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GENERALIZED α-ψ-φ-F-CONTRACTIVE MAPPINGS IN QUASI-b-METRIC-LIKE SPACES

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ABSTRACT. In this paper, we introduce some new generalized mappings in quasi-b-metric-like spaces and establish some fixed point theorems with concrete examples. Our results generalize fixed point results in the literature.

KEYWORDS: Fixed point, Quasi-b-metric-like space, Generalized α-ψ-Suzuki-contractive mapping, C-class function.
AMS Subject Classification: 47H10, 54H25.

1. Introduction

In an attempt to generalize Banach's fixed point theorem, Czerwik [7] in 1993 introduced b-metric space as a generalization of metric spaces. Later, many authors proved existence of fixed points for generalized contractions under b-metric space setting. Similiarly, the notion of metric-like space was introduced by Harandi[8] in 2012 under which many fixed point results were proved. In 2014, Ansari [2] introduced the concept of C-class functions which covers a large class of contractive conditions, and many researchers derived results using C-class functions. Recently, Afshari *et al.* [1] proved some fixed point results for generalized α-ψ-Suzuki-contractions in quasi-b-metric-like spaces. In this paper some fixed point results are derived for generalized α-ψ-Suzuki-contractions in quasi-b-metric-like spaces via C-class functions.

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2. Preliminaries

Definition 2.1. [6] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u, v) = 0 \implies u = v$, for all $u, v \in X$;
(ii) $d(u, u) = 0$, for all $u \in X$;
(iii) $d(u, v) = d(v, u)$, for all $u, v \in X$;
(iv) $d(u, v) \leq s[d(u, w) + d(w, v)]$, for all $u, v, w \in X$.

Then, d is a b-metric on X and the pair (X, d) is called a b-metric space, and s is its coefficient (see [5, 17] for more information on b-metric spaces).

If the conditions (i), (iii) and (iv) in Definition 2.1 are satisfied, then the space (X, d) is called a b-metric-like space. See [13] for more information on fixed points for some mappings in b-metric-like spaces.

Remark 2.2. Every b-metric space is a b-metric-like space, but the converse is not true.

Definition 2.3. [15] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u, v) = d(v, u) = 0 \iff u = v$, for all $u, v \in X$;
(ii) $d(u, v) \leq s[d(u, w) + d(w, v)]$, for all $u, v, w \in X$.

Then, d is a quasi-b-metric on X and the pair (X, d) is called a quasi-b-metric space.

Definition 2.4. [12] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a function $d : X \times X \longrightarrow [0, \infty)$ satisfies the following conditions:

(i) $d(u, v) = d(v, u) = 0 \implies u = v$, for all $u, v \in X$;
(ii) $d(u, v) \leq s[d(u, w) + d(w, v)]$, for all $u, v, w \in X$.

Then the pair (X, d) is called a quasi-b-metric-like space (or a dislocated quasi-b-metric space).

Remark 2.5. All b-metric-like spaces and quasi-b-metric spaces are obviously quasi-b-metric-like spaces, but the converse is not true.

See [9] for a generalization of b-metric-like spaces.

Example 2.6. Let $X = \{a_1, a_2, a_3\}$ be any set of three distinct elements.

Define $d : X \times X \longrightarrow [0, \infty)$ by
$$d(u, v) = \begin{cases} 0 & \text{if } (u, v) = (a_3, a_3); \\ 2 & \text{if } (u, v) \in \{(a_1, a_1), (a_2, a_1)\}; \\ 0.5 & \text{if } (u, v) \in (a_1, a_2); \\ 0.25 & \text{otherwise.} \end{cases}$$

Then (X, d) is a quasi-b-metric-like space with coefficient $s = 4$. Since $d(a_1, a_2) \neq d(a_2, a_1)$, it is clear that (X, d) is not a b-metric-like space; and since $d(a_1, a_1) \neq 0$, and $d(a_2, a_2) \neq 0$, it is also clear that (X, d) is not a quasi-b-metric space.

Definition 2.7. [1] Let (X, d) be a quasi-b-metric-like space. Let $\{u_n\}$ be a sequence in X and $u \in X$. The sequence $\{u_n\}$ converges to u if $\lim_{n \longrightarrow \infty} d(u_n, u) = d(u, u) = \lim_{n \longrightarrow \infty} d(u, u_n)$.

Definition 2.8. [1] Let (X, d) be a quasi-b-metric-like space. A sequence $\{u_n\}$ in X is said to be a left-Cauchy (respectively, right-Cauchy) sequence if $\lim_{n > m \longrightarrow \infty} d(u_n, u_m)$ (respectively, if $\lim_{m > n \longrightarrow \infty} d(u_n, u_m)$) exists and is finite. A sequence $\{u_n\}$ is said to be Cauchy if it is left-Cauchy and right-Cauchy.

Definition 2.9. [1] Let (X,d) be a quasi-b-metric-like space. We say that

(i) (X,d) is left-complete if each left-Cauchy sequence in X is convergent;
(ii) (X,d) is right-complete if each right-Cauchy sequence in X is convergent;
(iii) (X,d) is complete if and only if each Cauchy sequence in X is convergent.

Definition 2.10. [1] Let (X,d) be a quasi-b-metric-like space. A mapping $T: X \longrightarrow X$ is continuous if for any sequence $\{u_n\}$ in X converging to $u \in X$, the sequence $\{Tu_n\}$ converges to Tu.

For $s \geq 1$, let Ψ_s be the family of functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;
(ii) $\forall t > 0$, $\sum_{n=1}^{\infty} s^n \psi^n(t)$ is finite, where ψ^n is the n^{th} iterate of ψ.

It is clear that if $\psi \in \Psi_s$, then $\psi(t) < t$, for all $t > 0$. For $s \geq 1$, we have $\psi^n(t) \leq s^n \psi^n(t)$, and since $\sum_{n=1}^{\infty} s^n \psi^n(t) < \infty$, by comparison test, $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, and so we can conclude that $\Psi_s \subseteq \Psi_1$.

Samet *et al.* [14] introduced the concept of α-admissible mappings as follows.

Definition 2.11. [14] Let $\alpha : X \times X \longrightarrow [0,\infty)$ be a function and $T : X \longrightarrow X$ be a mapping. Then T is α-admissible if $\alpha(u,v) \geq 1$ implies $\alpha(Tu,Tv) \geq 1$.

Afshari *et al.* [1] introduced the concepts of right-α-orbital admissible mappings and left-α-orbital admissible mappings.

Definition 2.12. [1] Let $\alpha : X \times X \longrightarrow [0,\infty)$ be a function and $T : X \longrightarrow X$ be a mapping.

(i) T is right-α-orbital admissible if $\alpha(u,Tu) \geq 1 \implies \alpha(Tu,T^2u) \geq 1$.
(ii) T is left-α-orbital admissible if $\alpha(Tu,u) \geq 1 \implies \alpha(T^2u,Tu) \geq 1$.
(iii) T is α-orbital admissible if T is both right-α-admissible and left-α-admissible.

The notion of α-ψ-contractive mappings was defined by Samet [14] in the following way.

Definition 2.13. [14] Let (X,d) be a metric space and $T : X \longrightarrow X$ be a given mapping. Then T is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \longrightarrow [0,\infty)$ and $\psi \in \Psi_1$ such that $\alpha(u,v)d(Tu,Tv) \leq \psi(d(u,v))$, for all $u,v \in X$.

In 2008, Suzuki [16] proved the following theorem as a generalization of Banach contraction principle that characterizes metric completeness in which $\theta : [0,1) \longrightarrow (\frac{1}{2},1]$ is a nondecreasing function defined by
$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ (1+r)^{-1} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Theorem 2.14. [16] *Let (X,d) be a complete metric space. Then every mapping T on X satisfying the following:*
$$\exists\, r \in [0,1) \text{ such that } \forall\, u,v \in X,\ \theta(r)d(u,Tu) \leq d(u,v) \implies d(Tu,Tv) \leq r d(u,v),$$
has a unique fixed point.

Using Suzuki method, Afshari *et al.* [1] proved some fixed point results for generalized α-ψ-Suzuki contractive mappings in the setting of quasi-b-metric-like spaces as follows.

Definition 2.15. [1] Let (X, d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-Suzuki-contractive mapping of type A if there exist $\alpha : X \times X \longrightarrow [0, \infty)$, $\psi \in \Psi_s$ and $r \in [0, 1)$ such that

(i) $\forall\, u, v \in X, \theta(r)d(u, Tu) \leq d(u, v)$ implies $\alpha(u, v)d(Tu, Tv) \leq \psi(M(u, v))$;
(ii) $\forall\, u, v \in X, \theta(r)d(Tu, u) \leq d(v, u)$ implies $\alpha(v, u)d(Tv, Tu) \leq \psi(M'(u, v))$,

where

$$M(u, v) = \max\left\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tv)}{2s}\right\},$$
$$M'(u, v) = \max\left\{d(v, u), d(Tu, u), d(Tv, v), \frac{d(Tv, u)}{2s}\right\}.$$

Example 2.16. [1] Let $X = [-1, 1]$ and let $T : X \longrightarrow X$ be defined by $T(u) = u/2$. Define $d : X \times X \longrightarrow [0, \infty)$ by $d(u, v) = |u - v|^2 + 3u^2 + 2v^2$. Then (X, d) is a quasi-b-metric-like space and T is an α-ψ-Suzuki-contractive mapping of type A.

Theorem 2.17. [1] *Let (X, d) be a complete quasi-b-metric-like space and $T : X \longrightarrow X$ be an α-ψ-Suzuki-contractive mapping of type A. Suppose also that T is α-orbital admissible, continuous and there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$. Then T has a fixed point $u \in X$ and $d(u, u) = 0$.*

The following is the definition of a C-class function introduced by Ansari [2]. Many researchers then developed fixed point results and best proximity results using C-class functions. For example, see [3, 4, 10].

Definition 2.18. [2] A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if for any $p, q \in [0, \infty)$, the following conditions hold:

(1) $F(p, q) \leq p$;
(2) $F(p, q) = p$ implies that either $p = 0$ or $q = 0$.

The family of all C-class functions is denoted by $\mathcal{C}$.

Example 2.19. [2] The following are some C-class functions:

(i) $F(p, q) = p - q$, for all $p, q \in [0, \infty)$.
(ii) $F(p, q) = mp$, for all $p, q \in [0, \infty)$ and $m \in (0, 1)$.
(iii) $F(p, q) = \frac{p}{(1+q)^r}$, for all $p, q \in [0, \infty)$ and $r \in (0, \infty)$.
(iv) $F(p, q) = \log(q + a^p)/(1 + q)$, for all $p, q \in [0, \infty)$ and $a > 1$.

Definition 2.20. [11] An ultra altering distance function is a continuous, nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$, for $t > 0$ and $\varphi(0) = 0$. The set of all ultra altering distance functions is denoted by Φ_U.

3. Main results

The following definition is proposed in this paper.

Definition 3.1. Let (X, d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-φ-F-contractive mapping of type A if there exist $\alpha : X \times X \longrightarrow [0, \infty)$, $\psi \in \Psi_s$, $\varphi \in \Phi_U$, $F \in \mathcal{C}$ and $r \in [0, 1)$ such that the following are satisfied:

(A1) $\forall u, v \in X, \theta(r)d(u, Tu) \leq d(u, v) \implies$
$\alpha(u, v)\ M_A(u, v) \leq F\left(\psi(d(u, v)), \varphi(d(u, v))\right)$;

(A2) $\forall u, v \in X, \theta(r)d(Tu, u) \leq d(v, u) \implies$
$\alpha(v, u)\ M_A(u, v) \leq F\left(\psi(d(v, u)), \varphi(d(v, u))\right)$,

where $M_A(u, v) = \max\{d(u, Tv), d(v, Tv), d(v, Tu), d(Tv, v)\}$.

We have now our first main result.

Lemma 3.2. *Let (X, d) be a complete quasi-b-metric-like space with coefficient s and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type A. If T is α-orbital admissible, and if there exists $u_0 \in X$ such that $\alpha(u_0, Tu_0) \geq 1$ and $\alpha(Tu_0, u_0) \geq 1$, then $\lim_{n\to\infty} d(u_n, u_{n+1}) = \lim_{n\to\infty} d(u_{n+1}, u_n) = 0$, where $u_k = T^k u_0$, for $k \in \mathbb{N}$.*

Proof. If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then the proof is complete. If not, then $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Since T is right-α-orbital admissible, it can be derived that $\alpha(u_0, u_1) = \alpha(u_0, Tu_0) \geq 1 \implies \alpha(Tu_0, Tu_1) = \alpha(u_1, u_2) \geq 1$. Then by induction we get that

$$\alpha(u_{n-1}, u_n) \geq 1, \forall n \in \mathbb{N}. \tag{3.1}$$

Similarly, since T is left-α-orbital admissible, it can also be derived that $\alpha(u_1, u_0) = \alpha(Tu_0, u_0) \geq 1 \implies \alpha(Tu_1, Tu_0) = \alpha(u_2, u_1) \geq 1$.

Inductively, we get that

$$\alpha(u_n, u_{n-1}) \geq 1, \forall n \in \mathbb{N}. \tag{3.2}$$

Since T is an α-ψ-φ-F-contractive mapping of type A, by taking $u = u_{n-1}$ and $v = u_n$ in (A1) of Definition 3.1, we find that $\theta(r)d(u_{n-1}, Tu_{n-1}) \leq d(u_{n-1}, u_n)$ implies

$$\begin{aligned}
d(u_n, u_{n+1}) &\leq \alpha(u_{n-1}, u_n) d(u_n, u_{n+1}) \text{ by using (3.1)} \\
&\leq \alpha(u_{n-1}, u_n) \max\{d(u_{n-1}, u_{n+1}), d(u_n, u_{n+1}), d(u_n, u_n), d(u_{n+1}, u_n)\} \\
&= \alpha(u_{n-1}, u_n) \max\{d(u_{n-1}, Tu_n), d(u_n, Tu_n), d(u_n, Tu_{n-1}), d(Tu_n, u_n)\} \\
&= \alpha(u_{n-1}, u_n) M_A(u_{n-1}, u_n) \\
&\leq F\left(\psi(d(u_{n-1}, u_n)), \varphi(d(u_{n-1}, u_n))\right) \\
&\leq \psi(d(u_{n-1}, u_n)) \\
&< d(u_{n-1}, u_n).
\end{aligned}$$

Therefore, $d(u_n, u_{n+1}) \leq \psi(d(u_{n-1}, u_n))$ and $d(u_n, u_{n+1}) < d(u_{n-1}, u_n)$, for all $n \in \mathbb{N}$. Since $d(u_n, u_{n+1}) \leq \psi(d(u_{n-1}, u_n))$ for all $n \in \mathbb{N}$, inductively, we get $d(u_n, u_{n+1}) \leq \psi^n(d(u_0, u_1))$ for all $n \in \mathbb{N}$. Therefore, $\lim_{n\longrightarrow\infty} d(u_n, u_{n+1}) \leq \lim_{n\longrightarrow\infty} \psi^n(d(u_0, u_1)) = 0$, since $\psi \in \Psi_1$. Thus

$$\lim_{n\longrightarrow\infty} d(u_n, u_{n+1}) = 0.$$

Similarly, by taking $u = u_{n-1}$ and $v = u_n$ in (A2) of Definition 3.1, we find that $\theta(r)d(Tu_{n-1}, u_{n-1}) \leq d(u_n, u_{n-1})$ implies

$$\begin{aligned}
d(u_{n+1}, u_n) &\leq \alpha(u_n, u_{n-1}) d(u_{n+1}, u_n) \text{ by using (3.2)} \\
&\leq \alpha(u_n, u_{n-1}) \max\{d(u_{n-1}, u_{n+1}), d(u_n, u_{n+1}), d(u_n, u_n), d(u_{n+1}, u_n)\} \\
&= \alpha(u_n, u_{n-1}) \max\{d(u_{n-1}, Tu_n), d(u_n, Tu_n), d(u_n, Tu_{n-1}), d(Tu_n, u_n)\} \\
&= \alpha(u_n, u_{n-1}) M_A(u_{n-1}, u_n) \\
&\leq F\left(\psi(d(u_n, u_{n-1})), \varphi(d(u_n, u_{n-1}))\right) \\
&\leq \psi(d(u_n, u_{n-1})) \\
&< d(u_n, u_{n-1}).
\end{aligned}$$

Therefore, $d(u_{n+1}, u_n) \leq \psi(d(u_n, u_{n-1}))$ and $d(u_{n+1}, u_n) < d(u_n, u_{n-1})$, $\forall n \in \mathbb{N}$. Since $d(u_{n+1}, u_n) \leq \psi(d(u_n, u_{n-1}))$, for all $n \in \mathbb{N}$, inductively, we get that $d(u_{n+1}, u_n) \leq \psi^n(d(u_1, u_0))$, for all $n \in \mathbb{N}$.

Therefore, $\lim_{n\longrightarrow\infty} d(u_{n+1},u_n) \le \lim_{n\longrightarrow\infty} \psi^n(d(u_1,u_0)) = 0$.

$$\text{Thus, } \lim_{n\longrightarrow\infty} d(u_{n+1},u_n) = 0.$$

This completes the proof. □

Theorem 3.3. *Let (X,d) be a complete quasi-b-metric-like space with coefficient s, and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type A and continuous. If T is α-orbital admissible, and if there exists $u_0 \in X$ such that $\alpha(u_0,Tu_0) \ge 1$ and $\alpha(Tu_0,u_0) \ge 1$, then there exists an element $u \in X$ which is a fixed point of T and $d(u,u) = 0$.*

Proof. We have $\lim_{n\longrightarrow\infty} d(u_{n+1},u_n) = 0$ and $\lim_{n\longrightarrow\infty} d(u_n,u_{n+1}) = 0$, from Lemma 3.2. Now, we prove that the sequence $\{u_n\}$ is Cauchy. For $k \in \mathbb{N}$, we have

$$\begin{aligned} d(u_n,u_{n+k}) &\le sd(u_n,u_{n+1}) + s^2 d(u_{n+1},u_{n+2}) \cdots + s^k d(u_{n+k-1},u_{n+k}) \\ &\le \sum_{p=n}^{n+k-1} s^{p-n+1}\psi^p(d(u_0,u_1)) \\ &\le \sum_{p=n}^{\infty} s^p\psi^p(d(u_0,u_1)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Therefore, $\{u_n\}$ is right-Cauchy.
Similarly, $\{u_n\}$ is left-Cauchy, since we have

$$\begin{aligned} d(u_n,u_{n+k}) &\le sd(u_{n+k},u_{n+k-1}) + s^2 d(u_{n+k-1},u_{n+k-2}) \cdots + s^k d(u_{n+1},u_n) \\ &\le \sum_{p=n}^{n+k-1} s^{n+k-p}\psi^p(d(u_1,u_0)) \\ &\le \sum_{p=n}^{\infty} s^p\psi^p(d(u_1,u_0)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Consequently, $\{u_n\}$ is Cauchy in (X,d) since it is both right-Cauchy and left-Cauchy. Since X is complete, there exists $u \in X$ such that

$$d(u,u) = \lim_{n,m\to\infty} d(u_m,u_n) = \lim_{n\to\infty} d(u_n,u) = \lim_{n\to\infty} d(u,u_n) = \lim_{n,m\to\infty} d(u_n,u_m) = 0.$$

By the continuity of T,

$$u = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} Tu_n = Tu.$$

This completes the proof. □

We provide the following example as an illustration of Theorem 3.3.

Example 3.4. Let $X = \{0,1,2\}$ and define $d : X \times X \longrightarrow [0,\infty)$ by

$$d(u,v) = \begin{cases} 0 & \text{if } (u,v) \in \{(1,1),(2,2)\}; \\ 1 & \text{if } (u,v) = (0,1); \\ 2 & \text{if } (u,v) \in \{(0,0),(1,0)\}; \\ \frac{1}{4} & \text{elsewhere.} \end{cases}$$

Then, (X,d) is a quasi-b-metric-like space with coefficient $s = 4$.

Define $T : X \longrightarrow X$ by $Tu = \begin{cases} 0 & \text{if } u = 0; \\ 2 & \text{if } u \in \{1,2\}. \end{cases}$

Let

- $\alpha : X \times X \longrightarrow [0,\infty)$ be defined by $\alpha(u,v) = \begin{cases} 1 & \text{if } (u,v) = (2,2); \\ \frac{1}{128} & \text{elsewhere,} \end{cases}$
- $\psi \in \Psi_4$ be defined by $\psi(t) = \frac{t}{8}$, $\forall t \geq 0$,
- $\varphi \in \Phi_U$ be defined by $\varphi(t) = t$, $\forall t \geq 0$,
- $F \in \mathcal{C}$ be defined by $F(p,q) = \frac{p}{2}$, $\forall p, q \in [0,\infty)$, and
- $r = 0$.

Then, T becomes a generalized α-ψ-φ-F-contractive mapping of type A. Here, all the conditions of Theorem 3.3 are satisfied, and 2 is a fixed point of T and $d(2,2) = 0$.

Now, let us define a generalized α-ψ-φ-F-contractive mapping of type B.

Definition 3.5. Let (X,d) be a quasi-b-metric-like space with coefficient s. Then $T : X \longrightarrow X$ is a generalized α-ψ-φ-F-contractive mapping of type B if there exist $\alpha : X \times X \longrightarrow [0,\infty)$, $\psi \in \Psi_s$, $\varphi \in \Phi_U$, $F \in \mathcal{C}$ and $r \in [0,1)$ such that the following conditions are satisfied:

(B1) $\forall\, u,v \in X, \theta(r)d(u,Tu) \leq d(u,v) \implies$
$$\alpha(u,v)d(Tu,Tv) \leq F(\psi(M_B(u,v)), \varphi(M_B(u,v)));$$

(B2) $\forall\, u,v \in X, \theta(r)d(Tu,u) \leq d(v,u) \implies$
$$\alpha(v,u)\; d(Tv,Tu) \leq F\left(\psi(M'_B(u,v)), \varphi(M'_B(u,v))\right),$$

where
$$M_B(u,v) = \max\left\{d(u,v), d(u,Tu), d(v,Tv), \frac{d(u,Tv)}{2s}\right\},$$
$$M'_B(u,v) = \max\left\{d(v,u), d(Tu,u), d(Tv,v), \frac{d(Tv,u)}{2s}\right\}.$$

Lemma 3.6. *Let (X,d) be a complete quasi-b-metric-like space and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type B. If T is α-orbital admissible and if there exists $u_0 \in X$ such that $\alpha(u_0,Tu_0) \geq 1$ and $\alpha(Tu_0,u_0) \geq 1$, then $\lim_{n\to\infty} d(u_n,u_{n+1}) = \lim_{n\to\infty} d(u_{n+1},u_n) = 0$, where $u_k = T^k u_0$, for $k \in \mathbb{N}$.*

Proof. If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then the proof is complete. If not, then $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Then by Lemma 3.2, $d(u_{n-1},u_n) \geq 1$ and $d(u_n,u_{n-1}) \geq 1$ for all $n \in \mathbb{N}$. Since T is an α-ψ-φ-F-contractive mapping of type B, by taking $u = u_{n-1}$ and $v = u_n$ in (B1) of Definition 3.5, we find that $\theta(r)d(u_{n-1},Tu_{n-1}) \leq d(u_{n-1},u_n)$ implies

$$\begin{aligned}
d(u_n,u_{n+1}) &\leq \alpha(u_{n-1},u_n)d(u_n,u_{n+1}) \text{ by (3.1)} \\
&= \alpha(u_{n-1},u_n)d(Tu_{n-1},Tu_n) \\
&\leq F(\psi(M_B(u_{n-1},u_n)), \varphi(M(u_{n-1},u_n))) \\
&\leq \psi(M_B(u_{n-1},u_n)) \\
&= \psi\left(\max\left\{d(u_{n-1},u_n), d(u_{n-1},u_n), d(u_n,u_{n+1}), \frac{d(u_{n-1},u_n)+d(u_n,u_{n+1})}{2s}\right\}\right) \\
&= \psi\left(\max\left\{d(u_{n-1},u_n), d(u_n,u_{n+1}), \frac{d(u_{n-1},u_n)+d(u_n,u_{n+1})}{2}\right\}\right) \\
&= \psi\left(\max\left\{d(u_{n-1},u_n), d(u_n,u_{n+1})\right\}\right).
\end{aligned}$$

$$\text{Thus, } d(u_n,u_{n+1}) \leq \psi\left(\max\left\{d(u_{n-1},u_n), d(u_n,u_{n+1})\right\}\right). \tag{3.3}$$

If $\max\{d(u_{n-1},u_n), d(u_n,u_{n+1})\} = d(u_n,u_{n+1})$, then (3.3) implies that $d(u_n,u_{n+1}) \leq \psi(d(u_n,u_{n+1})) < d(u_n,u_{n+1})$, which is a contradiction.
So $\max\{d(u_{n-1},u_n), d(u_n,u_{n+1})\} = d(u_{n-1},u_n)$. Then we have

$$d(u_n,u_{n+1}) \leq \psi(d(u_{n-1},u_n)) < d(u_{n-1},u_n), \forall n \in \mathbb{N}.$$

Therefore, $\{d(u_n,u_{n+1})\}$ is a decreasing sequence and $d(u_n,u_{n+1}) \leq \psi(d(u_{n-1},u_n))$, for all $n \in \mathbb{N}$. Then inductively we get that

$$d(u_n,u_{n+1}) \leq \psi^n(d(u_0,u_1)), \forall n \in \mathbb{N}.$$

Hence $\lim_{n\longrightarrow\infty} d(u_n,u_{n+1}) \leq \lim_{n\longrightarrow\infty} \psi^n(d(u_0,u_1)) = 0$, since $\psi \in \Psi_1$. So

$$\lim_{n\longrightarrow\infty} d(u_n,u_{n+1}) = 0.$$

Similarly, by taking $u = u_{n-1}$ and $v = u_n$ in (B2) of Definition 3.5, we find that $\theta(r)d(Tu_{n-1},u_{n-1}) \leq d(u_n,u_{n-1})$ implies

$$\begin{aligned}
d(u_{n+1},u_n) &\leq \alpha(u_n,u_{n-1})d(u_{n+1},u_n) \quad \text{by using (3.2)} \\
&= \alpha(u_n,u_{n-1})d(Tu_n,Tu_{n-1}) \\
&\leq F(\psi(M'_B(u_{n-1},u_n)), \varphi(M'(u_{n-1},u_n))) \\
&\leq \psi(M'_B(u_{n-1},u_n)) \\
&= \psi\left(\max\left\{d(u_n,u_{n-1}), d(u_n,u_{n-1}), d(u_{n+1},u_n), \frac{d(u_n,u_{n-1})+d(u_{n+1},u_n)}{2s}\right\}\right) \\
&= \psi\left(\max\left\{d(u_n,u_{n-1}), d(u_{n+1},u_n), \frac{d(u_n,u_{n-1})+d(u_{n+1},u_n)}{2}\right\}\right) \\
&= \psi\left(\max\{d(u_n,u_{n-1}), d(u_{n+1},u_n)\}\right).
\end{aligned}$$

$$\text{So, } d(u_{n+1},u_n) \leq \psi\left(\max\{d(u_n,u_{n-1}), d(u_{n+1},u_n)\}\right). \tag{3.4}$$

If $\max\{d(u_n,u_{n-1}), d(u_{n+1},u_n)\} = d(u_{n+1},u_n)$, then (3.4) implies that $d(u_{n+1},u_n) \leq \psi(d(u_{n+1},u_n)) < d(u_{n+1},u_n)$, which is a contradiction.

Therefore, $\max\{d(u_n,u_{n-1}), d(u_{n+1},u_n)\} = d(u_n,u_{n-1})$. Thus we have $d(u_{n+1},u_n) \leq \psi(d(u_n,u_{n-1})) < d(u_n,u_{n-1})$ for all $n \in \mathbb{N}$. Therefore, $\{d(u_{n+1},u_n)\}$ is a decreasing sequence and $d(u_{n+1},u_n) \leq \psi(d(u_n,u_{n-1}))$, for all $n \in \mathbb{N}$. Then inductively we get that

$$d(u_{n+1},u_n) \leq \psi^n(d(u_1,u_0)), \forall n \in \mathbb{N}.$$

Therefore, $\lim_{n\longrightarrow\infty} d(u_{n+1},u_n) \leq \lim_{n\longrightarrow\infty} \psi^n(d(u_1,u_0)) = 0$, since $\psi \in \Psi_1$. So

$$\lim_{n\longrightarrow\infty} d(u_{n+1},u_n) = 0.$$

This completes the proof. □

The following theorem can easily be proved as that of Theorem 3.3.

Theorem 3.7. *Let (X,d) be a complete quasi-b-metric-like space and $T : X \to X$ be a generalized α-ψ-φ-F-contractive mapping of type B and continuous. If T is α-orbital admissible and if there exists $u_0 \in X$ such that $\alpha(u_0,Tu_0) \geq 1$ and $\alpha(Tu_0,u_0) \geq 1$, then there exists an element $u \in X$ which is a fixed point of T and $d(u,u) = 0$.*

We illustrate Theorem 3.7 with the following examples.

Example 3.8. The function $T : X \longrightarrow X$ defined on the quasi-b-metric-like space $X = \{0, 1, 2\}$ given in Example 3.4 is also a generalized α-ψ-φ-F-contractive mapping of type B, for the same α, ψ, φ, F and r given in Example 3.4. Here, all the conditions of Theorem 3.7 are satisfied, and 2 is a fixed point of T and $d(2, 2) = 0$.

Example 3.9. Let $T : X \longrightarrow X$ be the same function defined on the quasi-b-metric-like space $X = \{0, 1, 2\}$ given in Example 3.4.
Let

- $\alpha : X \times X \longrightarrow [0, \infty)$ be defined by $\alpha(u, v) = \begin{cases} 1 & \text{if } (u, v) = (2, 2); \\ \frac{1}{16} & \text{elsewhere,} \end{cases}$
- $\psi \in \Psi_4$ be defined by $\psi(t) = \frac{t}{8}$, $\forall t \geq 0$,
- $\varphi \in \Phi_U$ be defined by $\varphi(t) = t$, $\forall t \geq 0$,
- $F \in \mathcal{C}$ be defined by $F(p, q) = \frac{p}{2}$, $\forall p, q \in [0, \infty)$, and
- $r = 0$.

Then, T becomes a generalized α-ψ-φ-F-contractive mapping of type B, and not of type A. Here, all the conditions of Theorem 3.7 are satisfied, and 2 is a fixed point of T and $d(2, 2) = 0$.

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