &= \{z \in C : 2\langle x_n - y_n, Jz\rangle + \|y_n\|^2 - \|x_n\|^2 \le 0\} \cap C_n. \end{aligned}$$

for all $n \in \mathbb{N}$. The injectivity of J implies that

$$\begin{aligned} JC_{n+1} &= J\Big(\{z \in C : 2\langle x_n - y_n, Jz\rangle + \|y_n\|^2 - \|x_n\|^2 \le 0\} \cap C_n\Big) \\ &= J\{z \in C : 2\langle x_n - y_n, Jz\rangle + \|y_n\|^2 - \|x_n\|^2 \le 0\} \cap JC_n \\ &= \{z^* \in JC : 2\langle x_n - y_n, z^*\rangle + \|y_n\|^2 - \|x_n\|^2 \le 0\} \cap JC_n \end{aligned}$$

for all $n \in \mathbb{N}$. From the assumption for C, JC_1 is closed and convex. Suppose that JC_k is closed and convex for some $k \in \mathbb{N}$. Then, letting

$$D_k^* = \{z^* \in JC : 2\langle x_k - y_k, z^*\rangle + \|y_k\|^2 - \|x_k\|^2 \le 0\},$$

we have that D_k^* is obviously closed and convex and thus $JC_{k+1} = D_k^* \cap JC_k$ is also closed and convex. By Theorem 2.6, there exists a unique sunny generalized nonexpansive retraction of E onto C_n for each $n \in \mathbb{N}$ and hence $\{x_n\}$ is well defined.

Since $\{JC_n\}$ is a decreasing sequence of closed convex subsets of E^* such that $C_0^* = \bigcap_{n=1}^{\infty} JC_n$ is nonempty, it follows that

$$\text{M-}\lim_n JC_n = C_0^* = \bigcap_{n=1}^{\infty} JC_n \ne \emptyset.$$

By Theorem 3.3, $\{x_n\}$ converges strongly to $R_{C_0}x \in C_0$, where $C_0 = J^{-1}C_0^*$. The injectivity of J implies

$$JC_0 = C_0^* = \bigcap_{n=1}^{\infty} JC_n = J\bigcap_{n=1}^{\infty} C_n.$$

Therefore, we obtain that $x_0 = R_{C_0}x \in C_0 = J^{-1}C_0^* = \bigcap_{n=1}^{\infty} C_n$. From the definition of C_n, we have that

$$0 \le \limsup_{n\longrightarrow\infty} V(y_n, x_0) \le \lim_{n\longrightarrow\infty} V(x_n, x_0) = 0$$

and hence $\lim_{n\longrightarrow\infty} V(y_n, x_0) = 0$. From the property of the mapping V, we have that

$$0 \le \limsup_{n\longrightarrow\infty} (\|y_n\| - \|x_0\|)^2 \le \lim_{n\longrightarrow\infty} V(y_n, x_0) = 0,$$

and hence

$$\lim_{n\longrightarrow\infty} \|y_n\| = \|x_0\|. \tag{4.1}$$

Therefore, we also have that

$$\lim_{n\longrightarrow\infty} \langle y_n, Jx_0\rangle = \lim_{n\longrightarrow\infty} \frac{1}{2}\Big\{\|y_n\|^2 + \|x_0\|^2 - V(y_n, x_0)\Big\} = \|x_0\|^2.$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ converging weakly to some $y_0 \in E$. Using weak lower semicontinuity of the norm, we get from (4.1) that

$$\begin{aligned}\|x_0\|^2 &= \lim_{i\longrightarrow\infty} \langle y_{n_i}, Jx_0\rangle = \langle y_0, Jx_0\rangle \\ &\le \|y_0\|\|Jx_0\| = \|y_0\|\|x_0\| \\ &\le \|x_0\| \liminf_{i\longrightarrow\infty} \|y_{n_i}\| \\ &= \|x_0\| \lim_{i\longrightarrow\infty} \|y_{n_i}\| = \|x_0\|^2.\end{aligned}$$

Therefore, we have that $\|y_0\|^2 = \langle y_0, Jx_0\rangle = \|x_0\|^2 = \|Jx_0\|^2$ and hence $Jy_0 = Jx_0$. This implies that $y_0 = x_0$. Thus we have that $\{y_n\}$ converges weakly to x_0. Since (4.1) holds and E has the Kadec-Klee property, we have that $\{y_n\}$ converges strongly to x_0.

Put $S_n = \sum_{i=1}^{r} \delta_n^{(i)} T_i$. From the assumption that $\liminf_{n\longrightarrow\infty} \gamma_n < 1$, we may take a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$ such that $\lim_{j\longrightarrow\infty} \gamma_{n_j} = \gamma_0$ with $0 \le \gamma_0 < 1$. Then we have that

$$\begin{aligned}\|y_{n_j} - x_0\| &= \|\gamma_{n_j} x_{n_j} + (1-\gamma_{n_j}) S_{n_j} x_{n_j} - x_0\| \\ &\ge (1-\gamma_{n_j})\|S_{n_j} x_{n_j} - x_0\| - \gamma_{n_j}\|x_{n_j} - x_0\| \\ &\ge (1-\gamma_{n_j})\|S_{n_j} x_{n_j} - x_0\|\end{aligned}$$

for each $j \in \mathbb{N}$ and hence

$$\lim_{j\longrightarrow\infty} (1-\gamma_{n_j})\|S_{n_j} x_{n_j} - x_0\| = \lim_{j\longrightarrow\infty} (1-\gamma_0)\|S_{n_j} x_{n_j} - x_0\| = 0.$$

Since $\gamma_0 < 1$, we have that

$$\lim_{j\longrightarrow\infty} \|S_{n_j} x_{n_j} - x_0\| = 0.$$

Since $\{T_i x_{n_j}\}$ are bounded for each $i = 1, 2, \ldots, r$, there exists $\rho > 0$ such that $\{T_i x_{n_j}\} \subset B_\rho$ for each $i = 1, 2, \ldots, r$. Therefore, Lemma 2.1 is applicable. Then we obtain that, for each $k, l = 1, 2, \ldots, r$,

$$\begin{aligned}&V(S_{n_j} x_{n_j}, x_0) \\ &= \left\|\sum_{i=1}^{r} \delta_{n_j}^{(i)} T_i x_{n_j}\right\|^2 - 2\left\langle \sum_{i=1}^{r} \delta_{n_j}^{(i)} T_i x_{n_j}, Jx_0\right\rangle + \|x_0\|^2 \\ &\le \sum_{i=1}^{r} \delta_{n_j}^{(i)} \|T_i x_{n_j}\|^2 - \delta_{n_j}^{(k)} \delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \\ &\quad - 2\left\langle \sum_{i=1}^{r} \delta_{n_j}^{(i)} T_i x_{n_j}, Jx_0\right\rangle + \|x_0\|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \delta_{n_j}^{(i)} \left(\|T_i x_{n_j}\|^2 - 2\langle T_i x_{n_j}, Jx_0\rangle + \|x_0\|^2 \right) - \delta_{n_j}^{(k)}\delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \\
&= \sum_{i=1}^{r} \delta_{n_j}^{(i)} V(T_i x_{n_j}, x_0) - \delta_{n_j}^{(k)}\delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \\
&\le \sum_{i=1}^{r} \delta_{n_j}^{(i)} V(x_{n_j}, x_0) - \delta_{n_j}^{(k)}\delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \\
&= V(x_{n_j}, x_0) - \delta_{n_j}^{(k)}\delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|)
\end{aligned}
$$

and hence

$$
\delta_{n_j}^{(k)}\delta_{n_j}^{(l)} g(\|T_k x_{n_j} - T_l x_{n_j}\|) \le V(x_{n_j}, x_0) - V(S_{n_j} x_{n_j}, x_0). \tag{4.2}
$$

for each $k, l = 1, 2, \dots, r$. From the assumption that $\liminf_{n\longrightarrow\infty} \delta_n^{(k)} > 0$ for $k = 1, 2, \dots, r$, we may take subsequences, again denoted by $\{\delta_{n_j}^{(k)}\}$, such that $\lim_{j\longrightarrow\infty} \delta_{n_j}^{(k)} > 0$ for every $k = 1, 2, \dots, r$. Since

$$
\lim_{j\longrightarrow\infty} \|x_{n_j} - x_0\| = \lim_{j\longrightarrow\infty} \|S_{n_j} x_{n_j} - x_0\| = 0,
$$

we obtain from (4.2) that

$$
\lim_{j\longrightarrow\infty} g(\|T_k x_{n_j} - T_l x_{n_j}\|) = 0
$$

for $k, l = 1, 2, \dots, r$. Then the properties of g yield that

$$
\lim_{j\longrightarrow\infty} \|T_k x_{n_j} - T_l x_{n_j}\| = 0
$$

for each $k, l = 1, 2, \dots, r$. Therefore we obtain that

$$
\begin{aligned}
V(S_{n_j} x_{n_j}, T_k x_{n_j}) &= V\left(\sum_{i=1}^{r} \delta_{n_j}^{(i)} T_i x_{n_j}, T_k x_{n_j} \right) \\
&\le \sum_{i=1}^{r} \delta_{n_j}^{(i)} V(T_i x_{n_j}, T_k x_{n_j}) \\
&\le \sum_{i=1}^{r} \delta_{n_j}^{(i)} (V(T_i x_{n_j}, T_k x_{n_j}) + V(T_k x_{n_j}, T_i x_{n_j})) \\
&= 2\sum_{i=1}^{r} \delta_{n_j}^{(i)} \langle T_i x_{n_j} - T_k x_{n_j}, JT_i x_{n_j} - JT_k x_{n_j}\rangle \\
&\le 2\sum_{i=1}^{r} \delta_{n_j}^{(i)} \|T_i x_{n_j} - T_k x_{n_j}\| \|JT_i x_{n_j} - JT_k x_{n_j}\|
\end{aligned}
$$

for each $k = 1, 2, \dots, r$ and hence we have $\lim_{j\longrightarrow\infty} V(S_{n_j} x_{n_j}, T_k x_{n_j}) = 0$. From Lemma 2.2, we have $\lim_{j\longrightarrow\infty} \|S_{n_j} x_{n_j} - T_k x_{n_j}\| = 0$ for each $k = 1, 2, \dots, r$ and hence

$$
\lim_{j\longrightarrow\infty} \|T_k x_{n_j} - x_{n_j}\| \le \lim_{j\longrightarrow\infty} \|T_k x_{n_j} - S_{n_j} x_{n_j}\| + \lim_{j\longrightarrow\infty} \|S_{n_j} x_{n_j} - x_{n_j}\| = 0.
$$

for all $k = 1, 2, \dots, r$. By the assumption of T_k, it follows that $x_0 \in \bigcap_{i=1}^{r} F(T_i)$. Therefore we have

$$
x_0 \in \bigcap_{i=1}^{r} F(T_i) \subset \bigcap_{n=1}^{\infty} C_n
$$

and hence $x_0 = R_{\bigcap_{i=1}^{r} F(T_i)} x$, which completes the proof. $\square$

In the end of this section, we will discuss the assumptions for the coefficients used in Theorem 4.1. Plubtieng and Ungchittrakool [27] proved the following theorem:

Theorem 4.2 (Plubtieng-Ungchittrakool [27]). *Let E be a uniformly convex and uniformly smooth Banach space, C a nonempty closed convex subset of E, and $T_1, T_2, \dots, T_r$ relatively nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. For a point $x \in E$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{cases} y_n = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n)\sum_{i=0}^r \beta_n^{(i)} J T_i x_n\right), \\ C_{n+1} = \{z \in C_n : V(z, y_n) \le V(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x \end{cases}$$

for all $n \in \mathbb{N}$, where T_0 is the identity mapping on C, and $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0,1]$ are real sequences for $i = 0, 1, 2, \dots, r$ satisfying the following conditions:

(i) $\sup_{n\in\mathbb{N}} \alpha_n < 1$,

(ii) $\sum_{i=0}^r \beta_n^{(i)} = 1$ *for all* $n \in \mathbb{N}$, *and either*

  (a) $\liminf_{n\longrightarrow\infty} \beta_n^{(0)}\beta_n^{(i)} > 0$ *for all* $i = 1, 2, \dots, r$, *or*

  (b) $\lim_{n\longrightarrow\infty} \beta_n^{(0)} = 0$ *and* $\liminf_{n\longrightarrow\infty} \beta_n^{(k)}\beta_n^{(l)} > 0$ *for all* $k, l = 1, 2, \dots, r$ *with* $k \ne l$.

Then $\{x_n\}$ converges strongly to $\Pi_{\bigcap_{i=1}^r F(T_i)} x$, where Π_K is a generalized projection of E onto a nonempty closed convex subset K of E.

In order to compare our main result with the theorem above, we consider an analogous scheme to that in Theorem 4.2. For a given sequence $\{x_n\}$ in E, let $\{y_n\}$ be such that

$$y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^r \beta_n^{(i)} T_i x_n$$

for $n \in \mathbb{N}$, where $T_1, T_2, \dots, T_r$ are mappings of C into itself, T_0 is the identity mapping on C, and $\{\alpha_n\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for $i = 0, 1, 2, \dots, r$. Suppose that

(i) $\liminf_{n\longrightarrow\infty} \alpha_n < 1$,

(ii) $\liminf_{n\longrightarrow\infty} \beta_n^{(i)} > 0$ for each $i = 1, 2, \dots r$,

(iii) $\sum_{i=0}^r \beta_n^{(i)} = 1$ for each $n \in \mathbb{N}$.

Then, letting $\gamma_n = \alpha_n + (1-\alpha_n)\beta_n^{(0)}$ and

$$\delta_n^{(i)} = \begin{cases} 0 & (\beta_n^{(0)} = 1) \\ \beta_n^{(i)}/(1-\beta_n^{(0)}) & (\beta_n^{(0)} < 1) \end{cases}$$

for $i = 1, 2, \dots, r$ and $n \in \mathbb{N}$, we have that

$$\begin{aligned} y_n &= \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^r \beta_n^{(i)} T_i x_n \\ &= \gamma_n x_n + (1-\gamma_n)\sum_{i=1}^r \delta_n^{(i)} T_i x_n \end{aligned}$$

for every $n \in \mathbb{N}$. Then we can prove that the coefficients $\{\gamma_n\}$ and $\{\delta_n^{(i)}\}$ satisfy the conditions assumed in Theorem 4.1 such as

(i) $\liminf_{n\longrightarrow\infty}\gamma_n < 1$,
(ii) $\liminf_{n\longrightarrow\infty}\delta_n^{(i)} > 0$ for each $i = 1, 2, \dots r$,
(iii) $\sum_{i=1}^r \delta_n^{(i)} = 1$ for each $n \in \mathbb{N}$.

Indeed, since $\liminf_{n\longrightarrow\infty}\beta_n^{(i)} > 0$ for each $i = 1, 2, \dots, r$ and $\sum_{i=0}^r \beta_n^{(i)} = 1$, we have that

$$\liminf_{n\longrightarrow\infty}(1-\beta_n^{(0)}) = \liminf_{n\longrightarrow\infty}\sum_{i=1}^r \beta_n^{(i)} \ge \sum_{i=1}^r \liminf_{n\longrightarrow\infty}\beta_n^{(i)} > 0,$$

and thus $\limsup_{n\longrightarrow\infty}\beta_n^{(0)} < 1$. It follows that $\beta_n^{(0)} < 1$ for sufficiently large n and hence

$$\liminf_{n\longrightarrow\infty}\delta_n^{(i)} = \liminf_{n\longrightarrow\infty}\frac{\beta_n^{(i)}}{1-\beta_n^{(0)}} \ge \liminf_{n\longrightarrow\infty}\beta_n^{(i)} > 0$$

for each $i = 1, 2, \dots, r$. It is obvious that $\sum_{i=1}^r \delta_n^{(i)} = 1$ for each $n \in \mathbb{N}$ from the definition of $\{\delta_n^{(i)}\}$.

On the other hand, since $\liminf_{n\longrightarrow\infty}\alpha_n < 1$ and $\limsup_{n\longrightarrow\infty}\beta_n^{(0)} < 1$, there exist subsequences $\{\alpha_{n_j}\} \subset \{\alpha_n\}$ and $\{\beta_{n_j}^{(0)}\} \subset \{\beta_n^{(0)}\}$ converging $\alpha < 1$ and $\beta < 1$, respectively. Then we obtain that

$$\lim_{j\longrightarrow\infty}\gamma_{n_j} = \lim_{j\longrightarrow\infty}\alpha_{n_j} + (1-\alpha_{n_j})\beta_{n_j}^{(0)} = \alpha + (1-\alpha)\beta < 1$$

and hence $\liminf_{n\longrightarrow\infty}\gamma_n < 1$.

From the argument above, we obtain the following result.

Theorem 4.3. *Let E, C, and mappings $T_1, T_2, \dots, T_r$ be the same as in Theorem 4.1. Let T_0 be the identity mapping on C. For a point $x \in E$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=0}^r \beta_n^{(i)} T_i x_n, \\ C_{n+1} = \{z \in C_n : V(y_n, z) \le V(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n^{(i)}\} \subset [0, 1]$ are sequences for $i = 0, 1, 2, \dots, r$ satisfying the following conditions:

(i) $\liminf_{n\longrightarrow\infty}\alpha_n < 1$,
(ii) $\liminf_{n\longrightarrow\infty}\beta_n^{(i)} > 0$ *for each* $i = 1, 2, \dots, r$,
(iii) $\sum_{i=0}^r \beta_n^{(i)} = 1$ *for each* $n \in \mathbb{N}$.

Then $\{x_n\}$ converges strongly to $R_{\bigcap_{i=1}^r F(T_i)} x$.

Using the relations between generalized nonexpansive mappings and relatively nonexpansive mappings, and between sunny generalized nonexpansive retractions and generalized projections, we can see that the setting and the assumptions in Theorem 4.3 are more general than that of Theorem 4.2; see [19, 6, 26] for more details.

In particular, in the case where the underlying space E is a Hilbert space, the iterative schemes in Theorems 4.3 and 4.2 coincide with each other, except for the assumptions of mappings and the control coefficients. It is easy to see that the assumptions in Theorem 4.3 are milder than that of Theorem 4.2.

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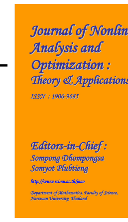

ON STATES WITH ABSORBING TENDENCIES IN SELF-ORGANIZING MAPS WITH INPUTS IN AN INNER PRODUCT SPACE

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ABSTRACT. We deal with self-organizing map models referred to as Kohonen type algorithm. In self-organizing maps, it is easy to observe some practical and interesting properties in the relation between the arrangement of the nodes and their values. We shall be concerned with behavior of ordering, state preserving properties and tendencies to be absorbed to a particular state in self-organizing maps with inputs taking values in an inner product space. We give a numerical example as its application and estimate ordering by simulation approach.

KEYWORDS : Self-organizing maps; Absorbing states.

MSC : 68T05.

1. FORMULATION OF SELF-ORGANIZING MAPS

We consider self-organizing map models referred to as Kohonen [8] type algorithm. Self-organizing map algorithm is very practical and has many useful applications, semantic map, diagnosis of speech voicing, solving traveling-salesman problem, and so on. There are some interesting phenomena between the array of nodes and the values of nodes in these models. Indeed practical properties in self-organizing map models are easy to observe, but they still remain

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without mathematical proof in general cases. Concerning the mathematical analysis of self-organizing maps, a proof of the convergence of the learning process in the one-dimensional case was first given by Cottrell and Fort [1] and convergence properties are more generally studied, e.g., in Erwin, Obermayer, and Schulten [2][3][4]. The purpose of this paper is to make a study of behavior of ordering and tendencies to be absorbed to a particular state in self-organizing maps with inputs taking values in an inner product space.

We consider to characterize a model $(I, V, X, \{m_k(\cdot)\}_{k=0}^{\infty})$ with four elements which consist of the *nodes*, the *values of nodes*, *inputs* and *model functions* with some *learning processes*, in this paper. There are several types of models with various spaces of nodes, spaces of their values and ways of learning for nodes. We suppose the followings.

(i) We suppose an array of *nodes*. Let I denote the set of all nodes, which is called the *node set*. We assume that I is a countable set metrized by a metric d. In many applications, usually, we use the following ones, a finite subset of the set $\mathbb{N}$ of all natural numbers, or a finite subset of $\mathbb{N}^2$.

(ii) We suppose that each node has its *value*. V is the space of values of nodes. We assume that V is a normed linear space with $\|\cdot\|$. A mapping $m : I \longrightarrow V$ transforming each node i to its value $m(i)$ is called a *model function*. Let M be the set of all model functions.

(iii) X is the *input set*. Let X be a subset of V. $x \in X$ is called an *input*.

(iv) The *learning process* is defined by the following. If an input is given, then the value of each node is renewed to a new value according to the input. If an input x is given, node i learns x and its value $m(i)$ changes to a new value $m'(i)$ determined by

$$m'(i) = (1 - \alpha_{m,x}(i))m(i) + \alpha_{m,x}(i)x \tag{1.1}$$

according to the rate $\alpha_{m,x}(i) \in [0, 1]$. If an initial model function m_0 and a sequence $x_0, x_1, x_2, \ldots \in X$ of inputs are given, then the model functions $m_1, m_2, m_3, \ldots$ are generated sequentially according to the following equation.

$$m_{k+1}(i) = (1 - \alpha_{m_k,x_k}(i))m_k(i) + \alpha_{m_k,x_k}(i)x_k. \tag{1.2}$$

There are several types of models with various spaces of nodes, spaces of their values and ways of learning for nodes. In this paper, we use two types of learning processes defined by the following.

Learning process $\mathrm{L_A}$:

(i) Areas of learning: for each $m_k \in M$ and $x_k \in X$,

$$I(m_k, x_k) = \left\{ i^* \in I \;\middle|\; \|m_k(i^*) - x_k\| = \inf_{i \in I} \|m_k(i) - x_k\| \right\},$$
$$N_\varepsilon(i) = \left\{ j \in I \;\middle|\; d(j,i) \le \varepsilon \right\} \quad (\varepsilon > 0 \text{ is the learning radius, } i \in I).$$

(ii) Learning-rate factor: $0 \le \alpha \le 1$.

(iii) Learning:

$$m_{k+1}(i) = \begin{cases} (1-\alpha)m_k(i) + \alpha x_k & \text{if } i \in \bigcup\limits_{i^* \in I(m_k,x_k)} N_\varepsilon(i^*), \\ m_k(i) & \text{if } i \notin \bigcup\limits_{i^* \in I(m_k,x_k)} N_\varepsilon(i^*), \end{cases} \qquad k = 0, 1, 2, \ldots.$$

We note that, if $I(m_k, x_k) = \varnothing$ then $m_{k+1} = m_k$.

Learning process L_m: This learning process is the same as Learning process L_A except that a node $J(m_k, x_k)$ is selected from $I(m_k, x_k)$ by a given rule. Let $J : M \times X \longrightarrow I$ be a mapping which satisfies $J(m_k, x_k) \in I(m_k, x_k)$. For example, if I is a totally ordered finite set, $J(m_k, x_k)$ may be defined by $J(m_k, x_k) = \min I(m_k, x_k)$. If $i \in N_\varepsilon(J(m_k, x_k))$ then $m_{k+1}(i) = (1-\alpha)m_k(i) + \alpha x_k$, otherwise $m_{k+1}(i) = m_k(i)$. We assume that $m_{k+1} = m_k$ if $I(m_k, x_k) = \varnothing$.

2. A FUNDAMENTAL SELF-ORGANIZING MAP AND ABSORBING STATES

In Section 2, we restrict our considerations to basic self-organizing maps with real-valued nodes and one dimensional array of nodes. Now, we suppose that set V of values of nodes is identified with $\mathbb{R}$ which is the set of all real numbers.

$$(I = \{1, 2, \ldots, n\}, V = \mathbb{R}, X \subset \mathbb{R}, \{m_k(\cdot)\}_{k=0}^{\infty})$$

(i) Let $I = \{1, 2, \ldots, n\}$ be the node set with metric $d(i,j) = |i - j|$. (ii) Assume $V = \mathbb{R}$, that is, each node is $\mathbb{R}$-valued. A model function m is written as $m = [m(1), m(2), \ldots, m(n)]$. (iii) $x_0, x_1, x_2, \ldots \in X \subset \mathbb{R}$ is an input sequence. (iv) Learning process L_A (1-dimensional array, $\mathbb{R}$-valued nodes and $\varepsilon = 1$): (a) areas of learning: $I(m_k, x_k) = \{i^* \in I \mid |m_k(i^*) - x_k| = \inf_{i \in I} |m_k(i) - x_k|\}$ and $N_1(i) = \{j \in I \mid |j - i| \le 1\}$; (b) learning-rate factor: $0 \le \alpha \le 1$; (c) learning: for each $k = 0, 1, 2, \ldots$, if $i \in \cup_{i^* \in I(m_k, x_k)} N_1(i^*)$ then $m_{k+1}(i) = (1-\alpha)m_k(i) + \alpha x_k$, otherwise $m_{k+1}(i) = m_k(i)$.

If an input $x_0 \in X$ is given, then we choose node i^* which has the most similar value to x_0 within $m_0(1), m_0(2), \ldots, m_0(n)$. Node i^* and the nodes which are in the neighborhood of i^* learn x_0 and their values change to new values $m_1(i) = (1-\alpha)m_0(i) + \alpha x_0$. The nodes which are not in the neighborhood of i^* do not learn and their values do not change. Repeating these updating for the inputs $x_1, x_2, x_3, \ldots$, the value of each node is renewed sequentially. Simultaneously, model functions $m_1, m_2, m_3, \ldots$ are also generated sequentially.

The following properties are well-known results and can be verified easily.

Theorem 2.1. *We consider a self-organizing map model* $(I = \{1, 2, \ldots, n\}, V = \mathbb{R}, X \subset \mathbb{R}, \{m_k(\cdot)\}_{k=0}^{\infty})$ *with Learning process* $L_A(\varepsilon = 1)$*. For model functions* m_1, m_2, m_3, ..., *the following statements hold:*

(i) *if* m_k *is increasing on* I*, then* m_{k+1} *is increasing on* I*;*
(ii) *if* m_k *is decreasing on* I*, then* m_{k+1} *is decreasing on* I*;*
(iii) *if* m_k *is strictly increasing on* I*, then* m_{k+1} *is strictly increasing on* I*;*
(iv) *if* m_k *is strictly decreasing on* I*, then* m_{k+1} *is strictly decreasing on* I*.*

Such properties as monotonicity are called *absorbing states* of self-organizing map models in the sense that once model function leads to increasing state, it never leads to other states for the learning by any input. Quasi-convexity or quasi-concavity is also an absorbing state of the previous self-organizing maps. Their details and definitions of quasi-convexity and quasi-concavity for model functions are in [6].

3. States with absorbing tendencies in 2-dimensional array model

In this section, we suppose the case of nodes with values in a normed linear space and 2-dimensional array.

$$\left(\{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\},\ V,\ X,\ \{m_k(\cdot, \cdot)\}_{k=0}^{\infty}\right)$$

(i) The node set. Let $I = \{1, 2, \ldots, n_1\} \times \{1, 2, \ldots, n_2\}$ with metric

$$d_I((i,j),(k,l)) = \sqrt{(i-k)^2 + (j-k)^2}, \quad (i,j),(k,l) \in I.$$

(ii) The values of nodes. Let $m : I \longrightarrow V$, where V is a normed linear space with an inner product $\langle \cdot, \cdot \rangle$.
(iii) $x_0, x_1, x_2, \ldots \in X \subset V$ is an input sequence.
(iv) Assume Larning process L_m (2-dimensional array and $\varepsilon = \sqrt{2}$)
  (a) areas of learning:

$$I(m,x) = \{(i^*, j^*) \in I \mid \|m(i^*, j^*) - x\| = \inf_{(i,j)\in I} \|m(i,j) - x\|\},$$

  $m \in M, x \in X$, let $J : M \times X \longrightarrow I$ be a mapping which satisfies $J(m,x) \in I(m,x)$, and

$$N_{\sqrt{2}}(i,j) = \{(k,l) \in I \mid d_I((i,j),(k,l)) \le \sqrt{2}\};$$

  (b) learning-rate factor: $0 \le \alpha \le 1$;
  (c) learning: if $(i,j) \in N_{\sqrt{2}}(J(m,x))$ then $m'(i,j) = (1 - \alpha)m(i,j) + \alpha x$, otherwise $m'(i,j) = m(i,j)$.

The following property can be use as measures of ordering and analysis of pre-ordered state in the learning processes.

Theorem 3.1. *We consider a self-organizing map model*

$$\big(\{1,2,\ldots,n_1\}\times\{1,2,\ldots,n_2\},\ V,\ X,\ \{m_k(\cdot,\cdot)\}_{k=0}^{\infty}\big)$$

with Learning process $L_m(\varepsilon=\sqrt{2})$*. Let* m *be an arbitrary model function and* x *an arbitrary input. Let* m' *be the renewed model function of* m *by* x*. For every node* (i,j) *with* $d_I((i,j),J(m,x))\neq\sqrt{2},2,\sqrt{5}$*, if*

$$\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle\leq 0 \tag{3.1}$$

and

$$\langle m(i,j-1)-m(i,j), m(i,j+1)-m(i,j)\rangle\leq 0 \tag{3.2}$$

hold, then

$$\langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle\leq 0 \tag{3.3}$$

and

$$\langle m'(i,j-1)-m'(i,j), m'(i,j+1)-m'(i,j)\rangle\leq 0 \tag{3.4}$$

hold.

Proof. We note that $d_I((i,j),J(m,x))=0,1,\sqrt{2},2,\sqrt{5},2\sqrt{2},\ldots$.

(A) For $d_I((i,j),J(m,x))\geq 2\sqrt{2}$, we have

$$m'(k,l)=m(k,l),\quad (k,l)=(i-1,j),(i,j),(i+1,j),(i,j-1),(i,j+1).$$

Therefore, (3.1) implies (3.3) and (3.2) implies (3.4).

(B) For $d_I((i,j),J(m,x))=0$, we have

$$m'(k,l)=(1-\alpha)m(k,l)+\alpha x,$$
$$(k,l)=(i-1,j),(i,j),(i+1,j),(i,j-1),(i,j+1).$$

Therefore, if (3.1) holds, then

$$\begin{aligned}
&\langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle\\
=\ &\langle (1-\alpha)m(i-1,j)+\alpha x-(1-\alpha)m(i,j)-\alpha x,\\
&(1-\alpha)m(i+1,j)+\alpha x-(1-\alpha)m(i,j)-\alpha x\rangle\\
=\ &(1-\alpha)^2\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle\\
\leq\ &0.
\end{aligned}$$

Similarly, (3.2) implies (3.4).

($C_{1,0}$) For $(i,j)=J(m,x)+(1,0)$, we have

$$m'(k,l)=\begin{cases}(1-\alpha)m(k,l)+\alpha x, & \text{if } (k,l)=(i-1,j),(i,j),(i,j-1),(i,j+1),\\ m(k,l)+\alpha x, & \text{if } (k,l)=(i+1,j).\end{cases}$$

Therefore

$$\langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle$$

$$\begin{aligned}
&= \langle (1-\alpha)m(i-1,j)+\alpha x-(1-\alpha)m(i,j)-\alpha x,\\
&\quad m(i+1,j)-(1-\alpha)m(i,j)-\alpha x\rangle\\
&= (1-\alpha)\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)+\alpha(m(i,j)-x)\rangle\\
&= (1-\alpha)\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle\\
&\quad -\alpha(1-\alpha)\langle m(i-1,j)-m(i,j), x-m(i,j)\rangle\\
&= (1-\alpha)\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle\\
&\quad -\frac{1}{2}\alpha(1-\alpha)\{\|m(i-1,j)-m(i,j)\|^2+\|m(i,j)-x\|^2\\
&\quad -\|m(i-1,j)-x\|^2\}.
\end{aligned}$$

Since $(i-1,j)=J(m,x)$, we have $\|m(i-1,j)-x\| \le \|m(i,j)-x\|$. It follows that

$$\begin{aligned}
&\langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle\\
\le\ &\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle.
\end{aligned}$$

Hence, if (3.1) holds, then (3.3) holds. Since

$$\begin{aligned}
&\langle m'(i,j-1)-m'(i,j), m'(i,j+1)-m'(i,j)\rangle\\
\le\ &\langle m(i,j-1)-m(i,j), m(i,j+1)-m(i,j)\rangle,
\end{aligned}$$

(3.2) implies (3.4).

($C_{0,1}$) For $(i,j)=J(m,x)+(0,1)$, we have
$m'(k,l)=$

$$\begin{cases}(1-\alpha)m(k,l)+\alpha x, & \text{if } (k,l)=(i-1,j),(i,j),(i+1,j),(i,j-1),\\ m(k,l)+\alpha x, & \text{if } (k,l)=(i,j+1).\end{cases}$$

By the same argument used in ($C_{1,0}$),

$$\begin{aligned}
&\langle m'(i,j-1)-m'(i,j), m'(i,j+1)-m'(i,j)\rangle\\
=\ &(1-\alpha)\langle m(i,j-1)-m(i,j), m(i,j+1)-m(i,j)\rangle\\
&-\frac{1}{2}\alpha(1-\alpha)\big\{\|m(i,j-1)-m(i,j)\|^2+\|m(i,j)-x\|^2\\
&-\|m(i,j-1)-x\|^2\big\}.
\end{aligned}$$

Since $(i,j-1)=J(m,x)$, we have $\|m(i,j-1)-x\| \le \|m(i,j)-x\|$. Hence, if (3.2) holds, then (3.4) holds. Since the left hand side of (3.1) equals the left hand side of (3.3) for $(i,j)=J(m,x)+(0,1)$, (3.1) implies (3.3).

($C_{-1,0}$) For $(i,j)=J(m,x)+(-1,0)$,
$m'(k,l)=$

$$\begin{cases}(1-\alpha)m(k,l)+\alpha x, & \text{if } (k,l)=(i,j),(i+1,j),(i,j-1),(i,j+1),\\ m(k,l)+\alpha x, & \text{if } (k,l)=(i-1,j).\end{cases}$$

Then, we have

$$\begin{aligned} & \langle m'(i-1,j)-m'(i,j), m'(i+1,j)-m'(i,j)\rangle \\ = \; & (1-\alpha)\langle m(i-1,j)-m(i,j), m(i+1,j)-m(i,j)\rangle \\ & -\frac{1}{2}\alpha(1-\alpha)\big\{\|x-m(i,j)\|^2+\|m(i+1,j)-m(i,j)\|^2 \\ & -\|x-m(i+1,j)\|^2\big\}. \end{aligned}$$

Since $(i+1,j)=J(m,x)$, we have $\|x-m(i+1,j)\|\leq\|x-m(i,j)\|$. Hence, if (3.1) holds, then (3.3) holds. Similarly, (3.2) implies (3.4) for $(i,j)=J(m,x)+(-1,0)$.

($C_{0,-1}$) For $(i,j)=J(m,x)+(0,-1)$, similarly, (3.3) and (3.4) hold. □

4. 1-DIMENSIONAL ARRAY CASE AND A NUMERICAL EXAMPLE

In this section, we suppose the case of nodes with values in a normed linear space and 1-dimensional array.

$$\big(\{1,2,\ldots,n\},\ V,\ X,\ \{m_k(\cdot)\}_{k=0}^{\infty}\big)$$

(i) The node set. Let $I=\{1,2,\ldots,n\}$ with metric $d_I(i,j)=|i-j|$, $i,j\in I$.

(ii) The values of nodes. Let $m: I\longrightarrow V$, where V is a normed linear space with an inner product $\langle\cdot,\cdot\rangle$.

(iii) $x_0,x_1,x_2,\ldots\in X\subset V$ is an input sequence.

(iv) Assume Larning process L_m (1-dimensional array and $\varepsilon=1$)

(a) areas of learning:

$$J(m,x)=\min\{i^*\in I \mid \|m(i^*)-x\|=\inf_{i\in I}\|m(i)-x\|\},\quad m\in M, x\in X$$

and $N_1(i)=\{j\in I \mid d_I(i,j)\leq 1\}$;

(b) learning-rate factor: $0\leq\alpha\leq 1$;

(c) learning: if $i\in N_1(J(m,x))$ then $m'(i)=(1-\alpha)m(i)+\alpha x$, otherwise $m'(i)=m(i)$.

For 1-dimensional array case, we have the following property and it is proved by the similar argument in the proof of Theorem 3.1.

Theorem 4.1. *We consider a self-organizing map model*

$$\big(\{1,2,\ldots,n\},\ V,\ X,\ \{m_k(\cdot)\}_{k=0}^{\infty}\big)$$

with Learning process $L_m(\varepsilon=1)$. Let m be an arbitrary model function and x an arbitrary input. Let m' be the renewed model function of m by x. For every node i with $d_I(i,J(m,x))\neq 2$, if

$$\langle m(i-1)-m(i), m(i+1)-m(i)\rangle\leq 0 \tag{4.1}$$

hold, then

$$\langle m'(i-1)-m'(i), m'(i+1)-m'(i)\rangle\leq 0$$

hold. Particularly, in case $0 \leq \alpha < 1$, *for every node* i *with* $d_I(i, J(m,x)) \neq 2$, *if*

$$\langle m(i-1) - m(i), m(i+1) - m(i)\rangle < 0$$

hold, then

$$\langle m'(i-1) - m'(i), m'(i+1) - m'(i)\rangle < 0$$

hold.

We give a numerical example of 1-dimensional array model with 2-dimensional inputs. We assume the following.

(i) Let $I = \{1, 2, 3, \ldots, 35\}$ be the node set with metric $d_I(i, j) = |i - j|$.

(ii) Initial model function is given by $m_0(1) = (3, 1), m_0(2) = (9, 8), \ldots, m_0(35) = (0, 7)$ (see Figure 2). For $x = (x_1, x_2), y = (y_1, y_2)$, $\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(iii) Suppose 19000 inputs as follows, randomly, generated by the distribution described in Figure 1. For example, the distribution in Figure 1 means a distribution map of a certain population, as demand. A densely distributed area has a large population.

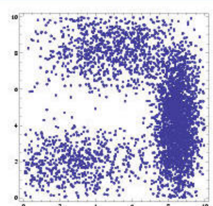
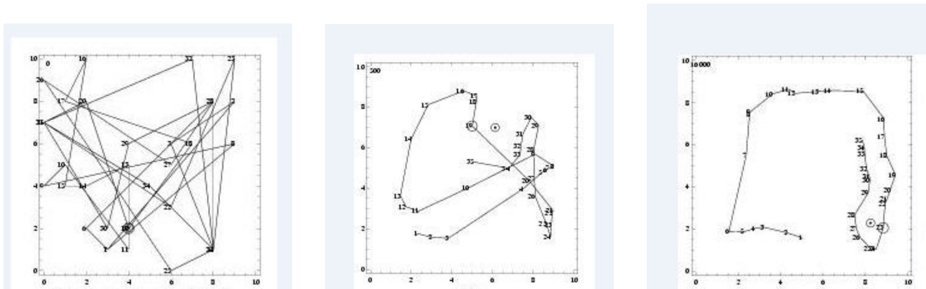


FIGURE 1. Distribution of inputs as demands

(iv) Assume Learning process $L_m(\varepsilon = 1)$ with learning-rate factor $\alpha = \frac{1}{3}$.

Figure 2 illustrates nodes and their values in each iteration steps. The position of every node means its value. The length of a path is defined by $\sum_{i=1}^{n-1} \|m(i) - m(i+1)\|$. By repeating learning, we can observe that the values of the nodes in each iteration step are ordered and that their values yield a well-balanced solution with a shorter path and satisfying demands. The path constructed by model function m_{16000} has more nodes in densely distributed areas on Figure 1.

Figure 3 shows relative frequencies of nodes satisfy condition (4.1) defined by



The length of path 182.92 43.28
29.01

FIGURE 2. Initial values of nodes (m_0), 500 renewals (m_{500}) and 16000 renewals (m_{16000})

$$r_{\text{inn}} = \frac{\text{the number of elements in } \{i \mid \langle m(i-1) - m(i), m(i+1) - m(i) \rangle \leq 0\}}{n-2},$$

where n is the number of nodes. This rate can be use as measures of ordering and estimations of convergence in the learning processes.

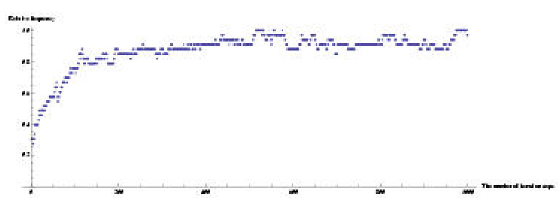


FIGURE 3. Transition of relative frequencies r_{inn}

Figures 4 and 5 show transitions of the length of path and the mean distance between a model function m and inputs defined by

$$\text{MD} = \frac{\sum_{x \in X} \min_{i \in I} \|m(i) - x\|}{N(X)},$$

where $N(X)$ is the number of elements in input set X.

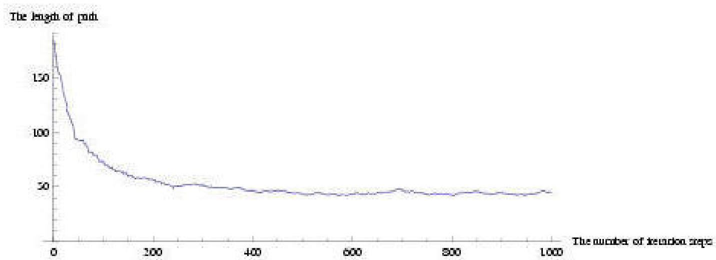


FIGURE 4. Transition of the length of path

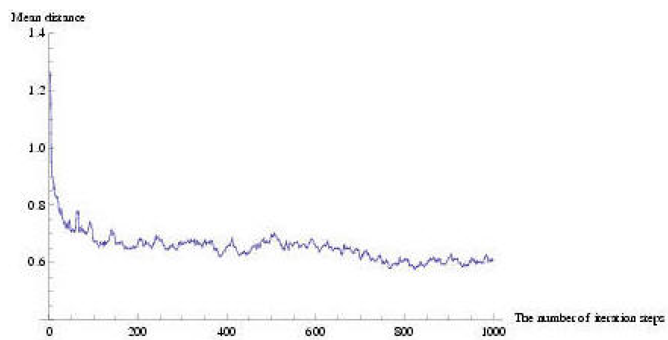


FIGURE 5. Transition of the mean distance MD

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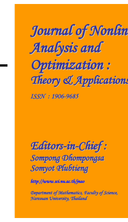

WEAK CONVERGENCE FOR A SEQUENCE OF NONEXPANSIVE MAPPINGS IN BANACH SPACES WITH GAUGE FUNCTIONS

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ABSTRACT. We investigate the convergence of Mann-type iteration for a sequence of nonexpansive mappings in the framework of a uniformly convex Banach space having the duality mapping j_φ, where φ is a gauge function on $[0, \infty)$. Our results improve and extend some well-known results in the literature.

KEYWORDS : Banach space; Common fixed point; Gauge function; Mann-type iteration.

1. INTRODUCTION AND PRELIMINARIES

Let K be a nonempty, closed and convex subset of a Banach space E. Let $T : K \to K$ be a mapping. Then T is said to be *nonexpansive* if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in K$. We denote $F(T) = \{x \in K : x = Tx\}$ by the fixed points set of T.

One classical way to approximate a fixed point of a nonexpansive mapping was introduced, in 1953, by Mann [1] as follows: a sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \ge 1, \tag{1.1}$$

where $\alpha_n \in (0, 1)$. Such a process is known as *Mann's iteration process*. Reich [2] showed that the sequence $\{x_n\}$ generated by (1.1) converges weakly to a fixed point of T if a real sequence $\{\alpha_n\}$ satisfies the condition $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. This is valid in a uniformly convex Banach space with a Fréchet differentiable norm. Since 1953, the convergence of nonlinear mappings by Mann iteration process has been extensively studied by many authors (see also [3, 4]). However, we note that

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the convergence results of the process (1.1) have been widely established in spaces having the normalized duality mappings. By the way, Browder [5] initiated the study of certain classes of nonlinear operators by means of the duality mapping associated to a gauge function which includes the generalized and the normalized duality mappings as special cases.

Motivated by Browder [5] and Reich [2], we continue the work to study the convergence of Mann-type iteration in a much more general setting, a uniformly convex Banach space having the duality mapping associated to a gauge function.

Let E be a Banach space which admits the duality mapping associated to a gauge function and K a nonempty, closed and convex subset of E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings of K. We consider the following Mann-type iteration: $x_1 \in K$ and

$$x_{n+1} \quad = \quad \alpha_n x_n + (1-\alpha_n)T_n x_n, \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$.

We recall that a Banach space E is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, $\|x+y\| \leq 2(1-\delta)$ holds. The modulus of convexity of E is defined by

$$\delta_E(\epsilon) = \inf\Big\{1 - \Big\|\frac{1}{2}(x+y)\Big\| : \ \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon\Big\},$$

for all $\epsilon \in [0,2]$. E is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in S(E)$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. See [6, 7, 8].

We need the following definitions and results which can be found in [5, 6, 7].

Definition 1.1. *A continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ is said to be gauge function if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.*

Definition 1.2. *Let E be a normed space and φ a gauge function. Then the mapping $J_\varphi : E \to 2^{E^*}$ defined by*

$$J_\varphi(x) = \big\{f^* \in E^* : \ \langle x, f^* \rangle = \|x\|\varphi(\|x\|), \ \|f^*\| = \varphi(\|x\|)\big\}, \quad x \in E$$

is called the duality mapping with gauge function φ.

In the case $\varphi(t) = t^{q-1}$, $q > 1$, the duality mapping $J_\varphi = J_q$ is called the generalized duality mapping.

Remark 1.3. *For the gauge function φ, the function $\Phi : [0,\infty) \to [0,\infty)$ defined by $\Phi(t) = \int_0^t \varphi(s)ds$ is a continuous convex and strictly increasing function on $[0,\infty)$. Therefore, Φ has a continuous inverse function Φ^{-1}.*

Remark 1.4. *For each x in a Banach space E, $J_\varphi(x) = \partial\Phi(\|x\|)$, where ∂ denotes the sub-differential.*

Remark 1.5. *For each $x, y \in E$, the following inequality holds:*

$$\Phi\big(\|x+y\|\big) \leq \Phi\big(\|x\|\big) + \big\langle y, j_\varphi(x+y)\big\rangle, \quad j_\varphi(x+y) \in J_\varphi(x+y). \tag{1.3}$$

Remark 1.6. A Banach space E is smooth if and only if each duality mapping J_φ with gauge function φ is single-valued; in this case

$$\frac{d}{dt}\Phi(\|x+ty\|)\Big|_{t=0} = \lim_{t\to 0} \frac{\Phi(\|x+ty\|) - \Phi(\|x\|)}{t} = \big\langle y, J_\varphi(x)\big\rangle, \quad \forall x, y \in E.$$

Lemma 1.7. [9] *Assume that a Banach space E has a weakly continuous duality mapping with gauge φ. Then for any sequence $\{x_n\}$ that converges weakly to x, we have for any $y \in E$, $\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|x - y\|)$.*

Let K be a subset of a real Banach space E and let $\{T_n\}$ be a family of mappings of K such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy *the AKTT-condition* [10] if for each bounded subset B of K, $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$.

Lemma 1.8. [10] *Let K be a nonempty and closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of K into itself which satisfies the AKTT-condition, then the mapping $T : K \to K$ defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in K$ satisfies*

$$\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0$$

for each bounded subset B of K.

In the sequel, we will write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 1.8 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

2. MAIN RESULTS

Proposition 2.1. *Let E be a reflexive Banach space having a weakly continuous duality mapping j_φ. Let $\{x_n\}$ be a sequence in E and $p, q \in \omega_w(\{x_n\})$. If $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} \|x_n - q\|$ exist. Then $p = q$.*

Proof. Suppose $p, q \in \omega_w(\{x_n\})$ and $p \neq q$. Then there exist subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ and $x_{n_j} \rightharpoonup q$. By Lemma 1.7 and the continuity of Φ, we have

$$\begin{aligned} \lim_{n\to\infty} \Phi(\|x_n - p\|) &= \lim_{k\to\infty} \Phi(\|x_{n_k} - p\|) < \lim_{k\to\infty} \Phi(\|x_{n_k} - q\|) \\ &= \lim_{j\to\infty} \Phi(\|x_{n_j} - q\|) < \lim_{j\to\infty} \Phi(\|x_{n_j} - p\|) = \lim_{n\to\infty} \Phi(\|x_n - p\|), \end{aligned}$$

which is a contradiction. Thus $p = q$. □

Proposition 2.2. *Let E be a Banach space having a Fréchet differentiable norm and a duality mapping j_φ. Then there exists an increasing function $b : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} \frac{b(t)}{t} = 0$ and*

$$\Phi(\|x\|) + \langle h, j_\varphi(x)\rangle \leq \Phi(\|x + h\|) \leq \Phi(\|x\|) + \langle h, j_\varphi(x)\rangle + b(\|h\|) \quad \forall x, h \in E. \tag{2.1}$$

Proof. Let $x \in E$ and define $b : [0,\infty) \to [0,\infty)$ by $b(0) = 0$ and

$$b(t) = \sup_{y \in S(E)} \left|\Phi(\|x + ty\|) - \Phi(\|x\|) - t\langle y, j_\varphi(x)\rangle\right|, \quad t > 0.$$

Then b is an increasing function. Since E has a Fréchet differentiable norm,

$$\lim_{t\to 0} \frac{b(t)}{t} = \lim_{t\to 0} \sup_{y\in S(E)} \left| \frac{\Phi(\|x + ty\|) - \Phi(\|x\|)}{t} - \langle y, j_\varphi(x)\rangle \right| = 0$$

and

$$\left|\Phi(\|x + ty\|) - \Phi(\|x\|) - t\langle y, j_\varphi(x)\rangle\right| \leq b(t) \quad \forall y \in S(E)$$

which implies

$$\Phi(\|x + ty\|) \leq \Phi(\|x\|) + t\langle y, j_\varphi(x)\rangle + b(t) \quad \forall y \in S(E). \tag{2.2}$$

Suppose $h \neq 0$. Put $y = \frac{h}{\|h\|}$ and $t = \|h\|$. By (2.2), we have

$$\Phi(\|x + h\|) \leq \Phi(\|x\|) + \langle h, j_\varphi(x)\rangle + b(\|h\|). \tag{2.3}$$

On the other hand, by (1.3), we have

$$\Phi(\|x\|) = \Phi(\|x+h-h\|) \leq \Phi(\|x+h\|) - \langle h, j_\varphi(x)\rangle \tag{2.4}$$

for each $h \in E$. Combining (2.3) and (2.4), we get the desired result. □

Following [2, 11], we can prove Proposition 2.3.

Proposition 2.3. *Let E be a uniformly convex Banach space having a Fréchet differentiable norm and a duality mapping j_φ and K a nonempty, closed and convex subset of E. Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of L_n-Lipschitzian self-mappings of K with $L_n \geq 1,\ n \geq 1$ and $\sum_{n=1}^{\infty}(L_n - 1) < \infty$. Assume that $F = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. If $x_1 \in E$ and $x_{n+1} = S_n x_n,\ n \geq 1$, then $\forall f_1, f_2 \in F$, $\lim_{n\to\infty}\langle x_n, j_\varphi(f_1 - f_2)\rangle$ exists.*

Proof. For each $f \in F$, we see that

$$\|x_{n+1} - f\| = \|S_n x_n - f\| \leq \big(1 + (L_n - 1)\big)\|x_n - f\|.$$

Hence $\lim_{n\to\infty}\|x_n - f\|$ exists for all $f \in F$. Now, taking $x = f_1 - f_2, h = t(x_n - f_1)$ in (2.1) and setting $a_n(t) = \|tx_n + (1-t)f_1 - f_2\|$, we obtain

$$\begin{aligned}\Phi(\|f_1 - f_2\|) \quad &+ \quad t\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle \leq \Phi(a_n(t)) \\ &\leq \quad \Phi(\|f_1 - f_2\|) + t\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle + b(t\|x_n - f_1\|) \\ &\leq \quad \Phi(\|f_1 - f_2\|) + t\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle + b(tM)\end{aligned}$$

for some $M > 0$. Since E is uniformly convex, by Lemma 2.2 of [11], we know that $\lim_{n\to\infty} a_n(t)$ exists. Hence $\lim_{n\to\infty}\Phi(a_n(t))$ also exists since Φ is continuous. Thus

$$\limsup_{n\to\infty}\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle \leq \liminf_{n\to\infty}\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle + b(tM)/t.$$

Since $b(tM)/t \to 0$ as $t \to 0$, $\lim_{n\to\infty}\langle x_n - f_1, j_\varphi(f_1 - f_2)\rangle$ exists. □

Theorem 2.4. *Let E be a Banach space having a duality mapping j_φ and K a nonempty, closed and convex subset of E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings of K such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition and let $\{x_n\}$ be defined by (1.2). Then,*
(i) For each $f \in \bigcap_{n=1}^{\infty} F(T_n)$, $\lim_{n\to\infty}\|x_n - f\|$ exists.
(ii) If E is uniformly convex and $\sum_{n=1}^{\infty}\min\{\alpha_n, 1-\alpha_n\} = \infty$, then

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Proof. (i) For any $f \in \bigcap_{n=1}^{\infty} F(T_n)$, we have

$$\|x_{n+1} - f\| \leq \alpha_n\|x_n - f\| + (1-\alpha_n)\|T_n x_n - f\| \leq \|x_n - f\|.$$

Hence $\{\|x_n - f\|\}$ is nonincreasing; consequently, $\lim_{n\to\infty}\|x_n - f\|$ exists.

(ii) Let $f \in \bigcap_{n=1}^{\infty} F(T_n)$ and assume $\|x_n - f\| > 0$. Since $\|T_n x_n - f\| \leq \|x_n - f\|$ and E is uniformly convex, it follows (see, for example, [12]) that

$$\|x_{n+1} - f\| \leq \|x_n - f\|\bigg\{1 - 2\min\{\alpha_n, 1-\alpha_n\}\delta_E\Big(\frac{\|x_n - T_n x_n\|}{\|x_n - f\|}\Big)\bigg\}.$$

Therefore

$$2\min\{\alpha_n, 1-\alpha_n\}\|x_n - f\|\delta_E\Big(\frac{\|x_n - T_n x_n\|}{\|x_n - f\|}\Big) \leq \|x_n - f\| - \|x_{n+1} - f\|.$$

Since $\lim_{n\to\infty}\|x_n - f\|$ exists and $\sum_{n=1}^{\infty}\min\{\alpha_n, 1-\alpha_n\} = \infty$, by the continuity of δ_E, we conclude that $\liminf_{n\to\infty}\|x_n - T_n x_n\| = 0$. Observe that

$$\|x_{n+1} - T_{n+1}x_{n+1}\| \quad \leq \quad \alpha_n\|x_n - T_{n+1}x_{n+1}\| + (1-\alpha_n)\|T_n x_n - T_{n+1}x_{n+1}\|$$

$$\begin{aligned}
&\leq \alpha_n\|x_n - x_{n+1}\| + \alpha_n\|x_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + (1-\alpha_n)\|T_nx_n - T_{n+1}x_n\| + (1-\alpha_n)\|T_{n+1}x_n - T_{n+1}x_{n+1}\| \\
&\leq \alpha_n(1-\alpha_n)\|x_n - T_nx_n\| + \alpha_n\|x_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + (1-\alpha_n)\sup_{z\in\{x_n\}}\|T_nz - T_{n+1}z\| + (1-\alpha_n)^2\|x_n - T_nx_n\| \\
&= (1-\alpha_n)\|x_n - T_nx_n\| + \alpha_n\|x_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + (1-\alpha_n)\sup_{z\in\{x_n\}}\|T_nz - T_{n+1}z\|,
\end{aligned}$$

which implies

$$\|x_{n+1} - T_{n+1}x_{n+1}\| \leq \|x_n - T_nx_n\| + \sup_{z\in\{x_n\}}\|T_nz - T_{n+1}z\|.$$

Since $\{T_n\}$ satisfies the AKTT-condition, $\lim_{n\to\infty}\|x_n - T_nx_n\|$ exists; consequently, $\lim_{n\to\infty}\|x_n - T_nx_n\| = 0$. On the other hand, we see that

$$\|x_n - Tx_n\| \leq \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\| \leq \|x_n - T_nx_n\| + \sup_{z\in\{x_n\}}\|T_nz - Tz\|.$$

Since $(\{T_n\}, T)$ satisfies the AKTT-condition, we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ by Lemma 1.8. This completes the proof. □

Theorem 2.5. *Let E be a uniformly convex Banach space having a duality mapping j_φ and K a nonempty, closed and convex subset of E. Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings of K such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition and let $\{x_n\}$ be defined by (1.2) with $\sum_{n=1}^\infty \min\{\alpha_n, 1-\alpha_n\} = \infty$. If one of the following statements holds:*

(i) E has a weakly continuous duality mapping j_φ;

(ii) E has a Fréchet differentiable norm.

Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}_{n=1}^\infty$.

Proof. Set $S_n = \alpha_nI + (1-\alpha_n)T_n$, $n \geq 1$. Then $x_{n+1} = S_nx_n$ and $F(T_n) = F(S_n)$ for all $n \geq 1$. By Theorem 2.4 (i) and (ii), we get that $\omega_w(\{x_n\}) \subset F(T)$ by the demiclosedness principle. Next, we show that $\omega_w(\{x_n\})$ is singleton. To this end, let $p, q \in \omega_w(\{x_n\})$. If E has a weakly continuous duality mapping j_φ, then $p = q$ by Proposition 2.1. Suppose that E has a Fréchet differentiable norm, and $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ and $x_{m_k} \rightharpoonup q$. Then

$$\|p-q\|\varphi(\|p-q\|) = \langle p-q, j_\varphi(p-q)\rangle = \lim_{k\to\infty}\langle x_{n_k} - x_{m_k}, j_\varphi(p-q)\rangle.$$

By Proposition 2.3, we conclude that $\|p-q\|\varphi(\|p-q\|) = 0$ and $p = q$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}_{n=1}^\infty$. □

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