Vol. 1 No. 1 (2010)

Journal of Nonlinear Analysis and Optimization: Theory & Applications

Editors-in-Chief:

Sompong Dhompongsa
Somyot Plubtieng

About the Journal

Journal of Nonlinear Analysis and Optimization: Theory & Applications is a peer-reviewed, open-access international journal, that devotes to the publication of original articles of current interest in every theoretical, computational, and applicational aspect of nonlinear analysis, convex analysis, fixed point theory, and optimization techniques and their applications to science and engineering. All manuscripts are refereed under the same standards as those used by the finest-quality printed mathematical journals. Accepted papers will be published in two issues annually in March and September, free of charge.

This journal was conceived as the main scientific publication of the Center of Excellence in Nonlinear Analysis and Optimization, Naresuan University, Thailand.

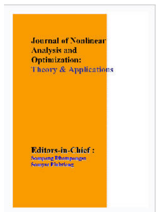
Contact

Narin Petrot (narinp@nu.ac.th)
Center of Excellence in Nonlinear Analysis and Optimization,
Department of Mathematics, Faculty of Science,
Naresuan University, Phitsanulok, 65000, Thailand.

Official Website: https://ph03.tci-thaijo.org/index.php/jnao

Table of Contents

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 1–7
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Stability of closedness of convex cones under linear mappings II

Jonathan M. Borwein[1], Warren B. Moors[2]

ABSTRACT: In this paper we revisit the question of when the continuous linear image of a fixed closed convex cone K is closed. Specifically, we improve the main result of [3] by showing that for all, except for at most a σ-porous set, of the linear mappings T from $\mathbb{R}^n$ into $\mathbb{R}^m$, not only is $T(K)$ closed, but there is also a neighbourhood around T whose members also preserve the closedness of K.

KEYWORDS: closed convex cone; linear mapping; stability; linear programming.

1. **Introduction**

In [3] we investigated when then continuous linear image of a closed convex cone in $\mathbb{R}^n$ is closed. This was motivated in part by the abstract versions of the Farkas lemma and the Krein-Rutman theorem as given in [2]. The closure of such conical images is central to duality theory in both semi-definite and conical linear programming [2, 3, 9, 10]. Recall that a nonempty set K of a vector space V is a *convex cone* if K is convex and for each $\lambda \in [0, \infty)$ and $x \in K$, $\lambda x \in K$. Although there are simple examples to show that the continuous linear image of a given closed convex cone K in $\mathbb{R}^n$ need not be closed (see [3, Example 1]), it was shown in [3] that in some sense, *for almost all* $T \in L(X, Y)$ —the space of all linear mappings acting between finite dimensional normed linear spaces X and Y, endowed with the operator norm—$T(K)$ is indeed closed in Y.

Specifically, in [3] we showed that for a given closed convex cone K in $\mathbb{R}^n$, $\text{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$ is dense in $L(\mathbb{R}^n, \mathbb{R}^m)$. We also showed that in general

$$\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$$

is not an open set. However, we did not address the question of the size of the set

$$L(\mathbb{R}^n, \mathbb{R}^m) \setminus \text{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$$

in terms of measure which, as shown in [7], can be quite distinct from being small in terms of category.

In this paper we remedy this situation by showing that

$$L(\mathbb{R}^n, \mathbb{R}^m) \setminus \text{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$$

Corresponding author: Warren B. Moors (moors@math.auckland.ac.nz).
Manuscript received March 18, 2010. Accepted July 2, 2010.
*This paper is dedicated to our colleague Claude Lemarechal on the occasion of his 65th birthday.

is *σ-porous*, a notion which is simultaneously small with regard to both measure and category. We were sure this was so at the time of writing [3] but the details are somewhat subtle. Along the way we shall show in Corollary 2.3 that

$$\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : \mathrm{rank}(T) < \min\{m, n\}\}$$

is σ-porous; a fact that is of independent interest since mappings with gradients of maximal rank admit inverse function theorems [5] and can be used to guarantee metric regularity [1].

2. Preliminaries

We start with some notation. For any x in a normed linear space $(X, \|\cdot\|)$ and $r \geq 0$ we shall denote by, $B(x;r)$ the set $\{y \in X : \|y - x\| < r\}$ and for any $M \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $R > 0$ we define $\gamma(x, R, M)$ to be the supremum over all $r \geq 0$ for which there exists $z \in \mathbb{R}^n$ such that $B(z, r) \subseteq B(x, R) \setminus M$. Then we define the *porosity of* M *at* x as

$$p(M, x) := \liminf_{R \to 0^+} \frac{\gamma(x, R, M)}{R}.$$

Further, we shall say that a set M is *porous at* x if $p(M, x) > 0$ and, moreover, if M is porous at each $x \in M$ then we shall say that M is *porous*. Finally, we shall say that M is *σ-porous* if it is a union of countably many porous sets. Porosity is a very natural geometric notion as the unfamiliar reader may discover by drawing some pictures in the plane.

It is easy to see that σ-porous sets enjoy some permanence properties. For example, if $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent norms on a vector space X then a subset M is σ-porous in $(X, \|\cdot\|)$ if, and only if, it is σ-porous in $(X, |\!|\!|\cdot|\!|\!|)$. In fact, an even stronger property is true.

Proposition 2.1. *Suppose If $(X, \|\cdot\|)$ and $(Y, |\!|\!|\cdot|\!|\!|)$ are normed linear spaces are normed spaces and $T : X \to Y$ is a continuous, open linear mapping. Then $T^{-1}(M)$ is σ-porous in $(X, \|\cdot\|)$ whenever $M \subseteq Y$ is σ-porous in $(Y, |\!|\!|\cdot|\!|\!|)$, .*

Since the notion of σ-porosity in finite dimensional normed linear spaces is insensitive to the particular choice of norm we shall henceforth (unless otherwise stated) assume that $\mathbb{R}^n$ is equipped with the Euclidean norm and that the space $L(X, Y)$ is equipped with the corresponding operator norm.

Our interest in σ-porosity stems from the fact that σ-porous sets are small in both a measure theoretic sense and in a Baire categorical sense, [11]. More precisely, a Lebesgue measurable set M in $\mathbb{R}^n$ that is σ-porous has Lebesgue measure zero and is at the same time a *first category* set (i.e., a countable union of nowhere dense sets). For further information—old and new—on Baire category the reader could consult [7] and [4].

In order to present our first theorem we need to introduce some matrix notation. For a $m \times n$ matrix A we shall denote by, $[A]_{ij}$ the $(ij)^{\text{th}}$ entry of the matrix A i.e., the entry in the i^{th} row and j^{th} column of the matrix A and by, A_{ij} the sub-matrix of A obtained by deleting the i^{th} row and j^{th} column of A. Finally, we shall denote by $M_{(m,n)}$ the set of all $m \times n$ matrices (over $\mathbb{R}$).

Theorem 2.2. *For each $n \in \mathbb{N}$, the set*

$$\{A \in M_{(n,n)} : \mathrm{rank}(A) = n - 1\}$$

is porous in $M_{(n,n)}$ with respect to any norm on $M_{(n,n)}$.

Proof: Let $\|\cdot\|$ be any norm on $M_{(n,n)}$, $M := \{A \in M_{(n,n)} : \mathrm{rank}(A) = n - 1\}$ and let $B \in M$. It will be sufficient, due to [11, page 517], to show that there is a neighbourhood U of B, subspaces H and F of $M_{(n,n)}$ such that (i) $\mathrm{Dim}(F) = 1$; (ii) $H \oplus F = M_{(n,n)}$ and (iii) a Lipschitz function $f : W \to F$ defined on a nonempty open subset W of H such that $U \cap M = \{x + f(x) : x \in W\}$.

Let $1 \leq i, j \leq n$ be chosen so that $\text{Det}(B_{ij}) \neq 0$. Let $H := \{A \in M_{(n,n)} : [A]_{ij} = 0\}$. Then H is a co-dimension 1 subspace $M_{(n,n)}$. Now define $E_{ij} \in M_{(n,n)}$ by,

$$[E_{ij}]_{i'j'} := \begin{cases} 1 & \text{if } (i,j) = (i',j') \\ 0 & \text{if } (i,j) \neq (i',j') \end{cases}$$

and let $F := \text{span}\{E_{ij}\}$. Finally, let $P_{ij} : M_{(n,n)} \to H$ be defined by, $P_{ij}(A) := A - [A]_{ij}E_{ij}$. Next, let W be any neighbourhood of $P_{ij}(B)$, with respect to the relative topology on H, such that $\text{Det}(A_{ij}) \neq 0$ for all $A \in W$ and let $U := (P_{ij})^{-1}(W)$. Note that for each $A \in U$, $\text{rank}(A) \geq n-1$. If, on the other hand, $A \in U$ and $\text{Det}(A) = 0$ (i.e., if $\text{rank}(A) = n-1$) then

$$\sum_{k=1}^{n} (-1)^{i+k}[A]_{ik}\text{Det}(A_{ik}) = 0$$

and so

$$[A]_{ij} = \frac{1}{\text{Det}(A_{ij})} \sum_{\substack{k=1 \\ k \neq j}}^{n} (-1)^{k+1-j}[A]_{ik}\text{Det}(A_{ik}).$$

Then we define $f : W \to F$ by,

$$f(A) := \left[\frac{1}{\text{Det}(A_{ij})} \sum_{\substack{k=1 \\ k \neq j}}^{n} (-1)^{k+1-j}[A]_{ik}\text{Det}(A_{ik}) \right] E_{ij}$$

and $g : W \to M$ by, $g(A) := A + f(A)$. Since f is C^1 on W, by possibly making W smaller, we can assume that f is Lipschitz on W with respect to $\|\cdot\|$. It is now routine to verify that $M \cap U = \{g(A) : A \in W\}$ since if $A \in M \cap U$ then $P_{ij}(A) \in W$ and $g(P_{ij}(A)) = A$. $\square$

In the following corollary we will repeatedly use the fact that if A' is a sub-matrix of a matrix A, obtained by deleting some rows and/or columns of A, then $\text{rank}(A') \leq \text{rank}(A)$.

We may now prove the result alluded to in the introduction:

Corollary 2.3 (Maximal Rank). *For each $(m,n) \in \mathbb{N}^2$, the set*

$$\{A \in M_{(m,n)} : \text{rank}(A) < \min\{m,n\}\}$$

is σ-porous in $M_{(m,n)}$ with respect to any norm on $M_{(m,n)}$.

Proof: Firstly, we may assume that $m, n \geq 2$. To show that $\{A \in M_{(m,n)} : \text{rank}(A) < \min\{m,n\}\}$ is σ-porous in $M_{(m,n)}$ it is sufficient to show that for each $1 \leq k < \min\{m,n\}$, $\{A \in M_{(m,n)} : \text{rank}(A) = k\}$ is σ-porous. Fix $1 \leq k < \min\{m,n\}$ and let Σ_k denote the set of all strictly increasing functions from $\{1, 2, \ldots, k+1\}$ into $\{1, 2, \ldots, m\}$ and let Σ_k^* denote the set of all strictly increasing functions from $\{1, 2, \ldots, k+1\}$ into $\{1, 2, \ldots, n\}$. For each $(\pi, \pi^*) \in \Sigma_k \times \Sigma_k^*$ and $A \in M_{(m,n)}$ let, $A^{(\pi,\pi^*)} \in M_{(k+1,k+1)}$ be the sub-matrix of A defined by, $[A^{(\pi,\pi^*)}]_{ij} := [A]_{\pi(i)\pi^*(j)}$ for each $1 \leq i, j \leq k+1$. Furthermore, let $N_k := \{A \in M_{(k+1,k+1)} : \text{rank}(A) = k\}$.

From Theorem 1 we know that N_k is σ-porous in $M_{(k+1,k+1)}$. For each $(\pi, \pi^*) \in \Sigma_k \times \Sigma_k^*$ let,

$$L_k^{(\pi,\pi^*)} := \{A \in M_{(m,n)} : A^{(\pi,\pi^*)} \in N_k\} = \{A \in M_{(m,n)} : \text{rank}(A^{(\pi,\pi^*)}) = k\}.$$

Since $L_k^{(\pi,\pi^*)}$ is the inverse image of N_k under the linear surjection $A \mapsto A^{(\pi,\pi^*)}$, $L_k^{(\pi,\pi^*)}$ is σ-porous in $M_{(m,n)}$. Now, from linear algebra we can deduce that

$$\{A \in M_{(m,n)} : \text{rank}(A) = k\} \subseteq \bigcup \{L_k^{(\pi,\pi^*)} : (\pi, \pi^*) \in \Sigma_k \times \Sigma_k^*\}$$

as required. $\square$

In order to expedite the proof of our main theorem we shall take this opportunity to record the following prerequisite result. To avoid confusion between scalars and vectors we shall, in the next lemma, denote vectors in bold; and the unit sphere in $\mathbb{R}^n$ by $S_{\mathbb{R}^n}$.

Lemma 2.4. *For any $\boldsymbol{a} := (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ and any $\boldsymbol{x} := (x_1, x_2, \ldots, x_n) \in S_{\mathbb{R}^n}$, there exists an operator $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that $T(\boldsymbol{x}) = \boldsymbol{a}$ and $\|T\| = \|\boldsymbol{a}\|$.*

Proof: Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by,

$$T(\boldsymbol{y}) := (a_1\boldsymbol{x} \cdot \boldsymbol{y}, a_2\boldsymbol{x} \cdot \boldsymbol{y}, \ldots, a_m\boldsymbol{x} \cdot \boldsymbol{y}).$$

Then for any $\boldsymbol{y} \in \mathbb{R}^n$ such that $\|\boldsymbol{y}\| \leq 1$,

$$\begin{aligned} \|T(\boldsymbol{y})\| &= \|(a_1\boldsymbol{x} \cdot \boldsymbol{y}, a_2\boldsymbol{x} \cdot \boldsymbol{y}, \ldots, a_m\boldsymbol{x} \cdot \boldsymbol{y})\| \\ &\leq \|(|a_1|, |a_2|, \ldots, |a_m|)\| \quad \text{since } |\boldsymbol{x} \cdot \boldsymbol{y}| \leq 1 \\ &= \|\boldsymbol{a}\|. \end{aligned}$$

Therefore $\|T\| \leq \|\boldsymbol{a}\|$. On the other hand,

$$\|T\| \geq \|T(\boldsymbol{x})\| = \|(a_1\boldsymbol{x} \cdot \boldsymbol{x}, a_2\boldsymbol{x} \cdot \boldsymbol{x}, \ldots, a_m\boldsymbol{x} \cdot \boldsymbol{x})\| = \|\boldsymbol{a}\|. \quad \square$$

There are various known sufficient conditions concerning when the continuous linear image of a closed convex cone K is closed. The best known is the classical result that it suffices that K be polyhedral [2]. The following is effectively a specialization of a *recession direction* [2] condition:

Proposition 2.5. [3, Proposition 3] *Let $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ and let K be a closed cone (not necessarily convex) in $\mathbb{R}^n$. If*

$$K \cap \ker(T) = \{0\}$$

then there exists a neighbourhood $\mathcal{N}$ of T in $L(\mathbb{R}^n, \mathbb{R}^m)$ such that $S(K)$ is closed in $\mathbb{R}^m$ for each $S \in \mathcal{N}$.

For a subset D of a vector space V, the *core* of D, denoted, $\mathrm{cor}(D)$, is the set of all points $d \in D$ where for each $x \in V \setminus \{d\}$ there exists an $0 < r < 1$ such that $\lambda x + (1 - \lambda)d \in D$ for all $0 \leq \lambda < r$. Clearly if the affine span $\mathrm{aff}(D) \neq V$ then $\mathrm{cor}(D) = \varnothing$. In this case the following concept is useful.

Given a subset C of a vector space V, the *intrinsic core* of C, denoted $\mathrm{icor}(A)$, is the set of all points $c \in C$ where for each $x \in \mathrm{aff}(C)$ there exists an $0 < r < 1$ such that $\lambda x + (1 - \lambda)c \in C$ for all $0 \leq \lambda < r$.

One of the most important properties of the intrinsic core is that if C is a convex subset of a finite dimensional vector space V then $\mathrm{icor}(C) \neq \varnothing$, [6, page 7]. In fact, if V is a finite dimensional topological vector space then $\mathrm{icor}(C)$ is dense in C for each convex subset C of the space V. Another important property of the core is that for a convex subset C of a finite dimensional topological vector space, $\mathrm{cor}(C) = \mathrm{int}(C)$, [2, Theorem 4.1.4].

The reason for our interest in the intrinsic core is that it provides the other sufficient condition that we shall need to apply:

Proposition 2.6 (Intrinsic core)**.** [3, Proposition 5] *Let Y be a normed linear space, $T : \mathbb{R}^n \to Y$ be a linear transformation and let K be a closed cone in $\mathbb{R}^n$. If*

$$\ker(T) \cap \mathrm{icor}(K) \neq \varnothing$$

then $T(K)$ is a finite dimensional linear subspace of Y and hence a closed convex cone.

Corollary 2.7. [3, Corollary 2] *The only way $T(K)$ can fail to be closed is if*

$$\ker(T) \cap K \subseteq K \setminus \mathrm{icor}(K)$$

and that at the same time $\ker(T) \cap K$ *is not a linear subspace. In particular,* $\ker(T) \cap K \neq \{0\}$.

3. **Main Results**

We require one more lemma:

Lemma 3.1. *Let Y be an n-dimensional normed linear space and let K be a closed convex cone in Y. Then $L(Y,\mathbb{R}^m)\setminus \text{int}\{T \in L(Y,\mathbb{R}^m) : T(K) \text{ is closed}\}$ is σ-porous in $L(Y,\mathbb{R}^m)$ if, and only if, $L(\mathbb{R}^n,\mathbb{R}^m)\setminus \text{int}\{T \in L(\mathbb{R}^n,\mathbb{R}^m) : T(\varphi(K)) \text{ is closed}\}$ is σ-porous in $L(\mathbb{R}^n,\mathbb{R}^m)$ where $\varphi : Y \to \mathbb{R}^n$ is any linear bijection from Y onto $\mathbb{R}^n$.*

Proof: Let $\varphi : Y \to \mathbb{R}^n$ be a linear bijection from Y onto $\mathbb{R}^n$ and let $\varphi^\# : L(\mathbb{R}^n,\mathbb{R}^m) \to L(Y,\mathbb{R}^m)$ be defined by, $\varphi^\#(T) := T \circ \varphi$. Then $\varphi^\#$ is an isomorphism from $L(\mathbb{R}^n,\mathbb{R}^m)$ onto $L(Y,\mathbb{R}^m)$ and

$$\{T \in L(Y,\mathbb{R}^m) : T(K) \text{ is closed}\} = \varphi^\#(\{T \in L(\mathbb{R}^n,\mathbb{R}^m) : T(\varphi(K)) \text{ is closed}\}). \quad \square$$

The next result shows—as promised—that although it is not true that, if $T(K)$ is closed for some closed convex cone K then $S(K)$ is closed for all S in some neighbourhood of T, it is "almost" true, in the sense that *for "almost all" $T \in L(\mathbb{R}^n,\mathbb{R}^m)$ if $T(K)$ is closed then there exists a neighbourhood $\mathcal{W}$ of T such that $S(K)$ is closed for all $S \in \mathcal{W}$*. More precisely:

Theorem 3.2. *Suppose that K is a closed convex cone in $\mathbb{R}^n$ then*

$$L(\mathbb{R}^n,\mathbb{R}^m)\setminus \text{int}\{T \in L(\mathbb{R}^n,\mathbb{R}^m) : T(K) \text{ is closed}\}$$

is a σ-porous set in $L(\mathbb{R}^n,\mathbb{R}^m)$.

Proof: Let $Y := K - K$. We shall consider first the case when $Y = \mathbb{R}^n$. Let $\mathcal{M}$ be the family of all linear mappings $T \in L(\mathbb{R}^n,\mathbb{R}^m)$ with maximal rank (i.e., $\text{rank}(T) = \min\{m,n\}$).

It is easy to verify that $\mathcal{M}$ is an open subset of $L(\mathbb{R}^n,\mathbb{R}^m)$. Let $\varphi : M_{(m,n)} \to L(\mathbb{R}^n,\mathbb{R}^m)$ be defined by, $\varphi(A)(\boldsymbol{x}) := A\boldsymbol{x}$. Then φ is an isomorphism from $M_{(m,n)}$ onto $L(\mathbb{R}^n,\mathbb{R}^m)$. Moreover, $\varphi^{-1}(\mathcal{M}) = \{A \in M_{(m,n)} : A \text{ has maximal rank}\}$. Therefore, from Corollary 1, $L(\mathbb{R}^n,\mathbb{R}^m)\setminus\mathcal{M}$ is σ-porous in $L(\mathbb{R}^n,\mathbb{R}^m)$. Hence to show that $L(\mathbb{R}^n,\mathbb{R}^m)\setminus \text{int}\{T \in L(\mathbb{R}^n,\mathbb{R}^m) : T(K) \text{ is closed}\}$ is σ-porous in $L(\mathbb{R}^n,\mathbb{R}^m)$ it is sufficient to show that $\mathcal{M}\setminus \text{int}\{T \in L(\mathbb{R}^n,\mathbb{R}^m) : T(K) \text{ is closed}\}$ is σ-porous in $L(\mathbb{R}^n,\mathbb{R}^m)$; which is what we shall do. There are two cases:

(i) If $n \leq m$ then each member of $\mathcal{M}$ is one-to-one and so $\ker(T) \cap K = \{0\}$ for each $T \in \mathcal{M}$ and thus we are done by Proposition 2.

(ii) Hence we shall assume that $m < n$. We now define $P : \mathcal{M} \to L(\mathbb{R}^n,\mathbb{R}^n)$ by, $P(S) := I - S^*(SS^*)^{-1}S$, where I is the identity mapping on $\mathbb{R}^n$ and S^* is the conjugate of S, i.e., $S^* \in L(\mathbb{R}^m,\mathbb{R}^n)$ and $S^*(\boldsymbol{y}) \cdot \boldsymbol{x} = \boldsymbol{y} \cdot S(\boldsymbol{x})$ for all $\boldsymbol{y} \in \mathbb{R}^m$ and all $\boldsymbol{x} \in \mathbb{R}^n$. It is routine to check that:

- (i) for each $S \in \mathcal{M}$, $P(S)$ is well-defined, i.e., $(SS^*)^{-1}$ exists;
- (ii) P is C^1 on $\mathcal{M}$ and hence locally Lipschitz on $\mathcal{M}$;
- (iii) for each $S \in \mathcal{M}$, $P(S)$ is the projection of $\mathbb{R}^n$ onto $\ker(S)$.

For each $n \in \mathbb{N}$, let

$$L_n := \{S \in \mathcal{M} : \text{ there exists an open neighbourhood } N \text{ of } S \text{ such that } P|_N \text{ is } n\text{-Lipschitz}\}$$

Now each L_n is an open subset of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{n=1}^{\infty} L_n$. So to show that

$$\mathcal{M}\setminus \text{int}\{T \in L(\mathbb{R}^n,\mathbb{R}^m) : T(K) \text{ is closed}\}$$

is σ-porous it is sufficient to show that for each $n \in \mathbb{N}$,

$$E_n := L_n \setminus \text{int}\{T \in L(\mathbb{R}^n,\mathbb{R}^m) : T(K) \text{ is closed}\}$$

is porous.

To this end, let us fix $n \in \mathbb{N}$ and consider $T \in E_n$. Since

$$T \notin \text{int}\{T \in L(\mathbb{R}^n,\mathbb{R}^m) : T(K) \text{ is closed}\},$$

it follows that

$$\{0\} \neq \ker(T) \cap K \subseteq K \setminus \text{icor}(K).$$

Choose $x \in \ker(T) \cap K$ such that $\|x\| = 1$. Note that this is possible since $\ker(T) \cap K$ is a nontrivial cone. Now select $y \in \text{int}(K) = \text{cor}(K) = \text{icor}(K) \neq \varnothing$ such that $\|y\| = 1$. Also choose $0 < r < 1$ such that $B(y; r) \subseteq \text{int}(K)$. We claim that

$$p(E_n, T) \geq \alpha := r/(8n[\|T\| + 1]) > 0.$$

Let $0 < R_0 < 1$ be chosen so that P is n-Lipschitz on $B(T; R_0)$ and for each $0 < R < R_0$ let, $\lambda_R := R/(8[\|T\| + 1])$ and $z_R := \lambda_R y + (1 - \lambda_R)x$. Now for each $0 < \lambda < 1$,

$$B(x + \lambda(y - x); \lambda r) = \lambda B(y, r) + (1 - \lambda)x \subseteq K$$

since K is convex and so $B(x + \lambda(y - x); \lambda r) \subseteq \text{int}(K)$. In particular, $B(z_R; \lambda_R r) \subseteq \text{int}(K)$. Now,

$$\|T(z_R)\| = \|T(x + \lambda_R(y - x)\| = \lambda_R \|T(y - x)\| \leq \lambda_R \|T\| \|y - x\| \leq 2\lambda_R \|T\| < R/4$$

and $1 \geq \|z_R\| = \|x + \lambda_R(y - x)\| \geq \|x\| - \lambda_R \|y - x\| \geq 1 - 2\lambda_R > 1 - 2(1/8) = 3/4$. By Lemma 1 there exists a $S'_R \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that $S'_R(z_R) = T(z_R)$ and $\|S'_R\| < R/3$. Let $S_R := T - S'_R$. Then $\|T - S_R\| < R/3$, $B(S_R; \alpha R) \subseteq B(T; R)$ (since $\alpha \leq 1/8$) and $S_R(z_R) = 0$. To complete the proof in this case we argue below that this holds for all $0 < R < R_0$, $p(E_n, T) \geq \alpha > 0$.

Claim: *For each* $0 < R < R_0$, $B(S_R; \alpha R) \subseteq$ *int*$\{S \in L(\mathbb{R}^n, \mathbb{R}^m) : S(K)$ *is closed*$\}$.

Proof of Claim: To see this, suppose that $S'' \in B(S_R; \alpha R)$, that is

$$\|S_R - S''\| < rR/(8n[\|T\| + 1]).$$

To show that

$$S'' \in \text{int}\{S \in L(\mathbb{R}^n, \mathbb{R}^m) : S(K) \text{ is closed}\}$$

it is sufficient to show that $\ker(S'') \cap \text{int}(K) \neq \varnothing$. In fact, since $P(S'')(z_R) \in \ker(S'')$ it is enough to show that $P(S'')(z_R) \in \text{int}(K)$. Now

$$\begin{aligned}
\|P(S'')(z_R) - z_R\| &= \|P(S'')(z_R) - P(S_R)(z_R)\| \quad \text{since } z_R \in \ker(S_R) \\
&\leq \|P(S'') - P(S_R)\| \|z_R\| \\
&\leq \|P(S'') - P(S_R)\| \quad \text{since } \|z_R\| \leq 1 \\
&\leq n\|S'' - S_R\| \quad \text{since } P \text{ is } n\text{-Lipschitz on } B(S_R, \alpha R) \\
&< \frac{nrR}{8n(\|t\| + 1)} = \frac{rR}{8(\|T\| + 1)} = \lambda_R r.
\end{aligned}$$

Therefore, $P(S'')(z_R) \in B(z_R; \lambda_R r) \subseteq \text{int}(K)$. Thus,

$$S'' \in \text{int}\{S \in L(\mathbb{R}^n, \mathbb{R}^m) : S(K) \text{ is closed}\};$$

which concludes the proof of the claim. □

In the case when Y is a proper subspace of $\mathbb{R}^n$, it follows from Lemma 2 and the previous case that $L(Y, \mathbb{R}^m) \setminus \text{int}\{T \in L(Y, \mathbb{R}^m) : T(K) \text{ is closed}\}$ is σ-porous in $L(Y, \mathbb{R}^m)$. To finish the proof we consider the linear surjection $R : L(\mathbb{R}^n, \mathbb{R}^m) \to L(Y, \mathbb{R}^m)$ defined by, $R(T) := T|_Y$. Then by setting $E := L(\mathbb{R}^n, \mathbb{R}^m) \setminus \text{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$ we have that:

$$\begin{aligned}
E &= L(\mathbb{R}^n, \mathbb{R}^m) \setminus \text{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T|_Y(K) \text{ is closed}\} \\
&= R^{-1}(L(Y, \mathbb{R}^m)) \setminus \text{int}[R^{-1}(\{S \in L(Y, \mathbb{R}^m) : S(K) \text{ is closed}\})] \\
&= R^{-1}(L(Y, \mathbb{R}^m)) \setminus R^{-1}(\text{int}\{S \in L(Y, \mathbb{R}^m) : S(K) \text{ is closed}\}) \quad (*) \\
&= R^{-1}(L(Y, \mathbb{R}^m) \setminus \text{int}\{S \in L(Y, \mathbb{R}^m) : S(K) \text{ is closed}\}).
\end{aligned}$$

The equality in line $(*)$ follows from the general fact that for any continuous and open mapping $R : X \to Y$ acting between topological spaces and any set $A \subseteq Y$, $R^{-1}(\text{int}(A)) = \text{int}(R^{-1}(A))$. The proof is now completed by appealing to Proposition 1. □

Further results on the images of closed convex cones under linear mappings may be found in [8, 9].

References

[1] H. H. Bauschke, J. M. Borwein and P. Tseng, “Metric regularity, strong CHIP, and CHIP are distinct properties,” J. Convex Analysis, 7 (2000), 395–412.

[2] J. M. Borwein and A. S. Lewis, Convex analysis and nonlinear optimization. Theory and examples. Second edition. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 3, Springer, New York, 2006.

[3] J. M. Borwein and W. B. Moors, “Stability of closedness of convex cones under linear mappings,” J. Convex Anal. 16 (2009), 699–705.

[4] J. Cao and W. B. Moors, “A survey on topological games and their applications in analysis,” RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 100 (2006), no. 1-2, 39–49.

[5] A. Dontchev and R. T. Rockafellar, Implicit Functions and Solution Mappings: A View from Variational Analysis, Springer-Verlag, 2009.

[6] John R. Giles, Convex analysis with applications in the differentiation of convex functions. Research Notes in Mathematics 58 Pitman (Advanced Publishing Program) Boston, Mass-London, 1982.

[7] John C. Oxtoby, Measure and Category. A survey of the analogies between topological and measure spaces, Second edition. Graduate Texts in Mathematics, 2. Springer-Verlag, New York-Berlin, 1980.

[8] Gabor Pataki, “On the closedness of the linear image of a closed convex cone,” Math of Oper. Res., **32** (2007), 395–412.

[9] Gabor Pataki and L. Tuncel, “On the Generic Properties of Convex Optimization Problems in Conic Form,” Mathematical Programming A, 89 (2001), 449–457.

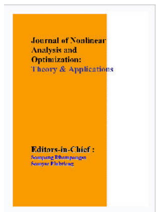
[10] H. Wolkowicz, R. Saigal, and L. Vandenberghe, Handbook of Semidefinite Programming: Theory, Algorithms, and Applications, International Series in Operations Research & Management Science, 27, Springer-Verlag, 2001.

[11] L. Zajíček, “On σ-porous sets in abstract spaces,” Abstract Applied Anal. 2005 (2005), 509–534.

[1]CENTRE FOR COMPUTER ASSISTED MATHEMATICS AND ITS APPLICATIONS (CARMA), UNIVERSITY OF NEWCASTLE, CALLAGHAN, NSW 2308, AUSTRALIA.
Email address: jborwein@newcastle.edu.au

[2] DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF AUCKLAND, PRIVATE BAG 92019, AUCKLAND, NEW ZEALAND.
Email address: moors@math.auckland.ac.nz

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 9–16
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Asymptotic stability of stochastic pantograph differential equations with Markovian switching

Feng Jiang[1], Hua Yang[2], Shenghua Wang[3]

ABSTRACT: In this paper, we investigate the almost surely asymptotic stability of the nonlinear stochastic pantograph differential equations (SPDEs) with Markovian switching under the weakened linear growth condition. Linear SPDEs with Markovian switching and nonlinear examples with Markovian switching will be discussed to illustrate the theory.

KEYWORDS: Stochastic pantograph differential equations; Asymptotic stability; Generalized Itô formula; Markov chain.

1. **Introduction**

Recently, the study of stochastic pantograph differential equations (SPDEs) has received a great deal of attention. For example, Baker and Buckwar [1] gave the necessary analytical theory for existence and uniqueness of a strong solution of the linear stochastic pantograph equation, and of strong approximations to the solution obtained by a continuous extension of the θ-Euler scheme. They also proved that the numerical solution produced by the continuous θ-method converges to the true solution with order $1/2$. Appleby and Buckwar [2] studied the asymptotic growth and delay properties of solutions of the linear stochastic pantograph equation. They give sufficient conditions on the parameters for solutions to grow at a polynomial rate on pth mean and in the almost sure sense. Fan et al. [3] investigated the existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations under the local Lipschitz condition and the linear growth condition. Fan et al. [4] investigated the αth moment asymptotical stability of the analytic solution and the numerical methods for the stochastic pantograph equation by using the Razumikhin technique. Li et al. [5] investigated the convergence of the Euler method of the stochastic pantograph differential equations with Markovian switching under the weaker conditions.

The classical stochastic stability theory deals with not only moment stability but also almost sure stability [6–10]. However, to the best of our knowledge, most of the existing results on the linear stochastic pantograph differential equations[1,2,4,11] are about the moment stability, while little is known on the almost surely asymptotic stability for SPDEs with Markovian switching under the non-linear growth condition which is the main topic of the present paper.

Corresponding author: Hua Yang (huay20@163.com)
Manuscript received March 16, 2010. Accepted July 2, 2010.

The paper is organised as follows. In Section 2, we introduce the SPDEs with Markovian switching. We investigate the almost surely asymptotic stability for the stochastic pantograph differential equations with Markovian switching under the non-linear growth condition in Section 3. In Section 4, Some examples are discussed to illustrate the theory.

2. SPDEs with Markovian switching

Throughout this paper,we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all P-null sets). Moreover, $|\cdot|$ is the Euclidean norm in $\mathcal{R}^n$. Let x_0 be an $\mathcal{F}_0-$measurable R^n-valued random variable such that $E|x_0|^2 < \infty$. Let $w(t) = (w_t^1, \cdots, w_t^m)^{\mathrm{T}}, t \geq 0$, be a m-dimensional Brownian motion defined on the probability space.

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta) & \text{if } i = j, \end{cases}$$

where $\delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is useful to recall that a continuous-time Markov chain $r(t)$ with generator $\Gamma = \{\gamma_{ij}\}_{N\times N}$ can be represented as a stochastic integral with respect to a Poisson random measure ([12, 13]).

$$dr(t) = \int_R \overline{h}(r(t-), y)v(dt, dy),\ t \geq 0 \tag{1}$$

with initial value $r(0) = i_0 \in S$, where $v(dt, dy)$ is a Poisson random measure with intensity $dt \times m(dy)$ in which m is the Lebesgue measure on R while the explicit definition of $\overline{h} : S \times R \to R$ can be found in ([12, 13]) but we will not need it in this paper.

Consider an n-dimensional stochastic pantograph differential equations with Markovian switching

$$dx(t) = f(t, x(t), x(qt))dt + f(t, x(t), x(qt))dw(t). \tag{2}$$

on $t \geq 0$ with initial data $x(0) = x_0, 0 < q < 1$ and $r(0) = i_0 \in S$, where $f : R_+ \times R^n \times R^n \times S \to R^n$ and $g : R_+ \times R^n \times R^n \times S \to R^{n\times m}$.

In this paper, the following hypothesis is imposed on the coefficients f and g.

Assumption H. Both f and g satisfy the local Lipschitz condition. For each integer $h \geq 1$ and $i \in S$, there exists a positive constant L_h such that

$$|f(t, x_1, x_2, i) - f(t, y_1, y_2, i)| \vee |g(t, x_1, x_2, i) - g(t, y_1, y_2, i)| \leq L_h(|x_1 - y_1| + |x_2 - y_2|)$$

for $x_1, x_2, y_1, y_2 \in R^n$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq h$. Moreover,

$$\sup\{|f(t, 0, 0, i)| \vee |g(t, 0, 0, i)| : t \geq 0, i \in S\} < \infty.$$

In general, this hypothesis will only guarantee a unique maximal local solution to Eq. (2) for any given initial value x_0 and i_0. However, the additional conditions imposed in our main result, Theorem 3.1, will guarantee that this maximal local solution is in fact a unique global solution (see Lemma 3.2), which is denoted by $x(t; x_0; i_0)$ in this paper. The main purpose of this paper is to discuss the almost surely asymptotic stability of the solution([6, 14]).

To state our main result, we will need a few more notations. Let $C(R^n; R_+)$ and $C(R_+ \times R^n; R_+)$ denote the families of all continuous nonnegative functions defined on R^n and $R_+ \times R^n$, respectively. Moreover, let $\mathcal{K}$ denote the class of continuous increasing functions μ from R_+ to R_+ with $\mu(0) = 0$. Let $\mathcal{K}_\infty$ denote the class of functions μ with $\mu(s) \to \infty$ as $s \to \infty$. Functions in $\mathcal{K}$ and $\mathcal{K}_\infty$ are called class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions, respectively. If $\mu \in \mathcal{K}$, its iverse function is denoted by μ^{-1} with domain $[0, \mu(\infty))$. We also denote by $L^1(R_+; R_+)$ the family of all functions $\gamma : R_+ \to R_+$ such that $\int_0^\infty \gamma(t)dt < \infty$. If E is a subset of R^n, denote by $d(x, E)$ the Haussdorff semi-distance between $x \in R^n$ and the set E,namely $d(x, E) = \inf_{y\in E} |x - y|$.

If W is a real-valued function defined on R^n, then its kernel is denoted by $Ker(W)$, namely $Ker(W) = \{x \in R^n : W(x) = 0\}$. Let $C^{1,2}(R_+ \times R^n \times S; R_+)$ denote the family of all nonnegative functions $V(t,x,i)$ on $R_+ \times R^n \times S$ which are continuously twice differentiable in x and once differentiable in t. If $V \in C^{1,2}(R_+ \times R^n \times S; R_+)$, define an operator LV from $R_+ \times R^n \times R^n \times S$ to R by

$$
\begin{aligned}
LV(t,x,y,i) &= V_t(t,x,i) + V_x(t,x,i)f(t,x,y,i) + \frac{1}{2}trace[g^T(t,x,y,i)V_{xx}(t,x,i)g(t,x,y,i)] \\
&\quad + \sum_{j=1}^{N} \gamma_{ij} V(t,x,j), \qquad (3)
\end{aligned}
$$

where
$V_t(t,x,i) = \frac{\partial V(t,x,i)}{\partial t}$, $V_x(t,x,i) = (\frac{\partial V(t,x,i)}{\partial x_1}, ..., \frac{\partial V(t,x,i)}{\partial x_n})$, $V_{xx}(t,x,i) = (\frac{\partial^2 V(t,x,i)}{\partial x_i \partial x_j})_{n\times n}$.
For the convenience of the reader we cite the generalized Itô's formula ([14]): If $V \in C^{1,2}(R_+ \times R^n \times S)$, then for any $t \geq 0$

$$
\begin{aligned}
V(t,x(t),r(t)) &= V(0,x(0),r(0)) + \int_0^t LV(s,x(s),x(qs),r(s))ds + \int_0^t V_x(s,x(s),r(s))dw(s) \\
&\quad + \int_0^t \int_R (V(s,x(s),i_0 + \overline{h}(r(s-),l)) - V(s,x(s),r(s)))u(ds,dl), \qquad (4)
\end{aligned}
$$

where $u(ds,dl) = v(ds,dl) - m(dl)ds$ is a martingale measure.

To establish our main result for locating limit sets of the solutions of the stochastic pantograph equations with Markovian switching, let us cite the useful convergence theorem of nonnegative semi-martingale ([15])as a lemma.

Lemma 2.1 *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ξ be a nonnegative $\mathcal{F}_0-$measurable random variable. Define*

$$X(t) = \xi + A(t) - U(t) + M(t) \text{ for } t \geq 0.$$

If $X(t)$ is nonnegative, then

$$\{\lim_{t\to\infty} A(t) < \infty\} \subset \{\lim_{t\to\infty} X(t) < \infty\} \cap \{\lim_{t\to\infty} U(t) < \infty\} \text{ a.s.},$$

where $B \subset D$ a.s. means $P(B \cap D_c) = 0$. In particular, if $\lim_{t\to\infty} A(t) < \infty$ a.s., then, with probability one,

$$\lim_{t\to\infty} X(t) < \infty, \ \lim_{t\to\infty} U(t) < \infty \text{ and } -\infty < \lim_{t\to\infty} M(t) < \infty.$$

That is , all of the three processes $X(t), U(t)$ and $M(t)$ converge to finite random variable.

3. **Asymptotic stability**

With the above notations, we can now state our main result.

Theorem 3.1 *Let (H) hold. Assume that there are functions $V \in C^{1,2}(R_+ \times R^n \times S; R_+), \gamma \in L^1(R_+; R_+)$ and $w_1, w_2 \in C(R^n; R_+)$ such that*

$$LV(t,x,y,i) \leq \gamma(t) - w_1(x) + qw_2(y) \qquad (5)$$

for all $(t,x,y,i) \in R_+ \times R^n \times R^n \times S$ and

$$w_1(0) = w_2(0) = 0, \ w_1(x) > w_2(x) \text{ for all } x \neq 0. \qquad (6)$$

and

$$\lim_{|x|\to\infty} \left[\inf_{(t,i)\in R_+ \times S} V(t,x,i) \right] = \infty. \qquad (7)$$

Then for any initial value x_0,

$$\lim_{t\to\infty} x(t; x_0, i_0) = 0 \text{ a.s.} \qquad (8)$$

That is, the solution of Eq. (2) is almost surely asymptotically stable.

To prove this theorem, we can also give the following lemma by the standard truncated technique (see e.g. [14]).

Lemma 3.2 *Under the conditions of Theorem 3.1, for any initial vulue x_0 and $r(0) = i_0 \in S$, Eq. (2) has a unique global solution.*

Let us now begin to prove our main result.

Proof of Theorem 3.1 We divide the proof into three steps.

Step 1. Fix any x_0 and i_0 and write $x(t; x_0, i_0) = x(t)$ for simplicity. By the generalized Itô's formula, (5) and (6) we derive that

$$\begin{aligned} V(t,x(t),r(t)) &\le V(0,x(0),r(0)) + M(t) + \int_0^t [\gamma(s) - w_1(x(s)) + qw_2(x(qs))]ds \\ &\le V(0,x(0),r(0)) + \int_0^t \gamma(s)ds - \int_0^t [w_1(x(s)) - w_2(x(s))]ds + M(t) \end{aligned} \tag{9}$$

where

$$M(t) = \int_0^t V_x(s,x(s),r(s))dw(s) + \int_0^t \int_R (V(s,x(s),i_0 + \bar{h}(r(s-),l)) - V(s,x(s),r(s)))u(ds,dl),$$

which is a continuous local martingale with $M(0) = 0$ a.s. Applying Lemma 2.1 we immediately obtain

$$\limsup_{t\to\infty} V(t,x(t),r(t)) < \infty \text{ a.s.} \tag{10}$$

Moreover, taking the expectations on both sides of (9) and letting $t \to \infty$, we obtain that

$$E\int_0^\infty [w_1(x(s)) - w_2(x(s))]ds < \infty \text{ a.s.} \tag{11}$$

This implies

$$\int_0^\infty [w_1(x(s)) - w_2(x(s))]ds < \infty \text{ a.s.} \tag{12}$$

Step 2. Set $\omega = w_1 - w_2$. Clearly, $\omega \in C(R^n; R_+)$. It is straightforward to see from (12) that

$$\liminf_{t\to\infty} \omega(x(t)) = 0 \text{ a.s.} \tag{13}$$

We now claim that

$$\lim_{t\to\infty} \omega(x(t)) = 0 \text{ a.s.} \tag{14}$$

If this is false, then

$$P\{\limsup_{t\to\infty} \omega(x(t)) > 0\} > 0.$$

Hence, there is a number $\varepsilon > 0$ such that

$$P(\Omega_1) \ge 3\varepsilon, \tag{15}$$

where

$$\Omega_1 = \{\limsup_{t\to\infty} \omega(x(t)) > 2\varepsilon\}.$$

It is easy to observe from (10) and continuity of both the solution $x(t)$ and the function $V(t,x,i)$ that

$$\sup_{0\le t<\infty} V(t,x(t),r(t)) < \infty \text{ a.s.}$$

Define $\rho : R_+ \to R_+$ by

$$\rho(r) = \inf_{|x|\ge r, 0\le t<\infty} V(t,x(t),i) \text{ for } r \ge 0.$$

Obviously,

$$\sup_{0\le t<\infty} \rho(|x(t)|) \le \sup_{0\le t<\infty} V(t,x(t),r(t)) < \infty \text{ a.s.}$$

On the other hand, by (7) we have

$$\lim_{r\to\infty} \rho(r) = \infty.$$

Therefore

$$\sup_{0\le t<\infty} |x(t) < \infty \text{ a.s.} \tag{16}$$

Recalling the boundedness of the initial value we can then find a positive number h, which depends on ε, sufficiently large for $|x_0| < h$, while

$$P(\Omega_2) \ge 1-\varepsilon, \tag{17}$$

where

$$\Omega_2 = \{\sup_{0\le t<\infty} |x(t)| < h\}.$$

It is easy to see from (15)and (17) that

$$P(\Omega_1 \cap \Omega_2) \ge 2\varepsilon. \tag{18}$$

We now define a sequence of stopping times,

$$\tau_h = \inf\{t \ge 0 : |x(t)| \ge h\},$$
$$\sigma_1 = \inf\{t \ge 0 : \omega(x(t)) \ge 2\varepsilon\},$$
$$\sigma_{2k} = \inf\{t \ge \sigma_{2k-1} : \omega(x(t)) \le \varepsilon\},\ k = 1,2,...,$$
$$\sigma_{2k+1} = \inf\{t \ge \sigma_{2k} : \omega(x(t)) \ge \varepsilon\},\ k = 1,2,...,$$

where throughout this paper e set $\inf \phi = \infty$. Note from (13) and the definition of Ω_1 and Ω_2 that if $\varpi \in \Omega_1 \cap \Omega_2$, then

$$\tau_h = \infty \text{ and} \sigma_k < \infty\ \forall k \ge 1. \tag{19}$$

Let I_A is denote the indicator function of set A, using the fact $\sigma_{2k} < \infty$ whenever $\sigma_{2k-1} < \infty$ and (13), by (11) we can compute

$$\begin{aligned} \infty &> E\int_0^\infty \omega(x(t))dt \\ &\ge \sum_{k=1}^\infty E\Big[I_{\{}\sigma_{2k-1} < \infty, \sigma_{2k} < \infty, \tau_h = \infty\} \int_{\sigma_{2k-1}}^{\sigma_{2k}} \omega(x(t))dt\Big] \\ &\ge \varepsilon \sum_{k=1}^\infty E\Big[I_{\{}\sigma_{2k-1} < \infty, \tau_h = \infty\}(\sigma_{2k} - \sigma_{2k-1})\Big]. \end{aligned} \tag{20}$$

On the other hand, by hypothesis (H),there exists a constant $K_h > 0$ such that $|f(t,x,y,i)|^2 \vee |g(t,x,y,i)|^2 \le K_h$ whenever $|x| \vee |y| \le h$. By Hölder's inequality and Doob's martingale inequality, we easily compute

$$E\Big[I_{\{}\sigma_{2k-1} \wedge \tau_h < \infty\} \sup_{0\le t\le T} |x(\tau_h \wedge (\sigma_{2k-1}+t)) - x(\tau_h \wedge \sigma_{2k-1})|^2\Big] \le 2K_h(T+4)T. \tag{21}$$

Since $\omega(\cdot)$ is continuous in R^n, it must be uniformly continuous in the closed ball $\overline{S}_h = \{x \in R^n : |x| \le h\}$. We can therefore choose $\delta = \delta(\varepsilon) > 0$ so small such that

$$|\omega(x) - \omega(x)| < \varepsilon \text{ whenever } x,y \in \overline{S}_h, |x-y| < \delta. \tag{22}$$

We furthermore chose $T = T(\varepsilon,\delta,h) > 0$ sufficiently small for $2K_h(T+4)T/\delta^2 < \varepsilon$. It then follows from (21) that

$$P\Big(\{\sigma_{2k-1} \wedge \tau_h < \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\tau_h \wedge (\sigma_{2k-1}+t)) - x(\tau_h \wedge \sigma_{2k-1})| \ge \delta\Big\}\Big) \le \frac{2K_h(T+4)T}{\delta^2} < \varepsilon.$$

This together with (18) and (19) yields

$$P\Big(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \Big\{ \sup_{0\le t\le T} |x(\sigma_{2k-1}+t) - x(\sigma_{2k-1})| \ge \delta \Big\}\Big) \le \varepsilon.$$

and

$$P\Big(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \Big\{ \sup_{0\le t\le T} |x(\sigma_{2k-1}+t) - x(\sigma_{2k-1})| < \delta \Big\}\Big) \ge \varepsilon.$$

Using (22), we derive that

$$(23)\quad P\Big(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \Big\{ \sup_{0\le t\le T} |\omega(x(\sigma_{2k-1}+t)) - \omega(x(\sigma_{2k-1}))| < \varepsilon Big\}\Big) \ge \varepsilon.$$

Set

$$\overline{\Omega}_k = \Big\{ \sup_{1\le t\le T} |\omega(x(\sigma_{2k-1}+t)) - \omega(x(\sigma_{2k-1}))| < \varepsilon \Big\}.$$

Noting that

$$\sigma_{2k}(\varpi) - \sigma_{2k-1}(\varpi) \ge T \text{ if } \varpi \in \{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \overline{\Omega}_k,$$

we derive from (20) and (23) that

$$\begin{aligned}
\infty &> \varepsilon \sum_{k=1}^{\infty} E[I_{\{}\sigma_{2k-1} < \infty, \tau_h = \infty\}(\sigma_{2k} - \sigma_{2k-1})] \\
&\ge \varepsilon \sum_{k=1}^{\infty} E[I_{\{\sigma_{2k-1}<\infty,\tau_h=\infty\}\cap\overline{\Omega}_k}(\sigma_{2k} - \sigma_{2k-1})] \\
&\ge \varepsilon T \sum_{k=1}^{\infty} P(\{\sigma_{2k-1} < \infty, \tau_h = \infty\} \cap \overline{\Omega}_k) \\
&\ge \varepsilon T \sum_{k=1}^{\infty} \varepsilon = \infty, \qquad (24)
\end{aligned}$$

which is a contradiction. So (14) must hold.

Step 3. We observe from (14) and (16) there is an $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that

$$(25)\qquad \lim_{t\to\infty} \omega(x(t,\varpi)) = 0 \text{ and } \sup_{0\le t<\infty} |x(t,\varpi)| < \infty \text{ for all } \varpi \in \Omega_0.$$

We shall now show that

$$(26)\qquad \lim_{t\to\infty} x(t,\varpi) = 0 \ \forall \varpi \in \Omega_0.$$

If this is false, then there is some $\hat{\sigma} \in \Omega_0$ such that

$$\limsup_{t\to\infty} |x(t,\hat{\sigma})| > 0,$$

whence there is a subsequence $\{x(t_k,\hat{\sigma})\}_{k\ge1}$ of $\{x(t,\hat{\sigma})\}_{t\ge0}$ such that

$$|x(t_k,\hat{\sigma})| \ge \alpha \forall\, k \ge 1$$

for some $\alpha > 0$. Since $\{x(t_k,\hat{\sigma})\}_{k\ge1}$ is bounded so there must be an increasing subsequence $\{\bar{t}_k\}_{k\ge1}$ such that $\{x(\bar{t}_k,\varpi)\}_{k\ge1}$ converges to some $z \in R^n$ with $|z| \ge \alpha$. Hence

$$\omega(z) = \lim_{k\to\infty} \omega(x(t_k,\varpi)) > 0.$$

However, by (25), $\omega(z) = 0$. This is a contradiction and hence (26) must hold. This implies that the solution of Eq. (2) is almost surely asymptotically stable and the proof is therefore complete.

It is not difficult to observe from the proof of Theorem 3.1 that the following more general result holds.

Theorem 3.3 *Assume that all the conditions of Theorem 3.1 hold except Condition (6) is replaced by*

$$w_1(x) \ge w_2(x),\ x \in R^n.$$

Then

$$Ker(w_1 - w_2) \neq \phi \text{ and } \lim_{t\to\infty} d(x(t;x_0,i_0), Ker(w_1 - w_2)) = 0 \text{ a.s.}$$

4. Examples

In this section we discuss a linear example and a nonlinear example to illustrate our theory. In the following examples we let $w(t)$ be a scalar Brownian motion.

Example 4.1 Let $r(t)$ be a right-continuous Markov chain. Assume that $w(t)$ and $r(t)$ are independent. Consider a one-dimensional linear autonomous stochastic pantograph differential with Markovian switching of the form

$$d(x(t)) = [A(r(t))x(t) + B(r(t))x(qt)]dt + [C(r(t))x(t) + D(r(t))x(qt)]dw(t) \tag{27}$$

on $t \geq 0$. For $i \in S$, we will write $A(r(t)) = A_i, B(r(t)) = B_i, C(r(t)) = C_i, D(r(t)) = D_i$, for simplicity. Let $V(t,x,i) = |x|^2$. Then

$$\begin{aligned} LV(t,x,y,i) &= 2x(A_i x + B_i y) + (C_i x + D_i y)^2 \\ &\leq (2A_i + |B_i| + |C_i D_i| + C_i^2)x^2 + (|B_i| + |C_i D_i| + D_i^2)y^2 \end{aligned}$$

By Theorem 4.1, if $1 + 2A_i + |B_i| + |C_i D_i| + C_i^2 = 0$ and $|B_i| + |C_i D_i| + D_i^2 < q$, we can conclude that the solution of Eq. (27) is almost surely asymptotically stable.

Example 4.2 Let $r(t)$ be a right-continuous Markov chain taking values in $S \in \{1,2\}$ with generator

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Assume that $w(t)$ and $r(t)$ are independent. Assume that $B(t)$ and $r(t)$ are independent. Consider a one-dimensional stochastic differential pantograph equation with Markov switching of the form

$$d(x(t)) = f(t,x(t),r(t))dt + g(t,x(qt),r(t))dw(t) \tag{28}$$

on $t \geq 0$ as $1/2 < q < 1$, where

$$f(t,x,1) = \frac{1}{4}x\sin t, f(t,x,2) = e^{-t} - 4x - 3x^3, g(t,x,1) = \frac{1}{8}x\cos t, g(t,x,2) = \frac{1}{\sqrt{2}}x\sin t,$$

Clearly

$$xf(t,x,1) \leq \frac{1}{4}|x|^2, xf(t,x,2) \leq |x|e^{-t} - 4x^2, g^2(t,x,1) \leq \frac{1}{64}|x|^2, g^2(t,x,2) = \frac{1}{2}|x|^2$$

for all $(t,x) \in (R_+, R)$. To examine the asymptotic stability, we construct a function $V : R \times S \to R_+$ by $V(x,i) = \beta_i|x|^2$ with $\beta_2 = 1$ and $\beta_1 = \beta$ a constant to be determined. It is easy to show that the operator LV from $R_+ \times R \times R \times S$ to R has the form

$$LV(t,x,y,i) = 2\beta_i x f(t,x,i) + \beta_i|g(t,y,i)|^2 + (\gamma_{i1}\beta + \gamma_{i2})|x|^2.$$

By the conditions, we then have

$$LV(t,x,y,1) \leq -(\frac{\beta}{2} - 1)x^2 + \frac{\beta}{64}y^2,$$

and

$$LV(t,x,y,2) \leq 2|x|e^{-t} + (2\beta - 10)x^2 + \frac{1}{2}y^2$$

Setting $\beta = 4$, and noting that $2|x|e^{-t} \leq x^2 + e^{-2t}$, we then have

$$LV(t,x,y,i) \leq e^{-2t} - x^2 + \frac{1}{2}y^2.$$

Although f does not satisfy the linear growth condition, by Theorem 3.1, we can also conclude that the solution of Eq. (28) is almost surely asymptotically stable.

Acknowledgements. The project reported here was supported by WHPF of China under Grant Nos.08Q33,06D19.

References

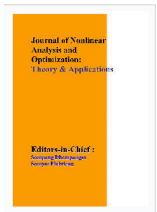
1. C.T.H. Baker, E. Buckwar, Continuous θ-methods for the stochastic pantograph equation, Electron. Trans. Numer. Anal. 11 (2000) 131–151.
2. J.A.D. Appleby, E. Buckwar, Sufficient Condition for Polynomial Asymptotic Behavior of the Stochastic Pantograph Equation. <http://www.dcu.ie/ maths/research/preprint.shtml>.
3. Z. Fan, M. Liu, W. Cao, Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations, J. Math. Anal. Appl. 325 (2007) 1142–1159.
4. Z. Fan, M. Song, M. Liu, The αth moment stability for the stochastic pantograph equation, Journal of Computational and Applied Mathematics 233 (2009) 109–120.
5. R. Li, M. Liu, W. Pang, Convergence of numerical solutions to stochastic pantograph equations with Markovian switching, Applied Mathematics and Computation 215 (2009) 414–422.
6. L. Arnold, Stochastic differential equations: Theory and applications, New York, Wiley, 1972.
7. X. Mao, Stability of stochastic differential equations with Markovian switching. Stochastics Processes and their Applications 79 (1999) 45–67.
8. X. Mao, A note on the LaSalle-type theorems for stochastic differential delay equations, Journal of Mathematical Analysis and Applications 268 (2002) 125C142.
9. C. Yuan, X. Mao, Robust stability and controllability of stochastic differential delay equations with Markovian switching, Automatica 40 (2004) 343–354.
10. Y. Shen, Q. Luo, X. Mao, The improved LaSalle-type theorems for stochastic functional differential equations, J. Math. Anal. Appl. 318 (2006) 134–154.
11. M.Z. Liu, Z.W. Yang, G.D. Hu, Asymptotical stability of the numerical methods with the constant stepsize for the pantograph equation, BIT 45 (2005) 743–759.
12. G.K. Basak, A. Bisi, M.K. Ghosh, Stability of a random diffusion with linear drift, J. Math. Anal. Appl. 202 (1996) 604–622.
13. M.K. Ghosh, A. Arapostathis, S.I. Marcus, Optimal control of switching diffusions with applications to flexible manufacturing systems, SIAM J. Control Optim. 31 (1993) 1183–1204.
14. X. Mao, C. Yuan, Stochastic differential equations with Markovian switching, Imperial College Press, 2006.
15. R.Sh. Lipster, A.N. Shiryayev, Theory of Martingales, Kluwer Academic Publishers, 1989.
16. I. Karatzas, S.E. Shreve, Brownian motion and stochastic calculus, Springer-Verlag, Berlin/New York, 1991.

[1]DEPARTMENT OF CONTROL SCIENCE AND ENGINEERING, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, WUHAN 430074, PR CHINA.
Email address: jeff20@163.com

[2]DEPARTMENT OF MATHEMATICS AND PHYSICS, WUHAN POLYTECHNIC UNIVERSITY, WUHAN 430023, PR CHINA.
Email address: huay20@163.com

[3]DEPARTMENT OF MATHEMATICS, GYEONGSANG NATIONAL UNIVERSITY, CHINJU 660-701, KOREA.

Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 1 No. 1 (2010), 17–22
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



A variant of the Nash equilibrium theorem in generalized convex spaces

Sehie Park

ABSTRACT: The existence theorem of pure-strategy Nash equilibrium due to H. Lu [Economics Letters 94 (2007) 459–462] is extended to generalized convex spaces. Consequently, our version can be applied to a broad class of abstract strategy spaces.

KEYWORDS: Generalized (G-) convex space; H-space; Acyclic map.

1. **Introduction**

In 1928, J. von Neumann obtained his celebrated minimax theorem, which is one of the fundamental results in the theory of games developed by himself.

The first remarkable one of generalizations of von Neumann's minimax theorem was Nash's theorem [1,2] on equilibrium points of non-cooperative games. The following formulation is given by Fan [3, Theorem 4]:

Theorem 1.1. [3] *Let $X_1, X_2, \cdots, X_n$ be n (≥ 2) nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \cdots, f_n$ be n real-valued continuous functions defined on $\prod_{i=1}^n X_i$. If for each $i = 1, 2, \cdots, n$ and for any given point $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \prod_{j \neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a quasi-concave function on X_i, then there exists a point $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \prod_{i=1}^n X_i$ such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \max_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

The original form of this theorem in [1,2] was for Euclidean spaces and its proofs were based on the Brouwer or Kakutani fixed point theorem. Since then there have appeared numerous generalizations and applications; see [4] and the references therein. Recently, H. Lu [5] obtained an existence theorem of pure-strategy Nash equilibrium where player's pure strategy spaces are topological vector spaces.

In the present paper, we show that such strategy spaces can be replaced by generalized convex spaces or G-convex spaces which are quite well-known in the fixed point theory and the KKM theory. Consequently, we obtained a very general version of Lu's existence theorem and our version can be applied to a broad class of abstract strategy spaces.

Corresponding author: Sehie Park (shpark@math.snu.ac.kr).
Manuscript received April 15, 2010. Accepted July 2, 2010.

Sections 2 and 3 are preliminaries on generalized convex spaces and fixed points of compositions of acyclic maps due to the present author. In Section 4, we give our main result which generalize Lu's theorem to G-convex spaces. Finally, we introduce some related generalizations of the Nash theorem.

2. Generalized convex spaces

Multimaps are also called simply maps. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. Recall the following in [6-9]:

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ is an abstract convex space such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

For details on G-convex spaces, see [10-15], where basic theory was extensively developed and lots of examples of G-convex spaces were given.

Example 2.3. The original KKM theorem is for the triple $(\Delta_n \supset V; \mathrm{co})$, where V is the set of vertices and $\mathrm{co} : \langle V \rangle \multimap \Delta_n$ the convex hull operation. This triple can be regarded as $(\Delta_n, N; \Gamma)$, where $N := \{0, 1, \ldots, n\}$ and $\Gamma_A := \mathrm{co}\{e_i \mid i \in A\}$ for each $A \subset N$.

Example 2.4. Fan's celebrated KKM lemma is for $(E \supset D; \mathrm{co})$, where D is a nonempty subset of a topological vector space E.

Example 2.5. A *convex space* $(X \supset D; \Gamma)$ is a triple where X is a subset of a vector space, $D \subset X$ such that $\mathrm{co}\, D \subset X$, and each Γ_A is the convex hull of $A \in \langle D \rangle$ equipped with the Euclidean topology. This concept generalizes the one due to Lassonde for $X = D$. However he obtained several KKM type theorems w.r.t. $(X \supset D; \Gamma)$. Note that any convex subset of a topological vector space is a convex space, but not conversely.

Example 2.6. If $X = D$ and Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$) and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then (X, Γ) becomes a C-space (or an H-space) due to Horvath. The hyperconvex metric spaces due to Aronszajn and Panitchpakti are examples of C-spaces.

Example 2.7. For other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, any continuous image of a G-convex space, L-spaces and B'-simplicial convexity due to Ben-El-Mechaiekh et al., Takahashi's convexity in metric spaces, Kulpa's simplicial structures, generalized H-spaces of Verma or Stachó, $P_{1,1}$-spaces of Forgo and Joó, mc-spaces of Llinares, FC-spaces of Ding, GFC-spaces of Khahn et al., and others.

Let $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ be any family of G-convex spaces. Let $X := \prod_{i\in I} X_i$ be equipped with the product topology and $D := \prod_{i\in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D\rangle$, define $\Gamma_A := \prod_{i\in I} \Gamma_i(\pi_i(A))$. Then the following is known:

Lemma 2.8. *$(X, D; \Gamma)$ is a G-convex space.*

Definition 2.9. Let E be a topological space and $(X, D; \Gamma)$ a G-convex space. A multimap $T : E \multimap X$ is called a *Φ-map* provided that there exists a multimap $S : E \multimap D$ satisfying
(a) for each $z \in E$, $M \in \langle S(z)\rangle$ implies $\Gamma_M \subset T(z)$; and
(b) $E = \bigcup\{\mathrm{Int}\, S^-(y) \mid y \in D\}$.

A *continuous selection* $f : E \to X$ of a map $T : E \multimap X$ is a continuous function such that $f(z) \in T(z)$ for all $z \in E$.
The following is given in [12]:

Lemma 2.10. *Let E be a Hausdorff space, $(X, D; \Gamma)$ a G-convex space, and $T : E \multimap X$ a Φ-map. Then for any nonempty compact subset K of E, $T|_K$ has a continuous selection $f : K \to X$ such that $f(K) \subset \Gamma_A$ for some $A \in \langle D\rangle$. More precisely, there exist two continuous functions $p : K \to \Delta_n$ and $\phi_A : \Delta_n \to \Gamma_A$ such that $f = \phi_A p$ for some $A \in \langle D\rangle$ with $|A| = n + 1$.*

From now on, we consider only G-convex spaces $(X, D; \Gamma)$ satisfying $X \supset D$.

3. Fixed points of compositions of acyclic maps

A topological space is said to be *acyclic* if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a topological vector space, convex $\Longrightarrow$ star-shaped $\Longrightarrow$ contractible $\Longrightarrow$ ω-connected $\Longrightarrow$ acyclic $\Longrightarrow$ connected, and not conversely in each stage.

For topological spaces X and Y, a multimap $F : X \multimap Y$ is called an *acyclic map* whenever F is u.s.c. with compact acyclic values.

In the proof of the main result of this paper, as in [5], we can apply a fixed point theorem due to Gorniewicz. But there are more general fixed point theorems on compositions of acyclic maps.

Let $\mathbb{V}(X, Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X, Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

The following theorems are only few examples of our previous works; for more general results, see [10,14,16].

Theorem 3.1. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and $T \in \mathbb{V}_c(X, X)$. If T is compact, then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

A nonempty subset X of a topological vector space E is said to be *admissible* (in the sense of Klee) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.
It is well-known that every nonempty convex subset of a locally convex Hausdorff topological vector space is admissible. Other examples of admissible topological vector spaces are ℓ^p, $L^p(0, 1)$, H^p for $0 < p < 1$, and many others; see [10,14,16] and references therein.

Theorem 3.2. *Let E be a Hausdorff topological vector space and X an admissible convex subset of E. Then any compact map $T \in \mathbb{V}_c(X, X)$ has a fixed point.*

4. Existence of pure-strategy Nash equilibrium

We follow [5]. Let $I := \{1, \ldots, n\}$ be a set of players. A non-cooperative n-person game of normal form is an ordered $2n$-tuple $\Lambda := \{X_1, \ldots, X_n; u_1, \ldots, u_n\}$, where the nonempty set X_i

is the ith player's pure strategy space and $u_i : X := \prod_{i=1}^n X_i \to \mathbb{R}$ is the ith player's payoff function. A point of X_i is called a strategy of the ith player. Let $X_{-i} := \prod_{j\in I\setminus\{i\}} X_j$ and denote by x and x_{-i} an element of X and X_{-i}, resp. A strategy n-tuple $(x_1^*, \dots, x_n^*)$ is called a *Nash equilibrium for the game* if the following inequality system holds:

$$u_i(x_i^*, x_{-i}^*) \geq u_i(y_i, x_{-i}^*) \text{ for all } y_i \in X_i \text{ and } i \in I.$$

As in [17], we define an aggregate payoff function $U : X \times X \to \mathbb{R}$ as follows:

$$U(x,y) := \sum_{i=1}^n [u_i(y_i, x_{-i}) - u_i(x)] \text{ for any } x = (x_i, x_{-i}), y = (y_i, y_{-i}) \in X.$$

The following is given in [5, Proposition 1]:

Lemma 4.1. *Let Λ be a non-cooperative game, K a nonempty subset of X, and $x^* = \{x_1^*, \dots, x_n^*\} \in K$. Then the following are equivalent:*
(a) *x^* is a Nash equilibrium;*
(b) *$\forall i \in I$, $\forall y_i \in X_i$, $u_i(x_i^*, x_{-i}^*) \geq u_i(y_i, x_{-i}^*)$;*
(c) *$\forall y \in X$, $U(x^*, y) \leq 0$.*

Note that (c) implies $U(x^*, y) \leq 0$ for all $y \in D \subset X$.

Recall that a real-valued function $f : X \to \mathbb{R}$ on a topological space is *lower* [resp., *upper*] *semicontinuous* (*l.s.c.*) [resp., *u.s.c.*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$ is open for each $r \in \mathbb{R}$. If X is a convex set in a vector space, then f is *quasiconcave* [resp., *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is convex for each $r \in \mathbb{R}$.
Now we have our main result:

Theorem 4.2. *Let $I = \{1, \dots, n\}$ be a set of players, $(X, D; \Gamma) = \prod_{i=1}^n (X_i, D_i; \Gamma_i)$ a Hausdorff product G-convex space, K a nonempty compact subset of X, and Λ a non-cooperative game. Suppose that*

(i) *the function $U : X \times X \to \mathbb{R}$ satisfies that*

$$\{(x,y) \in X \times X \mid U(x,y) > 0\}$$

is open;

(ii) *for each $x \in K$, $\{y \in X \mid U(x,y) > 0\}$ is Γ-convex [that is, $M \in \langle\{y \in D \mid U(x,y) > 0\}\rangle$ implies $\Gamma_M \subset \{y \in X \mid U(x,y) > 0\}$];*

(iii) *for each $y \in X$, the set $\{x \in K \mid U(x,y) \leq 0\}$ is acyclic.*

Then there exists a point $x^* \in K$ such that x^* is an equilibrium point for the non-cooperative game.

Proof. Suppose the conclusion does not hold. Then, by Lemma 4.1, for each $x \in K$, there exists a point $y \in D$ such that $U(x,y) > 0$. We define two multimaps $S : K \multimap D$ and $T : K \multimap X$ as follows:

$$T(x) := \{y \in X \mid U(x,y) > 0\} \text{ and } S(x) := \{y \in D \mid U(x,y) > 0\}$$

for each $x \in K$. Then each $T(x)$ is nonempty and, for each $x \in K$, $M \in \langle S(x)\rangle$ implies $\Gamma_M \subset T(x)$ by (ii). Moreover, for each $x \in K$, there exists $y \in D$ such that $x \in S^-(y) = \{x \in K \mid U(x,y) > 0\}$. Note that this $S^-(y)$ is open since $S^-(y)$ is homeomorphic to

$$\{(x,y) \in K \times \{y\} \mid U(x,y) > 0\} = \{(x,y) \in X \times Y \mid U(x,y) > 0\} \cap (K \times \{y\}).$$

This is relatively open in $K \times \{y\}$ which is homeomorphic to K.

Therefore $T : K \multimap X$ is a Φ-map on the compact subset K of X and, by Lemma 2.10, has a continuous selection $f : K \to X$ such that $f(K) \subset \Gamma_A$ for some $A \in \langle D\rangle$. More precisely, there exist two continuous functions $p : K \to \Delta_n$ and $\phi_A : \Delta_n \to \Gamma_A$ such that $f = \phi_A \circ p$ for some $A \in \langle D\rangle$ with $|A| = n+1$.

Here we define a multimap $F : X \multimap K$ by

$$F(y) := \{x \in K \mid U(x,y) \leq 0\} \text{ for } y \in X.$$

Then, by (i), $\{(x,y) \mid U(x,y) \leq 0\}$ is closed in $X \times X$ and hence

$$\mathrm{Gr}(F) := \{(x,y) \mid U(x,y) \leq 0\} \cap (X \times K)$$

is closed in $X \times K$ as the intersection of two closed sets. Hence F is a closed compact map with acyclic values by (iii) and hence an acyclic map. Then it is well-known that $pF\phi_A : \Delta_n \multimap \Delta_n$ has a fixed point $a_0 = pF\phi_A(a_0)$; see Theorem 3.2. Let $y_0 := \phi_A(a_0) \in \Gamma_A \subset X$. Then $y_0 = \phi_A(a_0) \in \phi_A pF(y_0) = fF(y_0)$ and hence $y_0 = f(x_0)$ for some $x_0 \in F(y_0) \subset K$, that is, $U(x_0, y_0) \leq 0$.

On the other hand, $x_0 = f(y_0) \in T(y_0)$ since f is a selection of T. Then, by the definition of T, we have $U(x_0, y_0) > 0$, which is a contradiction. □

Remark. Note that condition (i) can be replaced by one of the following:

(i)$'$ the function $U(x,y)$ is lower semicontinuous on $X \times X$.

(i)$''$ $\forall i \in I$, the function $u_i : X \to \mathbb{R}$ is continuous.

For the case (i)$'$, when $X = D$ is a topological vector space, Theorem 4.2 reduces to [5, Theorem 1]. Note that Nash's original theorem is a simple consequence of Theorem 4.1 under the case (i)$''$.

5. Other Nash type theorems

There are a large number of generalizations of the Nash theorem based on fixed point theorems. For example, based on a generalization of the Kakutani fixed point theorem due to Fan [18] and Glicksberg [19], certain generalizations of the Nash theorems were obtained; see [20,21].

Instead of the fixed point technique, we can apply the KKM theory. The first proof of the Nash theorem by the KKM method was given by Fan [3]. Applying the KKM method, we obtained some of the most general forms of the Nash theorem as follows:

Theorem 5.1. [22] *Let $\{(X_i;\Gamma_i)\}_{i=1}^n$ be a finite family of compact abstract convex spaces such that $(X;\Gamma) = (\prod_{i=1}^n X_i;\Gamma)$ satisfies the partial KKM principle and, for each i, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions such that*

(0) *$f_i(x) \leq g_i(x)$ for each $x \in X$;*
(1) *for each $x^i \in X^i$, $x_i \mapsto g_i[x^i, x_i]$ is quasiconcave on X_i;*
(2) *for each $x^i \in X^i$, $x_i \mapsto f_i[x^i, x_i]$ is u.s.c. on X_i; and*
(3) *for each $x_i \in X_i$, $x^i \mapsto f_i[x^i, x_i]$ is l.s.c. on X^i.*

Then there exists a point $\widehat{x} \in X$ such that

$$g_i(\widehat{x}) \geq \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad \text{for all } i = 1,2,\ldots,n.$$

Theorem 5.2. [23] *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of Hausdorff compact G-convex spaces and, for each $i \in I$, let $f_i, g_i : X = X^i \times X_i \to \mathbb{R}$ be real functions satisfying $(0)-(3)$. Then there exists a point $\widehat{x} \in X$ such that*

$$g_i(\widehat{x}) \geq \max_{y_i \in X_i} f_i[\widehat{x}^i, y_i] \quad \text{for all } \ i \in I.$$

References

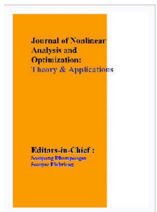
[1] J.F. Nash, *Equilibrium points in N-person games*, Proc. Nat. Acad. Sci. USA **36** (1950), 48–49.
[2] J. Nash, *Non-cooperative games*, Ann. Math. **54** (1951), 286–293.
[3] K. Fan, *Applications of a theorem concerning sets with convex sections*, Math. Ann. **163** (1966), 189–203.
[4] S. Park, *Generalizations of the Nash equilibrium theorem in the KKM theory*, Fixed Point Theory and Appl., vol. 2010, Article ID 234706, 23pp, doi:10.1155/2010/234706.
[5] H. Lu, *On the existence of pure-strategy Nash equilibrium*, Economics Letters 94 (2007), 459–462.
[6] S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. **45**(1) (2008), 1–27.

[7] S. Park, *Equilibrium existence theorems in KKM spaces*, Nonlinear Anal. **69** (2008), 4352–4364.
[8] S. Park, *New foundations of the KKM theory*, J. Nonlinear Convex Anal. **9**(3) (2008), 331–350.
[9] S. Park, *General KKM theorems for abstract convex spaces*, J. Inform. Math. Sci. **1**(1) (2009), 1–13.
[10] S. Park, *Fixed point theory of multimaps in abstract convex uniform spaces*, Nonlinear Anal. 71 (2009), 2468–2480.
[11] S. Park, *Five episodes related to generalized convex spaces*, Nonlinear Funct. Anal. & Appl. 2 (1997), 49–61.
[12] S. Park, *Continuous selection theorems in generalized convex spaces*, Numer. Funct. Anal. Optim. 25 (1999), 567–583.
[13] S. Park, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comp. Appl. Math. 7 (2000), 1–28.
[14] S. Park, *A unified fixed point theory in generalized convex spaces*, Acta Math. Sinica, English Series 23(8) (2007), 1509–1536.
[15] S. Park, *Generalized convex spaces, L-spaces, and FC-spaces*, J. Glob. Optim. 45(2) (2009), 203–210.
[16] S. Park, *Generalizations of the Himmelberg fixed point theorem*, Fixed Point Theory and Its Applications (Proc. ICFPTA-2007), 123–132, Yokohama Publ., 2008.
[17] H. Nikaido and K. Isoda, *Note on non-cooperative games*, Pacific J. Math. 5 (1955), 807–815.
[18] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 121–126.
[19] I.L. Glicksberg, *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. **3** (1952), 170–174.
[20] C.D. Aliprantis, D. Glycopantis, and D. Puzzello, *The joint continuity of the expected payoff functions*, J. Math. Econ. **42** (2006), 121–130.
[21] J.G. Becker and D.S. Damianov, *On the existence of symmetric mixed strategy equilibria*, Econ. Lett. **90** (2006), 84–87.
[22] S. Park, *From the KKM principle to the Nash equilibria*, Inter. J. Math. & Stat. **6**(S10) (2010), 77–88.
[23] S. Park, *Generalizations of the Nash equilibrium theorem on generalized convex spaces*, J. Korean Math. Soc. **38** (2001), 697–709.

THE NATIONAL ACADEMY OF SCIENCES, REPUBLIC OF KOREA, SEOUL 137–044; AND

DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SEOUL 151–747, KOREA.
Email address: shpark@math.snu.ac.kr; parkcha38@hanmail.net

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 23–33
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Existence and approximation of solution of the variational inequality problem with a skew monotone operator defined on the dual spaces of Banach spaces

Somyot Plubtieng[1,2], Wanna Sriprad[1]

ABSTRACT: In this paper, we first study an existence theorem of the variational inequality problem for a skew monotone operator defined on the dual space of a smooth Banach space. Secondary, we prove a weak convergence theorem for finding a solution of the variational inequality problem by using projection algorithm method with a new projection which was introduce by Ibaraki and Takahashi [T. Ibaraki, W. Takahashi, A new projection and convence theorems for the projections in Banach spaces, Journal of Approximation Theory 149 (2007), 1-14]. Further, we apply our convergence theorem to the convex minimization problem and the problem of finding a zero point of maximal skew monotone operator.

KEYWORDS: Generalized nonexpansive retraction; Inverse-strongly-skew-monotone operator; Variational inequality; p-uniformly smooth.

1. **Introduction**

Let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Let A be a monotone operator of C into H. The variational inequality problem [10, 16] is to find a point $u \in C$ such that

$$(1) \qquad \langle Au, v - u\rangle \geq 0, \quad \text{for all } v \in C.$$

Such a point $u \in C$ is called a solution of the problem and the set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Variational inequality theory has played a fundamental and powerful role in the study of a wide range of problems arising in differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, operations research, general equilibrium problems in economics and transportation, etc. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems. The projection operator technique, one usually establishes an equivalence between the variational inequalities and the fixed-point problem. This alternative equivalent formulation was used by Lions and Stampacchia [16] to study the existence of solutions of the variational inequalities. Projection method and its variant forms

Corresponding author: Wanna Sriprad (wanna_sriprad@hotmail.com)
Manuscript received May 19, 2010. Accepted July 5, 2010.
*Supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

represent important tools for finding the approximate solution of variational inequalities. This method starts with any $x_1 = x \in C$ and updates iteratively x_{n+1} according to the formula

$$x_{n+1} = P_C(x_n - \lambda_n A x_n) \tag{2}$$

for every $n = 1, 2, ...$, where A is a monotone operator of C into H, P_C, is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive numbers. An operator A of C into E^* is said to be *inverse-strongly-monotone* [2, 8, 12] if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in C$. In such a case A is said to be *α-inverse-strongly-monotone*. In the case where A is inverse-strongly-monotone, Iiduka, Takahashi and Toyoda [8] proved that the sequence $\{x_n\}$ generated by (2) converges weakly to some element of $VI(C, A)$.

Recently, Iiduka and Takahashi [7] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly-monotone operator A in a Banach space: $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n) \tag{3}$$

for every $n = 1, 2, ...$, where where A is an inverse-strongly-monotone operator of C into E^*, Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive numbers. They proved that the sequence $\{x_n\}$ generated by (3) converges weakly to some element of $VI(C, A)$. On the other hand, Ibaraki and Takahashi [5] introduced a new resolvent of a maximal monotone operator in a Banach space and the concept of a generalized nonexpansive mapping in a Banach space. Kohsaka and Takahashi [11], and Ibaraki and Takahashi [6] also studied some properties for generalized nonexpansive retractions in Banach spaces.

In this paper, motivated by Ibaraki and Takahashi [5] and Iiduka and Takahashi [7], we consider the following variational inequality problem: Let E be a smooth Banach space, let E^* be the dual space of E and let C be a nonempty and closed subset of E such that JC is closed and convex subset of E^*, where J is the duality mapping on E. Let A be a skew monotone operator of JC into E. Then, the variational inequality problem is to find

$$u \in C \text{ such that } \langle AJu, Jv - Ju\rangle \geq 0, \ \forall\, v \in C. \tag{4}$$

We denoted the set of solution of the variational inequality problem (4) by $VI(JC, A)$. If $E = H$ is Hilbert space and C is nonempty closed convex subset of H, then the variational inequality problem (4) is equivalent to the variational inequality problem (1). In this paper, we first prove existence theorem of the variational inequality problem for skew monotone operators defined on the dual space of E. Using the projection algorithm method with a new projection which was introduced by Ibaraki and Takahashi [5], we prove weak convergence theorem for finding a solution of the variational inequality problem (4) for an inverse-strongly-skew-monotone operator defined on the dual space of a uniformly convex and $2-$uniformly smooth Banach space. Further, using this result we consider the convex minimization problem and the problem of finding a zero point of maximal skew monotone operator.

2. **Preliminaries**

Let E be a real Banach space. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. An operator $T \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone T is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. It is known that a monotone operator T is maximal if and only if for $(x, x^*) \in E \times E^*$, $\langle x - y, x^* - y^*\rangle \geq 0$ for every $(y, y^*) \in G(T)$ implies $x^* \in T(x)$. If T is maximal monotone, then the solution set $T^{-1}0$ is closed and convex.

The normalized duality mapping J from E into E^* is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

We recall (see [17]) that E is reflexive if and only if J is surjective; E is smooth if and only if J is single-valued; E is strictly convex if and only if J is one-to-one; if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. We note that in a Hilbert space, H, J is the identity operator. The definitions of the strict (uniform) convexity, (uniformly) smoothness of Banach spaces and related properties can be found in [17].

The duality J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* [4] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Let E be a norm linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$

The space E is said to be *smooth* if $\rho_E(\tau) > 0$, $\forall \tau > 0$. E is called *uniformly smooth* if and only if $\lim_{t\to 0^+} \frac{\rho_E(\tau)}{t} = 0$. Let $p > 1$. E is said to be $p-$*uniformly smooth* (or to have a modulus of smoothness of power type p) if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^p$, $t > 0$. It is well known (see, for example, [18]) that

$$L_p\ (l_p) \text{ or } (W_m)^p \text{ is } \begin{cases} \text{2-uniformly smooth if } p \geq 2 \\ \text{p-uniformly smooth if } 1 < p \leq 2. \end{cases}$$

We observe that every p-uniformly smooth Banach space is uniformly smooth. Furthermore, from the proof of [18, Remark 5, p.208], we have the following lemma

Lemma 2.1. [18] *Let E be a 2-uniformly smooth Banach space. Then, for all $x, y \in E$, there exists a constant $c > 0$ such that*

$$\|Jx - Jy\| \leq c\|x - y\|, \tag{5}$$

where J is the normalized duality mapping of E.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x, y \in E,$$

is studied by Alber [1], Kamimura and Takahashi [9] and Reich [13]. It is obvious from the definition of ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$.

Lemma 2.2. *(see [9]) Let E be a uniformly convex, smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let E be a reflexive, strictly convex, smooth Banach space and J the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [1]:

$$V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{6}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x,x^*) = \phi(x, J^{-1}(x))$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.3. *(see [6]) Let E be a reflexive, strictly convex, smooth Banach space and let V be as in (6). Then*

$$V(x,x^*) + 2\langle y, Jx - x^*\rangle \leq V(x+y, x^*)$$

for all $x, y \in E$ and $x^* \in E^*$.

Let E be a smooth Banach space and let D be a nonempty closed subset of E. A mapping $R : D \to D$ is called generalized nonexpansive if $F(R) \neq \emptyset$ and $\phi(Rx, y) \leq \phi(x,y)$ for each $x \in D$ and $y \in F(R)$, where $F(R)$ is the set of fixed points of R. Let C be a nonempty closed subset of E. A mapping $R : E \to C$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E,\ \forall\, t \geq 0.$$

A mapping $R : E \to C$ is said to be a retraction if $Rx = x$, $\forall x \in C$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto C is uniquely determined if it exists (see [5]). We also know that if E is reflexive, smooth, and strictly convex and C is a nonempty closed subset of E, then there exists a sunny generalized nonexpansive retraction R_C of E onto C if and only if $J(C)$ is closed and convex. In this case R_C is given by $R_C = J^{-1}\Pi_{J(C)}J$

see [11]. Let C be a nonempty closed subset of a Banach space E. Then C is said to be a sunny generalized nonexpansive retract (resp. a generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp. a generalized nonexpansive retraction) of E onto C (see [5] for more detials). The set of fixed points of such a generalized nonexpansive retraction is C. The following Lemma was obtained in [5].

Lemma 2.4. *([5]) Let C be a nonempty and closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C, let $x \in E$ and let $z \in C$. Then the following hold:*
(a) $z = R_C x$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$
(b) $\phi(x, R_C x) + \phi(R_C x, z) \leq \phi(x, z)$.

3. **Variational inequalities for skew monotone operators defined on the dual space of a Banach space.**

In this section, we consider the variational inequalities for skew monotone operators defined on the dual space of a Banach space. Let E be a smooth Banach space and let C^* be the closed closed and convex subset of E^*. An operator $A : C^* \to E$ is said to be *skew monotone if $\langle Ax^* - Ay^*, x^* - y^* \rangle \geq 0$ for all $x^*, y^* \in C^*$. Let E^* be the dual space of E and let C be a nonempty and closed subset of E such that JC is closed and convex subset of E^*, where J is the duality mapping on E. Let A be a skew monotone operator of JC into E. Then, the variational inequality problem is to find*

$$u \in C \text{ such that } \langle AJu, Jv - Ju \rangle \geq 0, \ \forall\, v \in C. \tag{7}$$

Such a point u is called a solution of the problem. We denote the set of solution of the variational inequality problem (7) by $VI(JC, A)$. i.e.,

$$VI(JC, A) = \{u \in C : \langle AJu, Jv - Ju \rangle \geq 0, \ \forall\, v \in C\}$$

.

Lemma 3.1. *Let E be a Banach space with the dual space E^*. Let C^* be a nonempty, compact and convex subset of E^* and A is a skew monotone operator of C^* into E. Then there exists $x_0^* \in C^*$ such that*

$$\langle Ax^*, x^* - x_0^* \rangle \geq 0, \ \forall x^* \in C^*.$$

Proof. For any $y^* \in C^*$, we assume that the set $\{x^* \in C^* : \langle Ax^*, x^* - y^* \rangle < 0\}$ is nonempty. We also define two multi-valued mappings T and B of C^* into itself by

$$Tx^* = \{y^* \in C^* : \langle Ay^*, x^* - y^* \rangle < 0\} \text{ and } Bx^* = \{y^* \in C^* : \langle Ax^*, x^* - y^* \rangle < 0\}.$$

Then, for any $y^* \in C^*$, the set $T^{-1}y^* = \{x^* \in C^* : \langle Ay^*, x^* - y^* \rangle < 0\}$ is convex. Also, for any $y^* \in C^*$, the set $B^{-1}y^* = \{x^* \in C^* : \langle Ax^*, x^* - y^* \rangle < 0\}$ is nonempty. Since A is skew monotone, we have that $\langle Ax^*, x^* - y^* \rangle \geq \langle Ay^*, x^* - y^* \rangle$, for all $x^*, y^* \in C^*$. So, we have that $Bx^* \subset Tx^*$ for all $x^* \in C^*$. Since Bx^* is open for all $x^* \in C^*$, it follows by [17, Theorem 6.1.5] that there exists a point $x_0^* \in C^*$ such that $x_0^* \in Tx_0^*$. Thus, we have

$$0 = \langle Ax_0^*, x_0^* - x_0^* \rangle < 0$$

This is a contradiction. □

An operator $A : D(A) \subset E^* \to E$ is said to be hemicontinuous if for all $x^*, y^* \in D(A)$, the mapping f of $[0,1]$ into E defined by $f(t) = A(tx^* + (1-t)y^*)$ is continuous.

Lemma 3.2. *Let E be a Banach space with the dual space E^*. Let C^* be a nonempty and convex subset of E^* and let A be a skew monotone and hemicontinuous operator of C^* into E. Let $x_0^* \in C^*$. Then*

$$\langle Ax_0^*, x^* - x_0^* \rangle \geq 0, \ \forall x^* \in C^* \tag{8}$$

if and only if

$$\langle Ax^*, x^* - x_0^* \rangle \geq 0, \ \forall x^* \in C^*. \tag{9}$$

Proof. Suppose that $\langle Ax_0^*, x^* - x_0^*\rangle \geq 0,$ for all $x^* \in C^*$. By the skew monotonicity of A, we have

$$\langle Ax^*, x^* - x_0^*\rangle \geq \langle Ax_0^*, x^* - x_0^*\rangle \geq 0, \text{ for all } x^* \in C^*.$$

Conversely, suppose that $\langle Ax^*, x^* - x_0^*\rangle \geq 0,$ for all $x^* \in C^*$. Let $y^* \in C^*$ and $0 < t < 1$. Since C^* is convex, we have $y_t^* = (1-t)x_0^* + ty^* \in C^*$. This implies that

$$0 \leq \langle Ay_t^*, y_t^* - x_0^*\rangle = t\langle Ay_t^*, y^* - x_0^*\rangle.$$

Since $t > 0$, it follow that $0 \leq \langle Ay_t^*, y^* - x_0^*\rangle$. Thus by the hemicontinuity of A , we have

$$0 \leq \langle Ax_0^*, y^* - x_0^*\rangle \text{ as } t \to 0.$$

This completes the proof. □

Using Lemma 3.1 and Lemma 3.2, we obtained the following Theorem

Theorem 3.3. *Let E be a Banach space with the dual space E^*. Let C^* be a nonempty, compact and convex subset of E^* and let A be a skew monotone and hemicontinuous operator of C^* into E. Then there exists $x_0^* \in C^*$ such that*

$$\langle Ax_0^*, x^* - x_0^*\rangle \geq 0, \ \forall x^* \in C^*.$$

We note from Theorem 3.3 that if JC is compact and convex and A is a skew monotone and hemicontinuous operator of JC into E, then $VI(JC, A)$ is nonempty.

Lemma 3.4. *Let C be a nonempty and closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex set. Let A be a skew monotone operator of JC into E. Then*

$$u \in VI(JC, A) \text{ if and only if } u = R_C(u - \lambda AJu), \ \forall\, \lambda > 0,$$

where R_C is sunny generalized nonexpansive retraction of E onto C.

Proof. From the definition of $VI(JC, A)$ and Lemma 2.4, we have

$$\begin{aligned} u \in VI(JC, A) \ &\Leftrightarrow\ \langle AJu, Jy - Ju\rangle \geq 0\ \forall\, y \in C \\ &\Leftrightarrow\ \langle -\lambda AJu, Jy - Ju\rangle \leq 0\ \forall\, y \in C,\ \forall\, \lambda > 0 \\ &\Leftrightarrow\ \langle u - \lambda AJu - u, Jy - Ju\rangle \leq 0\ \forall\, y \in C,\ \forall\, \lambda > 0 \\ &\Leftrightarrow\ u = R_C(u - \lambda AJu),\ \forall\, \lambda > 0. \end{aligned}$$

□

Let E be a Banach space with the dual space E^* and let C be a nonempty and closed subset of E such that JC is closed and convex. Let i_{JC} be indicator function of JC. Since $i_{JC} : E^* \to (-\infty, \infty]$ is proper, lower semicontonuous and convex, the subdifferential ∂i_{JC} of i_{JC} defined by

$$\partial i_{JC}(x^*) = \{x \in E : i_{JC}(y^*) \geq i_{JC}(x^*) + \langle x, y^* - x^*\rangle\ (\forall\, y^* \in E^*)\ \},\ (\forall\, x^* \in E^*)$$

is a maximal skew monotone operator by [14, 15]. *Next, let C be a nonempty and closed subset of E such that JC is closed and convex set and $x^* \in JC$. Then we denote by $N_{JC}(x^*)$ the skew normal cone of JC at a point $x^* \in JC$, that is,*

$$N_{JC}(x^*) = \{x \in E : \langle x, x^* - y^*\rangle \geq 0 \ \textit{for all}\ y^* \in JC\}.$$

We note from [14, 15] *that*

$$\partial i_{JC}(x^*) = \begin{cases} N_{JC}(x^*), & \text{if } x^* \in JC, \\ \emptyset, & \text{if } x^* \notin JC. \end{cases}$$

Then we obtain the following theorem:

Theorem 3.5. *Let C be a nonempty and closed subset of a smooth Banach space E such that JC is closed and convex set and let A be a skew monotone and hemicontinuous operator of JC into E. Let $B \subset E^* \times E$ be an operator defined as follows:*

$$Bv^* = \begin{cases} Av^* + N_{JC}(v^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

Then B is maximal skew monotone and $(BJ)^{-1}0 = VI(JC, A)$.

Proof. We first show that B is skew monotone. Let $y_1 \in Ax_1^* + N_{JC}(x_1^*)$ and $y_2 \in Ax_2^* + N_{JC}(x_2^*)$. Then we can write them by $y_1 = Ax_1^* + z_1$ for some $z_1 \in N_{JC}(x_1^*)$ and $y_2 = Ax_2^* + z_2$ for some $z_2 \in N_{JC}(x_2^*)$. Since A is skew monotone, $z_1 \in N_{JC}(x_1^*)$ and $z_2 \in N_{JC}(x_2^*)$ it follows that

$$\begin{aligned}\langle y_1 - y_2, x_1^* - x_2^*\rangle &= \langle Ax_1^* + z_1 - (Ax_2^* + z_2), x_1^* - x_2^*\rangle \\ &= \langle Ax_1^* - Ax_2^*, x_1^* - x_2^*\rangle + \langle z_1 - z_2, x_1^* - x_2^*\rangle \\ &= \langle Ax_1^* - Ax_2^*, x_1^* - x_2^*\rangle - \langle z_1, x_2^* - x_1^*\rangle - \langle z_2, x_1^* - x_2^*\rangle \\ &\geq 0.\end{aligned}$$

This implies that B is skew monotone. Next, we shall show that B is maximal. Let $(x^*, x) \in E^* \times E$ such that $\langle y - x, y^* - x^*\rangle \geq 0$ for every $(y^*, y) \in G(B)$. Note that, for any $(y^*, y) \in G(B)$, we have $y \in By^* = Ay^* + N_{JC}(y^*)$. This implies that $y = Ay^* + z$ for some $z \in N_{JC}(y^*)$. So, the above inequality means that

$$\langle z, y^* - x^*\rangle + \langle Ay^* - x, y^* - x^*\rangle \geq 0. \tag{10}$$

for all $y^* \in JC$ and $z \in N_{JC}(y^*)$. It is clear from the definition of $N_{JC}(y^*)$ that if $z \in N_{JC}(y^*)$ and $\lambda \geq 0$, then $\lambda z \in N_{JC}(y^*)$. So from (10), we note that

$$\langle z, y^* - x^*\rangle \geq 0 \ \forall\, z \in N_{JC}(y^*). \tag{11}$$

In fact, if not, there exists $z \in N_{JC}(y^*)$ such that $\langle z, y^* - x^*\rangle < 0$. So, we have $\lambda\langle z, y^* - x^*\rangle \to -\infty$ as $\lambda \to \infty$. This is a contradiction. Then we got that (11) holds. Since $z \in N_{JC}(y^*) \Leftrightarrow z \in \partial i_{JC}(y^*)$, it follows from (11) that

$$\langle z - 0, y^* - x^*\rangle \geq 0 \ \forall\ (y^*, z) \in G(\partial i_{JC}).$$

Since ∂i_{JC} is maximal skew monotone, we have $0 \in \partial i_{JC}(x^*) = N_{JC}(x^*)$ and hence $x^* \in JC$. Define $x_t^* = tu^* + (1-t)x^*$, where $u^* \in JC$ and $t \in (0, 1)$. From the convexity of JC , we get $x_t^* \in JC$. By (10), we have from $0 \in N_{JC}(x_t^*)$ that

$$\langle 0, x_t^* - x^*\rangle + \langle Ax_t^* - x, x_t^* - x^*\rangle \geq 0,$$

and hence $\langle Ax_t^* - x, x_t^* - x^*\rangle \geq 0$. Since $x_t^* = tu^* + (1-t)x^*$, it follows that $t\langle Ax_t^* - x, u^* - x^*\rangle \geq 0$. Dividing this inequality by $t > 0$, we obtain

$$\langle Ax_t^* - x, u^* - x^*\rangle \geq 0.$$

So, letting $t \to 0$, we get

$$\langle x - Ax^*, x^* - u^*\rangle \geq 0 \ (\forall\ u^* \in JC)$$

and hence $x - Ax^* \in N_{JC}(x^*)$. This implies that $x \in Ax^* + N_{JC}(x^*) = Bx^*$. Therefore B is a maximal skew monotone operator. Finally, we will show that $(BJ)^{-1}0 = VI(JC, A)$.
We note that $(BJ)^{-1}0 = \{z \in C : 0 \in BJ(z)\}$. Thus, we have

$$\begin{aligned} z \in (BJ)^{-1}0 &\Leftrightarrow 0 \in AJz + N_{JC}(Jz) \\ &\Leftrightarrow -AJz \in N_{JC}(Jz) \\ &\Leftrightarrow \langle -AJz, Jz - y^*\rangle \geq 0 \ \forall\, y^* \in JC \\ &\Leftrightarrow \langle AJz, y^* - Jz\rangle \geq 0 \ \forall\, y^* \in JC \\ &\Leftrightarrow \langle AJz, Jy - Jz\rangle \geq 0 \ \forall\, y \in C \\ &\Leftrightarrow z \in VI(JC, A). \end{aligned}$$

□

Corollary 3.6. *Let E be a reflexive, strictly convex and smooth Banach space with a Fréchet differentiable norm and let C be a nonempty and closed subset of E such that JC is closed and convex and let A be a skew monotone and hemicontinuous operator of JC into E such that $VI(JC, A) \neq \emptyset$. Then $VI(JC, A)$ is closed and $JVI(JC, A)$ is closed and convex.*

Proof. Let $B \subset E^* \times E$ be an operator defined as follows:

$$Bv^* = \begin{cases} Av^* + N_{JC}(v^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

By Theorem 3.5, we have that B is maximal skew monotone operator and $(BJ)^{-1}0 = VI(JC, A)$. Since E is reflexive and strictly convex, it follows that J is bijective. Thus, we have $JVI(JC, A) = JJ^{-1}B^{-1}0 = B^{-1}0$. Since B is maximal skew monotone, it follows that $B^{-1}0$ is closed and convex and hence $JVI(JC, A)$ is closed and convex. Next, let $\{x_n\} \subset (BJ)^{-1}0$ with $x_n \to x$. From $x_n \in (BJ)^{-1}0$, we have $J(x_n) \in B^{-1}0$. Since J is norm to norm continuous and $B^{-1}0$ is closed, we have $J(x_n) \to J(x) \in B^{-1}0$. This implies that $x \in (BJ)^{-1}0$. Hence $(BJ)^{-1}0$ is closed and therefore $VI(JC, A)$ is closed. □

4. **Weak convergence theorem**

In this section, using the projection algorithm method we prove weak convergence theorem for finding a solution of the variational inequality for an inverse-strongly-skew-monotone operator defined on the dual space of a uniformly convex and 2-uniformly smooth Banach space.

Let E be a real Banach space with the dual space E^*. An operator $A : D(A) \subset E^* :\to E$ is said to be inverse-strongly-skew-monotone if there exists a positive real number α such that $\langle Ax^* - Ay^*, x^* - y^* \rangle \geq \alpha \|Ax^* - Ay^*\|^2$ for all $x^*, y^* \in D(A)$. In such a case A is said to be α-inverse-strongly-skew-monotone. An operator $A : D(A) \subset E^* :\to E$ is said to be Lipscitz continuous if there exists $L \geq 0$ such that $\|Ax^* - Ay^*\| \leq L\|x^* - y^*\|$, for all $x^*, y^* \in D(A)$. If A is α-inverse-strongly-skew-monotone, then A is Lipscitz continuous, that is, $\|Ax^* - Ay^*\| \leq (\frac{1}{\alpha})\|x^* - y^*\|$, for all $x^*, y^* \in D(A)$.

Before proving our theorem we need the following Lemma.

Lemma 4.1. *Let C be a nonempty and closed subset of a uniformly convex and smooth Banach space E such that JC is closed and convex. Let $\{x_n\}$ be a sequence in E such that, for all $u \in C$,*

$$\phi(x_{n+1}, u) \leq \phi(x_n, u) \tag{12}$$

for every $n = 1, 2, ...$ Then $\{R_C(x_n)\}$ is a Cauchy sequence, where R_C is sunny generalized nonexpansive retraction of E onto C.

Proof. Put $u_n = R_C(x_n)$ for all $n \in \mathbb{N}$. From (12), we note that

$$\begin{aligned} \phi(x_{n+m}, u) &\leq \phi(x_{n+m-1}, u) \\ &\leq \phi(x_{n+m-2}, u) \leq ... \leq \phi(x_n, u) \end{aligned}$$

for every $n = 1, 2, ...$ Thus, we have

$$\phi(x_{n+m}, u_n) \leq \phi(x_n, u_n) \tag{13}$$

Since $u_{n+m} = R_C(x_{n+m})$, it follows from Lemma 2.4 (b) and (13) that

$$\begin{aligned} \phi(u_{n+m}, u_n) &= \phi(R_C(x_{n+m}), u_n) \\ &\leq \phi(x_{n+m}, u_n) - \phi(x_{n+m}, u_{n+m}) \\ &\leq \phi(x_n, u_n) - \phi(x_{n+m}, u_{n+m}). \end{aligned} \tag{14}$$

Consequently, we have $\limsup_{l\to\infty} \phi(x_l, u_l) \leq \phi(x_n, u_n)$, which implies that $\{\phi(x_n, u_n)\}$ converges. By Lemma 2.2 and (14), we note that $\{u_n\}$ is a Cauchy sequence. □

Now, we can prove the following weak convergence theorem.

Theorem 4.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let C be a nonempty and closed subset of E such that JC is closed and convex and let A be an α-inverse-strongly-skew-monotone operator of JC into E such that*

$VI(JC,A) \neq \emptyset$ and $\|AJy\| \leq \|AJy - AJu\|$ for all $y \in C$ and $u \in VI(JC,A)$. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and

$$x_{n+1} = R_C(x_n - \lambda_n AJx_n), \tag{15}$$

for every $n = 1,2,...,$ where R_C is sunny generalized nonexpansive retraction of E onto C, $\{\lambda_n\} \subset [a,b]$ for some a,b with $0 < a < b < \frac{\alpha}{c}$, where c is a constant in (5). Then the sequence $\{x_n\}$ converges weakly to some element $z \in VI(JC,A)$. Further $z = \lim_{n\to\infty} R_{VI(JC,A)}(x_n)$.

Proof. Put $y_n = x_n - \lambda_n AJx_n$ for all $n = 1,2,....$ Let $u \in VI(JC,A)$. We first prove that $\{x_n\}$ is bounded. By Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\phi(x_{n+1},u) &= \phi(R_C y_n, u) \leq \phi(y_n,u) \\
&= V(y_n, Ju) \\
&= V(x_n - \lambda_n AJx_n, Ju) \\
&\leq V((x_n - \lambda_n AJx_n) + \lambda_n AJx_n, Ju) - 2\langle \lambda_n AJx_n, J(x_n - \lambda_n AJx_n) - Ju\rangle \\
&= V(x_n, Ju) - 2\lambda_n\langle AJx_n, Jy_n - Ju\rangle \\
&= \phi(x_n,u) - 2\lambda_n\langle AJx_n, Jx_n - Ju\rangle + 2\langle -\lambda_n AJx_n, Jy_n - Jx_n\rangle
\end{aligned} \tag{16}$$

for all $n \in \mathbb{N}$. Since A is α-inverse-strongly-skew-monotone and $u \in VI(JC,A)$, it follows that

$$\begin{aligned}
-2\lambda_n\langle AJx_n, Jx_n - Ju\rangle &= -2\lambda_n\langle AJx_n - AJu, Jx_n - Ju\rangle - 2\lambda_n\langle AJu, Jx_n - Ju\rangle \\
&\leq -2\alpha\lambda_n\|AJx_n - AJu\|^2
\end{aligned} \tag{17}$$

for all $n \in \mathbb{N}$. By Lemma 2.1 and our assumption, we obtain

$$\begin{aligned}
2\langle -\lambda_n AJx_n, Jy_n - Jx_n\rangle &= 2\langle -\lambda_n AJx_n, J(x_n - \lambda_n AJx_n) - Jx_n\rangle \\
&\leq 2\|\lambda_n AJx_n\|\|J(x_n - \lambda_n AJx_n) - Jx_n\| \\
&\leq 2c\|\lambda_n AJx_n\|\|(x_n - \lambda_n AJx_n) - x_n\| \\
&= 2c\lambda_n^2\|AJx_n\|^2 \leq 2c\lambda_n^2\|AJx_n - Ju\|^2
\end{aligned} \tag{18}$$

for all $n \in \mathbb{N}$. From (16), (17) and (18), we get

$$\begin{aligned}
\phi(x_{n+1},u) &\leq \phi(x_n,u) + 2\lambda_n(\lambda_n c - \alpha)\|AJx_n - AJu\|^2 \\
&\leq \phi(x_n,u) + 2a(bc - \alpha)\|AJx_n - AJu\|^2 \\
&\leq \phi(x_n,u)
\end{aligned} \tag{19}$$

for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty}\phi(x_n,u)$ exists and hence, $\{\phi(x_n,u)\}$ is bounded. It implies that $\{x_n\}$ is bounded. By (19), we note that

$$-2a(bc - \alpha)\|AJx_n - AJu\|^2 \leq \phi(x_n,u) - \phi(x_{n+1},u) \tag{20}$$

for all $n \in \mathbb{N}$ and hence $\lim_{n\to\infty}\|AJx_n - AJu\|^2 = 0$. From Lemma 2.3 , Lemma 2.4 and (18), we have

$$\begin{aligned}
\phi(x_{n+1},x_n) &= \phi(R_C y_n, x_n) \\
&\leq \phi(y_n,x_n) = \phi(x_n - \lambda_n AJx_n, x_n) \\
&= V(x_n - \lambda_n AJx_n, Jx_n) \\
&\leq V((x_n - \lambda_n AJx_n) + \lambda_n AJx_n, Jx_n) - 2\langle \lambda_n AJx_n, J(x_n - \lambda_n AJx_n) - Jx_n\rangle \\
&= \phi(x_n,x_n) + 2\langle -\lambda_n AJx_n, Jy_n - Jx_n\rangle \\
&\leq 2c\lambda_n^2\|AJx_n - Ju\|^2 \leq 2cb^2\|AJx_n - Ju\|^2
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\|AJx_n - AJu\|^2 = 0$, we have $\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0$. Applying Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{21}$$

From the uniform smoothness of E, we have

(22) $$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in E$. It follows that $x_{n_i+1} \rightharpoonup z$ as $i \rightharpoonup \infty$. We shall show that $z \in VI(JC, A)$. Let $B \subset E^* \times E$ be an operator as follows:

$$Bv^* = \begin{cases} Av^* + N_{JC}(v^*), & v^* \in JC, \\ \emptyset, & v^* \notin JC. \end{cases}$$

By Theorem 3.5, B is maximal skew monotone and $(BJ)^{-1}0 = VI(JC, A)$. Let $(v^*, w) \in G(B)$. Since $w \in Bv^* = Av^* + N_{JC}(v^*)$, it follows that $w - Av^* \in N_{JC}(v^*)$. From $Jx_{n+1} \in JC$, we have

(23) $$\langle w - Av^*, v^* - Jx_{n+1}\rangle \geq 0.$$

Since $w \in Bv^*$, we get $v^* \in JC$. This implies that there is $v \in C$ such that $Jv = v^*$. Thus it follow from (23) that

(24) $$\langle w - AJv, Jv - Jx_{n+1}\rangle \geq 0.$$

On the other hand, from $x_{n+1} = R_C(x_n - \lambda_n AJx_n)$ and Lemma 2.4, we have $\langle (x_n - \lambda_n AJx_n) - x_{n+1}, Jx_{n+1} - Jv\rangle \geq 0$ and hence

(25) $$\left\langle \frac{x_n - x_{n+1}}{\lambda_n} - AJx_n, Jv - Jx_{n+1} \right\rangle \leq 0.$$

From (24) and (25), we have

$$\begin{aligned}
\langle w, Jv - Jx_{n+1}\rangle &\geq \langle AJv, Jv - Jx_{n+1}\rangle \\
&\geq \langle AJv, Jv - Jx_{n+1}\rangle + \left\langle \frac{x_n - x_{n+1}}{\lambda_n} - AJx_n, Jv - Jx_{n+1} \right\rangle \\
&= \langle AJv - AJx_n, Jv - Jx_{n+1}\rangle + \left\langle \frac{x_n - x_{n+1}}{\lambda_n}, Jv - Jx_{n+1} \right\rangle \\
&= \langle AJv - AJx_{n+1}, Jv - Jx_{n+1}\rangle + \langle AJx_{n+1} - AJx_n, Jv - Jx_{n+1}\rangle \\
&\quad + \left\langle \frac{x_n - x_{n+1}}{\lambda_n}, Jv - Jx_{n+1} \right\rangle \\
&\geq -\frac{\|Jx_{n+1} - Jx_n\|}{\alpha}\|Jv - Jx_{n+1}\| - \frac{\|x_n - x_{n+1}\|}{a}\|Jv - Jx_{n+1}\| \\
&\geq -M\left(\frac{\|Jx_{n+1} - Jx_n\|}{\alpha} + \frac{\|x_n - x_{n+1}\|}{a}\right)
\end{aligned}$$

for all $n \in \mathbb{N}$, where $M = \sup\{\|Jv - Jx_{n+1}\| : n \in \mathbb{N}\}$. Taking $n = n_i$, from (21), (22) and the weakly sequential continuity of J, we obtain $\langle w, Jv - Jz\rangle \geq 0$ as $i \to \infty$. Hence, by the skew maximality of B, we obtain $Jz \in B^{-1}0$. That is $z \in (BJ)^{-1}0 = VI(JC, A)$.

From Corollary 3.6, we note that $VI(JC, A)$ is closed and $JVI(JC, A)$ is closed and convex. Put $u_n = R_{VI(JC,A)}(x_n)$ for all $n \in \mathbb{N}$. It holds from (19) and Lemma 4.1 that $\{u_n\}$ is Cauchy sequence. Since $VI(JC, A)$ is closed, $\{u_n\}$ converges strongly to $w \in VI(JC, A)$. By the uniform smoothness of E, we also have $\lim_{n\to\infty} \|Ju_n - Jw\| = 0$. Finally, we prove that $z = w$. From Lemma 2.4 and $z \in VI(JC, A)$, we have

(26) $$\langle x_n - u_n, Jz - Ju_n\rangle \leq 0$$

for all $n \in \mathbb{N}$. So, we get

$$\begin{aligned}
\langle x_n - u_n, Jz - Jw\rangle &= \langle x_n - u_n, Jz - Ju_n\rangle + \langle x_n - u_n, Ju_n - Jw\rangle \\
&\leq \|x_n - u_n\|\|Ju_n - Jw\| \\
&\leq K\|Ju_n - Jw\|
\end{aligned}$$

for all $n \in \mathbb{N}$, where $K = \sup\{\|x_n - u_n\| : n = 1, 2, ...\}$. Taking $n = n_i$, from $\lim_{n\to\infty} \|u_n - w\| = 0$ and $\lim_{n\to\infty} \|Ju_n - Jw\| = 0$, we obtain

$$\langle z - w, Jz - Jw\rangle \leq 0 \text{ as } i \to \infty.$$

This implies that $\langle z - w, Jz - Jw\rangle = 0$. Since E is strictly convex, it follows that $z = w$. Therefore the sequence $\{x_n\}$ converges weakly to $z = \lim_{n\to\infty} R_{VI(JC,A)}(x_n)$. This completes the proof. □

5. Application

In this section, we study the problem of finding a zero point of a maximal skew monotone operator of E^* into E and a minimizer of a continuously Fréchet differentiable and convex functional in a Banach space. To prove this, we need the following lemma:

Lemma 5.1. *(see* [3].*) Let E be a Banach space, f a continuously Fréchet differentiable and convex function on E^* and ∇f the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse-strongly-skew-monotone.*

Now, we consider the problem of finding a zero point of a maximal skew monotone operator of E^* into E and a zero point of an inverse-strongly-skew-monotone operator of E^* into E. In the case where $JC = E^*$.

Theorem 5.2. *Let E be a uniformly convex and $2-$uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let A be an α-inverse-strongly-skew-monotone of E^* into E with $A^{-1}0 \neq \emptyset$. Let $x_1 = x \in E$ and $\{x_n\}$ is given by*

$$x_{n+1} = x_n - \lambda_n AJx_n,$$

for every $n = 1, 2, ...$, where $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{\alpha}{c}$, where c is a constant in (2.1). Then the sequence $\{x_n\}$ converges weakly to some element z in $(AJ)^{-1}0$. Further $z = \lim_{n\to\infty} R_{(AJ)^{-1}(0)}(x_n)$.

Proof. From $R_E = I$, $VI(JE, A) = (AJ)^{-1}0$ and $\|AJy\| = \|AJy - 0\| = \|AJy - AJu\|$ for all $y \in E$ and $u \in (AJ)^{-1}0$, by using Theorem 4.2, $\{x_n\}$ converges weakly to some element z in $(AJ)^{-1}0$. □

Corollary 5.3. *Let E be a uniformly convex and $2-$uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Assume that f is a function on E^* such that f is a continuously Fréchet differentiable and convex function on E^*, ∇f is $1/\alpha$-Lipschitz continuous and $(\nabla f)^{-1}0 = \{z^* \in E^* : f(z^*) = \min_{y^*\in E^*} f(y^*)\} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$x_{n+1} = x_n - \lambda_n(\nabla f)Jx_n,$$

for every $n = 1, 2, ...$, where $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < \frac{\alpha}{c}$, where c is a constant in (2.1). Then the sequence $\{x_n\}$ converges weakly to some element z in $((\nabla f)J)^{-1}0$. Further $z = \lim_{n\to\infty} R_{((\nabla)fJ)^{-1}0}(x_n)$.

Proof. By Lemma 5.1, we have ∇f is an α-inverse-strongly-skew-monotone operator of E^* into E. Hence, by Theorem 5.2, $\{x_n\}$ converges weakly to some element z in $((\nabla f)J)^{-1}0$. □

Acknowledgments *The authors thank the National Centre of Excellence in Mathematics, PERDO, for the financial support. Moreover, the second author would like to thank the National Centre of Excellence in Mathematics, PERDO, under the Commission on Higher Education, Ministry of Education, Thailand.*

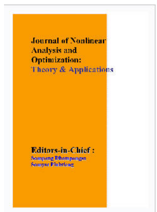
References

1. Y.I. Alber, Metric and generalized projection operators in Banach spaces: Properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear operators of Accretive and Monotone Type, in: Lecture Notes Pure Appl. Math., vol. 178, Dekker, New York, 1996, pp. 15-50.
2. F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197-228.
3. J.B. Baillon, G. Haddad, Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones, Israel J. Math. 115(1977), 137-150.
4. J. Diestel, Geometry of Banach Space-Selected Topics, Lecture Notes in Math., vol. 485, Springer-Verlag, Berlin, 1975.
5. T. Ibaraki, W. Takahashi, A new projection and convence theorems for the projections in Banach spaces, Journal of Approximation Theory 149 (2007), 1-14.
6. T. Ibaraki, W. Takahashi, Generalized nonexpansive mappings and a proximal-type algorithm in Banach spaces, to appear.
7. H. Iiduka, W. Takahashi, Weak convergence of a projection algorithm for variational inequalities in a Banach space, J. Math. Anal. Appl. 339(2008), 668-679.
8. H. Iiduka, W. Takahashi, M. Toyoda, Approximation of solutions of variational inequalities for monotone mappings, Panamer. Math. J. 14 (2004), 49-61.
9. S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in Banach a space, SIAM J. Optim. 13 (2002), 938-945.
10. D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.
11. F. Kohsaka and W. Takahashi, Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces, J. Nonlinear Convex Anal. 8 (2007), 197-209.
12. F. Liu, M.Z. Nashed, Regularization of nonlinear ill-posed variational inequalities and convergence rates, Set-Valued Anal. 6 (1998) 313-344.
13. S. Reich, A weak convergence theorem for the alternating method with Bregman distances, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in: Lecture Notes Pure Appl. Math., vol. 178, Dekker, New York, 1996, pp. 313-318.
14. R.T. Rockafellar, Maximal monotone operators and proximal point algorithm, SIAM J. Control Optim. 14 (1976), 887-898.
15. R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, Pacific J. Math. 33 (1970), 209-216.
16. J.L. Lions, G. Stampacchia, Variational inequalities, Comm. Pure Appl. Math. 20 (1967) 493-517.
17. W. Takahashi, Nonlinear Functional Analysis, Yokohama, 2000.
18. Z.B. Xu, G.F. Roach, Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl. 157 (1991) 189-210.

[1]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000,, Thailand.
Email address: somyotp@nu.ac.th(Somyot Plubtieng) and wanna_sriprad@hotmail.com(Wanna Sriprad).

[2] Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand.

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 35-43
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Remarks on the Gradient-Projection Algorithm

Meng Su[1], Hong-Kun Xu[2,*]

ABSTRACT: The gradient-projection algorithm (GPA) is a powerful method for solving constrained minimization problems in finite (and even infinite) dimensional Hilbert spaces. We consider GPA with variable stepsizes and show that if GPA generates a bounded sequence, then under certain assumptions, every accumulation point of the sequence is a solution of the minimization problem. We also look into the issue where the sequence of stepsizes is allowed to be the limiting case (e.g., approaching to zero).

KEYWORDS: Gradient-projection algorithm, constrained minimization, variational inequality, optimality condition, convex function, Lipschitz continuous gradient, Féjer-monotone.

1. **Introduction**

Consider the constrained minimization problem

$$\min_{x\in C} f(x) \tag{1}$$

where C is a nonempty closed convex set of $\mathbb{R}^n$, and the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Assume that (1) is consistent (i.e., it has a solution) and we use Γ to denote its solution set.

It is well known that the (necessary) optimality condition for a point $x^* \in C$ to be a solution of (1) is that

$$-\nabla f(x^*) \in N_C(x^*) \tag{2}$$

where $N_C(x^*)$ is the normal cone to C at x^*, namely,

$$N_C(x^*) = \{v \in \mathbb{R}^n : \langle v, x - x^*\rangle \le 0, \quad x \in C\}.$$

Condition (2) is equivalent to the following variational inequality (VI):

$$x^* \in C, \quad \langle \nabla f(x^*), x - x^*\rangle \ge 0, \quad x \in C. \tag{3}$$

Let P_C be the nearest point projection from $\mathbb{R}^n$ onto C. We can then rewrite VI (3) equivalently to a fixed point equation

$$x^* = P_C(I - \alpha\nabla f)x^*, \tag{4}$$

where $\alpha > 0$ is any (fixed) constant.

Corresponding author: Hong-Kun Xu (xuhk@math.nsysu.edu.tw).
Manuscript received July 11, 2010. Accepted July 31, 2010.
*Supported in part by NSC 97-2628-M-110-003-MY3 (Taiwan).

The gradient-projection algorithm (GPA, for short) is usually applied to solve the minimization problem (1). This algorithm generates a sequence $\{x^k\}$ through the recursion:

$$x^{k+1} = P_C(x^k - \alpha_k \nabla f(x^k)), \quad k = 0, 1, \cdots, \tag{5}$$

where the initial guess $x^0 \in C$ is chosen arbitrarily and $\{\alpha_k\}$ is a sequence of stepsizes which may be chosen in different ways.

GPA (5) has well been studied in the case of constant stepsizes $\alpha_k = \alpha$ for all k (see the books [4, 5], and the papers [1, 2, 3, 6, 7, 8]). A recent averaged mapping approach to GPA (5) can be found in [9].

A fundamental convergence result for GPA (5) is the following one which can be found in literature (cf. [5, Theorem 6.1] or [4, Theorem 1, Section 7.2] with constant stepsize).

Theorem 1.1. *Let $\{x^k\}$ be the sequence generated by GPA (5). Assume*

(i) *f is continuously differentiable and its gradient is Lipschitz continuous:*

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad x, y \in \mathbb{R}^n, \tag{6}$$

where $L \geq 0$ is a constant;

(ii) *the set $C_0 := \{x \in C : f(x) \leq f(x^0)\}$ is bounded;*

(iii) *the sequence $\{\alpha_k\}$ satisfies the condition:*

$$0 < \liminf_{k\to\infty} \alpha_k \leq \limsup_{k\to\infty} \alpha_k < \frac{2}{L}. \tag{7}$$

Then every accumulation point x^* of $\{x^k\}$ satisfies the optimality condition (2). Moreover, if f is also convex, then $\{x^k\}$ converges to a solution of the minimization (1).

In this paper we deal with the gradient-projection algorithm with variable stepsize. We show that if GPA (5) generates a bounded sequence $\{x^k\}$ (which is guaranteed by the boundedness of the set C_0), then under condition (7), every accumulation point of $\{x^k\}$ is a solution to the minimization problem (1). If the objective f is, in addition, convex, then $\{x^k\}$ indeed converges to a solution of (1). We also deal with the saturated situation where we can allow the sequence $\{\alpha_k\}$ to close zero (see the precise descriptions in Section 3).

2. Preliminaries

Let H be the real Euclidean n-space $\mathbb{R}^n$ and C be a nonempty closed convex subset of H.

Definition 2.1. The (metric or nearest-point or orthogonal) projection from H onto C is a mapping that assigns, to each point $x \in H$, a unique point in C, denoted $P_C x$, with the property:

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}. \tag{8}$$

The following characterizes the relation $z = P_C x$.

Proposition 2.2. *Given $z \in C$ and $x \in H$. Then $z = P_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad y \in C.$$

Consequently, P_C is firmly nonexpansive, that is,

$$\langle P_C x - P_C y, x - y, x - y\rangle \geq \|P_C x - P_C y\|^2, \quad x, y \in H.$$

In particular, P_C is nonexpansive:

$$\|P_C x - P_C y\| \leq \|x - y\|, \quad x, y \in H.$$

We next recall the optimality condition for the minimization problem (1).

Lemma 2.3. (Optimality Condition.) *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (1) is that x^* solves the variational inequality:*

$$x^* \in C, \quad \langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad x \in C. \tag{9}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C(x^* - \alpha \nabla f(x^*))$$

for every constant $\alpha > 0$.

If, in addition, f is convex, then the optimality condition (9) is also sufficient.

Lemma 2.4. *(cf.* [4, Lemma 2, Section 1.4]*) Suppose a continuously differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$ has a Lipschitz continuous gradient:*

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad x, y \in \mathbb{R}^n. \tag{10}$$

Then ∇f satisfies the so-called inversely strong (also known as co-coercive) monotonicity:

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2, \quad x, y \in \mathbb{R}^n. \tag{11}$$

3. The Gradient-Projection Algorithm

Recall that the gradient-projection method (GPM) for solving the minimization problem (1) generates a sequence $\{x^k\}$ according to the recursive process

$$x^{k+1} = P_C(x^k - \alpha_k \nabla f(x^k)), \quad k = 0, 1, 2, \cdots, \tag{12}$$

where $\{\alpha_k\}$ is a sequence of stepsizes with $\alpha_k \geq 0$ for each $k \geq 0$. The convergence of GPM (12) depends on the choice of the stepsize sequence $\{\alpha_k\}$ (and also on the behavior of the gradient ∇f). The purpose of this section is to discuss the convergence of GPM (12) under different choices of the stepsize sequence $\{\alpha_k\}$. (Note that in both books [5, 4], constant stepsize $\alpha_k \equiv \alpha$ is considered; here we deal with variable stepsize.)

3.1. Variable Stepsize

Theorem 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that the gradient ∇f satisfies the Lipschitz continuity condition (10). For a given initial guess $x^0 \in C$, let $\{x^k\}$ be the sequence generated by GPA (12). Assume*

- $(A1)$ *$\{\alpha_k\}$ satisfies condition (7).*
- $(A2)$ *$\{x^k\}$ is bounded. (This is guaranteed by the boundedness assumption (ii) of the set $C_0 = \{x \in C : f(x) \leq f(x^0)\}$ in Theorem 1.1, as assumed in* [5]*).*

Then we have

- (i) *Every accumulation point of $\{x^k\}$ is a solution of the minimization problem (1).*
- (ii) $\lim_{k\to\infty} f(x^k) = f_{\min} := \min\{f(x) : x \in C\}$.
- (iii) *If $\{x^k\}$ is Féjer-monotone with respect to the solution set Γ of the minimization problem (1), that is,*

$$\|x^{k+1} - x^*\| \leq \|x^k - x^*\|, \quad k \geq 0, \; x^* \in \Gamma, \tag{13}$$

then the sequence $\{x^k\}$ converges to a solution of (1).

Proof. We have, using (6),

$$\begin{aligned} f(x^{k+1}) - f(x^k) &= \int_0^1 \langle \nabla f(x^k + t(x^{k+1} - x^k)), x^{k+1} - x^k \rangle dt \\ &= \int_0^1 \langle \nabla f(x^k + t(x^{k+1} - x^k)) - \nabla f(x^k), x^{k+1} - x^k \rangle dt \\ &\quad + \langle \nabla f(x^k), x^{k+1} - x^k \rangle \\ &\le \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2}\|x^{k+1} - x^k\|^2. \end{aligned} \tag{14}$$

On the other hand, since x^{k+1} is the projection of $x^k - \alpha_k \nabla f(x^k)$ onto C, it follows from Proposition 2.2 that

$$\langle x^k - \alpha_k \nabla f(x^k) - x^{k+1}, x^k - x^{k+1} \rangle \le 0.$$

This implies that

$$\langle \nabla f(x^k), x^{k+1} - x^k \rangle \le -\frac{1}{\alpha_k}\|x^{k+1} - x^k\|^2. \tag{15}$$

Substituting (15) into (14), we get

$$f(x^{k+1}) \le f(x^k) - \left(\frac{1}{\alpha_k} - \frac{L}{2}\right)\|x^{k+1} - x^k\|^2. \tag{16}$$

Let $\underline{\alpha} = \liminf_{k\to\infty} \alpha_k$ and $\overline{\alpha} = \limsup_{k\to\infty} \alpha_k$. Then, by assumption (7), we see that there is an integer $N \ge 1$ such that

$$0 < \underline{\alpha} \le \alpha_k \le \overline{\alpha} < \frac{2}{L} \tag{17}$$

for all $k \ge N$. With no loss of generality, we may assume that (17) holds true for all $k \ge 0$. It then turns out from (16) that

$$f(x^{k+1}) \le f(x^k) - \frac{2 - \overline{\alpha}L}{2\overline{\alpha}}\|x^{k+1} - x^k\|^2. \tag{18}$$

The sequence $\{f(x^k)\}$ is therefore decreasing and $\lim_{k\to\infty} f(x^k)$ exists. Moreover,

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{19}$$

We next prove that every accumulation point of $\{x^k\}$ is a solution of the minimization problem (1). Note that since $\{x^k\}$ is assumed to be bounded, the set of accumulation points of $\{x^k\}$ is nonempty. Let $\hat{x}$ be an accumulation point of $\{x^k\}$ and let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ converging to $\hat{x}$. Due to (19), we also have that $\{x^{k_j+1}\}$ converges to $\hat{x}$.

We may also assume that the subsequence $\{\alpha_{k_j}\}$ is convergent to some number $\hat{\alpha}$ (say). According to (17), we get

$$0 < \underline{\alpha} \le \hat{\alpha} \le \overline{\alpha} < \frac{2}{L}.$$

Taking the limit as $j \to \infty$ in the relation

$$x^{k_j+1} = P_C(x^{k_j} - \alpha_{k_j} \nabla f(x^{k_j}))$$

yields $\hat{x} = P_C(\hat{x} - \hat{\alpha}\nabla f(\hat{x}))$ which exactly says that $\hat{x}$ solves the minimization problem (1) and hence, $\lim_{k\to\infty} f(x^k) = f(\hat{x}) = f_{\min}$.

Finally we show that the entire sequence $\{x^k\}$ converges if the Féjer monotonicity condition (13) holds. Let x^* be an accumulation point of $\{x^k\}$. Due to (13), the full limit as $k \to \infty$ of the sequence $\{\|x^k - x^*\|\}$, $\lim_{k\to\infty} \|x^k - x^*\|$, exists. However, a subsequence of it converges to zero. We therefore conclude that $\lim_{k\to\infty} \|x^k - x^*\| = 0$ and the full sequence $\{x^k\}$ converges to x^*. □

If f is also convex, then we can remove the boundedness assumption on $\{x^k\}$ in Theorem 3.1.

Theorem 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function such that the gradient ∇f satisfies the Lipschitz continuity condition (10). Assume the minimization problem (1) is consistent. For a given initial guess $x^0 \in C$, let $\{x^k\}$ be the sequence generated by GPA (12) where we assume $\{\alpha_k\}$ satisfies condition (7). Then $\{x^k\}$ converges to a solution of (1).*

Proof. We only need to prove the Féjer-monotonicity (13) holds when f is convex. To see this, we take $x^* \in \Gamma$. Observing $x^* = P_C(I - \alpha\nabla f)x^*$ for every $\alpha > 0$, and using the nonexpansivity of P_C and Lemma 2.4, we derive that

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|P_C(I - \alpha_k\nabla f)x^k - P_C(I - \alpha_k\nabla f)x^*\|^2 \\
&\le \|(I - \alpha_k\nabla f)x^k - (I - \alpha_k\nabla f)x^*\|^2 \\
&= \|(x^k - x^*) - \alpha_k(\nabla f(x^k) - \nabla f(x^*))\|^2 \\
&= \|x^k - x^*\|^2 - 2\alpha_k\langle x^k - x^*, \nabla f(x^k) - \nabla f(x^*)\rangle \\
&\quad + \alpha_k{}^2\|\nabla f(x^k) - \nabla f(x^*)\|^2 \\
&\le \|x^k - x^*\|^2 - \alpha_k\left(\frac{2}{L} - \alpha_k\right)\|\nabla f(x^k) - \nabla f(x^*)\|^2 \\
&\le \|x^k - x^*\|^2 - \frac{\underline{\alpha}(2 - \overline{\alpha})}{L}\|\nabla f(x^k) - \nabla f(x^*)\|^2.
\end{aligned}$$

It is now immediately clear that $\|x^{k+1} - x^*\| \le \|x^k - x^*\|$, namely, $\{x^k\}$ satisfies the Féjer-monotonicity condition (13). Consequently, $\{x^k\}$ is bounded, and moreover converges to a point in Γ. □

3.2. **Two-Slope Test**

The two-slope test for the unconstrained minimization problem

$$\min\{f(x) : x \in \mathbb{R}^n\} \tag{20}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, determines the stepsize sequence $\{\alpha_k\}$ in such a way that

$$f(x^k) - b\alpha_k\|\nabla f(x^k)\|^2 \le f(x^k - \alpha_k\nabla f(x^k)) \le f(x^k) - a\alpha_k\|\nabla f(x^k)\|^2, \tag{21}$$

where $0 < a < b < 1$ are two fixed constants.

The following theorem is known [5, Theorem 5.3].

Theorem 3.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and its gradient ∇f is Lipschitz with constant L. Let $\{x^k\}$ be the sequence generated by the steepest-descent method:*

$$x^{k+1} = x^k - \alpha_k\nabla f(x^k), \quad k = 0, 1, \cdots, \tag{22}$$

where the stepsize sequence $\{\alpha_k\}$ is determined by the two-slope test (21). Assume the set

$$M_0 := \{x \in \mathbb{R}^n : f(x) \le f(x^0)\}$$

is bounded. Then $\{x^k\}$ is bounded and every accumulation point x^* of $\{x^k\}$ satisfies the optimality condition $\nabla f(x^*) = 0$ for the unconstrained minimization (20).

Below we extend this two-slope test from unconstrained to constrained minimization.

The two-slope test for the constrained minimization problem (1) determines the stepsize for the $(k+1)$th iterate x^{k+1} in such a way that

$$f(x^k) - b\langle\nabla f(x^k), x^k - x^k(\alpha_k)\rangle \le f(x^k(\alpha_k)) \le f(x^k) - a\langle\nabla f(x^k), x^k - x^k(\alpha_k)\rangle, \tag{23}$$

where $0 < a < b < 1$ are two fixed constants, and where

$$x^k(\alpha) = P_C(x^k - \alpha\nabla f(x^k)), \quad \alpha \ge 0.$$

It is easily found that if C is the entire space $\mathbb{R}^n$ (i.e., the constrained minimization (1) is reduced to the unconstrained minimization (20)), then (23) is reduced to (21).

We have the following convergence result which extends Theorem 3.3 to the case of constrained minimization.

Theorem 3.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that its gradient ∇f satisfies the Lipschitz continuity condition (10). Let $\{x^k\}$ be the sequence generated by GPM (12), where the sequence of stepsizes, $\{\alpha_k\}$, satisfies the two-slope test (23) and $\liminf_{k\to\infty} \alpha_k > 0$. Assume, in addition, that the set*

$$C_0 := \{x \in C : f(x) \le f(x_0)\}$$

is bounded. Then $\{x^k\}$ is bounded and every accumulation point $\bar{x}$ of $\{x^k\}$ is a stationary point of the constrained minimization problem (1); namely, $\bar{x}$ satisfies the variational inequality

$$\bar{x} \in C, \quad \langle \nabla f(\bar{x}), x - \bar{x} \rangle \ge 0, \quad x \in C. \tag{24}$$

In particular, if f is also convex, then $\bar{x}$ is a solution of (1).

Proof. Noticing $x^{k+1} = x^k(\alpha_k)$, we get immediately from (23) that $f(x^{k+1}) \le f(x^k)$ for all $k \ge 0$. This implies that $\{x^k\} \subset C_0$; in particular, $\{x^k\}$ is bounded. Hence

$$\lim_{k\to\infty} f(x^k) \quad \text{exists.}$$

Also (23) implies that

$$\langle \nabla f(x^k), x^k - x^k(\alpha_k) \rangle \ge 0. \tag{25}$$

Since there also holds (again from (23))

$$a\langle \nabla f(x^k), x^k - x^k(\alpha_k) \rangle \le f(x^k) - f(x^{k+1}),$$

we get that

$$\lim_{k\to\infty} \langle \nabla f(x^k), x^k - x^{k+1} \rangle = 0. \tag{26}$$

Now since x^{k+1} is the projection of $x^k - \alpha_k \nabla f(x^k)$ onto C, we have by Proposition 2.2

$$\langle (x^k - \alpha_k \nabla f(x^k)) - x^{k+1}, y - x^{k+1} \rangle \le 0, \quad y \in C.$$

It turns out that

$$\langle x^k - x^{k+1}, y - x^{k+1} \rangle \le \alpha_k \langle \nabla f(x^k)), y - x^{k+1} \rangle, \quad y \in C.$$

Setting $y := x^k \in C$ we get

$$\|x^k - x^{k+1}\|^2 \le \alpha_k \langle \nabla f(x^k)), x^k - x^{k+1} \rangle. \tag{27}$$

Combining (26) and (27), we obtain

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{28}$$

Now let $\bar{x}$ be an accumulation point of $\{x^k\}$ and let $\{x^{k_j}\}$ be a subsequence of $\{x^k\}$ converging to $\bar{x}$. With no loss of generality, we may assume that $\alpha_{k_j} \to \bar{\alpha} > 0$ (for $\liminf_{k\to\infty} \alpha_k > 0$). Now taking the limit as $j \to \infty$ in the relation

$$x^{k_j+1} = P_C(x^{k_j} - \alpha_{k_j} \nabla f(x^{k_j}))$$

gives that

$$\bar{x} = P_C(\bar{x} - \bar{\alpha} \nabla f(\bar{x})).$$

This equivalently says that $\bar{x}$ satisfies VI (24) and is a stationary point of the minimization problem (1).

When f is convex, the optimality condition is also sufficient and $\bar{x}$ is therefore a solution of (1). □

3.3. **Strongly Monotone Gradient**

Assume that the objective function f is continuously differentiable such that its gradient is Lipschitzian and strongly monotone. Namely, there exist constants $\beta > 0$ and $L \geq 0$ satisfying, for all $x, y \in C$,

$$\langle \nabla f(x) - \nabla f(x), x - y \rangle \geq \beta \|x - y\|^2 \tag{29}$$

and

$$\|\nabla f(x) - \nabla f(x)\| \leq L\|x - y\|. \tag{30}$$

We then have that the mapping

$$T \equiv T_\gamma := P_C(I - \gamma \nabla f)$$

is a contraction provided $0 < \gamma < 2\beta/L^2$. As a matter of fact, we have

$$\begin{aligned}
\|Tx - Ty\|^2 &= \|P_C(I - \gamma\nabla f)x - P_C(I - \gamma\nabla f)y\|^2 \\
&\leq \|(I - \gamma\nabla f)x - (I - \gamma\nabla f)y\|^2 \\
&= \|(x - y) - \gamma(\nabla f(x) - \nabla f(y))\|^2 \\
&= \|x - y\|^2 - 2\gamma\langle \nabla f(x) - \nabla f(y), x - y\rangle + \gamma^2\|\nabla f(x) - \nabla f(y)\|^2 \\
&\leq \|x - y\|^2 - 2\gamma\beta\|x - y\|^2 + \gamma^2 L^2\|x - y\|^2 \\
&= (1 - \gamma(2\beta - \gamma L^2))\|x - y\|^2.
\end{aligned}$$

This shows that T is a contraction with constant $\sqrt{1 - \gamma(2\beta - \gamma L^2)}$.

Therefore, for such a choice of γ, we can apply Banach's contraction principle to get that for each $x^0 \in C$, the sequence $\{T_\gamma^k x^0\}$ converges to the unique fixed point of T_γ (or the unique solution of the minimization (1)).

We however look at the case where the stepsizes $\{\gamma_k\}$ are variable such that

$$0 < \liminf_{k\to\infty} \gamma_k < \limsup_{k\to\infty} \gamma_k < \frac{2\beta}{L^2}. \tag{31}$$

We have the following convergence result.

Theorem 3.5. *Let $x^0 \in C$ and define a sequence $\{x^k\}$ by the iterative algorithm:*

$$x^{k+1} = P_C(x^k - \gamma_k \nabla f(x^k)), \tag{32}$$

where the sequence $\{\gamma_k\}$ is selected according to the selection rule (31). Then $\{x^k\}$ converges to the unique solution x^* of the minimization (1).

Proof. By (31), there exist some natural number N and positive constants a and b such that

$$0 < a \leq b < 2\beta/L^2 \quad \text{and} \quad a \leq \gamma_k \leq b \ (k \geq N).$$

Set

$$h = \max_{a\leq\gamma\leq b} \sqrt{1 - \gamma(2\beta - \gamma L^2)}.$$

Then $0 \leq h < 1$ and it is easy to see that

$$0 \leq \sqrt{1 - \gamma_k(2\eta - \gamma_k L^2)} \leq h$$

for all $k \geq N$.

Denote by x^* the unique solution of the minimization (1). We now compute, for all $k \geq N$,

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &= \|P_C(I - \gamma_k \nabla f)x^k - P_C(I - \gamma_k \nabla f)x^*\|^2 \\ &\leq \|(I - \gamma_k \nabla f)x^k - (I - \gamma_k \nabla f)x^*\|^2 \\ &= \|(x^k - x^*) - \gamma_k(\nabla f(x^k) - \nabla f(x^*))\|^2 \\ &= \|x^k - x^*\|^2 + \gamma_k^2 \|\nabla f(x^k) - \nabla f(x^*)\|^2 \\ &\qquad - 2\gamma_k \langle x^k - x^*, \nabla f(x^k) - \nabla f(x^*) \rangle \\ &\leq [1 - \gamma_k(2\eta - \gamma_k L^2)] \|x^k - x^*\|^2 \\ &\leq h^2 \|x^k - x^*\|^2.\end{aligned}$$

Consequently

$$\|x^{k+1} - x^*\| \leq h\|x^k - x^*\| \leq \cdots \leq h^{k-N+1}\|x^N - x^*\|.$$

Therefore, we get $\lim_{k\to\infty} \|x^k - x^*\| = 0$. □

Theorem 3.5 asserts that if the parameter sequence $\{\gamma_k\}$ is bounded away below from zero and above from $2\beta/L^2$, then the sequence $\{x^k\}$ generated by GPA (32) converges to the unique solution of the minimization (1). The result below shows that we can allow $\{\gamma_k\}$ to close either zero or $2\beta/L^2$ and still keep the convergence of the sequence $\{x^k\}$.

Theorem 3.6. *Assume that the sequence $\{\gamma_k\}$ satisfies the condition*

$$0 < \gamma_k < \frac{2\eta}{L^2} \quad \textit{for all } k \textit{ and} \quad \sum_{k=0}^{\infty} \gamma_k \left(\frac{2\beta}{L^2} - \gamma_k \right) = \infty. \tag{33}$$

Then the sequence $\{x^k\}$ generated by GPA (32) converges to the unique solution x^* of the minimization (1).

Proof. The first part of condition (33) assures that that the mapping $P_C(I - \gamma_k \nabla f) : C \to C$ is a contraction with the coefficient $\sqrt{1 - \gamma_k(2\beta - \gamma_k L^2)}$ for all $k \geq 0$. Observing $x^* = P_C(I - \gamma_k \nabla f)x^*$ for all $k \geq 0$, we have

$$\begin{aligned}\|x^{k+1} - x^*\| &= \|P_C(I - \gamma_k \nabla f)x_n - P_C(I - \gamma_k \nabla f)x^*\| \\ &\leq \sqrt{1 - \gamma_k(2\beta - \gamma_k L^2)} \|x^k - x^*\| \\ &\leq \left(1 - \frac{1}{2}\gamma_k(2\beta - \gamma_k L^2)\right) \|x^k - x^*\| \qquad (34) \\ &\leq \|x^k - x^*\|.\end{aligned}$$

In particular, $\{x^k\}$ is bounded and $\lim_{k\to\infty} \|x^k - x^*\|$ exists. Also it follows from (34) that

$$\frac{L^2}{2}\gamma_k \left(\frac{2\beta}{L^2} - \gamma_k \right) \|x^k - x^*\| \leq \|x^k - x^*\| - \|x^{k+1} - x^*\|. \tag{35}$$

Put $r = \lim_{k\to\infty} \|x^k - x^*\|$. If $r > 0$, then by (35), we get (noticing $\|x^k - x^*\| \geq r$ for all k)

$$\frac{rL^2}{2}\gamma_k \left(\frac{2\beta}{L^2} - \gamma_k \right) \leq \|x^k - x^*\| - \|x^{k+1} - x^*\|.$$

Consequently,

$$\sum_{k=0}^{\infty} \gamma_k \left(\frac{2\beta}{L^2} - \gamma_k \right) < \infty$$

which contradicts the assumption (33). So we must have $r = 0$ and $x^k \to x^*$ as $k \to \infty$. □

References

[1] E.M. Gafni and D.P. Bertsekas, Two-metric projection methods for constained optimization, *SIAM J. Control Optim* 22 (1984), 936-964.

[2] P.H. Calamai and J.J. Moré, Projected gradient methods for linearly constained problems, *Mathematical Programming* 39 (1987), 93-116.

[3] E.S. Levitin and B.T. Polyak, Constrained minimization problems , *USSR Computationnal Mathematics and Mathematical Phsics* 6 (1966), 1-50.

[4] B.T. Polyak, "Introduction to Optimization," *Optimization Software*, New York, 1987.

[5] A. Ruszczynski, "Nonlinear optimization," *Princeton University Press*, New Jersey, 2006.

[6] C.Y. Wang and N.H. Xiu, Convergence of gradient projection methods for generalize convex minimization , *Computational Optim. Appl.* 16 (2000), 111-120.

[7] N.H. Xiu, C.Y. Wang, C.Y. and J.Z. Zhang, Convergence properties of projection and contraction methods for variational inequality problems, *Applied Math. Opt.* 43 (2001), 147-168.

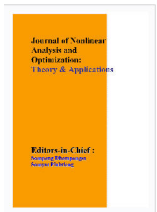
[8] N. Xiu, D. Wang and L. Kong, A note on the gradient projection method with exact stepsize rule, *Journal of Computational Mathematics* 25 (2007), 221-230.

[9] H.K. Xu, Averaged mappings and the gradient-projection algorithm. (Preprint.)

[1]Penn State University at Erie, The Behrend College, 4205 College Drive, Erie, PA 16563-0203, U.S.A.
Email address: mengsu@psu.edu

[2]Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan.
Email address: xuhk@math.nsysu.edu.tw

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 45-53
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Fixed point theorems for nonexpansive mappings with applications to generalized equilibrium and system of nonlinear variational inequalities problems*

Yeol Je Cho[1], Narin Petrot[2] and Suthep Suantai[3]

ABSTRACT: In this paper, a method for finding common element of two nonexpansive mappings are provided. Consequently, we provide some results for the fixed points of an infinite family of nonexpansive mappings and of an infinite family of strict pseudo-contraction mappings. Furthermore, we apply our main result to the problems of finding solution of generalized equilibrium and system of nonlinear variational inequalities problems. Some interesting remarks will be also pointed out and discussed.

KEYWORDS: Generalized equilibrium problem; system of nonlinear variational inequalities; nonexpansive mapping; strict pseudo-contraction mappings.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of $\mathcal{H}$. A mapping $D : K \to K$ is said to be *nonexpansive mapping* if

$$\|Dx - Dy\| \leq \|x - y\|, \quad \forall x, y \in K.$$

If $D : K \to K$ is a nonexpansive mapping, we denote the set of fixed points of D by $F(D)$, that is, $F(D) = \{x \in K : Dx = x\}$. By assuming that D is a nonexpansive mapping such that its fixed points set is not empty, a classical iterative method to find the fixed point of D of minimal norm was firstly studied by Halpern [7]. He introduced the following explicit iteration scheme($u = 0$): for fixed $u, x_0 \in K, a_n \in (0, 1)$

$$x_{n+1} = a_n u + (1 - a_n) D x_n, \quad n = 0, 1, 2, \ldots \tag{1}$$

and pointed out that the control conditions (C_1) $\lim_{n\to\infty} a_n = 0$ and (C_2) $\sum_{n=1}^{\infty} a_n = \infty$ are necessary for the convergence of the iteration scheme (1) to a fixed point of D. Since then, various extensions of Halpern result have been proposed. For examples, Lions [8] and Wittmann [21] established the strong convergence of the iteration scheme (1) under the control conditions

Corresponding author: Narin Petrot (narinp@nu.ac.th)
Manuscript received July 11, 2010. Accepted July 31, 2010.
*Supported by Faculty of Science, Naresuan University and the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

(C_1), (C_2) together with some additional control conditions. In 2007, Y. Yao et al.[22] introduced and studied the following implicit iterative scheme $\{x_n\}$:

(2) $$x_0 \in K, \quad x_n = a_n u + b_n x_{n-1} + c_n D x_n, \quad n \geq 1,$$

where u is an anchor and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three real sequences in $(0,1)$. They showed that under some suitable control conditions, the sequence $\{x_n\}$ converges to a fixed point D.

Until now, the problem of finding the fixed points of the nonlinear mappings is the one subject of current interest in functional analysis. Motivated by Y. Yao et al.[22], in this paper we introduce the following an explicit iterative scheme $\{x_n\}$:

***The Algorithm*:** Let $G, D : K \to K$ be two mappings. For any $u, x_1 \in K$, we define the sequence $\{x_n\}$ in K as following:

(3) $$x_{n+1} = a_n u + b_n x_n + c_n \left(\gamma G(x_n) + (1-\gamma) D(x_n)\right), \quad \forall n \geq 1,$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$ and $\gamma \in (0,1)$. One main goal of us is to provide the suitable control conditions for the convergence of $\{x_n\}$.

Related to the fixed point problems, we also have the equilibrium problems which have had a great impact and influence in the development of several branches of pure and applied sciences. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of the set of solutions of the equilibrium problems and the set of fixed points of nonlinear mappings (for examples, see [3, 12, 13, 15, 16, 23] and the references therein).

On the other hand, system of nonlinear variational inequalities problems were introduced by Verma [20]. Recently, Ceng et. al. [5] considered an iterative methods for a system of variational inequalities and obtained a strong convergence theorem for the such problem and a fixed point problem for a single nonexpansive mapping. For more examples, see [17, 18, 19] and the references therein.

All of above motivate us to apply the fixed point theory, our main result, to the problems of finding solution of generalized equilibrium and system of nonlinear variational inequalities problems. Also, some interesting remarks are also discussed. We would like to notice that, the results appeared in this paper can be viewed as an important improvement and extension of the previously known results.

Now we recall some well-known concepts and results.

Let K be a nonempty closed convex subset of $\mathcal{H}$. It is well known that, for any $z \in \mathcal{H}$, there exists a unique nearest point in K, denoted by $P_K z$, such that

$$\|z - P_K z\| \leq \|z - y\|, \quad \forall y \in K.$$

Such a mapping P_K is called the *metric projection* of $\mathcal{H}$ on to K. We know that P_K is nonexpansive. Furthermore, for any $z \in \mathcal{H}$ and $u \in K$,

(4) $$u = P_K z \Longleftrightarrow \langle u - z, w - u \rangle \geq 0, \quad \forall w \in K.$$

Lemma 1.1. [2] *Let K be a nonempty closed convex subset of a strictly convex Banach space E. If, for each $n \geq 1$, $S_n : K \to E$ is a nonexpansive mapping, then there exists a nonexpansive mapping $S : K \to E$ such that*

$$F(S) = \bigcap_{n=1}^{\infty} F(S_n).$$

In particular, if $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, then the mapping $S = \sum_{n=1}^{\infty} \mu_n S_n$ satisfies the above requirement, where $\{\mu_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \mu_n = 1$.

Lemma 1.2. [1] *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $S : K \to K$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero, i.e., if $\{x_n\}$ converges weakly to a point $x \in K$ and $\{x_n - Sx_n\}$ converges to zero, then $x = Sx$.*

Lemma 1.3. [14] *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space E and b_n be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1.$$

Suppose that $x_{n+1} = (1-b_n)l_n + b_n x_n$ for all $n \geq 1$ and

$$\limsup_{n\to\infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|l_n - x_n\| = 0$.

Lemma 1.4. [24] *Assume that $\{\theta_n\}$ is a sequence of nonnegative real numbers such that*

$$\theta_{n+1} \leq (1-a_n)\theta_n + \delta_n, \quad \forall n \geq 1,$$

where $\{a_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} a_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{a_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \theta_n = 0$.

2. **Main Results**

Of course, we will use the sequence $\{x_n\}$, generated by (3), to obtain our main results in this paper. To do so, the behaviors of the sequences $\{a_n\}, \{b_n\}$ or $\{c_n\}$ should be controlled. Here, we assume the following control condition:

Condition $(\mathcal{C})$: Let $\{a_n\}$ and $\{b_n\}$ be defined as in (3). We say that the condition $(\mathcal{C})$ is satisfied if

(i) $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$,

are hold true.

Our main result is as following.

Theorem 2.1. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $G, D : K \to K$ be two nonexpansive mappings such that $\Omega = F(G) \cap F(D) \neq \emptyset$. Let $u \in K$ be fixed and $\{x_n\}$ be a sequence in K generated by (3). If the condition $(\mathcal{C})$ is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_\Omega u$.*

Proof. Firstly, we must assert that the mapping P_Ω is well defined by showing that Ω is a closed convex set. Indeed, since G, D are nonexpansive mappings, we have the set $F(G)$ and $F(D)$ are closed convex subsets of $\mathcal{H}$. Therefore, it follows that $\Omega = F(G) \cap F(D)$ is a closed convex subset of $\mathcal{H}$.

Now the proof is divided into the five steps as following:

Step 1: The sequence $\{x_n\}$ is bounded.

Write $e_n = \gamma G(x_n) + (1-\gamma)D(x_n)$ for all $n \geq 1$. Let $x^* \in \Omega$. Let us consider the following computation:

$$\begin{aligned}
\|e_n - x^*\| &= \|\gamma G(x_n) + (1-\gamma)D(x_n) - x^*\| \\
&\leq \gamma\|G(x_n) - x^*\| + (1-\gamma)\|D(x_n) - x^*\| \\
&\leq \gamma\|x_n - x^*\| + (1-\gamma)\|x_n - x^*\| \\
&= \|x_n - x^*\|, \quad \forall n \geq 1.
\end{aligned}$$

Consequently,

$$\begin{aligned}\|x_2 - x^*\| &= \|a_1 u + b_1 x_1 + c_1 e_1 - x^*\| \\ &\le a_1\|u - x^*\| + b_1\|x_1 - x^*\| + c_1\|e_1 - x^*\| \\ &\le a_1\|u - x^*\| + b_1\|x_1 - x^*\| + c_1\|x_1 - x^*\| \\ &\le a_1\|u - x^*\| + (1-a_1)\|x_1 - x^*\| \\ &\le \max\{\|u - x^*\|, \|x_1 - x^*\|\}. \qquad (5)\end{aligned}$$

From (5) and induction, we know that the sequence $\{x_n\}$ is bounded, as required.

Step 2: $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.
To do this, let us put

$$l_n = \frac{x_{n+1} - b_n x_n}{1 - b_n}, \quad \forall n \ge 1, \qquad (6)$$

which implies that

$$x_{n+1} - x_n = (1 - b_n)(l_n - x_n), \quad \forall n \ge 1. \qquad (7)$$

Now, by (6), (7), Lemma 1.3 and the condition (ii), we show that

$$\limsup_{n\to\infty}\left(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\right) \le 0. \qquad (8)$$

Consider the following computation:

$$\begin{aligned}l_{n+1} - l_n &= \frac{a_{n+1}u + c_{n+1}e_{n+1}}{1 - b_{n+1}} - \frac{a_n u + c_n e_n}{1 - b_n} \\ &= \frac{a_{n+1}}{1 - b_{n+1}}u + \frac{1 - b_{n+1} - a_{n+1}}{1 - b_{n+1}}e_{n+1} - \frac{a_n}{1 - b_n}u - \frac{1 - b_n - a_n}{1 - b_n}e_n \\ &= \frac{a_{n+1}}{1 - b_{n+1}}(u - e_{n+1}) + \frac{a_n}{1 - b_n}(e_n - u) + e_{n+1} - e_n, \quad \forall n \ge 1, \qquad (9)\end{aligned}$$

and,

$$\begin{aligned}\|e_{n+1} - e_n\| &= \|\gamma G(x_{n+1}) + (1-\gamma)D(x_{n+1}) - (\gamma G(x_n) + (1-\gamma)D(x_n)\| \\ &\le \gamma\|G(x_{n+1}) - G(x_n)\| + (1-\gamma)\|D(x_{n+1}) - D(x_n)\| \\ &\le \gamma\|x_{n+1} - x_n\| + (1-\gamma)\|x_{n+1} - x_n\| \\ &= \|x_{n+1} - x_n\|, \quad \forall n \ge 1. \qquad (10)\end{aligned}$$

Using (9) and (10), we have

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \le \frac{a_{n+1}}{1 - b_{n+1}}\|u - e_{n+1}\| + \frac{a_n}{1 - b_n}\|e_n - u\|, \quad \forall n \ge 1.$$

Thus it follows from the conditions (i) and (ii) that

$$\limsup_{n\to\infty}\left(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\right) \le 0,$$

that is, (8) is satisfied.

Step 3: $x_n - e_n \to 0$ as $n \to \infty$.
From (3), we have

$$c_n(e_n - x_n) = x_{n+1} - x_n + a_n(x_n - u),$$

which implies that

$$c_n\|e_n - x_n\| \le \|x_{n+1} - x_n\| + a_n\|(x_n - u)\|$$

and so, from the conditions (i) and (ii), it follows that

$$\lim_{n\to\infty}\|e_n - x_n\| = 0. \qquad (11)$$

Step 4: $\limsup_{n\to\infty}\langle u - P_\Omega u, x_n - P_\Omega u\rangle \le 0$.

Write $\widetilde{x} = P_\Omega u$. Since $\{x_n\}$ is a bounded sequence, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in K$ such that $\{x_{n_j}\}$ converges weakly to a point z as $j \to \infty$ and

$$\limsup_{n\to\infty}\langle u - \widetilde{x}, x_n - \widetilde{x}\rangle = \limsup_{j\to\infty}\langle u - \widetilde{x}, x_{n_j} - \widetilde{x}\rangle. \tag{12}$$

Now, we show that $z \in \Omega = F(G) \cap F(D)$. To show this, we define a mapping $S : K \to K$ by

$$Sx = \gamma G(x) + (1-\gamma)D(x), \quad \forall x \in K.$$

From Lemma 1.1, it follows that S is a nonexpansive mapping such that

$$F(S) = F(G) \cap F(D).$$

Furthermore, from (11), we obtain

$$\lim_{j\to\infty}\|Sx_{n_j} - x_{n_j}\| = 0.$$

Thus, by Lemma 1.2, we have $z \in F(S) = \Omega$. Consequently, from (4) and (12), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - \widetilde{x}, x_n - \widetilde{x}\rangle &= \limsup_{j\to\infty}\langle u - \widetilde{x}, x_{n_j} - \widetilde{x}\rangle \\ &= \langle u - \widetilde{x}, p - \widetilde{x}\rangle \\ &\le 0.\end{aligned}$$

Step 5: $x_n \to \widetilde{x}$ as $n \to \infty$, where $\widetilde{x} = P_\Omega u$.

Notice that

$$\begin{aligned}\|x_{n+1} - \widetilde{x}\|^2 &= \|a_n u + b_n x_n + c_n e_n - \widetilde{x}\|^2 \\ &= \langle a_n(u-\widetilde{x}) + b_n(x_n - \widetilde{x}) + c_n(e_n - \widetilde{x}), x_{n+1} - \widetilde{x}\rangle \\ &\le a_n\langle u - \widetilde{x}, x_{n+1} - \widetilde{x}\rangle + b_n\|x_n - \widetilde{x}\|\|x_{n+1} - \widetilde{x}\| + c_n\|e_n - \widetilde{x}\|\|x_{n+1} - \widetilde{x}\| \\ &\le a_n\langle u - \widetilde{x}, x_{n+1} - \widetilde{x}\rangle + b_n\|x_n - \widetilde{x}\|\|x_{n+1} - \widetilde{x}\| + c_n\|x_n - \widetilde{x}\|\|x_{n+1} - \widetilde{x}\| \\ &= a_n\langle u - \widetilde{x}, x_{n+1} - \widetilde{x}\rangle + (1-a_n)\|x_n - \widetilde{x}\|\|x_{n+1} - \widetilde{x}\| \\ &\le a_n\langle u - \widetilde{x}, x_{n+1} - \widetilde{x}\rangle + \frac{(1-a_n)}{2}\left(\|x_n - \widetilde{x}\|^2 + \|x_{n+1} - \widetilde{x}\|^2\right).\end{aligned}$$

This implies that

$$\|x_{n+1} - \widetilde{x}\|^2 \le (1-a_n)\|x_n - \widetilde{x}\| + 2a_n\langle u - \widetilde{x}, x_{n+1} - \widetilde{x}\rangle. \tag{13}$$

Therefore, using (12) together with the conditions (i) and (ii), (13) and Lemma 1.4, it follows that $x_n \to \widetilde{x}$ as $n \to \infty$. This completes the proof. □

From Theorem 2.1, obviously, we can obtain the following result.

Corollary 2.2. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $D : K \to K$ be a nonexpansive mapping such that $F(D) \neq \emptyset$. Let $u \in K$ be fixed and $\{x_n\}$ be a sequence in K generated by*

$$x_{n+1} = a_n u + b_n x_n + c_n D(x_n), \quad \forall n \ge 1. \tag{14}$$

If the condition (C) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_{F(D)}u$.

Using Lemma 1.1, as an application of Corollary 2.2, we also have the following results:

Corollary 2.3. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\{S_n\}$ be a family of nonexpansive mappings from K into itself such that $\Theta =: \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Let $u \in C$ be fixed and $\{x_n\}$ be a sequence in K generated by (14) with $D = \sum_{n=1}^{\infty}\mu_n S_n$, where $\{\mu_n\}$ is a sequence of positive numbers with $\sum_{n=1}^{\infty}\mu_n = 1$. If the condition (C) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_\Theta u$.*

Remark 2.4. Recall that a mapping $W : C \to C$ is called a *τ-strict pseudo-contraction* with the coefficient $\tau \in [0, 1)$ if

$$\|Wx - Wy\|^2 \leq \|x - y\|^2 + \tau\|(I - W)x - (I - W)y\|^2, \quad \forall x, y \in C.$$

It is obvious that every nonexpansive mapping is a 0-strict pseudo-contraction. Furthermore, if a mapping $W^{(\zeta)} : C \to C$ is defined by $W^{(\zeta)}x = \zeta x + (1 - \zeta)Wx$ for all $x \in C$, where $\zeta \in [\tau, 1)$ is a fixed constant. Then $W^{(\zeta)}$ is a nonexpansive mapping such that $F(W^{(\zeta)}) = F(W)$, see [24]. Using this observation, in stead of the assumption that A and B are nonexpansive mappings, which were proposed in Theorem 2.1, we can further assume that the mappings A and B are strict pseudo-contractions.

Remark 2.5. If $f : C \to C$ is a contractive mapping and we replace u by $f(x_n)$ in the (3), then we can obtain the so-called viscosity iteration method (see [15] for more details).

3. **applications**

In this section, we will apply the Theorem 2.1 to some interesting problems as following:

3.1. **Generalized equilibrium problem**

Let $\varphi : K \to \mathbb{R}$ be a real-valued function, $Q : K \to \mathcal{H}$ be a mapping and $\Phi : \mathcal{H} \times K \times K \to \mathbb{R}$ be an equilibrium-like function. Let r be a positive number. For any $x \in K$, we consider the following problem:

$$\begin{cases} \text{Find } y \in K \text{ such that} \\ \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \quad \forall z \in K, \end{cases} \tag{15}$$

which is known as the *auxiliary generalized equilibrium problem*. We denote the set of solution of problem (15) by $GEP(K, Q, \Phi, \varphi)$. In order to studying the problem (15), we related to the following concept:

Let $T^{(r)} : K \to K$ be the mapping such that, for each $x \in K$, $T^{(r)}(x)$ is the solution set of the auxiliary problem (15), i.e.,

$$T^{(r)}(x) = \{y \in K : \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \ \forall z \in K\}, \quad \forall x \in K.$$

We will assume the following **Condition (Δ)**:

(a) $T^{(r)}$ is single-valued;
(b) $T^{(r)}$ is nonexpansive;
(c) $F(T^{(r)}) = GEP(K, Q, \Phi, \varphi)$;

Notice that the examples of showing the sufficient conditions for the existence of the Condition (Δ) can be found in [4].

Assuming that the Condition (Δ) is satisfied, then we can introduce the following algorithm: Let r be a fixed positive number and $D : K \to K$ be a mapping. For any $u, x_1 \in K$, there exist sequences $\{u_n\}$, and $\{x_n\}$ in K such that

$$\begin{cases} \Phi(Qx_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r}\langle u_n - x_n, v - u_n\rangle \geq 0, \quad \forall v \in K, \\ x_{n+1} = a_n u + b_n x_n + c_n \left[\gamma u_n + (1 - \gamma)D(x_n)\right], \quad \forall n \geq 1, \end{cases} \tag{16}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \geq 1$ and $\gamma \in (0, 1)$.

Theorem 3.1. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $D : K \to K$ be a nonexpansive mapping. Assume that the Condition (Δ) is satisfied and*

$$\Omega = GEP(K, Q, \Phi, \varphi) \cap F(D) \neq \emptyset.$$

Let $u \in K$ be fixed and $\{x_n\}$ is defined by (16). If the condition ($\mathcal{C}$) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_\Omega u$.

Proof. Notice that, for each $n \geq 1$, we have $u_n = T^{(r)}(x_n)$. Hence, by setting $T^{(r)} =: G$, we see that (3) and (16) are the same. Therefore, thanks to the condition (Δ), the result is followed from Theorem 2.1.

□

Remark 3.2. It is worth to mention that for appropriate and suitable choice of the mapping Q, the functions Φ, φ and the convex set K, one can obtain a number of the various classes of equilibrium problems as special cases. This means, evidently, the Theorem 3.1 is very useful. For further applications of the problem (15), interested readers may refer to [6, 10, 11], and the references therein.

3.2. **System of nonlinear variational inequalities problems**

For two nonlinear mappings $A, B : K \to \mathcal{H}$, we consider the following *system of nonlinear variational inequalities problems*: Find $(x^*, y^*) \in K \times K$ such that

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in K, \\ \langle \rho B x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in K, \end{cases} \tag{17}$$

where λ and ρ are fixed positive numbers. In recent years, the problem of type (17) and its applications have been studied and investigated by many authors, see [5, 17, 18, 19, 20] for examples.

Now we have the following result, as the technical lemma:

Lemma 3.3. *Let K be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $G : K \to K$ be a nonexpansive self-mapping and $T, V : K \to \mathcal{H}$ be two nonexpansive non-self mappings. Assume that*

$$\Omega = F(G) \cap F(D) \neq \emptyset,$$

where $D : K \to K$ is defined by

$$D(x) = P_K[T \circ (P_K \circ V)](x), \quad \forall x \in K. \tag{18}$$

Let $u \in K$ be fixed and $\{x_n\}$ be a sequences in K generated by (3). If the condition ($\mathcal{C}$) is satisfied then the sequence $\{x_n\}$ converges strongly to a point $\widetilde{x} = P_\Omega u$.

Proof. Since $T, V : K \to \mathcal{H}$ are nonexpansive non-self mappings, it is obvious that $D := P_K[T \circ (P_K \circ V)]$ is a nonexpansive mapping. Therefore, the conclusion is followed immediately from Theorem 2.1.

□

We also need the following well-known lemma:

Lemma 3.4. [5] *Let ρ and λ be positive numbers. For any $x^*, y^* \in C$ with $y^* = P_C(x^* - \rho B x^*)$, (x^*, y^*) is a solution of the problem (1.3) if and only if x^* is a fixed point of the mapping $D : C \to C$ defined by*

$$D(x) = P_K[P_K(x - \rho Bx) - \lambda A P_K(x - \rho Bx)], \quad \forall x \in K.$$

Now we give the purposed result.

Theorem 3.5. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $A, B : K \to \mathcal{H}$ and $G : K \to K$ be a nonexpansive mapping. Assume that*

$$\Omega := F(G) \cap F(D) \neq \emptyset,$$

where the mapping $D : K \to K$ is defined by

$$D(x) = [P_K(I - \lambda A) \circ P_K(I - \rho B)](x), \quad \forall x \in K, \tag{19}$$

when ρ and λ are positive constants appeared in the problem (17). Assume that

(i) $(I-\lambda A)$ *and* $(I-\rho B)$ *are nonexpansive mappings;*
(ii) *the condition (C) is satisfied.*

If $\{x_n\}$ *is a sequence in* K *generated by (3) then* $\{x_n\}$ *converges strongly to a point* $\widetilde{x} = P_\Omega u$. *Moreover, if* $\widetilde{y} = P_K(\widetilde{x} - \rho B\widetilde{x})$, *then* $(\widetilde{x}, \widetilde{y})$ *is a solution to the problem (17).*

Proof. Firstly, since $I-\lambda A$ and $I-\rho B$ are nonexpansive mappings, we know that D is a nonexpansive mapping. Thus, thanks to Lemma 3.3, we know that $\{x_n\}$ converges strongly to $\widetilde{x} := P_\Omega u$. Moreover, by the definition of the mapping D, we observe that

$$D = P_K(I-\lambda A)\circ P_K(I-\rho B) = P_K\left[P_K(I-\rho B) - \lambda A P_K(I-\rho B)\right].$$

Consequently, in view of Lemma 3.4, the second part of the required result is followed immediately.

□

Remark 3.6. Recall that a nonlinear mapping $A : K \to \mathcal{H}$ is said to be:

(1) *α-cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in K;$$

(2) *β-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in K.$$

Notice that if either

(A1) A is α-cocoercive mapping and $\lambda \in (0, 2\alpha]$ or,

(A2) A is β-strongly monotone and L-Lipschitz continuous mapping and $\lambda \in \left(0, \frac{2\beta}{L}\right]$;

is satisfied, then $I - \lambda A$ is a nonexpansive mapping. This means that the results obtained in the Theorem 3.5 can be viewed as an important extension of the previously known results.

Remark 3.7. In view of Corollary 2.3, one can apply Theorems 3.1 and 3.5 from a nonexpansive mapping to a family of nonexpansive mappings (or even a family of strict pseudo-contraction mappings).

References

1. F.E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, Arch. Ration. Mech. Anal. 24 (1967) 82-90.
2. R.E. Bruck, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. 179 (1973) 251-262.
3. S.S. Chang, H.W.J. Lee, C.K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, Nonlinear Anal., 70 (2009), 3307-3319.
4. L.C. Ceng, Q. H. Ansari, J.C. Yao, Viscosity approximation methods for generalized equilibrium problems and fixed point problems, J. Glob. Optim. 43 (2009) 487-502.
5. L.C. Ceng, C.Y. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Meth. Oper. Res. 67 (2008) 375-390.
6. F. Flores-Bazan, Existence theorems for generalized noncoercive equilibrium problems: the quasi-convex case, SIAM J. Optim. 11 (2000) 675-690.
7. B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957-961.
8. P.L. Lions, Approximation de points fixes de contraction, C.R.Acad. Sci. Ser. A-B Pairs (284) (1977) 1357-1359.
9. A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000) 46-55.
10. M.A. Noor, Variational-like inequalities, Optim. 30 (1994) 323-330.
11. M.A. Noor, W. Oettli, On general nonlinear complementarity problems and quasi equilibria, Matematiche (Catania) 49 (1994) 313-331.
12. X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilib- rium problems and fixed point problems in Banach spaces, J. Comput. Appl. Math. 225(1) (2009), 20-30.
13. X. Qin, M. Shang, Y. Su, Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems, Math. Comput. Model. 48 (2008) 1033-1046.
14. T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one- parameter nonexpansive semigroups without Bochne integrals, J. Math. Anal. Appl. 305 (2005) 227-239.
15. A. Tada, W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, J. Optim. Theory Appl. 133 (2007) 359-370.

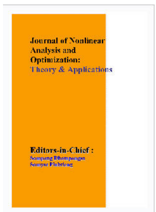
16. S. Takahashi, W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025-1033.
17. R.U. Verma, Generalized system for relaxed cocoercive variational inequalities and its projection methods, J. Optim. Theory Appl. 121 (2004) 203-210.
18. R.U. Verma, Generalized class of partial relaxed monotonicity and its connections, Adv. Nonlinear Var. Inequal. 7 (2004) 155-164.
19. R.U. Verma, General convergence analysis for two-step projection methods and applications to variational problems, Appl. Math. Lett. 18 (2005) 1286-1292.
20. R.U. Verma, On a new system of nonlinear variational inequalities and associated iterative algorithms, Math. Sci. Res. Hot-Line 3 (1999) 65-68.
21. R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 59 (1992) 486-491.
22. Y. Yao, Y.C. Liou, R. Chen, Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces, Nonlinear Analysis 67 (2007) 33113317.
23. Y. Yao, M.A. Noor, S. Zainab, Y.C. Liou, Mixed equilibrium problems and optimization problems, J. Math. Anal. Appl. 354 (2009) 319-329.
24. H.Y. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 69 (2008) 456-462.

[1]Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Korea.
Email address: yjcho@gsnu.ac.kr

[2]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand.
Email address: narinp@nu.ac.th

[3]Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand.
Email address: scmti005@chiangmai.ac.th

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 55-69
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



A new hybrid method for solving generalized equilibrium problem and common fixed points of asymptotically quasi nonexpansive mappings in Banach spaces*

Uthai Kamraksa, Rabian Wangkeeree

ABSTRACT: In this paper, we introduce a new hybrid projection iterative scheme based on the shrinking projection method for two asymptotically quasi-ϕ-nonexpansive mappings, for finding a common element of the set of solutions of the generalized mixed equilibrium problems and the set of common fixed points of two asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces. The results obtained in this paper improve and extend the recent ones announced by Matsushita and Takahashi [S. Matsushita, W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. (2004), 2004, 37-47], Qin et al. [X. Qin, S.Y. Cho, S.M Kang, On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings, Applied Mathematics and Computation 215 (2010) 38743883], and Chang, Lee and Chan [S.-s. Chang, H.W. Joseph Lee, C.K. Chan, A new hybrid method for solving generalized equilibrium problem variational inequality and common fixed point in banach spaces with applications, Nonlinear Analysis (2010), doi:10.1016/j.na.2010.06.006] and many others.

KEYWORDS: Generalized mixed equilibrium problem, Asymptotically quasi-ϕ-nonexpansive mapping, Strong convergence theorem, and Banach space.

1. Introduction

Let E be a real Banach space, and E^* the dual space of E. Let C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real-valued function, and $A : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem, is to find $x \in C$ such that

(1) $$f(x,y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0,\ \forall y \in C.$$

The set of solutions to (1) is denoted by EP, i.e.,

(2) $$f(x,y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0,\ \forall y \in C.$$

Corresponding author: Rabian Wangkeeree (rabianw@nu.ac.th).
Manuscript received May 11, 2010. Accepted June 14, 2010.
*Supported by Faculty of Science, Naresuan University, Thailand.

If $\varphi = 0$, the problem (1) reduces to the generalized equilibrium problem for f, denoted by $GEP(f)$, which is to find $x \in C$ such that

$$f(x,y) + \langle Ax, y-x\rangle \geq 0, \ \forall y \in C. \tag{3}$$

If $A = 0$, the problem (1) reduces to the mixed equilibrium problem for f, denoted by $MEP(f,\varphi)$, which is to find $x \in C$ such that

$$f(x,y) + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C. \tag{4}$$

If $f \equiv 0$, the problem (1) reduces to the mixed variational inequality of Browder type, denoted by $VI(C,A,\varphi)$, which is to find $x \in C$ such that

$$\langle Ax, y-x\rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C. \tag{5}$$

If $A = 0$ and $\varphi = 0$, the problem (1) reduces to the equilibrium problem for f, denoted by $EP(f)$, which is to find $x \in C$ such that

$$f(x,y) \geq 0, \ \forall y \in C. \tag{6}$$

Let $f(x,y) = \langle Ax, y-x\rangle$ for all $x,y \in C$. Then $p \in EP(f)$ if and only if $\langle Ap, y-p\rangle \geq 0$ for all $y \in C$, i.e., p is a solution of the variational inequality; there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is a unifying model for several problems arising in physical sciences. In the last two decades, many papers have appeared in the literature on the existence of solutions of the EP; see, for example [5, 17, 19, 20] and references therein. Some solution methods have been proposed to solve the EP; see, for example [9, 10, 15, 16, 22, 24, 30, 32, 34] and references therein.

Recall that a mappings $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x,y \in C.$$

T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \text{ for all } x \in C, y \in F(T).$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \text{ for all } x,y \in C.$$

T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - y\| \leq k_n\|x - y\|, \text{ for all } x \in C, y \in F(T).$$

T is called uniformly L-Lipschitzian continuous if there exists a $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \text{ for all } x,y \in C.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [18] in 1972. Since 1972, a host of authors have studied the weak and strong convergence of iterative processes for such a class of mappings.

If C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is a nonexpansive mapping. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [3] recently introduced a generalized projection operator C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{7}$$

for all $x,y \in E$, where J is the normalized duality mapping from E to E^*. Observe that, in a Hilbert space H, (7) reduces to $\phi(y,x) = \|x-y\|^2$ for all $x,y \in H$. The generalized projection

$\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y,x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y \in C} \phi(y,x). \tag{8}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y,x)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 9, 28]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y,x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$.
(2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$ for all $x,y,z \in E$.
(3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$.
(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x,y) = 0 \text{ if and only if } x = y.$$

For more detail see [14, 31]. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed point of T. A point p in C is said to be an *asymptotic fixed point* of T [29] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. Recall that the following :

(i) A mapping $T : C \to C$ is called *relatively nonexpansive* [7, 8, 11] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p,x)$ for all $x \in C$ and $p \in F(T)$.
The asymptotic behavior of relatively nonexpansive mappings were studied in [7, 8].
(ii) $T : C \to C$ is said to be *relatively asymptotically nonexpansive* [1, 28] if $\hat{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leqslant k_n \phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.
(iii) $T : C \to C$ is said to be *ϕ-nonexpansive* [26, 36] if $\phi(Tx, Ty) \leq \phi(x,y)$ for all $x, y \in C$.
(iv) $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* [26, 36] if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$.
(v) $T : C \to C$ is said to be *asymptotically ϕ-nonexpansive* [36] if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(T^n x, T^n y) \leq k_n \phi(x,y)$ for all $x, y \in C$.
(vi) $T : C \to C$ is said to be *asymptotically quasi-ϕ-nonexpansive* [36] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.
(vii) $T : C \to C$ is said to be *asymptotically regular* on C if, for any bounded subset D of C, there holds the following equality :

$$\lim_{n\to\infty} \sup_{x \in D} \|T^{n+1}x - T^n x\| = 0.$$

(viii) $T : C \to C$ is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Remark 1.1. The class of (asymptotically) quasi-ϕ-nonexpansive mappings is more general than the class of relatively (asymptotically) nonexpansive mappings which requires the strong restriction $\hat{F}(T) = F(T)$.

Remark 1.2. In real Hilbert spaces, the class of (asymptotically) quasi-ϕ-nonexpansive mappings is reduced to the class of (asymptotically) quasi-nonexpansive mappings.

We give some examples which are closed and asymptotically quasi-ϕ-nonexpansive.

Example 1.3. (1). Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then $J_r = (J + rA)^{-1}J$ is a closed and asymptotically quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

(2). Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then Π_C is a closed and asymptotically quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

Recently, Matsushita and Takahashi [25] obtained the following results in a Banach space.
Theorem MT. Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} < 1$. Suppose that $\{x_n\}$ is given by

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0,\ n = 0, 1, 2, \ldots, \end{cases} \tag{9}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

Recently, Qin et al. [27] further extended Theorem MT by considering a pair of asymptotically quasi-ϕ-nonexpansive mappings. To be more precise, they proved the following results.

Theorem QCK. Let E be a uniformly smooth and uniformly convex Banach space and C a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_s^{(t)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be real number sequences in $[0, 1]$. Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}(\beta_n J x_n + \gamma_n J(T^n x_n) + \delta_n J(S^n x_n)), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ C_{n+1} = \{w \in C_n : \phi(w, y_n) \leq \phi(w, x_n) + (k_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{10}$$

where $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, J is the duality mapping on E, $M_n = \sup\{\phi(z, x_n) : z \in \Omega\}$ for each $n \geq 1$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the following restrictions :

(a) $\beta_n + \gamma_n + \delta_n = 1,\ \forall n \geq 1$;
(b) $\liminf_{n\to\infty} \gamma_n \delta_n$, $\lim_{n\to\infty} \beta_n = 0$;
(c) $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$.

On the other hand, very recently, Chang, Lee and Chan [12] proved a strong convergence theorem for finding a common element of the set of solutions for a generalized equilibrium problem (3) and the set of common fixed points for a pair of relatively nonexpansive mappings in Banach spaces. They proved the following results.

Theorem CLC. Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a α-inverse-strongly monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1) - (A4)$. Let $S, T : C \to C$ be two relatively nonexpansive mappings such that $\Omega := F(T) \cap F(S) \cap GEP(f)$.

Let $\{x_n\}$ be the sequence generated by

$$(11)\quad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSx_n), \\ u_n \in C \textit{ such that} \\ \quad f(u_n, y) + \langle Au_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0,\ \forall y \in C, \\ H_n = \{v \in C : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)\phi(v, x_n)\}; \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0,\ \forall n \geq 0, \end{cases}$$

where $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [a, 1)$ for some $a > 0$. If the following conditions are satisfied:

(a) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(b) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$, where Π_Ω is the generalized projection of E onto Ω.

In this paper, motivated and inspired by the work of Matsushita and Takahashi [25], Qin et al. [27], and Chang, Lee and Chan [12], we introduce a new hybrid projection iterative scheme based on the shrinking projection method for two asymptotically quasi-ϕ-nonexpansive mappings, for finding a common element of the set of solutions of the generalized mixed equilibrium problems and the set of common fixed points of two asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces. The results obtained in this paper improve and extend the recent ones announced by Matsushita and Takahashi [25], Qin et al. [27], and Chang, Lee and Chan [12] and many others.

2. **Preliminaries**

For the sake of convenience, we first recall some definitions and conclusions which will be needed in proving our main results. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},\ x \in E.$$

is called the normalized duality mapping. By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in E$.

In the sequel, we denote the strong convergence, weak convergence and weak* convergence of a sequence $\{x_n\}$ by $x_n \to x$, $x_n \rightharpoonup x$ and $x_n \rightharpoonup^* x$, respectively.

A Banach space E is said to be strictly convex, if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. It is said to be uniformly convex, if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$.

It is well-known that a uniformly convex Banach space has the Kadec-Klee property, i.e., if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

The space E is said to be smooth, if the limit

$$(12)\quad \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. And E is said to be uniformly smooth, if the limit (12) exists uniformly in $x, y \in U$.

Remark 2.1. It is wellknown that if E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one and onto (see [14]).

Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Throughout this paper the Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall x, y \in E. \tag{13}$$

Following Alber [4], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = argmin_{y\in C}\phi(y,x),\ \forall x \in E. \tag{14}$$

If E is a real Hilbert space H, then $\phi(x,y) = \|x-y\|^2$ and Π_C is the metric projection of H onto C.

In order to our main results, we need the following concepts and lemmas.

Let E be a real Banach space, C a nonempty subset of E and $T : C \to C$ a nonlinear mapping. The mapping T is said to be uniformly asymptotically regular on C if

$$\lim_{n\to\infty}\left(\sup_{x\in C}\|T^{n+1}x - T^n x\|\right) = 0.$$

The mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Lemma 2.2. ([2, 4, 23]) *Let E be a reflexive, strictly convex and smooth Banach space, C a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y,x),\ \forall y \in C.$$

Lemma 2.3. ([4, 23]) *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset. Then the following conclusion hold:*

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y);\ \forall x \in C,\ y \in E;$
(2) *Let $x \in E$ and $z \in C$, then*

$$z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle,\ \forall y \in C.$$

Lemma 2.4. ([13]) *Let E be a uniformly convex Banach space and $B_r(0)$ a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \|\alpha x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|)$$

for all $x, y \in B_r(0)$ and $\alpha \in [0,1]$.

Lemma 2.5. ([27]) *Let E be a uniformly convex and smooth Banach space, C a nonempty closed convex subset of E and $T : C \to C$ a closed asymptotically quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.6. ([23]) *Let E be a smooth and uniformly convex Banach space. Let x_n and y_n be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $lim_{n\to\infty}\phi(x_n, y_n) = 0$, then $lim_{n\to\infty}\|x_n - y_n\| = 0$.*

For solving the generalized equilibrium problem, let us assume that the nonlinear mapping $A : C \to E^*$ is α-inverse strongly monotone and the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x,x) = 0\ \forall x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \le 0,\ \forall x, y \in C$;
(A3) $\limsup_{t\downarrow 0} f(x + t(z-x), y) \le f(x,y),\ \forall x, y, z \in C$;
(A4) the function $y \mapsto f(x,y)$ is convex and lower semicontinuous.

Lemma 2.7. ([5]) *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1) - (A4)$. Let $r > 0$ and $x \in E$, then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0,\ \forall y \in C. \tag{15}$$

Lemma 2.8. ([33]) *Let C be a closed convex subset of a uniform smooth, strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y-z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\},$$

for all $x \in C$. Then, the following conclusions holds:

(1) *T_r is single-valued ;*

(2) *T_r is a firmly nonexpansive-type mapping, i.e.;*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle,\ \forall x,y \in E;$$

(A3) *$F(T_r) = EP(f)$;*

(A4) *$EP(f)$ is a closed convex.*

Lemma 2.9. ([34]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r(x), x) \leq \phi(q, x).$$

Lemma 2.10. ([35]) *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$f(u,y) + \langle Au, y-u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y-u, Ju - Jx\rangle,\ \forall y \in C.$$

Define a mapping $K_r : C \to C$ as follows:

$$K_r(x) =$$

$$\left\{ u \in C : f(u,y) + \langle Au, y-u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y-u, Ju-Jx\rangle \geq 0,\ \forall y \in C \right\} \tag{16}$$

for all $x \in C$. Then, the following conclusions holds:

(a) K_r is single-valued ;

(b) K_r is a firmly nonexpansive-type mapping, *i.e.;*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle,\ \forall x,y \in E;$$

(c) $F(K_r) = \hat{F}(K_r) = EP$;

(d) EP is a closed convex,

(e) $\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, x),\ \forall p \in F(K_r),\ z \in E.$

Remark 2.11. ([35]) It follows from Lemma 2.10 that the mapping $K_r : C \to C$ defined by (16) is a relatively nonexpansive mapping. Thus, it is quasi-ϕ-nonexpansive.

3. **Main Results**

In this section, we shall prove a strong convergence theorem for finding a common element of the set of solutions for a generalized mixed equilibrium problem (1) and the set of common fixed points for a pair of asymptotically quasi-ϕ-nonexpansive mapping mappings in Banach spaces.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1,\infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1,\infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$.*

Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by

$$
(17) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n), \\ f(u_n, y) + \langle A u_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) k_n \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}
$$

where $\theta_n = (1-\beta_n)(k_n^2 - 1) M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Suppose that the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. First, we define two bifunctions $H : C \times C \to \mathbb{R}$ and $K_r : C \to C$ by

$$
(18) \qquad H(x, y) = f(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x), \quad \forall x, y \in C,
$$

and

$$
(19) \qquad K_r(x) = \{u \in C : H(u, y) + \frac{1}{r}\langle y - u, Ju - Jx \rangle \geq 0, \ \forall y \in C\}.
$$

By Lemma 2.10, we know that the function H satisfies conditions (A1) - (A4) and K_r has the properties (a)-(e). Therefore, (17) is equivalent to

$$
(20) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n), \\ H(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) k_n \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}
$$

(I) We show first that the sequence $\{x_n\}$ is well defined. By the same argument as in the proof of [36, Lemma 2.4], one can show that $F(T) \cap F(S)$ is closed and convex. Hence $\Omega := F(S) \cap F(T) \cap EP$ is a nonempty, closed and convex subset of C. Consequently, Π_Ω is well defined. Next, we prove that C_n is closed and convex for each $n \geq 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some positive integer h. Next, we prove that C_{h+1} is closed and convex. For $w \in C_{h+1}$, we see that

$$
\phi(w, u_h) \leq \beta_h \phi(w, x_n) + (1-\beta_h) k_h \phi(w, z_h)
$$

is equivalent to

$$
\begin{aligned} 2\langle w, (1-\beta_h) J z_h + \beta_h J x_h - J u_h \rangle \quad \leq \quad & (1-\beta_h) k_h \|z_h\|^2 - \|u_h\|^2 + \beta_h \|x_h\|^2 \\ & + (\beta_h + k_h - \beta_h k_h - 1)\|w\|^2, \end{aligned}
$$

and

$$
\beta_h \phi(w, x_h) + (1-\beta_h) k_h \phi(w, z_h) \leq \phi(w, x_h) + \theta_n
$$

is equivalent to

$$\phi(w, z_h) \leq \phi(w, x_h) + (k_h^2 - 1)M_h.$$

It is easy to see that C_{h+1} is closed and convex. Then, for each $n \geq 1$, we see C_n is closed and convex.

(II) Next we prove that $\Omega \subset C_n$ for each $n \geq 1$.

If $n = 1$, $\Omega \subset C_1 = C$ is obvious. Suppose that $\Omega \subset C_h$ for some positive integer h. Next, we claim that $\Omega \subset C_{h+1}$ for the same h. For every $w \in \Omega$, we obtain from the assumption that $w \in C_h$: On the other hand, we have

$$\begin{aligned}
\phi(w, z_h) &= \phi(w, J^{-1}(\alpha_h Jx_h + (1-\alpha_h)JT^h x_h)) \\
&= \|w\|^2 - 2\langle w, \alpha_h Jx_h + (1-\alpha_h)JT^h x_h\rangle + \|\alpha_h Jx_h + (1-\alpha_h)JT^h x_h\|^2 \\
&\leq \|w\|^2 - 2\alpha_h\langle w, Jx_h\rangle - 2(1-\alpha_h)\langle w, JT^h x_h\rangle \\
&\quad + \alpha_h\|x_h\|^2 + (1-\alpha_h)\|T^h x_h\|^2 \\
&= \alpha_h\phi(w, x_h) + (1-\alpha_h)\phi(w, T^h x_h) \\
&\leq \alpha_h\phi(w, x_h) + (1-\alpha_h)k_h^{(t)}\phi(w, x_h) \\
&\leq \alpha_n\phi(w, x_h) + (1-\alpha_h)k_h\phi(w, x_h) \\
&\leq \phi(w, x_h) + (k_h - 1)\phi(w, x_h). \qquad (21)
\end{aligned}$$

It follows that

$$\begin{aligned}
\phi(w, u_h) &= \phi(w, K_{r_h}y_h) \leq \phi(w, y_h) \\
&\leq \phi(w, J^{-1}(\beta_h Jx_h + (1-\beta_h)JS^h z_h)) \\
&= \|w\|^2 - 2\langle w, \beta_h Jx_h + (1-\beta_h)JS^h z_h\rangle + \|\beta_h Jx_h + (1-\beta_h)JS^h z_h\|^2 \\
&\leq \|w\|^2 - 2\beta_h\langle w, Jx_h\rangle - 2(1-\beta_h)\langle w, JS^h z_h\rangle + \beta_h\|x_h\|^2 + (1-\beta_h)\|S^h z_h\|^2 \\
&= \beta_h\phi(w, x_h) + (1-\beta_h)\phi(w, S^h z_h) \\
&\leq \beta_h\phi(w, x_h) + (1-\beta_h)k_h^{(s)}\phi(w, z_h) \\
&\leq \beta_h\phi(w, x_h) + (1-\beta_h)k_h\phi(w, z_h) \\
&= (1-(1-\beta_n))\phi(w, x_h) + (1-\beta_h)k_h\phi(w, z_h) \\
&= \phi(w, x_h) + (1-\beta_h)[k_h\phi(w, z_h) - \phi(w, x_h)] \\
&\leq \phi(w, x_h) + (1-\beta_h)[k_h(\phi(w, x_h) + (k_h-1)\phi(w, x_h)) - \phi(w, x_h)] \\
&= \phi(w, x_h) + (1-\beta_h)[k_h\phi(w, x_h) + (k_h^2 - k_h)\phi(w, x_h) - \phi(w, x_h))] \\
&= \phi(w, x_h) + (1-\beta_h)(k_h^2 - 1)\phi(w, x_h) \\
&\leq \phi(w, x_h) + (1-\beta_h)(k_h^2 - 1)M_h \\
&= \phi(w, x_h) + \theta_n. \qquad (22)
\end{aligned}$$

This shows that $w \in C_{h+1}$. This implies that $\Omega \subset C_n$ for each $n \geq 1$.

From $x_n = \Pi_{C_n}x_1$, we see that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \geq 0, \ \forall w \in C_n.$$

Since $\Omega \subset C_n$ for each $n \geq 1$, we arrive at

$$\langle x_n - w, Jx_1 - Jx_n\rangle \geq 0, \ \forall w \in \Omega. \qquad (23)$$

(III) Now we prove that $\{x_n\}$ is bounded.

In view of Lemma 2.2, we see that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \leq \phi(w, x_1) - \phi(w, x_n) \leq \phi(w, x_1),$$

for each $w \in C_n$. Therefore, we obtain that the sequence $\phi(x_n, x_1)$ is bounded, so are $\{x_n\}$, $\{y_n\}$, $\{T^n x_n\}$, $\{S^n x_n\}$ and $\{z_n\}$.

(IV) Now we prove that $\|x_n - T^n x_n\| \to 0$ and $\|z_n - S^n z_n\| \to 0$.
Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \ \forall n \geq 1.$$

This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing, and so the limit $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. By the construction of C_n, we have

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned} \tag{24}$$

Letting $m, n \to \infty$ in (24), we see that $\phi(x_m, x_n) \to 0$. It follows from Lemma 2.6 that $x_m - x_n \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E a Banach space and C is closed and convex, we can assume that

$$\lim_{n\to\infty} x_n = p \in C. \tag{25}$$

Now, we are in a position to state that $p \in \Omega = F(T) \cap F(S) \cap EP$. By taking $m = n+1$ in (24), we obtain that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{26}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, from definition of C_{n+1} we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \theta_n, \ \forall n \geq 1, \tag{27}$$

and

$$k_n \phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) + (k_n^2 - 1) M_n, \ \forall n \geq 1. \tag{28}$$

Since E is uniformly smooth and uniformly convex, from (26)-(28), $\theta_n \to 0$ as $n \to \infty$ and Lemma 2.6, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\|, \tag{29}$$

and so

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{30}$$

Since $u_n = K_{r_n} y_n$, from (22) we have

$$\phi(u, y_n) \leq \phi(u, x_n) + \theta_n, \ \forall u \in \Omega. \tag{31}$$

Since $\|x_n - u_n\| \to 0$ and J is uniformly continuous, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\ &\leq \phi(u, y_n) - \phi(u, K_{r_n} y_n) \\ &\leq \phi(u, x_n) - \phi(u, K_{r_n} y_n) + \theta_n \\ &= \phi(u, x_n) - \phi(u, u_n) + \theta_n \\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n \rangle + \theta_n \\ &\leq \|x_n - u_n\| \times (\|x_n\| + \|u_n\|) - 2\langle u, Jx_n - Ju_n \rangle + \theta_n \to 0. \end{aligned}$$

This implies that $\phi(y_n, u_n) \to 0$. Since E is smooth and uniformly convex, from Lemma 2.6, we have

$$\|y_n - u_n\| \to 0, \text{ and so } \|y_n - x_n\| \to 0. \tag{32}$$

From (17), we have

$$\|Jy_n - Jx_n\| = (1 - \beta_n) \|JS^n z_n - Jx_n\| \to 0, \tag{33}$$

and so $\|S^n z_n - x_n\| \to 0$. This together with $\|x_n - z_n\| \to 0$ yields

$$\|z_n - S^n z_n\| \to 0. \tag{34}$$

Again from (30) and (17) we have

$$\|Jz_n - Jx_n\| = (1 - \alpha_n) \|JT^n x_n - Jx_n\| \to 0. \tag{35}$$

This implies that $\|JT^n x_n - Jx_n\| \to 0$, and so

(36) $$\|T^n x_n - x_n\| \to 0 \text{ as } n \to \infty.$$

(V) Now we prove that $p \in F(T) \cap F(S) \cap EP = \Omega$.

From (36) and (30), we have

(37) $$\lim_{n\to\infty} \|T^n x_n - z_n\| = 0.$$

Note that

(38) $$\|T^n x_n - p\| \leq \|T^n x_n - z_n\| + \|z_n - x_n\| + \|x_n - p\|.$$

It follows from (37), (30) and (25) that

(39) $$\lim_{n\to\infty} \|T^n x_n - p\| = 0.$$

On other hand, we have

$$\|T^n x_n - p\| \leq \|T^{n+1} x_n - T^n x_n\| + \|T^n x_n - p\|.$$

Since T is uniformly asymptotically regular and (39), we obtain that

(40) $$\|T^{n+1} x_n - p\| = 0.$$

that is, $TT^n x_n \to p$ as $n \to \infty$. From the closedness of T, we see that $p \in F(T)$.
From (34) and (30), we have

(41) $$\lim_{n\to\infty} \|S^n x_n - z_n\| = 0.$$

Note that

(42) $$\|S^n x_n - p\| \leq \|S^n x_n - z_n\| + \|z_n - x_n\| + \|x_n - p\|.$$

It follows from (41), (30) and (25) that

(43) $$\lim_{n\to\infty} \|S^n x_n - p\| = 0.$$

On other hand, we have

$$\|S^n x_n - p\| \leq \|S^{n+1} x_n - S^n x_n\| + \|S^n x_n - p\|.$$

Since S is uniformly asymptotically regular and (43), we obtain that

(44) $$\|S^{n+1} x_n - p\| = 0.$$

that is, $SS^n x_n \to p$ as $n \to \infty$. From the closedness of S, we see that $p \in F(S)$.

Next we prove that $p \in EP$. Since $\lim_{n\to\infty} \|u_n - y_n\| = 0$ and J is uniformly norm-to-norm continuous on bounded sets, we have

(45) $$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0.$$

From the assumption $r_n > a$, we obtain

(46) $$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0.$$

Noticing that $u_n = K_{r_n} y_n$, we have

(47) $$H(u_n, y) + \frac{1}{r_n}\langle y - u_n, ju_n - Jy_n\rangle \geq 0, \ \forall y \in C.$$

From $(A2)$, we note that

(48) $$\|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -H(u_n, y) \geq H(y, u_n), \ \forall y \in C.$$

Taking the limit as $n \to \infty$ in the above inequality and from $(A4)$ and $u_n \to p$, we have $H(y,p) \leq 0, \forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, we obtain $y_t \in C$, which yields that $H(y_t, p) \leq 0$. It follows from $(A1)$ that

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1-t)H(y_t, p) \leq tH(y_t, y),$$

that is, $H(y_t, y) \geq 0$.

Let $t \downarrow 0$; from $(A3)$, we obtain $H(p, y) \geq 0, \forall y \in C$. Therefore $p \in EP$, and so $p \in \Omega$.

(VI) Finally, we prove that $p = \Pi_\Omega x_1$. Taking the limit as $n \to \infty$ in (23), we obtain that

$$\langle p - z, Jx_1 - Jp \rangle \geq 0, \ \forall z \in \Omega$$

and hence $p = \Pi_\Omega x_1$ by Lemma 2.3. This complete the proof. □

The following Theorems can be obtained from Theorem 3.1 immediately.

Corollary 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(49)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n) JS^n z_n), \\ f(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, Ju_n - Jx \rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) k_n \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2 - 1) M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. Putting $A = 0$ in Theorem 3.1, the conclusion of Theorem 3.2 can be obtained. □

Corollary 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that T and S are uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S) \cap VI(C, A, \varphi) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(50)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT^n x_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n) JS^n z_n), \\ \langle Au_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, Ju_n - Jx \rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) k_n \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2-1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. Putting $f = 0$ in Theorem 3.1, the conclusion of Theorem 3.3 can be obtained. □

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that S is uniformly asymptotically regular on C and $\Omega = F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(51) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n), \\ f(u_n, y) + \langle A u_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, J u_n - J x \rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) k_n \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2-1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. Taking $T = I$ in Theorem 3.1, then we have $z_n = x_n$, $\forall n \geq 1$. Hence the conclusion of Theorem 3.4 is obtained. □

Corollary 3.5. *Let E be a uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be a continuous and monotone mapping and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)$ - $(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(t)}\} \subset [1, \infty)$ such that $k_n^{(t)} \to 1$ as $n \to \infty$ and $S : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n^{(s)}\} \subset [1, \infty)$ such that $k_n^{(s)} \to 1$ as $n \to \infty$. Assume that T and S are closed relatively nonexpansive mappings such that $\Omega = F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(52) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T^n x_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S^n z_n), \\ u_n = K_{r_n} y_n, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n) k_n \phi(v, z_n) \leq \phi(v, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall n \geq 1, \end{cases}$$

where $\theta_n = (1-\beta_n)(k_n^2-1)M_n \to 0$ as $n \to \infty$, $k_n = \max\{k_n^{(t)}, k_n^{(s)}\}$ for each $n \geq 1$, $M_n = \sup\{\phi(v, x_n) : v \in \Omega\}$ for each $n \geq 1$, $J : E \to E^*$ is the normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;
(ii) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;
(iii) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is generalized projection of E onto Ω.

Proof. Since every closed relatively nonexpansive mapping is quasi-ϕ-nonexpansive, the result is implied by Theorem 3.1. □

Acknowledgements The first author is supported by grant from under the program Strategic Scholarships for Frontier Research Network for the Ph.D. Program Thai Doctoral degree from the Office of the Higher Education Commission, Thailand.

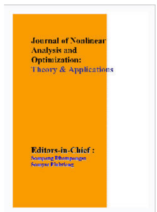
References

[1] R.P. Agarwal, Y.J. Cho, X. Qin, Generalized projection algorithms for nonlinear operators, Numer. Funct. Anal. Optim. 28 (2007) 11971215.
[2] Ya.I. Alber, S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, Panamer. Math. J. 4 (1994) 3954.
[3] Ya.I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 15-50.
[4] Ya.I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications.
[5] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Studient (1994), 63, 123-145.
[6] F.E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, Mathematische Zeitschrift, 100 (1967), 201-225.
[7] D. Butnariu, S. Reich, A.J. Zaslavski, Asymptotic behavior of relatively nonexpansive operators in Banach spaces, Journal of Applied Analysis 7 (2001) 151-174.
[8] D. Butnariu, S. Reich, A.J. Zaslavski, Weak convergence of orbits of nonlinear operators in reflexive Banach spaces, Numer. Funct. Anal. Optim. 24 (2003) 489-508.
[9] G. Cai, C.S. Hu, A hybrid approximation method for equilibrium and fixed point problems for a family of infinitely nonexpansive mappings and a monotone mapping, Nonlinear Analysis: Hybrid Systems 3 (2009) 395-407.
[10] L.C. Ceng, J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, Journal of Computational and Applied Mathematics 214 (2008) 186-201.
[11] Y. Censor, S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, Optimization 37 (1996) 323-339.
[12] S.-s. Chang, H.W. Joseph Lee, C.K. Chan, A new hybrid method for solving generalized equilibrium problem variational inequality and common fixed point in Banach spaces with applications, Nonlinear Analysis (2010), doi:10.1016/j.na.2010.06.006
[13] Y.J. Cho, H. Zhou, G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl. 47 (2004) 707717.
[14] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht (1990).
[15] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, Journal Nonlinear and Convex Analysis 6 (2005) 117-136.
[16] S.D. Flam, A.S. Antipin, Equilibrium programming using proximal-like algorithms, Mathematical Programming 78 (1997) 29-41.
[17] F. Flores-Bazan, Existence theory for finite-dimensional pseudomonotone equilibrium problems, Acta Applicandae Mathematicae 77 (2003) 249-297.
[18] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Am. Math. Soc. 35 (1972) 171-174.

[19] N. Hadjisavvas, S. Komlósi, S. Schaible, Handbook of Generalized Convexity and Generalized Monotonicity, Springer-Verlag, Berlin, Heidelberg, New York, 2005.
[20] N. Hadjisavvas, S. Schaible, From scalar to vector equilibrium problems in the quasimonotone case, Journal of Optimization Theory and Applications 96 (1998) 297-309.
[21] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc. 73 (1967) 957-961.
[22] C. Jaiboon, P. Kumam, U.W. Humphries, An extragradient method for relaxed cocoercive variational inequality and equilibrium problems, Analysis in Theory and Applications 25 (4) (2009) 381-400.
[23] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002) 938-945.
[24] P. Kumam, A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping, J. Appl. Math. Comput., 29 (2009) 263-280.
[25] S. Matsushita, W. Takahashi, Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. (2004), 2004, 37-47.
[26] X. Qin, Y.J. Cho, S.M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, Journal of Computational and Applied Mathematics 225 (2009) 20-30.
[27] X. Qin, S.Y. Cho, S.M Kang, On hybrid projection methods for asymptotically quasi-ϕ-nonexpansive mappings, Applied Mathematics and Computation 215 (2010) 38743883.
[28] X. Qin, Y. Su, C. Wu, K. Liu, Strong convergence theorems for nonlinear operators in Banach spaces, Commun. Appl. Nonlinear Anal. 14 (2007) 35-50.
[29] S. Reich, A weak convergence theorem for the alternating method with Bregman distance, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 313-318.
[30] A. Tada, W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, in: W. Takahashi, T. Tanaka (Eds.), Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, 2006, pp. 609-617.
[31] W. Takahashi, Convex Analysis and Approximation Fixed points, Yokohama-Publishers, 2000 (Japanese).
[32] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, Journal of Mathematical Analysis and Applications 331 (2007) 506-515.
[33] W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Analysis 70 (2009) 45-57.
[34] W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory and Applications (2008) Article ID 528476, 11 pages.
[35] S. Zhang, Generalized mixed equilibrium problem in Banach spaces, Applied Mathematics and Mechanics. English Edition 30 (2009) 1105-1112.
[36] H. Zhou, G. Gao, B. Tan, Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptotically nonexpansive mappings, J. Appl. Math. Comput. doi:10.1007/s12190-009-0263-4.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NARESUAN UNIVERSITY, PHITSANULOK 65000, THAILAND.
Email address: rabianw@nu.ac.th (Rabian Wangkeeree)

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 71-91
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



An extragradient type method for a system of equilibrium problems, variational inequality problems and fixed points of finitely many nonexpansive mappings

Thanyarat Jitpeera[1], **Poom Kumam**[1,2]

ABSTRACT: The purpose of this paper is to investigate the problem of finding the common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of a system of equilibrium problems and the set of solutions of the variational inequality problem for a monotone and k-Lipschitz continuous mapping in Hilbert spaces. Consequently, we obtain the strong convergence theorem of the proposed iterative algorithm to the unique solutions of variational inequality, which is the optimality condition for a minimization problem. The results presented in this paper generalize, improve and extend some well-known results in the literature.

KEYWORDS: Nonexpansive mapping; Monotone mapping, Variational inequality; Fixed points, System of equilibrium problems, Extragradient approximation method.

1. **Introduction**

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

(1) $$F(x, y) \geq 0, \quad \forall y \in C.$$

The set of solutions of (1) is denoted by $EP(F)$, that is,

(2) $$EP(F) = \{\, x \in C : F(x, y) \geq 0, \quad \forall y \in C \,\}.$$

Given a mapping $B : C \to H$, let $F(x, y) = \langle Bx, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Bz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality problems. Numerous problems in physics, optimization, saddle point problems, complementarity problems, mechanics and economics reduce to find a solution of (1). In 1997, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Some methods have

Corresponding author: Poom Kumam (poom.kum@kmutt.ac.th).
Manuscript received May 11, 2010. Accepted June 14, 2010.

been proposed to solve the problem (1); see, for instance, [10, 16, 21, 22, 23, 24, 29, 33, 38, 39] and the references therein.

Let $\Im = \{F_k\}_{k\in\Lambda}$ be a family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\Im = \{F_k\}_{k\in\Lambda}$ is to determine common equilibrium points for $\Im = \{F_k\}_{k\in\Lambda}$ such that

$$F_k(x,y) \geq 0, \quad \forall k \in \Lambda, \quad \forall y \in C. \tag{3}$$

where Λ is an arbitrary index set. The set of solutions of (3) is denoted by $SEP(\Im)$, that is,

$$SEP(\Im) = \left\{ x \in C : F_k(x,y) \geq 0, \quad \forall k \in \Lambda, \quad \forall y \in C \right\}. \tag{4}$$

If Λ is a singleton, then the problem (3) is reduced to the problem (1). The problem (3) is very general in the sense that it includes, as special case, some optimization , variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, economics and others (see, for instance, [2, 4, 5]). The classical variational inequality problem is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{5}$$

The set of solutions of (5) is denoted by $VI(C,B)$, that is,

$$VI(C,B) = \left\{ x \in C : \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C \right\}. \tag{6}$$

The variational inequality has been extensively studied in the literature; see, for instance [6, 7, 9, 12, 16, 29, 39]. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Recall the following definitions:

(1) A mapping B of C into H is called *monotone* if

$$\langle Bx - By, x - y\rangle \geq 0, \ \ \forall x, y \in C.$$

(2) B is called *β-strongly monotone* (see [3, 18]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in C.$$

(3) B is called *k-Lipschitz continuous* if there exists a positive real number k such that

$$\|Bx - By\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

(4) B is called *β-inverse-strongly monotone* (see [3, 18]) if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \beta\|Bx - By\|^2, \ \ \forall x, y \in C.$$

Remark 1.1. It is obvious that any β-inverse-strongly monotone mapping B is monotone and $\frac{1}{\beta}$-Lipschitz continuous.

(5) A mapping T of C into itself is called *nonexpansive* (see [30]) if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote $F(T) = \{x \in C : Tx = x\}$ be the set of fixed points of T.

(6) Let $f : C \to C$ is said to be a *α-contraction* if there exists a coefficient α $(0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

(7) An operator A is *strongly positive linear bounded operator* on H if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \ \ \forall x \in H.$$

(8) A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.

(9) A monotone mapping $T: H \to 2^H$ is *maximal* if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let B be a monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at $v \in C$, that is,

$$N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [27].

In 1976, Korpelevich [17] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda B x_n), \\ x_{n+1} = P_C(x_n - \lambda B y_n), \end{cases} \tag{7}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{k}), C$ is a closed convex subset of $\mathbb{R}^n$ and B is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C, B)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (7), converge to the same point $z \in VI(C, B)$. For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solution of variational inequalities for an β-inverse-strongly monotone, Takahashi and Toyoda [31] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n B x_n), \ \ \forall n \geq 0, \end{cases} \tag{8}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0,1) and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (8) converges weakly to some $z \in F(S) \cap VI(C, B)$. Recently, Iiduka and Takahashi [15] proposed a new iterative scheme as following

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x + (1-\alpha_n) S P_C(x_n - \lambda_n B x_n), \ \ \forall n \geq 0, \end{cases} \tag{9}$$

where B is β-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0, 1), and $\{\lambda_n\}$ is a sequence in $(0, 2\beta)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (9) converges strongly to some $z \in F(S) \cap VI(C, B)$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see e.g., [13, 35, 36, 37] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{10}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S on H and b is a given point in H. Moreover, it is shown in [19] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A) S x_n \tag{11}$$

converges strongly to $z = P_{F(S)}(I - A + \gamma f)(z)$. Recently, Plubtieng and Punpaeng [24] proposed the following iterative algorithm:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A)Su_n. \end{cases} \tag{12}$$

They proved that if the sequence $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \ \forall x \in F(S) \cap EP(F), \tag{13}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S) \cap EP(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{14}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2009, Peng and Yao [20] introduced an iterative scheme for finding a common element of the set of solutions of the system equilibrium problems (3), the set of solutions to the variational inequality for a monotone and Lipschitz continuous mapping and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert spaces and proved a strong convergence theorem.

Definition 1.2. [16]. For a finite family of nonexpansive mappings of $T_1, T_2, ..., T_N$ and sequence $\{\mu_{n,i}\}_{i=1}^N$ in $[0,1]$, we define the mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \mu_{n,1}T_1U_{n,0} + (1-\mu_{n,1})U_{n,0}, \\ U_{n,2} &= \mu_{n,2}T_2U_{n,1} + (1-\mu_{n,2})U_{n,1}, \\ &\vdots \\ U_{n,N-1} &= \mu_{n,N-1}T_{N-1}U_{n,N-2} + (1-\mu_{n,N-1})U_{n,N-2}, \\ W_n = U_{n,N} &= \mu_{n,N}T_NU_{n,N-1} + (1-\mu_{n,N})U_{n,N-1}. \end{aligned} \tag{15}$$

On the other hand, Colao et al. [10] introduced and considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem (1) and the set of common fixed points of a finite family of nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$ and defining a sequence $\{x_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + \beta x_n + \big((1-\beta)I - \epsilon_n A\big)W_n u_n, \end{cases} \tag{16}$$

where $\{\epsilon_n\}$ be a sequences in $(0,1)$. It is proved [10] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (16) converges strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap EP(F)$, where z is an equilibrium point for F and the unique solution of the variational inequality (13), i.e., $z = P_{\cap_{n=1}^{\infty} F(T_n) \cap EP(F)}(I - (A - \gamma f))z$.

In 2009, Colao et al. [11] introduced and considered an implicit iterative scheme for finding a common element of the set of solutions of the system equilibrium problems (3) and the set of common fixed points of an infinite family of nonexpansive mappings on C. Starting with an arbitrary initial $x_0 \in C$ and defining a sequence $\{z_n\}$ recursively by

$$z_n = \epsilon_n \gamma f(z_n) + (1 - \epsilon_n A)W_n J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} z_n, \tag{17}$$

where $\{\epsilon_n\}$ be a sequences in $(0,1)$. It is proved [11] that under certain appropriate conditions imposed on $\{\epsilon_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (17) converges strongly to $z \in \cap_{n=1}^{\infty} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big)$, where z is the unique solution of the variational inequality (13) and which is the optimality condition for the minimization problem (14). In 2010, He et al. [14] introduced an explicit iterative scheme for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of an infinite family of nonexpansive mappings.

In this paper, motivated by Colao et al. [10, 11], He et al. [14], and Peng and Yao [20, 21], we introduce a new iterative scheme in a Hilbert space H which is mixed the iterative schemes of (16) and (17). We prove that the sequence converges strongly to a common element of the set of solutions of the system equilibrium problems (3), the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of variational inequality (5) for be a monotone and k-Lipschitz continuous mapping in Hilbert spaces by using the extragradient approximation method. The results presented in this paper generalize, improve and extend some well-known results in the literature.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. When $\{x_n\}$ is a sequence in H, we denote strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

Lemma 2.1. [26] *Let $(C, \langle .,. \rangle)$ be an inner product space. Then for all $x, y, z \in C$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Recall that the metric (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point in $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y\in C} \|x-y\|.$$

The following characterizes the projection P_C.
In order to prove our main results, we need the following lemmas.

Lemma 2.2. *For a given $z \in H$, $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u\rangle \geq 0, \quad \forall v \in C.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \tag{18}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H, y \in C$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0. \tag{19}$$

It is easy to see that (19) is equivalent to the following inequality:

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2. \tag{20}$$

Using Lemma 2.2, one can see that the variational inequality (5) is equivalent to a fixed point problem.

It is easy to see that the following is true:

$$u \in VI(C,B) \Leftrightarrow u = P_C(u - \lambda Bu), \quad \lambda > 0. \tag{21}$$

Lemma 2.3. [25]. *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

hold for each $y \in H$ with $y \neq x$.

Lemma 2.4. [28] . *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.5. [32]. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - l_n)a_n + \sigma_n, \ n \geq 0,$$

where $\{l_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that
(1) $\sum_{n=1}^{\infty} l_n = \infty$,
(2) $\limsup_{n\to\infty} \frac{\sigma_n}{l_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. . *Let H be a real Hilbert space. Then for all $x, y \in H$,*
(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,
(2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.7. [19]. *Let C be a nonempty closed convex subset of H and let f be a contraction of H into itself with $\alpha \in (0,1)$, and A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Then , for $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,*

$$\Big\langle x - y, (A - \gamma f)x - (A - \gamma f)y \Big\rangle \geq (\bar{\gamma} - \alpha\gamma)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Lemma 2.8. [19]. *Assume A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.
The following lemma appears implicitly in [2].

Lemma 2.9. [2]. *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

The following lemma was also given in [5].

Lemma 2.10. [5]. *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $J_r^F : H \to C$ as follows:*

$$J_r^F(x) = \Big\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \Big\}$$

for all $z \in H$. Then, the following hold:
(1) *J_r^F is single- valued;*
(2) *J_r^F is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|J_r^F x - J_r^F y\|^2 \leq \Big\langle J_r^F x - J_r^F y, x - y \Big\rangle;$$

(3) *$F(J_r^F) = EP(F)$; and*
(4) *$EP(F)$ is closed and convex.*

3. **Main Results**

In this section, we deal with the strong convergence of extragradient approximation method (23) for finding a common element of the set of solutions of the system equilibrium problems (3), the set of common fixed points of a finite family of nonexpansive mappings and the set of solutions of variational inequality (5) for be a monotone and k-Lipschitz continuous mapping in Hilbert spaces.

First, let $T_i : C \to C$, where $i = 1, 2, \ldots, N$, be a family of finitely nonexpansive mappings. Let the mapping W_n be defined by

$$\text{(22)} \qquad \begin{cases} U_{n,0} = I, \\ U_{n,1} = \lambda_{n,1} T_1 U_{n,0} + (1 - \lambda_{n,1}) U_{n,0}, \\ U_{n,2} = \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) U_{n,1}, \\ \vdots \\ U_{n,N-1} = \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) U_{N-2}, \\ W_n = U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) U_{N-1}, \end{cases}$$

where $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\} \in (0, 1]$. Such a mapping W_n is called the *W-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Moreover, in ([1], Lemma 3.1), it is shown that $F(W_n) = \cap_{i=1}^{N} F(T_i)$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1, 2, 3, \ldots, M\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T_1, T_2, \ldots, T_N$ be a family of finitely nonexpansive mappings of C into itself and let W_n be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let B be a monotone and k-Lipschitz continuous mapping of C into H such that*

$$\Omega := \cap_{n=1}^{N} F(T_n) \cap \left(\cap_{k=1}^{M} SEP(F_k)\right) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\text{(23)} \qquad \begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J_{r_{M,n}}^{F_M} J_{r_{M-1,n}}^{F_{M-1}} J_{r_{M-2,n}}^{F_{M-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) W_n P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{k})$ and $\{r_{k,n}\}, k \in \{1, 2, 3, \ldots, M\}$ are a real sequence in $(0, \infty)$ satisfy the following conditions:

- (C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
- (C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
- (C3) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1, 2, 3, \ldots, M\}$,
- (C4) $\lim_{n\to\infty} \lambda_n = 0$,
- (C5) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Omega$ which is the unique solution of the variational inequality

$$\text{(24)} \qquad \left\langle (A - \gamma f)z, x - z \right\rangle \geq 0, \quad \forall x \in \Omega.$$

Equivalently, we have $z = P_\Omega(I - A + \gamma f)(z)$, which is the optimality condition for the minimization problem

$$\text{(25)} \qquad \min_{x \in \Omega} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Proof. Note that from the condition (C1), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. From Lemma 2.8, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I-\rho A\| \leq 1-\rho\bar{\gamma}$. We will assume that $\|I-A\| \leq 1-\bar{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\Big\{\big|\langle Ax, x\rangle\big| : x \in H, \|x\| = 1\Big\}.$$

Observe that

$$\begin{aligned}\Big\langle \big((1-\beta_n)I - \alpha_n A\big)x, x\Big\rangle &= 1-\beta_n-\alpha_n\langle Ax, x\rangle \\ &\geq 1-\beta_n-\alpha_n\|A\| \\ &\geq 0,\end{aligned}$$

this show that $(1-\beta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\alpha_n A\| &= \sup\Bigg\{\Big|\Big\langle \big((1-\beta_n)I-\alpha_n A\big)x, x\Big\rangle\Big| : x \in H, \|x\| = 1\Bigg\} \\ &= \sup\Big\{1-\beta_n-\alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\Big\} \\ &\leq 1-\beta_n-\alpha_n\bar{\gamma}.\end{aligned}$$

We will divide the proof of Theorem 3.1 into several steps.

Step 1. We claim that $\{x_n\}$ is bounded.

Indeed, pick any $x^* \in \Omega$. By taking $\Im_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}$ for $k \in \{1,2,3,\ldots,M\}$ and $\Im_n^0 = I$ for all n. The nonexpansivity of $J_{r_{k,n}}^{F_k}$ for each $k = 1,2,3,\ldots,M$ implies that $\Im_n^k$ is nonexpansive. Let $x^* = \Im_n^k x^*$, we note that $u_n = \Im_n^M x_n$, it follows that

$$\|u_n - x^*\| = \|\Im_n^M x_n - \Im_n^M x^*\| \leq \|x_n - x^*\|.$$

Put $v_n = P_C(u_n - \lambda_n B y_n)$. Then, from (20) and the monotonicity of B, we compute

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \|u_n - \lambda_n By_n - x^*\|^2 - \|u_n - \lambda_n By_n - v_n\|^2 \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, x^* - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 \\ &\quad +2\lambda_n\big(\langle By_n - Bx^*, x^* - y_n\rangle + \langle Bx^*, x^* - y_n\rangle + \langle By_n, y_n - v_n\rangle\big) \\ &\leq \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle By_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\ &\quad +2\lambda_n\langle By_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad +2\langle u_n - \lambda_n By_n - y_n, v_n - y_n\rangle.\end{aligned}$$

Moreover, since $y_n = P_C(u_n - \lambda_n B u_n)$ and (19), we have

$$\langle u_n - \lambda_n Bu_n - y_n, v_n - y_n\rangle \leq 0. \tag{26}$$

Since B is k-Lipschitz continuous and (26), we obtain

$$\begin{aligned}&\langle u_n - \lambda_n By_n - y_n, v_n - y_n\rangle \\ &\quad= \langle u_n - \lambda_n Bu_n - y_n, v_n - y_n\rangle + \langle \lambda_n Bu_n - \lambda_n By_n, v_n - y_n\rangle \\ &\quad\leq \langle \lambda_n Bu_n - \lambda_n By_n, v_n - y_n\rangle \\ &\quad\leq \lambda_n\|Bu_n - By_n\|\|v_n - y_n\| \\ &\quad\leq \lambda_n k\|u_n - y_n\|\|v_n - y_n\|.\end{aligned}$$

Since $\lambda_n \in (0, \frac{1}{k})$, we have

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|v_n - y_n\| \\ &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + \lambda_n^2 k^2\|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\ &= \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (27) \\ &\leq \|u_n - x^*\|^2, \end{aligned}$$

and hence

$$\|v_n - x^*\| \leq \|u_n - x^*\| \leq \|x_n - x^*\|. \qquad (28)$$

Thus, we can note that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \Big\|\alpha_n\big(\gamma f(x_n) - Ax^*\big) + \beta_n(x_n - x^*) + \big((1-\beta_n)I - \alpha_n A\big)(W_n v_n - x^*)\Big\| \\ &\leq (1 - \beta_n - \alpha_n\bar{\gamma})\|v_n - x^*\| + \beta_n\|x_n - x^*\| + \alpha_n\|\gamma f(x_n) - Ax^*\| \\ &\leq (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\| + \beta_n\|x_n - x^*\| + \alpha_n\|\gamma f(x_n) - Ax^*\| \\ &= (1 - \alpha_n\bar{\gamma})\|x_n - x^*\| + \alpha_n\gamma\|f(x_n) - f(x^*)\| + \alpha_n\|\gamma f(x^*) - Ax^*\| \\ &\leq (1 - \alpha_n\bar{\gamma})\|x_n - x^*\| + \alpha_n\gamma\alpha\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\| \\ &= (1 - (\bar{\gamma} - \gamma\alpha)\alpha_n)\|x_n - x^*\| + (\bar{\gamma} - \gamma\alpha)\alpha_n\frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma\alpha}. \qquad (29) \end{aligned}$$

By induction that

$$\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\bar{\gamma} - \gamma\alpha}\right\}, \quad n \in \mathbb{N}. \qquad (30)$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{Bu_n\}$, $\{Bv_n\}$, $\{W_n v_n\}$ and $\{f(x_n)\}$.

Step 2. We claim that

$$\lim_{n\to\infty}\|\Im_n^k x_n - \Im_{n+1}^k x_n\| = 0 \qquad (31)$$

for every $k \in \{1, 2, 3, \ldots, M\}$. From Step 2 of the proof in [10, Theorem 3.1], we have for $k \in \{1, 2, 3, \ldots, M\}$,

$$\lim_{n\to\infty}\|J_{r_{k,n+1}}^{F_k}x_n - J_{r_{k,n}}^{F_k}x_n\| = 0. \qquad (32)$$

Note that for every $k \in \{1, 2, 3, \ldots, M\}$, we obtain

$$\Im_n^k = J_{r_{k,n}}^{F_k}J_{r_{k-1,n}}^{F_{k-1}}J_{r_{k-2,n}}^{F_{k-2}}\ldots J_{r_{2,n}}^{F_2}J_{r_{1,n}}^{F_1} = J_{r_{k,n}}^{F_k}\Im_n^{k-1}.$$

So, we have

$$\begin{aligned} &\|\Im_n^k x_n - \Im_{n+1}^k x_n\| \qquad (33) \\ &= \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n+1}}^{F_k}\Im_{n+1}^{k-1}x_n\| \\ &\leq \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}x_n\| + \|J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n+1}}^{F_k}\Im_{n+1}^{k-1}x_n\| \\ &\leq \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}x_n\| + \|\Im_n^{k-1}x_n - \Im_{n+1}^{k-1}x_n\| \\ &\leq \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}x_n\| + \|J_{r_{k-1,n}}^{F_{k-1}}\Im_n^{k-2}x_n - J_{r_{k-1,n+1}}^{F_{k-1}}\Im_n^{k-2}x_n\| \\ &\quad + \|\Im_n^{k-2}x_n - \Im_{n+1}^{k-2}x_n\| \\ &\leq \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n+1}}^{F_k}\Im_n^{k-1}x_n\| + \|J_{r_{k-1,n}}^{F_{k-1}}\Im_n^{k-2}x_n - J_{r_{k-1,n+1}}^{F_{k-1}}\Im_n^{k-2}x_n\| \\ &\quad + \ldots + \|J_{r_{2,n}}^{F_2}\Im_n^1 x_n - J_{r_{2,n+1}}^{F_2}\Im_n^1 x_n\| + \|J_{r_{1,n}}^{F_1}x_n - J_{r_{1,n+1}}^{F_1}x_n\|. \end{aligned}$$

Now, apply (32) to (33), we conclude (31).

Step 3. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.
On the other hand, from $u_n = \Im_n^M x_n$ and $u_{n+1} = \Im_{n+1}^M x_{n+1}$, by the triangular inequality, we

have

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|\Im^M_{n+1}x_{n+1}-\Im^M_n x_n\|\\ &= \|\Im^M_{n+1}x_{n+1}-\Im^M_{n+1}x_n\|+\|\Im^M_{n+1}x_n-\Im^M_n x_n\|\\ &\le \|x_{n+1}-x_n\|+\|\Im^M_{n+1}x_n-\Im^M_n x_n\|. \end{aligned} \tag{34}$$

We note that

$$\begin{aligned}\|v_{n+1}-v_n\| &= \|P_C(u_{n+1}-\lambda_{n+1}By_{n+1})-P_C(u_n-\lambda_n By_n)\|\\ &\le \|u_{n+1}-\lambda_{n+1}By_{n+1}-(u_n-\lambda_n By_n)\|\\ &= \|(u_{n+1}-\lambda_{n+1}Bu_{n+1})-(u_n-\lambda_{n+1}Bu_n)\\ &\quad +\lambda_{n+1}(Bu_{n+1}-By_{n+1}-Bu_n)+\lambda_n By_n\|\\ &\le \|(u_{n+1}-\lambda_{n+1}Bu_{n+1})-(u_n-\lambda_{n+1}Bu_n)\|\\ &\quad +\lambda_{n+1}(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|)+\lambda_n\|By_n\|\\ &\le (1+\lambda_{n+1}k)\|u_{n+1}-u_n\|+\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|\big)\\ &\quad +\lambda_n\|By_n\|. \end{aligned} \tag{35}$$

Substituting (34) into (35), we have

$$\begin{aligned}\|v_{n+1}-v_n\| &\le (1+\lambda_{n+1}k)\|u_{n+1}-u_n\|+\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|\big)+\lambda_n\|By_n\|\\ &\le (1+\lambda_{n+1}k)\|x_{n+1}-x_n\|+(1+\lambda_{n+1}k)\|\Im^M_{n+1}x_n-\Im^M_n x_n\|\\ &\quad +\lambda_{n+1}\big(\|Bu_{n+1}\|+\|By_{n+1}\|+\|Bu_n\|\big)+\lambda_n\|By_n\|. \end{aligned} \tag{36}$$

Putting $z_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}=\frac{\alpha_n\gamma f(x_n)+\big((1-\beta_n)I-\alpha_n A\big)W_n v_n}{1-\beta_n}$ then, we get $x_{n+1}=(1-\beta_n)z_n+\beta_n x_n,\ n\ge 1$. It follows that

$$\begin{aligned} z_{n+1}-z_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1})+\big((1-\beta_{n+1})I-\alpha_{n+1}A\big)W_{n+1}v_{n+1}}{1-\beta_{n+1}}\\ &\quad -\frac{\alpha_n\gamma f(x_n)+\big((1-\beta_n)I-\alpha_n A\big)W_n v_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)+W_{n+1}v_{n+1}-W_n v_n\\ &\quad -\frac{\alpha_{n+1}}{1-\beta_{n+1}}AW_{n+1}v_{n+1}+\frac{\alpha_n}{1-\beta_n}AW_n v_n\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\gamma f(x_{n+1})-AW_{n+1}v_{n+1}\big)+\frac{\alpha_n}{1-\beta_n}\big(AW_n v_n-\gamma f(x_n)\big)\\ &\quad +W_{n+1}v_{n+1}-W_{n+1}v_n+W_{n+1}v_n-W_n v_n. \end{aligned} \tag{37}$$

It follows from (36) and (37) that

$$
\begin{aligned}
\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\
& + \frac{\alpha_n}{1-\beta_n}\big(\|AW_nv_n\| + \|\gamma f(x_n)\|\big) + \|W_{n+1}v_{n+1} - W_{n+1}v_n\| \\
& + \|W_{n+1}v_n - W_nv_n\| - \|x_{n+1}-x_n\| \\
\leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\
& + \frac{\alpha_n}{1-\beta_n}\big(\|AW_nv_n\| + \|\gamma f(x_n)\|\big) + \|v_{n+1}-v_n\| \\
& + \|W_{n+1}v_n - W_nv_n\| - \|x_{n+1}-x_n\| \\
\leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\
& + \frac{\alpha_n}{1-\beta_n}\big(\|AW_nv_n\| + \|\gamma f(x_n)\|\big) + \lambda_{n+1}k\|x_{n+1}-x_n\| \\
& + (1+\lambda_{n+1}k)\|\Im^M_{n+1}x_n - \Im^M_n x_n\| \\
& + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) \\
& + \lambda_n\|By_n\| + \|W_{n+1}v_n - W_nv_n\|. \qquad (38)
\end{aligned}
$$

By the definition of W_n that

$$
\begin{aligned}
\|W_{n+1}v_n - W_nv_n\| =\ & \|\lambda_{n+1,N}T_NU_{n+1,N-1}v_n + (1-\lambda_{n+1,N})v_n - \lambda_{n,N}T_NU_{n,N-1}v_n - (1-\lambda_{n,N})v_n\| \\
\leq\ & |\lambda_{n+1,N}-\lambda_{n,N}|\|v_n\| + \|\lambda_{n+1,N}T_NU_{n+1,N-1}v_n - \lambda_{n,N}T_NU_{n,N-1}v_n\| \\
\leq\ & |\lambda_{n+1,N}-\lambda_{n,N}|\|v_n\| + \|\lambda_{n+1,N}(T_NU_{n+1,N-1}v_n - T_NU_{n,N-1}v_n)\| \\
& +|\lambda_{n+1,N}-\lambda_{n,N}|\|T_NU_{n,N-1}v_n\| \\
\leq\ & 2M|\lambda_{n+1,N}-\lambda_{n,N}| + \lambda_{n+1,N}\|U_{n+1,N-1}v_n - U_{n,N-1}v_n\|, \qquad (39)
\end{aligned}
$$

where M is an approximate constant such that $M \geq \max\{\sup_{n\geq 1}\{\|v_n\|\}, \sup_{n\geq 1}\{\|T_mU_{n,m-1}v_n\|\} \mid m = 1,2,\ldots,N\}$.

Since $0 < \lambda_{n_i} \leq 1$ for all $n \geq 1$ and $i = 1,2,\ldots,N$, we compute

$$
\begin{aligned}
& \|U_{n+1,N-1}v_n - U_{n,N-1}v_n\| \\
=\ & \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}v_n + (1-\lambda_{n+1,N-1})v_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}v_n - (1-\lambda_{n,N-1})v_n\| \\
\leq\ & |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|v_n\| + \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}v_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}v_n\| \\
\leq\ & |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|v_n\| + \|\lambda_{n+1,N-1}(T_{N-1}U_{n+1,N-2}v_n - T_{N-1}U_{n,N-2}v_n)\| \\
& +|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|T_{N-1}U_{n,N-2}v_n\| \\
\leq\ & 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}| + \|U_{n+1,N-2}v_n - U_{n,N-2}v_n\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|U_{n+1,N-1}v_n - U_{n,N-1}v_n\| \leq\ & 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}| + 2M|\lambda_{n+1,N-2}-\lambda_{n,N-2}| \\
& +\|U_{n+1,N-3}v_n - U_{n,N-3}v_n\| \qquad (40) \\
\leq\ & 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| + \|U_{n+1,1}v_n - U_{n,1}v_n\| \\
=\ & 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| \\
& +\|\lambda_{n+1,1}T_1v_n + (1-\lambda_{n+1,1})v_n - \lambda_{n,1}T_1v_n - (1-\lambda_{n,1})v_n\| \\
\leq\ & 2M\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned}
$$

Substituting (40) into (39) yields that

$$\begin{aligned}\|W_{n+1}v_n - W_nv_n\| &\leq 2M|\lambda_{n+1,N} - \lambda_{n,N}| + 2\lambda_{n+1,N}M\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\ &\leq 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|, \quad (41)\end{aligned}$$

Applying (41) in (38), we get

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big(\|\gamma f(x_{n+1})\| + \|AW_{n+1}v_{n+1}\|\big) \\ &\quad + \frac{\alpha_n}{1-\beta_n}\big(\|AW_nv_n\| + \|\gamma f(x_n)\|\big) + \lambda_{n+1}k\|x_{n+1} - x_n\| \\ &\quad + (1+\lambda_{n+1}k)\|\Im_{n+1}^M x_n - \Im_n^M x_n\| \\ &\quad + \lambda_{n+1}\big(\|Bu_{n+1}\| + \|By_{n+1}\| + \|Bu_n\|\big) \\ &\quad + \lambda_n\|By_n\| + 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.\end{aligned}$$

By (31) and (C1)-(C5), imply that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \quad (42)$$

Applying (31), (42) and (C4) to (34) and (35), we obtain that

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = \lim_{n\to\infty}\|v_{n+1} - v_n\| = 0. \quad (43)$$

Step 4. We claim that $\lim_{n\to\infty}\|x_n - W_nv_n\| = 0$.
Since $x_{n+1} = \alpha_n\gamma f(x_n) + \beta_nx_n + ((1-\beta_n)I - \alpha_nA)W_nv_n$, we have

$$\begin{aligned}\|x_n - W_nv_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_nv_n\| \\ &= \|x_n - x_{n+1}\| + \Big\|\alpha_n\gamma f(x_n) + \beta_nx_n + \big((1-\beta_n)I - \alpha_nA\big)W_nv_n - W_nv_n\Big\| \\ &= \|x_n - x_{n+1}\| + \Big\|\alpha_n\big(\gamma f(x_n) - AW_nv_n\big) + \beta_n(x_n - W_nv_n)\Big\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\big(\|\gamma f(x_n)\| + \|AW_nv_n\|\big) + \beta_n\|x_n - W_nv_n\|,\end{aligned}$$

it follows that

$$\|x_n - W_nv_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\big(\|\gamma f(x_n)\| + \|AW_nv_n\|\big).$$

By (C1), (C2) and (42), we obtain

$$\lim_{n\to\infty}\|W_nv_n - x_n\| = 0. \quad (44)$$

Step 5. We claim that the following statements hold:

(i) $\lim_{n\to\infty}\|u_n - y_n\| = 0$;
(ii) $\lim_{n\to\infty}\|u_n - v_n\| = 0$;
(iii) $\lim_{n\to\infty}\|W_ny_n - y_n\| = 0$.

For any $x^* \in \Omega$ and (23), we compute

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \Big\|\big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - x^*) + \beta_n(x_n - x^*) + \alpha_n\big(\gamma f(x_n) - Ax^*\big)\Big\|^2 \\
&= \|\big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - x^*) + \beta_n(x_n - x^*)\|^2 + \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + 2\beta_n\alpha_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\alpha_n\Big\langle \big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - x^*), \gamma f(x_n) - Ax^*\Big\rangle \\
&\le \big[(1-\beta_n-\alpha_n\bar{\gamma})\|W_n v_n - x^*\| + \beta_n\|x_n - x^*\|\big]^2 + \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 \\
&\quad + 2\beta_n\alpha_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\alpha_n\Big\langle \big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - x^*), \gamma f(x_n) - Ax^*\Big\rangle \\
&\le \big[(1-\beta_n-\alpha_n\bar{\gamma})\|v_n - x^*\| + \beta_n\|x_n - x^*\|\big]^2 + c_n \\
&\le (1-\beta_n-\alpha_n\bar{\gamma})^2\|v_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + 2(1-\beta_n-\alpha_n\bar{\gamma})\beta_n\|v_n - x^*\|\|x_n - x^*\| + c_n \\
&\le (1-\beta_n-\alpha_n\bar{\gamma})^2\|v_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + (1-\beta_n-\alpha_n\bar{\gamma})\beta_n\big(\|v_n - x^*\|^2 + \|x_n - x^*\|^2\big) + c_n \\
&= \big[(1-\alpha_n\bar{\gamma})^2 - 2(1-\alpha_n\bar{\gamma})\beta_n + \beta_n^2\big]\|v_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 \\
&\quad + \big((1-\alpha_n\bar{\gamma})\beta_n - \beta_n^2\big)\big(\|v_n - x^*\|^2 + \|x_n - x^*\|^2\big) + c_n \\
&= (1-\alpha_n\bar{\gamma})^2\|v_n - x^*\|^2 - (1-\alpha_n\bar{\gamma})\beta_n\|v_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&= (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|v_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n, \qquad (45)
\end{aligned}$$

where

$$\begin{aligned}
c_n &= \alpha_n^2\|\gamma f(x_n) - Ax^*\|^2 + 2\beta_n\alpha_n\langle x_n - x^*, \gamma f(x_n) - Ax^*\rangle \\
&\quad + 2\alpha_n\Big\langle \big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - x^*), \gamma f(x_n) - Ax^*\Big\rangle.
\end{aligned}$$

It follows from the condition (C1) that

$$\lim_{n\to\infty} c_n = 0. \qquad (46)$$

Substituting (27) into (45), and (C4), we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|v_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\Big\{\|u_n - x^*\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2\Big\} \\
&\quad + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&\le (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\Big\{\|x_n - x^*\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2\Big\} \\
&\quad + (1-\alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\
&= (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})(\lambda_n^2k^2 - 1)\|u_n - y_n\|^2 + c_n \\
&\le \|x_n - x^*\|^2 + (\lambda_n^2k^2 - 1)\|u_n - y_n\|^2 + c_n.
\end{aligned}$$

Hence

$$\begin{aligned}
(1-\lambda_n^2k^2)\|u_n - y_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + c_n \\
&= (\|x_n - x^*\| - \|x_{n+1} - x^*\|)(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + c_n \\
&\le \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + c_n.
\end{aligned}$$

Since (46) and (42), we obtain

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \qquad (47)$$

By the same argument as in (27), we also have

$$\begin{aligned}\|v_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + 2\lambda_n k\|u_n - y_n\|\|v_n - y_n\| \\ &\leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 + \|u_n - y_n\|^2 + \lambda_n^2 k^2\|v_n - y_n\|^2 \\ &= \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 \\ &\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2.\end{aligned} \tag{48}$$

Substituting (48) into (45), and (C4), we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})\Big\{\|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2\Big\} \\ &\quad + (1 - \alpha_n\bar{\gamma})\beta_n\|x_n - x^*\|^2 + c_n \\ &= (1 - \alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + (1 - \alpha_n\bar{\gamma})(1 - \beta_n - \alpha_n\bar{\gamma})(\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 + c_n \\ &\leq \|x_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - v_n\|^2 + c_n.\end{aligned}$$

It follows that

$$\begin{aligned}(1 - \lambda_n^2 k^2)\|y_n - v_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + c_n \\ &\leq \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + c_n.\end{aligned}$$

Again from (46) and (42), we have

$$\lim_{n\to\infty}\|y_n - v_n\| = 0. \tag{49}$$

On the other hand, we note that

$$\|u_n - v_n\| \leq \|u_n - y_n\| + \|y_n - v_n\|.$$

Applying (47)and (49), we get

$$\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{50}$$

For any $x^* \in \Omega$, note that $J_{r_{k,n}}^{F_k}$ is firmly nonexpansive (Lemma 2.10) for $k \in \{1, 2, 3, \ldots, M\}$, then we get

$$\begin{aligned}\|\Im_n^k x_n - x^*\|^2 &= \|J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n}}^{F_k}x^*\|^2 \\ &\leq \Big\langle J_{r_{k,n}}^{F_k}\Im_n^{k-1}x_n - J_{r_{k,n}}^{F_k}x^*, \Im_n^{k-1}x_n - x^*\Big\rangle \\ &= \Big\langle \Im_n^k x_n - x^*, \Im_n^{k-1}x_n - x^*\Big\rangle \\ &= \frac{1}{2}\Big(\|\Im_n^k x_n - x^*\|^2 + \|\Im_n^{k-1}x_n - x^*\|^2 - \|\Im_n^k x_n - \Im_n^{k-1}x_n\|^2\Big)\end{aligned}$$

and hence

$$\|\Im_n^k x_n - x^*\|^2 \leq \|\Im_n^{k-1}x_n - x^*\|^2 - \|\Im_n^k x_n - \Im_n^{k-1}x_n\|^2, \quad k = 1, 2, 3, \ldots, M$$

which implies that for each $k \in \{1, 2, 3, \ldots, M\}$,

$$\begin{aligned}\|\Im_n^k x_n - x^*\|^2 &\leq \|\Im_n^0 x_n - x^*\|^2 - \|\Im_n^k x_n - \Im_n^{k-1}x_n\|^2 \\ &\quad - \|\Im_n^{k-1}x_n - \Im_n^{k-2}x_n\|^2 - \ldots - \|\Im_n^2 x_n - \Im_n^1 x_n\|^2 - \|\Im_n^1 x_n - \Im_n^0 x_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|\Im_n^k x_n - \Im_n^{k-1}x_n\|^2.\end{aligned}$$

Together with (27) and (48), we compute

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
\le\ & (1-\alpha_n\bar{\gamma})(1-\beta_n-\alpha_n\bar{\gamma})\|v_n-x^*\|^2+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
\le\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\Big\{\|u_n-x^*\|^2+(\lambda_n^2k^2-1)\|u_n-y_n\|^2\Big\}\\
&+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
=\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|u_n-x^*\|^2+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2\\
&+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
=\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-x^*\|^2+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2\\
&+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
\le\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\Big\{\|x_n-x^*\|^2-\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\Big\}\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
=\ & (1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|x_n-x^*\|^2-(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+(1-\alpha_n\bar{\gamma})\beta_n\|x_n-x^*\|^2+c_n\\
=\ & (1-\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2-(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n\\
=\ & \big[1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2\big]\|x_n-x^*\|^2-(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n\\
\le\ & \|x_n-x^*\|^2+(\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2-(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)\|\Im_n^k x_n-\Im_n^{k-1}x_n\|^2\\
\le\ & \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+(\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n\\
\le\ & \|x_n-x_{n+1}\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|)+(\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2\\
&+(1-\alpha_n\bar{\gamma})(1-\alpha_n\bar{\gamma}-\beta_n)(\lambda_n^2k^2-1)\|u_n-y_n\|^2+c_n.
\end{aligned}$$

Using (C1), (42) , (46) and (47), we get

(51) $$\lim_{n\to\infty}\|\Im_n^k x_n-\Im_n^{k-1}x_n\|=0,\forall k=1,2,...,M.$$

Observe that

$$\begin{aligned}
\|W_ny_n-y_n\| &\le \|W_ny_n-W_nv_n\|+\|W_nv_n-x_n\|+\|x_n-u_n\|+\|u_n-y_n\|\\
&\le \|y_n-v_n\|+\|W_nv_n-x_n\|+\|x_n-\Im_n^M x_n\|+\|u_n-y_n\|\\
&\le \|y_n-v_n\|+\|W_nv_n-x_n\|+\|\Im_n^0 x_n-\Im_n^1 x_n\|+\|\Im_n^1 x_n-\Im_n^2 x_n\|\\
&\quad+\ldots+\|\Im_n^{M-1}x_n-\Im_n^M x_n\|+\|u_n-y_n\|.
\end{aligned}$$

Applying (44), (47), (49) and (51) to the last inequality, we have

(52) $$\lim_{n\to\infty}\|W_ny_n-y_n\|=0.$$

We also have

$$\begin{aligned}
\|W_nu_n-u_n\| &\le \|W_nu_n-W_nv_n\|+\|W_nv_n-x_n\|+\|x_n-u_n\|\\
&\le \|W_nu_n-W_nv_n\|+\|W_nv_n-x_n\|+\|x_n-\Im_n^M x_n\|\\
&\le \|u_n-v_n\|+\|W_nv_n-x_n\|+\|\Im_n^0 x_n-\Im_n^1 x_n\|+\|\Im_n^1 x_n-\Im_n^2 x_n\|\\
&\quad+\ldots+\|\Im_n^{M-1}x_n-\Im_n^M x_n\|.
\end{aligned}$$

Applying (44), (50) and (51) to the last inequality, we have

$$\lim_{n\to\infty}\|W_nu_n - u_n\| = 0. \tag{53}$$

Step 6. We claim that $\limsup_{n\to\infty}\langle (A-\gamma f)z, z-x_n\rangle \leq 0$, which z is the unique solution of the variational inequality $\langle (A-\gamma f)z, x-z\rangle \geq 0,\ \forall x\in\Omega$.

Observe that $P_\Omega(I-A+\gamma f)$ is a contraction of H into itself. Indeed, for all $x,y\in H$, we have

$$\begin{aligned}\|P_\Omega(I-A+\gamma f)(x) - P_\Omega(I-A+\gamma f)(y)\| &\leq \|(I-A+\gamma f)(x) - (I-A+\gamma f)(y)\|\\ &\leq \|I-A\|\|x-y\| + \gamma\|f(x)-f(y)\|\\ &\leq (1-\bar\gamma)\|x-y\| + \gamma\alpha\|x-y\|\\ &= (1-(\bar\gamma-\gamma\alpha))\|x-y\|,\end{aligned}$$

since H is complete, there exists a unique fixed point $z\in H$ such that $z = P_\Omega(I-A+\gamma f)(z)$.

Since $z = P_\Omega(I-A+\gamma f)(z)$ is a unique solution of the variational inequality (24). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle (A-\gamma f)z, z-x_{n_i}\rangle = \limsup_{n\to\infty}\langle (A-\gamma f)z, z-x_n\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to $w\in C$. Without loss of generality, we can assume that $u_{n_i}\rightharpoonup w$. Since $\lim_{n\to\infty}\|\Im_n^k x_n - \Im_n^{k-1}x_n\| = 0$ for each $k = 1,2,...,M$, we have $\Im_{n_i}^k x_{n_i}\rightharpoonup w$ for each $k=1,2,...,M$. From $\|u_n - y_n\|\to 0$ and $\|u_n - v_n\|\to 0$, we also obtain $y_{n_i}\rightharpoonup w$ and $v_{n_i}\rightharpoonup w$. Since $\{u_{n_i}\}\subset C$ and C is closed and convex, we have $w\in C$.

Next, we show that $w\in\Omega$, where $\Omega := \cap_{n=1}^N F(T_n)\cap\left(\cap_{k=1}^M SEP(F_k)\right)\cap VI(C,B)$.

First, we show that $w\in\cap_{k=1}^M SEP(F_k)$. Since $u_n = \Im_n^k x_n$ for $k=1,2,3,\dots,M$, we also have

$$F_k(\Im_n^k x_n, y) + \frac{1}{r_n}\langle y - \Im_n^k x_n, \Im_n^k x_n - \Im_n^{k-1}x_n\rangle \geq 0,\quad \forall y\in C.$$

If follows from (A2) that,

$$\frac{1}{r_n}\langle y - \Im_n^k x_n, \Im_n^k x_n - \Im_n^{k-1}x_n\rangle \geq -F_k(\Im_n^k x_n, y) \geq F_k(y, \Im_n^k x_n)$$

and hence

$$\left\langle y - \Im_{n_i}^k x_{n_i}, \frac{\Im_{n_i}^k x_{n_i} - \Im_{n_i}^{k-1}x_{n_i}}{r_{n_i}}\right\rangle \geq F_k(y, \Im_{n_i}^k x_{n_i}).$$

Since $\frac{\Im_{n_i}^k x_{n_i} - \Im_{n_i}^{k-1}x_{n_i}}{r_{n_i}}\to 0$ and $u_{n_i} = \Im_{n_i}^k x_{n_i}\rightharpoonup w$, it follows by (A4) that

$$F_k(y,w)\leq 0\quad \forall y\in C,$$

for each $k = 1,2,3,\dots,M$.

For t with $0 < t\leq 1$ and $y\in H$, let $y_t = ty + (1-t)w$. Since $y\in C$ and $w\in C$, we have $y_t\in C$ and hence $F_k(y_t, w)\leq 0$. So, from (A1) and (A4) we have

$$0 = F_k(y_t,y_t) \leq tF_k(y_t,y) + (1-t)F_k(y_t,w) \leq tF_k(y_t,y)$$

and hence $F_k(y_t,y)\geq 0$. From (A3), we have $F_k(w,y)\geq 0$ for all $y\in C$ and hence $w\in EP(F_k)$ for $k=1,2,3,\dots,M$, that is, $w\in\cap_{k=1}^M SEP(F_k)$.

Next, we show that $w\in\cap_{i=1}^N F(T_i)$. By [1, Lemma 3.1], we have $F(W_n) = \cap_{i=1}^N F(T_n)$. Assume $w\notin F(W_n)$. Since $y_{n_i}\rightharpoonup w$, $\|W_n y_{n_i} - y_{n_i}\|\to 0$ and $w\neq W_n w$, it follows by the Opial's condition that

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i}-w\| &< \liminf_{i\to\infty}\|y_{n_i}-W_nw\|\\ &\le \liminf_{i\to\infty}\{\|y_{n_i}-W_ny_{n_i}\|+\|W_ny_{n_i}-W_nw\|\}\\ &\le \liminf_{i\to\infty}\|y_{n_i}-w\|\end{aligned}$$

which derives a contradiction. Thus, we have $w\in F(W_n)=\cap_{i=1}^N F(T_i)$.

Finally, we show that $w\in VI(C,B)$. Define

$$Tv=\begin{cases} Bv+N_Cv, & v\in C,\\ \emptyset, & v\notin C.\end{cases}$$

Then, T is maximal monotone. Let $(v,w_1)\in G(T)$. Since $w_1-Bv\in N_Cv$ and $v_n\in C$, we have $\langle v-v_n,w_1-Bv\rangle\ge 0$. On the other hand, $v_n=P_C(u_n-\lambda_nBy_n)$, we have

$$\langle v-v_n,v_n-(u_n-\lambda_nBy_n)\rangle\ge 0,$$

and hence

$$\left\langle v-v_n,\frac{v_n-u_n}{\lambda_n}+By_n\right\rangle\ge 0.$$

Therefore, we have

$$\begin{aligned}\langle v-v_{n_i},w\rangle &\ge \langle v-v_{n_i},Bv\rangle\\ &\ge \langle v-v_{n_i},Bv\rangle-\left\langle v-v_{n_i},\frac{v_{n_i}-u_{n_i}}{\lambda_{n_i}}+By_{n_i}\right\rangle\\ &= \left\langle v-v_{n_i},Bv-By_{n_i}-\frac{v_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle\\ &= \langle v-v_{n_i},Bv-Bv_{n_i}\rangle+\langle v-v_{n_i},Bv_{n_i}-By_{n_i}\rangle\\ &\quad-\left\langle v-v_{n_i},\frac{v_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle\\ &\ge \langle v-v_{n_i},Bv_{n_i}-By_{n_i}\rangle-\left\langle v-v_{n_i},\frac{v_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle.\end{aligned}$$

Since $\lim_{n\to\infty}\|v_n-u_n\|=\lim_{n\to\infty}\|v_n-y_n\|=0$, $u_{n_i}\rightharpoonup w$ and B is Lipschitz continuous, we obtain that $\lim_{n\to\infty}\|Bv_n-By_n\|=0$ and $v_{n_i}\rightharpoonup w$. From $\liminf_{n\to\infty}\lambda_n>0$ and $\lim_{n\to\infty}\|v_n-u_n\|=0$, we obtain

$$\langle v-w,w_1\rangle\ge 0.$$

Since T is maximal monotone, we have $w\in T^{-1}0$ and hence $w\in VI(C,B)$. Hence $w\in\Omega$, where $\Omega:=\cap_{i=1}^N F(T_i)\cap\left(\cap_{k=1}^M SEP(F_k)\right)\cap VI(C,B)$.

Since $z=P_\Omega(I-A+\gamma f)(z)$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\big\langle(A-\gamma f)z,z-x_n\big\rangle &= \limsup_{n\to\infty}\big\langle(A-\gamma f)z,z-x_n\big\rangle\\ &= \lim_{i\to\infty}\big\langle(A-\gamma f)z,z-x_{n_i}\big\rangle\\ &= \big\langle(A-\gamma f)z,z-w\big\rangle\le 0. \qquad (54)\end{aligned}$$

It follows from the last inequality and (44) that

$$\limsup_{n\to\infty}\big\langle\gamma f(z)-Az,W_nv_n-z\big\rangle\le 0. \qquad (55)$$

Step 7. Finally, we claim that $\{x_n\}$ converges strongly to $z=P_\Omega(I-A+\gamma f)(z)$.

Indeed, from (23) , we have

$$
\begin{aligned}
(56)\qquad & \|x_{n+1}-z\|^2 \\
= \;& \big\|\alpha_n\gamma f(x_n)+\beta_n x_n+\big((1-\beta_n)I-\alpha_n A\big)W_n v_n - z\big\|^2 \\
= \;& \|\big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - z)+\beta_n(x_n-z)+\alpha_n\big(\gamma f(x_n)-Az\big)\|^2 \\
= \;& \|\big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - z)+\beta_n(x_n-z)\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\langle x_n-z,\gamma f(x_n)-Az\rangle \\
& +2\alpha_n\Big\langle\big((1-\beta_n)I-\alpha_n A\big)(W_n v_n - z),\gamma f(x_n)-Az\Big\rangle \\
\le \;& \Big[(1-\beta_n-\alpha_n\bar{\gamma})\|W_n v_n - z\|+\beta_n\|x_n-z\|\Big]^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\gamma\langle x_n-z,f(x_n)-f(z)\rangle+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\gamma\alpha_n\langle W_n v_n - z,f(x_n)-f(z)\rangle+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& -2\alpha_n^2\langle A(W_n v_n - z),\gamma f(z)-Az\rangle \\
\le \;& \Big[(1-\beta_n-\alpha_n\bar{\gamma})\|W_n v_n - z\|+\beta_n\|x_n-z\|\Big]^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\gamma\|x_n-z\|\|f(x_n)-f(z)\|+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\gamma\alpha_n\|W_n v_n - z\|\|f(x_n)-f(z)\|+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& -2\alpha_n^2\langle A(W_n v_n - z),\gamma f(z)-Az\rangle \\
\le \;& \Big[(1-\beta_n-\alpha_n\bar{\gamma})\|x_n-z\|+\beta_n\|x_n-z\|\Big]^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\gamma\alpha\|x_n-z\|^2+2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\gamma\alpha_n\alpha\|x_n-z\|^2+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& -2\alpha_n^2\langle A(W_n v_n - z),\gamma f(z)-Az\rangle \\
= \;& \Big[(1-\alpha_n\bar{\gamma})^2+2\beta_n\alpha_n\gamma\alpha+2(1-\beta_n)\gamma\alpha_n\alpha\Big]\|x_n-z\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& -2\alpha_n^2\langle A(W_n v_n - z),\gamma f(z)-Az\rangle \\
\le \;& \big[1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\big]\|x_n-z\|^2+\bar{\gamma}^2\alpha_n^2\|x_n-z\|^2+\alpha_n^2\|\gamma f(x_n)-Az\|^2 \\
& +2\beta_n\alpha_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\alpha_n\langle W_n v_n - z,\gamma f(z)-Az\rangle \\
& +2\alpha_n^2\|A(W_n v_n - z)\|\|\gamma f(z)-Az\| \\
= \;& \big[1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\big]\|x_n-z\|^2+\alpha_n\Big\{\alpha_n\Big[\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2 \\
& +2\|A(W_n v_n - z)\|\|\gamma f(z)-Az\|\Big]+2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle \\
& +2(1-\beta_n)\langle W_n v_n - z,\gamma f(z)-Az\rangle\Big\}
\end{aligned}
$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{W_n v_n\}$ are bounded, we can take a constant $K>0$ such that

$$\bar{\gamma}^2\|x_n-z\|^2+\|\gamma f(x_n)-Az\|^2+2\|A(W_n v_n - z)\|\|\gamma f(z)-Az\|\le K,$$

for all $n\ge 0$. It then follows that

$$\|x_{n+1}-z\|^2\le\big[1-2(\bar{\gamma}-\alpha\gamma)\alpha_n\big]\|x_n-z\|^2+\alpha_n\sigma_n, \tag{57}$$

where

$$\sigma_n=2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n)\langle W_n v_n - z,\gamma f(z)-Az\rangle+\alpha_n K.$$

Using (C1), (54) and (55), we get $\limsup_{n\to\infty}\sigma_n \leq 0$. Applying Lemma 2.5 to (57), we conclude that $x_n \to z$ in norm. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1,2,3,\ldots,M\}$ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let B be a monotone and k-Lipschitz continuous mapping of C into itself such that*

$$\Omega := \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C,B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J^{F_M}_{r_{M,n}} J^{F_{M-1}}_{r_{M-1,n}} J^{F_{M-2}}_{r_{M-2,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big) P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$, $\{\lambda_n\} \subset [a,b] \subset (0,\frac{1}{k})$ and $\{r_{k,n}\}, k \in \{1,2,3,\ldots,M\}$ are real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,
(C3) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty}|r_{k,n+1} - r_{k,n}| = 0$,
(C4) $\lim_{n\to\infty}\lambda_n = 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Omega$ which is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)z, x - z \Big\rangle \geq 0, \ \ \forall x \in \Omega.$$

Equivalently, we have $z = P_\Omega(I - A + \gamma f)(z)$.

Proof. Put $T_n = I$ for all $n \in \mathbb{N}$ and for all $x \in C$. Then $W_n = I$ for all $x \in C$. The conclusion follows from Theorem 3.1. This completes the proof. □

If $A = I, \gamma \equiv 1$ and $\gamma_n = 1 - \alpha_n - \beta_n$ in Theorem 3.1, then we can obtain the following result immediately.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_k, k \in \{1,2,3,\ldots,M\}$ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let $\{T_n\}$ be a family of finitely nonexpansive mappings of C into itself and let B be a monotone and k-Lipschitz continuous mapping of C into H such that*

$$\Omega := \cap_{n=1}^{N} F(T_n) \cap \big(\cap_{k=1}^{M} SEP(F_k)\big) \cap VI(C,B) \neq \emptyset.$$

Let f be a contraction of H into itself with $\alpha \in (0,1)$. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ u_n = J^{F_M}_{r_{M,n}} J^{F_{M-1}}_{r_{M-1,n}} J^{F_{M-2}}_{r_{M-2,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C(u_n - \lambda_n B y_n), \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is the sequence generated by (15)and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$, $\{\lambda_n\} \subset [a,b] \subset (0,\frac{1}{k})$ and $\{r_{k,n}\}, k \in \{1,2,3,\ldots,M\}$ are a real sequence in $(0,\infty)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$,
(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,
(C4) $\liminf_{n\to\infty} r_{k,n} > 0$ *and* $\lim_{n\to\infty}|r_{k,n+1} - r_{k,n}| = 0$ *for each* $k \in \{1,2,3,\ldots,M\}$,
(C5) $\lim_{n\to\infty}\lambda_n = 0$.

(C6) $\lim_{n\to\infty}|\lambda_{n,i}-\lambda_{n-1,i}|=0$ *for all* $i=1,2,\ldots,N$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to a point* $z\in\Omega$ *which is the unique solution of the variational inequality*

$$\Big\langle z-f(z),x-z\Big\rangle\geq 0,\ \ \forall x\in\Omega.$$

Equivalently, we have $z=P_{\Omega}f(z)$.

Acknowledgement(s) : The authors would like to thanks the "National Centre of Excellence in Mathematics", under the Commission on Higher Education (CHE), Ministry of Education, Thailand. Moreover, we would like to thank Professor Dr. Somyot Plubiteng for providing valuable suggestions.

References

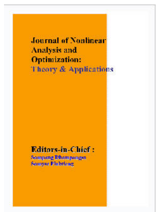
[1] S. Atsushiba and W. Takahashi, Strong convergence theorems for a finite family of nonexpansive mappings and applications, *Indian J. Math.*, 41 (1999), pp. 435-453.
[2] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, The Mathematics Student. vol. 63 no. 1–4, pp. 123–145, 1994.
[3] F.E. Browder and W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, Journal of Mathematical Analysis and Applications. vol. 20, pp. 197–228, 1967.
[4] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming using proximal-like algorithms, Mathematics Programming. vol.78, pp. 29–41, 1997.
[5] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. vol.6, pp. 29–41, 2005.
[6] L. C. Ceng and J. C. Yao, An Extragradient-Like Approximation Method for Variational Inequality Problems and Fixed Point Problems, Applied Mathematics and Computation. 190, pp. 205–215, 2007.
[7] L. C. Ceng and J. C. Yao, Relaxed Viscosity Approximation Methods for Fixed Point Problems and Variational Inequality Problems, Nonlinear Analysis Series A: Theory, Methods & Applications, 69, pp. 3299–3309, 2008.
[8] L. C. Ceng, C. Lee and J. C. Yao, Strong Weak Convergence Theorems of Implicit Hybrid Steepest-Descent Methods for Variational Inequalities, Taiwanese Journal of Mathematics, 12 (2008), 227-244.
[9] L. C. Ceng, A. Petruşel and J. C. Yao, Weak convergence theorem by a modified extragradient method for nonexpansive mappings and monotone mappings. Fixed Point Theory, 9, pp. 73–88, 2008.
[10] V. Colao, G. Marino and H.K. Xu, An iterative method for finding common solutions of equilibrium and fixed point problems, Journal of Mathematical Analysis and Applications, vol.344, pp. 340–352, 2008.
[11] V. Colao, G.L. Acedo and G. Marino, An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings, Nonlinear Analysis, vol.71, pp. 2708–2715, 2009.
[12] L. C. Ceng, P. Cubiotti and J. C. Yao, An Implicit Iterative Scheme for Monotone Variational Inequalities and Fixed Point Problems, Nonlinear Analysis Series A: Theory, Methods & Applications, 69, pp. 2445–2457, 2008.
[13] F. Deutsch, I. Yamada, Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings, Numerical Functional Analysis and Optimization, vol.19, pp. 33–56, 1998
[14] H. He, S. Liu and H. Zhou, An explicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of an infinite family of nonexpansive mappings, Nonlinear Analysis:Theory, Methods and Applications, vol.72 (2010) 341–350.
[15] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Analysis:Theory, Methods and Applications, vol.61 (2005) 341–350.
[16] A. Kangtunyakarn and S. Suantai, A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mapping, Nonlinear Analysis, vol.71, (2009) 4448–4460.
[17] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, E'konomika i Matematicheskie Metody, vol.12, no.4, (1976) 747–756.
[18] F. Liu and M.Z. Nashed, Regularization of nonlinear Ill-posed variational inequalities and convergence rates, Set-Valued Analysis, vol.6, pp. 313–344, 1998.
[19] G. Marino, H.K. Xu, A general iterative method for nonexpansive mapping in Hilbert spaces, Journal of Mathematical Analysis and Applications. vol.318, pp. 43–52, 2006.
[20] J.W. Peng and J.C. Yao, A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings, Nonlinear Analysis. 71 (2009), 6001-6010.
[21] J.-W. Peng and J.-C. Yao, Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems, *Math. and Comp. Model.*, 49 (2009), 1816-1828.

[22] J. W. Peng and J. C. Yao, A new hybrid-extragradient method for generalized mixed equilibrium problems and fixed point problems and variational inequality problems, Taiwanese Journal of Mathematics, 12 (2008), 1401-1433.
[23] J. W. Peng and J. C. Yao, Some new iterative algorithms for generalized mixed equilibrium problems with strict pseudo-contractions and monotone mappings, Taiwanese Journal of Mathematics, 2009, to appear.
[24] S. Plubtieng and R. Punpaenga, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, Journal of Mathematical Analysis and Applications, vol.336, no.1, pp. 455–469, 2007.
[25] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol.73, pp. 595–597, 1967.
[26] M.O. Osilike, D.I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comp. Math. Appl. 40:559–567(2000).
[27] R.T. Rockafellar, "On the maximality of sums of nonlinear monotone operators, Transacrions of the American Mathematical Society, vol.149, pp. 75–88, 1970.
[28] T. Suzuki, Strong convergence of krasnoselskii and manns type sequences for one-parameter nonexpansive semigroups without bochner integrals, Journal of Mathematical Analysis and Applications, vol.305, pp. 227–239, 2005.
[29] M. Shang, Y. Su and X. Qin, A General Iterative Method for Equilibrium Problems and Fixed Point Problems in Hilbert Spaces, Fixed Point Theory and Applications, vol.2007, Article ID 95412, 9 pages.
[30] W. Takahashi, Nonlinear functional analysis, Yokohama Publishers, Yokohama, 2000.
[31] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, Journal of Optimization Theory and Applications, vol. 118, pp. 417–428, 2003.
[32] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, Journal of Mathematical Analysis and Applications, vol. 298, pp. 279–291, 2004.
[33] Y. Yao, Y.C. Liou, J.C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, Fixed Point Theory and Applications, vol.2007, Article ID 64363, 12 pages.
[34] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwanese Journal of Mathematics, vol.5, pp. 387–404, 2001.
[35] H.K. Xu, Iterative algorithms for nonlinear operators, Journal of the London Mathematical Society, vol.66, pp. 240–256, 2002.
[36] H.K. Xu, An iterative approach to quadratic optimization, Journal of Optimization Theory and Applications, vol.116 no. 2, pp. 659–678, 2003.
[37] I. Yamada, " The hybrid steepest descent method for the variational inequality problem of the intersection of fixed point sets of nonexpansive mappings," in: D. Butnariu, Y. Censor, S. Reich (Eds.), Inherently Parallel Algorithm for Feasibility and Optimization, Elsevier, pp. 473–504, 2001.
[38] Y. Yao, M.A. Noor and Y.-C. Liou, On iterative methods for equilibrium problems, Nonlinear Analysis 70 (2009) 497509.
[39] L. C. Zeng and J. C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwan. J. Math., 10 (2006), 1293-1303.

[1]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangkok 10140,Thailand.
Email address: 52501403@st.kmutt.ac.th(Thanyarat Jitpeera) and poom.kum@kmutt.ac.th(Poom Kumam)

[2]Centre of Excellence in Mathematics, CHE, Si Ayutthaya Road, Bangkok 10400, Thailand.

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 93-96
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Mean nonexpansive mappings and Suzuki-generalized nonexpansive mappings*

Kittikorn Nakprasit

ABSTRACT: We give an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive, and vice versa. But in case of increasing mappings, we show that mean nonexpansiveness implies Suzuki-generalized nonexpansiveness.

KEYWORDS: Mean nonexpansive mapping; Suzuki-generalized nonexpansive mapping.

1. Introduction

Let C be a subset of a Banach space X. For nonnegative real numbers a and b such that $a + b \leq 1$, a mapping $T : C \to C$ is said to be (a, b)-*mean nonexpansive* if

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\| \quad \text{for all } x, y \in C.$$

We also say that T is *mean nonexpansive* if T is (a, b)-mean nonexpansive for some nonnegative real numbers a and b such that $a + b \leq 1$. This type of mappings is introduced in [4] and extensively studied in [2] and [3].

In [1], T. Suzuki introduced a weaker condition of nonpansiveness which is now known as Suzuki-generalized nonexpansive. We say that T is Suzuki-generalized nonexpansive if $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, then $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Incidentally, examples of mean nonexpansive mappings and Suzuki-generalized nonexpansive mappings in the known literature are essentially the same. So the question naturally arises whether there exists a subset relation between the class of mean nonexpansive mappings and the class of Suzuki-generalized nonexpansive mappings.

We find the answer negative by an example of a mapping that is mean nonexpansive but not Suzuki-generalized nonexpansive, and vice versa. However we prove that in case of increasing mappings, mean nonexpansiveness implies Suzuki-generalized nonexpansiveness.

Corresponding author: Kittikorn Nakprasit (kitnak@hotmail.com).
Manuscript received May 19, 2010. Accepted July 5, 2010.
*Supported by the Centre of Excellence in Mathematics, theCommission on Higher Education of Thailand.

2. A mean nonexpansive mapping that is not Suzuki-generalized nonexpansive

Lemma 2.1. *Suppose that $T : [0,5] \to [0,2]$ is a mapping defined by*

$$Tx = \begin{cases} 2 & \text{if } x \in [0,4], \\ 1 & \text{if } x \in (4,5), \\ 0 & \text{if } x = 5. \end{cases}$$

Then T is mean nonexpansive but not Suzuki-generalized nonexpansive.

Proof. Let $x = 4$ and $y = 5$. We have

$$\frac{1}{2}\|x - Tx\| = 1 = \|x - y\|.$$

But $\|Tx - Ty\| = 2 > \|x - y\|$. Thus T is not Suzuki-generalized nonexpansive.

Next we show that for each $0 \le x, y \le 5$,

$$\|Tx - Ty\| \le \frac{1}{2}\|x - y\| + \frac{1}{2}\|x - Ty\|.$$

Case 1: $x \in [0,5], y \in (4,5)$. Then

$$\begin{aligned} \|Tx - Ty\| &< \tfrac{3}{2} \\ &< \tfrac{1}{2}\|y - x + x - Ty\| \\ &= \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\|x - Ty\|. \end{aligned}$$

Case 2: $x \in [0,5], y = 5$. Then

$$\begin{aligned} \|Tx - Ty\| &< \tfrac{5}{2} \\ &= \tfrac{1}{2}\|y - x + x - Ty\| \\ &= \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\|x - Ty\|. \end{aligned}$$

Case 3: $x \in (4,5), y \in [0,4]$. Then

$$\begin{aligned} \|Tx - Ty\| &< \tfrac{3}{2} \\ &< \tfrac{1}{2}\|x - y + x - Ty\| \\ &\le \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\|x - Ty\|. \end{aligned}$$

Case 4: $x = 5, y \in [0,4]$. Then

$$\begin{aligned} \|Tx - Ty\| &< \tfrac{5}{2} \\ &\le \tfrac{1}{2}\|x - y + x - Ty\| \\ &\le \tfrac{1}{2}\|x - y\| + \tfrac{1}{2}\|x - Ty\|. \end{aligned}$$

We have $\|Tx - Ty\| = 0$ in a remaining case, so T is mean nonexpansive. □

3. A Suzuki-generalized nonexpansive mapping that is not mean nonexpansive

Lemma 3.1. *Suppose that $T : [0,11] \to [0,1]$ is a mapping defined by*

$$Tx = \begin{cases} 1 - x & \text{if } x \in [0,1], \\ 0 & \text{if } x \in (1,11) \\ 1 & \text{if } x = 11. \end{cases}$$

Then T is Suzuki-generalized nonexpansive but not mean nonexpansive.

Proof. Suppose T is mean nonexpansive. So there are nonnegative real numbers a and b such that $a + b \le 1$ and

$$\|Tx - Ty\| \le a\|x - y\| + b\|x - Ty\| \quad \text{for all } x, y \in [0,11].$$

But if $x = 0$ and $y = 1$, then

$$\begin{aligned} \|Tx - Ty\| &= 1 \\ &\leq a\|x - y\| + b\|x - Ty\| \\ &= a. \end{aligned}$$

So $a = 1$ and $b = 0$, i.e., T is nonexpansive. But this contradicts to the fact that T is not continuous. So T is not mean nonexpansive.

Next we show that T is Suzuki-generalized nonexpansive by contradiction. Suppose there are x and y such that

$$\|Tx - Ty\| > \|x - y\| \tag{1}$$

but

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \tag{2}$$

or

$$\frac{1}{2}\|y - Ty\| \leq \|x - y\|. \tag{3}$$

We may assume that $x = 11$ because T is nonexpansive on $[0, 11)$. Combining with $\|Tx - Ty\| \leq 1$ and (1), we have $y > 10$. But then

$$\begin{aligned} \tfrac{1}{2}\|x - Tx\| &= 5 \\ &> 1 \\ &\geq \|x - y\| \end{aligned}$$

and

$$\begin{aligned} \tfrac{1}{2}\|y - Ty\| &\geq \tfrac{9}{2} \\ &> 1 \\ &\geq \|x - y\| \end{aligned}$$

which contradict to (2) and (3). Thus T is Suzuki-generalized nonexpansive □

4. Increasing mean nonexpansive mapping is Suzuki-generalized nonexpansive

Lemma 4.1. *If T is an increasing mean nonexpansive mapping, then T is Suzuki-generalized nonexpansive.*

Proof. Let T be an increasing mean nonexpansive mapping.
Let $y < x$. We show that $\|Tx - Ty\| \leq \|x - y\|$ if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \tag{4}$$

or

$$\frac{1}{2}\|y - Ty\| \leq \|x - y\|. \tag{5}$$

We may assume

$$\|Tx - Ty\| < \min\{\|Tx - y\|, \|x - Ty\|\}. \tag{6}$$

Otherwise $\|Tx - Ty\| \leq \|x - y\|$ by mean nonexpansive condition of T.

Case 1: $Ty \leq Tx \leq y \leq x$.

Suppose x, y satisfy (4), we have

$$\|x - y\| + \|y - Tx\| = \|x - Tx\| \leq 2\|x - y\|.$$

So

$$\|y - Tx\| \leq \|x - y\|.$$

From (6), we have

$$\|Tx - Ty\| < \|Tx - y\|.$$

Thus

$$\|Tx - Ty\| < \|x - y\|.$$

Suppose x, y satisfy (5). Then

$$\|y - Tx\| + \|Tx - Ty\| = \|y - Ty\| \leq 2\|x - y\|.$$

So

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{or} \quad \|y - Tx\| \leq \|x - y\|.$$

Thus $\|Tx - Ty\| \leq \|x - y\|$ immediately or by (6).

Case 2: $Ty \leq y \leq Tx \leq x$.
This case does not satisfy $\|Tx - Ty\| < \|Tx - y\|$ in (6). Thus the case is impossible.

Case 3: $y \leq Ty \leq Tx \leq x$.
We have $\|Tx - Ty\| \leq \|x - y\|$ immediately.

Case 4:$y \leq Ty \leq x \leq Tx$.
This case does not satisfy $\|Tx - Ty\| < \|x - Ty\|$ in (6). Thus the case is impossible.

Case 5:$y \leq x \leq Ty \leq Tx$.

Suppose x, y satisfy (4). We have $\|Tx - Ty\| < \|x - Ty\|$ by (6), then

$$\begin{aligned} \|x - y\| &\geq \tfrac{1}{2}\|Tx - x\| \\ &= \tfrac{1}{2}\|Tx - Ty\| + \tfrac{1}{2}\|Ty - x\| \\ &> \tfrac{1}{2}\|Tx - Ty\| + \tfrac{1}{2}\|Tx - Ty\| \\ &= \|Tx - Ty\|. \end{aligned}$$

Suppose x, y satisfy (5). Then

$$\|Ty - x\| + \|x - y\| = \|Ty - y\| \leq 2\|x - y\|.$$

So

$$\|Ty - x\| \leq \|x - y\|.$$

From (6), we have

$$\|Tx - Ty\| < \|Ty - x\|.$$

Thus $\|Tx - Ty\| < \|x - y\|$. This completes the proof.

□

References

[1] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. J. Math. Anal. Appl. 340 (2008), 1088–1095.

[2] C. Wu and L.J. Zhang, Fixed points for mean non-expansive mappings. Acta Math. Appl. Sin. Engl. Ser. 23 (2007), no. 3, 489–494.

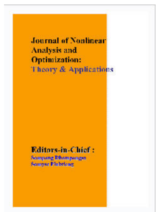
[3] Y. Yang and Y. Cui, Viscosity approximation methods for mean non-expansive mappings in Banach spaces. Appl. Math. Sci. (Ruse) 2 (2008), no. 13-16, 627–638.

[4] S. Zhang, About fixed point theory for mean nonexpansive mapping in Banach spaces. Journal of Sichuan University 2 (1975), 67-68.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KHON KAEN UNIVERSITY, KHON KAEN 40002, THAILAND.
Email address: kitnak@hotmail.com.

THE CENTRE OF EXCELLENCE IN MATHEMATICS, COMMISSION ON HIGHER EDUCATION (CHE), SRI AYUDTHAYA ROAD, BANGKOK 10400, THAILAND

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 97-110
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Shrinking projection methods for a family of relatively nonexpansive mappings, equilibrium problems and variational inequality problems in Banach spaces

Somyot Plubtieng, Tipphawan Thammathiwat

ABSTRACT: In this paper, we prove a strong convergence theorem by the shrinking projection method for finding a common element of the set of fixed points of a countable family of relatively nonexpansive mappings and the set of solutions of equilibrium problems and the set of solution of variational inequality problems in Banach spaces. Then, we apply our main theorem to the problem of finding a zero of a maximal monotone operator, the complementarity problems, and the convex feasibility problems.

KEYWORDS: Relatively nonexpansive mapping; Equilibrium problem; Variational inequality problem; Strong convergence.

1. Introduction

Let E be a real Banach space and let E^* be the dual space of E. Let C be a closed convex subset of E. Let $A : C \to E^*$ be a mapping. The classical variational inequality, denoted by $VI(A, C)$, is to find $x^* \in C$ such that

(1) $$\langle Ax^*, v - x^* \rangle \geq 0 \text{ for all} v \in C.$$

The variational inequality has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences; see, e.g. [9, 22, 33, 35, 37, 40] and the references therein. An operator A is called α-inverse-strongly monotone [7, 19] if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$.

In 2008, Iiduka and Takahashi [13] introduce the following algorithm for finding a solution of the variational inequality for an α-inverse-strongly monotone A in a 2-uniformly convex and uniformly smooth Banach space E. For an initial point $x_1 = x \in C$, define a sequence $\{x_n\}$ by

(2) $$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n),$$

Corresponding author: Somyot Plubtieng (somyotp@nu.ac.th).
Manuscript received July 11, 2010. Accepted July 31, 2010.
The authors would like to thanks The Thailand Research Fund for financial support. Moreover, T. Thammathiwat is also supported by the Royal Golden Jubilee Program under Grant PHD/0117/2549, Thailand.

where J is the duality mapping on E and Π_C is the generalized projection from E onto C. Then $\{x_n\}$ converges weakly to some element $z \in VI(A,C)$ where $z = \lim_{n\to\infty} \Pi_{VI(A,C)} x_n$.

Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $x \in C$ such that

$$f(x,y) \geq 0 \text{ for all } y \in C. \tag{3}$$

The set of solutions of (3) is denoted by $EP(f)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (3). In 1997 Combettes and Hirstoaga [12] introduced an iterative scheme of finding the best approximation to initial data when $EP(f)$ is nonempty and proved a strong convergence theorem. This equilibrium problem contains the fixed point problem, optimization problem, saddle point problem, variational inequality problem and Nash equilibrium problem as its special cases (see, e.g., Blum and Oettli [6], Combettes and Hirstoaga [12]).

A popular method is the hybrid projection method developed in Nakajo and Takahashi [22], Kamimura and Takahashi [14] and Martinez-Yanes and Xu [20]; see also Matsushita and Takahashi [21], Plubtieng and Ungchittrakool [25] and references therein. Recently Takahashi, et al. [34] introduced an alterative projection method, which is called the shrinking projection method, and they showed several strong convergence theorems for a family of nonexpansive mappings. In 2008, Takahashi and Zembayashi [36], introduced the following iterative scheme which is called the shrinking projection method:

$$\begin{cases} x_0 = x \in C, \;\; C_1 = C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \;\; \forall\, y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \;\; \forall n \geq 1, \end{cases} \tag{4}$$

where J is the duality mapping on E and Π_C is the generalized projection from E onto C. They proved that the sequence $\{x_n\}$ converges strongly to $q \in \Pi_{F(T)\cap EP(f)} x_0$. Recently, Cholamjiak [10] introduced a new hybrid projection algorithm and proved a strong convergence theorem for finding a common element of the set of solutions of the equilibrium problem and the set of the variational inequality for an inverse-strongly monotone operator and the set of fixed points of relatively quasinonexpansive mappings in a Banach space.

On the other hand, Aoyama, et al. [1] introduced a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T_n x_n \tag{5}$$

for all $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings with some condition. They proved that $\{x_n\}$ defined by (5) converges strongly to a common fixed point of $\{T_n\}$. Recently, Nakajo et al. [23] introduced the more general condition so-called the NST*-condition. A sequence $\{T_n\}$ is said to satisfy the NST*-condition if for every bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = \lim_{n\to\infty} \|z_n - z_{n+1}\| = 0 \text{ implies } \omega_w(z_n) \subset F.$$

where F is the set of common fixed point of $\{T_n\}$ and $\omega_w(z_n) = \{z : \exists z_{n_r} \rightharpoonup z\}$ denotes the weak ω-limit set of $\{z_n\}$. They also prove strong convergence theorems by the hybrid method for families of mappings in a uniformly convex Banach space E whose norm is G$\hat{a}$teaux differentiable.

Motivated and inspired by Takahashi and Zembayashi [36] and Cholamjiak [10], this paper is organized as follows. In section 2, we present some basic concepts and useful lemmas for proving the convergence result of this paper. In section 3, we introduce an iterative processes (15) below for finding a common element of the set of fixed points of a countable family of relatively nonexpansive mappings and the set of solutions of equilibrium problems and the set of solution of variational inequality problem. Then, we prove a strong convergence theorem. Moreover, we obtain corollary which extend the result of Takahashi and Zembayashi [36]. In section 4, we apply our main theorems to the problem of finding a zero of a maximal monotone operator, the complementarity problems, and the convex feasibility problems.

2. Preliminaries

Let E be a real Banach space and let $S = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if for any $x, y \in S$,

$$x \neq y \text{ implies } \|\frac{x+y}{2}\| < 1. \tag{6}$$

It is also said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in S$,

$$\|x - y\| \geq \varepsilon \text{ implies } \|\frac{x+y}{2}\| < 1 - \delta. \tag{7}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. We define a function $\delta : [0, 2] \to [0, 1]$, is called the *modulus of convexity of E*, as follows:

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}. \tag{8}$$

Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$; see [4, 5, 32] for more details. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{9}$$

exists for all $x, y \in S$. It is also said to be *uniformly smooth* if the limit (9) is attained uniformly for $x, y \in S$. One should note that no Banach space is p-uniformly convex for $1 < p < 2$; see [32]. It is well known that a Hilbert space is 2-uniformly convex, uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\} \tag{10}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. See [30, 31] for more details.

Lemma 2.1. [5] *Let p be a given real number with $p \geq 2$ and E a p-uniformly convex Banach space. Then, for all $x, y \in E$, $j_x \in J_p(x)$ and $j_y \in J_p(y)$,*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p, \tag{11}$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{12}$$

for all $x, y \in E$. In a Hilbert space H, we have $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$,
(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,
(3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$,

for all $x, y, z \in E$. Let E be a strictly convex, smooth, and reflexive Banach space, and let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [2]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{13}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping $V(x, \cdot) : E^* \to \mathbb{R}$ is a continuous and convex function from E^* into $\mathbb{R}$.

Lemma 2.2. [14] *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.3. [2, 14] *Let E be a a smooth, strictly convex, and reflexive Banach space and let V be as in (13). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \le V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, for any $x \in E$, there exists a point $x_0 \in C$ such that $\phi(x_0, x) = \min_{y\in C} \phi(y, x)$. The mapping $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$ is called the *generalized projection* [2, 14]. The following are well-known results. For example, see [2, 14].

Lemma 2.4. [2, 14] *Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$, and let $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for all $y \in C$.*

Lemma 2.5. [2, 14] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$ for all $y \in C$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let T be a mapping from C into itself, and let $F(T)$ be the set of all fixed points of T. Then a point $p \in C$ is said to be an *asymptotic fixed point* of T (see Reich [27]) if there exists a sequence $\{x_n\}$ in C converging weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$ and we say that T is a *relatively nonexpansive mapping* if the following conditions are satisfied:

(R1) $F(T)$ is nonempty;
(R2) $\phi(u, Tx) \leqslant \phi(u, x)$ for all $u \in F(T)$ and $x \in C$;
(R3) $\hat{F}(T) = F(T)$.

Lemma 2.6. [21] *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.7. [39] *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|tx + (1 - t)y\|^2 \le t\|x\|^2 + \|(1 - t)y\|^2 - t(1 - t)g(\|x - y\|), \tag{14}$$

for all $x, y \in B_r(0)$ and $t \in [0, 1]$.

Lemma 2.8. [26] *Let C be a closed convex subset of a smooth Banach space E and let $x, y \in E$. Then the set $K := \{v \in C : \phi(v, y) \leqslant \phi(v, x)\}$ is closed and convex.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbf{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\to 0} f(tz + (1 - t)x, y) \le f(x, y)$;
(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.9. [6] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.10. [36] *et C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single- valued;*
(2) *T_r is firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

Lemma 2.11. [36] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$$

An operator A of C into E^* is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0,1]$ into E^* defined by $F(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We define the *normal cone for C at a point $v \in C$*, $N_C(v)$ by

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \ \forall y \in C\}.$$

Lemma 2.12. [28] *Let C be a closed convex subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^*. Let $T_e \subset E \times E^*$ be an operator defined as follows:*

$$T_e v = \begin{cases} Av + N_C v, & v \in C; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then T_e is maximal monotone and $T_e^{-1}0 = VI(A, C)$.

Throughout the paper, we will use the notations:

(1) $\to$ for strong convergence.
(2) $\omega_w(x_n) = \{x : \exists x_{n_r} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

3. **Main result**

In this section, we prove the strong convergence theorems for finding a common element of the set of solutions of equilibrium problem, the set of the solutions of the variational inequality problem and the set of fixed point of a countable family of relatively nonexpansive mappings in Banach spaces by using the hybrid method in mathematical programming.

Theorem 3.1. *Let C be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse strongly monotone of E into E^* such that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A, C)$. Let $\{T_n\}$ be a family of relatively nonexpansive mappings of C into itself such that satisfies the NST^*-condition* *and* $F := \cap_{n=1}^{\infty} F(T_n) \cap EP(f) \cap VI(A,C) \neq \emptyset$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$, *define a sequence* $\{x_n\}$ *as follows:*

$$(15) \qquad \begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n z_n), \\ u_n \in C \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 1 \end{cases}$$

where J *is the duality mapping on* E. *Assume that* $\{\alpha_n\} \subset [0,1], \{\lambda_n\} \subset (0,\infty)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfying the retrictions*

(C1) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$,
(C2) $\{r_n\} \subset [s,\infty)$ *for some* $s > 0$,
(C3) $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < \frac{c^2\alpha}{2}$, *where* $\frac{1}{c}$ *is the 2-uniformly convexity of E.*

Then the sequence $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q = \Pi_F x_0$.

Proof. ***Step 1.*** Show that $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well-defined.

It is obvious that $VI(A,C)$ is closed convex subset of C. Thus, it follows form Lemma 2.6 and Lemma 2.10 that $\emptyset \neq F = \cap_{n=1}^{\infty} F(T_n) \cap EP(f) \cap VI(A,C)$ is closed and convex. This implies that $\Pi_F x_0$ is well-defined. Now, we claim that $F \subset C_n$ and C_n is closed convex for all $n \in \mathbb{N}$ .Obvious that $F \subset C = C_1$ is closed and convex. So $x_1 = \Pi_{C_1} x_0$ is well-defined. Next, suppose that $F \subset C_k$ and C_k is closed convex for some $k \in \mathbb{N}$. Thus $x_k = \Pi_{C_k} x_0$ is well-defined. We note from Lemma 2.8 that C_{k+1} is closed and convex. Consequently, C_n is closed and convex for all $n \in \mathbb{N}$. Set $v_n = J^{-1}(Jx_n - \lambda_n A x_n)$ and $u_n = T_{r_n} y_n$. For $u \in F \subset C_k$, we know from Lemma 2.3 and Lemma 2.5 that

$$\begin{aligned} \phi(u, z_k) &= \phi(u, \Pi_C v_k) \\ &\leq \phi(u, v_k) \\ &= \phi(u, J^{-1}(Jx_k - \lambda_k A x_k)) \\ &= V(u, Jx_k - \lambda_k A x_k) \\ &\leq V(u, (Jx_k - \lambda_k A x_k) + \lambda_k A x_k) - 2\langle J^{-1}(Jx_k - \lambda_k A x_k) - u, \lambda_k A x_k\rangle \\ &= V(u, Jx_k) - 2\lambda_k \langle v_k - u, A x_k\rangle \\ &= \phi(u, x_k) - 2\lambda_k\langle x_k - u, A x_k\rangle + 2\langle v_k - x_k, -\lambda_k A x_k\rangle. \qquad (16) \end{aligned}$$

Since $u \in VI(A,C)$ and A is an α-inverse strongly monotone, we have

$$\begin{aligned} -2\lambda_k\langle x_k - u, Ax_k\rangle &= -2\lambda_k\langle x_k - u, Ax_k - Au\rangle - 2\lambda_k\langle x_k - u, Au\rangle \\ &\leq -2\alpha\lambda_k \|Ax_k - Au\|^2. \qquad (17) \end{aligned}$$

Using Lemma 2.1 and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A,C)$, we obtain

$$\begin{aligned} 2\langle v_k - x_k, -\lambda_k A x_k\rangle &= 2\langle J^{-1}(Jx_k - \lambda_k A x_k) - J^{-1}(Jx_k), -\lambda_k A x_k\rangle \\ &\leq 2\|J^{-1}(Jx_k - \lambda_k A x_k) - J^{-1}(Jx_k)\|\|\lambda_k A x_k\| \\ &\leq \frac{4}{c^2}\|JJ^{-1}(Jx_k - \lambda_k A x_k) - JJ^{-1}(Jx_k)\|\|\lambda_k A x_k\| \\ &\leq \frac{4}{c^2}\lambda_k^2\|Ax_k\|^2 \\ &\leq \frac{4}{c^2}\lambda_k^2\|Ax_k - Au\|^2. \qquad (18) \end{aligned}$$

Replacing (17) and (18) into (16), we have

$$(19) \qquad \phi(u, z_k) \leq \phi(u, x_k) + 2\lambda_k(\frac{2}{c^2}\lambda_k - \alpha)\|Ax_k - Au\|^2 \leq \phi(u, x_k).$$

By convexity of $\|\cdot\|^2$ and (19) , for each $u \in F \subset C_k$, we obtain

$$\begin{aligned}
\phi(u,u_k) &= \phi(u, T_{r_k} y_k) \\
&\leq \phi(u, y_k) \\
&= \phi(u, J^{-1}(\alpha_k J x_k + (1-\alpha_k) J T_k z_k)) \\
&= \|u\|^2 - 2\alpha_k \langle u, Jx_k \rangle - 2(1-\alpha_k)\langle u, JT_k z_k\rangle + \|\alpha_k J x_k + (1-\alpha_k) JT_k z_k\|^2 \\
&\leq \|u\|^2 - 2\alpha_k \langle u, Jx_k \rangle - 2(1-\alpha_k)\langle u, JT_k z_k\rangle + \alpha_k \|Jx_k\|^2 + (1-\alpha_k)\|JT_k z_k\|^2 \\
&= \alpha_k \phi(u, x_k) + (1-\alpha_k)\phi(u, T_k z_k) \\
&\leq \alpha_k \phi(u, x_k) + (1-\alpha_k)\phi(u, z_k) \\
&\leq \alpha_k \phi(u, x_k) + (1-\alpha_k)\phi(u, x_k) \\
&= \phi(u, x_k). \qquad (20)
\end{aligned}$$

This show that $u \in C_{k+1}$ and so $F \subset C_{k+1}$. Consequently, $\emptyset \neq F \subset C_n$and C_n closed convex for all $n \geq 1$. This implies that $\Pi_{C_n} x_0$ for all $n \geq 1$ is well-defined.

Step 2. Show that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_n - u_n\| = 0$. Since $x_n = \Pi_{C_n} x_0$, it follows from Lemma 2.5 that

$$\phi(x_n, x_0) \leq \phi(u, x_0) - \phi(u, x_n) \leq \phi(u, x_0) \text{ for all } u \in C_n. \qquad (21)$$

From step 2 and (21), we get

$$\phi(x_n, x_0) \leq \phi(u, x_0) \text{ for all } u \in \mathbb{F} \text{ and for all } n \in \mathbb{N}. \qquad (22)$$

Therefore $\{\phi(x_n, x_0)\}$ is bounded and hence $\{x_n\}$ is bounded by (1). From $x_n = \Pi_{C_n} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) = min_{y\in C_n} \phi(y, x_0) \leq \phi(x_{n+1}, x_0) \text{ for all } n \in \mathbb{N}. \qquad (23)$$

Hence $\{\phi(x_n, x_0)\}$ is bounded and nondecreasing. This implies that there exists the limit of $\{\phi(x_n, x_0)\}$. It follows form Lemma 2.5 that

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) = \phi(x_{n+1}, x_0) - \phi(x_n, x_0), \qquad (24)$$

for all $n \in \mathbb{N}$. Thus, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \qquad (25)$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, it follows from the definition of C_{n+1} that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \to 0. \qquad (26)$$

By Lemma 2.2, 25 and 26, we note that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_n - u_n\| = 0. \qquad (27)$$

Since J is uniformly norm-to-norm continuous on bounded subset, we also obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \qquad (28)$$

Step 3. Show that $\{x_n\}$ is a Cauchy sequence.
Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$ for $m > n$, it follows from Lemma 2.5 that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) = \phi(x_m, x_0) - \phi(x_n, x_0). \qquad (29)$$

Taking $m, n \to \infty$, we obtain that $\phi(x_m, x_n) \to 0$. From Lemma 2.2, implies that $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence and so by the completeness of E and the closedness of C, we can assume that $x_n \to q \in C$ as $n \to \infty$.

Step 4. Show that $q \in \cap_{n=1}^{\infty} F(T_n)$.
Since $\{x_n\}$ and $\{T_n\}$ are bounded, there exists $r > 0$ such that $r = \sup_{n\geq 1}\{\|x_n\|, \|T_n\|\}$. Then, it

follows from (19), (20) and Lemma 2.7 that there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
\phi(u, u_n) &\le \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JT_nz_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JT_nz_n\|^2 \\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JT_nz_n\rangle + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|JT_nz_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&= \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, T_nz_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&\le \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, z_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&\le \alpha_n\phi(u, x_n) + (1-\alpha_n)[\phi(u, x_n) + 2\lambda_n(\frac{2}{c^2}\lambda_n - \alpha)\|Ax_k - Au\|^2] \\
&\quad -\alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&= \phi(u, x_n) + 2(1-\alpha_n)\lambda_n(\frac{2}{c^2}\lambda_n - \alpha)\|Ax_k - Au\|^2 - \alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) \\
&\le \phi(u, x_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|). \qquad (30)
\end{aligned}$$

This implies that

$$\begin{aligned}
\alpha_n(1-\alpha_n)g(\|Jx_n - JT_nz_n\|) &\le \phi(u, x_n) - \phi(u, u_n) \\
&= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle \\
&\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|. \qquad (31)
\end{aligned}$$

Using (27), (28), (31) and (C1), we get $\lim_{n\to\infty} g(\|Jx_n - JT_nz_n\|) = 0$. By the property of g, we have $\lim_{n\to\infty}\|Jx_n - JT_nz_n\| = 0$. Since J and J^{-1} are uniformly norm-to-norm continuous on bounded subset, it follows that

$$\lim_{n\to\infty}\|x_n - T_nz_n\| = \lim_{n\to\infty}\|J^{-1}(Jx_n) - J^{-1}(JT_nz_n)\| = 0 \qquad (32)$$

Again by 30, we have

$$\begin{aligned}
2a(\alpha - \frac{2}{c^2}b)\|Ax_k - Au\|^2 &\le \frac{1}{1-\alpha_n}\phi(u, x_n) - \phi(u, u_n) \\
&\le \frac{1}{1-\alpha_n}[\|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|]. \qquad (33)
\end{aligned}$$

It follows from (27), (28), (33) and (C1), we get that

$$\lim_{n\to\infty}\|Ax_k - Au\| = 0. \qquad (34)$$

From Lemma 2.3, Lemma 2.5 and (18), we have

$$\begin{aligned}
\phi(x_n, z_n) &= \phi(x_n, \Pi_C v_n) \\
&\le \phi(x_n, v_n) \\
&= \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&= V(x_n, Jx_n - \lambda_n Ax_n) \\
&\le V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\
&= V(x_n, Jx_n) - 2\lambda_n\langle v_n - x_n, Ax_n\rangle \\
&= \phi(x_n, x_n) + 2\langle v_n - x_n, -\lambda_n Ax_n\rangle \\
&\le \frac{4}{c^2}\lambda_n^2\|Ax_n - Au\|^2. \qquad (35)
\end{aligned}$$

By Lemma 2.2, (35) and J is uniformly norm-to-norm continuous on bounded subset, we note that

$$\lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|Jx_n - Jz_n\| = 0. \qquad (36)$$

Since $x_n \to q$ as $n \to \infty$, $z_n \to q$ as $n \to \infty$. Combining (27), (32) and (36), we also obtain

$$\|T_nz_n - z_n\| \le \|x_n - z_n\| + \|T_nz_n - x_n\| \to 0,$$

and hence

$$\|z_{n+1} - z_n\| \leq \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0.$$

Since $\{T_n\}$ satisfies the NST*-condition, we have $q \in \cap_{n=1}^{\infty} F(T_n)$.

Step 5. Show that $q \in EP(f)$.

From (20) and Lemma 2.11, we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\ &\leq \phi(u, y_n) - \phi(u, T_{r_n} y_n) \\ &\leq \phi(u, x_n) - \phi(u, T_{r_n} y_n) \\ &= \phi(u, x_n) - \phi(u, u_n). \end{aligned} \tag{37}$$

It follows from (31) that $\lim_{n\to\infty} \phi(u_n, y_n) = 0$. Hence, by Lemma 2.2, we note that $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded subset and (C2), we get

$$\lim_{n\to\infty} \|\frac{Ju_n - Jy_n}{r_n}\| = 0. \tag{38}$$

Using (A2), we note that, for each $y \in C$,

$$\begin{aligned} \|y - u_n\| \|\frac{Ju_n - Jy_n}{r_n}\| &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\ &\geq -f(u_n, y) \\ &\geq f(y, u_n). \end{aligned} \tag{39}$$

It follows from (A4) and $u_n \to q$ that $f(y, q) \leq 0$ for all $y \in C$. For each $0 < t < 1$ and $y \in C$, we define $y_t = ty + (1-t)q$. Hence $y_t \in C$ and therefore $f(y_t, q) \leq 0$. From (A1), we obtain $0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, q) \leq tf(y_t, y)$. Thus, $f(y_t, y) \geq 0$ and so from (A3) we get $f(q, y) \geq 0$. Since y is arbitrary element in C, we get $q \in EP(f)$.

Step 6. Show that $q \in VI(A, C)$.

Define $T_e \subset E \times E^*$ be as in Lemma 2.12 and let $(v, w) \in G(T_e)$. Then $w - Av \in N_C(v)$. Since $z_n \in C$ and by definition of $N_C(v)$, it follows that

$$\langle v - z_n, w - Av\rangle \geq 0. \tag{40}$$

On the other hand, by Lemma 2.5, we obtain

$$\langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n\rangle \leq 0. \tag{41}$$

Combining (40) and (41), we have

$$\begin{aligned} \langle v - z_n, w\rangle &\geq \langle v - z_n, Av\rangle \\ &\geq \langle v - z_n, Av\rangle + \langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n} - Ax_n\rangle \\ &= \langle v - z_n, Av - Ax_n\rangle + \langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n}\rangle \\ &= \langle v - z_n, Av - Az_n\rangle + \langle v - z_n, Az_n - Ax_n\rangle \\ &\quad + \langle v - z_n, \frac{Jx_n - Jz_n}{\lambda_n}\rangle \\ &\geq -\|v - z_n\| \frac{\|z_n - x_n\|}{\alpha} - \|v - z_n\| \frac{\|Jx_n - Jz_n\|}{a} \\ &\geq -M(\frac{\|z_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jz_n\|}{a}), \end{aligned} \tag{42}$$

where $M = \sup_{n\geq 1}\{\|v - z_n\|\}$. By taking the limit as $n \to \infty$ and from (36), we note that $\langle v - q, w\rangle \geq 0$. Since T_e is maximal monotone and $T_e^{-1}0 = VI(A, C)$, we obtain $q \in VI(A, C)$.

Step 7. Show that $q = \Pi_F x_0$.

Since $x_n = \Pi_{C_n} x_0$ and $F \subset C_n$ for all $n \in \mathbb{N}$, we obtain that

$$\langle Jx_0 - Jx_n, x_n - u\rangle \geq 0 \ \ \forall u \in F. \tag{43}$$

Taking the limit as $n \to \infty$ in (43), we get

$$\langle Jx_0 - Jq, q - u \rangle \geq 0 \ \ \forall u \in F. \tag{44}$$

By Lemma 2.4, we can conclude that $q = \Pi_F x_0$. This completes the proof. □

Setting $T_n = T$ for all $n \in \mathbb{N}$ and $A \equiv 0$ in Theorem 3.1, we have following result.

Corollary 3.2. [36] *Let C be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from* $C \times C \to \mathbb{R}$ *satisfying (A1)-(A4) and let T be a relatively nonexpansive mappings of C into itself* $F := F(T) \cap EP(f) \neq \emptyset$. *Assume that* $\{\alpha_n\} \subset [0,1]$ *satisfy* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\} \subset [s,\infty)$ *for some* $s > 0$. *Then the sequence* $\{x_n\}$ *generated by (4) converge strongly to* $q = \Pi_F x_0$.

Proof. Since $\hat{F}(T) = F(T)$, it follows that T satisfies the NST*-condition which is the desired result. □

Remark 3.3. It would be interesting to investigate convergent sequence when the countable family of relatively nonexpansive mappings of C into C is a semgroup, which is abelian or amenable. See: [17, 18].

4. Applications

4.1. Complementarity problems

Let K be a nonempty, closed convex cone in E, A an operator of K into E^*. We define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0, \forall x \in K\}. \tag{45}$$

Then the element $u \in K$ is called a solution of the *complementarity problem* if

$$Au \in K^*, \ \ \langle u, Au \rangle = 0. \tag{46}$$

The set of solutions of the complementarity problem is denoted by $C(K, A)$.

Theorem 4.1. *Let K be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from* $K \times K \to \mathbb{R}$ *satisfying (A1)-(A4) and let A be an* α*-inverse strongly monotone of E into* E^* *such that* $\|Ay\| \leq \|Ay - Au\|$ *for all* $y \in K$ *and* $u \in C(K, A)$. *Let* $\{T_n\}$ *be a family of relatively nonexpansive mappings of K into itself such that satisfies the NST*-condition and* $F := \cap_{n=1}^{\infty} F(T_n) \cap EP(f) \cap C(K, A) \neq \emptyset$. *For an initial point* $x_0 \in E$ *with* $x_1 = \Pi_{K_1} x_0$ *and* $K_1 = K$, *define a sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = \Pi_K J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n z_n), \\ u_n \in C \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \ \forall y \in K, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \ \forall n \geq 1 \end{cases} \tag{47}$$

where J is the duality mapping on E. Assume that $\{\alpha_n\} \subset [0,1], \{\lambda_n\} \subset (0,\infty)$ *and* $\{r_n\} \subset (0,\infty)$ *satisfying the condition (C1)-(C3) of Theorem 3.1. Then the sequence* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q = \Pi_F x_0$.

Proof. As in the proof of Takahashi [Lemma 7.11, [30]], we note that $VI(K, A) = C(K, A)$. Hence, we obtain the desired result. □

4.2. **Approximation of a zero of a maximal monotone operator**

Let B be a multivalued operator from E to E^* with domain $D(B) = \{z \in E : Az \neq \emptyset\}$ and range $R(B) = \cup\{Bz : z \in D(B)\}$. An operator B is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geqslant 0$ for each $x_i \in D(B)$ and $y_i \in Ax_i$, $i = 1,2$. A monotone operator B is said to be maximal if its graph $G(B) = \{(x,y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if B is a maximal monotone operator, then $B^{-1}(0)$ is closed and convex. Let E be a reflexive, strictly convex and smooth Banach space, and let B be a monotone operator from E to E^*, we known from Rockafellar [28] that B is maximal if and only if $R(J + rB) = E^*$ for all $r > 0$. Let $J_r : E \to D(B)$ defined by $J_r = (J + rB)^{-1}J$ and such a J_r is called the resolvent of B. We know that J_r is a relatively nonexpansive; see [21] and $B^{-1}(0) = F(J_r)$ for all $r > 0$; see [30, 31] for more details.

Theorem 4.2. *Let C be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse strongly monotone of E into E^* such that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A,C)$. Let B be a maximal monotone operator of E into E^* such that $F := B^{-1}(0) \cap EP(f) \cap VI(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:*

$$(48)\qquad \begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JJ_{t_n}z_n), \\ u_n \in C \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z,u_n) \leq \phi(z,x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \ \forall n \geq 1 \end{cases}$$

where J is the duality mapping on E and J_{t_n} is the resolvent of B. Assume that $\{\alpha_n\} \subset [0,1]$, $\{\lambda_n\} \subset (0,\infty)$ and $\{r_n\}, \{t_n\} \subset (0,\infty)$ satisfying the retrictions

(C1) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$,
(C2) $\{r_n\}, \{t_n\} \subset [s,\infty)$ *for some $s > 0$,*
(C3) $\{\lambda_n\} \subset [a,b]$ *for some a, b with $0 < a < b < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity of E.*

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. As in the proof of Nakajo et. al. [Theorem 4.2, [24]], we get that $\{J_{t_n}\}$ satisfies the NST*-condition. Hence, we obtain the desired result. □

4.3. **Convex feasibility problems**

Let I be a countable set and C_i be a nonempty closed convex subset of a Banach space E such that $C := \cap_{i\in I}C_i \neq \emptyset$. Then we are concerned with the *convex feasibility problem* (CFP)

$$\text{finding an } x \in C.$$

This problem is a frequently appearing problem in diverse areas of mathematical and physical sciences. There is a considerable investigation on (CFP) in the framework of Hilbert spaces which captures applications in various disciplines such as image restoration [11, 15, 16, 38], computer tomography [29], and radiation theraphy treatment planning [8]. In computer tomography with limited data, in which an unknown image has to be reconstructed from a *priori* knowledge and from measured results, each piece of information gives a constraint which in turn, gives rise to a convex set C_i to which the unknown image should belong (see [3]). It follows from Lemma 2.5 that the *generalized projection* Π_C is a relatively nonexpansive mapping. Then we get the following result by Theorem 3.1.

Theorem 4.3. *Let C be a closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E and let $\{\Omega_i\}_{i\in I}$ be a family of nonempty closed convex subset of C. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)-(A4) and let A be an α-inverse strongly monotone of E into E^* such that $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(A,C)$. Let $\Omega = \cap_{i\in I}\Omega_i \neq \emptyset$ and*

$F := \Omega \cap EP(f) \cap VI(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define a sequence $\{x_n\}$ as follows:

$$(49) \quad \begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J \Pi_{\Omega_{i(n)}} z_n), \\ u_n \in C \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall n \geq 1 \end{cases}$$

where J is the duality mapping on E and $\Pi_{\Omega_{i(n)}}$ is the generalized projection from E into $\Omega_{i(n)}$ and the index mapping $i : \mathbb{N} \cup \{0\} \to I$ satisfies for each $i \in I$, there exists $M_i > 0$, for all $n \in \mathbb{N} \cup \{0\}, i \in \{i(n), ..., i(n + M_i - 1)\}$. Assume that $\{\alpha_n\} \subset [0,1], \{\lambda_n\} \subset (0,\infty)$ and $\{r_n\} \subset (0,\infty)$ satisfying the condition (C1)-(C3) of Theorem 3.1. Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = \Pi_F x_0$.

Proof. We shall show that $\{\Pi_{\Omega_{i(n)}}\}$ satisfies the NST*-condition. Let $T_n = \Pi_{\Omega_{i(n)}}$ and $\{z_n\}$ be bounded sequence in E such that $\lim_{n\to\infty} \|z_{n+1} - z_n\| = \lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. Suppose that $z_{n_k} \rightharpoonup z$. Fixed $i \in I$. There exists a strictly increasing sequence $\{p_k\} \subset \mathbb{N} \cup \{0\}$ such that

$$n_k \leq p_k \leq n_k + M_i + 1 \text{ and } i(p_k), (\forall\ k \in \mathbb{N} \cup \{0\}).$$

Then we have,

$$\|z_{p_k} - z_{n_k}\| \leq \sum_{l=n_k}^{n_k+M_i-1} \|z_{l+1} - z_l\|.$$

for all $k \in \mathbb{N} \cup \{0\}$ which implies that $z_{p_k} \rightharpoonup z$. From $\|z_{p_k} - T_{p_k} z_{p_k}\| \to 0$, we get $T_{p_k} z_{p_k} = \Pi_{\Omega_i} z_{p_k} \rightharpoonup z$. So $z \in \Omega_i$, $\forall i$ and hence $z \in \Omega$. This implies that $\omega_w(z_n) \subset \Omega$. By Theorem 3.1, we obtain the result. □

References

[1] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal. 67 (2006), 2350-2360.

[2] Ya.I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 15- 50.

[3] A. Aleyner, S. Reich, Block-iterative algorithms for solving convex feasibility problems in Hilbert and in Banach spaces, J. Math. Anal. Appl., 343(2008), 427-435.

[4] K. Ball, E. A. Carlen, E. H. Lieb, Shap uniform convexity and smoothness inequalities for trace norms, Inventiones Mathematicae, 115(1994), 463-482.

[5] B. Beauzamy, Introduction to Banach Spaces and Their Geometry, vol.68 of North-Holland Mathematics Studies, North-Holland, Amsterdam, The Netherlands, 2nd edition, 1985.

[6] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Student, 63(1994), 123-145.

[7] Browder, F.E., Petryshyn, W.V., Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl., 20(1967), 197-228.

[8] Y. Censor and S.A. Zenios, Parallel Optimization, Theory, Algorithms, and Applications, Oxford University Press, New York, 1996.

[9] S.S. Chang, H.W. Joseph Lee, C.K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, Nonlinear Anal., 70(2009), 3307-3319.

[10] P. Cholamjiak, A hybrid iterative scheme for equilibrium problems, variational inequality problems, and fixed point problems in Banach spaces, Fixed Point Theory Appl., vol.2009, Article ID 719360, 18 pages doi: 10.1155/2009/719360.

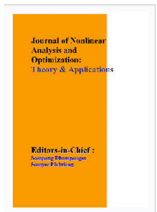
[11] P.L. Combettes, The convex feasibility problem in image recovery. In: P. Hawkes, Editor, Advances in Imaging and Electron Physics , Academic Press, New York, vol. 95(1996) 155-270

[12] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming using proximal-like algorithms, Math. Program, 78(1997), 29-41.
[13] H. Iiduka, W. Takahashi, Weak convergence of a projection algorithm for variational inequalities in Banach space, J. Math. Anal. Appl., 339(2008), 668-679.
[14] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim., 13(2002), 938-945.
[15] T. Kotzer, N. Cohen, J. Shamir, Image restoration by a novel method of parallel projection onto constraint sets, Opt. Lett., 20(1995), 1172-1174.
[16] T. Kotzer, N. Cohen, J. Shamir, Applications of serial- and parallel-projection methods to correlation filter design, Appl. Opt., 34(1995), 3883-3895.
[17] A. T.-M. Lau; H. Miyake, W. Takahashi, Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces, Nonlinear Anal., 67(2007), 1211–1225.
[18] A. T.-M. Lau; Y. Zhang, Fixed point properties of semigroups of non-expansive mappings, J. Funct. Anal., 254(2008), 2534–2554.
[19] F. Liu, M.Z. Nashed, Regularization of nonlinear Ill-posed variational inequalities and convergence rates, Set-Valued Anal., 6(1998), 313-344.
[20] C. Martinez-Yanes, H.-K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal., 64(2006), 2400-2411.
[21] S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theory, 134(2005), 257-266.
[22] N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl., 128(2006), 191-201.
[23] K. Nakajo, K. Shimoji, W. Takahashi, On strong convergence by the hybrid method for families of mappings in Hilbert spaces, Nonlinear Anal. 71(2009), 112-119.
[24] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence theorems by the hybrid method for families of mappings in Banach spaces, Nonlinear Anal., 71(2009), 812-818.
[25] S. Plubtieng, K. Ungchittrakool, Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space, J. Approx. Theory, 149(2007), 103-115.
[26] S. Plubtieng, K. Ungchittrakool, Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl., vol. 2008, Article ID 583082, 19 pages, 2008. doi:10.1155/2008/583082.
[27] S. Reich, A weak convergence theorem for the alternating method with Bregman distance, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 313-318.
[28] R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc., 149(1970), 75-88.
[29] M.I. Sezan, H. Stark, Application of convex projection theory to image recovery in tomography and related areas. In: H. Stark, Editor, Image Recovery Theory and Applications, Academic Press, Orlando, (1987), 415-462.
[30] W. Takahashi, Nonlinear Functional Analysis, Fixed Point Theory and Its Application, Yokohama, Yokohama, Japan, 2000.
[31] W. Takahashi, Convex Analysis and Approximation Fixed points, vol. 2 of Mathematical Analysis Series, Yokohama, Yokohama, Japan, 2000.
[32] Y. Takahashi, K. Hashimoto, M. Kato, On sharp uniform convexity, smoothness, and strong type, cotype inequalities, J. Nonlinear Convex Anal., 3(2002), 267-281.
[33] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl., 311(2007), 506-515.
[34] W. Takahashi, Y. Takeuchi, and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl., 341(2008), 276-286.
[35] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl., 118(2003), 417-428.
[36] W. Takahashi, K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory Appl., vol. 2008, Article ID 528476, 11 pages, 2008. doi:10.1155/2008/528476.
[37] J.C. Yao, O. Chadli, Pseudomonotone complementarity problems and variational inequalities, in: Crouzeix JP, Haddjissas N, Schaible S (Eds.), Handbook of Generalized Convexity and Monotonicity, (2005), pp. 501-558.

[38] D.C. Youla, H. Webb, Image restoration by the method of convex projections Part ITheory, IEEE Trans. Med. Imaging MI-1(1982), 81-94.
[39] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal., 16(1991), 1127-1138.
[40] L.C. Zeng, J.C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, Taiwanese J. Math., 10(2006), 1293-1303.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NARESUAN UNIVERSITY, PHITSANULOK 65000, THAILAND.
Email address: somyotp@nu.ac.th (Somyot Plubtieng)

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 111-129
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



The shrinking projection method for Generalized mixed Equilibrium Problems and Fixed Point Problems in Banach Spaces*

Rabian Wangkeeree, Uthai Kamraksa

ABSTRACT: The purpose of this paper is to introduce the iterative algorithms basing on the shrinking projection method for finding a common element of the set of common fixed points of two families of quasi-ϕ-nonexpansive mappings and the set of solutions of the generalized mixed equilibrium problems in the framework of Banach spaces. Our results improve and extend the corresponding results announced by many others.

KEYWORDS: Quasi-ϕ-nonexpansive mapping; Common fixed point; Shrinking projection method; Generalized mixed Equilibrium problems; Banach space.

1. Introduction

Let E be a Banach space and let E^* be the dual of E and let C be a closed convex subset of E. Let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \forall x \in E,$$

where E^* denoted the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. Some properties of the duality mapping have been given in [11, 32, 39].

Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be real-valued function, and $\Psi : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem is to find $u \in C$ such that

$$\Theta(u, y) + \langle \Psi u, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C. \tag{1}$$

The set of solutions to (1) is denoted by $GMEP(\Theta, \varphi, \Psi)$, i.e.,

$$GMEP(\Theta, \varphi, \Psi) = \{u \in C : \Theta(u, y) + \langle \Psi u, y - u \rangle + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C\}. \tag{2}$$

Special examples are as follows:

(I) If $\Psi = 0$, the problem (1) is equivalent to finding $u \in C$ such that

$$\Theta(u, y) + \varphi(y) - \varphi(u) \geq 0, \ \forall y \in C, \tag{3}$$

Corresponding author: Rabian Wangkeeree (rabianw@nu.ac.th).
Manuscript received May 11, 2010. Accepted June 14, 2010.
*Supported by Faculty of Science, Naresuan University, Thailand.

which is called the mixed equilibrium problem (see [6]). The set of solutions to (3) is denoted by MEP.

(II) If $\Theta = 0$, the problem (1) is equivalent to finding $u \in C$ such that

$$\langle \Psi u, y - u\rangle + \varphi(y) - \varphi(u) \geq 0,\ \forall y \in C, \tag{4}$$

which is called the mixed variational inequality of Browder type (see [3]). The set of solutions to (4) is denoted by $VI(C, A, \varphi)$.

If C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{5}$$

for all $x, y \in E$, where J is the normalized duality mapping from E to E^*. Observe that, in a Hilbert space H, (40) reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem:

$$\phi(x^*, x) = \inf_{y\in C} \phi(y, x). \tag{6}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 9, 28]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$ for all $x, y \in E$.
(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$ for all $x, y, z \in E$.
(3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$.
(4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \text{ if and only if } x = y.$$

For more detail see [11, 32]. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed point of T. A point p in C is said to be an *asymptotic fixed point* of T [29] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive [8, 10, 12] if $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leqslant \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings which was studied in [8, 10, 12] is of special interest in the convergence analysis of feasibility, optimization and equilibrium methods for solving the problems of image processing, rational resource allocation and optimal control. The most typical examples in this regard are the Bregman projections and the Yosida type operators which are the cornerstones of the common fixed point and optimization algorithms discussed in [9] (see also the references therein).

The mapping T is said to be *ϕ-nonexpansive* if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$. T is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 1.1. The class of quasi-ϕ-nonexpansive is more general than the class of relatively nonexpansive mappings [8, 10, 21, 24, 25] which requires the strong restriction $\hat{F}(T) = F(T)$.

Next, we give some examples which are closed quasi-ϕ-nonexpansive [27].

Example 1.2. (1). Let E be a uniformly smooth and strictly convex Banach space and A be a maximal monotone mapping from E to E such that its zero set $A^{-1}0$ is nonempty. Then $J_r = (J + rA)^{-1}$ is a closed quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

(2). Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E. Then Π_C is a closed and quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

On the other hand, One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping (see [4]). More precisely, let $t \in (0,1)$ and define a contraction $G_t : C \to C$ by $G_t x = tx_0 + (1-t)Tx$ for all $x \in C$, where $x_0 \in C$ is a fixed point in C. Applying Banach's Contraction Principle, there exists a unique fixed point x_t of G_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$ even if T has a fixed point. However, in the case of T having a fixed point, Browder [4] proved that the net $\{x_t\}$ defined by $x_t = tx_0 + (1-t)Tx_t$ for all $t \in (0,1)$ converges strongly to an element of $F(T)$ which is nearest to x_0 in a real Hilbert space. Motivated by Browder [4], Halpern [16] proposed the following innovation iteration process:

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad n \geqslant 0 \tag{7}$$

and proved the following theorem.

Theorem H. *Let C be a bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on C. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}, 0 < \theta < 1$. Define a sequence $\{x_n\}$ by (7). Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to u.*

Recently, Martinez-Yanes and Xu [20] has adapted Nakajo and Takahashi's [23] idea to modify the process (7) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 &= x \in C \text{chosen arbitrary}, \\ y_n &= \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \leqslant \|x_n - v\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n &= \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geqslant 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{cases} \tag{8}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H. They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ generated by (8) converges strongly to $P_{F(T)}x$.

In [25](see also [21]), Qin and Su improved the result of Martinez-Yanes and Xu [20] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem.

Theorem QS. *Let E be a uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 &= x \in C \text{chosen arbitrary}, \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, y_n) + (1-\alpha_n)\phi(v, x_n)\}, \\ Q_n &= \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{9}$$

where J is the single-valued duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.

Recently, Plubtieng and Ungchittrakool [24], still in the framework of Banach spaces, introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings:

$$\begin{cases} x_0 &= x \in C \text{chosen arbitrary}, \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n), \\ z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx\rangle)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= P_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases} \tag{10}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are sequences in $[0,1]$ with $\beta_n^{(1)}+\beta_n^{(2)}+\beta_n^{(3)}=1$ for all $n\in\mathbb{N}\cup\{0\}$ and T,S are relatively nonexpansive mappings and J is the single-valued duality mapping on E. They proved that the sequence $\{x_n\}$ generated by (10) converges strongly to a common fixed point of T and S.

Very recently, Qin, Cho, Kang and Zhou [26] introduced a new hybrid projection algorithm for two families of quasi-ϕ-nonexpansive mappings which more general than relatively nonexpansive mappings to have strong convergence theorems in the framework of Banach spaces. To be more precise, they proved the following theorem:

Theorem QCKZ. Let E be uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{S_i\}_{i\in I}$ and $\{T_i\}_{i\in I}$ be two families of closed quasi-ϕ-nonexpansive mappings of C into itself with $F:=\cap_{i\in I}F(T_i)\cap\cap_{i\in I}F(S_i)$ is nonempty, where I is an index set. Let the sequence $\{x_n\}$ be generated by the following manner:

$$(11)\quad\begin{cases} x_0 &= x\in C\text{chosen arbitrary},\\ z_{n,i} &= J^{-1}(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n),\\ y_{n,i} &= J^{-1}(\alpha_{n,i}Jx_0+(1-\alpha_{n,i})Jz_{n,i}),\\ C_{n,i} &= \{u\in C:\phi(u,y_{n,i})\leqslant\phi(u,x_n)+\alpha_{n,i}(\|x_0\|^2+2\langle u,Jx_n-Jx_n\rangle)\},\\ C_n &= \cap_{i\in I}C_{n,i},\\ Q_0 &= C,\\ Q_n &= \{u\in Q_{n-1}:\langle x_n-u,Jx_0-Jx_n\rangle\geq 0\},\\ x_{n+1} &= \Pi_{C_n\cap Q_n}x_0,\quad n=0,1,2,\dots,\end{cases}$$

where J is the duality mapping on E, $\{\alpha_{n,i}\},\{\beta_{n,i}^{(i)}\}(i=1,2,3,\dots)$ are sequences in $(0,1)$ such that

(i) $\beta_{n,i}^{(1)}+\beta_{n,i}^{(2)}+\beta_{n,i}^{(3)}=I$ for all $i\in I$
(ii) $\lim_{n\to\infty}\alpha_{n,i}=0$ for all $i\in I$; and
(iii) $\liminf_{n\to\infty}\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}>0$ and $\lim_{n\to\infty}\beta_{n,i}^{(1)}=0$ for all $i\in I$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

On the other hand, let $f:C\times C\to\mathbb{R}$ be a bifunction. The equilibrium problem for f is to find $\hat{x}\in C$ such that

$$(12)\qquad f(\hat{x},y)\ \geq\ 0,\quad\forall y\in C.$$

The set of solutions of (12) is denoted by $EP(f)$.

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; see, for instance, Blum and Oettli [5], Combettes and Hirstoaga [7], and Moudafi [22]. On the other hand, there are some methods for approximation of fixed points of Fixed Point Theory and Applications a nonexpansive mapping. Recently, Tada and Takahashi [30, 31] and Takahashi and Takahashi [37] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [31] established a strong convergence theorem for finding a common element of two sets by using the hybrid method introduced in Nakajo and Takahashi [23]. They also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space. Recently, Takahashi et al. [38] introduced a hybrid method which is different from Nakajo and Takahashis hybrid method. It is called the shrinking projection method. They obtained the strong convergence theorem in the frame work of Hilbert spaces. Based on the so-called shrinking projection method of Takahashi et al. [38], Takahashi and Zembayashi [36] introduced the following iterative scheme :

$$(13)\qquad \begin{cases} x_0 &= x \in C, \quad C_0 = C, \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} &= \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is the single-valued duality mapping on E and Π_C is the generalized projection from E onto C. They proved that the sequence $\{x_n\}$ defined by (13) converges strongly to $q = \Pi_{F(T)\cap EP(f)} x_0$ under appropriate conditions imposed on the parameters.

Motivated and inspired by Iiduka and Takahashi [17], Martinez-Yanes and Xu [20], S. Matsushita and W. Takahashi [21], Plubtieng and Ungchittrakool [24], Qin and Su [25], Qin, Cho, Kang and Zhou[26], Takahashi et al. [38] and Takahashi and Zembayashi [36], we introduce a new hybrid projection algorithm basing on the shrinking projection method for two families of quasi-ϕ-nonexpansive mappings which more general than relatively nonexpansive mappings to have strong convergence theorems for approximating the common element of the set of common fixed points of two families of quasi-ϕ-nonexpansive mappings and the set of solutions of the equilibrium problem in the framework of Banach spaces.

2. **Preliminaries**

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [14, 28, 32, 33]. A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [14, 32, 33] for more details. Let E be a smooth Banach space.

Now we collect some definitions and lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Lemma 2.1 (Kamimura and Takahashi [18]). *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, then $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

Lemma 2.2 (Alber [1], Alber and Reich [2], Kamimura and Takahashi [18]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0 \rangle \geqslant 0$ for $y \in C$.*

Lemma 2.3 (Alber [1], Alber and Reich [2], Kamimura and Takahashi [18]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$$

for all $y \in C$.

Lemma 2.4 (Qin et al. [26]). *Let E be a uniformly convex and smooth Banach space, C be a closed convex subset of E and T be a closed and quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Let E be a reflexive strictly convex, smooth and uniformly Banach space and the duality mapping from E to E^*. Then J^{-1} is also single-valued, one to one, surjective, and it is the duality mapping from E^* to E. We make use of the following mapping V studied in Alber [1],

$$V(x, x^*) = \|x^2\| - 2\langle x, x^*\rangle + \|x\|^2 \tag{14}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \phi(x, J^{-1}(x^*))$. We know the following lemma:

Lemma 2.5 (Kamimura and Takahashi [18]). *Let E be a reflexive, strictly convex and smooth Banach space, and let V be as in* (14). *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.6 ([13, Lemma 1.4]). *Let X be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leqslant \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|), \tag{15}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \quad \leq \quad f(x, y); \tag{16}$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x, y) = \langle Ax, y - x\rangle, \forall x, y \in C.$$

Then, f satisfies (A1)-(A4).

Lemma 2.7 (Blum and Oettli [5]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach spaces E, let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1) - (A4)$, and let $r > 0$ and $x \in E$. Then, there exists $u \in C$ such that*

$$f(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C. \tag{17}$$

Lemma 2.8 (Takahashi and Zembayashi [35]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. For all $r > 0$ and $x \in E$, define a mapping*

$$T_r x = \Big\{u \in C : f(z, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C\Big\}. \tag{18}$$

Then, the following hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive-type mapping* [19], *that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \quad \leq \quad \langle T_r x - T_r y, Jx - Jy\rangle; \tag{19}$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

Lemma 2.9 (Takahashi and Zembayashi [35]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, and let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \quad \leq \quad \phi(q, x).$$

Lemma 2.10. *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach spaces E, let Θ be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(A1)-(A4)$. Let $\Psi : C \to E^*$ be a continuous and monotone operator and $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $r > 0$ be any given number and $x \in E$ be any given point. Then, there exists $u \in C$ such that*

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C. \tag{20}$$

Proof. We define a bifunction $f : C \times C \to \mathbb{R}$ by

$$f(x,y) = \Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle, \quad \forall x, y \in C. \tag{21}$$

Next, we prove that the bifunction f satisfies condition (A1)-(A4):
(A1) $f(x,x) = 0$ for all $x \in C$.
Since $f(x,x) = \Theta(x,x) + \varphi(x) - \varphi(x) + \langle \Psi x, x - x\rangle = 0$, for all $x \in C$.
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$.
From the definition of f we have

$$\begin{aligned} f(x,y) + f(y,x) &= \Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle + \Theta(y,x) + \varphi(x) - \varphi(y) + \langle \Psi x, x - y\rangle \\ &= \Theta(x,y) + \Theta(y,x) \leq 0. \end{aligned}$$

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y).$$

Since

$$\begin{aligned} & \limsup_{t\downarrow 0} f(tz + (1-t)x, y) \\ = & \limsup_{t\downarrow 0}[\Theta(tz + (1-t)x, y) + \varphi(y) - \varphi(tz + (1-t)x) + \langle \Psi x, y - (tz + (1-t)x)\rangle] \\ \leq & \limsup_{t\downarrow 0} \Theta(tz + (1-t)x, y) + \varphi(y) - \liminf_{t\downarrow 0} \varphi(tz + (1-t)x) + \langle \Psi x, y\rangle - \liminf_{t\downarrow 0}\langle \Psi x, tz + (1-t)x\rangle \\ \leq & t\Theta(x,y) + (1-t)\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y\rangle - \langle \Psi x, x\rangle \\ = & \Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle = f(x,y). \end{aligned}$$

(A4) for each $x \in C$, $y \mapsto \Theta(x,y)$ is a convex and lower semicontinuous.
For each $x \in C$, $\forall t \in (0,1)$ and $\forall y, z \in C$, since G satisfies (A4), we have

$$\begin{aligned} f(x, ty + (1-t)z) &= \Theta(x, ty + (1-t)z) + \varphi(ty + (1-t)z) - \varphi(x) + \langle \Psi x, (ty + (1-t)z) - x\rangle \\ &\leq t\Theta(x,y) + (1-t)\Theta(x,z) + t\varphi(y) + (1-t)\varphi(z) - \varphi(x) + \langle \Psi x, (ty + (1-t)z) - x\rangle \\ &= t[\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle] \\ &\quad + (1-t)[\Theta(x,z) + \varphi(z) - \varphi(x) + \langle \Psi x, z - x\rangle] \\ &= tf(x,y) + (1-t)f(x,z). \end{aligned}$$

So, $y \mapsto f(x,y)$ is convex.
Similarly, we can prove that $y \mapsto f(x,y)$ is lower semicontinuous. Hence f satisfies condition (A1)-(A4). Applying Lemma 2.7, there exists $u \in C$ such that

$$f(u,y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

That is

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C.$$

This is completes the proof. □

3. **Main Results**

In this section, we prove two strong convergence theorems for approximating the common element of the set of common fixed points of two families of quasi-ϕ-nonexpansive mappings and the set of solutions of the generalized mixed equilibrium problem in the framework of a real Banach space.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $\Psi : C \to E^*$ be a continuous and monotone operator and $\varphi : C \to \mathbb{R}$ be a a lower semi-continuous and convex function. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, let $\{T_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ be two families of closed quasi-ϕ-nonexpansive mappings T_i, $S_i : C \to C$ such that the common fixed point set $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap GMEP(\Theta, \varphi, \Psi)$ is nonempty , where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$(22)\quad \begin{cases} x_0 \in C \text{ chosen arbitrary and } C_{0,i} = C, \quad \forall i \in I, \\ z_{n,i} = J^{-1}(\beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ u_{n,i} \in C \text{ such that } \Theta(u_{n_i}, y) + \varphi(y) - \varphi(u_{n,i}) + \\ \langle \Psi u_{n,i}, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \geq 0, \ \forall y \in C, \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, u_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is a duality mapping on E, $\{\alpha_{n,i}\}$, $\{\beta^{(i)}_{n,i}\}$ $(i = 1,2,3)$ and $\{r_{n,i}\}$ are sequences in $(0,1)$ satisfying

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each $i \in I$;*
(b) $\{r_{n,i}\} \subset [a, \infty)$ *for some $a > 0$ and for all $i \in I$;*
(c) $\beta^{(1)}_{n,i} + \beta^{(2)}_{n,i} + \beta^{(3)}_{n,i} = 1$ *for each $i \in I$ and if one of the following conditions is satisfied*
  (c-1) $\liminf_{n\to\infty} \beta^{(1)}_{n,i}\beta^{(l)}_{n,i} > 0$ *for all $l = 2,3$ and for all $i \in I$ and*
  (c-2) $\liminf_{n\to\infty} \beta^{(2)}_{n,i}\beta^{(3)}_{n,i} > 0$ *and* $\liminf_{n\to\infty} \beta^{(1)}_{n,i} = 0$ *for each $i \in I$.*

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. Let the bifunction $f : C \times C \to \mathbb{R}$ be defined by (21). Therefore, the mixed equilibrium problem (1) is equivalent to the following equilibrium problem: find $u \in C$ such that

$$f(u, y) \geq 0, \quad \forall y \in C,$$

and (66) can be written as:

$$(23)\quad \begin{cases} x_0 \in C \text{ chosen arbitrary and } C_{0,i} = C, \quad \forall i \in I, \\ z_{n,i} = J^{-1}(\beta^{(1)}_{n,i} Jx_n + \beta^{(2)}_{n,i} JT_i x_n + \beta^{(3)}_{n,i} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ u_{n,i} \in C \text{ such that } f(u_{n_i}, y) + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \geq 0, \ \forall y \in C, \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, u_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0. \end{cases}$$

Since the bifunction f satisfies conditions (A1) - (A4), from Lemma 2.10, for given $r > 0$ and $x \in C$, we define $T_r : C \to 2^C$ by

$$T_r(x) = \{u \in C : f(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

Moreover, T_r satisfies the conclusions in Lemma 2.8. We divide the proof of Theorem 3.1 into seven steps:

Step 1. Show that $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well defined.

By Lemma 2.4, we know that $\bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$ is closed and convex. From Lemma 2.8 (4), we also have $EP(f)$ is closed and convex. Hence $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap EP(f)$ is a nonempty, closed, and convex subset of C. Consequently, $\Pi_F x_0$ is well defined.

From the definition of C_n, it is obvious that C_n is closed for each $n \geq 0$. We show that C_{n+1} is convex for each $n \geq 0$. Notice that

$$C_{n+1,i} = \{u \in C_{n,i} : \phi(u, u_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}$$

is equivalent to

$$C'_{n+1,i} = \{u \in C_{n,i} : 2\langle u, Jx_n - Jy_{n,i}\rangle - 2\alpha_{n,i}\langle u, Jx_n - Jx_0\rangle \leq \|x_n\|^2 - \|y_{n,i}\|^2 + \alpha_{n,i}\|x_0\|^2\}.$$

It is easy to see that $C'_{n+1,i}$ is closed and convex for all $n \geq 0$ and $i \in I$. Therefore, $C_{n+1} = \bigcap_{i\in I} C_{n+1,i} = \bigcap_{i\in I} C'_{n+1,i}$ is closed and convex for every $n \geq 0$. This shows that $\Pi_{C_{n+1}} x_0$ is well-defined.

Step 2. Show that $F \subset C_n$ for all $n \geq 0$.

First, we observe that $u_{n,i} = T_{r_{n,i}} y_{n,i}$ for all $n \geq 1$ and $F \subset C_0 = C$. For any $w \in F$ and all $i \in I$, one has

$$\begin{aligned}
\phi(w, z_{n,i}) &= \phi(w, J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n)) \\
&= \|w\|^2 - 2\langle w, \beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n)\rangle \\
&\quad + \|\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n\|^2 \\
&\leq \|w\|^2 - 2\beta_{n,i}^{(1)}\langle w, Jx_n\rangle - 2\beta_{n,i}^{(2)}\langle w, JT_i x_n\rangle - 2\beta_{n,i}^{(3)}\langle w, JS_i x_n\rangle \\
&\quad + \beta_{n,i}^{(1)}\|x_n\|^2 + \beta_{n,i}^{(2)}\|T_i x_n\|^2 + \beta_{n,i}^{(3)}\|JS_i x_n\|^2 \\
&= \beta_{n,i}^{(1)}\phi(w, x_n) + \beta_{n,i}^{(2)}\phi(w, T_i x_n) + \beta_{n,i}^{(3)}\phi(w, S_i x_n) \\
&\leq \beta_{n,i}^{(1)}\phi(w, x_n) + \beta_{n,i}^{(2)}\phi(w, x_n) + \beta_{n,i}^{(3)}\phi(w, x_n) \\
&= \phi(w, x_n)
\end{aligned}$$

and hence

$$\begin{aligned}
\phi(w, u_{n,i}) &= \phi(w, T_{r_{n,i}} y_{n,i}) \\
&\leq \phi(w, y_{n,i}) \\
&= \phi(w, J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}) \\
&= \|w\|^2 - 2\langle w, \alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}\rangle + \|\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}\|^2 \\
&\leq \|w\|^2 - 2\alpha_{n,i}\langle w, Jx_0\rangle - 2(1-\alpha_{n,i})\langle w, Jz_{n,i}\rangle + \alpha_{n,i}\|x_0\|^2 + (1-\alpha_{n,i})\|z_{n,i}\|^2 \\
&= \alpha_{n,i}\phi(w, x_0) + (1-\alpha_{n,i})\phi(w, z_{n,i}) \\
&\leq \alpha_{n,i}\phi(w, x_0) + (1-\alpha_{n,i})\phi(w, x_n) \\
&= \phi(w, x_n) + \alpha_{n,i}[\phi(w, x_0) - \phi(w, x_n)] \\
&\leq \phi(w, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle w, Jx_n - Jx_0\rangle). \qquad (24)
\end{aligned}$$

This show that $w \in C_{n+1,i}$ for each $i \in I$. That is, $w \in C_{n+1} = \bigcap_{i\in I} C_{n+1,i}$ for all $n \geq 0$. Hence $F \subset C_n$ for all $n \geq 0$.

Step 3. Show that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists.

We note that $C_{n+1,i} \subset C_{n,i}$ for all $n \geq 0$ and for all $i \in I$. Hence

$$C_{n+1} = \cap_{i\in I} C_{n+1,i} \subset C_n = \cap_{i\in I} C_{n,i}.$$

From $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n} x_0 \in C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 1. \qquad (25)$$

This is, $\{\phi(x_n, x_0)\}$ is nondecreasing. On the other hand, from Lemma 2.3, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0). \tag{26}$$

for each $w \in F \subset C_n$. Combining (25) and (26), we obtain that limit $\{\phi(x_n, x_0)\}$ exists.

Step 4. Show that $\{x_n\}$ is a convergent sequence in C.

Since $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$ for $m \geq n$, by Lemma 2.3, We also have

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{Q_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{Q_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned} \tag{27}$$

Letting $m, n \to \infty$ in (27), one has $\phi(x_m, x_n) \to 0$. It follows from Lemma 2.1 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that

$$x_n \to p \in C \quad (n \to \infty). \tag{28}$$

Step 5. Show that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap GMEP(\Theta, \varphi, \Psi)$.

(a) We first will show that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$. Taking $m = n+1$ in (27), we obtain.

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{29}$$

From Lemma 2.1, one has

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{30}$$

Noticing that $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, from the definition of C_{n+1}, for every $i \in I$, we obtain

$$\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) + \alpha_{n,i}(\|x_0\|^2 + 2\langle x_{n+1}, Jx_n - Jx_0\rangle).$$

It follows from (29) and $\lim_{n\to\infty} \alpha_{n,i} = 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_{n,i}) = 0, \quad \forall i \in I. \tag{31}$$

From Lemma 2.1, we have $\lim_{n\to\infty} \|x_{n+1} - u_{n,i}\| = 0$. This together with (30) implies that

$$\lim_{n\to\infty} \|x_n - u_{n,i}\| = 0, \quad \forall i \in I. \tag{32}$$

Since J is uniformly norm-to-norm continuous on bounded sets, for every $i \in I$, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_{n,i}\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{33}$$

It follows from $x_n \to p$ as $n \to \infty$ that

$$u_{n,i} \to p \text{ as } n \to \infty, \quad \forall i \in I. \tag{34}$$

Let $r = \sup_{n\geq 1}\{\|x_n\|, \|T_i x_n\|, \|S_i x_n\|\}$ for every $i \in I$. Therefore Lemma 2.6 implies that there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$ and (15)

Case I. Assume that (c-1) holds. We observe that

$$\begin{aligned}
\phi(w, u_{n,i}) &= \phi(w, T_{r_{n,i}} y_{n,i}) \\
&\leq \phi(w, y_{n,i}) \\
&= \phi(w, J^{-1}(\beta_{n,i}^{(1)} J x_n + \beta_{n,i}^{(2)} J T_i x_n + \beta_{n,i}^{(3)} J S_i x_n)) \\
&= \|w\|^2 - 2\langle w, \beta_{n,i}^{(1)} J x_n + \beta_{n,i}^{(2)} J T_i x_n + \beta_{n,i}^{(3)} J S_i x_n \rangle \\
&\quad + \|\beta_{n,i}^{(1)} J x_n + \beta_{n,i}^{(2)} J T_i x_n + \beta_{n,i}^{(3)} J S_i x_n\|^2 \\
&\leq \|w\|^2 - 2\beta_{n,i}^{(1)} \langle w, J x_n \rangle - 2\beta_{n,i}^{(2)} \langle w, J T_i x_n \rangle - 2\beta_{n,i}^{(3)} \langle w, J S_i x_n \rangle \\
&\quad + \beta_{n,i}^{(1)} \|x_n\|^2 + \beta_{n,i}^{(2)} \|T_i x_n\|^2 + \beta_{n,i}^{(3)} \|S_i x_n\|^2 \\
&\quad - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|J x_n - J T_i x_n\|) \\
&= \beta_{n,i}^{(1)} \phi(w, x_n) + \beta_{n,i}^{(2)} \phi(w, T_i x_n) + \beta_{n,i}^{(3)} \phi(w, S_i x_n) \\
&\quad - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|J x_n - J T_i x_n\|) \\
&\leq \beta_{n,i}^{(1)} \phi(w, x_n) + \beta_{n,i}^{(2)} \phi(w, x_n) + \beta_{n,i}^{(3)} \phi(w, x_n) - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|J x_n - J T_i x_n\|) \\
&= \phi(w, x_n) - \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|J x_n - J T_i x_n\|).
\end{aligned}$$

This implies that

$$\beta_{n,i}^{(1)} \beta_{n,i}^{(2)} g(\|J x_n - J T_i x_n\|) \leq \phi(w, x_n) - \phi(w, u_{n,i}), \quad \forall i \in I. \tag{35}$$

On the other hand, for every $i \in I$, one has

$$\begin{aligned}
\phi(w, x_n) - \phi(w, u_{n,i}) &= \|x_n\|^2 - \|u_{n,i}\|^2 - 2\langle w, J x_n - J u_{n,i} \rangle \\
&\leq \|x_n - u_{n,i}\|(\|x_n\| + \|u_{n,i}\|) + 2\|w\| \|J x_n - J u_{n,i}\|.
\end{aligned}$$

It follows that (32) and (33) that

$$\phi(w, x_n) - \phi(w, u_{n,i}) \to 0 \quad (n \to \infty), \quad \forall i \in I. \tag{36}$$

Observing that assumption $\liminf_{n\to\infty} \beta_{n,i}^{(1)} \beta_{n,i}^{(2)} > 0$, (35) and (36), one has

$$g(\|J x_n - J T_i x_n\|) \to 0 \quad (n \to \infty), \quad \forall i \in I.$$

It follows from the property of the function g that

$$\|J x_n - J T_i x_n\| \to 0 \quad (n \to \infty), \quad \forall i \in I. \tag{37}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, for each $i \in I$, one has

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0. \tag{38}$$

In a similar way, one has

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0. \tag{39}$$

Noticing (28), (38), (39) and the closedness of T_i and S_i that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$.

Case II. Assume that (c-2) holds. We observe that

$$\begin{aligned}
\phi(w,u_{n,i}) &= \phi(w,T_{r_{n,i}}y_{n,i})\\
&\le \phi(w,y_{n,i})\\
&= \phi(w,J^{-1}(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n))\\
&= \|w\|^2-2\langle w,\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\rangle\\
&\quad +\|\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\|^2\\
&\le \|w\|^2-2\beta_{n,i}^{(1)}\langle w,Jx_n\rangle-2\beta_{n,i}^{(2)}\langle w,JT_ix_n\rangle-2\beta_{n,i}^{(3)}\langle w,JS_ix_n\rangle\\
&\quad +\beta_{n,i}^{(1)}\|x_n\|^2+\beta_{n,i}^{(2)}\|T_ix_n\|^2+\beta_{n,i}^{(3)}\|S_ix_n\|^2\\
&\quad -\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JS_ix_n-JT_ix_n\|)\\
&= \beta_{n,i}^{(1)}\phi(w,x_n)+\beta_{n,i}^{(2)}\phi(w,T_ix_n)+\beta_{n,i}^{(3)}\phi(w,S_ix_n)\\
&\quad -\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JT_ix_n-JS_ix_n\|)\\
&\le \beta_{n,i}^{(1)}\phi(w,x_n)+\beta_{n,i}^{(2)}\phi(w,x_n)+\beta_{n,i}^{(3)}\phi(w,x_n)-\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JT_ix_n-JS_ix_n\|)\\
&= \phi(w,x_n)-\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JT_ix_n-JS_ix_n\|).
\end{aligned}$$

This implies that

$$\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}g(\|JT_ix_n-JS_ix_n\|) \le \phi(w,x_n)-\phi(w,u_{n,i}), \quad \forall i\in I. \tag{40}$$

On the other hand, for every $i\in I$, one has

$$\begin{aligned}
\phi(w,x_n)-\phi(w,u_{n,i}) &= \|x_n\|^2-\|u_{n,i}\|^2-2\langle w,Jx_n-Ju_{n,i}\rangle\\
&\le \|x_n-u_{n,i}\|(\|x_n\|+\|u_{n,i}\|)+2\|w\|\|Jx_n-Ju_{n,i}\|.
\end{aligned}$$

It follows that (32) and (33) that

$$\phi(w,x_n)-\phi(w,u_{n,i})\to 0 \quad (n\to\infty), \quad \forall i\in I. \tag{41}$$

Observing that assumption $\liminf_{n\to\infty}\beta_{n,i}^{(2)}\beta_{n,i}^{(3)}>0$, (40) and (41), one has

$$g(\|JT_ix_n-JS_ix_n\|)\to 0 \quad (n\to\infty), \quad \forall i\in I.$$

It follows from the property of the function g that

$$\|JT_ix_n-JS_ix_n\|\to 0 \quad (n\to\infty), \quad \forall i\in I. \tag{42}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, for each $i\in I$, one has

$$\lim_{n\to\infty}\|T_ix_n-S_ix_n\|=0. \tag{43}$$

On the other hand, for each $i\in I$, one has

$$\begin{aligned}
\phi(T_ix_n,u_{n,i}) &= \phi(T_ix_n,T_{r_{n,i}}y_{n,i})\\
&\le \phi(T_ix_n,y_{n,i})\\
&= \phi(T_ix_n,J^{-1}(\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n))\\
&= \|T_ix_n\|^2-2\langle T_ix_n,\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\rangle\\
&\quad +\|\beta_{n,i}^{(1)}Jx_n+\beta_{n,i}^{(2)}JT_ix_n+\beta_{n,i}^{(3)}JS_ix_n\|^2\\
&\le \|T_ix_n\|^2-2\beta_{n,i}^{(1)}\langle T_ix_n,Jx_n\rangle-2\beta_{n,i}^{(2)}\langle T_ix_n,JT_ix_n\rangle\\
&\quad -2\beta_{n,i}^{(3)}\langle T_ix_n,JS_ix_n\rangle+\beta_{n,i}^{(1)}\|x_n\|^2+\beta_{n,i}^{(2)}\|T_ix_n\|^2+\beta_{n,i}^{(3)}\|S_ix_n\|^2\\
&\le \beta_{n,i}^{(1)}\phi(T_ix_n,x_n)+\beta_{n,i}^{(3)}\phi(T_ix_n,S_ix_n). \qquad (44)
\end{aligned}$$

Observe that

$$\begin{aligned}\phi(T_ix_n, S_ix_n) &= \|T_ix_n\|^2 - 2\langle T_ix_n, JS_ix_n\rangle + \|S_ix_n\|^2 \\ &= \|T_ix_n\|^2 - 2\langle T_ix_n, JT_ix_n\rangle + 2\langle T_ix_n, JT_ix_n - JS_ix_n\rangle + \|S_ix_n\|^2 \\ &\leq \|S_ix_n\|^2 - \|T_ix_n\|^2 + 2\|S_ix_n\|\|JT_ix_n - JS_ix_n\| \\ &\leq \|S_ix_n - T_ix_n\|(\|S_ix_n\| + \|T_ix_n\|) + 2\|S_ix_n\|\|JT_ix_n - JS_ix_n\|.\end{aligned}$$

It follows from (42) and (43) that

$$\lim_{n\to\infty} \phi(T_ix_n, S_ix_n) = 0, \quad \forall i \in I. \tag{45}$$

Noticing that $\beta_{n,i}^{(1)} \to 0$ as $n \to \infty$, (44) and (45), one arrives at

$$\lim_{n\to\infty} \phi(T_ix_n, u_{n,i}) = 0, \quad \forall i \in I. \tag{46}$$

From Lemma 2.1, one obtains

$$\lim_{n\to\infty} \|T_ix_n - u_{n,i}\| = 0, \quad \forall i \in I. \tag{47}$$

Hence

$$\|T_ix_n - x_n\| \leq \|T_ix_n - u_{n,i}\| + \|u_{n,i} - x_n\|, \quad \forall i \in I. \tag{48}$$

It follows from (32) and (47) that

$$\lim_{n\to\infty} \|T_ix_n - x_n\| = 0, \quad \forall i \in I. \tag{49}$$

Moreover, we observe that

$$\|S_ix_n - x_n\| \leq \|S_ix_n - T_ix_n\| + \|T_ix_n - x_n\|, \quad \forall i \in I. \tag{50}$$

Combining (43) with (49), one obtains $\lim_{n\to\infty} \|S_ix_n - x_n\| = 0$ for each $i \in I$. Noticing (28), it follows from the closedness of T_i and S_i and $x_n \to p$ that $p \in \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$.

(b) We next show that $p \in GMEP(\Theta, \varphi, \Psi)$.

From (40), we see

$$\phi(u, y_{n,i}) \leq \phi(u, x_{n,i}). \tag{51}$$

From $u_{n,i} = T_{r_{n,i}}y_{n,i}$ and Lemma 2.8, one has

$$\begin{aligned}\phi(u_n, y_{n,i}) &= \phi(T_{r_{n,i}}y_{n,i}, y_{n,i}) \\ &\leq \phi(w, y_{n,i}) - \phi(w, T_{r_{n,i}}y_{n,i}) \\ &\leq \phi(w, x_{n,i}) - \phi(w, T_{r_{n,i}}y_{n,i}) \\ &= \phi(w, x_{n,i}) - \phi(w, u_{n,i}).\end{aligned} \tag{52}$$

It follows from (41) that

$$\phi(u_{n,i}, y_{n,i}) \to 0 \text{ as } n \to \infty, \quad \forall i \in I. \tag{53}$$

Noticing Lemma 2.1, one sees

$$\|u_{n,i} - y_{n,i}\| \to 0 \text{ as } n \to \infty, \quad \forall i \in I. \tag{54}$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Ju_{n,i} - Jy_{n,i}\| = 0, \quad \forall i \in I. \tag{55}$$

From the assumption $r_{n,i} \geq a$, one sees

$$\lim_{n\to\infty} \frac{\|Ju_{n,i} - Jy_{n,i}\|}{r_{n,i}} = 0. \tag{56}$$

Noticing that $u_{n,i} = T_{r_{n,i}}y_{n,i}$, one obtains

$$f(u_{n,i}, y) + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy\rangle \geq 0, \quad \forall y \in C. \tag{57}$$

From (A2), one arrives at

$$(58)\quad \|y-u_{n,i}\|\frac{\|Ju_{n,i}-Jy_{n,i}\|}{r_{n,i}} \geq \frac{1}{r_{n,i}}\langle y-u_{n,i}, Ju_{n,i}-Jy_{n,i}\rangle \geq -f(u_{n,i},y) \geq f(y,u_{n,i}),\ \forall y \in C.$$

By taking the limit as $n \to \infty$ in the above inequality and from $(A4)$ and (34), one has

$$(59)\qquad f(y,p) \leq 0, \quad \forall y \in C.$$

For all $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, one obtains $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from $(A1)$ that

$$(60)\qquad 0 = f(y_t,y_t) \leq tf(y_t,y) + (1-t)f(y_t,p) \leq tf(y_t,y).$$

That is,

$$(61)\qquad f(y_t,y) \geq 0.$$

Let $t \downarrow 0$, from (A3), we obtain $f(p,y) \geq 0$, for all $y \in C$. We have $p \in EP(f)$ that is $p \in GMEP(\Theta,\varphi,\Psi)$. From (a) and (b), we conclude that $p \in F$.

Step 6. Show that $p = \Pi_F x_0$.

From $x_n = \Pi_{C_n} x_0$, we have

$$(62)\qquad \langle Jx_0 - Jx_n, x_n - z\rangle \geq 0, \quad \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$(63)\qquad \langle Jx_0 - Jx_n, x_n - u\rangle \geq 0, \quad \forall u \in F.$$

By taking limit in (63), we obtain that

$$(64)\qquad \langle Jx_0 - Jp, p - u\rangle \geq 0, \quad \forall u \in F.$$

By Lemma 2.2, we can conclude that $p = \Pi_F x_0$. This completes the proof. □

If $\beta_{n,i}^{(1)} = 0$ for all $n \geq 0$ and $T_i = S_i$ for all $i \in I$ in Theorem 3.1, then we have the following.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $\Psi : C \to E^*$ be a continuous and monotone operator and $\varphi : C \to \mathbb{R}$ be a real-valued function. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, let $\{T_i\}_{i\in I}$ be a family of closed quasi-ϕ-nonexpansive mappings $T_i : C \to C$ such that the common fixed point set $F := \bigcap_{i\in I} F(T_i) \cap GMEP(\Theta,\varphi,\Psi)$ is nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$(65)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrary and } C_{0,i} = C, \quad \forall i \in I,\\ y_{n,i} = J^{-1}(\alpha_{n,i}Jx_0 - (1-\alpha_{n,i})T_i x_n),\\ u_{n,i} \in C \text{ such that } \Theta(u_{n_i},y) + \varphi(y) - \varphi(u_{n,i}) + \\ \langle \Psi u_{n,i}, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \geq 0, \ \forall y \in C,\\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u,u_{n,i}) \leq \phi(u,x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\},\\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i},\\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where J is a duality mapping on E, $\{\alpha_{n,i}\}$, $(i = 1,2,3)$ and $\{r_{n,i}\}$ are sequences in $(0,1)$ satisfying

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each* $i \in I$;
(b) $\{r_{n,i}\} \subset [a,\infty)$ *for some* $a > 0$ *and for all* $i \in I$;

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 3.3. Corollary 3.2 improves Theorem 3.1 of Takahashi and Zembayashi [36] in the following senses:

(1) from the class of relatively nonexpansive mappings to the more general class of quasi-ϕ-nonexpansive mappings.

(2) from one mapping to a family of mappings.

(3) from the problem of finding the solutions of the equilibrium problem to the problem of finding the solutions of the generalized mixed equilibrium problem.

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $\{T_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ be two families of closed quasi-ϕ-nonexpansive mappings T_i, $S_i : C \to C$ such that the common fixed point set $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i)$ is nonempty, where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrary and } C_{0,i} = C, \quad \forall i \in I, \\ z_{n,i} = J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ C_{n+1,i} = \{u \in C_{n,i} : \phi(u, y_{n,i}) \le \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ C_{n+1} = \bigcap_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases} \tag{66}$$

where J is a duality mapping on E, $\{\alpha_{n,i}\}$ and $\{\beta_{n,i}^{(i)}\}$ $(i = 1,2,3)$ are sequences in $(0,1)$ satisfying

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each $i \in I$;*

(b) $\beta_{n,i}^{(1)} + \beta_{n,i}^{(2)} + \beta_{n,i}^{(3)} = 1$ *for each $i \in I$ and if one of the following is satisfied.*

(b-1) $\liminf_{n\to\infty} \beta_{n,i}^{(1)} \beta_{n,i}^{(l)} > 0$ *for all $l = 2,3$ and for all $i \in I$ and*

(b-2) $\liminf_{n\to\infty} \beta_{n,i}^{(2)} \beta_{n,i}^{(3)} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,i}^{(1)} = 0$ *for each $i \in I$.*

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. Put $f(x,y) = 0$, for all $x, y \in C$, $\Psi = \varphi = 0$ and $\{r_{n,i}\} = \{1\}, \forall i \in I$ in Theorem 3.1. Thus, we have $u_{n,i} = y_{n,i}$. Then the sequence $\{x_n\}$ generated in Corallary 3.4 converges strongly to $\Pi_F x_0$. □

Remark 3.5. (1) We note that the iterative method imposed in Corollary 3.4 bases on the shrinking projection method which is different from the iterative method imposed in Theorem QCKZ based on the hybrid method.

(2) We can obtain the Corollary 3.4 by using either the condition (b-1) or (b-2).

Theorem 3.6. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $\Psi : C \to E^*$ be a continuous and monotone operator and $\varphi : C \to \mathbb{R}$ be a real-valued function. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, let $\{T_i\}_{i\in I}$ and $\{S_i\}_{i\in I}$ be two families of closed quasi-ϕ-nonexpansive mappings T_i, $S_i : C \to C$ such that the common fixed point set $F := \bigcap_{i\in I} F(T_i) \cap \bigcap_{i\in I} F(S_i) \cap GMEP(\Theta, \varphi, \Psi)$ is nonempty , where I is an index set. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ z_{n,i} = J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ u_{n,i} \in C \text{ such that } \Theta(u_{n_i}, y) + \varphi(y) - \varphi(u_{n,i}) + \\ \langle \Psi u_{n,i}, y - u_{n,i}\rangle + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \ge 0, \ \forall y \in C, \\ H_{n,i} = \{u \in C : \phi(u, u_{n,i}) \le \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ H_n = \bigcap_{i\in I} H_{n,i}, \\ W_0 = C, \\ W_n = \{u \in W_{n-1} : \langle x_n - u, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad \forall n \ge 0, \end{cases} \tag{67}$$

where J is a duality mapping on E, $\{\alpha_{n,i}\}$, $\{\beta_{n,i}^{(i)}\}$ $(i = 1,2,3)$ are sequences in $(0,1)$ such that

(a) $\lim_{n\to\infty} \alpha_{n,i} = 0$ *for each $i \in I$;*

(b) $\{r_{n,i}\} \subset [a, \infty)$ *for some $a > 0$ and for all $i \in I$;*

(c) $\beta_{n,i}^{(1)} + \beta_{n,i}^{(2)} + \beta_{n,i}^{(3)} = 1$ *for each* $i \in I$ *and if either*

(c-1) $\liminf_{n\to\infty} \beta_{n,i}^{(1)}\beta_{n,i}^{(l)} > 0$ *for all* $l = 2, 3$ *and for all* $i \in I$ *or*

(c-2) $\liminf_{n\to\infty} \beta_{n,i}^{(2)}\beta_{n,i}^{(3)} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,i}^{(1)} = 0$ *for each* $i \in I$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

Proof. We define a bifunction $f : C \times C \to \mathbb{R}$ by

$$f(x,y) = \Theta(x,y) + \varphi(y) - \varphi(x) + \langle \Psi x, y - x\rangle, \quad \forall x, y \in C.$$

From Lemma 2.10, we have the bifunction f satisfies condition (A1)-(A4). Therefore, the mixed equilibrium problem (1) is equivalent to the following equilibrium problem: find $u \in C$ such that

$$f(u,y) \geq 0, \quad \forall y \in C,$$

and (67) can be written as:

$$(68) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ z_{n,i} = J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n), \\ y_{n,i} = J^{-1}(\alpha_{n,i} Jx_0 - (1-\alpha_{n,i}) Jz_{n,i}), \\ u_{n,i} \in C \textit{ such that } f(u_{n_i}, y) + \frac{1}{r_{n,i}}\langle y - u_{n,i}, Ju_{n,i} - Jy_{n,i}\rangle \geq 0, \ \forall y \in C, \\ H_{n,i} = \{u \in C : \phi(u, u_{n,i}) \leq \phi(u, x_n) + \alpha_{n,i}(\|x\|^2 + 2\langle u, Jx_n - Jx_0\rangle)\}, \\ H_n = \bigcap_{i\in I} H_{n,i}, \\ W_0 = C, \\ W_n = \{u \in W_{n-1} : \langle x_n - u, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad \forall n \geq 0, \end{cases}$$

It is obvious that $H_n \cap W_n$ is closed and convex. Now we show that $F \subset H_n \cap W_n$ for all $n \geq 0$. First, we show that $F \subset H_n$ for all $n \geq 0$. For $\forall w \in F$ and all $i \in I$, one has

$$\begin{aligned} \phi(w, z_{n,i}) &= \phi(w, J^{-1}(\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n)) \\ &= \|w\|^2 - 2\langle w, \beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n)\rangle \\ &\quad + \|\beta_{n,i}^{(1)} Jx_n + \beta_{n,i}^{(2)} JT_i x_n + \beta_{n,i}^{(3)} JS_i x_n\|^2 \\ &\leq \|w\|^2 - 2\beta_{n,i}^{(1)}\langle w, Jx_n\rangle - 2\beta_{n,i}^{(2)}\langle w, JT_i x_n\rangle - 2\beta_{n,i}^{(3)}\langle w, JS_i x_n\rangle \\ &\quad + \beta_{n,i}^{(1)}\|x_n\|^2 + \beta_{n,i}^{(2)}\|T_i x_n\|^2 + \beta_{n,i}^{(3)}\|JS_i x_n\|^2 \\ &= \beta_{n,i}^{(1)}\phi(w, x_n) + \beta_{n,i}^{(2)}\phi(w, T_i x_n) + \beta_{n,i}^{(3)}\phi(w, S_i x_n) \\ &\leq \beta_{n,i}^{(1)}\phi(w, x_n) + \beta_{n,i}^{(2)}\phi(w, x_n) + \beta_{n,i}^{(3)}\phi(w, x_n) \\ &= \phi(w, x_n) \end{aligned}$$

and then

$$\begin{aligned}
\phi(w,u_{n,i}) &= \phi(w,T_{r_{n,i}}y_{n,i})\\
&\leq \phi(w,y_{n,i})\\
&= \phi(w,J^{-1}(\alpha_{n,i}Jx_0-(1-\alpha_{n,i})Jz_{n,i})\\
&= \|w\|^2-2\langle w,\alpha_{n,i}Jx_0-(1-\alpha_{n,i})Jz_{n,i}\rangle+\|\alpha_{n,i}Jx_0-(1-\alpha_{n,i})Jz_{n,i}\|^2\\
&\leq \|w\|^2-2\alpha_{n,i}\langle w,Jx_0\rangle-2(1-\alpha_{n,i})\langle w,Jz_{n,i}\rangle+\alpha_{n,i}\|x_0\|^2+(1-\alpha_{n,i})\|z_{n,i}\|^2\\
&= \alpha_{n,i}\phi(w,x_0)+(1-\alpha_{n,i})\phi(w,z_{n,i})\\
&\leq \alpha_{n,i}\phi(w,x_0)+(1-\alpha_{n,i})\phi(w,x_n)\\
&= \phi(w,x_n)+\alpha_{n,i}[\phi(w,x_0)-\phi(w,x_n)]\\
&\leq \phi(w,x_n)+\alpha_{n,i}(\|x_0\|^2+2\langle w,Jx_n-Jx_0\rangle). \qquad (69)
\end{aligned}$$

This show that $w\in H_{n,i}$ for each $i\in I$. That is, $w\in H_n=\bigcap_{i\in I}H_{n,i}$ for all $n\geq 0$.

Next, we show that $F\subset W_n$ for all $n\geq 0$. In fact, we prove this by induction. For $n=0$, we have $F\subset C=W_0$. Assume that $F\subset H_{n-1}$ for some $n\geq 1$, we will show that $F\subset W_n$ for the same $n\geq 1$. Since x_n is the projection of x_0 onto $H_{n-1}\cap W_{n-1}$, by Lemma 2.2, we have

$$\langle x_n-z,Jx_0-Jx_n\rangle \geq 0, \quad \forall z\in C_{n-1}\cap Q_{n-1}. \qquad (70)$$

Since $F\subset H_{n-1}\cap W_{n-1}$ by the induction assumptions, the last inequality holds, in particular, for all $w\in F$. This together with the definition of W_n implies that $F\subset W_n$. Thus we proved that $F\subset H_n\cap W_n$, $\forall n\geq 0$. This means that $\{x_n\}$ is well define.

From the definition of W_n, we know that

$$\langle x_n-z,Jx-Jx_n\rangle \geq 0, \quad \forall z\in W_n.$$

So by Lemma 2.2 we have $x_n=\Pi_{W_n}x$. If we instead C_n by W_n and C_{n+1} by H_n in the proof of Theorem 3.1, and notice that $x_{n+1}=\Pi_{H_n\cap W_n}x\in H_n\cap W_n\subset W_n$, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\|x_n-u_{n,i}\|=0. \qquad (71)$$

Since J is uniformly norm-to-norm continuous on bounded set, we have

$$\lim_{n\to\infty}\|Jx_n-Ju_{n,i}\|=0. \qquad (72)$$

Thus the proof that $\{x_n\}$ converges strongly to $\Pi_F x$ follows on the lines of Theorem 3.1. □

Acknowledgements. The authors would like to thank the referees for comments which improve the manuscript.

References

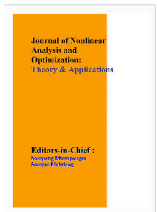
1. Ya.I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operator of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 15-50.
2. Ya.I. Alber, S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, Panamer. Math. J. 4 (1994) 39-54.
3. F. E. Browder, Existence and approximation of solutions of nonlinear variational inequalities. Proc. Natl. Acad. Sci. USA 56(4), 10801086 (1966)
4. F. E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces. Proc. Natl. Acad. Sci. USA 53, 1272-1276 (1965).
5. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, The Mathematics Student, vol. 63, no. 1-4, pp. 123-145, 1994.
6. L-C Ceng, J-C Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. J. Comput. Appl. Math. 214, 186201 (2008)
7. P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, Journal of Nonlinear and Convex Analysis, vol. 6, no. 1, pp. 117-136, 2005.
8. D. Butnariu, S. Reich, A.J. Zaslavski, Asymptotic behavior of relatively nonexpansive operators in Banach spaces, J. Appl. Anal. 7 (2001) 151-174.

9. D. Butnariu, A. N. Iusem, Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization. Kluwer, Dordrecht (2000).
10. D. Butnariu, S. Reich, A.J. Zaslavski, Weak convergence of orbits of nonlinear operators in reflexive Banach spaces, Numer. Funct. Anal. Optim. 24 (2003) 489-508.
11. I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Kluwer, Dordrecht (1990)
12. Y. Censor, S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, Optimization 37 (1996) 323-339.
13. Y.J. Cho, H.Y. Zhou, G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl. 47 (2004) 707-717.
14. I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht, 1990.
15. A. Genel, J. Lindenstrass, An example concerning fixed points, Israel J. Math. 22 (1975) 81-86.
16. B. Halpern, Fixed points of nonexpanding maps, Bull. Am. Math. Soc. 73 (1967) 957-961.
17. H. Iiduka andW. Takahashi, Weak convergence of a projection algorithm for variational inequalities in a Banach space, Journal of Mathematical Analysis and Applications, vol. 339, no. 1, pp. 668-679, 2008.
18. S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, SIAM J. Optim. 13 (2002) 938-945.
19. F. Kohsaka and W. Takahashi, Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces, SIAM Journal on Optimization, vol. 19, no. 2, pp. 824-835, 2008.
20. C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal. 64 (2006) 2400-2411.
21. S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theory 134 (2005) 257-266.
22. A. Moudafi, Second-order differential proximal methods for equilibrium problems, Journal of Inequalities in Pure and Applied Mathematics, vol. 4, no. 1, article 18, pp. 1-7, 2003.
23. K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372-379.
24. S. Plubtieng, K. Ungchittrakool, Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space. J. Approx. Theory 149, 103-115 (2007).
25. X. Qin, Y. Su, Strong convergence theorems for relatively nonexpansive mappings in a Banach space. Nonlinear Anal. 67, 1958-1965 (2007)
26. X.L. Qin, Y.J. Cho, S.M. Kang and H.Y. Zhou, Convergence of a Hybrid Projection Algorithm in Banach Spaces, Acta Appl. Math. DOI 10.1007/s10440-008-9313-4.
27. X. Qin, Y. J. Cho, and S. M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, Journal of Computational and Applied Mathematics, vol. 225, no. 1, pp. 20-30, 2009.
28. S. Reich, Review of Geometry of Banach spaces, Duality Mappings and Nonlinear Problems by loana Cioranescu, Kluwer Academic Publishers, Dordrecht, 1990, Bull. Amer. Math. Soc. 26 (1992) 367-370.
29. S. Reich, A weak convergence theorem for the alternating method with Bregman distance, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 313-318.
30. A. Tada and W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, in Nonlinear Analysis and Convex Analysis, W. Takahashi and T. Tanaka, Eds., pp. 609-617, Yokohama Publishers, Yokohama, Japan, 2007.
31. A. Tada and W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, Journal of Optimization Theory and Applications, vol. 133, no. 3, pp. 359-370, 2007.
32. W. Takahashi, Convex Analysis and Approximation Fixed points, Yokohama-Publishers, 2000 (Japanese).
33. W. Takahashi, Nonlinear Functional Analysis, Fixed point Theory and Its Applications, Yokohama-Publishers, 2000.
34. W. Takahashi, K. Hashimoto, M. Kato, On sharp uniform convexity, smoothness, and strong type, cotype inequalities , J. Nonlinear Convex Anal., 3(2002), 267-281.
35. W. Takahashi, K. Zembayashi, Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces, Nonlinear Analysis: Theory, Methods and Applications, vol. 70, no. 1, pp. 45-57, 2009.
36. W. Takahashi and K. Zembayashi, Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings, Fixed Point Theory and Applications, vol. 2008, Article ID 528476, 11 pages, 2008.
37. S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, Journal of Mathematical Analysis and Applications, vol. 331, no. 1, pp. 506515, 2007.
38. W. Takahashi, Y. Takeuchi, and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, Journal of Mathematical Analysis and Applications, vol. 341, no. 1, pp. 276286, 2008.

39. M. M. Vainberg, Variational Methods and Method of Monotone Operators. Wiley, New York (1973)
40. R. Wittmann, Approximation of fixed points of nonexpansive mappings. Arch. Math. 58 (1992) 486-491.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NARESUAN UNIVERSITY, PHITSANULOK 65000 THAILAND.
Email address: rabianw@nu.ac.th(Rabian Wangkeeree) and uthaikam@hotmail.com(Uthai Kamraksa).

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 131–137
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Strong convergence of a new two-step iterative scheme for two quasi-nonexpansive multi-valued maps in Banach spaces

W. Cholamjiak[1] and S. Suantai[1]

ABSTRACT: In this paper, a new two-step iterative scheme is introduced for two quasi-nonexpansive multi-valued maps in Banach spaces. Strong convergence theorem of the purposed iterative scheme is established for quasi-nonexpansive multi-valued maps in Banach spaces. The result obtained in this paper improve and extend the corresponding one announced by Shahzad and Zegeye [N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Analysis 71 (2009) 838-844.].

KEYWORDS: Quasi-nonexpansive multi-valued map; Nonexpansive multi-valued map; Common fixed point; Strong convergence; Banach space.

1. Introduction

Let D be a nonempty convex subset of a Banach spaces E. The set D is called *proximinal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x - y\| = d(x, D)$, where $d(x, D) = \inf\{\|x - z\| : z \in D\}$. Let $CB(D), K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

for $A, B \in CB(D)$. A single-valued map $T : D \to D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued map $T : D \to CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to D$ (respectively, $T : D \to CB(D)$) if $p = Tp$ (respectively, $p \in Tp$). The set of fixed points of T is denoted by $F(T)$. The mapping $T : D \to CB(D)$ is called *quasi-nonexpansive*[13] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$. It is clear that every nonexpansive multi-valued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive, see [12].

The mapping $T : D \to CB(D)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \to CB(D)$ is *hemicompact*.

Corresponding author: scmti005@chiangmai.ac.th (S. Suantai)
Manuscript received March 16, 2010. Accepted July 31, 2010.

A mapping $T: D \to CB(D)$ is said to satisfy *Condition (I)* if there is a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x,Tx) \geq f(d(x,F(T)))$$

for all $x \in D$.

A family $\{T_i : D \to CB(D), i = 1,2,...,N\}$ is said to satisfy *Condition (II)* if there is a nondecreasing function $f: [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x,T_ix) \geq f(d(x,\bigcap_{i=1}^{N} F(T_i)))$$

for all $i = 1,2,...,N$ and $x \in D$.

In 1953, Mann [6] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in [0,1].

However, we note that Mann's iteration process (1) has only weak convergence, in general; for instance, see [1, 3, 9].

In 2005, Sastry and Babu [10] proved that the Mann and Ishikawa iteration schemes for multi-valued map T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. More precisely, they proved the following result for nonexpansive multi-valued map with compact domain.

In 2007, Panyanak [8] extended the above result of Sastry and Babu [10] to uniformly convex Banach spaces but the domain of T remains compact.

Later, Song and Wang [14] noted that there was a gap in the proofs of Theorem 3.1(see [8]) and Theorem 5 (see [12]). They further solved/revised the gap and also gave the affirmative answer to Panyanak [8] question using the following Ishikawa iteration scheme. In the main results, domain of T is still compact, which is a strong condition (see [14], Theorem 1) and T satisfies condition(I) (see [14], Theorem 1).

In 2009, Shahzad and Zegeye [10] extended and improved the results of Panyanak [8], Sastry and Babu [12] and Song and Wang [14] to quasi-nonexpansive multi-valued maps. They also relaxed compactness of the domain of T. The results provided an affirmative answer to Panyanak [8] question in a more general setting. They introduced a new iteration as follows: Let D be a nonempty convex subset of a Banach space E and $\alpha_n, \alpha'_n \in [0,1]$. Let $T: D \to P(D)$ and $P_Tx = \{y \in Tx : \|x-y\| = d(x,Tx)\}$. The sequence of Ishikawa iterates is defined by $x_0 \in D$,

$$\begin{aligned} y_n &= \alpha'_n z'_n + (1-\alpha'_n)x_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_n z_n + (1-\alpha_n)x_n, \quad n \geq 0, \end{aligned} \tag{2}$$

where $z'_n \in P_Tx_n$ and $z_n \in P_Ty_n$.

Since 2003, the iterative schemes with errors for a single-valued map in Banach spaces have been studied by many authors, see [2, 4, 5, 7].

Question: How can we modify Mann and Ishikawa iterative schemes with errors to obtain convergence theorems for finding a common fixed point of two multi-valued nonexpansive maps ?

Motivated by Shahzad and Zegeye [12], we purpose a new two-step iterative scheme for two multi-valued quasi-nonexpansive maps in Banach spaces and prove strong convergence theorems of the purposed iteration.

2. **Main Results**

We use the following iteration scheme:
Let D be a nonempty convex subset of a Banach space E, $\alpha_n, \beta_n, \alpha'_n, \beta'_n \in [0,1]$ and $\{u_n\}, \{v_n\}$ are bounded sequences in D.

Let T_1, T_2 be two quasi-nonexpansive multi-valued maps from D into $P(D)$ and $P_{T_i}x = \{y \in T_i x : \|x - y\| = d(x, T_i x)\}$, $i = 1, 2$. Let $\{x_n\}$ be the sequence defined by $x_0 \in D$,

$$
\begin{aligned}
y_n &= \alpha'_n z'_n + \beta'_n x_n + (1 - \alpha'_n - \beta'_n)u_n, \quad n \geq 0, \\
x_{n+1} &= \alpha_n z_n + \beta_n x_n + (1 - \alpha_n - \beta_n)v_n, \quad n \geq 0, \qquad (3)
\end{aligned}
$$

where $z'_n \in P_{T_1}x_n$ and $z_n \in P_{T_2}y_n$.

We shall make use of the following results.

Lemma 2.1. [15] *Let $\{s_n\}, \{t_n\}$ be two nonnegative sequences satisfying*

$$s_{n+1} \leq s_n + t_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$ then $\lim_{n\to\infty} s_n$ exists.

Lemma 2.2. [11] *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Theorem 2.3. *Let E be a uniformly convex Banach space, D a nonempty, closed and convex subset of E, and T_1, T_2 be two multi-valued maps from D into $P(D)$ with $F(T_1) \cap F(T_2) \neq \emptyset$ such that P_{T_1}, P_{T_2} are nonexpansive. Assume that*
(i) *$\{T_1, T_2\}$ satisfies condition (II);*
(ii) *$\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$ and $\sum_{n=1}^{\infty}(1 - \alpha'_n - \beta'_n) < \infty$;*
(iii) *$0 < \ell \leq \alpha_n, \alpha'_n \leq k < 1$.*
Then the sequence $\{x_n\}$ generated by (3) converges strongly to some elements in $F(T_1) \cap F(T_2)$.

Proof. We split the proof into three steps.

Step 1. Show that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2)$.

Let $p \in F(T_1) \cap F(T_2)$. Then $P_{T_1}p = \{p\}$ and $P_{T_2}p = \{p\}$. Since u_n, v_n are bounded, therefore exists $M > 0$ such that $\max\{\sup_{n\in\mathbb{N}} \|u_n - p\|, \sup_{n\in\mathbb{N}} \|v_n - p\|\} \leq M$. Then

$$
\begin{aligned}
\|y_n - p\| &\leq \alpha'_n \|z'_n - p\| + \beta'_n \|x_n - p\| + (1 - \alpha'_n - \beta'_n)\|u_n - p\| \\
&\leq \alpha'_n d(z'_n, P_{T_1}p) + \beta'_n \|x_n - p\| + (1 - \alpha'_n - \beta'_n)M \\
&\leq \alpha'_n H(P_{T_1}x_n, P_{T_1}p) + \beta'_n \|x_n - p\| + (1 - \alpha'_n - \beta'_n)M \\
&\leq (\alpha'_n + \beta'_n)\|x_n - p\| + (1 - \alpha'_n - \beta'_n)M \\
&\leq \|x_n - p\| + (1 - \alpha'_n - \beta'_n)M. \qquad (4)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \|z_n - p\| + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n)\|v_n - p\| \\
&= \alpha_n d(z_n, P_{T_2}p) + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n)M \\
&\leq \alpha_n H(P_{T_2}y_n, P_{T_2}p) + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n)M \\
&\leq \alpha_n \|y_n - p\| + \beta_n \|x_n - p\| + (1 - \alpha_n - \beta_n)M \\
&\leq \alpha_n \big(\|x_n - p\| + (1 - \alpha'_n - \beta'_n)M\big) + \beta_n \|x_n - p\| \\
&\quad + (1 - \alpha_n - \beta_n)M \\
&= (\alpha_n + \beta_n)\|x_n - p\| + \big(\alpha_n(1 - \alpha'_n - \beta'_n) + (1 - \alpha_n - \beta_n)\big)M \\
&\leq \|x_n - p\| + \big(\alpha_n(1 - \alpha'_n - \beta'_n) + (1 - \alpha_n - \beta_n)\big)M \\
&= \|x_n - p\| + \varepsilon_n, \qquad (5)
\end{aligned}
$$

where $\varepsilon_n = \big(\alpha_n(1 - \alpha'_n - \beta'_n) + (1 - \alpha_n - \beta_n)\big)M$. By (ii), we have $\varepsilon_n \to 0$ as $n \to \infty$. Thus by Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2)$.

Step 2. Show that $\lim_{n\to\infty} \|z_n - x_n\| = 0 = \lim_{n\to\infty} \|z'_n - x_n\|$.

Let $p \in F(T_1) \cap F(T_2)$. By Step 1, By Step 1, there is a real number $c > 0$ such that $\lim_{n\to\infty} \|x_n - p\| = c$. Let $S = \max\{\sup_{n\in\mathbb{N}} \|v_n - y_n\|, \sup_{n\in\mathbb{N}} \|u_n - x_n\|\}$. From 4, we get

$$\limsup_{n\to\infty} \|y_n - p\| \leq c. \qquad (6)$$

Next, we consider

$$\begin{aligned}\|z_n - p + (1-\alpha_n-\beta_n)(v_n - x_n)\| &\leq \|z_n - p\| + (1-\alpha_n-\beta_n)\|v_n - x_n\| \\ &\leq d(z_n, P_{T_2}p) + (1-\alpha_n-\beta_n)S \\ &\leq H(P_{T_2}y_n, P_{T_2}p) + (1-\alpha_n-\beta_n)S \\ &\leq \|y_n - p\| + (1-\alpha_n-\beta_n)S\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \|z_n - p + (1-\alpha_n-\beta_n)(v_n - x_n)\| \leq c.$$

Also

$$\begin{aligned}\|x_n - p + (1-\alpha_n-\beta_n)(v_n - x_n)\| &\leq \|x_n - p\| + (1-\alpha_n-\beta_n)\|v_n - x_n\| \\ &\leq \|x_n - p\| + (1-\alpha_n-\beta_n)S\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \|x_n - p + (1-\alpha_n-\beta_n)(v_n - x_n)\| \leq c.$$

Since

$$\begin{aligned}\lim_{n\to\infty} \| &\alpha_n\big(z_n - p + (1-\alpha_n-\beta_n)(v_n - x_n)\big) \\ &+ (1-\alpha_n)\big(x_n - p + (1-\alpha_n-\beta_n)(v_n - x_n)\big)\| = \lim_{n\to\infty}\|x_{n+1} - p\| = c,\end{aligned}$$

by Lemma 2.2, we obtain that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{7}$$

By the nonexpansiveness of P_{T_2}, we have

$$\begin{aligned}\|x_n - p\| &\leq \|x_n - z_n\| + \|z_n - p\| \\ &= \|x_n - z_n\| + d(z_n, P_{T_2}p) \\ &\leq \|x_n - z_n\| + H(P_{T_2}y_n, P_{T_2}p) \\ &\leq \|x_n - z_n\| + \|y_n - p\|\end{aligned}$$

which implies

$$c \leq \liminf_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq c.$$

Hence $\lim_{n\to\infty} \|y_n - p\| = c$. Since

$$\begin{aligned}y_n - p &= \alpha'_n\big(z'_n - p + (1-\alpha'_n-\beta'_n)(u_n - x_n)\big) \\ &\quad + (1-\alpha'_n)\big(x_n - p + (1-\alpha'_n-\beta'_n)(u_n - x_n)\big),\end{aligned}$$

we have

$$\begin{aligned}\lim_{n\to\infty} \| &\alpha'_n\big(z'_n - p + (1-\alpha'_n-\beta'_n)(u_n - x_n)\big) \\ &+ (1-\alpha'_n)\big(x_n - p + (1-\alpha'_n-\beta'_n)(u_n - x_n)\big)\| = c.\end{aligned}$$

Moreover, we get

$$\begin{aligned}\|z'_n - p + (1-\alpha'_n-\beta'_n)(u_n - x_n)\| &\leq \|z'_n - p\| + (1-\alpha'_n-\beta'_n)\|u_n - x_n\| \\ &\leq d(z'_n, P_{T_1}p) + (1-\alpha'_n-\beta'_n)S \\ &\leq H(P_{T_1}x_n, P_{T_1}p) + (1-\alpha'_n-\beta'_n)S \\ &\leq \|x_n - p\| + (1-\alpha'_n-\beta'_n)S.\end{aligned}$$

This yields that

$$\limsup_{n\to\infty} \|z'_n - p + (1-\alpha'_n-\beta'_n)(u_n - x_n)\| \leq c.$$

Also

$$\begin{aligned}\|x_n - p + (1-\alpha'_n - \beta'_n)(u_n - x_n)\| &\leq \|x_n - p\| + (1-\alpha'_n - \beta'_n)\|u_n - x_n\| \\ &\leq \|x_n - p\| + (1-\alpha'_n - \beta'_n)S.\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \|x_n - p + (1-\alpha'_n - \beta'_n)(u_n - x_n)\| \leq c.$$

Again by Lemma 2.2, we have

$$\lim_{n\to\infty} \|z'_n - x_n\| = 0. \tag{8}$$

Step 3. Show that $\{x_n\}$ converges strongly to q for some $q \in F(T_1) \cap F(T_2)$

From Step 2, we know that $\lim_{n\to\infty} \|z_n - x_n\| = 0 = \lim_{n\to\infty} \|z'_n - x_n\|$. Also $d(x_n, T_1x_n) \leq d(x_n, P_{T_1}x_n) \leq \|z'_n - x_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}, \{u_n\}$ are bounded, so is $\{u_n - z'_n\}$. Now, let $K = \sup_{n\in\mathbb{N}} \|u_n - z'_n\|$. By assumption and (8), we get

$$\begin{aligned}\|y_n - z'_n\| &\leq \|\alpha'_n z'_n + \beta'_n x_n + (1-\alpha'_n - \beta'_n)u_n - z'_n\| \\ &\leq \beta'_n\|x_n - z'_n\| + (1-\alpha'_n - \beta'_n)\|u_n - z'_n\| \\ &\leq \beta'_n\|x_n - z'_n\| + (1-\alpha'_n - \beta'_n)K \\ &\to 0 \qquad (9)\end{aligned}$$

as $n \to \infty$. It follows from (8) and (9) that

$$\begin{aligned}\|y_n - x_n\| &\leq \|y_n - z'_n\| + \|z'_n - x_n\| \\ &\to 0 \qquad (10)\end{aligned}$$

as $n \to \infty$. It follows from (7) and (10) that

$$\begin{aligned}d(x_n, T_2x_n) &\leq d(x_n, P_{T_2}x_n) \\ &\leq d(x_n, P_{T_2}y_n) + H(P_{T_2}y_n, P_{T_2}x_n) \\ &\leq \|x_n - z_n\| + \|y_n - x_n\| \\ &\to 0.\end{aligned}$$

Since that $\{T_1, T_2\}$ satisfies the condition (II), we have $d(x_n, F(T_1) \cap F(T_2)) \to 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F(T_1) \cap F(T_2)$ such that

$$\|x_{n_k} - p_k\| < \frac{1}{2^k} \tag{11}$$

for all k. From (5), we obtain

$$\begin{aligned}\|x_{n_{k+1}} - p\| &\leq \|x_{n_{k+1}-1} - p\| + \varepsilon_{n_{k+1}-1} \\ &\leq \|x_{n_{k+1}-2} - p\| + \varepsilon_{n_{k+1}-2} + \varepsilon_{n_{k+1}-1} \\ &\;\;\vdots \\ &\leq \|x_{n_k} - p\| + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}\end{aligned}$$

for all $p \in F(T_1) \cap F(T_2)$. This implies that

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i} < \frac{1}{2^k} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}.$$

Next, we shall show that $\{p_k\}$ is Cauchy sequence in D. Notice that

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i} \\ &< \frac{1}{2^{k-1}} + \sum_{i=0}^{n_{k+1}-n_k-1} \varepsilon_{n_k+i}. \end{aligned}$$

This implies that $\{p_k\}$ is Cauchy sequence in D and thus converges to $q \in D$. Since

$$d(p_k, T_i q) \leq d(p_k, P_{T_i} q) \leq H(P_{T_i} q, P_{T_i} p_k) \leq \|q - p_k\|$$

for all $i = 1, 2$ and $p_k \to q$ as $n \to \infty$, it follows that $d(q, T_i q) = 0$ for all $i = 1, 2$ and thus $q \in F(T_1) \cap F(T_2)$. It implies by (11) that $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, it follows that $\{x_n\}$ converges strongly to q. This completes the proof. □

For $T_1 = T_2 = T$ and $\alpha_n + \beta_n = 1 = \alpha'_n + \beta'_n$ in Theorem 2.3, we obtain the following result.

Theorem 2.4. (See [12], Theorem 2.7) *Let E be a uniformly convex Banach space, D a nonempty, closed and convex subset of E, and $T : D \to P(D)$ a multi-valued map with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the Ishikawa iterates defined by (B). Assume that T satisfies condition (I) and $\alpha_n, \alpha'_n \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

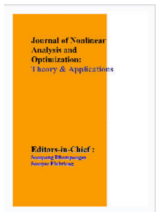
Acknowledgements. This research is supported by the Centre of Excellence in Mathematics, the commission on Higher Education, Thailand, the Thailand Research Fund and the Graduate School of Chiang Mai University for the financial support.

References

1. H. H. Bauschke, E. Matoušková and S. Reich, Projection and proximal point methods: convergence results and counterexamples, Nonlinear Anal. 56 (2004) 715-738.
2. H. Fukhar-ud-din, S. H. Khan, Convergence of iterates with errors of asymtotically quasi-nonexpansive mappings and applications, J. Math. Anal. Appl. 328 (2007) 821-829.
3. A. Genal, J. Lindenstrass, An example concerning fixed points, Israel J. Math. 22 (1975) 81-86.
4. S. H. Khan, H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, Nonlinear Anal. 61 (2005) 1295-1301.
5. D. Lei, L. Shenghong, Ishikawa iteration process with errors for nonexpansive mappings in uniformly convex Banach spaces, Internat. J. Math. &Math. Sci. 24 (2000) 49-53.
6. W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506-510.
7. K. Nammanee, M. A. Noor, S. Suantai, Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 314 (2006) 320-334.
8. B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, Comput. Math. Appl. 54 (2007) 872-877.
9. S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979) 274-276.
10. K.P.R. Sastry, G.V.R. Babu, Convergence of Ishikawa iterates for a multivalued mappings with a fixed point, Czechoslovak Math. J. 55 (2005) 817-826.
11. J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991) 153-159.
12. N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Analysis 71 (2009) 838-844.
13. C. Shiau, K.K. Tan, C.S. Wong, Quasi-nonexpansive multi-valued maps and selection, Fund. Math. 87 (1975) 109-119.
14. Y. Song, H. Wang, Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces"[Comput. Math. Appl. 54 (2007) 872-877]. Comput. Math. Appl. 55 (2008) 2999-3002.
15. K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301-308.

[1]DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY, CHIANG MAI 50200, THAILAND.
Email address: c-wchp007@hotmail.com (W. Cholamjiak), scmti005@chiangmai.ac.th (S. Suantai)

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 139-150
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



A general iterative algorithm for the solution of variational inequalities for a nonexpansive semigroup in Banach spaces

Pitipong Sunthrayuth and Poom Kumam

ABSTRACT: Let X be a uniformly convex and smooth Banach space which admits a weakly sequentially continuous duality mapping, C a nonempty bounded closed convex subset of X. Let $\mathcal{S} = \{T(s) : 0 \leq 0 < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and $f : C \to C$ is a contraction mapping with coefficient $\alpha \in (0,1)$, A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. We prove that the sequences $\{x_t\}$ and $\{x_n\}$ are generated by the following iterative algorithms, respectively

$$x_t = t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds$$

and

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds$$

where $\{t\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ and $\{\lambda_t\}_{0<t<1}$, $\{t_n\}$ are positive real divergent sequences, converging strongly to a common fixed point $x^* \in F(\mathcal{S})$, which solves variational inequality $\langle(\gamma f - A)x^*, J(x - x^*)\rangle \leq 0$ for $x \in F(\mathcal{S})$. Our results presented in this paper extend and improve the corresponding results announced by many others.

1. Introduction

Let X be a real Banach space, and let C a nonempty closed convex subset of X. Mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x,\ y \in C$. We denote $F(T)$ as the set of fixed points of T. We know that $F(T)$ is nonempty if C is bounded; for more detail see [3]. A one-parameter family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ from C of X into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$;
(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous; and
(iv) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$.

Corresponding author: poom.kum@kmutt.ac.th (Poom Kumam)
Manuscript received March 16, 2010. Accepted July 31, 2010.
The authors would like to thank the Faculty of Science KMUTT Research Fund for their financial support.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is $F(\mathcal{S}) = \bigcap_{s\geq 0} F(T(s))$. We know that $F(\mathcal{S})$ is nonempty if C is bounded, see [4]. Recall that a self mapping $f : C \to C$ is a *contraction* if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for each $x,\ y \in C$.

Iterative methods for nonexpansive mappings have recently been applied to solve minimization problems; see, e.g. [8, 20, 21, 23, 24]. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1}$$

where F is the fixed point set of a nonexpansive mapping T on H, and u is a given point in H.

Assume A is strongly positive; that is, there is a constant $\bar{\gamma}$ with the property such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \tag{2}$$

for all $x \in H$.

In 2003, Xu [20] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n u + (I - \alpha_n A)Tx_n,\ \forall n \geq 0, \tag{3}$$

converges strongly to the unique solution of the minimization problem (1), provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [15] introduced the viscosity approximation method for nonexpansive mappings (see [22] for further developments in both Hilbert and Banach spaces). Starting with an arbitrary initial $x_0 \in H$, defined the sequence $\{x_n\}$ recursively by

$$x_{n+1} = \sigma_n f(x_n) + (1 - \sigma_n)Tx_n,\ \forall n \geq 0, \tag{4}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$. It is proved in [15, 22] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (4) strongly converges to the unique solution x^* of the variational inequality

$$\langle (f - I)x^*, x - x^*\rangle \leq 0,\ \forall x \in F(T). \tag{5}$$

Recently, Marino and Xu [14] combined the iterative method (3) with the viscosity approximation method (4) considering the following general iterative process:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n,\ \forall n \geq 0, \tag{6}$$

where $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. They proved that the sequence $\{x_n\}$ generated by (6) converges strongly to a unique solution x^* of the variational inequality

$$\langle (\gamma f - A)x^*, x - x^*\rangle \leq 0,\ \forall x \in F(T). \tag{7}$$

On the other hand, Browder [2] proved that if X is a Hilbert space for a nonexpansive mapping from C into itself, then the net sequence $\{x_t\}$ with $t \in (0,1)$, generated by

$$x_t = tu + (1 - t)Tx_t, \tag{8}$$

converges strongly to the element of $F(T)$, which is nearest to $x \in F(T)$ as $t \to 0$. Moudafi [15] and Xu [22] used the viscosity approximation method for a nonexpanive mapping T. It proved that the net sequence $\{x_t\}$ with $t \in (0,1)$, generated by

$$x_t = tf(x_t) + (1 - t)Tx_t, \tag{9}$$

converges strongly to the element in $F(T)$ which is the unique solution to the variational inequality (5). Later, Bailon and Brezis [1] proved that if $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on C, then the continuous scheme with $t \in (0,1)$

$$x_t = \frac{1}{t}\int_0^t T(s)x_t ds, \tag{10}$$

converges weakly to a common fixed point of $\mathcal{S}$. Those results have been generalized by many authors; see, for instance Takahashi [19]. Shioji and Takahashi [18] introduced the implicit iteration

$$x_n = \alpha_n u + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds,\ \forall n \in \mathbb{N}. \tag{11}$$

In 2007, Chen and Song [6] proposed the explicit iterative process $\{x_n\}$ in a Banach space, as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \tag{12}$$

where $\{t_n\}$ is a positive real divergent sequence. They proved, under certain appropriate conditions $\{\alpha_n\}$ be a real sequence in $(0,1)$, that $\{x_n\}$ converges strongly to a unique solution x^* of the variational inequality

$$\langle (f-I)x^*, J(x-x^*)\rangle,\ \forall x \in F(T). \tag{13}$$

Recently, Li et al [12] and Plubtieng and Wangkeeree [16] considered the iterative process $\{x_n\}$, in a Hilbert space H, $x_0 \in H$ is arbitrary and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds,\ \forall n \geq 0, \tag{14}$$

where A is a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$, $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\{t_n\}$ is a positive real divergent sequence. They proved, under certain appropriate conditions $\{\alpha_n\} \subset (0,1)$ and $\{t_n\}$ is a positive real divergent sequence, that $\{x_n\}$ converges strongly to a unique solution x^* of the variational inequality (7). Moreover, Plubtieng and Wangkeeree [16], also considered and studied the continuous scheme $\{x_t\}$ with $t \in (0,1)$ defined as follows:

$$x_t = t\gamma f(x_t) + (I-tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds, \tag{15}$$

where A is a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$, $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\{\lambda_t\}$ is a positive real divergent net. They proved, under certain appropriate conditions $\{\lambda_t\} \subset (0,1)$, that $\{x_t\}$ converges strongly to a unique solution x^* of the variational inequality (7).

Very recently, Kang et al.[11] considered the iterative process $\{x_n\}$ in a Hilbert space as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds,\ \forall n \geq 0, \tag{16}$$

They proved, under certain appropriate condition $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequence in $(0,1)$, that $\{x_n\}$ converges strongly to a unique solution of the variational inequality (7).

Question 1.1. Can Theorem of Kang et al. [11] and Plubtieng and Wangkeeree [16] be extend from Hilbert spaces to a general Banach space? such as uniformly convex Banach space.

Question 1.2. Can we extend the iterative method of algorithm (14) to a general iterative process?

The purpose of this paper is to give affirmative answer to these questions mentioned above. In this paper, motivated and inspired by Chen and Song [6] and Kang et al.[11], we consider the iterative schemes defined by (15) and (16) for a nonexpansive semigroup in a Banach space. We proved that both schemes converge strongly to a common fixed point of $\mathcal{S}$. The results in this paper extend and improve the main results of Kang et al.[11], Li et al. [12] and Plubtieng and Wangkeeree [16] and some others to Banach spaces.

2. **Preliminaries**

Throughout this paper, let X be a real Banach space, C be a closed convex subset of X. Let $J : X \to 2^{X^*}$ be a normalized duality mapping by $J(x) = \{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \}$, where X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ denotes the generalized duality paring. In the following, the notation $\rightharpoonup$ and $\to$ denote the weak and strong convergence, respectively. Also, a mapping $I : C \to C$ denotes the identity mapping.

The norm of a Banach space X is said to be *Gâteaux differentiable* if the $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x,\ y \in C$ on the unit sphere $S(X)$ of X. In this case X is smooth. Moreover, if for each y in $S(X)$ the limit above is uniformly attained for $x \in S(X)$, we say that the norm X is *uniformly Gâteaux differentiable*.

Recall that the Banach space X is said to be *smooth* if duality mapping J is single valued. In a smooth Banach space, we always assume that A is strongly positive (see [5]), that is, a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J(x) \rangle \geq \bar{\gamma}\|x\|^2,\ \|aI - bA\| = \sup_{\|x\|\leq 1} \|\langle (aI - bA)x, J(x) \rangle\|\ a \in [0,1],\ b \in [-1,1]. \tag{17}$$

A Banach space X is said to be *strictly convex* if $\|x\| = \|y\| = 1,\ x \neq y$ implies $\frac{\|x+y\|}{2} < 1$. A Banach space X is said to be *uniformly convex* if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$, where $\delta_X(\epsilon)$ is *modulus of convexity* of X defined by $\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x+y\| \geq \varepsilon\right\}$, $\forall \epsilon \in [0, 2]$. A uniformly convex Banach space X is reflexive and strictly convex (see Theorem 4.1.6, Theorem 4.1.2 of [19]).

In the sequel we will use the following lemmas, which will be used in the proofs for the main results in the next section.

Lemma 2.1. (Cai and Hu [5]) *Assume that A is a strongly positive linear bounded operator on a smooth Banach space X with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \bar{\gamma}$.*

Lemma 2.2. (Chen and Song [6]) *Let C be a closed convex subset of a uniformly convex Banach space X and let $\Gamma = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty. Then for each $r > 0$ and $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C\cap B_r} \|\frac{1}{t}\int_0^t T(s)xds - T(h)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.3. (Liu [13]) *Let X be a real Banach space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$$

for all $j(x + y) \in J(x + y)$ with $x \neq y$.

If a Banach space X admits a sequentially continuous duality mapping J from weak topology to weak star topology, then by Lemma 1 of [9], we have that duality mapping J is a single value. In this case, the duality mapping J is said to be a weakly sequentially continuous duality mapping, i.e. for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, we have $J(x_n) \rightharpoonup^* J(x)$ (see [9, 10, 17] for more detail).

A Banach space X is said to be satisfying Opial's condition if for any sequence $x_n \rightharpoonup x$ for all $x \in X$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|\ \forall y \in X, \text{ with } x \neq y.$$

By Theorem 1 in [9], it is well known that if X admits a weakly sequentially continuous duality mapping, then X satisfies Opial's condition, and X is smooth.

Lemma 2.4. ([10] Demiclosed Principle) *Let C be a nonempty closed convex subset of a reflexive Banach space X which satisfies Opial's condition, and suppose $T : C \to X$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 2.5. (Xu [20]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} = (1-\gamma_n)a_n + \delta_n,\ n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that:*

(i) $\lim_{n\to\infty} \gamma_n = 0$;
(ii) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(iii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. **Main Results**

In this section, we prove our main results.

Theorem 3.1. *Let* C *be a nonempty bounded closed convex subset of a uniformly convex, smooth Banach space* X *which admits a weakly sequentially continuous duality mapping* J *from* X *into* X^*, $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ *be a nonexpansive semigroup such that* $F(\mathcal{S}) \neq \emptyset$, $f : C \to C$ *is a contraction mapping with coefficient* $\alpha \in (0,1)$, A *a strongly positive bounded linear operator with coefficient* $\bar{\gamma} > 0$. *Let* $t \in (0,1)$ *such that* $t \leq \|A\|^{-1}$ *and* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ *which satisfies* $t \to 0$. *Then the sequence* $\{x_t\}$ *defined by* (15) *converges strongly to the common fixed point* x^* *as* $t \to 0$, *where* x^* *is a unique solution in* $F(\mathcal{S})$ *of the variational inequality*

(18) $$\langle (\gamma f - A)x^*, J(x - x^*)\rangle \leq 0,\ \forall x \in F(\mathcal{S}).$$

Proof. First, we show the uniqueness of a solution of the variational inequality. Supposing $\tilde{x},\ x^* \in F(\mathcal{S})$ satisfy the inequality, we have

(19) $$\langle (\gamma f - A)\tilde{x}, J(x^* - \tilde{x})\rangle \leq 0,$$

and

(20) $$\langle (\gamma f - A)x^*, J(\tilde{x} - x^*)\rangle \leq 0.$$

Adding up (19) and (20), we get that

$$\begin{aligned} 0 &\geq \langle (\gamma f - A)\tilde{x} - (\gamma f - A)x^*, J(x^* - \tilde{x})\rangle \\ &= \langle A(x^* - \tilde{x}), J(x^* - \tilde{x})\rangle - \gamma\langle f(x^*) - f(\tilde{x}), J(x^* - \tilde{x})\rangle \\ &\geq \bar{\gamma}\|x^* - \tilde{x}\|^2 - \gamma\|f(x^*) - f(\tilde{x})\|\|J(x^* - \tilde{x})\| \\ &\geq \bar{\gamma}\|x^* - \tilde{x}\|^2 - \gamma\alpha\|x^* - \tilde{x}\|^2 \\ &= (\bar{\gamma} - \gamma\alpha)\|x^* - \tilde{x}\|^2. \end{aligned}$$

Since $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ this implies that $\bar{\gamma} - \gamma\alpha > 0$, which is a contradiction. Hence $\tilde{x} = x^*$ and the uniqueness is proved.

Next, we show that $\{x_t\}$ is bounded. Indeed, for any $p \in F(\mathcal{S})$, we have

$$\begin{aligned}
\|x_t - p\| &= \|t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p\| \\
&= \|t(\gamma f(x_t) - Ap) + (I - tA)(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p)\| \\
&\leq t\|\gamma f(x_t) - Ap\| + \|I - tA\|\frac{1}{\lambda_t}\|\int_0^{\lambda_t} T(s)x_t - p\|ds \\
&\leq t\|\gamma f(x_t) - Ap\| + (1 - t\bar{\gamma})\|x_t - p\| \\
&\leq t\|\gamma(f(x_t) - f(p)) + \gamma f(p) - Ap\| + (1 - t\bar{\gamma})\|x_t - p\| \\
&\leq t(\gamma\alpha\|x_t - p| + \|\gamma f(p) - Ap\|) + (1 - t\bar{\gamma})\|x_t - p\| \\
&= (1 - t(\bar{\gamma} - \gamma\alpha))\|x_t - p\| + t\|\gamma f(p) - Ap\|.
\end{aligned}$$

It follows that $\|x_t - p\| \leq \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}$. Hence $\{x_t\}$ is bounded.
Next, we show that $\|x_t - T(h)x_t\| \to 0$ as $t \to 0$. We observe that

$$\begin{aligned}
\|x_t - T(h)x_t\| &= \|x_t - \tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| + \|\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - T(h)\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| \\
&\quad + \|T(h)\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - T(h)x_t\| \\
&\leq 2\|x_t - \tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| + \|\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - T(h)\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\|
\end{aligned} \tag{21}$$

for every $0 \leq h \leq \infty$. On the other hand, we note that

$$\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - x_t\| = t\|A(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds) - \gamma f(x_t)\| \tag{22}$$

for every $t > 0$. Define the set $K = \{\|z - p\| \leq \frac{1}{\bar{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\|\}$, then K is a nonempty bounded closed convex subset of C which is $T(s)-$ invariant for each $s \in [0, \infty]$. Since $\{x_t\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$, and it follows by Lemma 2.2 that

$$\lim_{\lambda_t \to \infty} \|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - T(h)(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds)\| = 0 \tag{23}$$

for every $0 \leq h < \infty$. From (21)-(23) and let $t \to 0$, then

$$\|x_t - T(h)x_t\| \to 0, \tag{24}$$

for every $0 \leq h < \infty$. Assume $\{t_n\}_{n=1}^{\infty} \subset (0, 1)$ is such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $\lambda_n := \lambda_{t_n}$. We will show that $\{x_n\}$ contains a subsequence converging strongly to x^*, where $x^* \in F(\mathcal{S})$. Since $\{x_n\}$ is bounded sequence and Banach space X is uniformly convex, hence it is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $x^* \in C$ as $n \to \infty$. Again since Banach space X has a weakly sequentially continuous duality mapping satisfying Opial's condition. It follows by Lemma 2.4 and noting 24, we have $x^* \in F(\mathcal{S})$. For each $n \geq 1$, we note that

$$\begin{aligned} x_n - x^* &= t_n\gamma f(x_n) + (I - t_nA)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - x^* \\ &= t_n(\gamma f(x_n) - Ax^*) + (I - t_nA)(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - x^*). \end{aligned}$$

Thus, we have

$$\begin{aligned} \|x_n - x^*\|^2 &= t_n\langle \gamma f(x_n) - Ax^*, J(x_n - \tilde{x})\rangle + \langle (I - t_nA)(\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds), J(x_n - x^*)\rangle \\ &\leq t_n\langle \gamma f(x_n) - Ax^*, J(x_n - x^*)\rangle + \|I - t_nA\|\|\frac{1}{\lambda_n}\int_0^{\lambda_n}(T(s)x_n - \tilde{x})ds\|\|J(x_n - x^*)\| \\ &\leq t_n\langle \gamma f(x_n) - Ax^*, J(x_n - x^*)\rangle + (1 - t_n\bar{\gamma})\|x_n - z\|(\frac{1}{\lambda_n}\int_0^{\lambda_n}\|T(s)x_n - x^*\|ds) \\ &\leq t_n\langle \gamma f(x_n) - Ax^*, J(x_n - x^*)\rangle + (1 - t_n\bar{\gamma})\|x_n - \tilde{x}\|(\frac{1}{\lambda_n}\int_0^{\lambda_n}\|x_n - x^*\|ds) \\ &\leq t_n\langle \gamma f(x_n) - Az, J(x_n - x^*)\rangle + (1 - t_n\bar{\gamma})\|x_n - x^*\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_n - x^*\|^2 &\leq \frac{1}{\bar{\gamma}}\langle \gamma f(x_n) - Ax^*, J(x_n - x^*)\rangle \\ &= \frac{1}{\bar{\gamma}}[\langle \gamma f(x_n) - \gamma f(x^*), J(x_n - x^*)\rangle + \langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle] \\ &\leq \frac{1}{\bar{\gamma}}[\gamma\alpha\|x_n - x^*\|^2 + \langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle]. \end{aligned}$$

This implies that

$$\|x_n - x^*\|^2 \leq \frac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(x^*) - Ax^*, J(x_n - \tilde{x})\rangle.$$

In particular, we have

$$\|x_{n_j} - x^*\|^2 \leq \tfrac{1}{\bar{\gamma} - \gamma\alpha}\langle \gamma f(x^*) - Ax^*, J(x_{n_j} - x^*)\rangle. \tag{25}$$

Since $\{x_n\}$ is bounded and the duality mapping J is single-valued and weakly sequentially continuous from X into X^*, it follows (25), we have that $x_{n_j} \to x^*$ as $j \to \infty$. Next, we show that x^* solves the variational inequality (18). Since $x_t = t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds$. Thus, we have

$$(\gamma f - A)x_t = -\frac{1}{t}(I - tA)(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - x_t).$$

We notice that

$$\begin{aligned}
\langle \frac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))xds - \frac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))x_t ds, J(x-x_t)\rangle &\geq \|x-x_t\|^2 \\
&\quad -\|\frac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)x_t - T(s)x)ds\|\|J(x-x_t)\| \\
&\geq \|x-x_t\|^2 - \|x-x_t\|\|x-x_t\| \\
&= \|x-x_t\|^2 - \|x-x_t\|^2 \\
&= 0,
\end{aligned}$$

for each $x \in F(\mathcal{S})$ and for all $t > 0$,

$$\begin{aligned}
\langle(\gamma f - A)x_t, J(x-x_t)\rangle &= -\tfrac{1}{t}\langle(I-tA)(\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - x_t), J(x-x_t)\rangle \\
&= -\tfrac{1}{t}\langle(I-tA)(\tfrac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - \tfrac{1}{\lambda_t}\int_0^{\lambda_t} x_t ds), J(x-x_t)\rangle \\
&= -\tfrac{1}{t}\langle \tfrac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))xds - \tfrac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))x_t ds, J(x-x_t)\rangle \\
&\quad +\langle A(\tfrac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))x_t ds), J(x-x_t)\rangle \\
&\leq \langle A(\tfrac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)-I)x_t ds), J(x-x_t)\rangle.
\end{aligned} \tag{26}$$

Now replacing t and λ_t with t_{n_j} and λ_{n_j}, respectively in (26), and letting $j \to \infty$, we notice that $(T(s)-I)x_{n_j} \to (T(s)-I)x^* = 0$ for $x^* \in F(\mathcal{S})$, we obtain $\langle(\gamma f - A)x^*, J(x-x^*)\rangle \leq 0$. That is, x^* is a solution of variational inequality (18). By uniqueness, as $x^* = \tilde{x}$, we have shown that each cluster point of the net sequence $\{x_t\}$ is equal to x^*. Then, we conclude that $x_t \to x^*$ as $t \to 0$. This proof is completes.

If X is a Hilbert space, we can get the following corollary easily.

Corollary 3.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S}) \neq \emptyset$. Let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and let $t \in (0,1)$ such that $t \leq \|A\|^{-1}$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, which satisfies $t \to 0$. Then the sequence $\{x_t\}$ defined by* (15) *converges strongly to the common fixed point x^* as $t \to 0$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality* (7).

Remark 3.3. Theorem 3.1 improves and extends Theorem 3.1 of Plubtieng and Wangkeeree [16] from a Hilbert space to a Banach space.

Theorem 3.4. *Let C be a nonempty bounded closed convex subset of a uniformly convex, smooth Banach space X which admits a weakly sequentially continuous duality mapping J from X into X^*, $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S}) \neq \emptyset$, $f : C \to C$ is a contraction mapping with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$. Assume the following control conditions are hold:*

(C1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ defined by (16) *converges strongly to the common fixed point x^* as $n \to \infty$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality* (18).

Proof. First, we show $\{x_n\}$ is bounded. By the control condition $(C1)$, we may assume, with no loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$. Since A is a linear bounded operator on X, by

(17), we have $\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in X,\ \|u\| = 1\}$. Observe that

$$\begin{aligned} \langle((1-\beta_n)I - \alpha_n A)u, J(u)\rangle &= 1 - \beta_n - \alpha_n\langle Au, J(u)\rangle \\ &\geq 1 - \beta_n - \alpha_n\|A\| \\ &\geq 0. \end{aligned}$$

It follows that

$$\begin{aligned} \|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle((1-\beta_n)I - \alpha_n A)u, J(u)\rangle : u \in X,\ \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Au, J(u)\rangle : u \in X,\ \|u\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n\bar{\gamma}. \end{aligned}$$

Taking, $p \in F(\mathcal{S})$ we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p)\| \\ &\leq \alpha_n[\gamma\|f(x_n) - f(p)\| + \|\gamma f(p) - Ap\|] + \beta_n\|x_n - p\| + \\ &\quad \|(1-\beta_n)I - \alpha_n A\|\frac{1}{t_n}\int_0^{t_n} \|T(s)x_n - p\| ds \\ &\leq \alpha_n[\gamma\alpha\|x_n - p\| + \|\gamma f(p) - Ap\|] + \beta_n\|x_n - p\| + ((1-\beta_n) - \alpha_n\bar{\gamma})\|x_n - p\| \\ &= \alpha_n\|\gamma f(p) - Ap\| + [1 - (\bar{\gamma} - \gamma\alpha)\alpha_n]\|x_n - p\| \\ &= (\bar{\gamma} - \gamma\alpha)\alpha_n\frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha} + [1 - (\bar{\gamma} - \gamma\alpha)\alpha_n]\|x_n - p\| \end{aligned}$$

By induction, we get

$$\|x_{n+1} - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}\right\},$$

for $n \geq 0$. Hence $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{T(t_n)x_n\}$. It follows from Theorem 3.1 that there is a unique solution $x^* \in F(\mathcal{S})$ of the variational inequality (18).

Next, we show $\|x_n - T(h)x_n\| \to 0$ as $n \to \infty$. We note that

(27)

$$\begin{aligned} \|x_{n+1} - T(h)x_{n+1}\| &= \|x_{n+1} - \tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds\| + \|\tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds - T(h)\tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds\| \\ &\quad + \|T(h)\tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds - T(h)x_{n+1}\| \\ &\leq 2\|x_{n+1} - \tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds\| + \|\tfrac{1}{t_n}\int_0^{t_n} T(s)x_t ds - T(h)\tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds\| \\ &\leq 2\alpha_n\|\gamma f(x_n) - A(\tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds)\| + \beta_n\|x_n - \tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds\| \\ &\quad + \|\tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds - T(h)\tfrac{1}{t_n}\int_0^{t_n} T(s)x_n ds\|. \end{aligned}$$

Define the set $K = \{z \in C : \|z - z_0\| \leq \|x - x_0\| + \frac{\|\gamma f(x_0) - Az_0\|}{\bar{\gamma} - \gamma\alpha}\}$. Then K is a nonempty closed bounded convex subset of C which is $T(s)-$invariant for each $s \in [0, \infty]$ and contains $\{x_n\}$; it follows by Lemma 2.2 that

$$\lim_{n\to\infty}\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - T(h)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds)\| = 0, \tag{28}$$

for every $0 \leq h < \infty$. Since $\{x_n\}$, $\{f(x_n)\}$ and $\{T(s)x_n\}$ are bounded, by control conditions $(C1)$ and (28), into (27), we get that $\|x_{n+1} - T(h)x_{n+1}\| \to 0$ as $n \to \infty$, and hence

$$\|x_n - T(h)x_n\| \to 0 \text{ as } n \to \infty. \tag{29}$$

Let x^* be the unique solution in $F(\mathcal{S})$ of the variational inequality (18).

Now, we show that $\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle \leq 0$. We can take subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{j\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_{n_j} - x^*)\rangle = \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle. \tag{30}$$

Since X is uniformly convex, hence it is reflexive, and $\{x_n\}$ is bounded then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $x \in C$ as $j \to \infty$. Again, since Banach space X has a weakly sequentially continuous duality mapping satisfying Opial's condition. By Lemma 2.4, and noting (29), we have $x \in F(\mathcal{S})$. Hence by (18), we obtain

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle = \langle \gamma f(x^*) - Ax^*, J(x - x^*)\rangle \leq 0. \tag{31}$$

Finally, we show that $x_n \to x^*$ as $n \to \infty$. For each $n \geq 0$, by Lemma 2.3 we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*)\|^2 \\
&\leq \|((1-\beta_n)I - \alpha_n A)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*) + \beta_n(x_n - x^*)\|^2 \\
&\quad +2\alpha_n\langle \gamma f(x_n) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&\leq [(1-\beta_n - \alpha_n\bar{\gamma})\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x^*\| + \beta_n\|x_n - x^*\|]^2 \\
&\quad +2\alpha_n\langle \gamma f(x_n) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(x^*), J(x_{n+1} - x^*)\rangle \\
&\quad +2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\|\gamma f(x_n) - \gamma f(x^*)\|\|J(x_{n+1} - x^*)\| \\
&\quad +2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&= (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad +2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + \alpha_n\gamma\alpha(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad +2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \left[\frac{(1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\right]\|x_n-x^*\|^2 \\ &\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle \\ &= \left[\frac{1-2\alpha_n\bar{\gamma}+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\right]\|x_n-x^*\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2 \\ &\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle \\ &= \left[1-\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\right]\|x_n-x^*\|^2+\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2 \\ &\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle.\end{aligned}$$

Put $\gamma_n=\frac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}$ and $\delta_n=\frac{\alpha_n^2\bar{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle$. Then the above reduces to formula $\|x_{n+1}-x^*\|^2\leq(1-\gamma_n)\|x_n-x^*\|^2+\delta_n$. By control conditions $(C1)$, $(C2)$ and (31) it is easily seen that $\lim_{n\to\infty}\gamma_n=0$, $\sum_{n=0}^{\infty}\gamma_n=\infty$ and

$$\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}=\limsup_{n\to\infty}\left[\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\alpha)}\|x_n-x^*\|^2+\frac{1}{\bar{\gamma}-\gamma\alpha}\langle\gamma f(x^*)-Ax^*,J(x_{n+1}-x^*)\rangle\right]\leq 0.$$

By Lemma 2.5, we conclude that $x_n\to x^*$ as $n\to\infty$. This completes the proof.

If X is a Hilbert space, we can get the following corollary easily.

Corollary 3.5. *Let C be a nonempty bounded closed convex subset of a Hilbert space H, $\mathcal{S}=\{T(s):0\leq s<\infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S})\neq\emptyset$, $f:C\to C$ is a contraction mapping with coefficient $\alpha\in(0,1)$, A a strongly positive bounded linear operator with coefficient $\bar{\gamma}>0$ such that $0<\gamma<\frac{\bar{\gamma}}{\alpha}$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $(0,1)$. Assume the following control conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$;
(C2) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

Then the sequence $\{x_n\}$ defined by (16) *converges strongly to the common fixed point x^* as $n\to\infty$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality* (7).

Remark 3.6. Theorem 3.4 improves and extends Theorem 3.1 of Kang et al.[11] from a Hilbert space to a Banach space.

Corollary 3.7. *Let C be a nonempty bounded closed convex subset of a Hilbert space H, $\mathcal{S}=\{T(s):0\leq s<\infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S})\neq\emptyset$, $f:C\to C$ is a contraction mapping $\alpha\in(0,1)$, A a strongly positive bounded linear operator with coefficient $\bar{\gamma}>0$ such that $0<\gamma<\frac{\bar{\gamma}}{\alpha}$ and $\{\alpha_n\}$ be a sequence in $(0,1)$. Assume the following control conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n=0$;
(C2) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

Then the sequence $\{x_n\}$ defined by (14) *converges strongly to the common fixed point x^* as $n\to\infty$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality* (7).

Remark 3.8. Theorem 3.4 improves and extends Theorem 3.2 of Plubtieng and Wangkeeree [16] and Li et al [12] from a Hilbert space to a Banach space for a nonexpansive semigroup.

If taking $A=I$ and $\gamma=1$ in Theorem 3.4, we get the following corollary easily.

Corollary 3.9. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X which admits a weakly sequentially continuous duality mapping J from X into X^*, $\mathcal{S}=\{T(s):0\leq s<\infty\}$ be a nonexpansive semigroup such that $F(\mathcal{S})\neq\emptyset$, $f:C\to C$ is a contraction mapping with coefficient $\alpha\in(0,1)$ and $\{\alpha_n\}$ be a sequence in $(0,1)$. Assume the following control conditions are hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

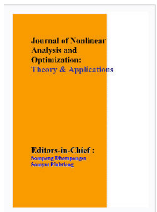
Then the sequence $\{x_n\}$ *defined by* (12) *converges strongly to the common fixed point* x^* *as* $n \to \infty$, *where* x^* *is a unique solution in* $F(\mathcal{S})$ *of the variational inequality* (13).

References

1. J.B. Baillon, H. and Brezis, Une Remarque Sur Le Comportement Asymptotique Des Semigruopes Non Lineairs, *Houston Journal of Mathematics* 2 (1976) 5-7.
2. F.E. Browder, Convergence of approximates to fixed points of nonexpansive nonlinear mappings in Banach spaces, Arch. Ration. Mech. Anal. 24 (1967) 82-90.
3. F.E. Browder, Fixed-point theorems for noncompact mappings in Hilbert space, *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, no. 6, pp. 12721276, 1965.
4. F.E. Browder, Nonexpansive nonlinear operators in a Banach space, *Proceedings of the National Academy of Sciences of the United States of America*, vol. 54, no. 4, pp. 10411044, 1965.
5. G. Cai, C.S. Hu, Strong convergence theorems of a general iterative process for a finite family of λ_i-strict pseudo-contraction in q-uniformly smooth Banach spaces, *Comput. Math. Appl.* 59 (2010) 149-160.
6. R. Chen, Y. Song, Convergence to common fixed point of nonexpansive semigroup, *J. Comput. Appl. Math.* 200 (2007) 566-575.
7. F. Deutsch, I. Yamada, Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings, *Numer. Funct. Anal. Optim.* 19 (1998) 33-56.
8. K. Geobel, W.A. Kirk, Topics in Metric Fixed point Theory, in: Cambridge Stud. Adv. Math., vol. 28, Cambridge university Press, 1990.
9. J.P. Gossez, E. Lami Dozo, Some geometric properties related to the fixed points theory for nonexpansive mappings. *Pacific J. Math.* 40 (1972) 565-573.
10. J.S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, *J. Math Anal. Appl.* 302 (2005) 509-520.
11. J. Kang, Y. Su, and X. Zhang, General iterative algorithm for nonexpansive semigroups and variational inequalities in Hilbert spaces, *Journal of Inequalities and Applications* vol. 2010, Article ID 264052, 10 pages, 2010.
12. S. Li, L. Li and Y. Su, General iterative methods for one-parameter nonexpansive semigroup in Hilbert space, *Nonlinear Anal.* 70 (2009) 3065-3071.
13. L.S. Liu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.* 194 (1995) 114-125.
14. G. Marino, H.K. Xu, A generalized iterative for nonexpansive mapping in Hilbert spaces, *J. Math. Anal. Appl.* 318 (2006) 43-52.
15. A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.* 241 (2000) 46-55.
16. S. Plubtieng, R. Wangkeeree, A general viscosity approximation method of fixed point solution of variational inequalities for nonexpansive semigroups in Hilbert spaces, *Bull. Korean Math. Soc.* 45 (2008) No.4, pp. 717-728.
17. J. Schu, Approximation of fixed points of asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* 112 (1991) 143-151.
18. N. Shioji, W. Takahashi, Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces, *Nonlinear Analysis*, vol. 34, no. 1, pp. 8799, 1998.
19. W. Takahashi, *Nonlinear functional analysis*, Fixed point theory and its applications. Yokohama Publisher, Yokohama, 2000.
20. H.K. Xu, An iterative approach to quadratic optimization, *J. Optim Theory Appl.* 116 (2003) 659-678.
21. H.K. Xu, Iterative algorithms for nonlinear operators, *J. London Math Soc.* 66 (2002) 240-256.
22. H.K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* 298 (2004) 279-291.
23. I. Yamada, The hybrid steepest descent method for the variational inequality problem of the intersection of fixed point sets for nonexpansive mappings, in: K. Butnariu, Y. Censor, S. Reich (Eds.), Inherently Parallel Algorithms for Feasibility and Optimization, Elsevier, 2001, 473-504.
24. I. Yamada, N. Ogura, Y. Yamashita and K. Sakaniwa, Quadratic approximation of fixed points of nonexpansive mappings in Hilbert spaces, *Numer. Funct. Anal. Optim.* 19 (1998) 165-190.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KING MONGKUT'S UNIVERSITY OF TECHNOLOGY THONBURI (KMUTT), BANGKOK 10140,THAILAND.
Email address: poom.kum@kmutt.ac.th (Poom Kumam), sci_math@hotmail.com (P. Sunthrayuth)

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 151-167
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



An iterative method for finding common solutions of generalized mixed equilibrium problems and fixed point problems

Benjawan Rodjanadid

ABSTRACT: In this paper, we introduce an iterative method for finding a common element of the set of solutions of a generalized mixed equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Then, we prove that the sequence converges strongly to a common element of the above two sets. Furthermore, we apply our result to prove three new strong convergence theorems in fixed point problems, mixed equilibrium problems, generalized equilibrium problems and equilibrium problems.

1. **Introduction**

Let H be a real Hilbert space, C a nonempty closed convex subset of H, $\varphi : C \to \mathbb{R}$ a real value function, $A : C \to H$ a nonlinear mapping and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction, i.e., $\Phi(x, x) = 0$ for each $x \in C$. Then, we consider the following mixed equilibrium problem :

Find $x^* \in C$ such that

$$(1) \qquad (GMEP): \qquad \Phi(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0, \qquad \forall y \in C.$$

The set of solutions for problem (1) is denoted by Ω, i.e.,

$$(2) \qquad \Omega = \{x^* \in C : \Phi(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C\}.$$

If $A \equiv 0$ in (1), then (GMEP) (1) reduces to the classical mixed equilibrum problem (for short, MEP) and Ω is denoted by $MEP(\Phi, \varphi)$, that is,

$$(3) \qquad MEP(\Phi, \varphi) = \{x^* \in C : \Phi(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C\}.$$

If $\varphi \equiv 0$ in (1), then (GMEP) (1) reduces to the generalized equilibrium problem (for short, GEP) and Ω is denoted by EP, that is,

$$(4) \qquad EP = \{x^* \in C : \Phi(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C\}.$$

If $\varphi \equiv 0$ and $A \equiv 0$ in (1), then (GMEP) (1) reduces to the classical equilibrium problem (for short, EP) and Ω is denoted by $EP(\Phi)$, that is,

$$(5) \qquad EP(\Phi) = \{x^* \in C : \Phi(x^*, y) \geq 0, \quad \forall y \in C\}.$$

*Corresponding author: benjawan@sut.ac.th (Benjawan Rodjanadid)
Manuscript received March 16, 2010. Accepted July 31, 2010.

If $\Phi \equiv 0$ and $\varphi \equiv 0$ in (1), then (GMEP) (1) reduces to the classical variational inequality and Ω is denoted by $VI(A,C)$, that is,

$$VI(A,C) = \{x^* \in C : \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C\}. \tag{6}$$

In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $EP(\Phi) \neq \emptyset$ and proved a strong convergence theorem.

In 2006, Takahashi and Takahashi [14] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of nonexpansive mapping in a Hilbert space and proved a strong convergence theorem.

In 2007, Tada and Takahashi [12] introduced two iterative schemes for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem. In 2008, Takahashi and Takahashi [13] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and then obtain that the sequence converges strongly to a common element of two sets. Moreover they proved three new strong convergence theorems in fixed point problems, variational inequalities and equilibrium problems.

Recently, Ceng and Yao [2] introduced a hybrid iterative scheme for finding a common element of the set of solutions of mixed equilibrium problem (3) and the set of common fixed points of finitely many nonexpansive mappings and they proved that the sequences generated by the hybrid iterative scheme converge strongly to a common element of the set of solutions of mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings.

In 2008, Peng and Yao [9] obtained some strong convergence theorems for iterative schemes based on the hybrid method and the extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality.

In this paper, we introduced another iterative method for finding an element of the set of solutions of problem (1) and the set of common fixed points of finitely many nonexpansive mappings in real Hilbert space, where $A : C \to H$ is also an $\alpha-$inverse strongly monotone mapping and then obtain a strong convergence theorem. Moreover we using this theorem to the problem for finding a common elements of $\cap_{i=1}^{N} F(T_i) \cap MEP(\Phi, \varphi)$, $\cap_{i=1}^{N} F(T_i) \cap EP$ and $\cap_{i=1}^{N} F(T_i) \cap EP(\Phi)$, respectively.

2. **Preliminaries**

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let symbols $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $\|x - P_C(x)\| \leq \|x - y\|, \ \ \forall y \in C$. The mapping $P_C : x \to P_C(x)$ is called the *metric projection* of H onto C. We know that P_C is nonexpansive.

The following characterizes the projection P_C.

Lemma 2.1. *(See* [11]*) Given $x \in H$ and $y \in C$. Then $P_C(x) = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \ \ \forall z \in C.$$

Recall that the following definitions.

(1) A mapping $T : C \to C$ is called **nonexpansive** if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Next, we denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$.

(2) A mapping $f : H \to H$ is said to be a **contraction** if there exists a constant $\rho \in (0,1)$ such that $\|f(x) - f(y)\| \leq \rho \|x - y\|$ for all $x, y \in H$.

(3) A mapping $A : C \to H$ is called **monotone** if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in C$ and it is called $\alpha-$**inverse strongly monotone** if there exists a positive real number α such that $\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C$. We can see that if A is $\alpha-$**inverse strongly monotone**, then A is monotone mapping.

The following lemmas will be useful for proving our main results.

Lemma 2.2. *(See* [11]*) For all $x, y \in H$, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3. *(See* [11]*) In a strictly convex Banach space E, if*

$$\|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\|,$$

for all $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Lemma 2.4. *(See* [16]*) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying $a_{n+1} = (1 - \alpha_n)a_n + \alpha_n\beta_n, \ \forall n \geq 0$ where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions*

(i) $\{\alpha_n\} \subset [0, 1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty;$

(ii) $\limsup_{n\to\infty} \beta_n \leq 0.$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *(See* [10]*) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n,$$

for all integer $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.6. *(See* [15]*) Let C be a nonempty closed convex subset of H , $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function and let Φ be a bifunction of $C \times C$ in to $\mathbb{R}$ satisfy*

(A1) *$\Phi(x, x) = 0$ for all $x \in C$;*

(A2) *Φ is monotone, i.e., $\Phi(x, y) + \Phi(y, x) \leq 0, \quad \forall x, y \in C$;*

(A3) *for all $x, y, z \in C, \quad \lim_{t\to 0} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y)$;*

(A4) *for all $x \in C, \quad y \mapsto \Phi(x, y)$ is convex and lower semicontinuous;*

(B1) *for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C\backslash D_x$,*

$$\Phi(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z).$$

(B2) *C is bounded set.*

Assume that either (B1) or (B2) holds. For $x \in C$ and $r > 0$, define a mapping $T_r : H \to C$ as follows.

$$T_r(x) := \{z \in C : \Phi(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \ \forall y \in C\}$$

for all $x \in H$. Then , the following conditions hold:

(i) *For each $x \in H, T_r(x) \neq \emptyset$;*

(ii) *T_r is single-valued;*

(iii) *T_r is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \ \forall x, y \in H;$$

(iv) *$F(T_r) = MEP(\Phi, \varphi)$;*

(v) *$MEP(\Phi, \varphi)$ is closed and convex.*

Lemma 2.7. *(see* [1]*) Let C be a nonempty closed convex subset of H, and let Φ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.8. *(see* [3]*) Assume that $\Phi : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \{z \in C : \Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $x \in H$. Then, the following hold:

(i) *S_r is single-valued;*
(ii) *S_r is firmly nonexpansive;*
(iii) *$F(S_r) = EP(\Phi)$;*
(iv) *$EP(\Phi)$ is closed and convex.*

Let X be a real Hilbert space and C a nonempty closed convex subset of X. For a finite family of nonexpansive mappings $T_1, T_2, \ldots, T_N$ and sequence $\{\lambda_{n,i}\}_{i=1}^N$ in $[0, 1]$, Kangtunyakarn and Suantai [6] defined the mapping $K_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I,\\
U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})U_{n,1},\\
U_{n,3} &= \lambda_{n,2}T_3U_{n,2} + (1 - \lambda_{n,3})U_{n,2},\\
&\vdots\\
U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})U_{n,N-2},\\
K_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})U_{n,N-1} \qquad (7)
\end{aligned}$$

Such a mapping K_n is called the $K-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$.

Definition 2.9. (See [6]) Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mapping of C into itself, and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, \ldots, N$. They define a mapping $K : C \to C$ as follows:

$$\begin{aligned}
U_1 &= \lambda_1T_1 + (1 - \lambda_1)I,\\
U_2 &= \lambda_2T_2U_1 + (1 - \lambda_2)U_1,\\
U_3 &= \lambda_3T_3U_2 + (1 - \lambda_3)U_2,\\
&\vdots\\
U_{N-1} &= \lambda_{N-1}T_{N-1}U_{N-2} + (1 - \lambda_{N-1})U_{N-2},\\
K = U_N &= \lambda_NT_NU_{N-1} + (1 - \lambda_N)U_{N-1}.
\end{aligned}$$

Such a mapping K is called the *$K-$mapping* generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$.

Lemma 2.10. *(See* [6]*) Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\cap_{i=1}^N F(T_i) \neq \emptyset$ and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the $K-$mapping generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$. Then $F(K) = \cap_{i=1}^N F(T_i)$.*

Lemma 2.11. *(See* [6]*) Let C be a nonempty closed convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0, 1]$ such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$ $(i = 1, 2, \ldots, N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the $K-$mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$, respectively. Then, for every $x \in C$,*

$$\lim_{n\to\infty} \|K_nx - Kx\| = 0.$$

Lemma 2.12. *Let $\{x_n\}$ be a bounded sequence in a Hilbert space H. Then there exits $L > 0$ such that*

$$\|K_{n+1}x_{n+1} - K_n x_n\| \le \|x_{n+1} - x_n\| + L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|, \ \ \forall n \ge 0. \tag{8}$$

Proof. From (7) and the nonexpansivity of T_N and $U_{n,N}$, we obtain

$$\begin{aligned}
\|K_{n+1}x_n - K_n x_n\| &= \|\lambda_{n+1,N}T_N U_{n+1,N-1}x_n + (1-\lambda_{n+1,N})U_{n+1,N-1}x_n \\
&\quad - \lambda_{n,N}T_N U_{n,N-1}x_n - (1-\lambda_{n,N})U_{n,N-1}x_n\| \\
&= \|\lambda_{n+1,N}T_N U_{n+1,N-1}x_n + U_{n+1,N-1}x_n - \lambda_{n+1,N}U_{n+1,N-1}x_n \\
&\quad - \lambda_{n,N}T_N U_{n,N-1}x_n - U_{n,N-1}x_n + \lambda_{n,N}U_{n,N-1}x_n\| \\
&\le \|\lambda_{n+1,N}T_N U_{n+1,N-1}x_n - \lambda_{n,N}T_N U_{n,N-1}x_n\| + \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad + \|\lambda_{n+1,N}U_{n+1,N-1}x_n - \lambda_{n,N}U_{n,N-1}x_n\| \\
&= \|\lambda_{n+1,N}T_N U_{n+1,N-1}x_n - \lambda_{n+1,N}T_N U_{n,N-1}x_n + \lambda_{n+1,N}T_N U_{n,N-1}x_n \\
&\quad - \lambda_{n,N}T_N U_{n,N-1}x_n\| + \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + \|\lambda_{n+1,N}U_{n+1,N-1}x_n \\
&\quad - \lambda_{n+1,N}U_{n,N-1}x_n + \lambda_{n+1,N}U_{n,N-1}x_n - \lambda_{n,N}U_{n,N-1}x_n\| \\
&\le \lambda_{n+1,N}\|T_N U_{n+1,N-1}x_n - T_N U_{n,N-1}x_n\| + |\lambda_{n+1,N} - \lambda_{n,N}|\|T_N U_{n,N-1}x_n\| \\
&\quad + \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + \lambda_{n+1,N}\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad + |\lambda_{n+1,N} - \lambda_{n,N}|\|U_{n,N-1}x_n\| \\
&\le \lambda_{n+1,N}\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + \|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad + \lambda_{n+1,N}\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + |\lambda_{n+1,N} - \lambda_{n,N}|\|U_{n,N-1}x_n\| \\
&\quad + |\lambda_{n+1,N} - \lambda_{n,N}|\|T_N U_{n,N-1}x_n\| \\
&\le (2\lambda_{n+1,N} + 1)\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + 2L_1|\lambda_{n+1,N} - \lambda_{n,N}|, \qquad (9)
\end{aligned}$$

where $L_1 = \sup_{n\ge 0}\{\|U_{n,j-1}x_n\|, \|T_N U_{n,j-1}x_n\|\}, \ \ j = 1, 2, \ldots, N.$
Again, from (7), we have

$$\begin{aligned}
\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| &= \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n + (1-\lambda_{n+1,N-1})U_{n+1,N-2}x_n \\
&\quad - \lambda_{n,N-1}T_{N-1}U_{n,N-2}x_n - (1-\lambda_{n,N-1})U_{n,N-2}x_n\| \\
&\le \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}x_n\| \\
&\quad + \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| + \|\lambda_{n+1,N-1}U_{n+1,N-2}x_n \\
&\quad - \lambda_{n,N-1}U_{n,N-2}x_n\|
\end{aligned}$$

$$\begin{aligned}
&\le \lambda_{n+1,N-1}\|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| + |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \\
&\quad \times \|T_{N-1}U_{n,N-2}x_n\| + \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| \\
&\quad + \lambda_{n+1,N-1}\|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| + |\lambda_{n+1,N-1} - \lambda_{n,N-1}| \\
&\quad \times \|U_{n,N-2}x_n\| \\
&\le (2\lambda_{n+1,N-1} + 1)\|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| + 2L_1|\lambda_{n+1,N-1} - \lambda_{n,N-1}|. \qquad (10)
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad \le (2\lambda_{n+1,N-1}+1)(2\lambda_{n+1,N-2}+1)\|U_{n+1,N-3}x_n - U_{n,N-3}\| \\
&\qquad + (2\lambda_{n+1,N-1}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| \\
&\quad \le \prod_{i=N-1}^{2}(2\lambda_{n+1,i}+1)\|U_{n+1,1}x_n - U_{n,1}x_n\| + \prod_{i=N-1}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| \\
&\qquad + \prod_{i=N-1}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| + \prod_{i=N-1}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| \\
&\quad + \ldots + \prod_{i=N-1}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| \\
&\quad = \prod_{i=N-1}^{2}(2\lambda_{n+1,i}+1)\|\lambda_{n+1,1}T_1x_n + (1-\lambda_{n+1,1})x_n - \lambda_{n,1}T_1x_n - (1-\lambda_{n,1})x_n\| \\
&\qquad + \prod_{i=N-1}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| + \prod_{i=N-1}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| \\
&\qquad + \prod_{i=N-1}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| + \ldots + \\
&\qquad + \prod_{i=N-1}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}|,
\end{aligned}
$$

then

$$
\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad \le \prod_{i=N-1}^{2}(2\lambda_{n+1,i}+1)(|\lambda_{n+1,1}-\lambda_{n,1}|\|T_1x_n\| + |\lambda_{n+1,1}-\lambda_{n,1}|\|x_n\|) \\
&\qquad + \prod_{i=N-1}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| + \prod_{i=N-1}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| \\
&\qquad + \prod_{i=N-1}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| + \ldots + \\
&\qquad + \prod_{i=N-1}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| \\
&\quad \le \prod_{i=N-1}^{2}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,1}-\lambda_{n,1}| \\
&\qquad + \prod_{i=N-1}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| + \prod_{i=N-1}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| \\
&\qquad + \prod_{i=N-1}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| + \ldots + \\
&\qquad + \prod_{i=N-1}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| + 2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| \qquad (11)
\end{aligned}
$$

Substituting (11) in (9), we have

$$
\begin{aligned}
&\|K_{n+1}x_n - K_n x_n\| \\
&\quad \le \prod_{i=N}^{2}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,1}-\lambda_{n,1}| + \prod_{i=N}^{3}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,2}-\lambda_{n,2}| \\
&\qquad + \prod_{i=N}^{4}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,3}-\lambda_{n,3}| + \prod_{i=N}^{5}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,4}-\lambda_{n,4}| \\
&\qquad + \ldots + \prod_{i=N}^{N-1}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-2}-\lambda_{n,N-2}| \\
&\qquad + \prod_{i=N}^{N}(2\lambda_{n+1,i}+1)2L_1|\lambda_{n+1,N-1}-\lambda_{n,N-1}| + 2L_1|\lambda_{n+1,N}-\lambda_{n,N}| \\
&\quad \le L\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|, \qquad (12)
\end{aligned}
$$

where $L = \prod_{i=N}^{2}(2\lambda_{n+1,i}+1)2L_1$. It follows that

$$
\begin{aligned}
\|K_{n+1}x_{n+1} - K_n x_n\| &\le \|K_{n+1}x_{n+1} - K_{n+1}x_n\| + \|K_{n+1}x_n - K_n x_n\| \\
&\le \|x_{n+1}-x_n\| + L\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned}
$$

□

3. **Main Results**

In this section, we deal with an iterative scheme by the approximation method for finding a common element of the set of common fixed points of finite family of nonexpansive mappings and the set of solutions of GMEP (1) in real Hilbert spaces.

Theorem 3.1. *Let H be a Hilbert space, C a closed convex nonempty subset of H, $\varphi : C \to \mathbb{R}$ a proper lower semicontinuous and convex functional, A an $\alpha-$inverse strongly monotone mapping of C into H, $\Phi : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)-(A4), $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^N F(T_i) \cap \Omega \neq 0$ and f a $\rho-$contraction of C into itself. Moreover, let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ a sequence in $[a,b]$ with $0 < a \le b < 1$ and $\{r_n\}$ a sequence in $[0, 2\alpha]$ for all $n \in \mathbb{N}$. Assume that:*

(i) *either (B1) or (B2) holds;*
(ii) *the sequence $\{r_n\}$ satisfies*
  (C1) $0 < c \le r_n \le d < 2\alpha$*; and*
  (C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*;*
(iii) *the sequence $\{\alpha_n\}$ satisfies*
  (D1) $\lim_{n\to\infty} \alpha_n = 0$*; and*
  (D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(iv) *the sequence $\{\beta_n\}$ satisfies*
  (E1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*;*
(v) *the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfies*
  (F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every $i \in \{1, 2, \ldots, N\}$.*

For every $n \in \mathbb{N}$, let K_n be a $K-$mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$ and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0 \in C$ and

$$\begin{cases} \Phi(y_n, x) + \varphi(x) - \varphi(y_n) + \langle Ax_n, x - y_n\rangle + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, \ \ \forall x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n. \end{cases} \tag{13}$$

Then both $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* = P_\Gamma f(x^*)$ where $\Gamma = \cap_{i=1}^N F(T_i) \cap \Omega$

Proof. Let $x, y \in C$. Since A is $\alpha-$strongly monotone and $r_n \in (0, 2\alpha)\ \ \forall n \in \mathbb{N}$, we have

$$\begin{aligned} \|(I - r_n A)x - (I - r_n A)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r_n\alpha\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + r_n(r_n - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2, \end{aligned}$$

which implies that $I - r_n A$ is nonexpansive.

Next we prove that the sequences $\{x_n\}, \{y_n\}, \{Ax_n\}, \{f(x_n)\}$ and $\{K_n y_n\}$ are bounded. Since

$$\Phi(y_n, x) + \varphi(x) - \varphi(y_n) + \langle Ax_n, x - y_n\rangle + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, \ \ \forall x \in C,$$

we have

$$\Phi(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r_n}\langle x - y_n, y_n - (x_n - r_n Ax_n)\rangle \geq 0, \ \ \forall x \in C.$$

It follows from Lemma 2.6 that $y_n = T_{r_n}(x_n - r_n Ax_n), \ \ \forall n \in \mathbb{N}$.
Let $p \in \cap_{i=1}^N F(T_i) \cap \Omega$. Then we have

$$\Phi(p, y) + \varphi(y) - \varphi(p) + \langle Ap, y - p\rangle \geq 0, \ \ \forall y \in C,$$

so

$$\Phi(p, y) + \varphi(y) - \varphi(p) + \frac{1}{r_n}\langle y - p, p - (p - r_n Ap)\rangle \geq 0, \ \ \forall y \in C.$$

By Lemma 2.6, we have $p = T_{r_n}(p - r_n Ap)$.
Since T_{r_n} and $(I - r_n A)$ are nonexpansive, we have

$$\begin{aligned} \|y_n - p\| &= \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(p - r_n Ap)\| \\ &\leq \|(x_n - r_n Ax_n) - (p - r_n Ap)\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{14}$$

From (13) and (14), we deduce that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n - p\| \\ &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n - (\alpha_n + \beta_n + \gamma_n)p\| \\ &\leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|K_n y_n - p\| \\ &\leq \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\leq \alpha_n(\rho\|x_n - p\| + \|f(p) - p\|) + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \alpha_n\rho\|x_n - p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= \alpha_n\rho\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= (1 - \alpha_n(1 - \rho))\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= (1 - \alpha_n(1 - \rho))\|x_n - p\| + \alpha_n(1 - \rho) \cdot \frac{1}{1 - \rho}\|f(p) - p\| \end{aligned} \tag{15}$$

It follows from (15) induction that

$$\|x_n - p\| \leq M, \quad \forall n \geq 0$$

where $M = \max\{\|x_0 - p\|, \frac{1}{1-\rho}\|f(p) - p\|\}$. So $\{x_n\}$ is bounded. Therefore $\{y_n\}, \{Ax_n\}, \{f(x_n)\}$ and $\{K_n y_n\}$ are also bounded.
Next we shall show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
Define

$$z_n = \frac{\alpha_n}{1-\beta_n} f(x_n) + \frac{\gamma_n}{1-\beta_n} K_n y_n, \tag{16}$$

we have

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \quad \forall n \geq 0. \tag{17}$$

Consider

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}} f(x_{n+1}) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} K_{n+1} y_{n+1} - \frac{\alpha_n}{1-\beta_n} f(x_n) - \frac{\gamma_n}{1-\beta_n} K_n y_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(x_{n+1}) - f(x_n)\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}| \|f(x_n)\| \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|K_{n+1} y_{n+1} - K_n y_n\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}| \|K_n y_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \rho \|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}| (\|f(x_n)\| + \|K_n y_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|K_{n+1} y_{n+1} - K_n y_n\|. \qquad (18)
\end{aligned}$$

Substituting (8) from Lemma 2.12 into (18), we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \rho \|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}| (\|f(x_n)\| + \|K_n y_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (\|y_{n+1} - y_n\| + L \sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|). \qquad (19)
\end{aligned}$$

Putting $u_n = x_n - r_n A x_n$. Then we have $y_{n+1} = T_{r_{n+1}} u_{n+1}, \quad y_n = T_{r_n} u_n$. Hence from the nonexpansivity of $T_{r_{n+1}}$ we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|T_{r_{n+1}} u_{n+1} - T_{r_n} u_n\| \\
&\leq \|T_{r_{n+1}} u_{n+1} - T_{r_{n+1}} u_n\| + \|T_{r_{n+1}} u_n - T_{r_n} u_n\| \\
&\leq \|u_{n+1} - u_n\| + \|T_{r_{n+1}} u_n - T_{r_n} u_n\|. \qquad (20)
\end{aligned}$$

Since $I - r_n A$ is nonexpansive for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|x_{n+1} - r_{n+1} A x_{n+1} - x_n + r_n A x_n\| \\
&\leq \|(I - r_{n+1} A) x_{n+1} - (I - r_{n+1} A) x_n\| + |r_n - r_{n+1}| \|A x_n\| \\
&\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n| \|A x_n\|. \qquad (21)
\end{aligned}$$

By Lemma 2.6, we obtain

$$\Phi(T_{r_n} u_n, y) + \varphi(y) - \varphi(T_{r_n} u_n) + \frac{1}{r_n} \langle y - T_{r_n} u_n, T_{r_n} u_n - u_n \rangle \geq 0, \ \ \forall y \in C, \tag{22}$$

and

$$\Phi(T_{r_{n+1}} u_n, y) + \varphi(y) - \varphi(T_{r_{n+1}} u_n) + \frac{1}{r_{n+1}} \langle y - T_{r_{n+1}} u_n, T_{r_{n+1}} u_n - u_n \rangle \geq 0, \ \ \forall y \in C. \tag{23}$$

Putting $y = T_{r_{n+1}} u_n$ in (22) and $y = T_{r_n} u_n$ in (23), we have

$$\Phi(T_{r_n} u_n, T_{r_{n+1}} u_n) + \varphi(T_{r_{n+1}} u_n) - \varphi(T_{r_n} u_n) + \frac{1}{r_n} \langle T_{r_{n+1}} u_n - T_{r_n} u_n, T_{r_n} u_n - u_n \rangle \geq 0, \tag{24}$$

and

$$\Phi(T_{r_{n+1}} u_n, T_{r_n} u_n) + \varphi(T_{r_n} u_n) - \varphi(T_{r_{n+1}} u_n) + \frac{1}{r_{n+1}} \langle T_{r_n} u_n - T_{r_{n+1}} u_n, T_{r_{n+1}} u_n - u_n \rangle \geq 0. \tag{25}$$

Summing up (24) and (25) and using (A2), we have

$$\frac{1}{r_{n+1}}\langle T_{r_n}u_n - T_{r_{n+1}}u_n, T_{r_{n+1}}u_n - u_n\rangle + \frac{1}{r_n}\langle T_{r_{n+1}}u_n - T_{r_n}u_n, T_{r_n}u_n - u_n\rangle \geq 0,$$

and

$$\langle T_{r_n}u_n - T_{r_{n+1}}u_n, \frac{T_{r_{n+1}}u_n - u_n}{r_{n+1}} - \frac{T_{r_n}u_n - u_n}{r_n}\rangle \geq 0,$$

and hence

$$\begin{aligned}
0 &\leq \langle T_{r_{n+1}}u_n - T_{r_n}u_n, T_{r_n}u_n - u_n - \frac{r_n}{r_{n+1}}(T_{r_{n+1}}u_n - u_n)\rangle \\
&= \langle T_{r_{n+1}}u_n - T_{r_n}u_n, T_{r_n}u_n - T_{r_{n+1}}u_n + (1 - \frac{r_n}{r_{n+1}})(T_{r_{n+1}}u_n - u_n)\rangle \\
&\leq \|T_{r_{n+1}}u_n - T_{r_n}u_n\|(\|T_{r_{n+1}}u_n - T_{r_n}u_n\| + |1 - \frac{r_n}{r_{n+1}}|\|T_{r_{n+1}}u_n - u_n\|).
\end{aligned}$$

From (C1), we can find a real number a such that $r_n \geq a > 0$ for all $n \in \mathbb{N}$.
Then, we have

$$\|T_{r_{n+1}}u_n - T_{r_n}u_n\|^2 \leq |1 - \frac{r_n}{r_{n+1}}|\|T_{r_{n+1}}u_n - T_{r_n}u_n\|(\|T_{r_{n+1}}u_n\| + \|u_n\|),$$

and hence

$$\begin{aligned}
\|T_{r_{n+1}}u_n - T_{r_n}u_n\| &\leq |1 - \frac{r_n}{r_{n+1}}|(\|T_{r_{n+1}}u_n\| + \|u_n\|) \\
&\leq \frac{1}{a}|r_{n+1} - r_n|\hat{L}, \qquad (26)
\end{aligned}$$

where $\hat{L} = \sup\{\|T_{r_{n+1}}u_n\| + \|u_n\| \ : n \in \mathbb{N}\}$.
By (20), (21) and (26), we have

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| + \frac{1}{a}|r_{n+1} - r_n|\hat{L}. \qquad (27)$$

Combining (19) and (27), we deduce

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\rho\|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}|(\|f(x_n)\| + \|K_n y_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(\|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| + \frac{1}{a}|r_{n+1} - r_n|\hat{L} \\
&\quad + L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|) \\
&\leq \|x_{n+1} - x_n\| + |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}|(\|f(x_n)\| + \|K_n y_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}|r_{n+1} - r_n|\|Ax_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \cdot \frac{1}{a}|r_{n+1} - r_n|\hat{L} \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\leq |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}|(\|f(x_n)\| + \|K_n y_n\|) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}|r_{n+1} - r_n|\|Ax_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \cdot \frac{1}{a}|r_{n+1} - r_n|\hat{L} \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}L\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|. \qquad (28)
\end{aligned}$$

Applying the conditions (C2), (D1), (E1) and (F1) and taking the superior limit as $n \to \infty$ to (28), we have

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)=0.$$

Hence, by Lemma 2.5, we have $\lim_{n\to\infty}\|z_n-x_n\|=0$. This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{29}$$

Using (C2), (27) and (29), we have

$$\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{30}$$

Next we show that $\lim_{n\to\infty}\|x_n-y_n\|=0$ and $\lim_{n\to\infty}\|K_ny_n-y_n\|=0$.
Since $x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+\gamma_nK_ny_n$, we obtain

$$\begin{aligned}\|x_n-K_ny_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-K_ny_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_nf(x_n)+\beta_nx_n+\gamma_nK_ny_n-K_ny_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_nf(x_n)+\beta_nx_n-(1-\gamma_n)K_ny_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_nf(x_n)+\beta_nx_n-(\alpha_n+\beta_n)K_ny_n\|\\ &\le \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-K_ny_n\|+\beta_n\|x_n-K_ny_n\|\end{aligned}$$

and hence

$$\|x_n-K_ny_n\|\le\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-K_ny_n\|. \tag{31}$$

Since $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, (31) implies that

$$\lim_{n\to\infty}\|x_n-K_ny_n\|=0. \tag{32}$$

From (14) and monotonicity of A and nonexpansivity of T_{r_n}, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_nf(x_n)+\beta_nx_n+\gamma_nK_ny_n-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|K_ny_n-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &= \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|T_{r_n}(x_n-r_nAx_n)-T_{r_n}(p-r_nAp)\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|(x_n-r_nAx_n)-(p-r_nAp)\|^2\\ &= \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|(x_n-p)-r_n(Ax_n-Ap)\|^2\\ &= \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n(\|x_n-p\|^2-2r_n\langle x_n-p,Ax_n-Ap\rangle\\ &\quad +r_n^2\|Ax_n-Ap\|^2)\\ &\le \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n(\|x_n-p\|^2-2r_n\alpha\|Ax_n-Ap\|^2\\ &\quad +r_n^2\|Ax_n-Ap\|^2)\\ &= \alpha_n\|f(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|x_n-p\|^2-2r_n\gamma_n\alpha\|Ax_n-Ap\|^2\\ &\quad +\gamma_nr_n^2\|Ax_n-Ap\|^2\\ &= \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2+\gamma_nr_n(r_n-2\alpha)\|Ax_n-Ap\|^2. \qquad (33)\end{aligned}$$

By (33), we have

$$\begin{aligned}\gamma_nr_n(2\alpha-r_n)\|Ax_n-Ap\|^2 &\le \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &= \alpha_n\|f(x_n)-p\|^2-\alpha_n\|x_n-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\le \alpha_n\|f(x_n)-p\|^2-\alpha_n\|x_n-p\|^2+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|). \qquad (34)\end{aligned}$$

Since, $0 < c \le r_n \le d < 2\alpha$, we have

(35)
$$\gamma_n c(2\alpha - d)\|Ax_n - Ap\|^2 \le \alpha_n\|f(x_n) - p\|^2 - \alpha_n\|x_n - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).$$

From $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$ and the boundedness of $\{x_n\}$ and $\{f(x_n)\}$, we have

(36)
$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0.$$

Since T_{r_n} is a firmly nonexpansive, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp)\|^2\\ &\le \langle T_{r_n}(x_n - r_nAx_n) - T_{r_n}(p - r_nAp), (x_n - r_nAx_n) - (p - r_nAp)\rangle\\ &= \langle y_n - p, (x_n - r_nAx_n) - (p - r_nAp)\rangle\\ &= \frac{1}{2}(\|y_n - p\|^2 + \|(x_n - r_nAx_n) - (p - r_nAp)\|^2 - \|(y_n - p)\\ &\quad - ((x_n - r_nAx_n) - (p - r_nAp))\|^2)\\ &\le \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|(x_n - y_n) - r_n(Ax_n - Ap)\|^2)\\ &= \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2r_n\langle x_n - y_n, Ax_n - Ap\rangle\\ &\quad - r_n^2\|Ax_n - Ap\|^2) \qquad (37)\end{aligned}$$

and hence

(38)
$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2r_n\|x_n - y_n\|\|Ax_n - Ap\|.$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2\\ &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n(\|x_n - p\|^2 - \|x_n - y_n\|^2\\ &\quad + 2r_n\|x_n - y_n\|\|Ax_n - Ap\|)\\ &= \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 - \gamma_n\|x_n - y_n\|^2\\ &\quad + 2\gamma_nr_n\|x_n - y_n\|\|Ax_n - Ap\|\\ &= \alpha_n\|f(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \gamma_n\|x_n - y_n\|^2\\ &\quad + 2\gamma_nr_n\|x_n - y_n\|\|Ax_n - Ap\|.\end{aligned}$$

This implies

$$\begin{aligned}\gamma_n\|x_n - y_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - \alpha_n\|x_n - p\|^2\\ &\quad + 2\gamma_nr_n\|x_n - y_n\|\|Ax_n - Ap\|\\ &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|)\\ &\quad - \alpha_n\|x_n - p\|^2 + 2\gamma_nr_n\|x_n - y_n\|\|Ax_n - Ap\|. \qquad (39)\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, $\|Ax_n - Ap\| \to 0$ and the sequences $\{x_n\}, \{y_n\}$ and $\{f(x_n)\}$ are bounded, it follows from (39) that

(40)
$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

From $\|K_ny_n - y_n\| \le \|K_ny_n - x_n\| + \|x_n - y_n\|$
by (32) and (40), we have

(41)
$$\lim_{n\to\infty} \|K_ny_n - y_n\| = 0.$$

Next, we show that

(42)
$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0,$$

where $x^* = P_{\cap_{i=1}^N F(T_i)\cap\Omega} f(x^*)$. To show this inequality, we can choose a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{i\to\infty}\langle f(x^*)-x^*, y_{n_i}-x^*\rangle = \limsup_{n\to\infty}\langle f(x^*)-x^*, y_n-x^*\rangle. \tag{43}$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{ij}}\}$ of $\{y_{n_i}\}$ which converges weakly to ω. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \omega$. From $\|K_n y_n - y_n\| \to 0$, so we have $K_n y_{n_i} \rightharpoonup \omega$. Let us show $\omega \in \cap_{i=1}^N F(T_i)\cap\Omega$.
First, we show $\omega\in\Omega$. Since $y_n = T_{r_n}(x_n - r_n A x_n)$, for any $z\in C$ we have

$$\Phi(y_n, z) + \varphi(z) - \varphi(y_n) + \langle Ax_n, z-y_n\rangle + \frac{1}{r_n}\langle z-y_n, y_n-x_n\rangle \geq 0.$$

From (A2) we have

$$\varphi(z) - \varphi(y_n) + \langle Ax_n, z-y_n\rangle + \frac{1}{r_n}\langle z-y_n, y_n-x_n\rangle \geq -\Phi(y_n,z) \geq \Phi(z, y_n),$$

and hence

$$\varphi(z) - \varphi(y_{n_i}) + \langle Ax_{n_i}, z-y_{n_i}\rangle + \langle z-y_{n_i}, \frac{y_{n_i}-x_{n_i}}{r_{n_i}}\rangle \geq \Phi(z, y_{n_i}), \tag{44}$$

Put $y_t = tz + (1-t)\omega$ for all $t\in(0,1]$ and $z\in C$. Then we have $y_t \in C$. From (44) we have

$$\begin{aligned}
&\varphi(y_t) - \varphi(y_{n_i}) + \langle y_t - y_{n_i}, Ay_t\rangle \\
&\quad\geq \langle y_t - y_{n_i}, Ay_t\rangle - \langle y_t - y_{n_i}, Ax_{n_i}\rangle - \langle y_t - y_{n_i}, \frac{y_{n_i}-x_{n_i}}{r_{n_i}}\rangle + \Phi(y_t, y_{n_i}) \\
&\quad= \langle y_t - y_{n_i}, Ay_t - Ay_{n_i}\rangle + \langle y_t - y_{n_i}, Ay_{n_i} - Ax_{n_i}\rangle - \langle y_t - y_{n_i}, \frac{y_{n_i}-x_{n_i}}{r_{n_i}}\rangle + \Phi(y_t, y_{n_i}).
\end{aligned}$$

Since $\|y_{n_i} - x_{n_i}\| \to 0$, we have $\|Ay_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle y_t - y_{n_i}, Ay_t - Ay_{n_i}\rangle \geq 0$.
Thus from the weakly semicontinuity of φ and (A4), we have

$$\varphi(y_t) - \varphi(\omega) + \langle y_t - \omega, Ay_t\rangle \geq \Phi(y_t, \omega) \ \text{ as } \ i\to\infty. \tag{45}$$

From (A1), (A4), (45) and the convexity of φ , we also have

$$\begin{aligned}
0 &= \Phi(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\
&= \Phi(y_t, (tz + (1-t)\omega)) + \varphi(tz + (1-t)\omega) - \varphi(y_t) \\
&\leq t\Phi(y_t, z) + (1-t)\Phi(y_t,\omega) + t\varphi(z) + (1-t)\varphi(\omega) - \varphi(y_t) \\
&\leq t\Phi(y_t, z) + (1-t)(\varphi(y_t) - \varphi(\omega) + \langle y_t-\omega, Ay_t\rangle) + t\varphi(z) + (1-t)\varphi(\omega) - \varphi(y_t) \\
&= t\Phi(y_t, z) - t\varphi(y_t) + (1-t)\langle y_t - \omega, Ay_t\rangle + t\varphi(z) \\
&= t[\Phi(y_t, z) - \varphi(y_t) + \varphi(z)] + (1-t)t\langle z-\omega, Ay_t\rangle
\end{aligned} \tag{46}$$

Dividing by t, we have

$$\Phi(y_t, z) - \varphi(y_t) + \varphi(z) + (1-t)\langle z-\omega, Ay_t\rangle \geq 0, \quad \forall z\in C.$$

Letting $t\to 0$, it follows from (A3) and the weakly semicontinuity of φ that

$$\Phi(\omega, z) - \varphi(\omega) + \varphi(z) + \langle z-\omega, A\omega\rangle \geq 0, \quad \forall z\in C. \tag{47}$$

Therefore $\omega\in\Omega$. Next, we show that $\omega\in\cap_{i=1}^N F(T_i)$. Assume that there exists $j\in\{1,2,\ldots,N\}$ such that $\omega \neq T_j\omega$. By Lemma 2.10, we have $\omega\neq K\omega$.
Since $y_{n_i}\rightharpoonup\omega$ and $\omega\neq K\omega$, by Opial's condition [8] and (41) and Lemma 2.11, we have

$$\begin{aligned}
\liminf_{i\to\infty}\|y_{n_i} - \omega\| &< \liminf_{i\to\infty}\|y_{n_i} - K\omega\| \\
&\leq \liminf_{i\to\infty}(\|y_{n_i} - K_{n_i}y_{n_i}\| + \|K_{n_i}y_{n_i} - K_{n_i}\omega\| + \|K_{n_i}\omega - K\omega\|) \\
&\leq \liminf_{i\to\infty}\|y_{n_i} - \omega\|,
\end{aligned}$$

which is a contradiction. Thus $\omega = K\omega$ and $\omega \in F(K) = \cap_{i=1}^{N} F(T_i)$. Hence $\omega \in \cap_{i=1}^{N} F(T_i) \cap \Omega$. Since $x^* = P_{\cap_{i=1}^{N} F(T_i)\cap\Omega} f(x^*)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle f(x^*) - x^*, x_{n_i} - x^*\rangle \\ &= \lim_{i\to\infty}\langle f(x^*) - x^*, y_{n_i} - x^*\rangle \\ &= \langle f(x^*) - x^*, \omega - x^*\rangle \le 0. \qquad (48)\end{aligned}$$

Finally, we prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to x^*. From (13), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n - x^*, x_{n+1} - x^*\rangle \\ &= \alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle + \beta_n\langle x_n - x^*, x_{n+1} - x^*\rangle + \gamma_n\langle K_n y_n - x^*, x_{n+1} - x^*\rangle \\ &\le \alpha_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \frac{1}{2}\beta_n(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \frac{1}{2}\gamma_n(\|K_n y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\ &\le \frac{1}{2}(1-\alpha_n)(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \frac{1}{2}\alpha_n(\|f(x_n) - f(x^*)\|^2 + \|x_{n+1} - x^*\|^2) \\ &\le \frac{1}{2}(1-\alpha_n)(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &\quad + \frac{1}{2}\alpha_n\rho^2\|x_n - x^*\|^2 + \frac{1}{2}\alpha_n\|x_{n+1} - x^*\|^2 \\ &= \frac{1}{2}(1-\alpha_n(1-\rho^2))\|x_n - x^*\|^2 + \frac{1}{2}\|x_{n+1} - x^*\|^2 + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle, \qquad (49)\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n(1-\rho^2))\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &= (1-\alpha_n(1-\rho^2))\|x_n - x^*\|^2 + \alpha_n(1-\rho^2)\cdot\frac{2}{(1-\rho^2)}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\ &= (1-\delta_n)\|x_n - x^*\|^2 + \delta_n\sigma_n, \qquad (50)\end{aligned}$$

where $\delta_n = \alpha_n(1-\rho^2)$ and $\sigma_n = \frac{2}{(1-\rho^2)}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle$. It is easy to see that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$. Applying Lemma 2.4 to (50), we conclude that $x_n \to x^*$ as $n \to \infty$. Consequently, $\{y_n\}$ converge strongly to x^*. This completes the proof. □

Corollary 3.2. *Let H be a Hilbert space, C a closed convex nonempty subset of H, $\varphi : C \to \mathbb{R}$ a proper lower semicontinuous and convex functional, $\Phi : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)-(A4), $\{T_i\}_{i=1}^{N}$ a finite family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{N} F(T_i) \cap MEP(\Phi, \varphi) \neq 0$ and f a $\rho-$contraction of C into itself. Moreover, let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^{N}$ a sequence in $[a,b]$ with $0 < a \le b < 1$ and $\{r_n\}$ a sequence in $[0, 2\alpha]$ for all $n \in \mathbb{N}$. Assume that:*

(i) *either (B1) or (B2) holds;*
(ii) *the sequence $\{r_n\}$ satisfies*
  (C1) $0 < c \le r_n \le d < 2\alpha$*; and*
  (C2) $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$*;*
(iii) *the sequence $\{\alpha_n\}$ satisfies*
  (D1) $\lim_{n\to\infty}\alpha_n = 0$*; and*
  (D2) $\sum_{n=0}^{\infty}\alpha_n = \infty$*;*
(iv) *the sequence $\{\beta_n\}$ satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(v) *the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^N$ *satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

For every $n \in \mathbb{N}$, *let* K_n *be a* $K-$*mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} \Phi(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \ge 0, \quad \forall x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n. \end{cases}$$

Then both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\cap_{i=1}^N F(T_i)\cap MEP(\Phi,\varphi)} f(x^*)$.

Proof. Put $A \equiv 0$. Then, for all $\alpha \in (0, \infty)$, we have that

$$\langle x - y, Ax - Ay\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Hence all the conditions of Theorem 3.1 are satisfied. Therefore the corollary is obtained by Theorem 3.1. □

Corollary 3.3. *Let* H *be a Hilbert space,* C *a closed convex nonempty subset of* H, A *an* $\alpha-$*inverse strongly monotone mapping of* C *into* H, $\Phi : C \times C \to \mathbb{R}$ *a bifunction satisfying (A1)-(A4),* $\{T_i\}_{i=1}^N$ *a finite family of nonexpansive mappings of* C *into itself such that* $\cap_{i=1}^N F(T_i) \cap EP \ne 0$ *and* f *a* $\rho-$*contraction of* C *into itself. Moreover, let* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $(0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ *a sequence in* $[a, b]$ *with* $0 < a \le b < 1$ *and* $\{r_n\}$ *a sequence in* $[0, 2\alpha]$ *for all* $n \in \mathbb{N}$. *Assume that:*

(i) *the sequence* $\{r_n\}$ *satisfies*

(C1) $0 < c \le r_n \le d < 2\alpha$; *and*

(C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(ii) *the sequence* $\{\alpha_n\}$ *satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$; *and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *the sequence* $\{\beta_n\}$ *satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iv) *the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^N$ *satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

For every $n \in \mathbb{N}$, *let* K_n *be a* $K-$*mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} \Phi(y_n, x) + \langle Ax_n, x - y_n\rangle + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \ge 0, \quad \forall x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n. \end{cases}$$

Then both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\cap_{i=1}^N F(T_i)\cap EP} f(x^*)$.

Proof. Put $\varphi \equiv 0$ in Theorem 3.1. Hence all the conditions of Theorem 3.1 are satisfied. Therefore the corollary is obtained by Theorem 3.1. □

Corollary 3.4. *Let* H *be a Hilbert space,* C *a closed convex nonempty subset of* H, $\Phi : C \times C \to \mathbb{R}$ *a bifunction satisfying (A1)-(A4),* $\{T_i\}_{i=1}^N$ *a finite family of nonexpansive mappings of* C *into itself such that* $\cap_{i=1}^N F(T_i) \cap EP(\Phi) \ne 0$ *and* f *a* $\rho-$*contraction of* C *into itself. Moreover, let* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are three sequences in* $(0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $\{\lambda_{n,i}\}_{i=1}^N$ *a sequence in* $[a, b]$ *with* $0 < a \le b < 1$ *and* $\{r_n\}$ *a sequence in* $[0, 2\alpha]$ *for all* $n \in \mathbb{N}$. *Assume that:*

(i) *the sequence* $\{r_n\}$ *satisfies*

(C1) $0 < c \le r_n \le d < 2\alpha$; *and*

(C2) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(ii) *the sequence* $\{\alpha_n\}$ *satisfies*

(D1) $\lim_{n\to\infty} \alpha_n = 0$; *and*

(D2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *the sequence* $\{\beta_n\}$ *satisfies*

(E1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) *the finite family of sequences* $\{\lambda_{n,i}\}_{i=1}^{N}$ *satisfies*

(F1) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$ *for every* $i \in \{1, 2, \ldots, N\}$.

For every $n \in \mathbb{N}$, *let* K_n *be a* $K-$*mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ *and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by* $x_0 \in C$ *and*

$$\begin{cases} \Phi(y_n, x) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, & \forall x \in C \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n K_n y_n. \end{cases}$$

Then both $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $x^* = P_{\cap_{i=1}^{N} F(T_i) \cap EP(\Phi)} f(x^*)$.

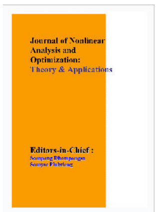
Proof. Put $\varphi \equiv 0$ and $A \equiv 0$ in Theorem3.1. Hence the corollary is obtained by Theorem 3.1. □

References

1. E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems,* The Mathematics Student. **63**(1994) 123-145.
2. L. C. Ceng and J. C. Yao, *A hybrid iterative scheme for mixed equilibriun problems and fixed point problem,* J. Comput. Appl. Math. **214**(2008) 186-201.
3. P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces,* J. Nonlinear Convex Anal. **6**(2005)117-136.
4. A. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge Stud. Adv. Math., vol. 28, Cambridge Univ. Prees, Cambridge, 1990.
5. B. Halpern, *Fixed points of nonexpansive maps,* Bull. J. Math. Amer. Math. Soc. **73** (1967)957–961.
6. A. Kangtunyakarn and S. Suantai, *A new mapping for findding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings,* Nonlinear Analysis (2009), doi:10.1016/j.na.2009.03.003.
7. A. Moudafi, *Viscosity approximation methods for fixed points problems*, J. Math. Anal. Appl. **241** (2000)46–55.
8. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings,* Bull. Amer. Math. Soc. **73** (1967)591–597.
9. J. W. Peng and J. C. Yao, *A new hybrid-extragradient method for generalized mixed equilibrium problems and variational inequality problems,* Taiwanese J. Math. **12(6)**(2008)1401-1432.
10. T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals,* J. Math. Anal. Appl. **305**(2008) 227-239.
11. W. Takahashi, Nonlinear Functional Analysis : Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, 2000.
12. A. Tada and W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem,* J. Optim. Theory Appl. **133** (2007) 359-370.
13. S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space,* Nonlinear Analysis. **69**(2008) 1025-1033.
14. S. Takahashi and W. Takahashi, *Vicosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces,* J. Math. Anal. Appl. **331**(2006) 506-515.
15. R. Wangkeeree and R. Wangkeeree, *strong convergence of the iterative scheme based on the extragradient method for mixed equlibrium problems and fixed point problems of an infinite family of nonexpansive mappings,* Nonlinear Analysis: Hybrid Systems, (2009), doi:10.1016/j.nahs2009.06.009.
16. H. K. Xu, *An iterative approach to quadratic optimization,* J. Optim. Theory Appl. **116(3)**(2003)659-678.

SCHOOL OF MATHEMATICS, INSTITUTE OF SCIENCE, SURANAREE UNIVERSITY OF TECHNOLOGY, NAKHON RATCHASIMA, 30000, THAILAND.
Email address: benjawan@sut.ac.th

Journal of Nonlinear Analysis and Optimization
Vol. 1 No. 1 (2010), 169-182
ISSN : 1906-9685
http://www.sci.nu.ac.th/jnao



Common Fixed Points of a New Three-Step Iteration with Errors of Asymptotically Quasi-Nonexpansive Nonself-Mappings in Banach spaces

Utith Inprasit[1,2] and Hathaikarn Wattanataweekul[1,2]

ABSTRACT: In this paper, we study a new three-step iterative scheme for approximating a common fixed point of three asymptotically quasi-nonexpansive nonself-mappings with errors and prove several strong and weak convergence results of the iterative sequences with errors in a uniformly convex Banach space. We also extend and improve some recent corresponding results in the literature.

1. **Introduction**

We assume that X is a normed space and C is a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim\limits_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $x, y \in C$ and each $n \geq 1$. The class of asymptotically nonexpansive mappings is a natural generalization of the important class of nonexpansive mappings. Goebel and Kirk [3] proved that if C is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point. A mapping $T : C \to C$ is called *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim\limits_{n\to\infty} k_n = 1$ such that $\|T^n x - q\| \leq k_n\|x - q\|$ for all $x \in C, q \in F(T), n \geq 1$, where $F(T)$ is the set of fixed points of T. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in C$ and each $n \geq 1$. It is easy to see that an asymptotically nonexpansive mapping must be uniformly L-Lipschitzian as well as asymptotically quasi-nonexpansive but the converse does not hold.

In 2000, Noor [9] introduced a three-step iterative sequence and studied the approximate solutions of variational inclusions in Hilbert spaces. Glowinski and Le Tallec [4] applied three-step iterative sequences for finding the approximate solutions of the elastoviscoplasticity problem, eigenvalue problems and in the liquid crystal theory. It has been shown in [1], that three-step method performs better than two-step and one-step methods for solving variational inequalities. The three-step schemes are natural generalization of the splitting methods to solve partial differential equations; see, Noor [9, 10, 11]. This signifies that Noor three-step methods are robust and more efficient than the Mann (one-step) and Ishikawa (two-step) type

Corresponding author: utith@sci.ubu.ac.th(U. Inprasit)
Manuscript received March 16, 2010. Accepted July 31, 2010.

iterative methods to solve problems of pure and applied sciences.

In 2001, Khan and Takahashi [5] have approximated common fixed points of two asymptotically nonexpansive mappings by the modified Ishikawa iteration. Recently Shahzad and Udomene [15] established convergence theorems for the modified Ishikawa iteration process of two asymptotically quasi-non expansive mappings to a common fixed point of the mappings. For related results with error terms, we refer to [2, 6, 13] and [15].

The purpose of this paper is to establish strong and weak convergence theorems of a new three-step iteration for three asymptotically quasi-nonexpan sive non-self mappings in a uniformly convex Banach space. This scheme can be viewed as an extension of Xu and Noor [18], Suantai [16] and Nilsrakoo and Saejung [8].

Let X be a normed space. A subset C of X is said to be a *retract* of X if there exists a continuous map $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex set of a uniformly convex Banach space is a retract. A map $P : X \to C$ is said to be a *retraction* if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. A mapping $T : C \to X$ is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}x - q\| \leq k_n\|x - q\|$$

for all $x \in C, q \in F(T), n \geq 1$, where $F(T)$ is the set of fixed points of T and $(PT)^0 = I$, the identity operator on C.

The mapping $T : C \to X$ is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Let C be a nonempty closed convex subset of X and $P : X \to C$ a nonexpansive retraction of X onto C, and let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings and F is the set of all common fixed points of T_i i.e., $F = \bigcap_{i=1}^{3} F(T_i)$, where $F(T_i) = \{x \in C : T_i x = x\}$ for all $i = 1, 2, 3$. Then, for arbitrary $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= P[a_nT_1(PT_1)^{n-1}x_n + (1 - a_n - \delta_n)x_n + \delta_n u_n], \\ y_n &= P[b_nT_2(PT_2)^{n-1}z_n + c_nT_1(PT_1)^{n-1}x_n + (1 - b_n - c_n - \sigma_n)x_n + \sigma_n v_n], \\ x_{n+1} &= P[\alpha_nT_3(PT_3)^{n-1}y_n + \beta_nT_2(PT_2)^{n-1}z_n + \gamma_nT_1(PT_1)^{n-1}x_n \\ &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] \end{aligned} \tag{1}$$

for all $n \geq 1$, where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C.

Without errors ($\delta_n = \sigma_n = \rho_n \equiv 0$), and T_1, T_2, T_3 are self-maps of C, the iterative scheme (1) reduces to the following iterative scheme:

$$\begin{aligned} z_n &= a_nT_1^n x_n + (1 - a_n)x_n, \\ y_n &= b_nT_2^n z_n + c_nT_1^n x_n + (1 - b_n - c_n)x_n, \\ x_{n+1} &= \alpha_nT_3^n y_n + \beta_nT_2^n z_n + \gamma_nT_1^n x_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n, \quad n \geq 1, \end{aligned} \tag{2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

If $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Nilsrakoo and Saejung [8].

If $\gamma_n \equiv 0$ and $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Suantai [16].

If $c_n = \beta_n = \gamma_n \equiv 0$ and $T := T_1 = T_2 = T_3$, then (2) reduces to the iterative scheme defined by Xu and Noor [18].

If $a_n = b_n = c_n \equiv 0$, then (2) reduces to the following iterative scheme:

$$x_{n+1} = \alpha_n T_3^n x_n + \beta_n T_2^n x_n + \gamma_n T_1^n x_n + (1 - \alpha_n - \beta_n - \gamma_n) x_n \tag{3}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are appropriate sequences in $[0,1]$.

To study strong and weak convergence theorems of the iterative scheme 1, we recall some useful well-known concepts and results.

Recall that a Banach space X is said to satisfy *Opial's condition* [12] if for each sequence $\{x_n\}$ and $x, y \in X$ with $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|.$$

In what follows, we shall make use of the following lemmas.

Lemma 1.1. [17, Lemma 1]. *Let $\{a_n\}, \{b_n\}, \{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n \qquad \text{for all } n = 1, 2, \dots.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ *exists, and*

(ii) $\lim_{n\to\infty} a_n = 0$ *whenever* $\lim_{n\to\infty} \inf a_n = 0$.

Lemma 1.2. [7, Lemma 1.4]. *Let X be a uniformly convex Banach space and let $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$ be a closed ball of X. Then there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \vartheta w\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \xi \|z\|^2 + \vartheta \|w\|^2 - \lambda \mu g\big(\|x - y\|\big)$$

for all $x, y, z, w \in B_r$ and all $\lambda, \mu, \xi, \vartheta \in [0,1]$ with $\lambda + \mu + \xi + \vartheta = 1$.

Similar to Lemma 1.2, we can prove the next lemma.

Lemma 1.3. *Let X be a uniformly convex Banach space and let B_r be a closed ball of X. Then there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty), g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \vartheta w + \zeta s\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \xi \|z\|^2 + \vartheta \|w\|^2 + \zeta \|s\|^2 - \lambda \mu g\big(\|x - y\|\big)$$

for all $x, y, z, w, s \in B_r$ and all $\lambda, \mu, \xi, \vartheta, \zeta \in [0,1]$ with $\lambda + \mu + \xi + \vartheta + \zeta = 1$.

Lemma 1.4. [16, Lemma 2.7]. *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be so that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

2. Main Results

In this section, we prove strong and weak convergence theorems for the iterative scheme (1) for asymptotically quasi-nonexpansive nonself-mappings in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset$, $k_n \geq 1$, $l_n \geq 1, m_n \geq 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, $\sum_{n=1}^{\infty} (l_n - 1) < \infty$ $\sum_{n=1}^{\infty} (m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$, $\{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0,1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0,1]$ for all $n \geq 1$, $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the* *sequences defined as in* (1). *Then*

(i) $\lim_{n\to\infty} \|x_n - q\|$ *exists for all* $q \in F$.

(ii) *If one of the following conditions* $(a), (b), (c)$ *and* (d) *holds, then* $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

(a) $\lim_{n\to\infty} \inf \beta_n > 0$ *and* $0 < \lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \sup(a_n + \delta_n) < 1$.

(b) $\lim_{n\to\infty} \inf \alpha_n, \lim_{n\to\infty} \inf b_n > 0$ *and* $0 < \lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \sup(a_n + \delta_n) < 1$.

(c) $0 < \lim_{n\to\infty} \inf \gamma_n \le \lim_{n\to\infty} \sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(d) $0 < \lim_{n\to\infty} \inf \alpha_n$ *and* $0 < \lim_{n\to\infty} \inf c_n \le \lim_{n\to\infty} \sup(b_n + c_n + \sigma_n) < 1$.

(iii) *If either* (a) $0 < \lim_{n\to\infty} \inf \beta_n \le \lim_{n\to\infty} \sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$ *or* (b) $\lim_{n\to\infty} \inf \alpha_n > 0$ *and* $0 < \lim_{n\to\infty} \inf b_n \le \lim_{n\to\infty} \sup(b_n + c_n + \sigma_n) < 1$, *then* $\lim_{n\to\infty} \|T_2(PT_2)^{n-1}z_n - x_n\| = 0$.

(iv) *If* $0 < \lim_{n\to\infty} \inf \alpha_n \le \lim_{n\to\infty} \sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$, *then* $\lim_{n\to\infty} \|T_3(PT_3)^{n-1}y_n - x_n\| = 0$.

Proof. (i) Let $q \in F$. By (1), we obtain

$$\begin{aligned} \|z_n - q\| &= \|P[a_nT_1(PT_1)^{n-1}x_n + (1 - a_n - \delta_n)x_n + \delta_n u_n] - P(q)\| \\ &\le a_n\|T_1(PT_1)^{n-1}x_n - q\| + (1 - a_n - \delta_n)\|x_n - q\| + \delta_n\|u_n - q\| \\ &\le (1 + a_n(k_n - 1) - \delta_n)\|x_n - q\| + \delta_n\|u_n - q\| \end{aligned} \tag{4}$$

and

$$\begin{aligned} \|y_n - q\| &= \|P[b_nT_2(PT_2)^{n-1}z_n + c_nT_1(PT_1)^{n-1}x_n + (1 - b_n - c_n - \sigma_n)x_n \\ &\quad + \sigma_n v_n] - P(q)\| \\ &\le b_n\|T_2(PT_2)^{n-1}z_n - q\| + c_n\|T_1(PT_1)^{n-1}x_n - q\| \\ &\quad + (1 - b_n - c_n - \sigma_n)\|x_n - q\| + \sigma_n\|v_n - q\| \\ &\le b_n l_n\|z_n - q\| + c_n k_n\|x_n - q\| + (1 - b_n - c_n - \sigma_n)\|x_n - q\| \\ &\quad + \sigma_n\|v_n - q\|. \end{aligned} \tag{5}$$

By (4) and (5), we obtain

$$\begin{aligned} \|x_{n+1} - q\| &= \|P[\alpha_nT_3(PT_3)^{n-1}y_n + \beta_nT_2(PT_2)^{n-1}z_n + \gamma_nT_1(PT_1)^{n-1}x_n \\ &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)x_n + \rho_n w_n] - P(q)\| \\ &\le \alpha_n\|T_3(PT_3)^{n-1}y_n - q\| + \beta_n\|T_2(PT_2)^{n-1}z_n - q\| \\ &\quad + \gamma_n\|T_1(PT_1)^{n-1}x_n - q\| + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| \\ &\quad + \rho_n\|w_n - q\| \\ &\le \alpha_n m_n\|y_n - q\| + \beta_n l_n\|z_n - q\| + \gamma_n k_n\|x_n - q\| \\ &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| + \rho_n\|w_n - q\| \\ &\le (\alpha_n m_n b_n l_n + \beta_n l_n)\|z_n - q\| + \alpha_n m_n c_n k_n\|x_n - q\| \\ &\quad + (\alpha_n m_n - \alpha_n m_n b_n - \alpha_n m_n c_n - \alpha_n m_n \sigma_n)\|x_n - q\| \\ &\quad + \alpha_n m_n \sigma_n\|v_n - q\| + \gamma_n k_n\|x_n - q\| \\ &\quad + (1 - \alpha_n - \beta_n - \gamma_n - \rho_n)\|x_n - q\| + \rho_n\|w_n - q\| \\ &\le \|x_n - q\| + ((l_n - 1)(\alpha_n m_n b_n + \beta_n) + (k_n - 1)(\gamma_n + \alpha_n m_n c_n \\ &\quad + (\alpha_n m_n b_n l_n + \beta_n l_n)a_n) + \alpha_n(m_n - 1))\|x_n - q\| \\ &\quad + (m_n l_n + l_n)\delta_n\|u_n - q\| + m_n\sigma_n\|v_n - q\| + \rho_n\|w_n - q\|. \end{aligned}$$

Since $\{l_n\}, \{m_n\}, \{u_n\}, \{v_n\}, \{w_n\}$ are bounded, there exists a constant $K > 0$ such that $\alpha_n m_n b_n + \beta_n \leq K$, $\gamma_n + \alpha_n m_n c_n + (\alpha_n m_n b_n l_n + \beta_n l_n)a_n \leq K$, $(m_n l_n + l_n)\|u_n - q\| \leq K, m_n\|v_n - q\| \leq K, \|w_n - q\| \leq K$ and $\alpha_n \leq K$ for all $n \geq 1$. Then

$$\begin{aligned}\|x_{n+1} - q\| \leq &\Big(1 + K\big((k_n - 1) + (l_n - 1) + (m_n - 1)\big)\Big)\|x_n - q\| \\ &+ K(\delta_n + \sigma_n + \rho_n)\end{aligned} \tag{6}$$

By Lemma 1.1, we obtain $\lim_{n\to\infty} \|x_n - q\|$ exists.

Next, we want to prove (ii), (iii) and (iv). It follows from (i) that $\{x_n - q\}, \{T_1(PT_1)^{n-1}x_n - q\}, \{y_n - q\}, \{T_3(PT_3)^{n-1}y_n - q\}, \{z_n - q\}$ and $\{T_2(PT_2)^{n-1}z_n - q\}$ are all bounded. Let

$$\begin{aligned}M = \max\Big\{ &\sup_{n\geq 1}\|x_n - q\|, \sup_{n\geq 1}\|T_1(PT_1)^{n-1}x_n - q\|, \sup_{n\geq 1}\|y_n - q\|, \\ &\sup_{n\geq 1}\|T_3(PT_3)^{n-1}y_n - q\|, \sup_{n\geq 1}\|z_n - q\|, \sup_{n\geq 1}\|u_n - q\|, \\ &\sup_{n\geq 1}\|T_2(PT_2)^{n-1}z_n - q\|, \sup_{n\geq 1}\|v_n - q\|, \sup_{n\geq 1}\|w_n - q\|\Big\}.\end{aligned}$$

By Lemma 1.3, there exists a continuous, strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned}\|\lambda x + \mu y + \xi z + \vartheta w + \zeta s\|^2 \leq &\lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \vartheta\|w\|^2 + \zeta\|s\|^2 \\ &- \lambda\mu g\big(\|x - y\|\big)\end{aligned} \tag{7}$$

for all $x, y, z, w, s \in B_r$ and all $\lambda, \mu, \xi, \vartheta, \zeta \in [0, 1]$ with $\lambda + \mu + \xi + \vartheta + \zeta = 1$. By (7), we have

$$\begin{aligned}\|z_n - q\|^2 &= \|P[a_n T_1(PT_1)^{n-1}x_n + (1 - a_n - \delta_n)x_n + \delta_n u_n] - P(q)\|^2 \\ &\leq \|a_n(T_1(PT_1)^{n-1}x_n - q) + (1 - a_n - \delta_n)(x_n - q) + \delta_n(u_n - q)\|^2 \\ &\leq a_n\|T_1(PT_1)^{n-1}x_n - q\|^2 + (1 - a_n - \delta_n)\|x_n - q\|^2 \\ &\quad + \delta_n\|u_n - q\|^2 - a_n(1 - a_n - \delta_n)g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) \\ &\leq a_n k_n^2\|x_n - q\|^2 + (1 - a_n - \delta_n)\|x_n - q\|^2 + \delta_n\|u_n - q\|^2 \\ &\quad - a_n(1 - a_n - \delta_n)g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) \\ &\leq (1 + a_n(k_n^2 - 1) - \delta_n)\|x_n - q\|^2 + \delta_n\|u_n - q\|^2\end{aligned} \tag{8}$$

and

$$\begin{aligned}\|y_n - q\|^2 &= \|P[b_n T_2(PT_2)^{n-1}z_n + c_n T_1(PT_1)^{n-1}x_n + (1 - b_n - c_n - \sigma_n) \\ &\quad x_n + \sigma_n v_n] - P(q)\|^2 \\ &\leq \|b_n(T_2(PT_2)^{n-1}z_n - q) + c_n(T_1(PT_1)^{n-1}x_n - q) \\ &\quad + (1 - b_n - c_n - \sigma_n)(x_n - q) + \sigma_n(x_n - q)\|^2 \\ &\leq b_n\|T_2(PT_2)^{n-1}z_n - q\|^2 + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 \\ &\quad + c_n\|T_1(PT_1)^{n-1}x_n - q\|^2 + \sigma_n\|v_n - q\|^2 \\ &\quad - b_n(1 - b_n - c_n - \sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n - x_n\|\big) \\ &\leq b_n l_n^2\|z_n - q\|^2 + (1 - b_n - c_n - \sigma_n)\|x_n - q\|^2 + c_n k_n^2\|x_n - q\|^2 \\ &\quad + \sigma_n\|v_n - q\|^2 - b_n(1 - b_n - c_n - \sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n - x_n\|\big)\end{aligned} \tag{9}$$

By (7) , (8) and (9), we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|P[\alpha_n T_3(PT_3)^{n-1}y_n+\beta_n T_2(PT_2)^{n-1}z_n+\gamma_n T_1(PT_1)^{n-1}x_n \\
&\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n+\rho_n w_n]-P(q)\|^2 \\
&\leq \alpha_n\|T_3(PT_3)^{n-1}y_n-q\|^2+\beta_n\|T_2(PT_2)^{n-1}z_n-q\|^2 \\
&\quad +\gamma_n\|T_1(PT_1)^{n-1}x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) \\
&\quad +\rho_n\|w_n-q\|^2 \\
&\leq \alpha_n m_n^2\|y_n-q\|^2+\beta_n l_n^2\|z_n-q\|^2+\gamma_n k_n^2\|x_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) \\
&\leq \alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2+\alpha_n m_n^2(1-b_n-c_n-\sigma_n)\|x_n-q\|^2 \\
&\quad +\gamma_n k_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\
&\quad +(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)\|z_n-q\|^2+\alpha_n m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_n m_n^2 b_n(1-b_n-c_n-\sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) \\
&\leq \alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2+\alpha_n m_n^2(1-b_n-c_n)\|x_n-q\|^2 \\
&\quad +\gamma_n k_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n)\|x_n-q\|^2+(\alpha_n m_n^2 b_n l_n^2 \\
&\quad +\beta_n l_n^2)\|x_n-q\|^2+(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)(a_n(k_n^2-1))\|x_n-q\|^2 \\
&\quad -(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n(1-a_n-\delta_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad +(m_n^2 l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_n m_n^2 b_n(1-b_n-c_n-\sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) \\
&= \|x_n-q\|^2+\big((k_n^2-1)(\alpha_n m_n^2 c_n+\gamma_n+(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n) \\
&\quad +(l_n^2-1)(\alpha_n m_n^2 b_n+\beta_n)+\alpha_n(m_n^2-1)\big)\|x_n-q\|^2 \\
&\quad +(m_n^2 l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\
&\quad -\alpha_n m_n^2 b_n l_n^2 a_n(1-a_n-\delta_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad -\beta_n l_n^2 a_n(1-a_n-\delta_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\
&\quad -\alpha_n m_n^2 b_n(1-b_n-c_n-\sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big)
\end{aligned}$$

Since $\{k_n\},\{l_n\},\{m_n\},\{u_n\},\{v_n\},\{w_n\}$ are bounded and $\{x_n\}$ is bounded, there exist constants $K_0>0$ such that
$(\alpha_n m_n^2 c_n+\gamma_n+(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n)\|x_n-q\|^2\leq K_0$,
$(\alpha_n m_n^2 b_n+\beta_n)\|x_n-q\|^2\leq K_0, \alpha_n\|x_n-q\|^2\leq K_0, (m_n^2 l_n^2+l_n^2)\|u_n-q\|^2\leq K_0$,
$m_n^2\|v_n-q\|^2\leq K_0$ and $\|w_n-q\|^2\leq K_0$ for all $n\geq 1$. Thus

$$\begin{aligned}
\alpha_n m_n^2 b_n l_n^2 a_n(1-a_n-\delta_n)g\big(||T_1(PT_1)^{n-1}x_n-x_n||\big) &\leq ||x_n-q||^2-||x_{n+1}-q||^2 \\
&\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big) \\
&\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (10)
\end{aligned}$$

$$\begin{aligned}
\beta_n l_n^2 a_n(1-a_n-\delta_n)g\big(||T_1(PT_1)^{n-1}x_n-x_n||\big) &\leq ||x_n-q||^2-||x_{n+1}-q||^2 \\
&\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big) \\
&\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (11)
\end{aligned}$$

$$\begin{aligned} \alpha_n m_n^2 b_n(1-b_n-c_n-\sigma_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2 \\ &\quad + K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big) \\ &\quad + K_0(\delta_n+\sigma_n+\rho_n). \end{aligned} \tag{12}$$

$$\begin{aligned} \alpha_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_3(PT_3)^{n-1}y_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2 \\ &\quad + K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big) \\ &\quad + K_0(\delta_n+\sigma_n+\rho_n). \end{aligned} \tag{13}$$

Again (7), we obtain

$$\begin{aligned} \|y_n-q\|^2 &= \|P[b_nT_2(PT_2)^{n-1}z_n+c_nT_1(PT_1)^{n-1}x_n+(1-b_n-c_n-\sigma_n)x_n \\ &\quad +\sigma_n v_n]-P(q)\|^2 \\ &\leq c_n\|T_1(PT_1)^{n-1}x_n-q\|^2+(1-b_n-c_n-\sigma_n)\|x_n-q\|^2 \\ &\quad + b_n\|T_2(PT_2)^{n-1}z_n-q\|^2+\sigma_n\|v_n-q\|^2 \\ &\quad - c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\ &\leq c_nk_n^2\|x_n-q\|^2+(1-b_n-c_n-\sigma_n)\|x_n-q\|^2+b_nl_n^2\|z_n-q\|^2 \\ &\quad +\sigma_n\|v_n-q\|^2-c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \end{aligned} \tag{14}$$

By (7), (8) and (14), we have

$$\begin{aligned} \|x_{n+1}-q\|^2 &= \|P[\alpha_nT_3(PT_3)^{n-1}y_n+\beta_nT_2(PT_2)^{n-1}z_n+\gamma_nT_1(PT_1)^{n-1}x_n \\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n+\rho_nw_n]-P(q)\|^2 \\ &\leq \alpha_n\|T_3(PT_3)^{n-1}y_n-q\|^2+\beta_n\|T_2(PT_2)^{n-1}z_n-q\|^2+\rho_n\|w_n-q\|^2 \\ &\quad +\gamma_n\|T_1(PT_1)^{n-1}x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\ &\quad -\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\ &\leq \alpha_nm_n^2\|y_n-q\|^2+\beta_nl_n^2\|z_n-q\|^2+\gamma_nk_n^2\|x_n-q\|^2 \\ &\quad +\rho_n\|w_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\ &\quad +\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\ &\leq \alpha_nm_n^2c_nk_n^2\|x_n-q\|^2+\alpha_nm_n^2(1-b_n-c_n-\alpha_n)\|x_n-q\|^2 \\ &\quad +\gamma_nk_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2 \\ &\quad +(\alpha_nm_n^2b_nl_n^2+\beta_nl_n^2)\|z_n-q\|^2+\alpha_nm_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\ &\quad -\alpha_nm_n^2c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\ &\quad -\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\ &\leq \alpha_nm_n^2c_nk_n^2\|x_n-q\|^2+\alpha_nm_n^2(1-b_n-c_n)\|x_n-q\|^2 \\ &\quad +\gamma_nk_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n)\|x_n-q\|^2+(\alpha_nm_n^2b_nl_n^2 \\ &\quad +\beta_nl_n^2)\|x_n-q\|^2+(\alpha_nm_n^2b_nl_n^2+\beta_nl_n^2)(a_n(k_n^2-1))\|x_n-q\|^2 \\ &\quad +(m_n^2l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\ &\quad -\alpha_nm_n^2c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\ &\quad -\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \\ &= \|x_n-q\|^2+\big((k_n^2-1)(\alpha_nm_n^2c_n+\gamma_n+(\alpha_nm_n^2b_nl_n^2+\beta_nl_n^2)a_n) \\ &\quad +(l_n^2-1)(\alpha_nm_n^2b_n+\beta_n)+\alpha_n(m_n^2-1)\big)\|x_n-q\|^2 \\ &\quad +(m_n^2l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2 \\ &\quad -\alpha_nm_n^2c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) \\ &\quad -\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) \end{aligned}$$

Thus

$$\begin{aligned}\alpha_n m_n^2 c_n(1-b_n-c_n-\sigma_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big)\\ &\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (15)\end{aligned}$$

$$\begin{aligned}\beta_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_2(PT_2)^{n-1}z_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big)\\ &\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (16)\end{aligned}$$

By (7) , (8) and (9), we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|P[\alpha_n T_3(PT_3)^{n-1}y_n+\beta_n T_2(PT_2)^{n-1}z_n+\gamma_n T_1(PT_1)^{n-1}x_n\\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)x_n+\rho_n w_n]-P(q)\|^2\\ &\leq \gamma_n\|T_1(PT_1)^{n-1}x_n-q\|^2+\alpha_n\|T_3(PT_3)^{n-1}y_n-q\|^2+\rho_n\|w_n-q\|^2\\ &\quad +(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2+\beta_n\|T_2(PT_2)^{n-1}z_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big)\\ &\leq \gamma_n k_n^2\|x_n-q\|^2+\alpha_n m_n^2\|y_n-q\|^2+\beta_n l_n^2\|z_n-q\|^2\\ &\quad +\rho_n\|w_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big)\\ &\leq \gamma_n k_n^2\|x_n-q\|^2+\beta_n l_n^2\|z_n-q\|^2+\alpha_n m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2\\ &\quad +\alpha_n m_n^2 b_n l_n^2\|z_n-q\|^2+\alpha_n m_n^2(1-b_n-c_n-\sigma_n)\|x_n-q\|^2\\ &\quad +\alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2+(1-\alpha_n-\beta_n-\gamma_n-\rho_n)\|x_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big)\\ &\leq \gamma_n k_n^2\|x_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2\\ &\quad +(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)\|x_n-q\|^2+(m_n^2 l_n^2+l_n^2)\delta_n\|u_n-q\|^2\\ &\quad +\big((\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n(k_n^2-1)\big)\|x_n-q\|^2+\alpha_n m_n^2 c_n k_n^2\|x_n-q\|^2\\ &\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n-q\|^2+\alpha_n m_n^2(1-b_n-c_n)\|x_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big)\\ &= \|x_n-q\|^2+\big((k_n^2-1)(\alpha_n m_n^2 c_n+\gamma_n+(\alpha_n m_n^2 b_n l_n^2+\beta_n l_n^2)a_n)\\ &\quad +(l_n^2-1)(\alpha_n m_n^2 b_n+\beta_n)+\alpha_n(m_n^2-1)\big)\|x_n-q\|^2\\ &\quad +(m_n^2 l_n^2+l_n^2)\delta_n\|u_n-q\|^2+m_n^2\sigma_n\|v_n-q\|^2+\rho_n\|w_n-q\|^2\\ &\quad -\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big).\end{aligned}$$

Thus

$$\begin{aligned}\gamma_n(1-\alpha_n-\beta_n-\gamma_n-\rho_n)g\big(\|T_1(PT_1)^{n-1}x_n-x_n\|\big) &\leq \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad +K_0\big((k_n^2-1)+(l_n^2-1)+(m_n^2-1)\big)\\ &\quad +K_0(\delta_n+\sigma_n+\rho_n). \qquad (17)\end{aligned}$$

(ii) (a) Let $\liminf_{n\to\infty}\beta_n>0$ and $0<\liminf_{n\to\infty}a_n\leq\limsup_{n\to\infty}(a_n+\delta_n)<1$. Then there exists a positive integer n_0 and $\eta,\eta^{'}\in(0,1)$ such that $0<\eta<\beta_n$, $0<\eta<a_n$ and $a_n+\delta_n<\eta^{'}<1$

for all $n \geq n_0$. This implies by (11) that

$$\eta^2(1-\eta')g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + K_0\big((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1)\big) + K_0(\delta_n + \sigma_n + \rho_n) \tag{18}$$

for all $n \geq n_0$. It follows from (18) that for $r \geq n_0$,

$$\begin{aligned}\sum_{n=n_0}^{r} g\Big(\|T_1(PT_1)^{n-1}x_n - x_n\|\Big) &\leq \frac{1}{\eta^2(1-\eta')}\Big(\sum_{n=n_0}^{r} (\|x_n - q\|^2 - \|x_{n+1} - q\|^2) \\ &\quad + K_0 \sum_{n=n_0}^{r} \big((k_n^2 - 1) + (l_n^2 - 1) + (m_n^2 - 1) \\ &\quad + \delta_n + \sigma_n + \rho_n)\Big) \\ &\leq \frac{1}{\eta^2(1-\eta')}\Big(\|x_{n_0} - q\|^2 + K_0 \sum_{n=n_0}^{r} \big((k_n^2 - 1) \\ &\quad + (l_n^2 - 1) + (m_n^2 - 1)\big)\Big).\end{aligned} \tag{19}$$

Since $0 \leq t^2 - 1 \leq 2t(t-1)$ for all $t \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, we get $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n^2 - 1) < \infty$. By inequality (19), let $r \to \infty$. We get $\sum_{n=n_0}^{\infty} g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) < \infty$. Thus $\lim_{n\to\infty} g\big(\|T_1(PT_1)^{n-1}x_n - x_n\|\big) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0$.

By using a similar method as in (ii) part (a) together with (10), (17), (15), (16), (12) and (13), the results in (ii) (b,c,d), (iii) (a,b) and (iv), respectively, can be proved. □

Lemma 2.2. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1$, $l_n \geq 1, m_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0,1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n, \alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0,1]$ for all $n \geq 1$, $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (1). Suppose T_1, T_2, T_3 are uniformly L-Lipschitzian. If $\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_2(PT_2)^{n-1}z_n - x_n\| = 0$, $\lim_{n\to\infty}\|T_3(PT_3)^{n-1}y_n - x_n\| = 0$, then*

(i) $\lim_{n\to\infty}\|T_1x_n - x_n\| = 0$,
(ii) $\lim_{n\to\infty}\|T_2x_n - x_n\| = 0$, *and*
(iii) $\lim_{n\to\infty}\|T_3x_n - x_n\| = 0$.

Proof. Since

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \alpha_n\|T_3(PT_3)^{n-1}y_n - x_n\| + \beta_n\|T_2(PT_2)^{n-1}z_n - x_n\| \\ &\quad + \gamma_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \rho_n\|w_n - x_n\| \\ &\longrightarrow 0 \text{ as } n \to \infty,\end{aligned}$$

we obtain

$$\begin{aligned}\|T_1(PT_1)^{n-1}x_{n+1} - x_{n+1}\| &\le \|T_1(PT_1)^{n-1}x_{n+1} - T_1(PT_1)^{n-1}x_n\| \\ &\quad + \|T_1(PT_1)^{n-1}x_n - x_n\| + \|x_{n+1} - x_n\| \\ &\le L\|x_{n+1} - x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\| \\ &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{20}$$

By (20), we get

$$\begin{aligned}\|T_1x_n - x_n\| &\le \|T_1(PT_1)^{n-1}x_n - x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1x_n\| \\ &\le \|T_1(PT_1)^{n-1}x_n - x_n\| + L\|T_1(PT_1)^{n-2}x_n - x_n\| \\ &\longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Thus $\lim_{n\to\infty} \|T_1x_n - x_n\| = 0$. Next, we want to prove (ii). Since

$$\|z_n - x_n\| \le a_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \delta_n\|u_n - x_n\| \longrightarrow 0 \text{ as } n \to \infty,$$

we obtain

$$\begin{aligned}\|T_2(PT_2)^{n-1}x_{n+1} - x_{n+1}\| &\le \|T_2(PT_2)^{n-1}x_{n+1} - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}z_n - x_n\| + \|x_{n+1} - x_n\| \\ &\le L\|x_{n+1} - x_n\| + L\|z_n - x_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\| \\ &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|T_2x_n - x_n\| &\le \|T_2(PT_2)^{n-1}x_n - x_n\| + \|T_2(PT_2)^{n-1}x_n - T_2x_n\| \\ &\le \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\| \\ &\quad + L\|T_2(PT_2)^{n-2}x_n - x_n\| \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Thus $\lim_{n\to\infty} \|T_2x_n - x_n\| = 0$, so (ii) is obtained. Since

$$\begin{aligned}\|y_n - x_n\| &\le b_n\|T_2(PT_2)^{n-1}z_n - x_n\| + c_n\|T_1(PT_1)^{n-1}x_n - x_n\| \\ &\quad + \sigma_n\|v_n - x_n\| \longrightarrow 0\end{aligned}$$

and $\|T_3(PT_3)^{n-1}y_n - x_n\| \longrightarrow 0$ as $n \to \infty$, we obtain

$$\begin{aligned}\|T_3(PT_3)^{n-1}x_n - x_n\| &\le \|T_3(PT_3)^{n-1}y_n - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - x_n\| \\ &\le L\|y_n - x_n\| + \|T_3(PT_3)^{n-1}y_n - x_n\| \\ &\longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Thus

$$\begin{aligned}\|T_3(PT_3)^{n-1}x_{n+1} - x_{n+1}\| &\le \|T_3(PT_3)^{n-1}x_{n+1} - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}y_n - x_n\| + \|x_{n+1} - x_n\| \\ &\le L\|x_{n+1} - x_n\| + L\|y_n - x_n\| + \|T_3(PT_3)^{n-1}y_n - x_n\| \\ &\quad + \|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

It follows that,

$$\begin{aligned}\|T_3x_n - x_n\| &\leq \|T_3(PT_3)^{n-1}x_n - x_n\| + \|T_3(PT_3)^{n-1}x_n - T_3x_n\| \\ &\leq \|T_3(PT_3)^{n-1}x_n - x_n\| + L\|T_3(PT_3)^{n-2}x_n - x_n\| \\ &\longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Hence (iii) is satisfied. □

Theorem 2.3. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1$, $l_n \geq 1, m_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. Assume that T_1, T_2, T_3 are uniformly L-Lipschitzian. If one of $T_i (i = 1, 2, 3)$ is a completely continuous and one of the following conditions (C1)-(C5) is satisfied:*

(C1) $0 < \lim_{n\to\infty}\inf a_n \leq \lim_{n\to\infty}\sup(a_n + \delta_n) < 1$,
$0 < \lim_{n\to\infty}\inf b_n \leq \lim_{n\to\infty}\sup(b_n + c_n + \sigma_n) < 1$, *and*
$0 < \lim_{n\to\infty}\inf \alpha_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(C2) $0 < \lim_{n\to\infty}\inf b_n, \lim_{n\to\infty}\inf c_n \leq \lim_{n\to\infty}\sup(b_n + c_n + \sigma_n) < 1$, *and*
$0 < \lim_{n\to\infty}\inf \alpha_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(C3) $0 < \lim_{n\to\infty}\inf b_n \leq \lim_{n\to\infty}\sup(b_n + c_n + \sigma_n) < 1$, *and*
$0 < \lim_{n\to\infty}\inf \alpha_n, \lim_{n\to\infty}\inf \gamma_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(C4) $\lim_{n\to\infty}\inf b_n > 0$, *and* $0 < \lim_{n\to\infty}\inf a_n \leq \lim_{n\to\infty}\sup(a_n + \delta_n) < 1$, *and*
$0 < \lim_{n\to\infty}\inf \alpha_n, \lim_{n\to\infty}\inf \beta_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

(C5) $0 < \lim_{n\to\infty}\inf \alpha_n, \lim_{n\to\infty}\inf \beta_n, \lim_{n\to\infty}\inf \gamma_n \leq \lim_{n\to\infty}\sup(\alpha_n + \beta_n + \gamma_n + \rho_n) < 1$.

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ defined as in (1) converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. Suppose one of the conditions (C1)-(C5) is satisfied. By Lemma 2.2, we obtain $\lim_{n\to\infty}\|T_ix_n - x_n\| = 0$ for $i = 1, 2, 3$. Assume one of T_1, T_2 and T_3 says T_1 is completely continuous. Since $\{x_n\}$ is a bounded sequence in C, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T_1x_{n_k}\}$ converges to $q \in C$. Since $\|x_{n_k} - q\| \leq \|T_1x_{n_k} - x_{n_k}\| + \|T_1x_{n_k} - q\|$, we get $\lim_{k\to\infty}\|x_{n_k} - q\| = 0$. Thus $\{x_{n_k}\}$ converges to $q \in C$. By continuity of T_i, we have $T_ix_{n_k} \to T_iq$ as $k \to \infty$. Since

$$\|T_iq - q\| \leq \|T_ix_{n_k} - T_iq\| + \|T_ix_{n_k} - x_{n_k}\| + \|x_{n_k} - q\| \to 0 \text{ as } k \to \infty,$$

we obtain $T_iq = q$ $(i = 1, 2, 3)$. Thus $q \in F$. By Lemma 2.1 (i), $\lim_{n\to\infty}\|x_n - q\|$ exists. This implies $\lim_{n\to\infty}\|x_n - q\| = 0$. By Lemma 2.1, we have

$$\|T_1(PT_1)^{n-1}x_n - x_n\| \to 0 \text{ and } \|T_2(PT_2)^{n-1}z_n - x_n\| \to 0 \text{ as } n \to \infty.$$

It follows that

$$\|y_n - x_n\| \leq b_n\|T_2(PT_2)^{n-1}z_n - x_n\| + c_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \sigma_n\|v_n - x_n\|$$

$\longrightarrow 0$ and $\|z_n - x_n\| \leq a_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \delta_n\|u_n - x_n\| \to 0$ as $n \to \infty$. These imply $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. □

Remark 2.4. In Theorem 2.3, assume that T_1, T_2 and T_3 are asymptotically nonexpansive self-mappings of C such that one of them is completely continuous and thus T_1, T_2, T_3 are uniformly

L-Lipschitzian and $\delta_n = \sigma_n = \rho_n \equiv 0$. We obtain the following results.

(1) If one of the conditions $(C1) - (C5)$ is satisfied, then the sequences $\{x_n\}$, $\{y_n\}, \{z_n\}$ defined as in (2) converge strongly to a common fixed point of T_1, T_2 and T_3.

(2) If $T := T_1 = T_2 = T_3$ and one of the conditions $(C1) - (C5)$ is satisfied, then we obtain the results of Nilsrakoo and Saejung [8].

(3) If $T := T_1 = T_2 = T_3$ and one of the conditions $(C1), (C2), (C4)$ is satisfied and $\gamma_n \equiv 0$, then we obtain the results of Suantai [16].

(4) If $T := T_1 = T_2 = T_3$ with condition $(C1)$ is satisfied and $c_n = \beta_n = \gamma_n \equiv 0$, then we obtain the results of Xu and Noor [18].

(5) If the condition $(C5)$ is satisfied and $a_n = b_n = c_n \equiv 0$, then the sequence $\{x_n\}$ defined as in (3) converges strongly to a common fixed point of T_1, T_2 and T_3.

The mapping $T : C \to X$ with $F(T) \neq \emptyset$ is said to satisfy *Condition A* [14] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - q\| : q \in F(T)\}$. As Tan and Xu [17] pointed out, the Condition A is weaker than the compactness of C.

The following result gives a strong convergence theorem for asymptotically quasi-nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Condition A.

Theorem 2.5. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1$, $l_n \geq 1, m_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\sigma_n < \infty$, $\sum_{n=1}^{\infty}\rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. Suppose T_1 satisfies Condition A and T_2, T_3 are uniformly L-Lipschitzian and one of the conditions (C1)-(C5) in Theorem 2.3 is satisfied. Then the sequence $\{x_n\}$ defined as in (1) converges strongly to a common fixed point of T_1, T_2 and T_3.*

Proof. Let $q \in F$. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - q\|$ exists. Thus $\{x_n - q\}$ is bounded. Then there is a constant H such that $\|x_n - q\| \leq H$ for all $n \geq 1$. This together with (6), we have

$$\|x_{n+1} - q\| \leq \|x_n - q\| + D_n, \tag{21}$$

where $D_n = KH\big((k_n - 1) + (l_n - 1) + (m_n - 1)\big) + K(\delta_n + \sigma_n + \rho_n) < \infty$ for all $n \geq 1$. By Lemma 2.2, we have $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ $(i = 1, 2, 3)$. Since T_1 satisfies Condition A, we obtain $\lim_{n\to\infty} d(x_n, F(T_1)) = 0$. Next, we want to show $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, F(T_1)) = 0$ and $\sum_{n=1}^{\infty} D_n < \infty$, for any $\epsilon > 0$, there exists a positive integer n_0 such that $d(x_n, F(T_1)) < \epsilon/4$ and $\sum_{k=n_0}^{n} D_k < \epsilon/2$ for all $n \geq n_0$. Now, let $n \in \mathbb{N}$ be such that $n \geq n_0$. Then we can find $q^* \in F$ such that $\|x_n - q^*\| < \epsilon/4$. This implies by (21) that for $m \geq 1$,

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q^*\| + \|x_n - q^*\| \\
&\leq 2\|x_n - q^*\| + \sum_{k=n}^{n+m-1} D_k \\
&= 2\|x_n - q^*\| + \sum_{k=n_0}^{n+m-1} D_k < 2\Big(\frac{\epsilon}{4}\Big) + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence and so it is convergent. Let $\lim_{n\to\infty} x_n = p$. Since $d(x_n, F(T_1)) \to 0$ as $n \to \infty$, it follows that $d(p, F(T_1)) = 0$ and hence $p \in F(T_1)$. Next, we want to show $p \in F(T_2) \cap F(T_3)$. Since T_2, T_3 are uniformly L-Lipschitzian and by Lemma 2.2, we obtain

$$\begin{aligned} \|T_i p - p\| &\leq \|T_i x_n - T_i p\| + \|T_i x_n - x_n\| + \|x_n - p\| \\ &\leq L\|x_n - p\| + \|T_i x_n - x_n\| + \|x_n - p\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $T_i p = p$ $(i = 2, 3)$. Therefore $p \in F$. □

In the next result, we prove weak convergence for the iterative scheme (1) for asymptotically quasi-nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, T_3 : C \to X$ be asymptotically quasi-nonexpansive mappings with respect to sequences $\{k_n\}, \{l_n\}, \{m_n\}$, respectively, such that $F \neq \emptyset, k_n \geq 1, l_n \geq 1, m_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(m_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\sigma_n\}, \{\rho_n\}$ be real sequences in $[0, 1]$ such that $a_n + \delta_n$, $b_n + c_n + \sigma_n$ and $\alpha_n + \beta_n + \gamma_n + \rho_n$ are in $[0, 1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ and let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in C. Suppose T_1, T_2, T_3 are uniformly L-Lipschitzian and $I - T_i$ $(i = 1, 2, 3)$ is demiclosed at 0. If one of the following conditions (C1)-(C5) in Theorem 2.3 is satisfied, then the sequence $\{x_n\}$ defined as in (1) converges weakly to a common fixed point of T_1, T_2 and T_3.*

Proof. Assume one of the conditions (C1)-(C5) is satisfied. By Lemma 2.1 and Lemma 2.2, we have $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ $(i = 1, 2, 3)$. Since X is uniformly convex and $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_n \to u$ weakly as $n \to \infty$. Since $I - T_i$ is demiclosed at 0, we obtain $u \in F$. Suppose subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. Also, since $I - T_i (i = 1, 2, 3)$ is demiclosed at 0, we have u and $v \in F$. By Lemma 2.1, we obtain $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 1.4 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. □

References

[1] A. Bnouhachem, M.A. Noor, Th.M. Rassias, Three-steps iterative algorithms for mixed variational inequalities, *Appl. Math. Comput.* 183: 436-446 (2006).

[2] H. Fukhar-ud-din, S.H. Khan, Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications, *J. Math. Anal. Appl.* 328: 821-829 (2007).

[3] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* 35: 171-174 (1972).

[4] R. Glowinski, P. Le Tallec, Augmented Lagrangian and Operator-Splittting Methods in Nonlinear Mechanics, SIAM, Philadelphia, 1989.

[5] S.H. Khan, W.Takahashi, Approximating common fixed points of two asymptotically nonexpansive mappings, *Sci. Math. Jpn.* 53: 133-138 (2001).

[6] S.H. Khan, H. Fukhar-ud-din, Weak and strong convergence of a scheme with errors for two nonexpansive mappings, *Nonlinear Anal.* 8: 1295-1301 (2005).

[7] K. Nammanee, M.A. Noor, S. Suantai, Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.*, in press.

[8] W. Nilsrakoo and S. Saejung, A new three-step fixed point iteration scheme for asymptotically nonexpansive mappings, *J. Appl. Math. Comput.* 181: 1026-1034 (2006).

[9] M.A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.* 251: 217-229 (2000).

[10] M.A. Noor, Three-step iterative algorithms for multivalued quasi variational inclusions, *J. Math. Anal. Appl.* 255: 589-604 (2001).

[11] M.A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.* 152: 199-277 (2004).

[12] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* 73: 591-597 (1967).
[13] L. Qihou, Iteration sequences for asymptotically quasi-nonexpansive mapping with an error member of uniform convex Banach space, *J. Math. Anal. Appl.* 266: 468-471 (2002).
[14] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.* 44: 375-380 (1974).
[15] N. Shahzad, A. Udomene, Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces, *Fixed Point Theory Appl.* (2006), article ID 18909, 10 pp.
[16] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* 311: 506-517 (2005).
[17] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* 178: 301-308 (1993).
[18] B.L. Xu, M. Aslam Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* 267: 444-453 (2002).

[1,2]DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER, FACULTY OF SCIENCE,, UBON RATCHATHANI UNIVERSITY, UBON RATCHATHANI 34190, THAILAND.

[1,2]CENTRE OF EXCELLENCE IN MATHEMATICS, CHE, SI AYUTTHAYA RD., BANGKOK 10400, THAILAND.
Email address: utith@sci.ubu.ac.th(U. Inprasit) and hathaikarn@sci.ubu.ac.th (H. Wattanataweekul)