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FIXED POINT ASSOCIATED WITH A NEW CLASS OF CONDENSING OPERATORS AND SOLVABILITY OF VOLTERRA INTEGRAL EQUATION HAVING DELAY

V. E. NIKAM[1] AND J. D. MASHALE*[2]

[1] Department of Mathematics, Arts Commerce and Science College, Vikramgad 401605, India, Email: nikamvishal832@gmail.com

[2] Department of Mathematics, PAH, Solapur University, Solapur, 413255, India Email: jdthenge@sus.ac.in

ABSTRACT. This work presents a novel class of condensing operators to explore the possibility of solutions for the Volterra integral equation with a singular kernel and proportional delay. These equations are significant in many domains, including engineering and physics, yet conventional solution techniques face substantial difficulties due to single kernels and delays. To solve this, we provide a more flexible method of handling such equations by creating a class of condensing operators based on pairs of functions that satisfy specific local requirements. We define these operators and also show some fixed point theorems that expand the application of Darbo's fixed point theorem to a broader class of situations. At the end we provide examples to illustrate our theoretical results and show that the suggested approach works well.

KEYWORDS: Condensing operators, Measure of noncompactness, Volterra integral equations.
AMS Subject Classification: 47H09; 45G10.

1. Introduction

Integral and differential equations play significant roles in mathematical modeling, physics, and engineering [3, 5, 4]. In Particular, the integral equation with singular kernels having proportional delay appeared in the study of the motion of particles in a liquid, population dynamics, and many other branches of science and engineering [3, 8, 9, 12, 7]. On the other hand, the approach of Darbo [6] fixed point theorem is a very effective tool to deal with such problems. Despite the applicability, the Darbo fixed point theorem generated much interest from researchers

* *Corresponding author.*
Email address :jdthenge@sus.ac.in.
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in this domain. The Darbo fixed point theorem has been used, generalized, and extended in various directions. Here we list some relevant literature on this topic [3, 5, 9, 1, 14, 10, 11].

Following this direction of research work, we define a new class of condensing operators and establish corresponding fixed point results. The obtained results are then applied to prove existence of the solution of nonlinear integral equation with singular kernels having proportional delay. The integral equation discussed in our work is a generalized form of the equation discussed in [8].

1.1. **Preliminaries.** Through out this paper, we will denote the family of nonempty, closed and bounded convex sets by $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$, the term measure of non-compactness by $\mathcal{M}.\mathcal{N}.\mathcal{C}$, B_r means the open ball with radius r, $conv(.)$ as convex hull, $\mathfrak{R}_\Omega$ means the class of all bounded subsets of the space Ω and the symbols $\mathbb{R}$, $\mathbb{R}_+$ & $\mathbb{N}$ are used to denote the set of all real numbers, the set of all set of positive real numbers and the set of all set of positive integers respectively.

Definition 1.1. [3] A mapping $\mathcal{M} : \mathfrak{R}_\Omega \longrightarrow \mathbb{R}_+$ is termed as $\mathcal{M}.\mathcal{N}.\mathcal{C}$, if it satisfy the following condition:

i) $\mathcal{M}(\Lambda) = 0 \;\Leftrightarrow\; \bar{\Lambda}$ is compact;
ii) The family $ker(M) = \{\Lambda \in \mathfrak{R}_\Omega : \; M(\Lambda) = 0\}$, is non-empty and $ker\ M \subseteq \mathfrak{S}_\Omega$;
iii) $\Lambda \subset \Lambda_1 \implies M(\Lambda) \leq \mathcal{M}(\Lambda_1)$;
iv) $\mathcal{M}(\Lambda) = \mathcal{M}(\overline{\Lambda})$;
v) $\mathcal{M}(\Lambda) = \mathcal{M}(conv\Lambda)$;
vi) $\mathcal{M}\big(\lambda\ \Lambda + (1-\alpha)\Lambda_1\big) \leq \lambda\ \mathcal{M}(\Lambda) + (1-\lambda)\ \mathcal{M}(\Lambda_1),\ \forall\ \lambda \in [0,1]$;
vii) If the non-increasing sequence of closed subsets $\langle \Lambda_n \rangle \subset \mathfrak{R}_\Omega$ for $n \in \mathbb{N}$ of Ω with $\lim\limits_{n\longrightarrow\infty} \mathcal{M}(\Lambda_n) = 0$, then $\Lambda_\infty = \bigcap\limits_{n=1}^{\infty} \Lambda_n$ is non-empty.

The family of sets $ker\mathcal{M} = \{\Lambda \in \mathfrak{R}_\Omega : \mathcal{M}(\Lambda) = 0\}$, mentioned in (ii) is termed as the kernel of $\mathcal{M}$. Indeed the condition (vi) validates $\mathcal{M}(\Lambda_\infty) \leq \mathcal{M}(\Lambda_n)$, for any n thus $\mathcal{M}(\Lambda_\infty) \longrightarrow 0$. This conformed that $\Lambda_\infty \in ker\mathcal{M}$.

The following results are some fundamental theorems in the direction of Dardo fixed point theorem.

Theorem 1.1. [13] *Let Ξ be any arbitrary set from the family of $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of a Banach space Ω, then for any compact and continuous mapping on Ξ admit a fixed point in Ξ.*

Theorem 1.2. [6] *Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω and a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$ satisfying*

$$\mathcal{M}(\mathcal{Q}(\Lambda)) \leq \lambda\mathcal{M}(\Lambda),$$

for $\phi \neq \Lambda \subset \Xi$, where $0 \leq \lambda < 1$ and $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$. Then the mapping $\mathcal{Q}$ admit a fixed point in Ξ.

Theorem 1.3. [1] *Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω and a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$ satisfying*

$$\mathcal{M}(\mathcal{Q}(\Lambda)) \leq \phi(\mathcal{M}(\Lambda)),$$

for $\phi \neq \Lambda \subset \Xi$, where $\phi : [0,\infty) \longrightarrow [0,\infty)$ is non decreasing function such that $\lim\limits_{n\longrightarrow\infty} \phi^n(t) = 0$, and $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$. Then $\mathcal{Q}$ admit a fixed point in Ξ.

Theorem 1.4. [2] *Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω and a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$ satisfying*

$$\mathcal{M}(\mathcal{Q}\Lambda) \leqslant \alpha(\mathcal{M}(\Lambda))\mathcal{M}(\Lambda), \tag{1.1}$$

for $\phi \neq \Lambda \subset \Xi$, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and $\alpha : (0,\infty) \longrightarrow [0,1)$ such that $\alpha(t_n) \longrightarrow 1 \Rightarrow t_n \longrightarrow 0$, for any $t > 0$. Then the mapping $\mathcal{Q}$ admit a fixed point in Ξ.

Theorem 1.5. [14] *Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω and a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$ satisfying*

$$\beta(u, \mathcal{Q}u)\mathcal{M}(\mathcal{Q}\Lambda) \leq \phi(\mathcal{M}(\Lambda)), \tag{1.2}$$

for $\phi \neq \Lambda \subset \Xi$, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and $\beta : \Omega \times \Omega \longrightarrow [0,+\infty)$ & $\phi : [0,\infty) \longrightarrow [0,\infty)$, having following conditions,

(i) *$\beta(u,v) \geq 1 \Rightarrow \beta(\mathcal{Q}u, \mathcal{Q}v) \geq 1$, for all $u, v \in \Omega$;*
(ii) *ϕ is monotonic increasing such that $\phi(t) < t$, for all $t > 0$;*
(iii) *There exist closed and convex $\Lambda_0 \subset \Xi$, and $u_0 \in \Lambda_0$, such that*

$$\mathcal{Q}\Lambda_0 \subset \Lambda_0,\ \beta(u_0, \mathcal{Q}u_0) \geq 1.$$

Then the mapping $\mathcal{Q}$ admit a fixed point in Ξ.

2. Main Results

Theorem 2.1. *Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω. If there exists a continuous mapping $\mathcal{Q} : \Xi \longrightarrow \Xi$ satisfying*

$$\varpi(\mathcal{M}(\mathcal{Q}\Lambda)) \leq \vartheta(\mathcal{M}(\Lambda)), \tag{2.1}$$

for nonempty subset Λ of Ξ, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and the functions ϖ & ϑ : $[0,\infty) \longrightarrow [0,\infty)$, assumes the following properties,

(i) *ϖ is monotonic increasing and ϑ is lower semi-continuous;*
(ii) *$\vartheta(t) < \varpi(t)$, $\forall t \in (0.\infty)$.*

Then $\mathcal{Q}$ admits a fixed point in Ξ.

Proof. We begins the proof by the construction of the sequence of sets $\langle\Lambda_n\rangle$ of Λ by the following rule:

$$\mathcal{Q}\Lambda_n \subset \Lambda_n \subset \Lambda_{n-1},\ \forall\ n \in \mathbb{N}.$$

Let $\Lambda_0 = \Lambda$, we construct a sequence $\langle\Lambda_n\rangle$ by the rule $\Lambda_{n+1} = conv\mathcal{Q}(\Lambda_n)$ for $n \in \{0\} \cup \mathbb{N}$. For $n = 0$, we can easily check that $\mathcal{Q}\Lambda_0 \subset \mathcal{Q}\Lambda \subset \Lambda = \Lambda_0$. Now assume that the rule holds for $k = 1, 2, 3, \cdots n$. Now by the construction of $\langle\Lambda_n\rangle$ we deduce that,

$$\mathcal{Q}\Lambda_n \subset \Lambda_n \text{ implies } \Lambda_{n+1} = conv(\mathcal{Q}\Lambda_n) \subset \Lambda_n,$$

therefore $\mathcal{Q}\Lambda_{n+1} \subset \mathcal{Q}\Lambda_n \subset \Lambda_{n+1}$. If $\mathcal{M}(\Lambda_K) = 0$ for some $K \in \mathbb{N}$, then Λ_K is pre-compact sets. Since $\mathcal{Q}(\Lambda_K) \subseteq conv(\mathcal{Q}\Lambda_K) = \Lambda_{K+1} \subseteq \Lambda_K$, i.e. $\mathcal{Q}$ has a fixed point in $\Lambda_K \subset \Lambda$.

Assume that $\mathcal{M}(\Lambda_n) > 0, \forall n \geq 1$. Now , we shall prove that $\mathcal{M}(\Lambda_n) \longrightarrow 0$ as $n \longrightarrow +\infty$. From v) of definition 1.1 and equation 2.1 we have,

$$\begin{aligned}\varpi(\mathcal{M}(\Lambda_{n+1})) &= \varpi(\mathcal{M}(conv\mathcal{Q}\Lambda_n))\\ &= \varpi(\mathcal{M}(\mathcal{Q}\Lambda_n))\\ &\leqslant \vartheta(\mathcal{M}(\Lambda_n)),\end{aligned}$$

i.e.,

$$\varpi\left(\mathcal{M}\left(\Lambda_{n+1}\right)\right) \leq \vartheta\left(\mathcal{M}\left(\Lambda_{n}\right)\right) \quad \text{holds for all n.} \tag{2.2}$$

Note that ϖ and ϑ satisfies assumption (a). Then it follows from (2.2) that

$$\varpi\left(\mathcal{M}\left(\Lambda_{n+1}\right)\right) \leq \vartheta\left(\mathcal{M}\left(\Lambda_{n}\right)\right) < \varpi\left(\mathcal{M}\left(\Lambda_{n}\right)\right) \text{ holds for all n.} \tag{2.3}$$

Since, the function ϖ is non-decreasing and $\langle\varpi\left(\mathcal{M}\left(\Lambda_{n}\right)\right)\rangle$ is non-increasing sequence hence $\langle\mathcal{M}\left(\Lambda_{n}\right)\rangle$ is non-increasing sequence of positive real numbers.
Equation (2.3) deduce that sequences $\langle\vartheta\left(\mathcal{M}\left(\Lambda_{n}\right)\right)\rangle$ and $\langle\varpi\left(\mathcal{M}\left(\Lambda_{n}\right)\right)\rangle$, are non-negative and dominating to each other, so both are bounded below and these sequences have sub-sequences (may or many not be identical) converges to $r \geq 0$ with the following relation

$$\lim_{k\longrightarrow\infty}\langle\varpi\left(\mathcal{M}\left(\Lambda_{n_k}\right)\right)\rangle \leq \lim_{k\longrightarrow\infty}\left\langle\vartheta\left(\mathcal{M}\left(\Lambda_{n_{k'}}\right)\right)\right\rangle.$$

On the contrary assume that $r > 0$. Since ϖ is non decreasing function, so it has only jump types of discontinuity and by assumption (b), the function ϑ is semi-continuous, which leads to following expression;

$$\begin{aligned}
\varpi(r) &\leq \lim_{t\longrightarrow r^+}\sup\varpi(t)\\
&\leq \lim_{k\longrightarrow\infty}\langle\varpi\left(\mathcal{M}\left(\Lambda_{n_k}\right)\right)\rangle\\
&\leq \lim_{k\longrightarrow\infty}\left\langle\vartheta\left(\mathcal{M}\left(\Lambda_{n_{k'}}\right)\right)\right\rangle\\
&\leq \lim_{t\longrightarrow r^+}\sup\vartheta(t)\\
&\leq \vartheta(r).
\end{aligned} \tag{2.4}$$

Which is a contradiction to assumption (b). Therefore, we conformed that $r = 0$, and the sequence $\mathcal{M}\left(\Lambda_{n_k}\right) \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore from the assumption (vii) of definition 1.1, the set $\Lambda_\infty = \bigcap_{n=1}^{\infty}\Lambda_n$, is non-empty, convex, closed & relatively compact also invariant under the mapping $\mathcal{Q}$. Thus in the view of Schauder fixed point theorem [13] on Λ_∞, the mapping $\mathcal{Q}$ has at-least one fixed point theorem in Λ_∞. □

Example 2.1. Let us define the functions $\varpi, \vartheta : [0,\infty) \longrightarrow [0,\infty)$ as follows:

$$\varpi(t) = t, \quad \vartheta(t) = \frac{t}{1+t}.$$

It is easy to verify that:

- ϖ is continuous and monotonic increasing on $[0,\infty)$,
- ϑ is continuous (hence lower semi-continuous) on $[0,\infty)$,
- $\vartheta(t) < \varpi(t)$ for all $t > 0$.

Thus, the pair (ϖ, ϑ) satisfies all assumptions of Theorem 2.1.

Theorem 2.2. *Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω. If there exists a continuous mapping $\mathcal{Q} : \Xi \longrightarrow \Xi$ satisfying the condition*

$$\varpi(\mathcal{M}(\mathcal{Q}\Lambda)) \leq \vartheta(\mathcal{M}(\Lambda)), \tag{2.5}$$

for $\phi \neq \Lambda \subset \Xi$, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and the functions ϖ & $\vartheta : [0,\infty) \longrightarrow [0,\infty)$ posses the following properties,

(i) *ϖ is non-decreasing and $\lim_{t\longrightarrow\varepsilon_+}\sup\vartheta(t) < \varpi(\varepsilon_+)$, for any $\epsilon > 0$.*

(ii) $\lim_{t\longrightarrow\epsilon_+}\sup\vartheta(t) < \lim_{t\longrightarrow\epsilon_+}\sup\varpi(t)$ *for any $\epsilon > 0$.*

Then $\mathcal{Q}$ admits at least one fixed point in Ξ.

Proof. The proof follows similar lines of the proof of Theorem 2.1 up to the case, when $\mathcal{M}(\Lambda_K) = 0$ for some $K \in \mathbb{N}$.

So, consider the case when $\mathcal{M}(\Lambda_n) > 0, \forall\, n \geq 1$ Note that $\langle \mathcal{M}(\Lambda_n)\rangle$ is a sequence of non-negative real numbers and so bounded below. Therefore, it has a convergent sub-sequence $\langle \mathcal{M}(\Lambda_{n_k})\rangle$, which converges to non-negative real number ϵ as $k \longrightarrow \infty$.

Claim that $\epsilon = 0$, suppose contrary $\epsilon > 0$. Then by v) of definition 1.1 and equation 2.5 for each natural number n we have,

$$\begin{aligned} \varpi\left(\mathcal{M}\left(\Lambda_{n+1}\right)\right) &= \varpi\left(\mathcal{M}\left(conv\mathcal{Q}\Lambda_n\right)\right) \\ &= \varpi\left(\mathcal{M}\left(\mathcal{Q}\Lambda_n\right)\right) \\ &\leqslant \vartheta\left(\mathcal{M}\left(\Lambda_n\right)\right). \end{aligned} \tag{2.6}$$

Utilizing the convergence of $\{\mathcal{M}(\Lambda_{n_k}\}$ to 2.6, we obtain following expressions,

$$\varpi\left(\epsilon_+\right) = \lim_{n\longrightarrow\infty} \varpi\left(\mathcal{M}\left(\Lambda_{n+1}\right)\right) \leqslant \lim_{n\longrightarrow\infty}\sup \vartheta\left(\mathcal{M}\left(\Lambda_n\right)\right) \leqslant \lim_{t\longrightarrow\epsilon_+}\sup \vartheta\left(t\right), \tag{2.7}$$

and

$$\liminf_{t\longrightarrow\varepsilon} \varpi\left(t\right) \leq \liminf_{n\longrightarrow\infty} \varpi\left(\mathcal{M}\left(\Lambda_n\right)\right) \leq \limsup_{n\longrightarrow\infty} \vartheta\left(\mathcal{M}\left(\Lambda_n\right)\right) \leq \limsup_{t\longrightarrow\epsilon_+} \vartheta\left(t\right), \tag{2.8}$$

equation 2.7 is contradiction to assumption (a) and equation 2.8 is contradiction to assumption (b), this contradiction arises due to the wrong assumption $\epsilon \neq 0$. Hence the sequence converges to zero i.e., ϵ must be zero.

Now form assumption (vi) of the definition 1.1 , the countable intersection $\Lambda_\infty = \bigcap_{n=1}^{\infty} \Lambda_n$, is a non-empty set which is closed, convex invariant under $\mathcal{Q}$ and relatively compact. In the view of Theorem 1.1 to the set $\Lambda_\infty = \bigcap_{n=1}^{\infty} \Lambda_n$, we get required result. □

Theorem 2.3. *Let Ξ be the member of a family of $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of Banach space Ω and a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$, satisfies*

$$\varpi(\mathcal{M}(\mathcal{Q}\Lambda)) \leq \vartheta(\mathcal{M}(\Lambda)), \tag{2.9}$$

for nonempty subset Λ of Ξ, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and ϖ & $\vartheta : [0,\infty) \longrightarrow [0,\infty)$, are such that $\vartheta(t) < \varpi(t)$, for any $t > 0$ with $\varpi(0) = \vartheta(0) = 0$, satisfying at least one of the following conditions:

(i) *ϖ & ϑ are continuous and if $\langle \varpi(t_n)\rangle$, be a non-increasing sequence then $\langle t_n\rangle$, is bounded;*
(ii) *ϖ is increasing, continuous and ϖ^{-1}, ϑ are semi-continuous functions;*
(iii) *ϖ is increasing, continuous and ϑ is continuous at 0 with $\liminf_{t\longrightarrow\infty} \left(t - \varpi^{-1}\left(\vartheta\left(t\right)\right)\right) > 0$, and $\lim_{s\longrightarrow t}\sup \vartheta\left(s\right) < \vartheta\left(t\right)$, for each $t > 0$.*

Then $\mathcal{Q}$ admits at least one fixed point in Ξ.

Proof. We begins the proof by the construction of the sequence of sets $\langle \Lambda_n\rangle$ of Λ having following property;

$$\mathcal{Q}\Lambda_n \subset \Lambda_n \subset \Lambda_{n-1}, \text{ for all } n \in \mathbb{N}.$$

Let $\Lambda_0 = \Lambda$, we define the sequence by $\Lambda_{n+1} = conv\mathcal{Q}(\Lambda_n)$ for $n \in \{0\} \cup \mathbb{N}$. If $n = 0$, then we can effortlessly validate $\mathcal{Q}\Lambda_0 \subset \mathcal{Q}\Lambda \subset \Lambda = \Lambda_0$. Next, let the rule holds for $k = 1, 2, 3, \cdots n$. Now , by the construction of $\langle \Lambda_n \rangle$ we deduce that,

$$\mathcal{Q}\Lambda_n \subset \Lambda_n \text{ implies } \Lambda_{n+1} = conv\left(\mathcal{Q}A_n\right) \subset A_n,$$

i.e., $\mathcal{Q}\Lambda_{n+1} \subset \mathcal{Q}\Lambda_n \subset \Lambda_n$. Thus in the view of mathematical induction $\mathcal{Q}\Lambda_{n+1} \subset \Lambda_n$, hold for all $n \in \mathbb{N}$. If $\mathcal{M}(\Lambda_K) = 0$ for some $K \in \mathbb{N}$, then Λ_K is pre-compact set. Since $\mathcal{Q}(\Lambda_K) \subseteq conv\left(\mathcal{Q}\Lambda_K\right) = \Lambda_{K+1} \subseteq \Lambda_K$, i.e., $\mathcal{Q}$ is a self-mapping on Λ_K. Then Theorem 2.1 concludes that $\mathcal{Q}$ has a fixed point in $\Lambda_K \subset \Lambda$.
On the other hand, we assume that $\mathcal{M}(\Lambda_n) > 0, \forall n \geq 1$ and prove that $\mathcal{M}(\Lambda_n) \longrightarrow 0$ as $n \longrightarrow +\infty$. Now by using assumption $v)$ of definition 1.1 and equation 2.9 we have,

$$\begin{aligned}\varpi\left(\mathcal{M}\left(\Lambda_{n+1}\right)\right) &= \varpi\left(\mathcal{M}\left(conv\mathcal{Q}\Lambda_n\right)\right)\\ &\leqslant \varpi\left(\mathcal{M}\left(\mathcal{Q}\Lambda_n\right)\right)\\ &\leqslant \vartheta\left(\mathcal{M}\left(\Lambda_n\right)\right),\end{aligned}$$

i.e.,

$$\varpi\left(\mathcal{M}\left(\Lambda_{n+1}\right)\right) \leq \varpi\left(\mathcal{M}\left(\Lambda_n\right)\right), \quad \text{holds for all n.} \tag{2.10}$$

From equation (2.10) we conformed that $\langle \varpi\left(\mathcal{M}\left(\Lambda_n\right)\right)\rangle$ is non-increasing sequence of non negative real numbers and consequently there exist a real number $r \geq 0$ with

$$\lim_{n \longrightarrow \infty} \varpi\left(\mathcal{M}\left(\Lambda_n\right)\right) = r. \tag{2.11}$$

Claim that

$$\lim_{n \longrightarrow \infty} \mathcal{M}\left(\Lambda_n\right) = 0. \tag{2.12}$$

Assume that ϖ & ϑ satisfies assumption (1). From equation (2.10) the sequence $\langle \varpi\left(\mathcal{M}\left(\Lambda_n\right)\right)\rangle$, is non-increasing sequence therefore by the virtue of assumption (a) the sequence $\langle \mathcal{M}\left(\Lambda_n\right)\rangle$, is bounded. Hence for some sub-sequence $\langle \mathcal{M}\left(\Lambda_{n_k}\right)\rangle$ of $\langle \mathcal{M}\left(\Lambda_n\right)\rangle$, there exist a real number s such that

$$\lim_{k \longrightarrow \infty} \mathcal{M}\left(\Lambda_{n_k}\right) = s. \tag{2.13}$$

Assume the contradiction that $s > 0$. Using the continuity of ϖ & ϑ and equations (2.10), (2.11) and (2.13) we will deduce the following expression

$$\varpi(s) = \lim_{k \longrightarrow \infty} \varpi\left(\mathcal{M}\left(\Lambda_{n_k}\right)\right) \leq \lim_{k \longrightarrow \infty} \vartheta\left(\mathcal{M}\left(\Lambda_{n_{k-1}}\right)\right) = \vartheta(s).$$

Hence we get $s = 0$, which is contradiction, thus equation (2.12) holds.
Now assume that the functions ϖ & ϑ satisfies assumptions (b) and (c). In both the assumptions ϖ is increasing and continuous function hence equation (2.9) deduce that $\langle \mathcal{M}\left(\Lambda_n\right)\rangle$, is a non-increasing sequence of positive real numbers therefore this sequence has at least one convergent sub-sequence say $\langle \mathcal{M}\left(\Lambda_{n_k}\right)\rangle$ i.e., $\lim_{k \longrightarrow \infty} \mathcal{M}\left(\Lambda_{n_k}\right) = s$, for some $s \geq 0$.
Now, using the continuity of functions we obtain $\lim_{k \longrightarrow \infty} \varpi\left(\mathcal{M}\left(\Lambda_{n_k}\right)\right) = \varpi(s)$, and subsequently we get

$$s \leqslant \varpi^{-1}\left(\vartheta(s)\right).$$

This is possible only when $s = 0$. Hence equation (2.12) holds in any case. Now the assumption (vi) of the definition 1.1 deduce that the countable intersection $\Lambda_\infty = \bigcap_{n=1}^{\infty} \Lambda_n$, is a non-empty, closed, convex and invariant under $\mathcal{Q}$ and relatively compact set. In the view of Theorem 1.1 to the set $\Lambda_\infty = \bigcap_{n=1}^{\infty} \Lambda_n$, we get required result. ☐

2.1. **Consequences.**

Remark 2.2. Theorem 2.1 and 2.2 reduces to Theorem A (page 90) given in [6], if we choose $\varpi(t) = t$ and $\vartheta(t) = \lambda t$, where $0 \leq \lambda < 1$, in Theorem 2.1 and 2.2 respectively.

Remark 2.3. Theorem 2.1 and 2.2 reduces to theorem 2 of [1], if we choose $\varpi(t) = t \ \forall \ t$, in Theorem 2.1 and 2.2 respectively.

Corollary 2.1. Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω. If there exists a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$, such that

$$\varpi(\mathcal{M}(\mathcal{Q}C)) \leq \varphi(\varpi(\mathcal{M}(C))), \tag{2.14}$$

for $\phi \neq \Lambda \subset \Xi$, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and the functions φ & $\varpi : [0, \infty) \longrightarrow [0, \infty)$, possess the following properties

(i) ϖ is monotonic increasing;
(ii) φ is continuous from right side and $\varphi(t) < t \ \forall t \in (0, \infty)$ such that $\lim\limits_{t \longrightarrow \infty} \inf (t - \varphi(t)) > 0$.

Then mapping $\mathcal{Q}$ admits at least one fixed point in Ξ.

Proof. Define $\vartheta(t) = \varphi(\varpi(t))$, where $\varpi : [0, \infty) \longrightarrow [0, \infty)$ is upper semi continuous function such that $\varpi(t) < t \ \forall \ t > 0$, in Theorem 2.1. Then we assures the mapping $\mathcal{Q}$ admits a fixed point in Ξ. ☐

Corollary 2.2. [14] Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω. If there exists a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$, such that

$$\beta(u, \mathcal{Q}u) \mathcal{M}(\mathcal{Q}\Lambda) \leq \vartheta(\mathcal{M}(\Lambda)), \tag{2.15}$$

for $\phi \neq \Lambda \subset \Xi$, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and the functions $\beta : \Omega \times \Omega \longrightarrow [0, +\infty)$ & $\vartheta : [0, \infty) \longrightarrow [0, \infty)$, satisfies following conditions,

(i) $\beta(u, v) \geq 1 \Rightarrow \beta(\mathcal{Q}u, \mathcal{Q}v) \geq 1$ for all $u, v \in \Omega$.
(ii) ϑ is monotonic increasing with $\vartheta(t) < t \ \forall \ t > 0$.
(iii) There exist closed and convex $\Lambda_0 \subset \Xi$ and $u_0 \in \Lambda_0$, such that

$$\mathcal{Q}\Lambda_0 \subset \Lambda_0, \ \beta(u_0, \mathcal{Q}u_0) \geq 1.$$

Then mapping $\mathcal{Q}$ admits at least one fixed point in Ξ.

Proof. Let us define the function $\varpi(t) = \beta(u, v)t$, then using the properties (1) of the function $\beta : \Omega \times \Omega \longrightarrow [0, +\infty)$, we say that ϖ is non-decreasing for $t \in [0, \infty)$. Hence by Theorem 2.1 we assures that the mapping $\mathcal{Q}$ admits a fixed point in Ξ. ☐

Corollary 2.3. Let Ξ be any arbitrary set from the $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ family of Banach space Ω. If there exists a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$, such that

$$\varpi(\mathcal{M}(\mathcal{Q}\Lambda)) \leqslant \alpha(\mathcal{M}(\Lambda)) \varpi(\mathcal{M}(\Lambda)), \tag{2.16}$$

for $\phi \neq \Lambda \subset \Xi$, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and the $\mathcal{M}(\Lambda) > 0$, and the functions $\varpi : (0, \infty) \longrightarrow (0, \infty)$, and $\alpha : (0, \infty) \longrightarrow (0, 1)$, are such that;

(i) ϖ is monotonic increasing;
(ii) $\alpha(t_n) \longrightarrow 1 \Rightarrow t_n \longrightarrow 0$.

Then mapping $\mathcal{Q}$ admits at least one fixed point in Ξ.

Proof. Define a function ϑ as α as $\vartheta(t) = \alpha(t)\varpi(t)$, in Theorem 2.1 then the given conditions produces following equivalences;

- ϖ is monotonic increasing implies ϖ is upper semi continuous;
- Since $\alpha(t) < 1\ \forall\ t$ hence $\vartheta(t) = \alpha(t)\varpi(t) < \varpi(t)$.

Hence by Theorem 2.1 the mapping $\mathcal{Q}$ admits a fixed point in Ξ. □

Remark 2.4. In the corollary 2.3 if we take $\varpi(t) = t$, then it reduces to a fixed point theorem in [2].

Corollary 2.4. [1] Let Ξ be any arbitrary set of the family $\mathcal{N}.\mathcal{B}.\mathcal{C}.\mathcal{C}$ of Banach space Ω and a continuous map $\mathcal{Q} : \Xi \longrightarrow \Xi$, satisfies

$$\varpi\left(\mathcal{M}\left(\mathcal{Q}\Lambda\right)\right) \leqslant \varpi\left(\mathcal{M}\left(\Lambda\right)\right) - \vartheta\left(\mathcal{M}\left(\Lambda\right)\right), \tag{2.17}$$

for $\phi \neq \Lambda \subset \Xi$, where $\mathcal{M}$ is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ and ϑ & $\varpi : [0, \infty) \longrightarrow [0, \infty)$, satisfying the following conditions;

(i) ϖ is continuous function;
(ii) ϑ is a continuous from left and $\varpi(t) = 0 \Leftrightarrow t = 0$.

Then mapping $\mathcal{Q}$ admits at least one fixed point in Ξ.

Proof. Let us define $\vartheta : [0, \infty) \longrightarrow [0, \infty)$ by $\vartheta = \varpi - \vartheta$, in Theorem 2.3 then all the conditions are satisfied and we assure fixed point of mapping $\mathcal{Q} : \Xi \longrightarrow \Xi$. □

3. Application

From the last few decades many researchers showed that the concept of $\mathcal{M}.\mathcal{N}.\mathcal{C}$ brings into play a sparkling role in the study of existence and uniqueness of solution of an integral equations [4].
In this section, we will use the $\mathcal{M}.\mathcal{N}.\mathcal{C}$ in the space $C([0, a])$ contains all continuous functions $x : [0, a] \longrightarrow \mathbb{R}$ having the norm,

$$||x|| = \max\left\{|x\left(r\right)| : r \in [0, a]\right\}; x \in C\left([0, a]\right),$$

Let $\Lambda \neq \phi$ be any subset of $C([0, a]; \mathbb{R})$. Now for $\epsilon > 0$ we define modulus of continuity $\omega(x, \epsilon)$ of x on $[0, a]$ as;

$$\omega\left(x, \varepsilon\right) = \max\left\{|x\left(r\right) - x\left(s\right)|; r, s \in [0, a], |r - s| \leq \varepsilon\right\},$$

and further we define the term $\omega\left(W, \varepsilon\right)$ as follows;

$$\omega\left(\Lambda, \varepsilon\right) = \sup\left\{\omega\left(x, \varepsilon\right); x \in \Lambda\right\}.$$

Note that modulus of continuity $\omega\left(\Lambda, \varepsilon\right)$, is non-negative and increasing, hence we assure that there exists a finite limit of $\lim_{\varepsilon \longrightarrow 0} \omega\left(\Lambda, \varepsilon\right)$, and finally we obtain the expression for the term $\mathcal{M}(\Lambda)$ in the form of limit as;

$$\mathcal{M}(\Lambda) = \lim_{\varepsilon \longrightarrow 0} \omega\left(\Lambda, \varepsilon\right). \tag{3.1}$$

In [4, 3] it is proved that the term $\mathcal{M}$ mentioned in equation (3.1) is $\mathcal{M}.\mathcal{N}.\mathcal{C}$ in the Banach space $C([0, a])$.

3.1. **On existence of solutions of non-linear proportional delay Volterra integral equation with singular kernel in the space $C[0,1]$.** In this section, we use the results form section 2 to prove a theorem which ensure the solutions of nonlinear proportional delay Volterra integral equation. The integral equation considered in this section generalizes the integral equation given in [8].

$$u(t) = \Re\left(t, u(t), \int_0^{qt} \frac{\mathrm{K}\left(t,\tau,u\left(\tau\right)\right)}{\left(qt-\tau\right)^{\gamma}} d\tau\right), \tag{3.2}$$

$$t \in I = [0,1],\ 0 < q \leq 1,\ 0 < \gamma < 1 \ \&\ \gamma \in \mathbb{Q}.$$

i) $\Re : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, is continuous function satisfies following inequality;

$$\left|\Re\left(t,x_1,y_1\right) - \Re\left(t,x_2,y_2\right)\right| \leq \sqrt{\xi\left(|x_1-x_2|\right)} + |y_1-y_2|,$$

where $\xi : [0,\infty) \longrightarrow [0,\infty)$, is a function of real numbers satisfies $\xi\left(t\right) < t^2$.

ii) $\mathrm{K} : I \times [0,A] \times \mathbb{R} \longrightarrow \mathbb{R}$, is continuous function and a there exist non-decreasing $\zeta : \mathbb{R} \longrightarrow \mathbb{R}_+$, such that;

$$\left|\mathrm{K}\left(t,\tau,u\left(\tau\right)\right)\right| \leq \zeta\left(||u||\right).$$

iii) there is at lest one real number $r_0 > 0$, which sacrifices the inequality

$$\left(1-\gamma\right)\sqrt{\xi\left(r_0\right)} + \zeta\left(r_0\right) + \left(1-\gamma\right)M \leq \left(1-\gamma\right)r_0.$$

where $M \geq 0$, is constant which satisfies the relation $\left|\Re\left(t,0,0\right)\right| \leq M\ \forall\ t \in I$.

Theorem 3.1. *If assumptions $(i)-(iii)$ satisfied by the Volterra integral equation (3.2) then it has at least one solution in the space $C([0,1])$.*

Proof. For $u \in C(I)$ define an mapping Q on the Banach space $C(I)$ in the following manner;

$$Q\left(u(t)\right) = \Re\left(t, u(t), \int_0^{qt} \frac{\mathrm{K}\left(t,\tau,u\left(\tau\right)\right)}{\left(qt-\tau\right)^{\gamma}} d\tau\right). \tag{3.3}$$

Here, we can easily show that Q, is a self-mapping on the space $C(I)$. To do this fix, $\epsilon > 0$, and choose random numbers $t, s \in I$ such that $|t-s| < \epsilon$, without loss of generality, we take $s < t$, and obtain following expression;

$$
\begin{aligned}
&\left|Q\left(u(t)\right)-Q\left(u(s)\right)\right| \\
&=\left|\ \Re\left(t,u(t),\int_0^{qt}\frac{\mathrm{K}(t,\tau,u(\tau))}{(qt-\tau)^{\gamma}}d\tau\right)-\Re\left(s,u(s),\int_0^{qs}\frac{\mathrm{K}(s,\tau,u(\tau))}{(qs-\tau)^{\gamma}}d\tau\right)\ \right| \\
&\leq\left|\Re\left(t,u(t),\int_0^{qt}\frac{\mathrm{K}\left(t,\tau,u\left(\tau\right)\right)}{\left(qt-\tau\right)^{\gamma}}d\tau\right)-\Re\left(t,u(s),\int_0^{qt}\frac{\mathrm{K}\left(s,\tau,u\left(\tau\right)\right)}{\left(qt-\tau\right)^{\gamma}}d\tau\right)\right| \\
&+\left|\Re\left(t,u(s),\int_0^{qt}\frac{\mathrm{K}\left(s,\tau,u\left(\tau\right)\right)}{\left(qt-\tau\right)^{\gamma}}d\tau\right)-\Re\left(s,u(s),\int_0^{qs}\frac{\mathrm{K}\left(s,\tau,u\left(\tau\right)\right)}{\left(qs-\tau\right)^{\gamma}}d\tau\right)\right| \\
&\leq\sqrt{\xi\left(\left|u(t)-u(s)\right|\right)}+\left|\int_0^{qt}\frac{\mathrm{K}\left(t,\tau,u\left(\tau\right)\right)}{\left(qt-\tau\right)^{\gamma}}d\tau-\int_0^{qt}\frac{\mathrm{K}\left(s,\tau,u\left(\tau\right)\right)}{\left(qt-\tau\right)^{\gamma}}d\tau\right| \\
&+\omega_{\Re}\left(I,\varepsilon\right)+\left|\int_0^{qt}\frac{\mathrm{K}\left(s,\tau,u\left(\tau\right)\right)}{\left(qt-\tau\right)^{\gamma}}d\tau-\int_0^{qs}\frac{\mathrm{K}\left(s,\tau,u\left(\tau\right)\right)}{\left(qs-\tau\right)^{\gamma}}d\tau\right| \\
&\leq\sqrt{\xi\left(\left|u(t)-u(s)\right|\right)}+\int_0^{qt}\frac{\left|\mathrm{K}\left(t,\tau,u\left(\tau\right)\right)-\mathrm{K}\left(s,\tau,u\left(\tau\right)\right)\right|}{\left(qt-\tau\right)^{\gamma}}d\tau \\
&+\omega_{\Re}\left(I,\varepsilon\right)+\left|\begin{array}{l}\int_0^{qt}\left|\mathrm{K}\left(s,\tau,u\left(\tau\right)\right)\right|\left(\frac{1}{(qt-\tau)^{\gamma}}-\frac{1}{(qs-\tau)^{\gamma}}\right)d\tau\\ +\int_0^{qt}\frac{\mathrm{K}(s,\tau,u(\tau))}{(qs-\tau)^{\gamma}}d\tau-\int_0^{qs}\frac{\mathrm{K}(s,\tau,u(\tau))}{(qs-\tau)^{\gamma}}d\tau\end{array}\right| \\
&\leq\sqrt{\xi\left(\left|u(t)-u(s)\right|\right)}+\omega_{\mathrm{K}}\left(I,\varepsilon\right)\left|\int_0^{qt}\frac{1}{\left(qt-\tau\right)^{\gamma}}d\tau\right|+\omega_{\Re}\left(I,\varepsilon\right) \\
&+\zeta\left(||u||\right)\left(\left|\int_{qs}^{qt}\left(\frac{1}{\left(qs-\tau\right)^{\gamma}}\right)d\tau\right|+\left|\int_0^{qt}\left(\frac{1}{\left(qt-\tau\right)^{\gamma}}-\frac{1}{\left(qs-\tau\right)^{\gamma}}\right)d\tau\right|\right) \\
&\leq\sqrt{\xi\left(\omega\left(u,\varepsilon\right)\right)}+\omega_{\mathrm{K}}\left(I,\varepsilon\right)\left(\frac{1}{1-\gamma}\right)+\omega_{\Re}\left(I,\varepsilon\right)+\zeta\left(||u||\right)\left(\frac{\varepsilon^{1-\gamma}}{1-\gamma}+\frac{\varepsilon^{1-\gamma}}{1-\gamma}-\frac{\varepsilon^{1-\gamma}}{1-\gamma}\right),
\end{aligned}
\tag{3.4}
$$

where,

$$
\begin{aligned}
\omega_{\Re}\left(I,\varepsilon\right)&=\sup\left\{\left|\Re\left(t,x,y\right)-\Re\left(s,x,y\right)\right|:t,s\in I,\left|t-s\right|\leq\varepsilon\right\},\\
\omega_{\mathrm{K}}\left(I,\varepsilon\right)&=\sup\left\{\left|\mathrm{K}\left(t,x,y\right)-\mathrm{K}\left(s,x,y\right)\right|:t,s\in I,\left|t-s\right|\leq\varepsilon\right\},\\
\omega\left(u,\varepsilon\right)&=\sup\left\{\left|u\left(t\right)-u\left(s\right)\right|:t,s\in I,\left|t-s\right|\leq\varepsilon\right\}.
\end{aligned}
\tag{3.5}
$$

From the above estimate and assumptions (i) & (ii) we ensure that Qu, is a continuous function on the interval I. Now, considering above established expression and assumption (iii), we ensure that the mapping Q maps the space $C(I)$ into itself.
For $r_0 > 0$, consider

$$B_{r_0}=\left\{u\in C\left([0,1]\right)||u||\leq r_0\right\},$$

be the closed ball centered at origin.
We claim Q is a continuous map from B_{r_0} into itself. Indeed for a random but fixed

element $u \in C(I)$ and $t \in I$, we secure following inequality;

$$\begin{aligned} |Q(u(t))| &= \left| \Re\left(t, u(t), \int_0^{qt} \frac{\mathrm{K}(s,\tau,u(\tau))}{(qt-\tau)^\gamma} d\tau\right) - \Re(t,0,0,) + \Re(t,0,0,) \right| \\ &\leq \sqrt{\xi(||u||)} + \zeta(||u||) \left| \int_0^{qt} \frac{1}{(qt-\tau)^\gamma} d\tau \right| + M \\ &\leq \sqrt{\xi(||u||)} + \zeta(||u||)\left(\frac{1}{1-\gamma}\right) + M. \end{aligned} \tag{3.6}$$

By the virtue of assumption (iii) and equation (3.6), we conformed Q maps form the B_{r_0} into itself.
Now, in order to show the continuity of Q on B, fix $\epsilon > 0$ and choose $\delta > 0$, at the same time select an arbitrary pair $u, v \in B$, such that $||u - v|| < \delta$. With this consideration, for a random $t \in I$, we obtain;

$$\begin{aligned} &|Q(u(t)) - Q(v(t))| \\ &= \left| \Re\left(t, u(t), \int_0^{qt} \tfrac{\mathrm{K}(s,\tau,u(\tau))}{(qt-\tau)^\gamma} d\tau\right) - \Re\left(t, v(t), \int_0^{qt} \tfrac{\mathrm{K}(s,\tau,v(\tau))}{(qt-\tau)^\gamma} d\tau\right) \right| \\ &\leq \sqrt{\xi(|u(t)-v(t)|)} + \left| \int_0^{qt} \tfrac{\mathrm{K}(t,\tau,u(\tau)) - \mathrm{K}(t,\tau,v(\tau))}{(rt-\tau)^\gamma} d\tau \right| \\ &\leq \sqrt{\xi(|u(t)-v(t)|)} + (\omega_{\mathrm{K}}(I,\varepsilon)(1/1-\gamma)). \end{aligned} \tag{3.7}$$

The expression (3.7) confirm that the mapping Q enjoys continuity property on B_{r_0}. Let B be a collection of all the functions from closed ball $u \in B_{r_0}$, having the property that $u(t) \geq 0$ for $t \in I$. Obviously B is non-empty since $r_0 > 0$. Let $D \neq \phi$, be a non-empty subset of B and $u \in D$. For a fixed real number $\epsilon > 0$, choose the pair $t, s \in I$, such that $|t - s| \leq \epsilon$. Without loss of generality, assume $s < t$, equation (3.4), produces the following expression;

$$\omega(Qu,\varepsilon) \leq \sqrt{\xi(\omega(u,\varepsilon))} + \left(\omega_{\mathrm{K}}(I,\varepsilon)\left(\tfrac{1}{1-\gamma}\right) + \zeta(||u||)\left(\tfrac{\varepsilon^{1-\gamma}}{1-\gamma}\right)\right) + \omega_{\Re}(I,\varepsilon). \tag{3.8}$$

By the virtue of assumptions (i) and (ii) the term $\omega_{\mathrm{K}}(I,\varepsilon) \longrightarrow 0$ & $\omega_{\Re}(I,\varepsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$ also ζ is bounded function on its domain, hence applying $\epsilon \longrightarrow 0$ and from 3.1, equation (3.8) remain with the following inequality;

$$\mathcal{M}(Q(D)) \leq \sqrt{\xi(\mathcal{M}(D))},$$

where $\mathcal{M}$ is $\mathcal{M.N.C}$. Now, define the functions ϖ & $\vartheta : [0,\infty) \longrightarrow [0,\infty)$, by $\varpi(t) = t^2$ and $\vartheta(t) = \xi(t)$, where the map ξ is mentioned in assumption (i). Considering above functions we get the following estimate;

$$\varpi(\mathcal{M}(Q(D))) \leq \vartheta(\mathcal{M}(D)). \tag{3.9}$$

Utilizing theorem (2.1) of Section (2) with above estimation we confirm that the map Q admits a fixed point in $B \subset C([0,1])$. This proves the integral equation (3.3) admit at-least one solution in Banach space $C([0,1])$. □

3.2. **Example.**

Example 3.1. Consider the non-linear functional integral equation:

$$u(t) = \cos(t) - \frac{1}{6}t^{\frac{1}{3}} + \frac{t}{4(t+1)}\cos(u(t)) + \frac{1}{18}\int_0^{qt} \frac{\sin^2(\tau) + u^2(\tau)}{(qt-\tau)^{\gamma}}d\tau, \tag{3.10}$$

$$\text{where } t \in I = [0,1],\ 0 < q \leq 1,\ 0 < \gamma < 1 .$$

Eq. (3.10) is obtained from (3.2) by the following substitution

$$\Re(t,u,v) = \cos(t) - \frac{1}{6}t^{\frac{1}{3}} + \frac{t}{4(t+1)}\cos(u(t)) + \frac{1}{18}\int_0^{qt} \frac{\sin^2(\tau) + u^2(\tau)}{(qt-\tau)^{\gamma}}d\tau,$$

where

$$K(t,\tau,u(\tau)) = \frac{\sin^2(\tau) + u^2(\tau)}{18}.$$

For $t, \tau \in I$ and $u \in C(I)$, we estimate following expression;

$$\begin{aligned} |K(t,\tau,u(\tau))| &= \left|\frac{\sin^2(\tau) + u^2(\tau)}{18}\right| \\ &\leq \frac{1 + ||u||^2}{18} = \zeta(u). \end{aligned}$$

Consider $\Re(t,u,v) = \cos(t) - \frac{1}{6}t^{\frac{1}{3}} + \frac{t}{4(t+1)}\cos(u(t)) + v$, in (3.10). For $t \in I$, and $u, v \in C(I)$, we have the following expression;

$$\begin{aligned} |\Re(t,u,v) - \Re(t,x,y)| &= \left|\frac{t}{4(t+1)}\cos(u(t)) - \frac{t}{4(t+1)}\cos(x(t))\right| + |v-y| \\ &\leq \sqrt{\xi|u-x|} + |v-y|, \end{aligned}$$

where, $\sqrt{\xi|u-x|} = \frac{1}{8}|u-x|$. Moreover, $M = \max\{|\Re(t,0,0)| : t \in [0,1]\} \approx 1.048$.

Now, using the above functions viz. $\zeta(s) = \frac{1+s^2}{18}$, $\sqrt{\xi|s|} = \frac{1}{8}s$ and $M = 1.048$, in the existing inequality of assumption (iii), we get

$$\begin{aligned} &(1-\gamma)\sqrt{\xi(r_0)} + \zeta(r_0) + (1-\gamma)M \leq (1-\gamma)r_0 \\ &\Rightarrow (1-\gamma)\left(\frac{r_0}{8}\right) + \frac{1+{r_0}^2}{18} + (1-\gamma)(1.048) \leq (1-\gamma)r_0. \end{aligned} \tag{3.11}$$

It is easily seen that the above inequality have a positive solution for suitable choice of $0 < \gamma < 1$. In particular, if we choose $\gamma = 0.5$ and $r_0 = 1.8$, then the inequality (3.11) is satisfied. Moreover, we define a function $A(\gamma, r)$ using the inequality (3.11) for $0 < \gamma < 1$ and $1 \leq r \leq 5$, by

$$A(\gamma, r) = 7(1-\gamma)\left(\frac{r}{8}\right) + \frac{1+r^2}{18} + (1-\gamma)(1.048) \tag{3.12}$$

and plot is given in the following figure 1 for $\gamma = 0.2, 0.4, 0.6, 0.8$ & 1. Figure 1, shows that the values of $A(\gamma, r)$, lies in first quadrent for $0 \leq r \leq 5$, and $0 < \gamma \leq 1$. Consequently, we can easily verfies that there are some mores of γ and r that satisfies the inequality (3.11). Thus from all the above observations we see that the integral equation (3.10) satisfies all the assumptions of Theorem 3.1. Hence by Theorem 3.1, we ensures that the integral (3.10) has atleast one solution.

Plot of equation (3.12) in the view of assumption (iii).

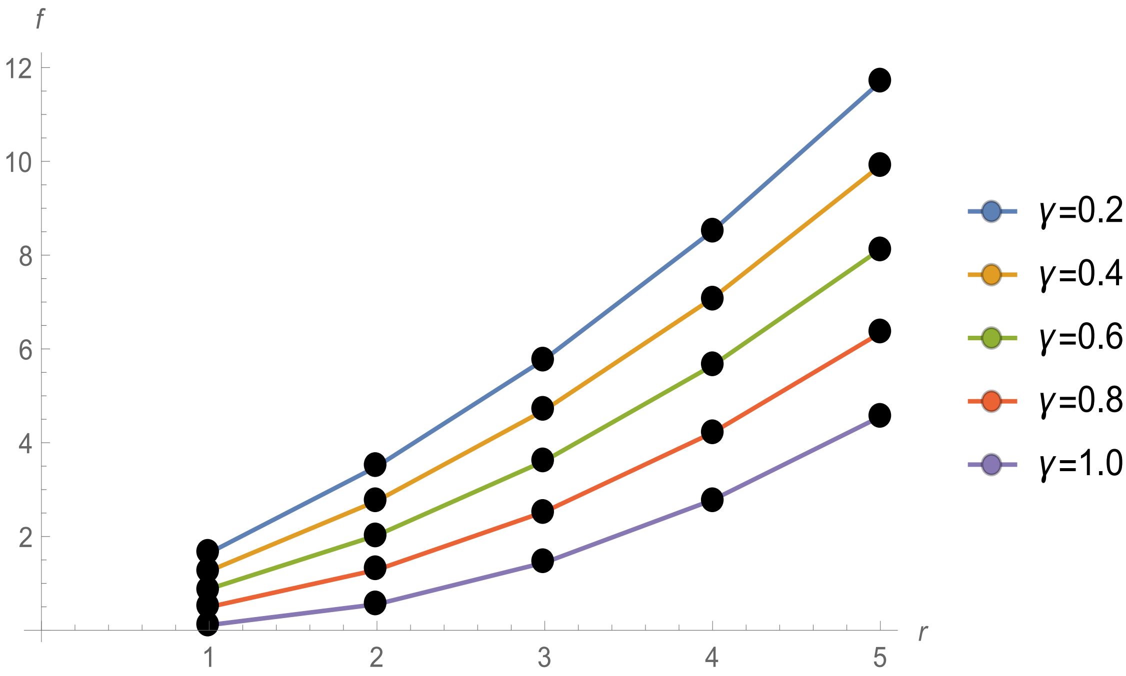

FIGURE 1.

Example 3.2.

$$u(t) = f(t) + \chi u(t) + t \int_0^{qt} \frac{e^{\tau} \arcsin(u(\tau))}{(qt-\tau)^{\gamma}} d\tau, \tag{3.13}$$

where $\chi \in \mathbb{R}$, $f : [0,1] \longrightarrow \mathbb{R}$ is continuous function and $t \in I = [0,1], 0 < q \leq 1,\ \&\ 0 < \gamma < 1$.

If we take following substitutions in (3.2) we obtain equation (3.13),

$$\Re(t,u,v) = f(t) + \chi u(t) + \int_0^{qt} \frac{u^3(\tau)}{(qt-\tau)^{\gamma}} d\tau.$$

where

$$K(t,\tau,u(\tau)) = u^3(\tau).$$

Now, for $t, \tau \in I$ and $u \in C(I)$, we estimate following expression;

$$\begin{aligned} |K(t,\tau,u(\tau))| &= \left|u^3(\tau)\right| \\ &\leqslant \left\|u^3\right\| = \zeta(u) \ \text{(say)}, \end{aligned}$$

Consider $\Re(t,u,v) = f(t) + \chi u(t) + v$, in (3.13). For $t \in I$, and $u, v \in C(I)$, we have the following expression;

$$\begin{aligned} |\Re(t,u,v) - \Re(t,x,y)| &= \chi |u(t) - x(t)| + |v-y| \\ &\leqslant \sqrt{\xi |u-x|} + |v-y|, \end{aligned}$$

Plot of equation (3.15) in the view of assumption (iii) for $\chi = 0.1$.

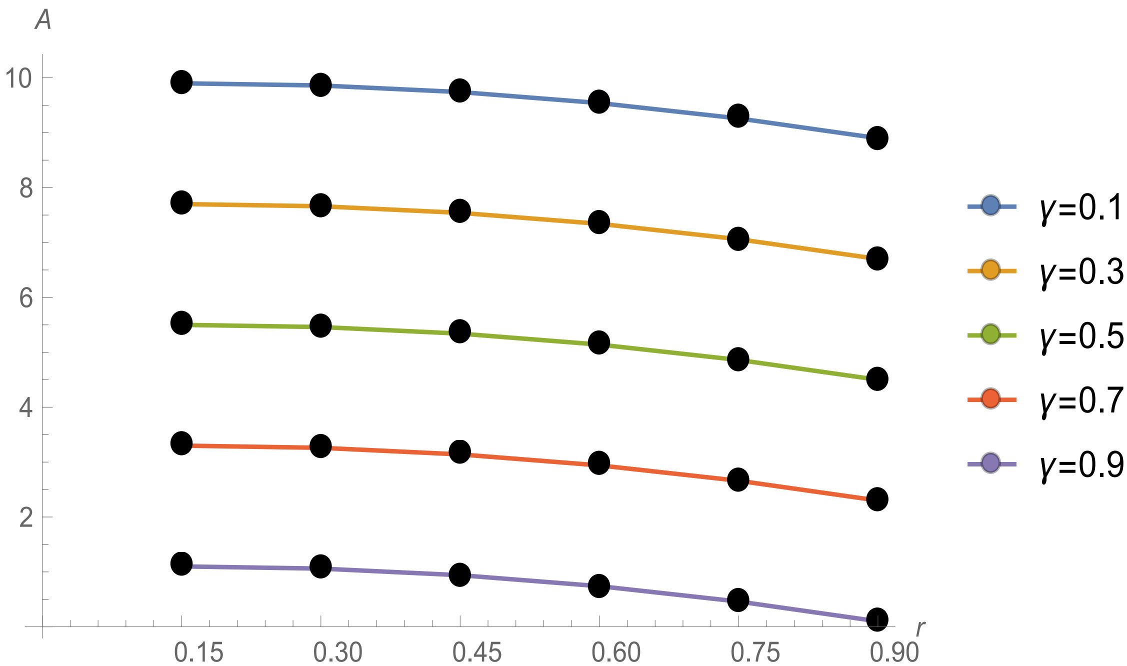

FIGURE 2.

where, $\sqrt{\xi\,|u-x|} = \chi\,|u(t)-x(t)|$. Moreover, $M = \max\{|\Re\,(t,0,0)| : t \in [0,1]\} \approx 1.048$.
Now, using the above functions viz. $\zeta\,(s) = s^3$, $\sqrt{\xi\,(|s|)} = \chi s$ and $M = 0$, in the existing inequality of assumption (iii), we get

$$\begin{aligned} &(1-\gamma)\sqrt{\xi\,(r_0)} + \zeta\,(r_0) + (1-\gamma)\,M \leqslant (1-\gamma)\,r_0 \\ &\Rightarrow (1-\gamma)\,\chi r_0 + {r_0}^3 \leqslant (1-\gamma)\,r_0. \end{aligned} \tag{3.14}$$

It is easily seen that the above inequality have a positive solution for suitable choice of $0 < \gamma < 1$. In particular, if we choose $\gamma = 0.2$ and $r_0 = 0.8$ and $\chi = 0.1$, then the inequality (3.11) is satisfied. Moreover, we define a function $A(\gamma, r)$ using the inequality (3.11) for $0 < \gamma < 1$ and $0 \leq r < 1$, by

$$A(\gamma, r) = (1-\gamma)\,(1-\chi) - {r_0}^2, \tag{3.15}$$

and plot is given in the following figure 2.

Figure 2, shows that the values of $A(\gamma, r)$ for $\chi = -10$ lies in first quadrent for $0 < r < 1$, and $\gamma = 0.1, 0.3, 0.5, 0.7$ & 0.9. Consequently, we will easily check that there are more combinations for values of γ, r and χ that satisfies the inequality (3.14). Thus from all the above discussion we see that the integral equation (3.13) satisfies all the assumptions of Theorem 3.1. Hence Theorem 3.1, ensures the existance of solution for integral equation (3.13).

4. CONCLUSION

In this work we proved Volterra integral equation with weakly singular kernel having proportional delay is solvable under certain conditions. The solvability is proved by the Darbo type fixed point theorem defined with the help of new class of condensing operators. The Darbo type fixed point theorem derived in this paper generalizes many of the existing results. To support our reslts we discussed two problems and proved existance conditions. The exact solution of 3.2 is given in [8]. The problem discussed in [8] will be obtain by substituting $\chi = 0$ in (3.13). In future, we expect to study the stability behavior to the solutions (3.13).

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