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NEW APPLICATIONS OF THE METATHEOREM IN ORDERED FIXED POINT THEORY

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In memory of 65 years with Gyoung

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ABSTRACT. Our aim in this paper is to find new applications of our long-standing 2023 Metatheorem. In fact, a certain particular form of Metatheorem on fixed point theorems characterizes metric completeness. Moreover, classical theorems due to Banach, Rus-Hicks-Rhoades, Nadler, Covitz-Nadler, Oettli-Théra, Edelstein, Turinici, Tasković, Khamsi and ourselves are equivalently formulated or improved by applying Metatheorem.

KEYWORDS: Quasi-metric space, fixed point, RHR contraction principle, orbitally complete, orbitally continuous.
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1. Introduction

Our Metatheorem in Ordered Fixed Point Theory has a long history and many applications. Its more than one hundred applications produce old and new theorems and clarify mutual relations among them. One of the main applications of them is closely related to the Banach contraction principle — the origin of Metric Fixed Point Theory.

Let (X, d) be a metric space. A Banach contraction $T : X \longrightarrow X$ is a map satisfying

$$d(Tx, Ty) \leq \alpha\, d(x, y) \quad \text{for all } x, y \in X$$

with some $\alpha \in [0, 1)$. There have been appeared thousands of articles related to the Banach contraction. It is well-known that the Banach contraction does not characterize the metric completeness.

Recently, we introduced the Rus-Hicks-Rhoades (RHR) map $T : X \longrightarrow X$ [31], [7] satisfying

$$d(Tx, T^2x) \leq \alpha\, d(x, Tx) \quad \text{for all } x \in X$$

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with some $\alpha \in [0, 1)$. See our recent works [27], [28], [30]. The RHR maps are also known as graphic contractions, iterative contractions, weakly contractions, or Banach mappings; see Berinde et al. [2],[3]. Moreover, it is recently known that well-known metric fixed point theorems related to the RHR maps hold for quasi-metric spaces (without assuming the symmetry); see [28], [30].

Our aim in the present paper is to find new applications of our Metatheorem. In fact, certain classical theorems due to Banach, Rus-Hicks-Rhoades [26], [27], [29], Nadler [11], Covitz-Nadler [5], Oettli-Théra [12], Edelstein [6], Turinici [34], [35], Tasković [33], and Khamsi [10] are equivalently formulated or improved by applying our Metatheorem. Especially, the completeness of quasi-metric spaces are equivalent to several fixed point or other theorems due to Rus-Hicks-Rhoades, Nadler, Covitz-Nadler, Oettli-Théra, and others.

This paper is organized as follows: Section 2 is to introduce our long-standing Metatheorem. In Section 3, basic terminology on quasi-metric spaces are given as preliminaries. Section 4 is to introduce our recent versions of the Rus-Hicks-Rhoades (RHR) contraction principle and the new Banach contraction principle. Section 5 devotes to a certain particular form of Metatheorem on fixed point theorems which characterizes metric completeness. In Sections 6-10, several theorems due to Edelstein, Turinici, Tasković, and Khamsi are equivalently formulated by applying our Metatheorem. Finally, Section 11 is for the epilogue.

In this paper, multimaps are always non-empty valued.

2. Our 2023 Metatheorem

Our Metatheorem has a long history. We obtained the following form called the new 2023 Metatheorem in [19],[24],[26]:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x, y)$ a sentence formula for $x, y \in X$. Then the following are equivalent:*

(α) *There exists an element $v \in A$ such that the negation of $G(v, w)$ holds for any $w \in X \backslash \{v\}$.*

($\beta 1$) *If $f : A \longrightarrow X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $G(x, y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*

($\beta 2$) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $G(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

($\gamma 1$) *If $f : A \longrightarrow X$ is a map such that $G(x, fx)$ for any $x \in A$ with $x \neq fx$, then f has a fixed element $v \in A$, that is, $v = fv$.*

($\gamma 2$) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow X$ satisfying $G(x, fx)$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

($\delta 1$) *If $F : A \multimap X$ is a multimap such that, for any $x \in A \backslash Fx$ there exists $y \in X \backslash \{x\}$ satisfying $G(x, y)$, then F has a fixed element $v \in A$, that is, $v \in Fv$.*

($\delta 2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A \backslash Fx$ there exists $y \in X \backslash \{x\}$ satisfying $G(x, y)$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*

($\epsilon 1$) *If $F : A \multimap X$ is a multimap satisfying $G(x, y)$ for any $x \in A$ and any $y \in Fx \backslash \{x\}$, then F has a stationary element $v \in A$, that is, $\{v\} = Fv$.*

(ϵ2) If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $G(x, y)$ holds for any $x \in A$ and any $y \in Fx\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.

(η) If Y is a subset of X such that for each $x \in A\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $G(x, z)$, then there exists a $v \in A \cap Y$.

For the proof, see Park [19], [24], [26]. Each item in Metatheorem has a long history. Especially, (η) is originated from Oettli-Théra [12].

This Metatheorem guarantees the truth of all items when one of them is true. Since 1985, we have shown nearly one hundred cases of such situation. See [13]–[26].

3. Quasi-metric Spaces

It is well-known that some key-results in Metric Fixed Point Theory hold for quasi-metric spaces. For example, Banach contraction principle, Nadler or Covitz-Nadler fixed point theorem, Ekeland variational principle, Caristi fixed point theorem, Takahashi minimization principle, and many others.

We recall the following:

Definition 3.1. A *quasi-metric* on a nonempty set X is a function $\delta : X \times X \longrightarrow \mathbb{R}^+ = [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $\delta(x, y) = \delta(y, x) = 0 \Longleftrightarrow x = y$;
(b) (triangle inequality) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

A *metric* on a set X is a quasi-metric satisfying

(c) (symmetry) $\delta(x, y) = \delta(y, x)$ for all $x, y \in X$.

The convergence and completeness in a quasi-metric space (X, δ) are defined as follows:

Definition 3.2. ([1], [8])
(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n \longrightarrow \infty} \delta(x_n, x) = \lim_{n \longrightarrow \infty} \delta(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $n > m > N$.
(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $m > n > N$.
(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $\delta(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 3.3. ([1], [8])
(1) (X, δ) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, δ) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, δ) is *complete* if every Cauchy sequence in X is convergent.

Definition 3.4. Let (X, δ) be a quasi-metric space and $T : X \longrightarrow X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, Tx, \cdots, T^n x, \cdots\}.$$

The space X is said to be *T-orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\longrightarrow\infty} T^n x = x_0 \Longrightarrow \lim_{n\longrightarrow\infty} T^{n+1} x = T x_0$$

for any $x \in X$.

4. The Rus-Hicks-Rhoades Contraction Principle

For quasi-metric spaces (X, δ), simply δ is not symmetric.

Definition. The *orbit* of a selfmap $T : X \longrightarrow X$ at $x \in X$ is the set $O(x, T) = \{T^n x : n = 0, 1, 2, \dots\}$. The space X is said to be *T-orbitally complete* if every (right)-Cauchy sequence in $O(x, T)$ is convergent in X. A selfmap T of X is said to be *T-orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\longrightarrow\infty} T^n(x) = x_0 \implies \lim_{n\longrightarrow\infty} T^{n+1}(x) = T(x_0)$$

for any $x \in X$.

The following in Park [27], [28], [30] is called the Rus-Hicks-Rhoades (RHR) Contraction Principle:

Theorem P. *Let (X, δ) be a quasi-metric space and let $T : X \longrightarrow X$ be an RHR map; that is,*

$$\delta(T(x), T^2(x)) \le \alpha\, \delta(x, T(x)) \text{ for every } x \in X,$$

where $0 \le \alpha < 1$.

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\longrightarrow\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \le \frac{\alpha^n}{1-\alpha}\delta(x, T(x)), \quad n = 1, 2, \cdots,$$

$$\delta(T^n(x), x_0) \le \frac{\alpha}{1-\alpha}\delta(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) *x_0 is a fixed point of T, and, equivalently,*

(iii) *$T : X \longrightarrow X$ is orbitally continuous at $x_0 \in X$.*

This was proved in [30] by analyzing a typical proof of the Banach Contraction Principle.

For the condition: there exists $0 < \alpha < 1$ such that $d(T(x), T^2(x)) \le \alpha \cdot d(x, T(x))$, for all $x \in X$, we meet the following names: graphic contraction, iterative contraction, weakly contraction, Banach mapping, etc.

Moreover, the following consequence of Theorem P in Park [30] extends the usual Banach Contraction Principle:

Theorem Q. *Let (X, δ) be a quasi-metric space and let $T : X \longrightarrow X$ be an improved Banach contraction, that is, for each $x \in X$, there exists a $y \in X$ such that*

$$\delta(T(x), T(y)) \le \alpha\, \delta(x, y) \text{ where } 0 \le \alpha < 1.$$

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\longrightarrow\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha}\delta(x, T(x)), \quad n = 1, 2, \cdots,$$

$$\delta(T^n(x), x_0) \leq \frac{\alpha}{1-\alpha}\delta(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) *x_0 is the unique fixed point of T (equivalently, $T : X \longrightarrow X$ is orbitally continuous at $x_0 \in X$).*

The Banach Contraction Principle appeared in thousands of publications should be corrected as in Theorem Q.

We began our study on RHR maps in [27] and [28]. Later we found a large number of examples of RHR maps in [29], [30], where we showed a large number of metric fixed point theorems can be extended or improved.

5. Completeness of Quasi-metric Spaces

In our previous work [30], we obtained the following RHR theorem:

Theorem H(γ1). *Let (X, δ) be a quasi-metric space, $0 < \alpha < 1$, and $f : X \longrightarrow X$ be a map satisfying*

$$\delta(f(x), f^2(x)) \leq \alpha\, \delta(x, f(x)) \; for \; all \; x \in X \backslash \{f(x)\}.$$

Then f has a fixed point $v \in X$ if and only if X is f-orbitally complete.

Let (X, δ) be a quasi-metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A, B) = \max\{\sup\{\delta(a, B) : a \in A\}, \; \sup\{\delta(b, A) : b \in B\}\},$$

where $\delta(a, B) = \inf\{\delta(a, b) : b \in B\}$. Then H is called a generalized Hausdorff quasi-metric since it may have infinite values.

Recently, as a basis of Ordered Fixed Point Theory [19], [24], [26], we obtained the 2023 Metatheorem and Theorem H including Nadler's fixed point theorem [11] in 1969 and its extended version by Covitz-Nadler [5] in 1970.

From Theorem H(γ1) and Metatheorem, we have the following new version:

Theorem H. ([24], [26], [30]) *Let (X, δ) be a quasi-metric space and $0 < r < 1$. Then the following statements are equivalent:*

(0) *(X, δ) is complete.*

(α) *For a multimap $T : X \longrightarrow \mathrm{Cl}(X)$, there exists an element $v \in X$ such that $H(Tv, Tw) > r\, \delta(v, w)$ for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : X \longrightarrow X$ such that, for any $x \in X \backslash \{fx\}$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(fx, fy) \leq r\, \delta(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : X \longrightarrow X$ satisfying $\delta(fx, f^2x) \leq r\, d(x, fx)$ for all $x \in X \backslash \{fx\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $T : X \longrightarrow \mathrm{Cl}(X)$ such that, for any $x \in X \backslash Tx$, there exists $y \in X \backslash \{x\}$ satisfying $H(Tx, Ty) \leq r\, \delta(x, y)$. Then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v \in Tv$ for all $T \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : X \longrightarrow \mathrm{Cl}(X)$ satisfying $H(Tx, Ty) \leq r\, \delta(x, y)$ for all $x \in X$ and any $y \in Tx \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = Tv$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in X \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $H(Tx, Tz) \leq r\,\delta(x, z)$ for a $T : X \longrightarrow \mathrm{Cl}(X)$, then there exists a $v \in X \cap Y = Y$.*

Proof. The equivalency (α)-(η) follows from Metatheorem. When $\mathfrak{F}$ is a singleton, (β)-(ϵ) are denoted by (β1)-(ϵ1), respectively. They are also logically equivalent to (α)-(η) by Metatheorem. Note that (γ1) follows from Theorem H(γ1). The equivalency of (0) and (γ1) is given in [29], [30]. Then Theorem H holds. □

Remark 5.1. (1) The completeness in (0) can be replaced by f-orbitally or T-orbitally completeness according to the corresponding situation.

(2) (β1) properly extends the Banach contraction principle.

(3) (γ1) is the Rus-Hicks-Rhoades theorem and equivalent to (0).

(4) Further, (δ1) and (ϵ1) extend the well-known theorems of Nadler [11] and Covitz-Nadler [5] on multi-valued contraction.

(5) Actually, the proof of Theorem H covers the corresponding ones of Banach, Rus [31], Hicks-Rhoades [7], Nadler [11], Covitz-Nadler [5], and Oettli-Théra [12].

(6) There are a large number of characterizations of metric completeness. It is well-known that the Banach contraction does not characterize. However, so does its slight generalized form (β1) and the RHR map in (γ1).

We have a single-valued version of Theorem H(α) as follows:

Theorem H(α1). *Let (X, δ) be a quasi-metric space, $f : X \longrightarrow X$ a map and $0 < r < 1$. Then X is f-orbitally complete if and only if there exists an element $v \in X$ such that $\delta(fv, fw) > r\,\delta(v, w)$ for any $w \in X \backslash \{v\}$.*

This is also equivalent to all items in Theorem H. In some sense, this shows that the Banach contraction principle does not characterize the metric completeness. But so does the RHR theorem or Theorem H(γ1).

6. Edelstein [6] in 1962

In this section, we apply Metatheorem to a particular situation when $f : X \longrightarrow X$ is a map and $G(x, y)$ means $\delta(x, fx) \leq \delta(y, fy)$ for $x, y \in X$.

Definition 6.1. A map $f : X \longrightarrow X$ on a quasi-metric space (X, δ) is said to be *contractive* if

$$\delta(fx, fy) < \delta(x, y)$$

for all $x, y \in X$ with $x \neq y$.

We recall the well-known Edelstein fixed point theorem:

Theorem 6.2. (Edelstein) *Let (X, d) be a compact metric space and $f : X \longrightarrow X$ be a contractive map. Then f has a unique fixed point $v \in X$, and moreover, for each $x \in X$, we have $\lim_{n \longrightarrow \infty} f^n(x) = v$.*

Motivated by Theorem 6.2, we have the following from our Metatheorem:

Theorem 6.3. *Let (X, δ) be a compact quasi-metric space. Then the following statements are equivalent:*

(α) *For a map $f : X \longrightarrow X$, there exists a point $v \in X$ such that $\delta(fv, fw) \geq \delta(v, w)$ for any $w \in X \backslash \{v\}$.*

($\beta 1$) *For a map $f : X \longrightarrow X$ such that, for any $x \in X$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(fx, fy) < \delta(x, y)$, then f has a fixed point $v \in X$, that is, $v = fv$.*

($\beta 2$) *If $\mathfrak{F}$ is a family of maps $f : X \longrightarrow X$ such that, for any $x \in X$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(fx, fy) < \delta(x, y)$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

($\gamma 1$) *If $f : X \longrightarrow X$ is a map such that, for any $x \in X$ satisfying $\delta(fx, f^2x) < \delta(x, fx)$ for all $x \in X$ with $x \neq fx$, then f has a fixed point $v \in X$, that is, $v = fv$.*

($\gamma 2$) *If $\mathfrak{F}$ is a family of maps $f : X \longrightarrow X$ satisfying $\delta(fx, f^2x) < \delta(x, fx)$ for all $x \in X$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

($\delta 1$) *If $T : X \longrightarrow \mathrm{Cl}(X)$ is a multimap such that for any $x \in X \backslash Tx$ there exists a $y \in X \backslash \{x\}$ satisfying $H(Tx, Ty) < \delta(x, y)$, then T has a fixed point $v \in X$, that is, $v \in T(v)$.*

($\delta 2$) *If $\mathfrak{F}$ is a family of multimaps $T : X \longrightarrow \mathrm{Cl}(X)$ such that for any $x \in X \backslash Tx$ there exists a $y \in X \backslash \{x\}$ satisfying $H(Tx, Ty) < \delta(x, y)$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v \in Tv$ for all $T \in \mathfrak{F}$.*

($\epsilon 1$) *If $T : X \longrightarrow \mathrm{Cl}(X)$ is a multimap such that $H(Tx, Ty) < \delta(x, y)$ holds for any $x \in X$ and any $y \in Tx \backslash \{x\}$, then T has a stationary point $v \in X$, that is, $\{v\} = Tv$.*

($\epsilon 2$) *If $\mathfrak{F}$ is a family of multimaps $T : X \longrightarrow \mathrm{Cl}(X)$ such that $H(Tx, Ty) < \delta(x, y)$ holds for any $x \in X$ and any $y \in Tx \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary point $v \in X$, that is, $\{v\} = Tv$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in X \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $H(Tx, Tz) < \delta(x, z)$ for a multimap $T : X \longrightarrow \mathrm{Cl}(X)$, then there exists a $v \in X \cap Y = Y$.*

Proof. Equivalency follows from Metatheorem. □

Remark 6.4. (1) Theorem 6.3 means the equivalency of the items (α)-(η). Therefore, each items are conjecture.

(2) Each item implies the Edelstein Theorem 6.2. This is clear for ($\beta 1$), ($\gamma 1$), ($\delta 1$), and ($\epsilon 1$),

(3) In case f is continuous in (α), all (α), ($\beta 1$), ($\gamma 1$), ($\delta 1$), and ($\epsilon 1$) are true. In fact, let a map $\varphi : X \longrightarrow \mathbb{R}^+$ by putting

$$\varphi(x) = \delta(x, fx), \quad x \in X.$$

Then φ is continuous and bounded below, so it has a minimum value at a point $v \in X$. Hence (α) holds. Moreover, ($\beta 1$)–(η) also hold by Metatheorem.

From Theorem 6.3, we can deduce several fixed point theorems on a compact quasi-metric space (X, δ) extending the Edelstein Theorem 6.1.

For example, we have the following:

Theorem 6.5. ($\beta 1$) *If $f : X \longrightarrow X$ is a continuous map such that for any $x \in X$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(x, fx) > d(y, fy)$, then f has a fixed point $v \in X$, that is, $v = fv$.*

($\gamma 2$) *If* $\mathfrak{F}$ *is a family of continuous maps* $f : X \longrightarrow X$ *satisfying* $d(x, fx) > d(fx, f^2x)$ *for all* $x \in X$ *with* $x \neq fx$, *then* $\mathfrak{F}$ *has a common fixed point* $v \in X$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

($\epsilon 1$) *If* $T : X \multimap X$ *is a multimap and* $f : X \longrightarrow X$ *is a continuous selection of* T *such that* $d(x, fx) > d(y, fy)$ *holds for any* $x \in X$ *and any* $y \in T(x)\backslash\{x\}$, *then* T *has a stationary point* $v \in X$, *that is,* $\{v\} = T(v)$.

Recently, Kirk and Shahzad raised one open question on Edelstein's fixed point theorem. In 2018, Suzuki [32] gave a negative answer to this question, and extended Edelstein's theorem to semimetric spaces.

7. Turinici [34] in 1980

Turinici's main result ([34], Theorem 3.1) is as follows:

Theorem 7.1. *Let* (X, d) *be a metric space, and* $\preccurlyeq$ *an ordering on* X *such that*

(1) $\preccurlyeq$ *is a closed ordering on* X,
(2) (X, d) *is a* $\preccurlyeq$*-asymptotic metric space, and*
(3) (X, d) *is a* $\preccurlyeq$*-complete metric space.*

Then, for every $x \in X$ *there is a maximal element* $z \in X$ *such that* $x \preccurlyeq z$.

This can be applied to our Metatheorem as follows:

Theorem 7.2. *Let* (X, d) *be a metric space, and* $\preccurlyeq$ *an ordering on* X *satisfying* (1)-(3). *Let* $z \in X$ *and* $A := \{x \in X : z \preccurlyeq x\}$.

Then the following equivalent statements hold:

(α) *There exists an element* $v \in A$ *such that* $w \prec v$ *for any* $w \in X\backslash\{v\}$.

(β) *If* $\mathfrak{F}$ *is a family of maps* $f : A \longrightarrow X$ *such that for any* $x \in A$ *with* $x \neq fx$, *there exists a* $y \in X\backslash\{x\}$ *satisfying* $x \preccurlyeq y$), *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(γ) *If* $\mathfrak{F}$ *is a family of maps* $f : A \longrightarrow X$ *satisfying* $x \preccurlyeq fx$ *for all* $x \in A$ *with* $x \neq fx$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(δ) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : A \multimap X$ *such that, for any* $x \in A\backslash Fx$ *there exists* $y \in X\backslash\{x\}$ *satisfying* $x \preccurlyeq y$. *Then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v \in Fv$ *for all* $F \in \mathfrak{F}$.

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $F : A \multimap X$ *such that* $x \preccurlyeq y$ *holds for any* $x \in A$ *and any* $y \in Fx\backslash\{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in A$, *that is,* $\{v\} = Fv$ *for all* $F \in \mathfrak{F}$.

(η) *If* Y *is a subset of* X *such that for each* $x \in A\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $x \preccurlyeq y$, *then there exists a* $v \in A \cap Y$.

Proof. Under the hypothesis, the conclusion of ([34], Theorem 3.1) is "for every $x \in X$, there is a maximal element $z \in X$." Replacing (x, z) by (z, v), we obtain α. The equivalency is obtained from Metatheorem, where $G(x, y)$ is replaced by $x \preccurlyeq y$. □

Note that (α) and ($\gamma 1$) are [34], Theorems 3.1 and 3.2, respectively.

8. Tasković [33] in 1986

Recall that Tasković [33] showed that Zorn's lemma is equivalent to the following:

Theorem 8.1. *Let $\mathfrak{F}$ be a family of selfmaps defined on a partially ordered set A such that $x \leq fx$ (resp. $fx \leq x$), for all $x \in A$ and all $f \in \mathfrak{F}$. If each chain in A has an upper bound (resp. lower bound), then the family $\mathfrak{F}$ has a common fixed point.*

This can be applied to the following:

Theorem 8.2. *Let A be a partially ordered set such that each chain in A has an upper bound. Then the following equivalent statements hold:*

(α) *There exists an element $v \in A$ such that $w \prec v$ for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow A$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in A \backslash \{x\}$ satisfying $x \preccurlyeq y$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow A$ satisfying $x \preccurlyeq fx$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap A$ such that, for any $x \in A \backslash Fx$ there exists $y \in A \backslash \{x\}$ satisfying $x \preccurlyeq y$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap A$ such that $x \preccurlyeq y$ holds for any $x \in A$ and any $y \in Fx \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of A such that for each $x \in A \backslash Y$ there exists a $z \in A \backslash \{x\}$ satisfying $x \preccurlyeq y$, then there exists a $v \in Y$.*

Proof. Note that (α) is a form of Zorn's lemma and (γ) is the theorem due to Tasković. Therefore Theorem 7.2 holds by Metatheorem. □

Other true statements (β1)-(ϵ1) can be also obtained.

9. Khamsi [10] in 2009

In [10], Khamsi gave a characterization of the existence of minimal elements in partially ordered sets in terms of fixed point of multimaps.

Let A be an abstract set partially ordered by $\prec$. We will say that $a \in A$ is a minimal element of A if and only if $b \prec a$ implies $b = a$. The concept of minimal element is crucial in the proofs given to Caristi's fixed point theorem.

The following is [10], Theorem 1:

Theorem 9.1. *Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent.*

(1) *A contains a minimal element,*

(2) *Any multimap T defined on A such that for any $x \in A$, there exists $y \in Tx$ with $y \prec x$, has a fixed point, i.e there exists $a \in A$ such that $a \in Ta$.*

According to our method in the present paper, Theorem 9.1 can be extended as follows:

Theorem 9.2. *Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent:*

(α) *There exists an element $v \in A$ such that $w \prec v$ for any $w \in A \backslash \{v\}$.*

(β) *If* $\mathfrak{F}$ *is a family of maps* $f : A \longrightarrow A$ *such that for any* $x \in A$ *with* $x \neq fx$, *there exists a* $y \in A \backslash \{x\}$ *satisfying* $x \preccurlyeq y$), *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(γ) *If* $\mathfrak{F}$ *is a family of maps* $f : A \longrightarrow A$ *satisfying* $x \preccurlyeq fx$ *for all* $x \in A$ *with* $x \neq fx$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = fv$ *for all* $f \in \mathfrak{F}$.

(δ) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : A \multimap A$ *such that, for any* $x \in A \backslash Fx$ *there exists* $y \in A \backslash \{x\}$ *satisfying* $x \preccurlyeq y$. *Then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v \in Fv$ *for all* $F \in \mathfrak{F}$.

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $F : A \multimap X$ *such that* $x \preccurlyeq y$ *holds for any* $x \in A$ *and any* $y \in Fx \backslash \{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in A$, *that is,* $\{v\} = Fv$ *for all* $F \in \mathfrak{F}$.

(η) *If* Y *is a subset of* A *such that for each* $x \in A \backslash Y$ *there exists a* $z \in A \backslash \{x\}$ *satisfying* $x \preccurlyeq y$, *then there exists a* $v \in Y$.

Proof. Let $G(x, y)$ means $x \prec y$. Then Theorem 8.2 follows from Metatheorem. □

Note that (α) and ($\delta 1$) are (1) and (2) of Theorem 9.1. Therefore Theorem 9.2 extends Theorem 9.1.

In what follows Khamsi assumes that $\eta : [0, \infty) \longrightarrow [0, \infty)$ is nondecreasing, continuous, such that there exist $c > 0$ and $\delta_0 > 0$ such that for any $t \in [0, \delta_0]$ we have $\eta(t) \geq c\,t$. Under these assumptions we have the following result.

Theorem 9.3. *Let* M *be a complete metric space. Define the relation* $\prec$ *by*

$$x \prec y \iff \eta(d(x, y)) \leq \phi(y) - \phi(x)$$

where η *and* ϕ *satisfy all the above assumptions.*
Then the following equivalent statement hold:

(α) $(M, \prec)$ *has a minimal element* x_*, *i.e. if* $x \prec x_*$ *then we must have* $x = x_*$.

($\gamma 1$) *If* $f : M \longrightarrow M$ *is a map such that* $fx \prec x$ *for any* $x \in X$, *then* f *has a fixed element* $v \in A$, *that is,* $v = fv$.

($\delta 1$) *If* $F : M \multimap M$ *is a multimap such that, for any* $x \in M \backslash Fx$ *there exists* $y \in X \backslash \{x\}$ *satisfying* $y < x$, *then* F *has a fixed element* $v \in M$, *that is,* $v \in Fv$.

Note that (α) - ($\delta 1$) are due to Khamsi ([10], Theorems 2-4), respectively. Applying our Metatheorem, we can make some more as for ($\beta 2$)-($\epsilon 2$) and (η).

10. Turinici [35] in 2009

In [35], some pseudometric versions of the Brézis-Browder ordering principle [4] are discussed. An application of these facts to equilibrium points is also included.

Let $(M, \preccurlyeq)$ be a quasi-ordered structure; and $x \mapsto \varphi(x)$ stand for a function between M and $R_+ \cup \{\infty\} = [0, \infty]$. The following is ([35], Proposition 1):

Proposition 10.1. ([35]) *Assume*

(1a) $(M, \preccurlyeq)$ *is sequentially inductive: each ascending sequence has an upper bound (modulo* $(\preccurlyeq)$*),*

(1b) ψ *is* $(\preccurlyeq)$*-decreasing* $(x \preccurlyeq y \Longrightarrow \psi(x) \geq \psi(y))$, *and*

(2a) $(M, \preccurlyeq)$ *is almost regular (modulo* φ*):*

$$\forall x \in M,\ \forall \varepsilon > 0,\ \exists y = y(x, \varepsilon) \succcurlyeq x \text{ with } \varphi(y) \leq \varepsilon.$$

Let $u \in M$ and $A = \{y \in M : u \preccurlyeq y\}$. Then there exists $v \in A$ with $\varphi(v) = 0$ (hence v is $(\preccurlyeq, \varphi)$- maximal).

From Proposition 10.1 and our Metatheorem, we have the following extended version:

Theorem 10.2. *Under the hypothesis of Proposition 9.1, the following equivalent statements hold:*

(α) *There exists an element $v \in A$ such that $w \prec v$ for any $w \in M \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow M$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in M \backslash \{x\}$ satisfying $x \preccurlyeq y$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \longrightarrow M$ satisfying $x \preccurlyeq fx$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap M$ such that, for any $x \in A \backslash Fx$ there exists $y \in M \backslash \{x\}$ satisfying $x \preccurlyeq y$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap M$ such that $x \preccurlyeq y$ holds for any $x \in A$ and any $y \in Fx \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of M such that for each $x \in A \backslash Y$ there exists a $z \in M \backslash \{x\}$ satisfying $x \preccurlyeq y$, then there exists a $v \in A \cap Y$.*

Turinici stated that the following ordering principle ([35], Proposition 2) is then available (cf. Kang and Park [13]):

Proposition 10.3. *Assume that $(M, \preccurlyeq)$ is sequentially inductive and weakly regular (modulo d). Then, for each $u \in M$, there exists a $(\preccurlyeq, d)$- maximal $v \in M$ with $u \preccurlyeq v$.*

In [35], Propositions 2, 3, 4, 5 and Theorems 2, 3, 4 are all maximality statements and can be also equivalently formulated by Metatheorem.

11. Conclusion

In this paper, by applying Metatheorem, we obtain equivalent forms of some known theorems. Most of them are new and useful as the original theorems. Therefore our Metatheorem is the way to lead new truth from the equivalent old one. This method was already applied almost one hundred times by the author in [13]-[30].

Recall that there are many articles characterizing metric completeness. Recall that the Banach contraction does not characterize the completeness. In the present article, we introduced a surprising result. Theorem H shows that certain general forms of theorems of Banach, Nadler, Covitz-Nadler, and others characterize completeness of quasi-metric spaces. More precisely, the Rus-Hicks-Rhoades theorem is the one of such theorems.

In our previous works [27]–[30], we concentrated the study of RHR maps and found the close relation between such maps and completeness.

References

[1] H. Aydi, M. Jellali, E. Karapinar, *On fixed point results for α-implicit contractions in quasi-metric spaces and consequences*, Nonlinear Anal. Model. Control, 21(1) (2016) 40–56.

[2] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math. 19(1) (2003) 7– 22.

[3] V. Berinde, A. Petrusel, I. Rus, *Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces*, Fixed Point Theory, 24(2) (2023) 525–540. DOI: 10.24193/fpt-ro.2023.2.05

[4] H. Brézis and F.E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Adv. Math. 21 (1976) 355–364.

[5] H. Covitz, S.B. Nadler, Jr. *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. 8 (1970) 5–11.

[6] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37 (1962) 74–79.

[7] T.L. Hicks, B.E. Rhoades, *A Banach type fixed point theorem*, Math. Japon. 24 (1979) 327–330.

[8] M. Jleli, B. Samet, *Remarks on G-metric spaces and fixed point theorems*, Fixed Point Theory Appl. (2012) 2012:210.

[9] B.G. Kang, S. Park, *On generalized ordering principles in nonlinear analysis*, Nonlinear Anal. 14 (1990) 159–165.

[10] M.A. Khamsi, *Remarks on Caristi's fixed point theprem*, Nonlinear Anal., Preprint.

[11] S.B. Nadler, Jr. *Multi-valued contraction mappings*, Pacific J. Math. 30 (1969) 475-488.

[12] W. Oettli, M. Théra, *Equivalents of Ekeland's principle*, Bull. Austral. Math. Soc. 48 (1993) 385-392.

[13] S. Park, *Equivalents of various maximum principles*, Results in Nonlinear Analysis 5(2) (2022) 169–174.

[14] S. Park, *Applications of various maximum principles*, J. Fixed Point Theory (2022) 2022-3, 1–23. ISSN:2052–5338.

[15] S. Park, *Equivalents of maximum principles for several spaces*, Top. Algebra Appl. 10 (2022) 68–76.

[16] S. Park, *Equivalents of ordered fixed point theorems of Kirk, Caristi, Nadler, Banach, and others*, Adv. Th. Nonlinear Anal. Appl. 6(4) (2022) 420–432.

[17] S. Park, *Extensions of ordered fixed point theorems*, Nonlinear Funct. Anal. Appl. 28(3) (2023) 831–850.

[18] S. Park, *Variations of ordered fixed point theorems*, Linear Nonlinear Anal. 8(3) (2022) 225–237.

[19] S. Park, *Foundations of Ordered Fixed Point Theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **61**(2) (2022), 247–287.

[20] S. Park, *Applications of generalized Zorn's Lemma*, J. Nonlinear Anal. Optim. 13(2) (2022) 75 -84. ISSN : 1906-9605

[21] S. Park, *Applications of several minimum principles*, Adv. Th. Nonlinear Anal. Appl. 7(1) (2023) 52–60. ISSN: 2587-2648

[22] S. Park, *Equivalents of certain minimum principles*, Letters Nonlinear Anal. Appl. 1(1) (2023) 1–11.

[23] S. Park, *Equivalents of some ordered fixed point theorems*, J. Advances Math. Comp, Sci. 38(1) (2023) 52–67.

[24] S. Park, *Remarks on the Metatheorem in Ordered Fixed Point Theory*, Advanced Mathematical Analysis and Its Applications (Edited by P. Debnath, D.F.M. Torres, Y.J. Cho) CRC Press (2023), 11–27. DOI : 10.1201/9781003388678-2

[25] S. Park, *Equivalents of various theorems of Zermelo, Zorn, Ekeland, Caristi and others*, J. Advances Math. Comp, Sci. 38(5) (2023) 60-73. DOI: 10.9734/jamcs/2023/v38i51761

[26] S. Park, *History of the metatheorem in ordered fixed point theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 62(2) (2023) 373–410.

[27] S. Park, *Relatives of a theorem of Rus-Hicks-Rhoades*, Letters Nonlinear Anal. Appl. 1 (2023) 57–63. ISSN 2985-874X

[28] S. Park, *Almost all about Rus-Hicks-Rhoades maps in quasi-metric spaces*, Adv. Th. Nonlinear Anal. Appl. 7(2) (2023) 455–471. DOI 0.31197/atnaa.1185449

[29] S. Park, *Comments on the Suzuki type fixed point theorems*, Adv. Theory Nonlinear Anal. Appl. 7(3) (2023) 67–78. https://doi.org/10.17762/atnaa.v7.i3.275

[30] S. Park, *The realm of the Rus-Hicks-Rhoades maps in the metric fixed point theory*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 63(1) (2024) 1–50.

[31] I.A. Rus, *Teoria punctului fix, II*, Univ. Babes-Bolyai, Cluj, 1973.

[32] T. Suzuki, *Edelstein's fixed point theorems in symmetric spaces*, J. Nonlinear Var. Anal. 2 (2018) 165–175.

[33] M. R. Tasković, *On an equivalent of the axiom of choice and its applications*, Math. Japonica 31(6) (1986) 979–991.

[34] M. Turinici, *Maximal elements in a class of order complete metric spaces*, Math. Japonica 25(5) (1980) 511–517.

[35] M. Turinici, *Brézis-Browder principles and equilibrium points,* Libertas Mathematika, 29(1) (2009) 37–46.