



THE NAKAJO–TAKAHASHI TYPE PROJECTION METHOD FOR A NONEXPANSIVE MAPPING ON A HADAMARD SPACE

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ABSTRACT. In this work, we propose a new iterative scheme on a Hadamard space to find a fixed point of a nonexpansive mapping. For this purpose, we deal with a projection method introduced by Nakajo and Takahashi, and we use a tangent space on a Hadamard space to generate an iterative set-sequence.

KEYWORDS: Hadamard space, fixed point approximation, projection method.

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1. INTRODUCTION

Fixed point approximation theory is one of the most important topics in nonlinear analysis. In the situation of Hilbert spaces, many kinds of iterative schemes have been proposed and have shown their strong or weak convergence to a fixed point. For instance, Mann type [11] and Halpern type [5, 15] iterative schemes are typical. In 2003, Nakajo and Takahashi [12] introduced an iteration using a sequence of closed convex sets and metric projections onto them, which is motivated by Solodov and Svaiter [14].

Theorem 1.1 (Nakajo–Takahashi [12]). *Let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping on C having a fixed point. For a given point $u = x_1 \in C$, generate a sequence $\{x_n\}$ of C as follows:*

$$\begin{aligned}C_n &= \{z \in C \mid \|Tx_n - z\| \leq \|x_n - z\|\}; \\Q_n &= \{z \in C \mid \langle z - x_n, u - x_n \rangle \leq 0\}; \\x_{n+1} &= P_{C_n \cap Q_n} u\end{aligned}$$

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for $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ converges strongly to the closest fixed point of T to u .

Recently, some iterative schemes have been carried over from the situation of Hilbert spaces to the CAT(0) space setting. A convergence theorem with the Mann type iterative scheme is shown by He, Fang, López and Li [6], and the Halpern type one is shown by Saejung [13]. In addition, a fixed point approximation theorem with the shrinking projection method has been proved by Kimura [8].

In this work, we prove a convergence theorem with the Nakajo–Takahashi type projection method to find a fixed point of a nonexpansive mapping on a complete geodesic space having nonpositive bounded curvature, where such a space is called a Hadamard space. On a general Hadamard space, its half spaces used for projection methods are not convex. For this reason, we need to suppose an additional assumption to show a convergence theorem with a projection method. We investigate some kinds of half spaces on a Hadamard space and show their admissibility. Particularly, in our purpose, we define a half space using tangent spaces on a CAT(0) space. Tangent spaces on a geodesic space are first defined by Bridson and Haefliger [2], and modified by Chaipunya, Kohsaka and Kumam [3] in the setting of CAT(0) spaces.

2. PRELIMINARIES

We call a mapping T on a metric space H a nonexpansive mapping if

$$d(Tx, Ty) \leq d(x, y)$$

for $x, y \in H$. We call a point $p \in H$ a fixed point of T if $Tx = x$. We denote the set of all fixed points of T by $\text{Fix}T$.

Let (H, d) be a metric space. For $x, y \in H$ and $l = d(x, y)$, we call an isometric mapping γ_{xy} from $[0, l]$ into H a geodesic from x to y if $\gamma_{xy}(0) = x$ and $\gamma_{xy}(l) = y$. Additionally, H is said to be uniquely geodesic if for all $x, y \in H$, there exists a unique geodesic. In a uniquely geodesic space H , for $x, y \in H$ and $t \in [0, 1]$, we define convex combination of x and y with a ratio t by

$$tx \oplus (1 - t)y = \gamma_{xy}((1 - t)d(x, y)).$$

We define a CAT(0) space. The classical definition of CAT(0) spaces uses the notion of comparison triangles. However, we know an equivalent proposition to the definition as follows: A uniquely geodesic space H is a CAT(0) space if and only if

$$d(tx \oplus (1 - t)y, z)^2 \leq td(x, z)^2 + (1 - t)d(y, z)^2 - t(1 - t)d(x, y)^2$$

for $x, y, z \in H$ and $t \in [0, 1]$. For details about this inequality, refer to [1, Theorem 1.3.3]. In particular, we call H a Hadamard space if it is a complete CAT(0) space.

Let H be a CAT(0) space. We say that a subset C of H is convex if

$$tx \oplus (1 - t)y \in C$$

for $x, y \in C$ and $t \in [0, 1]$. The fixed point set of a nonexpansive mapping T on H is closed and convex. Let C be a nonempty closed convex subset of a Hadamard space H . For $x \in H$, we can find a unique point $P_Cx \in C$ such that

$$d(x, P_Cx) = \inf_{y \in C} d(x, y).$$

We call a mapping P_C the metric projection onto C .

Let H be a metric space and $\{x_n\}$ a bounded sequence of H . We call $x \in H$ an asymptotic centre of $\{x_n\}$ if

$$\limsup_{n \rightarrow \infty} d(x, x_n) = \inf_{y \in H} \limsup_{n \rightarrow \infty} d(y, x_n).$$

We say that $\{x_n\}$ Δ -converges to a Δ -limit x if x is a unique asymptotic centre for any subsequence of $\{x_n\}$. It is well known that a bounded sequence of a Hadamard space has a unique asymptotic centre, and that such a sequence has a Δ -convergent subsequence. Suppose that H is a Hadamard space. If a bounded sequence $\{x_n\}$ of H Δ -converges to $x \in H$, then

$$d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y)$$

for $y \in H$. If a sequence $\{x_n\}$ of H Δ -converges to $x \in H$ and

$$\lim_{n \rightarrow \infty} d(x_n, y) = d(x, y)$$

for some $y \in H$, then $\{x_n\}$ converges to x . For more details about Δ -convergence, refer to [1, 4] for instance.

In what follows, we define tangent spaces on a CAT(0) space. Let H be a CAT(0) space. For $p, x, y \in H$, we define the Alexandrov angle A_p at p as follows:

$$A_p(x, y) = \lim_{t \rightarrow 0^+} \arccos \left(1 - \frac{d(\gamma_{px}(t), \gamma_{py}(t))^2}{2t^2} \right) \in [0, \pi]$$

if $p \neq x$ and $p \neq y$; $A_p(x, p) = A_p(p, x) = \pi/2$ if $p \neq x$; $A_p(p, p) = 0$. We have known the following:

Theorem 2.1 (Bridson–Haefliger [2]). *Let H be a CAT(0) space and $p, x, y \in H$ with $p \neq y$. Then, a function $A_p(x, \cdot)$ on H is continuous at y .*

For more details, refer to [2, Proposition 1.14 in Chapter I.1 and Proposition 3.1 in Chapter II.3] for instance.

Let H be a CAT(0) space and $p \in H$. We define an equivalence relation \sim_p on H by $x \sim_p y$ if

$$A_p(x, y) = 0.$$

For the simplicity, we denote an equivalence class $[x]_{\sim_p}$ of $x \in H$ by $[x]_p$. Notice that $[p]_p = \{p\}$ since $A_p(p, x) = \pi/2$ if $p \neq x$. Let

$$D_p H = H / \sim_p = \{[x]_p \mid x \in H\}.$$

Then, $(D_p H, A_p)$ is a metric space, where the distance A_p is defined by

$$A_p([x]_p, [y]_p) = A_p(x, y)$$

for $[x]_p, [y]_p \in D_p H$. We define a function ζ on $D_p H$ by

$$\zeta([x]_p) = \begin{cases} 0 & ([x]_p = [p]_p); \\ 1 & ([x]_p \neq [p]_p) \end{cases}$$

for $[x]_p \in D_p H$. We define an equivalence relation \simeq_p on a Cartesian product

$$[0, \infty[\times D_p H$$

by $(r_1, [x]_p) \simeq_p (r_2, [y]_p)$ if one of the following conditions is satisfied:

- $r_1 \zeta([x]_p) = r_2 \zeta([y]_p) = 0$;
- $r_1 \zeta([x]_p) = r_2 \zeta([y]_p) > 0$ and $[x]_p = [y]_p$.

Let

$$T_p H = ([0, \infty[\times D_p H) / \simeq_p.$$

For the simplicity, we denote an element $[(r, [x]_p)]_{\simeq_p}$ of $T_p H$ by $r[x]_p$. In particular, we denote $0[p]_p$ by 0_p . We define a distance function d_p on $T_p H$ by

$$d_p(r[x]_p, s[y]_p) = \sqrt{r^2 \zeta([x]_p) + s^2 \zeta([y]_p) - 2rs \zeta([x]_p) \zeta([y]_p) \cos A_p(x, y)}$$

for $r[x]_p, s[y]_p \in T_p H$. We call $T_p H$ the tangent space of H at p .

Let H be a CAT(0) space and $p \in H$. We define a logarithmic mapping \log_p from H to $T_p H$ by

$$\log_p x = d(p, x)[x]_p \in T_p H$$

for $x \in H$. We notice that $\log_p p = 0_p$. We define a function g_p by

$$g_p(u_p, v_p) = \frac{d_p(u_p, 0_p)^2 + d_p(v_p, 0_p)^2 - d_p(u_p, v_p)^2}{2}$$

for $u_p, v_p \in T_p H$. We notice that for $p, x, y \in H$,

$$g_p(\log_p x, \log_p y) = d(p, x)d(p, y) \cos A_p(x, y).$$

If H is a Hilbert space, then $T_p H$ is isometric to

$$\{x - p \in H \mid x \in H\},$$

and hence it is isometric to H . Furthermore, for $x, y \in H$,

$$g_p(\log_p x, \log_p y) = \langle x - p, y - p \rangle,$$

where $\langle \cdot, \cdot \rangle$ is an inner product on H .

We further know the following propositions:

Theorem 2.2 (Kimura–Sudo [10]). *Let H be a CAT(0) space. Then,*

$$2g_p(\log_p x, \log_p y) \geq d(p, x)^2 + d(p, y)^2 - d(x, y)^2$$

for $p, x, y \in H$.

Theorem 2.3 (Chaipunya–Kohsaka–Kumam [3], Kimura–Sudo [10]). *Let H be a CAT(0) space. Then,*

$$\lim_{t \rightarrow 0^+} \frac{d(p, y)^2 - d(tx \oplus (1-t)p, y)^2}{t} = 2g_p(\log_p x, \log_p y)$$

for $p, x, y \in H$.

The identity of the second theorem is called the first variation formula. For more details about tangent spaces on a geodesic space, refer to [2, 3, 10].

3. CONVEXITY OF HALF SPACES ON A HADAMARD SPACE

In this section, we discuss the convexity of two kinds of half spaces on a CAT(0) space. We first prove the following theorem:

Theorem 3.1. *Let H be a CAT(0) space. Then, the following hold:*

(i) *A subset*

$$\{z \in H \mid g_x(\log_x z, \log_x y) \leq 0\}$$

is closed for $x, y \in H$;

(ii) if a subset

$$\{z \in H \mid d(u, z) \leq d(v, z)\}$$

is convex for any $u, v \in H$, then a subset

$$\{z \in H \mid g_x(\log_x z, \log_x y) \leq 0\}$$

is convex for $x, y \in H$.

Proof. Fix $x, y \in H$ and let

$$Q = \{z \in H \mid g_x(\log_x z, \log_x y) \leq 0\}.$$

Take a sequence $\{z_n\}$ of Q converging to $z_0 \in H$. If $z_0 = x$, then we have

$$g_x(\log_x z_0, \log_x y) = g_x(0_x, \log_x y) = 0$$

and hence $z_0 \in Q$. We assume $z_0 \neq x$. From the definition of g_x , we get

$$0 \geq g_x(\log_x z_n, \log_x y) = d(z_n, x)d(y, x) \cos A_x(z_n, y)$$

for $n \in \mathbb{N}$. Since a function $A_x(\cdot, y)$ is continuous at z_0 , we have

$$\begin{aligned} 0 &\geq \lim_{n \rightarrow \infty} d(z_n, x)d(y, x) \cos A_x(z_n, y) \\ &= d(z_0, x)d(y, x) \cos A_x(z_0, y) = g_x(\log_x z_0, \log_x y), \end{aligned}$$

and thus $z_0 \in Q$. It means that Q is closed.

We next show (ii). If $x = y$, then since

$$Q = \{z \in H \mid g_x(\log_x z, \log_x y) \leq 0\} = \{z \in H \mid g_x(\log_x z, 0_x) \leq 0\} = H,$$

it is convex. We assume that $x \neq y$. Fix $z_1, z_2 \in Q$ and $t \in [0, 1]$. Put

$$z_0 = tz_1 \oplus (1-t)z_2,$$

and show $z_0 \in Q$. Fix $s \in]0, 1[$ arbitrarily. Then, we define a convex subset H_s of H by

$$H_s = \{z \in H \mid d(x, z) \leq d(sy \oplus (1-s)x, z)\}.$$

We first show $z_0 \in H_s$. To show this, we prove $z_1, z_2 \in H_s$. If $z_1 = x$, then we immediately get $z_1 \in H_s$. Suppose $z_1 \neq x$. From the definition of g_x ,

$$g_x(\log_x z_1, \log_x(sy \oplus (1-s)x)) = sd(x, z_1)d(x, y) \cos A_x(z_1, sy \oplus (1-s)x).$$

We show that

$$\cos A_x(z_1, sy \oplus (1-s)x) = \cos A_x(z_1, y).$$

Since $x \neq y$, for $w = sy \oplus (1-s)x$, we know that

$$\cos A_x(z_1, w) = 1 - \lim_{t \rightarrow 0^+} \frac{d(\gamma_{xz_1}(t), \gamma_{xw}(t))^2}{2t^2}$$

and

$$\cos A_x(z_1, y) = 1 - \lim_{t \rightarrow 0^+} \frac{d(\gamma_{xz_1}(t), \gamma_{xy}(t))^2}{2t^2}.$$

Then, for a fixed real number t such that

$$0 < t < \min\{d(x, z_1), sd(x, y)\},$$

we have $\gamma_{xy}(t) = \gamma_{xw}(t)$, and hence

$$1 - \frac{d(\gamma_{xz_1}(t), \gamma_{xw}(t))^2}{2t^2} = 1 - \frac{d(\gamma_{xz_1}(t), \gamma_{xy}(t))^2}{2t^2}.$$

Letting $t \rightarrow 0^+$, we have $\cos A_x(z_1, w) = \cos A_x(z_1, y)$. Thus, since $z_1 \in Q$,

$$\begin{aligned} g_x(\log_x z_1, \log_x(sy \oplus (1-s)x)) &= sd(x, z_1)d(x, y) \cos A_x(z_1, sy \oplus (1-s)x) \\ &= sd(x, z_1)d(x, y) \cos A_x(z_1, y) \end{aligned}$$

$$= sg_x(\log_x z_1, \log_x y) \leq 0.$$

Therefore,

$$\begin{aligned} 0 &\geq 2g_x(\log_x z_1, \log_x (sy \oplus (1-s)x)) \\ &\geq d(x, z_1)^2 + d(x, sy \oplus (1-s)x)^2 - d(z_1, sy \oplus (1-s)x)^2 \\ &\geq d(x, z_1)^2 - d(z_1, sy \oplus (1-s)x)^2, \end{aligned}$$

and thus

$$d(z_1, x) \leq d(z_1, sy \oplus (1-s)x).$$

It means that $z_1 \in H_s$. Similarly, we get $z_2 \in H_s$. Therefore, $z_0 \in H_s$ since H_s is convex. We notice that for $s \in]0, 1[$,

$$d(z_0, x) \leq d(z_0, sy \oplus (1-s)x),$$

or equivalently

$$d(z_0, x)^2 - d(z_0, sy \oplus (1-s)x)^2 \leq 0.$$

Dividing both sides by s , and letting $s \rightarrow 0+$, we conclude from the first variation formula that

$$0 \geq \lim_{s \rightarrow 0+} \frac{d(z_0, x)^2 - d(z_0, sy \oplus (1-s)x)^2}{s} = 2g_x(\log_x y, \log_x z_0),$$

which implies that

$$g_x(\log_x z_0, \log_x y) \leq 0.$$

Thus, $z_0 \in Q$, and therefore Q is convex. \square

Let H be a CAT(0) space. We consider the following three conditions on H :

- (H1) For all $x, y \in H$, a set $\{z \in H \mid d(x, z) \leq d(y, z)\}$ is convex;
- (H2) for all $x, y \in H$, a set $\{z \in H \mid g_x(\log_x z, \log_x y) \leq 0\}$ is convex;
- (H3) for all $x, y \in H$, a set $\{z \in H \mid d(x, z)^2 + d(x, y)^2 \leq d(y, z)^2\}$ is convex.

We notice that if H is a Hilbert space, then

$$\{z \in H \mid \langle z - x, y - x \rangle \leq 0\} = \{z \in H \mid \|x - z\|^2 + \|x - y\|^2 \leq \|y - z\|^2\}$$

for $x, y \in H$. From Theorem 3.1, in a general CAT(0) space, the condition (H1) implies the condition (H2). According to results in [4, 7], a CAT(0) glueing space containing two spaces of constant but different curvature does not enjoy the condition (H1).

Kimura [8] assumed the condition (H1) to prove an approximation theorem with the shrinking projection method as follows:

Theorem 3.2 (Kimura [8]). *Let H be a Hadamard space satisfying the condition (H1). Let T be a nonexpansive mapping on H having a fixed point. For given points $u, x_1 \in H$ and $C_1 = H$, generate a sequence $\{x_n\}$ of H as follows:*

$$\begin{aligned} C_{n+1} &= \{z \in H \mid d(Tx_n, z) \leq d(x_n, z)\} \cap C_n; \\ x_{n+1} &= P_{C_{n+1}} u \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to $P_{\text{Fix} T} u$.

We notice that many Hadamard spaces enjoy the condition (H1). For instance, nonempty closed convex subsets of a Hilbert space, hyperbolic spaces, real-trees, and so forth.

On the other hand, Kimura and Naiki [9] assumed the conditions (H1) and (H3) to prove a convergence theorem for an equilibrium problem with the Nakajo–Takahashi type projection method as follows:

Theorem 3.3 (Kimura–Naiki [9]). *Let H be a Hadamard space satisfying the conditions (H1) and (H3). Suppose that H satisfies the convex hull finite property. Let K be a nonempty closed convex subset of H and f a real-valued function on K^2 satisfying the following conditions:*

- For $y \in K$, $f(y, y) = 0$;
- for $y, z \in K$, $f(z, y) + d(y, z) \leq 0$;
- for $z \in K$, $f(z, \cdot)$ is lower semicontinuous and convex;
- for $y, z \in K$, $\limsup_{t \rightarrow 0^+} f(ty \oplus (1-t)z, y) \leq f(z, y)$;
- a set $\text{Equil } f = \{z \in K \mid \inf_{y \in K} f(z, y) \geq 0\}$ is nonempty.

Let $\{r_n\}$ be a positive real sequence such that

$$0 < \inf_{k \in \mathbb{N}} r_k \leq \sup_{k \in \mathbb{N}} r_k < \infty.$$

For a given point $u = x_1 \in K$, generate a sequence $\{x_n\}$ of H as follows:

$$\begin{aligned} J_{r_n f} x_n &= \left\{ z \in K \mid \inf_{y \in K} \left(r_n f(z, y) + \frac{1}{2} d(y, x)^2 \right) - \frac{1}{2} d(z, x)^2 \geq 0 \right\}; \\ C_n &= \{z \in K \mid d(J_{r_n f} x_n, z) \leq d(x_n, z)\}; \\ Q_n &= \{z \in K \mid d(x_n, z)^2 + d(x_n, u)^2 \leq d(u, z)^2\}; \\ x_{n+1} &= P_{C_n \cap Q_n} u \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to $P_{\text{Equil } f} u$.

Nevertheless, we obtain an example of a CAT(0) space which enjoys the condition (H1), and does not enjoy the condition (H3).

Example 3.4. Let

$$H = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\},$$

and define a function d on H^2 by

$$d(z_1, z_2) = \text{arcosh} \left(\frac{|z_1 - z_2|^2}{2 \text{Im}(z_1) \text{Im}(z_2)} + 1 \right)$$

for $z_1, z_2 \in H$. Then, (H, d) is a complete metric space. It further implements one of the models of hyperbolic space, and hence it is a CAT(0) space, namely, it is a Hadamard space. Moreover, we immediately notice that H enjoys the condition (H1), and therefore it enjoys the condition (H2). Let $x = 10i$ and $y = 8 + 6i$. Then, we know that $x, y \in H$. Put

$$Q = \{z \in H \mid d(x, z)^2 + d(x, y)^2 \leq d(y, z)^2\}.$$

Let $z_1 = 1 + i$ and $z_2 = 1 + 25i$. Then, we have

$$\begin{aligned} \cosh d(x, y) &= \frac{5}{3}; \\ \cosh d(x, z_1) &= \frac{51}{10}; \\ \cosh d(y, z_1) &= \frac{43}{6}; \\ \cosh d(x, z_2) &= \frac{363}{250}; \\ \cosh d(y, z_2) &= \frac{71}{30}, \end{aligned}$$

and hence $z_1, z_2 \in Q$. Let $z_0 = 1 + 5i$. We notice that it is the midpoint of z_1 and z_2 , namely,

$$z_0 = \frac{1}{2}z_1 \oplus \frac{1}{2}z_2.$$

In contrast, z_0 does not belong to Q . Indeed, we get

$$\begin{aligned} \cosh d(x, z_0) &= \frac{63}{50}; \\ \cosh d(y, z_0) &= \frac{11}{6}, \end{aligned}$$

and therefore $z_0 \notin Q$. This implies that Q is not convex. Consequently, this space H satisfies the condition (H1), and does not satisfy the condition (H3). \square

4. THE NAKAJO–TAKAHASHI TYPE PROJECTION METHOD

In this section, we prove an approximation theorem with a projection method proposed by Nakajo and Takahashi [12].

Theorem 4.1. *Let H be a Hadamard space. Suppose that for all $x, y \in H$, a subset*

$$\{z \in H \mid d(x, z) \leq d(y, z)\}$$

of H is convex. Let T be a nonexpansive mapping on H having a fixed point. Let P_K be the metric projection onto a nonempty closed convex subset K of H . For a given anchor point $u = x_1 \in H$, generate a sequence $\{x_n\}$ of H as follows:

$$\begin{aligned} C_n &= \{z \in H \mid d(Tx_n, z) \leq d(x_n, z)\}; \\ Q_n &= \{z \in H \mid g_{x_n}(\log_{x_n} z, \log_{x_n} u) \leq 0\}; \\ x_{n+1} &= P_{C_n \cap Q_n} u \end{aligned}$$

for $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ converges to $P_{\text{Fix} T} u$.

Proof. To show that $\{x_n\}$ is well defined, we prove $C_n \cap Q_n$ is closed and convex, and

$$\text{Fix} T \subset C_n \cap Q_n$$

for $n \in \mathbb{N}$. Let $p \in \text{Fix} T$. If x_n is defined, then since T is nonexpansive,

$$d(Tx_n, p) = d(Tx_n, Tp) \leq d(x_n, p),$$

which implies that $p \in C_n$. Therefore, $\text{Fix} T \subset C_n$ if x_n is defined. We prove that

$$\text{Fix} T \subset C_n \cap Q_n$$

for $n \in \mathbb{N}$ by induction. Since $u = x_1$,

$$Q_1 = \{z \in H \mid g_{x_1}(\log_{x_1} z, 0_{x_1}) \leq 0\} = H$$

and thus $\text{Fix} T \subset C_1 \cap Q_1$. We assume that $\text{Fix} T \subset C_k \cap Q_k$ for fixed $k \in \mathbb{N}$ and show

$$\text{Fix} T \subset C_{k+1} \cap Q_{k+1}.$$

From the assumptions and Theorem 3.1, $C_k \cap Q_k$ is nonempty, closed and convex. Hence, x_{k+1} is defined since $x_{k+1} = P_{C_k \cap Q_k} u$. Let $q \in \text{Fix} T$ and $t \in]0, 1[$. Then, since $q \in C_k \cap Q_k$, we deduce from the definition of x_{k+1} that

$$d(u, x_{k+1}) \leq d(u, tq \oplus (1-t)x_{k+1}),$$

and hence

$$\frac{d(u, x_{k+1})^2 - d(u, tq \oplus (1-t)x_{k+1})^2}{t} \leq 0.$$

Letting $t \rightarrow 0+$, we get

$$\begin{aligned} 0 &\geq \lim_{t \rightarrow 0+} \frac{d(u, x_{k+1})^2 - d(u, tq \oplus (1-t)x_{k+1})^2}{t} \\ &= 2g_{x_{k+1}}(\log_{x_{k+1}} q, \log_{x_{k+1}} u). \end{aligned}$$

It means that $q \in Q_{k+1}$, and therefore $\text{Fix } T \subset Q_{k+1}$. Thus,

$$\text{Fix } T \subset C_{k+1} \cap Q_{k+1}.$$

Consequently,

$$\text{Fix } T \subset C_n \cap Q_n$$

for $n \in \mathbb{N}$. From the assumption and Theorem 3.1, $C_n \cap Q_n$ is closed and convex for $n \in \mathbb{N}$. Therefore, the sequence $\{x_n\}$ is well defined.

We next show that $\{x_n\}$ converges to $P_{\text{Fix } T}u$. Since

$$d(u, x_{n+1}) = d(u, P_{C_n \cap Q_n} u) \leq d(u, P_{\text{Fix } T} u)$$

for $n \in \mathbb{N}$, the sequence $\{x_n\}$ is bounded. For $n \in \mathbb{N}$, since $x_{n+1} \in Q_n$, we obtain

$$\begin{aligned} 0 &\geq 2g_{x_n}(\log_{x_n} x_{n+1}, \log_{x_n} u) \\ &\geq d(x_n, x_{n+1})^2 + d(x_n, u)^2 - d(x_{n+1}, u)^2 \\ &\geq d(x_n, u)^2 - d(x_{n+1}, u)^2, \end{aligned}$$

which implies that $d(x_n, u) \leq d(x_{n+1}, u)$, and that

$$d(x_n, x_{n+1})^2 + d(x_n, u)^2 - d(x_{n+1}, u)^2 \leq 0.$$

Therefore, $\{d(x_n, u)\}$ is a convergent real sequence, and therefore

$$\limsup_{n \rightarrow \infty} d(x_n, x_{n+1})^2 \leq \limsup_{n \rightarrow \infty} (d(x_{n+1}, u)^2 - d(x_n, u)^2) = 0.$$

We obtain $\{d(x_n, x_{n+1})\}$ converges to 0. Further, since $x_{n+1} \in C_n$,

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(Tx_n, x_{n+1}) \leq 2d(x_n, x_{n+1})$$

for $n \in \mathbb{N}$. Letting $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Then, $\{x_{n_i}\}$ has a Δ -convergent subsequence $\{x_{n_{i_j}}\}$. Let $w \in H$ be its Δ -limit. Since T is nonexpansive,

$$\begin{aligned} \limsup_{j \rightarrow \infty} d(x_{n_{i_j}}, Tw) &\leq \limsup_{j \rightarrow \infty} (d(x_{n_{i_j}}, Tx_{n_{i_j}}) + d(Tx_{n_{i_j}}, Tw)) \\ &= \limsup_{j \rightarrow \infty} d(Tx_{n_{i_j}}, Tw) \\ &\leq \limsup_{j \rightarrow \infty} d(x_{n_{i_j}}, w). \end{aligned}$$

Since w is a unique asymptotic centre of $\{x_{n_{i_j}}\}$, we obtain $Tw = w$, namely, w is a fixed point of T . Then,

$$d(u, P_{\text{Fix } T} u) \leq d(u, w) \leq \liminf_{j \rightarrow \infty} d(u, x_{n_{i_j}}) \leq \limsup_{j \rightarrow \infty} d(u, x_{n_{i_j}}) \leq d(u, P_{\text{Fix } T} u),$$

which means that $w = P_{\text{Fix } T} u$, and that

$$\lim_{i \rightarrow \infty} d(u, x_{n_{i_j}}) = d(u, P_{\text{Fix } T} u).$$

Since $\{x_{n_{i_j}}\}$ Δ -converges to $P_{\text{Fix } T} u$, the sequence $\{x_{n_{i_j}}\}$ converges to $P_{\text{Fix } T} u$. Consequently, all subsequences of $\{x_n\}$ have a subsequence converging to $P_{\text{Fix } T} u$. It means that $\{x_n\}$ converges to $P_{\text{Fix } T} u$. \square

CONCLUSION

In this work, we obtain a convergence theorem to approximate a fixed point of a nonexpansive mapping defined on a Hadamard space. The advantage of this result is that the underlying space does not need to be a Hilbert space. Thus, we should consider numerical experiments in the setting of non-Hilbert spaces such as hyperbolic spaces.

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