T. Yuting et al., J. Nonlinear Anal. Optim. Vol. 15(1) (2024), 13-22

Journal of Nonlinear Analysis and Optimization
Volume 15(1) (2024)
http://ph03.tci-thaijo.org
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

BEST PROXIMITY POINT AND FIXED POINT THEOREMS IN COMPLEX VALUED RECTANGULAR b-METRIC SPACES

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ABSTRACT. The aim of this paper, by using the concept of continuity of $\phi : [0,\infty)^2 \to [0,\infty)^2$ which satisfies $\phi(t) \prec t$ and $\phi(0) = 0$ to define some contraction condition of T introduced by G. Meena [12], we prove the unique best proximity point of A and fixed point of T in complex valued rectangular b-metric space. Our results extend and improve the results of G. Meena [12], and many others.

KEYWORDS: best proximity point, rectangular b-metric spaces, rectangular complex valued b-metric spaces.
AMS Subject Classification: : 46C05, 47D03, 47H09, 47H10, 47H20.

1. Introduction

Fixed point theorems in metric spaces were introduced in 1906 [8] by Fréchet. After that, many mathematicians studied and proved the existence theorem of fixed points for using the Banach contraction principle in metric spaces and every generalized metric space [6, 7, 13] and [15].

The notion of b-metric spaces was introduced in 1989 by Bakhtin [3]. After, many mathematicians extended the fixed point theorems from metric spaces to b-metric spaces, for example in [1, 2]

In 2000, A. Branciari [5], he gave a fixed point theorem related to the contraction mapping principle of Banach and Caccioppoli; here we have considered generalized metric spaces, that is, metric spaces with the triangular inequality replaced by similar ones, which involve four or more points instead of three.

In 2011, A. Azam, B. Fisher and M. Khan [2] defined the definition of the notion of complex valued metric spaces and proved the common fixed point theorems in complex valued metric spaces of a pair of mappings satisfying a contractive condition.

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Article history : Received 13 June 2023 Accepted 13 March 2024.

In the same year, S. Bhatt, S. Chaukiyal and R. C. Dimri [4] proved a common fixed point theorem for weakly compatible maps in complex valued metric spaces without using the notion of continuity.

In 2015, O. Ege [6] introduced complex valued rectangular b-metric spaces. We prove an analogue of the Banach contraction principle and prove a different contraction principle with a new condition and a fixed point theorem in this space.

In 2018, G. Meena [12] introduced the best proximity points for non-self mappings between two subsets in the setting of complex valued rectangular metric spaces by using the concept of P-property.

The aim of this paper, we introduce [6, 12], study and suppose some contractive conditions and prove the best proximity point result in b-metric space. Therefore, our results are comprehensive the results in [10].

2. Preliminaries

In this section, we let X be a nonempty set and recalled some definitions and lemmas for using in Section 3.

Definition 2.1. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric if for $x, y, z \in X$, the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.
The pair (X, d) is called a metric space, and d is called a metric on X.

Next, we suppose the definition of b-metric space, this space is more generalized than metric spaces.

Definition 2.2. [3] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied.
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

The following is an example of b-metric spaces.

Example 2.3. [3] Let (X, d) be a metric space. The funcion $\rho(x, y)$ is defined by $\rho(x, y) = (d(x, y))^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

In 2000, A. Branciari [5] presented the notion of rectangular metric space as follows:

Definition 2.4. [5] Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is called a rectangular metric on X if for any $x, y \in X$ and all distinct points $u, v \in X - \{x, y\}$, it satisfies the following conditions:
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, u) + d(u, v) + d(v, y)$.
In this case, the pair (X, d) is called a rectangular metric space.

There is a completeness property in real numbers but on order relation is not welll-defined in complex numbers. Before giving the definition of complex valued metric spaces and complex-valued b-metric spaces, we define partial order in complex numbers (see [11]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order relation $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

This means that we would have $z_1 \precsim z_2$ if and only if one of the following conditions holds:
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
If one of the conditions (ii), (iii), and (iv) holds, then we write $z_1 \prec z_2$. From the above partial order relation, we have the following remark:

Remark 2.5. We can easily check the following:
(i) If $a, b \in \mathbb{R}, 0 \leq a \leq b$ and $z_1 \precsim z_2$ then $az_1 \precsim bz_2, \forall z_1, z_2 \in \mathbb{C}$.
(ii) If $0 \precsim z_1 \prec z_2$ then $|z_1| < |z_2|$.
(iii) If $z_1 \precsim z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$.
(iv) If $z \in \mathbb{C}$, for $a, b \in \mathbb{R}$ and $a \leq b$, then $az \precsim bz$.

A b-metric on a b-metric space is a function having real value. Based on the definition of partial order on complex numbers, real-valued b-metric can be generalized into complex-valued b-metric as folllows:

Definition 2.6. [2] Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \precsim d(x, y) + d(y, z)$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Next, we give the definition of complex valued b-metric space.

Definition 2.7. [13] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:
(i) $0 \precsim d(x, y)$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, z) \precsim s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a complex valued b-metric space. We see that if $s = 1$ then (X, d) is a complex valued metric space, which is defined in Definition 2.6. The following example is an example of complex valued b-metric space.

Example 2.8. [13] Let $X = \mathbb{C}$. Define the mapping $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbb{C}, d)$ is a complex valued b-metriic space with $s = 2$.

From A. Branciari [5] and [13], we can define the notion of rectangular b-metric space as follows:

Definition 2.9. [6] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued rectangular b-metric on X if for any $x, y \in X$ and all distinct points $u, v \in X - \{x, y\}$, it satisfies the following conditions:
(i) $0 \preccurlyeq d(x, y)$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$;
(iv) $d(x, z) \preccurlyeq s[d(x, u) + d(u, v) + d(v, y)]$.
In this case, the pair (X, d) is called a complex valued rectangular b-metric space.

Example 2.10. [6] Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $B = \mathbb{Z}^+$ and $d : X \times X \to \mathbb{C}$ defined as follows:

$$g(x, y) = d(y, x)$$

for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0 & \text{if } \; x = y \\ 2t & \text{if } \; x, y \in A \\ \frac{t}{2n} & \text{if } \; x \in A \text{ and } y \in \{2, 3\} \\ t & \text{otherwise,} \end{cases}$$

where $t > 0$ is a constant. Then (X, d) is a complex valued rectangular b-metric space with coefficient $s = 2 > 1$.

Definition 2.11. [6] Let (X, d) be a complex valued rectangular b-metric space.
(i) A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.$$

(ii) A point $x \in X$ is called limit point of a set A if for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - x) \neq \emptyset$
(iii) A subset $A \subseteq X$ is open if each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is closed if each limit point of A is contained in A.

Definition 2.12. [6] Let (X, d) be complex valued rectangular b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.
(i) The sequence $\{x_n\}$ is converges to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x) \prec r$. Thus x is the limit of (x_n) and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(ii) The sequence $\{x_n\}$ is said to be a Cauchy sequence if for ever $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.
(iii) If for every Cauchy sequence in$X$ is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 2.13. [6] *Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.14. [6] *Let (X, d) be a complex valued rectangular b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Definition 2.15. [12] The max function for complex numbers with partial order relation $\preccurlyeq$ is defined as
(i) $\max\{z_1, z_2\} = z_2 \Rightarrow z_1 \preccurlyeq z_2$;

(ii) $z_1 \precsim \max\{z_1, z_2\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.
On the similar lines Singh et al. [14] defined min function as
(i) $\min\{z_1, z_2\} = z_1 \Rightarrow z_1 \precsim z_2$;
(ii) $\min\{z_1, z_2\} \precsim z_3 \Rightarrow z_1 \precsim z_3$ or $z_2 \precsim z_3$. Now we introduce the best proximity point and some related concept in complex valued rectangular metric space.

Definition 2.16. [16] Let A and B be two nonempty bounded subsets of a complex valued rectangular b-metric space (X, d). Then $\{d(x, y) : x \in A, y \in B\}$ is always bounded below by $z_0 = 0 + 0i$ and hence, $\inf\{d(x, y) : x \in A, y \in B\}$ exists. Here we define

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$
$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

From the above definition, it is clear that for every $x \in A_0$ there exists $y \in B_0$ such that $d(x, y) = d(A, B)$ and conversely, for every $y \in B_0$ there exists $x \in A_0$ such that $d(x, y) = d(A, B)$.

Definition 2.17. [16] Let A and B be two nonempty bounded subsets of a complex valued rectangular b-metric space (X, d) and $T : A \to B$ be a non-self-mapping. A point $x \in A$ is called a best proximity point of T if $d(x, Tx) = d(A, B)$.

The definition of P-property was introduced in [17]. Now we define them in complex valued rectangular b-metric space.

Definition 2.18. [17] Let A and B be two nonempty subsets of a complex valued rectangular b-metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if, for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ such that

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

3. Main Results

In this section, we consider the context of Matkowski [9] the function $\phi : [0, \infty)^2 \to [0, \infty)^2$ such that $\phi(t) \prec t$ and $\phi(0) = 0$ [where $t = (t_1, t_2) \in [0, \infty)^2$]. We denote Φ the family of function of ϕ.

Theorem 3.1. *Let A and B be two nonempty bounded subsets of a complete complex valued rectangular b-metric space (X, d) with a pair (A, B) satisfies the P-property. Let a continuous mapping $T : A \to B$ with $T(A_0) \subset B_0$, where A_0 is nonempty, if there exist $L > 0$ and a continuous $\phi \in \Phi$, such that*

$$d(Tx, Ty) \precsim k\phi\Big(\max\Big\{ \frac{(d(x,Ty)-d(A,B))(d(y,Tx)-d(A,B))(d(x,Tx)+d(y,Ty)-2d(A,B))}{1+d(x,y)}, d(x, y)\Big\}\Big)$$
$$+L \min\Big\{(d(x, Tx)-d(A, B)), (d(y, Ty)-d(A, B)), (d(x, Ty)-d(A, B)), (d(y, Tx)-d(A, B))\Big\} \tag{3.1}$$

for all $x, y \in X$, where $0 < k < \frac{1}{s} \leq 1$. Then T has a unique best proximity point in A.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subset B_0$ we have $Tx_0 \in B_0$ then there exists $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Again $Tx_1 \in B_0$, then there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

By continuing this process, we can form a sequence $\{x_n\}$ in A_0, with

$$d(x_{n+1}, Tx_n) = d(A, B), \ \forall n \in \mathbb{N}.$$

From a pair (A, B) satisfying P-property, we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0-1} = x_{n_0}$, then we have

$$d(x_{n_0}, Tx_{n_0-1}) = d(A, B) = d(x_{n_0-1}, Tx_{n_0-1}). \tag{3.2}$$

This proof is complete.

Assume that $x_{n-1} \neq x_n$, for all $n \in \mathbb{N}$. We replace $x = x_{n-1}$ and $y = x_n$ in (3.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preccurlyeq k\phi\Big(\max\Big\{ \tfrac{(d(x_{n-1},Tx_n)-d(A,B))(d(x_n,Tx_{n-1})-d(A,B))(d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)-2d(A,B))}{1+d(x_{n-1},x_n)}, \\ &\qquad d(x_{n-1}, x_n)\Big\}\Big) \\ &\quad + L\min\Big\{(d(x_{n-1}, Tx_{n-1}) - d(A, B)), (d(x_n, Tx_n) - d(A, B)), \\ &\qquad (d(x_{n-1}, Tx_n) - d(A, B)), (d(x_n, Tx_{n-1}) - d(A, B))\Big\}. \end{aligned}$$

It follows that,

$$d(x_n, x_{n+1}) \preccurlyeq k\phi\left(d(x_{n-1}, x_n)\right).$$

From the definition of ϕ, we have

$$d(x_n, x_{n+1}) \preccurlyeq kd(x_{n-1}, x_n).$$

It follows that

$$d(x_n, x_{n+1}) \preccurlyeq kd(x_{n-1}, x_n) \preccurlyeq k^2 d(x_{n-2}, x_{n-1}) \preccurlyeq \cdots \preccurlyeq k^n d(x_0, x_1).$$

For any $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\preccurlyeq s[d(x_n, x_{n-1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\preccurlyeq sd(x_n, x_{n-1}) + sd(x_{n+1}, x_{n+2}) + s^2[d(x_{n+2}, x_{n+3}) \\ &\quad + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_m)] \\ &\preccurlyeq sd(x_n, x_{n-1}) + sd(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) \\ &\quad + s^3[d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) + d(x_{n+6}, x_m)] \\ &\preccurlyeq sd(x_n, x_{n-1}) + sd(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) + s^2 d(x_{n+3}, x_{n+4}) \\ &\quad + s^3 d(x_{n+4}, x_{n+5}) + s^3 d(x_{n+5}, x_{n+6}) + \cdots \\ &\quad + s^{\frac{(m-n-1)}{2}}[d(x_{n+(m-n-3)}, x_{n+(m-n-2)}) + d(x_{n+(m-n-2)}, x_{n+(m-n-1)}) \\ &\quad + d(x_{n+(m-n-1)}, x_m)] \\ &\preccurlyeq \Big[sk^n + sk^{n+1} + s^2k^{n+2} + s^2k^{n+3} + s^3k^{n+4} + s^3k^{n+5} + \cdots \\ &\quad + s^{\frac{(m-n-1)}{2}} k^{m-1}\Big] d(x_0, x_1) \\ &\preccurlyeq \Big[(sk)^n + (sk)^{n+1} + (sk)^{n+2} + (sk)^{n+3} + (sk)^{n+4} + (sk)^{n+5} + \cdots + \\ &\quad (sk)^{n+(m-n-1)}\Big] d(x_0, x_1) \\ &= (sk)^n[1 + (sk) + (sk)^2 + (sk)^3 + (sk)^4 + \cdots + (sk)^{m-n-1}] d(x_0, x_1) \\ &\preccurlyeq (sk)^n[1 + (sk) + (sk)^2 + (sk)^3 + (sk)^4 + \cdots] d(x_0, x_1) \\ &= \frac{(sk)^n}{1 - sk} d(x_0, x_1) \to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Thus $\{x_n\}$ is a cauchy sequence in A. Since X is complete, so there exists $u \in X$ such that $x_n \to u \in X$. Since A is closed subset of X, we have $u \in A$. Next, we show that u is a best proximity point in A. Using the rectangular b-metric, we get

$$\begin{aligned} d(u,Tu) &\precsim s[d(u,x_{n+1}) + d(x_{n+1},Tx_n) + d(Tx_n,Tu)] \\ &\precsim sd(u,x_{n+1}) + sd(x_{n+1},Tx_n) \end{aligned}$$

$$+sk\phi\Big(\max\Big\{\tfrac{(d(x_n,Tu)-d(A,B))(d(u,Tx_n)-d(A,B))(d(x_n,Tx_n)+d(u,Tu)-2d(A,B))}{1+d(x_n,u)}, d(x,y)\Big\}\Big)$$
$$+L\min\Big\{(d(x_n,Tx_n)-d(A,B)),(d(u,Tu)-d(A,B)),(d(x_n,Tu)-d(A,B)),$$
$$(d(u,Tx_n)-d(A,B))\Big\}$$

From (3.3), taking $n \to \infty$, we get

$$d(u,Tu) \precsim d(A,B).$$

Since $u \in A, Tu \in B$ and the definition of $d(A,B)$, it follows that

$$d(u,Tu) = d(A,B).$$

Hence, u is the best proximity point of T.

Finally, we show that u is a unique best proximity point of T. Let $u^* \in A$ is another best proximity point of T. Then

$$d(u^*,Tu^*) = d(A,B).$$

Assume $u \neq u^*$, by using P-property, we have

$$d(u,u^*) = d(Tu,Tu^*)$$
$$\precsim k\phi\Big(\max\Big\{\tfrac{(d(u,Tu^*)-d(A,B))(d(u^*,Tu)-d(A,B))(d(u,Tu)+d(u^*,Tu^*)-2d(A,B))}{1+d(u,u^*)}, d(u,u^*)\Big\}\Big)$$
$$+L\min\Big\{(d(u,Tu)-d(A,B)),(d(u^*,Tu^*)-d(A,B)),(d(u,Tu^*)-d(A,B)),$$
$$(d(u^*,Tu)-d(A,B))\Big\}$$
$$\begin{aligned} &\precsim k\phi(d(u,u^*)) \\ &\precsim kd(u,u^*). \end{aligned}$$

A contradiction. Hence, $d(u,u^*) = 0$ or $u = u^*$ is a unique best proximity point of T. □

From Theorem 3.1, we have the parallel result with the result of G. Meena [12], as follows.

Corollary 3.2. [12] *Let A and B be two nonempty bounded subsets of a complete complex valued rectangular metric space (X,d) with a pair (A,B) satisfies the P-property. Let a continuous mapping $T : A \to B$ with $T(A_0) \subset B_0$, where A_0 is nonempty, if there exist $L > 0$ and a continuous $\phi \in \Phi$, such that*

$$d(Tx,Ty) \precsim k\phi\Big(\max\Big\{\tfrac{(d(x,Ty)-d(A,B))(d(y,Tx)-d(A,B))(d(x,Tx)+d(y,Ty)-2d(A,B))}{1+d(x,y)}, d(x,y)\Big\}\Big)$$
$$+L\min\Big\{(d(x,Tx)-d(A,B)),(d(y,Ty)-d(A,B)),(d(x,Ty)-d(A,B)),$$
$$(d(y,Tx)-d(A,B))\Big\},$$

for all $x, y \in X$, where $0 < k < 1$. Then T has a unique best proximity point in A.

Theorem 3.3. *Let (X,d) be a complete complex valued rectangular b-metric space. Let a mapping $T : X \to X$ and a continuous $\phi \in \Phi$, such that*

$$d(Tx,Ty) \precsim k\phi\left(\max\left\{\frac{d(x,Ty)d(y,Tx)(d(x,Tx)+d(y,Ty))}{1+d(x,y)}, d(x,y)\right\}\right)$$

$$+L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\},$$

for all $x,y\in X$, and k is any real number with $0<k<\frac{1}{s}\leq 1$. Then T has a unique fixed point in X.

Proof. Let $x_0\in X$, from $T:X\to X$ there exists $x_1\in X$ such that $x_1=Tx_0$. From $x_1\in X$ there exists $x_2\in X$ such that $x_2=Tx_1$. By the following method, we have a sequence $\{x_n\}\subseteq X$ such that $x_{n+1}=Tx_n$. Consider,

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n)\\ &\preccurlyeq k\phi\Big(\max\Big\{\tfrac{d(x_{n-1},Tx_n)d(x_n,Tx_{n-1})(d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n))}{1+d(x_{n-1},x_n)},d(x_{n-1},x_n)\Big\}\Big)\\ &\quad +L\min\Big\{d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,Tx_{n-1}\Big\}. \end{aligned}$$

It follows that,

$$d(x_n,x_{n+1})\preccurlyeq k\phi\left(d(x_{n-1},x_n)\right).$$

From the definition of ϕ, we have

$$d(x_n,x_{n+1})\preccurlyeq kd(x_{n-1},x_n).$$

It follows that

$$d(x_n,x_{n+1})\preccurlyeq kd(x_{n-1},x_n)\preccurlyeq k^2d(x_{n-2},x_{n-1})\preccurlyeq\cdots\preccurlyeq k^nd(x_0,x_1).$$

For any $m>n$, we have

$$\begin{aligned} d(x_n,x_m) &\preccurlyeq s[d(x_n,x_{n-1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_m)]\\ &\preccurlyeq sd(x_n,x_{n-1})+sd(x_{n+1},x_{n+2})+s^2[d(x_{n+2},x_{n+3})+d(x_{n+3},x_{n+4})\\ &\quad +d(x_{n+4},x_m)]\\ &\preccurlyeq sd(x_n,x_{n-1})+sd(x_{n+1},x_{n+2})+s^2d(x_{n+2},x_{n+3})+s^2d(x_{n+3},x_{n+4})\\ &\quad +s^3[d(x_{n+4},x_{n+5})+d(x_{n+5},x_{n+6})+d(x_{n+6},x_m)]\\ &\preccurlyeq sd(x_n,x_{n-1})+sd(x_{n+1},x_{n+2})+s^2d(x_{n+2},x_{n+3})+s^2d(x_{n+3},x_{n+4})\\ &\quad +s^3d(x_{n+4},x_{n+5})+s^3d(x_{n+5},x_{n+6})+\cdots\\ &\quad +s^{\frac{(m-n-1)}{2}}\Big[d(x_{n+(m-n-3)},x_{n+(m-n-2)})+d(x_{n+(m-n-2)},x_{n+(m-n-1)})\\ &\quad +d(x_{n+(m-n-1)},x_m)\Big]\\ &\preccurlyeq \Big[sk^n+sk^{n+1}+s^2k^{n+2}+s^2k^{n+3}+s^3k^{n+4}+s^3k^{n+5}+\cdots\\ &\quad +s^{\frac{(m-n-1)}{2}}k^{m-1}\Big]d(x_0,x_1)\\ &\preccurlyeq [(sk)^n+(sk)^{n+1}+(sk)^{n+2}+(sk)^{n+3}+(sk)^{n+4}+(sk)^{n+5}+\cdots\\ &\quad +(sk)^{n+(m-n-1)}]d(x_0,x_1)\\ &= (sk)^n[1+(sk)+(sk)^2+(sk)^3+(sk)^4+\cdots+(sk)^{m-n-1}]d(x_0,x_1)\\ &\preccurlyeq (sk)^n[1+(sk)+(sk)^2+(sk)^3+(sk)^4+\cdots]d(x_0,x_1)\\ &= \frac{(sk)^n}{1-sk}d(x_0,x_1)\to 0 \text{ as } m,n\to\infty. \end{aligned}$$

Thus $\{x_n\}$ is a cauchy sequence in A. Since X is complete, so there exists $u\in X$ such that $x_n\to u\in X$. Since A is closed subset of X, we have $u\in A$. Next, we show that u is a fixed point of T. Using the rectangular b-metric, we get

$$\begin{aligned} d(u,Tu) &\preccurlyeq s[d(u,x_{n+1})+d(x_{n+1},Tx_n)+d(Tx_n,Tu)]\\ &\preccurlyeq sd(u,x_{n+1})+sd(x_{n+1},Tx_n) \end{aligned}$$

$$+sk\phi\left(\max\left\{\frac{d(x_n,Tu)d(u,Tx_n)(d(x_n,Tx_n)+d(u,Tu))}{1+d(x_n,u)},d(x,y)\right\}\right)$$
$$+L\min\{d(x_n,Tx_n),d(u,Tu),d(x_n,Tu),d(u,Tx_n)\} \tag{3.3}$$

From (3.3), taking $n\to\infty$, we get

$$d(u,Tu)\precsim skd(u,Tu).$$

Hence, u is fixed point of T.

Finally, we show that u is a unique fixed point of T. Let $u^*\in A$ is anotherfixed point of T. Then $u^*=Tu^*$. Assume $u\neq u^*$, consider

$$\begin{aligned} d(u,u^*) &= d(Tu,Tu^*)\\ &\precsim k\phi\left(\max\left\{\frac{d(u,Tu^*)d(u^*,Tu)(d(u,Tu)+d(u^*,Tu^*)}{1+d(u,u^*)},d(u,u^*)\right\}\right)\\ &\quad +L\min\{d(u,Tu),d(u^*,Tu^*),d(u,Tu^*),d(u^*,Tu)\}\\ &\precsim k\phi(d(u,u^*))\\ &\precsim kd(u,u^*).\end{aligned}$$

A contradiction. Hence, $u=u^*$ is a unique fixed point of T. □

Corollary 3.4. [12] *Let (X,d) be a complete complex valued rectangular b-metric space. Let a mapping $T:X\to X$ and a continuous $\phi\in\Phi$, such that*

$$\begin{aligned} d(Tx,Ty) &\precsim k\phi\left(\max\left\{\frac{d(x,Ty)d(y,Tx)(d(x,Tx)+d(y,Ty))}{1+d(x,y)},d(x,y)\right\}\right)\\ &\quad +L\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\},\end{aligned}$$

for all $x,y\in X$, and k is any real number with $0<k<\frac{1}{s}<1$. Then T has a unique fixed point in X.

4. Acknowledgements

The authors would like to thank Department of Mathematics, faculty of Science and Technology. Moreover, we would like to thank Thailand Science Research and Innovation for financial support.

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