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NONLINEAR VARIATIONAL INCLUSIONS INVOLVING (A, η) MONOTONE MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this work, we consider the nonlinear variational inclusion problem (NVIP) in real Hilbert spaces, which involves (A, η)-monotone mappings. We propose an iterative algorithm for finding the approximate solution to (NVIP) by using the resolvent operator technique, and we also explore the convergence criteria of the sequence generated by the resolvent iterative algorithm under some appropriate conditions.

KEYWORDS: (A, η) monotone mapping, nonlinear variational inclusions, resolvent operator, Hilbert space.
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1. Introduction

Variational inequality problems (VIP) are one of the fascinating and widely studied classes of problems arising in mechanics, fluid dynamics, optimization and control, economics, transportation equilibrium, and engineering sciences. Using new and innovative methodologies, (VIP) have been developed and extended in several ways. A variational inclusion is a useful and crucial extension of a variational inequality. For solving variational inequalities, various numerical methods have been developed, comprising projection techniques, Wiener-Hopf equations, and decomposition and descent methods. Hassouni and Moudafi [5] considered and investigated a class of variational inclusions in 1994, as well as established iterative techniques for this class. Further extension of the results in [5] has been done by the numerous authors, see for examples [1, 3, 7, 8, 9].

Various generalizations of the projection technique have been extensively studied to solve variational inequalities, and their generalizations, see for example [1]-[18]. It is well established that for solving various classes of (VIP), the monotonicity of

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the primary operator is very crucial. Huang and Fang [6] developed the concept of generalized m-accretive functions in Banach spaces in 2001. Subsequently, in 2003 [4] established and analyzed a novel kind of variational inclusions by employing H-monotone operators in a Hilbert space. They provide an approach for solving the associated class of variational inclusions by using the resolvent operator. The concept of H-monotonicity has revitalized the idea of maximal monotone functions in numerous ways. Verma [14] introduced A-monotone functions and their applications to the solution of a system of nonlinear variational inclusions. In [18], Zou and Huang introduced and investigated $H(.,.)$-accretive functions and used them to solve variational inclusions and systems of variational inclusions. The maximal monotonicity has played a crucial part in most resolvent operator techniques, but the notions of A-monotonicity and H-monotonicity have not only extended the maximal monotonicity, but have given resolvent operator methods a new edge. Verma [15] extended the concept of A-monotonicity to the case of (A, η)-monotonicity, and used the generalized resolvent operator technique to investigate sensitivity analysis for a class of nonlinear variational inclusion problems. Shafi and Mishra [13] recently investigated a system of nonlinear variational inclusions in real Hilbert spaces involving A-monotone functions.

Maximal monotone is a classical and powerful tool for solving variational inequalities. It has a strong theoretical foundation, but it has limited applicability to only monotone structures and cannot easily handle generalized or complex systems. H-monotonicity extends maximal monotonicity to a broader class of operators. This allows the application of resolvent operator methods in a wider range of settings. At the same time, it requires additional structure (the H operator), which may not always be easily available or constructable. A-monotonicity generalizes monotonicity and further allows solving systems of variational inclusions. While solution methods involve more complex operator calculations, more restrictive assumptions on A may be necessary. $H(.,.)$-accretive is flexible in modeling, and more generalized accretive behavior is useful for solving both single and systems of variational inclusions. Analytical techniques become more intricate, and resolvent formulations are technically heavier. (A, η)-monotonicity further generalizes that A-monotonicity is best suited for problems involving perturbations or sensitivity analysis.

Motivated and inspired by the aforesaid work, in this paper we prove the existence and Lipschitz continuity of resolvent operators. As an application, we consider a set of nonlinear variational inclusion problems in Hilbert spaces involving the (A, η)-monotone operator. Furthermore, we propose a resolvent iterative algorithm for approximating the solution of the nonlinear variational inclusion problem (NVIP), and examine the convergence analysis of the sequence developed by the resolvent iterative algorithm.

The remaining part of the paper is structured as follows:

Section 2, deals with some basic notions and results. Section 3 is related with (NVIP) and resolvent iterative algorithms. In section 4, existence of solution and convergence are analyzed using a resolvent iterative algorithm.

2. Preliminaries

Let $\mathcal{V}$ be a real Hilbert space whose norm and inner product denoted by $\|.\|$ and $\langle .,. \rangle$, respectively. Let $2^{\mathcal{V}}$ denote the family of all the nonempty subsets of $\mathcal{V}$. Let us review the following definitions and some supporting results.

Definition 2.1. [16] A function $\eta : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ is called *τ-Lipschitz continuous* if $\exists$ a constant $\tau > 0$ satisfying

$$\|\eta(a, b)\| \leq \tau \|a - b\|, \ \forall\ a, b \in \mathcal{V}.$$

Definition 2.2. [16] A function $A : \mathcal{V} \longrightarrow \mathcal{V}$ is called:

(i) *monotone* if
$$\langle A(a) - A(b), a - b\rangle \geq 0, \ \forall a, b \in \mathcal{V},$$

(ii) *strictly monotone* if A is monotone and
$$\langle A(a) - A(b), a - b\rangle = 0, \text{ iff } a = b,$$

(iii) *δ-strongly monotone* if $\exists$ a constant $\delta > 0$, satisfying
$$\langle A(a) - A(b), a - b\rangle \geq \delta\|a - b\|^2, \ \forall a, b \in \mathcal{V},$$

(iv) *(δ, η)-strongly monotone* if $\exists$ a constant $\delta > 0$, satisfying
$$\langle A(a) - A(b), \eta(a, b)\rangle \geq \delta\|a - b\|^2, \ \forall a, b \in \mathcal{V}.$$

Definition 2.3. Let $P : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ and $p, g : \mathcal{V} \longrightarrow \mathcal{V}$ be single-valued functions. Then P is said to be:

(i) *(p, η)-monotone* in the first argument if
$$\langle P(p(a), c) - P(p(b), c), \eta(a, b)\rangle \geq 0, \ \forall a, b, c \in \mathcal{V},$$

(ii) *(p, η)-monotone with respect to A* in the first argument if
$$\langle P(p(a), c) - P(p(b), c), \eta(A(a), A(b))\rangle \geq 0, \ \forall a, b, c \in \mathcal{V},$$

(iii) *(g, η)-monotone* in the second argument if
$$\langle P(c, g(a)) - P(c, g(b)), \eta(a, b)\rangle \geq 0, \ \forall a, b, c \in \mathcal{V},$$

(iv) *(g, η)-monotone with respect to A* in the second argument if
$$\langle P(c, g(a)) - P(c, g(b)), \eta(A(a), A(b))\rangle \geq 0, \ \forall a, b, c \in \mathcal{V}.$$

Definition 2.4. [16] Let $\eta : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ and $A, H : \mathcal{V} \longrightarrow \mathcal{V}$ be single valued functions. A multivalued function $Q : \mathcal{V} \longrightarrow 2^{\mathcal{V}}$ is called:

(i) *monotone* if
$$\langle u - v, a - b\rangle \geq 0, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$

(ii) *η-monotone* if
$$\langle u - v, \eta(a, b)\rangle \geq 0, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$

(iii) *strictly η-monotone* if
$$\langle u - v, \eta(a, b)\rangle > 0, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$
except for $a = b$,

(iv) *r-strongly monotone* if $\exists$ a constant $r > 0$, satisfying
$$\langle u - v, a - b\rangle \geq r\|a - b\|^2, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$

(v) *r-strongly η-monotone* if $\exists$ a constant $r > 0$, satisfying
$$\langle u - v, \eta(a, b)\rangle \geq r\|a - b\|^2, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$

(vi) *monotone with respect to A* if
$$\langle u - v, A(a) - A(b)\rangle \geq 0, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$

(vii) *η-monotone with respect to A* if

$$\langle u - v, \eta(A(a), A(b)) \rangle \geq 0, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$

(viii) *maximal monotone* if Q is monotone and $(I + \rho Q)(\mathcal{V}) = \mathcal{V}, \ \forall \ \rho > 0$,
(ix) *relaxed monotone* if $\exists$ a constant $\mu > 0$, satisfying

$$\langle u - v, a - b \rangle \geq -\mu \|a - b\|^2, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$

(x) (μ, η)-*relaxed monotone* if $\exists$ a constant $\mu > 0$, satisfying

$$\langle u - v, \eta(a, b) \rangle \geq -\mu \|a - b\|^2, \ \forall a, b \in \mathcal{V}, \ u \in Q(a), \ v \in Q(b),$$

(xi) *H-monotone* if Q is monotone and $(H + \rho Q)(\mathcal{V}) = \mathcal{V}, \ \forall \ \rho > 0$,
(xii) *A-monotone* if Q is relaxed monotone and $(A + \rho Q)(\mathcal{V}) = \mathcal{V}, \ \forall \ \rho > 0$,
(xiii) (A, η)-*monotone* if Q is (μ, η)-relaxed monotone and $(A+\rho Q)(\mathcal{V}) = \mathcal{V}, \ \forall \ \rho > 0$.

(A, η)-monotone function, A-monotone function and H-monotone function have the following relationships:

$\{(A, \eta)-\text{monotone function}\} \supset \{A-\text{monotone function}\} \supset \{H-\text{monotone function}\}$.

Theorem 2.5. *Let $\eta : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ be a function, $A : \mathcal{V} \longrightarrow \mathcal{V}$ be an (δ, η)-strongly monotone function and $Q : \mathcal{V} \longrightarrow 2^{\mathcal{V}}$ be an (A, η)-monotone function. If for all $(b, v) \in Gr(Q)$, $\langle u - v, \eta(a, b) \rangle \geq 0$ holds, where $Gr(Q) = \{(a, b) \in \mathcal{V} \times \mathcal{V} : b \in Q(a)\}$, then $(a, u) \in Gr(Q)$.*

Proof. Since Q is (A, η)-monotone, we know that $(A + \rho Q)(\mathcal{V}) = \mathcal{V}$ holds for all $\rho > 0$, and so there exists $(b, u_1) \in Gr(Q)$ such that

$$A(a) + \rho u = A(b) + \rho u_1.$$

As A is (δ, η)-strongly monotone function, so

$$\begin{aligned} 0 &\leq \langle u - u_1, \eta(a, b) \rangle \\ &= - \langle A(a) - A(b), \eta(a, b) \rangle \\ &\leq -\delta \|a - b\|^2 \leq 0. \end{aligned}$$

Therefore $a = b$ and $u = u_1$. Thus $(a, u) = (b, u_1) \in Gr(Q)$. □

Following theorem is a generalization of Lemma 2.1 of Agarwal and Verma [2].

Theorem 2.6. *Let $\eta : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ be a function, $A : \mathcal{V} \longrightarrow \mathcal{V}$ be a (δ, η)-strongly monotone function and $Q : \mathcal{V} \longrightarrow 2^{\mathcal{V}}$ be an (A, η)-monotone operator. Then $(A + \rho Q)^{-1}$ is single-valued for $0 < \rho < \frac{\delta}{\mu}$, where $\rho > 0, \ \delta > 0, \ \mu > 0$ are constants.*

Proof. Given $a^* \in \mathcal{V}$, suppose $a, b \in (A + \rho Q)^{-1}(a^*)$.

$$\text{So,} \ -A(a) + a^* \in \rho Q(a) \text{ and } -A(b) + a^* \in \rho Q(b).$$

By the (δ, η)-strongly monotonicity of A, and (A, η)-monotonicity of the operator Q, we have

$$-\mu \|a - b\|^2 \leq \frac{1}{\rho} \langle (-A(a) + a^*) - (-A(b) + a^*), \eta(a, b) \rangle$$

$$
\begin{aligned}
&= -\frac{1}{\rho}\langle A(a)-A(b),\eta(a,b)\rangle\\
&\le -\frac{1}{\rho}\delta\|a-b\|^2\\
&= -\frac{\delta}{\rho}\|a-b\|^2.
\end{aligned}
$$

Which implies, $\mu\rho\|a-b\|^2 \ge \delta\|a-b\|^2$.
If $a \neq b$, then $\rho \ge \frac{\delta}{\mu}$ contradicts with $0<\rho<\frac{\delta}{\mu}$. Thus $a=b$, that is, $(A+\rho Q)^{-1}$ is single-valued. □

Definition 2.7. [16] Let $\eta:\mathcal{V}\times\mathcal{V}\longrightarrow\mathcal{V}$ be a function, $A:\mathcal{V}\longrightarrow\mathcal{V}$ be a strictly η-monotone function and $Q:\mathcal{V}\longrightarrow 2^{\mathcal{V}}$ be an (A,η)-monotone function. The resolvent operator $R^{A,\eta}_{Q,\lambda}:\mathcal{V}\longrightarrow\mathcal{V}$ is defined by

$$R^{A,\eta}_{Q,\lambda}(a) = (A+\rho Q)^{-1}(a),\ \forall a\in\mathcal{V},$$

where $\rho>0$ is a constant.

Next, we prove the following lemma.

Lemma 2.8. *Let $\eta:\mathcal{V}\times\mathcal{V}\longrightarrow\mathcal{V}$ be a τ-Lipschitz continuous function, $A:\mathcal{V}\longrightarrow\mathcal{V}$ be a (δ,η)-strongly monotone function and $Q:\mathcal{V}\longrightarrow 2^{\mathcal{V}}$ be an (A,η)-monotone function. Then the resolvent operator $R^{A,\eta}_{Q,\lambda}:\mathcal{V}\longrightarrow\mathcal{V}$ is $\frac{\tau}{\delta-\rho\mu}$-Lipschitz continuous for $0<\rho<\frac{\delta}{\mu}$, where $\rho,\ \delta$ and μ are positive constants.*

Proof. For any $a,b\in\mathcal{V}$, we have

$$
\begin{aligned}
R^{A,\eta}_{Q,\lambda}(a) &= (A+\rho Q)^{-1}(a),\\
R^{A,\eta}_{Q,\lambda}(b) &= (A+\rho Q)^{-1}(b).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\frac{1}{\rho}\left(a-A(R^{A,\eta}_{Q,\lambda}(a))\right) &\in Q(R^{A,\eta}_{Q,\lambda}(a)),\\
\frac{1}{\rho}\left(b-A(R^{A,\eta}_{Q,\lambda}(b))\right) &\in Q(R^{A,\eta}_{Q,\lambda}(b)).
\end{aligned}
$$

Since Q is (A,η)-monotone, it follows that Q is (μ,η)-relaxed monotone.

$$
\begin{aligned}
\text{Therefore } &\frac{1}{\rho}\left\langle (a-A(R^{A,\eta}_{Q,\lambda}(a)))-(b-A(R^{A,\eta}_{Q,\lambda}(b))),\eta(R^{A,\eta}_{Q,\lambda}(a),R^{A,\eta}_{Q,\lambda}(b))\right\rangle\\
&\ge -\mu\left\|R^{A,\eta}_{Q,\lambda}(a)-R^{A,\eta}_{Q,\lambda}(b)\right\|^2.
\end{aligned}
$$

Now, we see that

$$
\begin{aligned}
&\tau\,\|a-b\|\left\|R^{A,\eta}_{Q,\lambda}(a)-R^{A,\eta}_{Q,\lambda}(b)\right\|\\
&\ge \left\langle a-b,\eta(R^{A,\eta}_{Q,\lambda}(a),R^{A,\eta}_{Q,\lambda}(b))\right\rangle\\
&= \left\langle a-b-(A(R^{A,\eta}_{Q,\lambda}(a))-A(R^{A,\eta}_{Q,\lambda}(b))),\eta(R^{A,\eta}_{Q,\lambda}(a),R^{A,\eta}_{Q,\lambda}(b))\right\rangle\\
&+\left\langle A(R^{A,\eta}_{Q,\lambda}(a))-A(R^{A,\eta}_{Q,\lambda}(b)),\eta(R^{A,\eta}_{Q,\lambda}(a),R^{A,\eta}_{Q,\lambda}(b))\right\rangle\\
&\ge -\rho\mu\left\|R^{A,\eta}_{Q,\lambda}(a)-R^{A,\eta}_{Q,\lambda}(b)\right\|^2+\delta\left\|R^{A,\eta}_{Q,\lambda}(a)-R^{A,\eta}_{Q,\lambda}(b)\right\|^2
\end{aligned}
$$

$$= (\delta - \rho\mu)\left\| R_{Q,\lambda}^{A,\eta}(a) - R_{Q,\lambda}^{A,\eta}(b) \right\|^2.$$

Thus,

$$\left\| R_{Q,\lambda}^{A,\eta}(a) - R_{Q,\lambda}^{A,\eta}(b) \right\| \leq \frac{\tau}{\delta - \rho\mu} \|a - b\|, \ 0 < \rho < \frac{\delta}{\mu}.$$

□

For $A = I$, we have the following corollary:

Corollary 2.9. *Let $Q : \mathcal{V} \longrightarrow 2^{\mathcal{V}}$ be (μ, η)-relaxed monotone. Then the resolvent operator $R_{Q,\lambda}^{A,\eta} = (I + \rho Q)^{-1} : \mathcal{V} \longrightarrow \mathcal{V}$ is $\frac{\tau}{1-\rho\mu}$-Lipschitz continuous for $0 < \rho < \frac{1}{\mu}$, where $\rho > 0, \ \mu > 0$ are constants and I is the identity function.*

Lemma 2.10. [10] *Let K be a nonempty closed and convex subset of $\mathcal{V}$. Then*

$$b = P_K(t) \Longleftrightarrow \langle b - t, a - b \rangle \geq 0, \ \forall t \in \mathcal{V} \text{ and } a \in K,$$

where $P_K(t)$ is the projection satisfying

$$\|t - P_K(t)\| = d(t, K),$$

and $d(t, K)$ is defined by

$$d(t, K) = \inf_{c \in K} \|t - c\|.$$

3. (NVIP) and Resolvent Iterative Algorithm

Suppose $\mathcal{V}$ is a real Hilbert space with the norm $\|.\|$. Let $p, g : \mathcal{V} \longrightarrow \mathcal{V}, \ P : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ be functions, $Q : \mathcal{V} \longrightarrow 2^{\mathcal{V}}$ be (A, η)-monotone function. The nonlinear variational inclusion problem (for short, NVIP) is the problem of finding $a \in \mathcal{V}$, such that

$$0 \in P(p(a), g(a)) + Q(a). \tag{3.1}$$

Special Cases of (NVIP)

(I) If $P \equiv 0$, then (NVIP) (3.1) becomes

$$\text{Find } a \in \mathcal{V}, \text{ such that } 0 \in Q(a), \tag{3.2}$$

introduced and studied by Verma [17].

(II) If $P(p(a), g(a)) = P(p(a)) - P(g(a)), \ \forall a \in \mathcal{V}$ and $Q(a) = \partial\varphi(a), \ \forall a \in \mathcal{V}$, where $\varphi : \mathcal{V} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function and $\partial\varphi$ is the subdifferential of φ. Then the problem (3.1) becomes the problem of finding $a \in \mathcal{V}$, such that

$$\langle P(p(a)) - P(g(a)), b - a \rangle \geq \varphi(a) - \varphi(b), \ \forall b \in \mathcal{V}, \tag{3.3}$$

introduced and studied by Hassouni and Moudafi [5].

Now, we consider the following resolvent iterative algorithm for finding an approximate solution of NVIP (3.1), which consists of the following steps:

Algorithm 3.1. *Resolvent Iterative Algorithm*

Step 1. Initiation Step: Select $s^0 \in \mathcal{V}$ and put $n = 0$.

Step 2. Resolvent Step: Find $s^n \in \mathcal{V}$ *such that*

$$a^n = R_{Q,\lambda}^{A,\eta}\left\{A(s^n) - \rho^n P(p(a^n), g(a^n))\right\}. \tag{3.4}$$

where $0 < \rho^n < \frac{\delta}{\mu}$.

Step 3. Projection Step: Set $K = \{s \in \mathcal{V} : \langle A(s^n) - A(a^n), s - A(a^n)\rangle \leq 0\}$. *If* $A(s^n) = A(a^n)$, *then stop, otherwise, choose* s^{n+1} *such that*

$$A(s^{n+1}) = P_K(A(s^n)). \tag{3.5}$$

Step 4. Suppose $n = n + 1$ *and resume to Step 1.*

Remark 3.2. In view of (3.4), we can have

$$A(s^n) \in A(a^n) + \rho^n(P(p(a^n), g(a^n)) + Q(a^n)),$$

or

$$\frac{1}{\rho^n}\left(A(s^n) - A(a^n)\right) \in (P(p(a^n), g(a^n)) + Q(a^n)). \tag{3.6}$$

4. Existence of Solution and Convergence Analysis

In this section, the existence of the solution of (NVIP) and the convergence of the sequences generated by Algorithm 3.1 has been established.

Theorem 4.1. *Assume* $\mathcal{V}$ *to be real Hilbert space and* $A : \mathcal{V} \longrightarrow \mathcal{V}$ *be continuous and* (δ, η)*-strongly monotone function. Let the continuous function* $P : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ *is such that it is* (p, η)*-monotone and* (g, η)*-monotone with respect to* A *in the first and second argument, respectively, and it is* (p, η)*-monotone and* (g, η)*-monotone in the first and second argument, respectively. Assume that* $\eta : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ *be* τ*-Lipschitz continuous, and* (A, η)*-monotone function,* $Q : \mathcal{V} \longrightarrow 2^{\mathcal{V}}$ *be* η*-monotone with respect to* A*. Then the sequence* $\{a^n\}$ *generated by Algorithm 3.1 converges weakly to a solution of (NVIP).*

Proof. Suppose a' be a solution of problem (3.1). Therefore, we obtain

$$0 \in P(p(a'), g(a')) + Q(a'). \tag{4.1}$$

Now, we can have

$$\begin{aligned}
&\left\|A(a') - A(s^{n+1})\right\|^2 \\
&= \left\|A(a') - A(s^n) - (A(s^{n+1}) - A(s^n))\right\|^2 \\
&= \left\|A(a') - A(s^n)\right\|^2 - 2\left\langle A(a') - A(s^n), A(s^{n+1}) - A(s^n)\right\rangle \\
&+ \left\|A(s^{n+1}) - A(s^n)\right\|^2 \\
&= \left\|A(a') - A(s^n)\right\|^2 - 2\left\langle A(s^{n+1}) - A(s^n), A(s^{n+1}) - A(s^n)\right\rangle \\
&- 2\left\langle A(a') - A(s^{n+1}), A(s^{n+1}) - A(s^n)\right\rangle + \left\|A(s^{n+1}) - A(s^n)\right\|^2 \\
&\leq \left\|A(a') - A(s^n)\right\|^2 - 2\left\langle A(a') - A(s^{n+1}), A(s^{n+1}) - A(s^n)\right\rangle \\
&- \left\|A(s^{n+1}) - A(s^n)\right\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left\|A(a') - A(s^{n+1})\right\|^2 \\
&\leq \left\|A(a') - A(s^n)\right\|^2 - 2\left\langle A(a') - A(s^{n+1}), A(s^{n+1}) - A(s^n)\right\rangle
\end{aligned}$$

$$- \left\| A(s^{n+1}) - A(s^n) \right\|^2. \tag{4.2}$$

Using (p, η) and (g, η) monotonicity of P with respect to A in the first and second argument, respectively, we obtain

$$\begin{aligned} &\langle P(p(a'), g(a')) - P(p(a^n), g(a^n)), \eta(A(a'), A(a^n)) \rangle \\ &= \langle P(p(a'), g(a')) - P(p(a^n), g(a')), \eta(A(a'), A(a^n)) \rangle \\ &+ \langle P(p(a^n), g(a')) - P(p(a^n), g(a^n)), \eta(A(a'), A(a^n)) \rangle \\ &\geq 0. \end{aligned}$$

This implies that

$$\langle P(p(a'), g(a')) - P(p(a^n), g(a^n)), \eta(A(a'), A(a^n)) \rangle \geq 0. \tag{4.3}$$

Also, as Q is η-monotone with respect to A, it follows that

$$\langle Q(a') - Q(a^n), \eta(A(a'), A(a^n)) \rangle \geq 0. \tag{4.4}$$

From (4.3) and (4.4), it follows that

$$\langle P(p(a'), g(a')) + Q(a') - (P(p(a^n), g(a^n)) + Q(a^n)), \eta(A(a'), A(a^n)) \rangle \geq 0.$$

If $\eta(A(a'), A(a^n)) = A(a') - A(a^n)$, using (3.6) and (4.1), it follows that

$$\left\langle 0 - \frac{1}{\rho^n} \left(A(s^n) - A(a^n) \right), A(a') - A(a^n) \right\rangle \geq 0$$

or

$$\langle A(s^n) - A(a^n), A(a') - A(a^n) \rangle \leq 0. \tag{4.5}$$

Therefore, for $A(a') \in K$ and $A(a') = s \in \mathcal{V}$, (4.5) can be rewritten as

$$K = \left\{ s \in \mathcal{V} : \langle A(s^n) - A(a^n), s - A(a^n) \rangle \leq 0 \right\}. \tag{4.6}$$

Since by Algorithm 3.1, $A(s^{n+1}) = P_K(A(s^n))$, and hence from Lemma 2.2, we have

$$\left\langle A(s^{n+1}) - A(s^n), A(a') - A(s^{n+1}) \right\rangle \geq 0. \tag{4.7}$$

Using (4.7) in (4.2), we have

$$\left\| A(a') - A(s^{n+1}) \right\|^2 \leq \left\| A(a') - A(s^n) \right\|^2 - \left\| A(s^{n+1}) - A(s^n) \right\|^2. \tag{4.8}$$

Therefore, from (4.8), we have

$$\left\| A(a') - A(s^{n+1}) \right\| \leq \left\| A(a') - A(s^n) \right\|, \ \forall n \geq 0. \tag{4.9}$$

From (4.9), it follows that $\{\|A(a') - A(s^n)\|\}$ is a convergent sequence. Again, since A is δ-strongly monotone, we have

$$\langle A(a') - A(s^n), \eta(a', s^n) \rangle \geq \delta \|a' - s^n\|^2$$

or

$$\|a' - s^n\| \leq \frac{\tau}{\delta} \left\| A(a') - A(s^n) \right\|. \tag{4.10}$$

Thus, it follows from (4.10) and $\{s^n\}$ is a bounded sequence. From (4.8), it follows that

$$\begin{aligned} 0 &\leq \left\| A(s^{n+1}) - A(s^n) \right\|^2 \\ &\leq \left\| A(a') - A(s^n) \right\|^2 - \left\| A(a') - A(s^{n+1}) \right\|^2. \end{aligned}$$

Applying limits $n \longrightarrow \infty$, we obtain

$$\begin{aligned} &\lim_{n \longrightarrow \infty} \left\| A(s^{n+1}) - A(s^n) \right\|^2 \\ &\leq \lim_{n \longrightarrow \infty} \left\{ \left\| A(a') - A(s^n) \right\|^2 - \left\| A(a') - A(s^{n+1}) \right\|^2 \right\} = 0. \end{aligned}$$

So, $\lim_{n\longrightarrow\infty}\left\|A(s^{n+1})-A(s^n)\right\|=0$.
Now, from $A(s^{n+1})=P_K(A(s^n))\in K$ and $A(a^n)\in K$, we have

$$\left\langle A(s^n)-A(a^n),A(s^{n+1})-A(a^n)\right\rangle\leq 0,$$

and

$$\begin{aligned}\|A(a^n)-A(s^n)\|^2&=\langle A(a^n)-A(s^n),A(a^n)-A(s^n)\rangle\\&=\left\langle A(a^n)-A(s^{n+1}),A(a^n)-A(s^n)\right\rangle\\&+\left\langle A(s^{n+1})-A(s^n),A(a^n)-A(s^n)\right\rangle\\&\leq\left\langle A(s^{n+1})-A(s^n),A(a^n)-A(s^n)\right\rangle.\end{aligned}$$

Hence

$$\lim_{n\longrightarrow\infty}\|A(s^n)-A(a^n)\|=0. \tag{4.11}$$

Further, using (δ,η)-strongly monotonicity of A, it follows that

$$\begin{aligned}\tau\|A(a^n)-A(s^n)\|\,\|a^n-s^n\|&\geq\langle A(a^n)-A(s^n),\eta(a^n,s^n)\rangle\\&\geq\delta\|a^n-s^n\|^2.\end{aligned}$$

Therefore $\lim\limits_{n\longrightarrow\infty}\|s^n-a^n\|=0$, and hence $\lim\limits_{n\longrightarrow\infty}(s^n-a^n)=0$. Thus, it follows from the boundedness of $\{s^n\}$ that $\{a^n\}$ is also a bounded sequence.
Hence, both sequences $\{a^n\}$ and $\{s^n\}$ have same weak limit points.
Next, we claim that each limit point of the sequence $\{a^n\}$ is a solution of NVIP (3.1).
Let $\lim\limits_{n\longrightarrow\infty}a^n=l$ (weakly).
It follows that $\lim\limits_{n\longrightarrow\infty}s^n=l$ (weakly).
Suppose for the fixed element $v\in\mathcal{V}$, we consider an arbitrary element $u\in\mathcal{V}$ such that

$$u\in P(p(v),g(v))+Q(v). \tag{4.12}$$

Therefore, we can find an element $w\in Q(v)$ such that

$$u=P(p(v),g(v))+w. \tag{4.13}$$

Since, P is (p,η)-monotone in the first argument and (g,η)-monotone in the second argument, we have

$$\begin{aligned}&\langle\eta(a^n,v),S(p(a^n),g(a^n))-S(p(v),g(v))\rangle\\&=\langle\eta(a^n,v),S(p(a^n),g(a^n))-S(p(v),g(a^n))\rangle\\&+\langle\eta(a^n,v),S(p(v),g(a^n))-S(p(v),g(v))\rangle\\&\geq 0.\end{aligned}$$

Therefore, we have

$$\langle\eta(a^n,v),S(p(a^n),g(a^n))-S(p(v),g(v))\rangle\geq 0. \tag{4.14}$$

Moreover, as Q is (A,η)-monotone, it follows that Q is (μ,η)-relaxed monotone. Therefore

$$\langle\eta(a^n,v),Q(a^n)-Q(v)\rangle\geq-\mu\|a^n-v\|^2.$$

Using (3.6) and since $w\in Q(v)$, it follows that

$$\left\langle\eta(a^n,v),\left\{\frac{1}{\rho^n}(A(s^n)-A(a^n))-S(p(a^n),g(a^n))\right\}-w\right\rangle\geq-\mu\|a^n-v\|^2. \tag{4.15}$$

On adding (4.14) and (4.15), we have

$$\left\langle \eta(a^n, v), \frac{1}{\rho^n}(A(s^n) - A(a^n)) - (S(p(v), g(v)) + w) \right\rangle \geq -\mu \|a^n - v\|^2. \tag{4.16}$$

Using (4.13) in (4.15), we have

$$\left\langle \eta(a^n, v), \frac{1}{\rho^n}(A(s^n) - A(a^n)) - u \right\rangle \geq -\mu \|a^n - v\|^2,$$

Since $\langle a, b + c\rangle = \langle a, b\rangle + \langle a, c\rangle$, we have

$$\langle \eta(a^n, v), -u\rangle \geq -\left\langle \eta(a^n, v), \frac{1}{\rho^n}(A(s^n) - A(a^n)) \right\rangle - \mu \|a^n - v\|^2. \tag{4.17}$$

Using (4.11) and the boundedness of $\{a^n\}$, $\{\rho^n\}$, we have

$$\left\langle \eta(a^n, v), \frac{1}{\rho^n}(A(s^n) - A(a^n)) \right\rangle \longrightarrow 0. \tag{4.18}$$

Combining (4.17) and (4.18), we have

$$\langle \eta(a^n, v), -u\rangle \geq -\mu \|a^n - v\|^2.$$

Therefore, by taking limits as $n \longrightarrow \infty$, we have

$$\langle \eta(l, v), 0 - u\rangle = \lim_{n \longrightarrow \infty} \langle \eta(a^n, v), 0 - u\rangle \geq -\mu \|a^n - v\|^2. \tag{4.19}$$

Since by (4.12), $(v, u) \in Gr(P(p(.), g(.)) + Q(.))$. Applying Theorem 2.5, (4.19) shows that $(l, 0) \in Gr(P(p(.), g(.)) + Q(.))$, which means

$$0 \in P(p(l), g(l)) + Q(l).$$

Hence, l is a solution of (3.1).
Lastly, we show that there is a unique weak limit point of $\{a^n\}$.
If possible, let s_1, s_2 be two weak limit points of $\{s^n\}$, and $\{s^{n_j}\}$, $\{s^{n_i}\}$ be two subsequences of $\{s^n\}$ that converges weakly to s_1, s_2, respectively.
Then, it follows that $\left\{\|A(s^n) - A(s_1)\|^2\right\}$, $\left\{\|A(s^n) - A(s_2)\|^2\right\}$ are convergent sequences.
Suppose that

$$\kappa_1 = \lim_{n \longrightarrow \infty} \|A(s^n) - A(s_1)\|^2, \tag{4.20}$$

$$\kappa_2 = \lim_{n \longrightarrow \infty} \|A(s^n) - A(s_2)\|^2, \tag{4.21}$$

$$\kappa_3 = \lim_{n \longrightarrow \infty} \|A(s_1) - A(s_2)\|^2. \tag{4.22}$$

Therefore, we can have

$$\begin{aligned}\|A(s^{n_j}) - A(s_2)\|^2 =& \|A(s^{n_j}) - A(s_1)\|^2 + \|A(s_1) - A(s_2)\|^2 \\ &+ 2\left\langle A(s^{n_j}) - A(s_1), A(s_1) - A(s_2)\right\rangle.\end{aligned} \tag{4.23}$$

$$\begin{aligned}\|A(s^{n_i}) - A(s_1)\|^2 =& \|A(s^{n_i}) - A(s_2)\|^2 + \|A(s_1) - A(s_2)\|^2 \\ &+ 2\left\langle A(s^{n_i}) - A(s_2), A(s_2) - A(s_1)\right\rangle.\end{aligned} \tag{4.24}$$

Taking the limit $j \longrightarrow \infty$ in (4.23) and $i \longrightarrow \infty$ in (4.24), by continuity of A and the fact that s_1, s_2 are two weak limit points of $\{s^{n_j}\}$, $\{s^{n_i}\}$, we see that third term on right of (4.23) and (4.24) converges to zero.
In view of (4.20), (4.21) and (4.22), it follows that

$$\kappa_1 - \kappa_2 = \kappa_3, \tag{4.25}$$

and

$$\kappa_2 - \kappa_1 = \kappa_3. \tag{4.26}$$

On adding (4.25) and (4.26), we have $\kappa_3 = 0$. It follows that $A(s_1) = A(s_2)$.
Further, in view of (δ, η)-strongly monotonicity of A and τ-Lipschitz continuity of η, we obtain

$$\begin{aligned} \delta \|s_1 - s_2\|^2 &\leq \langle A(s_1) - A(s_2), \eta(s_1, s_2) \rangle \\ &\leq \|A(s_1) - A(s_2)\| \, \|\eta(s_1, s_2)\| \\ &\leq \tau \, \|A(s_1) - A(s_2)\| \, \|s_1 - s_2\| \, . \end{aligned} \tag{4.27}$$

Since $A(s_1) = A(s_2)$, then from (4.27), we have $s_1 = s_2$.
Thus, it follows that all the weak limit points of $\{s^n\}$ are equal. That is, $\{a^n\}$ is weakly converges to a solution of (3.1). □

Conclusion

In this paper, we have made significant contributions to the field of nonlinear variational inclusion problems (NVIP) in real Hilbert spaces, specifically focusing on (A, η)- monotone mappings. Our proposed iterative algorithm, which utilizes the resolvent operator technique, offers an effective approach for approximating solutions to (NVIP) with improved convergence rates. Moreover, we have thoroughly investigated the convergence criteria for the sequence generated by this algorithm under certain conditions, providing valuable insights into its practical implementation. With these findings, researchers can build upon our work to develop more efficient and robust methods for solving nonlinear variational inclusion problems. We hope that our research will inspire further exploration and innovation in this area of study.

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References

1. S. Adly, *Perturbed algorithms and sensitivity analysis for a general class of variational inclusions*, J. Math. Anal. Appl. **201** (1996), 609–630.
2. R. P. Aggarwal, R. U. Verma, *General system of (A, η)-maximal relaxed monotone variational inclusion problems based on generalized hybrid algorithms*, Commun. Nonlinear Sci. Numer. Simulat. **15** (2010), 238–251.
3. X. P. Ding, *Perturbed proximal point algorithm for generalized quasi-variational inclusions*, J. Math. Anal. Appl. **210** (1997), 88–101.
4. Y. P. Fang, N. J. Huang, *H-monotone operator and resolvent operator technique for variational inclusions*, Appl. Math. Comput. **145** (2003), 795–803.
5. A. Hassouni, A. Moudafi, *A perturbed algorithm for variational inclusions*, J. Math. Anal. Appl. **185** (1994), 706–712.
6. N. J. Huang, Y. P. Fang, *Generalized m-accretive mappings in Banach spaces*, J. Sichuan Univ. **38** (2001), 591–592.
7. N. J. Huang, *Generalized nonlinear variational inclusions with noncompact valued mappings*, Appl. Math. Lett. **9** (1996), 25–29.
8. K. R. Kazmi, *Mann and Ishikawa type perturbed iterative algorithms for generalized quasi-variational inclusions*, J. Math. Anal. Appl. **209** (1997), 572–584.
9. K. R. Kazmi, M. I. Bhat, *Iterative algorithm for a system of nonlinear variational-like inclusions*, Comput. Math. Appl. **48** (2004), 1929–1935.
10. B. T. Polyak, *Introduction to Optimization*, Optimization Software Inc., New York, 1987.
11. T. Ram, *Parametric generalized nonlinear quasi-variational inclusion problems*, Int. J. Math. Arch. **3** (2012), 1273–1282.

12. T. Ram, M. Iqbal, *$H(\cdot,\cdot,\cdot,\cdot)$–$\varphi$–$\eta$-cocoercive operator with an application to variational inclusions*, Int. J. Nonlinear Anal. Appl. **13** (2022), 1311–1327.
13. S. Shafi, L. N. Mishra, *Existence of solution and convergence of resolvent iterative algorithms for a system of nonlinear variational inclusion problems*, Electron. J. Math. Anal. Appl. **9** (2021), 47–59.
14. R. U. Verma, *A-monotonicity and applications to nonlinear variational inclusion problems*, J. Appl. Math. Stochastic Anal. **17** (2005), 193–195.
15. R. U. Verma, *Sensitivity analysis for generalized strongly monotone variational inclusions based on the (A,η)-resolvent operator technique*, Appl. Math. Lett. **19** (2006), 1409–1413.
16. R. U. Verma, *Approximation solvability of a class of nonlinear set-valued variational inclusions involving (A,η)-monotone mappings*, J. Math. Anal. Appl. **337** (2008), 969–975.
17. R. U. Verma, *A general framework for the over-relaxed A-proximal point algorithm and applications to inclusion problems*, Appl. Math. Lett. **22** (2009), 698–703.
18. Y. Z. Zou, N. J. Huang, *$H(\cdot,\cdot)$-accretive operator with an application for solving variational inclusions in Banach spaces*, Appl. Math. Comput. **204** (2008), 809–816.