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ON THE REALM OF WEAKLY NONEXPANSIVE MAPS ON QUASI-METRIC SPACES

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ABSTRACT. Let (X, d) be a metric space. There have appeared thousands of works on nonexpansive maps $f : X \to X$ and the way to get their fixed points. Recently, many maps of the type satisfying $d(fx, f^2x) \leq d(x, fx)$ on $x \in X$ appeared in the literature and they are called the weakly nonexpansive maps. There are a large number of fixed point theorems on weakly nonexpansive maps on metric spaces. Their proofs are different each other. In the present article, we trace the history of their applications to ordered fixed point theory. Certain new proofs are also given for known theorems.

KEYWORDS: fixed point, quasi-metric space, fixed point, (weak) contraction, (weakly) nonexpansive, (weakly) contractive.
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1. Prologue

Let (X, d) be a metric space. A (Banach) *contraction* $T : X \to X$ is a map satisfying

$$d(Tx, Ty) \leq \alpha\, d(x, y) \quad \text{for all } x, y \in X$$

with some $\alpha \in [0, 1)$. There have appeared thousands of articles related to the Banach contraction.

Recently, we introduced the Rus-Hicks-Rhoades (RHR) map or a *weak contraction* $T : X \to X$ satisfying

$$d(Tx, T^2x) \leq \alpha\, d(x, Tx) \quad \text{for all } x \in X$$

with some $\alpha \in [0, 1)$. See our recent works [35, 36, 38, 39].

For a long period, there have appeared thousands of another works on nonexpansive maps $T : X \to X$ satisfying

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all } x, y \in X.$$

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Such works were concerned with the existences of fixed points and the methods to find them.

Recently, as a generalization of the RHR maps, Golshan [17] in 2025 introduced a weakly nonexpansive map $T : X \to X$ satisfying

$$d(Tx, T^2x) \le d(x, Tx) \quad \text{for all } x \in X.$$

Note that any translation is a (weakly) nonexpansive map without fixed point. Moreover, there have appeared generalizations of weakly nonexpansive maps.

One of the well-known fixed point theorems for nonexpansive maps due to Kirk or Kirk-Browder-Göhde in 1965 was equivalently formulated in [32] and [33] by our well-known old Metatheorem. See also on the history of Metatheorem in [34]. Moreover Karapinar and Taş [23] showed a large number of maps extending the above classes.

Recall that the (Edelstein) *contractive map* $T : X \to X$ is the one satisfying

$$d(Tx, Ty) < d(x, y) \quad \text{for all } x, y \in X \text{ with } x \ne y.$$

In the present article, we define a *weakly contractive* map $T : X \to X$ satisfying

$$d(Tx, T^2x) < d(x, Tx) \quad \text{for all } x \in X \text{ with } x \ne Tx.$$

Our main aim in the present article is to collect the contents of articles concerning weakly nonexpansive maps, weakly contractive maps and their extensions on quasi-metric spaces.

Recall that the Banach contraction principle gives a unified proof for fixed points for the family of contractions on complete metric spaces. Moreover, recently we extended the principle to the weak contraction principle or the Rus-Hicks-Rhoades contraction principle for the family of RHR maps on complete quasi-metric spaces.

There are a large number of fixed point theorems on weakly nonexpansive maps on quasi-metric spaces. Their proofs are different each other. Therefore, some new principle for them giving unified proofs would be anticipated. At present, we have no such principle.

As a preparation of such principle, in the present paper, we collect the families of weakly nonexpansive maps and others.

Let T be a selfmap on a subset C of a Banach space E. Then Suzuki [47] defined Condition (C) for T as

$$\frac{1}{2}||x - Tx|| \le ||x - y|| \text{ implies } ||Tx - Ty|| \le ||x - y||$$

for all $x, y \in C$. There have appeared many papers related to Condition (C).

However, the Condition (C) implies that T is weakly nonexpansive, that is,

$$||Tx - T^2x|| \le ||x - Tx||$$

for all $x \in C$. Therefore, this condition can be applicable instead of Condition (C). There also appeared many papers on the so- called generalized (C) conditions,

This paper is organized as follows: Section 2 is preliminaries for quasi-metric spaces. In Section 3, we introduce the RHR contraction principle (Theorem P) and the basic Theorem A for weakly contractive maps. Based on Sections 2 and 3, in Section 4, we review history of weakly nonexpansive maps and their extensions with a large number of various examples. There we can found various types of their proofs which causes the difficulty of making a unified proof or a principle for them. Finally, Section 5 is for epilogue.

2. Preliminaries

Recall the following:

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $q : X \times X \to [0, \infty)$ verifying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $q(x, y) = q(y, x) = 0 \Longleftrightarrow x = y$;
(b) (triangle inequality) $q(x, z) \leq q(x, y) + q(y, z)$.

A *metric* on a set X is a quasi-metric d satisfying that for all $x, y \in X$,

(c) (symmetry) $d(x, y) = d(y, x)$.

Definition 2.2. ([6], [27]) Let (X, q) be a quasi-metric space.
(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $n > m > N$.
(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n > N$.
(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 2.3. ([6], [27]) Let (X, q) be a quasi-metric space.
(1) (X, q) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, q) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, q) is *complete* if every Cauchy sequence in X is convergent.

Definition 2.4. Let $f : X \to X$ be a selfmap. The *orbit* of f at $x \in X$ is the set

$$O_f(x) = \{x, fx, \cdots, f^n x, \cdots\}.$$

The space (X, q) is said to be *f-orbitally complete* if every right-Cauchy sequence in $O_f(x)$ is convergent in X. A selfmap f of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} f^n x = x_0 \Longrightarrow \lim_{n\to\infty} f^{n+1} x = f x_0$$

for any $x \in X$.

Remark 2.5. Definition 2.4 also works for a topological space X and a function $q : X \times X \to [0, \infty)$ such that $q(x, y) = 0$ implies $x = y$ for $x, y \in X$.

Every quasi-metric induces a metric, that is, if (X, q) is a quasi-metric space, then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \max\{q(x, y), q(y, x)\}$$

is a metric on X; see Jleli et al. [27].
The following is given in [27]:

Theorem 2.6. *A selfmap $T : X \to X$ of a quasi-metric space (X, q) has a fixed point $z \in X$ if and only if z is a fixed point of the selfmap T of the induced metric space (X, d).*

3. Our Basic Theorems on Quasi-metric Spaces

The following in [40] is a correct form of the weak contraction principle or the Rus-Hicks-Rhoades (RHR) contraction principle given previously in several papers, e.g. [36], [37].

Theorem P. *Let (X, q) be a quasi-metric space and let $f : X \to X$ be an RHR map for $0 \leq \alpha < 1$; that is,*

$$q(fx, f^2x) \leq \alpha\, q(x, fx) \text{ for every } x \in X,$$

such that X is f-orbitally complete, Then

(i) *for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} f^n x = x_0,$$

$$q(f^n x, x_0) \leq \frac{\alpha^n}{1-\alpha} q(x, fx), \quad n = 1, 2, \cdots ,$$

$$q(f^n x, x_0) \leq \frac{\alpha}{1-\alpha} q(f^{n-1}x, f^n x), \quad n = 1, 2, \cdots ,$$

and

(ii) *x_0 is a fixed point of f if and only if f is orbitally continuous at x_0.*

This was proved in [37] by analyzing a typical proof of the Banach contraction principle given by Art Kirk ([25], Theorem 2.2).=

The original Rus-Hicks-Rhoades theorem can be extended to the following consequence of the Caristi type fixed point theorem; see [37].

Corollary P.1. *Let f be a continuous selfmap of a complete quasi-metric space (M, q) satisfying*

$$q(fx, f^2x) \leq \alpha\, q(x, fx) \quad \text{for every } x \in M,$$

where $0 < \alpha < 1$. Then f has a fixed point and the statement (i) *of Theorem P holds.*

The following was given as a main result in [39] and its original version for metric spaces was given in [31]:

Theorem A. *Let f be a selfmap of a topological space X and $d : X \times X \to [0, \infty)$ a function such that $d(x, y) = 0$ implies $x = y$ for $x, y \in X$. If*

(i) *there exists a point $u \in X$ such that $\overline{O_f(u)}$ has a cluster point $\xi \in X$,*
(ii) *f satisfies*

$$d(fx, f^2x) < d(x, fx)$$

for all $x \in \overline{O_f(u)}$, $x \neq fx$.

Then ξ is a fixed point of f if and only if f is orbitally continuous at ξ and $f\xi$.

Remark 3.1. (1) The sufficiency of Theorem A was originally given Theorem 1 in [31] for metric spaces without using the symmetry. For a contractive map f, the condition (i) is needed in order to ensure that every such f possesses a fixed point (Rhoades [43], Theorem 2).

(2) The requirement (i) is implied by the compactness of X. In fact, if $\overline{O_f(u)}$ or X is compact, the condition (i) is not necessary.

(3) In Theorem A, if f is contractive, then f has a unique fixed point. Hence, we obtain Edelstein's theorem on contractive maps [14].

(4) In [31], we listed a large number of historically well-known consequences of Theorem A for metric spaces due to Pal-Maiti, Pal-Maiti-Achari, Rhoades, Wong, Meir-Keeler, Ćirić, Husain-Sehgal, and Taskovitz.

(5) Billy E. Rhoades [44] in 2007 noted that the original form of Theorem A (in Park [31]) and "other one contain as special cases a number of papers involving contractive conditions not covered by my Transactions paper [43]."

Corollary A.1. *Let f be a selfmap of a quasi-metric space (X, q). Assume that*

(i) *there exists a point $u \in X$ such that $\overline{O_f(u)}$ has a cluster point $\xi \in X$, and*
(ii) *f satisfies*

$$q(fx, f^2x) < q(x, fx)$$

for all $x \in O_f(u)$, $x \neq fx$.

Then ξ is a fixed point of f if and only if f is orbitally continuous at ξ and $f\xi$.

Corollary A.2. *Let (X, q) be a compact quasi-metric space and $f : X \to X$ be an orbitally continuous weakly contractive selfmap. Then f has a fixed point.*

4. History of weakly nonexpansive maps

In the metric fixed point theory, a large number of weakly contractive maps or weakly nonexpansive maps exist. In this section the numbers attached to Theorems, Definitions, etc. are the same to the original sources.

Edelstein [14] in 1962

Edelstein established a relative of the Banach contraction principle for contractive maps and state the following version of the Banach contraction principle:

Theorem 1. ([14]) *Let (X, ρ) be a metric space and $f : X \to X$ be a contractive map. If there exists $x \in X$ such that the sequence of iterates $f^n(x)$ has a cluster point $\zeta \in X$ (that is, $\zeta = \lim_{k\to\infty} f^{n_k}(x)$ for some subsequence of $\{f^n(x)\}$), then ζ is the unique fixed point of f.*

This follows from Theorem A. Note that f is continuous.

Theorem 2. ([14]) *Let (X, ρ) be a compact metric space and $f : X \to X$ be a contractive map. If there exists $x \in X$ such that the sequence of iterates $f^n(x)$ has a limit point $\zeta \in X$, then ζ is the unique fixed point of f.*

Comments. This is same to Theorem of V. Nemitzki [28] in 1936, and holds any topological space X with a function $\delta : X \times X \to [0, \infty)$ such that $\delta(x, y) = 0$ implies $x = y$ for $x, y \in X$.

Note that contractive maps satisfy all requirements of Theorem A for compact metric spaces.

Kannan [19] in 1969

Theorem. ([19]) *Let (X, d) be a complete metric space. Let T be a Kannan mapping on X, i.e., there exists $\alpha \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$. Then T has a unique fixed point.

Comments. This is a simple consequence of Theorem P for quasi-metric spaces. In fact, for $y = Tx$, we have

$$d(Tx, T^2x) \leq \frac{\lambda}{1-\lambda} d(x, Tx) \text{ and } 0 \leq \frac{\lambda}{1-\lambda} < 1.$$

Kannan's example does not require the continuity of the map at every point, although maps satisfying his condition are continuous at fixed points. Hence Kannan's example follows from Theorem P.

Meir and Keeler [26] in 1969

Theorem. ([26]) *Let T be a self map of a complete metric space (X, d). If for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x,\ y \in X$,*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Longrightarrow d(Tx, Ty) < \varepsilon,$$

then T has a unique fixed point.

Comments. The map $T : X \to X$ is called a Meir-Keeler contraction on X (in short, MK-contraction). An MK contraction T is continuous since $d(Tx, Ty) < d(x, y)$ for every $x, y \in X$ with $x \neq y$. Moreover, an MK contraction T is a weakly contractive map.

We show that the MK theorem follows from Theorem A. In fact, for any $x \in X$, $\{f^n x\}$ is a Cauchy sequence. Since X is complete, $\overline{O(x)}$ has a cluster point. Moreover, f is continuous. Hence, by Theorem A, f has a fixed point.

It is known that this theorem does not hold for quasi-metric spaces. See an example of Romaguerra and Tirado [45] in 2019.

Goebel, Kirk, and Shimi [15] in 1973

The following is the main theorem:

Theorem. ([15]) *Let X be a uniformly convex Banach space, K a nonempty bounded closed and convex subset of X, and $F : K \to K$ a continuous mapping satisfying for each $x, y \in K$:*

$$||Fx - Fy|| \leq a_1||x - y|| + a_2||x - Fx|| + a_3||y - Fy|| + a_4||x - Fy|| + a_5||y - Fx||$$

where $a_i > 0$ and $\sum_{i=1}^5 a_i = 1$. Then F has a fixed point in K.

Comment. For $y = Fx$, since $a_4 = a_5$, F is a weakly nonexpansive map.

Bogin [7] in 1976

Theorem 1. *Let (X, d) be a complete metric space and $F : X \to X$ a mapping satisfying for each $x, y \in X$:*

$$d(Fx, Fy) \leq ad(x, y) + b[d(x, Fx) + d(y, Fy)] + c[d(x, Fy) + d(y, Fx)]$$

where $a, b, c > 0$ and $a + 2b + 2c = 1$.

Then F has a unique fixed point.

Comments. In [7], it is noted that $d(Fx, F^2x) \leq d(x, Fx)$ for each $x \in X$, that is, F is a weakly nonexpansive map. Bogin extends the previous one in [15].

Greguš [16] in 1980

Let X be a Banach space and C a closed convex subset of X. Greguš proved the following result:

Theorem 1. ([16]) *Let $T : C \to C$ be a mapping satisfying*

$$||Tx - Ty|| \leq a||x - y|| + b||Tx - x|| + c||Ty - y||$$

for all $x, y \in C$ where $0 < a < 1,\ b \geq 0$ and $a + b + c = 1$. Then T has a unique fixed point.

Comment. Many theorems which are closely related to Greguš's Theorem have appeared in recent years.

For $y = Tx$ and any $a,\ b,\ c \in \mathbb{R}$ satisfying $a + b + c = 1$, we have $||Tx - T^2x|| \leq ||x - Tx||$, and hence T is a weakly nonexpansive map.

Ćirić [9] in 1991

It is proved that if T and E (E continuous) are two compatible selfmaps of a complete convex metric space X such that the condition

$$d(Tx, Ty) \leq ad(Ex, Ey) + (1 - a)\max\{d(Ex, Tx), d(Ey, Ty)\}$$

holds for all $x, y \in X$, where $0 < a < 1$ and $\mathrm{Co}[T(X)] \subset E(X)$, then T and E have a unique common fixed point.

Comment. When E is the identity map 1_X, then T is a weakly nonexpansive map.

Ćirić [10] in 1993

Theorem 2. ([10]) *Let K be a closed convex subset of a complete convex metric space X and $T : K \to K$ a mapping satisfying*

$$d(Tx, Ty) \leq ad(x, y) + (1 - a)\max\{d(x, Tx), d(y, Ty), b[d(x, Ty) + d(y, Tx)]\}$$

where $0 < a < 1$ and $0 < b \leq \frac{1}{2} - \frac{1-a^2}{10+6a^2}$ for all $x, y \in K$. Then T has a unique fixed point.

For $y = Tx$, we recognize that T is not a weakly nonexpansive map.

Ćirić [11] in 2000

Theorem 3. *Let C be a closed convex subset of a complete convex metric space X and $T : C \to C$ a mapping satisfying*

$$d(Tx, Ty) \leq a\max\{d(x, y), c[d(x, Ty) + d(y, Tx)]\} + b\max\{d(x, Tx), d(y, Ty)\}$$

where

$$0 < a < 1,\ a + b = 1,\ c \leq \frac{4 - a}{8 - a},$$

for all $x, y \in C$. Then T has a unique fixed point.

It is easy to check that T is not a weakly nonexpansive map.

Suzuki [49] in 2005

In this paper, Suzuki extend the generalized Caristi's fixed point theorems proved by Bae [J. Math. Anal. Appl. 284 (2003) 690–697] and others.

From the original Caristi's theorem, Suzuki deduced the following:

Theorem 2. ([49]) *Let X be a complete metric space with metric d. Let T be a mapping from X into itself and let f be a lower semicontinuous function from X into $[0, \infty)$. Let φ be a function from X into $[0, \infty)$ satisfying*

$$\sup\{\varphi(x) : x \in X,\ f(x) \leq \inf_{w \in X} f(w) + \eta\} < \infty$$

for some $\eta > 0$*. Assume that*

$$d(x, Tx) \leq \varphi(x)(f(x) - f(Tx))$$

for all $x \in X$*. Then there exists a fixed point* $x_0 \in X$ *of* T*.*

Comment. Since the lower semicontinuity in the Caristi theorem can be extended to the one *from above*, so are Suzuki's theorem and all of its consequences due to Bae et al. Moreover, they may be extended to quasi-metric spaces. Consequently, all 11 theorems in Suzuki [49] can be improved.

Hussain [18] in 2008

The ordered pair (T, I) of two self maps of a metric space (X, d) is called a Banach operator pair, if the set $F(I)$ is T-invariant, namely $T(F(I)) \subseteq F(I)$. Obviously commuting pair (T, I) is a Banach operator pair but not conversely in general. If (T, I) is a Banach operator pair then (I, T) need not be Banach operator pair. If the self-maps T and I of X satisfy

$$d(ITx, Tx) \leq kd(Ix, x) \text{ for all } x \in X \text{ and } k \geq 0,$$

then (T, I) is a Banach operator pair. In particular, when $I = T$ and X is a normed space, the above inequality can be rewritten as

$$||T^2x - Tx|| \leq k||Tx - x|| \text{ for all } x \in X.$$

Such T is called a Banach operator of type k in 1977 by P.V. Subrahmanyam [46].

Comment. A weakly nonexpansive map is a Banach operator of type 1.

Suzuki [47] in 2008

Suzuki introduced some condition on maps which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. He presented fixed point theorems and convergence theorems for maps satisfying the condition.

Definition. Let T be a selfmap on a subset C of a Banach space E. Then T is said to satisfy Condition (C) if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \text{ implies } ||Tx - Ty|| \leq ||x - y||$$

for all $x, y \in C$.

We recall that a selfmap T on a subset C of a Banach space E is called quasi-nonexpansive if $||Tx - z|| \leq ||x - z||$ for all $x \in C$ and $z \in \mathrm{Fix}(T)$, the fixed point set of T. All nonexpansive maps with a fixed point are quasi-nonexpansive.

The following propositions are obvious.

Proposition 1. *Every nonexpansive map satisfies Condition* (C).

Proposition 2. *Assume that a map* T *satisfies Condition* (C) *and has a fixed point. Then* T *is a quasinonexpansive map.*

Lemma 5. *Let* T *be a selfmap on a subset* C *of a Banach space* E*. Assume that* T *satisfies Condition* (C)*. Then for* $x, y \in C$*, the following hold:*

(i) $||Tx - T^2x|| \leq ||x - Tx||$.
(ii) *Either* $(1/2)||x - Tx|| \leq ||x - y||$ *or* $(1/2)||Tx - T^2x|| \leq ||Tx - y||$ *holds.*
(iii) *Either* $||Tx - Ty|| \leq ||x - y||$ *or* $||T^2x - Ty|| \leq ||Tx - y||$ *holds.*

Lemma 7. *Let T be a selfmap on a subset C of a Banach space E. Assume that T satisfies Condition* (C)*. Then*

$$||x - Ty|| \leq 3||Tx - x|| + ||x - y||$$

holds for all $x, y \in C$.

Theorem 4. *Let T be a selfmap on a convex subset C of a Banach space E. Assume that T satisfies Condition* (C)*. Assume also that either of the following holds:*

- *C is compact;*
- *C is weakly compact and E has the Opial property.*

Then T has a fixed point.

Comments. A map T satisfying Lemma 5(i) $||Tx - T^2x|| \leq ||x - Tx||$ for all $x \in C$ is a weakly nonexpansive map. Hence Condition (C) for a map T implies that T is weakly nonexpansive. Note that a weakly nonexpansive map T satisfies an improved form of Lemma 5.

Note that Lemma 7 implies $||x - T^2x|| \leq 4||x - Tx||$ for all $x \in C$. However, we have

$$||x - T^2x|| \leq ||x - Tx|| + ||Tx - T^2x|| \leq 2||x - Tx||.$$

Dhompongsa et al. [12] **in 2009**

A new condition for mappings, called condition (C), which is more general than nonexpansiveness, was recently introduced by Suzuki [47].

Lemma 3.2. ([12]) *Let T be a mapping on a subset E of a Banach space X. Assume that T satisfies condition (C). Then*

$$||x - Tx|| \leq 2||x - y|| + 3||Ty - y||$$

holds for all $x, y \in E$.

Comment. Note that Lemma 3.2 implies a trivial fact

$$0 \leq ||x - Tx|| + 3||Tx - T^2x|| \text{ for } x \in E.$$

Suzuki [49] **in 2009**

The author proves a generalization of Edelstein's fixed point theorem. Though there are thousands of fixed point theorems in metric spaces, his theorem is a new type of theorem.

Theorem 3. *Let (X, d) be a compact metric space and let T be a mapping on X. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y)$$

for $x, y \in X$. Then T has a unique fixed point.

We next prove that $1/2$ in Theorem 3 is the best constant.

Theorem 4. *For every $\eta \in (1/2, \infty)$, there exist a compact metric space (X, d) and a mapping T on X satisfying the following:*

- *T has no fixed points.*
- *$\eta d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$.*

Comment. From Theorem 3, we have

$$\forall x \in X, x \neq Tx \Longrightarrow \frac{1}{2}d(x, Tx) < d(x, Tx) \Longrightarrow d(Tx, T^2x) < d(x, Tx).$$

Hence, T is a weakly contractive map.

Note that, by Theorem 3, we have

$$\exists x \in X \text{ such that } d(x, Tx) \leq d(Tx, T^2x) \Longrightarrow x = Tx.$$

Altun and Erduran [4] in 2011

The authors present a fixed-point theorem for a single-valued map in a complete metric space using implicit relation, which is a generalization of several previously stated results including that of Suzuki [48] in 2008.

The aim of this paper is to generalize the above results using the implicit relation technique in such a way that

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0,$$

for $x, y \in X$, where $F : [0, \infty)^6 \to \mathbb{R}$ is a function as given as follows:

Let Ψ be the set of all continuous functions $F : [0, \infty)^6 \to \mathbb{R}$ satisfying the following conditions:

F1: $F(t_1, \ldots, t_6)$ is nonincreasing in variables $t_2, \ldots, t_6$,

F2: there exists $r \in [0, 1)$, such that $F(u, v, v, u, u + v, 0) \leq 0$ or $F(u, v, 0, u + v, u, v) \leq 0$ or $F(u, v, v, v, v, v) \leq 0$ implies $u \leq rv$,

F3: $F(u, 0, 0, u, u, 0) > 0$, for all $u > 0$.

Comment. From this, the authors showed $d(Tx, T^2x) \leq d(x, Tx)$, that is, T is a weakly nonexpansive map.

Karapınar [20] in 2011

Abstract. Recently, Suzuki [49] published a paper on which Edelstein's fixed theorem was generalized. In this manuscript, we give some theorems which are the generalization of the fixed theorem of Suzuki's Theorems and thus Edelstein's result [14].

Theorem 2.1. ([20]) *Let T be a self mapping on a compact metric space (X, d). Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Longrightarrow d(Tx, Ty) < M(x, y) \text{ for all } x, y \in X,$$

where $M(x, y) = \max\{d(x, y), d(Tx, x), d(y, Ty), \frac{1}{2}d(Tx, y), \frac{1}{2}d(x, Ty)\}$.

Then, T has a unique fixed point $z \in X$, that is, $Tz = z$.

Comment. For $x \neq y = Tx$, it can be assumed

$$d(Tx, T^2x) < M(x, Tx) = \max\{d(x, Tx), \frac{1}{2}d(x, T^2x)\} \text{ or } d(Tx, T^2x) < d(x, Tx).$$

Hence T is a weakly contractive map.

Karapınar and Taş [23] in 2011

In this manuscript, the notion of C-condition of Suzuki [47] is generalized. Some new fixed point theorems are obtained.

Let $F(T)$ be the set of all fixed points of a mapping T. A mapping T on a subset K of a Banach space E is called a *quasi-nonexpansive mapping* if $||Tx - z|| \leq ||x - z||$ for all $x \in K$ and $z \in F(T)$.

We suggest new definitions which are modifications of Suzuki's C-condition:

Definition 5. Let T be a mapping on a subset K of a Banach space E. Then T is said to satisfy *Suzuki-Ćirić-(C)-condition* (in short, (SCC)-condition) if

$$\frac{1}{2}||x-Tx|| \le ||x-y|| \implies ||Tx-Ty|| \le M(x,y)$$

where $M(x,y) = \max\{||x-y||, ||x-Tx||, ||Ty-y||, ||Tx-y||, ||x-Ty||\}$ for all $x,\ y \in K$.

Moreover, T is said to satisfy *Suzuki-(KC)-condition* (in short, (SKC)-condition) if

$$\frac{1}{2}||x-Tx|| \le ||x-y|| \implies ||Tx-Ty|| \le N(x,y)$$

where $N(x,y) = \max\{||x-y||, \frac{1}{2}[||x-Tx|| + ||Ty-y||], \frac{1}{2}[||Tx-y|| + ||x-Ty||]\}$ for all $x,\ y \in K$.

Definition 6. Let T be a mapping on a subset K of a Banach space E. Then T is said to satisfy (for all $x,\ y \in K$)

(i) *Kannan-Suzuki-(C)-condition* (in short, (KSC)-condition) if

$$\frac{1}{2}||x-Tx|| \le ||x-y|| \implies ||Tx-Ty|| \le \frac{1}{2}[||Tx-x|| + ||y-Ty||].$$

(ii) *Chatterjea-Suzuki-(C)-condition* (in short, (CSC)-condition) if

$$\frac{1}{2}||x-Tx|| \le ||x-y|| \implies ||Tx-Ty|| \le \frac{1}{2}[||Tx-y|| + ||x-Ty||].$$

In this manuscript, we modify some results and suggest some new theorems.

Comments. Note that the SCC-condition reduces to $0 \le ||x-Tx||$ for $y = Tx$.

All of the maps satisfying any one of the SKC, KSC, CSC-conditions reduce to weakly nonexpansive maps, that is,

$$||Tx - T^2x|| \le ||x-Tx|| \quad \text{for } y = Tx.$$

Popescu [42] in 2011

The aim of this paper is to generalize two classical fixed point theorems given by Bogin (1976) and Greguš (1980). The author also complement and extend some very recent results by Suzuki (2008).

Here the RHR map can be extended to the weakly nonexpansive map satisfying

$$d(fx, f^2x) \le d(x, fx) \quad \text{for every } x \in X.$$

Popescu [42] studied such type of maps.

Theorem 2.1. ([42]) *Let (X,d) be a complete metric space and $T : X \to X$ satisfy $\frac{1}{2}d(x,Tx) \le d(x,y)$ implies*

$$d(Tx,Ty) \le ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)],$$

where $a \ge 0,\ b > 0,\ c > 0$ and $a + 2b + 2c = 1$.

Then T has a unique fixed point.

Comments. In the proof, the author showed

$$d(Tx, T^2x) \le \frac{a+b+c}{1-b-c}d(x,Tx) = d(x,Tx).$$

Hence T is a weakly nonexpansive map. Similarly, Popescu [42] proved a generalization of Greguš's theorem.

Karapınar [21] in 2012

Abstract. In 2012, keeping Suzuki (C) condition in mind, Karapınar introduced the notion of (RCSC) condition. A self-map T of a subset D of a Banach space is said to have (C) condition (sometimes called Reich-Chatterjea-Suzuki (C) condition) if every two elements $v,\ w \in D$, we have

$$\frac{1}{2}||v - Tv|| \leq ||v - w|| \Longrightarrow ||Tv - Tw|| \leq \frac{1}{3}(||v - w|| + ||w - Tv|| + ||v - Tw||).$$

Comment. Note that, for $w = Tv$, this RCSC becomes

$$||Tv - T^2v|| \leq \frac{1}{3}(||v - Tv|| + ||v - T^2v||) \text{ or } ||Tv - T^2v|| \leq ||v - Tv||$$

for all $v \in D$. Therefore F is a weakly nonexpansive map.

Djafari-Rouhani and Moradi [13] in 2014

The authors obtained the following improvement of Ćirić's result:

Theorem 1.3. ([13]) *Let (X, d) be a complete convex metric space and $T : X \to X$ be a mapping satisfying*

$$d(T(x), T(y)) \leq a \max\{d(x, y), c[d(x, T(y))+d(y, T(x))]\}+b \max\{d(x, T(x)), d(y, T(y))\},$$

for all $x, y \in X$, where $0 < a < 1,\ a + b = 1$ and $0 \leq c < 1/2$.
Then T has a unique fixed point.

Comment. In fact, the authors gave a simple example which shows that the conclusion of Theorem 1.3 does not hold if $c > 1/2$ and asked whether its conclusion holds when $c = 1/2$. This problem is still open.

Y.J. Cho [8] in 2017

In 1973, Bruck introduced a class of nonexpansive mappings which he called firmly nonexpansive mappings as follows:

Let C be a nonempty closed convex subset of a Banach space X. A mapping $T : C \to X$ is said to be *firmly nonexpansive* if, for all $x, y \in C$ and for $t \geq 0$,

$$||Tx - Ty|| \leq ||t(x - y) + (1 - t)(Tx - Ty)||.$$

Recall that a mapping $T : C \to X$ is said to be *generalized nonexpansive* if there exist nonnegative constants $a_1, a_2, \ldots, a_5$ with $a_1 + a_2 + \cdots + a_5 \leq 1$ such that, for all $x, y \in C$,

$$||Tx-Ty|| \leq a_1||x-y||+a_2||x-Tx||+a_3||y-Ty||+a_4||x-Ty||+a_5||y-Tx||. \text{(GNM1)}$$

Since the distance function is symmetric, we can replace $a_2,\ a_3$ with $(a_2 + a_3)/2$ and $a_4,\ a_5$ with $(a_4 + a_5)/2$ and so the generalized nonexpansive mapping (GNM1) is equivalent to the following: There exist nonnegative constants a, b, c with $a+2b+2c \leq 1$ such that, for all $x, y \in C$,

$$||Tx - Ty|| \leq a||x - y|| + b(||x - Tx|| + ||y - Ty||) + c(||x - Ty|| + ||y - Tx||). \text{(GNM2)}$$

Comments. In this paper, it is shown that the realm of nonexpansive maps is quite complicated. Especially, we have

firmly nonexpansive $\Longrightarrow$ nonexpansive $\Longrightarrow$ generalized nonexpansive
$\Longrightarrow$ weakly nonexpansive.

Pant and Shukla [41] in 2017

Definition. Let B be a uniformly convex nonempty subset of a Banach space H. A mapping $G : B \to B$ is called a generalized α-nonexpansive mapping if for any real number $\alpha \in [0, 1)$,

$$\frac{1}{2}||x-Gx|| \leq ||x-y|| \text{ implies } ||Gx-Gy|| \leq \alpha||Gx-y||+\alpha||Gy-x||+(1-2\alpha)||x-y||$$

for all $x, y \in B$.

Remark. Every class of generalized α-nonexpansive mapping fully contains the mapping satisfying condition (C).

Comment. A generalized α-nonexpansive map is weakly nonexpansive.

Abdeljawad et al. [1] in 2020

Abstract. Let K be a nonempty subset of a Banach space E. A mapping $T : K \to K$ is said to satisfy (RCSC) condition if each $a, b \in K$,

$$(1/2)||a - Fa|| \leq ||a - b|| \Longrightarrow ||Fa - Fb|| \leq (1/3)(||a - b|| + ||a - Fb|| + ||b - Fa||).$$

In this paper, we study, under some appropriate conditions, weak and strong convergence for this class of maps through M iterates in uniformly convex Banach space. We also present a new example of mappings with condition (RCSC). We connect M iteration and other well-known processes with this example to show the numerical efficiency of our results. The presented results improve and extend the corresponding results of the literature.

The authors give the necessary and sufficient condition for the existence of a fixed point for a map with (RCSC) defined on a nonempty closed convex subset of a complete uniformly convex Banach space.

Comment. Recall that a map with (RCSC) is a weakly nonexpansive map.

Karapınar [22] in 2021

The author revisit the renowned contractions of Meir-Keeler by involving the interpolation theory in the context of complete metric space. He provides a simple example to illustrate the validity of the observed result.

Definition 2.1. Let (X, d) be a complete metric space. A mapping $T : X \to X$ is said to be an interpolative Kannan-Meir-Keeler type contraction on X (on short, KMK-contraction), if there exists $\gamma \in (0, 1)$ such that for every $x, y \in X\backslash\text{Fix}(T)$ we have

(1) given $\varepsilon > 0$, there exists $\delta > 0$ so that

$$\varepsilon < [d(x, Tx)]^{\gamma}[d(y, Ty)]^{1-\gamma} < \varepsilon + \delta \Longrightarrow d(Tx, Ty) \leq \varepsilon,$$

(2) $$d(Tx, Ty) < [d(x, Tx)]^{\gamma}[d(y, Ty)]^{1-\gamma}.$$

Theorem 2.2. *On a complete metric space (X, d), any interpolative KMK-contraction $T : X \to X$ has a fixed point.*

Comment. The assumption (2) seems to be not necessary.

Karapınar et al. [24] in 2021

Abstract. In this paper, we propose two new contractions via simulation function that involves rational expression in the setting of partial b-metric space. The obtained results not only extend, but also generalize and unify the existing results in two senses: in the sense of contraction terms and in the sense of the abstract setting. We present an example to indicate the validity of the main theorem.

Noorwali [29] in 2021

Abstract. The aim of this study is to introduce a new interpolative contractive mapping combining the Hardy-Rogers contractive mapping of Suzuki type and $\mathcal{Z}$-contraction. We investigate the existence of a fixed point of this type of mappings and prove some corollaries. The new results of the paper generalize a number of existing results which were published in the last two decades.

Ahmad et al. [3] in 2022

Abstract. In this research, we suggest some convergence results for operators having (RCSC) condition in Banach space setting under F iterative scheme. We establish weak convergence under Opials condition and also establish some important strong convergence results under some appropriate assumptions on the domain or on the applying. We furnish a non-trivial example of mappings having (RCSC) condition and show that its F iterative scheme is more effective than the corresponding well known iterative schemes on this particular example.

On the other hand, Suzuki suggested the notion of generalized nonexpansive mappings. A self-map T of a subset D of a Banach space is said to have Suzuki (C) condition (also called Suzuki nonexpansive) if every two elements $v,\ v' \in D$, follow that

$$\frac{1}{2}||v - Tv|| \leq ||v - v'|| \Longrightarrow ||Tv - Tv'|| \leq ||v - v'||.$$

In 2012, keeping Suzuki (C) condition in mind, Karapinar [25] introduced the notion of (RCSC) condition. A self-map T of a subset D of a Banach space is said to have (C) condition (sometimes called Reich-Chatterjea-Suzuki (C) condition) if every two elements $v,\ v' \in D$, follow that

$$\frac{1}{2}||v - Tv|| \leq ||v - v'|| \Longrightarrow ||Tv - Tv'|| \leq \frac{1}{3}(||v - v'|| + ||v' - Tv|| + ||v - Tv'||).$$

Comment. For $v' = Tv$, the RCSC implies

$$||Tv - T^2v|| \leq ||v - Tv||.$$

Hence T is weakly nonexpansive.

Abbas, Anjum, and Rakočević [2] in 2022

Abstract. We introduce a large class of contractive mappings, called Suzuki-Berinde type contraction. We show that any Suzuki-Berinde type contraction has a fixed point and characterizes the completeness of the underlying normed space. A fixed point theorem for multivalued mapping is also obtained. These results unify, generalize and complement various known comparable results in the literature.

We start with the following theorem which is a generalization of the Theorem 2 ([34]) in the setting of a Banach space.

Theorem 5. *Let $(X, ||\cdot||)$ be Banach space and T a selfmapping on X. If there exists $b \in [0, \infty)$ and $\theta \in [0, b+1)$ with $\lambda = \frac{1}{b+1}$ such that for any $x, y \in X$*

$$\psi(r)||x - Tx|| \leq ||x - y|| \tag{1}$$

implies that

$$||b(x - y) + Tx - Ty|| \leq \theta||x - y||, \tag{2}$$

where, $\theta\lambda = r$ and ψ is a nonincreasing function from $[0, 1)$ onto $[0, 1)$ given by

$$\psi(r) = \begin{cases} \lambda & \text{if } 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ \frac{\lambda(1-r)}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} < r < \frac{1}{\sqrt{2}}, \\ \frac{\lambda}{1+r} & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Then T has unique fixed point.

We prove the following theorem, which is the generalization of the Theorem 4 in the setting of compact normed spaces.

Theorem 6. ([2]) *Let $(X, ||\cdot||)$ be compact normed space and $T : X \to X$. If there exists $b \in [0, \infty)$ with $\lambda = \frac{1}{b+1}$ such that for any $x, y \in X$*

$$\frac{\lambda}{2}||x - Tx|| < ||x - y|| \tag{10}$$

implies that

$$||b(x - y) + Tx - Ty|| < ||x - y||. \tag{11}$$

Then T has a fixed point.

Comment. There are some senseless statements.

Ali, Hussain, Karapinar, and Cholamjiak [5] 2022

Abstract. The aim of this article is to design a new iteration process for solving certain fixed-point problems. In particular, we prove weak and strong convergence theorems for generalized nonexpansive mappings in the framework of uniformly convex Banach spaces. In addition, we discuss the stability of the solution under mild conditions. Further, we provide some numerical examples to indicate that the proposed method works properly.

Let M be a nonempty subset of a Banach space X and $F : M \to M$. We denote by $\text{Fix}(F)$ the fixed-point set of F. A mapping $F : M \to M$ is said to be a contraction if there exists $k \in (0, 1)$ such that for all $r,\ s \in M$, $||Fr-Fs|| \le k||r-s||$. If $k = 1$, then F is called nonexpansive and quasi nonexpansive if for all $r \in M$ and $p \in \text{Fix}(F)$, $||Fr - p|| \le ||r - p||$. A mapping F is said to be generalized nonexpansive if for all $r, s \in X$,

$$\frac{1}{2}||r - Fr|| \le ||r - s|| \implies ||Fr - Fs|| \le ||r - s||.$$

Definition 2. (See, e.g., [25]) A mapping $F : M \to M$ is said to satisfy condition (C) if for all $\xi, \eta \in M$, we have

$$\frac{1}{2}||\xi - F\xi|| \le ||\xi - \eta|| \implies ||F\xi - F\eta|| \le ||\xi - \eta||.$$

Indeed, this notion of Suzuki [25] was improved in [31]. Also, he proved the following lemma (see Lemma 7 in [25]).

Lemma 5. [25] *Let M be a nonempty subset of a Banach space X and $F : M \to M$ be a Suzuki generalized nonexpansive mapping. Then, for all $r,\ s \in X$, we have*

$$||Fr - Fs|| \le 3||Fr - r|| + ||r - s||.$$

Consequently, $||Fr - F^2r|| \le 4||Fr - r||$.

Comment. A generalized nonexpansive map is weakly nonexpansive for all $r \in X$,

$$||Fr - F^2r|| \le ||r - Fr||.$$

Okeke, Udo, and Alqahtani [30] in 2025

Abstract. In this paper, we use an existing fixed point iterative scheme to approximate a class of generalized α-nonexpansive mapping in Banach spaces. We also prove weak and strong convergence results for the mapping using the AG iterative scheme. An example of a generalized α-nonexpansive mapping is given to show the validity of the claims. We apply the main results to the approximation of solution of a mixed type Voltera-Fredholm functional nonlinear integral equation and to the spread of HIV modeled in terms of a fractional differential equation of the Caputo type.

Comment. As a generalization of the nonexpansive mapping, Suzuki in 2008, introduced a new class of mapping which is said to be a mapping satisfying condition (C). Further extensions are added.

5. Epilogue

After Banach's fixed point theorem appeared in 1922, peoples formulated the Banach contraction principle. This is not adequate enough since it contains only small number of examples and is not characterize the metric completeness.

One hundred years later in 2022, we formulated the Rus-Hicks-Rhoades contraction principle for quasi-metric spaces (Theorem P). It contains a large number of examples and characterize the metric completeness.

There are many families of mappings on quasi-metric spaces other than the Rus-Hicks-Rhoades contraction. One of the large families is the nonexpansive maps on Banach spaces for which thousands of works appeared to find their fixed points. For such maps, there have appeared some useful results.

In this paper, we collected works on weakly nonexpansive maps on quasi-metric spaces. Our hope is to make a fixed point principle which can be commonly applicable to such maps. At present, it is hopeless even for any small part of them.

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