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SOLVING FREDHOLM INTEGRAL EQUATION VIA FIXED POINT THEOREM IN CONTROLLED METRIC TYPE SPACES

ANKIT BAMEL[1] AND VIZENDER SIHAG*[2]

[1] Research Scholar, Department of Mathematics, Guru Jambheshwar University of Science & Technology, Hisar, Haryana(INDIA)

[2] Assistant Professor, Department of Mathematics, Guru Jambheshwar University of Science & Technology, Hisar, Haryana(INDIA)

ABSTRACT. In this paper, we present the concepts of hesitant fuzzy mapping, contraction of hesitant fuzzy mapping, and generalized contraction of hesitant fuzzy mapping within controlled metric-type spaces. These concepts are used to establish fixed-point theorems. Finally, the results are applied to solve a Fredholm-type integral equation.

KEYWORDS: Fixed point, Hesitant fuzzy set, Controlled metric type space, Hesitant fuzzy mapping.
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1. Introduction

Fixed point theory is a foundational branch of mathematical analysis with broad applications in nonlinear analysis, optimization, differential and integral equations, game theory, and computer science. One of the earliest contributions to this theory was the Banach Contraction Principle (BCP), introduced by Banach [3] in 1922, which guarantees the existence and uniqueness of fixed points for contraction mappings in complete metric spaces. This result has since inspired many generalizations and extensions to accommodate more complex structures and mapping behaviors.

The classical concept of a metric space, formalized by Fréchet [13], was generalized over time to reflect diverse real-world phenomena. For instance, quasi-metric spaces introduced by Wilson [30] drop the symmetry condition, while b-metric spaces (Czerwik, 1993 [5]) relax the triangle inequality. These generalizations have motivated a deeper investigation into fixed point theory in non-traditional metric-like structures.

* *Vizender Sihag* .
Email address : ankitbamel2015@gmail.com, vsihag3@gmail.com.
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One such recent development is the *controlled metric-type space*, introduced by Mlaiki [20] in 2018. This framework utilizes a control function $\psi(\zeta, \kappa)$ to modify the standard triangle inequality, leading to a more flexible and generalized space where:

$$d(\zeta, \varrho) \leq \psi(\zeta, \kappa)d(\zeta, \kappa) + \psi(\kappa, \varrho)d(\kappa, \varrho), \quad \forall \zeta, \kappa, \varrho \in W.$$

Unlike traditional metric or even b-metric spaces, the use of a control function allows the modeling of distance with adaptive sensitivity, particularly useful in systems influenced by varying local dynamics, uncertainty, or partial observability. Such extensions provide a meaningful foundation for analyzing mappings in abstract and applied contexts.

Parallel to developments in generalized spaces, the concept of fuzziness in mathematical modeling has played a key role in capturing uncertainty. Since Zadeh's introduction of fuzzy sets in 1965 [32], many variants have emerged, including hesitant fuzzy sets (HFS), proposed by Torra [29], which allow multiple possible membership values for a single element. This feature makes HFS suitable for modeling indecision or hesitation in complex systems, such as those found in social networks, decision theory, and artificial intelligence.

It is important to emphasize that the proposed methodology is largely independent of whether the underlying equations are fractional, integral, or classical differential in nature, and is also unaffected by the specific form of nonlinearity embedded within the integral operators. In contrast to several existing approaches that impose stringent conditions on the input data to ensure the well-posedness of integral equations—often through advanced fixed point theory—our framework operates under more general assumptions. Numerous studies have addressed existence, uniqueness, and optimal control of integral and fractional differential equations using such tools (see [6, 7, 8, 9, 16, 24, 28, 27]). Furthermore, integral equations, while mathematically rich, pose additional difficulties in deriving approximate or optimal solutions, which have been extensively explored in recent literature through the development of numerical and approximation methods. Many of these methods rely on transforming differential problems into integral forms to apply fixed point results, with detailed convergence analyses available in the latest works (see [1, 10, 11, 12, 26, 23, 15, 17, 18, 25]).

Hesitant fuzzy mappings have recently been studied in the context of fixed point theory. Osawaru [21] and Bamel and Sihag [2] examined fixed points of such mappings in various general metric settings, including b-metric spaces. However, to the best of our knowledge, no work has yet combined hesitant fuzzy mappings with the controlled metric-type space structure, nor linked this setting to the solution of Fredholm integral equations, which are central to modeling a wide range of physical systems, such as heat conduction, wave propagation, and quantum mechanics.

This paper aims to fill this gap by developing a new class of fixed point results for hesitant fuzzy mappings within the framework of controlled metric-type spaces. Building upon and extending the foundational ideas of Mlaiki [20] and Bamel and Sihag [2], we introduce a novel contraction condition based on deviation degrees, which is specifically designed to measure variations in hesitant fuzzy membership values while incorporating control-based distance measures. Within this generalized setting, we further establish the existence of solutions to Fredholm integral equations by applying the proposed framework.

The novelty of our work lies in the integration of hesitant fuzzy logic with controlled metric structures and the application to integral equations, which has not been addressed in previous literature. Furthermore, the use of deviation degrees provides a new approach to comparing membership functions and analyzing convergence behavior in fuzzy settings.

Our motivation is twofold: from a theoretical perspective, to enrich fixed point theory in abstract metric-type environments; and from an applied viewpoint, to provide tools for solving real-world problems involving ambiguity, partial knowledge, and complex relational structures features common in control systems, data science, and engineering models governed by integral equations.

2. Preliminaries

Definition 2.1. [20] Let $W(\neq \emptyset)$ a set and $\psi : W \times W \to [1, \infty)$. A distance function $d_\psi : W \times W \to [0, \infty)$ satisfying the following conditions, if $\forall\ \zeta, \kappa, \varrho \in W$:

(i) $d(\zeta, \kappa) = 0$ if and only if $\zeta = \kappa$,
(ii) $d(\zeta, \kappa) = d(\kappa, \zeta)$,
(iii) $d(\zeta, \varrho) \leq \psi(\zeta, \kappa)d(\zeta, \kappa) + \psi(\kappa, \varrho)d(\kappa, \varrho)$.

then (W, d) is known as controlled metric type space. If for all $\zeta, \varrho \in\ W,\ \psi(\zeta, \varrho) = s$ and $s \geq 1$, then it is known as b-metric space and if $\psi(\zeta, \varrho) = s = 1$, then it is called metric space.

Remark 2.2. (i) If, $for\ all\ \psi(\zeta, \kappa) = s \geq 1$, then (W, d) is a b-metric space, which leads us to conclude that every b-metric space is a controlled metric type space. In addition, a controlled metric type space is not in general an extended b-metric space when taking the same function.
(ii) If, $for\ all\ \psi(\zeta, \kappa) = s = 1$, then (W, d) is a metric space.

Torra [29] generalized fuzzy sets due to Zadeh [32] in 1965 by introducing the new notion of hesitant fuzzy logic and hesitant fuzzy sets.

Definition 2.3. [29] Let $W(\neq \emptyset)$ denote a set, and let S represent a collection of finite subsets within the interval $[0, 1]$. A hesitant fuzzy set defined on W is a function $h : W \to S$, where for each element ζ in W, $h(\zeta)$ belongs to S. When h is single-valued for every ζ in W, a hesitant fuzzy set simplifies to a fuzzy set.
Further, we represent $H(W)$ by a collection of hesitant fuzzy set on W.

Xia and Xu [31] introduced a method for comparing hesitant fuzzy memberships by evaluating their scores. They characterized the score of a hesitant membership values $A_1 \in S$ as follows:

$$s(A_1) = \frac{1}{n(A_1)} \sum_{a \in A_1} a.$$

where $n(A_1)$ denotes the cardinality of A_1 and $s(A_1) \in [0, 1]$.

Definition 2.4. [22] Let h be represent a hesitant fuzzy set on W. The α-cut of a hesitant fuzzy set A is defined as:

$$h^A_\alpha = \{\zeta \in W : s(h^A(\zeta)) \geq \alpha\} \text{ for any } \alpha \in (0, 1],$$

and

$$h^A_{\{0\}} = C(\{\zeta \in E : s(h^A(\zeta)) > \{0\}\}) = C(B).$$

with $\alpha = \{0\} \in S$ is known as α-cut (level set) of a hesitant fuzzy set, where $C(B)$ denotes the closure of B.

A relation on hesitant fuzzy membership values is established such that if $s(A_1) > s(A_2)$, then $A_1 > A_2$. Moreover, A_1 is considered similar to A_2 if $s(A_1) = s(A_2)$ for all $A_1, A_2 \in S$. Liao and Xu [19] pointed out that this relationship may not hold true in certain special cases. To address this concern, Chen [4] introduced the concept of deviation degree. The deviation degree of a hesitant fuzzy membership value $A_1 \in S$ is defined as:

$$d(A_1) = \sqrt{\frac{1}{n(A_1)} \sum_{a \in A_1} (a - s(A_1))^2}.$$

They also proposed a comparison for sets of hesitant fuzzy membership values based on the following criteria:

(i) $A_1 < A_2$ if $s(A_1) < s(A_2)$ or if $s(A_1) = s(A_2)$ and $d(A_1) > d(A_2)$,
(ii) $A_1 = A_2$ if $s(A_1) = s(A_2)$ and $d(A_1) = d(A_2)$,
(iii) $A_1 > A_2$ if $s(A_1) = s(A_2)$ and $d(A_1) < d(A_2)$.

Example 2.5. [2] Let $(\mathbb{Z}, d)$ represent a b-metric space and the distance function d is defined as $d(\zeta, \kappa) = |\zeta - \kappa|^2 \ \forall \ \zeta, \kappa \in \mathbb{Z}$. Suppose $h' : W = \{1 \leq \zeta \leq 7\} \to S$ is a hesitant fuzzy map, where

$$h'(\zeta) = \{\frac{1}{s} \in [0,1], s \text{ is a multiple of } \zeta,\ s \leq 12\}.$$

Then, we prove the comparison on sets of hesitant fuzzy membership values using deviation degrees.

Solution: First, we find the value of the hesitant fuzzy map on the interval $[1, 7]$ and then get the score of the hesitant fuzzy membership values and deviation degree of a hesitant fuzzy membership on the interval $[1, 7]$. Finally, we compare the values.

$$h'(1) = \{1, 0.5, 0.33, 0.25, 0.2, 0.17, 0.14, 0.13, 0.11, 0.1, 0.09, 0.08\},$$
$$h'(2) = \{0.5, 0.25, 0.17, 0.13, 0.1, 0.08\},$$
$$h'(3) = \{0.33, 0.17, 0.11, 0.08\},$$
$$h'(4) = \{0.25, 0.13, 0.08\},$$
$$h'(5) = \{0.2, 0.1\},$$
$$h'(6) = \{0.17, 0.08\},$$
$$h'(7) = \{0.14\}.$$

Then,

$$\begin{aligned} s(h'(1)) &= \frac{1}{n(h'(1))} \sum_{a \in h'(1)} a, \\ &= \frac{1}{12}(1 + 0.5 + 0.33 + 0.25 + 0.2 + 0.17 + 0.14 + 0.13 + 0.11 \\ &\quad + 0.1 + 0.09 + 0.08) = 0.26, \end{aligned}$$

$$s(h'(2)) = \frac{1}{6}(0.5 + 0.25 + 0.17 + 0.13 + 0.1 + 0.08) = 0.21,$$
$$s(h'(3)) = \frac{1}{4}(0.33 + 0.17 + 0.11 + 0.08) = 0.17,$$
$$s(h'(4)) = \frac{1}{3}(0.25 + 0.13 + 0.08) = 0.15,$$
$$s(h'(5)) = \frac{1}{2}(0.2 + 0.1) = 0.15,$$
$$s(h'(6)) = \frac{1}{2}(0.17 + 0.08) = 0.13,$$
$$s(h'(7) = 0.14.$$

and

$$d(h^{'}(1)) = \sqrt{\frac{1}{n(h^{'}(1))}\sum_{a\in h^{'}(1)}(a-s(h^{'}(1)))^2},\ d(h^{'}(2)) = \sqrt{\frac{1}{n(h^{'}(2))}\sum_{a\in h^{'}(2)}(a-s(h^{'}(2)))^2},$$

$$d(h^{'}(3)) = \sqrt{\frac{1}{n(h^{'}(3))}\sum_{a\in h^{'}(3)}(a-s(h^{'}(3)))^2},\ d(h^{'}(4)) = \sqrt{\frac{1}{n(h^{'}(4))}\sum_{a\in h^{'}(4)}(a-s(h^{'}(4)))^2},$$

$$d(h^{'}(5)) = \sqrt{\frac{1}{n(h^{'}(5))}\sum_{a\in h^{'}(5)}(a-s(h^{'}(5)))^2},\ d(h^{'}(6)) = \sqrt{\frac{1}{n(h^{'}(6))}\sum_{a\in h^{'}(6)}(a-s(h^{'}(6)))^2},$$

$$d(h^{'}(7)) = \sqrt{\frac{1}{n(h^{'}(7))}\sum_{a\in h^{'}(7)}(a-s(h^{'}(7)))^2}.$$

Let $\alpha = \{0.1, 0.3\}$, then

$$s(\alpha) \ = \ \frac{1}{n(\alpha)}\sum_{a\in\alpha} a \ = \ \frac{1}{2}(0.1+0.3) \ = \ 0.2$$

If we take $\alpha = \{0.1\}$, then $s(0.1) = 0.2$ and $d(0.1) = \frac{1}{\sqrt{5}}$. If we take $\alpha = \{0.3\}$, then $s(0.3) = 0.2$ and $d(0.3) = \frac{1}{\sqrt{15}}$. Therefore, if $\alpha = 0.1 \leq \alpha = 0.3 \implies d(\alpha = 0.1) \geq d(\alpha = 0.3)$.

Definition 2.6. [21] A hesitant fuzzy subset h of W is classified as a hesitant fuzzy approximate quantity iff its α level set is a convex subset of W, for all $\alpha \in [0,1]$ and $sup_{\zeta\in W}\{h(\zeta)^+\} = \{1\}$.

Example 2.7. Let the pair $(\mathbb{Z}, d)$ denote a b-metric space, with the distance function d defined as $d(\zeta,\kappa) = |\zeta-\kappa|^2 \ \forall\ \zeta,\kappa \in \mathbb{Z}$. Suppose $h^{'} : W = \{1 \leq \zeta \leq 4\} \to S$ is a hesitant fuzzy map, where,

$$h^{'}(\zeta) = \{\frac{1}{s} \in [0,1], s \text{ is a multiple of } \zeta,\ s \leq 7\}.$$

Then, $sup_{\zeta\in W}\{h^{'}(\zeta)^+\} = \{1\}$.
Solution: Firstly, we compute the values of the hesitant fuzzy map over the interval $[1,4]$.

$$h^{'}(1) = \{1, 0.5, 0.33, 0.25, 0.2, 0.17, 0.14\},\ h^{'}(2) = \{0.5, 0.25, 0.17\},$$
$$h^{'}(3) = \{0.33, 0.17\},\ h^{'}(4) = \{0.25\}$$

Then, $sup_{\zeta\in W}\{h^{'}(\zeta)^+\} = \{1\}$.

Definition 2.8. [21] Assume s is the coefficient of the b-metric space (W, d) and h is a hesitant fuzzy set on W. The α cut of a hesitant fuzzy set is defined as:

$$h_\alpha = \{\zeta \in W : s(h(\zeta)) \geq \alpha\},$$

for any $\alpha \in (0,1]$, and

$$h_{\{0\}} = C(\{\zeta \in W : s(h(\zeta)) > \{0\}\}),$$

with $\alpha = \{0\} \in S$ is known as α cut of a hesitant fuzzy set, and $C(B)$ denotes the closure of B.

Definition 2.9. [21] Let $U(W) \subset H(W)$ represent a collection of hesitant fuzzy approximate quantities on W. For $h, k \in U(W)$ and $\alpha \in S$, the α set-space of h and k is defined as:

$$p_\alpha(h,k) = inf_{\zeta \in h_\alpha, \kappa \in k_\alpha} d(\zeta, \kappa),$$
$$p(h,k) = sup_\alpha \; p_\alpha(h,k).$$

If $h, k \in U(W)$, then the fuzzy approximate quantity h is deemed more precise than k if $h \subset k$, which is equivalent to $h(\zeta) \leq k(\zeta)$ for every $\zeta \in W$.

Definition 2.10. [21] For $h, k \in U(W)$ and $\alpha = [0,1] \in S$, the α set-distance of h and k is defined as:

$$D_\alpha(h,k) = HD(h_\alpha, k_\alpha),$$

where HD signifies the Hausdorff distance.

Let $h, k \in U(W)$ and $\alpha \in S$. Thus overall distance between h and k is given by

$$D(h,k) = sup_\alpha \; D_\alpha(h,k).$$

Definition 2.11. [21] Let $W(\neq \emptyset)$ be a set with (W,d) be a metric space. The collection $H(W)$ of hesitant fuzzy sets on W have a sub-collection $U(W)$ of hesitant approximate quantities. The hesitant mapping is defined as $H_F : W \to U(W)$ such that $H_F(\zeta) \in U(W)$ for every $\zeta \in W$.

Definition 2.12. [21] Assume $W(\neq \emptyset)$ be a set with (W,d) a metric space. The collection $H(W)$ of hesitant fuzzy sets on W contains a sub-collection $U(W)$ of hesitant approximate quantities. The pair of hesitant fuzzy maps $H_{F_1}, H_{F_2} : W \to U(W)$ is defined such that:

$$\begin{aligned} D(H_{F_1}(\zeta), H_{F_2}(\kappa)) \leq\; & a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\kappa, H_{F_2}(\kappa)) + a_3 p(\kappa, H_{F_1}(\zeta)) \\ & + a_4 p(\zeta, H_{F_2}(\kappa)) + a_5 d(\zeta, \kappa), \end{aligned}$$

for any $\zeta, \kappa \in E$, where $\sum_{i=1}^{5} a_i < 1,\ a_1 = a_2$ or $a_3 = a_4$ $(a_i \in \mathbb{R}^+)$.

Theorem 2.1. [21] Let (W,d) be a metric space and $H_{F_1}, H_{F_2} : W \to U(W)$ hesitant maps such that

$$\begin{aligned} D(H_{F_1}(\zeta), H_{F_2}(\kappa)) \leq\; & a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\kappa, H_{F_2}(\kappa)) + a_3 p(\kappa, H_{F_1}(\zeta)) \\ & + a_4 p(\zeta, H_{F_2}(\kappa)) + a_5 d(\zeta, \kappa), \end{aligned}$$

for any $\zeta, \kappa \in W$, where $\sum_{i=1}^{5} a_i < 1$, and $a_1 = a_2$ or $a_3 = a_4$ $(a_i \in \mathbb{R}^+)$. Then there exists $\zeta^* \in W$ such that $\{\zeta^*\} \in H_{F_1}(\zeta^*)$ and $\{\zeta^*\} \in H_{F_2}(\zeta^*)$ also hold.

Definition 2.13. [21] Assume $W(\neq \emptyset)$ be a set and (W,d) represents a b-metric space. The collection $H(W)$ of hesitant fuzzy sets on W have a sub-collection $U(W)$ of hesitant approximate quantities. The hesitant fuzzy mapping on b-metric space is defined as:

$$H_F : W \to U(W),$$

such that $H_F(\zeta) \in U(W)$ for every $\zeta \in W$.

Definition 2.14. [21] Let s be the coefficient of a b-metric space (W,d). The hesitant fuzzy map $H_F : W \to U(W)$ is known as the contraction of hesitant fuzzy map on b-metric space. If

$$D(H_{F_\zeta}, H_{F_\kappa}) \leq a d(\zeta, \kappa),$$

for any $\zeta, \kappa \in W$, where $a \in (0, \frac{1}{s})$ and $s \geq 1$.

Definition 2.15. [21] Assume $W(\neq \emptyset)$ is a set and s is the coefficient of a b-metric space (W, d). The collection $H(W)$ of hesitant fuzzy sets on W contains a sub-collection $U(W)$ of hesitant approximate quantities. The generalized contraction of hesitant fuzzy maps on b-metric space is defined as:

$$H_{F_1}, H_{F_2} : W \to U(W),$$

such that

$$D(H_{F_1}(\zeta), H_{F_2}(\kappa)) \leq \frac{1}{s}\Big[a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\kappa, H_{F_2}(\kappa)) + a_3 p(\kappa, H_{F_1}(\zeta)) + a_4 p(\zeta, H_{F_2}(\kappa)) + a_5 d(\zeta, \kappa)\Big],$$

for any $\zeta, \kappa \in W$ where $a_1+a_2+s[a_3+a_4]+a_5 < 1$, and $a_1 = a_2$ *or* $a_3 = a_4$ $(a_i \in \mathbb{R}^+)$.

Theorem 2.2. [14] Assume $W(\neq \emptyset)$ be a set and (W, d) be a complete controlled metric space and d is a continuous functional. Assume $T : W \to W$ and $\exists\ \zeta_0 \in W$ such that

$$d(T\kappa, T^2\kappa) \leq kd(\kappa, T\kappa) \text{ for each } \kappa \in O(\zeta_0) = \text{orbit of } \zeta_0,$$

where $k \in (0, 1)$ be such that for $\zeta_0 \in W$, $sup_{m\geq 1} lim_{i\to\infty} \frac{\psi(\zeta_{i+1},\zeta_{i+2})}{\psi(\zeta_i,\zeta_{i+1})}\psi(\zeta_{i+1}, \zeta_m) < \frac{1}{k}$, here $\zeta_n = T^n(\zeta_0), n = 1, 2, 3, \ldots$. Then $T^n\zeta_0 \to \zeta_0 \in W$. Therefore, ζ is a fixed point of T iff $G(\zeta) = d(\zeta, T\zeta)$ is T-"orbitally lower semi-continuous" at ζ.

3. Main Results

In this paper, we examine the novel concept of hesitant and contractive fuzzy mapping within the context of controlled metric-type spaces.

Definition 3.1. Let $W(\neq \emptyset)$ be a set and let (W, d_ψ) denote a controlled metric type space. The collection $H(W)$ of hesitant fuzzy sets on W have a sub-collection $U(W)$ of hesitant approximate quantities. A hesitant fuzzy mapping on a controlled metric type space is defined as $H_F : W \to U(W)$ such that $H_F(\zeta) \in U(W)$ for each $\zeta \in W$.

Definition 3.2. Let (W, d_ψ) be a controlled metric-type space characterized by a coefficient $\psi(\zeta, \kappa)$ where $\psi : W \times W \to [1, \infty)$. Let D be a distance function defined as $D : W \times W \to [0, \infty)$. The hesitant fuzzy map $H_F : W \to U(W)$ is termed a contraction hesitant fuzzy map on controlled metric-type space. If

$$D(H_{F_\zeta}, H_{F_\kappa}) \leq ad_\psi(\zeta, \kappa),$$

for any $\zeta, \kappa \in W$, where $a \in (0, \frac{1}{\psi(\zeta,\kappa)})$.

Remark 3.3.
(i) If $\psi(\zeta, \kappa) = s$ and s ≥ 1, then a contraction hesitant fuzzy map in a controlled metric-type space reduces to a contraction hesitant fuzzy map in a b-metric space.
(ii) If $\psi(\zeta, \kappa) = s$ and s $= 1$, then a contraction hesitant fuzzy map in a controlled metric-type space becomes a contraction hesitant fuzzy map in a metric space.

Definition 3.4. Let $W(\neq \phi)$ be a set and let $\psi(\zeta, \kappa)$ denote the coefficient of a controlled metric type space (W, d_ψ). The collection $H(W)$ of hesitant fuzzy sets on

W has a sub-collection $U(W)$ of hesitant approximate quantities. The generalized contraction of hesitant fuzzy maps is defined for $H_{F_1}, H_{F_2} : W \to U(W)$ such that

$$D(H_{F_1}(\zeta), H_{F_2}(\kappa)) \leq \frac{1}{\psi(\zeta,\kappa)}\Big[a_1 p(\zeta, H_{F_1}(\zeta)) + a_2 p(\kappa, H_{F_2}(\kappa)) + a_3 p(\kappa, H_{F_1}(\zeta)) + a_4 p(\zeta, H_{F_2}(\kappa)) + a_5 d_\psi(\zeta,\kappa)\Big],$$

for any $\zeta, \kappa \in W$, where $a_1 + a_2 + \psi(\zeta,\kappa)[a_3 + a_4] + a_5 < 1$, and $a_1 = a_2$ *or* $a_3 = a_4$ $(a_i \in \mathbb{R}^+)$.

Remark 3.5.
(i) If $\psi(\zeta,\kappa) = s$ and s ≥ 1, then the generalized contraction hesitant fuzzy map in a controlled metric-type space becomes a generalized hesitant fuzzy contraction in a b-metric space.
(ii) If $\psi(\zeta,\kappa) = s$ and s $= 1$, then the generalized contraction hesitant fuzzy map in a controlled metric-type space transforms into a generalized hesitant fuzzy contraction in a metric space.

Lemma 3.6. Assume W be a controlled metric type space with $\zeta \in W, h \in U(W)$ and $\{\zeta\}$ is a hesitant fuzzy set whose hesitant membership function is equal to the hesitant characteristic function of the set $\{\zeta\}$. If $\{\zeta\} \subset h$ then $p_\alpha(\zeta, h) = 0$ for each $\alpha \in S$.
Proof If $\{\zeta\} \subset h$ then $\zeta \in h_\alpha$ for each $\alpha \in S$ and h is an approximate quantity. So, $p_\alpha(\zeta, h) = inf_{\kappa \in h_\alpha} d(\zeta,\kappa) = 0$.

Lemma 3.7. Assume (W, d_ψ) is a controlled metric-type space with coefficient $\psi(\zeta,\kappa)$. Then,

$$p_\alpha(\zeta, h) \leq \psi(\zeta,\kappa) d(\zeta,\kappa) + \psi(\kappa,\varrho) p_\alpha(\kappa, h)],$$

for any $\zeta, \kappa, \varrho \in W$.
Proof We know that

$$\begin{aligned} p_\alpha(\zeta, h) &= inf_{\varrho \in h_\alpha} d(\zeta, \varrho) \\ &\leq inf_{\varrho \in h_\alpha}\ [\psi(\zeta,\kappa) d(\zeta,\kappa) + \psi(\kappa,\varrho) d(\kappa,\varrho)] \\ &\leq \psi(\zeta,\kappa) d(\zeta,\kappa) + \psi(\kappa,\varrho) inf_{\varrho \in h_\alpha} d(\kappa,\varrho) \\ &= \psi(\zeta,\kappa) d(\zeta,\kappa) + \psi(\kappa,\varrho) p_\alpha(\kappa, h). \end{aligned}$$

Lemma 3.8. Let (W, d_ψ) be a controlled metric-type space with coefficient $\psi(\zeta,\kappa)$. If $\{\zeta_0\} \subset h$ and $h \in U(W)$, then for every $k \in U(W)$, we have $p_\alpha(\zeta_0, k) \leq D_\alpha(h, k)$.
Proof We know that

$$\begin{aligned} p_\alpha(\zeta, k) &= inf_{\kappa \in k_\alpha}\ d(\zeta,\kappa) \\ &\leq sup_{\zeta \in h_\alpha} inf_{\kappa \in k_\alpha}\ d(\zeta,\kappa) \\ &\leq D_\alpha(h, k). \end{aligned}$$

Lemma 3.9. Assume (W, d_ψ) is a complete controlled metric-type space with coefficient $\psi(\zeta,\kappa)$ and $h \in U(W)$. Then,

$$p_\alpha(\zeta, h) \leq \psi(\zeta,\kappa) d(\zeta,\kappa),$$

if $\{\kappa\} \subset h$.
Proof: By lemma (3.7), we have:

$$p_\alpha(\zeta, h) \leq \psi(\zeta,\kappa) d(\zeta,\kappa) + \psi(\kappa,\varrho) p_\alpha(\kappa, h).$$

Since $\kappa \in h$, by lemma (3.6), we find that $p_\alpha(\kappa, h) = 0$. Therefore, we conclude:

$$p_\alpha(\zeta, h) \leq \psi(\zeta,\kappa) d(\zeta,\kappa).$$

Theorem 3.1. Assume $W(\neq \emptyset)$ is a complete controlled metric type space. Let H_{F_1} *and* H_{F_2} be hesitant fuzzy mapping from W into $U(W)$. If $\exists$ a constant $a \in [0, 1)$, such that for each $\zeta, \kappa \in W$,

$$D(H_{F_1}(\zeta), H_{F_2}(\kappa)) \leq a\ max\{d(\zeta, \kappa), p_\alpha(\zeta, H_{F_1}(\zeta)), p_\alpha(\kappa, H_{F_2}(\kappa)), \frac{p_\alpha(\zeta, H_{F_2}(\kappa)) + p_\alpha(\kappa, H_{F_1}(\zeta))}{1 + \psi(\zeta, \kappa)}\} \tag{3.1}$$

then $\exists\ \zeta^* \in W$ such that $\zeta^* \subset H_{F_1}(\zeta^*)$ *and* $\zeta^* \subset H_{F_2}(\zeta^*)$.

Proof Let $\zeta_0 \in W$ and $\zeta_1 \subset H_{F_1}(\zeta_0)$. Then $\exists\ \zeta_2 \in W \subset H_{F_2}(\zeta_1)$, and

$$\begin{aligned} d(\zeta_1, \zeta_2) &\leq D_1(H_{F_1}(\zeta_0), H_{F_2}(\zeta_1)) \\ &\leq D(H_{F_1}(\zeta_0), H_{F_2}(\zeta_1)) \\ &\leq \frac{a}{\psi(\zeta_0, \zeta_1)}\ max\{d(\zeta_0, \zeta_1), p_\alpha(\zeta_0, H_{F_1}(\zeta_0)), p_\alpha(\zeta_1, H_{F_2}(\zeta_1)), \\ &\frac{p_\alpha(\zeta_0, H_{F_2}(\zeta_1)) + p_\alpha(\zeta_1, H_{F_1}(\zeta_0))}{1 + \psi(\zeta_0, \zeta_1)}\} \\ &\leq \frac{a}{\psi(\zeta_0, \zeta_1)}\ max\{d(\zeta_0, \zeta_1), d(\zeta_0, \zeta_1), d(\zeta_1, \zeta_2), \frac{d(\zeta_0, \zeta_2) + d(\zeta_1, \zeta_1)}{1 + \psi(\zeta_0, \zeta_1)}\} \\ &\leq \frac{a}{\psi(\zeta_0, \zeta_1)}\ max\{d(\zeta_0, \zeta_1), d(\zeta_1, \zeta_2), \frac{d(\zeta_0, \zeta_2)}{1 + \psi(\zeta_0, \zeta_1)}\} \\ &\leq \frac{a}{\psi(\zeta_0, \zeta_1)}\ max\{d(\zeta_0, \zeta_1), d(\zeta_1, \zeta_2), \frac{\psi(\zeta_0, \zeta_1)d(\zeta_0, \zeta_1) + \psi(\zeta_1, \zeta_2)d(\zeta_1, \zeta_2)]}{1 + \psi(\zeta_0, \zeta_1)}\}. \end{aligned}$$

But we know that

$$\frac{\psi(\zeta_0, \zeta_1)a + \psi(\zeta_1, \zeta_2)b}{1 + \psi(\zeta_0, \zeta_1)} \leq a + b,\ \forall\ a, b \in \mathbb{R}^+ \text{ and } \psi(\zeta_0, \zeta_1) \geq 1.$$

Set,

$$\begin{aligned} \psi(\zeta_i, \zeta_{i+1}) &= \psi(\zeta, \kappa)\ \forall\ i = 0, 1, 2, 3, ... \\ d(\zeta_1, \zeta_2) &\leq \frac{a}{\psi(\zeta, \kappa)}\ max\{d(\zeta_0, \zeta_1), d(\zeta_1, \zeta_2)\} \\ &\leq \frac{a}{\psi(\zeta, \kappa)} d(\zeta_0, \zeta_1). \end{aligned}$$

Also, since $\zeta_1 \in W$ and $\zeta_2 \subset H_{F_2}(\zeta_1)$. Then $\exists\ \zeta_3 \in W$ such that $\zeta_3 \subset H_{F_1}(\zeta_2)$ and

$$\begin{aligned} d(\zeta_2, \zeta_3) &\leq D_1(H_{F_2}(\zeta_1), H_{F_1}(\zeta_2)) \\ &\leq D(H_{F_2}(\zeta_1), H_{F_1}(\zeta_2)) \\ &\leq \frac{a}{\psi(\zeta_1, \zeta_2)}\ max\{d(\zeta_1, \zeta_2), p_\alpha(\zeta_1, H_{F_2}(\zeta_1)), p_\alpha(\zeta_2, H_{F_1}(\zeta_2)), \\ &\frac{p_\alpha(\zeta_1, H_{F_1}(\zeta_2)) + p_\alpha(\zeta_2, H_{F_2}(\zeta_1))}{1 + \psi(\zeta_1, \zeta_2)}\} \\ &\leq \frac{a}{\psi(\zeta_1, \zeta_2)}\ max\{d(\zeta_1, \zeta_2), d(\zeta_1, \zeta_2), d(\zeta_2, \zeta_3), \frac{d(\zeta_1, \zeta_3)}{1 + \psi(\zeta_1, \zeta_2)}\} \\ &\leq \frac{a}{\psi(\zeta, \kappa)} d(\zeta_1, \zeta_2) \\ &\leq \frac{a^2}{[\psi(\zeta, \kappa)]^2} d(\zeta_0, \zeta_1). \end{aligned}$$

Continue this process, $\{\zeta_n \in W\}$ be a sequence with $n \geq 0$ such that

$$\zeta_{2n+1} \subset H_{F_1}(\zeta_{2n}),$$

and

$$\zeta_{2n+2} \subset H_{F_2}(\zeta_{2n+1}).$$

such that

$$\begin{aligned} d(\zeta_n, \zeta_{n+1}) &\leq D_1(H_{F_2}(\zeta_{n-1}), H_{F_1}(\zeta_n)) \\ &\leq D(H_{F_2}(\zeta_{n-1}), H_{F_1}(\zeta_n)) \\ &\leq \frac{a}{\psi(\zeta_{n-1}, \zeta_n)}\ max\{d(\zeta_{n-1}, \zeta_n), p_\alpha(\zeta_n, H_{F_1}(\zeta_n)), p_\alpha(\zeta_{n-1}, H_{F_2}(\zeta_{n-1})), \\ &\frac{p_\alpha(\zeta_{n-1}, H_{F_1}(\zeta_n)) + p_\alpha(\zeta_n, H_{F_2}(\zeta_{n-1}))}{1 + \psi(\zeta_{n-1}, \zeta_n)}\} \\ &\leq \frac{a}{\psi(\zeta_{n-1}, \zeta_n)}\ max\{d(\zeta_{n-1}, \zeta_n), d(\zeta_n, \zeta_{n+1}), d(\zeta_{n-1}, \zeta_n), \\ &\frac{d(\zeta_{n-1}, \zeta_{n+1}) + d(\zeta_n, \zeta_n)}{1 + \psi(\zeta_{n-1}, \zeta_n)}\} \\ &\leq \frac{a}{\psi(\zeta, \kappa)}\ max\{d(\zeta_{n-1}, \zeta_n), d(\zeta_n, \zeta_{n+1})\} \\ &\leq \frac{a}{\psi(\zeta, \kappa)}\ d(\zeta_{n-1}, \zeta_n) \\ &\leq (\frac{a}{\psi(\zeta, \kappa)})^n\ d(\zeta_0, \zeta_1). \end{aligned}$$

To prove that every sequence in W is Cauchy. Let $p, q \in \mathbb{N}$ with (p is less than q). Then,

$$\begin{aligned} d(\zeta_p, \zeta_q) &\leq \psi(\zeta_p, \zeta_{p+1})d(\zeta_p, \zeta_{p+1}) + \psi(\zeta_{p+1}, \zeta_q)d(\zeta_{p+1}, \zeta_q) \\ &\leq \psi(\zeta_p, \zeta_{p+1})d(\zeta_p, \zeta_{p+1}) + \psi(\zeta_{p+1}, \zeta_q)\psi(\zeta_{p+1}, \zeta_{p+2})d(\zeta_{p+1}, \zeta_{p+2}) \\ &+ \psi(\zeta_{p+1}, \zeta_q)\psi(\zeta_{p+2}, \zeta_q)d(\zeta_{p+2}, \zeta_q) \\ &\leq \psi(\zeta_p, \zeta_{p+1})d(\zeta_p, \zeta_{p+1}) + \psi(\zeta_{p+1}, \zeta_q)\psi(\zeta_{p+1}, \zeta_{p+2})d(\zeta_{p+1}, \zeta_{p+2}) \\ &+ \psi(\zeta_{p+1}, \zeta_q)\psi(\zeta_{p+2}, \zeta_q)\psi(\zeta_{p+2}, \zeta_{p+3})d(\zeta_{p+2}, \zeta_{p+3}) \\ &+ \psi(\zeta_{p+1}, \zeta_q)\psi(\zeta_{p+2}, \zeta_q)\psi(\zeta_{p+3}, \zeta_q)d(\zeta_{p+3}, \zeta_q) \\ &\leq \ldots\ldots\ldots\ldots. \\ &\leq \psi(\zeta_p, \zeta_{p+1})d(\zeta_p, \zeta_{p+1}) + \sum_{i=p+1}^{q-2} \Big(\prod_{j=p+1}^{i} \psi(\zeta_j, \zeta_k) \Big) \psi(\zeta_i, \zeta_{i+1})d(\zeta_i, \zeta_{i+1}) \\ &+ \prod_{w=p+1}^{q-1} \psi(\zeta_w, \zeta_q)d(\zeta_{q-1}, \zeta_q) \\ &\leq \psi(\zeta_p, \zeta_{p+1})(\frac{a}{\psi(\zeta, \kappa)})^p d(\zeta_0, \zeta_1) + \prod_{w=p+1}^{q-1} \psi(\zeta_w, \zeta_q)(\frac{a}{\psi(\zeta, \kappa)})^{q-1} d(\zeta_0, \zeta_1) \\ &+ \sum_{i=p+1}^{q-2} \Big(\prod_{j=p+1}^{i} \psi(\zeta_j, \zeta_k) \Big) \psi(\zeta_i, \zeta_{i+1})(\frac{a}{\psi(\zeta, \kappa)})^i d(\zeta_0, \zeta_1). \end{aligned}$$

We also know that $0 < (\frac{a}{\psi(\zeta, \kappa)}) < 1$. So, for the n^{th} term $(\frac{a}{\psi(\zeta, \kappa)})^n \to 0$ as $n \to \infty$. Then,

$$d(\zeta_p, \zeta_q) < \epsilon$$

Consequently, the sequence $\{\zeta_n\}$ in W is a Cauchy sequence. This implies that there exists $\zeta^* \in W$, such that $\{\zeta_n\} \to \zeta^*$ *as* n approaches ∞. Thus, (W, d) is a complete space. Now,

$$\begin{aligned} p_0(\zeta^*, H_{F_2}(\zeta^*)) &\leq \psi(\zeta^*, \zeta_{2n+1})d(\zeta^*, \zeta_{2n+1}) + \psi(\zeta_{2n+1}, \zeta^*)H_{F_2}(\zeta_{2n+1}, H_{F_1}(\zeta^*)) \\ &\leq \psi(\zeta^*, \zeta_{2n+1})d(\zeta^*, \zeta_{2n+1}) + \psi(\zeta_{2n+1}, \zeta^*)D(\zeta_{2n}, H_{F_2}(\zeta^*)). \quad (3.2) \end{aligned}$$

$$\begin{aligned} D(\zeta_{2n}, H_{F_2}(\zeta^*)) \leq\ & a\ max\{d(\zeta_{2n}, \zeta^*), p_\alpha(\zeta_{2n}, H_{F_1}(\zeta_{2n})), p_\alpha(\zeta^*, H_{F_2}(\zeta^*)), \\ & \frac{p_\alpha(\zeta_{2n}, H_{F_2}(\zeta^*)) + p_\alpha(\zeta^*, H_{F_1}(\zeta_{2n}))}{1 + \psi(\zeta_{2n}, \zeta^*)}\} \\ & \leq a\ max\{d(\zeta_{2n}, \zeta^*), d(\zeta_{2n}, \zeta_{2n+1}), \\ & \psi(\zeta^*, \zeta_{2n+1})d(\zeta^*, \zeta_{2n+1}) + \psi(\zeta_{2n+1}, \zeta^*)D(\zeta_{2n}, H_{F_2}(\zeta^*)) \\ & \frac{\psi(\zeta_{2n}, \zeta_{2n+1})d(\zeta_{2n}, \zeta_{2n+1}) + \psi(\zeta_{2n+1}, \zeta^*)D(\zeta_{2n}, H_{F_2}(\zeta^*)) + d(\zeta^*, \zeta_{2n+1})}{1 + \psi(\zeta_{2n}, \zeta^*)}\} \\ & \leq a\ max\{d(\zeta_{2n}, \zeta^*), d(\zeta^*, \zeta_{2n+1})\} \\ & \leq a\ d(\zeta_{2n}, \zeta^*). \end{aligned}$$

By the above inequality and equation (3.2), we write

$$p_0(\zeta^*, H_{F_2}(\zeta^*)) \leq \psi(\zeta^*, \zeta_{2n+1})d(\zeta^*, \zeta_{2n+1}) + \psi(\zeta_{2n+1}, \zeta^*)a\ d(\zeta_{2n}, \zeta^*)$$

$$p_0(\zeta^*, H_{F_2}(\zeta^*)) \to 0\ as\ n \to \infty.$$

Therefore, from lemma (3.6), $\{\zeta^*\} \subset H_{F_2}(\zeta^*)$.
Similarly, we can prove $\{\zeta^*\} \subset H_{F_1}(\zeta^*)$, and the proof is complete.

Remark 3.10. (i) If we put $\psi(\zeta, \kappa) = s$ and s ≥ 1 in Theorem (3.1), then this result also holds for the hesitant fuzzy map on b-metric space.
(ii) If we put $\psi(\zeta, \kappa) = s$ and s $= 1$ in Theorem (3.1), then this result also holds for the hesitant fuzzy map on metric space.

4. Application

In this section, we apply the fixed point theorem to a Fredholm-type integral equation of the form:

$$\zeta(u) = \int_i^j N(u, v, \zeta(v))\, dv + h(u), \quad u, v \in [i, j]. \quad (4.1)$$

Let $W = C([i, j], \mathbb{R})$ be the space of all continuous real-valued functions on $[i, j]$. Define the metric $d : W \times W \to [0, \infty)$ and the auxiliary function $\psi : W \times W \to [1, \infty)$ by:

$$d(\zeta, \kappa) = \sup_{u \in [i,j]} |\zeta(u) - \kappa(u)|^2, \quad \psi(\zeta, \kappa) = |\zeta(u)| + |\kappa(u)| + 4.$$

It is easy to verify that (W, d) forms a complete controlled metric-type space.

Now define an operator $T : W \to W$ by

$$(T\zeta)(u) = \int_i^j N(u, v, \zeta(v))\, dv + h(u), \quad \forall\, u \in [i, j].$$

Hence, the integral equation (4.1) has a solution if and only if T has a fixed point.

Existence and Uniqueness Statement. Assume the following conditions are satisfied:

(i) $N : [i,j] \times [i,j] \times \mathbb{R} \to \mathbb{R}$ and $h : [i,j] \to \mathbb{R}$ are continuous functions.

(ii) There exists $a \in (0,1)$ such that

$$|N(u,v,\zeta(v)) - N(u,v,T\zeta(v))| \leq a\,|\zeta(v) - T\zeta(v)|, \quad \forall u,v \in [i,j],\ \zeta \in W.$$

Remark on Nonlinear Condition: The above condition is a relaxed version of the classical Lipschitz condition. It does not require a uniform Lipschitz constant for the function $\zeta \mapsto N(u,v,\zeta)$, but rather compares the function evaluated at $\zeta(v)$ and $T\zeta(v)$. This form accommodates a wider class of nonlinearities, particularly those where the dependence is operator-based rather than pointwise.

Condition on h: Since $h \in C([i,j],\mathbb{R})$, it ensures the inhomogeneous part of the integral equation remains smooth and bounded, contributing to the overall continuity and solvability of the equation. For uniqueness, the boundedness and continuity of h play a role in keeping the mapping T within the function space W.

Smoothness of Solution: Due to the continuity of both N and h, and the integral operator preserving continuity, the resulting fixed point (solution) $\zeta \in W$ is continuous. Further smoothness (e.g., differentiability) would require additional smoothness assumptions on N and h, such as partial differentiability.

Extension to Integro-Differential Operators: The current framework can be extended to more generalized forms such as Volterra-type or integro-differential equations by incorporating derivative terms in the operator T, e.g.,

$$T\zeta(u) = \frac{d}{du}\left(\int_i^u K(u,v,\zeta(v))\,dv\right) + h(u),$$

provided that the modified operator still maps a complete metric-type space into itself and satisfies a suitable contractive or generalized contractive condition. Such extensions have been rigorously studied in recent literature, including fixed point-based approaches for fractional and integro-differential equations.

Proof. Since (W,d) is a complete controlled metric-type space and $T : W \to W$ is defined by

$$T\zeta(u) = \int_i^j N(u,v,\zeta(v))\,dv + h(u),$$

we use the given assumption:

$$|N(u,v,\zeta(v)) - N(u,v,T\zeta(v))| \leq \frac{1}{2}|\zeta(v) - T\zeta(v)|.$$

Then, for any $\zeta \in W$,

$$|T\zeta(u) - T(T\zeta(u))|^2 \leq \left(\int_i^j |N(u,v,\zeta(v)) - N(u,v,T\zeta(v))|\,dv\right)^2 \leq \frac{1}{4}|\zeta(v) - T\zeta(v)|^2.$$

That is,

$$|T\zeta(u) - T(T\zeta(u))| \leq \frac{1}{2}|\zeta(v) - T\zeta(v)|. \tag{4.2}$$

Further, from the contraction mapping condition in controlled metric-type spaces (Theorem 2.2), we also have:

$$D(H_F(\zeta), H_F(\kappa)) \leq a\,d(\zeta,\kappa), \tag{4.3}$$

for any $\zeta,\kappa \in W$, with $a \in \left(0, \frac{1}{\psi(\zeta,\kappa)}\right)$ and $\psi(\zeta,\kappa) \geq 1$.

Combining inequalities (4.2) and (4.3), it follows that T has a unique fixed point in W, and hence the integral equation (4.1) has a unique solution.

Remark 4.1. (i) If we put $\psi(\zeta, \kappa) = s$ and s ≥ 1 in inequality (4.3), then this result also holds for the hesitant fuzzy map on b-metric space.

(ii) If we put $\psi(\zeta, \kappa) = s$ and s $= 1$ in inequality (4.3), then this result also holds for the hesitant fuzzy map on metric space.

5. Conclusion

In this article, we presented new results concerning hesitant fuzzy mappings, their contractions, and generalized contractions within the framework of controlled metric-type spaces—a generalization that includes b-metric spaces and classical metric spaces as particular cases. By choosing $\psi(\zeta, \kappa) = s$ with $s = 1$, our results naturally reduce to known fixed point theorems in b-metric and metric spaces.

The methodology established here has promising potential for further extension. In particular, it can be generalized to systems involving multiple integral and differential operators, where each operator acts on a component of a vector-valued function. Such systems arise naturally in coupled physical, biological, and engineering models. In these cases, the controlled metric-type space can be replaced with a suitable product space equipped with a vector-valued metric structure, and the contraction conditions can be extended component-wise or in a coupled form.

Moreover, initial or boundary conditions associated with differential or integral operators play a crucial role in ensuring the uniqueness and smoothness of solutions. In the methodology developed in Section 3, these conditions influence the structure of the operator T and ensure it maps into a function space where fixed point results are applicable. Specifically, the inclusion of initial or boundary conditions is often encoded within the kernel or structure of the operator itself and must be reflected in the selection of the function space W and the definition of the control function ψ.

Future work will involve adapting the current framework to handle integro-differential and boundary value problems in fractional and nonlocal contexts, using appropriate modifications of the contraction principle in more generalized functional spaces.

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