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GENERALIZED α-ψ-φ-F-CONTRACTIVE MAPPINGS IN QUASI-b-METRIC-LIKE SPACES

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ABSTRACT. In this paper, we introduce some new generalized mappings in quasi-b-metric-like spaces and establish some fixed point theorems with concrete examples. Our results generalize fixed point results in the literature.

1. INTRODUCTION

In an attempt to generalize Banach's fixed point theorem, Czerwik [7] in 1993 introduced b-metric space as a generalization of metric spaces. Later, many authors proved existence of fixed points for generalized contractions under b-metric space setting. Similiarly, the notion of metric-like space was introduced by Harandi[8] in 2012 under which many fixed point results were proved. In 2014, Ansari [2] introduced the concept of C-class functions which covers a large class of contractive conditions, and many researchers derived results using C-class functions. Recently, Afshari *et al.* [1] proved some fixed point results for generalized α-ψ-Suzuki-contractions in quasi-b-metric-like spaces. In this paper some fixed point results are derived for generalized α-ψ-Suzuki-contractions in quasi-b-metric-like spaces via C-class functions.

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2. Preliminaries

Definition 2.1. [6] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u,v) = 0 \implies u = v$, for all $u, v \in X$;

(ii) $d(u,u) = 0$, for all $u \in X$;

(iii) $d(u,v) = d(v,u)$, for all $u, v \in X$;

(iv) $d(u,v) \leq s[d(u,w) + d(w,v)]$, for all $u, v, w \in X$.

Then, d is a b-metric on X and the pair (X, d) is called a b-metric space, and s is its coefficient (see [5, 17] for more information on b-metric spaces).

If the conditions (i), (iii) and (iv) in Definition 2.1 are satisfied, then the space (X, d) is called a b-metric-like space. See [13] for more information on fixed points for some mappings in b-metric-like spaces.

Remark 2.2. Every b-metric space is a b-metric-like space, but the converse is not true.

Definition 2.3. [15] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u,v) = d(v,u) = 0 \iff u = v$, for all $u, v \in X$;

(ii) $d(u,v) \leq s[d(u,w) + d(w,v)]$, for all $u, v, w \in X$.

Then, d is a quasi-b-metric on X and the pair (X, d) is called a quasi-b-metric space.

Definition 2.4. [12] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u,v) = d(v,u) = 0 \implies u = v$, for all $u, v \in X$;

(ii) $d(u,v) \leq s[d(u,w) + d(w,v)]$, for all $u, v, w \in X$.

Then the pair (X, d) is called a quasi-b-metric-like space (or a dislocated quasi-b-metric space).

Remark 2.5. All b-metric-like spaces and quasi-b-metric spaces are obviously quasi-b-metric-like spaces, but the converse is not true.

See [9] for a generalization of b-metric-like spaces.

Example 2.6. Let $X = \{a_1, a_2, a_3\}$ be any set of three distinct elements.

Define $d : X \times X \longrightarrow [0, \infty)$ by $d(u,v) = \begin{cases} 0 & \text{if } (u,v) = (a_3, a_3); \\ 2 & \text{if } (u,v) \in \{(a_1, a_1), (a_2, a_1)\}; \\ 0.5 & \text{if } (u,v) \in (a_1, a_2); \\ 0.25 & \text{otherwise.} \end{cases}$

Then (X, d) is a quasi-b-metric-like space with coefficient $s = 4$. Since $d(a_1, a_2) \neq d(a_2, a_1)$, it is clear that (X, d) is not a b-metric-like space; and since $d(a_1, a_1) \neq 0$, and $d(a_2, a_2) \neq 0$, it is also clear that (X, d) is not a quasi-b-metric space.

Definition 2.7. [1] Let (X, d) be a quasi-b-metric-like space. Let $\{u_n\}$ be a sequence in X and $u \in X$. The sequence $\{u_n\}$ converges to u if $\lim_{n \to \infty} d(u_n, u) = d(u, u) = \lim_{n \to \infty} d(u, u_n)$.

Definition 2.8. [1] Let (X, d) be a quasi-b-metric-like space. A sequence $\{u_n\}$ in X is said to be a left-Cauchy (respectively, right-Cauchy) sequence if $\lim_{n > m \to \infty} d(u_n, u_m)$ (respectively, if $\lim_{m > n \to \infty} d(u_n, u_m)$) exists and is finite. A sequence $\{u_n\}$ is said to be Cauchy if it is left-Cauchy and right-Cauchy.

Definition 2.9. [1] Let (X, d) be a quasi-b-metric-like space. We say that

(i) (X, d) is left-complete if each left-Cauchy sequence in X is convergent;

(ii) (X, d) is right-complete if each right-Cauchy sequence in X is convergent;

(iii) (X, d) is complete if and only if each Cauchy sequence in X is convergent.

Definition 2.10. [1] Let (X, d) be a quasi-b-metric-like space. A mapping $T : X \longrightarrow X$ is continuous if for any sequence $\{u_n\}$ in X converging to $u \in X$, the sequence $\{Tu_n\}$ converges to Tu.

For $s \geq 1$, let Ψ_s be the family of functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) $\forall t > 0$, $\sum\limits_{n=1}^{\infty} s^n \psi^n(t)$ is finite, where ψ^n is the nth iterate of ψ.

It is clear that if $\psi \in \Psi_s$, then $\psi(t) < t$, for all $t > 0$. For $s \geq 1$, we have $\psi^n(t) \leq s^n \psi^n(t)$, and since $\sum\limits_{n=1}^{\infty} s^n \psi^n(t) < \infty$, by comparison test, $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$, and so we can conclude that $\Psi_s \subseteq \Psi_1$.

Samet *et al.* [14] introduced the concept of α-admissible mappings as follows.

Definition 2.11. [14] Let $\alpha : X \times X \longrightarrow [0, \infty)$ be a function and $T : X \longrightarrow X$ be a mapping. Then T is α-admissible if $\alpha(u, v) \geq 1$ implies $\alpha(Tu, Tv) \geq 1$.

Afshari *et al.* [1] introduced the concepts of right-α-orbital admissible mappings and left-α-orbital admissible mappings.

Definition 2.12. [1] Let $\alpha : X \times X \longrightarrow [0, \infty)$ be a function and $T : X \longrightarrow X$ be a mapping.

(i) T is right-α-orbital admissible if $\alpha(u, Tu) \geq 1 \implies \alpha(Tu, T^2u) \geq 1$.

(ii) T is left-α-orbital admissible if $\alpha(Tu, u) \geq 1 \implies \alpha(T^2u, Tu) \geq 1$.

(iii) T is α-orbital admissible if T is both right-α-admissible and left-α-admissible.

The notion of α-ψ-contractive mappings was defined by Samet [14] in the following way.

Definition 2.13. [14] Let (X, d) be a metric space and $T : X \longrightarrow X$ be a given mapping. Then T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \longrightarrow [0, \infty)$ and $\psi \in \Psi_1$ such that $\alpha(u, v)d(Tu, Tv) \leq \psi(d(u, v))$, for all $u, v \in X$.

In 2008, Suzuki [16] proved the following theorem as a generalization of Banach contraction principle that characterizes metric completeness in which $\theta : [0, 1) \longrightarrow (\frac{1}{2}, 1]$ is a nondecreasing function defined by $\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ (1+r)^{-1} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$

Theorem 2.14. [16] *Let (X, d) be a complete metric space. Then every mapping T on X satisfying the following:*
$\exists\, r \in [0, 1)$ such that $\forall\, u, v \in X$, $\theta(r)d(u, Tu) \leq d(u, v) \implies d(Tu, Tv) \leq rd(u, v)$,
has a unique fixed point.

Using Suzuki method, Afshari *et al.* [1] proved some fixed point results for generalized α-ψ-Suzuki contractive mappings in the setting of quasi-b-metric-like spaces as follows.

Definition 2.15. [1] Let (X, d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-Suzuki-contractive mapping of type A if there exist $\alpha : X \times X \longrightarrow [0, \infty)$, $\psi \in \Psi_s$ and $r \in [0, 1)$ such that

(i) $\forall u, v \in X, \theta(r)d(u, Tu) \leq d(u, v)$ implies $\alpha(u, v)d(Tu, Tv) \leq \psi(M(u, v))$;

(ii) $\forall u, v \in X, \theta(r)d(Tu, u) \leq d(v, u)$ implies $\alpha(v, u)d(Tv, Tu) \leq \psi(M'(u, v))$,

where

$$M(u, v) = \max \left\{ d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tv)}{2s} \right\},$$

$$M'(u, v) = \max \left\{ d(v, u), d(Tu, u), d(Tv, v), \frac{d(Tv, u)}{2s} \right\}.$$

Example 2.16. [1] Let $X = [-1, 1]$ and let $T : X \longrightarrow X$ be defined by $T(u) = u/2$. Define $d : X \times X \longrightarrow [0, \infty)$ by $d(u, v) = |u - v|^2 + 3u^2 + 2v^2$. Then (X, d) is a quasi-b-metric-like space and T is an α-ψ-Suzuki-contractive mapping of type A.

Theorem 2.17. [1] *Let (X, d) be a complete quasi-b-metric-like space and $T : X \longrightarrow X$ be an α-ψ-Suzuki-contractive mapping of type A. Suppose also that T is α-orbital admissible, continuous and there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$. Then T has a fixed point $u \in X$ and $d(u, u) = 0$.*

The following is the definition of a C-class function introduced by Ansari [2]. Many researchers then developed fixed point results and best proximity results using C-class functions. For example, see [3, 4, 10].

Definition 2.18. [2] A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if for any $p, q \in [0, \infty)$, the following conditions hold:

(1) $F(p, q) \leq p$;

(2) $F(p, q) = p$ implies that either $p = 0$ or $q = 0$.

The family of all C-class functions is denoted by $\mathcal{C}$.

Example 2.19. [2] The following are some C-class functions:

(i) $F(p, q) = p - q$, for all $p, q \in [0, \infty)$.

(ii) $F(p, q) = mp$, for all $p, q \in [0, \infty)$ and $m \in (0, 1)$.

(iii) $F(p, q) = \frac{p}{(1+q)^r}$, for all $p, q \in [0, \infty)$ and $r \in (0, \infty)$.

(iv) $F(p, q) = \log(q + a^p)/(1 + q)$, for all $p, q \in [0, \infty)$ and $a > 1$.

Definition 2.20. [11] An ultra altering distance function is a continuous, nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$, for $t > 0$ and $\varphi(0) = 0$. The set of all ultra altering distance functions is denoted by Φ_U.

3. MAIN RESULTS

The following definition is proposed in this paper.

Definition 3.1. Let (X, d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-φ-F-contractive mapping of type A if there exist $\alpha : X \times X \longrightarrow [0, \infty)$, $\psi \in \Psi_s$, $\varphi \in \Phi_U$, $F \in \mathcal{C}$ and $r \in [0, 1)$ such that the following are satisfied:

(A1) $\forall u, v \in X, \theta(r)d(u, Tu) \leq d(u, v) \implies$
$\quad \alpha(u, v) M_A(u, v) \leq F(\psi(d(u, v)), \varphi(d(u, v)))$;

(A2) $\forall u, v \in X, \theta(r)d(Tu, u) \leq d(v, u) \implies$
$\quad \alpha(v, u) M_A(u, v) \leq F(\psi(d(v, u)), \varphi(d(v, u)))$,

where $M_A(u, v) = \max\{d(u, Tv), d(v, Tv), d(v, Tu), d(Tv, v)\}$.

We have now our first main result.

Lemma 3.2. *Let (X, d) be a complete quasi-b-metric-like space with coefficient s and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type A. If T is α-orbital admissible, and if there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$, then $\lim_{n \to \infty} d(u_n, u_{n+1}) = \lim_{n \to \infty} d(u_{n+1}, u_n) = 0$, where $u_k = T^k u_0$, for $k \in \mathbb{N}$.*

Proof. If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then the proof is complete. If not, then $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Since T is right-α-orbital admissible, it can be derived that $\alpha(u_0, u_1) = \alpha(u_0, Tu_0) \geq 1 \implies \alpha(Tu_0, Tu_1) = \alpha(u_1, u_2) \geq 1$. Then by induction we get that

$$\alpha(u_{n-1}, u_n) \geq 1, \forall n \in \mathbb{N}. \tag{3.1}$$

Similarly, since T is left-α-orbital admissible, it can also be derived that $\alpha(u_1, u_0) = \alpha(Tu_0, u_0) \geq 1 \implies \alpha(Tu_1, Tu_0) = \alpha(u_2, u_1) \geq 1$.

Inductively, we get that

$$\alpha(u_n, u_{n-1}) \geq 1, \forall n \in \mathbb{N}. \tag{3.2}$$

Since T is an α-ψ-φ-F-contractive mapping of type A, by taking $u = u_{n-1}$ and $v = u_n$ in (A1) of Definition 3.1, we find that $\theta(r) d(u_{n-1}, Tu_{n-1}) \leq d(u_{n-1}, u_n)$ implies

$$
\begin{aligned}
d(u_n, u_{n+1}) &\leq \alpha(u_{n-1}, u_n) d(u_n, u_{n+1}) \text{ by using (3.1)} \\
&\leq \alpha(u_{n-1}, u_n) \max\{d(u_{n-1}, u_{n+1}), d(u_n, u_{n+1}), d(u_n, u_n), d(u_{n+1}, u_n)\} \\
&= \alpha(u_{n-1}, u_n) \max\{d(u_{n-1}, Tu_n), d(u_n, Tu_n), d(u_n, Tu_{n-1}), d(Tu_n, u_n)\} \\
&= \alpha(u_{n-1}, u_n) M_A(u_{n-1}, u_n) \\
&\leq F(\psi(d(u_{n-1}, u_n)), \varphi(d(u_{n-1}, u_n))) \\
&\leq \psi(d(u_{n-1}, u_n)) \\
&< d(u_{n-1}, u_n).
\end{aligned}
$$

Therefore, $d(u_n, u_{n+1}) \leq \psi(d(u_{n-1}, u_n))$ and $d(u_n, u_{n+1}) < d(u_{n-1}, u_n)$, for all $n \in \mathbb{N}$. Since $d(u_n, u_{n+1}) \leq \psi(d(u_{n-1}, u_n))$ for all $n \in \mathbb{N}$, inductively, we get $d(u_n, u_{n+1}) \leq \psi^n(d(u_0, u_1))$ for all $n \in \mathbb{N}$. Therefore, $\lim_{n \to \infty} d(u_n, u_{n+1}) \leq \lim_{n \to \infty} \psi^n(d(u_0, u_1)) = 0$, since $\psi \in \Psi_1$. Thus

$$\lim_{n \to \infty} d(u_n, u_{n+1}) = 0.$$

Similarly, by taking $u = u_{n-1}$ and $v = u_n$ in (A2) of Definition 3.1, we find that $\theta(r) d(Tu_{n-1}, u_{n-1}) \leq d(u_n, u_{n-1})$ implies

$$
\begin{aligned}
d(u_{n+1}, u_n) &\leq \alpha(u_n, u_{n-1}) d(u_{n+1}, u_n) \text{ by using (3.2)} \\
&\leq \alpha(u_n, u_{n-1}) \max\{d(u_{n-1}, u_{n+1}), d(u_n, u_{n+1}), d(u_n, u_n), d(u_{n+1}, u_n)\} \\
&= \alpha(u_n, u_{n-1}) \max\{d(u_{n-1}, Tu_n), d(u_n, Tu_n), d(u_n, Tu_{n-1}), d(Tu_n, u_n)\} \\
&= \alpha(u_n, u_{n-1}) M_A(u_{n-1}, u_n) \\
&\leq F(\psi(d(u_n, u_{n-1})), \varphi(d(u_n, u_{n-1}))) \\
&\leq \psi(d(u_n, u_{n-1})) \\
&< d(u_n, u_{n-1}).
\end{aligned}
$$

Therefore, $d(u_{n+1}, u_n) \leq \psi(d(u_n, u_{n-1}))$ and $d(u_{n+1}, u_n) < d(u_n, u_{n-1})$, $\forall n \in \mathbb{N}$. Since $d(u_{n+1}, u_n) \leq \psi(d(u_n, u_{n-1}))$, for all $n \in \mathbb{N}$, inductively, we get that $d(u_{n+1}, u_n) \leq \psi^n(d(u_1, u_0))$, for all $n \in \mathbb{N}$.

Therefore, $\lim\limits_{n \longrightarrow \infty} d(u_{n+1}, u_n) \leq \lim\limits_{n \longrightarrow \infty} \psi^n(d(u_1, u_0)) = 0$.

$$\text{Thus, } \lim\limits_{n \longrightarrow \infty} d(u_{n+1}, u_n) = 0.$$

This completes the proof. $\hfill\square$

Theorem 3.3. *Let (X, d) be a complete quasi-b-metric-like space with coefficient s, and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type A and continuous. If T is α-orbital admissible, and if there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$, then there exists an element $u \in X$ which is a fixed point of T and $d(u, u) = 0$.*

Proof. We have $\lim\limits_{n \longrightarrow \infty} d(u_{n+1}, u_n) = 0$ and $\lim\limits_{n \longrightarrow \infty} d(u_n, u_{n+1}) = 0$, from Lemma 3.2. Now, we prove that the sequence $\{u_n\}$ is Cauchy. For $k \in \mathbb{N}$, we have

$$
\begin{aligned}
d(u_n, u_{n+k}) &\leq sd(u_n, u_{n+1}) + s^2 d(u_{n+1}, u_{n+2}) \cdots + s^k d(u_{n+k-1}, u_{n+k}) \\
&\leq \sum_{p=n}^{n+k-1} s^{p-n+1} \psi^p(d(u_0, u_1)) \\
&\leq \sum_{p=n}^{\infty} s^p \psi^p(d(u_0, u_1)) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Therefore, $\{u_n\}$ is right-Cauchy.

Similarly, $\{u_n\}$ is left-Cauchy, since we have

$$
\begin{aligned}
d(u_n, u_{n+k}) &\leq sd(u_{n+k}, u_{n+k-1}) + s^2 d(u_{n+k-1}, u_{n+k-2}) \cdots + s^k d(u_{n+1}, u_n) \\
&\leq \sum_{p=n}^{n+k-1} s^{n+k-p} \psi^p(d(u_1, u_0)) \\
&\leq \sum_{p=n}^{\infty} s^p \psi^p(d(u_1, u_0)) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Consequently, $\{u_n\}$ is Cauchy in (X, d) since it is both right-Cauchy and left-Cauchy. Since X is complete, there exists $u \in X$ such that

$$d(u, u) = \lim\limits_{n,m \to \infty} d(u_m, u_n) = \lim\limits_{n \to \infty} d(u_n, u) = \lim\limits_{n \to \infty} d(u, u_n) = \lim\limits_{n,m \to \infty} d(u_n, u_m) = 0.$$

By the continuity of T,

$$u = \lim\limits_{n \to \infty} u_{n+1} = \lim\limits_{n \to \infty} Tu_n = Tu.$$

This completes the proof. $\hfill\square$

We provide the following example as an illustration of Theorem 3.3.

Example 3.4. Let $X = \{0, 1, 2\}$ and define $d : X \times X \longrightarrow [0, \infty)$ by

$$
d(u, v) = \begin{cases} 0 & \text{if } (u, v) \in \{(1, 1), (2, 2)\}; \\ 1 & \text{if } (u, v) = (0, 1); \\ 2 & \text{if } (u, v) \in \{(0, 0), (1, 0)\}; \\ \frac{1}{4} & \text{elsewhere.} \end{cases}
$$

Then, (X, d) is a quasi-b-metric-like space with coefficient $s = 4$.

Define $T : X \longrightarrow X$ by $Tu = \begin{cases} 0 & \text{if } u = 0; \\ 2 & \text{if } u \in \{1, 2\}. \end{cases}$

Let

- $\alpha : X \times X \longrightarrow [0, \infty)$ be defined by $\alpha(u, v) = \begin{cases} 1 & \text{if } (u, v) = (2, 2); \\ \frac{1}{128} & \text{elsewhere,} \end{cases}$
- $\psi \in \Psi_4$ be defined by $\psi(t) = \frac{t}{8}, \forall t \geq 0$,
- $\varphi \in \Phi_U$ be defined by $\varphi(t) = t, \forall t \geq 0$,
- $F \in \mathcal{C}$ be defined by $F(p, q) = \frac{p}{2}, \forall p, q \in [0, \infty)$, and
- $r = 0$.

Then, T becomes a generalized α-ψ-φ-F-contractive mapping of type A. Here, all the conditions of Theorem 3.3 are satisfied, and 2 is a fixed point of T and $d(2, 2) = 0$.

Now, let us define a generalized α-ψ-φ-F-contractive mapping of type B.

Definition 3.5. Let (X, d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-φ-F-contractive mapping of type B if there exist $\alpha : X \times X \longrightarrow [0, \infty)$, $\psi \in \Psi_s$, $\varphi \in \Phi_U$, $F \in \mathcal{C}$ and $r \in [0, 1)$ such that the following conditions are satisfied:

(B1) $\forall u, v \in X, \theta(r) d(u, Tu) \leq d(u, v) \implies$
$\alpha(u, v) d(Tu, Tv) \leq F(\psi(M_B(u, v)), \varphi(M_B(u, v)))$;

(B2) $\forall u, v \in X, \theta(r) d(Tu, u) \leq d(v, u) \implies$
$\alpha(v, u) \, d(Tv, Tu) \leq F(\psi(M'_B(u, v)), \varphi(M'_B(u, v)))$,

where
$$M_B(u, v) = \max \left\{ d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tv)}{2s} \right\},$$
$$M'_B(u, v) = \max \left\{ d(v, u), d(Tu, u), d(Tv, v), \frac{d(Tv, u)}{2s} \right\}.$$

Lemma 3.6. *Let (X, d) be a complete quasi-b-metric-like space and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type B. If T is α-orbital admissible and if there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$, then $\lim\limits_{n \to \infty} d(u_n, u_{n+1}) = \lim\limits_{n \to \infty} d(u_{n+1}, u_n) = 0$, where $u_k = T^k u_0$, for $k \in \mathbb{N}$.*

Proof. If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then the proof is complete. If not, then $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Then by Lemma 3.2, $d(u_{n-1}, u_n) \geq 1$ and $d(u_n, u_{n-1}) \geq 1$ for all $n \in \mathbb{N}$. Since T is an α-ψ-φ-F-contractive mapping of type B, by taking $u = u_{n-1}$ and $v = u_n$ in (B1) of Definition 3.5, we find that $\theta(r) d(u_{n-1}, Tu_{n-1}) \leq d(u_{n-1}, u_n)$ implies

$d(u_n, u_{n+1}) \leq \alpha(u_{n-1}, u_n) d(u_n, u_{n+1})$ by (3.1)
$= \alpha(u_{n-1}, u_n) d(Tu_{n-1}, Tu_n)$
$\leq F(\psi(M_B(u_{n-1}, u_n)), \varphi(M(u_{n-1}, u_n)))$
$\leq \psi(M_B(u_{n-1}, u_n))$
$= \psi \left(\max \left\{ d(u_{n-1}, u_n), d(u_{n-1}, u_n), d(u_n, u_{n+1}), \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1})}{2s} \right\} \right)$
$= \psi \left(\max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}), \frac{d(u_{n-1}, u_n) + d(u_n, u_{n+1})}{2} \right\} \right)$
$= \psi \left(\max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}) \right\} \right).$

Thus, $d(u_n, u_{n+1}) \leq \psi \left(\max \left\{ d(u_{n-1}, u_n), d(u_n, u_{n+1}) \right\} \right).$ (3.3)

If $\max\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\} = d(u_n, u_{n+1})$, then (3.3) implies that $d(u_n, u_{n+1}) \leq \psi(d(u_n, u_{n+1})) < d(u_n, u_{n+1})$, which is a contradiction. So $\max\{d(u_{n-1}, u_n), d(u_n, u_{n+1})\} = d(u_{n-1}, u_n)$. Then we have

$$d(u_n, u_{n+1}) \leq \psi(d(u_{n-1}, u_n)) < d(u_{n-1}, u_n), \forall n \in \mathbb{N}.$$

Therefore, $\{d(u_n, u_{n+1})\}$ is a decreasing sequence and $d(u_n, u_{n+1}) \leq \psi(d(u_{n-1}, u_n))$, for all $n \in \mathbb{N}$. Then inductively we get that

$$d(u_n, u_{n+1}) \leq \psi^n(d(u_0, u_1)), \forall n \in \mathbb{N}.$$

Hence $\lim_{n \to \infty} d(u_n, u_{n+1}) \leq \lim_{n \to \infty} \psi^n(d(u_0, u_1)) = 0$, since $\psi \in \Psi_1$. So

$$\lim_{n \to \infty} d(u_n, u_{n+1}) = 0.$$

Similarly, by taking $u = u_{n-1}$ and $v = u_n$ in (B2) of Definition 3.5, we find that $\theta(r)d(Tu_{n-1}, u_{n-1}) \leq d(u_n, u_{n-1})$ implies

$$d(u_{n+1}, u_n) \leq \alpha(u_n, u_{n-1})d(u_{n+1}, u_n) \text{ by using (3.2)}$$

$$= \alpha(u_n, u_{n-1})d(Tu_n, Tu_{n-1})$$

$$\leq F(\psi(M'_B(u_{n-1}, u_n)), \varphi(M'(u_{n-1}, u_n)))$$

$$\leq \psi(M'_B(u_{n-1}, u_n))$$

$$= \psi\left(\max\left\{d(u_n, u_{n-1}), d(u_n, u_{n-1}), d(u_{n+1}, u_n), \frac{d(u_n, u_{n-1}) + d(u_{n+1}, u_n)}{2s}\right\}\right)$$

$$= \psi\left(\max\left\{d(u_n, u_{n-1}), d(u_{n+1}, u_n), \frac{d(u_n, u_{n-1}) + d(u_{n+1}, u_n)}{2}\right\}\right)$$

$$= \psi\left(\max\{d(u_n, u_{n-1}), d(u_{n+1}, u_n)\}\right).$$

$$\text{So, } d(u_{n+1}, u_n) \leq \psi\left(\max\{d(u_n, u_{n-1}), d(u_{n+1}, u_n)\}\right). \tag{3.4}$$

If $\max\{d(u_n, u_{n-1}), d(u_{n+1}, u_n)\} = d(u_{n+1}, u_n)$, then (3.4) implies that $d(u_{n+1}, u_n) \leq \psi(d(u_{n+1}, u_n)) < d(u_{n+1}, u_n)$, which is a contradiction.

Therefore, $\max\{d(u_n, u_{n-1}), d(u_{n+1}, u_n)\} = d(u_n, u_{n-1})$. Thus we have $d(u_{n+1}, u_n) \leq \psi(d(u_n, u_{n-1})) < d(u_n, u_{n-1})$ for all $n \in \mathbb{N}$. Therefore, $\{d(u_{n+1}, u_n)\}$ is a decreasing sequence and $d(u_{n+1}, u_n) \leq \psi(d(u_n, u_{n-1}))$, for all $n \in \mathbb{N}$. Then inductively we get that

$$d(u_{n+1}, u_n) \leq \psi^n(d(u_1, u_0)), \forall n \in \mathbb{N}.$$

Therefore, $\lim_{n \to \infty} d(u_{n+1}, u_n) \leq \lim_{n \to \infty} \psi^n(d(u_1, u_0)) = 0$, since $\psi \in \Psi_1$. So

$$\lim_{n \to \infty} d(u_{n+1}, u_n) = 0.$$

This completes the proof.   □

The following theorem can easily be proved as that of Theorem 3.3.

Theorem 3.7. *Let (X, d) be a complete quasi-b-metric-like space and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type B and continuous. If T is α-orbital admissible and if there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$, then there exists an element $u \in X$ which is a fixed point of T and $d(u, u) = 0$.*

We illustrate Theorem 3.7 with the following examples.

Example 3.8. The function $T : X \longrightarrow X$ defined on the quasi-b-metric-like space $X = \{0, 1, 2\}$ given in Example 3.4 is also a generalized α-ψ-φ-F-contractive mapping of type B, for the same α, ψ, φ, F and r given in Example 3.4. Here, all the conditions of Theorem 3.7 are satisfied, and 2 is a fixed point of T and $d(2, 2) = 0$.

Example 3.9. Let $T : X \longrightarrow X$ be the same function defined on the quasi-b-metric-like space $X = \{0, 1, 2\}$ given in Example 3.4.
Let

- $\alpha : X \times X \longrightarrow [0, \infty)$ be defined by $\alpha(u, v) = \begin{cases} 1 & \text{if } (u, v) = (2, 2); \\ \frac{1}{16} & \text{elsewhere,} \end{cases}$
- $\psi \in \Psi_4$ be defined by $\psi(t) = \frac{t}{8}$, $\forall t \geq 0$,
- $\varphi \in \Phi_U$ be defined by $\varphi(t) = t$, $\forall t \geq 0$,
- $F \in \mathcal{C}$ be defined by $F(p, q) = \frac{p}{2}$, $\forall p, q \in [0, \infty)$, and
- $r = 0$.

Then, T becomes a generalized α-ψ-φ-F-contractive mapping of type B, and not of type A. Here, all the conditions of Theorem 3.7 are satisfied, and 2 is a fixed point of T and $d(2, 2) = 0$.

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