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SHRINKING PROJECTION METHOD WITH ALLOWABLE RANGES FOR ZERO POINT PROBLEMS IN A BANACH SPACE

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ABSTRACT. In this paper, we study the shrinking projection method with allowable ranges introduced by Takeuchi [29] for the zero point problem. We obtain strong convergence theorems for finding a zero point of a maximal monotone operator in a Banach space. Using our results, we discuss the convex minimization problem.

KEYWORDS: Shrinking projection method, allowable range, zero point, maximal monotone operator, resolvent
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1. Introduction

In 2008, Takahashi, Takeuchi, and Kubota [31] introduced the shrinking projection method, which is an iterative method for finding a common fixed point of some families of nonlinear mappings in a Hilbert space. In 2009, Kimura and Takahashi [17] improved this method in a Banach space. The shrinking projection method is a very useful method in the fixed point approximation theory and has been studied extensively; see [8, 10, 12, 14–16] and others.

In the shrinking projection method, we need to obtain the exact value of the metric projection to generate a sequence in every step, and it is a task of difficulty. To solve this problem, in 2012, Kimura [14] presented the shrinking projection method with nonsummable errors in a geodesic space; see also [15, 16]. Motivated by Kimura [14], in 2019, Takeuchi [29] proposed another method called the shrinking projection method with allowable ranges and obtained a strong convergence theorem for finding a fixed point of a nonlinear mapping in a Banach space.

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Theorem 1.1 ([29]). *Let E be a reflexive, smooth, and strictly convex Banach space which has the Kadec–Klee property. Let C be a nonempty closed convex subset of E. Let $B : C \longrightarrow E$ be a mapping of type (P) such that $F(B) := \{p \in C : Bp = p\}$ is nonempty. Consider an iterative procedure as below: Let $x_0 \in E$, $w_1 \in C$, $D_1 = \{y \in C : \langle Bw_1 - y, J(w_1 - Bw_1)\rangle \geq 0\}$ and $x_1 = P_{D_1}x_0$. Let $A_1 = C \setminus (D_1 \cup \{w_1\})$ and let $y_1 \in A_1$. For each $n \in \mathbb{N}$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and y_{n+1} by*

$$D_{n+1} = \{y \in D_n : \langle By_n - y, J(y_n - By_n)\rangle \geq 0\}, \quad x_{n+1} = P_{D_{n+1}}x_0,$$
$$A_{n+1} = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq y_n\}, \quad y_{n+1} \in A_{n+1}$$

where P_K is the metric projection of E onto a nonempty closed convex subset K of E. Then, either of following holds:

- (i) *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{F(B)}x_0$*
- (ii) *$A_k = \emptyset$ for some $k \in \mathbb{N}$; the procedure is stopped. In this case, $y_{k-1} \in F(B)$ or $w_1 \in F(B)$ holds.*

Note that the original result of the theorem above deals with nonlinear mappings related to mappings of type (P).

On the other hand, we consider the zero point problem: Let H be a real Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Then, the zero point problem is to find $u \in H$ such that

$$0 \in Au. \tag{1.1}$$

Such a $u \in H$ is called a zero point of A. This problem is connected with many problems in Nonlinear Analysis and Optimization, such as convex minimization problems, variational inequality problems, equilibrium problems, and so on. A well-known method for solving (1.1) is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n}x_n, \quad n = 1, 2, \ldots, \tag{1.2}$$

where $\{r_n\} \subset]0, \infty[$ and $J_{r_n} = (I + r_nA)^{-1}$. This algorithm was first introduced by Martinet [20]. In 1976, Rockafellar [27] proved that if $\liminf_{n\longrightarrow\infty} r_n > 0$ and $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a solution of the zero point problem. Later, many researchers have studied the convergence of the proximal point algorithm; see [6, 8, 10, 12, 17–19, 28] and others. In particular, in 2009, Inoue, Takahashi and Zembayashi [12] studied another proximal-type algorithm in a Banach space by using the original shrinking projection method and obtained a strong convergence theorem for finding a zero point of a maximal monotone operator. In 2016, Ibaraki [8] studied the shrinking projection method with nonsummable error, introduced by Kimura [14], for the zero point problem in a Banach space; see also [9, 10].

Motivated by the aforementioned works, we study the shrinking projection method with allowable ranges, introduced by Takeuchi [29], for the zero point problem. We obtain strong convergence theorems for finding a zero point of a maximal monotone operator in a Banach space. As an application of our results, we discuss a convex minimization problem. Finally, we provide some examples to support our results.

2. Preliminaries

Let E be a real Banach space with its dual E^*. The normalized duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in E$. A Banach space E is said to have the Kadec–Klee property if a sequence $\{x_n\}$ of E converges strongly to x_0 whenever $\{x_n\}$ converges weakly to x_0 and $\{\|x_n\|\}$ converges to $\|x_0\|$. We know the following properties; see, for instance, [4, 30].

- if E is smooth and strictly convex, then $\langle x-y, Jx-Jy\rangle = 0$ if and only if $x = y$,
- if E is reflexive, smooth and strictly convex, then J is surjective and the duality mapping on E^* is J^{-1},
- E^* has a Fréchet differential norm if and only if E is reflexive, strictly convex, and has the Kadec–Klee property.

An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x-y, x^*-y^*\rangle \geq 0$ for each $(x, x^*), (y, y^*) \in A$. A monotone operator A is said to be maximal if $A = A'$ whenever $A' \subset E \times E^*$ is a monotone operator such that $A \subset A'$. We denote by $A^{-1}0$ the set $\{z \in D(A) : 0 \in Az\}$. We know that J is monotone.

Let E be a smooth Banach space and consider the following function $V : E \times E \longrightarrow \mathbb{R}$ defined by

$$V(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.1}$$

for each $x, y \in E$. We know the following properties; see [1, 11, 13].

- $(\|x\| - \|y\|)^2 \leq V(x,y) \leq (\|x\| + \|y\|)^2$ for each $x, y \in E$,
- $V(x,y) + V(y,x) = 2\langle x-y, Jx-Jy\rangle$ for each $x, y \in E$,
- $V(x,y) = V(x,z) + V(z,y) + 2\langle x-z, Jz-Jy\rangle$ for each $x, y, z \in E$,
- if E is additionally assumed to be strictly convex, then $V(x,y) = 0$ if and only if $x = y$.

Let E be a reflexive and strictly convex Banach space, and let C be a nonempty closed convex subset of E. It is known that for each $x \in E$, there exists a unique point $z \in C$ such that $\|x-z\| = \min\{\|x-y\| : y \in C\}$. Such a point z is denoted by $P_C x$, and P_C is called the metric projection of E onto C. The following result is well known; see, for instance, [30].

Lemma 2.1. *Let E be a reflexive, smooth, and strictly convex Banach space, let C be a nonempty closed convex subset of E, let P_C be the metric projection of E onto C, let $x \in E$ and let $x_0 \in C$. Then $x_0 = P_C x$ if and only if*

$$\langle x_0 - y, J(x-x_0)\rangle \geq 0$$

for all $y \in C$.

The following result was obtained in [29]; see also [21].

Lemma 2.2 ([29])**.** *Let E be a reflexive and strictly convex Banach space. Let $x_0 \in E$ and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E satisfying $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$ and $C_0 := \cap_{n\in\mathbb{N}} C_n \neq \emptyset$. Let $x_1 = P_{C_1} x_0$. For each $n \in \mathbb{N}$, define x_{n+1}, K_n and u_n by*

$$x_{n+1} = P_{C_{n+1}} x_0, \quad K_n = \{z \in C_n : \|x_0 - z\| \leq \|x_0 - x_{n+1}\|\}, \quad u_n \in K_n.$$

Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $P_{C_0} x_0$. Furthermore, when E has the Kadec–Klee property, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{C_0} x_0$.

3. Approximation theorem for the resolvents of Type (P)

Let $r > 0$ and let $A \subset E \times E^*$ be a maximal monotone operator. Then it is known that $R(I + rJ^{-1}A) = E$ for all $r > 0$; see [5,26]. We can define the following single-valued mapping

$$P_r := (I + rJ^{-1}A)^{-1} : E \longrightarrow D(A)$$

for each $r > 0$. It is called the resolvent of type (P) of A for $r > 0$; see [4]. It is known that

$$\frac{J(x - P_r x)}{r} \in AP_r x \tag{3.1}$$

for each $x \in E$ and

$$A^{-1}0 = F(P_r) := \{p \in E : P_r p = p\}. \tag{3.2}$$

To obtain our main result in this section, we need the following two lemmas. Compare these lemmas with the results in Takeuchi [29].

Lemma 3.1. *Let E be a reflexive, smooth, and strictly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$ and let C be a nonempty closed convex subset of E such that $A^{-1}0 \subset C$. Let $r > 0$ and let $u \in E$. Let M be a subset of E defined by*

$$M := \{z \in C : \langle P_r u - z, J(u - P_r u)\rangle \geq 0\}.$$

Then, M is closed and convex, and $A^{-1}0 \subset M$.

Proof. It is clear that M is closed and convex. Let $r > 0$ and let $u \in E$. Set $M_E := \{z \in E : \langle P_r u - z, J(u - P_r u)\rangle \geq 0\}$. We show $A^{-1}0 \subset M_E$. Let $v \in A^{-1}0$. From the monotonicity of A and (3.1), we have

$$0 \leq \left\langle P_r u - v, \frac{J(u - P_r u)}{r} - 0 \right\rangle = \frac{1}{r}\langle P_r u - v, J(u - P_r u)\rangle.$$

Since $r > 0$, we get $\langle P_r u - v, J(u - P_r u)\rangle \geq 0$ and hence $v \in M_E$. This implies $A^{-1}0 \subset M_E$. Since $A^{-1}0 \subset C$, we have $A^{-1}0 \subset M_E \cap C = M$. □

Lemma 3.2. *Let E be a reflexive, smooth, and strictly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0, \infty[$ satisfying $\liminf_{n \longrightarrow \infty} r_n > 0$. Let $\{u_n\}$ be a sequence in E. Define $\{C_n\}$ as a sequence of subsets of E by $C_1 = E$ and*

$$y_n = P_{r_n} u_n,$$
$$C_{n+1} = \{z \in C_n : \langle y_n - z, J(u_n - y_n)\rangle \geq 0\}$$

for each $n \in \mathbb{N}$. Let $C_0 = \cap_{n\in\mathbb{N}} C_n$. Then the following hold:

(i) *C_n is closed and convex, and $A^{-1}0 \subset C_n$ for each $n \in \mathbb{N} \cup \{0\}$.*
(ii) *Suppose that $\{u_n\}$ converges strongly to some $u_0 \in C_0$, then $u_0 \in A^{-1}0$.*

Proof. We first show (1) by induction. It is clear that C_1 is closed and convex, and $A^{-1}0 \subset C_1$. Suppose that C_k is closed and convex, and $A^{-1}0 \subset C_k$ for some $k \in \mathbb{N}$. Using Lemma 3.1 with $u = u_k$, $r = r_k$, $C = C_k$, $M = C_{k+1}$, we see that C_{k+1} is closed and convex, and $A^{-1}0 \subset C_{k+1}$. This implies that C_n is closed and convex, and $A^{-1}0 \subset C_n$ for each $n \in \mathbb{N}$. Hence, it follows that C_0 is closed and convex, and $A^{-1}0 \subset \cap_{n\in\mathbb{N}} C_n = C_0$. Thus, we conclude that (1) holds.

Next, we show (2). Suppose that $\{u_n\}$ converges strongly to some $u_0 \in C_0$. Since $C_0 \subset C_n$ for each $n \in \mathbb{N}$, we see that

$$0 \leq \langle y_n - u_0, J(u_n - y_n)\rangle = \langle y_n - u_n, J(u_n - y_n)\rangle + \langle u_n - u_0, J(u_n - y_n)\rangle$$

for each $n \in \mathbb{N}$ and thus

$$\begin{aligned}\|u_n - y_n\|^2 &\leq \langle u_n - u_0, J(u_n - y_n)\rangle \\ &\leq \|u_n - u_0\|\|J(u_n - y_n)\| \\ &= \|u_n - u_0\|\|u_n - y_n\|\end{aligned}$$

for each $n \in \mathbb{N}$. Therefore we have $\|u_n - y_n\| \leq \|u_n - u_0\|$. Since $\{u_n\}$ converges strongly to u_0, we obtain that

$$\lim_{n\longrightarrow\infty} \|u_n - y_n\| = 0 \tag{3.3}$$

and hence $\{y_n\}$ also converges strongly to u_0. Furthermore, since E is smooth, J is norm-to-weak continuous. So, we see that $\{J(u_n - y_n)\}$ converges weakly to 0. For each $(z, z^*) \in A$, we obtain from (3.1) that

$$\left\langle z - y_n, z^* - \frac{J(u_n - y_n)}{r_n} \right\rangle \geq 0$$

for each $n \in \mathbb{N}$. Letting $n \longrightarrow \infty$, from $\liminf_{n\longrightarrow\infty} r_n > 0$, we get $\langle z - u_0, z^* - 0\rangle \geq 0$. From the maximality of A, we obtain $u_0 \in A^{-1}0$. □

Theorem 3.3. *Let E be a reflexive, smooth, and strictly convex Banach space which has the Kadec–Klee property. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0, \infty[$ satisfying $\liminf_{n\longrightarrow\infty} r_n > 0$. Consider an iteration procedure as below: Let $x_0 \in E$, $C_1 = D_1 = E$, $x_1 = P_{C_1}x_0$ and $u_1 \in D_1$. For each $n \in \mathbb{N}$, define y_n, C_{n+1}, x_{n+1}, D_{n+1} and u_{n+1} by*

$$\begin{aligned}y_n &= P_{r_n}u_n, \\ C_{n+1} &= \{z \in C_n : \langle y_n - z, J(u_n - y_n)\rangle \geq 0\}, \\ x_{n+1} &= P_{C_{n+1}}x_0, \\ D_{n+1} &= \{y \in C_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq u_n\}, \\ u_{n+1} &\in D_{n+1}.\end{aligned}$$

Then, either of the following holds:

(i) *$D_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{A^{-1}0}x_0$.*

(ii) *$D_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in A^{-1}0$ holds.*

Proof. From $C_1 = D_1 = E$, C_1 and D_1 are nonempty. Let $u_1 \in D_1 = E$. Then we generate y_1 and C_2. From Lemma 3.1, C_2 is nonempty, closed, and convex, and also generates x_2 and D_2. D_2 may be empty. In the case of $D_2 \neq \emptyset$, we can find $u_2 \in D_2$ and generate y_2 and C_3. From Lemma 3.1, C_3 is nonempty, closed, and convex, and also generates x_3 and D_3. D_3 may be empty. In the case of $D_3 \neq \emptyset$, we can find $u_3 \in D_3$ and continue this process. So, the procedure is stopped when we meet $k \in \mathbb{N} \setminus \{1\}$ satisfying $D_k = \emptyset$.

In the case of (1), we can generate the sequences $\{y_n\}$, $\{C_n\}$, $\{x_n\}$, $\{D_n\}$, and $\{u_n\}$ inductively. By our generating method, from Lemma 3.2 (1), $\{C_n\}$ is a sequence of closed convex subsets of E satisfying $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$ and $\emptyset \neq A^{-1}0 \subset C_0 = \cap_{n\in\mathbb{N}}C_n$. For each $n \in \mathbb{N}$, let K_n be as in Lemma 2.2, that is,

$$K_n = \{z \in C_n : \|x_0 - z\| \leq \|x_0 - x_{n+1}\|\}.$$

Then, $u_{n+1} \in D_{n+1} \subset K_n$ for each $n \in \mathbb{N}$. By Lemma 2.2, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{C_0}x_0$. Since $\{u_n\}$ converges strongly to $P_{C_0}x_0 \in C_0$, by Lemma 3.2

(2), we see $P_{C_0}x_0 \in A^{-1}0$. From $A^{-1}0 \subset C_0$, we get $P_{C_0}x_0 = P_{A^{-1}0}x_0$, which completes the proof of (1).

We show (2). Suppose that we generated u_{k-1}, y_{k-1}, C_k, x_k, and $D_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$. Then, C_k and C_{k-1} are nonempty, closed, and convex. Since $K_{k-1} = \{z \in C_{k-1} : \|x_0 - z\| \le \|x_0 - x_k\|\}$, we see $x_{k-1}, x_k \in K_{k-1}$ and $K_{k-1} \neq \emptyset$. By $K_{k-1}\backslash\{u_{k-1}\} = D_k = \emptyset$, we see that $u_{k-1} \in K_{k-1}$ and K_{k-1} is a singleton. From these, $u_{k-1} = x_{k-1} = x_k$ holds. So, by $u_{k-1} = x_k \in C_k$, we see that

$$0 \le \langle y_{k-1} - u_{k-1}, J(u_{k-1} - y_{k-1})\rangle = -\|y_{k-1} - u_{k-1}\|^2.$$

This implies $u_{k-1} = y_{k-1} = P_{r_{k-1}}u_{k-1}$. By (3.2), we get $u_{k-1} \in A^{-1}0$. This completes the proof. □

4. Approximation theorem for the resolvents of Type (Q)

Let $r > 0$ and let $A \subset E \times E^*$ be a maximal monotone operator. Then it is known that $R(J + rA) = E^*$ for all $r > 0$; see [5, 26]. We can define the following single-valued mapping

$$Q_r := (J + rA)^{-1}J : E \longrightarrow D(A)$$

for each $r > 0$. It is called the resolvent of type (Q) of A for $r > 0$; see [5]. It is known that

$$\frac{Jx - JQ_rx}{r} \in AQ_rx \tag{4.1}$$

for each $x \in E$ and

$$A^{-1}0 = F(Q_r). \tag{4.2}$$

To obtain our main result in this section, we need the following two lemmas, which can be proved in the same way of proofs as the Lemmas 3.1 and 3.2.

Lemma 4.1. *Let E be a reflexive, smooth, and strictly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$ and let C be a nonempty closed convex subset of E such that $A^{-1}0 \subset C$. Let $r > 0$ and let $u \in E$. Let N be a subset of E defined by*

$$N := \{z \in C : \langle Q_ru - z, Ju - JQ_ru\rangle \ge 0\}.$$

Then, N is closed and convex, and $A^{-1}0 \subset N$.

Proof. It is clear that N is closed and convex. Let $r > 0$ and let $u \in E$. Set $N_E = \{z \in E : \langle Q_ru - z, Ju - JQ_ru\rangle \ge 0\}$. We show $A^{-1}0 \subset N_E$. Let $v \in A^{-1}0$. From the monotonicity of A and (4.1), we have

$$0 \le \left\langle Q_ru - v, \frac{Ju - JQ_ru}{r} - 0\right\rangle = \frac{1}{r}\langle Q_ru - v, Ju - JQ_ru\rangle.$$

Since $r > 0$, we get $\langle Q_ru - v, Ju - JQ_ru\rangle \ge 0$ and hence $v \in N_E$. This implies $A^{-1}0 \subset N_E$. Since $A^{-1}0 \subset C$, we have $A^{-1}0 \subset N_E \cap C = N$. □

Lemma 4.2. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec–Klee property. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0, \infty[$ satisfying $\liminf_{n\longrightarrow\infty} r_n > 0$. Let $\{u_n\}$ be a sequence in E. Define $\{C_n\}$ as a sequence of subsets of E by $C_1 = E$ and*

$$y_n = Q_{r_n}u_n,$$
$$C_{n+1} = \{z \in C_n : \langle y_n - z, Ju_n - Jy_n\rangle \ge 0\}$$

for each $n \in \mathbb{N}$. Let $C_0 = \cap_{n\in\mathbb{N}}C_n$. Then the following hold:

(i) *C_n is closed and convex, and $A^{-1}0 \subset C_n$ for each $n \in \mathbb{N} \cup \{0\}$.*
(ii) *Suppose that $\{u_n\}$ converges strongly to some $u_0 \in C_0$, then $u_0 \in A^{-1}0$.*

Proof. We first show (1) by induction. It is clear that C_1 is closed and convex, and $A^{-1}0 \subset C_1$. Suppose that C_k is closed and convex, and $A^{-1}0 \subset C_k$ for some $k \in \mathbb{N}$. Using Lemma 4.1 with $u = u_k$, $r = r_k$, $C = C_k$, $N = C_{k+1}$, we see that C_{k+1} is closed and convex, and $A^{-1}0 \subset C_{k+1}$. This implies that C_n is closed and convex, and $A^{-1}0 \subset C_n$ for each $n \in \mathbb{N}$. Hence, it follows that C_0 is closed and convex, and $A^{-1}0 \subset \cap_{n\in\mathbb{N}} C_n = C_0$. Thus, we see that (1) holds.

Next, we show (2). Suppose that $\{u_n\}$ converges strongly to some $u_0 \in C_0$. Since $C_0 \subset C_n$ for each $n \in \mathbb{N}$ and the property of V, we see that

$$\begin{aligned} 0 &\le 2\langle y_n - u_0, Ju_n - Jy_n\rangle \\ &= 2\langle u_0 - y_n, Jy_n - Ju_n\rangle = V(u_0, u_n) - V(u_0, y_n) - V(y_n, u_n) \end{aligned}$$

for each $n \in \mathbb{N}$. Therefore, we obtain that

$$V(y_n, u_n) \le V(u_0, u_n) \tag{4.3}$$

and

$$0 \le (\|y_n\| - \|u_0\|)^2 \le V(u_0, y_n) \le V(u_0, u_n) \tag{4.4}$$

for each $n \in \mathbb{N}$. Since $\{u_n\}$ converges strongly to u_0 and (4.4), $\{\|y_n\|\}$ converges to $\|u_0\|$ and hence $\{y_n\}$ is bounded. Since E has a Fréchet differentiable norm, J is norm-to-norm continuous. Thus, we see that $\{Ju_n\}$ converges strongly to Ju_0. From the boundedness of $\{y_n\}$, there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to some $y_0 \in E$. Since $\|\cdot\|$ is weakly lower semicontinuous and (4.3), we see that

$$\begin{aligned} V(y_0, u_0) &= \|y_0\|^2 - 2\langle y_0, Ju_0\rangle + \|u_0\|^2 \\ &\le \liminf_{i\longrightarrow\infty}\{\|y_{n_i}\|^2 - 2\langle y_{n_i}, Ju_{n_i}\rangle + \|u_{n_i}\|^2\} \\ &= \liminf_{i\longrightarrow\infty} V(y_{n_i}, u_{n_i}) \\ &\le \liminf_{i\longrightarrow\infty} V(u_0, u_{n_i}) = \lim_{i\longrightarrow\infty} V(u_0, u_{n_i}) = 0. \end{aligned}$$

From the property of V, we obtain $y_0 = u_0$. This implies that $\{y_n\}$ converges weakly to u_0. By the argument as above, since E has the Kadec–Klee property, we immediately see that $\{y_n\}$ converges strongly to u_0. Furthermore, since J is norm-to-norm continuous, we also see that $\{Ju_n - Jy_n\}$ converges strongly to 0. For each $(z, z^*) \in A$, we obtain from (4.1) that

$$\left\langle z - y_n, z^* - \frac{Ju_n - Jy_n}{r_n}\right\rangle \ge 0$$

for each $n \in \mathbb{N}$. Letting $n \longrightarrow \infty$, from $\liminf_{n\longrightarrow\infty} r_n > 0$, we get $\langle z - u_0, z^* - 0\rangle \ge 0$. From the maximality of A, we obtain $u_0 \in A^{-1}0$. ☐

Theorem 4.3. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec–Klee property. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0, \infty[$ satisfying $\liminf_{n\longrightarrow\infty} r_n > 0$. Consider an iteration procedure as below: Let $x_0 \in E$, $C_1 = D_1 = E$, $x_1 = P_{C_1}x_0$ and $u_1 \in D_1$. For each $n \in \mathbb{N}$, define y_n, C_{n+1}, x_{n+1}, D_{n+1} and u_{n+1} by*

$$\begin{aligned} y_n &= Q_{r_n}u_n, \\ C_{n+1} &= \{z \in C_n : \langle y_n - z, Ju_n - Jy_n\rangle \ge 0\}, \end{aligned}$$

$$x_{n+1} = P_{C_{n+1}} x_0,$$
$$D_{n+1} = \{y \in C_n : \|x_0 - y\| \le \|x_0 - x_{n+1}\|, y \ne u_n\},$$
$$u_{n+1} \in D_{n+1}.$$

Then, either of the following holds:

(i) $D_n \neq \emptyset$ *for each* $n \in \mathbb{N}$*; the procedure is not stopped. In this case,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{A^{-1}0} x_0$*.*

(ii) $D_k = \emptyset$ *for some* $k \in \mathbb{N} \setminus \{1\}$*; the procedure is stopped. In this case,* $u_{k-1} \in A^{-1}0$ *holds.*

Proof. From $C_1 = D_1 = E$, C_1 and D_1 are nonempty. Let $u_1 \in D_1 = E$. Then we generate y_1 and C_2. From Lemma 4.1, C_2 is nonempty, closed, and convex, and also generates x_2 and D_2. D_2 may be empty. In the case of $D_2 \neq \emptyset$, we can find $u_2 \in D_2$ and generate y_2 and C_3. From Lemma 4.1, C_3 is nonempty, closed, and convex, and also generates x_3 and D_3. D_3 may be empty. In the case of $D_3 \neq \emptyset$, we can find $u_3 \in D_3$ and continue this process. So, the procedure is stopped when we meet $k \in \mathbb{N} \setminus \{1\}$ satisfying $D_k = \emptyset$.

In the case of (1), we can generate the sequences $\{y_n\}$, $\{C_n\}$, $\{x_n\}$, $\{D_n\}$, and $\{u_n\}$ inductively. By our generating method, from Lemma 4.2 (1), $\{C_n\}$ is a sequence of closed convex subsets of E satisfying $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$ and $\emptyset \neq A^{-1}0 \subset C_0 = \cap_{n\in\mathbb{N}} C_n$. For each $n \in \mathbb{N}$, let K_n be as in Lemma 2.2, that is,

$$K_n = \{z \in C_n : \|x_0 - z\| \le \|x_0 - x_{n+1}\|\}.$$

Then, $u_{n+1} \in D_{n+1} \subset K_n$ for each $n \in \mathbb{N}$. By Lemma 2.2, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{C_0} x_0$. Since $\{u_n\}$ converges strongly to $P_{C_0} x_0 \in C_0$, by Lemma 4.2 (2), we see $P_{C_0} x_0 \in A^{-1}0$. From $A^{-1}0 \subset C_0$, we get $P_{C_0} x_0 = P_{A^{-1}0} x_0$, which completes the proof of (1).

We show (2). Suppose that we generated u_{k-1}, y_{k-1}, C_k, x_k and $D_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$. Then, C_k and C_{k-1} are nonempty, closed, and convex. Since $K_{k-1} = \{z \in C_{k-1} : \|x_0 - z\| \le \|x_0 - x_k\|\}$, we see $x_{k-1}, x_k \in K_{k-1}$ and $K_{k-1} \neq \emptyset$. By $K_{k-1} \setminus \{u_{k-1}\} = D_k = \emptyset$, we see that $u_{k-1} \in K_{k-1}$ and K_{k-1} is a singleton. From these, $u_{k-1} = x_{k-1} = x_k$ holds. Thus, by $u_{k-1} = x_k \in C_k$, we see

$$0 \le \langle y_{k-1} - u_{k-1}, Ju_{k-1} - Jy_{k-1} \rangle.$$

Since J is monotone, we also see

$$0 \ge \langle y_{k-1} - u_{k-1}, Ju_{k-1} - Jy_{k-1} \rangle.$$

Hence, we have $\langle u_{k-1} - y_{k-1}, Ju_{k-1} - Jy_{k-1} \rangle = 0$. From the property of J, we see that $u_{k-1} = y_{k-1} = Q_{r_{k-1}} u_{k-1}$. By (4.2), we get $u_{k-1} \in A^{-1}0$. This completes the proof. □

5. Approximation theorem for the resolvents of Type (R)

Let $r > 0$ and $B \subset E^* \times E$ be a maximal monotone operator. Then it is known that $R(I + rBJ) = E$ for all $r > 0$; see [5, 11, 26]. We can define the following single-valued mapping

$$R_r := (I + rBJ)^{-1} : E \longrightarrow D(BJ)$$

for each $r > 0$. It is called the resolvent of type (R) of B for $r > 0$; see [11]. It is known that the following: Let Q_r^* be a mapping from E^* to E^* defined by

$$Q_r^* := JR_rJ^{-1}.$$

Then Q_r^* is the resolvent of type (Q) of B for $r > 0$ on E^*; see [3, 7].

Note that the metric projection of E^* onto a nonempty closed convex subset K^* of E^* is denoted by $P^*_{K^*}$.

Theorem 5.1. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec–Klee property. Let $B \subset E^* \times E$ be a maximal monotone operator such that $B^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0, \infty[$ satisfying $\liminf_{n\longrightarrow\infty} r_n > 0$. Consider an iteration procedure as below: Let $x_0 \in E$, $C_1 = D_1 = E$, $x_1 = J^{-1}P^*_{JC_1}Jx_0$ and $u_1 \in D_1$. For each $n \in \mathbb{N}$, define y_n, C_{n+1}, x_{n+1}, D_{n+1} and u_{n+1} by*

$$\begin{aligned}
y_n &= R_{r_n} u_n,\\
C_{n+1} &= \{z \in C_n : \langle u_n - y_n, Jy_n - Jz\rangle \geq 0\},\\
x_{n+1} &= J^{-1}P^*_{JC_{n+1}}Jx_0,\\
D_{n+1} &= \{y \in C_n : \|Jx_0 - Jy\| \leq \|Jx_0 - Jx_{n+1}\|, y \neq u_n\},\\
u_{n+1} &\in D_{n+1}.
\end{aligned}$$

Then, either of the following holds:

(i) *$D_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $J^{-1}P^*_{B^{-1}0}Jx_0$.*

(ii) *$D_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in (BJ)^{-1}0$ holds.*

Proof. Put $u^*_n := Ju_n$, $y^*_n := Jy_n$, $x^*_n := Jx_n$, $C^*_n := JC_n$ and $D^*_n := JD_n$ for each $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
y^*_n &= Q^*_r u^*_n,\\
C^*_{n+1} &= \{z^* \in C^*_n : \langle J^{-1}u^*_n - J^{-1}y^*_n, y^*_n - z^*\rangle \geq 0\},\\
x^*_{n+1} &= P^*_{C^*_{n+1}}Jx_0,\\
D^*_{n+1} &= \{y^* \in C^*_n : \|Jx_0 - y^*\| \leq \|Jx_0 - x^*_{n+1}\|, y^* \neq u^*_n\},\\
u^*_{n+1} &\in D^*_{n+1}
\end{aligned}$$

for each $n \in \mathbb{N}$. Then $\{u^*_n\}$, $\{y^*_n\}$, $\{x^*_n\}$, $\{C^*_n\}$, and $\{D^*_n\}$ satisfy the conditions of Theorem 4.3 on E^*.

In the case of (1), if $D_n \neq \emptyset$ for each $n \in \mathbb{N}$, then $D^*_n \neq \emptyset$. From Theorem 4.3 (1), we obtain that $\{x^*_n\}$ and $\{u^*_n\}$ converge strongly to $P^*_{B^{-1}0}Jx_0$. Since E is reflexive, strictly convex, and has the Kadec–Klee property, E^* has Fréchet differentiable norm and hence the duality mapping J^{-1} on E^* is norm-to-norm continuous. Therefore, we see that

$$u_n = J^{-1}u^*_n \longrightarrow J^{-1}P^*_{B^{-1}0}Jx_0 \quad \text{and} \quad x_n = J^{-1}x^*_n \longrightarrow J^{-1}P^*_{B^{-1}0}Jx_0.$$

Next, in the case of (2), if $D_k = \emptyset$ for some $k \in \mathbb{N}\setminus\{1\}$, then $D^*_k = \emptyset$. From Theorem 4.3 (2), we have $Ju_{k-1} = u^*_{k-1} \in B^{-1}0$. This implies $u_{k-1} \in (BJ)^{-1}0$. □

6. Applications

In this section, we consider the convex minimization problem: Let E be a reflexive, smooth, and strictly convex Banach space and its dual E^*. Let $f : E \longrightarrow]-\infty, \infty]$ and $f^* : E^* \longrightarrow]-\infty, \infty]$ be proper lower semicontinuous convex functions. Then, the subdifferentials of f and f^* are defined as follows:

$$\begin{aligned}
\partial f(x) &= \{x^* \in E^* : f(x) + \langle y - x, x^*\rangle \leq f(y),\ \forall y \in E\} \quad (\forall x \in E),\\
\partial f^*(x^*) &= \{x \in E : f^*(x^*) + \langle x, y^* - x^*\rangle \leq f^*(y^*),\ \forall y^* \in E^*\} \quad (\forall x^* \in E^*).
\end{aligned}$$

By Rockafellar's theorem [23,24], the subdifferentials $\partial f \subset E\times E^*$ and $\partial f^* \subset E^*\times E$ are maximal monotone. It is easy to see that

$$(\partial f)^{-1}0 = \operatorname{argmin}\{f(x) : x \in E\} \text{ and } (\partial f^*)^{-1}0 = \operatorname{argmin}\{f^*(x^*) : x^* \in E^*\}.$$

Fix $r > 0$ and $z \in E$. Let P_r and Q_r be the resolvent of ∂f, and let R_r be the resolvent of ∂f^*. Then we know that

$$\begin{aligned}
P_r z &= (I + rJ^{-1}\partial f)^{-1} z = \operatorname*{argmin}_{y\in E} \left\{ f(y) + \frac{1}{2r}\|y - z\|^2 \right\},\\
Q_r z &= (J + r\partial f)^{-1} Jz = \operatorname*{argmin}_{y\in E} \left\{ f(y) + \frac{1}{2r}\|y\|^2 - \frac{1}{r}\langle y, Jz\rangle \right\},\\
R_r z &= (I + r\partial f^*)^{-1} z = J^{-1} \operatorname*{argmin}_{y^*\in E^*} \left\{ f^*(y^*) + \frac{1}{2r}\|y^*\|^2 - \frac{1}{r}\langle z, y^*\rangle \right\}.
\end{aligned}$$

See, for instance, [8,10,25]. As a direct consequence of theorems 3.3, 4.3, and 5.1, we can show the following applications.

Corollary 6.1. *Let E be a reflexive, smooth, and strictly convex Banach space which has the Kadec–Klee property. Let $f : E \longrightarrow]-\infty,\infty]$ be a proper lower semicontinuous convex function such that $(\partial f)^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0,\infty[$ satisfying $\liminf_{n\longrightarrow\infty} r_n > 0$. Consider an iteration procedure as below: Let $x_0 \in E$, $C_1 = D_1 = E$, $x_1 = P_{C_1}x_0$ and $u_1 \in D_1$. For each $n \in \mathbb{N}$, define y_n, C_{n+1}, x_{n+1}, D_{n+1} and u_{n+1} by*

$$\begin{aligned}
y_n &= \operatorname*{argmin}_{y\in E} \left\{ f(y) + \frac{1}{2r_n}\|y - u_n\|^2 \right\},\\
C_{n+1} &= \{z \in C_n : \langle y_n - z, J(u_n - y_n)\rangle \geq 0\},\\
x_{n+1} &= P_{C_{n+1}}x_0,\\
D_{n+1} &= \{y \in C_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq u_n\},\\
u_{n+1} &\in D_{n+1}.
\end{aligned}$$

Then, either of the following holds:

- (i) *$D_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{(\partial f)^{-1}0}x_0$.*
- (ii) *$D_k = \emptyset$ for some $k \in \mathbb{N}\setminus\{1\}$; the procedure is stopped. In this case, $u_{k-1} \in (\partial f)^{-1}0$ holds.*

Corollary 6.2. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec–Klee property. Let $f : E \longrightarrow]-\infty,\infty]$ be a proper lower semicontinuous convex function such that $(\partial f)^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0,\infty[$ satisfying $\liminf_{n\longrightarrow\infty} r_n > 0$. Consider an iteration procedure as below: Let $x_0 \in E$, $C_1 = D_1 = E$, $x_1 = P_{C_1}x_0$ and $u_1 \in D_1$. For each $n \in \mathbb{N}$, define y_n, C_{n+1}, x_{n+1}, D_{n+1} and u_{n+1} by*

$$\begin{aligned}
y_n &= \operatorname*{argmin}_{y\in E} \left\{ f(y) + \frac{1}{2r_n}\|y\|^2 - \frac{1}{r_n}\langle y, Ju_n\rangle \right\},\\
C_{n+1} &= \{z \in C_n : \langle y_n - z, Ju_n - Jy_n\rangle \geq 0\},\\
x_{n+1} &= P_{C_{n+1}}x_0,\\
D_{n+1} &= \{y \in C_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq u_n\},\\
u_{n+1} &\in D_{n+1}.
\end{aligned}$$

Then, either of the following holds:

(i) *$D_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{(\partial f)^{-1}0} x_0$.*

(ii) *$D_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in (\partial f)^{-1}0$ holds.*

Corollary 6.3. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec–Klee property. Let $f^* : E^* \longrightarrow]-\infty, \infty]$ be a proper lower semicontinuous convex function such that $(\partial f^*)^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a sequence in $]0, \infty[$ satisfying $\liminf_{n\longrightarrow\infty} r_n > 0$. Consider an iteration procedure as below: Let $x_0 \in E$, $C_1 = D_1 = E$, $x_1 = J^{-1}P^*_{JC_1} Jx_0$ and $u_1 \in D_1$. For each $n \in \mathbb{N}$, define y_n, C_{n+1}, x_{n+1}, D_{n+1} and u_{n+1} by*

$$\begin{aligned}
y_n &= J^{-1} \operatorname*{argmin}_{y^* \in E^*} \left\{ f^*(y^*) + \frac{1}{2r_n}\|y^*\|^2 - \frac{1}{r_n}\langle u_n, y^*\rangle \right\},\\
C_{n+1} &= \{z \in C_n : \langle u_n - y_n, Jy_n - Jz\rangle \geq 0\},\\
x_{n+1} &= J^{-1}P^*_{JC_{n+1}} Jx_0,\\
D_{n+1} &= \{y \in C_n : \|Jx_0 - Jy\| \leq \|Jx_0 - Jx_{n+1}\|, y \neq u_n\},\\
u_{n+1} &\in D_{n+1}.
\end{aligned}$$

Then, either of the following holds:

(i) *$D_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $J^{-1}P^*_{(\partial f^*)^{-1}0} Jx_0$.*

(ii) *$D_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in (\partial f^* J)^{-1}0$ holds.*

7. Examples

In this section, we present some examples to support the main issue. In advance, recall that weak and strong topologies on a Euclidean space coincide. We first provide a numerical example for the case where the procedure is not stopped.

Example 7.1. In Theorem 3.3 let $E = \mathbb{R}$ and let $A : \mathbb{R} \longrightarrow \mathbb{R}$ be an operator defined by $Ax = x$ for each $x \in \mathbb{R}$. It is easy to see that A is maximal monotone with $A^{-1}0 = \{0\}(\neq \emptyset)$ and the resolvent of type (P) for $r > 0$ is

$$P_r x = \frac{1}{1+r}x$$

for each $x \in \mathbb{R}$. For the sequence $\{r_n\}$, we consider the following three cases:

$$(1)\ r_n := 1, \quad (2)\ r_n := 1 + \frac{1}{n}, \quad (3)\ r_n := 1 - \frac{1}{n+1}$$

for each $n \in \mathbb{N}$. It is also easy to see that $\lim_{n\longrightarrow\infty} r_n = 1$ in all cases. Let $x_0 \in]0, \infty[$, $C_1 = D_1 = \mathbb{R}$, $x_1 (= P_{C_1}x_0) = x_0$, and $u_1 \in]0, x_0]$. Then, in the case where $\{r_n\}$ is (j) $(j = 1, 2, 3)$, we see that for each $n \in \mathbb{N}$,

$$\begin{cases}
y_n = \alpha_n^{(j)} u_n, \quad C_{n+1} = \big]-\infty, \alpha_n u_n\big], \qquad x_{n+1} = \alpha_n^{(j)} u_n < \alpha_{n-1}^{(j)} u_{n-1},\\
D_{n+1} = \begin{cases} \big[\alpha_1^{(j)} u_1, u_1\big[\cup \big]u_1, 2x_0 - \alpha_1^{(j)} u_1\big] & (n = 1),\\ \big[\alpha_n^{(j)} u_n, \alpha_{n-1}^{(j)} u_{n-1}\big] & (n = 2, 3, 4, \ldots),\end{cases}\\
u_{n+1} = c^{(j)} u_n \in D_{n+1}, \quad 0 < u_{n+1} < u_n \leq x_0
\end{cases}$$

where $c^{(1)} = \dfrac{2}{3}$, $c^{(2)} = \dfrac{1}{2}$, $c^{(3)} = \dfrac{9}{13}$, $\alpha_n^{(1)} = \dfrac{1}{2}$, $\alpha_n^{(2)} = \dfrac{n}{2n+1}$ and $\alpha_n^{(3)} = \dfrac{n+1}{2n+1}$ for each $n \in \mathbb{N}$.

Now, we present the following algorithm for finding a zero point of A as demonstrated in Example 7.1.

Algorithm for the case where $\{r_n\}$ is (j) $(j = 1, 2, 3)$	
Step 1:	Choose $x_0 \in]0, \infty[$ and $u_1 \in]0, x_0]$. Put $x_1 := x_0$.
Step 2:	Update $x_{n+1} := \alpha_n^{(j)} u_n$ and $u_{n+1} := c^{(j)} u_n$.
Step 3:	Put $n := n + 1$ and return to Step 2.

We first test for differences in the behavior of x_n and u_n for each case (j) of the sequence $\{r_n\}$, where $j = 1, 2, 3$. The initial point is chosen to be $(x_0, u_1) = (10, 10)$, and the results of this algorithm are shown in Figures 1 to 3.

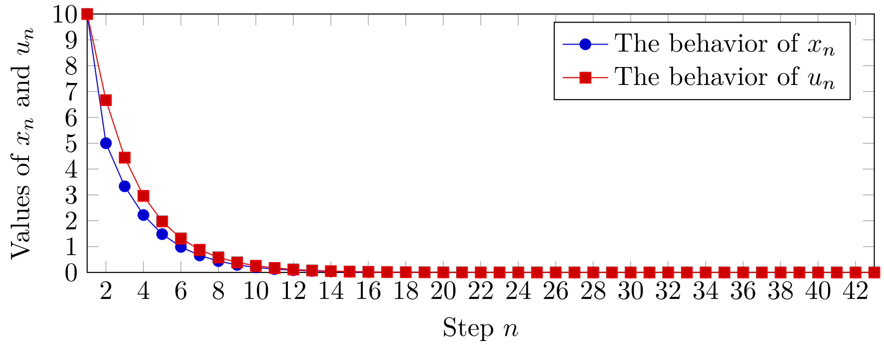

FIGURE 1. In the case where $r_n = 1$ and $(x_0, u_1) = (10, 10)$

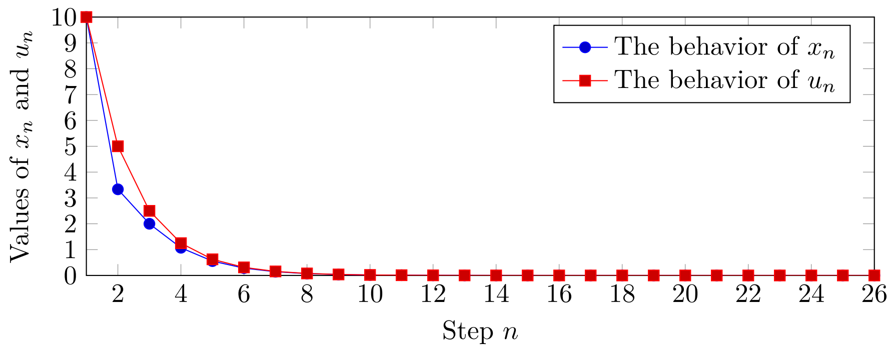

FIGURE 2. In the case where $r_n = 1 + \frac{1}{n}$ and $(x_0, u_1) = (10, 10)$

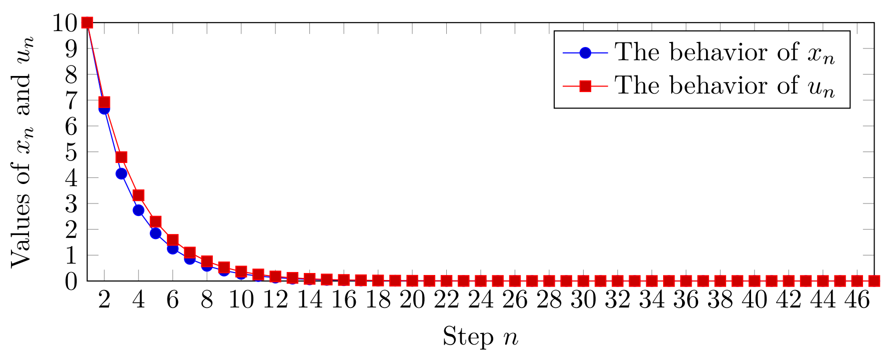

FIGURE 3. In the case where $r_n = 1 - \frac{1}{n+1}$ and $(x_0, u_1) = (10, 10)$

Furthermore, the respective behaviors of x_n and u_n are compared in three cases of the sequence $\{r_n\}$, and the results of this algorithm comparison are shown in Figures 4 and 5.

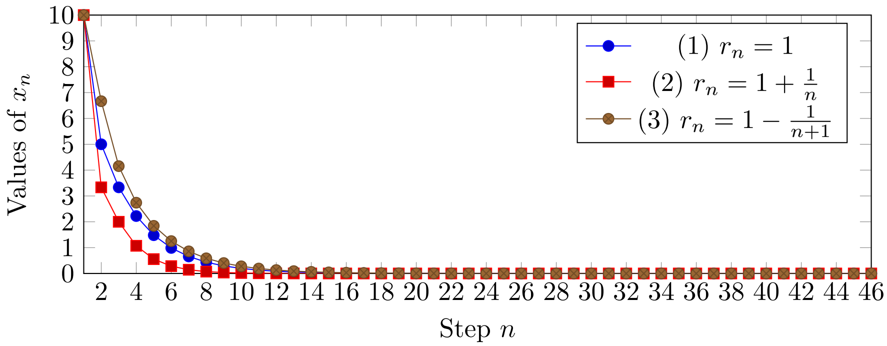

FIGURE 4. The behavior of x_n in the case $(x_0, u_1) = (10, 10)$

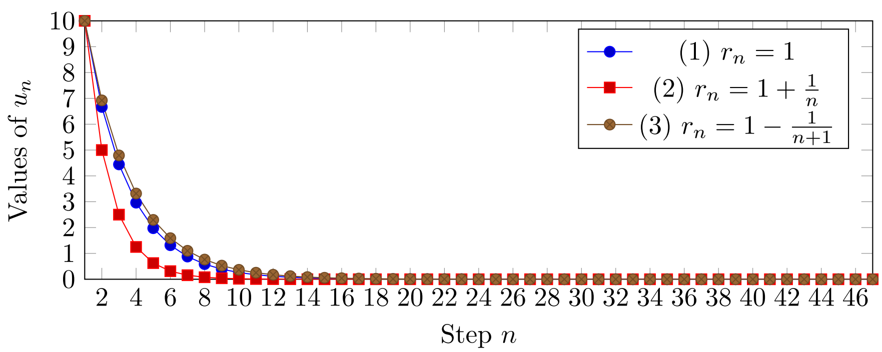

FIGURE 5. The behavior of u_n in the case $(x_0, u_1) = (10, 10)$

Table 1 shows the calculation results for sequences $\{x_n\}$ and $\{u_n\}$ in the case where $\{r_n\}$ is (j) $(j = 1, 2, 3)$, where the initial point is $(x_0, u_1) = (10, 10)$. The numbers in Table 1 are rounded to the seventh decimal place.

TABLE 1. The values of sequences $\{x_n\}$ and $\{u_n\}$

Step	x_n			u_n		
	case (1)	case (2)	case (3)	case (1)	case (2)	case (3)
1	10.000000	10.000000	10.000000	10.000000	10.000000	10.000000
2	5.000000	3.333333	6.666667	6.666667	5.000000	6.923077
3	3.333333	2.000000	4.153846	4.444444	2.500000	4.792899
4	2.222222	1.071429	2.738800	2.962963	1.250000	3.318161
5	1.481481	0.555556	1.843423	1.975309	0.625000	2.297188
⋮	⋮	⋮	⋮	⋮	⋮	⋮
25	0.000446	0.000001	0.001083	0.000594	0.000001	0.001470
26	0.000297	0.000000	0.000749	0.000396	0.000000	0.001017
⋮	⋮	⋮	⋮	⋮	⋮	⋮
41	0.000001	0.000000	0.000003	0.000001	0.000000	0.000004
42	0.000000	0.000000	0.000002	0.000001	0.000000	0.000003
43	0.000000	0.000000	0.000001	0.000000	0.000000	0.000002
44	0.000000	0.000000	0.000001	0.000000	0.000000	0.000001
45	0.000000	0.000000	0.000001	0.000000	0.000000	0.000001
46	0.000000	0.000000	0.000000	0.000000	0.000000	0.000001
47	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮

From these comparisons, both x_n and u_n converge in the fewest number of steps in case (2). Next, for case (2) of the sequence $\{r_n\}$, we choose the initial points $(x_0, u_1) = (10, 10), (5, 5), (1, 1)$ to test the effect of the initial points on the solution. The behavior of x_n and u_n for this algorithm is shown in Figures 6 and 7.

Table 2 shows the calculation results for sequences $\{x_n\}$ and $\{u_n\}$ in the case where $\{r_n\}$ is (2) with initial points $(x_0, u_1) = (10, 10), (5, 1), (1, 1)$. The numbers in Table 2 are rounded to the seventh decimal place.

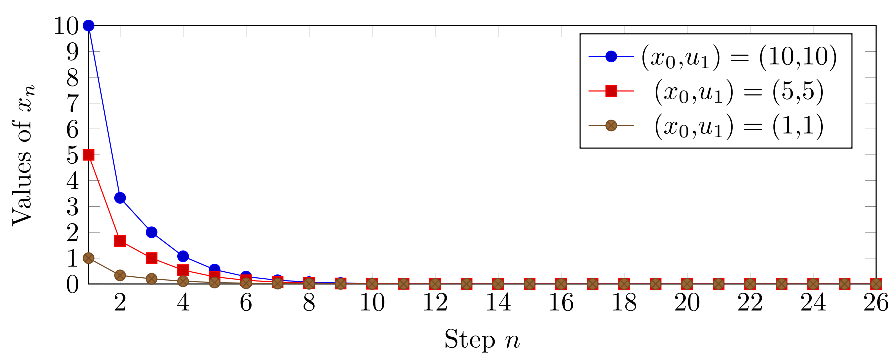

FIGURE 6. The behavior of x_n in the case $r_n = 1 + \frac{1}{n}$

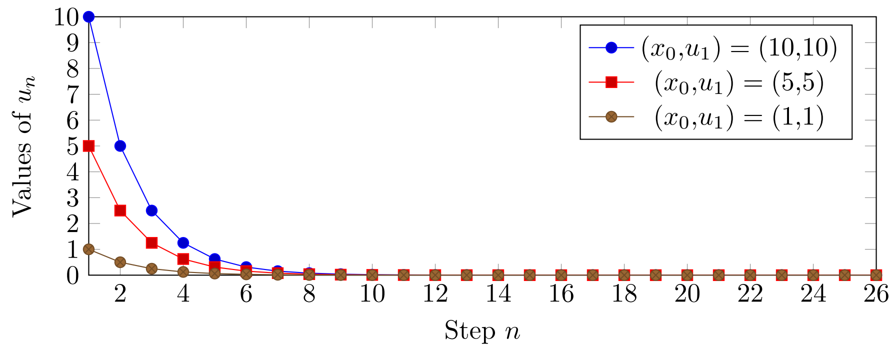

FIGURE 7. The behavior of x_n in the case $r_n = 1 + \frac{1}{n}$

TABLE 2. The values of sequences $\{x_n\}$ and $\{u_n\}$ in case (2)

Step	x_n			u_n		
	$(10, 10)$	$(5, 5)$	$(1, 1)$	$(10, 10)$	$(5, 5)$	$(1, 1)$
1	10.000000	5.000000	1.000000	10.000000	5.000000	1.000000
2	3.333333	1.666667	0.333333	5.000000	2.500000	0.500000
3	2.000000	1.000000	0.200000	2.500000	1.250000	0.250000
4	1.071429	0.535714	0.107143	1.250000	0.625000	0.125000
5	0.555556	0.277778	0.055556	0.625000	0.312500	0.062500
⋮	⋮	⋮	⋮	⋮	⋮	⋮
21	0.000009	0.000005	0.000001	0.000010	0.000005	0.000001
22	0.000005	0.000002	0.000000	0.000005	0.000002	0.000000
23	0.000002	0.000001	0.000000	0.000002	0.000001	0.000000
24	0.000001	0.000001	0.000000	0.000001	0.000001	0.000000
25	0.000001	0.000000	0.000000	0.000001	0.000000	0.000000
26	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Finally, we provide an example for the case where the procedure is stopped.

Example 7.2. In Theorem 3.3 let $E = \mathbb{R}$, let $A \subset \mathbb{R} \times \mathbb{R}$ be an operator defined by

$$Ax := \begin{cases} 1 & x > 0 \\ [-1, 1] & x = 0 \\ -1 & x < 0 \end{cases} \tag{7.1}$$

and let $\{r_n\}$ be a sequence defined by $r_n = 1$ for each $n \in \mathbb{N}$. It is to see that A is maximal monotone with $A^{-1}0 = \{0\}(\neq \emptyset)$ and the resolvent of type (P) for 1 is

$$P_1 x = \begin{cases} x - 1 & x > 1 \\ 0 & -1 \leq x \leq 1 \\ x + 1 & x < -1 \end{cases}. \tag{7.2}$$

Let $x_0 \in]1, \infty[$, $C_1 = D_1 = \mathbb{R}$, $x_1 (= P_{C_1} x_0) = x_0$ and $u_1 \in \left]\frac{1}{2}, 1\right]$. From (7.2), we see that

$$y_1 = 0,\ C_2 = \left]-\infty, 0\right],\ x_2 = 0,\ D_2 = \left[0, u_1\right[\cup \left]u_1, 2x_0\right].$$

We can find $u_2 \in \left]0, u_1\right[\subset D_2$. Similarly, we see that

$$y_2 = 0,\ C_3 = \left]-\infty, 0\right],\ x_3 = 0,\ D_3 = \{0\}.$$

We can only find $u_3 = 0 \in D_3$. Finally, we see that

$$y_3 = 0,\ C_4 = \left]-\infty, 0\right],\ x_4 = 0,\ D_4 = \emptyset.$$

So, the procedure is stopped. In fact, we see that $u_3 = 0 \in A^{-1}0$. Furthermore, as in the proof of Theorem 3.3, we find that $u_{k-1} = x_{k-1} = x_k$ when $D_k \neq \emptyset$ for some $k \in \mathbb{N}$. We also see that $u_3 = x_3 = x_4 = 0$.

Remark 7.3. We note that Examples 7.1 and 7.2 are instances of Theorems 4.3 and 5.1. In Example 7.1, the resolvents of types (P) and (Q) are identical, which means that Example 7.1 is an instance of Theorem 4.3. Furthermore, the duality mapping J is the identity mapping on $\mathbb{R}$. If $B := A$, then the resolvents of types (P) and (R) coincide, which means that Example 7.1 is an instance of Theorem 5.1. Example 7.2 can be shown in the same way.

We also note that Examples 7.1 and 7.2 are instances of Corollaries 6.1, 6.2 and 6.3. In fact, let $E = \mathbb{R}$ and define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \frac{1}{2}x^2$ for each $x \in \mathbb{R}$. We see that $\partial f = A$, where the operator A is the one that appears in Example 7.1, which means that Example 7.1 is an instance of Corollaries 6.1, 6.2 and 6.3. Furthermore, let $E = \mathbb{R}$ and define a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = |x|$ for each $x \in \mathbb{R}$. We also see that $\partial f = A$, where the operator A is the one that appears in Example 7.2, which means that Example 7.1 is an instance of Corollaries 6.1, 6.2 and 6.3.

8. Conclusions

In this paper, we considered proximal-type algorithms for finding a zero point of maximal monotone operators defined on a Banach space. We deal with three different types of resolvent operators, which are called the resolvents of types (P), (Q), and (R). We obtained strong convergences for each iterative scheme generated by the shrinking projection method with allowable ranges [29] in a Banach space. Using our results, we discussed our results for finding a minimizer of a convex function defined on a Banach space. Finally, we provide some examples to support the main theorems.

In the original result [29] of the shrinking projection method with allowable ranges, a strong convergence theorem was obtained for finding a fixed point of mappings of type (P) in a Banach space. The resolvents of type (P), (Q), and (R) are mappings of type (P), (Q), and (R), respectively. It would be interesting to use our work for fixed point approximations for mappings of types (Q) and (R), which are generalizations of firmly nonexpansive mappings defined on a Hilbert space.

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