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SOLUTION OF MIXED ORDERDED SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS VIA LAPALCE DECOMPOSITION METHOD

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ABSTRACT. In this paper, In the Caputo sense, we examine a system of partial differential equations with mixed fractional-order derivatives. We use the Laplace Decomposition Method (LDM), which successfully integrates the Laplace transform with Adomian decomposition method, to get approximate semi-analytical solutions. To illustrate the effectiveness and validity of the suggested approach, it is used on a number of illustrative problems. The correctness of the approach is validated by graphical comparisons between the LDM solutions and exact solutions. Additionally, it is noted that when the order becomes closer to unity, the solutions of the fractional-order system converge to those of the equivalent integer-order system. According to these findings, LDM is a solid and dependable method for resolving intricate fractional differential systems that appear in mathematical and engineering models.

KEYWORDS: Lapalce Transform, Laplace Decomposition Method, Fractional Pratial Differential equations, System of Equations.
AMS Subject Classification:44A10,45D05,45J05.

1. Introduction

One of the most well-known topics in nonlinear science is the system of partial differential equations. Numerous attempts have been undertaken to investigate different nonlinear partial differential equations throughout the last several decades [7]. Inverse scattering theory [1], Backlund transformation [21, 11], Darboux transformation [17], and Painlev expansion method [2] are some of the conventional techniques for resolving nonlinear wave equations. The homogeneous balance method [22] and Jafari [14], which discussed numerical solutions of telegraph and laplace equations on cantor sets using the local fractional laplace decomposition method,

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are two examples of the new, potent solving techniques that have emerged with the rapid development of nonlinear science. Kumar [12] employed the RDT approach to solve the coupled Klein-Gordon equation and the coupled Burgers' equation system. Pirzada [19] spoke about the Adomian Decomposition approach for solving fuzzy heat equations. Two types of systems of equations with mixed Caputo fractional order partial derivatives were solved in this study. These systems were examined in [6, 3, 8]. They resolved these issues for single partial derivatives or partial derivatives of integer order.

In this study we expand the Caputo derivatives to mixed order type system of partial differential equations. The expanding complexity of engineering and physical models that need for the construction and solution of fractional and mixed-order partial differential equations (PDEs) is what motivated this project. Nonlinearities, fractional derivatives, and multivariate systems are frequently difficult for traditional numerical techniques to handle. A potential semi-analytical method in this regard is the Laplace Decomposition Method (LDM), which combines the Laplace transform and Adomian Decomposition Method (ADM) in a synergistic manner. The use of LDM to solve a system of non-homogeneous, mixed-order partial differential equations with two to three independent variables and several dependent variables is what makes this study innovative.

The Caputo fractional derivative is employed because of its advantageous treatment of handling of initial conditions, which are stated in the same format as those for traditional integer-order differential equations the Caputo fractional derivative is used. As initial values are typically described in terms of classical derivatives, the Caputo derivative is hence more suited for physical and technical challenges. The memory and heredity characteristics of different materials and processes, which are not well represented by conventional integer-order models, may be well modeled using fractional calculus in general. A more precise and adaptable mathematical model of biological systems, anomalous diffusion, viscoelasticity, and signal processing is made possible by the use of fractional derivatives. Because of these features, fractional differential equations are very useful for describing intricate dynamics seen in actual systems. Next we provide classical defination of Caputo derivative and integration.

Definition 1.1. The Caputo fractional partial derivative of order α with respect to t, where $n-1<\alpha<n$ and $n\in\mathbb{N}$, is defined as:

$$ {}^{C}D_t^{\alpha}u(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\partial^n u(x,\tau)}{\partial t^n}\frac{d\tau}{(t-\tau)^{\alpha-n+1}}. \tag{1.1}$$

Definition 1.2. The Caputo fractional partial derivative of order β with respect to x, where $m-1<\beta<m$, is given by:

$$ {}^{C}D_x^{\beta}u(x,t)=\frac{1}{\Gamma(m-\beta)}\int_0^x \frac{\partial^m u(\xi,t)}{\partial x^m}\frac{d\xi}{(x-\xi)^{\beta-m+1}}. \tag{1.2}$$

Definition 1.3. Let $u(x,t)$ be a sufficiently smooth function defined on the domain $[0,a]\times[0,b]$. The Riemann–Liouville fractional integral of order $\alpha>0$ with respect to the time variable t is defined as:

$$ I_t^{\alpha}u(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(x,\tau)\,d\tau, \quad t>0. \tag{1.3}$$

Definition 1.4. The Riemann–Liouville fractional integral of order $\beta > 0$ with respect to the spatial variable x is given by:

$$I_x^{\beta} u(x,t) = \frac{1}{\Gamma(\beta)} \int_0^x (x-\xi)^{\beta-1} u(\xi,t)\, d\xi, \quad x > 0. \tag{1.4}$$

The present paper has the following structure. We define Lapalce decomposition method in part 2, show the comparison results with two instances in section 3, and provide a conclusion in section 4.

2. Methods description

The Laplace Adomian Decomposition Method (LADM) has drawn special attention because of its effectiveness and simplicity among the different analytical and semi-analytical techniques for solving differential equations, including the Homotopy Analysis Method (HAM), Variational Iteration Method (VIM), and finite difference schemes [20]. The LADM is a hybrid approach that combines the Adomian Decomposition Method (ADM) and the Laplace transform, providing a strong foundation for solving both linear and nonlinear problems. While the ADM uses Adomian polynomials to methodically break down the nonlinear components, the Laplace transform simplifies the equation by handling beginning conditions naturally and transforming derivatives into algebraic terms. This combination eliminates the requirement for perturbation, linearization, and discretization all of which are sometimes necessary in other approaches.

Moreover, LADM frequently produces series solutions that converge quickly while requiring fewer computing steps. These benefits make LADM especially appropriate for the class of nonlinear (or fractional) differential equations that are the subject of this investigation. Compared to other current approaches, it is a great choice because to its proven ability to produce correct analytical or semi-analytical results.

2.1. **Laplace Decomposition Method.** Consider the system of mixed order partial differential equations in operator form

$$\begin{aligned} \mathcal{D}_t^{\alpha} u + \Im_1(u,v) + \mathcal{N}_1(u,v) &= \mathfrak{F}_1 \\ \mathcal{D}_t^{\beta} u + \Im_2(u,v) + \mathcal{N}_2(u,v) &= \mathfrak{F}_2. \end{aligned} \tag{2.1}$$

with initial conditions

$$\begin{aligned} u(x,0) &= h_1, \\ v(x,0) &= h_2. \end{aligned} \tag{2.2}$$

The fractional order partial differential operators are $\mathcal{D}^{\alpha}$ & $\mathcal{D}^{\beta}$. The linear operators are $\Im_1$ and $\Im_2$, the nonlinear operators are $\mathcal{N}_1$ and $\mathcal{N}_2$, and the inhomogenous terms are $\mathfrak{F}_1$ and $\mathfrak{F}_2$. The Laplace Decomposition method (LDM) is another name for the methodology that combines the Adomian Decomposition and Laplace Transform approaches. Finding the exact or approximate solution to a nonlinear equation is one of this method's primary advantages [13]. Suheil A. Khuri first introduced the Laplace Decomposition method (LDM) [9, 10], which is an effective technique for solving differential equations. Using initial conditions (2.2) and applying the Laplace transform to each side of Eq. (2.1), it yields

$$\begin{aligned} \mathcal{L}\{\mathcal{D}_t^{\alpha} u\} + \mathcal{L}\{\Im_1(u,v)\} + \mathcal{L}\{\mathcal{N}_1(u,v)\} &= \mathcal{L}\{\mathfrak{F}_1\} \\ \mathcal{L}\left\{\mathcal{D}_t^{\beta} u\right\} + \mathcal{L}\{\Im_2(u,v)\} + \mathcal{L}\{\mathcal{N}_2(u,v)\} &= \mathcal{L}\{\mathfrak{F}_2\} \end{aligned} \tag{2.3}$$

Using the differentiation property of Laplace transform, it gives

$$\begin{aligned}\mathcal{L}\{u\} &= \frac{h_1}{s} + \frac{1}{s^{\alpha}}\mathcal{L}\{\mathfrak{F}_1\} - \frac{1}{s^{\alpha}}\mathcal{L}\{\Im_1(u,v)\} - \frac{1}{s^{\alpha}}\mathcal{L}\{\mathcal{N}_1(u,v)\},\\ \mathcal{L}\{v\} &= \frac{h_2}{s} + \frac{1}{s^{\beta}}\mathcal{L}\{\mathfrak{F}_2\} - \frac{1}{s^{\beta}}\mathcal{L}\{\Im_2(u,v)\} - \frac{1}{s^{\beta}}\mathcal{L}\{\mathcal{N}_2(u,v)\}.\end{aligned} \tag{2.4}$$

The Laplace Adomian decomposition method decomposes the unknown functions $u(x,t)$ and $v(x,t)$ by an infinite series of components as

$$\begin{aligned}u(x,t) &= \sum_{k=0}^{\infty} u_k(x,t).\\ v(x,t) &= \sum_{k=0}^{\infty} v_k(x,t),\end{aligned} \tag{2.5}$$

and the nonlinear operators $\mathcal{N}_1(u,v)$ and $\mathcal{N}_2(u,v)$ can be represented by an infinite series so called Adomian polynomials

$$\begin{aligned}\mathcal{N}_1(u,v) &= \sum_{k=0}^{\infty} A_k,\\ \mathcal{N}_2(u,v) &= \sum_{k=0}^{\infty} B_k.\end{aligned} \tag{2.6}$$

The Adomian polynomials [23, 5, 24] can be generated for all forms of nonlinearity. They are determined by the following relations

$$\begin{aligned}A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathcal{N}_1\left(\sum_{j=0}^{n}\lambda^j v_j\right)\right]_{\lambda=0,n=0,1,2,\ldots},\\ B_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathcal{N}_2\left(\sum_{j=0}^{n}\lambda^j v_j\right)\right]_{\lambda=0,n=0,1,2,\ldots}.\end{aligned} \tag{2.7}$$

Substituting Eqs. (2.6) and (2.5) into Eq. (2.4) and doing some mathematical steps we get

$$\begin{aligned}\mathcal{L}\{u_0\} &= \frac{h_1}{s} + \frac{1}{s^{\alpha}}\mathcal{L}\{\mathfrak{F}_1\},\\ \mathcal{L}\{v_0\} &= \frac{h_2}{s} + \frac{1}{s^{\beta}}\mathcal{L}\{\mathfrak{F}_2\}.\end{aligned} \tag{2.8}$$

From (2.8) we obtain first or inital interation in the solution process by applying inverse laplace trasformation. Furthe the iterative scheme for the solution process is

$$\begin{aligned}\mathcal{L}\{u_{k+1}\} &= -\frac{1}{s^{\alpha}}\mathcal{L}\{\Im_1(u,v)\} - \frac{1}{s^{\alpha}}\mathcal{L}\{A_k\},\\ \mathcal{L}\{v_{k+1}\} &= -\frac{1}{s^{\beta}}\mathcal{L}\{\Im_2(u,v)\} - \frac{1}{s^{\beta}}\mathcal{L}\{B_k\}.\end{aligned} \tag{2.9}$$

Finally, by applying the inverse Laplace transform, we can evaluate u_k and v_k, obtaining the solutions in the original time domain. This step converts the transformed expressions back to their corresponding functions in the physical domain, providing the final approximations for $u(x,t)$ and $v(x,t)$.

2.2. **Convergence Analysis.** The convergence of a proposed analytical or semi-analytical method is a crucial component in validating its reliability and applicability to complex systems. In the Laplace Decomposition Method (LDM), the solution to a system of partial differential equations is constructed as an infinite series:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \qquad v(x,t) = \sum_{k=0}^{\infty} v_k(x,t),$$

where each term $u_k(x,t)$ and $v_k(x,t)$ is generated iteratively using the inverse Laplace transform and Adomian polynomials. To ensure that this series representation leads to an accurate and valid solution, it is important to examine the convergence of the series under appropriate assumptions.

Definition 2.1. Let $\{u_k(x,t)\}$ and $\{v_k(x,t)\}$ be sequences of functions defined on a closed and bounded domain $D \subset \mathbb{R}^2$. The Laplace Decomposition Method is said to converge if the series

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \quad v(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$$

converge uniformly and absolutely to the functions $u(x,t)$ and $v(x,t)$, respectively, on D.

Theorem 2.1. *Let $u(x,t)$ and $v(x,t)$ be the solutions obtained using the Laplace Decomposition Method (LDM) for the system of equations defined in Eq. (2.1). Assume that the nonlinear operators $\mathcal{N}_1(u,v)$ and $\mathcal{N}_2(u,v)$ satisfy a Lipschitz condition. Then the series*

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \quad v(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$$

converge uniformly and absolutely to the exact solution of the system on a finite domain $D \subset \mathbb{R}^2$.

Proof. Let us assume that the nonlinear operators $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy the following Lipschitz conditions:

$$\|\mathcal{N}_1(u,v) - \mathcal{N}_1(\tilde{u},\tilde{v})\| \leq L_1(\|u-\tilde{u}\| + \|v-\tilde{v}\|),$$
$$\|\mathcal{N}_2(u,v) - \mathcal{N}_2(\tilde{u},\tilde{v})\| \leq L_2(\|u-\tilde{u}\| + \|v-\tilde{v}\|),$$

where $0 < L_1, L_2 < 1$, and $\|\cdot\|$ denotes an appropriate norm in a Banach space.

Under this condition, the Adomian polynomials A_k and B_k, representing the nonlinear terms, generate bounded sequences. The Laplace transform $\mathcal{L}\{\cdot\}$ and its inverse are linear and bounded operators that preserve convergence. The recursive construction of u_{k+1} and v_{k+1} from the previous terms,

$$\mathcal{L}\{u_{k+1}\} = -\frac{1}{s^\alpha}\mathcal{L}\{\Im_1(u,v)\} - \frac{1}{s^\alpha}\mathcal{L}\{A_k\},$$
$$\mathcal{L}\{v_{k+1}\} = -\frac{1}{s^\beta}\mathcal{L}\{\Im_2(u,v)\} - \frac{1}{s^\beta}\mathcal{L}\{B_k\},$$

ensures that the sequences $\{u_k\}$ and $\{v_k\}$ form Cauchy sequences in the Banach space. Therefore, the series

$$\sum_{k=0}^{\infty} u_k(x,t), \quad \sum_{k=0}^{\infty} v_k(x,t)$$

converge uniformly and absolutely on D, completing the proof. □

3. Main Result

Example 3.1. Consider the mixed order system of partial differential equations,

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial \tau^\alpha} - v\frac{\partial u}{\partial \xi} - \frac{\partial v}{\partial \tau}\frac{\partial u}{\partial \eta} = 1 - \xi + \eta + \tau \\ &\frac{\partial^\beta v}{\partial \tau^\beta} - u\frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \eta} = 1 - \xi - \eta - \tau, 0 \leqslant \alpha, \beta \leqslant 1 \end{aligned} \tag{3.1}$$

with initial conditions $u(\xi, \eta, 0) = \xi + \eta - 1, v(\xi, \eta, 0) = \xi - \eta + 1$, with exact solutions $u(\xi, \eta, \tau) = \xi + \eta + \tau - 1, v(\xi, \eta, \tau) = \xi - \eta - \tau + 1$ given in Example 1 of the article [4].

By applying the Laplace transformation to both mixed-order partial differential equations (3.1), we convert them into algebraic equations in the Laplace domain. This transformation simplifies the analysis and solution of the given equations.

$$\begin{aligned} L\left\{\frac{\partial^\alpha u}{\partial \tau^\alpha}\right\} &= L\left\{v\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \tau}\frac{\partial u}{\partial \eta} + 1 - \xi + \eta + \tau\right\} \\ L\left\{\frac{\partial^\beta v}{\partial \tau^\beta}\right\} &= L\left\{u\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \eta} + 1 - \xi - \eta - \tau\right\}. \end{aligned}$$

Using the property of the Laplace transformation for derivatives, we obtain an algebraic equation in the Laplace domain. This helps in transforming differential equations into a more manageable form for analysis and solution.

$$\begin{aligned} s^\alpha L\left[u(\xi, \eta, \tau)\right] - s^{\alpha-1}\left(u(\xi, \eta, 0)\right) &= L\left\{v\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \tau}\frac{\partial u}{\partial \eta} + 1 - \xi + \eta + \tau\right\} \\ s^\beta L\left[v(\xi, \eta, \tau)\right] - s^{\beta-1}\left(v(\xi, \eta, 0)\right) &= L\left\{u\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \eta} + 1 - \xi - \eta - \tau\right\}. \end{aligned}$$

Next, we apply the inverse Laplace transformation, obtaining the solution in the domain. This step converts the transformed equations back to their initial form,

$$\begin{aligned} u(\xi, \eta, \tau) &= L^{-1}\left\{\frac{u(\xi, \eta, 0)}{s} + \frac{1}{s^\alpha}L\left\{v\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \tau}\frac{\partial u}{\partial \eta} + 1 - \xi + \eta + \tau\right\}\right\} \\ v(\xi, \eta, \tau) &= L^{-1}\left\{\frac{v(\xi, \eta, 0)}{s} + \frac{1}{s^\beta}L\left\{u\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \eta} + 1 - \xi - \eta - \tau\right\}\right\}. \end{aligned}$$

Using the given initial conditions, we obtain specific expressions for the transformed equations. These conditions help determine the first iteration of the solution process, providing a foundation for further computations. $u(\xi, \eta, 0) = \xi + \eta - 1, v(\xi, \eta, 0) = \xi - \eta + 1$, we get

$$\begin{aligned} u(\xi, \eta, \tau) &= \xi + \eta - 1 + L^{-1}\left\{\frac{1}{s^\alpha}L\left\{v\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \tau}\frac{\partial u}{\partial \eta} + 1 - \xi + \eta + \tau\right\}\right\} \\ v(\xi, \eta, \tau) &= \xi - \eta + 1 + L^{-1}\left\{\frac{1}{s^\beta}L\left\{u\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \eta} + 1 - \xi - \eta - \tau\right\}\right\}. \end{aligned} \tag{3.2}$$

Next, we apply the concept of Adomian polynomials to handle the nonlinear term, expressing it as a series expansion that simplifies the solution process and allows for iterative approximation.

$$\sum_{k=0}^{\infty} u_k(\xi,\eta,\tau) = \xi+\eta-1+L^{-1}\left\{\frac{1}{s^{\alpha}}L\left\{\sum_{k=0}^{\infty} A_k(v,u)+\sum_{k=0}^{\infty} B_k(v,u)+1-\xi+\eta+\tau\right\}\right\},$$
$$\sum_{k=0}^{\infty} v_k(\xi,\eta,\tau) = \xi-\eta+1+L^{-1}\left\{\frac{1}{s^{\beta}}L\left\{\sum_{k=0}^{\infty} C_k(u,v)+\sum_{k=0}^{\infty} D_k(u,v)+1-\xi-\eta-\tau\right\}\right\}, \tag{3.3}$$

where the Adomian polynomials, A_k, B_k, C_k, and D_k, are defined in equation (2.7) and may be represented as follows. The answer may be computed efficiently by breaking down the nonlinear terms into a sequence of iterative terms using these polynomials.

$$\begin{aligned}
A_0 &= v_0\frac{\partial u_0}{\partial \xi}\\
A_1 &= v_0\frac{\partial u_1}{\partial \xi}+v_1\frac{\partial u_0}{\partial \xi}\\
A_2 &= v_0\frac{\partial u_2}{\partial \xi}+v_1\frac{\partial u_1}{\partial \xi}+v_2\frac{\partial u_0}{\partial \xi}\\
B_0 &= \frac{\partial v_0}{\partial \tau}\frac{\partial u_0}{\partial \eta}\\
B_1 &= \frac{\partial v_0}{\partial \tau}\frac{\partial u_1}{\partial \eta}+\frac{\partial v_1}{\partial \tau}\frac{\partial u_0}{\partial \eta}\\
B_2 &= \frac{\partial v_0}{\partial \tau}\frac{\partial u_2}{\partial \eta}+\frac{\partial v_1}{\partial \tau}\frac{\partial u_1}{\partial \eta}+\frac{\partial v_2}{\partial \tau}\frac{\partial u_0}{\partial \eta}\\
C_0 &= u_0\frac{\partial v_0}{\partial \xi}\\
C_1 &= u_0\frac{\partial v_1}{\partial \xi}+u_1\frac{\partial v_0}{\partial \xi}\\
C_2 &= u_0\frac{\partial v_2}{\partial \xi}+u_1\frac{\partial v_1}{\partial \xi}+u_2\frac{\partial v_0}{\partial \xi}\\
D_0 &= \frac{\partial u_0}{\partial \tau}\frac{\partial v_0}{\partial \eta}\\
D_1 &= \frac{\partial u_0}{\partial \tau}\frac{\partial v_1}{\partial \eta}+\frac{\partial u_1}{\partial \tau}\frac{\partial v_0}{\partial \eta}\\
D_2 &= \frac{\partial u_0}{\partial \tau}\frac{\partial v_2}{\partial \eta}+\frac{\partial u_1}{\partial \tau}\frac{\partial v_1}{\partial \eta}+\frac{\partial u_2}{\partial \tau}\frac{\partial v_0}{\partial \eta}.
\end{aligned}$$

Now, the initial apprximation is

$$\begin{aligned}
u_0(\xi,\eta,0) &= \xi+\eta-1,\\
v_0(\xi,\eta,0) &= \xi-\eta+1.
\end{aligned}$$

Further we obtain

$$u_{k+1}(\xi,\eta,\tau) = L^{-1}\left\{\frac{1}{s^{\alpha}}L\left\{\sum_{k=0}^{\infty} A_k(v,u)+\sum_{k=0}^{\infty} B_k(v,u)+1-\xi+\eta+\tau\right\}\right\},$$
$$v_{k+1}(\xi,\eta,\tau) = L^{-1}\left\{\frac{1}{s^{\beta}}L\left\{\sum_{k=0}^{\infty} C_k(u,v)+\sum_{k=0}^{\infty} D_k(u,v)+1-\xi-\eta-\tau\right\}\right\}. \tag{3.4}$$

where u_{k+1} and v_{k+1} represent the current approximations at the k^{th} iteration, and A_k, B_k, C_k and D_k are the Adomian polynomials for the respective nonlinear terms. These iterative schemes allow for successive approximations to converge to the solution of the nonlinear system. Thus, the first terms A_0, B_0, C_0 and D_0 provide the initial approximations of the nonlinear terms, which will be used in the first iteration of the scheme.

$$\begin{aligned} u_1(\xi,\eta,\tau) &= L^{-1}\left\{\frac{2}{s^{\alpha+1}}+\frac{1}{s^{\alpha+2}}\right\} = \frac{2\tau^{\alpha}}{\Gamma(\alpha+1)}+\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)} \\ v_1(\xi,\eta,\tau) &= L^{-1}\left\{\frac{-1}{s^{\beta+2}}\right\} = \frac{-\tau^{\beta+1}}{\Gamma(\beta+2)}. \end{aligned} \tag{3.5}$$

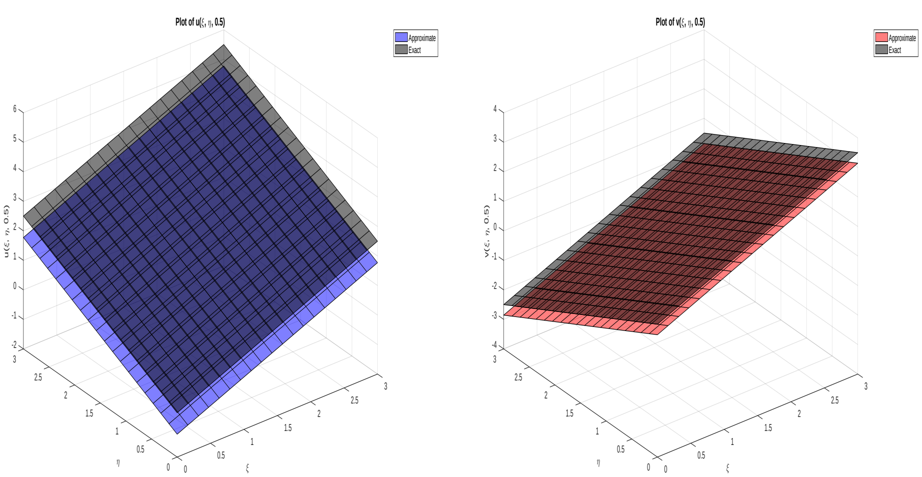

FIGURE 1. Plot of the approximate and exact solution of (Example 3.1) for $u(\xi,\eta,\tau(=0.5))$ and $v(\xi,\eta,\tau(=0.5))$ for $\alpha=\beta=1$.

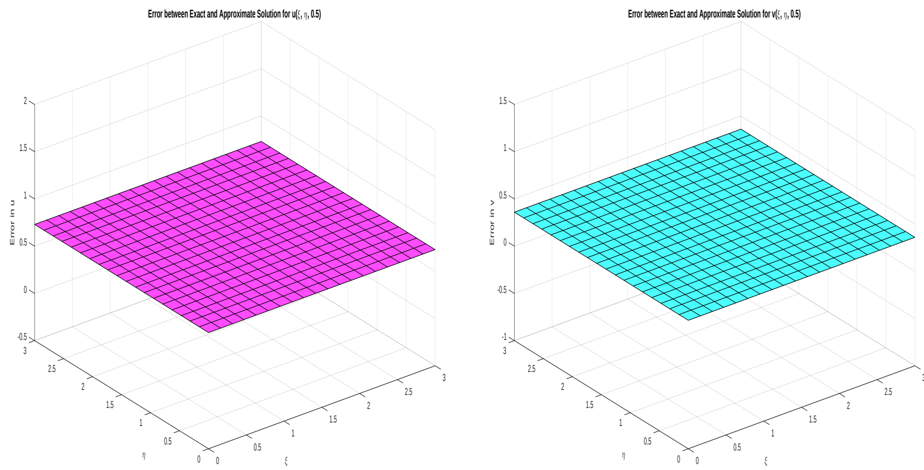

FIGURE 2. Plot of error of estimation $|u_{exact}(\xi,\eta,\tau(=0.5)) - u_{approx}(\xi,\eta,\tau(=0.5))|$ and $|v_{exact}(\xi,\eta,\tau(=0.5)) - v_{approx}(\xi,\eta,\tau(=0.5))|$ for Example 3.1 taking $\alpha=\beta=1$.

Applying the scheme in same fashion and taking the sum of the iterations we obtain the approximate semi-analytic solution for Example 3.1. This approach allows for a systematic refinement of the solution with each iteration, progressively improving the accuracy of the approximation as more terms are included in the series.

$$\begin{aligned} u(\xi,\eta,\tau) &= \xi+\eta-1+\frac{2\tau^{\alpha}}{\Gamma(\alpha+1)}+\frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{-\tau^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{\tau^{2\alpha}}{\Gamma(\alpha+2)} \\ &+\frac{\tau^{3\alpha+1}}{\Gamma(3\alpha+2)}+\left(\frac{2\Gamma(\alpha+2)-(\alpha+1)\Gamma(\alpha+1)}{\Gamma(3\alpha+1)\Gamma(\alpha+2)}\right)\tau^{3\alpha}+\frac{\tau^{3\alpha}}{\Gamma(2\alpha+2)} \\ &-\frac{2\alpha\Gamma(\alpha)\tau^{3\alpha-1}}{\Gamma(\alpha+1)\Gamma(3\alpha)}+\left(\frac{(2\Gamma(\alpha+2))-(\alpha+1)\Gamma(\alpha+1)2\alpha\Gamma(\alpha)}{\Gamma(3\alpha)\Gamma(2\alpha+1)\Gamma(\alpha+2)}\right)\tau^{3\alpha-1} \\ &-\left(\frac{2\alpha\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(\alpha+1)}\right)\tau^{3\alpha-2}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} v(\xi,\eta,\tau) &= \xi-\eta+1+\frac{-\tau^{\beta+1}}{\Gamma(\beta+2)}+\left(\frac{2\Gamma(\beta+2)-(\beta+1)\Gamma(\beta+1)}{\Gamma(2\beta+1)\Gamma(\beta+2)}\right)\tau^{2\alpha} \\ &+\frac{\tau^{2\beta+1}}{\Gamma(2\beta+2)}-\frac{2\beta\Gamma(\beta)\tau^{2\beta-1}}{\Gamma(\beta+1)\Gamma(2\beta)}+\frac{-\tau^{3\beta+1}}{\Gamma(3\beta+2)} \\ &+\left(\frac{(\beta+1)\Gamma(\beta+1)2\beta\Gamma(2\beta)}{\Gamma(3\beta)\Gamma(\beta+2)\Gamma(\beta+2)}\right)\tau^{3\beta-1}. \end{aligned} \tag{3.7}$$

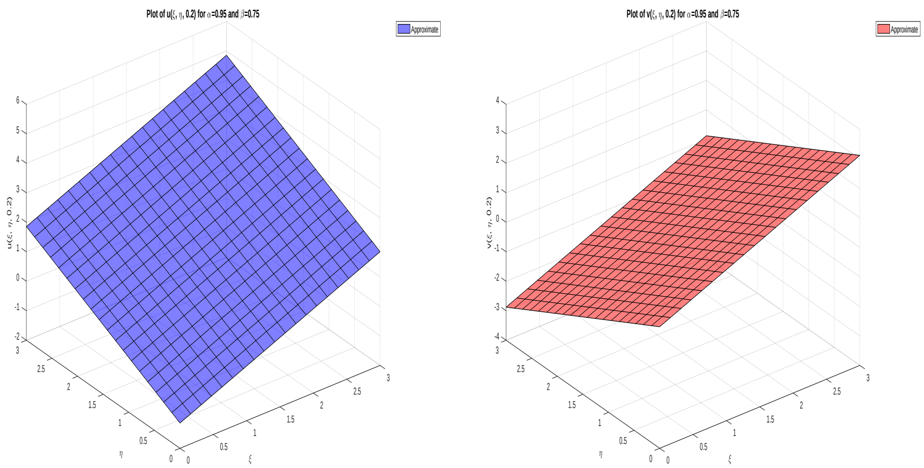

FIGURE 3. Plot of the approximate and exact solution of (Example 3.1) for $u(\xi, \eta, \tau(=0.2))$ and $\alpha = 0.95$ and $\beta = 0.75$.

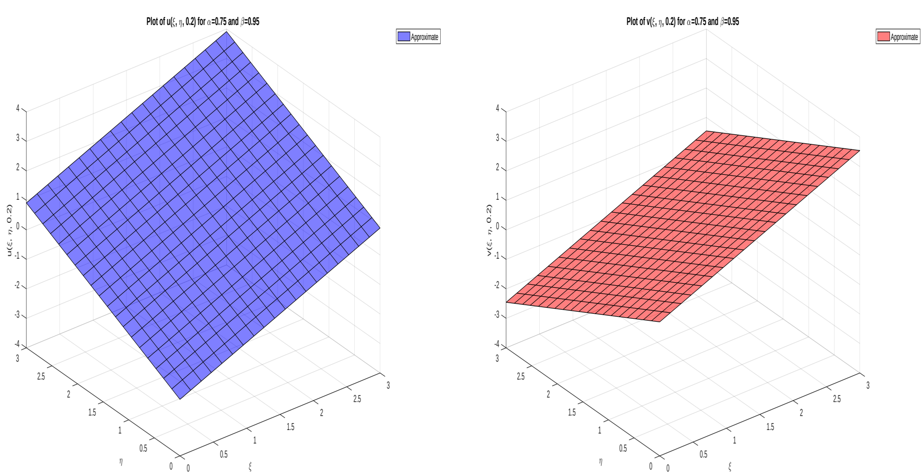

FIGURE 4. Plot of the approximate and exact solution of (Example 3.1) for $u(\xi, \eta, \tau(=0.2))$ and $\alpha = 0.75$ and $\beta = 0.95$.

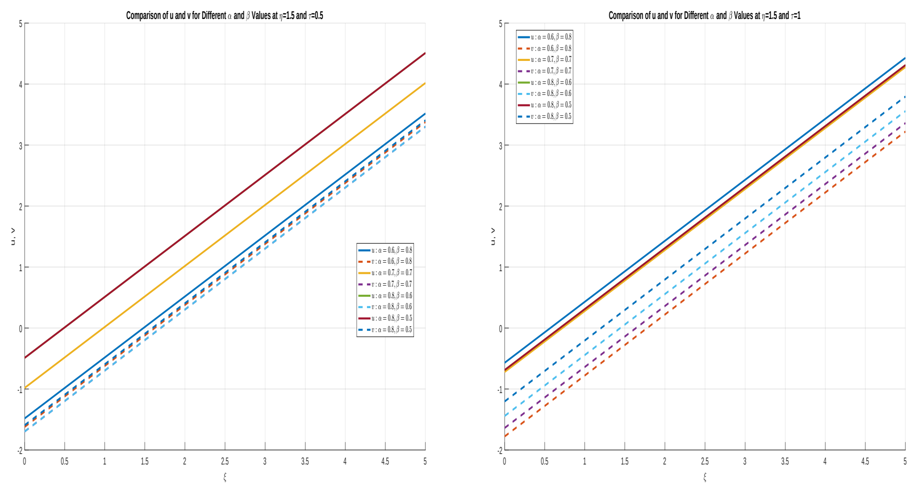

FIGURE 5. Plot of the comparision approximate of solutions of Example 3.1 for $u(\xi,\eta,\tau)$ and $v(\xi,\eta,\tau)$ for different values of α and β.

Next, we provide the comparision table for Example 3.1 for $\alpha = 0.9$, $\beta = 0.95$, $\xi = 0.5$, $\eta = 0.5$.

TABLE 1. Comparison of Exact and Approximate Solutions for $\alpha = 0.9$, $\beta = 0.95$, $\xi = 0.5$, $\eta = 0.5$

τ	u_{exact}	u_{approx}	$\lvert u_{\text{error}}\rvert$	v_{exact}	v_{approx}	$\lvert v_{\text{error}}\rvert$
0.01	0.01	-0.053082	0.063082	0.99	0.967223	0.022777
0.02	0.02	-0.079156	0.099156	0.98	0.939118	0.040882
0.03	0.03	-0.104809	0.134809	0.97	0.910646	0.059354
0.04	0.04	-0.130043	0.170043	0.96	0.881849	0.078151
0.05	0.05	-0.154858	0.204858	0.95	0.852767	0.097233
0.06	0.06	-0.179255	0.239255	0.94	0.823437	0.116563
0.07	0.07	-0.203234	0.273234	0.93	0.793893	0.136107
0.08	0.08	-0.226796	0.306796	0.92	0.764165	0.155835
0.09	0.09	-0.249942	0.339942	0.91	0.734283	0.175717
0.10	0.10	-0.272672	0.372672	0.90	0.704274	0.195726

Example 3.2. Consider the mixed order system of partial differential equations

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} + v\frac{\partial u}{\partial x} + u = 1\\ \frac{\partial^\beta v}{\partial t^\beta} - u\frac{\partial v}{\partial x} - v = 1, 0 \leqslant \alpha, \beta \leqslant 1\end{aligned} \tag{3.8}$$

with the initial conditions $u\left(x,0\right) = e^{x}$, $v\left(x,0\right) = e^{-x}$, with exact solutions $u\left(x,t\right) = e^{x-t}$, $v\left(x,t\right) = e^{-x+t}$ provided in Example 3.3 from the article [6].

By applying the Laplace transformation to both partial differential equations, we convert them into algebraic equations in the Laplace domain. This transformation simplifies the analysis and solution of the given equations.

$$L\left\{\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\right\}=L\left\{1-v\frac{\partial u}{\partial x}-u\right\}$$
$$L\left\{\frac{\partial^{\beta}v}{\partial t^{\beta}}\right\}=L\left\{1+u\frac{\partial v}{\partial x}+v\right\}$$

Using the property of the Laplace transformation for derivatives, we obtain an algebraic equation in the Laplace domain. This helps in transforming differential equations into a more manageable form for analysis and solution.

$$s^{\alpha}L\left\{u\left(x,t\right)\right\}-s^{\alpha-1}u\left(x,0\right)=L\left\{1\right\}-L\left\{v\frac{\partial u}{\partial x}\right\}-L\left\{u\right\}$$
$$s^{\beta}L\left\{v\left(x,t\right)\right\}-s^{\beta-1}v\left(x,0\right)=L\left\{1\right\}+L\left\{u\frac{\partial v}{\partial x}\right\}+L\left\{v\right\}$$

Using the given initial conditions, we obtain specific expressions for the transformed equations. These conditions help determine the first iteration of the solution process, providing a foundation for further computations.

$$s^{\alpha}L\left\{u\left(x,t\right)\right\}=s^{\alpha-1}e^{x}+\frac{1}{s}-L\left\{v\frac{\partial u}{\partial x}\right\}-L\left\{u\right\}$$
$$s^{\beta}L\left\{v\left(x,t\right)\right\}=s^{\beta-1}e^{-x}+\frac{1}{s}+L\left\{u\frac{\partial v}{\partial x}\right\}+L\left\{v\right\}$$
$$L\left\{u\left(x,t\right)\right\}=\frac{e^{x}}{s}+\frac{1}{s^{\alpha+1}}-\frac{1}{s^{\alpha}}L\left\{v\frac{\partial u}{\partial x}\right\}-\frac{1}{s^{\alpha}}L\left\{u\right\}$$
$$L\left\{v\left(x,t\right)\right\}=\frac{e^{-x}}{s}+\frac{1}{s^{\beta+1}}+\frac{1}{s^{\beta}}L\left\{u\frac{\partial v}{\partial x}\right\}+\frac{1}{s^{\beta}}L\left\{v\right\}$$

Next, we apply the inverse Laplace transformation, obtaining the solution in the domain. This step converts the transformed equations back to their initial form,

$$u\left(x,t\right)=L^{-1}\left\{\frac{e^{x}}{s}+\frac{1}{s^{\alpha+1}}-\frac{1}{s^{\alpha}}L\left\{v\frac{\partial u}{\partial x}\right\}-\frac{1}{s^{\alpha}}L\left\{u\right\}\right\}$$
$$v\left(x,t\right)=L^{-1}\left\{\frac{e^{-x}}{s}+\frac{1}{s^{\beta+1}}+\frac{1}{s^{\beta}}L\left\{u\frac{\partial v}{\partial x}\right\}+\frac{1}{s^{\beta}}L\left\{v\right\}\right\}$$

Now, the first iteration is obtained by substituting the initial conditions into the transformed equations, providing an approximation that serves as the starting point for further iterations.

$$\begin{aligned}u_0\left(x,t\right)&=L^{-1}\left\{\frac{e^{x}}{s}+\frac{1}{s^{\alpha+1}}\right\}=e^{x}+\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)}\\ v_0\left(x,t\right)&=L^{-1}\left\{\frac{e^{-x}}{s}+\frac{1}{s^{\beta+1}}\right\}=e^{-x}+\frac{t^{\beta}}{\Gamma\left(\beta+1\right)}\end{aligned}\tag{3.9}$$

Next, we apply the concept of Adomian polynomials to handle the nonlinear term, expressing it as a series expansion that simplifies the solution process and allows for iterative approximation.

$$v\frac{\partial u}{\partial x}=\sum_{k=0}^{\infty}A_k \;\&\; u\frac{\partial v}{\partial x}=\sum_{k=0}^{\infty}B_k\tag{3.10}$$

Where A_k and B_k are the Adomian polynomials, defined in equation (2), and are expressed in the following manner. These polynomials are used to decompose the nonlinear terms into a series of iterative terms, which allows for efficient computation of the solution.

$$\begin{aligned}
A_0 &= v_0 \frac{\partial u_0}{\partial x}, \\
A_1 &= v_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_0}{\partial x}, \\
A_2 &= v_0 \frac{\partial u_2}{\partial x} + v_1 \frac{\partial u_1}{\partial x} + v_2 \frac{\partial u_0}{\partial x} \\
&\;\;\vdots \\
B_0 &= u_0 \frac{\partial v_0}{\partial x}, \\
B_1 &= u_0 \frac{\partial v_1}{\partial x} + u_1 \frac{\partial v_0}{\partial x}, \\
B_2 &= u_0 \frac{\partial v_2}{\partial x} + u_1 \frac{\partial v_1}{\partial x} + u_2 \frac{\partial v_0}{\partial x}. \\
&\;\;\vdots
\end{aligned}$$

Using all the above estimations, the iterative scheme is given by:

$$\begin{aligned}
u_{k+1}(x,t) &= -L^{-1}\left\{\frac{1}{s^{\alpha}} L\{A_k\}\right\} - L^{-1}\left\{\frac{1}{s^{\alpha}} L\{u_k\}\right\} \\
v_{k+1}(x,t) &= L^{-1}\left\{\frac{1}{s^{\beta}} L\{B_k\}\right\} + L^{-1}\left\{\frac{1}{s^{\beta}} L\{v_k\}\right\}.
\end{aligned} \tag{3.11}$$

where u_{k+1} and v_{k+1} represent the current approximations at the k^{th} iteration, and A_k and B_k are the Adomian polynomials for the respective nonlinear terms. These iterative schemes allow for successive approximations to converge to the solution of the nonlinear system. Thus, the first terms A_0 & B_0 provide the initial approximations of the nonlinear terms, which will be used in the first iteration of the scheme.

$$\begin{aligned}
A_0 = v_0 \frac{\partial u_0}{\partial x} &= \left(e^{-x} + \frac{t^{\beta}}{\Gamma(\beta+1)}\right) \frac{\partial}{\partial x}\left(e^{x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) \\
&= \left(1 + \frac{t^{\beta} e^{x}}{\Gamma(\beta+1)}\right)
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
B_0 = u_0 \frac{\partial v_0}{\partial x} &= \left(e^{x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) \frac{\partial}{\partial x}\left(e^{-x} + \frac{t^{\beta}}{\Gamma(\beta+1)}\right) \\
&= \left(-1 - \frac{t^{\alpha} e^{-x}}{\Gamma(\alpha+1)}\right).
\end{aligned} \tag{3.13}$$

Now, the next iteration is obtained by applying the recursive relation for the Adomian polynomials to the nonlinear terms as follows

$$\begin{aligned} u_1(x,t) &= -L^{-1}\left\{\frac{1}{s^{\alpha}}L\{A_0\}\right\} - L^{-1}\left\{\frac{1}{s^{\alpha}}L\{u_0\}\right\} \\ &= -\frac{t^{\alpha}}{\Gamma(\alpha+1)}(1+e^{x}) - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}e^{x} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} v_1(x,t) &= L^{-1}\left\{\frac{1}{s^{\beta}}L\{B_0\}\right\} + L^{-1}\left\{\frac{1}{s^{\beta}}L\{v_0\}\right\} \\ &= -\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{\beta}}{\Gamma(\beta+1)}e^{-x} - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}e^{-x} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} \end{aligned} \tag{3.15}$$

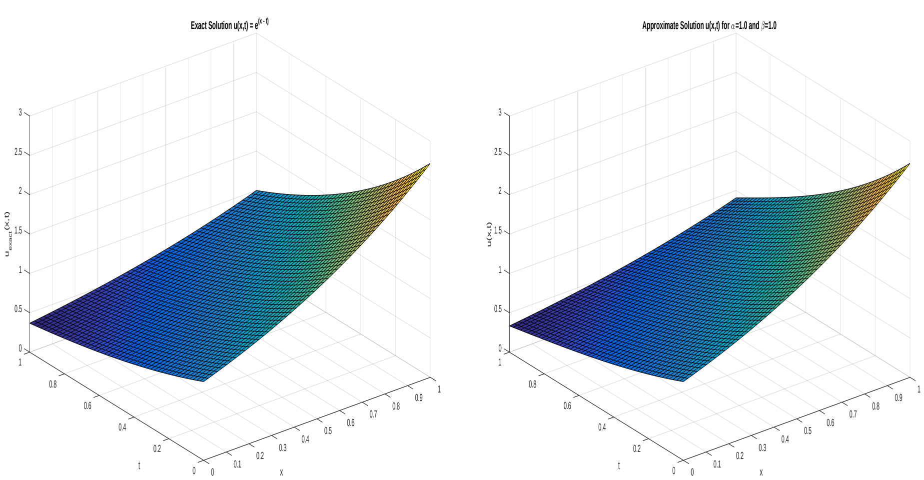

Figure 6. Plot of the approximate and exact solution of (Example 3.2) for $u(x,t)$ and $\alpha = \beta = 1$.

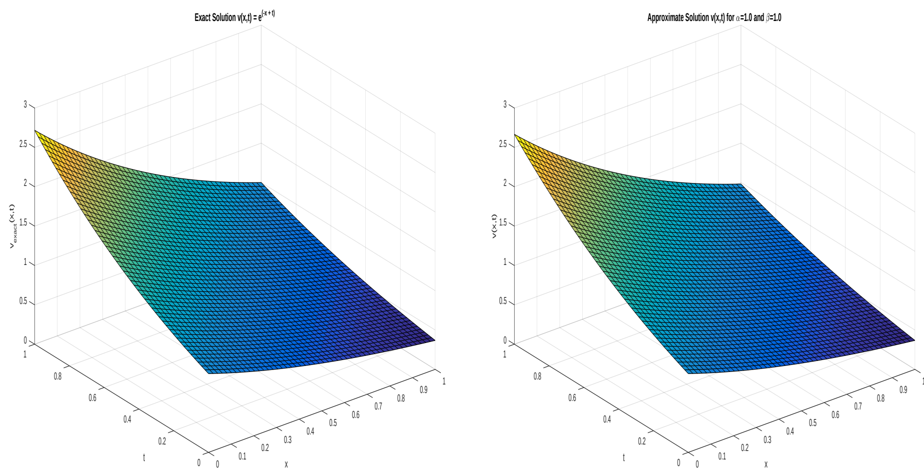

FIGURE 7. Plot of the approximate and exact solution of (Example 3.2) for $v(x,t)$ and $\alpha = \beta = 1$.

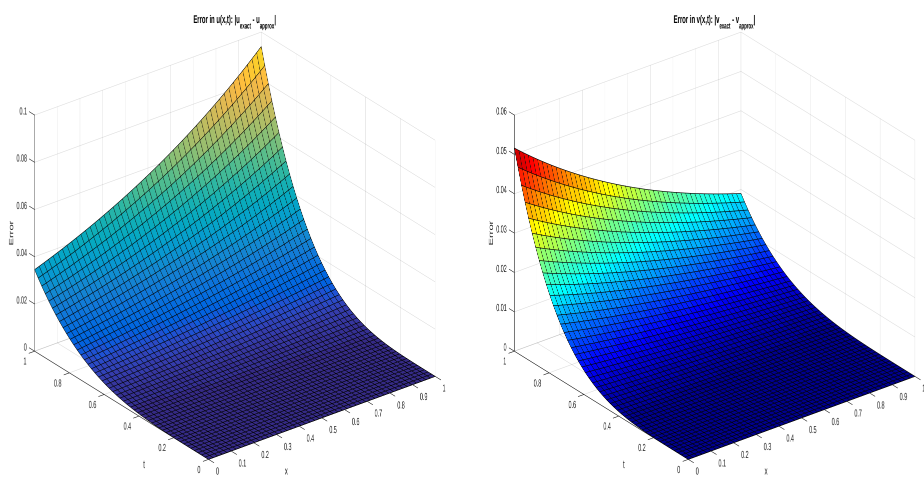

FIGURE 8. Plot of error of estimation $|u_{exact}(x,t) - u_{approx}(x,t)|$ and $|v_{exact}(x,t) - v_{approx}(x,t)|$ for Example 3.2 for $\alpha = \beta = 1$.

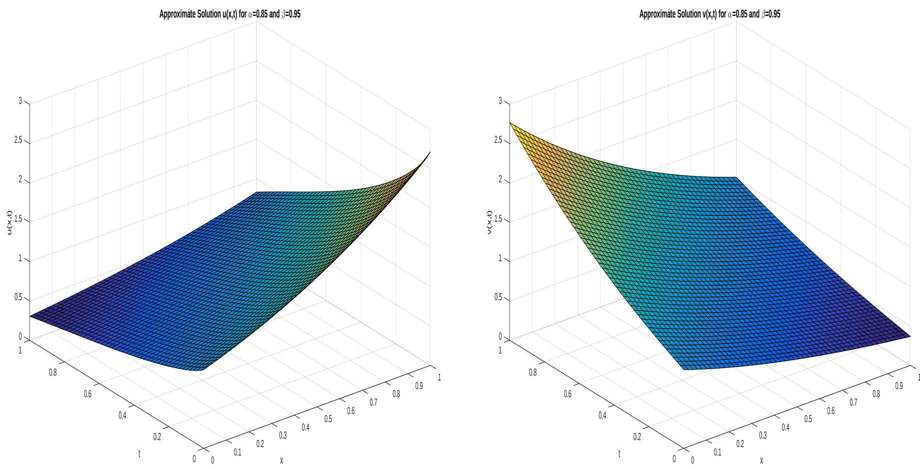

FIGURE 9. Plot of $u(x,t)$ and $v(x,t)$ to Example 3.2 for $\alpha = 0.85\beta = 0.95$.

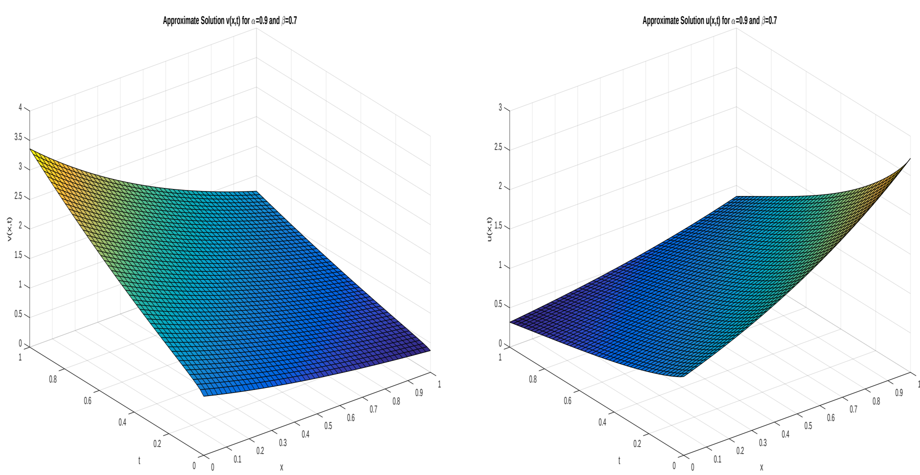

FIGURE 10. Plot of the approximate solution of (Example 3.2) for $u(x,t)$ & $v(x,t)$ and $\alpha = 0.9, \beta = 0.7$.

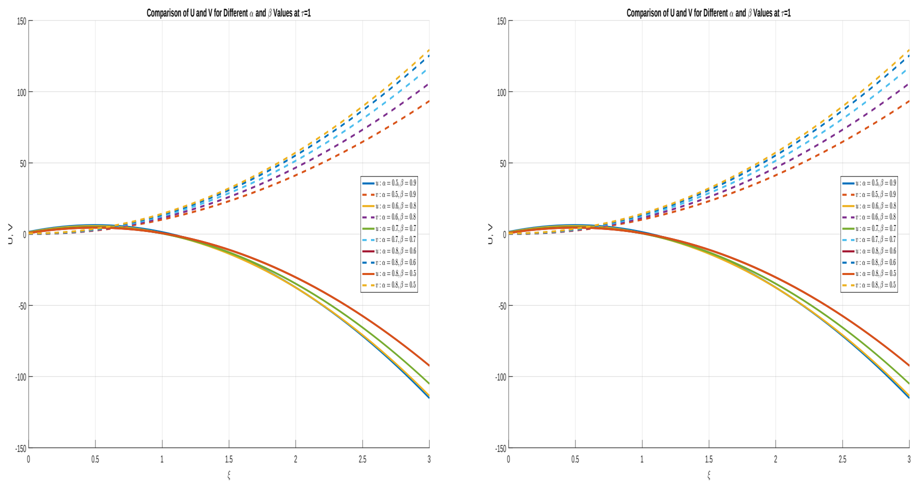

FIGURE 11. Plot of the comparision approximate of solutions of Example 3.2 for $u(x,t)$ and $v(x,t)$ for different values of α and β by fixing $t=1$ & $t=3$

In the simillar fashion we evaulate the next iterations and the sum of the iterations provides the approximate semi-analytic solution for Example 3.2. This approach allows for a systematic refinement of the solution with each iteration, progressively improving the accuracy of the approximation as more terms are included in the series.

$$\begin{aligned} u(x,t) &= e^{x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)} - e^{x}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}e^{x} \\ &- \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{x}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ &+ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{t^{3\alpha}e^{x}}{\Gamma(3\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{t^{2\alpha+\beta}e^{x}}{\Gamma(2\alpha+\beta+1)} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} v(x,t) &= e^{-x} + \frac{t^{\beta}}{\Gamma(\beta+1)} - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{\beta}}{\Gamma(\beta+1)}e^{-x} - \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}e^{-x} \\ &+ \frac{t^{2\beta}}{\Gamma(2\beta+1)} - \frac{t^{2\beta}}{\Gamma(2\beta+1)} + e^{-x}\frac{t^{2\beta}}{\Gamma(2\beta+1)} + e^{-x}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\ &+ \frac{t^{3\beta}}{\Gamma(3\beta+1)} - \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \frac{t^{3\beta}e^{-x}}{\Gamma(3\beta+1)} - \frac{t^{\alpha+2\beta}e^{-x}}{\Gamma(\alpha+2\beta+1)} + \frac{t^{4\beta}}{\Gamma(4\beta+1)} \end{aligned} \tag{3.17}$$

Which seems to be a complex structure of the solution, let us simplify. After rearranging the terms, we get a more manageable form that allows for easier computation and clearer interpretation of the solution.

$$\begin{aligned} u(x,t) &= e^{x} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}e^{x} + e^{x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}e^{x}}{\Gamma(3\alpha+1)} + \cdots \\ v(x,t) &= e^{-x} + \frac{t^{\beta}}{\Gamma(\beta+1)}e^{-x} + e^{-x}\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)}e^{-x} + \cdots \end{aligned} \tag{3.18}$$

This approach converges to the exact solution $u(x,t) = e^{x-t}, v(x,t) = e^{-x+t}$ for the limiting case when $\alpha = 1$ & $\beta = 1$. In this case, the iterative process becomes exact, yielding the known analytical solution at limiting case.

TABLE 2. Exact vs Approximate Solutions for $u(x,t) = e^{x-t}$, $v(x,t) = e^{-x+t}$, with $\alpha = 0.9$, $\beta = 0.95$, $x = 1$

t	u_{exact}	u_{approx}	$\lvert u_{\text{error}}\rvert$	v_{exact}	v_{approx}	$\lvert v_{\text{error}}\rvert$
0.01	1.788	1.788	0.0004	0.406	0.406	0.0003
0.02	1.768	1.767	0.0013	0.414	0.414	0.0010
0.03	1.748	1.747	0.0029	0.423	0.422	0.0019
0.04	1.729	1.726	0.0047	0.431	0.430	0.0029
0.05	1.710	1.705	0.0068	0.440	0.438	0.0040
0.06	1.691	1.684	0.0091	0.449	0.446	0.0052
0.07	1.673	1.662	0.0115	0.458	0.455	0.0065
0.08	1.655	1.641	0.0140	0.468	0.464	0.0079
0.09	1.638	1.620	0.0170	0.477	0.472	0.0095
0.10	1.622	1.598	0.0201	0.487	0.481	0.0111

4. Conclusion

This work demonstrates the successful application of the Laplace decomposition method to solve a system of non-homogeneous mixed-order partial differential equations with specified initial conditions. In Examples 3.1 and 3.2, we addressed problems involving three and two independent variables, respectively, with two dependent variables. Through this approach, we obtained approximate semi-analytic solutions and analyzed the behavior of these solutions for various parameter values. In this work we compared the approximate solution with exact solution by means of graphical represntation. Additionally, we computed the errors between the exact and approximate solutions for different fractional orders. The results indicate that the Laplace decomposition method provides an effective and reliable means for obtaining approximate solutions to such complex problems. This method is particularly advantageous for handling nonlinearities and fractional orders, making it a valuable tool for solving similar types of differential equations in mathematical and engineering applications.

When it comes to solving nonlinear systems of mixed-order fractional partial differential equations, the suggested Laplace Decomposition Method (LDM) has various advantages. It offers speedily convergent semi-analytical solutions, effectively manages nonlinearities, and avoids discretization and linearization. The complex processes of producing Adomian polynomials for extremely nonlinear systems and the reliance on invertible Laplace transforms are obstacles, though. The approach may be expanded in future research to include stochastic models, variable-order systems, and boundary value issues. Its performance might also be confirmed by automated symbolic computation and compared with other techniques like HAM or q-HATM.

References

1. M. J. Ablowitz, P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge Univ. Press, 1991.
2. F. Cariello, M. Tabor, *Painlevé expansions for nonintegrable evolution equations*, Physica D **39** (1989), 77–94.

3. J. Biazar, M. Eslami, *A new homotopy perturbation method for solving systems of partial differential equations*, Comput. Math. Appl. **62** (2011), 225–234.
4. J. Biazar, M. Eslami, *A new homotopy perturbation method for solving systems of partial differential equations*, Comput. Math. Appl. **62** (2011), 225–234.
5. A. Elsaid, *Adomian polynomials: A powerful tool for iterative methods of series solution of nonlinear equations*, J. Appl. Anal. Comput. **2** (2012).
6. J. Fadaei, *Application of Laplace–Adomian decomposition method on linear and nonlinear systems of PDEs*, Appl. Math. Sci. **5** (2011), 1307–1315.
7. S. S. Handibag, J. H. Bhosale, *Solution of nonlinear integro-differential equations by using Laplace decomposition method and series solution method*, Indian J. Sci. Technol. **18** (2025), no. 12, 974–982.
8. H. Khan, R. Shah, P. Kumam, D. Baleanu, M. Arif, *Laplace decomposition for solving nonlinear systems of fractional-order partial differential equations*, Adv. Difference Equ. **2020** (2020), 1–18.
9. S. A. Khuri, *A Laplace decomposition algorithm applied to a class of nonlinear differential equations*, J. Math. Anal. Appl. **4** (2001).
10. S. A. Khuri, *A new approach to the Bratu problem*, Appl. Math. Comput. **147** (2004).
11. K. Konno, M. Wadati, *Simple derivation of Bäcklund transformation from Riccati form of inverse method*, Prog. Theor. Phys. **53** (1975), 1652–1656.
12. A. Kumar, R. Arora, *Solutions of the coupled system of Burgers' equations and coupled Klein–Gordon equation by RDT method*, Int. J. Adv. Appl. Math. Mech. **1** (2013), 103–115.
13. F. Hussain, *Laplace decomposition method for systems of linear and nonlinear ordinary differential equations*, Math. Theory Model. **5** (2015), No. 12.
14. H. Jafari, H. K. Jassim, *Numerical solutions of telegraph and Laplace equations on Cantor sets using local fractional Laplace decomposition method*, Int. J. Adv. Appl. Math. Mech. **2** (2015), 144–151.
15. J. Li, L. Zhang, *Bifurcations of traveling wave solutions in generalized Pochhammer–Chree equations*, Chaos Solitons Fractals **14** (2002), 581–593.
16. S. Liu, Z. Fu, S. Liu, Q. Zhao, *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Phys. Lett. A **289** (2001), 69–74.
17. V. B. Matveev, M. A. Salle, *Darboux transformations and solitons*, Springer, 1991.
18. V. Nikam, D. Gopal, H. ur Rehman, T. Bantaojai, *New Geraghty-type condensing operators and solvability of nonlinear quadratic Volterra–Stieltjes integral equations*, Nonlinear Funct. Anal. Appl. **25** (2020), 769–797.
19. U. M. Pirzada, D. C. Vakaskar, *Solution of fuzzy heat equations using Adomian decomposition method*, Int. J. Adv. Appl. Math. Mech. **3** (2015), 87–91.
20. L. Verma, R. Meher, D. P. Pandya, *Parameter estimation study of temporal fractional HIV/AIDS transmission model with fractal dimensions using real data in India*, Math. Comput. Simul. **234** (2025), 135–150.
21. M. Wadati, H. Sanuki, K. Konno, *Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws*, Prog. Theor. Phys. **53** (1975), 419–436.
22. M. Wang, Y. Zhou, Z. Li, *Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics*, Phys. Lett. A **216** (1996), 67–75.
23. A.-M. Wazwaz, *Linear and nonlinear integral equations: Methods and applications.*
24. A. N. Wazwaz, *A new technique for calculating Adomian polynomials for nonlinear problems*, Appl. Math. Comput. **111** (2000).