



ON SOME COMMON FIXED POINT RESULTS IN HYPERBOLIC SPACES

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ABSTRACT. The aim of this manuscript is to establish a common fixed point theorem for two uniformly L -Lipschitzian and asymptotically quasi-nonexpansive non-self maps with respect to retraction \mathcal{P} via implicit algorithm and to prove common fixed point results of two weakly inward and asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} satisfying condition (\mathcal{A}) and condition (\mathcal{B}) , in a more general set up of hyperbolic space. Our results generalize, extend and improve some related results in the existing literature.

KEYWORDS: common fixed point, asymptotically quasi-nonexpansive mapping, Banach space, CAT(0) space, hyperbolic space.

AMS Subject Classification: 47H10; 54H25.

1. INTRODUCTION

Let $(\mathbb{X}, \|\cdot\|)$ be a real **normed linear space**, and let \mathbb{E} be a nonempty closed convex subset of \mathbb{X} . Let $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{E}$ be a self-mapping. Let $F(\mathbb{T})$ denote the set of fixed points of \mathbb{T} , that is, $F(\mathbb{T}) = \{x \in \mathbb{E} : \mathbb{T}x = x\}$. A self-mapping $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{E}$ is said to be

(1). asymptotically nonexpansive [8] if there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ and such that

$$\|\mathbb{T}^n x - \mathbb{T}^n y\| \leq a_n \|x - y\|, \forall x, y \in \mathbb{E}, n \geq 1. \quad (1.1)$$

(2). asymptotically quasi-nonexpansive if $F(\mathbb{T}) \neq \emptyset$ and there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ and such that

$$\|\mathbb{T}^n x - p\| \leq a_n \|x - p\|, \forall x \in \mathbb{E}, p \in F(\mathbb{T}) \text{ and } n \geq 1. \quad (1.2)$$

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Article history : Received 07/01/2025 Accepted 31/05/2025.

(3). uniformly L -Lipschitzian if there exists constant $L \geq 0$ such that

$$\|\mathbb{T}^n x - \mathbb{T}^n y\| \leq L\|x - y\|, \forall x, y \in \mathbb{E}, n \geq 1. \quad (1.3)$$

Note that an asymptotically nonexpansive mapping must be uniformly L -Lipschitzian as well as asymptotically quasi-nonexpansive but the converse does not hold true in general.

In 2003, Chidume *et al.*[7] introduced the notion of asymptotically nonexpansive non-self mappings as a generalization of asymptotically nonexpansive self-mappings as follows.

Definition 1.1. [7] Let \mathbb{E} be a nonempty subset of real normed linear space \mathbb{X} . Let $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{X}$ be a nonself mapping and $\mathcal{P} : \mathbb{X} \rightarrow \mathbb{E}$ be the nonexpansive retraction of \mathbb{X} into \mathbb{E} . \mathbb{T} is said to be

(1). asymptotically nonexpansive if there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ and such that

$$\|\mathbb{T}(\mathcal{P}\mathbb{T})^{n-1} x - \mathbb{T}(\mathcal{P}\mathbb{T})^{n-1} y\| \leq a_n\|x - y\|, \forall x, y \in \mathbb{E}, n \geq 1. \quad (1.4)$$

(2). uniformly L -Lipschitzian if there exists constant $L \geq 0$ such that

$$\|\mathbb{T}(\mathcal{P}\mathbb{T})^{n-1} x - \mathbb{T}(\mathcal{P}\mathbb{T})^{n-1} y\| \leq L\|x - y\|, \forall x, y \in \mathbb{E}, n \geq 1. \quad (1.5)$$

Chidume *et al.*[7] established a demiclosed principle, weak and strong convergence results for such mappings in a uniformly convex Banach space via the following algorithm:

$$x_1 \in \mathbb{E}, x_{n+1} = \mathcal{P}((1 - \alpha_n)x_n + \alpha_n \mathbb{T}(\mathcal{P}\mathbb{T})^{n-1} x_n), n \geq 1.$$

After Chidume *et al.*[7], a number of authors have studied the weak and strong convergence for such mappings (see [10, 11, 15, 12, 26, 29, 33] for examples).

Later, in 2007 Zhou *et al.*[34] introduced the following generalized definition.

Definition 1.2. [34] Let \mathbb{E} be a nonempty subset of real normed linear space \mathbb{X} . Let $\mathcal{P} : \mathbb{X} \rightarrow \mathbb{E}$ be the nonexpansive retraction of \mathbb{X} into \mathbb{E} . A nonself mapping $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{X}$ is said to be

(1). asymptotically nonexpansive with respect to \mathcal{P} if there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ and such that

$$\|(\mathcal{P}\mathbb{T})^n x - (\mathcal{P}\mathbb{T})^n y\| \leq a_n\|x - y\|, \forall x, y \in \mathbb{E}, n \geq 1. \quad (1.6)$$

(2). uniformly L -Lipschitzian with respect to \mathcal{P} if there exists constant $L \geq 0$ such that

$$\|(\mathcal{P}\mathbb{T})^n x - (\mathcal{P}\mathbb{T})^n y\| \leq L\|x - y\|, \forall x, y \in \mathbb{E}, n \geq 1. \quad (1.7)$$

Zhou *et al.* [34] introduced the following iterative process,

$$x_1 \in \mathbb{E}, x_{n+1} = \alpha_n x_n + \beta_n (\mathcal{P}\mathbb{T}_1)x_n + \gamma_n (\mathcal{P}\mathbb{T}_2)x_n, n \geq 1, \quad (1.8)$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are three sequences in $[a, 1 - a]$ for some $a \in (0, 1)$, satisfying $\alpha_n + \beta_n + \gamma_n = 1$. And, they established some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to \mathcal{P} in uniformly convex Banach spaces.

In 2007, Agarwal *et al.*[2] introduced the iterative algorithm:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)\mathbb{T}^n x_n + \alpha_n \mathbb{T}^n y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n \mathbb{T}^n x_n, n \geq 1. \end{aligned} \quad (1.9)$$

Later, in 2011 Turkmen *et al.*[32] used the following iterative process to establish common fixed point results of two asymptotically nonexpansive mappings.

$$\begin{aligned} x_1 &\in \mathbb{E}, \\ x_{n+1} &= (1 - \alpha_n)(\mathcal{P}\mathbb{T}_1)y_n + \alpha_n(\mathcal{P}\mathbb{T}_2)y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n(\mathcal{P}\mathbb{T}_1)x_n, \quad n \geq 1, \end{aligned} \quad (1.10)$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ satisfying certain conditions.

Inspired by the work mentioned above, Khan *et al.*[16] generalized the definition of nonself asymptotically nonexpansive mappings with respect to \mathcal{P} to nonself asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} . Thus, a nonself mapping $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{X}$ is said to be an asymptotically quasi-nonexpansive with respect to \mathcal{P} if $F(\mathbb{T}) \neq \emptyset$ and there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ such that

$$\|(\mathcal{P}\mathbb{T})^n x - p\| \leq a_n \|x - p\|, \quad \forall x \in \mathbb{E}, \quad n \geq 1. \quad (1.11)$$

Khan *et al.*[16] introduced the following iterative process. Let \mathbb{E} be a nonempty closed convex subset of a real **normed linear space** \mathbb{X} with retraction \mathcal{P} . Let $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two nonself asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} . Their iterative scheme reads as follows:

$$\begin{aligned} x_1 &\in \mathbb{E}, \\ x_{n+1} &= (1 - \alpha_n)(\mathcal{P}\mathbb{T}_1)^n x_n + \alpha_n(\mathcal{P}\mathbb{T}_2)^n y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n(\mathcal{P}\mathbb{T}_1^n)x_n, \quad n \geq 1, \end{aligned} \quad (1.12)$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$.

Khan *et al.*[16] obtained strong convergence theorems for two asymptotically quasi-nonexpansive mappings using a general and independent two-step iterative process (1.12) assuming compactness of only one of the two mappings in smooth Banach space. They also proved a weak convergence result under Opial's condition.

Question: Can we can extend and improve those results in [16] from a uniformly convex and smooth Banach space to a more general set up of a uniformly convex hyperbolic space? The answer is affirmative.

We recall the following definition.

Definition 1.3. [6] Let $(\mathbb{X}, \mathfrak{d})$ be a metric space and \mathbb{E} be a nonempty subset of \mathbb{X} and $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{E}$ be a self-mapping. Then \mathbb{T} is said to be

(i). asymptotically nonexpansive if there exists a sequence $\{a_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ such that

$$\mathfrak{d}(\mathbb{T}^n x, \mathbb{T}^n y) \leq a_n \mathfrak{d}(x, y), \quad \forall x, y \in \mathbb{E} \text{ and } \forall n \geq 1. \quad (1.13)$$

(ii). asymptotically quasi-nonexpansive if $F(\mathbb{T}) \neq \emptyset$ and there exists a sequence $\{a_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ such that

$$\mathfrak{d}(\mathbb{T}^n x, p) \leq a_n \mathfrak{d}(x, p), \quad \forall x, y \in \mathbb{E}, \quad \forall p \in F(\mathbb{T}) \text{ and } \forall n \geq 1. \quad (1.14)$$

(ii). uniformly L -Lipschitzian if there exists a constant $L > 0$ such that

$$\mathfrak{d}(\mathbb{T}^n x, \mathbb{T}^n y) \leq L \mathfrak{d}(x, y), \quad \forall x, y \in \mathbb{E} \text{ and } \forall n \geq 1. \quad (1.15)$$

Note that if $F(\mathbb{T})$ is nonempty, then nonexpansive mapping, quasi-nonexpansive mapping, asymptotically nonexpansive mapping all are the special cases of asymptotically quasi-nonexpansive type mappings.

The purpose of this paper is to extend and improve the results of Khan *et al.*[16] from the setting of a real uniformly convex and smooth Banach space to the setting of a uniformly convex hyperbolic space, a more general setting.

2. PRELIMINARIES

In this section, we recall an important definition of hyperbolic space which will be crucial for our main results.

Definition 2.1. [18] Let $(\mathbb{X}, \mathfrak{d})$ is a metric space. A hyperbolic space is a triple $(\mathbb{X}, \mathfrak{d}, W)$, where $W : \mathbb{X} \times \mathbb{X} \times [0, 1] \rightarrow \mathbb{X}$ is such that

- (W1). $\mathfrak{d}(W(x, y, \alpha), z) \leq (1 - \alpha)\mathfrak{d}(z, x) + \alpha\mathfrak{d}(z, y)$,
- (W2). $\mathfrak{d}(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|\mathfrak{d}(x, y)$,
- (W3). $W(x, y, \alpha) = W(y, x, (1 - \alpha))$,
- (W4). $\mathfrak{d}(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)\mathfrak{d}(x, y) + \alpha\mathfrak{d}(z, w)$ for all $x, y, z, w \in \mathbb{X}$, where $\alpha, \beta \in [0, 1]$.

It follows from (W1) that, for each $x, y \in \mathbb{X}$ and $\alpha \in [0, 1]$,

$$\mathfrak{d}(W(x, y, \alpha), x) \leq \alpha\mathfrak{d}(x, y), \quad \mathfrak{d}(W(x, y, \alpha), y) \leq (1 - \alpha)\mathfrak{d}(x, y).$$

A subset \mathbb{E} of a hyperbolic space \mathbb{X} is convex if $W(x, y, \alpha) \in \mathbb{E}$ for all $x, y \in \mathbb{E}$ and $\alpha \in [0, 1]$. For more detail of convex structure of a metric space, see [27].

We note that the class of hyperbolic spaces also contains Hadamard manifolds [5], and Cartesian products of Hilbert balls, the Hilbert open unit ball equipped with the hyperbolic metric[9], as special cases. It is wellknown that spaces like CAT(0) spaces (in the sense of Gromov) and R -tree (in the sense of Tits) are special cases of hyperbolic spaces. Some remarkable results in CAT(0) spaces and hyperbolic spaces in [14, 19, 17, 20, 22, 23, 24, 25, 30, 31] are examples of nonlinear structures which play a major role in recent research in metric fixed point theory.

A hyperbolic space $(\mathbb{X}, \mathfrak{d}, W)$ is said to be uniformly convex [27] if for all $x, y, u \in \mathbb{X}, r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left. \begin{array}{l} \mathfrak{d}(x, u) \leq r \\ \mathfrak{d}(y, u) \leq r \\ \mathfrak{d}(x, y) \geq \epsilon r \end{array} \right\} \Rightarrow \mathfrak{d}(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r.$$

A map $\eta : (0, \infty) \times (0, 2] \rightarrow (0, 1]$ which gives such a number $\delta = \eta(r, \epsilon)$, for a given $r > 0$ and $\epsilon \in (0, 2]$, is called **modulus of uniform convexity**. The modulus of uniform convexity η is said to be monotone if it decreases with r (for a fixed ϵ). A uniformly convex hyperbolic space is strictly convex (see [20]).

For more interesting results on hyperbolic spaces, we refer readers to [13, 20, 21]. The following lemma is essential for our main results.

Lemma 2.2. [13] Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η . Let $u \in \mathbb{X}$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in \mathbb{X} such that

$$\limsup_{n \rightarrow \infty} \mathfrak{d}(x_n, u) \leq r, \quad \limsup_{n \rightarrow \infty} \mathfrak{d}(y_n, u) \leq r$$

and

$$\lim_{n \rightarrow \infty} \mathfrak{d}(W(x_n, y_n, \alpha_n), u) = r$$

for some $r \geq 0$. Then

$$\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, y_n) = 0.$$

Recall that a subset \mathbb{E} of \mathbb{X} is said to be retract if there exists a mapping $\mathcal{P} : \mathbb{X} \rightarrow \mathbb{E}$ which is continuous and such that $\mathcal{P}x = x$ for all $x \in \mathbb{E}$. A mapping $\mathcal{P} : \mathbb{X} \rightarrow \mathbb{X}$ is said to be a retraction if $\mathcal{P}^2 = \mathcal{P}$. Let \mathbb{K} and \mathbb{E} be subsets of a hyperbolic space \mathbb{X} . A mapping \mathcal{P} from \mathbb{K} into \mathbb{E} is called sunny if $\mathcal{P}(W(\mathcal{P}x, x, \alpha)) = \mathcal{P}x$ for $x \in \mathbb{K}$ with $W(\mathcal{P}x, x, \alpha) \in \mathbb{K}$ and $\alpha \in [0, 1]$. Note that, if \mathcal{P} is a retraction, then $\mathcal{P}z = z$ for every $z \in R(\mathcal{P})$, the range of \mathcal{P} . We note that every closed convex subset of a uniformly convex hyperbolic space is a retract.

For each $x \in \mathbb{E}$, the **inward set** $I_{\mathbb{E}}(x)$ is defined by

$$I_{\mathbb{E}}(x) = \{y \in \mathbb{X} : y = W(x, z, \lambda), z \in \mathbb{E}, \lambda \in [0, 1]\}.$$

A mapping $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{X}$ is said to satisfy the **inward condition** if $\mathbb{T}x \in I_{\mathbb{E}}(x)$ for all $x \in \mathbb{E}$. \mathbb{T} is said to be weakly inward if, for each $x \in \mathbb{E}$, $\mathbb{T}x \in cl[I_{\mathbb{E}}(x)]$, where $cl[I_{\mathbb{E}}(x)]$ is the closure of $I_{\mathbb{E}}(x)$.

A Hyperbolic space $(\mathbb{X}, \mathfrak{d}, W)$ is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in \mathbb{X} , $x_n \rightharpoonup x$ (i.e. $\{x_n\}_{n=1}^{\infty}$ converges weakly to x) implies that

$$\limsup_{n \rightarrow \infty} \mathfrak{d}(x_n, x) < \limsup_{n \rightarrow \infty} \mathfrak{d}(x_n, y),$$

for all $y \in \mathbb{X}$ with $y \neq x$.

From now on, let \mathbb{T}_1 and \mathbb{T}_2 be two maps on \mathbb{E} , we denote $F(\mathbb{T}_1) = \{x : \mathbb{T}_1x = x\}$, $F(\mathbb{T}_2) = \{x : \mathbb{T}_2x = x\}$ the set of fixed point of \mathbb{T}_1 and \mathbb{T}_2 respectively, and $\mathbb{F} = (F(\mathbb{T}_1) \cap F(\mathbb{T}_2)) \neq \emptyset$.

Recall that a sequence $\{x_n\}$ in a metric space \mathbb{X} is said to be **Fejér monotone** with respect to \mathbb{E} (a subset of \mathbb{X}) if $\mathfrak{d}(x_{n+1}, x) \leq \mathfrak{d}(x_n, x)$ for all $x \in \mathbb{E}$ and for all $n \geq 1$.

A map $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{E}$ is said to be **semi-compact** if any bounded sequence $\{x_n\}$ satisfying $\mathfrak{d}(x_n, \mathbb{T}x_n) \rightarrow 0$ as $n \rightarrow \infty$, has a convergent subsequence.

Let f be a nondecreasing selfmap on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ and let $\mathfrak{d}(x, A) = \inf\{\mathfrak{d}(x, y) : y \in A\}$.

Let \mathbb{T}_1 and \mathbb{T}_2 be two mappings on \mathbb{E} with $\mathbb{F} \neq \emptyset$. Then the two mappings are said to satisfy:

(i). **condition (A)** on \mathbb{E} if

$$f(\mathfrak{d}(x, \mathbb{F})) \leq \mathfrak{d}(x, \mathbb{T}_1x) \text{ or } f(\mathfrak{d}(x, \mathbb{F})) \leq \mathfrak{d}(x, \mathbb{T}_2x)$$

for all $x \in \mathbb{E}$, holds for at least one $\mathbb{T}_i, i = 1, 2$.

(ii). **condition (B)** on \mathbb{E} if

$$f(\mathfrak{d}(x, \mathbb{F})) \leq \frac{1}{2} [\mathfrak{d}(x, \mathbb{T}_1x) + \mathfrak{d}(x, \mathbb{T}_2x)]$$

holds for all $x \in \mathbb{E}$.

Lemma 2.3. [3] Let $(\mathbb{X}, \mathfrak{d})$ be a complete metric space and \mathbb{E} be a nonempty closed subset of \mathbb{X} , and $\{x_n\}$ be Fejér monotone with respect to \mathbb{E} . Then $\{x_n\}$ converges to some $p \in \mathbb{E}$ if and only if $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{E}) = 0$.

The following lemma is very useful.

Lemma 2.4. [28] Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i). $\lim_{n \rightarrow \infty} a_n$ exists.

(ii). In particular, if $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n \rightarrow \infty} a_n = 0$.

For more details of results obtained by Khan *et al.* in the context of uniformly convex and smooth Banach spaces, we refer readers to [16].

Inspired and motivated by the works of Khan *et al.*[16] and some related results, we establish common fixed point theorems for two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} in the setting of hyperbolic space, a more general set up. Our results extend and improve the results obtained by Khan *et al.*[16], as well as many related results in CAT(0) spaces and uniformly Banach spaces.

3. MAIN RESULTS

Let $\{a_n^{(1)}\}_{n=1}^{\infty} \subset [1, \infty)$ and $\{a_n^{(2)}\}_{n=1}^{\infty} \subset [1, \infty)$ be sequences satisfying the asymptotically quasi-nonexpansive mappings \mathbb{T}_1 and \mathbb{T}_2 with $\sum_{n=1}^{\infty} (a_n^{(i)} - 1) < \infty$, ($i = 1, 2$). Let $a_n = \max\{a_n^{(1)}, a_n^{(2)}\}$, and throughout this section, we will take only sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$.

Let $(\mathbb{X}, \mathfrak{d}, W)$ be a hyperbolic space and \mathbb{E} be a nonempty closed convex subset of \mathbb{X} with retraction \mathcal{P} . Let $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two non-self asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} . We define the sequence $\{x_n\}_{n=1}^{\infty}$ in a hyperbolic space as follows:-

$$\begin{aligned} x_1 &\in \mathbb{E}, \\ x_{n+1} &= W((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), \\ y_n &= W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), \forall n \geq 1. \end{aligned} \tag{3.1}$$

We introduce the following definition.

Definition 3.1. Let $(\mathbb{X}, \mathfrak{d})$ be a metric space and \mathbb{E} be a nonempty subset of \mathbb{X} and $\mathbb{T} : \mathbb{E} \rightarrow \mathbb{X}$ be a nonself mapping with respect to retraction \mathcal{P} . Then \mathbb{T} said to be:

(i). asymptotically nonexpansive with respect to \mathcal{P} if there exists a sequence $\{a_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ such that

$$\mathfrak{d}((\mathcal{P}\mathbb{T})^n x, (\mathcal{P}\mathbb{T})^n y) \leq a_n \mathfrak{d}(x, y), \forall x, y \in \mathbb{E} \text{ and } \forall n \geq 1. \tag{3.2}$$

(ii). asymptotically quasi-nonexpansive with respect to \mathcal{P} if $F(\mathbb{T}) \neq \emptyset$ and there exists a sequence $\{a_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} a_n = 1$ such that

$$\mathfrak{d}((\mathcal{P}\mathbb{T})^n x, p) \leq a_n \mathfrak{d}(x, p), \forall x, y \in \mathbb{E}, \forall p \in F(\mathbb{T}) \text{ and } \forall n \geq 1. \tag{3.3}$$

(ii). uniformly L -Lipschitzian with respect to \mathcal{P} if there exists a constant $L > 0$ such that

$$\mathfrak{d}((\mathcal{P}\mathbb{T})^n x, (\mathcal{P}\mathbb{T})^n y) \leq L \mathfrak{d}(x, y), \forall x, y \in \mathbb{E} \text{ and } \forall n \in \mathbb{N}. \tag{3.4}$$

We first prove two technical lemmas.

Lemma 3.2. Let $(\mathbb{X}, \mathfrak{d}, W)$ be a hyperbolic space with monotone modulus of uniform convexity η and \mathbb{E} be a nonempty closed convex subset of \mathbb{X} which is also a nonexpansive retract of \mathbb{X} . Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. Then

- (i). $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p)$ exists, $\forall p \in \mathbb{F}$.
- (ii). There exists a constant $\mathcal{C} > 0$ such that $\mathfrak{d}(x_{n+m}, p) \leq \mathcal{C}\mathfrak{d}(x_n, p)$, $\forall m, n \in \mathbb{N}$ and $p \in \mathbb{F}$.

Proof. (i). Let $p \in \mathbb{F}$. From (3.1), we have

$$\begin{aligned}
\mathfrak{d}(y_n, p) &= \mathfrak{d}(W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), p) \\
&\leq (1 - \beta_n)\mathfrak{d}(x_n, p) + \beta_n\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) \\
&\leq (1 - \beta_n)\mathfrak{d}(x_n, p) + \beta_n a_n \mathfrak{d}(x_n, p) \\
&= (1 + \beta_n(a_n - 1))\mathfrak{d}(x_n, p) \\
&\leq (1 + (a_n - 1))\mathfrak{d}(x_n, p) \\
&= a_n \mathfrak{d}(x_n, p).
\end{aligned} \tag{3.5}$$

From (3.1) and (3.5), we have

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, p) &= \mathfrak{d}(W((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), p) \\
&\leq (1 - \alpha_n)\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) + \alpha_n\mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, p) \\
&\leq (1 - \alpha_n)a_n \mathfrak{d}(x_n, p) + \alpha_n a_n \mathfrak{d}(y_n, p) \\
&\leq (1 - \alpha_n)a_n \mathfrak{d}(x_n, p) + \alpha_n a_n^2 \mathfrak{d}(x_n, p) \\
&= (1 + \alpha_n a_n(a_n - 1))\mathfrak{d}(x_n, p) \\
&\leq (1 + a_n(a_n - 1))\mathfrak{d}(x_n, p) \\
&\leq (1 + (a_n^2 - 1))\mathfrak{d}(x_n, p).
\end{aligned} \tag{3.6}$$

Note that $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. This implies $\sum_{n=1}^{\infty} (a_n^2 - 1) < \infty$. Thus, by Lemma 2.4, $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p)$ exists, $\forall p \in \mathbb{F}$.

(ii). From (3.6), we have

$$\mathfrak{d}(x_{n+1}, p) \leq (1 + (a_n^2 - 1))\mathfrak{d}(x_n, p). \tag{3.7}$$

We know that $1 + x \leq e^x$ for all $x \geq 0$. Using it for the above inequality (3.7), we have

$$\begin{aligned}
\mathfrak{d}(x_{n+m}, p) &\leq (1 + (a_{n+m-1}^2 - 1))\mathfrak{d}(x_{n+m-1}, p) \\
&\leq e^{a_{n+m-1}^2 - 1} \mathfrak{d}(x_{n+m-1}, p) \\
&\leq [e^{a_{n+m-1}^2 - 1}] (1 + (a_{n+m-2}^2 - 1))\mathfrak{d}(x_{n+m-2}, p) \\
&\leq [e^{(a_{n+m-1}^2 - 1) + (a_{n+m-2}^2 - 1)}] \mathfrak{d}(x_{n+m-2}, p) \\
&\vdots \\
&\leq [e^{\sum_{j=n}^{n+m-1} (a_j^2 - 1)}] \mathfrak{d}(x_n, p) \\
&= \mathcal{C}\mathfrak{d}(x_n, p),
\end{aligned} \tag{3.8}$$

where $\mathcal{C} = e^{\sum_{j=n}^{n+m-1} (a_j^2 - 1)}$. That is, $\mathfrak{d}(x_{n+m}, p) \leq \mathcal{C}\mathfrak{d}(x_n, p)$ for all $n, m \in \mathbb{N}$ and $p \in \mathbb{F}$. \square

Lemma 3.3. *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a hyperbolic space with monotone modulus of uniform convexity η and \mathbb{E} be a nonempty closed convex subset of \mathbb{X} which is also a nonexpansive retract of \mathbb{X} . Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. Suppose*

that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. Then

$$\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n) = 0 = \lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)^n x_n)$$

and also

$$\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n) = 0 = \lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n).$$

Proof. By Lemma 3.2, $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p)$ exists. Assume that $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p) = \zeta$. Taking \limsup on both sides in the inequality (3.5), we obtain

$$\limsup_{n \rightarrow \infty} \mathfrak{d}(y_n, p) \leq \limsup_{n \rightarrow \infty} \mathfrak{d}(x_n, p) = \zeta. \quad (3.9)$$

Next, $\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) \leq a_n \mathfrak{d}(x_n, p)$ for all $n \in \mathbb{N}$ implies that

$$\limsup_{n \rightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) \leq \zeta. \quad (3.10)$$

Also, by (3.9) we get

$$\limsup_{n \rightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, p) \leq \limsup_{n \rightarrow \infty} \mathfrak{d}(y_n, p) \leq \zeta. \quad (3.11)$$

Moreover, from (3.6) we have

$$\lim_{n \rightarrow \infty} \mathfrak{d}(x_{n+1}, p) = \lim_{n \rightarrow \infty} \mathfrak{d}(W((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), p) = \zeta. \quad (3.12)$$

From (3.10), (3.11), (3.12), and Lemma 2.2, we have

$$\lim_{n \rightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n) = 0. \quad (3.13)$$

Observe that

$$\begin{aligned} \mathfrak{d}(x_{n+1}, p) &= \mathfrak{d}(W((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), p) \\ &\leq (1 - \alpha_n) \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) + \alpha_n \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, p) \\ &\leq (1 - \alpha_n) \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) + \alpha_n [\mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, (\mathcal{P}\mathbb{T}_1)^n x_n) \\ &\quad + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p)]. \end{aligned}$$

Taking the limit inf. as $n \rightarrow \infty$ in the above inequality, and applying (3.13) we get

$$\zeta \leq \liminf_{n \rightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p). \quad (3.14)$$

From (3.10) and (3.14), we get

$$\lim_{n \rightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) = \zeta.$$

Furthermore,

$$\begin{aligned} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, p) &\leq \mathfrak{d}(\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n) + \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, p) \\ &\leq \mathfrak{d}(\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n) + a_n \mathfrak{d}(y_n, p). \end{aligned} \quad (3.15)$$

This implies

$$\zeta \leq \liminf_{n \rightarrow \infty} \mathfrak{d}(y_n, p). \quad (3.16)$$

By (3.9) and (3.16), we obtain

$$\lim_{n \rightarrow \infty} \mathfrak{d}(y_n, p) = \zeta. \quad (3.17)$$

Therefore,

$$\begin{aligned}
\zeta &= \lim_{n \rightarrow \infty} d(y_n, p) = \lim_{n \rightarrow \infty} \mathfrak{d}(W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), p) \\
&\leq \lim_{n \rightarrow \infty} \{(1 - \beta_n)\mathfrak{d}(x_n, p) + \beta_n a_n \mathfrak{d}(x_n, p)\} \\
&= \lim_{n \rightarrow \infty} [1 + \beta_n(a_n - 1)]\mathfrak{d}(x_n, p) \\
&\leq \lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p) = \zeta.
\end{aligned} \tag{3.18}$$

That is,

$$\lim_{n \rightarrow \infty} \mathfrak{d}(W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), p) = \zeta. \tag{3.19}$$

We know that $\limsup_{n \rightarrow \infty} \mathfrak{d}(x_n, p) \leq \zeta$, together with (3.10), (3.19) and Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, x_n) = 0. \quad (*) \tag{3.20}$$

In addition, we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \mathfrak{d}(y_n, x_n) &= \lim_{n \rightarrow \infty} \mathfrak{d}(W(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n, \beta_n), x_n) \\
&\leq \lim_{n \rightarrow \infty} [\beta_n \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, x_n)].
\end{aligned} \tag{3.21}$$

Hence, by (3.20) we get

$$\lim_{n \rightarrow \infty} \mathfrak{d}(y_n, x_n) = 0. \tag{3.22}$$

Also

$$\mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n) \leq \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, (\mathcal{P}\mathbb{T}_1)^n x_n) + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, x_n)$$

which implies, by (3.13) and (3.20), that

$$\lim_{n \rightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n) = 0. \tag{3.23}$$

And, from (3.22) and (3.23) we have

$$\begin{aligned}
\mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n x_n, x_n) &\leq \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n) + \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n) \\
&\leq L\mathfrak{d}(x_n, y_n) + \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n) \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned} \tag{3.24}$$

That is,

$$\lim_{n \rightarrow \infty} \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n x_n, x_n) = 0. \quad (*) \tag{3.25}$$

Using (3.20) and (3.23), we obtain that

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, x_n) &= \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_2)^n y_n, \alpha_n), x_n) \\
&\leq (1 - \alpha_n)\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, x_n) + \alpha_n \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n y_n, x_n) \\
&\rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned} \tag{3.26}$$

Therefore, by (3.22) and (3.26) we obtain

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, y_n) &\leq \mathfrak{d}(x_{n+1}, x_n) + d(x_n, y_n) \\
&\rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned} \tag{3.27}$$

Consider

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_1)^n y_n) &\leq \mathfrak{d}(x_{n+1}, x_n) + \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n) \\
&\quad + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_1)^n y_n) \\
&\leq \mathfrak{d}(x_{n+1}, x_n) + \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n) \\
&\quad + L\mathfrak{d}(x_n, y_n).
\end{aligned} \tag{3.28}$$

This implies, by (3.20), (3.22) and (3.26), that

$$\lim_{n \rightarrow \infty} \mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_1)^n y_n) = 0. \tag{3.29}$$

Next, consider

$$\begin{aligned}
\mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n) &\leq \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n) + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n x_n, (\mathcal{P}\mathbb{T}_1)^n y_{n-1}) \\
&\quad + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)^n y_{n-1}, (\mathcal{P}\mathbb{T}_1)x_n) \\
&\leq \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)^n x_n) + L\mathfrak{d}(x_n, y_{n-1}) \\
&\quad + L\mathfrak{d}((\mathcal{P}\mathbb{T}_1)^{n-1} y_{n-1}, x_n).
\end{aligned} \tag{3.30}$$

Using (3.20), (3.27) and (3.29), we obtain

$$\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n) = 0. \quad (**)$$

Now,

$$\mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_2)^n x_n) \leq \mathfrak{d}(x_{n+1}, x_n) + \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)^n x_n). \tag{3.31}$$

This implies, by (3.25), (3.26) that

$$\lim_{n \rightarrow \infty} \mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_2)^n x_n) = 0. \tag{3.32}$$

Next, we consider

$$\begin{aligned}
\mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_2)x_{n+1}) &\leq \mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_2)^{n+1} x_{n+1}) \\
&\quad + \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^{n+1} x_{n+1}, (\mathcal{P}\mathbb{T}_2)^{n+1} x_n) \\
&\quad + \mathfrak{d}((\mathcal{P}\mathbb{T}_2)^{n+1} x_n, (\mathcal{P}\mathbb{T}_2)x_{n+1}) \\
&\leq \mathfrak{d}(x_{n+1}, (\mathcal{P}\mathbb{T}_2)^{n+1} x_{n+1}) + L\mathfrak{d}(x_{n+1}, x_n) \\
&\quad + L\mathfrak{d}((\mathcal{P}\mathbb{T}_2)^n x_n, x_{n+1}))
\end{aligned} \tag{3.33}$$

Again, this implies, by (3.25), (3.26) and (3.32), that

$$\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n) = 0. \quad (**)$$

Our proof is finished. \square

Theorem 3.4. *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space. Let \mathbb{E} be a nonempty closed convex subset of \mathbb{X} with monotone modulus of uniform convexity η and \mathcal{P} as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$.*

Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. If \mathbb{T}_1 and \mathbb{T}_2 are weakly inward and one of \mathbb{T}_1 and \mathbb{T}_2 is compact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of \mathbb{T}_1 and \mathbb{T}_2 .

Proof. By Lemma 3.2, $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p)$ exists for any $p \in \mathbb{F}$. It is sufficient to show that $\{x_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to a common fixed point of \mathbb{T}_1 and \mathbb{T}_2 . By Lemma 3.3, $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n) = 0 = \lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n)$. Suppose that \mathbb{T}_1 is compact. Since \mathcal{P} is nonexpansive, there exists a subsequence $\{(\mathcal{P}\mathbb{T}_1)x_{n_j}\}$ of $\{(\mathcal{P}\mathbb{T}_1)x_n\}$ such that $(\mathcal{P}\mathbb{T}_1)x_{n_j} \rightarrow p$. Thus

$$\mathfrak{d}(x_{n_j}, p) \leq \mathfrak{d}(x_{n_j}, (\mathcal{P}\mathbb{T}_1)x_{n_j}) + \mathfrak{d}((\mathcal{P}\mathbb{T}_1)x_{n_j}, p) \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

This means $x_{n_j} \rightarrow p$ as $j \rightarrow \infty$. Again $\lim_{j \rightarrow \infty} \mathfrak{d}(x_{n_j}, (\mathcal{P}\mathbb{T}_1)x_{n_j}) = 0$ yields by continuity of \mathcal{P} and \mathbb{T}_1 that $p = (\mathcal{P}\mathbb{T}_1)p$. Similarly, $p = (\mathcal{P}\mathbb{T}_2)p$. Noting that $F(\mathcal{P}\mathbb{T}) = F(\mathbb{T})$. Therefore $p = F(\mathbb{T}_1) = F(\mathbb{T}_2)$, and so $p \in \mathbb{F}$. Thus $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point p of \mathbb{T}_1 and \mathbb{T}_2 . \square

Corollary 3.5. *Let \mathbb{X} be a complete CAT(0) space and \mathbb{E} be a nonempty closed convex subset of \mathbb{X} with \mathcal{P} as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. If \mathbb{T}_1 and \mathbb{T}_2 are weakly inward and one of \mathbb{T}_1 and \mathbb{T}_2 is compact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of \mathbb{T}_1 and \mathbb{T}_2 .*

Theorem 3.6. *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space. Let \mathbb{E} be a nonempty closed convex subset of \mathbb{X} with monotone modulus of uniform convexity η and \mathcal{P} as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1-a]$ for some $a \in (0, 1)$. If \mathbb{T}_1 and \mathbb{T}_2 are weakly inward and satisfy condition (\mathcal{B}) , then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of \mathbb{T}_1 and \mathbb{T}_2 .*

Proof. By Lemma 3.2 $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p)$ exists and so $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{F})$ exists for all $p \in \mathbb{F}$. Again, by Lemma 3.3, $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n) = 0 = \lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n)$. It follows from condition (\mathcal{B}) and Lemma 3.3 that

$$\lim_{n \rightarrow \infty} f(\mathfrak{d}(x_n, \mathbb{F})) \leq \lim_{n \rightarrow \infty} \left(\frac{1}{2} [\mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n)) + \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n)] \right) = 0.$$

That is,

$$\lim_{n \rightarrow \infty} f(\mathfrak{d}(x_n, \mathbb{F})) = 0.$$

Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$.

Next, we show that x_n is a Cauchy sequence. Let $\epsilon > 0$. Since $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$, there exists a positive number n_0 such that for all $n \geq n_0$, we have

$$\mathfrak{d}(x_n, \mathbb{F}) < \frac{\epsilon}{2C},$$

where $C > 0$ is the constant in Lemma 3.2(ii). So we can find $p' \in \mathbb{F}$ such that

$$\mathfrak{d}(x_n, p') < \frac{\epsilon}{2C}.$$

Using Lemma 3.2(ii), we have for all $n \geq n_0$ and $m \in \mathbb{N}$ that

$$\begin{aligned} \mathfrak{d}(x_{n+m}, x_n) &\leq \mathfrak{d}(x_{n+m}, p') + \mathfrak{d}(p', x_n) \\ &\leq C\mathfrak{d}(x_n, p') + \mathfrak{d}(x_n, p') \\ &\leq C\mathfrak{d}(x_n, p') + C\mathfrak{d}(x_n, p') \\ &= 2C\mathfrak{d}(x_n, p') < \epsilon. \end{aligned} \tag{3.34}$$

Hence, $\{x_n\}$ is a Cauchy sequence in a closed convex subset \mathbb{E} of a hyperbolic space \mathbb{X} , therefore, it must converge to a point in \mathbb{E} . Let $\lim_{n \rightarrow \infty} x_n = q$. Now, $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$ yields that $\mathfrak{d}(q, \mathbb{F}) = 0$. Since the set of fixed points of asymptotically quasi-nonexpansive mappings is closed, we have $q \in \mathbb{F}$. \square

Corollary 3.7. *Let \mathbb{X} be a complete CAT(0) space and \mathbb{E} be a nonempty closed convex subset of \mathbb{X} with \mathcal{P} as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive*

mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If \mathbb{T}_1 and \mathbb{T}_2 are weakly inward and satisfy **condition (B)**, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of \mathbb{T}_1 and \mathbb{T}_2 .

We shall use condition **(A)** to prove strong convergence of the algorithm (3.1). Before that, we prove the following technical lemma.

Lemma 3.8. *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space. Let \mathbb{E} be a nonempty closed convex subset of \mathbb{X} with monotone modulus of uniform convexity η and \mathcal{P} as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If \mathbb{T}_1 and \mathbb{T}_2 are weakly inward and satisfy condition **(A)**. Then the sequence $\{x_n\}$ converges strongly to $p \in \mathbb{F}$ if and only if $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$.*

Proof. We know from Lemma 3.2 that $\mathfrak{d}(x_{n+1}, p) \leq \mathfrak{d}(x_n, p)$. It follows that $\{x_n\}$ is Fejér monotone with respect to \mathbb{F} and $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{F})$ exists. Hence, the result follows from Lemma 2.3. \square

Applying Lemma 3.8, we obtain following strong convergence theorem.

Theorem 3.9. *Let $(\mathbb{X}, \mathfrak{d}, W)$ be a uniformly convex hyperbolic space. Let \mathbb{E} be a nonempty closed convex subset of \mathbb{X} with monotone modulus of uniform convexity η and \mathcal{P} as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If \mathbb{T}_1 and \mathbb{T}_2 are weakly inward and satisfy condition **(A)**, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of \mathbb{T}_1 and \mathbb{T}_2 .*

Proof. It follows from Lemma 3.2 that $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{F})$ exists. Moreover, by Lemma 3.3 we have that $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_1)x_n) = \lim_{n \rightarrow \infty} \mathfrak{d}(x_n, (\mathcal{P}\mathbb{T}_2)x_n) = 0$. Applying condition **(A)**, we obtain

$$\lim_{n \rightarrow \infty} f(\mathfrak{d}(x_n, \mathbb{F})) = 0.$$

Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, \mathbb{F}) = 0$. Therefore, Lemma 3.8 implies that $\{x_n\}$ converges strongly to a point $p \in \mathbb{F}$. \square

Corollary 3.10. *Let \mathbb{X} be a complete $CAT(0)$ space and \mathbb{E} be a nonempty closed convex subset of \mathbb{X} with \mathcal{P} as a sunny nonexpansive retraction. Let $\mathbb{F} \neq \emptyset$ and $\mathbb{T}_1, \mathbb{T}_2 : \mathbb{E} \rightarrow \mathbb{X}$ be two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} with sequence $\{a_n\}_{n=1}^{\infty} \subset [1, \infty)$, $\lim_{n \rightarrow \infty} a_n = 1$ satisfying $\sum_{n=1}^{\infty} (a_n - 1) < \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is defined by (3.1), where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[a, 1 - a]$ for some $a \in (0, 1)$. If \mathbb{T}_1 and \mathbb{T}_2 are weakly inward and satisfy **condition (A)**, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a common fixed point of \mathbb{T}_1 and \mathbb{T}_2 .*

4. CONCLUSION

In this manuscript, we have established new common fixed point results for two uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} and two weakly inward and asymptotically quasi-nonexpansive mappings with respect to \mathcal{P} satisfying condition (\mathcal{A}) and condition (\mathcal{B}) , in a more general set up of hyperbolic space. Our results significantly extend and improve the results obtained by Khan et al.[16], as well as many related results in $CAT(0)$ spaces and uniformly Banach spaces. As consequences of our main results, we obtain the corresponding corollaries which are valid in $CAT(0)$ spaces.

5. ACKNOWLEDGEMENTS

The authors are grateful to the referee(s) for helpful comments for the improvement of this manuscript and to Maejo University, Lampang Rajabhat University for moral support.

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