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NEW ASPECT FOR FIXED POINT THEORY ON ULTRAMETRIC SPACE

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ABSTRACT. In this paper, we prove some fixed point theorems using different types of contractions with special functions and also give an example to illustrate our results.

KEYWORDS: Fixed point, integral type mapping, ultrametric space.
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1. Introduction and Preliminaries

A metric space (X, d) is called an ultrametric space if the metric d satisfied the strong triangle inequality; i.e., for all $x, y, z \in X$:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

In this case, d is called ultrametric [2].

We denote by $\mathfrak{B}(x, r)$, the closed ball

$$\mathfrak{B}(x, r) = \{y \in X : d(x, y) \leq r\}$$

where $x \in X$ and $r \geq 0$ with $\mathfrak{B}(x, 0) = \{x\}$. Also, we have the characteristic property of ultrametric spaces is the following:

If $x, y \in X$, $0 \leq r \leq s$ and $\mathfrak{B}(x, r) \bigcap \mathfrak{B}(r, s) \neq \emptyset$, then $\mathfrak{B}(x, r) \subset \mathfrak{B}(y, s)$.

An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection. In 2001, Gajic [2] obtained a fixed point theorem with uniqueness for the mapping having the property:

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\}, \;\; x, y \in X, \;\; x \neq y.$$

Then, in [3], Gajic extended this result for multivalued mapping such as:

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Theorem 1.1. *Let (X,d) spherically complete ultrametric space. If $T : X \to 2^X_C$ is a mapping such that for any $x, y \in X, x \neq y$,*

$$H(Tx,Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Then T has a fixed point [3].

After Gajic and Popa [10] presented a general fixed point theorem in ultrametric spaces for mappings satisfying an implicit relation. Kirk and Shahzad [4] gave several similarities between ultrametric space and hyperconvex metric space. Also, they give some fixed point results. Furthermore, Mamghaderi and Masiha [6] introduced some generalizations of several well-known fixed point theorems for mappings defined on an ultrametric space and non-Archimedean normed space which are endowed with a graph.

Then, in [1], Acar gave some fixed point results combined $F-$contraction and rational type contraction. For more, information see ([7], [9], [8]).

In this paper, we give some fixed point results using different types of contractions with special functions.

2. Main Results

In this section, we give some fixed point results for integral type mapping on spherically complete ultrametric space. The functions of class ϕ that we use in the following theorems are taken from the article [5].

Theorem 2.1. *Let (X,d) be a spherically complete ultrametric space and $T : X \to 2^X_C$ is a generalized multivalued integral type I mapping if, $\forall x, y \in X$,*

$$\psi\left(\int_0^{H(Tx,Ty)} \varphi(t)dt\right) \leq \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right), \tag{2.1}$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$
$\Phi_1 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0\}$,
$\Phi_2 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, satisfied that $\liminf_{n\to\infty} \varphi(a_n) > 0 \Leftrightarrow \liminf_{n\to\infty} a_n > 0$ for each $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+\}$,
$\Phi_3 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$
and

$$M(x,y) = \max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Tx)D(y,Ty)}{1+d(x,y)}, \frac{D(x,Tx)D(y,Ty)}{1+H(Tx,Ty)}\right\}.$$

Then T has a fixed point.

Proof. Let $\mathfrak{B}_\alpha = B(\alpha, D(\alpha, T\alpha))$ is the closed sphere with centered at α and radius $D(\alpha, T\alpha) = \inf_{s\in T\alpha} d(\alpha, s)$ and let $\mathfrak{A}$ be the collection of these spheres for all $\alpha \in X$. The relation

$$\mathfrak{B}_\alpha \preceq \mathfrak{B}_\beta \text{ iff } \mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$$

is a partial order on $\mathfrak{A}$. Let $\mathfrak{A}_1$ be a totally ordered subfamily of $\mathfrak{A}$. Since X is spherically complete,

$$\bigcup_{\mathfrak{B}_\alpha \in \mathfrak{A}_1} \mathfrak{B}_\alpha = B \neq \emptyset.$$

Let $\beta \in B$ and $\mathfrak{B}_\alpha \in \mathfrak{A}_1$. Clearly, $\beta \in \mathfrak{B}_\alpha$ thus

$$d(\beta, \alpha) \leq D(\alpha, T\alpha).$$

Take $u \in T\alpha$ such that $d(\alpha, u) = D(\alpha, T\alpha)$. (since $T\alpha$ is a nonempty compact set). Now, if $\alpha = \beta$, it is obvious that $\mathfrak{B}_\alpha = \mathfrak{B}_\beta$. We can say $\alpha \neq \beta$, $x \in \mathfrak{B}_\beta$ and

$$\begin{aligned} d(x, \beta) \leq D(\beta, T\beta) &\leq \inf_{v \in T\beta} d(\beta, v) \\ &\leq \max\left\{ d(\beta, \alpha), d(\alpha, u), \inf_{v \in T\beta} d(u, v) \right\} \\ &\leq \max\{D(\alpha, T\alpha), H(T\alpha, T\beta)\}. \end{aligned}$$

Using (2.1) and $(\varphi, \phi, \psi) \in \mathit{\Phi}_1 \times \mathit{\Phi}_2 \times \mathit{\Phi}_3$, we conclude that, all $\alpha, \beta \in X$

$$\begin{aligned} &\psi\left(\int_0^{H(T\alpha, T\beta)} \varphi(t)dt \right) \\ \leq &\psi\left(\int_0^{M(\alpha,\beta)} \varphi(t)dt \right) - \phi\left(\int_0^{M(\alpha,\beta)} \varphi(t)dt \right) \\ < &\psi\left(\int_0^{M(\alpha,\beta)} \varphi(t)dt \right) \\ < &\psi\left(\int_0^{\max\left\{ d(\alpha,\beta), D(\alpha,T\alpha), D(\beta,T\beta), \frac{D(\alpha,T\alpha)D(\beta,T\beta)}{1+d(\alpha,\beta)}, \frac{D(\alpha,T\alpha)D(\beta,T\beta)}{1+H(T\alpha,T\beta)} \right\}} \varphi(t)dt \right) \\ < &\psi\left(\int_0^{\max\{D(\alpha,T\alpha), D(\beta,T\beta)\}} \varphi(t)dt \right). \end{aligned}$$

So, we get

$$\psi\left(\int_0^{D(\beta,T\beta)} \varphi(t)dt \right) \leq \psi\left(\int_0^{H(T\alpha,T\beta)} \varphi(t)dt \right) < \psi\left(\int_0^{\max\{D(\alpha,T\alpha), D(\beta,T\beta)\}} \varphi(t)dt \right).$$

If $D(\alpha, T\alpha) \leq D(\beta, T\beta)$, then we obtain

$$\psi\left(\int_0^{D(\beta,T\beta)} \varphi(t)dt \right) < \psi\left(\int_0^{D(\beta,T\beta)} \varphi(t)dt \right),$$

which is a contradiction. Hence, it must be

$$\psi\left(\int_0^{H(T\alpha,T\beta)} \varphi(t)dt \right) < \psi\left(\int_0^{D(\alpha,T\alpha)} \varphi(t)dt \right).$$

Also, using the $\varphi \in \Phi_1$ and $\psi \in \Phi_3$, we get

$$H(T\alpha, T\beta) < D(\alpha, T\alpha).$$

Now, for any $x \in \mathfrak{B}_\beta$,

$$d(x, \beta) \leq H(T\alpha, T\beta) < D(\alpha, T\alpha),$$

so $x \in \mathfrak{B}_\alpha$, $\mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$ for any $\mathfrak{B}_\alpha$ in $\mathfrak{A}_1$. Thus $\mathfrak{B}_\beta$ is the upper bound for the collection $\mathfrak{A}_1$ in $\mathfrak{A}$ and we can say that, from Zorn's lemma, $\mathfrak{A}$ has a maximum element say $\mathfrak{B}_z$ for some $z \in X$. Now, show that $z \in Tz$. Assume that $z \notin Tz$, $z^\sim \in Tz$, $z \neq z^\sim$ and $d(z, z^\sim) = d(z, Tz)$. We must show that $\mathfrak{B}_{z^\sim} \subseteq \mathfrak{B}_z$. Using

(2.1) and $(\varphi, \phi, \psi) \in \varPhi_1 \times \varPhi_2 \times \varPhi_3$, we conclude that

$$\begin{aligned}
&\psi\bigg(\int_0^{D(z^{\sim},Tz^{\sim})} \varphi(t)dt\bigg)\\
=&\psi\bigg(\int_0^{H(Tz,Tz^{\sim})} \varphi(t)dt\bigg)\\
\leq&\psi\bigg(\int_0^{M(z,z^{\sim})} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{M(z,z^{\sim})} \varphi(t)dt\bigg)\\
<&\psi\bigg(\int_0^{M(z,z^{\sim})} \varphi(t)dt\bigg)\\
<&\psi\bigg(\int_0^{\max\left\{d(z,z^{\sim}),D(z,Tz),D(z^{\sim},Tz^{\sim}),\frac{D(z,Tz)D(z^{\sim},Tz^{\sim})}{1+d(z,z^{\sim})},\frac{D(z,Tz)D(z^{\sim},Tz^{\sim})}{1+H(Tz,Tz^{\sim})}\right\}} \varphi(t)dt\bigg)\\
<&\psi\bigg(\int_0^{\max\{D(z,Tz),D(z^{\sim},Tz^{\sim})\}} \varphi(t)dt\bigg).
\end{aligned}$$

So, we have

$$\psi\bigg(\int_0^{D(z^{\sim},Tz^{\sim})} \varphi(t)dt\bigg) \leq \psi\bigg(\int_0^{H(Tz,Tz^{\sim})} \varphi(t)dt\bigg) < \psi\bigg(\int_0^{\max\{D(z,Tz),D(z^{\sim},Tz^{\sim})\}} \varphi(t)dt\bigg).$$

If $D(z, Tz) \leq D(z^{\sim}, Tz^{\sim})$, then

$$\psi\bigg(\int_0^{D(z^{\sim},Tz^{\sim})} \varphi(t)dt\bigg) < \psi\bigg(\int_0^{D(z^{\sim},Tz^{\sim})} \varphi(t)dt\bigg),$$

which is a contradiction. Hence, it must be

$$\psi\bigg(\int_0^{H(Tz,Tz^{\sim})} \varphi(t)dt\bigg) < \psi\bigg(\int_0^{D(z,Tz)} \varphi(t)dt\bigg).$$

Since, using the $\varphi \in \Phi_1$ and $\psi \in \Phi_3$, we have

$$D(z^{\sim}, Tz^{\sim}) < D(z, Tz).$$

Now if $y \in \mathfrak{B}_{z^{\sim}}$, then

$$d(y, z^{\sim}) \leq D(z^{\sim}, Tz^{\sim}) < D(z, Tz)$$

and, for $z \in X$,

$$d(y, z) \leq \max\{d(y, z^{\sim}), d(z^{\sim}, z)\} \leq D(z, Tz),$$

which implies $y \in \mathfrak{B}_z$. Hence $\mathfrak{B}_{z^{\sim}} \subseteq \mathfrak{B}_z$. Since

$$D(z^{\sim}, Tz^{\sim}) < D(z, Tz)$$

implies $z \notin \mathfrak{B}_{z^{\sim}}$. Therefore $\mathfrak{B}_z \subsetneq \mathfrak{B}_{z^{\sim}}$. This is a contradiction to the maximality of $\mathfrak{B}_{z^{\sim}}$. Then $z \in Tz$ and z is the fixed point of T. This completes the proof. □

Corollary 2.1. *Let (X, d) be a spherically complete ultrametric space and $T : X \to 2^X_C$ is a generalized multivalued integral type I mapping if*

$$\psi\bigg(\int_0^{H(Tx,Ty)} \varphi(t)dt\bigg) \leq \psi\bigg(\int_0^{d(x,y)} \varphi(t)dt\bigg) - \phi\bigg(\int_0^{d(x,y)} \varphi(t)dt\bigg),\ \forall x, y \in X,$$

where $(\varphi, \phi, \psi) \in \varPhi_1 \times \varPhi_2 \times \varPhi_3$. Then T has a fixed point.

Theorem 2.2. *Let (X,d) be a spherically complete ultrametric space and $T : X \to 2^X_C$ is a generalized multivalued integral type II mapping if, $\forall x, y \in X$,*

$$\psi\Big(\int_0^{H(Tx,Ty)} \varphi(t)dt\Big) \leq \alpha^\sim(M(x,y))\psi\Big(\int_0^{M(x,y)} \varphi(t)dt\Big), \tag{2.2}$$

where $(\varphi, \psi, \alpha^\sim) \in \Phi_1 \times \Phi_3 \times \Phi_5$
$\Phi_1 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0\}$,
$\Phi_3 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$
$\Phi_5 = \{\alpha^\sim : \alpha^\sim : \mathbb{R}^+ \to [0,1)$ satisfies that $\limsup_{s\to t} \alpha^\sim(s) < 1$ for each $t > 0\}$ and

$$M(x,y) = \max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Tx)D(y,Ty)}{1+d(x,y)}, \frac{D(x,Tx)D(y,Ty)}{1+H(Tx,Ty)}\right\}.$$

Then T has a fixed point.

Proof. Let $\mathfrak{B}_\alpha = B(\alpha, D(\alpha, T\alpha))$ is the closed sphere with centered at α and radius $D(\alpha, T\alpha) = \inf_{s\in T\alpha} d(\alpha, s)$ and let $\mathfrak{A}$ be the collection of these spheres for all $\alpha \in X$. The relation

$$\mathfrak{B}_\alpha \preceq \mathfrak{B}_\beta \text{ iff } \mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$$

is a partial order on $\mathfrak{A}$. Let $\mathfrak{A}_1$ be a totally ordered subfamily of $\mathfrak{A}$. Since X is spherically complete,

$$\bigcup_{\mathfrak{B}_\alpha \in \mathfrak{A}_1} \mathfrak{B}_\alpha = B \neq \emptyset.$$

Let $\beta \in B$ and $\mathfrak{B}_\alpha \in \mathfrak{A}_1$. Clearly, $\beta \in \mathfrak{B}_\alpha$ and

$$d(\beta, \alpha) \leq D(\alpha, T\alpha).$$

Take $u \in T\alpha$ such that $d(\alpha, u) = D(\alpha, T\alpha)$. (since $T\alpha$ is a nonempty compact set). Now, if is $\alpha = \beta$, it is obvious that $\mathfrak{B}_\alpha = \mathfrak{B}_\beta$. We can say $\alpha \neq \beta$, $x \in \mathfrak{B}_\beta$ and

$$\begin{aligned} d(x,\beta) \leq D(\beta, T\beta) &\leq \inf_{v\in T\beta} d(\beta, v) \\ &\leq \max\left\{d(\beta,\alpha), d(\alpha,u), \inf_{v\in T\beta} d(u,v)\right\} \\ &\leq \max\{D(\alpha,T\alpha), H(T\alpha,T\beta)\}. \end{aligned}$$

Using (2.2) and $(\varphi, \psi, \alpha^\sim) \in \Phi_1 \times \Phi_3 \times \Phi_5$, if follows that

$$\begin{aligned} &\psi\Big(\int_0^{H(T\alpha,T\beta)} \varphi(t)dt\Big) \\ \leq&\alpha^\sim(M(\alpha,\beta))\psi\Big(\int_0^{M(\alpha,\beta)} \varphi(t)dt\Big) \\ \leq&\alpha^\sim(M(\alpha,\beta))\psi\Big(\int_0^{\max\left\{d(\alpha,\beta), D(\alpha,T\alpha), D(\beta,T\beta) \frac{D(\alpha,T\alpha)D(\beta,T\beta)}{1+d(\alpha,\beta)}, \frac{D(\alpha,T\alpha)D(\beta,T\beta)}{1+H(T\alpha,T\beta)}\right\}} \varphi(t)dt\Big) \\ <&\psi\Big(\int_0^{\max\{D(\alpha,T\alpha), D(\beta,T\beta)\}} \varphi(t)dt\Big) \end{aligned}$$

for all $\alpha, \beta \in X$. So, we get

$$\psi\Big(\int_0^{D(\beta,T\beta)} \varphi(t)dt\Big) \leq \psi\Big(\int_0^{H(T\alpha,T\beta)} \varphi(t)dt\Big) < \psi\Big(\int_0^{\max\{D(\alpha,T\alpha), D(\beta,T\beta)\}} \varphi(t)dt\Big).$$

If $D(\alpha, T\alpha) \leq D(\beta, T\beta)$, then

$$\psi\bigg(\int_0^{D(\beta,T\beta)}\varphi(t)dt\bigg) < \psi\bigg(\int_0^{D(\beta,T\beta)}\varphi(t)dt\bigg),$$

which is a contradiction. Hence, we have

$$\psi\bigg(\int_0^{H(T\alpha,T\beta)}\varphi(t)dt\bigg) < \psi\bigg(\int_0^{D(\alpha,T\alpha)}\varphi(t)dt\bigg).$$

Since $\varphi \in \Phi_1$ and $\psi \in \Phi_3$,

$$H(T\alpha, T\beta) < D(\alpha, T\alpha).$$

Now, for any $x \in \mathfrak{B}_\beta$,

$$d(x, \beta) \leq H(T\alpha, T\beta) < D(\alpha, T\alpha).$$

This follows that $x \in \mathfrak{B}_\alpha$ and $\mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$ for any $\mathfrak{B}_\alpha$ in $\mathfrak{A}_1$. Thus $\mathfrak{B}_\beta$ is the upper bound for the collection $\mathfrak{A}_1$ in $\mathfrak{A}$ and we can say that, from Zorn's lemma, $\mathfrak{A}$ has a maximum element say $\mathfrak{B}_z$ for some $z \in X$. Now, let's show that $z \in Tz$. Assume that $z \notin Tz$, $z^\sim \in Tz$, $z \neq z^\sim$ and $d(z, z^\sim) = d(z, Tz)$. We must show that $\mathfrak{B}_{z^\sim} \subseteq \mathfrak{B}_z$. Using (2.2) and $(\varphi, \psi, \alpha^\sim) \in \varPhi_1 \times \varPhi_3 \times \varPhi_5$, we conclude that

$$\begin{aligned}
&\psi\bigg(\int_0^{D(z^\sim,Tz^\sim)}\varphi(t)dt\bigg)\\
=&\psi\bigg(\int_0^{H(Tz,Tz^\sim)}\varphi(t)dt\bigg)\\
\leq&\alpha^\sim(M(z,z^\sim))\psi\bigg(\int_0^{M(z,z^\sim)}\varphi(t)dt\bigg)\\
<&\psi\bigg(\int_0^{\max\left\{d(z,z^\sim),D(z,Tz),D(z^\sim,Tz^\sim),\frac{D(z,Tz)D(z^\sim,Tz^\sim)}{1+d(z,z^\sim)},\frac{D(z,Tz)D(z^\sim,Tz^\sim)}{1+H(Tz,Tz^\sim)}\right\}}\varphi(t)dt\bigg)\\
<&\psi\bigg(\int_0^{\max\{D(z,Tz),D(z^\sim,Tz^\sim)\}}\varphi(t)dt\bigg).
\end{aligned}$$

So, we obtain

$$\psi\bigg(\int_0^{D(z^\sim,Tz^\sim)}\varphi(t)dt\bigg) \leq \psi\bigg(\int_0^{H(Tz,Tz^\sim)}\varphi(t)dt\bigg) < \psi\bigg(\int_0^{\max\{D(z,Tz),D(z^\sim,Tz^\sim)\}}\varphi(t)dt\bigg).$$

If $D(z, Tz) \leq D(z^\sim, Tz^\sim)$, then

$$\psi\bigg(\int_0^{D(z^\sim,Tz^\sim)}\varphi(t)dt\bigg) < \psi\bigg(\int_0^{D(z^\sim,Tz^\sim)}\varphi(t)dt\bigg),$$

which is a contradiction. Hence, it must be

$$\psi\bigg(\int_0^{H(Tz,Tz^\sim)}\varphi(t)dt\bigg) < \psi\bigg(\int_0^{D(z,Tz)}\varphi(t)dt\bigg).$$

Since $\varphi \in \Phi_1$ and $\psi \in \Phi_3$, we have

$$D(z^\sim, Tz^\sim) < D(z, Tz).$$

Now if $y \in \mathfrak{B}_{z^\sim}$, then

$$d(y, z^\sim) \leq D(z^\sim, Tz^\sim) < D(z, Tz)$$

and, for $z \in X$,

$$d(y, z) \leq \max\{d(y, z^\sim), d(z^\sim, z)\} \leq D(z, Tz)$$

which implies $y \in \mathfrak{B}_z$. Hence $\mathfrak{B}_{z^\sim} \subseteq \mathfrak{B}_z$. Since

$$D(z^\sim, Tz^\sim) < D(z, Tz),$$

then $z \notin \mathfrak{B}_{z^\sim}$. Therefore $\mathfrak{B}_z \subsetneq \mathfrak{B}_{z^\sim}$. This is a contradiction to the maximality of $\mathfrak{B}_{z^\sim}$. Then $z \in Tz$ and z is the fixed point of T. This completes the proof.

□

Corollary 2.2. *Let (X, d) be a spherically complete ultrametric space and $T : X \to 2^X_C$ is a generalized multivalued integral type II mapping if*

$$\psi\Big(\int_0^{H(Tx,Ty)} \varphi(t)dt\Big) \leq \alpha^\sim(d(x,y))\psi\Big(\int_0^{d(x,y)} \varphi(t)dt\Big), \ \forall x, y \in X,$$

where $(\varphi, \psi, \alpha^\sim) \in \Phi_1 \times \Phi_3 \times \Phi_5$. Then T has a fixed point.

Theorem 2.3. *Let (X, d) be a spherically complete ultrametric space and $T : X \to 2^X_C$ is a generalized multivalued integral type III mapping if, $\forall x, y \in X$,*

$$\begin{aligned} \psi\Big(\int_0^{H(Tx,Ty)} \varphi(t)dt\Big) &\leq \alpha^\sim(M(x,y))\phi\Big(\int_0^{D(x,Tx)} \varphi(t)dt\Big) \\ &\quad + \beta^\sim(M(x,y))\psi\Big(\int_0^{D(y,Ty)} \varphi(t)dt\Big), \end{aligned} \tag{2.3}$$

where $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$ and $(\alpha^\sim, \beta^\sim) \in \Phi_6$,
$\Phi_1 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$, is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0\}$,
$\Phi_3 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$
$\Phi_4 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that $\varphi(0) = 0\}$,
$\Phi_6 = \{(\alpha^\sim, \beta^\sim) : \alpha^\sim, \beta^\sim : \mathbb{R}^+ \to [0, 1)$ satisfy that

$$\limsup_{s\to 0^+} \beta^\sim(s) < 1, \quad \limsup_{s\to t^+} \tfrac{\alpha^\sim(s)}{1-\beta^\sim(s)} < 1$$

and $\alpha^\sim(t) + \beta^\sim(t) < 1$ for each $t > 0\}$ and

$$M(x,y) = \max\Big\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Tx)D(y,Ty)}{1+d(x,y)}, \frac{D(x,Tx)D(y,Ty)}{1+H(Tx,Ty)}\Big\}.$$

$$\phi(t) < \psi(t), \tag{2.4}$$

for all $t > 0$. Then T has a fixed point.

Proof. Let $\mathfrak{B}_\alpha = B(\alpha, D(\alpha, T\alpha))$ is the closed sphere with centered at α and radius $D(\alpha, T\alpha) = \inf_{s\in T\alpha} d(\alpha, s)$ and let $\mathfrak{A}$ be the collection of these spheres for all $\alpha \in X$. The relation

$$\mathfrak{B}_\alpha \preceq \mathfrak{B}_\beta \text{ iff } \mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$$

is a partial order on $\mathfrak{A}$. Let $\mathfrak{A}_1$ be a totally ordered subfamily of $\mathfrak{A}$. Since X is spherically complete,

$$\bigcup_{\mathfrak{B}_\alpha \in \mathfrak{A}_1} \mathfrak{B}_\alpha = B \neq \emptyset.$$

Let $\beta \in B$ and $\mathfrak{B}_\alpha \in \mathfrak{A}_1$. Clearly, $\beta \in \mathfrak{B}_\alpha$ thus

$$d(\beta, \alpha) \leq D(\alpha, T\alpha).$$

Take $u \in T\alpha$ such that $d(\alpha, u) = D(\alpha, T\alpha)$. (since $T\alpha$ is a nonempty compact set). Now, if is $\alpha = \beta$, it is obvious that $\mathfrak{B}_\alpha = \mathfrak{B}_\beta$. We can say $\alpha \neq \beta$ and $x \in \mathfrak{B}_\beta$ which implies that

$$\begin{aligned} d(x, \beta) \leq D(\beta, T\beta) &\leq \inf_{v \in T\beta} d(\beta, v) \\ &\leq \max\left\{d(\beta, \alpha), d(\alpha, u), \inf_{v \in T\beta} d(u, v)\right\} \\ &\leq \max\{D(\alpha, T\alpha), H(T\alpha, T\beta)\}. \end{aligned}$$

Using features of multivalued, (2.3), (2.4) and $(\varphi, \psi, \phi) \in \mathit{\Phi}_1 \times \mathit{\Phi}_3 \times \mathit{\Phi}_4$ and $(\alpha^\sim, \beta^\sim) \in \Phi_6$, we conclude that,

$$\begin{aligned} &\psi\left(\int_0^{H(T\alpha,T\beta)} \varphi(t)dt\right) \\ \leq&\alpha^\sim(M(\alpha,\beta))\phi\left(\int_0^{D(\alpha,T\alpha)} \varphi(t)dt\right) + \beta^\sim(M(\alpha,\beta))\psi\left(\int_0^{D(\beta,T\beta)} \varphi(t)dt\right) \\ \leq&\alpha^\sim(M(\alpha,\beta))\phi\left(\int_0^{D(\alpha,T\alpha)} \varphi(t)dt\right) + \beta^\sim(M(\alpha,\beta))\psi\left(\int_0^{H(T\alpha,T\beta)} \varphi(t)dt\right) \\ \leq&\frac{\alpha^\sim(M(\alpha,\beta))}{1-\beta^\sim(M(\alpha,\beta))}\phi\left(\int_0^{D(\alpha,T\alpha)} \varphi(t)dt\right) \\ <&\frac{\alpha^\sim(M(\alpha,\beta))}{1-\beta^\sim(M(\alpha,\beta))}\psi\left(\int_0^{D(\alpha,T\alpha)} \varphi(t)dt\right) \\ <&\psi\left(\int_0^{D(\alpha,T\alpha)} \varphi(t)dt\right). \end{aligned}$$

Since $\varphi \in \Phi_1$ and $\psi \in \Phi_3$,

$$H(T\alpha, T\beta) < D(\alpha, T\alpha).$$

Now, for any $x \in \mathfrak{B}_\beta$,

$$d(x, \beta) \leq H(T\alpha, T\beta) < D(\alpha, T\alpha).$$

So $x \in \mathfrak{B}_\alpha$ and $\mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$ for any $\mathfrak{B}_\alpha$ in $\mathfrak{A}_1$. Thus $\mathfrak{B}_\beta$ is the upper bound for the collection $\mathfrak{A}_1$ in $\mathfrak{A}$ and we can say that, from Zorn's lemma, $\mathfrak{A}$ has a maximum element say $\mathfrak{B}_z$ for some $z \in X$. Now, show that $z \in Tz$. Assume that $z \notin Tz$, $z^\sim \in Tz$, $z \neq z^\sim$ and $d(z, z^\sim) = d(z, Tz)$. We must show that $\mathfrak{B}_{z^\sim} \subseteq \mathfrak{B}_z$. Using (2.3), (2.4) and $(\varphi, \psi, \phi) \in \mathit{\Phi}_1 \times \mathit{\Phi}_3 \times \mathit{\Phi}_4$ and $(\alpha^\sim, \beta^\sim) \in \Phi_6$, we conclude that

$$\begin{aligned} &\psi\left(\int_0^{D(z^\sim,Tz^\sim)} \varphi(t)dt\right) \\ =&\psi\left(\int_0^{H(Tz,Tz^\sim)} \varphi(t)dt\right) \\ \leq&\alpha^\sim(M(z,z^\sim))\phi\left(\int_0^{D(z,Tz)} \varphi(t)dt\right) + \beta^\sim(M(z,z^\sim))\psi\left(\int_0^{D(z^\sim,Tz^\sim)} \varphi(t)dt\right) \\ \leq&\alpha^\sim(M(z,z^\sim))\phi\left(\int_0^{D(z,Tz)} \varphi(t)dt\right) + \beta^\sim(M(z,z^\sim))\psi\left(\int_0^{H(Tz,Tz^\sim)} \varphi(t)dt\right) \\ \leq&\frac{\alpha^\sim(M(z,z^\sim))}{1-\beta^\sim(M(z,z^\sim))}\phi\left(\int_0^{D(z,Tz)} \varphi(t)dt\right) \end{aligned}$$

$$
\begin{aligned}
&<\frac{\alpha^{\sim}(M(z,z^{\sim}))}{1-\beta^{\sim}(M(z,z^{\sim}))}\psi\bigg(\int_0^{D(z,Tz)}\varphi(t)dt\bigg)\\
&<\psi\bigg(\int_0^{D(z,Tz)}\varphi(t)dt\bigg).
\end{aligned}
$$

Since is $\varphi \in \Phi_1$ and $\psi \in \Phi_3$, we have

$$D(z^{\sim}, Tz^{\sim}) < D(z, Tz).$$

Now if $y \in \mathfrak{B}_{z^{\sim}}$, then

$$d(y, z^{\sim}) \leq D(z^{\sim}, Tz^{\sim}) < D(z, Tz)$$

and, for $z \in X$,

$$d(y, z) \leq \max\{d(y, z^{\sim}), d(z^{\sim}, z)\} \leq D(z, Tz)$$

which implies $y \in \mathfrak{B}_z$. Hence $\mathfrak{B}_{z^{\sim}} \subseteq \mathfrak{B}_z$. Since

$$D(z^{\sim}, Tz^{\sim}) < D(z, Tz)$$

implies $z \notin \mathfrak{B}_{z^{\sim}}$. Therefore $\mathfrak{B}_z \subsetneq \mathfrak{B}_{z^{\sim}}$. This is a contradiction to the maximality of $\mathfrak{B}_{z^{\sim}}$. Then $z \in Tz$ and z is the fixed point of T. This completes the proof. □

Corollary 2.3. *Let (X, d) be a spherically complete ultrametric space and $T : X \to 2_C^X$ is a generalized multivalued integral type III mapping if*

$$
\begin{aligned}
\psi\bigg(\int_0^{H(Tx,Ty)}\varphi(t)dt\bigg) \leq{}& \alpha^{\sim}(d(x,y))\phi\bigg(\int_0^{D(x,Tx)}\varphi(t)dt\bigg)\\
&+\beta^{\sim}(d(x,y))\psi\bigg(\int_0^{D(y,Ty)}\varphi(t)dt\bigg), \quad \forall x, y \in X,
\end{aligned}
$$

where $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$ and $(\alpha^{\sim}, \beta^{\sim}) \in \Phi_6$. Then T has a fixed point.

Example 2.4. Let $X = [0, (1/2)] \cup \{1\} \cup \{3\}$ be endowed with the ultrametric

$$
\begin{aligned}
&d(x, y) = \max\{x, y\}, \ \ x \neq y,\\
&d(x, y) = 0 \Leftrightarrow x = y,\\
&d(x, 0) = x.
\end{aligned}
$$

Assume that $T : X \to 2_C^X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ are defined by

$$T(x) = \begin{cases} \{0, \frac{x}{2}\} & , \quad \forall x \in [0, \frac{1}{2}],\\ \{0\} & , \quad x = 1,\\ \{1\} & , \quad x = 3. \end{cases} \quad ,$$

$$\varphi(t) = \begin{cases} 1/2 & , \quad \forall t \in [0, 1],\\ 1 & , \quad \forall t \in (1, \infty). \end{cases} \quad ,$$

$$\phi(t) = \begin{cases} \frac{t^2}{4} & , \quad \forall t \in [0, 1],\\ \frac{t^2}{8} & , \quad \forall t \in (1, \infty). \end{cases} \quad ,$$

$$\psi(t) = \begin{cases} t & , \quad \forall t \in [0, 1],\\ \frac{t^2+1}{2} & , \quad \forall t \in (1, \infty). \end{cases} \quad .$$

Clearly, d is an ultrametric on X and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. Let $x, y \in X$ with $x < y$. In order to verify (2.1), we have to consider following four cases.

Case I: Let $x, y \in [0, \frac{1}{2}]$. Then we have

$$\begin{aligned}
\psi\left(\int_0^{H(Tx,Ty)} \varphi(t)dt\right) &= \psi\left(\int_0^{\frac{y}{2}} \varphi(t)dt\right) \\
&= \psi\left(\frac{y^2}{16}\right) \\
&= \frac{y^2}{16} \\
&\leq \frac{y^2}{4} - \frac{y^4}{64} \\
&= \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right).
\end{aligned}$$

Case II: Let $x \in [0, \frac{1}{2}]$ and $y = 1$. Then we have

$$\begin{aligned}
\psi\left(\int_0^{H(Tx,Ty)} \varphi(t)dt\right) &= \psi\left(\int_0^{\frac{y}{2}} \varphi(t)dt\right) \\
&= \psi\left(\frac{y^2}{16}\right) \\
&= \frac{y^2}{16} \\
&\leq \frac{y^2}{4} - \frac{y^4}{64} \\
&= \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right).
\end{aligned}$$

Case III: Let $x \in [0, \frac{1}{2}]$ and $y = 3$. Then we have

$$\begin{aligned}
\psi\left(\int_0^{H(Tx,Ty)} \varphi(t)dt\right) &= \psi\left(\int_0^{\frac{y}{2}} \varphi(t)dt\right) \\
&= \psi\left(\frac{3}{2}\right) \\
&= \frac{13}{8} \\
&\leq \frac{y^2+1}{2} - \frac{y^2}{8} \\
&= \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right).
\end{aligned}$$

Case IV: Let $x = 1$ and $y = 3$. Then we have

$$\begin{aligned}\psi\left(\int_0^{H(Tx,Ty)} \varphi(t)dt\right) &= \psi\left(\int_0^{x} \varphi(t)dt\right)\\ &= \psi\left(\frac{1}{4}\right)\\ &= \frac{1}{4}\\ &\le \frac{y^2+1}{2} - \frac{y^2}{8}\\ &= \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right) - \phi\left(\int_0^{M(x,y)} \varphi(t)dt\right).\end{aligned}$$

As a result, (2.1) holds. So, T is a generalized multivalued integral type I mapping and also T has a fixed point such that $x = 0 \in T\{0\}$.

Example 2.5. Let $X = [\frac{1}{2}, 1] \cup [\frac{3}{2}, 2]$ be endowed with the ultrametric

$$\begin{aligned}d(x,y) &= \max\left\{\frac{x}{2}, \frac{y}{2}\right\}, \ x \neq y,\\ d(x,y) &= 0 \Leftrightarrow x = y,\\ d(x,0) &= \frac{x}{2}.\end{aligned}$$

Assume that $T : X \to 2^X_C$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\alpha^\sim, \beta^\sim : \mathbb{R}^+ \to [0,1)$ are defined by

$$T(x) = \begin{cases} \{0,1\} & , \ \forall x \in \left[\frac{1}{2}, 1\right], \\ \{\frac{x}{2}\} & , \ \forall x \in \left[\frac{3}{2}, 3\right]. \end{cases} ,$$

$$\varphi(t) = \Big\{2t, \quad \forall t \in \mathbb{R}^+ \quad , \quad \phi(t) = \Big\{16t^2, \quad \forall t \in \mathbb{R}^+ \quad , \quad \psi(t) = \Big\{4t^2, \quad \forall t \in \mathbb{R}^+$$

and

$$\alpha^\sim(t) = \left\{\frac{t}{(\frac{1}{2}+t)^3}, \quad \forall t \in \mathbb{R}^+ \quad , \quad \beta^\sim(t) = \left\{\frac{t^2}{(\frac{1}{2}+t)^3}, \quad \forall t \in \mathbb{R}^+.\right.\right.$$

Clearly, d is an ultrametric on X and $(\varphi, \phi, \psi) \in \varPhi_1 \times \varPhi_3 \times \varPhi_4$, $(\alpha^\sim, \beta^\sim) \in \Phi_6$. Let $x, y \in X$ with $x < y$. In order to verify (2.3), we have to consider following three cases.

Case I: Let $x, y \in [\frac{1}{2}, 1]$. Then we have

$$\begin{aligned}\psi\left(\int_0^{H(Tx,Ty)} \varphi(t)dt\right) &= \psi\left(\int_0^{\frac{1}{2}} 2tdt\right)\\ &= \psi\left(\frac{1}{4}\right)\\ &= \frac{1}{4}\\ &\le \frac{1}{2} + \frac{1}{16}\\ &= \frac{1}{2}16\left(\frac{1}{4}\right)^2 + \frac{1}{4}\frac{1}{4}\end{aligned}$$

$$
\begin{aligned}
&= \alpha^{\sim}\Big(\frac{1}{2}\Big)\phi\Big(\int_0^{\frac{1}{2}} 2tdt\Big) + \beta^{\sim}\Big(\frac{1}{2}\Big)\psi\Big(\int_0^{\frac{1}{2}} 2tdt\Big) \\
&\leq \alpha^{\sim}(M(x,y))\phi\Big(\int_0^{D(x,Tx)} \varphi(t)dt\Big) \\
&+ \beta^{\sim}(M(x,y))\psi\Big(\int_0^{D(y,Ty)} \varphi(t)dt\Big).
\end{aligned}
$$

Case II: Let $x, y \in [\frac{3}{2}, 2]$. Then we have

$$
\begin{aligned}
\psi\Big(\int_0^{H(Tx,Ty)} \varphi(t)dt\Big) &= \psi\Big(\int_0^{\frac{y}{4}} 2tdt\Big) \\
&= \psi\Big(\frac{y^2}{16}\Big) \\
&= \frac{y^4}{64} \\
&\leq \frac{4x^4 y}{(y+1)^3} + \frac{y^6}{2(y+1)^3} \\
&= \alpha^{\sim}\Big(\frac{y}{2}\Big)\phi\Big(\int_0^{\frac{x}{2}} 2tdt\Big) + \beta^{\sim}\Big(\frac{y}{2}\Big)\psi\Big(\int_0^{\frac{y}{2}} 2tdt\Big) \\
&\leq \alpha^{\sim}(M(x,y))\phi\Big(\int_0^{D(x,Tx)} \varphi(t)dt\Big) \\
&+ \beta^{\sim}(M(x,y))\psi\Big(\int_0^{D(y,Ty)} \varphi(t)dt\Big).
\end{aligned}
$$

Case III: $x \in [\frac{1}{2}, 1]$ and $y \in [\frac{3}{2}, 2]$. Then we have

$$
\begin{aligned}
\psi\Big(\int_0^{H(Tx,Ty)} \varphi(t)dt\Big) &= \psi\Big(\int_0^{\frac{1}{2}} 2tdt\Big) \\
&= \psi\Big(\frac{1}{4}\Big) \\
&= \frac{1}{4} \\
&\leq \frac{4y}{(y+1)^3} + \frac{y^6}{2(y+1)^3} \\
&= \alpha^{\sim}\Big(\frac{y}{2}\Big)\phi\Big(\int_0^{\frac{1}{2}} 2tdt\Big) + \beta^{\sim}\Big(\frac{y}{2}\Big)\psi\Big(\int_0^{\frac{y}{2}} 2tdt\Big) \\
&\leq \alpha^{\sim}(M(x,y))\phi\Big(\int_0^{D(x,Tx)} \varphi(t)dt\Big) \\
&+ \beta^{\sim}(M(x,y))\psi\Big(\int_0^{D(y,Ty)} \varphi(t)dt\Big).
\end{aligned}
$$

As a result, (2.3) holds. So, T is a generalized multivalued integral type III mapping and also T has a fixed point such that $x = 1 \in T\{1\}$.

3. Conclusion

In this paper, we obtain some fixed point theorems using different type contractions with special functions and give an example to illustrate our results. Moreover, we also showed that the obtained results are a generalization of the results existing in the literature.

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