M. Chaudhary et al., J. Nonlinear Anal. Optim. Vol. 16(2) (2025), 107-118

Journal of Nonlinear Analysis and Optimization
Volume 16(2) (2025)
http://ph03.tci-thaijo.org
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

GENERALISED α-m MONOTONICITY AND β-WELL POSEDNESS OF SET VARIATIONAL INEQUALITY PROBLEM

MAMTA CHAUDHARY[1*], MANSI DHINGRA[2] AND UMESH KUMAR[3]

[1]Department of Mathematics, Satyawati College (M), University of Delhi, Ashok Vihar, New Delhi, India 110052, India
[2]Department of Mathematics, Rajdhani College, University of Delhi, Raja Garden, New Delhi, India 110015, India
[3]Department of Mathematics, Rajdhani College, University of Delhi, Raja Garden, New Delhi, India 110015, India

ABSTRACT. This paper attempts to introduce two new concepts of monotonicty defined on semi-inner product spaces to analyse well-posedness for variational inequality problems. These concepts are generalised α-m monotonicity and generalised α-m pseudomonotonicity for set maps. We also present a new concept of well-posedness, namely β-well-posedness for set variational inequality problem (VI). We further study the relationship of these monotone maps along with (VI). Then we demonstrate a gap function for the above (VI). Beneath the assumption of the said pseudo-monotonicity, a result is obtained showing the relation between the solution and gap function of the said (VI) problem. Finally, with the help of this gap function, we formulate said variational inequality problem into a corresponding mathematical programming problem (MP) and establish the relations between the β-well-posedness of both problems.

KEYWORDS: Variational inequality, generalised α-m monotonicity, β-well-posedness, Gap function.
AMS Subject Classification: 47J20, 54C60.

1. Introduction

Tykhonov [21] proposed the concept of well-posedness for unconstrained optimization problems. This concept is helpful because it guarantees the convergence of a series of approximations to the exact solution of specific optimization problems. Different forms of well-posedness, such as Levitin-Polyak well-posedness etc have been researched for various inequalities over the past few decades as can be seen in references [3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23]. Jayswal

* *Corresponding author.*
Email address : mam.gupta18@gmail.com, mansidhingra7@gmail.com, umesh.kumar@rajdhani.du.ac.in.
Article history : Received 29/12/2024 Accepted 01/10/2025.

and Shalini [10] recently published a study of the above types of problems for both vector variational inequalities and mathematical programming problems.

In studying variational inequalities, monotone maps and their generalisations are crucial. In the literature, mathematicians have employed various types of monotonicities to generalise well-posedness and results related to variational inequality problems, as seen in [1, 2, 11, 13, 14].

Semi-inner products are of two types, the first one is an inner product not required to be strictly positive, and the second one is an inner product not required to be conjugate symmetric. It was formulated by Günter Lumer [18] to extend Hilbert space-type arguments to Banach spaces in functional analysis. Fundamental properties were later explored by Giles [7]. These concepts were further explored in the form of a book by Dragomir [6]. In this paper, we took the definition of the first type. To analyse well-posedness for variational inequality problems, this paper primarily aims to introduce two new concepts of monotonicity: generalised α-m monotonicity and generalised α-m pseudo monotonicity. The paper is divided into four sections. Sections 1 and 2 deal with the introduction and preliminaries, respectively. The third section presents the concepts of generalised α-m monotonicity, generalised α-m pseudo monotonicity and β-well-posedness of the problem (VI). Further, using generalised α-m monotonicity, the β-well-posedness is investigated for the problem (VI) and a property is derived from this well-posedness of (VI). Sufficient requirements are demonstrated for β-well-posedness for the set of approximate solutions of (VI). We employ the optimization problem and provide a gap function for (VI) in section 4. Assuming generalised α-m pseudo monotonicity of the set-valued map F, one result is proved related to the gap function of (VI). Further, we establish the relations of β-well-posedness of (VI) and the corresponding (MP).

2. Preliminaries

Suppose X is a linear space and p is a real valued function defined on R_+. Remember p as semi-norm on X if for all x, y belonging to X

(i) $p(x)$ is non negative;
(ii) homogeneity property holds ("$p(\alpha x) = |\alpha| p(x) \ \forall \ \alpha$");
(iii) triangle inequality holds ("$p(x + y) \leq p(x) + p(y)$").

p is referred to as a norm provided $p(x)$ is equal to zero implies x is equal to zero.

Definition 2.1. [18] A function $s(\cdot, \cdot) : X \times X \to K$ is called a semi-inner product on a linear space X over field K if

(i) $s(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 s(x_1 + y) + \lambda_2 s(x_2 + y), \forall \ x_1, x_2, y \in X$ and $\lambda_1, \lambda_2 \in K$;
(ii) $s(x, x)$ is greater than equal to zero and it is equal to zero iff x is zero;
(iii) $s(x, y) = s(y, x)$.

A linear space X along with $s(\cdot, \cdot)$ is known as a semi-inner product space. Furthermore, X can be assigned a topology associated with the semi-inner product. A semi-norm in X is described as

$$p(x) = \sqrt{s(x, x)}.$$

Cauchy Schwarz Inequality theorem holds for $(X, s(\cdot, \cdot))$. This means,

$$s(x, y) \leq \sqrt{s(x, x)} \sqrt{s(y, y)}.$$

Let $(X, s(\cdot, \cdot))$ be a semi-inner product space and S be a nonempty closed convex subset of X. The below relaxed variational inequality is studied in this paper in its set-valued form.

Find x of S and x^* of $F(x)$ for which

$$\text{(VI)}: \quad s(x^*, x-y) \ \leq 0, \ \forall \ y \in S,$$

where $F : S \to X$ is a set-valued map.

Remember that for every x, y of S, x^* of $F(x)$ and y^* of $F(y)$,

$$s(x^* - y^*, x - y) \geq 0, \text{ then } F \text{ is referred as monotone map on } S \text{ [14]}.$$

The concepts of upper semi-continuity and closedness for map F employed in our research are below.

Definition 2.2. [14]

(i) For any x of S and for any $\{x_n\}$ in $S \mapsto x$ and $\{y_n\}$ in $X \mapsto y$ so that $y_n \in F(x_n)$, if $y \in F(x)$; then F is referred to as *Closed*;

(ii) When for any x of S and for any $\{y_n\} \mapsto y$ in X there is a $\{x_n\}$ in $S \mapsto x$ with y_n of $F(x_n)$ such that $y \in F(x)$; then F is called as *Upper Semi-continuous* on S;

(iii) F is referred to as *Upper hemi-continuous* on S, if it is upper semi-continuous when restricted to S's line segments.

3. Links between generalised α-m monotonicity and β-well posedness of (VI)

First, we shall define a generalised α-m monotone set-valued map on S

Definition 3.1. Suppose $\alpha > 0$, $F : S \to X$ is called a generalised α-m monotone map on S provided at every x, y of S, x^* of $F(x)$ and y^* of $F(y)$

$$s(x^* - y^*, x - y) + \alpha[p(x - y)]^m \geq 0,$$

or

$$s(y^*, x - y) - \alpha[p(x - y)]^m \leq s(x^*, x - y).$$

where $m > 1$ is a constant.

Every monotone map is a generalised α-m monotone. However, the converse is false as shown below.

Example 3.2. For $S = [0.1, 0.5]$, suppose $F : S \to R$ is denoted by

$$F(x) = \{(1 - x)\}, \quad \text{for } x \in S.$$

Then for $\alpha = 1$, $m = 2$, F is generalised α-m monotone map. However, F is not monotone since

$$s(x^* - y^*, x - y) < 0, \ \ \forall \ x, y \text{ of } S, \ x^* \text{ of } F(x) \text{ and } y^* \text{ of } F(y).$$

Definition 3.3. Assume $\alpha > 0$, map F is called generalised α-m pseudomonotone. provided at every $x, y \in S$, $s(x^*, x - y) \leq 0$ for some $x^* \in F(x) \Rightarrow s(y^*, x - y) - \alpha[(p(x - y)]^m \leq 0$ for all $y^* \in F(y)$.

Every generalised α-m monotone map is a generalised α-m pseudo monotone with the same α and m. however, the opposite does not hold as demonstrated below

Example 3.4. For $S = R^+$, F is real valued function on R_+ described as

$$F(x) = [x^2, 2x^2], \quad \forall \ x \in R^+.$$

Then for $\alpha = 2$ and $m = 3$, F is generalised α-m pseudomonotone. But for $x = 1$, $y = 6/5$, $x^* = 2$ and $y^* = \frac{36}{25}$.

$$s(x^* - y^*, x - y) + \alpha[p(x - y)]^m \ \text{ is positive.}$$

Hence F is not generalised α-m monotone.

On the lines of Theorem 3.3 mentioned by Lalitha and Bhatia in [14], it can be proved that "If F is a generalised α-m pseudo monotone set-valued map and upper hemi-continuous having compact values and if there exists $y \in S$ such that $p(x-y) > 0$ and $s(x^*, x-y) > 0$.

for every $x^* \in F(x)$, then the problem (VI) is solvable."

We will now introduce β-well-posedness for the problem (VI) on S.

Definition 3.5. Let $\beta > 0$. $\{x_n\}$ is said to be β-approximating for (VI) provided

(i) x_n belonging to S for all natural number n;
(ii) there exist $x_n^* \in F(x_n)$ for all natural number n and $\{\varepsilon_n\}_{n=1}^{\infty}, \varepsilon_n > 0$ for each $n, \varepsilon_n \to 0$ as $n \to \infty$ at which

$$s(x_n^*, x_n - y) - \beta[p(x_n - y)]^m \leq \varepsilon_n, \quad \forall\, y \in S, \ \forall\, n \in N.$$

Definition 3.6. (VI) is β-well-posed provided

(i) it has only one solution x_0;
(ii) x_n approaches to x_0 as n tends to ∞ provided $\{x_n\}$ is a β-approximating sequence.

Definition 3.7. The variational inequality problem (VI) is said to be generalised β-well-posed if

(i) $X^0 \neq \varphi$;
(ii) For each β-approximating sequence $\{x_n\}$ there is a $\{x_{n_k}\} \subseteq \{x_n\}$ so that $x_{n_k} \to x_0$ for some x_0 of X^0, where X^0 is the solution set for (VI).

By generalised β-well-posedness, the solution set X^0 for (VI) is non-empty and compact.

We now use F's generalised α-m-monotonicity to get existence findings for (VI).

An important lemma in deriving existence results for (VI) is the one that follows.

Lemma 3.8. *For $\gamma > 0$, let's study the following inequality issue of the Minty type.*

(MVI) *Find $x_0 \in S$ for which*

$$s(y^*, x_0 - y) - \gamma[p(x_0 - y)]^r \leq 0, \ \text{at every } y \text{ of } S \text{ and } y^* \text{ of } F(y),$$

where X^0 and $M(X^0)$ signify the corresponding solution sets of (VI) and (MVI) respectively, then $M(X^0)$ is a subset of X^0 provided F is upper hemi-continuous and has a compact value.

Proof. Suppose $y_0 \in M(X^0)$ but $y_0 \notin X^0$. Then there exist $y^* \in S$ so that for all $y \in F(y_0)$.

$$s(y, y_0 - y^*) > 0.$$

Since $F(y_0)$ is compact therefore existence of $\varepsilon > 0$ is guaranteed so that at every $y \in F(y_0)$.

$$s(y, y_0 - y^*) < \varepsilon.$$

Let $\tau = \{y : s(y, y_0 - y^* > \varepsilon\}$. Then set τ is open and $F(y_0) \subseteq \tau$. By the convexity of S we get

$$y_\lambda = \lambda_y + (1-\lambda)y_0 \in S, \quad \text{for } \lambda \in [0,1] \text{ and } y_\lambda \to y_0 \text{ as } \lambda \to 0.$$

Also, F is upper hemi-continuous on S, so $\exists\ \delta \in (0,1)$ such that $F(y_\lambda) \subseteq \tau$, for every $\lambda \in (0, \delta)$.

This implies for $\lambda \in (0, \delta)$ and $t_\lambda \in F(y_\lambda)$,

$$s(t_\lambda, y_0 - y^*) > \varepsilon \tag{3.1}$$

Now $y_0 \in M(X^0)$ implies for every $\lambda \in (0, \delta)$ and $t_\lambda \in F(y_\lambda)$, $s(t_\lambda, y_0 - y_\lambda) - \gamma p(y_0 - y_\lambda) \leq 0$.

This leads to $s(t_\lambda, \lambda(y - y_0)) - \gamma[p(\lambda(y - y_0))]^m \leq 0$ (as $y_\lambda = \lambda y + (1 - \lambda)y_0$), which implies $\lambda s(t_\lambda, y - y_0) - \gamma\lambda^m[p(y - y_0)]^m \leq 0$, which further implies $s(t_\lambda, y - y_0) - \gamma\lambda^{m-1}[p(y - y_0)]^m \leq 0$.

As $\lambda \to 0$, this gives us

$$s(t_\lambda, y - y_0) \leq 0,$$

which contradicts (3.1). Hence $M(X^0) \subseteq X^0$. □

Let $X^0_{\beta,\varepsilon} = \{x \in S : \exists\, x^* \in F(x) \text{ so that } s(x^*, x-y) - \beta[p(x-y)]^m \leq \varepsilon,\ \forall\, y \in S\}$, where ε is non-negative.

Using Lemma 3.8 we have $X^0 = X^0_{\beta,\varepsilon}$ when $\varepsilon = 0$. Also, $X^0 \subseteq X^0_{\beta,\varepsilon}$ $\forall\, \varepsilon > 0$.

Remember that for the set A

$$\operatorname{diam} A = \sup_{a,b \in A} p(a - b),$$

where $\operatorname{diam} A$ is the diameter of A.

The β-well-posedness of (VI) is considered by the behaviour of $X^0_{\beta\varepsilon}$ as shown in the following theorem.

Theorem 3.9. *(VI) is β-well-posed iff*

$$X^0_{\beta,\varepsilon} \neq \phi \quad \forall\, \varepsilon > 0 \text{ and } \operatorname{diam} X^0_{\beta,\varepsilon} \text{ approaches to } 0 \text{ as } \varepsilon \text{ tends to } 0, \tag{3.2}$$

provided F is generalised α-m-monotone along with upper hemi-continuous having compact values on S.

Proof. Let (VI) be β-well-posed. This implies there is only one solution x_0 of X^0. Hence

$$X^0_{\beta,\varepsilon} \neq \phi, \text{ for all positive } \varepsilon \text{ as } X^0 \subseteq X^0_{\beta,\varepsilon} \text{ for all positive } \varepsilon.$$

Let if possible $\operatorname{diam} X^0_{\beta,\varepsilon} \not\to |0$ as ε tends to 0. This implies a positive number r, a natural number m, and a positive number ε_n exist for all n where $\varepsilon_n \to 0$ as n tends to ∞ and x_n, x'_n for which

$$p(x_n - x'_n) > r, \quad \forall\, n \geq m. \tag{3.3}$$

Because $x_n, x'_n \in X^0_{\beta,\varepsilon}$, therefore there exists $z_n \in F(x_n)$, $z'_n \in F(x_n)$ such that

$$s(z_n, x_n - y) - \beta[p(x_n - y)]^m \leq \varepsilon_n, \quad \forall\, y \in S$$

and

$$s(z'_n, x'_n - y) - \beta[p(x'_n - y)]^m \leq \varepsilon_n, \quad \forall\, y \in S.$$

This implies $\{x_n\}$ and $\{x'_n\}$ are β-approximating sequence for (VI). Both sequences converge to the only solution $x_0$since (VI) is β-well-posed, which defies (3.3).

Hence $\operatorname{diam} X^0_{\beta,\varepsilon}$ approaches to 0 as ε tends to 0

Conversely, let $X^0_{\beta,\varepsilon} \neq \phi$ for every $\varepsilon \to 0$ and condition (3.2) holds.

Suppose $\{x_n\}$ is a β-approximating for (VI). This implies $x^*_n \in F(x_n)$ and $\{\varepsilon_n\}$, $\varepsilon_n > 0$ exist for every n so that

$$s(x^*_n, x_n - y) - \beta\,[p(x_n - y)]^m \leq \varepsilon_n, \quad \forall\, y \in S,\ \forall\, n \in N. \tag{3.4}$$

This implies $x_n \in X^0_{\beta,\varepsilon_n}$. Since condition (3.2) holds, so we have

$$\operatorname{diam} X^0_{\beta,\varepsilon_n} \to 0, \quad \varepsilon_n \to 0.$$

This implies $\{x_n\}$ is a Cauchy sequence and S is compact also, so it converges to some x_0. Further $x_0 \in S$ as S is closed.

Now F is generalised α-m monotone on S, so for any $y^* \in F(y)$ and $y \in S$,

$$\begin{aligned} s(y^*, x_0 - y) - \alpha[p(x_0 - y)]^m &= \lim[s(y^*, x_n - y) - \alpha[p(x_n - y)]^m] \\ &\leq \lim[s(x_n^*, x_n - y)] \\ &\leq \beta \lim[s(x_n^*, x_n - y)]^m + \lim \varepsilon_n \quad \text{(by (3.4))} \\ &= \beta[p(x_0 - y)]^m. \end{aligned}$$

This implies

$$s(y^*; x_0 - y) - (\alpha + \beta)[p(x_0 - y)]^m \leq 0.$$

Here $\gamma = \alpha + \beta > 0$, so by Lemma 3.8, we have

$$s(y^*, x_0 - y) \leq 0 \text{ at every point } y \text{ of } S.$$

Thus x_0 solves (VI). □

Corollary 3.10. *If F is generalised α-m monotone, upper hemi-continuous and compact valued on S, prove that (VI) is β-well posed iff X^0 is non-empty and* $\operatorname{diam} X^0_{\beta,\varepsilon}$ *approaches to 0 as ε tends to 0.*

Theorem 3.11. *Let $\alpha > \beta > 0$. Let F be generalised α-m monotone and upper hemi-continuous with compact values on S, prove that (VI) is β-well-posed iff it has only one solution.*

Proof. Let (VI) be β-well-posed. This implies that (VI) has only one solution. On the other hand, suppose (VI) has only one solution x_0. Let if possible (VI) is not β-well-posed. This means $\exists$ a β-approximating $\{x_n\}$ for (VI) so that $x_n \not\to x_0$. As $\{x_n\}$ is a β-approximating sequence so a sequence $x_n^* \in F(x_n)$ and $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, $\varepsilon_n > 0$ exist for each n so that

$$s(x_n^*, x_n - y) - \beta[p(x_n - y)]^m \leq \varepsilon_n, \quad \forall\, y \in S,\ \forall\, n \in N. \tag{3.5}$$

Claim: $\{x_n\}$ is bounded.

Let, if possible, $\{x_n\}$ be unbounded. We can presume without losing generality that $p(x_n) \longrightarrow \infty$ as $n \to \infty$.

Let $w_n = x_0 + \lambda_n(x_n - x_0)$, where $\lambda_n = \frac{1}{p(x_n - x_0)}$.

We can proceed by assuming that $\lambda_n \in (0, 1)$ and $w_n \to w \neq x_0$.

At every y of S and y^* of $F(y)$,

$$\begin{aligned} s(y^*, w - y) &= s(y^*, w - w_n) + s(y^*, w_n - x_0) + s(y^*, x_0 - y) \\ &= s(y^*, w - w_n) + s(y^*, x_0 + \lambda_n)(x_n - x_0) - x_0) + s(y^*, x_0 - y) \\ &= s(y^*, w - w_n) + \lambda_n s(y^*, x_n - x_0) + s(y^*, x_0 - y) \\ &= s(y^*, w - w_n) + \lambda_n s(y^*, x_n - y) + (1 - \lambda_n) s(y^*, x_0 - y). \end{aligned} \tag{3.6}$$

Since x_0 is the only solution of (VI), there exists $x^* \in F(x_0)$ that

$$s(x^*, x_0 - y) \leq 0, \quad \forall\, y \in S. \tag{3.7}$$

Again as F is generalised α-m monotone on S, we have

$$s(y^*, x_0 - y) - \alpha[p(x_0 - y)]^m \leq s(x^*, x_0 - y)$$

and

$$s(y^*, x_n - y) - \alpha[p(x_n - y)]^m \leq s(x^*, x_n - y), \tag{3.8}$$

where $x^* \in F(x_0)$ and $y^* \in F(y)$.

Equations (3.6) and (3.8) give us

$$s(y^*, w - y) \leq s(y^*, w - w_n) + \lambda_n[s(x_n^*, x_n - y) + \alpha[p(x_n - y)]^m]$$

$$
\begin{aligned}
&+ (1-\lambda_n)[s(x^*, x_0 - y) + \alpha[p(x_0 - y)]^m] \\
&\le s(y^*, w - w_n) + \lambda_n[\varepsilon_n + \beta[p(x_n - y)]^m \\
&\quad + \alpha[p(x_n - y)]^m] + (1-\lambda_n)[\alpha[p(x_0 - y)]^m] \\
&\qquad\qquad \text{(using (3.5) and (3.7))} \\
&= s(y^*, w - w_n) + \lambda_n[\varepsilon_n + (\alpha+\beta)[p(x_n - y)]^m] \\
&\quad + (1-\lambda_n)[\alpha[p(x_0 - w_n + w_n - y)]^m] \\
&\le s(y^*, w - w_n) + \lambda_n[\varepsilon_n + (\alpha+\beta)[p(x_n - y)]^m] \\
&\quad + (1-\lambda_n)\alpha[p(x_0 - x_0 - \lambda_n(x_n - x_0))]^m] \\
&\quad + (1-\lambda_n)\alpha[p(w_n - y)]^m
\end{aligned}
$$

Upon interpreting the limit as $n \to \infty$ on both sides of the aforementioned inequality, we get

$$s(y^*, w - y) \le \alpha[p(w - y)]^m.$$

That is,

$$s(y^*,\ w - y) - \gamma[p(w - y)]^m \le 0,$$

where $\gamma = \alpha$.

Lemma 3.8 suggests that w solves (VI), which contradicts the solution's exclusivity. As a result, $\{x_n\}$ is not an unbounded sequence and hence there is $\{x_{n_k}\} \subseteq \{x_n\}$ with $x_{n_k} \to \bar{x}$ as $k \to \infty$.

Now consider for all $y \in S$ and for $y^* \in F(y)$.

$$
\begin{aligned}
s(y^*, \bar{x} - y) - x[p(x - y)]^m &= \lim_{k\to\infty} [s(y^*, x_{n_k} - y) - \alpha[p(x_{n_k} - y)]^m] \\
&\le \lim_{k\to\infty} s(x^*_{n_k}, x_{n_k} - y) \\
&\qquad \text{(because } F \text{ is generalised } \alpha\text{-m monotone)} \\
&\le \lim_{k\to\infty} (\varepsilon_{n_k} + \beta[p(x_{n_k} - y)]^m \quad \text{(from (3.5))} \\
&= \beta[p(\bar{x} - y)]^m.
\end{aligned}
$$

So, we have

$$s(y^*, \bar{x} - y) - (\alpha+\beta)[p(\bar{x} - y)]^m \le 0.$$

Thus by Lemma 3.8, $\bar{x}$ solves (VI). Since x_0 is the only one which solves (VI), therefore $\bar{x} = x_0$. It is true for any convergent $\{x_{n_k}\} \subseteq \{x_n\}$, since $\{x_n\} \to x_0$ so we deduce that (VI) is β-well-posed. □

Theorem 3.12. *(VI) is generalised β-well-posed iff $X^0 \neq \varphi$ whenever F is generalised α-m-monotone and upper hemi-continuous on S and S is a compact set.*

Proof. Let (VI) be generalised β-well-posed. This gives $X^0 \neq \varphi$. Suppose $X^0 \neq \varphi$. Further, assume that $\{x_n\}$ is a β-approximating sequence for the problem (VI). This means $x_n^* \in F(x_n)$ at every natural number n and $\{\varepsilon_n\}$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$ exist for every n, we get

$$s(x_n^*, x_n - y) - \beta[p(x_n - y)]^m \le \varepsilon_n, \quad \forall\ y \in S,\ \forall\ n \in N. \tag{3.9}$$

Since S is compact, therefore $\exists\ \{x_{n_k}\} \subseteq \{x_n\}$ so that $\{x_{n_k}\} \to x_0$ for some $x_0 \in S$.

Consider at every y of S and y^* of $F(y)$,

$$
\begin{aligned}
s(y^*, x_0 - y) - \alpha[p(x_0 - y)]^m &= \lim_{k\to\infty} [s(y^*, x_{n_k} - y) - \alpha[p(x_{n_k} - y)]^m \\
&\le \lim_{k\to\infty} s(x^*_{n_k}, x_{n_k} - y) \\
&\qquad \text{(because } F \text{ is generalised } \alpha\text{-m monotone)}
\end{aligned}
$$

$$\leq \lim_{k\to\infty} (\varepsilon_{n_k} + \beta[p(x_{n_k} - y)]^m) \quad \text{(from (3.9))}$$
$$= \beta\,[p(x_0 - y)]^m\,.$$

Therefore
$$s(y^*, x_0 - y) - (\alpha + \beta)[p(x_0 - y)]^m \leq 0$$
from Lemma 3.8 this implies $x_0 \in X^0$. □

Theorem 3.13. *If existence of some $\epsilon > 0$ is guaranteed so that bounded set $X^0_{\beta,\varepsilon} \neq \varphi$, then (VI) is generalised β-well-posed provided F is generalised α-m-monotone and upper hemi-continuous with compact values on S.*

Proof. Suppose the bounded set $X^0_{\beta,\varepsilon} \neq \varphi$ for some $\varepsilon > 0$. Further, assume $\{x_n\}$ ia a β-approximating for (VI). This means ∃ a natural number m for which $x_n \in X^0_{\beta,\varepsilon}$ $\forall\ n > m$. This gives $\{x_n\}$ is a bounded sequence. Therefore $\exists\ \{x_{n_k}\} \subseteq \{x_n\}$ so that $x_{n_k} \to x_0$ where $x_0 \in S$. Using the same steps as given in Theorem 3.12, it may be shown that$x_0 \in X^0$. □

4. Link between β-well-posedness of associated mathematical programming problem (MP) and (VI)

This section presents our gap function proposal for (VI) and examines the connection between (VI)'s β-well-posedness and that of a related (MP) problem.

Suppose F from S to R is a set-valued map with compact values and $\varphi \neq S \subseteq X$ is a convex closed set.

For each $x \in S$, g_β is a real valued function defined on $F(x)$ by
$$g_\beta(x^*) = \sup_{y\in S}\{s(x^*, x - y) - \beta[p(x - y)]^m\}$$
and h is a real valued function defined on S by
$$h(x) = \inf_{x^*\in F(x)} g_\beta(x^*). \tag{4.1}$$
The function $h(x)$ is valid as $F(x)$ is a compact set for each $x \in S$.

Lemma 4.1. *If h is described by (4.1), then it is the gap function for (VI).*

Proof. It is known that h has to meet the following criteria to become a gap function of (VI):

(i) $h(x)$ is non-negative at every point of S;
(ii) $h(x_0) = 0$ iff x_0 solves (VI).

At every point $x \in S$and $x^* \in F(x)$, we have
$$g_\beta(x^*) = \sup_{y\in S}\{s(x^*, x - y) - \beta[p(x - y)]^m\}$$
$$\geq s(x^*, x - y) - \beta[p(y - y)]^m = 0 \tag{4.2}$$
Thus
$$g_\beta(x^*) \geq 0, \quad \forall\ x \in S.$$
Hence
$$h(x) \geq 0, \quad \forall\ x \in S.$$
Therefore (i) holds.

Let $h(x_0)$ be equal to 0. Then $\exists\ y_0 \in F(x_0)$ so $g_\beta(y_0) = 0$. Thus $s(y_0, x_0 - y) - \beta[p(x - y)]^m \leq 0$, for all $y \in S$. Hence $x_0 \in M(X^0)$. So by Lemma 3.8, x_0 is a solution of (VI).

Conversely: Let x_0 be a solution of (VI). Then we have that for some $y_0 \in F(x_0)$, $g_\beta(y_0) = 0$. Also, from (4.2), we have at every $y \in F(x_0)$, $g_\beta(y) \geq 0$.

Now

$$\begin{aligned} h(x_0) &= \inf_{y \in F(x_0)} g_\beta(y) \\ &= g_\beta(y_0) \\ &= 0. \end{aligned}$$

Therefore, (ii) also holds. Thus h is a gap function for (VI). □

Theorem 4.2. *At every $x \in S$,*

$$h(x) \geq -r[p(x - x_0)]^m,$$

whenever $x_0 \in S$ solves (VI) and F is a generalised $\alpha - m$, pseudo monotone map on S, where $r > 0$.

Proof. For $x \in S$ then for any $x^* \in F(x)$.

$$\begin{aligned} g_\beta(x^*) &= \sup_{y \in S}\{s(x^*, x - y) - \beta[p(x - y)]^m\} \\ &\geq s(x^*, x - y) - \beta[p(x - y)]^m\,. \end{aligned} \tag{4.3}$$

Because $x_0 \in S$ is a solution of (VI), there exists $y_0 \in F(x_0)$ such that

$$s(y_0, x_0 - x) \leq 0.$$

Using the generalised α-m pseudo-monotonicity of F, we get that at each $x^* \in F(x)$.

$$\begin{aligned} & s(x^*, x_0 - x) - \alpha[p(x - x_0]^m \leq 0 \\ \Rightarrow \quad & - s(x^*, x - x_0) - \alpha[p(x - x_0]^m \leq 0 \\ \Rightarrow \quad & s(x^*, x - x_0) \geq -\alpha[p(x - x_0]^m \end{aligned} \tag{4.4}$$

Equations (4.3) and (4.4) give

$$g_\beta(x^*) \geq -(a + \beta)[\beta(x - x_0)]^m, x^* \in F(x).$$

This implies

$$h(x) \geq -\gamma[p(x - x_0)]^m, \quad \text{for all } x \in S,$$

where $\gamma = \alpha + \beta$. □

The gap function mentioned above aids in the formulation of (VI) into an analogous mathematical programming problem denoted as follows:

$$(\text{MP}): \quad \min_{x \in S} h(x).$$

Suppose τ^0 is the set of all those points of S which solve (MP).

The concept of β-well-posedness for (OP) is now introduced.

Definition 4.3. Let $\beta \geq 0$. $\{x_n\}$ is called a β-minimizing sequence for (MP) provided

(i) $x_n \in S$, $\forall\, n \in N$;
(ii) there is $x_n^* \in h(x_n)$ at every n in N, $\{\varepsilon_n\}_{n=1}^{\infty}$, $\varepsilon_n > 0\ \forall\, n, \varepsilon_n \to 0$ for which

$$s(x_n^*, x_n - y) - \beta[p(x_n, y)]^m \leq \varepsilon_n, \quad \forall\, y \in S,\ \forall\, n \in N.$$

Definition 4.4. The (MP) would be β-well-posed provided

(i) if (MP) has only solution x_0;
(ii) every β-minimizing sequence $\{x_n\}$ for (MP) approaches to x_0 as n tends to ∞.

Definition 4.5. (MP) is called generalised β-well-posed provided:

(i) $\tau^0 \neq \varphi$;

(ii) every β-minimizing sequence $\{x_n\}$ for (MP) $\exists$ $\{x_{n_k}\} \subseteq \{x_n\}$so that $x_{n_k} \to x_0$ for some x_0 of τ^0.

The association between the β-well-posedness of (MP) and (VI) is as follows.

Theorem 4.6. *(VI) is β-well-posed iff (MP) is β-well-posed.*

Proof. Let (VI) be β-well-posed. Then there exists a unique solution x_0 for (VI). Since h is a gap function, therefore $h(x_0) = 0$. Further, $h(x)$ is non-negative at every x of S. So x_0 minimizes the point for h. To prove that (OP) has a unique solution, consider $x' \in S$ be so that $h(x') = h(x_0) = 0$.

To every $y \in S$, consider the point $w = \lambda x' + (1-\lambda)y$, $\lambda \in [0, 1]$.

The point w belongs to S as S is a convex set.

$$
\begin{aligned}
&s(x^*, x' - w) - \beta[p(x' - w)]^m \\
&\quad = s(x^*, x' - [\lambda x' + (1-\lambda)y]) - \beta[p(x' - (\lambda x' + (1-\lambda)y]^m \\
&\quad = s(x^*, (1-\lambda)x' + (1-\lambda)y) - \beta[p(x' - \lambda x' - (1-\lambda)y]^m \\
&\quad = (1-\lambda)s(x^*, x' - y) - \beta[p(x' - \lambda x' - (1-\lambda)y)]^m \\
&\quad = s(x^*, x' - y) - (1-\lambda)^{m-1}\beta[p(x' - y')]^m \\
&\quad \leq 0 \qquad \text{(because } h(x') = 0 \text{ so } g_\beta(x') = 0).
\end{aligned}
$$

This implies

$$(1-\lambda)s(x^*, x' - y) - \beta(1-\lambda)^m[p(x' - y')]^m \leq 0, \quad \forall\, \lambda \in [0, 1].$$

So when $\lambda \to 1$, this implies

$$\langle x^*, x' - y\rangle \leq 0, \quad \forall\, y \in S.$$

Thus x' solves (VI). Therefore $x' = x_0$ as x_0 solves (VI) and is unique. The first part is proved because the family of β-minimizing sequences for (MP) coincides with the family of β-approximating sequences for (VI).

Conversely: Let (MP) be β-well-posed then $\exists$ only one solution x_0 of (MP). Therefore x_0 minimizes h and it is unique also. As h is the gap function it means $h(x) \geq 0$ for all $y \in S$. This gives $h(x_0) = 0$. Thus x_0 is a solution of (VI) also. To establish x_0 is the only solution of (VI), let x' be another solution to (VI). Since h is a gap function so we have $h(x') = 0$. Thus x' should be the solution of (MP). But (MP) has the only solution x_0, so x' must be equal to x_0. Thus, the result can be established as in the first part. □

Theorem 4.7. *(VI) is generalised β-well-posed if and only if (MP) is generalised β-well-posed in the sense.*

5. Conclusion

In this paper, we established the relationship of α-m monotone and generalised α-m pseudo monotone maps with set variational inequality problem (VI). Then we constructed a gap function for (VI) problem and used this gap function to formulate (VI) problem into a corresponding Mathematical Programming Problem (MP). Finally, we established the relations between the β-well-pseudoness of both problems. In future, one can explore new well-posedness concepts for spaces with a semi-inner product.

Statements and Declarations

(i) On behalf of all authors, the corresponding author states that there is no conflict of interest.
(ii) No funding was received for conducting this research work.
(iii) The authors have no financial or proprietary interests in any material discussed in this article.

References

1. T. O. Alakoya, V. A. Uzor, O. T. Mewomo, J. C. Yao, *On a system of monotone variational inclusion problems with fixed-point constraint*, J. Inequal. Appl. **2022** (2022), Article 47.
2. T. O. Alakoya, V. A. Uzor, O. T. Mewomo, *A new projection and contraction method for solving split monotone variational inclusion, pseudo-monotone variational inequality and common fixed-point problems*, Comput. Appl. Math. **42** (2023), Article 33.
3. D. Aussel, D. T. Luc, *Existence conditions in general quasimonotone variational inequalities*, Bull. Aust. Math. Soc. **71** (2005), 285–303.
4. L. C. Ceng, J. C. Yao, *Well-posedness of generalized mixed variational inequalities, inclusion problems and fixed-point problems*, Nonlinear Anal. **69** (2008), 4585–4603.
5. L. C. Ceng, N. Hadjisavvas, S. Schaible, J. C. Yao, *Well-posedness for mixed quasivariational-like inequalities*, J. Optim. Theory Appl. **139** (2008), 109–125.
6. S. S. Dragomir, *Semi-inner products and applications*, Nova Science Publishers, New York, 2004.
7. J. R. Giles, *Classes of semi-inner product spaces*, Trans. Amer. Math. Soc. **116** (1967), 436–446.
8. R. Hu, Y. P. Fang, *Levitin–Polyak well-posedness by perturbations of inverse variational inequalities*, Optim. Lett. **7** (2013), 343–359.
9. X. X. Huang, *Extended and strongly extended well-posedness of set-valued optimization problems*, Math. Methods Oper. Res. **53** (2001), 101–116.
10. A. Jayswal, S. Jha, *Well-posedness for generalized mixed vector variational-like inequality problems in Banach spaces*, Math. Commun. **22** (2017), 287–302.
11. S. Jha, P. Das, S. Bandyopadhyay, *Well-posedness for multi-time variational inequality problems via generalized monotonicity and for variational problems with multi-time variational inequality constraints*, J. Comput. Appl. Math. **407** (2022), Article 114033.
12. B. Jiang, J. Zhang, X. X. Huang, *Levitin–Polyak well-posedness of generalized quasi-variational inequalities with functional constraints*, Nonlinear Anal. TMA **70** (2009), 1492–1503.
13. C. S. Lalitha, M. Mehta, *Vector variational inequalities with cone pseudo-monotone bifunctions*, Optimization **54** (2005), 327–338.
14. C. S. Lalitha, G. Bhatia, *Well-posedness for variational inequality problems with generalized set-valued maps*, Numer. Funct. Anal. Optim. **30** (2009), 548–565.
15. C. S. Lalitha, G. Bhatia, *Levitin–Polyak well-posedness for parametric quasi-variational inequality problems of the Minty type*, Positivity **16** (2012), 527–541.
16. M. B. Lignola, *Well-posedness and L-well-posedness for quasi-variational inequalities*, J. Optim. Theory Appl. **128** (2006), 119–138.
17. R. Lucchetti, F. Patrone, *A characterization of Tikhonov well-posedness for minimum problems, with applications to variational inequalities*, Numer. Funct. Anal. Optim. **3** (1981), 461–476.
18. G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. **100** (1961), 29–43.
19. I. D. Prete, M. B. Lignola, J. Morgan, *New concepts of well-posedness for optimization problems with variational inequality constraints*, J. Inequal. Pure Appl. Math. **3** (2002), Article 5.
20. S. Treanţă, *On well-posed isoperimetric-type constrained variational control problems*, J. Differ. Equ. **298** (2021), 480–499.
21. S. Treanţă, S. Singh, *Weak sharp solutions associated with a multidimensional variational-type inequality*, Positivity **25** (2021), 329–351.
22. A. N. Tikhonov, *On the stability of functional optimization problems*, USSR Comput. Math. Math. Phys. **6** (1966), 631–634.
23. G. Virmani, M. Srivastava, *On Levitin–Polyak well-posedness of perturbed variational–hemivariational inequalities*, Optimization **64** (2015), 1153–1172.