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ON SOME COMMON FIXED POINT RESULTS IN HYPERBOLIC SPACES

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ABSTRACT. The aim of this manuscript is to establish a common fixed point theorem for two uniformly L-Lipschitzian and asymptotically quasi-nonexpansive non-self maps with respect to retraction $\mathcal{P}$ via implicit algorithm and to prove common fixed point results of two weakly inward and asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ satisfying condition $(\mathcal{A})$ and condition $(\mathcal{B})$, in a more general set up of hyperbolic space. Our results generalize, extend and improve some related results in the existing literature.

KEYWORDS: common fixed point, asymptotically quasi-nonexpansive mapping, Banach space, CAT(0) space, hyperbolic space.
AMS Subject Classification: 47H10; 54H25.

1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a real **normed linear space**, and let $\mathbb{E}$ be a nonempty closed convex subset of $\mathbb{X}$. Let $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{E}$ be a self-mapping. Let $F(\mathbb{T})$ denote the set of fixed points of $\mathbb{T}$, that is, $F(\mathbb{T}) = \{x \in \mathbb{E} : \mathbb{T}x = x\}$. A self-mapping $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{E}$ is said to be

(1). asymptotically nonexpansive [8] if there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} a_n = 1$ and such that

$$\|\mathbb{T}^n x - \mathbb{T}^n y\| \le a_n \|x - y\|, \forall x, y \in \mathbb{E}, n \ge 1. \tag{1.1}$$

(2). asymptotically quasi-nonexpansive if $F(\mathbb{T}) \neq \emptyset$ and there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} a_n = 1$ and such that

$$\|\mathbb{T}^n x - p\| \le a_n \|x - p\|, \forall x, \in \mathbb{E}, p \in F(\mathbb{T}) \text{ and } n \ge 1. \tag{1.2}$$

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(3). uniformly L-Lipschitzian if there exists constant $L \geq 0$ such that

$$\|\mathbb{T}^n x - \mathbb{T}^n y\| \leq L\|x - y\|, \forall x, y \in \mathbb{E},\, n \geq 1. \tag{1.3}$$

Note that an asymptotically nonexpansive mapping must be uniformly L-Lipschitzian as well as asymptotically quasi-nonexpansive but the converse does not hold true in general.

In 2003, Chidume *et al.*[7] introduced the notion of asymptotically nonexpansive non-self mappings as a generalization of asymptotically nonexpansive self-mappings as follows.

Definition 1.1. [7] Let $\mathbb{E}$ be a nonempty subset of real normed linear space $\mathbb{X}$. Let $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{X}$ be a nonself mapping and $\mathcal{P} : \mathbb{X} \longrightarrow \mathbb{E}$ be the nonexpansive retraction of $\mathbb{X}$ into $\mathbb{E}$. $\mathbb{T}$ is said to be

(1). asymptotically nonexpansive if there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} a_n = 1$ and such that

$$\|\mathbb{T}(\mathcal{P}\mathbb{T})^{n-1}x - \mathbb{T}(\mathcal{P}\mathbb{T})^{n-1}y\| \leq a_n\|x - y\|, \forall x, y \in \mathbb{E}, n \geq 1. \tag{1.4}$$

(2). uniformly L-Lipschitzian if there exists constant $L \geq 0$ such that

$$\|\mathbb{T}(\mathcal{P}\mathbb{T})^{n-1}x - \mathbb{T}(\mathcal{P}\mathbb{T})^{n-1}y\| \leq L\|x - y\|, \forall x, y \in \mathbb{E}, n \geq 1. \tag{1.5}$$

Chidume *et al.*[7] established a demiclosed principle, weak and strong convergence results for such mappings in a uniformly convex Banach space via the following algorithm:

$$x_1 \in \mathbb{E}, x_{n+1} = \mathcal{P}\big((1 - \alpha_n)x_n + \alpha_n \mathbb{T}(\mathcal{P}\mathbb{T})^{n-1}x_n\big),\, n \geq 1.$$

After Chidume *et al.*[7], a number of authors have studied the weak and strong convergence for such mappings (see [10, 11, 15, 12, 26, 29, 33] for examples).

Later, in 2007 Zhou *et al.*[34] introduced the following generalized definition.

Definition 1.2. [34] Let $\mathbb{E}$ be a nonempty subset of real normed linear space $\mathbb{X}$. Let $\mathcal{P} : \mathbb{X} \longrightarrow \mathbb{E}$ be the nonexpansive retraction of $\mathbb{X}$ into $\mathbb{E}$. A nonself mapping $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{X}$ is said to be

(1). asymptotically nonexpansive with respect to $\mathcal{P}$ if there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} a_n = 1$ and such that

$$\|(\mathcal{P}\mathbb{T})^n x - (\mathcal{P}\mathbb{T})^n y\| \leq a_n\|x - y\|, \forall x, y \in \mathbb{E},\, n \geq 1. \tag{1.6}$$

(2). uniformly L-Lipschitzian with respect to $\mathcal{P}$ if there exists constant $L \geq 0$ such that

$$\|(\mathcal{P}\mathbb{T})^n x - (\mathcal{P}\mathbb{T})^n y\| \leq L\|x - y\|, \forall x, y \in \mathbb{E},\, n \geq 1. \tag{1.7}$$

Zhou *et al.* [34] introduced the following iterative process,

$$x_1 \in \mathbb{E},\, x_{n+1} = \alpha_n x_n + \beta_n(\mathcal{P}\mathbb{T}_1)x_n + \gamma_n(\mathcal{P}\mathbb{T}_2)x_n,\, n \geq 1, \tag{1.8}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are three sequences in $[a, 1-a]$ for some $a \in (0, 1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$. And, they established some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to $\mathcal{P}$ in uniformly convex Banach spaces.

In 2007, Agarwal *et al.*[2] introduced the iterative algorithm:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)\mathbb{T}^n x_n + \alpha_n \mathbb{T}^n y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n \mathbb{T}^n x_n,\, n \geq 1. \end{aligned} \tag{1.9}$$

Later, in 2011 Turkmen *et al.*[32] used the following iterative process to establish common fixed point results of two asymptotically nonexpansive mappings.

$$\begin{aligned} x_1 &\in \mathbb{E}, \\ x_{n+1} &= (1-\alpha_n)(\mathcal{P}\mathbb{T}_1)y_n + \alpha_n(\mathcal{P}\mathbb{T}_2)y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n(\mathcal{P}\mathbb{T}_1)x_n,\ n \geq 1, \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying certain conditions.

Inspired by the work mentioned above, Khan *et al.*[16] generalized the definition of nonself asymptotically nonexpansive mappings with respect to $\mathcal{P}$ to nonself asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$. Thus, a nonself mapping $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{X}$ is said to be an asymptotically quasi-nonexpansive with respect to $\mathcal{P}$ if $F(\mathbb{T}) \neq \emptyset$ and there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1,\infty)$ with $\lim\limits_{n\longrightarrow\infty} a_n = 1$ such that

$$\|(\mathcal{P}\mathbb{T})^n x - p\| \leq a_n\|x-p\|, \forall x \in \mathbb{E},\ n \geq 1. \tag{1.11}$$

Khan *et al.*[16] introduced the following iterative process. Let $\mathbb{E}$ be a nonempty closed convex subset of a real **normed linear space** $\mathbb{X}$ with retraction $\mathcal{P}$. Let $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ be two nonself asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$. Their iterative scheme reads as follows:

$$\begin{aligned} x_1 &\in \mathbb{E}, \\ x_{n+1} &= (1-\alpha_n)(\mathcal{P}\mathbb{T}_1)^n x_n + \alpha_n(\mathcal{P}\mathbb{T}_2)^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n(\mathcal{P}\mathbb{T}_1^n)x_n,\ n \geq 1, \end{aligned} \tag{1.12}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0,1)$.

Khan *et al.*[16] obtained strong convergence theorems for two asymptotically quasi-nonexpansive mappings using a general and independent two-step iterative process (1.12) assuming compactness of only one of the two mappings in smooth Banach space. They also proved a weak convergence result under Opial's condition.

Question: Can we can extend and improve those results in [16] from a uniformly convex and smooth Banach space to a more general set up of a uniformly convex hyperbolic space? The answer is affirmative.

We recall the following definition.

Definition 1.3. [6] Let $(\mathbb{X}, \mathfrak{d})$ be a metric space and $\mathbb{E}$ be a nonempty subset of $\mathbb{X}$ and $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{E}$ be a self-mapping. Then $\mathbb{T}$ is said to be

(i). asymptotically nonexpansive if there exists a sequence $\{a_n\} \subset [1,\infty)$ with $\lim\limits_{n\longrightarrow\infty} a_n = 1$ such that

$$\mathfrak{d}(\mathbb{T}^n x, \mathbb{T}^n y) \leq a_n \mathfrak{d}(x,y),\ \forall\, x, y \in \mathbb{E} \text{ and } \forall n \geq 1. \tag{1.13}$$

(ii). asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{a_n\} \subset [1,\infty)$ with $\lim\limits_{n\longrightarrow\infty} a_n = 1$ such that

$$\mathfrak{d}(\mathbb{T}^n x, p) \leq a_n \mathfrak{d}(x,p), \forall x, y \in \mathbb{E}, \forall p \in F(\mathbb{T}) \text{ and } \forall n \geq 1. \tag{1.14}$$

(ii). uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\mathfrak{d}(\mathbb{T}^n x, \mathbb{T}^n y) \leq L\mathfrak{d}(x,y),\ \forall x, y \in \mathbb{E} \text{ and } \forall n \geq 1. \tag{1.15}$$

Note that if $F(\mathbb{T})$ is nonempty, then nonexpansive mapping, quasi-nonexpansive mapping, asymptotically nonexpansive mapping all are the special cases of asymptotically quasi-nonexpansive type mappings.

The purpose of this paper is to extend and improve the results of Khan *et al.*[16] from the setting of a real uniformly convex and smooth Banach space to the setting of a uniformly convex hyperbolic space, a more general setting.

2. Preliminaries

In this section, we recall an important definition of hyperbolic space which will be crucial for our main results.

Definition 2.1. [18] Let $(\mathbb{X}, \mathfrak{d})$ is a metric space. A hyperbolic space is a triple $(\mathbb{X}, \mathfrak{d}, W)$, where $W : \mathbb{X} \times \mathbb{X} \times [0, 1] \longrightarrow \mathbb{X}$ is such that

(W1). $\mathfrak{d}(W(x, y, \alpha), z) \leq (1 - \alpha)\mathfrak{d}(z, x) + \alpha\mathfrak{d}(z, y)$,
(W2). $\mathfrak{d}(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|\mathfrak{d}(x, y)$,
(W3). $W(x, y, \alpha) = W(y, x, (1 - \alpha))$,
(W4). $\mathfrak{d}(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)\mathfrak{d}(x, y) + \alpha\mathfrak{d}(z, w)$ for all $x, y, z, w \in \mathbb{X}$,

where $\alpha, \beta \in [0, 1]$.

It follows from (W1) that, for each $x, y \in \mathbb{X}$ and $\alpha \in [0, 1]$,

$$\mathfrak{d}(W(x, y, \alpha), x) \leq \alpha\mathfrak{d}(x, y), \ \mathfrak{d}(W(x, y, \alpha), y) \leq (1 - \alpha)\mathfrak{d}(x, y).$$

A subset $\mathbb{E}$ of a hyperbolic space $\mathbb{X}$ is convex if $W(x, y, \alpha) \in \mathbb{E}$ for all $x, y \in \mathbb{E}$ and $\alpha \in [0, 1]$. For more detail of convex structure of a metric space, see [27].

We note that the class of hyperbolic spaces also contains Hadamard manifolds [5], and Cartesian products of Hilbert balls, the Hilbert open unit ball equipped with the hyperbolic metric[9], as special cases. It is wellknown that spaces like CAT(0) spaces (in the sense of Gromov) and R-tree (in the sense of Tits) are special cases of hyperbolic spaces. Some remarkable results in CAT(0) spaces and hyperbolic spaces in [14, 19, 17, 20, 22, 23, 24, 25, 30, 31] are examples of nonlinear structures which play a major role in recent research in metric fixed point theory.

A hyperbolic space $(\mathbb{X}, \mathfrak{d}, W)$ is said to be uniformly convex [27] if for all $x, y, u \in \mathbb{X}, r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$such that

$$\left.\begin{array}{l}\mathfrak{d}(x, u) \leq r \\ \mathfrak{d}(y, u) \leq r \\ \mathfrak{d}(x, y) \geq \epsilon r\end{array}\right\} \quad \Rightarrow \quad \mathfrak{d}(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r.$$

A map $\eta : (0, \infty) \times (0, 2] \longrightarrow (0, 1]$ which gives such a number $\delta = \eta(r, \epsilon)$, for a given $r > 0$ and $\epsilon \in (0, 2]$, is called ***modulus of uniform convexity***. The modulus of uniform convexity η is said to be monotone if it decreases with r (for a fixed ϵ). A uniformly convex hyperbolic space is strictly convex (see [20]).

For more interesting results on hyperbolic spaces, we refer readers to [13, 20, 21]. The following lemma is essential for our main results.

Lemma 2.2. [13] *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $u \in \mathbb{X}$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{X}$ such that*

$$\limsup_{n\longrightarrow\infty} \mathfrak{d}(x_n, u) \leq r, \quad \limsup_{n\longrightarrow\infty} \mathfrak{d}(y_n, u) \leq r$$

and

$$\lim_{n\longrightarrow\infty} \mathfrak{d}(W(x_n, y_n, \alpha_n), u) = r$$

for some $r \geq 0$. Then

$$\lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, y_n) = 0.$$

Recall that a subset $\mathbb{E}$ of $\mathbb{X}$ is said to be retract if there exists a mapping $\mathcal{P} : \mathbb{X} \longrightarrow \mathbb{E}$ which is continuous and such that $\mathcal{P}x = x$ for all $x \in \mathbb{E}$. A mapping $\mathcal{P} : \mathbb{X} \longrightarrow \mathbb{X}$ is said to be a retraction if $\mathcal{P}^2 = \mathcal{P}$. Let $\mathbb{K}$ and $\mathbb{E}$ be subsets of a hyperbolic space $\mathbb{X}$. A mapping $\mathcal{P}$ from $\mathbb{K}$ into $\mathbb{E}$ is called sunny if $\mathcal{P}(W(\mathcal{P}x, x, \alpha)) = \mathcal{P}x$ for $x \in \mathbb{K}$ with $W(\mathcal{P}x, x, \alpha) \in \mathbb{K}$ and $\alpha \in [0, 1]$. Note that, if $\mathcal{P}$ is a retraction, then $\mathcal{P}z = z$ for every $z \in R(\mathcal{P})$, the range of $\mathcal{P}$. We note that every closed convex subset of a uniformly convex hyperbolic space is a retract.

For each $x \in \mathbb{E}$, the ***inward set*** $I_{\mathbb{E}}(x)$ is defined by

$$I_{\mathbb{E}}(x) = \{y \in \mathbb{X} : y = W(x, z, \lambda), z \in \mathbb{E}, \lambda \in [0, 1]\}.$$

A mapping $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{X}$ is said to satisfy the ***inward condition*** if $\mathbb{T}x \in I_{\mathbb{E}}(x)$ for all $x \in \mathbb{E}$. $\mathbb{T}$ is said to be weakly inward if, for each $x \in \mathbb{E}, \mathbb{T}x \in cl[I_{\mathbb{E}}(x)]$, where $cl[I_{\mathbb{E}}(x)]$ is the closure of $I_{\mathbb{E}}(x)$.

A Hyperbolic space $(\mathbb{X}, \mathfrak{d}, W)$ is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ (i.e. $\{x_n\}_{n=1}^{\infty}$ converges weakly to x) implies that

$$\limsup_{n \longrightarrow \infty} \mathfrak{d}(x_n, x) < \limsup_{n \longrightarrow \infty} \mathfrak{d}(x_n, y),$$

for all $y \in \mathbb{X}$ with $y \neq x$.

From now on, let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two maps on $\mathbb{E}$, we denote $F(\mathbb{T}_1) = \{x : \mathbb{T}_1 x = x\}$, $F(\mathbb{T}_2) = \{x : \mathbb{T}_2 x = x\}$ the set of fixed point of $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively, and $\mathbb{F} = \big(F(\mathbb{T}_1) \cap F(\mathbb{T}_2)\big) \neq \emptyset$.

Recall that a sequence $\{x_n\}$ in a metric space $\mathbb{X}$ is said to be **Fejér monotone** with respect to $\mathbb{E}$ (a subset of $\mathbb{X}$) if $\mathfrak{d}(x_{n+1}, x) \leq \mathfrak{d}(x_n, x)$ for all $x \in \mathbb{E}$ and for all $n \geq 1$.

A map $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{E}$ is said to be ***semi-compact*** if any bounded sequence $\{x_n\}$ satisfying $\mathfrak{d}(x_n, \mathbb{T}x_n) \longrightarrow 0$ as $n \longrightarrow \infty$, has a convergent subsequence.

Let f be a nondecreasing selfmap on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ and let $\mathfrak{d}(x, A) = \inf\{\mathfrak{d}(x, y) : y \in A\}$.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two mappings on $\mathbb{E}$ with $\mathbb{F} \neq \emptyset$. Then the two mappings are said to satisfy:

(i). **condition** $(\mathcal{A})$ on $\mathbb{E}$ if

$$f(\mathfrak{d}(x, \mathbb{F})) \leq \mathfrak{d}(x, \mathbb{T}_1 x) \ \text{ or } \ f(\mathfrak{d}(x, \mathbb{F})) \leq \mathfrak{d}(x, \mathbb{T}_2 x)$$

for all $x \in \mathbb{E}$, holds for at least one $\mathbb{T}_i, i = 1, 2$.

(ii). **condition** $(\mathcal{B})$ on $\mathbb{E}$ if

$$f(\mathfrak{d}(x, \mathbb{F})) \leq \frac{1}{2}\big[\mathfrak{d}(x, \mathbb{T}_1 x) + \mathfrak{d}(x, \mathbb{T}_2 x)\big]$$

holds for all $x \in \mathbb{E}$.

Lemma 2.3. [3] *Let $(\mathbb{X}, \mathfrak{d})$ be a complete metric space and $\mathbb{E}$ be a nonempty closed subset of $\mathbb{X}$, and $\{x_n\}$ be Fejér monotone with respect to $\mathbb{E}$. Then $\{x_n\}$ converges to some $p \in \mathbb{E}$ if and only if $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, \mathbb{E}) = 0$.*

The following lemma is very useful.

Lemma 2.4. [28] *Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n,\ n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i). $\lim_{n \longrightarrow \infty} a_n$ *exists.*

(ii). *In particular, if* $\{a_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to zero, then* $\lim\limits_{n\longrightarrow\infty} a_n = 0$.

For more details of results obtained by Khan *et al.* in the context of uniformly convex and smooth Banach spaces, we refer readers to [16].

Inspired and motivated by the works of Khan *et al.*[16] and some related results, we establish common fixed point theorems for two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ in the setting of hyperbolic space, a more general set up. Our results extend and improve the results obtained by Khan *et al.*[16], as well as many related results in CAT(0) spaces and uniformly Banach spaces.

3. Main Results

Let $\{a_n^{(1)}\}_{n=1}^{\infty} \subset [1,\infty)$ and $\{a_n^{(2)}\}_{n=1}^{\infty} \subset [1,\infty)$ be sequences satisfying the asymptotically quasi-nonexpansive mappings $\mathbb{T}_1$ and $\mathbb{T}_2$ with $\sum_{n=1}^{\infty}(a_n^{(i)}-1) < \infty$, $(i=1,2)$. Let $a_n = \max\{a_n^{(1)}, a_n^{(2)}\}$, and throughout this section, we will take only sequence $\{a_n\}_{n=1}^{\infty} \subset [1,\infty)$ satisfying $\sum_{n=1}^{\infty}(a_n-1) < \infty$.

Let $(\mathbb{X},\mathfrak{d},W)$ be a hyperbolic space and $\mathbb{E}$ be a nonempty closed convex subset of $\mathbb{X}$ with retraction $\mathcal{P}$. Let $\mathbb{T}_1,\mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ be two non-self asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$. We define the sequence $\{x_n\}_{n=1}^{\infty}$ in a hyperbolic space as follows:-

$$\begin{aligned} x_1 &\in \mathbb{E}, \\ x_{n+1} &= W\big((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n\big), \\ y_n &= W\big(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n\big),\ \forall n \geq 1. \end{aligned} \tag{3.1}$$

We introduce the following definition.

Definition 3.1. Let $(\mathbb{X},\mathfrak{d})$ be a metric space and $\mathbb{E}$ be a nonempty subset of $\mathbb{X}$ and $\mathbb{T} : \mathbb{E} \longrightarrow \mathbb{X}$ be a nonself mapping with respect to retraction $\mathcal{P}$. Then $\mathbb{T}$ said to be:

(i). asymptotically nonexpansive with respect to $\mathcal{P}$ if there exists a sequence $\{a_n\} \subset [1,\infty)$ with $\lim\limits_{n\longrightarrow\infty} a_n = 1$ such that

$$\mathfrak{d}((\mathcal{P}\mathbb{T})^n x, (\mathcal{P}\mathbb{T})^n y) \leq a_n \mathfrak{d}(x,y),\ \forall\, x,y \in \mathbb{E}\,\text{and}\ \forall\ n \geq 1. \tag{3.2}$$

(ii). asymptotically quasi-nonexpansive with respect to $\mathcal{P}$ if $F(\mathbb{T}) \neq \emptyset$ and there exists a sequence $\{a_n\} \subset [1,\infty)$ with $\lim\limits_{n\longrightarrow\infty} a_n = 1$ such that

$$\mathfrak{d}((\mathcal{P}\mathbb{T})^n x, p) \leq a_n \mathfrak{d}(x,p), \forall x,y \in \mathbb{E}, \forall p \in F(\mathbb{T})\,\text{and}\ \forall n \geq 1. \tag{3.3}$$

(ii). uniformly L-Lipschitzian with respect to $\mathcal{P}$ if there exists a constant $L > 0$ such that

$$\mathfrak{d}((\mathcal{P}\mathbb{T})^n x, (\mathcal{P}\mathbb{T})^n y) \leq L\mathfrak{d}(x,y),\ \forall x,y \in \mathbb{E}\ \text{and}\ \ \forall n \in \mathbb{N}. \tag{3.4}$$

We first prove two technical lemmas.

Lemma 3.2. *Let* $(\mathbb{X},\mathfrak{d},W)$ *be a hyperbolic space with monotone modulus of uniform convexity* η *and* $\mathbb{E}$ *be a nonempty closed convex subset of* $\mathbb{X}$ *which is also a nonexpansive retract of* $\mathbb{X}$. *Let* $\mathbb{F} \neq \emptyset$ *and* $\mathbb{T}_1,\mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ *be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to* $\mathcal{P}$ *with sequence* $\{a_n\}_{n=1}^{\infty} \subset [1,\infty)$, $\lim\limits_{n\longrightarrow\infty} a_n = 1$ *satisfying* $\sum_{n=1}^{\infty}(a_n-1) < \infty$. *Suppose that* $\{x_n\}_{n=1}^{\infty}$ *is defined by* (3.1), *where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequences in* $[a, 1-a]$ *for some* $a \in (0,1)$. *Then* (i). $\lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, p)$ *exists,* $\forall p \in \mathbb{F}$.

(ii). *There exists a constant* $\mathcal{C} > 0$ *such that* $\mathfrak{d}(x_{n+m}, p) \leq \mathcal{C}\mathfrak{d}(x_n, p)$, $\forall m, n \in \mathbb{N}$ *and* $p \in \mathbb{F}$.

Proof. (i). Let $p \in \mathbb{F}$. From (3.1), we have

$$\begin{aligned}
\mathfrak{d}(y_n, p) &= \mathfrak{d}\big(W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), p\big) \\
&\leq (1-\beta_n)\mathfrak{d}(x_n, p) + \beta_n \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) \\
&\leq (1-\beta_n)\mathfrak{d}(x_n, p) + \beta_n a_n \mathfrak{d}(x_n, p) \\
&= (1+\beta_n(a_n-1))\mathfrak{d}(x_n, p) \\
&\leq (1+(a_n-1))\mathfrak{d}(x_n, p) \\
&= a_n \mathfrak{d}(x_n, p).
\end{aligned} \tag{3.5}$$

From (3.1) and (3.5), we have

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, p) &= \mathfrak{d}\big(W((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), p\big) \\
&\leq (1-\alpha_n)\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) + \alpha_n \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, p) \\
&\leq (1-\alpha_n)a_n \mathfrak{d}(x_n, p) + \alpha_n a_n \mathfrak{d}(y_n, p) \\
&\leq (1-\alpha_n)a_n \mathfrak{d}(x_n, p) + \alpha_n a_n^2 \mathfrak{d}(x_n, p) \\
&= (1+\alpha_n a_n(a_n-1))\mathfrak{d}(x_n, p) \\
&\leq (1+a_n(a_n-1))\mathfrak{d}(x_n, p) \\
&\leq (1+(a_n^2-1))\mathfrak{d}(x_n, p).
\end{aligned} \tag{3.6}$$

Note that $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. This implies $\sum_{n=1}^{\infty}(a_n^2 - 1) < \infty$. Thus, by Lemma 2.4, $\lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, p)$ exists, $\forall p \in \mathbb{F}$.

(ii). From (3.6), we have

$$\mathfrak{d}(x_{n+1}, p) \leq (1+(a_n^2-1))\mathfrak{d}(x_n, p). \tag{3.7}$$

We know that $1 + x \leq e^x$ for all $x \geq 0$. Using it for the above inequality (3.7), we have

$$\begin{aligned}
\mathfrak{d}(x_{n+m}, p) &\leq (1+(a_{n+m-1}^2-1))\mathfrak{d}(x_{n+m-1}, p) \\
&\leq e^{a_{n+m-1}^2-1}\mathfrak{d}(x_{n+m-1}, p) \\
&\leq [e^{a_{n+m-1}^2-1}]\big(1+(a_{n+m-2}^2-1)\big)\mathfrak{d}(x_{n+m-2}, p) \\
&\leq [e^{(a_{n+m-1}^2-1+a_{n+m-2}^2-1)}]\mathfrak{d}(x_{n+m-2}, p) \\
&\vdots \\
&\leq [e^{\sum_{j=n}^{n+m-1}(a_j^2-1)}]\mathfrak{d}(x_n, p) \\
&= \mathcal{C}\mathfrak{d}(x_n, p),
\end{aligned} \tag{3.8}$$

where $\mathcal{C} = e^{\sum_{j=n}^{n+m-1}(a_j^2-1)}$. That is, $\mathfrak{d}(x_{n+m}, p) \leq \mathcal{C}\mathfrak{d}(x_n, p)$ for all $n, m \in \mathbb{N}$ and $p \in \mathbb{F}$. □

Lemma 3.3. *Let* $(\mathbb{X}, \mathfrak{d}, W)$ *be a hyperbolic space with monotone modulus of uniform convexity* η *and* $\mathbb{E}$ *be a nonempty closed convex subset of* $\mathbb{X}$ *which is also a nonexpansive retract of* $\mathbb{X}$. *Let* $\mathbb{F} \neq \emptyset$ *and* $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ *be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to* $\mathcal{P}$ *with sequence* $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n\longrightarrow\infty} a_n = 1$ *satisfying* $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. *Suppose* *that* $\{x_n\}_{n=1}^{\infty}$ *is defined by* (3.1)*, where* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are sequences in* $[a, 1-a]$ *for some* $a \in (0,1)$*. Then*

$$\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n) = 0 = \lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)^n x_n)$$

and also

$$\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1) x_n) = 0 = \lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2) x_n).$$

Proof. By Lemma 3.2, $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, p)$ exists. Assume that $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, p) = \zeta$. Taking lim sup on both sides in the inequality (3.5), we obtain

$$\limsup_{n \longrightarrow \infty} \mathfrak{d}(y_n, p) \leq \limsup_{n \longrightarrow \infty} \mathfrak{d}(x_n, p) = \zeta. \tag{3.9}$$

Next, $\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) \leq a_n \mathfrak{d}(x_n, p)$ for all $n \in \mathbb{N}$ implies that

$$\limsup_{n \longrightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) \leq \zeta. \tag{3.10}$$

Also, by (3.9) we get

$$\limsup_{n \longrightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, p) \leq \limsup_{n \longrightarrow \infty} \mathfrak{d}(y_n, p) \leq \zeta. \tag{3.11}$$

Moreover, from (3.6) we have

$$\lim_{n \longrightarrow \infty} \mathfrak{d}(x_{n+1}, p) = \lim_{n \longrightarrow \infty} \mathfrak{d}\big(W((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), p\big) = \zeta. \tag{3.12}$$

From (3.10), (3.11), (3.12), and Lemma 2.2, we have

$$\lim_{n \longrightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n) = 0. \tag{3.13}$$

Observe that

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, p) &= \mathfrak{d}\big(W((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), p\big) \\
&\leq (1-\alpha_n)\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) + \alpha_n \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, p) \\
&\leq (1-\alpha_n)\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) + \alpha_n [\mathfrak{d}\big((\mathcal{P}\mathbb{T}_2)^n y_n, (\mathcal{P}\mathbb{T}_1)^n x_n\big) \\
&\quad + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n) x_n, p)].
\end{aligned}$$

Taking the limit inf. as $n \longrightarrow \infty$ in the above inequality, and applying (3.13) we get

$$\zeta \leq \liminf_{n \longrightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p). \tag{3.14}$$

From (3.10) and (3.14), we get

$$\lim_{n \longrightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) = \zeta.$$

Furthermore,

$$\begin{aligned}
\mathfrak{d}\big((\mathcal{P}\mathbb{T}_1)^n x_n, p\big) &\leq \mathfrak{d}\big(\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n\big) + \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, p) \\
&\leq \mathfrak{d}\big(\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n\big) + a_n \mathfrak{d}(y_n, p).
\end{aligned} \tag{3.15}$$

This implies

$$\zeta \leq \liminf_{n \longrightarrow \infty} \mathfrak{d}(y_n, p). \tag{3.16}$$

By (3.9) and (3.16), we obtain

$$\lim_{n \longrightarrow \infty} \mathfrak{d}(y_n, p) = \zeta. \tag{3.17}$$

Therefore,

$$\begin{aligned}\zeta = \lim_{n\longrightarrow\infty} d(y_n, p) &= \lim_{n\longrightarrow\infty} \mathfrak{d}\big(W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), p\big)\\ &\leq \lim_{n\longrightarrow\infty} \{(1-\beta_n)\mathfrak{d}(x_n, p) + \beta_n a_n \mathfrak{d}(x_n, p)\}\\ &= \lim_{n\longrightarrow\infty} [1+\beta_n(a_n-1)]\mathfrak{d}(x_n, p)\\ &\leq \lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, p) = \zeta.\end{aligned} \tag{3.18}$$

That is,

$$\lim_{n\longrightarrow\infty} \mathfrak{d}(W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), p) = \zeta. \tag{3.19}$$

We know that $\limsup_{n\longrightarrow\infty} \mathfrak{d}(x_n, p) \leq \zeta$, together with (3.10), (3.19) and Lemma 2.2, we obtain

$$\lim_{n\longrightarrow\infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, x_n) = 0. \qquad (*) \tag{3.20}$$

In addition, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} \mathfrak{d}(y_n, x_n) &= \lim_{n\longrightarrow\infty} \mathfrak{d}\big(W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), x_n\big)\\ &\leq \lim_{n\longrightarrow\infty} [\beta_n \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, , x_n)].\end{aligned} \tag{3.21}$$

Hence, by (3.20) we get

$$\lim_{n\longrightarrow\infty} \mathfrak{d}(y_n, x_n) = 0. \tag{3.22}$$

Also

$$\mathfrak{d}((P\mathbb{T}_2)^n y_n, x_n) \leq \mathfrak{d}\big((\mathcal{P}\mathbb{T}_2)^n y_n, (\mathcal{P}\mathbb{T}_1)^n x_n\big) + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, x_n)$$

which implies, by (3.13) and (3.20), that

$$\lim_{n\longrightarrow\infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n) = 0. \tag{3.23}$$

And, from (3.22) and (3.23) we have

$$\begin{aligned}\mathfrak{d}\big((\mathcal{P}\mathbb{T}_2)^n x_n, x_n\big) &\leq \mathfrak{d}\big((\mathcal{P}\mathbb{T}_2)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n\big) + \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n)\\ &\leq L\mathfrak{d}(x_n, y_n) + \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n) \longrightarrow 0, \text{ as } n \longrightarrow \infty.\end{aligned} \tag{3.24}$$

That is,

$$\lim_{n\longrightarrow\infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n x_n, x_n) = 0. \qquad (*) \tag{3.25}$$

Using (3.20) and (3.23), we obtain that

$$\begin{aligned}\mathfrak{d}(x_{n+1}, x_n) &= \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), x_n)\\ &\leq (1-\alpha_n)\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, x_n) + \alpha_n \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n)\\ &\longrightarrow 0, \text{ as } n \longrightarrow \infty.\end{aligned} \tag{3.26}$$

Therefore, by (3.22) and (3.26) we obtain

$$\begin{aligned}\mathfrak{d}(x_{n+1}, y_n) &\leq \mathfrak{d}(x_{n+1}, x_n) + d(x_n, y_n)\\ &\longrightarrow 0, \text{ as } n \longrightarrow \infty.\end{aligned} \tag{3.27}$$

Consider

$$\begin{aligned}\mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_1)^n y_n) &\leq \mathfrak{d}(x_{n+1}, x_n) + \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n)\\ &\quad + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_1)^n y_n)\\ &\leq \mathfrak{d}(x_{n+1}, x_n) + \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)^n) x_n)\\ &\quad + L\mathfrak{d}(x_n, y_n).\end{aligned} \tag{3.28}$$

This implies, by (3.20), (3.22) and (3.26), that

$$\lim_{n\longrightarrow\infty} \mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_1)^n y_n) = 0. \tag{3.29}$$

Next, consider

$$\begin{aligned}\mathfrak{d}(x_n,(\mathcal{P}\mathbb{T}_1)x_n) &\le \mathfrak{d}(x_n,(\mathcal{P}\mathbb{T}_1)^n x_n)+\mathfrak{d}\big((\mathcal{P}\mathbb{T}_1)^n x_n,(\mathcal{P}\mathbb{T}_1)^n y_{n-1}\big)\\ &\quad +\mathfrak{d}\big((\mathcal{P}\mathbb{T}_1)^n y_{n-1},(\mathcal{P}\mathbb{T}_1)x_n\big)\\ &\le \mathfrak{d}\big(x_n,(\mathcal{P}\mathbb{T}_1)^n x_n\big)+L\mathfrak{d}(x_n,y_{n-1})\\ &\quad +L\mathfrak{d}\big((\mathcal{P}\mathbb{T}_1)^{n-1}y_{n-1},x_n\big).\end{aligned}\tag{3.30}$$

Using (3.20), (3.27) and (3.29), we obtain

$$\lim_{n\longrightarrow\infty}\mathfrak{d}(x_n,(\mathcal{P}\mathbb{T}_1)x_n)=0. \qquad (**)$$

Now,

$$\mathfrak{d}(x_{n+1},(\mathcal{P}\mathbb{T}_2)^n x_n)\le \mathfrak{d}(x_{n+1},x_n)+\mathfrak{d}(x_n,(\mathcal{P}\mathbb{T}_2)^n x_n).\tag{3.31}$$

This implies, by (3.25), (3.26) that

$$\lim_{n\longrightarrow\infty}\mathfrak{d}(x_{n+1},(\mathcal{P}\mathbb{T}_2)^n x_n)=0.\tag{3.32}$$

Next, we consider

$$\begin{aligned}\mathfrak{d}\big(x_{n+1},(\mathcal{P}\mathbb{T}_2)x_{n+1}\big) &\le \mathfrak{d}\big(x_{n+1},(\mathcal{P}\mathbb{T}_2)^{n+1}x_{n+1}\big)\\ &\quad +\mathfrak{d}\big((\mathcal{P}\mathbb{T}_2)^{n+1}x_{n+1},(\mathcal{P}\mathbb{T}_2)^{n+1}x_n\big)\\ &\quad +\mathfrak{d}\big((\mathcal{P}\mathbb{T}_2)^{n+1}x_n,(\mathcal{P}\mathbb{T}_2)x_{n+1})\big)\\ &\le \mathfrak{d}\big(x_{n+1},(\mathcal{P}\mathbb{T}_2)^{n+1}x_{n+1}\big)+L\mathfrak{d}(x_{n+1},x_n)\\ &\quad +L\mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n x_n,x_{n+1}))\end{aligned}\tag{3.33}$$

Again, this implies, by (3.25), (3.26) and (3.32), that

$$\lim_{n\longrightarrow\infty}\mathfrak{d}(x_n,(\mathcal{P}\mathbb{T}_2)x_n)=0. \qquad (**)$$

Our proof is finished. □

Theorem 3.4. *Let $(\mathbb{X},\mathfrak{d},W)$ be a uniformly convex hyperbolic space. Let $\mathbb{E}$ be a nonempty closed convex subset of of $\mathbb{X}$ with monotone modulus of uniform convexity η and $\mathcal{P}$ as a sunny nonexpansive retraction. Let $\mathbb{F}\neq\emptyset$ and $\mathbb{T}_1,\mathbb{T}_2:\mathbb{E}\longrightarrow\mathbb{X}$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ with sequence $\{a_n\}_{n=1}^{\infty}\subset[1,\infty)$, $\lim_{n\longrightarrow\infty}a_n=1$ satisfying $\sum_{n=1}^{\infty}(a_n-1)<\infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a,1-a]$ for some $a\in(0,1)$. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are weakly inward and one of $\mathbb{T}_1$ and $\mathbb{T}_2$ is compact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\mathbb{T}_1$ and $\mathbb{T}_2$.*

Proof. By Lemma 3.2, $\lim_{n\longrightarrow\infty}\mathfrak{d}(x_n,p)$ exists for any $p\in\mathbb{F}$. It is sufficient to show that $\{x_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to a common fixed point of $\mathbb{T}_1$ and $\mathbb{T}_2$. By Lemma 3.3, $\lim_{n\longrightarrow\infty}\mathfrak{d}(x_n,(\mathcal{P}\mathbb{T}_1)x_n)=0=\lim_{n\longrightarrow\infty}\mathfrak{d}(x_n,(\mathcal{P}\mathbb{T}_2)x_n)$. Suppose that $\mathbb{T}_1$ is compact. Since $\mathcal{P}$ is nonexpansive, there exists a subsequence $\{(\mathcal{P}\mathbb{T}_1)x_{n_j}\}$ of $\{(\mathcal{P}\mathbb{T}_1)x_n\}$ such that $(\mathcal{P}\mathbb{T}_1)x_{n_j}\longrightarrow p$. Thus

$$\mathfrak{d}(x_{n_j},p)\le\mathfrak{d}\big(x_{n_j},(\mathcal{P}\mathbb{T}_1)x_{n_j}\big)+\mathfrak{d}\big((\mathcal{P}\mathbb{T}_1)x_{n_j},p\big)\longrightarrow 0,\quad \text{as } j\longrightarrow\infty.$$

This means $x_{n_j}\longrightarrow p$ as $j\longrightarrow\infty$. Again $\lim_{j\longrightarrow\infty}\mathfrak{d}\big(x_{n_j},(\mathcal{P}\mathbb{T}_1)x_{n_j}\big)=0$ yields by continuity of $\mathcal{P}$ and $\mathbb{T}_1$ that $p=(\mathcal{P}\mathbb{T}_1)p$. Similarly, $p=(\mathcal{P}\mathbb{T}_2)p$. Noting that $F(\mathcal{P}\mathbb{T})=F(\mathbb{T})$. Therefore $p=F(\mathbb{T}_1)=F(\mathbb{T}_2)$, and so $p\in\mathbb{F}$. Thus $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of $\mathbb{T}_1$ and $\mathbb{T}_2$. □

Corollary 3.5. *Let $\mathbb{X}$ be a complete CAT(0) space and $\mathbb{E}$ be a nonempty closed convex subset of of $\mathbb{X}$ with $\mathcal{P}$ as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \longrightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are weakly inward and one of $\mathbb{T}_1$ and $\mathbb{T}_2$ is compact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\mathbb{T}_1$ and $\mathbb{T}_2$.*

Theorem 3.6. *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space. Let $\mathbb{E}$ be a nonempty closed convex subset of of $\mathbb{X}$ with monotone modulus of uniform convexity η and $\mathcal{P}$ as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \longrightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are weakly inward and satisfy condition $(\mathcal{B})$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\mathbb{T}_1$ and $\mathbb{T}_2$..*

Proof. By Lemma 3.2 $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, p)$ exists and so $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, \mathbb{F})$ exists for all $p \in \mathbb{F}$. Again, by Lemma 3.3, $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n) = 0 = \lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n)$. It follows from condition $(\mathcal{B})$ and Lemma 3.3 that

$$\lim_{n \longrightarrow \infty} f(\mathfrak{d}(x_n, \mathbb{F})) \leq \lim_{n \longrightarrow \infty} \big(\frac{1}{2}[\mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n)) + \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n)]\big) = 0.$$

That is,

$$\lim_{n \longrightarrow \infty} f(\mathfrak{d}(x_n, \mathbb{F})) = 0.$$

Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$.

Next, we show that x_n is a Cauchy sequence. Let $\epsilon > 0$. Since $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$, there exists a positive number n_0 such that for all $n \geq n_0$, we have

$$\mathfrak{d}(x_n, \mathbb{F}) < \frac{\epsilon}{2\mathcal{C}},$$

where $\mathcal{C} > 0$ is the constant in Lemma 3.2(ii). So we can find $p' \in \mathbb{F}$ such that

$$\mathfrak{d}(x_n, p') < \frac{\epsilon}{2\mathcal{C}}.$$

Using Lemma 3.2(ii), we have for all $n \geq n_0$ and $m \in \mathbb{N}$ that

$$\begin{aligned} \mathfrak{d}(x_{n+m}, x_n) &\leq \mathfrak{d}(x_{n+m}, p') + \mathfrak{d}(p', x_n) \\ &\leq \mathcal{C}\mathfrak{d}(x_n, p') + \mathfrak{d}(x_n, p') \\ &\leq \mathcal{C}\mathfrak{d}(x_n, p') + \mathcal{C}\mathfrak{d}(x_n, p') \\ &= 2\mathcal{C}\mathfrak{d}(x_n, p') < \epsilon. \end{aligned} \tag{3.34}$$

Hence, $\{x_n\}$ is a Cauchy sequence in a closed convex subset $\mathbb{E}$ of a hyperbolic space $\mathbb{X}$, therefore, it must converge to a point in $\mathbb{E}$. Let $\lim_{n \longrightarrow \infty} x_n = q$. Now, $\lim_{n \longrightarrow \infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$ yields that $\mathfrak{d}(q, \mathbb{F}) = 0$. Since the set of fixed points of asymptotically quasi-nonexpansive mappings is closed, we have $q \in \mathbb{F}$. □

Corollary 3.7. *Let $\mathbb{X}$ be a complete CAT(0) space and $\mathbb{E}$ be a nonempty closed convex subset of of $\mathbb{X}$ with $\mathcal{P}$ as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive* *mappings with respect to $\mathcal{P}$ with sequence $\{a_n\}_{n=1}^{\infty} \subset [1,\infty)$, $\lim_{n\longrightarrow\infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are weakly inward and satisfy* **condition** *$(\mathcal{B})$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\mathbb{T}_1$ and $\mathbb{T}_2$.*

We shall use condition $(\mathcal{A})$ to prove strong convergence of the algorithm (3.1). Before that, we prove the following technical lemma.

Lemma 3.8. *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space. Let $\mathbb{E}$ be a nonempty closed convex subset of of $\mathbb{X}$ with monotone modulus of uniform convexity η and $\mathcal{P}$ as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ with sequence $\{a_n\}_{n=1}^{\infty} \subset [1,\infty)$, $\lim_{n\longrightarrow\infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are weakly inward and satisfy condition $(\mathcal{A})$. Then the sequence $\{x_n\}$ converges strongly to $p \in \mathbb{F}$ if and only if $\lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$.*

Proof. We know from Lemma 3.2 that $\mathfrak{d}(x_{n+1}, p) \leq \mathfrak{d}(x_n, p)$. It follows that $\{x_n\}$ is Fejér monotone with respect to $\mathbb{F}$ and $\lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, \mathbb{F})$ exists. Hence, the result follows from Lemma 2.3. □

Applying Lemma 3.8, we obtain following strong convergence theorem.

Theorem 3.9. *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space. Let $\mathbb{E}$ be a nonempty closed convex subset of of $\mathbb{X}$ with monotone modulus of uniform convexity η and $\mathcal{P}$ as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ with sequence $\{a_n\}_{n=1}^{\infty} \subset [1,\infty)$, $\lim_{n\longrightarrow\infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are weakly inward and satisfy condition $(\mathcal{A})$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\mathbb{T}_1$ and $\mathbb{T}_2$.*

Proof. It follows from Lemma 3.2 that $\lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, \mathbb{F})$ exists. Moreover, by Lemma 3.3 we have that $\lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n) = \lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n) = 0$. Applying condition $(\mathcal{A})$, we obtain

$$\lim_{n\longrightarrow\infty} f(\mathfrak{d}(x_n, \mathbb{F})) = 0.$$

Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n\longrightarrow\infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$. Therefore, Lemma 3.8 implies that $\{x_n\}$ converges strongly to a point $p \in \mathbb{F}$. □

Corollary 3.10. *Let $\mathbb{X}$ be a complete CAT(0) space and $\mathbb{E}$ be a nonempty closed convex subset of of $\mathbb{X}$ with $\mathcal{P}$ as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \longrightarrow \mathbb{X}$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ with sequence $\{a_n\}_{n=1}^{\infty} \subset [1,\infty)$, $\lim_{n\longrightarrow\infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty}(a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0,1)$. If $\mathbb{T}_1$ and $\mathbb{T}_2$ are weakly inward and satisfy* **condition** *$(\mathcal{A})$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of $\mathbb{T}_1$ and $\mathbb{T}_2$.*

4. Conclusion

In this manuscript, we have established new common fixed point results for two uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ and two weakly inward and asymptotically quasi-nonexpansive mappings with respect to $\mathcal{P}$ satisfying condition $(\mathcal{A})$ and condition $(\mathcal{B})$, in a more general set up of hyperbolic space. Our results signifigantly extend and improve the results obtained by Khan et al.[16], as well as many related results in CAT(0) spaces and uniformly Banach spaces. As consequences of our main results, we obtain the corresponding corollaries which are valid in CAT(0) spaces.

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