



## QUALITATIVE BEHAVIOR OF THIRD-ORDER HALF LINEAR ADVANCED DIFFERENCE EQUATIONS

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**ABSTRACT.** Improved oscillation conditions for third-order half linear advanced difference equations with canonical and non-canonical operators are established in this study. Our results reinforce and enhance existing ones. Examples are constructed to demonstrate the reliability of the proposed results.

**KEYWORDS:** Half-linear, difference equation, canonical, non-canonical, advanced arguments.

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### 1. INTRODUCTION

In this paper, the emphasis is on the oscillatory behavior of the half linear advanced difference equation of the form

$$\Delta(a(k)\Delta(b(k)(\Delta x(k))^\alpha)) + p(k)(\Delta x(k+1))^\alpha - q(k)g(x(\tau(k))) = 0, \quad k \geq k_0 \quad (1.1)$$

where  $k_0$  is a positive integer and  $\alpha$  is a quotient of odd positive integers. While delivering the results, we take account of the following circumstances:

- H1)  $a(k)$ ,  $b(k)$ ,  $p(k)$  and  $q(k)$  are nonnegative real sequence for all  $k \geq k_0$ ;
- H2)  $\tau(k)$  is a nondecreasing sequence of integers such that  $\tau(k) \geq k+1$  for all  $k \geq k_0$ ;
- H3)  $ug(u) > 0$ ,  $g$  is nondecreasing for  $u \neq 0$ , and  $g(uv) \geq g(u)g(v)$  for  $uv > 0$ .

By a solution of (1.1), we mean a real sequence  $\{x(k)\}$  that satisfies (1.1) for all  $k \geq k_0$ . A nontrivial solution of (1.1) is called nonoscillatory if it is either eventually positive or eventually negative, otherwise it is called oscillatory.

Let us characterized the operators

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$$\begin{aligned} L_0 x(k) &= x(k), \\ L_1 x(k) &= b(k)(\Delta x(k))^\alpha, \\ L_2 x(k) &= a(k)\Delta(L_1 x(k)), \\ L_3 x(k) &= \Delta(L_2 x(k)), \forall k \geq k_0. \end{aligned}$$

Then, (1.1) can be represented as

$$L_3 x(k) + \frac{p(k)}{b(k+1)} L_1 x(k+1) - q(k)g(x(\tau(k))) = 0, \quad k \geq k_0. \quad (1.2)$$

The oscillatory properties of difference equations have been the subject of recent research in [6],[9],[11],[12],[18],[23]. A few applications of biomathematics, economics, and physics may also be found in [10],[13],[16]. However, compared to second-order equations, third-order difference equations receive less attention from researchers. We review several research that are relevant to the findings in this investigation.

In [14], Srinivasan examined the new oscillatory result for third-order advanced difference equation of the form

$$\Delta(a_n \Delta(b_n (\Delta x_n)^\alpha)) - p_n f(x_{\sigma(n)}) = 0, \quad n \geq n_0,$$

with the condition

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \sum_{n=n_0}^{\infty} \frac{1}{b_n} = \infty.$$

In [15], author used the comparison method to establish asymptotic behavior of delay difference equation with negative middle term

$$\Delta(a_n \Delta(\Delta w_n)^\alpha) - p_n (\Delta w_{n+1})^\alpha - q_n h(w_{n-l}) = 0, \quad n \geq n_0.$$

In [19], the author discussed oscillatory and asymptotic properties of third-order delay difference equation

$$\Delta(a_n \Delta(b_n (\Delta x_n)^\alpha)) + p_n (\Delta x_{n+1})^\alpha + q_n f(x_{n-l}) = 0,$$

with the condition

$$\sum_{s=n_0}^{n-1} \frac{1}{a_s} = \infty \text{ as } n \rightarrow \infty.$$

In the recent work [22], the author discussed the oscillatory behavior of advanced difference equation of the form

$$\Delta(a(l)\Delta(d(l)(\Delta x(l))^\alpha)) - p(l)(\Delta x(l+1))^\alpha - q(l)g(x(\sigma(l))) = 0, \quad l \geq l_0,$$

where  $p(l)$  and  $q(l)$  are positive sequences and provided the open problem for researchers to get the oscillation criteria for the equation (1.1), where  $\sigma(l) \geq l+1$ .

To accomplish this, the aim of our paper is to examine the new oscillatory results for (1.1) under non-canonical and canonical conditions.

$$\sum_{s=k_0}^{\infty} \frac{1}{a(s)} < \infty, \quad \sum_{s=k_0}^{\infty} \frac{1}{b(s)} < \infty. \quad (1.3)$$

$$\sum_{s=k_0}^{\infty} \frac{1}{a(s)} = \infty, \quad \sum_{s=k_0}^{\infty} \frac{1}{b(s)} = \infty. \quad (1.4)$$

We can see that the properties of solutions to equation (1.1) are closely related to nonoscillatory solutions of an auxiliary second order difference equation

$$\Delta(a(k)\Delta y(k)) + \frac{p(k)}{b(k+1)} y(k+1) = 0. \quad (1.5)$$

## 2. AUXILIARY LEMMAS

Let us define

$$A_1(k, k_1) = \sum_{s=k_1}^{k-1} \frac{1}{a(s)}, A_2(k, k_1) = \sum_{s=k_1}^{k-1} \frac{1}{b(s)},$$

$$A_{11}(k, k_1) = \sum_{s=k_1}^{k-1} \frac{1}{a(s)y(s)y(s+1)}, A_{12}(k, k_1) = \sum_{s=k_1}^{k-1} \frac{y(s)}{b(s)}, R(k, k_1) = \sum_{s=k_1}^{k-1} \left( \frac{y(s)}{b(s)} \right)^{\frac{1}{\alpha}} A_{11}^{\frac{1}{\alpha}}(s, k_1)$$

**Lemma 2.1.** *Assume that  $y(k)$  is a positive solution of (1.5) for all  $k \geq k_0$ . Then (1.1) can be written in the form*

$$\begin{aligned} & \Delta(a(k)\Delta(b(k)(\Delta x(k))^\alpha)) + p(k)(\Delta x(k+1))^\alpha \\ &= \Delta\left(a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)}(\Delta x(k))^\alpha\right)\right) = q(k)y(k+1)g(x(\tau(k))). \end{aligned} \quad (2.1)$$

*Proof.* Using an elementary computation, we can observe that

$$\begin{aligned} & \frac{1}{y(k+1)}\Delta\left(a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)}(\Delta x(k))^\alpha\right)\right) \\ &= \frac{1}{y(k+1)}\Delta(a(k)\Delta(b(k)(\Delta x(k))^\alpha))y(k+1) - a(k)b(k+1)(\Delta y(k+1))^\alpha \Delta y(k) \\ &= \Delta(a(k)\Delta(b(k)(\Delta x(k))^\alpha)) + \frac{1}{y(k+1)}(a(k+1)\Delta(b(k+1)(\Delta x(k))^\alpha)\Delta y(k+1)) \\ &\quad - \frac{1}{y(k+1)}(b(k+1)(\Delta y(k+1))^\alpha \Delta(a(k)\Delta y(k))) - \frac{1}{y(k+1)}(a(k+1)\Delta(b(k+1)(\Delta x(k))^\alpha)\Delta y(k+1)) \\ &= \Delta(a(k)\Delta(b(k)(\Delta x(k))^\alpha)) + p(k)(\Delta x(k+1))^\alpha. \end{aligned}$$

This completes the proof. □

In view of ([1], Theorem 6.3.4) Equation (1.5) has a couple of nonoscillatory solutions  $\{y(k)\}$  such that either

$$y(k)\Delta y(k) > 0,$$

or

$$y(k)\Delta y(k) < 0, \quad (2.2)$$

for all  $k \geq k_0$ .

In order to reveal the structure of potential nonoscillatory solutions for (1.1), a nonoscillatory solution  $\{y(k)\}$  must satisfy (2.2) with the following property.

**Lemma 2.2.** *(see [2]) If*

$$a(k) \geq 1, \quad \sum_{n=n_0}^{\infty} \frac{p(n)}{b(n+1)} < \infty, \quad \limsup_{n \rightarrow \infty} n \sum_{s=n}^{\infty} \frac{p(s)}{b(s+1)} < \frac{1}{4}, \quad (2.3)$$

*then the solution of equation (1.5) is a positive solution.*

**Lemma 2.3.** *(see [2]) Assume that (2.3) holds. Then  $\exists$  a non-oscillatory solution to the equation (1.5) fulfilling*

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)y(n)y(n+1)} = \infty. \quad (2.4)$$

Now let us assume that

$$\sum_{n=n_0}^{\infty} \left( \frac{y(n)}{b(n)} \right) = \infty, \quad (2.5)$$

where  $\{y(n)\}$  is a positive solution which exists by Lemma 2.2.

**Lemma 2.4.** (see [8]) *Let  $\{q(k)\}$  be a positive sequence,  $\tau$  be a positive integer. If the difference inequality*

$$\Delta y(k) - q(k)g(y(\tau(k))) \geq 0,$$

*has a positive solution, then the difference equation  $\Delta y(k) - q(k)g(y(\tau(k))) = 0$  also has a positive solution.*

Now we introduce the following classes of nonoscillatory solutions:

$$\begin{aligned} x(k) \in I_1 : x(k) > 0, \frac{b(k)}{y(k)} (\Delta x(k))^\alpha > 0, a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right) < 0, \\ \Delta \left( a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right) \right) > 0, \\ x(k) \in I_3 : x(k) > 0, \frac{b(k)}{y(k)} (\Delta x(k))^\alpha > 0, a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right) > 0, \\ \Delta \left( a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right) \right) > 0, \end{aligned}$$

for all  $k \geq k_1 \geq k_0$ .

**Lemma 2.5.** *Assume that  $x(k)$  is a positive solution of (2.1) such that  $x(k) \in I_3$ ,  $\forall k \geq k_1 \geq k_0$ . If*

$$\sum_{k=k_1}^{\infty} q(k)y(k+1)g(R(\tau(k), k_1)) = \infty, \quad (2.6)$$

*then  $\exists k_2 \geq k_1$ , such that*

$$\frac{x(k)}{R(k, k_1)} \text{ is nondecreasing for } k \geq k_2.$$

*Proof.* Assume that  $x(k) \in I_3$  be a positive solution of (2.1)  $\forall k \geq k_1$ . Since  $a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right)$  is increasing,  $\exists$  a constant  $C > 0$  such that

$$a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right) \geq C, \quad \forall k \geq k_1.$$

Now, summing the above inequality from  $k_1$  to  $k-1$ , we get

$$\frac{b(k)}{y(k)} (\Delta x(k))^\alpha \geq CA_{11}(k, k_1).$$

Again summing, we obtain

$$x(k) \geq C^{\frac{1}{\alpha}} R(k, k_1). \quad (2.7)$$

We claim that the condition (2.6), implies

$$\lim_{k \rightarrow \infty} a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right) = \infty.$$

Using (2.7) in (2.1), we obtain

$$\Delta \left( a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right) \right) \geq g(C^{\frac{1}{\alpha}})q(k)y(k+1)g(R(\tau(k), k_1)). \quad (2.8)$$

Summing (2.8), we get the claim to hold. Therefore, for  $k \geq k_2 \geq k_1$ , we obtain

$$\begin{aligned}
 \frac{b(k)}{y(k)} (\Delta x(k))^\alpha &= \frac{b(k_2)}{y(k_2)} (\Delta x(k_2))^\alpha + \sum_{s=k_2}^{k-1} \frac{a(s)y(s)y(s+1)\Delta\left(\frac{b(s)}{y(s)} (\Delta x(s))^\alpha\right)}{a(s)y(s)y(s+1)} \\
 &\leq \frac{b(k_2)}{y(k_2)} (\Delta x(k_2))^\alpha + A_{11}(k, k_2)a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)} (\Delta x(k))^\alpha\right) \\
 &= \frac{b(k_2)}{y(k_2)} (\Delta x(k_2))^\alpha - A_{11}(k_2, k_1)a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)} (\Delta x(k))^\alpha\right) \\
 &\quad + A_{11}(k, k_1)a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)} (\Delta x(k))^\alpha\right) \\
 &\leq A_{11}(k, k_1)a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)} (\Delta x(k))^\alpha\right),
 \end{aligned}$$

which gives

$$\Delta\left(\frac{\frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{A_{11}(k, k_1)}\right) = \frac{A_{11}(k, k_1)a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)} (\Delta x(k))^\alpha\right) - \frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{a(k)y(k)y(k+1)A_{11}(k, k_1)A_{11}(k+1, k_1)} \geq 0,$$

and hence  $\frac{\frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{A_{11}(k, k_1)}$  is nondecreasing for  $k \geq k_2$ . Again for any  $k \geq k_3 \geq k_2$ , we obtain

$$\begin{aligned}
 x(k) &= x(k_3) + \sum_{s=k_3}^{k-1} \left( \frac{A_{11}(s, k_1) \frac{b(s)}{y(s)} (\Delta x(s))^\alpha}{\frac{b(s)}{y(s)} A_{11}(s, k_1)} \right)^{\frac{1}{\alpha}} \\
 &\leq x(k_3) + R(k, k_3) \left( \frac{\frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{A_{11}(k, k_1)} \right)^{\frac{1}{\alpha}} \\
 &\leq x(k_3) - R(k_3, k_1) \left( \frac{\frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{A_{11}(k, k_1)} \right)^{\frac{1}{\alpha}} + R(k, k_1) \left( \frac{\frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{A_{11}(k, k_1)} \right)^{\frac{1}{\alpha}}.
 \end{aligned}$$

From the discrete L'Hospital rule that

$$\lim_{k \rightarrow \infty} \frac{\frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{A_{11}(k, k_1)} = \lim_{k \rightarrow \infty} a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)} (\Delta x(k))^\alpha\right) = \infty$$

and so we have

$$x(k) \leq R(k, k_1) \left( \frac{\frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{A_{11}(k, k_1)} \right)^{\frac{1}{\alpha}}, \quad k \geq k_3.$$

Then,

$$\Delta\left(\frac{x(k)}{R(k, k_1)}\right) = \frac{R(k, k_1) \left( \frac{\frac{b(k)}{y(k)} (\Delta x(k))^\alpha}{A_{11}(k, k_1)} \right)^{\frac{1}{\alpha}} - x(k)A_{11}^{\frac{1}{\alpha}}(k, k_1)}{R(k, k_1)R(k+1, k_1)} \geq 0$$

Thus,  $\frac{x(k)}{R(k, k_1)}$  is nondecreasing for all  $k \geq k_3$ . Hence, the proof.  $\square$

## 3. OSCILLATION CRITERIA UNDER (1.3)

**Theorem 3.1.** *Let (H1)- (H3) and (1.3) hold and assume that  $y(k)$  is positive solution of (1.5). If*

$$\limsup_{k \rightarrow \infty} \frac{1}{R(\tau(k), k_1)} \sum_{s=k_1}^{\tau(k)-1} \left( \frac{y(s)}{b(s)} A_{11}(s, k_1) \sum_{u=k}^{s-1} y(u+1)q(u)(R(\tau(u), k_1))^\alpha \right)^{\frac{1}{\alpha}} > 1, \quad (3.1)$$

and

$$\sum_{s=k}^{\infty} \left( \frac{y(s)}{b(s)} \sum_{u=k}^{\infty} \frac{1}{a(u)y(u)y(u+1)} \sum_{v=k}^{\infty} q(v)y(v+1) \right)^{\frac{1}{\alpha}} = \infty, \quad (3.2)$$

then (1.1) is either oscillatory or tends to zero as  $k \rightarrow \infty$ .

*Proof.* Assume that  $x(k)$  be a positive solution of (2.1) and hence (1.1) for  $k \geq k_0$  such that  $x(k) \in I_1$  or  $x(k) \in I_3$ . If  $x(k) \in I_3$ , then summing (2.1) from  $k$  to  $i-1$ , we get

$$a(i)y(i)y(i+1)\Delta \left( \frac{b(i)}{y(i)} (\Delta x(i))^\alpha \right) \geq \sum_{s=k}^{i-1} y(s+1)q(s)g(x(\tau(s))).$$

From Lemma 2.5,  $\frac{x(k)}{R(k, k_1)}$  is nondecreasing for all  $k \geq k_2$ , and so

$$a(i)y(i)y(i+1)\Delta \left( \frac{b(i)}{y(i)} (\Delta x(i))^\alpha \right) \geq g \left( \frac{x(\tau(k))}{R(\tau(k), k_1)} \right) \sum_{s=k}^{i-1} y(s+1)q(s)g(R(\tau(s), k_1)).$$

Again summing, one obtains

$$\frac{b(i)}{y(i)} (\Delta x(i))^\alpha \geq g \left( \frac{x(\tau(k))}{R(\tau(k), k_1)} \right) \sum_{s=k}^{i-1} \frac{1}{a(s)y(s)y(s+1)} \sum_{t=k}^{s-1} y(t+1)q(t)g(R(\tau(t), k_1)).$$

Summing the aforementioned inequality from  $k_1$  to  $i-1$ , we find

$$x(i) \geq g^{\frac{1}{\alpha}} \left( \frac{x(\tau(k))}{R(\tau(k), k_1)} \right) \sum_{s=k_1}^{i-1} \left( \frac{y(s)}{b(s)} A_{11}(s, k_1) \sum_{u=k}^{s-1} y(u+1)q(u)g(R(\tau(u), k_1)) \right)^{\frac{1}{\alpha}}.$$

Setting  $i = \tau(k)$  and  $g(u) = u^\alpha$

$$x(\tau(k)) \geq \frac{x(\tau(k))}{R(\tau(k), k_1)} \sum_{s=k_1}^{\tau(k)-1} \left( \frac{y(s)}{b(s)} A_{11}(s, k_1) \sum_{u=k}^{s-1} y(u+1)q(u)(R(\tau(u), k_1))^\alpha \right)^{\frac{1}{\alpha}},$$

$$1 \geq \frac{1}{R(\tau(k), k_1)} \sum_{s=k_1}^{\tau(k)-1} \left( \frac{y(s)}{b(s)} A_{11}(s, k_1) \sum_{u=k}^{s-1} y(u+1)q(u)(R(\tau(u), k_1))^\alpha \right)^{\frac{1}{\alpha}}.$$

Taking limsup on both side we get contradiction with (3.1). Hence  $I_3 = \emptyset$ . Now  $x(k) \in I_1$ , then from  $\frac{b(k)}{y(k)} (\Delta x(k))^\alpha > 0$ , we have  $x$  is increasing, that is,  $\exists$  a constant  $e \geq 0$ , such that  $\lim_{k \rightarrow \infty} x(k) = e$ .

We claim that  $e = 0$ . Contrary suppose that  $e > 0$ . Then  $(x(\tau(k))) \geq e$  for  $k \geq k_2$ ,  $k_2 \geq k_1$ . Hence,

$$\Delta \left( a(k)y(k)y(k+1)\Delta \left( \frac{b(k)}{y(k)} (\Delta x(k))^\alpha \right) \right) = q(k)y(k+1)g(x(\tau(k))) \geq g(e)q(k)y(k+1). \quad (3.3)$$

Summing (3.3) from  $k$  to  $\infty$ , gives

$$-a(k)y(k)y(k+1)\Delta\left(\frac{b(k)}{y(k)}(\Delta x(k))^\alpha\right) \geq g(e)\sum_{s=k_1}^{k-1} q(s)y(s+1).$$

Again summing from  $k$  to  $\infty$ , we get

$$\frac{b(k)}{y(k)}(\Delta x(k))^\alpha \geq g(e)\sum_{s=k}^{\infty} \frac{1}{a(s)y(s)y(s+1)}\sum_{u=k}^{\infty} q(u)y(u+1).$$

Once more summing, we obtain

$$-x(k) \geq g(e)^{\frac{1}{\alpha}} \sum_{s=k}^{\infty} \left( \frac{y(s)}{b(s)} \sum_{u=k}^{\infty} \frac{1}{a(u)y(u)y(u+1)} \sum_{v=k}^{\infty} q(v)y(v+1) \right)^{\frac{1}{\alpha}}.$$

By (3.2)  $x(k) \leq -\infty$  which contradicts positivity of  $x(k)$ . Thus  $\lim_{k \rightarrow \infty} x(k) = 0$ .  $\square$

#### 4. OSCILLATION CRITERIA UNDER (1.4)

**Theorem 4.1.** Assume that (H1)-(H3) and (1.4) hold. Let  $p(k) = 0$ ,  $\Delta a(k) \geq 0$  and there is a positive sequence  $\rho(k)$  such that

$$\limsup_{k \rightarrow \infty} \sum_{s=k}^{\tau(k)-1} \left( \frac{1}{b(s)} \sum_{u=k}^{s-1} \frac{1}{a(u)} \sum_{v=k}^{u-1} q(v) \right)^{\frac{1}{\alpha}} > 1, \quad (4.1)$$

and

$$\sum_{s=k}^{\infty} \left( \frac{1}{b(s)} \sum_{u=s}^{\infty} \frac{1}{a(u)} \sum_{v=u}^{\infty} q(v) \right)^{\frac{1}{\alpha}} = \infty. \quad (4.2)$$

Then every solution of (1.1) is either oscillatory or tends to zero as  $k \rightarrow \infty$ .

*Proof.* Let  $x(k)$  be nonoscillatory solution of (1.2) and hence (1.1). Without loss of generality, we may assume that  $x(k)$  is eventually positive. Then,  $\exists$  an integer  $k_1 \geq k_0$  such that  $x(k) > 0$  and  $x(\tau(k)) > 0$  for all  $k \geq k_1$ . From (1.2)

$$L_3 x(k) = q(k)g(x(\tau(k))) > 0. \quad (4.3)$$

Now, we introduce the following two cases of possible nonoscillatory solutions of (4.3)

(I)  $x(k) > 0$ ,  $L_1 x(k) > 0$ ,  $L_2 x(k) > 0$ ,  $L_3 x(k) > 0$ .

(II)  $x(k) > 0$ ,  $L_1 x(k) < 0$ ,  $L_2 x(k) > 0$ ,  $L_3 x(k) > 0$ .

Case I: Summing (4.3) from  $k$  to  $i-1$  and also using the monotonicity of  $x(k)$ , we get

$$\begin{aligned} L_2 x(i) &\geq L_2 x(i) - L_2 x(k) \geq \sum_{s=k}^{i-1} q(s)g(x(\tau(s))) \\ &\geq g(x(\tau(k))) \sum_{s=k}^{i-1} q(s). \end{aligned}$$

Once more summing, we have

$$L_1 x(i) \geq g(x(\tau(k))) \sum_{s=k}^{i-1} \frac{1}{a(s)} \sum_{u=k}^{s-1} q(u).$$

Once again summing from  $k$  to  $i - 1$ , we obtain

$$x(i) \geq g^{\frac{1}{\alpha}}(x(\tau(k))) \sum_{s=k}^{i-1} \left( \frac{1}{b(s)} \sum_{u=k}^{s-1} \frac{1}{a(u)} \sum_{v=k}^{u-1} q(v) \right)^{\frac{1}{\alpha}}.$$

Setting  $i = \tau(k)$  and  $g(u) = u^\alpha$ , so we have

$$\begin{aligned} x(\tau(k)) &\geq x(\tau(k)) \sum_{s=k}^{\tau(k)-1} \left( \frac{1}{b(s)} \sum_{u=k}^{s-1} \frac{1}{a(u)} \sum_{v=k}^{u-1} q(v) \right)^{\frac{1}{\alpha}} \\ 1 &\geq \sum_{s=k}^{\tau(k)-1} \left( \frac{1}{b(s)} \sum_{u=k}^{s-1} \frac{1}{a(u)} \sum_{v=k}^{u-1} q(v) \right)^{\frac{1}{\alpha}}. \end{aligned}$$

Taking the lim sup on both side we get a contradiction to (4.1).

Case II: From the monotonicity of  $x(k) \exists$  a constant  $c \geq 0$  such that

$$\lim_{k \rightarrow \infty} x(k) = c.$$

We claim that  $c = 0$ , contrary suppose that  $c > 0$ . Then  $x(\tau(k)) \geq c$  for  $k \geq k_2 \geq k_1$ . Hence,

$$L_3 x(k) \geq g(c)q(k). \quad (4.4)$$

Summing (4.4) from  $k$  to  $\infty$ , we have

$$-L_2 x(k) \geq g(c) \sum_{s=k}^{\infty} q(s).$$

Again summing from  $k$  to  $\infty$ , gives

$$L_1 x(k) \geq g(c) \sum_{s=k}^{\infty} \frac{1}{a(s)} \sum_{u=s}^{\infty} q(u).$$

Finally summing the aforementioned inequality from  $k$  to  $\infty$ , we obtain

$$-x(k) \geq g^{\frac{1}{\alpha}}(c) \sum_{s=k}^{\infty} \left( \frac{1}{b(s)} \sum_{u=s}^{\infty} \frac{1}{a(u)} \sum_{v=u}^{\infty} q(v) \right)^{\frac{1}{\alpha}}.$$

By (4.2),  $\lim_{k \rightarrow \infty} x(k) = -\infty$ , which contradicts the positivity of  $x(k)$ . Thus  $\lim_{k \rightarrow \infty} x(k) = 0$ .

□

## 5. EXAMPLES

**Example 5.1.** Consider

$$\Delta \left( k^2 \Delta \left( \frac{k+1}{k+2} (\Delta x(k))^3 \right) \right) + \frac{1}{k+3} (\Delta(x(k+1)))^3 - 4k^5(k+1)^2 g(x(2k)), \quad k \geq 1 \quad (5.1)$$

Here,  $a(k) = k^2$ ,  $b(k) = \frac{k+1}{k+2}$ ,  $p(k) = \frac{1}{(k+3)}$ ,  $q(k) = 4k^5(k+1)^2$ ,  $\alpha = 3$ ,  $g(u) = u^\alpha$ ,  $\tau(k) = 2k$ . Now (1.5) can be written as

$$\Delta(k \Delta y(k)) + \frac{1}{k+2} y(k+1) = 0. \quad (5.2)$$

Equation (5.2) has a nonoscillatory solution  $y(k) = \frac{1}{k}$ . Now

$$\sum_{s=1}^{\infty} \frac{1}{a(s)} = \sum_{s=1}^{\infty} \frac{1}{s^2} < \infty, \quad \sum_{s=1}^{\infty} \frac{1}{b(s)} = \sum_{s=1}^{\infty} \frac{(s+2)}{s+1} < \infty,$$

Therefore (H1)-(H3) and (1.3) are fulfilled. Also

$$\limsup_{r \rightarrow \infty} \frac{2r^{\frac{2}{3}}}{(2r+1)^{\frac{1}{3}}} \sum_{s=1}^{2r-1} (s+2)^{\frac{1}{3}} s^{\frac{1}{3}} (2s+2)^{\frac{2}{3}} (s+1)^{\frac{1}{3}} > 1,$$

and

$$\sum_{s=1}^{\infty} \left( \frac{s+2}{s(s+1)} \sum_{u=s}^{\infty} \frac{(u+1)}{u} \sum_{v=u}^{\infty} v^3 (2v)^2 (v+1) \right)^{\frac{1}{3}} = \infty$$

Hence (3.1) and (3.2) hold. Thus all the conditions of Theorem 3.1 are satisfied. So every solution of (5.1) is oscillatory.

## 6. CONCLUSION

In this study, we use the conditions (1.3) and (1.4) to develop new oscillatory behavior of (1.1). The current study can be viewed as an advancement over the earlier findings. The main results were illustrated with examples.

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