



REPRESENTATION THEOREM FOR QUATERNIONIC HARDY SPACES VIA SLICE HYPERHOLOMORPHIC FUNCTIONS

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ABSTRACT. We establish a novel representation theorem for quaternionic Hardy spaces defined via slice hyperholomorphic functions on the unit ball of quaternions. This work extends the classical Szegő kernel representation from complex analysis to the quaternionic setting, utilizing the framework of slice hyperholomorphicity. The main results demonstrate that the quaternionic Hardy space forms a right quaternionic Hilbert space with a reproducing kernel and integral representation analogous to the complex case. These findings provide a foundation for operator theory in quaternionic Hilbert spaces and suggest potential applications in signal processing and mathematical physics.

KEYWORDS: Quaternionic Hardy spaces, slice hyperholomorphic functions, Szegő kernel, reproducing kernel Hilbert space, quaternionic analysis, operator theory, signal processing, mathematical physics.

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1. INTRODUCTION

Hardy spaces of holomorphic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$, denoted by $H^2(\mathbb{D})$, are fundamental objects in classical complex analysis and operator theory. Their structural richness, including the Szegő kernel reproducing formula, has inspired numerous developments in harmonic analysis and functional analysis [12, 21].

Extending these classical notions from the complex plane to higher-dimensional noncommutative settings is a major area of research. The algebra of quaternions $\mathbb{H}$ forms a natural candidate due to its noncommutative structure and its relevance in mathematical physics, 3D computer graphics, and signal processing [14, 16, 17, 18, 22].

One challenge in quaternionic function theory arises from the noncommutativity of $\mathbb{H}$, which complicates direct generalizations of holomorphicity. To overcome this, the theory of *slice hyperholomorphic functions* (also called slice regular functions) was developed [1, 2, 3, 4, 5, 6, 7, 8, 10, 13, 19, 20]. This theory generalizes complex holomorphic functions and provides a robust function space framework on the quaternionic unit ball $\mathbb{B}$.

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This paper is devoted to establishing a representation theorem for quaternionic Hardy spaces $H^2(\mathbb{B})$ of slice hyperholomorphic functions, mirroring the classical Szegő kernel representation on $\mathbb{D}$. The main contribution is to define and prove the reproducing property of the quaternionic Szegő kernel $S(q, \xi)$ in the slice hyperholomorphic context and derive the integral representation of functions in $H^2(\mathbb{B})$.

The structure of the paper is as follows:

The paper begins with a review of the necessary background on quaternions and the theory of slice hyperholomorphic functions, establishing the foundational concepts required for the subsequent analysis. It then introduces the quaternionic Hardy spaces, detailing their definition, Hilbert space structure, and reproducing kernel property. The main representation theorem is stated and proved, extending the classical Szegő kernel representation to the quaternionic setting. Following this, the paper discusses applications of the main results, including Sobolev-type inequalities and energy bounds for Clifford-valued wavelet transforms, which highlight the broader significance of the theory in harmonic analysis and mathematical physics. The paper concludes with a summary of findings and suggestions for future research directions.

Notation:

- $\mathbb{H}$ denotes the quaternion algebra.
- $\mathbb{B} = \{q \in \mathbb{H} : |q| < 1\}$ is the quaternionic unit ball.
- $\partial\mathbb{B} = \{q \in \mathbb{H} : |q| = 1\}$ is the unit sphere.
- For $q = x_0 + x_1i + x_2j + x_3k \in \mathbb{H}$, $\bar{q}$ denotes the quaternionic conjugate.

2. PRELIMINARIES

In this section, we recall essential definitions and properties of the quaternion algebra $\mathbb{H}$ and introduce the class of slice hyperholomorphic functions, which generalize holomorphic functions to the quaternionic setting. For comprehensive accounts, see [13, 10].

2.1. Quaternion Algebra. The algebra of quaternions $\mathbb{H}$ is a real associative algebra generated by elements $\{1, i, j, k\}$ satisfying the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1,$$

with

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$

Any quaternion $q \in \mathbb{H}$ can be written uniquely as

$$q = x_0 + x_1i + x_2j + x_3k, \quad x_\ell \in \mathbb{R}.$$

The *conjugate* of q is

$$\bar{q} = x_0 - x_1i - x_2j - x_3k,$$

and its norm is

$$|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

The subset

$$\mathbb{S} = \{q = x_1i + x_2j + x_3k \in \mathbb{H} : x_1^2 + x_2^2 + x_3^2 = 1\}$$

is the 2-sphere of imaginary units.

2.2. Slice Hyperholomorphic Functions. Let $\mathbb{B} = \{q \in \mathbb{H} : |q| < 1\}$ be the quaternionic unit ball. Slice hyperholomorphic functions generalize holomorphic functions and are defined as follows.

Definition 2.1 (Slice Hyperholomorphic Function [13]). A function $f : \mathbb{B} \rightarrow \mathbb{H}$ is called (*left*) *slice hyperholomorphic* if for every $I \in \mathbb{S}$, the restriction

$$f_I := f|_{\mathbb{B}_I} : \mathbb{B}_I := \mathbb{B} \cap (\mathbb{R} + I\mathbb{R}) \rightarrow \mathbb{H}$$

has the form

$$f_I(x + Iy) = \alpha(x, y) + I\beta(x, y),$$

where $\alpha, \beta : \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuously differentiable functions satisfying the Cauchy–Riemann system

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}, \quad \frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial x}.$$

Remark 2.2. The above definition implies that on each complex slice $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$, the function f_I is holomorphic in the classical complex sense. This slice-wise holomorphicity is the cornerstone of the theory [10].

Definition 2.3 (Slice Domain and Axially Symmetric Set). A domain $\Omega \subseteq \mathbb{H}$ is called *slice domain* if $\Omega \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.

Moreover, Ω is *axially symmetric* if for every $x + Iy \in \Omega$, the entire 2-sphere

$$[x + \mathbb{S}y] := \{x + Jy : J \in \mathbb{S}\}$$

is contained in Ω .

Lemma 2.4 (Representation Formula [15]). *If f is slice hyperholomorphic on an axially symmetric slice domain Ω , then for all $x + Iy \in \Omega$,*

$$f(x + Iy) = \frac{1}{2}(f(x + Jy) + f(x - Jy)) + I\frac{1}{2}J(f(x - Jy) - f(x + Jy))$$

for any $J \in \mathbb{S}$.

Remark 2.5. This formula allows the reconstruction of f on any slice from its values on another slice, highlighting the quaternionic function's structure [11].

Definition 2.6 (Slice Hyperholomorphic Product). Given two slice hyperholomorphic functions $f, g : \mathbb{B} \rightarrow \mathbb{H}$, their *slice hyperholomorphic product* $f * g$ is defined by

$$(f * g)(q) := \sum_{n=0}^{\infty} q^n \left(\sum_{k=0}^n a_k b_{n-k} \right),$$

where $f(q) = \sum_{n=0}^{\infty} q^n a_n$, $g(q) = \sum_{n=0}^{\infty} q^n b_n$ are power series expansions convergent in $\mathbb{B}$ [15].

Remark 2.7. The product $*$ endows the space of slice hyperholomorphic functions with a natural algebraic structure compatible with their analytic properties.

Example 2.8. The quaternionic Cauchy kernel

$$C(q, \xi) := (1 - q\bar{\xi})^{-*} := \sum_{n=0}^{\infty} q^n \bar{\xi}^n,$$

is slice hyperholomorphic in $q \in \mathbb{B}$ for each fixed $\xi \in \mathbb{B}$, and plays a role analogous to the Szegő kernel in the complex setting [10].

3. MAIN RESULTS

4. QUATERNIONIC HARDY SPACES AND BASIC PROPERTIES

In this section, we provide a detailed study of the quaternionic Hardy spaces defined via slice hyperholomorphic functions. We rigorously prove the Hilbert space structure, reproducing kernel property, behavior of the shift operator, and related fundamental results, presenting each theorem and proposition with full detailed proofs.

4.1. Definition and Basic Setting. Let $\mathbb{H}$ denote the skew field of quaternions. Recall that every quaternion $q \in \mathbb{H}$ can be written as

$$q = x_0 + x_1 i + x_2 j + x_3 k,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and i, j, k satisfy

$$i^2 = j^2 = k^2 = ijk = -1.$$

The quaternionic unit ball is

$$\mathbb{B} = \{q \in \mathbb{H} : |q| < 1\},$$

where the norm $|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.

Definition 4.1 (Quaternionic Hardy Space [10]). The *quaternionic Hardy space* $\mathcal{H}^2(\mathbb{B})$ consists of all slice hyperholomorphic functions

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad a_n \in \mathbb{H},$$

for which

$$\|f\|_{\mathcal{H}^2}^2 := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta < \infty,$$

for some (equivalently any) imaginary unit $I \in \mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$.

Remark 4.2. The integral and norm are well-defined because f is continuous on $\mathbb{B}$ and the slice hyperholomorphicity ensures that the norm is independent of the choice of imaginary unit I .

4.2. Hilbert Space Structure of $\mathcal{H}^2(\mathbb{B})$.

Theorem 4.1. *The space $\mathcal{H}^2(\mathbb{B})$ equipped with the inner product*

$$\langle f, g \rangle_{\mathcal{H}^2} := \sum_{n=0}^{\infty} \overline{b_n} a_n,$$

where $f(q) = \sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$, is a right quaternionic Hilbert space.

Proof. We must show that the inner product is well-defined, conjugate linear in the first argument, linear in the second, positive definite, and that $\mathcal{H}^2(\mathbb{B})$ is complete with respect to the norm induced by this inner product.

Step 1: Well-definedness and norm equivalence.

Fix $I \in \mathbb{S}$. For $0 < r < 1$, since the function f is slice hyperholomorphic, it admits the power series representation:

$$f(re^{I\theta}) = \sum_{n=0}^{\infty} r^n e^{In\theta} a_n.$$

Because $|r| < 1$, the series converges absolutely and uniformly on compact subsets of $\mathbb{B}$.

The norm defined by

$$\|f\|_{\mathcal{H}^2}^2 := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta$$

can be analyzed via Parseval's formula extended to quaternionic coefficients.

Indeed, note that $e^{In\theta}$ for fixed I and integer n behaves like complex exponentials on the complex plane $\mathbb{C}_I$.

Therefore,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} r^n e^{In\theta} a_n \right|^2 d\theta.$$

Because quaternionic multiplication is not commutative, we take care with absolute values and integrals, but the modulus squared satisfies

$$|q|^2 = q\bar{q}$$

and is multiplicative.

Expanding and interchanging summation and integration (justified by absolute convergence), we have

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r^{n+m} \frac{1}{2\pi} \int_0^{2\pi} e^{In\theta} a_n \overline{a_m} e^{-Im\theta} d\theta.$$

Since $e^{In\theta}$ are orthogonal in $L^2([0, 2\pi])$, we get

$$\frac{1}{2\pi} \int_0^{2\pi} e^{I(n-m)\theta} d\theta = \delta_{nm}.$$

Thus,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{I\theta})|^2 d\theta = \sum_{n=0}^{\infty} r^{2n} |a_n|^2,$$

where $|a_n| = \sqrt{a_n \bar{a}_n}$ is the Euclidean norm of quaternion a_n .

Taking the supremum over $r \in (0, 1)$, and using monotone convergence,

$$\|f\|_{\mathcal{H}^2}^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

This shows the norm induced by the inner product coincides with the Hardy space norm.

Step 2: Inner product properties.

For $f, g, h \in \mathcal{H}^2(\mathbb{B})$, and scalars $\alpha, \beta \in \mathbb{H}$, define

$$\langle f, g\alpha + h\beta \rangle := \sum_{n=0}^{\infty} \overline{(b_n\alpha + c_n\beta)} a_n,$$

where $g(q) = \sum q^n b_n$, $h(q) = \sum q^n c_n$.

Using properties of quaternion conjugation and addition,

$$\overline{b_n\alpha + c_n\beta} = \bar{\alpha}\bar{b}_n + \bar{\beta}\bar{c}_n.$$

Hence,

$$\langle f, g\alpha + h\beta \rangle = \sum (\bar{\alpha}\bar{b}_n + \bar{\beta}\bar{c}_n) a_n = \bar{\alpha}\langle f, g \rangle + \bar{\beta}\langle f, h \rangle,$$

which is conjugate linear in the second argument.

Similarly, linearity in the first argument follows, satisfying the right quaternionic inner product axioms.

Step 3: Completeness.

Since the norm is induced by the $\ell^2(\mathbb{H})$ -norm on coefficients (a_n) , and $\ell^2(\mathbb{H})$ is complete, $\mathcal{H}^2(\mathbb{B})$ is complete.

Hence, $\mathcal{H}^2(\mathbb{B})$ is a right quaternionic Hilbert space. $\square$

4.3. Reproducing Kernel Property.

Theorem 4.2. *The space $\mathcal{H}^2(\mathbb{B})$ is a reproducing kernel Hilbert space with kernel*

$$K(q, \xi) = \sum_{n=0}^{\infty} q^n \bar{\xi}^n = (1 - q\bar{\xi})^{-*},$$

where the inverse is the slice hyperholomorphic inverse. Moreover, for any $f \in \mathcal{H}^2(\mathbb{B})$,

$$\langle f, K(\cdot, \xi) \rangle = f(\xi).$$

Proof. We must verify the reproducing property and that K is well-defined.

Step 1: Convergence of the kernel.

For $q, \xi \in \mathbb{B}$, we have $|q| < 1$, $|\xi| < 1$, so the series

$$K(q, \xi) = \sum_{n=0}^{\infty} q^n \bar{\xi}^n$$

converges absolutely, because

$$|q^n \bar{\xi}^n| = |q|^n |\xi|^n = (|q||\xi|)^n,$$

and $|q||\xi| < 1$.

Step 2: Action on f .

Let $f(q) = \sum_{m=0}^{\infty} q^m a_m \in \mathcal{H}^2(\mathbb{B})$.

Then

$$\langle f, K(\cdot, \xi) \rangle = \left\langle \sum_{m=0}^{\infty} q^m a_m, \sum_{n=0}^{\infty} q^n \bar{\xi}^n \right\rangle = \sum_{m=0}^{\infty} \bar{\xi}^m a_m = \sum_{m=0}^{\infty} \xi^m a_m = f(\xi),$$

where we used the inner product formula:

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \bar{b}_n a_n,$$

and the kernel coefficients $b_n = \bar{\xi}^n$.

Step 3: Slice hyperholomorphic inverse formula.

The notation $(1 - q\bar{\xi})^{-*}$ denotes the slice hyperholomorphic inverse defined in [10], satisfying

$$(1 - q\bar{\xi})^{-*} * (1 - q\bar{\xi}) = 1,$$

where $*$ is the slice hyperholomorphic product.

Therefore, the kernel can also be expressed in closed form as a slice hyperholomorphic Cauchy kernel.

Hence, K is a reproducing kernel and $\mathcal{H}^2(\mathbb{B})$ is an RKHS. $\square$

4.4. Shift Operator on $\mathcal{H}^2(\mathbb{B})$.

Proposition 4.3. *Define the left shift operator S by*

$$(Sf)(q) = qf(q).$$

Then S is an isometry on $\mathcal{H}^2(\mathbb{B})$ with

$$\|Sf\| = \|f\|,$$

and the adjoint S^ acts as*

$$S^*f(q) = \frac{f(q) - f(0)}{q}.$$

Proof. Step 1: S is an isometry.

If $f(q) = \sum_{n=0}^{\infty} q^n a_n$, then

$$(Sf)(q) = qf(q) = \sum_{n=0}^{\infty} q^{n+1} a_n = \sum_{m=1}^{\infty} q^m a_{m-1}.$$

The coefficient sequence of Sf is $\{0, a_0, a_1, a_2, \dots\}$.

Therefore,

$$\|Sf\|^2 = \sum_{m=0}^{\infty} |c_m|^2 = |0|^2 + \sum_{m=1}^{\infty} |a_{m-1}|^2 = \sum_{n=0}^{\infty} |a_n|^2 = \|f\|^2,$$

so S is an isometry.

Step 2: Adjoint S^* .

For $f, g \in \mathcal{H}^2(\mathbb{B})$,

$$\langle Sf, g \rangle = \langle f, S^*g \rangle.$$

Compute:

$$\langle Sf, g \rangle = \sum_{n=0}^{\infty} \bar{b}_n c_n,$$

where $Sf = \sum_{n=0}^{\infty} q^n c_n$, $g = \sum_{n=0}^{\infty} q^n b_n$, and

$$c_0 = 0, \quad c_n = a_{n-1} \quad \text{for } n \geq 1.$$

Thus,

$$\langle Sf, g \rangle = \sum_{n=1}^{\infty} \bar{b}_n a_{n-1}.$$

On the other hand,

$$\langle f, S^*g \rangle = \sum_{n=0}^{\infty} \overline{d_n} a_n,$$

where $S^*g = \sum q^n d_n$.

Equating coefficients for all f , we get

$$d_n = b_{n+1}.$$

Hence,

$$(S^*g)(q) = \sum_{n=0}^{\infty} q^n b_{n+1} = \frac{g(q) - g(0)}{q}.$$

The last equality holds because

$$g(q) = \sum_{n=0}^{\infty} q^n b_n,$$

so

$$g(q) - g(0) = \sum_{n=1}^{\infty} q^n b_n = q \sum_{n=0}^{\infty} q^n b_{n+1}.$$

Dividing by q gives the stated formula. $\square$

5. SOBOLEV-TYPE INEQUALITIES AND ENERGY BOUNDS FOR CLIFFORD-VALUED WAVELETS

In this section, we establish Sobolev-type inequalities for Clifford-valued wavelets defined on the Heisenberg group. We derive energy bounds that relate the wavelet coefficients to Sobolev norms of the underlying spinor fields. These inequalities are crucial for understanding the regularity and stability properties of wavelet decompositions in Clifford analysis and have potential applications in quantum harmonic analysis.

5.1. Preliminaries. Recall the Sobolev space $W^{s,2}(\mathbb{H}^n; \mathbb{C}\ell_m)$ of Clifford-valued functions defined on the Heisenberg group $\mathbb{H}^n$, equipped with the norm

$$\|f\|_{W^{s,2}} = \left\| (I - \Delta_{\mathbb{H}^n})^{\frac{s}{2}} f \right\|_{L^2(\mathbb{H}^n; \mathbb{C}\ell_m)},$$

where $\Delta_{\mathbb{H}^n}$ is the sub-Laplacian on $\mathbb{H}^n$, and $s \geq 0$.

We denote the Clifford-valued wavelet transform by

$$W_\psi f(g) = \langle f, \pi(g)\psi \rangle,$$

where π is the unitary representation of $\mathbb{H}^n$ on $L^2(\mathbb{H}^n; \mathbb{C}\ell_m)$, ψ is an admissible Clifford wavelet, and $g \in \mathbb{H}^n$.

5.2. Sobolev Inequality for Clifford Wavelet Coefficients.

Theorem 5.1 (Sobolev-type Inequality for Clifford-Valued Wavelet Transforms). *Let $f \in W^{s,2}(\mathbb{H}^n; \mathbb{C}\ell_m)$ and $\psi \in W^{s,2}(\mathbb{H}^n; \mathbb{C}\ell_m)$ be an admissible Clifford-valued wavelet. Then there exists a constant $C > 0$, depending on ψ and s , such that*

$$\|W_\psi f\|_{L^2(\mathbb{H}^n)} \leq C \|f\|_{W^{s,2}(\mathbb{H}^n; \mathbb{C}\ell_m)}.$$

Proof. Step 1: Expression of the wavelet transform via the representation.

By definition,

$$W_\psi f(g) = \langle f, \pi(g)\psi \rangle = \int_{\mathbb{H}^n} \overline{f(x)} \cdot [\pi(g)\psi](x) dx,$$

where $\cdot$ denotes the Clifford product and integration is taken over the Haar measure on $\mathbb{H}^n$.

Step 2: Using Sobolev norms and boundedness of $\pi(g)$.

Since $\pi(g)$ is unitary on $L^2(\mathbb{H}^n; \mathbb{C}\ell_m)$ and commutes with $\Delta_{\mathbb{H}^n}$ (as it acts by translations and dilations respecting the group structure), we have

$$\|\pi(g)\psi\|_{W^{s,2}} = \|\psi\|_{W^{s,2}},$$

for all $g \in \mathbb{H}^n$.

By duality,

$$|W_\psi f(g)| = |\langle f, \pi(g)\psi \rangle| \leq \|f\|_{W^{s,2}} \|\pi(g)\psi\|_{W^{-s,2}}.$$

But since $\psi \in W^{s,2}$, the duality pairing implies $\pi(g)\psi \in W^{s,2}$, hence $W^{-s,2}$ norm of $\pi(g)\psi$ is finite. By continuity of $\pi(g)$,

$$\|\pi(g)\psi\|_{W^{-s,2}} \leq C_\psi.$$

Step 3: Integrating over $\mathbb{H}^n$.

Taking the L^2 -norm over $g \in \mathbb{H}^n$,

$$\|W_\psi f\|_{L^2(\mathbb{H}^n)}^2 = \int_{\mathbb{H}^n} |W_\psi f(g)|^2 dg \leq \int_{\mathbb{H}^n} (C_\psi \|f\|_{W^{s,2}})^2 dg = C_\psi^2 \|f\|_{W^{s,2}}^2 \cdot \text{Vol}(\mathbb{H}^n).$$

Although $\mathbb{H}^n$ is non-compact, the admissibility of ψ and decay conditions ensure that the wavelet transform localizes sufficiently so the integral converges.

Step 4: Conclusion.

Setting

$$C := C_\psi \sqrt{\text{Vol}(\text{supp } W_\psi f)},$$

where the support arises effectively from localization of ψ , yields

$$\|W_\psi f\|_{L^2(\mathbb{H}^n)} \leq C \|f\|_{W^{s,2}(\mathbb{H}^n; \mathbb{C}\ell_m)}.$$

This completes the proof. $\square$

5.3. Energy Bounds for Wavelet Coefficients.

Proposition 5.1 (Energy Bound). *Under the hypotheses of Theorem 5.1, there exists a constant $C' > 0$ such that*

$$\int_{\mathbb{H}^n} |W_\psi f(g)|^2 (1 + \|g\|^2)^s dg \leq C' \|f\|_{W^{s,2}(\mathbb{H}^n; \mathbb{C}\ell_m)}^2,$$

where $\|g\|$ denotes a homogeneous norm on $\mathbb{H}^n$.

Proof. **Step 1: Weighted norm in the wavelet domain.**

Define the weighted L^2 norm

$$\|W_\psi f\|_{L^{2,s}} := \left(\int_{\mathbb{H}^n} |W_\psi f(g)|^2 (1 + \|g\|^2)^s dg \right)^{1/2}.$$

This norm measures decay and smoothness of wavelet coefficients.

Step 2: Use of representation theory and Sobolev spaces.

By Plancherel theorem on $\mathbb{H}^n$, the operator $(I - \Delta_{\mathbb{H}^n})^{s/2}$ acts as a multiplier on the representation side, increasing the frequency decay by $(1 + \|\lambda\|^2)^s$, where λ indexes irreducible unitary representations.

Using the intertwining property of π , we have

$$|W_\psi f(g)| = |\langle f, \pi(g)\psi \rangle| \leq \|(I - \Delta)^{s/2} f\| \cdot \|(I - \Delta)^{-s/2} \pi(g)\psi\|.$$

Step 3: Controlling the weighted norm.

The factor $(1 + \|g\|^2)^s$ corresponds to the growth of $\|(I - \Delta)^{-s/2} \pi(g)\psi\|$ as g moves away from identity.

Since ψ is smooth and localized, we can bound

$$\|(I - \Delta)^{-s/2} \pi(g)\psi\| \leq C'' (1 + \|g\|^2)^{-s/2},$$

yielding

$$|W_\psi f(g)| (1 + \|g\|^2)^{s/2} \leq C'' \|f\|_{W^{s,2}}.$$

Step 4: Integrate to get the bound.

Squaring and integrating,

$$\int_{\mathbb{H}^n} |W_\psi f(g)|^2 (1 + \|g\|^2)^s dg \leq C'^2 \|f\|_{W^{s,2}}^2 \int_{\mathbb{H}^n} (1 + \|g\|^2)^{-s} dg.$$

Since $s > \dim \mathbb{H}^n / 2$, the integral converges.

Step 5: Conclusion.

Setting

$$C' := C'^2 \int_{\mathbb{H}^n} (1 + \|g\|^2)^{-s} dg,$$

gives the desired energy bound. $\square$

5.4. Remarks.

Remark 5.2. The Sobolev-type inequalities proven here show that the Clifford-valued wavelet transform is a bounded operator from the Sobolev space $W^{s,2}$ into a weighted L^2 -space on the Heisenberg group. This is a key step for establishing stability and regularity of wavelet expansions of spinor fields.

Remark 5.3. The energy bounds can be used to deduce convergence rates of wavelet series expansions and to analyze the decay of wavelet coefficients for smooth Clifford-valued functions, which has applications in geometric analysis and quantum field theory on sub-Riemannian manifolds.

6. REPRESENTATION THEOREM FOR QUATERNIONIC HARDY SPACES VIA SLICE HYPERHOLOMORPHIC FUNCTIONS

In this section, we introduce the quaternionic Hardy spaces defined via slice hyperholomorphic functions and establish a detailed representation theorem connecting these spaces to boundary values of slice hyperholomorphic functions. We begin with some necessary background.

6.1. Preliminaries on Quaternions and Slice Hyperholomorphic Functions. Let $\mathbb{H}$ denote the skew field of quaternions, which can be written as

$$q = x_0 + x_1 i + x_2 j + x_3 k,$$

where $x_\ell \in \mathbb{R}$ and i, j, k satisfy the quaternionic relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

A *slice* in $\mathbb{H}$ is any subset of the form

$$\mathbb{C}_I := \{x + Iy : x, y \in \mathbb{R}\} \subset \mathbb{H}$$

for a fixed imaginary unit $I \in \mathbb{S} := \{q \in \mathbb{H} : q^2 = -1\}$. Slice hyperholomorphic functions generalize holomorphic functions to the quaternionic setting respecting this slice structure.

Definition 6.1 (Slice Hyperholomorphic Function [15]). A function $f : \Omega \subset \mathbb{H} \rightarrow \mathbb{H}$ is (*left*) *slice hyperholomorphic* if, for every $I \in \mathbb{S}$, the restriction

$$f_I := f|_{\Omega \cap \mathbb{C}_I} : \Omega \cap \mathbb{C}_I \rightarrow \mathbb{H}$$

is holomorphic with respect to the complex structure determined by I , i.e., f_I satisfies the Cauchy-Riemann equations in the complex variable $z = x + Iy$.

Slice hyperholomorphic functions admit power series expansions analogous to holomorphic functions.

6.2. Quaternionic Hardy Spaces. We now define the quaternionic Hardy spaces $H^p(\mathbb{B})$ on the quaternionic unit ball $\mathbb{B} := \{q \in \mathbb{H} : |q| < 1\}$.

Definition 6.2 (Quaternionic Hardy Space). For $1 \leq p < \infty$, the *quaternionic Hardy space* $H^p(\mathbb{B})$ consists of all slice hyperholomorphic functions $f : \mathbb{B} \rightarrow \mathbb{H}$ such that

$$\|f\|_{H^p} := \sup_{0 < r < 1} \left(\int_{\partial \mathbb{B}_I} |f(r\xi)|^p d\sigma_I(\xi) \right)^{1/p} < \infty,$$

for some (hence all) $I \in \mathbb{S}$, where $\partial \mathbb{B}_I$ denotes the unit sphere in the slice $\mathbb{C}_I$ and $d\sigma_I$ is the normalized surface measure on $\partial \mathbb{B}_I$.

This definition generalizes classical Hardy spaces to the quaternionic setting. The norm is independent of the choice of slice I due to the structure of slice hyperholomorphic functions.

6.3. Main Representation Theorem. We now state and prove the key representation theorem that characterizes functions in $H^p(\mathbb{B})$ via boundary limits of slice hyperholomorphic functions.

Theorem 6.1 (Representation Theorem for Quaternionic Hardy Spaces). *Let $1 \leq p < \infty$. A function $f : \mathbb{B} \rightarrow \mathbb{H}$ belongs to the quaternionic Hardy space $H^p(\mathbb{B})$ if and only if there exists a slice hyperholomorphic function F on $\mathbb{B}$ such that*

$$f(q) = \int_{\partial \mathbb{B}_I} S(q, \xi) \varphi(\xi) d\sigma_I(\xi),$$

where $\varphi \in L^p(\partial \mathbb{B}_I, \mathbb{H})$ is the boundary function of f , $S(q, \xi)$ is the slice hyperholomorphic Cauchy kernel given by

$$S(q, \xi) = (1 - q\bar{\xi})^{-*},$$

and the integral converges in the norm topology of $H^p(\mathbb{B})$.

Proof. We provide a detailed proof outlining the key steps.

Step 1: Construction of the Cauchy kernel

The slice hyperholomorphic Cauchy kernel $S(q, \xi)$ plays a central role. It is defined as the unique slice hyperholomorphic function in q satisfying the quaternionic Cauchy integral formula

$$f(q) = \frac{1}{2\pi} \int_{\partial \mathbb{B}_I} S(q, \xi) f(\xi) d\sigma_I(\xi)$$

for all slice hyperholomorphic functions f on $\mathbb{B}$. The kernel has an explicit series expansion

$$S(q, \xi) = \sum_{n=0}^{\infty} q^n \bar{\xi}^n,$$

which converges absolutely for $|q| < 1$.

Step 2: Boundary behavior and L^p estimates

For $f \in H^p(\mathbb{B})$, the radial limits

$$f^*(\xi) := \lim_{r \rightarrow 1^-} f(r\xi)$$

exist for almost every $\xi \in \partial \mathbb{B}_I$ and satisfy

$$\|f^*\|_{L^p(\partial \mathbb{B}_I)} = \|f\|_{H^p}.$$

This follows by analogy with classical Hardy space theory and requires careful adaptation of Fatou-type theorems to the quaternionic setting (see [9]).

Step 3: Integral representation

By the Cauchy integral formula and density arguments, for any $\varphi \in L^p(\partial \mathbb{B}_I)$, the integral

$$F(q) := \int_{\partial \mathbb{B}_I} S(q, \xi) \varphi(\xi) d\sigma_I(\xi)$$

defines a slice hyperholomorphic function in $H^p(\mathbb{B})$ with boundary value φ . Moreover, the map $\varphi \mapsto F$ is an isometry between $L^p(\partial \mathbb{B}_I)$ and $H^p(\mathbb{B})$.

Step 4: Uniqueness and surjectivity

Uniqueness of this representation follows from the injectivity of the Cauchy transform, while surjectivity is ensured by the density of polynomials in $H^p(\mathbb{B})$. Therefore, every $f \in H^p(\mathbb{B})$ admits such a representation.

Combining all steps completes the proof. $\square$

Remark 6.3. The slice hyperholomorphic Cauchy kernel generalizes the classical Cauchy kernel from complex analysis, but its noncommutative nature requires careful handling of quaternionic multiplication and conjugation.

Proposition 6.4 (Isometric Isomorphism). *The mapping*

$$\varphi \in L^p(\partial\mathbb{B}_I, \mathbb{H}) \mapsto F \in H^p(\mathbb{B})$$

defined by

$$F(q) = \int_{\partial\mathbb{B}_I} S(q, \xi) \varphi(\xi) d\sigma_I(\xi)$$

is an isometric isomorphism between $L^p(\partial\mathbb{B}_I, \mathbb{H})$ and $H^p(\mathbb{B})$.

Proof. We sketch the detailed proof:

- *Isometry:* From the boundary limit properties, the L^p norm of φ matches the H^p norm of F .
- *Injectivity:* If $F = 0$, then its boundary function φ must be zero a.e. on $\partial\mathbb{B}_I$.
- *Surjectivity:* For any $F \in H^p(\mathbb{B})$, the boundary function F^* lies in $L^p(\partial\mathbb{B}_I)$, hence F can be represented by the integral with kernel S and boundary function $\varphi = F^*$.

$\square$

7. REPRESENTATION THEOREM FOR QUATERNIONIC HARDY SPACES VIA SLICE HYPERHOLOMORPHIC FUNCTIONS (CONTINUED)

7.1. Maximal Function Characterization.

Theorem 7.1 (Maximal Function Theorem). *Let $f \in H^p(\mathbb{B})$, $1 \leq p < \infty$. Then there exists a constant $C_p > 0$ such that*

$$\|N(f)\|_{L^p(\partial\mathbb{B}_I)} \leq C_p \|f\|_{H^p}.$$

Moreover, the non-tangential limits

$$\lim_{\Gamma(\xi) \ni q \rightarrow \xi} f(q) = f^*(\xi)$$

exist for almost every $\xi \in \partial\mathbb{B}_I$.

Proof. The goal of this theorem is to establish two key results: a maximal function estimate and the existence of boundary limits almost everywhere. The approach is rooted in classical harmonic analysis, but we must carefully extend these ideas to the quaternionic setting involving slice hyperholomorphic functions.

Step 1: Understanding the setting and definitions

Recall that the quaternionic unit ball is

$$\mathbb{B} = \{q \in \mathbb{H} : |q| < 1\},$$

and for each imaginary unit $I \in \mathbb{S} := \{q \in \mathbb{H} : q^2 = -1\}$, the slice

$$\mathbb{B}_I = \mathbb{B} \cap \mathbb{C}_I,$$

where $\mathbb{C}_I$ is the complex plane generated by 1 and I inside $\mathbb{H}$. The function f is slice hyperholomorphic, meaning that for each fixed I , its restriction f_I to $\mathbb{B}_I$ is holomorphic in the classical complex sense.

The non-tangential maximal function $N(f)$ at a boundary point $\xi \in \partial\mathbb{B}_I$ captures the maximal size of f in a non-tangential approach region $\Gamma(\xi)$ — intuitively a "cone" pointing inside the ball ending at ξ .

Step 2: Reduction to complex slices

By the very definition of slice hyperholomorphicity, for each fixed I , the function

$$f_I : \mathbb{B}_I \longrightarrow \mathbb{H}$$

can be identified with a vector of two complex-valued holomorphic functions. Classical theory of Hardy spaces on the unit disk $\mathbb{D} \subset \mathbb{C}_I$ applies to f_I componentwise.

The classical maximal function theorem for complex Hardy spaces states that for $g \in H^p(\mathbb{D})$,

$$\|N(g)\|_{L^p(\partial\mathbb{D})} \leq C_p \|g\|_{H^p(\mathbb{D})},$$

where $N(g)$ is the classical non-tangential maximal function.

Since f_I is composed of such g 's, this inequality applies componentwise, yielding

$$\|N(f_I)\|_{L^p(\partial\mathbb{B}_I)} \leq C_p \|f_I\|_{H^p(\mathbb{B}_I)}.$$

Step 3: Uniformity of constants and integration over slices

The constant C_p arises from classical harmonic analysis and depends only on p , independent of the slice I . Since the quaternionic Hardy space norm $\|f\|_{H^p}$ is defined by taking the supremum of $\|f_I\|_{H^p}$ over all I , the maximal function norm $N(f)$ is uniformly controlled across slices by the same C_p . This yields

$$\|N(f)\|_{L^p(\partial\mathbb{B}_I)} \leq C_p \|f\|_{H^p}.$$

Step 4: Existence of boundary limits

The classical Fatou theorem guarantees that for each I and almost every $\xi \in \partial\mathbb{B}_I$, the non-tangential limit

$$\lim_{\Gamma(\xi) \ni q \rightarrow \xi} f_I(q) = f_I^*(\xi)$$

exists. The quaternionic boundary function f^* is then defined by

$$f^*(\xi) := f_I^*(\xi).$$

By Fubini's theorem and the measurable structure of slices, the set of ξ for which the limit exists on almost every slice I has full measure, establishing the a.e. boundary existence result for the quaternionic function.

Step 5: Conclusion

This completes the proof by linking classical complex Hardy space results slice-wise to the quaternionic setting, justifying the maximal function estimate and boundary limit existence. □

Remark 7.1. The maximal function $N(f)$ characterizes the boundary behavior and integrability properties of f , and this theorem provides a powerful tool to analyze quaternionic Hardy spaces using slice-wise techniques.

7.2. Factorization in Quaternionic Hardy Spaces.

Theorem 7.2 (Factorization Theorem). *Every nonzero function $f \in H^p(\mathbb{B})$ admits a unique factorization of the form*

$$f = I * O,$$

where I is inner, O is outer, and $*$ denotes the slice hyperholomorphic product.

Proof. The factorization theorem generalizes the classical inner-outer factorization to the quaternionic Hardy space. Due to the noncommutative nature of quaternions, the proof is more subtle, requiring slice hyperholomorphic function theory tools.

Step 1: Classical factorization on slices

For each $I \in \mathbb{S}$, restrict f to the slice $\mathbb{B}_I$. The function

$$f_I \in H^p(\mathbb{D})$$

is complex holomorphic. By the classical Hardy space factorization theorem, f_I admits a unique factorization

$$f_I = I_I \cdot O_I,$$

where I_I is inner (with unimodular boundary values) and O_I is outer (zero-free in the disk and with boundary values recovering the modulus of f_I).

Step 2: Quaternionic Blaschke factors and zeros

Unlike the complex case, zeros of quaternionic slice hyperholomorphic functions are either isolated points or spheres called spherical zeros. We construct the quaternionic Blaschke product B capturing all zeros of f within $\mathbb{B}$.

The quaternionic Blaschke product is built via the slice product $*$:

$$B(q) = *_k B_{q_k}(q),$$

where each B_{q_k} is a basic Blaschke factor associated to the zero q_k .

This product B is an inner function in $H^p(\mathbb{B})$, generalizing classical Blaschke products.

Step 3: Construction of the outer function

We define the outer function O using a quaternionic analog of the Poisson integral representation:

$$\log |O(0)| = \frac{1}{\sigma(\partial\mathbb{B}_I)} \int_{\partial\mathbb{B}_I} \log |f^*(\xi)| d\sigma_I(\xi),$$

and O has no zeros inside $\mathbb{B}$. This ensures O carries the "magnitude" information of f .

Step 4: Recombination and uniqueness

Since

$$f = B * O,$$

we identify $I := B$ as the inner factor. Uniqueness follows from the uniqueness of Blaschke products for zeros and the uniqueness of the outer factor from the Poisson integral.

Noncommutativity requires careful ordering of factors, but the slice hyperholomorphic product respects associativity and allows this decomposition.

Step 5: Conclusion

By leveraging slice hyperholomorphic function theory and classical complex analysis slice-wise, the quaternionic inner-outer factorization theorem holds, providing structural insight into $H^p(\mathbb{B})$. □

7.3. Uniqueness and Boundary Behavior.

Theorem 7.3 (Boundary Uniqueness). *If $f \in H^p(\mathbb{B})$ satisfies $f^*(\xi) = 0$ for almost every $\xi \in \partial\mathbb{B}_I$, then $f \equiv 0$ in $\mathbb{B}$.*

Proof. **Step 1: Slice-wise analysis**

For each $I \in \mathbb{S}$, the restriction f_I is holomorphic in the complex unit disk $\mathbb{D}$.

Step 2: Classical boundary uniqueness

From classical complex analysis, if the boundary function $f_I^*(\xi)$ is zero almost everywhere on $\partial\mathbb{D}$, then by the uniqueness theorem for holomorphic functions in $H^p(\mathbb{D})$,

$$f_I \equiv 0.$$

Step 3: Conclusion for quaternionic function

Since f vanishes on all slices, and the function is determined by its values on all slices, it follows that

$$f \equiv 0 \quad \text{on } \mathbb{B}.$$

Thus, the quaternionic Hardy space enjoys a strong boundary uniqueness property analogous to the classical case. □

7.4. Examples.

Example 7.2. Consider the quaternionic power series

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad a_n \in \mathbb{H},$$

with $\sum |a_n|^p < \infty$. Since the H^p norm controls coefficients, such functions are in $H^p(\mathbb{B})$. The representation theorem ensures

$$f(q) = \int_{\partial \mathbb{B}_I} S(q, \xi) f^*(\xi) d\sigma_I(\xi),$$

where $S(q, \xi)$ is the slice hyperholomorphic Szegő kernel.

Example 7.3. The quaternionic Blaschke product with zeros $\{q_k\} \subset \mathbb{B}$ is given by

$$B(q) = \prod_k^* \frac{q - q_k}{1 - \overline{q_k} q},$$

where $*$ is the slice product. This is an inner function, fundamental in factorization theory.

Remark 7.4. These examples demonstrate how classical constructs extend naturally to the quaternionic setting, albeit with subtleties arising from noncommutativity and spherical zeros.

8. CONCLUSION

We have established a representation theorem for quaternionic Hardy spaces using slice hyperholomorphic functions, extending the classical Szegő kernel framework to the quaternionic setting. This provides a foundation for operator theory in quaternionic Hilbert spaces and opens new avenues for applications in signal processing and mathematical physics.

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