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LEVITIN-POLYAK WELL-POSEDNESS OF MIXED VARIATIONAL INEQUALITIES INVOLVING A BIFUNCTION

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ABSTRACT. In this article, we analyze Levitin–Polyak (**LP**) well-posedness of a mixed variational inequality problem involving a bifunction. Sufficient criteria are derived that assert the solution existence. We explore the connection between saddle points of the associated Lagrangian and solutions to both the original variational inequality and its Minty counterpart. The study establishes key results on **LP** well-posedness and generalized **LP** well-posedness, characterizing them through the behavior of approximate solution sets. A notable aspect of this work is the well-posedness analysis based on the gap function approach. In particular, we establish suitable criteria for the **LP** well-posedness of the mixed variational inequality problem by examining the level boundedness of its associated gap function. Furthermore, the **LP** well-posedness of the mixed variational inequality problem is reduced to verifying the well-posedness of a related optimization problem.

KEYWORDS: mixed variational inequality problem, existence theorem, Lagrangian, saddle point, gap function, well-posedness.

1. INTRODUCTION

Variational inequalities hold significant importance in mathematical modeling, particularly in the study of equilibrium problems. They provide a framework for formulating and analyzing equilibrium conditions, addressing aspects such as solvability, uniqueness, stability, parameter dependence, and computational procedures. In a broad sense, a generalized directional derivative can be viewed as an extended real valued bifunction $\varphi(w; d)$, where w refers to a point in the domain C and d refers to a given direction in $\mathbb{R}^n$. A common characteristic of most generalized directional derivatives is their positive homogeneity as a function of the direction d. In optimization problems where the objective function is not necessarily differentiable, necessary optimality conditions can be expressed using a generalized directional derivative. Inspired by these optimality conditions, researchers have

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investigated the following variational inequality problem formulated using the bifunction φ, as discussed in previous studies [[1], [14]-[16]].
Find $\hat{w} \in C$ such that

(SVI)$_\varphi$ $$\varphi(\hat{w}; z - \hat{w}) \geq 0, \quad \forall z \in C.$$

If $\varphi(\hat{w}; z - \hat{w}) = \langle T(\hat{w}), z - \hat{w} \rangle$, where T is an operator on $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ represents the standard inner product on $\mathbb{R}^n$, then the problem (SVI)$_\varphi$ simplifies to the classical variational inequality problem initially proposed by Hartman and Stampacchia [10].

In this paper, we study an extension of the above problem defined as follows:
Find $\hat{w} \in C$ such that

(MSVI)$_\varphi$ $$\varphi(\hat{w}; z - \hat{w}) \geq l(\hat{w}) - l(z), \quad \forall z \in C,$$

where C is a subset of $\mathbb{R}^n$ that is, closed and convex, $\varphi : C \times \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$, and $l : C \longrightarrow \mathbb{R}$. This formulation encompasses a variety of generalized variational inequalities, including the generalized mixed variational inequality studied in [20]. Notably, when $l \equiv 0$, the problem reduces to **(SVI)$_\varphi$**.

The mixed variational inequality framework is particularly useful in handling problems with additional constraints, non-monotonicity, and coupled interactions, which arise naturally in fields such as optimization, economics, mechanics, and game theory. The study of mixed variational inequalities has drawn much attention in recent studies, particularly in dealing with nonlinear problems, leading to new solution methods and theoretical insights [[9], [11], [20]].

Given the importance of mixed variational inequalities in various applications, an essential aspect of their study involves the idea of well-posedness, which ensures stability and convergence of solutions. The analysis of well-posedness is central to the convergence theory of numerical methods, ensuring that iterative approximations reliably approach the true solution of the problem. Tikhonov [24] was the first to introduce the notion of well-posedness for minimization problems, characterizing it through the requirement that every minimizing sequence converges to a unique minimizer. As the theory evolved, researchers recognized the importance of studying well-posedness in cases where solutions are not unique. In such scenarios, well-posedness is established if the set of minimizers is non-empty and if a subsequence of the minimizing sequence converges to an element within this set.

Levitin and Polyak [18] later proposed a generalized concept of well-posedness, known as Levitin-Polyak **(LP)** well-posedness, which extended Tikhonov's concept by requiring that every sequence derived from a broader class of optimizing sequences converges to the optimal solution. Lucchetti and Patrone [21] were the first to introduce the notion of well-posedness for variational inequality problems, drawing motivation from the fact that a minimization problem can be expressed as a variational inequality involving the gradient of the objective function. Since then, the study of well-posedness in variational inequalities has advanced significantly, with numerous researchers [[12], [19], [22],[23]] contributing to its development. More recently, well-posedness analysis has been extended to generalized mixed variational inequalities [[4] - [8], [12], [13]].

This paper undertakes a detailed study of the **LP** well-posedness of the mixed variational inequality problem **(MSVI)$_\varphi$** and is structured as follows. In Section 2, we establish existence results using the KKM lemma, addressing both compact and noncompact settings. Section 3 develops gap functions for the problem and its Minty counterpart, and explores their role in characterizing saddle points of an associated Lagrangian function. In Section 4, we formally present the definitions of well-posedness and its generalized variant for **(MSVI)$_\varphi$** and provide distance-based criteria in terms of approximate solution sets. It is shown that under suitable assumptions, **LP** well-posedness follows from existence and uniqueness, while generalized well-posedness of the problem follows from the bounded nature of a specific approximate solution set. Furthermore, Section 4 also presents a characterization of the **LP** well-posedness of the mixed variational inequality problem in terms of the **LP** well-posedness of a corresponding optimization problem formulated via a gap function.

2. EXISTENCE THEOREM

This section establishes criteria that ensure the admissibility of solutions to $(\mathbf{MSVI})_\varphi$.

Definition 2.1 ([17]). A mapping $K : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is termed as *KKM mapping* if, for every finite collection of points $\{u_1, u_2, \ldots, u_m\} \subset D$, the convex hull of these points satisfies:

$$\mathrm{co}\{u_1, u_2, \ldots, u_m\} \subseteq \bigcup_{i=1}^{m} K(u_i).$$

Lemma 2.1 ([17]) (KKM Lemma). Let $D \subseteq \mathbb{R}^n$, and for each $z \in D$, $K(z)$ be a subset of $\mathbb{R}^n$ which is closed. Further, let K be a KKM mapping and assume that there exists at least one point $z_0 \in D$ such that $K(z_0)$ is bounded. Then

$$\bigcap_{z \in D} K(z) \neq \emptyset.$$

The ensuing theorem establishes the existence result by imposing compactness on the feasible set C.

Theorem 2.1. Let C be a compact subset of $\mathbb{R}^n$ which is also convex. Further, suppose that φ and l fulfill the assumptions stated below:

(i) φ exhibits positive homogeneity in the second component;

(ii) φ exhibits proper subodd property in the second component, that is, for each $v \in C$
$$\varphi(v; d_1) + \varphi(v; d_2) + \cdots + \varphi(v; d_p) \geq 0$$
whenever $\sum_{i=1}^{p} d_i = 0$ for $d_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, p$;

(iii) l satisfies convexity and lower semicontinuity on C;

(iv) φ exhibits upper semicontinuity in both the components;

then the problem $(\mathbf{MSVI})_\varphi$ admits a solution.

Proof. Consider the set-valued operator S on C given as

$$S(z) = \{w \in C \mid \varphi(w; z - w) \geq l(w) - l(z)\}.$$

Clearly, $z \in S(z)$, since by positive homogeneity of φ, we have $\varphi(z; 0) = 0$. Therefore, $S(z) \neq \emptyset$. Next, we show that S satisfies the KKM property. In contrast, assume that there exist points $z_1, z_2, ..., z_p \in C$ and non-negative scalars λ_i, $i = 1, 2, ..., p$ with $\sum_{i=1}^{p} \lambda_i = 1$ such that for $w = \sum_{i=1}^{p} \lambda_i z_i$ we have $w \notin \bigcup_{i=1}^{p} S(z_i)$. Consequently,

$$\varphi(w; z_i - w) < l(w) - l(z_i), \quad \forall\ i = 1, 2, ..., p.$$

As φ is positively homogeneous, multiplying the above inequalities by $\lambda_i \geq 0$ and summing, we get

$$\sum_{i=1}^{p} \varphi(w; \lambda_i(z_i - w)) < l(w) - \sum_{i=1}^{p} \lambda_i l(z_i).$$

Note that $\sum_{i=1}^{p} \lambda_i(z_i - w) = w - w = 0$. By applying the proper subodd property of φ, we conclude

$$0 \leq \sum_{i=1}^{p} \varphi(w; \lambda_i(z_i - w)) < l(w) - \sum_{i=1}^{p} \lambda_i l(z_i),$$

which yields

$$l(w) > \sum_{i=1}^{p} \lambda_i l(z_i),$$

a contradiction to the convexity of l. Consequently, S is a KKM map. Moreover, the lower semicontinuity of l together with the upper semicontinuity of φ guarantees that each $S(z)$ is closed in C. Since C is compact and $S(z)$ is a closed subset of C, it follows that $S(z)$ is compact, for each $z \in C$. Applying **Lemma 2.1**, we conclude that $\bigcap_{z\in C} S(z) \neq \emptyset$ that is, the problem $(\mathbf{MSVI})_\varphi$ is solvable. □

The next example serves to illustrate the necessity of the convexity assumption on l in the above theorem.

Example 2.1. Let $C = [-1, 2]$, and define $\varphi : C \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\varphi(w; d) = -w^2 d$, and $l : C \longrightarrow \mathbb{R}$ by $l(w) = w^3$. We note that l is not convex on C, but all other conditions of **Theorem 2.1** are satisfied. It can be verified that $(\mathbf{MSVI})_\varphi$ does not admit any solution.

Remark 2.1. The existence theorem for the problem $(\mathbf{MSVI})_\varphi$ can also be deduced from Theorem 4.4 in Aussel and Luc [3] by taking $f(w, z) = \varphi(w; z-w) + l(z) - l(w)$, but under a different set of conditions.

The following example demonstrates that while **Theorem 4.4** of [3] is not applicable, the hypotheses of **Theorem 2.1** are fulfilled.

Example 2.2. Let $C = [-1, 1]$, define $\varphi\colon C \times \mathbb{R} \longrightarrow \mathbb{R}$ as $\varphi(w; d) = |wd|$, and $l\colon C \longrightarrow \mathbb{R}$ by $l(w) = |w|$. It can be seen that all the assumptions of **Theorem 2.1** are met. Taking $f(w, z) = \varphi(w; z - w) + l(z) - l(w)$, we note that $f(w, w) = f(z, z) = 0$ but the quasi-monotonicity assumption in **Theorem 4.4** of [3] is not satisfied because for $w = 1$ and $z = -1$, we have

$$\begin{aligned} &\min\left\{f(z, w) - f(z,z), f(w, z) - f(w, w)\right\} \\ &\qquad = \min\Big\{\varphi\big(z; w - z\big) + l(w) - l(z),\ \varphi\big(w; z - w\big) + l(z) - l(w)\Big\} > 0. \end{aligned}$$

However, the set of solutions of the problem $(\mathbf{MSVI})_\varphi$ is precisely $\{-1, 0, 1\}$.

The subsequent theorem provides sufficient conditions for existence of solution to $(\mathbf{MSVI})_\varphi$ for the case when C is an unbounded.

Theorem 2.2. Let C be a nonempty closed and convex but unbounded set in $\mathbb{R}^n$. Furthermore, let φ and l satisfy all the assumptions of **Theorem 2.1**, along with the condition

(v) there is nonempty set $D \subset C$ which is convex and compact, such that for each $w \in C \setminus D$, we can find an element $\hat{z} \in D$ satisfying

$$\varphi(w; \hat{z} - w) < l(w) - l(\hat{z});$$

then the problem $(\mathbf{MSVI})_\varphi$ is solvable and the solution is included in D.

Proof. Define the set valued map $S\colon C \longrightarrow 2^C$ as

$$S(z) = \{w \in C \mid \varphi(w; z - w) \geq l(w) - l(z)\}.$$

Then, repeating the same argument as in Theorem 2.1, it can be shown that for every $z \in C$, $S(z)$ is nonempty and closed. Moreover, S is a KKM map, and from condition (v), we deduce that $S(\hat{z})$ is a compact set. Therefore, applying Lemma 2.1 on this map, we have $\bigcap_{z\in C} S(z) \neq \emptyset$. It follows that any solution of $(\mathbf{MSVI})_\varphi$ belongs to this intersection and hence, it also included in $S(\hat{z}) \subseteq D$. □

Next consider the mixed Minty variational inequality problem $(\mathbf{MMVI})_\varphi$ defined as:

Find $\hat{w} \in C$ such that

$$\textbf{(MMVI)}_{\varphi} \qquad \varphi(\hat{w}; \hat{w} - z) \leq l(z) - l(\hat{w}), \quad \forall\, z \in C$$

where C, φ and l are as introduced earlier in the context of $\textbf{(MSVI)}_{\varphi}$.

We proceed to demonstrate an association between the solution sets of $\textbf{(MSVI)}_{\varphi}$ and $\textbf{(MMVI)}_{\varphi}$, providing a refinement of the classical Minty Lemma.

Definition 2.2. An extended real valued function $s : C \subseteq \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is said to be *hemicontinuous* on a convex set C if, for every pair of points, $u, v \in C$ and for all $\lambda \in [0, 1]$, the mapping $\lambda \to s(u + \lambda(v - u))$ is continuous from the right at $\lambda = 0$.

Theorem 2.3. Let C be a nonempty convex subset of $\mathbb{R}^n$ which is also closed. Then, the following statements hold:

(i) If the bifunction φ satisfies monotonicity criteria then every solution to $\textbf{(MSVI)}_{\varphi}$ also solves $\textbf{(MMVI)}_{\varphi}$;

(ii) If φ is hemicontinuous in the first component, subodd and positively homogeneous in the second component and l is a convex function on C then every solution of $\textbf{(MMVI)}_{\varphi}$ also solves $\textbf{(MSVI)}_{\varphi}$.

Proof. (i) It follows from monotonicity of φ.

(ii) Let $w \in C$ solve $\textbf{(MMVI)}_{\varphi}$. Let $z \in C$ be arbitrary but fixed and $r \in]0, 1[$. Then, owing to convexity of C we have

$$\varphi(w + r(z - w); w - (w + r(z - w))) \leq l(w + r(z - w)) - l(w).$$

Given that φ is positively homogeneous in the second variable and l is convex, we infer that

$$r\varphi(w + r(z - w); w - z) \leq r(l(z) - l(w)),$$

which, dividing by $r > 0$, gives

$$\varphi(w + r(z - w); u - z) \leq l(z) - l(w).$$

Taking the limit as $r \longrightarrow 0^+$ and making use of the hemicontinuity of φ in the first variable, we obtain the following.

$$\varphi(w; w - z) \leq l(z) - l(w),$$

which invoking the suboddness of φ leads to

$$\varphi(w; z - w) \geq l(w) - l(z).$$

Since $z \in C$ was arbitrary, the result follows. □

Remark 2.2. It can be observed from Example 2.2 that the solution set to $\textbf{(MMVI)}_{\varphi}$ is $\{0\}$. Consequently, the solution set of $\textbf{(MSVI)}_{\varphi}$ is not contained within that of $\textbf{(MMVI)}_{\varphi}$. This non-inclusion occurs due to the failure of the monotonicity condition for the bifunction φ on C.

3. GAP FUNCTION AND LAGRANGIAN SADDLE SOLUTIONS

One common technique for addressing variational inequality problems involves utilizing a gap function. This technique reformulates the original problem as an optimization problem, which then allows the use of established optimization algorithms and methods to efficiently determine solutions.

We now introduce an Auslender-type gap function (see [2]) for the problem $(\textbf{MSVI})_{\varphi}$. To this end, we begin by defining the concept of a gap function for this problem.

Definition 3.1. A function $\psi : C \longrightarrow \mathbb{R} \cup \{-\infty\}$ is called a *gap function* for the problem $(\mathbf{MSVI})_\varphi$ if it satisfies the following conditions:

(i) $\psi(\hat{w}) = 0 \Leftrightarrow \hat{w}$ is a solution to $(\mathbf{MSVI})_\varphi$;
(ii) $\psi(w) \leq 0 \;\forall\; w \in C$.

Obviously, if $\varphi(w; 0) = 0 \;\forall\; w \in C$, then the function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{-\infty\}$ given by

$$\psi(w) = \inf_{z \in C} \big\{\varphi(w; z - w) + l(z) - l(w)\big\}$$

is a gap function for the problem $(\mathbf{MSVI})_\varphi$.

Similarly, assuming $\varphi(w; 0) = 0 \;\forall\; w \in C$, the function $\xi : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined as

$$\xi(z) = \sup_{w \in C} \big\{\varphi(w; z - w) + l(z) - l(w)\big\}$$

is a gap function for the problem $(\mathbf{MMVI})_\varphi$, that is, it satisfies the conditions:

(i) $\xi(z) \geq 0, \quad \forall\, z \in C$;
(ii) $\xi(z^*) = 0 \Leftrightarrow z^*$ solves $(\mathbf{MMVI})_\varphi$.

One important application of gap functions lies in establishing error bounds, which offer upper estimates on the distance between a feasible point and the solution set of a variational inequality problem. In the following, we establish an upper estimate for the gap function ψ by leveraging the notion of strong monotonicity.

Definition 3.2. The bifunction $g : C \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is called *strongly monotone* with modulus $\rho > 0$ if, for every $w_1, w_2 \in C$ the following inequality holds:

$$g(w_1; w_2 - w_1) + g(w_2; w_1 - w_2) \leq -\rho\|w_1 - w_2\|^2.$$

Theorem 3.1. Suppose the assumptions listed below hold :

(i) φ is positively homogeneous with respect to the second argument;
(ii) φ is strongly monotone on C;
(iii) $\varphi(w; 0) = 0,\; \forall w \in C$;
(iv) l is a convex function on C.

Then, for any solution $\hat{w} \in C$ of $(\mathbf{MSVI})_\varphi$, there exists a positive constant ρ such that

$$\psi(w) \leq -\rho\|\hat{w} - w\|^2, \quad \forall w \in C.$$

Proof. Since $\hat{w} \in C$ is a solution of $(\mathbf{MSVI})_\varphi$, we have

$$\varphi(\hat{w}; z - \hat{w}) + l(z) - l(\hat{w}) \geq 0,\; \forall z \in C. \tag{3.1}$$

Given that φ is strongly monotone, there exists a constant $\mu > 0$ satisfying

$$\varphi(\hat{w}; z - \hat{w}) + \varphi(z; \hat{w} - z) \leq -\mu\|z - \hat{w}\|^2,\; \forall z \in C.$$

Combining this with (3.1) we get

$$\varphi(z; \hat{w} - z) + l(\hat{w}) - l(z) \leq -\mu\|z - \hat{w}\|^2. \tag{3.2}$$

For any $w \in C$, consider $z = \hat{w} + t(w - \hat{w}),\; t \in]0, 1[$ then

$$\psi(w) \leq \varphi(w; \hat{w} + t(w - \hat{w}) - w) + l(\hat{w} + t(w - \hat{w})) - l(w),$$

As φ exhibits positive homogeneity in the second variable and l is convex, we derive

$$\psi(w) \leq (1 - t)\varphi(w; \hat{w} - w) + (1 - t)(l(\hat{w}) - l(w)).$$

Making use of inequality (3.2) we obtain

$$\begin{aligned} \psi(w) &\leq -\mu(1 - t)\|w - \hat{w}\|^2 \\ &= -\rho\|w - \hat{w}\|^2, \quad \text{where } \rho = \mu(1 - t) > 0. \end{aligned}$$

□

Define the *Lagrangian function* $\mathcal{L} : C \times C \longrightarrow \bar{\mathbb{R}}$ as

$$\mathcal{L}(w, z) = \varphi(w; z - w) + l(z) - l(w). \tag{3.3}$$

Then the condition $w^* \in C$ solves $\textbf{(MSVI)}_\varphi$ is equivalent to $\mathcal{L}(w^*, z) \geq 0,\ \forall\ z \in C$ and $z^* \in C$ solves $\textbf{(MMVI)}_\varphi$ is equivalent to $\mathcal{L}(w, z^*) \leq 0,\ \forall\ w \in C$.

Definition 3.3. A pair $(w^*, z^*) \in C \times C$ is referred to as a *saddle point* of the Lagrangian $\mathcal{L}$ if the following condition holds for all $w, z \in C$

$$\mathcal{L}(w, z^*) \leq \mathcal{L}(w^*, z^*) \leq \mathcal{L}(w^*, z).$$

Moreover, it is well established that a saddle point $(w^*, z^*) \in C \times C$ of the Lagrangian $\mathcal{L}$ can be characterized by the equality

$$\sup_{w\in C} \inf_{z\in C} \mathcal{L}(w, z) = \inf_{z\in C} \sup_{w\in C} \mathcal{L}(w, z) = \mathcal{L}(w^*, z^*).$$

Theorem 3.2. Assume that $\varphi(w; 0) = 0,\ \forall w \in C$ and let $\mathcal{L}$ be the Lagrangian function defined as in (3.3). Then the following results hold:

(i) $\textbf{(MSVI)}_\varphi$ has a solution $w^* \in C$ if and only if

$$\sup_{w\in C} \inf_{z\in C} \mathcal{L}(w, z) = 0$$

and the supremum is attained at w^*;

(ii) $\textbf{(MMVI)}_\varphi$ has a solution $z^* \in C$ if and only if

$$\inf_{z\in C} \sup_{w\in C} \mathcal{L}(w, z) = 0$$

and the infimum is attained at z^*;

(iii) w^* and z^* solve $\textbf{(MSVI)}_\varphi$ and $\textbf{(MMVI)}_\varphi$ respectively if and only if (w^*, z^*) is a saddle point of $\mathcal{L}$ on $C \times C$.

Proof. (i) The function $\psi(w) = \inf_{z\in C} \mathcal{L}(w, z)$ serves as a gap function for the problem $\textbf{(MSVI)}_\varphi$. Therefore, $w^* \in C$ solves $\textbf{(MSVI)}_\varphi$ if and only if $\psi(w^*) = 0$, that is,

$$0 = \psi(w^*) = \sup_{w\in C} \inf_{z\in C} \mathcal{L}(w, z).$$

(ii) Similarly, since $\xi(z) = \sup_{w\in C} \mathcal{L}(w, z)$ acts as a gap function for $\textbf{(MMVI)}_\varphi$, we have that $z^* \in C$ solves $\textbf{(MMVI)}_\varphi$, which holds true when

$$0 = \xi(z^*) = \inf_{z\in C} \sup_{w\in C} \mathcal{L}(w, z).$$

(iii) Let w^* and z^* solve $\textbf{(MSVI)}_\varphi$ and $\textbf{(MMVI)}_\varphi$ respectively. Consequently, in view of (i) and (ii), one concludes that (w^*, z^*) is a saddle point of $\mathcal{L}$ on $C \times C$.
Conversely, if $(w^*, z^*) \in C \times C$ is a saddle point of $\mathcal{L}$ on $C \times C$. Then, for all $w, z \in C$ we have

$$\begin{aligned} \varphi(w; z^* - w) + (z^*) - l(w) &\leq \varphi(w^*; z^* - w^*) + l(z^*) - l(w^*) \\ &\leq \varphi(w^*; z - w^*) + l(z) - l(w^*). \end{aligned}$$

By choosing $w = z^*$ and $z = w^*$ in the above inequalities, we obtain

$$\varphi(w^*; z^* - w^*) + l(z^*) - l(w^*) = 0,$$

thereby establishing that w^* is a solution to $\textbf{(MSVI)}_\varphi$ while z^* is a solution to $\textbf{(MMVI)}_\varphi$. □

4. WELL-POSEDNESS CRITERIA FOR $(\mathbf{MSVI})_\varphi$

This section focuses on analyzing the well-posedness of $(\mathbf{MSVI})_\varphi$. Denote by S_φ the set of solutions of $(\mathbf{MSVI})_\varphi$, that is,

$$S_\varphi := \Big\{ \hat{w} \in C \ : \ \varphi(\hat{w};\, z - \hat{w}) \geq l(\hat{w}) - l(z), \quad \forall\, z \in C \Big\}.$$

We proceed by introducing the concept of an **LP** approximating solution sequence, which plays a key role in defining **LP** well-posedness of $(\mathbf{MSVI})_\varphi$.

Definition 4.1. A sequence $\{w_n\} \in \mathbb{R}^n$ is referred to as **LP** *approximating sequence* for $(\mathbf{MSVI})_\varphi$ provided there exists positive real sequence $\{\varepsilon_n\}$ with $\varepsilon_n \longrightarrow 0$ satisfying the following:

(i) $d(w_n, C) \leq \varepsilon_n$;
(ii) $\varphi(w_n; z - w_n) \geq l(w_n) - l(z) - \varepsilon_n, \quad \forall\, z \in C.$

Definition 4.2. The problem $(\mathbf{MSVI})_\varphi$ is **LP** *well-posed* if

(i) it admits one and only one solution $\hat{w}$;
(ii) each **LP** approximating sequence must converge to $\hat{w}$.

For each positive ε, consider the set of approximate solutions to $(\mathbf{MSVI})_\varphi$ given by

$$S_\varphi(\varepsilon) := \Big\{ \hat{w} \in \mathbb{R}^n : d(\hat{w}, C) \leq \varepsilon,\, \varphi(\hat{w};\, z - \hat{w}) \geq l(\hat{w}) - l(z) - \varepsilon,\, \forall\, z \in C \Big\}.$$

Clearly, $S_\varphi \subseteq S_\varphi(\varepsilon)$ for all $\varepsilon > 0$.

The upcoming theorem characterizes the **LP** well-posedness of $(\mathbf{MSVI})_\varphi$ using a metric approach, by examining the nature of the approximate solution set. For any nonempty set $S \subseteq X = \mathbb{R}^n$, the diameter is defined as the maximum distance between any two points in S, that is,

$$\operatorname{diam} S := \sup_{s_1, s_2 \in S} \|s_1 - s_2\|.$$

Theorem 4.1. Assume that the conditions imposed in Theorem 2.3 hold. In addition, let φ be lower semicontinuous in the second argument and l be lower semicontinuous on C. Then $(\mathbf{MSVI})_\varphi$ is **LP** well-posed precisely if and only if

$$S_\varphi(\varepsilon) \neq \emptyset, \quad \forall\, \varepsilon > 0, \quad \text{and} \quad \operatorname{diam} S_\varphi(\varepsilon) \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{4.1}$$

Proof. If $(\mathbf{MSVI})_\varphi$ is **LP** well-posed, then it has one and only one solution $\hat{w} \in S_\varphi$ and hence $S_\varphi(\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. Suppose, in contrast, that $\operatorname{diam} S_\varphi(\varepsilon) \nrightarrow 0$ as $\varepsilon \longrightarrow 0$. Then there exist $m > 0$, a positive integer k, a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \longrightarrow 0$, and $w_n, \bar{w}_n \in S_\varphi(\varepsilon_n)$ such that

$$\|w_n - \bar{w}_n\| > m, \quad \forall\, n \geq k. \tag{4.2}$$

Since $w_n \in S_\varphi(\varepsilon_n)$, it follows that

$$d(w_n, C) \leq \varepsilon_n \ , \quad \varphi(w_n; z - w_n) \geq l(w_n) - l(z) - \varepsilon_n, \quad \forall\, z \in C.$$

Similarly, for $\bar{w}_n \in S_\varphi(\varepsilon_n)$,

$$d(\bar{w}_n, C) \leq \varepsilon_n, \quad \varphi(\bar{w}_n; z - \bar{w}_n) \geq l(\bar{w}_n) - l(z) - \varepsilon_n, \quad \forall\, z \in C.$$

Thus, both $\{w_n\}$ and $\{\bar{w}_n\}$ are **LP** approximating sequences for $(\mathbf{MSVI})_\varphi$. By **LP** well-posedness, they must converge to the unique solution $\hat{w} \in S_\varphi$, which contradicts (4.2).

Conversely, let $\{w_n\} \in \mathbb{R}^n$ be an **LP** approximating sequence for $(\mathbf{MSVI})_\varphi$. Then there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \longrightarrow 0$ such that

$$d(w_n, C) \leq \varepsilon_n \quad \varphi(w_n; z - w_n) \geq l(w_n) - l(z) - \varepsilon_n, \quad \forall\, z \in C.$$

This implies that for each $n \in N$, there exists $w_n' \in C$ such that

$$\|w_n - w_n'\| \leq \varepsilon_n.$$

Since $\operatorname{diam} S_\varphi(\varepsilon_n) \longrightarrow 0$ as $\varepsilon_n \longrightarrow 0$, it is evident that $\{w_n\}$ is a Cauchy sequence in $\mathbb{R}^n$ and hence converges to some $\hat{w} \in \mathbb{R}^n$. Then it follows that

$$d(\hat{w}, C) \leq \|\hat{w} - w'_n\| \leq \|\hat{w} - w_n\| + \|w_n - w'_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Moreover, since φ is monotone and lower semicontinuous in its second component,l is lower semicontinuous, we have for any $z \in C$ that

$$\begin{aligned}
&\varphi\big(z; \hat{w} - z\big) + l(\hat{w}) - l(z) \\
&\quad \leq \liminf_{n \longrightarrow \infty} \big[\varphi\big(z; w_n - z\big) + l(w_n) - l(z)\big] \\
&\quad \leq \liminf_{n \longrightarrow \infty} \big[- \varphi\big(w_n; z - w_n\big) + l(w_n) - l(z)\big] \\
&\quad \leq \liminf_{n \longrightarrow \infty} (-\varepsilon_n) = 0.
\end{aligned}$$

Hence, according to Theorem 2.3, $\hat{w}$ is a solution of $(\mathbf{MSVI})_\varphi$. Its uniqueness follows from the condition in (4.1). □

Example 4.1. Let $C = [0, \infty)$, define φ and l by $\varphi(w; d) = |d|$ and $l(w) = 2w$. Observe that

$$S_\varphi(\varepsilon) = [-\varepsilon, \varepsilon], \forall\, \varepsilon > 0, \text{and hence if } \varepsilon \longrightarrow 0, \text{ then diameter } S_\varphi(\varepsilon) \text{ approaches zero.}$$

Thus, from Theorem 4.1 it is evident that $(\mathbf{MSVI})_\varphi$ is **LP** well-posed.

The subsequent result provides a characterization of **LP** well-posedness for the mixed variational inequality problem by linking it to the uniqueness of its solution.

Theorem 4.2. Suppose that the assumptions of Theorem 4.1 are satisfied. Then $(\mathbf{MSVI})_\varphi$ is **LP** well-posed if and only if it admits a unique solution.

Proof. Suppose $(\mathbf{MSVI})_\varphi$ is **LP** well-posed. Then, as defined, solution of $(\mathbf{MSVI})_\varphi$ is unique.

Conversely, let $(\mathbf{MSVI})_\varphi$ admit a unique solution $\hat{w}$. Assume, to the contrast, that $(\mathbf{MSVI})_\varphi$ is not **LP** well-posed. Then, we can choose a **LP** approximating sequence $\{w_n\} \in \mathbb{R}^n$ such that $w_n \nrightarrow \hat{w}$. By definition of **LP** approximating sequence, we can find $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ and $\varepsilon_n \longrightarrow 0$ such that

$$d(w_n, C) \leq \varepsilon_n, \tag{4.3}$$

$$\varphi(w_n; z - w_n) \geq l(w_n) - l(z) - \varepsilon_n, \quad \forall\, z \in C. \tag{4.4}$$

By (4.3), for each $n \in N$, there exists $w'_n \in C$ such that

$$\|w_n - w'_n\| \leq \varepsilon_n. \tag{4.5}$$

We assert that $\{w_n\}$ is bounded. If $\{w_n\}$ fails to be bounded then without loss of generality, assume $\|w_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Fix any $z \in C$, define $\lambda_n = \frac{1}{\|w_n - \hat{w}\|}$ and $v_n = \hat{w} + \lambda_n(z - \hat{w})$. Without loss of generality, let $\lambda_n \in (0, 1)$ so that $v_n \in C$ and $v_n \longrightarrow v \neq \hat{w}$. By the lower semicontinuity and positive homogeneity of φ in the second argument, the lower semi-continuity of l, it follows that for any $z \in C$,

$$\begin{aligned}
\varphi(z; v - z) + l(v) - l(z) &\leq \liminf_{n \longrightarrow \infty} [\varphi(z; v_n - z) + l(v_n) - l(z)] \\
&\leq \liminf_{n \longrightarrow \infty} [-\varphi(z; (1 - \lambda_n)(\hat{w} - z)) + (1 - \lambda_n)(l(\hat{w}) - l(z))] \\
&= -\varphi(z; \hat{w} - z) + l(\hat{w}) - l(z) \leq 0.
\end{aligned}$$

Since z was arbitrary, Theorem 2.3 ensures that v is a solution of $(\mathbf{MSVI})_\varphi$, which counters the uniqueness of $\hat{w}$. Hence, $\{w_n\}$ must be bounded and therefore, has a convergent subsequence $\{w_{n_k}\}$ with limit w_0. Then, it follows from above and on using (4.5) that

$$d(w_0, C) \leq \|w_0 - w'_{n_k}\| \leq \|w_0 - w_{n_k}\| + \|w_{n_k} - w'_{n_k}\| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Thus, $w_0 \in C$. Using the lower semi-continuity of φ and l, monotonicity of φ and (4.4), we get for any $z \in C$,

$$\begin{aligned}\varphi(z; w_0 - z) + l(w_0) - l(z) &\leq \liminf_{k \longrightarrow \infty} [\varphi(z; w_{n_k} - z) + l(w_{n_k}) - l(z)] \\ &\leq \liminf_{k \longrightarrow \infty} [-\varphi(w_{n_k}; z - w_{n_k}) + l(w_{n_k}) - l(z)] \\ &\leq \liminf_{k \longrightarrow \infty} (-\varepsilon_{n_k}) = 0.\end{aligned}$$

Hence, by Theorem 2.3, w_0 is a solution of $(\mathbf{MSVI})_\varphi$. By uniqueness, $w_0 = \hat{w}$. Therefore, every convergent subsequence of $\{w_n\}$ must converge to $\hat{w}$, implying $w_n \longrightarrow \hat{w}$. This leads us to the well-posedness of $(\mathbf{MSVI})_\varphi$. □

We now relax the notion of well-posedness to discuss the case when $(\mathbf{MSVI})_\varphi$ does not have a unique solution.

Definition 4.3. $(\mathbf{MSVI})_\varphi$ is said to be **LP** *well-posed in the generalized sense* if:

(i) S_φ is nonempty;
(ii) every **LP** approximating sequence associated with $(\mathbf{MSVI})_\varphi$ possesses a subsequence converging to some point of S_φ.

Consider the following condition:

$$\{S_\varphi \neq \emptyset, \text{ and for any } \mathbf{LP} \text{ approximating sequence } \{w_n\},\ d(w_n, S_\varphi) \longrightarrow 0.\} \tag{4.6}$$

Proposition 4.1. If $(\mathbf{MSVI})_\varphi$ is **LP** well-posed in the generalized sense, then Condition (4.6) holds. Conversely, if (4.6) holds and S_φ is compact, then $(\mathbf{MSVI})_\varphi$ is **LP** well-posed in the generalized sense.

Proof. The result follows immediately from the definitions of **LP** approximating sequence and generalized **LP** well-posedness. □

Theorem 4.3. Let C be a nonempty compact convex subset of $\mathbb{R}^n$ and suppose that assumptions of Theorem 4.1 hold. Then the variational inequality problem $(\mathbf{MSVI})_\varphi$ is generalized **LP** well-posed if and only if $S_\varphi \neq \emptyset$.

Proof. Suppose that $(\mathbf{MSVI})_\varphi$ is generalized **LP** well-posed. By Definition 4.3, its solution set S_φ must be nonempty.

Conversely, assume that $\{w_n\} \in \mathbb{R}^n$ is an **LP** approximating sequence for $(\mathbf{MSVI})_\varphi$. Then there exists a sequence $\{\varepsilon_n\}$ of positive real numbers with $\varepsilon_n \longrightarrow 0$ such that

$$d(w_n, C) \leq \varepsilon_n, \quad \text{and} \quad \varphi(w_n; z - w_n) \geq l(w_n) - l(z) - \varepsilon_n, \quad \forall\, z \in C.$$

Then, for each $n \in N$, there exists $w'_n \in C$ such that

$$\|w_n - w'_n\| \leq \varepsilon_n.$$

By the compactness of C, the sequence $\{w'_n\}$ has a convergent subsequence $\{w'_{n_k}\}$ such that $\{w'_{n_k}\} \longrightarrow \hat{w} \in C$. Consequently, the corresponding subsequence $\{w_{n_k}\}$ also converges to $\hat{w}$. Then, applying similar process as in the proof of Theorem 4.2, we conclude that $\hat{w}$ solves $(\mathbf{MSVI})_\varphi$.

□

The following example serves to validate the preceding theorem.

Example 4.2. Let $C = [-1, 1]$, define $\varphi(w; d) = |d|$ and $l(w) = |w|$. It can be verified that

$$S_\varphi = [-1, 1] \quad \text{and} \quad S_\varphi(\varepsilon) = [-1 - \varepsilon, 1 + \varepsilon], \quad \forall\, \varepsilon > 0.$$

Hence, by Theorem 4.3, $(\mathbf{MSVI})_\varphi$ is generalized LP well-posed.

The corollary below establishes generalized LP well-posedness by relaxing the compactness assumption on the feasible set.

Corollary 4.1. Let C be a subset of $\mathbb{R}^n$ that is nonempty closed convex and suppose that all conditions of Theorem 4.1 hold. If there exists $\varepsilon > 0$ such that $S_\varphi(\varepsilon)$ is nonempty and bounded, then $(\mathbf{MSVI})_\varphi$ is generalized LP well-posed.

The above corollary is demonstrated by the example that follows.

Example 4.3. Let $C = [0, \infty)$ and define

$$\varphi(w; d) = |d|, \quad l(w) = \begin{cases} w & \text{if } w \leq 1, \\ 2w & \text{if } w > 1. \end{cases}$$

It can be verified that $S_\varphi = [0, 1]$ and $S_\varphi(\varepsilon) = [-\varepsilon, 1]$ for $\varepsilon > 0$. Hence, by Corollary 4.1, $(\mathbf{MSVI})_\varphi$ is generalized **LP** well-posed.

The gap function defined in Section 3 allows us to reformulate the variational inequality problem as the following optimization problem:

$$(\mathbf{OP}) \quad \max_{w \in C} \psi(w),$$

where Ω is the solution set of $(\mathbf{OP})$.

Subsequently, this gap functions enables us to conduct a rigorous analysis of well-posedness. In particular,the gap function plays a pivotal role in this framework by establishing a connection between the well-posedness of variational inequalities and its associated optimization problems and by facilitating the derivation of sufficient conditions for well-posedness.

We proceed to define well-posedness in the context of **(OP)**.

Definition 4.4. A sequence $\{w_n\} \in \mathbb{R}^n$ is termed to be **LP** *maximizing* for $(\mathbf{OP})$ if there exists $\{\varepsilon_n\}$ in $\mathbb{R}$, $\varepsilon_n > 0$, with $\varepsilon_n \longrightarrow 0$ such that:

(i) $d(w_n, C) \leq \varepsilon_n$;
(ii) $\liminf_{n \longrightarrow \infty} \psi(w_n) \geq \psi(\hat{w})$ and $\psi(\hat{w}) = 0$.

Definition 4.5. The problem $(\mathbf{OP})$ is termed as **LP** *well-posed* if:

(i) a unique solution $\hat{w}$ of $(\mathbf{OP})$ exists;
(ii) each LP maximizing sequence associated with $(\mathbf{OP})$ converges to $\hat{w}$.

Definition 4.6. $(\mathbf{OP})$ is said to be **LP** *well-posed in the generalized sense* if:

(i) Ω is nonempty;
(ii) every **LP** maximizing sequence associated with $(\mathbf{OP})$ admits a subsequence converging to some element of Ω.

For the upcoming results, take $\varphi(w; 0) = 0$ for all $w \in C$.

Theorem 4.4. **LP** well-posedness (or **LP** well-posedness in the generalized sense) of the mixed variational inequality problem $(\mathbf{MSVI})_\varphi$ holds if and only if the corresponding optimization problem $(\mathbf{OP})$ is **LP** well-posed (respectively, **LP** well-posed in the generalized sense).

Proof. Since ψ is a gap function, it is an immediate consequence that

$$\hat{w} \in S_\varphi \iff \psi(\hat{w}) = 0 \quad \forall\, \hat{w} \in \Omega.$$

To establish the result, it suffices to show that $\{w_n\}$ is an **LP** approximating sequence for $(\mathbf{MSVI})_\varphi$ if and only if it is an **LP** maximizing sequence for $(\mathbf{OP})$.

Suppose $\{w_n\}$ is an **LP** approximating sequence for $(\mathbf{MSVI})_\varphi$. Then there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0$, $\varepsilon_n \longrightarrow 0$ such that $d(w_n, C) \leq \varepsilon_n$ and for every $z \in C$,

$$\varphi(w_n; z - w_n) + l(z) - l(w_n) \geq -\varepsilon_n.$$

It follows that $\psi(w_n) \geq -\varepsilon_n$, which implies $\liminf_{n \longrightarrow \infty} \psi(w_n) \geq 0 = \psi(\hat{w})$.

Conversely, let $\{w_n\}$ be an **LP** maximizing sequence for (**OP**).Then it follows that $\liminf_{n\longrightarrow\infty}\psi(w_n)\geq 0$. Hence, there exists $\{\varepsilon_n\}$ with $\varepsilon_n>0$ and $\varepsilon_n\longrightarrow 0$ such that $\psi(w_n)\geq -\varepsilon_n$. Therefore, for all $z\in C$,

$$\varphi(w_n;z-w_n)+l(z)-l(w_n)\geq -\varepsilon_n,$$

which shows that $\{w_n\}$ is an **LP** approximating sequence for $(\mathbf{MSVI})_\varphi$. □

Definition 4.7. A function $g:Y\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}\cup\{-\infty\}$ is termed as *level bounded* if either of the following conditions hold:

(i) the domain Y is bounded;
(ii) $\lim_{\|y\|\longrightarrow\infty} g(y)=-\infty$.

Theorem 4.5. Suppose that assumptions of Theorem 4.1 hold and the extended gap function ψ is level bounded over C, then the problem $(\mathbf{MSVI})_\varphi$ is **LP** well-posed in the generalized sense.

Proof. Let $\{w_n\}\in\mathbb{R}^n$ be an **LP** approximating sequence for $(\mathbf{MSVI})_\varphi$. Then there exists $\{\varepsilon_n\}$ with $\varepsilon_n>0$ and $\varepsilon_n\longrightarrow 0$ such that $d(w_n,C)\leq\varepsilon_n$ and

$$\varphi(w_n;z-w_n)\geq l(w_n)-l(z)-\varepsilon_n,\quad \forall\, z\in C,$$

which implies

$$\psi(w_n)\geq -\varepsilon_n. \tag{4.7}$$

Suppose $\{w_n\}$ is unbounded. Then, without loss of generality, assume $\|w_n\|\longrightarrow\infty$. Applying the level boundedness of ψ, $\lim_{n\longrightarrow\infty}\psi(w_n)=-\infty$, which contradicts (4.7). Hence, $\{w_n\}$ is bounded, so it has a convergent subsequence $\{w_{n_k}\}$ converging to $\tilde{w}$. Following the arguments in Theorem 4.2, $\tilde{w}\in C$ and subsequently, $\tilde{w}$ solves $(\mathbf{MSVI})_\varphi$. □

Remark 4.1. For the problem $(\mathbf{MSVI})_\varphi$ in Example 4.3, the gap function

$$\psi(w)=\begin{cases}0 & \text{if } w\leq 1,\\ -w & \text{if } w>1\end{cases}$$

is level bounded. Hence, $(\mathbf{MSVI})_\varphi$ is **LP** well-posed.

Consider a real-valued function $\theta:\mathbb{R}^2_+\longrightarrow\mathbb{R}$ satisfying:

$$\theta(p,q)\geq 0,\quad \forall\, p,q\geq 0;\quad \theta(0,0)=0 \tag{4.8}$$

and

$$p_n\geq 0,\ q_n\longrightarrow 0,\ \theta(p_n,q_n)\longrightarrow 0 \implies p_n\longrightarrow 0. \tag{4.9}$$

Theorem 4.6.

(i) $(\mathbf{MSVI})_\varphi$ is **LP** well-posed (or **LP** well-posed in the generalized sense), then there exists a function θ satisfying (4.8) and (4.9) such that for every $w\in C$,

$$|\psi(w)|\geq\theta(d(w,S_\varphi),d(w,C)). \tag{4.10}$$

(ii) Suppose that S_φ is nonempty and compact and (4.10) holds for some θ satisfying (4.8) and (4.9). Then $(\mathbf{MSVI})_\varphi$ is **LP** well-posed in the generalized sense.

Proof. (i) Let

$$\theta(p,q)=\inf\{\,|\psi(w)|:d(w,S_\varphi)=p,\ d(w,C)=q\}.$$

Then $\theta(p,q)\geq 0$ for all $p,q\geq 0$ and $\theta(0,0)=0$ since ψ is a gap function for $(\mathbf{MSVI})_\varphi$. Let $p_n\geq 0$, $q_n\longrightarrow 0$, and $\theta(p_n,q_n)\longrightarrow 0$. Then there exists a sequence $\{w_n\}\in R^n$ with $d(w_n,S_\varphi)=p_n$ and $d(w_n,C)=q_n$ such that $\psi(w_n)\longrightarrow 0$. Hence, $\{w_n\}$ is an **LP** maximizing sequence for (**OP**). By Theorem 4.4, $\{w_n\}$ is also an **LP** approximating sequence for $(\mathbf{MSVI})_\varphi$. Since $(\mathbf{MSVI})_\varphi$ is **LP** well-posed, Proposition 4.1 yields $p_n=d(w_n,S_\varphi)\longrightarrow 0$. Thus, the function θ satisfies (4.8) and (4.9), and inequality (4.10) holds by its definition.

(ii) Let $\{w_n\}$ be an **LP** approximating sequence for $(\mathbf{MSVI})_\varphi$. Then, we can choose $\{\varepsilon_n\}$ with $\varepsilon_n > 0$, $\varepsilon_n \longrightarrow 0$, such that

$$d(w_n, C) \leq \varepsilon_n, \quad \varphi(w_n; z - w_n) \geq l(w_n) - l(z) - \varepsilon_n, \quad \forall\, z \in C.$$

By (4.10),

$$|\psi(w_n)| \geq \theta(d(w_n, S_\varphi), d(w_n, C)).$$

Let $p_n = d(w_n, S_\varphi)$ and $q_n = d(w_n, C)$. Then $q_n \longrightarrow 0$. Moreover, $\{w_n\}$ is an **LP** maximizing sequence for (**OP**) and hence, $\psi(w_n) \longrightarrow 0$. Using (4.9), it follows that $p_n = d(w_n, S_\varphi) \longrightarrow 0$. The compactness of S_φ and Proposition 4.1 then yield that $(\mathbf{MSVI})_\varphi$ is **LP** well-posed in the generalized sense. □

Remark 4.2. It is worth noting that analogous definitions of well-posedness can be formulated for the Minty problem $(\mathbf{MMVI})_\varphi$. Consequently, the results established in this section for $(\mathbf{MSVI})_\varphi$ may be extended to the Minty formulation by employing the Minty Lemma. This observation also suggests a potential equivalence between the well-posedness of the two problems.

5. CONCLUSION

This paper studied the existence and **LP** well-posedness of a mixed variational inequality problem involving a bifunction. Existence results were established using the KKM lemma and gap functions were introduced to relate the problem to an equivalent optimization formulation. Characterizations for **LP** well-posedness were developed by observing the behavior of approximate solution sets, as well as using the gap function approach. Suitable examples were provided to illustrate the theoretical findings. These results contribute to the foundation for further research in non-smooth variational analysis. Future work may explore algorithmic approaches and extensions to broader classes of problems.

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